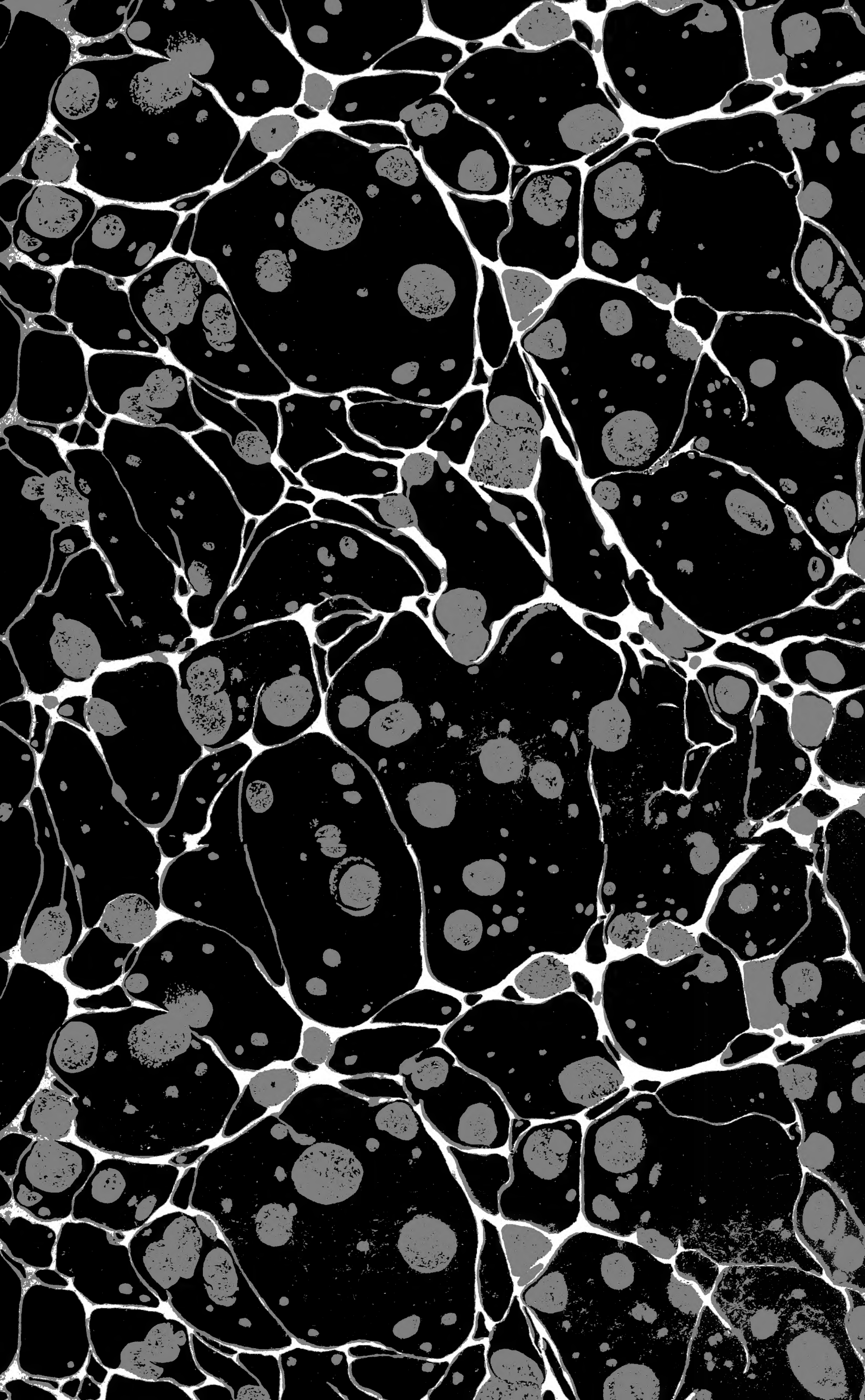

THE

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AND

JOURNAL OF SCIENCE.

CONDUCTED BY

LORD KELVIN, G.C.V.O. D.C.L. LL.D. F.R.S. &c.
JOHN JOLY, M.A. D.Sc. F.R.S. F.G.S.
AND
WILLIAM FRANCIS, Ph.D. F.L.S. F.R.A.S. F.C.S.

"Nec aranearum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex alienis libamus ut apes." Just. Lips. *Polit.* lib. i. cap. 1. Not.

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"Meditationis est perscrutari occulta; contemplationis est admirari perspicua .... Admiratio generat quæstionem, quæstio investigationem, investigatio inventionem."—*Hugo de S. Victore.*

——"Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus ferrugine condat,
Quid toties diros cogat flagrare cometas,
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu."

J. B. Pinelli ad Mazonium.

ALERE FLAMMAM.

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THE
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[SIXTH SERIES.]

JANUARY 1902.

I. *On the Clustering of Gravitational Matter in any part of the Universe.* *By* Lord KELVIN, *G.C.V.O., F.R.S.*[*]

GRAVITATIONAL matter, according to our ideas of universal gravitation, would be all matter. Now is there any matter which is not subject to the law of gravitation? I think I may say with absolute decision that there is. We are all convinced, with our President, that ether is matter, but we are forced to say that the properties of molar matter are not to be looked for in ether as generally known to us by action resulting from force between atoms and matter, ether and ether, and atoms of matter and ether. Here I am illogical when I say between matter and ether, as if ether were not matter. It is to avoid an illogical phraseology that I use the title 'gravitational matter.' Many years ago I gave strong reason to feel certain that ether was outside the law of gravitation. We need not absolutely exclude, as an idea, the possibility of there being a portion of space occupied by ether beyond which there is absolute vacuum—no ether and no matter. We admit that that is something that one could think of; but I do not believe any living scientific man considers it in the slightest degree probable that there is a boundary around our universe beyond which there is no ether and no matter. Well, if ether extends through all space, then it is certain that ether cannot be subject to the law of mutual gravitation between its parts, because if it were subject to

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[*] Communicated by the Author, having been read before the British Association at the Glasgow meeting.

mutual attraction between its parts its equilibrium would be unstable, unless it were infinitely incompressible. But here, again, I am reminded of the critical character of the ground on which we stand in speaking of properties of matter beyond what we see or feel by experiment. I am afraid I must here express a view different from that which Professor Rücker announced in his Address, when he said that continuity of matter implied absolute resistance to condensation. We have no right to bar condensation as a property of ether. While admitting ether not to have any atomic structure, it is postulated as a material which performs functions of which we know something, and which may have properties allowing it to perform other functions of which we are not yet cognisant. If we consider ether to be matter, we postulate that it has rigidity enough for the vibrations of light, but we have no right to say that it is absolutely incompressible. We must admit that sufficiently great pressure all round could condense the ether in a given space, allowing the ether in surrounding space to come in towards the ideal shrinking surface. When I say that ether must be outside the law of gravitation, I assume that it is not infinitely incompressible. I admit that if it were infinitely incompressible, it might be subject to the law of mutual gravitation between its parts; but to my mind it seems infinitely improbable that ether is infinitely incompressible, and it appears more consistent with the analogies of the known properties of molar matter, which should be our guides, to suppose that ether has not the quality of exerting an infinitely great force against compressing action of gravitation. Hence, if we assume that it extends through all space, ether must be outside the law of gravitation—that is to say, truly imponderable. I remember the contempt and self-complacent compassion with which sixty years ago—I myself, I am afraid—and most of the teachers of that time looked upon the ideas of the elderly people who went before us, who spoke of 'the imponderables.' I fear that in this, as in a great many other things in science, we have to hark back to the dark ages of fifty, sixty, or a hundred years ago, and that we must admit there is something which we cannot refuse to call matter, but which is not subject to the Newtonian law of gravitation. That the sun, stars, planets, and meteoric stones are all of them ponderable matter is true, but the title of my paper implies that there is something else. Ether is not any part of the subject of this paper: what we are concerned with is gravitational matter, ponderable matter.

Ether we relegate, not to a limbo of imponderables, but to distinct species of matter which have inertia, rigidity, elasticity, compressibility, but not heaviness. In a paper I have already published I gave strong reasons for limiting to a definite amount the quantity of matter in space known to astronomers. I can scarcely avoid using the word 'universe,' but I mean our universe, which may be a very small affair after all, occupying a very small portion of all the space in which there is ponderable matter.

Supposing a sphere of radius $3{\cdot}09.10^{16}$ kilometres (being the distance at which a star must be to have parallax $0''{\cdot}001$) to have within it, uniformly distributed through it, a quantity of matter equal to one thousand million times the sun's mass, the velocity acquired by a body placed originally at rest at the surface would, in five million years, be about 20 kilometres per second, and in twenty-five million years would be 108 kilometres per second (if the acceleration remained sensibly constant for so long a time). Hence, if the thousand million suns had been given at rest twenty-five million years ago, uniformly distributed throughout the supposed sphere, many of them would now have velocities of 20 or 30 kilometres per second, while some would have less and some probably greater velocities than 108 kilometres per second; or, if they had been given thousands of million years ago at rest so distributed that now they were equally spaced throughout the supposed sphere, their mean velocity would now be about 50 kilometres per second*. This is not unlike the measured velocities of stars, and hence it seems probable that there might be as much matter as one thousand million suns within the distance $3{\cdot}09.10^{16}$ kilometres. The same reasoning shows that ten thousand million suns in the same sphere would produce velocities far greater than the known star velocities, and hence there is probably much less than ten thousand million times the sun's mass in the sphere considered. A general theorem discovered by Green seventy-three years ago regarding force at a surface of any shape, due to matter (gravitational, or ideal electric, or ideal magnetic) acting according to the Newtonian law of the inverse square of the distance, shows that a non-uniform distribution of the same total quantity of matter would give greater velocities than would the uniform distribution. Hence we cannot, by any non-uniform distribution of matter within the supposed sphere of $3{\cdot}09.10^{16}$ kilometres radius,

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* Phil. Mag. August 1901, pp. 169, 170.

escape from the conclusion limiting the total amount of the matter within it to something like one thousand million times the sun's mass.

If we compare the sunlight with the light from the thousand million stars, each being supposed to be of the same size and brightness as our sun, we find that the ratio of the apparent brightness of the star-lit sky to the brightness of our sun's disc would be $3{\cdot}87.10^{-13}$. This ratio * varies directly with the radius of the containing sphere, the number of equal globes per equal volume being supposed constant; and hence to make the sum of the apparent areas of discs 3·87 per cent. of the whole sky, the radius must be $3{\cdot}09.10^{27}$ kilometres. With this radius light would take $3\frac{1}{4}.10^{14}$ years to travel from the outlying stars to the centre. Irrefragable dynamics proves that the life of our sun as a luminary is probably between 25 and 100 million years; but to be liberal, suppose each of our stars to have a life of 100 million years as a luminary: and it is found that the time taken by light to travel from the outlying stars to the centre of the sphere is three and a quarter million times the life of a star. Hence it follows that to make the whole sky aglow with the light of all the stars at the same time the commencements of the stars must be timed earlier and earlier for the more and more distant ones, so that the time of the arrival of the light of every one of them at the earth may fall within the durations of the lights of all the others at the earth. My supposition as to uniform density is quite arbitrary; but nevertheless I think it highly improbable that there can be enough of stars (bright or dark) to make a total of star-disc area more than 10^{-12} or 10^{-11} of the whole sky.

To help to understand the density of the supposed distribution of 1000 million suns in a sphere of $3{\cdot}09.10^{16}$ kilometres radius, imagine them arranged exactly in cubic order, and the volume per sun is found to be $123{\cdot}5.10^{39}$ cubic kilometres, and the distance from one star to any one of its six nearest neighbours would be $4{\cdot}98.10^{13}$ kilometres The sun seen at this distance would probably be seen as a star of between the first and second magnitude; but supposing our 1000 million suns to be all of such brightness as to be stars of the first magnitude at distance corresponding to parallax $1''{\cdot}0$, the brightness at distance $3{\cdot}09.10^{16}$ kilometres would be one one-millionth of this; and the most distant of our stars would be seen through powerful telescopes as stars of the

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* Phil. Mag. August 1901, p. 175.

sixteenth magnitude. Newcomb estimated from thirty to fifty million as the number of stars visible in modern telescopes. Young estimated at 100 million the number visible through the Lick telescope. This larger estimate is only one tenth of our assumed 1000 million masses equal to the sun, of which, however, 900 million might be either non-luminous, or, though luminous, too distant to be seen by us at their actual distances from the earth. Remark, also, that it is only for facility of counting that we have reckoned our universe as 1000 million suns; and that the meaning of our reckoning is that the total amount of matter within a sphere of $3{\cdot}09.10^{16}$ kilometres radius is 1000 million times the sun's mass. The sun's mass is $1{\cdot}99.10^{27}$ metric tons, or $1{\cdot}99.10^{33}$ grammes. Hence our reckoning of our supposed spherical universe is that the ponderable part of it amounts to $1{\cdot}99.10^{42}$ grammes, or that its average density is $1{\cdot}61.10^{-23}$ of the density of water.

Let us now return to the question of sum of apparent areas. The ratio of this sum to 4π, the total apparent area of the sky viewed in all directions, is given by the formula*: $\alpha = \frac{3N}{4}\left(\frac{a}{r}\right)^2$, provided its amount is so small a fraction of unity that its diminution by eclipses, total or partial, may be neglected. In this formula, N is a number of globes of radius a uniformly distributed within a spherical surface of radius r. For the same quantity of matter in N′ globes of the same density, uniformly distributed through the same sphere of radius r, we have $\frac{N'}{N} = \left(\frac{a}{a'}\right)^3$ and therefore $\frac{a'}{a} = \frac{\alpha}{\alpha'}$. With $N=10^9$, $r=3{\cdot}09.10^{16}$ kilometres; and a (the sun's radius) $=7.10^5$ kilometres; we had $\alpha=3{\cdot}87.10^{-13}$. Hence $a'=7$ kilometres gives $\alpha'=3{\cdot}87.10^{-8}$; and $a''=1$ centimetre gives $\alpha''=1/36{\cdot}9$. Hence if the whole mass of our supposed universe were reduced to globules of density 1·4 (being the sun's mean density), and of 2 centimetres diameter, distributed uniformly through a sphere of $3{\cdot}09.10^{16}$ kilometres radius, an eye at the centre of this sphere would lose only 1/36·9 of the light of a luminary outside it! The smallness of this loss is easily understood when we consider that there is only one globule of 2 centimetres diameter per 364,000,000 cubic kilometres of space, in our supposed universe reduced to globules of 2 centimetres diameter. Contrast with the

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* Phil. Mag. August 1901, p. 175.

total eclipse of the sun by a natural cloud of water spherules, or by the cloud of smoke from the funnel of a steamer.

Let now all the matter in our supposed universe be reduced to atoms (literally brought back to its probable earliest condition). Through a sphere of radius r let atoms be distributed uniformly in respect to gravitational quality. It is to be understood that the condition 'uniformly' is fulfilled if equivoluminal globular or cubic portions, small in comparison with the whole sphere, but large enough to contain large numbers of the atoms, contain equal total masses, reckoned gravitationally, whether the atoms themselves are of equal or unequal masses, or of similar or dissimilar chemical qualities. As long as this condition is fulfilled, each atom experiences very approximately the same force as if the whole matter were infinitely fine-grained, that is to say, utterly homogeneous.

Let us therefore begin with a uniform sphere of matter of density ρ, gravitational reckoning, with no mutual forces except gravitation between its parts, given with every part at rest at the initial instant; and let it be required to find the subsequent motion. Imagining the whole divided into infinitely thin concentric spherical shells, we see that every one of them falls inwards, as if attracted by the whole mass within it collected at the centre. Hence our problem is reduced to the well known students' exercise of finding the rectilinear motion of a particle attracted according to the inverse square of the distance from a fixed point. Let x_0 be the initial distance, $\frac{4\pi\rho}{3}x_0^3$ the attracting mass, v and x velocity and distance from the centre at time t. The solution of the problem for the time during which the particle is falling towards the centre is

$$\tfrac{1}{2}v^2 = \frac{4\pi\rho}{3}x_0^3\left(\frac{1}{x} - \frac{1}{x_0}\right)$$

and

$$t = \sqrt{\frac{3}{8\pi\rho}}\left(\frac{\pi}{2} - \theta + \tfrac{1}{2}\sin 2\theta\right) = \frac{\pi}{2}\sqrt{\frac{3}{8\pi\rho}}\left[1 - \frac{2\theta}{\pi}\left(1 - \frac{\sin 2\theta}{2\theta}\right)\right]$$

where θ denotes the acute angle whose sine is $\sqrt{\frac{x}{x_0}}$. This shows that the time of falling through any proportion of the initial distance is the same whatever be the initial distance; and that the time (which we shall denote by T) of falling to the centre is $\tfrac{1}{2}\pi\sqrt{\frac{3}{8\pi\rho}}$. Hence in our problem of homogeneous gravitational matter given at rest within a spherical surface and left to fall inwards, the augmenting density remains homogeneous, and the time of shrinkage to any stated proportion of the initial radius is inversely as the square root of the density.

To apply this result to the supposed spherical universe of radius $3{\cdot}09.10^{16}$ kilometres, and mass equal to a thousand million times the mass of our sun, we find the gravitational attraction on a body at its surface gives acceleration of $1{\cdot}37.10^{-13}$ kilometres per second per second. This therefore is the value of $\frac{4\pi\rho}{3}x_0$, with one second as the unit of time and one kilometre as the unit of distance; and we find $T=52{\cdot}8.10^{13}$ seconds $=16{\cdot}8$ million years. Thus our formulas become

$$\tfrac{1}{2}v^2=1{\cdot}37.10^{-13}\,x_0\left(\frac{x_0}{x}-1\right)$$

giving

$$v=5{\cdot}23.10^{-7}\sqrt{x_0\left(\frac{x_0}{x}-1\right)}$$

and

$$t=52{\cdot}8.10^{13}\left[1-\frac{2\theta}{\pi}\left(1-\frac{\sin 2\theta}{2\theta}\right)\right];$$

whence, when $\sin\theta$ is very small,

$$t=52{\cdot}8.10^{13}\left(1-\frac{4\theta^3}{3\pi}\right).$$

Let now, for example, $x_0=3{\cdot}09.10^{16}$ kilometres, and $\frac{x_0}{x}=10^7$; and, therefore, $\sin\theta=\theta=3{\cdot}16.10^{-4}$; whence, $v=291{,}000$ kilometres per second, and $t=T-7080$ seconds $=T-2$ hours approximately.

By these results it is most interesting to know that our supposed sphere of perfectly compressible fluid, beginning at rest with density $1{\cdot}61.10^{-23}$ of that of water, and of any magnitude large or small, and left unclogged by ether to shrink under the influence of mutual gravitation of its parts, would take nearly seventeen million years to reach ·0161 of the density of water, and about two hours longer to shrink to infinite density at its centre. It is interesting also to know that if the initial radius is $3{\cdot}09.10^{16}$ kilometres, the inward velocity of the surface is 291,000 kilometres per second at the instant when its radius is $3{\cdot}09.10^{9}$ and its density ·0161 of that of water. If now, instead of an ideal compressible fluid, we go back to atoms of ordinary matter of all kinds as the primitive occupants of our sphere of $3{\cdot}09.10^{16}$ kilometres radius, all these conclusions, provided all the velocities are less than the velocity of light, would still hold, notwithstanding the ether occupying the space through which the atoms move. This would, I believe [*], exercise no resistance whatever to uniform motion of an atom through it; but it would certainly add quasi-inertia to the intrinsic Newtonian inertia of the atom itself moving through ideal space void of ether; which, according to the Newtonian law, would be exactly in proportion to the amount of its gravitational quality. The additional quasi-inertia must be exceedingly small in comparison with the Newtonian inertia, as is demonstrated by the Newtonian proofs, including that founded on Kepler's laws for the groups of atoms constituting the planets, and movable bodies experimented on at the earth's surface.

In one thousand seconds of time after the density ·0161 of the density of water is reached, the inward surface velocity would be 305,000 kilometres per second, or greater than the velocity of light; and the whole surface of our condensing globe of gas or vapour or crowd of atoms would begin to glow, shedding light inwards and outwards. All this is absolutely realistic, except the assumption of uniform distribution through a sphere of the enormous radius of $3{\cdot}09.10^{16}$ kilometres, which we adopted temporarily for illustrational purpose. The enormously great velocity (291,000 kilometres per second) and rate of acceleration (13·7 kilometres per second per second) of the boundary inwards, which we found at the instant of density ·0161 of that of water, are due to greatness of the primitive radius, and the uniformity of density in the primitive distribution.

To come to reality, according to the most probable judgment present knowledge allows us to form, suppose at many millions, or thousands of millions, or millions of millions of years ago, all the matter in the universe to have been atoms very nearly at rest [†] or quite at rest; more densely dis-

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[*] 'On the Motion produced in an Infinite Elastic Solid by the Motion through the Space occupied by it of a Body acting on it only by Attraction or Repulsion,' Cong. International de Physique, Paris, Volume of Reports; Phil. Mag., August 1900.

[†] 'On Mechanical Antecedents of Motion, Heat, and Light,' Brit. Assoc. Rep., Part 2, 1854; Edin. New. Phil. Jour. vol. i. 1855; *Comptes Rendus,* vol. xl. 1855; Kelvin's Collected Math. and Phys. Papers vol. ii. art. lxix.

tributed in some places than in others, of infinitely small average density through the whole of infinite space. In regions where the density was then greater than in neighbouring regions, the density would become greater still; in places of less density, the density would become less; and large regions would quickly become void or nearly void of atoms. These large void regions would extend so as to completely surround regions of greater density. In some part or parts of each cluster of atoms thus isolated, condensation would go on by motions in all directions not generally convergent to points, and with no perceptible mutual influence between the atoms until the density becomes something like 10^{-6} of our ordinary atmospheric density, when mutual influence by collisions would begin to become practically effective. Each collision would give rise to a train of waves in ether. These waves would carry away energy, spreading it out through the void ether of infinite space. The loss of energy, thus taken away from the atoms, would reduce large condensing clusters to the condition of gas in equilibrium [*] under the influence of its own gravity only, or rotating like our sun or moving at moderate speeds as in spiral nebulas, &c. Gravitational condensation would at first produce rise of temperature, followed later by cooling and ultimately freezing, giving solid bodies; collisions between which will produce meteoric stones such as we see them. We cannot regard as probable that these lumps of broken-looking solid matter (something like the broken stones used on our macadamised roads) are primitive forms in which matter was created. Hence we are forced, in this twentieth century, to views regarding the atomic origin of all things closely resembling those presented by Democritus, Epicurus, and their majestic Roman poetic expositor, Lucretius.

II. *On the Michelson-Morley Experiment relating to the Drift of the Æther.* *By* W. M. Hicks, *F.R.S.*[†]

[Plate I.]

IN the following pages it is proposed to consider in detail the general theory of the experiment by which Messrs. Michelson & Morley [‡] attempted to decide the question of the rest or motion of the æther when a material body moves through it. The theory is not so simple as it may appear at

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[*] Homer Lane, American Journal of Science, 1870, p. 57; Sir W Thomson, Phil. Mag. March 1887, p. 287.

[†] Communicated by the Author.

[‡] Phil. Mag. Dec. 1887.

first sight owing to the changes produced by actual reflexion at a moving surface. The correction due to alteration in the angle of reflexion was first introduced by Lorentz, and was taken account of in the joint paper by Michelson & Morley in 1887. But reflexion produces also a change in the wave-length of the reflected light. Further, when the source of light moves with the apparatus, the light incident at any instant on a plate does not come from the position occupied by the source at that instant, but from a point which it occupied at some interval before; consequently the angle of incidence alters by a small quantity of the first order as the direction of drift of the apparatus changes.

The present investigation is undertaken with the view of making these corrections. As in a theory of this kind it does not seem legitimate to assume beforehand that the ratio of the velocity of the earth through the æther to that of light is extremely small, the general theory is worked out without approximation. It is interesting to see how comparatively simple this complete theory is. The principal result of the correction is that in the experiment of Michelson & Morley the effect to be expected is the reverse of that hitherto supposed.

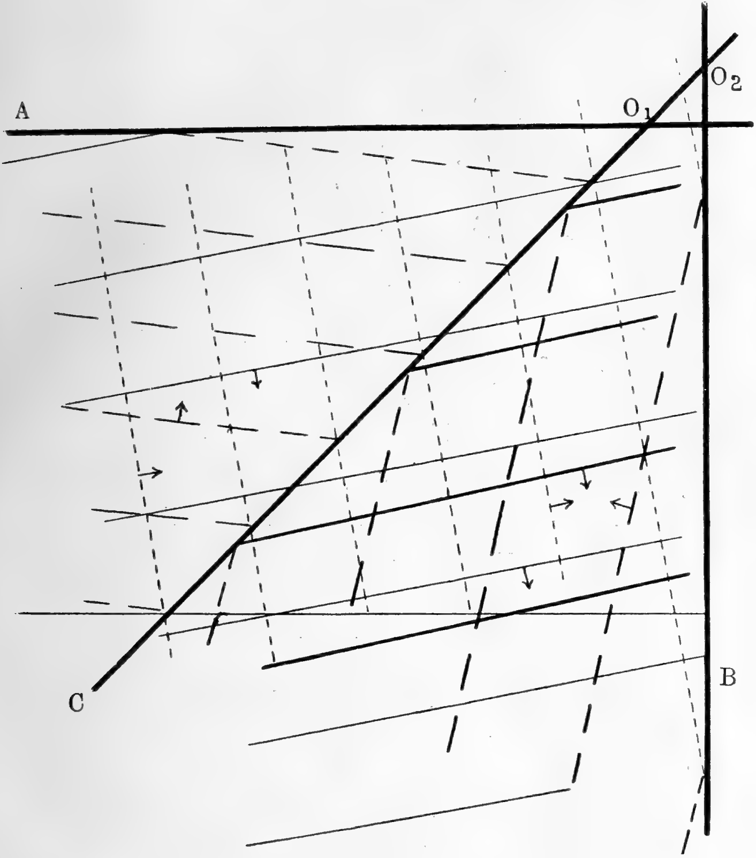
In the actual experiment the light from a flame in the focal plane of a lens falls on a small plane-glass plate by which it is divided, and the two portions are afterwards reflected back to the plate by two small plane mirrors. We shall suppose in the first place that the source is a point of light or a vertical slit; so that the incident light consists of a single train of parallel waves.

We have thus two mirrors inclined at any given angle, and between them a transparent semi-reflecting plate, whose plane does not in general pass through the intersection of the mirrors. In practice we utilize only small portions of these planes, but in considering the general theory it is best to deal with the complete system of planes and to consider the phenomena taking place in the angular space between them.

1. Consider the state of the æther at any given instant, and draw the wave diagram, that is a diagram of lines representing all the crests (or every *n*th crest).

In fig. 1, A, B represent the planes of the mirrors, C that of the plate. On the plate is incident a train of waves (dotted lines), which produces at the plate a train of reflected waves (thin broken lines) and of transmitted waves (dotted lines). The first set are incident on the mirror plane A and produce a train of reflected waves (thin continuous lines). The transmitted set fall on the mirror plane B, and produce a reflected train (thick broken lines), these, after reflexion from the glass plate, produce another train (thick continuous lines). The two sets of waves, represented by the thin and thick lines, interfere and produce the fringes whose laws we have to investigate. The figure gives an instantaneous picture of the configuration of the waves in the æther for a special case of motion and disposition of the planes*. The diagram is complicated by the number of lines. But the different trains can be followed easily, and afterwards the eye fixed on the continuous lines only. The essential point

Fig. 1.

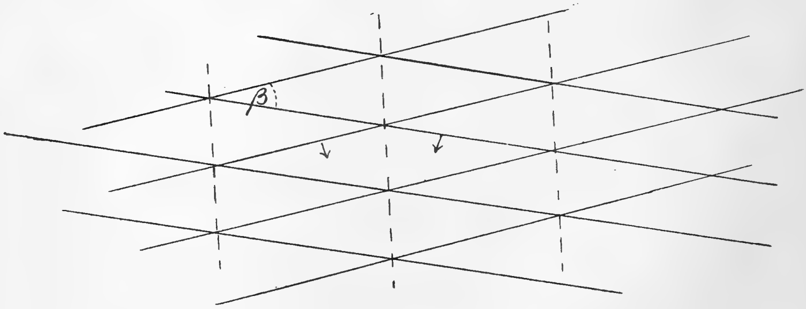


[*] The figure is drawn to scale for the case of $U = \frac{1}{5}V$, drift makes angle of 53° 8′ ($\tan a = \frac{4}{3}$) with OA. $CO_1A = 44° 30'$, mirrors at 90° to one another.

to be noticed about this wave-diagram is that precisely the same diagram will serve for every instant of the motion; only in this case the lines do not represent crests, but all belong to the same phase of the respective waves, this common phase differing with the instant for which the diagram is drawn. This is because the change produced by reflexion so modifies the reflected waves (as will be seen shortly) that the incident crests and their corresponding reflected crests always intersect along the surface as it moves through the æther. If the diagram were drawn with angles of reflexion and incidence equal at one instant, at the next they would be thrown into confusion. This permanence of type is the basis of the development of the theory adopted in the following general reasoning. The diagram gives an instantaneous view. If an observer is fixed in the æther he sees that in this picture the waves have different wave-lengths, but that they advance with the same velocity V, and the apparatus moves with U. If the observer is fixed to the apparatus he sees that in this picture the apparatus is fixed but that each system of waves advances with a different velocity, yet in all cases such, that as they reach him their frequencies are the same—the longer waves have the greater apparent velocity of propagation.

2. It will be best to consider first the case of no drift of the æther. Here the æther between B and C will be mapped out by a network of lines representing crests of two sets of waves, the meshes of which, in case of no motion, are equal-sided parallelograms. The diagonals which lie between the advancing sides of the waves are points of maximum disturbance. They form a system of parallel straight lines bisecting the angles between the wave-fronts, and at a

Fig. 2.



distance from one another $c=\lambda/\left(2\sin\frac{\beta}{2}\right)$, where λ is the common wave-length and β the angle between the two sets of wave-fronts. For convenience we shall call these lines maximal lines. If a screen be placed perpendicularly to them we shall get a fringe, the intersections with the maximals being the centres of the bright bands, and the distance between the bands being c. If the screen is placed at an angle γ to these lines, the breadth of the bands is $c/\sin\gamma$.

All the waves interfering at points along a given maximal differ in phase by the same integral number of wave-lengths. We can therefore designate any maximal by this number. It is important to be able to determine it.

If the plate and the first mirror intersect at O_1 (fig. 1), the phase of the reflected wave of the first train at O_1 is the same as that of the incident. Let p_1 denote the perpendicular distance of any point P from the wave-front through O_1. Then the phase of the first train at $P=p_1/\lambda+$ phase of incident light at O_1. Similarly if the plate and the second mirror intersect at O_2, and p_2 denote the distance of P from the wave-front of the second train through O_2, the phase of the second train at $P=p_2/\lambda+$phase of incident light at O_2. If then the distance $O_1\,O_2$ resolved perpendicularly to the incident system be q, the interfering waves at P differ in phase by $(p_2+q-p_1)/\lambda$. This is the amount by which the second train is ahead of the first. If this be zero we have the central maximal, which is the same for all colours. Consequently, when white light is used it is only near this maximal that fringes will be visible.

It follows from this that if the planes of the plate and the mirrors intersect in a point the central maximal will pass through that point. There is no need to consider specially the quantitative theory for no drift, as it will be included in the more general case of motion to follow.

3. Taking now the case where the apparatus moves in an æther at rest, the wave-diagram gives as before an instantaneous state of the æther. But now the conditions are very different. In the first place, as will be shown later, when the two reflected systems are inclined their wave-lengths will in general be different, whilst if the wave-lengths are equal the two systems must be parallel. Thus to a person fixed in the æther, no fringes can *in general* be seen, either because the waves which ought to interfere have different frequencies, or because when they have the same, the interference produces a uniform change of illumination and not bands of finite breadth. This is not the case, however, to a person who

moves with the apparatus. For consider a point P fixed relatively to it. At every instant the phases of the two wave-fronts at P (each phase measured as a fraction of its corresponding period) differ by the same amount. Consequently as P moves through the æther the two waves at P interfere. The *apparent* frequency to an observer at P of each system must consequently be the same. A formal proof of this is given below.

So far then as the phenomena at points moving with the apparatus are considered, we may consider the space as divided up into a network of parallelograms (not now equal-sided) with a corresponding set of maximal lines. These maximal lines are not fixed in the æther, but move as if fixed to the apparatus. The intersections of these maximals with a screen moving with the apparatus give the fringes, as in the case of no motion. The maximal lines will not now be equally inclined to the two wave-fronts. If as before c denote the distance between the maximals, it can easily be shown that

$$c=\frac{\lambda_1\lambda_2}{\sqrt{\{(\lambda_1-\lambda_2)^2+4\lambda_1\lambda_2\sin^2\frac{1}{2}\beta\}}}.$$

The maximal lines move as a rigid system attached to the apparatus. They give the phenomena taking place in the fixed æther as well. If a screen be fixed in the æther the intersections with the maximals move, and we get fringes which drift along it, and which therefore in general produce the effect of white light on the eye owing to their rapid motion. If, however, the direction of drift be parallel to a maximal line, there is no drift of fringes on the screen, and the eye will see a fixed fringe. If the drift slightly alters, the fringe will begin to move slowly along the screen. The observation naturally cannot be made, but the result is important, because, if the fringes exist, the interfering waves must have equal frequencies. Hence the two sets of wave-lengths will be equal when the drift is parallel to a maximal. That this is actually the case the formula to be developed later will show.

4. If a photographic plate be exposed in any position, an image of the fringe there situated will be impressed. If, however, the eye be focussed on the same plane, either directly or by any optical instrument, the fringe observed will not be the same as that impressed on the plate. The rays through any point of the object fringe all pass through the optical image on the retina, and all traverse the same optical distance,

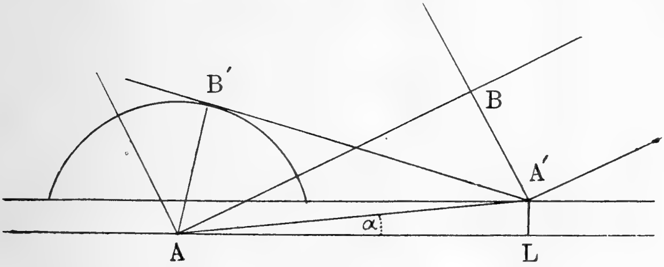
say D′. At the object the two interfering waves have a certain phase-difference, but in space their wave-lengths are different, and in travelling over the same distance D′ their phase-differences are changed by the amount $D'\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right)$. Consequently the optical image of a bright line is not itself a bright line; or, conversely, the bright central band in the image is the optical image of a point on the original fringe to one side of its central band. This result is important, as in certain cases it may, as will be shown later, entirely modify the nature of the changes produced as the direction of drift of the apparatus alters. Again, superposed on this there will be an aberration effect.

So much for the general qualitative theory of the phenomenon. It remains now to obtain the quantitative laws.

Reflexion of a Plane Wave at a Moving Mirror.

5. *Angle of Reflexion.*—In the figure let AB denote a wave-front incident at A on the mirror AL. Let V denote the

Fig. 3.



velocity of light, v that of the mirror perpendicular to itself. When the mirror has advanced $A'L=vt$ suppose AB to have advanced so as to be incident at A′. During the interval t, *provided the æther be at rest*, the disturbance produced at A will have spread to the spherical surface B′, which lies outside the new position of the mirror. What happens between the old and new positions of the mirror does not concern us. Hence, with the usual reasoning, the reflected wave-front through A′ is A′B′ (the fig. explains itself). Let ϕ, ϕ' be the angles of incidence and of reflexion, and let α denote the angle A′AL.

Then
$$BAA' = B'A'A,$$

i. e.
$$\phi - \alpha = \phi' + \alpha,$$

or
$$\alpha = \frac{\phi - \phi'}{2},$$

$$BAA' = \phi - \alpha = \frac{\phi + \phi'}{2}.$$

Again,
$$\frac{\sin BAA'}{\sin \alpha} = \frac{A'B}{A'L} = \frac{V}{v}$$

or
$$\frac{\sin \frac{\phi + \phi'}{2}}{\sin \frac{\phi - \phi'}{2}} = \frac{V}{v},$$

whence
$$\tan \tfrac{1}{2} \phi' = \frac{V - v}{V + v} \tan \tfrac{1}{2} \phi, \quad . \quad . \quad . \quad . \quad (1)$$

which gives the law of reflexion. In this it is to be remembered that v is to be regarded as positive when the plane moves towards the incident light.

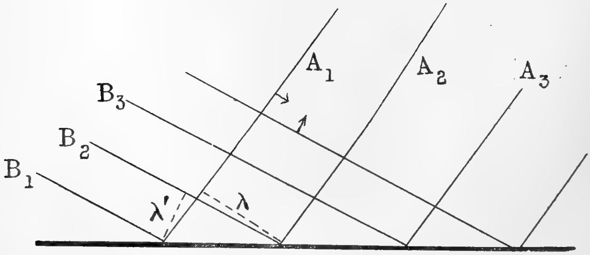
It can easily be shown from the above that if D_v denotes $\sqrt{\{V^2 + v^2 + 2Vv \cos \phi\}}$,

$$e^{\frac{\phi' \iota}{2}} = \frac{V e^{\frac{\phi \iota}{2}} + v e^{-\frac{\phi \iota}{2}}}{D_v},$$

a formula which later will be found very useful.

Change in Wave-length.—In fig. 4 let A_1 ... represent a train of wave-crests incident on the mirror at any moment.

Fig. 4.



B_1 . . . the corresponding reflected wave-crests. The figure shows at once that, λ', λ denoting the wave-lengths of the reflected and incident waves respectively,

$$\frac{\lambda'}{\lambda}=\frac{\sin\phi'}{\sin\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This result is true whatever be the law of reflexion.

Applying the actual law of reflexion above, it follows that

$$\frac{\lambda'}{\lambda}=\frac{V^2-v^2}{V^2+v^2+2Vv\cos\phi} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Modification produced on the Light at any point by motion of the source.—Owing to motion of the source the light reaching a point P will not come from the instantaneous position of the source but from the position occupied by it at some time previously. Consequently if S denote the source, P this position, and v the velocity of S perpendicular to SP, the light at P makes an angle θ with SP where

$$V\sin\theta=v.$$

Again, if λ denote the wave-length at P (or V/λ the frequency), and if Λ denote the wave-length if the source is at rest, and u denote the velocity of the source towards P,

$$\lambda=\frac{V\cos\theta-u}{V}\,\Lambda\cos\theta. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

6. *Specification of the Configuration.*—We shall call the mirror which receives the light after reflexion at the plate, the first mirror; that which receives it after transmission through the plate the second mirror, and we shall refer all directions to the line through the source parallel to the first mirror.

The normal to the plate makes an angle ϕ (fig. 5) with this datum line, and the normal to the second plate an angle χ (positive as in fig.). Let θ be the inclination of the incident light to this datum line. The angles of incidence on the plate and the second mirror are $\phi+\theta$ and $\chi+\theta$.

Let the angles of reflexion be ϕ' and χ'. Let the angles of incidence and reflexion on the first mirror be ϕ_1, ϕ_1', and on the plate after reflexion at the second mirror be ϕ_2 and ϕ_2'. Also let A, B denote the angles the final wave-fronts make with the first mirror (or the datum line). This specifies the light. Let the apparatus drift with velocity U in a direction making α with the datum, its velocities along and perpendicular thereto being denoted by u, v.

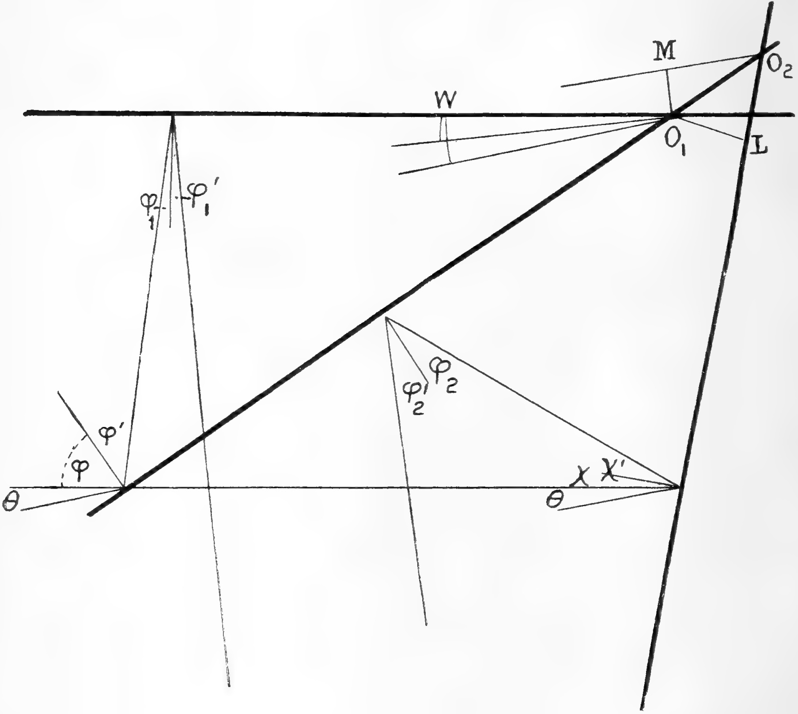
The velocity of the plate perpendicular to itself, and towards the side on which is the source of light, is

$$w = v\sin\phi - u\cos\phi = -\mathrm{U}\cos(\phi+\alpha). \quad . \quad . \quad (5)$$

Similarly that of the second mirror is

$$w' = v\sin\chi - u\cos\chi = -\mathrm{U}\cos(\chi+\alpha). \quad . \quad . \quad (6)$$

Fig. 5.



7. *The Wave-lengths.*—With the above notations, the final wave-lengths will be by (2),

$$\lambda_1 = \frac{\sin\phi'}{\sin(\phi+\theta)} \cdot \frac{\sin\phi_1'}{\sin\phi_1}\lambda,$$

$$\lambda_2 = \frac{\sin\chi'}{\sin(\chi+\theta)} \cdot \frac{\sin\phi_2'}{\sin\phi_2}\lambda,$$

where λ is the wave-length incident on the plate.

Whence by (3)

$$\left.\begin{aligned}\frac{\lambda_1}{\lambda} &= \frac{(\mathrm{V}^2-w^2)(\mathrm{V}^2-v^2)}{\{\mathrm{V}^2+w^2+2w\mathrm{V}\cos(\phi+\theta)\}\{\mathrm{V}^2+v^2-2v\mathrm{V}\cos\phi_1\}} \\ \frac{\lambda_2}{\lambda} &= \frac{(\mathrm{V}^2-w^2)(\mathrm{V}^2-w'^2)}{\{\mathrm{V}^2+w^2-2w\mathrm{V}\cos\phi_2\}\{\mathrm{V}^2+w'^2+2w'\mathrm{V}\cos(\chi+\theta)\}}\end{aligned}\right\} \quad (7)$$

It will be convenient to denote these denominators by D_1^2 and D_2^2.

Now $$\phi_1=\phi+\phi'-\tfrac{1}{2}\pi.$$

Hence

$$\begin{aligned}\mathrm{D}_1^2 = {} & (\mathrm{V}^2+v^2)\{\mathrm{V}^2+w^2+2w\mathrm{V}\cos(\phi+\theta)\} \\ & -2\mathrm{V}v\sin(\phi+\phi')\{\mathrm{V}^2+w^2+2w\mathrm{V}\cos(\phi+\theta)\}.\end{aligned}$$

ϕ' is the reflected angle for incident angle $(\phi+\theta)$ and for vel. $=w$. Therefore

$$\{\mathrm{V}^2+w^2+2w\mathrm{V}\cos(\phi+\theta)\}\begin{cases}\sin\phi'=(\mathrm{V}^2-w^2)\sin(\phi+\theta), \\ \cos\phi'=(\mathrm{V}^2+w^2)\cos(\phi+\theta)+2\mathrm{V}w.\end{cases}$$

Substituting it can be easily shown, remembering the values of w, w' given by (5, 6), that

$$\begin{aligned}\mathrm{D}_1^2 = {} & \mathrm{V}^4-2\mathrm{V}^3\cos\phi\{u\cos(\phi+\theta)+v\sin(\phi+\theta)\} \\ & +\mathrm{V}^2(v^2+w^2-4vw\sin\phi)+2vw\mathrm{V}\cos\phi(v\cos\theta-u\sin\theta)+v^2w^2.\end{aligned}$$

Similarly it may be shown that

$$\begin{aligned}\mathrm{D}_2^2 = {} & \mathrm{V}^4+2\mathrm{V}^3\sin(\phi-\chi)\{u\sin(\theta+\chi-\phi)-v\cos(\theta+\chi-\phi)\} \\ & +\mathrm{V}^2\{w^2+w'^2-4ww'\cos(\phi-\chi)\} \\ & -2ww'\mathrm{V}\sin(\phi-\chi)(u\sin\theta-v\cos\theta)+w^2w'^2.\end{aligned}$$

The difference of wave-lengths is given by $d\lambda=\lambda_1-\lambda_2$, where

$$\frac{d\lambda}{\lambda}=(\mathrm{V}^2-w^2)\frac{(\mathrm{V}^2-v^2)\mathrm{D}_2^2-(\mathrm{V}^2-w'^2)\mathrm{D}_1^2}{\mathrm{D}_1^2\mathrm{D}_2^2}.$$

After some reduction this becomes

$$\frac{d\lambda}{\lambda}=\frac{\mathrm{U}(\mathrm{V}^2-w^2)\mathrm{PQ}}{\mathrm{D}_1^2\mathrm{D}_2^2}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where

$$\left.\begin{aligned}
&\mathrm{P}\equiv\mathrm{V}\{\mathrm{V}^2\cos(\chi-\alpha+\theta)-\mathrm{VU}\cos\chi+\mathrm{U}^2\sin\alpha\cos(\chi+\alpha)\sin(\alpha-\theta)\}\\
&\text{or}\\
&\mathrm{P}\equiv(\mathrm{V}\cos\theta-u)\{\mathrm{V}^2\cos(\chi-\alpha)+v^2\cos(\chi+\alpha)\}\\
&\qquad\qquad-(\mathrm{V}\sin\theta-v)\{\mathrm{V}^2\sin(\chi-\alpha)+uv\cos(\chi+\alpha)\}\\
&\mathrm{Q}\equiv2\mathrm{V}^2\cos(2\phi-\chi)-\mathrm{U}^2(\cos2\phi+\cos2\alpha)\cos\chi\\
&\qquad\qquad+4\mathrm{U}^2\sin\alpha\sin\chi\cos\phi\cos(\phi+\alpha)\\
&\text{or}\\
&\mathrm{Q}\equiv(2\mathrm{V}^2-\mathrm{U}^2)\cos(2\phi-\chi)+\mathrm{U}^2\{\sin(2\alpha+2\phi)\sin\chi-\cos(2\alpha+\chi)\}
\end{aligned}\right\}\quad(9)$$

where it is to be noticed that Q is independent of the direction of the incident light and P of the inclination of the plate. If the source is fixed in space, $\theta=0$. If it is fixed to the apparatus, $\mathrm{V}\sin\theta-v=0$, and

$$\frac{d\lambda}{\lambda}=\frac{\mathrm{U}(\mathrm{V}^2-w^2)(\mathrm{V}\cos\theta-u)\{\mathrm{V}^2\cos(\chi-\alpha)+v^2\cos(\chi+\alpha)\}\mathrm{Q}}{\mathrm{D}_1{}^2\mathrm{D}_2{}^2}\ .\ (10)$$

8. *The Angles.*

$$\mathrm{A}=\phi_1',$$

$$\mathrm{B}=\tfrac{1}{2}\pi-(\phi+\phi_2').$$

By § 5 and writing $[x]$ for $e^{x\iota}$,

$$e^{\frac{\phi_1'\iota}{2}}=\frac{\mathrm{V}\left[\frac{\phi_1}{2}\right]-v\left[-\frac{\phi_1}{2}\right]}{\mathrm{D}_v}$$

But $$\phi_1=\phi+\phi'-\frac{\pi}{2};$$

hence

$$\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+w\left[\frac{\phi+\theta}{2}\right]\Big)\\
e^{\frac{\mathrm{A}\iota}{2}}&=\frac{\left[\frac{\phi}{2}-\frac{\pi}{4}\right]\mathrm{V}\left\{\mathrm{V}\left[\frac{\phi+\theta}{2}\right]+w\left[-\frac{\phi+\theta}{2}\right]\right\}-v\left[\frac{\pi}{4}-\frac{\phi}{2}\right]\left\{\mathrm{V}\left[-\frac{\phi+\theta}{2}\right]\right.}{\mathrm{D}_v\mathrm{D}_w}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-vw\left[\frac{\pi}{4}+\frac{\theta}{2}\right].\\
&=\frac{\mathrm{V}^2\left[-\frac{\pi}{4}+\phi+\frac{\theta}{2}\right]+\mathrm{V}\left\{w\left[-\frac{\pi}{4}-\frac{\theta}{2}\right]-v\left[\frac{\pi}{4}-\phi-\frac{\theta}{2}\right]\right\}}{\mathrm{D}_1}
\end{aligned}\quad . \ . \ . \ (11)$$

Again,

$$e^{\frac{\phi_2'\iota}{2}} = \frac{V\left[\frac{\phi_2}{2}\right] - w\left[-\frac{\phi_2}{2}\right]}{D_{-w}}$$

and $\phi_2 = \phi - \chi - \chi'$;

hence

$$e^{\frac{B\iota}{2}} = e^{\left(\frac{\pi}{4} - \frac{\phi}{2} - \frac{\phi_2'}{2}\right)\iota}$$

$$= \left[\frac{\pi}{4} - \frac{\phi}{2}\right] \frac{\left[-\frac{\phi-\chi}{2}\right] V\left\{V\left[\frac{\chi+\theta}{2}\right] + w'\left[-\frac{\chi+\theta}{2}\right]\right\} - w\left[\frac{\phi-\chi}{2}\right]\left\{V\left[-\frac{\chi+\theta}{2}\right] + w'\left[\frac{\chi+\theta}{2}\right]\right\}}{D_{-w}D_{w'}}$$

$$= \frac{V^2\left[\frac{\pi}{4} - \phi + \chi + \frac{\theta}{2}\right] - V\left\{-w'\left[\frac{\pi}{4} - \phi - \frac{\theta}{2}\right] + w\left[\frac{\pi}{4} - \chi - \frac{\theta}{2}\right]\right\} - ww'\left[\frac{\pi}{4} + \frac{\theta}{2}\right]}{D_2}. \quad . \quad . \quad . \quad (12)$$

Hence after multiplication and reduction

$$D_1D_2e^{\frac{A-B}{2}\iota}$$

$$= V^4\left[-\frac{\pi}{2} + 2\phi - \chi\right]$$

$$+ V^3\left\{w\left[-\frac{\pi}{2} + \phi - -\chi - \theta\right] - w\left[-\frac{\pi}{2} + \phi + \chi + \theta\right] - v[-\chi - \theta] + w'\left[-\frac{\pi}{2} + 2\phi + \theta\right]\right\}$$

$$+ V^2\left\{-vw[\phi - \chi] + vw[\chi - \phi] - w'v[0] - w^2\left[-\frac{\pi}{2} + \chi\right]\right\}$$

$$+ V\left\{w^2v[\chi + \theta] + ww'v[-\phi - \theta] - ww'v[\phi + \theta] - w'w^2\left[-\frac{\pi}{2} - \theta\right]\right\}$$

$$+ vw^2w'[0]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Therefore

$$\begin{aligned} D_1D_2 \sin\frac{A-B}{2} = \\ & -V^4\cos(2\phi-\chi)+V^3\{-2w\sin\phi\sin(\chi+\theta) \\ & \qquad +v\sin(\chi+\theta)-w'\cos(2\phi+\theta)\} \\ & +V^2w\{w\cos\chi-2v\sin(\phi-\chi)\} \\ & +Vw\{-2vw'\sin(\phi+\theta)+vw\sin(\chi+\theta)+ww'\cos\theta\}. \end{aligned}$$

After reduction this is found to give

$$\sin\frac{A-B}{2} = -\frac{1}{2}\,\frac{V\{V-U\cos(\theta-\alpha)\}Q}{D_1D_2}; \quad . \quad . \quad . \qquad (14)$$

where Q has the value given above.
When the source is fixed to the apparatus ($V\sin\theta=v$)

$$\begin{aligned} V\{V-U\cos(\theta-\alpha)\} &= V^2-Vu\cos\theta-v^2=V^2\cos^2\theta-Vu\cos\theta \\ &= V\cos\theta(V\cos\theta-u), \end{aligned}$$

and then

$$\sin\frac{A-B}{2} = -\frac{1}{2}\,\frac{V\cos\theta(V\cos\theta-u)\,Q}{D_1D_2}.$$

This gives the angle between the wave-fronts.

9. *The distance between the maximal lines is*

$$\begin{aligned} p &= \frac{\lambda_1\lambda_2}{\sqrt{\{\lambda_1{}^2+\lambda_2{}^2-2\lambda_1\lambda_2\cos(A-B)\}}} \\ &= \frac{\lambda_1\lambda_2}{\sqrt{\left\{(\lambda_1-\lambda_2)^2+4\lambda_1\lambda_2\sin^2\frac{A-B}{2}\right\}}} \\ &= \lambda\frac{(V^2-w^2)^2(V^2-v^2)(V^2-w'^2)}{(V^2-w^2)\sqrt{\{U^2P^2Q^2+(V^2-v^2)(V^2-w'^2)V^2(V-U\cos\overline{\theta-\alpha})^2Q^2\}}} \\ &= \frac{\lambda}{Q}\cdot\frac{(V^2-v^2)(V^2-w'^2)(V^2-w^2)}{\sqrt{\{U^2P^2+V^2(V^2-v^2)(V^2-w'^2)(V-U\cos\overline{\theta-\alpha})^2\}}}; \quad . \quad . \quad . \qquad (15) \end{aligned}$$

for source fixed to apparatus

$$\begin{aligned} p &= \frac{\lambda}{Q}\,\frac{(V^2-v^2)(V^2-w'^2)(V^2-w^2)}{(V\cos\theta-u)\sqrt{\{(V^2\cos\overline{\chi-\alpha}+v^2\cos\overline{\chi+\alpha})^2U^2+(V^2-v^2)^2(V^2-w'^2)\}}} \\ &= \frac{\Lambda\cos\theta}{VQ}\cdot\frac{(V^2-v^2)(V^2-w'^2)(V^2-w^2)}{\sqrt{\{(V^2\cos\overline{\chi-\alpha}+v^2\cos\overline{\chi+\alpha})^2U^2+(V^2-v^2)^2(V^2-w'^2)\}}}, \end{aligned}$$

where Λ is actual wave-length of light emitted by the source.

The denominator may be simplified so that in this case

$$p=\frac{\Lambda\cos\theta}{\mathrm{V}^2\mathrm{Q}}\frac{(\mathrm{V}^2-v^2)(\mathrm{V}^2-w^2)(\mathrm{V}^2-w'^2)}{\sqrt{[(\mathrm{V}^2-v^2)^2\cos^2\chi+\{(\mathrm{V}^2-u^2)\sin\chi+2uv\cos\chi\}^2]}}$$

10. This seems the proper place to give a formal proof of the statement made above from general considerations that the apparent frequencies of each wave-train with respect to a point moving with the apparatus are the same.

The velocity of such a point relative to the first train is

$$\mathrm{V}+\mathrm{U}\sin(\alpha-\mathrm{A}).$$

Its apparent frequency is therefore

$$\frac{\mathrm{V}+\mathrm{U}\sin(\alpha-\mathrm{A})}{\lambda_1}.$$

Now $\mathrm{A}=\phi_1'$. Hence

$$\mathrm{V}+\mathrm{U}\sin(\alpha-\mathrm{A})=\mathrm{V}+v\cos\phi_1'-u\sin\phi_1'.$$

Substituting for ϕ_1' in terms of ϕ_1, it will be seen that

$$\mathrm{V}+\mathrm{U}\sin(\alpha-\mathrm{A})=\frac{\mathrm{V}^2-v^2}{\mathrm{D}_v}\{\mathrm{V}-v\cos\phi_1-u\sin\phi_1\}.$$

Again, $\phi_1=\phi+\phi'-\frac{1}{2}\pi$. Substituting this and expressing ϕ' in terms of its angle of incidence $\phi+\theta$,

$$\mathrm{V}-v\cos\phi_1-u\sin\phi_1$$
$$=\frac{\mathrm{V}^2-w^2}{\mathrm{D}_w}\{\mathrm{V}+w\cos(\phi+\theta)-(v\cos\phi+u\sin\phi)\sin(\phi+\theta)\};$$

whence

$$\mathrm{V}+\mathrm{U}\sin(\alpha-\mathrm{A})=\frac{\mathrm{V}^2-v^2)(\mathrm{V}^2-w^2)}{\mathrm{D}_1^2}(\mathrm{V}-v\sin\theta-u\cos\theta)$$
$$=\frac{\lambda_1}{\lambda}\{\mathrm{V}-\mathrm{U}\cos(\theta-\alpha)\}.$$

Similarly, it can be shown that

$$\mathrm{V}+\mathrm{U}\sin(\alpha-\mathrm{B})=\frac{\lambda_2}{\lambda}\{\mathrm{V}-\mathrm{U}\cos(\theta-\alpha)\};$$

whence

$$\frac{\mathrm{V}+\mathrm{U}\sin(\alpha-\mathrm{A})}{\lambda_1}=\frac{\mathrm{V}+\mathrm{U}\sin(\alpha-\mathrm{B})}{\lambda_2}=\frac{\mathrm{V}-\mathrm{U}\cos(\theta-\alpha)}{\lambda}, \quad (16)$$

which shows that the frequencies are equal.

But further,

$$\mathrm{V}-\mathrm{U}\cos(\theta-\alpha)=\mathrm{V}-u\cos\theta-v\sin\theta.$$

Hence when the source is fixed to the apparatus, or $v = \mathrm{V}\sin\theta$,

$$\mathrm{V} - \mathrm{U}\cos(\theta - \alpha) = (\mathrm{V}\cos\theta - u)\cos\theta;$$

and the apparent frequency becomes

$$\frac{(\mathrm{V}\cos\theta - u)\cos\theta}{\lambda} = \frac{\mathrm{V}}{\Lambda};$$

or the apparent frequency is the same as if the whole apparatus were at rest. Whatever, therefore, be the velocity of drift, the apparent frequency remains the same.

11. *Direction of Maximal Lines.*—Let the common direction of the maximal lines make an angle ψ with the first mirror. Then, considering a mesh of the network formed by the two trains of waves, the sides are inclined to the diagonal at angles $180 - (\psi + \mathrm{A})$ and $\psi + \mathrm{B}$, supposing $\mathrm{A} > \mathrm{B}$. The sides are proportional to the sines of these angles and also to the wave-lengths. Consequently,

$$\frac{\sin(\psi + \mathrm{A})}{\sin(\psi + \mathrm{B})} = \frac{\lambda_1}{\lambda_2}$$
$$= \frac{\mathrm{V} + \mathrm{U}\sin(\alpha - \mathrm{A})}{\mathrm{V} + \mathrm{U}\sin(\alpha - \mathrm{B})};$$

whence

$$\tan\psi = \frac{\mathrm{V}\cos\frac{\mathrm{A}+\mathrm{B}}{2} + v\cos\frac{\mathrm{A}-\mathrm{B}}{2}}{\mathrm{V}\sin\frac{\mathrm{A}+\mathrm{B}}{2} - u\cos\frac{\mathrm{A}-\mathrm{B}}{2}}.$$

From the value of $e^{\frac{\mathrm{A}-\mathrm{B}}{2}\iota}$ above,

$$\mathrm{D}_1\mathrm{D}_2\cos\frac{\mathrm{A}-\mathrm{B}}{2} = \mathrm{V}^4\sin(2\phi - \chi) + \mathrm{V}^3\{-2w\cos\phi\sin(\chi + \theta)$$
$$- v\cos(\chi + \theta) + w'\sin(2\phi + \theta)\}$$
$$+ \mathrm{V}^2(-w'v - w^2\sin\chi) + \mathrm{V}\{w'w^2\sin\theta + vw^2\cos(\chi + \theta)\} + vw^2w'.$$

Also

$$\mathrm{D}_1\mathrm{D}_2 e^{\frac{\mathrm{A}+\mathrm{B}}{2}\iota}$$
$$= \mathrm{V}^4[\chi + \theta] + \mathrm{V}^3\left\{w[-\phi + \chi] - w[\phi - \chi] - v\left[\frac{\pi}{2} - 2\phi + \chi\right] + w'[0]\right\}$$
$$+ \mathrm{V}^2\left\{vw\left[\frac{\pi}{2} - \phi - \chi - \theta\right] - vw\left[\frac{\pi}{2} - \phi + \chi + \theta\right] + ww[-\phi - \theta]\right.$$
$$\left. - ww'[\phi + \theta] - vw'\left[\frac{\pi}{2} - 2\phi - \theta\right] - w^2[-\chi - \theta]\right\}$$
$$+ \mathrm{V}\left\{vw^2\left[\frac{\pi}{2} - \chi\right] - w'w^2[0]\right\} + vw^2w'\left[\frac{\pi}{2} + \theta\right].$$

Hence

$$\begin{aligned} D_1D_2\sin\frac{A+B}{2} &= V^4\sin(\chi+\theta)+V^3\{-2w\sin(\phi-\chi)-v\cos(2\phi-\chi)\} \\ &\quad +V^2\{-2vw\sin\phi\sin(\chi+\theta)-2ww'\sin(\phi+\theta) \\ &\qquad -vw'\cos(2\phi+\theta)+w^2\sin(\chi+\theta)\} \\ &\quad +Vvw^2\cos\chi+vw^2w'\cos\theta; \end{aligned}$$

$$\begin{aligned} D_1D_2\cos\frac{A+B}{2} &= V^4\cos(\chi+\theta)+V^3\{w'-v\sin(2\phi-\chi)\} \\ &\quad +V^2\{2vw\cos\phi\sin(\chi+\theta)-vw'\sin(2\phi+\theta) \\ &\qquad -w^2\cos(\chi+\theta)\} \\ &\quad +V\{vw^2\sin\chi-w'w^2\}-vw^2w'\sin\theta. \end{aligned}$$

From these it can be shown, after easy reduction, that

$$\begin{aligned} D_1D_2&\left\{V\cos\frac{A+B}{2}+v\cos\frac{A-B}{2}\right\} \\ &=(V^2-v^2)(V^2-w^2)\{V\cos(\chi+\theta)+w'\} \\ &=(V^2-v^2)(V^2-w^2)\{(V\cos\theta-u)\cos\chi-(V\sin\theta-v)\sin\chi\} \end{aligned}$$

$$\begin{aligned} D_1D_2&\left\{V\sin\frac{A+B}{2}-u\cos\frac{A-B}{2}\right\} \\ &=(V^2-w^2)[(V\cos\theta-u)\{(V^2-v^2)\sin\chi+2uv\cos\chi\} \\ &\qquad +(V\sin\theta-v)(V^2-u^2)\cos\chi]. \end{aligned}$$

Hence

$$\tan\psi=\frac{(V^2-v^2)\{(V\cos\theta-u)\cos\chi-(V\sin\theta-v)\sin\chi\}}{(V\cos\theta-u)\{(V^2-v^2)\sin\chi+2uv\cos\chi\}+(V\sin\theta-v)(V^2-u^2)\cos\chi} \qquad (17)$$

When the source is fixed to the apparatus ($V\sin\theta=v$),

$$\cot\psi=\tan\chi+\frac{2uv}{V^2-v^2}.$$

That is, if the ratio U/V is very small,

$$\psi=\frac{\pi}{2}-\chi;$$

so that the maximals make for all directions of drift the same angle with the first mirror as the second does, but oppositely inclined.

12. *Position of Central Maximal.*—Let the central maximal

(fig. 5) cut the plane of the first mirror in W, and denote O_1W by x. Draw through O_1 lines parallel to the respective wave-fronts.

The phase of that belonging to the first set at O_1 is the same as that of the incident at O_1, say Π.

The phase of wave-front through W

$$=\Pi-\frac{x\sin A}{\lambda_1}.$$

The phase of the wave-front through O_2 of the second set is the same as that of the incident at O_2, that is of the

$$\text{incident at } O_1+\frac{O_1O_2\sin(\phi+\theta)}{\lambda}=\Pi+\frac{a\sin(\phi+\theta)}{\lambda},$$

since the incident ray makes $\frac{1}{2}\pi-\phi-\theta$ with O_1O_2.

The wave-front through O_1 belonging to this set has phase

$$\Pi+\frac{a\sin(\phi+\theta)}{\lambda}+\frac{O_1M}{\lambda_2}=\Pi+a\left\{\frac{\sin(\phi+\theta)}{\lambda}+\frac{\cos(\phi+B)}{\lambda_2}\right\},$$

and phase of that through W

$$=\Pi+a\left\{\frac{\sin(\phi+\theta)}{\lambda}+\frac{\cos(\phi+B)}{\lambda_2}\right\}-\frac{x\sin B}{\lambda_2}.$$

Since W is on the central maximal these phases are equal. Therefore

$$x\left\{\frac{\sin B}{\lambda_2}-\frac{\sin A}{\lambda_1}\right\}=a\left\{\frac{\cos(\phi+B)}{\lambda_2}+\frac{\sin(\phi+\theta)}{\lambda}\right\}.$$

Now $\lambda_1:\lambda_2=V+U\sin(\alpha-A):V+U\sin(\alpha-B)$.

Hence

$$x[\{V+U\sin(\alpha-A)\}\sin B-\{V+V\sin(\alpha-B)\}\sin A]$$

$$=a\{V+U\sin(\alpha-A)\}\left\{\cos(\phi+B)+\frac{(V^2-w^2)(V^2-w'^2)}{D_2^2}\sin(\phi+\theta)\right\},$$

$$2x\sin\frac{B-A}{2}\left\{V\cos\frac{A+B}{2}+v\cos\frac{A-B}{2}\right\}$$

$$=\frac{(V^2-v^2)(V^2-w^2)}{D_1^2D_2^2}\{V-U\cos(\theta-\alpha)\}\{D_2^2\cos(\phi+B)$$

$$+(V^2-w^2)(V^2-w'^2)\sin(\phi+\theta)\}.$$

Substituting the values for the factors on the left already obtained, and dividing out,

$$x\mathrm{V}\{\mathrm{V}\cos(\chi+\theta)+w'\}\mathrm{Q}=a\{\mathrm{D}_2^2\cos(\phi+\mathrm{B})$$
$$+(\mathrm{V}^2-w^2)(\mathrm{V}^2-w'^2)\sin(\phi+\theta)\}.$$

From the value of $e^{\frac{\mathrm{B}\iota}{2}}$ in § 8 it easily follows that

$$\mathrm{D}_2^2e^{(\phi+\mathrm{B})\iota}=\mathrm{V}^4\left[\frac{\pi}{2}-\phi+2\chi+\theta\right]+\mathrm{V}^3\left\{2w'\left[\frac{\pi}{2}-\phi+\chi\right]-2w\left[\frac{\pi}{2}\right]\right\}$$
$$+\mathrm{V}^2\left\{-2ww'\left[\frac{\pi}{2}+\chi+\theta\right]-2ww'\left[\frac{\pi}{2}-\chi-\theta\right]\right.$$
$$\left.+w'^2\left[\frac{\pi}{2}-\phi-\theta\right]+w^2\left[\frac{\pi}{2}+\phi-2\chi-\theta\right]\right\}$$
$$+\mathrm{V}\left\{-2ww'^2\left[\frac{\pi}{2}\right]+2w^2w'\left[\frac{\pi}{2}+\phi-\chi\right]\right\}+w^2w'^2\left[\frac{\pi}{2}+\phi+\theta\right];$$

whence after a short reduction,

$$\mathrm{D}_2^2\cos(\phi+\mathrm{B})+(\mathrm{V}^2-w^2)(\mathrm{V}^2-w'^2)\sin(\phi+\theta)$$
$$=2\mathrm{V}(\mathrm{V}^2-w^2)\{\mathrm{V}\cos(\chi+\theta)+w'\}\sin(\phi-\chi).$$

Therefore

$$x=2a\frac{\mathrm{V}^2-w^2}{\mathrm{Q}}\sin(\phi-\chi). \quad . \quad . \quad . \quad . \quad (18)$$

Since this is independent of θ, *the position of the central point is the same whether the source is fixed in the æther or to the apparatus.*

13. We have now obtained the magnitudes of the various quantities required for the discussion of the experiment. For this purpose, however, it will be convenient to express them in terms of quantities defining the configuration of the apparatus. Let **C** denote the angle between the two mirrors, and let **A**, **B** denote the angles which the first and second mirrors respectively make with the plate. Then

$$\mathbf{C}=\tfrac{1}{2}\pi-\chi,$$
$$\mathbf{A}=\tfrac{1}{2}\pi-\phi,$$
$$\mathbf{B}=\phi-\chi.$$

Further, denote the distance of O_1 from the plane of the second mirror by b. Then

$$b=a\sin\mathbf{B}=a\sin(\phi-\chi).$$

It will be convenient to collect the formulæ expressed in these new quantities :—

$$\frac{\lambda_1-\lambda_2}{\lambda}=\frac{\mathrm{U}(\mathrm{V}^2-w^2)\mathrm{PQ}}{\mathrm{D}_1^2\mathrm{D}_2^2},$$

$$\sin\tfrac{1}{2}(\mathrm{B}-\mathrm{A})=\frac{1}{2}\,\frac{\mathrm{V}\{\mathrm{V}-\mathrm{U}\cos(\theta-\alpha)\}}{\mathrm{D}_1\mathrm{D}_2}\mathrm{Q},$$

$$p=\frac{\lambda}{\mathrm{Q}}\cdot\frac{(\mathrm{V}^2-v^2)(\mathrm{V}^2-w^2)(\mathrm{V}^2-w'^2)}{\surd\{\mathrm{U}^2\mathrm{P}^2+\mathrm{V}^2(\mathrm{V}^2-v^2)(\mathrm{V}^2-w'^2)(\mathrm{V}-\mathrm{U}\cos\overline{\theta-\alpha})^2\}},$$

$$\tan\psi=\frac{(\mathrm{V}^2-v^2)\{(\mathrm{V}\cos\theta-u)\sin\mathbf{C}-(\mathrm{V}\sin\theta-v)\cos\mathbf{C}\}}{(\mathrm{V}\cos\theta-u)\{\mathrm{V}^2-v^2)\cos\mathbf{C}+2uv\sin\mathbf{C}\}+(\mathrm{V}\sin\theta-v)(\mathrm{V}^2-u^2)\sin\mathbf{C}}.$$

$$x=2b\,\frac{\mathrm{V}^2-w^2}{\mathrm{Q}};$$

where

$$\begin{aligned}\mathrm{P}&=\mathrm{V}\{\mathrm{V}^2\sin(\mathbf{C}+\alpha-\theta)-\mathrm{VU}\sin\mathbf{C}+\mathrm{U}^2\sin\alpha\sin(\alpha-\theta)\sin(\mathbf{C}-\alpha)\}\\&=(\mathrm{V}\cos\theta-u)\{\mathrm{V}^2\sin(\mathbf{C}+\alpha)+v^2\sin(\mathbf{C}-\alpha)\}\\&\qquad-(\mathrm{V}\sin\theta-v)\{\mathrm{V}^2\cos(\mathbf{C}+\alpha)+uv\sin(\mathbf{C}-\alpha)\},\end{aligned}$$

$$\mathrm{Q}=2\mathrm{V}^2\sin(\mathbf{B}-\mathbf{A})-\mathrm{U}^2\{\sin(\mathbf{B}-\mathbf{A})+\sin(2\alpha-2\mathbf{A})\cos\mathbf{C}+\sin(\mathbf{C}-2\alpha$$

For the case of $\mathrm{V}\sin\theta=v$, source fixed to apparatus,

$$\sin\tfrac{1}{2}(\mathrm{B}-\mathrm{A})=\frac{1}{2}\,\frac{\mathrm{V}\cos\theta(\mathrm{V}\cos\theta-u)}{\mathrm{D}_1\mathrm{D}_2}\mathrm{Q},$$

$$p=\frac{\Lambda\cos\theta}{\mathrm{V}^2\mathrm{Q}}\,\frac{(\mathrm{V}^2-v^2)(\mathrm{V}^2-w^2)(\mathrm{V}^2-w'^2)}{\surd[(\mathrm{V}^2-v^2)^2\sin^2\mathbf{C}+\{(\mathrm{V}^2-u^2)\cos\mathbf{C}+2uv\sin\mathbf{C}\}^2]}$$

$$\cot\psi=\cot\mathbf{C}+\frac{\mathrm{U}^2}{\mathrm{V}^2-v^2}\sin2\alpha,$$

$$\mathrm{P}=(\mathrm{V}\cos\theta-u)\{\mathrm{V}^2\sin(\mathbf{C}+\alpha)+v^2\sin(\mathbf{C}-\alpha)\}.$$

When there is no motion,

$$\lambda_1=\lambda_2=\lambda,$$

$$\sin\tfrac{1}{2}(\mathrm{B}-\mathrm{A})=\sin(\mathbf{B}-\mathbf{A}),$$

$$p=\frac{\lambda}{2\sin(\mathbf{B}-\mathbf{A})},$$

$$\psi=\mathbf{C},$$

$$x=\frac{b}{\sin(\mathbf{B}-\mathbf{A})}.$$

If b be measured in wave-lengths $=n\lambda$ say,

$$x = \frac{n\lambda}{\sin(\mathbf{B}-\mathbf{A})} = 2np;$$

or there are twice as many bands from O_1 to the central band as there are wave-lengths in O_1L.

When there is motion Q has the same value as if the apparatus were at rest when the direction of drift is given by

$$\sin(\mathbf{B}-\mathbf{A}) + \sin(2\alpha-2\mathbf{A})\cos\mathbf{C} + \sin(\mathbf{C}-2\alpha) = 0, \quad . \quad (19)$$

which gives two real values for $\tan\alpha$. In the actual case $\mathbf{C}$ is nearly 90° and $\mathbf{B}$ nearly $=\mathbf{A}$; whence α is nearly $\frac{\pi}{4}$ or $\pi+\frac{\pi}{4}$, i. e. *the drift produces no effect when it is nearly parallel to the plane of the plate.*

The interfering waves have the same frequencies when either

(1) $U=0$,
(2) $w=\pm V$,
(3) $Q=0$,
(4) $P=0$.

(1) is the case of no motion.

(2) makes the velocity of the plate perpendicular to itself equal to that of light—a case which may be put aside.

(3) is the case which will have to be discussed immediately.

(4) gives the direction of drift to be that of the maximal lines. It is the case, indeed, already discovered by general reasoning in § 3. To prove the statement it is necessary to show that when $\alpha=\pi-\psi$, $P=0$. It may be sufficient to do this for the case of a fixed source. The equations are:—

$$-\cot\alpha = \cot\mathbf{C} + \frac{2uv}{V^2-v^2},$$

$$P = (V\cos\theta - u)\{V^2\sin(\mathbf{C}+\alpha) + v^2\sin(\mathbf{C}-\alpha)\}$$
$$= (V\cos\theta - u)\{(V^2+v^2)\sin\mathbf{C}\cos\alpha + (V^2-v^2)\cos\mathbf{C}\sin\alpha\}.$$

Now the first equation gives at once

$$(V^2-v^2)\cos\mathbf{C}\sin\alpha + \cos\alpha\sin\mathbf{C}(V^2-v^2) + 2uv\sin\mathbf{C}\sin\alpha = 0,$$

and $$u\sin\alpha = v\cos\alpha,$$

which makes the second factor of P vanish.

14. The preceding formulæ hold whatever the velocity of drift may be. It is, however, extremely unlikely that this

velocity amounts to anything approximating to 100 times the velocity of the earth in its orbit, which latter is roughly 10^{-4} times that of light. Even in this extreme case the square of the ratio of the velocities may be neglected in all terms except those which are divided by a small quantity. In the discussion which follows we shall suppose the conditions to be those of the actual apparatus of Michelson and Morley, and that the square of the velocity ratios may be neglected except under the circumstances mentioned above. The ratio U/V will be denoted by ξ.

The breadth of a band is then given by

$$p=\frac{\Lambda}{2\sin(\mathbf{B}-\mathbf{A})-\xi^2\{\sin(\mathbf{B}-\mathbf{A})+\sin(2\alpha-2\mathbf{A})\cos\mathbf{C}+\sin(\mathbf{C}-2\alpha)\}}.$$

In the particular apparatus **A**, **B** were nearly 45° and **C** nearly 90°. Hence in terms which multiply ξ^2 we may put $\mathbf{A}=\mathbf{B}=45^\circ$, $\mathbf{C}=90^\circ$ exactly. Then

$$p=\frac{\Lambda}{2\sin(\mathbf{B}-\mathbf{A})-\xi^2\cos 2\alpha}.$$

If $\mathbf{B}-\mathbf{A}$ is $<\frac{1}{2}\xi^2$, the breadths of the bands of a fringe will change from the breadth they would have without drift to infinity, as the direction of the drift alters.

In the case of $\xi=10^{-4}$, $\Lambda=5\,.\,10^{-5}$ cm., and $\mathbf{B}-\mathbf{A}=\frac{1}{2}\xi^2$, the breadth of a band when the drift is parallel to the plate is 50 metres; so that such a disposition would be impossible to observe. If the velocity of drift were 100 times that of the earth's orbit, this minimum breadth would be 5 mm., and under these circumstances the observations would have shown enormous variations in the breadth of the bands.

It may be taken as certain then that $\mathbf{B}-\mathbf{A}$ was considerably larger than ξ^2, although in itself exceedingly small. If c denote the breadth of band when there is no drift or when $\alpha=\frac{\pi}{4}$, then in general

$$p=\frac{c}{1-\frac{\xi^2}{2\sin(\mathbf{B}-\mathbf{A})}\cos 2\alpha}.$$

It has been seen that for all probable values of ξ, $\sin(\mathbf{B}-\mathbf{A})$ will be considerably greater than $\frac{1}{2}\xi^2$, so that $\xi^2/2\sin(\mathbf{B}-\mathbf{A})$ will be a small quantity; whence

$$p=c\left\{1+\frac{1}{2}\frac{\xi^2\cos 2\alpha}{\sin(\mathbf{B}-\mathbf{A})}\right\}.$$

Also to this order $\psi = C$, or the direction of the maximals is constant.

15. It remains to determine the magnitude of the effect referred to in § 4. By means of the lenses of the eye, aided or not by an optical instrument, an image is thrown on the retina of points on a certain plane on which the instrument is focussed. Of any such point P an image is formed at Q (say) on the retina. Through P two rays (belonging to the two systems of waves) pass, which by the optical apparatus are again brought to pass through Q, having passed over the same optical distance D′, say. The waves travel with the same velocity but have different wave-lengths. Suppose at P the phase-difference is Π, *i. e.* the λ_2 phase greater by Π than the λ_1 phase. Then in the wave diagram the two paths (D′) are occupied by waves of different lengths λ_1, λ_2, and therefore at Q the phase-difference of λ_2 over λ_1 is $\Pi + \frac{D'}{\lambda_2} - \frac{D'}{\lambda_1}$, or there is an increased difference of phase of amount $D'\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)$. Consequently the resulting intensity at Q will not be the same as that at P—the maxima and minima on the retina are not the images of the maxima and minima of the object viewed. Now the λ_2 waves are ahead of the λ_1 on the right hand of the maximals. Hence in order that Q may be the central bright band it must be the optical image of a point on the left of the centre of the original fringe, for at Q the λ_2 waves lead still more than at P, and the distance from it must be $D'\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)$ bands.

Now by (8)

$$\frac{\lambda_1 - \lambda_2}{\lambda} = \frac{U(V^2 - w^2)P\,.\,Q}{D_1^2 D_2^2},$$

$$\frac{\lambda_1 \lambda_2}{\lambda^2} = \frac{(V^2 - w^2)^2(V^2 - v^2)(V^2 - w'^2)}{D_1^2 D_2^2};$$

$$\therefore \quad D' \frac{\lambda_1 - \lambda_2}{\lambda_1 \lambda_2} = \frac{D'}{\lambda} \cdot \frac{UPQ}{(V^2 - w^2)(V^2 - v^2)(V^2 - w'^2)}$$

$$= \frac{D'(V \cos\theta - u)}{\lambda} \cdot \frac{U \cos\alpha\,.\,Q}{V^4}$$

$$= \frac{D'}{\Lambda} \frac{Q}{V^2} \xi \cos\alpha$$

$$= \frac{D'}{p} \xi \cos\alpha\,;$$

and $D'\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2}$ bands $=D'\xi\cos\alpha$ cm.

Hence the distance of the observed central fringe is

$$x=D+\frac{D\xi^2}{\Lambda}c\cos 2\alpha+D'\xi\cos\alpha. \quad . \quad . \quad . \qquad (20)$$

In the actual experiment D was about 11 metres, and, so far as can be estimated from the description, D′ nearly 3 metres.

The relative importance of these terms depends for given ξ on the value chosen for the breadth of the band. No indication is given of this, but it must probably have been adjusted to something between $\frac{1}{10}$ mm. and 10 mm. The following are a few cases, taking $\Lambda=5\,.\,10^{-5}$ cm. and writing $x=a+b\cos 2\alpha+c\cos\alpha$, and measuring in millims.,

	Band	·1	1	10
$\xi=10^{-4}$ Earth's orbital velocity.	b	·2	2	20
	c	·3	·3	·3
$\xi=10^{-5}$.	b	·002	·02	·2
	c	·03	·03	·03

For $\xi=10^{-4}$ a breadth of ·1 mm. would therefore make both terms equally sensible; whilst for $\xi=10^{-5}$, the second would overpower the first, but would be of equal weight for a breadth of 1 mm. If $D'=D$, *i. e.* if the instrument was focussed on the actual first mirror, then for $\xi=10^{-5}$ a breadth of band of 1 cm. would make both terms about equally sensible.

Finally, aberration would displace the apparent position to the right. Hence

$$x=a+b\cos 2\alpha+c\cos\alpha-d\sin\alpha,$$

where d involves the first power of ξ.

16. From conversation with Professor Morley I learn that in the actual experiment no slit was used, and that the reading-telescope was focussed on the first mirror. To complete the theory therefore it is necessary to take account of the finite breadth of the flame. The flame may be regarded as built up of vertical narrow strips of light, from each of which, as has been seen, there results a system of fringes on whatever point the reading-telescope is focussed. It is necessary then to see whether these fringes overlap, or whether there are places where the superposed fringes exactly fit. Now the fringes are given by the maximals; and the

question is, Are there *loci* where the maximals for all the various component slits coincide?

The general formulæ developed above involve the angle θ, —the angle which the incident light makes with the first mirror. If a point on the flame makes an angle θ with the datum line, it will be sufficient to replace θ by $\theta_1+\theta$ in the various formulæ in order to obtain the result for this part of the flame.

The direction of drift now makes an angle $\alpha-\theta_1$ with the line from the slit to the centre of the plate. Hence

$$\begin{aligned}\mathrm{V}\sin\theta &= \mathrm{U}\sin(\alpha-\theta_1)\\ &= v\cos\theta_1-u\sin\theta_1.\end{aligned}$$

The position of the white point—*i. e.* where the central maximal cuts the first mirror, is given by

$$x=2b\,.\,\frac{\mathrm{V}^2-w^2}{\mathrm{Q}},$$

θ_1 only enters through w. Hence when ξ is small x is independent of θ_1—that is, the central maximal cuts the plane of the first mirror in a point which is the same for all points of the plane.

Next the common direction of the maximals is given by (17). In this we put $\theta_1+\theta$ for θ. Now

$$\mathrm{V}\cos(\theta+\theta_1)-u=\mathrm{V}\cos\theta\cos\theta_1-\mathrm{V}\sin\theta_1\sin\theta-u,$$

where the source is fixed to the apparatus,

$$\mathrm{V}\sin\theta=v\cos\theta_1-u\sin\theta_1\,;$$

whence

$$\begin{aligned}\mathrm{V}\cos(\theta_1+\theta)-u&=\cos\theta_1\{\mathrm{V}\cos\theta-u\cos\theta_1-v\sin\theta_1\},\\ \mathrm{V}\sin(\theta_1+\theta)-v&=\sin\theta_1\{\mathrm{V}\cos\theta-u\cos\theta_1-v\sin\theta_1\}.\end{aligned}$$

Hence

$$\cot\psi=\frac{\mathrm{V}^2\sin(\theta_1+\chi)-v^2\sin\chi\cos\theta_1+2uv\cos\chi\cos\theta_1-u^2\cos\chi\sin\theta_1}{(\mathrm{V}^2-v^2)\cos(\theta_1+\chi)},$$

or

$$\begin{aligned}\cot\psi&=\tan(\chi+\theta_1)+\frac{v^2\sin\theta_1+2uv\cos\theta_1-u^2\sin\theta_1}{\mathrm{V}^2-v^2}\cdot\frac{\cos\chi}{\cos(\chi+\theta_1)}\\ &=\tan(\chi+\theta_1)+\frac{\mathrm{U}^2\sin(2\alpha-\theta_1)}{\mathrm{V}^2-v^2}\cdot\frac{\cos\chi}{\cos(\chi+\theta_1)}.\end{aligned}$$

When ξ^2 is small

$$\psi=\tfrac{1}{2}\pi-(\chi+\theta_1).$$

The maximals for different small strips of the flame are therefore differently inclined. It has been seen that these intersections with the plane of the first mirror coincide. Hence they coincide nowhere else. Consequently a fringe will be seen if the telescope is focussed on the first mirror, which will gradually become more and more indistinct as the plane on which it is focussed recedes more and more from it.

If γ denote the angular breadth of the flame as seen from the point on the plate where the datum line meets it, the maximals for the various points of the flame will form a system of pencils of angular breadth γ, whose vertices pass through the maximal points on the first mirror which have just been shown to be the same for every point on the flame. The fringe on a screen parallel to the first mirror will then completely fade into white light when its distance from the first mirror is such that the pencils intersect, each, the succeeding one. This takes place at a distance y such that $y\gamma$=breadth of a band. At distances greater than y no fringes can be seen at all. At distances nearly as great as y we should expect dark lines on a more or less uniformly bright background.

Discussion of the Displacement of Fringe.

17. The position of the central band is given by

$$x = \frac{b}{\sin(\mathbf{B}-\mathbf{A}) - \frac{1}{2}\xi^2 \cos 2\boldsymbol{\alpha}}.$$

If then $\mathbf{B}>\mathbf{A}$, b must be positive, that is the plane of the second mirror must lie to the right of the intersection of the first mirror and the plate. If on the other hand $\mathbf{A}>\mathbf{B}$, b must be negative, or the second plane must lie to the left of the same intersection. It will be convenient to distinguish these two cases as the B and the A type respectively.

Suppose an experiment to start with the drift in the same direction as the incident light. Then as the drift alters from this position in either direction, the central band is displaced to the right in the B type and to the left in the A type. As $\mathbf{A}-\mathbf{B}$ is exceedingly small—of order 10^{-5} (or 2 sec.) at most—the adjustment of the mirrors can easily change from one type to the other on consecutive days. It follows that averaging the results of different days in the usual manner is not allowable unless the types are all the same. If this is not attended to the average displacement may be expected to come out zero—at least if a large number are averaged.

If D denote the distance of the central band from the intersection of the plate and mirrors, when there is no drift—or when the drift is parallel to the plate—

$$D = \frac{b}{\sin(\mathbf{B}-\mathbf{A})}$$

$$x = \frac{D\sin(\mathbf{B}-\mathbf{A})}{\sin(\mathbf{B}-\mathbf{A})-\frac{1}{2}\xi^2\cos 2\alpha}.$$

$$\left.\begin{matrix}\text{Displacement to}\\ \text{left of this position}\end{matrix}\right\} = x - D = \frac{\frac{1}{2}\xi^2 D\cos 2\alpha}{\sin(\mathbf{B}-\mathbf{A})-\frac{1}{2}\xi^2\cos 2\alpha}.$$

In any given case suppose $\sin(\mathbf{B}-\mathbf{A}) = \frac{1}{2}k\xi^2$, then

$$\text{Displacement} = \frac{D\cos 2\alpha}{k-\cos 2\alpha}.$$

Fig. 6.

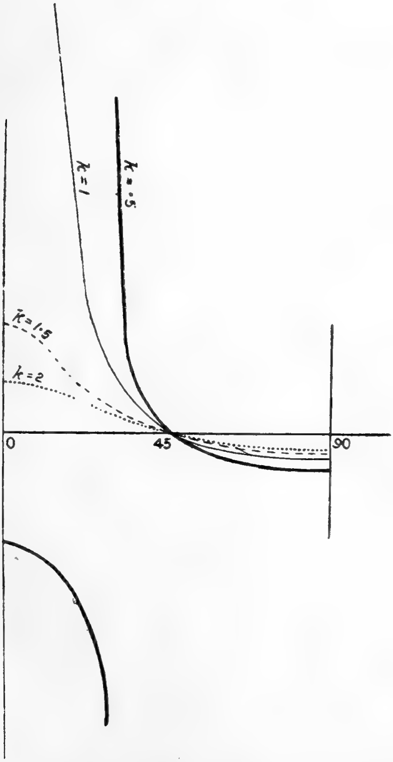


Fig. 6 shows how the displacement changes with α, for the four cases of $k=\cdot 5$, 1, 1·5, and 2. When $k<1$, these curves have an asymptote which as k increases from 0 to 1 moves up to the origin. The negative infinity branch becomes narrower and at the same time moves off to infinity. When $k=1$ there is no negative branch, but a series of curves with positive infinite peaks. As k increases beyond 1, these peaks shorten until when k is large the curves become the ordinary harmonic curve $y=k\cos 2\alpha$. In Michelson and Morley's experiment k was apparently always large.

Discussion of Michelson and Morley's Observations.

18. The result of §17, that it is not allowable to average the results of different sets of observations until the type of each has been determined, naturally leads us to a reconsideration of the numerical data obtained by Michelson and Morley, who did lump together the observations taken on different days. I propose to show that, instead of giving a null result, the numerical data published in their paper show distinct evidence of an effect of the kind to be expected.

It may here be recalled that in taking an observation, the apparatus was rotated in its mercury bath and readings taken at 16 equidistant points as the reading-telescope passed them. On each occasion this was repeated six times, and the means of the six readings in each position taken. These means are the numbers printed in their paper. They are given for noon on July 8, 9, and 11, and for 6 P.M. on July 8, 9, and 12. The means of these three days are taken and then the means of the first eight and of the second eight, thus eliminating any effect depending on $\cos\alpha$ alone. The result is that there is no apparent displacement of the fringe.

In looking at the sets of readings, one is struck at once with the fact that all the readings continuously increase or decrease. This is evidently the effect of temperature changes. For short intervals, it is extremely likely that the temperature disturbances will be a linear function of the time. If this is exactly so, and if the readings were taken at equal intervals of time, it is possible to eliminate the disturbances due to this. For the readings at the beginning and at the end of a complete revolution ought (in absence of temperature effects) to be the same, whilst on the supposition made above there would be a temperature error altering by equal steps for each successive reading—in a way to be indicated immediately. The readings for each set of complete revolutions should first be corrected in this way and then the average of the six taken to eliminate accidental and personal errors. I have applied this correction to the published means. The result is the same as if the correction had first been applied to each series and the results then averaged. This would have been preferable if possible, as a comparison of each set with the average would have given data for a measure of the probable error.

To illustrate the method of applying this correction, I give here the working of the 6 P.M. observations on July 8 :—

I.	61·2,	63·3,	63·3*,	68·2,	67·7,	69·3,	70·3,	69·8,	69·0
II.	61·2,	62·1,	63·0,	63·9,	64·8,	65·7,	66·6,	67·5,	68·4
III.	0	+1·2	+·3	+4·3	+2·9	+3·6	+3·7	+2·3	+·6
I.	69·0,	71·3,	71·3,	70·5,	71·2,	71·2,	70·5,	72·5,	75·7
II.	68·4,	69·3,	70·2,	71·1,	72·0,	72·9,	73·8,	74·7,	75·7
III.	+·6	+2·0	+1·1	−·6	−·8	−1·7	−3·3	−2·2	+0
	0	+1·2	+·3	+4·3	+2·9	+3·6	+3·7	+2·3	+·6
IV.	·6	+3·2	+1·4	+3·7	+2·1	+1·9	+·4	+·1	+·6
Deduct 1·7. } V.	−1·1	+1·5	−·3	+2·0	+·4	+·2	−1·3	−1·6	−1·1

In the above, line I. gives the published observations. Line II. gives a series of numbers, increasing by equal steps from 61·2 to 75·7. Line III. gives the differences of I. & II., and therefore, according to our supposition, freed from the temperature errors. Line IV. gives the mean of the first eight and the last eight, and therefore eliminates effects depending on cos α (see § 15). Line V. is the result obtained by deducting 1·7 from each, so as to reduce readings to give deviations from the mean, 1·7=sum/8.

The other sets are treated in the same way. The results are as follows :—

Noon Observations.

July 8. −1·0+0·2+1·2+0·7+2·8−1·1+0·2+2·6−1·0
July 9. −1·7−1·8−0·2+0·3+0·4+1·1+2·2−1·4−1·7
July 11. 0·9−2·2−2·1−2·7+0·3+2·0+1·9+1·7+0·9

P.M. Observations.

July 8. −1·1+1·5−0·3+2·0+0·4+0·2−1·3−1·6−1·1
July 9. −1·3−0·7+0·1−0·5+1·1+1·6+0·3−0·6−1·3
July 12. 0·4−1·0−2·2−1·6−2·0+1·4+3·3+1·5+0·4

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* If this were a MS. misreading for 65·3, the correct numbers in lines III., IV., and V. would be 2·3, 3·4, and 1·7, which, as a glance at the curve in Plate I. will show, would make the curve regular.

These results are represented graphically on Plate I. All the curves give distinct evidence of a cos 2α effect, except noon of July 8, and possibly of 6 P.M. of July 8; the latter certainly if the supposition of the footnote is correct. Moreover, the curves clearly show that the observations of July 9, 11, 12 belong to one type (B or A), and those of July 8 to the other (A or B). The last curve represents the average of all, on this supposition. That is the value of the ordinates are one-third of July 9 + July 11 − July 8 and July 9 + July 12 − July 8.

The evidence is also strengthened by comparison of the noon and P.M. curves. The drift at 6 P.M would be at right-angles to that at noon, consequently we should expect the curve to be shifted half a period (*i. e.* 90°) with reference to the first. Now in the P.M. observations, the rotation of the apparatus was in the opposite direction to that at noon. Consequently, in order to compare the curves, the numbers for the P.M. curves should be read backwards. This is done in the dotted curves in Pl. I. A glance at the curves renders evident the fact that the shift shown is of the right character. The amplitude of the P.M. curve should be less than that of the noon in the ratio $\sin^2\lambda$* where λ is the latitude.

The preceding attempt to get rid of the temperature effect is not proposed as one which gives an accurate result. The object is to show that the observations of Michelson and Morley do give an affirmative answer to the question "Is there a drift of æther past the earth?" The argument is sufficient to show that the experiments should be repeated with extreme care to eliminate temperature errors, and if possible *in vacuo*. If possible the absence of observers would be desirable, and, for the reason stated in § 15, also the absence of reading-telescopes. We have seen that if a slit is used for the light source, fringes are formed on a screen placed in any position. This points to a method in which a photographic film on a rotating drum is exposed automatically.

The FitzGerald-Lorentz Effect.

19. Amongst the various explanations advanced to account for the supposed null result of Michelson and Morley's experiment, the best known and accepted is that first proposed, I believe, by G. F. FitzGerald, viz., that the very motion of a

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* That is supposing the component of the drift perpendicular to the plane of the apparatus to produce no effect. It probably does produce an effect, however.

solid through the æther produces a small extension perpendicular to the direction of drift, or contraction in the direction, the amount being proportional to ξ^2. This has received some justification from the fact that on the theories of Larmor and of Lorentz as to the connexion between matter and æther, a contraction such as is indicated should be expected. At first sight—especially in the theory of the experiment as given by Michelson and Morley—an effect of this kind of the proper amount might seem capable of annulling any observed displacement. If, however, the effect of such a contraction on the displacement of the fringe be worked out on the lines of the rigorous theory developed in the present paper, it will be found that not only is it incapable of explaining the null result, but that in fact it should increase the displacement to be observed. To produce annulment an extension along the line of drift is required. It is the object of the present section to prove this and to show how by a suitable modification of Michelson and Morley's experiment it lies at our disposal to test the truth or otherwise of Larmor and Lorentz's result, and, if such contraction exists, to measure its amount.

We shall assume then that when a solid moves through the æther, it suffers a contraction along the direction of the drift such that a length l parallel to it is contracted by an amount $k\xi^2 l$. The effective changes in the apparatus are those which occur in the horizontal plane. If then the total drift makes an angle I with the plane of the apparatus, lines in the plane will be distorted as if there were a contraction $k\xi^2 \cos \mathrm{I} \, . \, l$ in the direction of the component of the drift in the plane, or—if ξ represent this component as in the preceding pages—a contraction $k\xi^2 l/\cos \mathrm{I}$. In what immediately follows we shall replace $k/\cos \mathrm{I}$ by k.

20. Owing to the distortion produced both the lengths of lines and the magnitudes of angles lying in the plane will be altered. It is first necessary to determine this alteration.

In fig. 7 (p. 40) AB denotes any line, LB the direction of drift. Draw AL perpendicular to BL.

Through the contraction B comes to B′ where

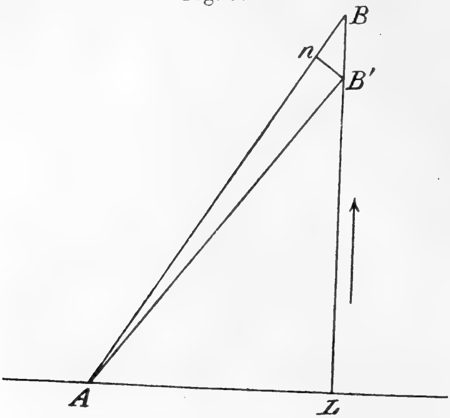
$$\mathrm{BB}' = k\xi^2 \, . \, \mathrm{BL}$$

and the line AB is displaced to AB′. If $\mathrm{AB} = r$, $\mathrm{ABL} = \theta$,

$$\begin{aligned}\delta r &= \mathrm{B}n = k\xi^2 \, . \, \mathrm{BL} \cos \mathrm{ABL} \\ &= k\xi^2 r \cos^2 \theta = \tfrac{1}{2} k\xi^2 r (1 + \cos 2\theta), \\ \delta\theta &= \mathrm{BAB}' = \frac{\mathrm{B}'n}{r} = \frac{k\xi^2 \, . \, \mathrm{BL} \sin \theta}{r} \\ &= \tfrac{1}{2} k\xi^2 \sin 2\theta.\end{aligned}$$

These formulæ hold only when $k\xi^2$ is small. Hence the following results are not applicable to cases where the velocity of drift is comparable with that of light. Further, we shall suppose ξ^2 so small that it may be neglected in comparison with unity. In this case we write V^2 for V^2-w^2 and

Fig. 7.



$$x=\frac{2b\sin\mathbf{C}}{Q}$$

$$=\frac{b\sin\mathbf{C}}{\sin(\mathbf{B}-\mathbf{A})-\frac{1}{2}\xi^2.R},$$

where
$$R=(1-\cos\mathbf{C})\sin(\mathbf{C}-2\alpha)$$
$$=\cos 2\alpha \text{ when } \mathbf{C}=90^\circ.$$

The change in $b\sin\mathbf{C}$ will be of order ξ^2 and may be neglected in comparison with it. So, too, the change in ξ^2R will be of order ξ^4 and may be neglected. There remains only the change in $\mathbf{B}-\mathbf{A}$.

In fig. 8, O_1X, O_2Z denote the directions of the planes of the mirrors and O_1Y that of the plate. XYZ is any line perpendicular to the direction of drift. Then to compute the deformation we have

$$\mathbf{A}=XO_1Y=XO_1L-YO_1L,$$
$$\delta\mathbf{A}=\delta\,.\,XO_1L-\delta\,.\,YO_1L$$
$$=\tfrac{1}{2}k\xi^2\{\sin 2\alpha-\sin 2(\alpha-\mathbf{A})\}.$$

So also
$$\delta\mathbf{B}=\tfrac{1}{2}k\xi^2\{\sin 2(\alpha-\mathbf{A})+\sin 2(\mathbf{C}-\alpha)\};$$
$$\therefore\quad \delta(\mathbf{B}-\mathbf{A})=\tfrac{1}{2}k\xi^2\{\sin 2(\mathbf{C}-\alpha)+2\sin 2(\alpha-\mathbf{A})-\sin 2\alpha\}.$$

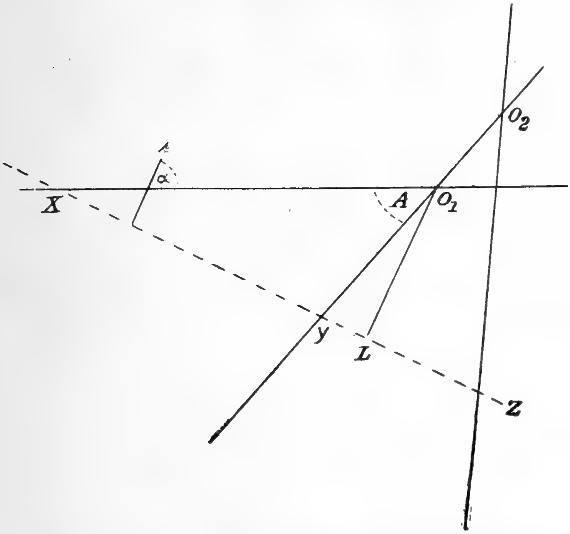
If any fringes are to be seen at all, **B**−**A** must be exceedingly small. Hence in the small term multiplied by ξ^2 we may put **B**=**A**, **C**=2**A**. Then it can be shown that

$$\delta(\mathbf{B}-\mathbf{A})=-k\xi^2(1-\cos\mathbf{C})\sin(\mathbf{C}-2\alpha)=-k\xi^2\mathrm{R}\,;$$

and, since $\cos(\mathbf{B}-\mathbf{A})=1$,

$$x=\frac{b}{\sin(\mathbf{B}-\mathbf{A})-(k+\frac{1}{2})\mathrm{R}\xi^2}.$$

Fig. 8.



21. In Michelson and Morley's experiment **C**=90°, $k=\cos 2\alpha$,

$$x=\frac{b}{\sin(\mathbf{B}-\mathbf{A})-(k+\frac{1}{2})\xi^2\cos 2\alpha}.$$

Hence to annul the effect k should be $-\frac{1}{2}$, and consequently an expansion instead of a contraction is necessary.

If now λ denote the true coefficient of contraction due to drift $k=\lambda/\cos\mathrm{I}$, and if the vertical component of drift produce (owing to motion of source) no direct effect in displacing the fringe

$$x=\frac{b}{\sin(\mathbf{B}-\mathbf{A})-(\lambda/\cos\mathrm{I}+\frac{1}{2})\xi^2\cos 2\alpha}.$$

Observations at noon and at 6 P.M. give the direction of the projection of the total drift on the plane of the apparatus —that is, we have the projections of the same direction on two known planes. These planes are the position of the horizontal at noon and the same plane turned 90° round the axis of the earth. Consequently the absolute *direction* of drift is determined, and the values of I for noon and 6 P.M. can be calculated—say I_1, I_2.

The magnitudes of the shift of the fringes then give

$$(\lambda/\cos I_1+\tfrac{1}{2})\Xi^2\cos^2 I_1 \quad\text{and}\quad (\lambda/\cos I_2+\tfrac{1}{2})\Xi^2\cos^2 I_2,$$

where Ξ is the ratio of total drift to velocity of light. Hence λ and Ξ can be separately determined.

III. *The Decomposition of Hydrogen Peroxide by Light, and the Electrical Discharging Action of this Decomposition.* By R. F. D'Arcy, *M.A.*[*]

THE progress of knowledge on the subject of the probable modes of development of electrical phenomena in the atmosphere during the last few years has been rapid. The difference in behaviour of positive and negative ions as nuclei of condensation, combined with the proved existence of ions as being normally present in the air, may be considered as establishing the precipitation theory of atmospheric electricity.

The object of the present paper is to give an account of experiments which were undertaken with the view of investigating a chemical action to account for a possible origin of these charged particles. It may be stated that the view adopted is that such a possible action is the decomposition of hydrogen peroxide *by light*[†]. The water formed by the decomposition being positively charged, and the oxygen (whatever its atomicity may be) being negatively charged.

The formation of hydrogen peroxide in nature is a well recognized fact. The probable formation of hydrogen peroxide by ultra-violet light in moist oxygen has been indicated by C. T. R. Wilson (Phil. Trans. 1899); the minute drops of hydrogen-peroxide solution being uncharged. The main ideas of the present paper are to suggest that hydrogen peroxide *when split up* yields two parts which are oppositely charged, and that in nature this splitting up may be brought about by the action of light.

The experiments made were of two kinds. Some were made to investigate the conditions under which hydrogen peroxide is decomposed by light, as, although this effect has

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[*] Communicated by the Author.

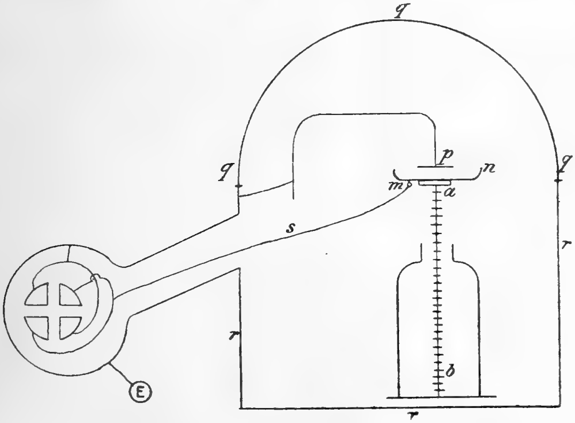
[†] The possible formation of ions by supersaturation has been considered by C. T. R. Wilson, Phil. Trans. vol. cxciii. A. 1900.

been very briefly described by Downes and Blunt[*], the author has been unable to find any account of a more varied examination of this effect. The details of these experiments are given in the Appendix. The results would seem to be of some importance in considering any chemical action of hydrogen peroxide. The other experiments were undertaken to investigate whether a surface of hydrogen peroxide undergoing this decomposition is capable of discharging electrification or not as compared with a water surface. Experiments were made with the surface both positively and negatively electrified. No clear indications were obtained in the case of positive electrification, so the experiments are not given; but evidence in the case of negative electrification seems definite.

Description of Experiments to show that a surface of Hydrogen Peroxide exposed to Sunshine in the open air loses a Negative charge of Electricity more rapidly than a Water surface.

In these experiments a shallow dish containing H_2O_2 or water was placed on an insulating support. Over the centre of the dish a horizontal piece of wire gauze permanently connected to the earth was fixed, the whole being screened electrically by being surrounded with a wire cage and box coated with tinfoil. This part of the apparatus was arranged on a shelf outside a window on the south side of the house. The general arrangement is shown diagrammatically in fig. 1.

Fig. 1.



n is the dish; *ab*, insulating stand; *qq*, wire cage; *r*, tin-coated box; *p*, wire gauze.

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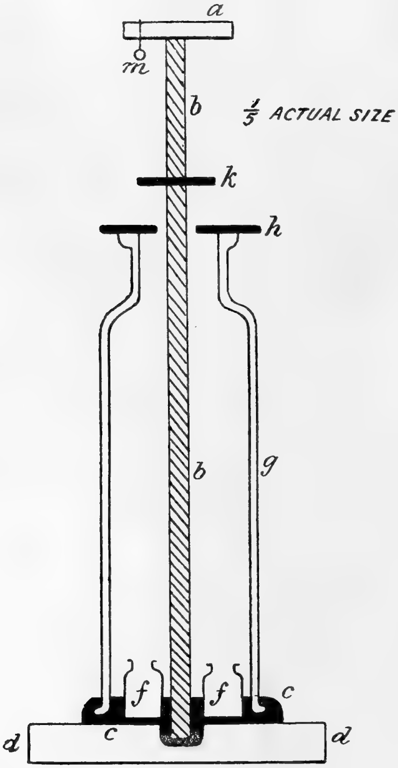
[*] 'Nature,' xx. p. 521.

The dish was connected by means of a wire *s* with one pair of quadrants of a quadrant-electrometer, the other quadrants being earthed. The needle had an independent positive charge. The following experiments required the instrument to be in a comparatively insensitive condition.

The only part of the apparatus of which a detailed description need be given is the insulating stand. A vertical section of this, drawn to scale, is given in fig. 2. Here *a* is a disk of varnished wood supported by *bb* an ebonite rod held in position by being cemented with sulphur *c* to the base board *d*. Let into the sulphur also are the drying-bottles *ff* (of which there are four, all containing sulphuric acid), and the tall glass gas-jar *g*, which was cleaned so as to have a good insulating surface internally when dry, this jar being cemented on with melted sulphur. An ebonite disk *h* was sealed to the top of the jar through a hole in the centre of which the ebonite rod passed; this hole was about 6 mm. wider than

Fig. 2.



the ebonite rod, and so an annular air-space of about 3 mm. was left surrounding the ebonite rod at this part. The small ebonite plate *k* could be pushed down when the apparatus was not in use to close this space, and so to exclude dust and damp.

The dish used in the experiments described was a guttapercha photographic dish, measuring about 10 cm. by 12 cm. The wire gauze was 6 cm. square. Its distance from the surface of the liquid was about one centimetre in the earlier experiments, about ·8 cm. in those of August 17 & 19. Its position during each day was fixed throughout that day.

In making an experiment a definite volume of water, generally 75 c.c., was placed in the dish. The dish and the quadrants connected with it were electrified by touching the wire *s* (fig. 1) with a wire connected to the negative pole of a water-battery (the positive pole being earthed). The touching wire was removed, leaving the dish charged—the deflexion noted (the fourth column in the tables), another reading taken in twenty minutes' time (the fifth column in the tables). Many preliminary experiments seemed to show that this was a convenient method of procedure, especially with the apparatus available. The sensitiveness of the electrometer remained sensibly constant for several days in these experiments, the needle requiring only a small charge. A convenient deflexion could always be obtained by varying the number of cells used in the water-battery. An experiment with water would be followed generally by one with hydrogen peroxide, the same volume of this being used and all the arrangements being just the same; or sometimes two experiments consecutively were made with the same liquid.

The difference of potential between the liquid and the wire gauze varied from 20 to 30 volts. The sensitiveness of the electrometer was found approximately by using a Clark's cell.

The experiments were made at the author's house at Kessingland, on the coast of Suffolk.

The results of all the experiments made with this apparatus between August 8 and August 19 are given in the Tables. These results are also plotted. An ordinate in each case shows the loss of electrification during an experiment, as given by the sixth column of the tables, made at the time of day given by the abscissa. The experiments with water are indicated thus ×, those with H_2O_2 ⊕.

In annotation of these results, it may be remarked that those on August 8, 9, 11, 17 are in agreement with the general statement that a surface of H_2O_2 exposed to sunshine discharges negative electrification more rapidly than a water surface.

The irregularities of August 10 are to be accounted for by the changing weather conditions. The large loss with water in exp. vii. was with a wind blowing which did some damage in the way of blowing down tents, &c. Observations on days such as August 13 are useful for showing that the effect as observed in these experiments is not pronounced except in bright sunshine.

The observations of August 19 have to be considered. The day was a very fine one. The dish on this day was very completely sheltered from the light north wind by the house. The explanation given by the author is that the liquid being kept in an unstirred condition, the decomposing molecules of H_2O_2 were not brought to the surface in anything like as large numbers as on other days when the liquid was kept in a state of slight agitation by wind. The surface conditions may possibly affect the rate of discharge. Moreover, it must be borne in mind that the electric field in these experiments was very weak.

The effect observed seemed quite independent of the formation of bubbles of oxygen.

In some of these experiments, the partial decomposition of a dilute solution of hydrogen peroxide in a flask and exposed to sunshine for two and a half hours, was used as a convenient "sunshine recorder" during the electrical experiments. The relative values of the decomposition on August 11, 13, 17, and 19 being 90, 65, 85, 65; numbers which indicate that the days on which the radiation is most effective in decomposing H_2O_2 are also the days on which the discharging action is most pronounced.

If the origin of the positive and negative ions in the air is due to the decomposition of H_2O_2,—this substance being itself formed by the interaction of water and free atmospheric oxygen,—it would seem that the phenomena of atmospheric electricity may be traced to the action of plants, if the view is correct that the free oxygen of the atmosphere is to be attributed to the action of plants (Stevenson, Phil. Mag. September & October 1900).

These experiments possibly raise the question whether some of the experiments which have been described on the action of light, especially ultra-violet light, in discharging negative electrification may not be due to a moisture-film containing some H_2O_2 being constantly formed and decomposed. Would all these experiments give the same results if the surrounding dielectric was dry nitrogen instead of atmospheric air?

Perhaps the results of some observers on the alleged loss of electrification in the evaporation of water (in air) might possibly be similarly explained.

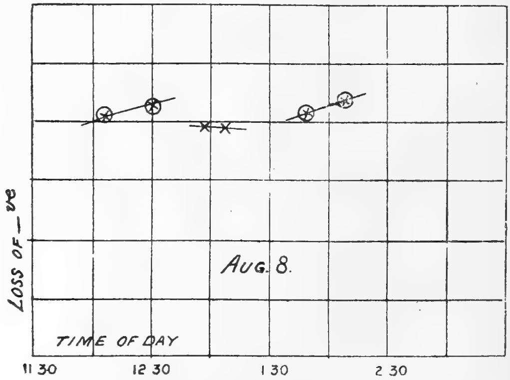
Experiment.	Liquid.	Time of starting.	Electrometer Deflexions.		Loss.	Weather Notes.
			Initial.	Final.		
Aug. 8.		h m				
I.	$H_2O_2(a)$	11 55	265	164	101	S.W. breeze. Bright sunshine, though sun occasionally obscured by passing clouds.
II.	$H_2O_2(a)$	12 19	287	182	105	
III.	Water.	12 47	284	187	97	
IV.	Water.	1 10	286	190	96	
V.	$H_2O_2(a)$	1 38	287	184	103	
VI.	$H_2O_2(a)$	2 0	288	181	107	
Aug. 9.						
I.	Water.	9 40	271	182	89	Calm to slight S. breeze. Continuous bright sunshine.
II.	Water.	10 2	272	194	78	
III.	$H_2O_2(a)$	10 30	277	190	87	
IV.	$H_2O_2(a)$	10 52	272	189	83	
V.	Water.	11 21	278	205	73	
VI.	Water.	11 44	276	210	66	
VII.	$H_2O_2(a)$	12 9	277	189	88	
VIII.	$H_2O_2(a)$	12 31	272	189	83	
IX.	Water.	12 59	260	193	67*	
X.	$H_2O_2(a)$	1 26	266	198	68	
XI.	$H_2O_2(a)$	1 49	274	207	67	
XII.	Water.	2 20	273	219	54	* A rapid loss at first was noticed in this experiment.
XIII	$H_2O_2(a)$	2 51	278	194	84	
Aug. 10.						
I.	Water.	9 52	272	170	102	Wind S.E. slight. Sky overcast with thin clouds. No bright sunshine, but fairly continuous feeble sunshine, sufficient to cast definite shadows.
II.	Water.	10 17	267	164	103	
III.	$H_2O_2(b)$	10 45	267	161	106	
IV.	$H_2O_2(b)$	11 13	273	184	89	
V.	Water.	11 50	274	184	90	Weather conditions seemed strictly comparable.
VI.	$H_2O_2(b)$	12 20	274	167	107	
VII.	Water.	12 50	275	155	120	Wind freshening to a stiff breeze. Some bright sunshine.
VIII.	$H_2O_2(b)$	1 23	272	130	142	
Aug. 11.						
I.	Water.	10 15	274	209	65	Bright sunshine. S.S.E. breeze, freshening as day went on.
II.	$H_2O_2(b)$	10 46	275	184	91	
III.	$H_2O_2(b)$	11 9	275	88	187	
IV.	Water.	11 44	274	185	89	
V.	Water.	12 10	275	197	78	
VI	$H_2O_2(b)$	12 40	275	82	193	
VII.	Water.	1 11	273	174	99	
VIII.	H_2O_2	1 37	260	72	188	
Aug. 13.						
I.	Water.	10 33	271	185	86	Wind W. backing to S., overcast.
II.	$H_2O_2(c)$	11 20	275	193	82	
Aug. 14.						
I.	Water.	10 45	274	187	87	S.E. breeze, bright sunshine.
II.	$H_2O_2(c)$	11 20	274	196	78	
III.	$H_2O_2(c)$	11 47	276	190	86	
IV.	Water.	12 20	273	212	61	
V.	$H_2O_2(c)$	12 59	270	43	227	

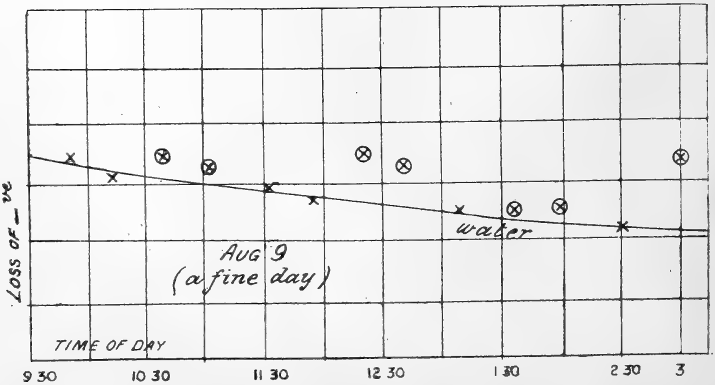
(*a*) 75 c.c. of a "20-vol." H_2O_2. (*b*) 75 c.c. of a fresh supply of "20-vol. puriss." H_2O_2. (*c*) Another 75 c.c. of the same H_2O_2.

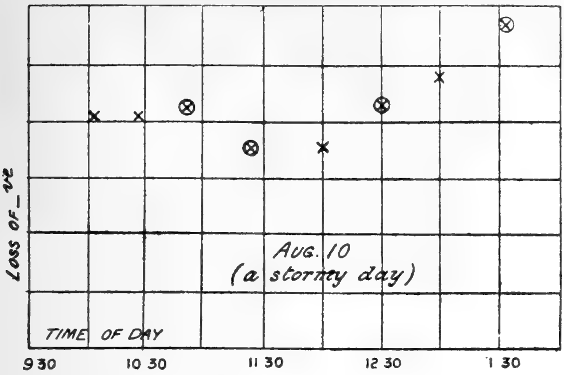
Experiment.	Liquid.	Time of starting.	Electrometer Deflexions.		Loss.	Weather Notes.
			Initial.	Final.		
Aug. 17.		h m				
I.	Water.	10 19	267	223	44	Light S. breeze. Bright but thin clouds occasionally partially obscuring sun.
II.	H_2O_2(*d*	10 53	264	119	145	
III.	Water.)	11 31	267	222	45	
IV.	H_2O_2(*d*)	12 0	267	46	221	
V.	Water.	12 35	266	223	43	
Aug. 19.						
I.	Water.	10 11	280	216	64	Very light N.N.E. breeze. A very fine day, continuous sunshine.
II.	H_2O_2 (*e*)	10 41	281	222	59	
III.	H_2O_2 (*e*)	11 16	278	230	48	
IV.	Water.	11 52	280	227	53	
V.	H_2O_2(*f*)	12 26	280	227	53	
VI.	H_2O_2(*f*)	12 57	279	225	54	
VII.	Water.	1 28	273	216	57	

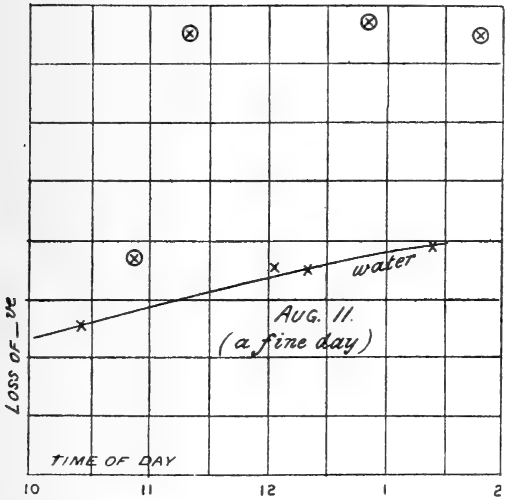
(*d*) Consisted of (*c*) made up to 80 c.c. in test-mixer with water. (*e*) 40 c.c. H_2O_2, 40 c.c. water. (*f*) 80 c.c. of the 20-vol. H_2O_2 from same stock as used August 14 & 17.

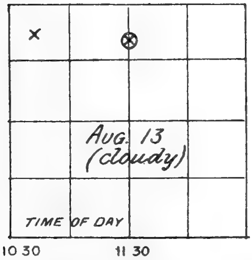
In experiments August 8 to August 14, 75 c.c. of water used; August 17 and August 19, 80 c.c.

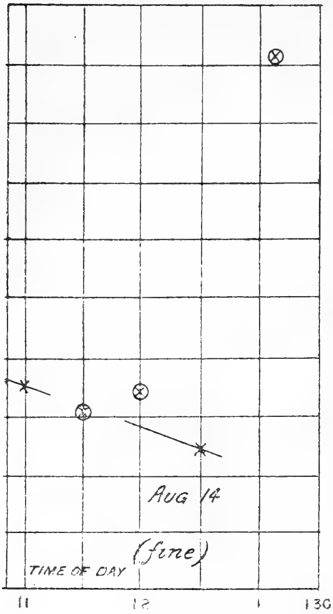


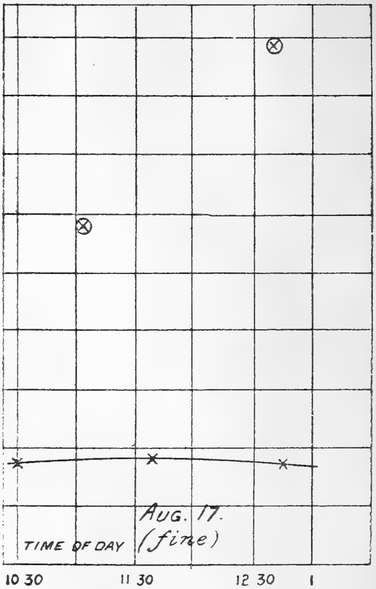


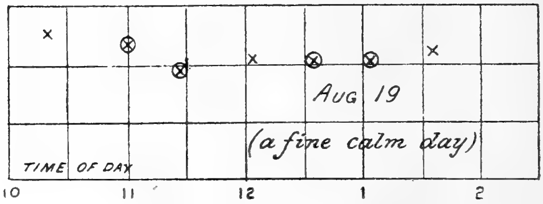

LOSS OF _ ve
AUG. 10
(a stormy day)
TIME OF DAY
9 30
10 30
11 30
12 30
1 30


LOSS OF _ ve
water
AUG. 11.
(a fine day)
TIME OF DAY
10
11
12
1
2


AUG. 13
(cloudy)
TIME OF DAY
10 30
11 30







Appendix.

On the Decomposition of Hydrogen Peroxide by Light.

The following is a brief record of experiments made on this subject. In those made between April 1 and June 20, the H_2O_2 solutions were in open dishes; the conditions of the solutions exposed to sunshine being much the same as in the electrical experiments. The inability of the author to find any account of this decomposition led him to make the flask and other experiments recorded. The instability of H_2O_2, even in dilute solutions, when exposed to light would seem to be a property which should always be considered in any experiments with it.

Experiments on the Action of Light on H_2O_2 *Solutions*:—

April 1.—25 c.c. H_2O_2 ["20-vol. puriss."] diluted to 250 c.c.

100 c.c. of this solution exposed to sunshine in a dish from 8 A.M. to 4 P.M. On April 2, 50 c.c. water added and dish exposed from 9 A.M. to 5 P.M.

100 c.c. of same solution kept in dish in laboratory.

The contents of the dishes, on analysis with $KMnO_4$, showed that the ratio of H_2O_2 in unexposed dish to that in the exposed dish was

415 : 288.

April 4.—10 c.c. H_2O_2 diluted to 250 c.c.

100 c.c. of this exposed on April 4 and 5 to sunshine.

100 c.c. of this in dish in laboratory.

The temperature of the solutions in each constantly observed; in some cases temperature of exposed dish lower than other. Extreme variations not more than 3 degrees. Average temperature did not differ by more than a degree.

Analysis of contents showed ratio of H_2O_2 in unexposed dish to H_2O_2 in exposed dish 159 : 125.

April 9.—10 c.c. H_2O_2 diluted to 250 c.c.

100 c.c. exposed in dish April 9, 8 A.M. to 3.40 P.M. April 10, 7.45 A.M. to 6 P.M. 25 c.c. water being added.

100 c.c. in dish in laboratory in large water-bath kept at a temperature 2–5 degrees higher than the exposed dish.

H_2O_2 in unexposed but warmed dish : H_2O_2 in exposed dish

159 : 121.

April 12 to April 16.—A similar experiment to the last one gave

144 : 121.

June 16.—40 c.c. H_2O_2 ["20-vol. puriss.," but containing free H_2SO_4, namely ·000011 grm. acid hydrogen per litre] diluted to 1000 c.c.

(*a*) 100 c.c. of this diluted with another 100 c.c. of water and exposed in dish to available sunshine between June 16—June 20, but kept diluted by successive additions of water.

(*b*) 100 c.c. of this solution exposed between June 16—June 20 as above, but without dilution.

(*c*) 100 c.c. of this in dish, covered with a thin glass plate, but exposed to sunshine, June 16—June 20, as above.

(*d*) 100 c.c. of this put in a basin in desiccator over strong sulphuric acid and kept in the dark June 16—July 15, when its volume was found to be 62 c.c.

H_2O_2 in 100 c.c. of this solution : H_2O_2 in (*a*) : H_2O in (*b*) : H_2O_2 in (*c*) : H_2O_2 in (*d*) :—

164 : 85 : 77 : 103 : 142.

June 20.—100 c.c. of the same H_2O_2 solution put in each of two flasks, labelled A and B, both being well corked.

A exposed to available sunshine on June 21, 22, 24, 25, 27.

B flask in laboratory.

Analysis of contents showed that H_2O_2 in B : H_2O_2 in A
146 : 35.

June 27.—Fresh lots of solution put in flasks. Sensitive and compared thermometers in each.

A kept in the laboratory in the dark in a water-bath at a temperature above that of B—which was exposed to available sunshine. A always 5–2 degrees higher than B. B exposed to available sunshine June 27—July 13, when contents analysed.

$$H_2O_2 \text{ in A} : H_2O_2 \text{ in B} = 10 : 1.$$

In this experiment the average temperature of A was undoubtedly higher than B. Moreover, A was subjected to a considerably higher temperature (20 degrees or so) on two occasions during the period than was ever reached by B.

Caius College, Cambridge.

IV. *Dielectric Constant of Paraffins.*
By WILL G. HORMELL*.

THE following paper presents some experiments in which, by means of a modified form of the Blondlot oscillator, the dielectric constants of four commercially different paraffins were obtained for 40-, 60-, and 80-centimetre waves. The paper also contains an investigation of the relative velocities of 80-centimetre waves along magnetic and non-magnetic wires of different diameters. And, finally, there is given in the table by way of comparison the light-index of refraction of the different paraffins for the D_1 line as obtained by means of the Abbe Refractometer.

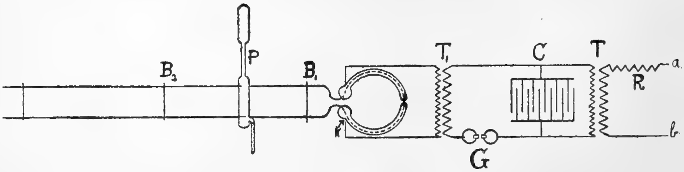


Figure I.

Apparatus.

Fig. 1 is a diagram of the connexions made to the various pieces of apparatus used in the experiment. The terminals

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* Communicated by Prof. J. Trowbridge.

a and *b* are connected to a commercial line delivering an alternating current with a pressure of 52 volts. The resistance R is used to regulate the quantity of current flowing through the circuit. T is a transformer whose primary of about 200 turns is wound around a core of soft iron wire. The secondary of the coil contains about 2000 turns. C is a battery of four leyden-jars each having a capacity of 5900 electrostatic units. T_1 is a Tesla coil made after the formula given by Tesla, and is capable when fed to its full capacity of giving a ten-inch spark. In all the following experiments a current of about 7 amperes was taken from the commercial line, which would give in the Tesla coil a spark of not more than 3 cm. G is a spark-gap consisting of two small iridium[*] cylinders inserted in $\frac{3}{4}$-in. brass spheres. The length of the spark-gap can be nicely adjusted by means of a screw-thread on the axle of one of the spheres.

The constancy of the spark at this gap is of the greatest importance in securing the best possible results from the oscillator. A rapidly-rotating disk presenting projections to the opposite terminal of the gap gave very indifferent results. Zinc balls, polished brass balls, and brass balls with platinum terminals, inserted after the fashion of the iridium cylinders, were used, but none of these gave the constancy secured through the use of the iridium. Platinum terminals gave the next best results, but on account of their rapid deterioration they required almost constant attention in order to keep their surfaces well polished. The iridium gap needs no attention except for occasional adjustment of length.

The secondary terminals of the Tesla coil are joined to the primary terminals E′E (fig. 2, p. 54) of theBlondlot oscillator. The remaining portion of fig. 1 is shown in detail in fig. 3.

The Blondlot oscillator (fig. 2) is the modified form used by W. D. Coolidge[†]. Some additions, however, have been made in order to bring the adjustments more completely under the control of the observer.

The two semicircles composing the primary coil are firmly fastened to two hard rubber uprights which are pivoted on the crossbar L. The screw D regulates the spark-gap *m*. A spiral spring, not shown in the figure, wound round D, keeps the two uprights separated. The wire E′ is passed through a thick-walled glass tube, and is inserted in one of the ball-terminals of the primary. The other wire E forms a spark-gap at *k*, which is regulated by the screw G. The

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[*] G. W. Pierce, American Journal of Science, vol. ix. p. 256.

[†] *Annalen der Physik*, lxvii. p. 578 (1899).

glass tubes are held in place by pressure-screws not shown in the figure. This framework, holding the primary, is placed in a 4 × 5 in. glass jar. Immediately under the primary a secondary of the same diameter is placed, and its terminals are led out through two corks inserted in the side of the jar (fig. 3). This coil is firmly held in position by three rubber posts sent up from a rubber plate placed in the bottom of the jar. A mica plate is placed between the primary and secondary, the distance between which is regulated by the screw A (fig. 2). The screw-clamp F (fig. 3) holds the framework

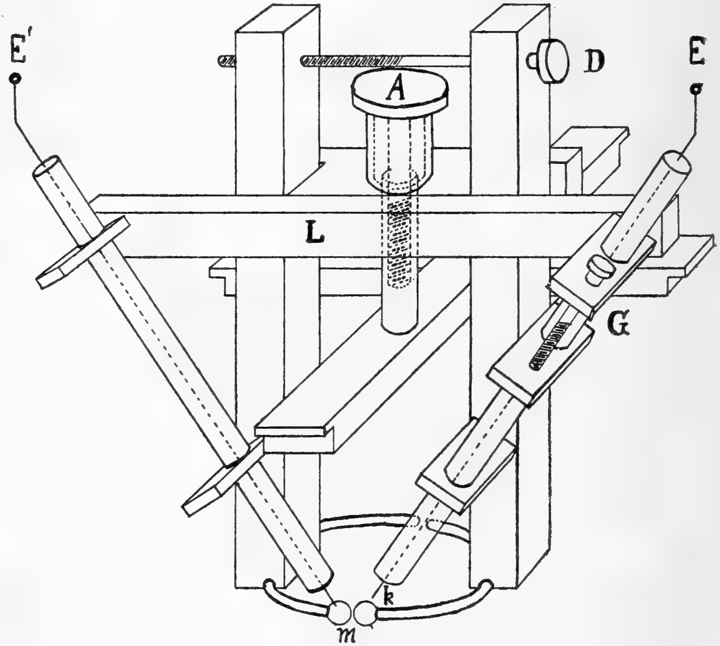

Figure 2.

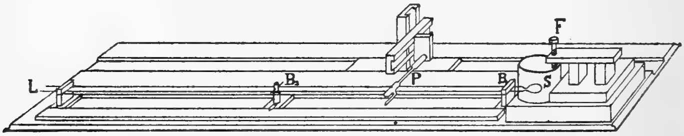

Figure 3.

of the primary rigidly in place. When the coils are in use the jar is filled with kerosene.

The first bridge B_1 (fig. 3) rests in a groove in the rubber support. It is 30 mm. long, 1 mm. thick, and 3 mm. wide. It has two holes in which to receive the ends of the secondary, and these, when in place, are soldered to the bridge. Two grooves symmetrically placed are filed on top of the bridge, 1·8 cm. apart. The Lecher wires L rest in these grooves and are clamped firmly to the bridge by means of the top of the rubber support, which is furnished with two strong brass screws. The other ends of the Lecher wires 1·8 cm. apart are clamped in a second rubber support. The supports, about 150 cm. apart, are strong enough to allow a tension of several pounds on the wires. Unless otherwise designated, the Lecher wires are made of no. 18 copper wire.

The second and movable bridge B_2 is a brass wire 1 mm. in diameter and 4 cm. long. The post holding the bridge works in a socket containing a spring which presses the bridge against the two wires. The base of the bridge-holder has an index extending over a metre-rod, thus indicating the position of the bridge.

P is a Plücker argon tube, mounted in such a way that it may be moved at will along the three mutually perpendicular directions. Tubes filled with other gases were tried, but none of these glowed with such brilliancy. Professors John Trowbridge and T. W. Richards* have called attention to the unusual sensitiveness of an argon tube to oscillating discharges. In order to observe the glow of the tube to the best advantage, it is necessary to work in a darkened room.

Some Preliminary Experiments.

(*a*) Determination of the Line Constants.

In this adjustment of the oscillator the primary was made of copper wire 2 mm. in diameter, and was furnished with discharging balls 4 mm. in diameter. The diameter of the coil was 4 cm. The total length of the secondary, including the leading-out wires to the first bridge, was about 25 cm. In every case the secondary is made of no. 18 copper wire. The wave-length given out by such a combination when the primary was ·3 mm. from the secondary was 61·33 cm. In order to determine the exact wave-length one may proceed as follows :—Place the argon tube under the wires, in contact with them, and about 14 cm. from the first bridge. Slide the movable bridge along the wires until the first

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* American Journal of Science, vol. cliii. p. 19 (1897).

position of maximum glow is reached; this will be 27·80 cm. from the first bridge. When the bridge is in the proper position, the maximum glow of the tube may be easily noted from the extension of the glow into the sealing-off tube.

Coolidge * found it necessary to connect the first bridge with the earth in order to lead off the irregular vibrations of the oscillator which otherwise masked the position of the node of the fundamental wave.

I have not found this necessary, however, and attribute it largely to the accurately symmetrical arrangement of the primary and secondary.

Table I. contains a set of readings which were taken for the determination of the line constants. In this, as in all other cases, ten readings were taken in locating the position of a node. The average of two or three such groups for the same node never varied beyond ½ of 1 per cent.

TABLE I.

1st Node.	2nd Node.	2nd Node with balance-tube.	3rd Node.	4th Node.
centim.	centim.	centim.	centim.	centim.
27·65	58·85	58·15	89·25	120·75
27·65	58·95	58·00	89·55	120·45
27·80	59·15	58·30	89·50	120·65
27·75	58·75	58·00	89·50	120·55
27·80	59·00	58·20	89·25	121·00
27·80	58·80	58·20	89·60	120·80
27·90	59·15	58·35	89·55	120·80
27·70	59·15	58·15	89·40	120·50
28·00	58·60	58·10	89·55	120·30
27·95	59·00	58·00	89·75	120·50
27·80	58·94	58·14	89·49	120·60

The readings for the nodes most distant from the first bridge are not so concordant, but the percentage of error is no greater. Generally a careful adjustment of the three spark-gaps will lead to a more brilliant and a more constant glow of the tube. Frequently, for various reasons, the tube will cease to glow in the dark. If on these occasions white light from an incandescent lamp or a lighted match be permitted to fall on the tube, it will instantly begin to glow. This is no doubt "the effect of ultra-violet light upon the

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* *Annalen der Physik*, lxvii. p. 582 (1899).

electric discharge" which Hertz * has so thoroughly investigated.

Positive and negative surgings start at the middle of the first bridge and meet in opposite phases at the middle of the second bridge, thus forming, according to Hertz, potential loops at points midway between the two bridges and potential nodes at the middle of the bridges. The distance between the middle points of the bridges is a half wave-length, the value of which can be found by adding the distance between the two wires to the scale-reading. It is doubtful just what this bridge correction ought to be, but the determination of the wave-length by another method (given later) seems to confirm the above correction.

This half-wave is not the length of a free half-wave along the wires. The presence of the tube has added capacity to the wires and thereby shortened the wave. The tube correction must therefore be determined. To do this find the position of the second node. The table shows this to be 58·94 cm. Now place a tube, which we shall call the balance-tube, in contact with the wires at the middle of the second potential loop. The wave is now shortened by an amount which is equal to the tube effect. The balance-tube ought to be of the same diameter as the argon tube, but its length is not essential provided it extends beyond the two wires. The balance-tube as well as the argon tube ought to be within at most 1½ cm. of the middle of the loop, otherwise the capacity effect is lessened. One-half of this last reading (*i. e.* the distance between the first bridge and the second node) will give what may be called *the half tube-wave*, in which one node rests on the wires and the other rests on the centre of the bridge. We shall call the former the *wire node*, and the latter the *bridge node*. The balance-tube may now be removed and the third and fourth nodes located, leaving the argon tube at the first potential loop, though as far as the position of the nodes is concerned the tube may be placed at any other potential loop. If we add the tube effect to the average reading in the second column, and also the distance between the two wires, we shall get what may be called a *wave-length in air*. (To this is added also a scale-correction of 3 mm.) This is in fact the length of a free wave along the wires, but since the experiments of Sarasin and De La Rive have † shown that the velocity of a rapidly oscillating electrical disturbance along a wire is the same as that in air, we have good authority for calling the above the *wave-length in air*.

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* 'Electric Waves,' p. 63.

† *Archives des Sciences et Naturelles Genève*, 1890, t. xxiii. p. 113.

The same quantity may be secured by subtracting the first reading from the third and the second from the fourth.

Summary of the Line Constants.

(1) $58{\cdot}94 - 58{\cdot}14 = {\cdot}80$ cm. = tube effect.

(2) $\dfrac{58{\cdot}14 + {\cdot}3}{2} = 29{\cdot}22$ cm. = 1/2 tube wave.

(3) $58{\cdot}94 + {\cdot}3 + {\cdot}80 + 1{\cdot}80 = 61{\cdot}84$ cm. = wave-length in air.
(4) $89{\cdot}49 - 27{\cdot}80$ . . . $= 61{\cdot}69$ cm. = „ „
(5) $120{\cdot}61 - 58{\cdot}94$ . . . $= 61{\cdot}67$ cm. = „ „

61·73 cm. = average wave-length in air.

It will be seen that the wave-length in air determined by the first method (3) is very approximately the same as that secured by the second method (4) and (5).

(*b*) Determination of the Dependence of the Wave-length upon the Distance between the Primary and the Secondary Coils.

In this experiment the coils were 5 cm. in diameter. The primary semicircles were made of 3 mm. wire, and had discharging balls 5 mm. in diameter. The pitch of the screw controlling the primary was 1·27 mm. The primary was removed and replaced a number of times in order to test the accuracy with which the relative positions of the coils could be reproduced. In all the cases the agreement of the wave-lengths was very satisfactory.

TABLE II.

Distance between centres of coils.	Wave-length.
2·84 millim.	89·80 millim.
3·15 „	87·25 „
3·46 „	84·80 „
3·77 „	80·20 „
4·08 „	76·10 „
4·39 „	73·05 „
4·69 „	71·15 „

Table II. shows the results of this experiment. It will be observed that as the distance between the coils increases the wave-length decreases, showing that a decrease in the capacity and mutual induction of the system decreases the period of oscillation.

(*c*) Determination of the Relative Velocities of Electrical Waves along Magnetic and Non-Magnetic Wires of Different Diameters.

In this experiment the coils were 5 cm. in diameter, and the length of the secondary was 48·5 cm. The coils were kept at the same distance, and the Lecher wires were changed.

TABLE III.

Kind of Wire.	Diameter of Wire.	Corrected Wave-length.
	millim.	centim.
Copper	1·00	81·52
,, ...	2·00	80·84
,, ...	3·40	80·80
Iron	1·00	81·70
,, ...	1·25	81·56
,, ...	1·80	80·36
,, ...	2·60	80·86
Copper..............	1·00	81·08
Nickel	0·93	80·90
German-silver......	1·00	80·88
Brass	1·00	81·10
Copper Secondary	1·00	79·26
Iron Secondary	1·00	79·44

Table III. shows the final results. The last two sets of readings show the comparison of waves produced when the Lecher wires were kept constant and the secondary of the Blondlot oscillator changed. In the first case the secondary was made of copper wire, and in the second it was made of iron. Great care was taken to make the two secondaries of the same size and shape, and to keep the primary and secondary the same distance apart in both cases.

Allowing less than 1 per cent. for error of observation, it is readily seen that the wave-lengths for different kinds of wires are practically the same. One concludes from these observations that conductors offer no resistance to the passage of short electrical waves, and that the magnetization of iron is not able to follow oscillations as high as 800,000,000 per second. These conclusions agree with those of Hertz * and Lodge †. But St. John ‡ found a difference of wave-length in iron and copper of from $1\frac{1}{2}$ to 3 per cent. This difference, however, was for oscillations of 115,000,000 reversals per second.

Dielectric Constant of Paraffin.

Maxwell has shown theoretically that the dielectric constant of a substance is equal to the square of the index of refraction. The index of refraction of a given substance for electrical waves, as well as for light waves, is the ratio of their relative velocities in air and in the given substance. But since the ratio of the velocity is equal to the ratio of the wave-lengths, we have

$$K=\mu^2=\left(\frac{\lambda_a}{\lambda_s}\right)^2$$

where λ_a = the wave-length of a given electrical disturbance in air;

λ_s = the wave-length of the same electrical disturbance in the given substance;

μ = index of refraction;

K = dielectric constant.

The Blondlot oscillator offers a unique and very satisfactory means of measuring the wave-length in air and in the given substance. This means is satisfactory, because one is able to keep the oscillator in a constant condition, thus making the wave-length in air a constant for a given combination. The method of securing the wave-length in air has been described, and it now remains to indicate the way in which the wave-length in paraffin may be obtained.

Let us suppose that the oscillator is adjusted for a half wave-length of about 40 cm. Take a block of paraffin 40 cm. long, 6 cm. wide, and 8 cm. thick. It can be readily sawed from a commercial cake by means of a table rip-saw. With this same saw make two grooves 1·8 cm. apart and half-way through one side. Bring the block up under the two Lecher wires, making them lie in the grooves and placing the end of the block against the support of the first bridge. Using a hot iron and a piece of paraffin, run the melted wax into the

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* 'Electric Waves,' p. 113.

† 'Modern Views of Electricity,' p. 101 (1889).

‡ Proceedings of American Academy, p. 218 (1894).

grooves, thus making a solid column around the wires. Place the argon tube between the free end of the column of wax and the movable bridge. When the oscillator is in action and the tube located, it will be noticed that the distance between the end of the column and the bridge is less than the *half tube-wave*. The column may now be cut off piece by piece until the *wire node* of the half tube-wave rests on the end of the column. The length of the column, plus half the distance between the Lecher wires (·9 cm.), is very approximately the half wave-length in paraffin. In the actual experiment the length of this column was 26·4 cm. And if to this we add ·9 cm., we shall get 27·3 cm. as the approximate wave-length. If the base of the apparatus be long enough for three half-waves, a column of paraffin 85 or 90 cm. long ought to be moulded. For this purpose a box was prepared 90 cm. long, 8 cm. wide, and 8 cm. deep, and was lined with tinfoil. It was put together with screws, so that it could be easily taken apart. By a suitable contrivance, two copper wires 1·8 cm. apart could be stretched through the middle of the box and made long enough to be fastened in the two supports of the base. The melted paraffin was poured into the box and allowed to cool partially. Then a closely-fitting top was placed on the paraffin and the whole subjected to a pressure brought into play by three iron clamps. A gauge inserted in the side of the box recorded the pressure. All this was necessary in order to secure columns uniformly dense and free from air-holes. Having a column thus suitably prepared, one must decide two questions before a final determination of the dielectric constant is made.

1st. How much beyond a node may the end of the paraffin column rest without changing the position of the second bridge?

2nd. How large in area must the section of the column be in order to include the whole of the effective field surrounding the Lecher wires?

The first question can be quickly answered. Fasten the moulded column between the two supports, making one end rest against the first support. It will be about 3 mm. from the first bridge, since the bridge rests in a groove 3 mm. from the outer surface. Putting the argon tube and the movable bridge beyond the column of paraffin and locating the first external node (the fourth from the first bridge), it will be found that a node (the third) will rest in the column 6 or 8 cm. from the end. Saw off a 2 cm. slab from the end of the column and again locate the node. Continue this until the wire node of the half tube-wave rests upon the end of the column. Frequent experiments have shown that if the column be as much as 1·5 cm. too long or too short, the position of the fourth node is not affected. The conclusion, therefore, is that no error will be made if the column of paraffin be 1·5 cm. too long or too short, or the end be not against the first bridge. This fact makes it very convenient when using the same column for different wave-lengths.

This same column of proper length may be used to answer the second question. After the position of the fourth node has been determined, the column may be removed, and by means of the table-saw the section reduced to one 6 × 8 cm. The process of locating the node and reducing the section may be continued until the column is too thin to allow further reduction.

Table IV. shows the result of this experiment. From it we learn that the effective field, as far as it can be determined by means of an argon tube, certainly lies within a column whose cross section is 5 × 7 cm. By a comparison of the values in the fifth column with those in the first, we find that the effect on the half-wave varies approximately inversely as the square of the distance of the wires from the outer surface.

Table IV.

Size of Column.	Area of Cross-section.	Position of Node.	Change in Position of Node.	Effect on one-half Wave-length.
cm.	sq. cm.	cm.	cm.	cm.
7×8	56	120·34		
6×8	48	120·20		
5×7	35	120·35		
4×6	24	120·28		
3×5	15	120·83	·5	·2
2×4	8	123·60	3·2	1·1
1·5×3·5	5·2	126·68	6·4	2·3
1×3	3	131·14	11·8	4·4

Four specimens of the paraffin were secured from a wholesale dealer. They were known as numbers 120, 125, 130, and 135, their numbers representing their melting-point on the Fahrenheit scale. Two columns of each kind were prepared and the melting-point on the Centigrade scale noted. The density was determined from four different specimens of each column. The total pressure to which the paraffin was subjected when in the mould was 2140 lbs., or 7 lbs. per sq. in. Moulding the columns under pressure was the only way in which masses of this size could be secured of anything like uniform density. The dielectric constant of a column moulded without pressure differed as much as 3 per cent. from that of a column of the same kind of paraffin moulded under pressure.

In Table VI. (p. 65) will be found the dielectric constant of two kinds of oil, a paraffin oil and a "transformer-oil." This grade of paraffin oil is sometimes used for transformers; but the so-called "transformer-oil" is said by the wholesale dealers to be the best oil procurable for transformer work. In order to make the necessary measurements with these oils, a telescoping box was made and lined with tinfoil. The ends were constructed of thin rubber plates, and through them the Lecher wires were stretched. When the nodes rested on or near the rubber ends, the wave-length in air was not changed by the pressure of the box. The box was filled with oil, and the length adjusted until the outer end rested on the third node. The box was also filled with melted paraffin, and the temperature effect on the position of the node was noted.

Table V. shows how the index of refraction, and therefore the dielectric constant, increased as the temperature decreased. It is very probable that the change of the dielectric constant can be accounted for on the ground of the change in density. The densities at 79° C. and 18° C. are given in the table.

TABLE V.

Temperature.	Index of Refraction.	Dielectric Constant.	Density.
° C.			
79	1·44 (7)	2·09 (5)	·764
71	1·45 (6)	2·12 (5)	
65	1·46 (3)	2·14 (0)	
56·5	1·47 (1)	2·16 (3)	
53·7	1·47 (5)	2·17 (6)	
18	1·48 (2)	2·19 (8)	·908

In order to illustrate the method by which the dielectric constant of each column for each wave-length was determined, I have selected a set of readings at random. They are the data secured for column 1 of the no. 120 paraffin.

Diameter of primary	5·00 cm.
Length of wave in air	40·85 ,,
Length of tube-wave	39·24 ,,
Length of column	83·00 ,,

Position of Fourth Node.

First Set of Readings.	Second Set of Readings.
121·90	122·10
121·45	121·60
122·05	122·30
122·00	121·70
122·05	122·30
121·65	122·00
122·10	122·05
122·05	122·30
122·40	121·80
122·15	122·50
121·99	122·07

122·07
121·99
122·03
39·24
82·79 = scale-reading for position of node.
·30 = scale-correction.
·90 = bridge-correction.
3) 83·99
27·99 = length of half-wave in paraffin.

$$\frac{40{\cdot}85}{27{\cdot}99} = \mu = 1{\cdot}46(1).$$

$$\frac{[40{\cdot}85]^2}{[27{\cdot}99]^2} = K = 2{\cdot}13(0).$$

By a very simple device the primary of the oscillator was replaced by one 4 cm. in diameter, and under this was placed a secondary of corresponding diameter. With this combination the half-wave in air was 30·66 cm. With a 3 cm. primary the half-wave in air was 20·85 cm. By a happy coincidence, which was not altogether fortuitous, the columns were long enough to contain within the limit of error three half-waves of the 5-cm. primary, four half-waves of the 4-cm. primary, and six half-waves of the 3-cm. primary.

Table VI. exhibits a comparison of the melting-points, densities, indices of refraction, and dielectric constants of the paraffins and the two kinds of oils for the three different wave-lengths. The dielectric constant of the oils, however, was determined for only one wave-length. In the last two columns will be found the light-index of refraction for the D_1 line and the corresponding dielectric constant. A comparison of the dielectric constants for any given substance

TABLE VI.

Kind of Paraffin.	Observed Melting-point.	Density.	1/2λ=40·84 cm.		1/2λ=30·66 cm.		1/2λ=20·84 cm.		λ=·00005896 cm.	
	°C.		μ.	K.	μ.	K.	μ.	K.	μ.	K.
No. 120	47·6	·894	1·46 (2)	2·13 (7)	1·47 (0)	2·16 (1)	1·48 (8)	2·21 (7)	1·5380	2·3655
No. 125	50·2	·905	1·47 (3)	2·18 (0)	1·48 (9)	2·21 (8)	1·50 (4)	2·26 (2)	1·5430	2·3808
No. 130	53·3	·907	1·48 (3)	2·19 (8)	1·49 (3)	2·22 (0)	1·50 (2)	2·25 (6)	1·5505	2·4044
No. 135	56·2	·897	1·49 (0)	2·22 (1)	1·50 (1)	2·25 (4)	1·51 (2)	2·28 (6)	1·5523	2·4098
Paraffin Oil	309 Flash	·905	1·54 (3)	2·38 (1)	......	......	......	......	1·5050	2·2650
Transformer Oil..	....	·823	1·48 (7)	2·21 (3)	......	......	......	......	1·4565	2·1711

will reveal the fact that in every case there is a gradual increase as the wave-length decreases, even including the light-wave. These results certainly indicate the analogous behaviour of electrical and light waves. A comparison of the dielectric constants of the different paraffins shows that these constants increase with the melting-point. From these experiments it seems necessary, therefore, in tabulating the dielectric constant of paraffin to give its density, its melting-point, and the wave-length for which the constant was obtained. The third decimal place is bracketed because the accuracy of measurements does not warrant its unqualified use, and yet the quantities of which the tabulated numbers are the average are of such uniformity that some credence must be given to this third place.

Light-Index of Refraction.

The light-index of refraction was obtained by means of an Abbe * refractometer. This is constructed upon the principle of total reflexion, and is graduated to read directly the index of refraction of the D_1 line of any transparent liquid or solid which can be placed between its prisms. Films of paraffin were easily procured by floating bits of wax on hot water and allowing the melted wax to solidify. In order to ensure close contact, both the film and the prisms were gently heated before the wax was placed between the prisms. The refractometer gave data also from which by means of a table one could calculate the dispersion-factor between the D_1 and the F lines.

Having the index of refraction for two lines in the spectrum, one may apply Cauchy's formula, and thus obtain the index for infinitely long waves. When the constants A and B of this formula are eliminated it takes, according to Gordon †, the following form:—

$$\mu_{\lambda=\infty}=\frac{\mu\lambda^2-\mu_1\lambda_1^2}{\lambda^2-\lambda_1^2},$$

where $\mu_{\lambda=\infty}$ = index of refraction for infinitely long waves,
μ = ,, ,, a given long wave,
μ_1 = ,, ,, a shorter wave,
λ and λ_1 represent the corresponding wave-lengths.

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* *Physikalischer Fraktikum*, Wiedemann-Ebert.
† Phil. Trans. pt. i. p. 441 (1879).

TABLE VII.

μ_1 for $\lambda=81\cdot64$ cm.	μ_2 for $\lambda=41\cdot64$ cm.	μ_0 for $\lambda=\infty$.	μ_1' for D_1 line.	μ_2' for F line.	μ_0 for $\lambda=\infty$.
1·46 (2)	1·48 (8)	1·45 (6)	1·5380	1·5567	1·498
1·47 (3)	1·50 (4)	1·46 (2)	1·5430	1·5589	1·509
1·48 (3)	1·50 (2)	1·47 (7)	1·5505	1·5665	1·513
1·49 (0)	1·51 (2)	1·48 (5)	1·5523	1·5684	1·519

Table VII. shows the results obtained from the application of this formula to the light-indices, and also to the electrical indices. The values of the sixth column should agree with those of the third. They should at least be less than the index found for the longest wave. This formula does not appear to fit the experimental facts.

Conclusions.

1. For reversals as high as 800,000,000 per second the velocity of electrical disturbances along magnetic and non-magnetic wires of different diameters is the same, thus showing that the magnetic properties of iron are not able to follow such rapid changes.

2. The effective field around the Lecher wires, as far as it can be detected by means of an argon tube, does not extend more than 3 cm. from the wires; and the effect on the half-wave within this region varies approximately inversely as the square of the distance from the wires.

3. The dielectric constant of a given paraffin increases with the density of the paraffin. It increases rapidly from a temperature 20° above the melting-point to a temperature 30° below the melting-point. Among different paraffins the dielectric constant increases as the melting-point of the paraffin increases.

4. The dielectric constant increases as the wave-length decreases. It is greater for short light-waves than it is for short electrical waves. Cauchy's formula as a means of obtaining the index of refraction for infinitely long waves does not meet the experimental facts.

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V. *Absorption of Röntgen Rays by Aqueous Solutions.* By R. K. McClung, *M.A.*, *Demonstrator in Physics, and* D. McIntosh, *M.A.*, *Demonstrator in Chemistry, McGill University*[*].

THE question of the absorption of Röntgen rays by different substances is one which has occupied the attention of various investigators since the publication of Röntgen's first paper. The permeability of various salts, both in the solid form and in solution, has been studied to some extent. The method employed by those who have paid the most attention to this subject, up to the present time, has been a photographic one[†]. This consisted, in general, in allowing the Röntgen rays to fall upon a photographic plate, after having passed through the substance to be investigated, and then comparing the impression made on the plate when different substances were interposed in the path of the rays. A comparison of the intensities of these photographs was in general used to obtain a comparison of the permeability of the substances in question.

The most careful investigation of liquids, up to the present time, by this photographic method, has been that conducted by Lord Blythswood and Dr. Marchant (Proc. Roy. Soc. vol. lxv. 1899), who, after obtaining photographs, measured the intensity by a special form of photometer. However, as we believed that this method could be replaced by a more accurate and convenient one, we undertook to investigate the question by an electrical method, which admits of more rapid measurements and definite quantitative comparisons.

Various objections may be raised to the photographic method. In the first place, it is almost impossible to obtain photographic plates in which the films are exactly the same thickness on every portion of the plate. Therefore, if rays of the same intensity fall upon different portions of a plate, it would not necessarily be darkened to the same extent. The intensity of an X-ray photograph would depend then not only upon the intensity of the rays, but also upon the film. Besides this, even though the film might be quite homogeneous, the intensity of the photograph would depend upon

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[*] Communicated by Prof. E. Rutherford.

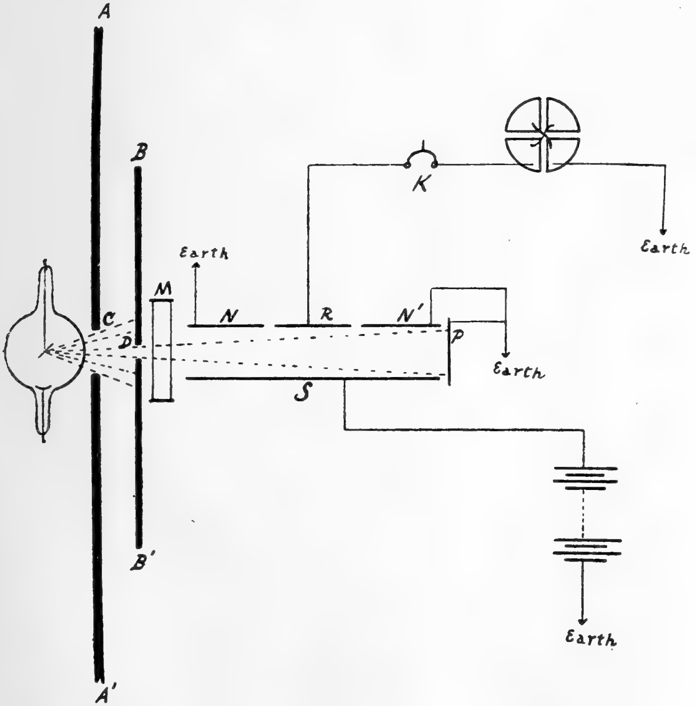
[†] Bouchard, *Comptes Rendus*, vol. cxxiii. pp. 967 & 1042. Gladstone & Hibbert, B. A. Reports, 1896, 1897, 1898. Meslans, *Comptes Rendus*, vol. cxxii. p. 307. Bleunard & Labasse, *Comptes Rendus*, vol. cxxii. pp. 527 & 723. Novak & Sulc, *Zeit. Phys. Chem.* vol. xix. p. 489. F. Rè, *Elettricità*, p. 261 (1898). Van Aubel, *Journ. de Physique*, vol. v. p. 511. Batelli & Garbasso, *Nuov. Cim.* [4] iii. p. 40. Winkelmann & Strantiel, Jena. Arnold, *Zeit. für Elektrochem.* p. 602 (1896). Chabaud, *Comptes Rendus*, vol. cxxii. p. 237.

the time of developing and also upon the strength of the developing solution. The individual photographic plate and also the developing of the photograph are therefore important factors, which enter to a great extent into the intensity of the impression made on the plate.

Again, after the photographs have been taken, it is a somewhat difficult matter to compare the intensities of two impressions with any great degree of accuracy, and especially to obtain definite numerical comparisons.

In taking a photograph, it is necessary to run the X-ray bulb a considerable time; usually about ten minutes are necessary to give an impression of sufficient intensity. In order to make a large number of comparisons, the taking of photographs for each measurement and developing the plates must necessarily involve a great deal of time and labour. It is desirable then to obtain a more rapid and also a more accurate method for determining these relative absorptions.

Fig. 1.



For these reasons the following method was adopted. The arrangement of the apparatus used and the connexions are shown in fig. 1.

If two metal plates, which are at different potentials, are arranged parallel to each other, and the X-rays allowed to pass between them, a current will pass from one plate to the other. The amount of this current will be proportional to the intensity of the rays, provided other conditions remain the same. If the current, which passes between two such plates, is measured for different intensities of rays, a direct comparison of the intensities is obtained. Such a current may be measured accurately by means of an electrometer, and thus the relative amount of rays which pass through different substances can be ascertained.

The bulb used to generate the rays, along with the induction-coil by which the bulb was run, was contained in a small metal room connected to earth. The rays from the bulb passed through a small orifice C in a thick lead plate AA′. A few centimetres in front of this, another thick lead plate BB′ was placed, and the rays passed through a rectangular orifice D, 6 cms. by 0·6 cm., in it. After emerging from D, the cone of rays then passed between a set of insulated, parallel zinc plates, arranged in the form of a guard-ring. The large plate S was charged to about six hundred volts, by being connected to one pole of a battery of small accumulators, while the other pole was to earth. The central plate R of the guard-ring was connected through suitable keys to one pair of quadrants of the electrometer, while the other pair was connected to earth. The outside plates N and N′ were also to earth. When the rays passed between these plates, a current passed from S to R proportional to the intensity of the rays, provided the voltage was large enough to produce a saturation or maximum current. The potential gradient employed was about seventy volts per centimetre. When the pair of quadrants connected to R is insulated, the deflexion of the electrometer-needle per second is proportional to the current passing between S and R; and since the current is proportional to the intensity of the rays, the intensity of the rays is therefore proportional to the deflexion of the electrometer-needle.

The cell, which contained the liquid whose absorption was to be measured, consisted of a rectangular box, of which the parallel sides through which the rays passed were of thin glass. The capacity of the box was about 300 c. c. This box, containing the liquid, was placed in the path of the rays at M, the thickness of the liquid traversed by the rays being 2·6 cms.

In this investigation, the usual difficulty which arises in making a series of measurements on X-rays, namely, the variations in the intensity of the radiation, presented itself. In order to overcome this difficulty, another box, similar to the one used to contain the liquid, was made; and this box, filled with water, was used as a standard with which all the other measurements were compared. The method of taking the observations was as follows:—The standard cell, filled with water, was placed in position, and the rate of leak per second, as indicated by the deflexion of the electrometer-needle, taken. Several readings in succession were made and the mean of these taken. As the X-ray bulb usually gives out rays more uniform in intensity when run regularly, the observations were taken at regular intervals of about a minute and a half. The bulb was run for ten seconds before taking the deflexion, so as to insure that the ionization would reach a maximum, and at the end of this time, the one pair of quadrants of the electrometer were insulated, and the deflexion of the needle in a certain time observed, the time being taken by means of a stop-watch. Now, if the intensity of the radiation remains constant, this deflexion should remain constant also; however, if the intensity changes, this deflexion will change in the same ratio.

After taking the deflexion for the standard cell filled with water, it was removed, and the other cell (which we may call Cell II.) was placed in exactly the same position and a corresponding set of deflexions observed. This was again removed, and a set of readings taken with the standard in place. Replacing the standard by Cell II., filled with water, the operation was repeated. This method of alternate sets of readings was continued throughout a whole series of observations. Each set of observations being thus followed by a set taken with the standard in place, any variation in the rays would be indicated by the difference in deflexion produced by the standard, and therefore all the readings could be compared with one another by simply reducing them to the same basis. A sample of such a set of readings is shown below.

Cell.	Mean deflexion per sec.	
Standard..................	15·0	
Cell II. with water ...	16·6	15·8
Standard..................	16·6	
Cell II. empty	48·2	16·3
Standard..................	16·0	

A deflexion of 48·2 for Cell II. empty corresponds to a deflexion of 16·3 for the standard. It is reduced to the basis 15·8 for the standard, and can therefore be compared with the reading 16·6 for Cell II. full of water. The percentage absorption of the given thickness of water is thus easily obtained. The percentage absorption of any other liquid can be obtained in a similar way. By this method of taking observations, a considerable number of solutions could be compared with one another during the course of a few hours.

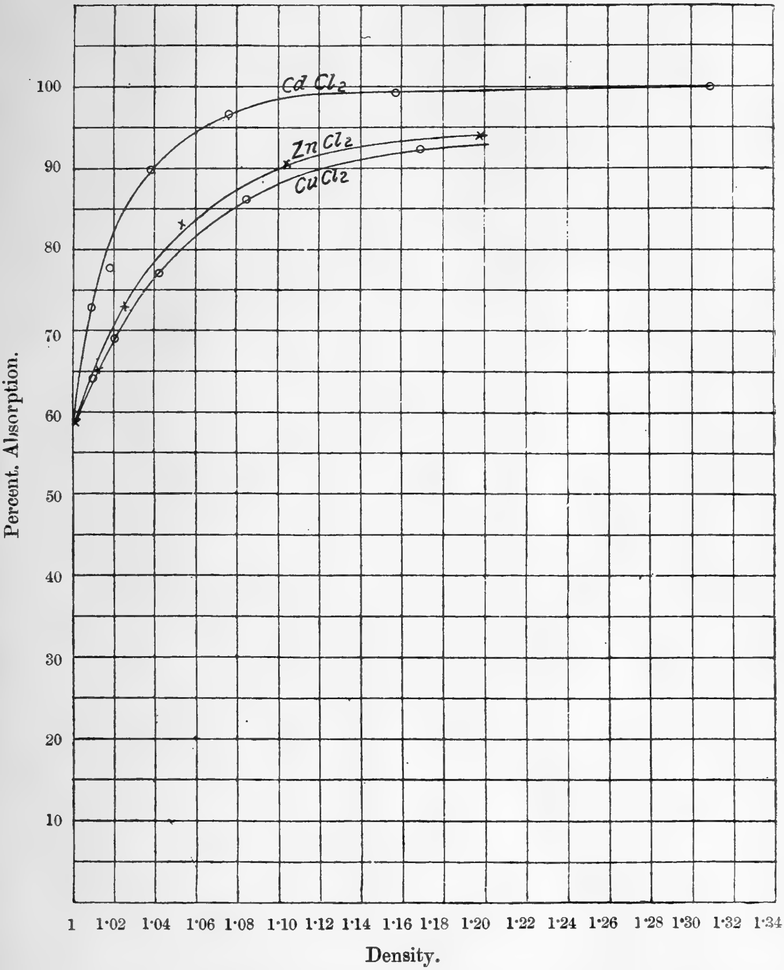
The percentage of the rays absorbed by the given thickness of water kept fairly constant from day to day, except in one or two cases. These determinations are shown in Table I., the ones which vary widely from the mean being given also.

TABLE I.

Electrometer Deflexions per sec.		Percentage Absorption.
Cell II.	Cell II. with water.	
63·4	26·3	58·6
47·0	18·1	61·5
55·0	22·6	58·9
38·9	15·1	61·2
35·6	17·4	51·1
45·2	16·7	63·1
44.7	16·4	66·3
49·6	17·2	65·3
95·3	42·6	55·3
74·9	31·7	57·6
64·7	27·8	56·9
77·5	31·4	59·5
49·1	16·5	66·4
		Mean = 59·9

The variation is well within the limits of experimental error, and the agreement is so close that it indicates that the absorptive power of water for the rays at different times is practically the same. This agrees with the results found by Lord Blythswood and Dr. Marchant, namely, that the permeability of any given substance is the same for rays of different intensities.

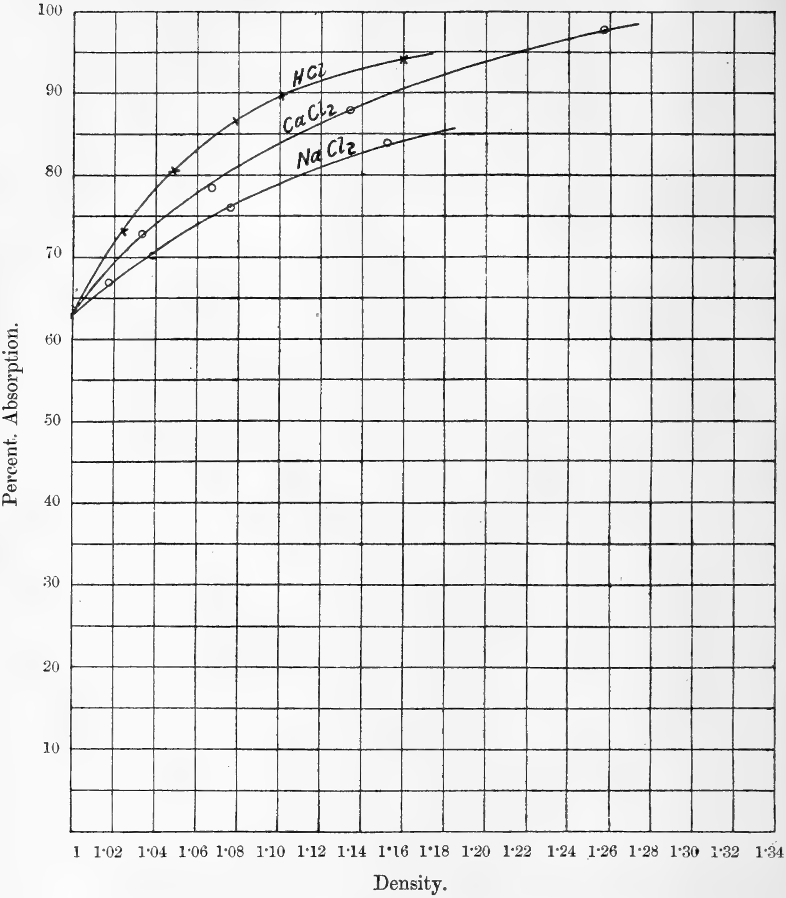
Fig. 2.



From an extended series of experiments on water, made at various temperatures, we proved that the absorption is independent of the temperature. All subsequent experiments were made at room temperature, which varied between 20° and 25° C.

In order to find the effect of concentration on the power of absorption, a number of salt solutions at different concentrations were investigated. In each case a concentrated solution

Fig. 3.



of the salt in question was made up, and the percentage of the rays which it absorbed determined in the manner already explained for water. The density as compared with water was determined by means of a Mohr-Wesphal specific-gravity balance. The solution was then diluted to a certain extent and the absorption again determined, the density being measured as before. This was done for several densities of the liquid until the absorption approached that of water. This relation between density and absorption was investigated for several salts in solution, and the same general relation was found to exist in each case. By plotting percentage absorptions as ordinates and densities as abscissæ, the curve showing the variation of absorption with density is obtained. Several of these curves for chlorides are shown in figs. 2 and 3. The results shown by these curves, along with a number of other chlorides, which are not plotted, are given in Tables II., III., and IV. Then, for comparison, a determination for zinc sulphate is shown in fig. 4, the corresponding numbers being given in Table V.

Absorption of Chlorides.

TABLE II.

Hydrochloric Acid.		Calcium Chloride.		Sodium Chloride.	
Density.	Percentage Absorption.	Density.	Percentage Absorption.	Density.	Percentage Absorption.
1·160	94·2	1·259	98·0	1·153	84·2
1·112	89·9	1·135	88·3	1·078	76·1
1·079	86·9	1·069	78·8	1·039	70·3
1·049	80·8	1·034	73·0	1·019	67·0
1·026	73·4	1·018	65·7	1·000	63·0
1·000	63·0	1·000	63·0		

TABLE III.

Cadmium Chloride.		Zinc Chloride.		Copper Chloride.	
Density.	Percentage Absorption.	Density.	Percentage Absorption.	Density.	Percentage Absorption.
1·310	99·9	1·198	94·1	1·169	92·3
1·157	99·3	1·104	90·5	1·085	86·2
1·077	96·6	1·053	83·1	1·042	77·0
1·038	89·8	1·026	73·2	1·020	69·0
1·018	77·6	1·012	65·2	1·009	64·3
1·009	72·9	1·000	58·9	1·000	58·9
1·000	58·9				

TABLE IV.

Magnesium Chloride.		Ammonium Chloride.		Manganese Chloride.	
Density.	Percentage Absorption.	Density.	Percentage Absorption.	Density.	Percentage Absorption.
1·140	81·8	1·062	76·4	1·194	96·9
1·071	69·1	1·033	68·3	1·097	89·6
1·036	64·3	1·017	60·8	1·048	83·0
1·017	59·5	1·004	57·2	1·023	70·7
1·000	56·9	1·000	56·7	1·010	63·5
				1·000	61·2

Barium Chloride.		Strontium Chloride.	
Density.	Percentage Absorption.	Density.	Percentage Absorption.
1·175	96·7	1·250	94·9
1·088	93·0	1·130	90·2
1·044	84·6	1·032	83·8
1·022	74·2	1·015	78·0
1·010	71·3	1·007	75·6
1·000	55·3	1·000	65·3

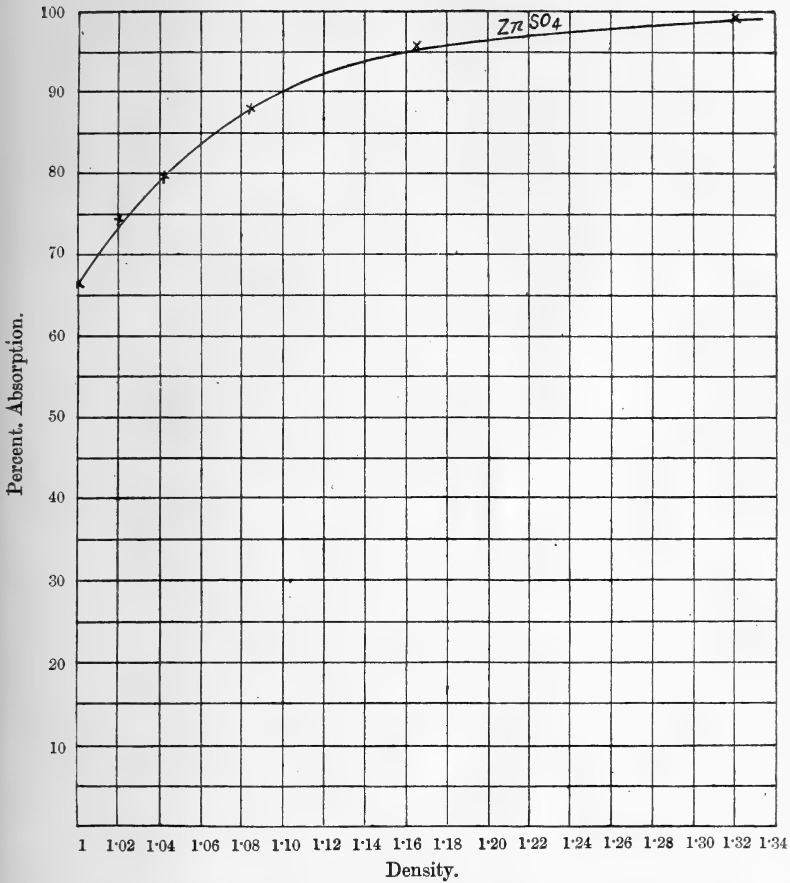
Mercuric Chloride.		Lead Chloride.	
Density.	Percentage Absorption.	Density.	Percentage Absorption.
1·052	96·6	1·009	78·1
1·026	89·9	1·004	70·3
1·011	79·8	1·002	65·2
1·004	68·8	1·000	57·7
1·002	58·1		
1·000	51·1		

TABLE V.

Absorption of Zinc Sulphate.

Density.	Percentage Absorption.
1·321	99·0
1·165	95·7
1·085	88·0
1·043	79·7
1·021	74·4
1·000	66·4

Fig. 4.



These determinations show clearly that the effect of an increase in the amount of salt in solution produces an increase in the absorption. These, however, are not directly proportional; but, as seen from the form of the curves, the relation appears to be a logarithmic one. This agrees with what has been pointed out by Lord Blythswood and Dr. Marchant.

Having investigated the variation in absorption with variation of density in solutions, a comparison of several solutions of a given specific gravity was then made. The result of the comparison is shown in Tables VI. and VII., the former containing a series of chlorides and the latter a series of sulphates, with a few other salts, all of a specific gravity of 1·06.

TABLE VI.

Absorption of Chlorides.

Solution.	Percentage Absorption.
Water	62·0
Magnesium chloride	75·0
Sodium ,,	75·1
Calcium ,,	77·8
Potassium ,,	78·6
Ammonium ,,	80·2
Hydrochloric acid	80·6
Manganese chloride	82·4
Copper ,,	85·6
Zinc ,,	88·3
Barium ,,	91·5
Strontium ,,	93·4
Cadmium ,,	94·8

TABLE VII.

Absorption of Sulphates, &c.

Solution.	Percentage Absorption.
Water	66·1
Ammonium sulphate	69·2
Magnesium ,,	69·5
Sugar	69·5
Sodium sulphate	69·7
Sulphuric acid	72·0
Potassium nitrate	75·8
,, sulphate	76·0
Manganese ,,	81·9
Copper ,,	82·5
Zinc ,,	84·4
Cadmium ,,	93·9
Potassium iodide	96·1
,, bromide	96·2

This particular density was chosen because, in the majority of cases, the absorption could here be measured with the greatest accuracy.

From Tables VI. and VII., and in the case of the salts of lead and mercury from the previous tables, it is seen that the absorption depends upon two factors, namely, the kation and the anion. With one or two exceptions, an increase of atomic weight is accompanied by an increase of absorption. For the chlorides and sulphates respectively the order of absorption is as follows, the first-named being the least absorbent:—

Chlorides.

$$\left\{\begin{array}{l}\text{Sodium.}\\ \text{Magnesium.}\end{array}\right.$$
$$\left\{\begin{array}{l}\text{Calcium.}\\ \text{Potassium.}\end{array}\right.$$
$$\left\{\begin{array}{l}\text{Ammonium.}\\ \text{Hydrogen.}\end{array}\right.$$

Manganese.
Copper.
Zinc.
Barium.
Strontium.
Cadmium.
Mercury.
Lead.

Sulphates.

$$\left\{\begin{array}{l}\text{Ammonium.}\\ \text{Magnesium.}\\ \text{Sodium.}\end{array}\right.$$

Hydrogen.
Potassium.
Manganese.
Copper.
Zinc.
Cadmium.

For the anion, although the number of salts investigated is hardly sufficient to generalize, yet the determinations indicate that the absorbing-power is in the following order:—

$$\left\{\begin{array}{l}SO_4,\\ NO_3\end{array}\right.$$
$$Cl.$$
$$\left\{\begin{array}{l}I,\\ Br.\end{array}\right.$$

These results, with one or two exceptions, agree with those obtained by previous investigators working in this field.

Since even very thin sheets of metals absorb the rays almost completely, we thought it would be interesting to investigate the effect of such metals held in solution in a finely divided state. With this idea in view, the metals lead, mercury, and silver were investigated in the form of colloidal solutions, as described by Bredig (*Zeit. für Electrochem.* iv. pp. 514 & 547). With the possible exception of mercury, it was found that these metals in solution differed very slightly from water in their power of absorption.

Owing to the short time at our disposal, only a few of the solutions which we had hoped to investigate have been examined, and the determinations are in some cases not so full as could be desired. We hope, however, to continue this research and make more complete determinations. The question of the variation of absorption with the increase of thickness of the layer of solution traversed by the rays, we also hope to investigate by a somewhat similar but improved method to that used.

In conclusion we wish to express our thanks to Prof. Rutherford for the valuable suggestions which he offered during the course of this research.

McDonald Physical and Chemical Laboratories,
McGill University, Montreal. July 15th, 1901.

VI. *The Behaviour of the Phosphorus Emanation in Spherical Condensers**.—V. *By* C. Barus†.

1. IN the preceding papers it was assumed that the conditions could be so chosen (swift air-currents, highly active ionizer, &c.) that the decay of the ionization might be temporarily disregarded. Such an assumption is naturally precarious, and the following series of experiments are planned with particular reference to the factor ignored. Accordingly a closed spherical condenser was installed with its outer surface put to earth and its inner (concentric) surface, always very small, in contact with the charged electrometer. The intervening space was suitably ionized by a small piece of phosphorus, about as large as a pea, suspended at the centre From such a condenser no ions can escape; it therefore offers greater theoretical simplicity than the plate-condenser, as discussed in an earlier paper, while the two discriminating variables, radius and voltage, may be changed at will. Practically, however, the experiments are very difficult, not only because of the baffling irregularity of the ionizer, but because it is not easy to lead a charge to or from the centre, without conduction.

As a whole, I think, the results may be interpreted as showing that decay due to the mutual destruction of ions is not in direct evidence, and may be considered negligible in investigations of the present order of precision. The enclosed air at a distance from the phosphorus rather behaves as though it contained more ions than reach it from the source.

2. A series of König's resonators seemed very suitable for the present purposes, since they were at hand in a large range of diameters, and fig. 1 shows the original adjustment. R is the brass resonator; B the curl of wire making the inner surface of the condenser, and supporting the piece of phosphorus P; C is an insulating glass tube 30 centims. long, through which the charge is conveyed by the wire *ab*, to be dissipated in the condenser. The tube C is grasped by the hard-rubber sheath F, fig. 2. B is thus in contact with the electrometer, and the capacity of the latter (about 60 cm.) is always large as compared with the condenser (negligible here and less than 1 cm. in the experiments below). D is a perforated copper plate closing the condenser and putting it to earth at E.

3. The early experiments made with this apparatus (Resonator, K 6; $2\,R = 8{\cdot}6$ cm.) seemed to indicate a linear

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* *Cf.* Am. Journ. Sci. [4] xi. p. 310 (1901).

† Communicated by the Author.—For previous communications, see Phil. Mag. [6] ii. pp. 477–488 and references there given.

relation of potential and time, at least within the first 7 minutes of discharge. This constant current was not appreciably modified by stopping the lower tubulure of the resonator with cotton, nor after vigorously airing it out; neither was there any marked change of current even after 30 minutes.

This interesting result was not, however, borne out by longer single periods of observation (20 minutes). Marked curvature was eventually in evidence, which did not seem referable to leakage errors. Moreover, the conduction decreased in succeeding curves, apparently indicating a diminution of the ionization due to the phosphorus. Possibly the activity of phosphorus may be quenched by its own emanation in the lapse of time, but the discrepancy is liable to be hygrometric.

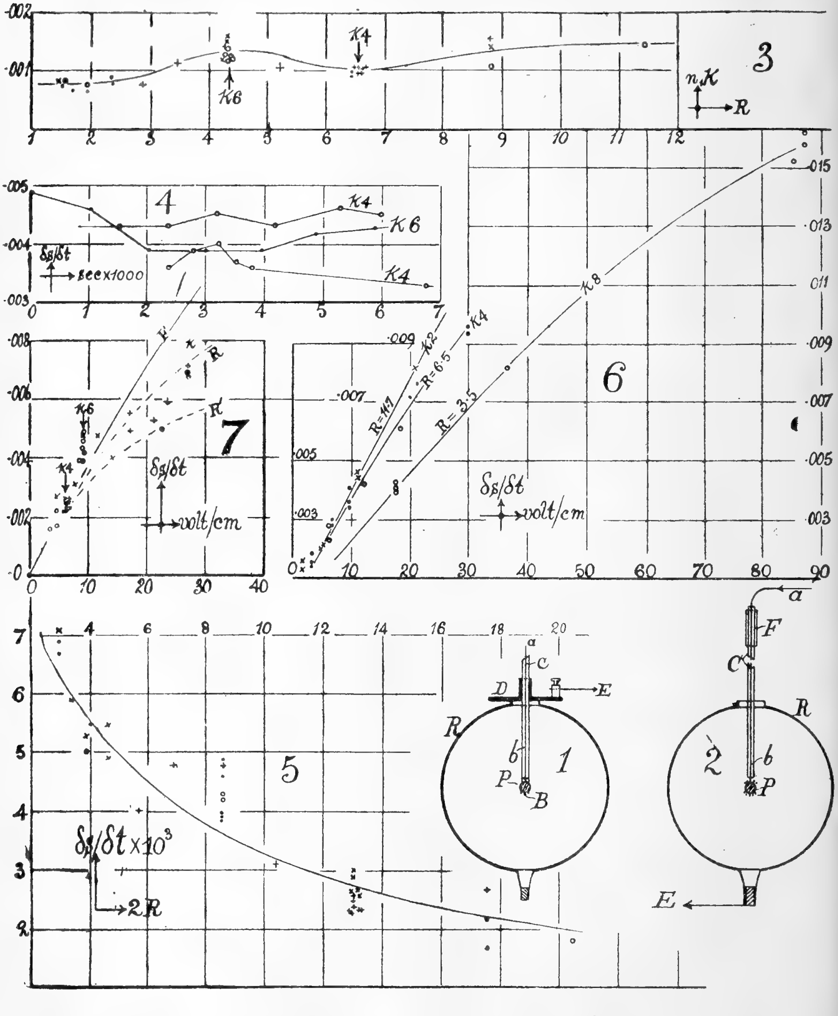
4. A systematic series of experiments was now carried through with receivers ranging in diameter from $2R=3$ to $2R=23$ centims. The results were surprisingly irregular seeing that some pains was taken with the work, but they mapped out a graph of an exponential character to which different interpretations may be given. To repeat them here would carry me too far.

5. Before continuing the work it seemed advisable to make trials to secure more perfect insulation. No means were found for removing leakage in the apparatus perfectly (paraffin, rubber, &c. were tried in succession with no advantage); and a dry clean glass stem, C, fig. 1, held in place by a wide hard-rubber sheath F, seemed the best remedy.

Experiments were then tried to ascertain the difference between a condenser closed as above and one left open for the stem to pass through, as shown in fig. 2. Here the wire and plug attachment puts the outer surface to earth at E. The closed condenser always introduced the larger leakage. Care must be taken to avoid friction in manipulating the stem.

6. Conformably with the experience gained, the condenser in the following experiments is left open above, as seen in fig. 2, and is put to earth at E by a brass plug wedged into the neck. The glass tube, C, terminating in the ball of phosphorus P, is kept clean and dry, and extends to 30 centims. above the condenser. Unfortunately there is some escape of ions around the neck and the access of air here is a menace, but there seems to be no way out of the dilemma.

To estimate the loss of ionizing activity of the phosphorus pellicle one of the condensers (K 6 and K 4 respectively) was treated as a standard, and observations were made upon this before and after the observations of each of the other condensers. Table I. is obtained in this way. The scheme adopted limited all observations to an interval of 5 minutes. Readings, s, at the electrometer were taken per minute, and



by dividing them into two parts the rates $\delta s/\delta t$ were obtained linearly from data 3 minutes apart. The table also contains other constants to be discussed below. The first measurement is given in full as an example of the method, the others are abbreviated.

TABLE I.—Leakage of Spherical Condensers with a medium ionized by Phosphorus. Initial potential-difference, $V_0=40$ volts. Electrometer deflexions, *s*.

No. Radius R. Field.	Time *t*.	Deflexion *s*.	$10^5\times n_1K'$.
K 6. 4·3 cm. 9·3 v./cm.	$29^{\rm m}$ $0^{\rm s}$	6·60	117
	60	6·40	121
	120	6·15	124
	180	5·90	—
	240	5·70	121
	300	5·45	
K *. 1·5 cm. 27 v./cm.	$37^{\rm m}$—$42^{\rm m}$	...	79
K 6. 4·3 cm. 9·3 v./cm.	$45^{\rm m}$—$50^{\rm m}$	...	122
K 3. 8·8 cm. 4·5 v./cm.	$53^{\rm m}$—$58^{\rm m}$	...	102

No. Radius R. Field.	Time *t*.	Deflexion *s*.	$10^5\times n_1K'$.
K 6. 4·3 cm. 9·3 v./cm.	$61^{\rm m}$—$66^{\rm m}$	...	119
K 17. 1·95 cm. 20·5 v./cm.	$69^{\rm m}$—$74^{\rm m}$	...	73
K 6. 4·3 cm. 9·5 v./cm.	$76^{\rm m}$—$81^{\rm m}$	...	113
K 2. 11·7 cm. 3·4 v./cm.	$84^{\rm m}$—$89^{\rm m}$	...	143
K 6. 4·3 cm. 9·3 v./cm.	$93^{\rm m}$- $98^{\rm m}$	...	135

* The following data are abbreviated by omitting the individual deflexions *s* and times *t*. Again $2·3\ n_2K'=n_1K$.

Curiously, K6 showed a linear diminution of potential or constant current, while in other cases the currents usually fell off. Again, the values of $\delta s/\delta t$ for K6 in the lapse of time, fig. 4, first show a decrease (·0049 to ·0039), thereafter a constant variation, finally actually an increase (·0039 to ·0043). Hence the factors which determine the activity of phosphorus are extremely complicated. Moreover, if the observations for the other condensers are corrected by the results for K6, by putting the current for the latter ·0050 throughout, the new values show worse agreement than the original values. It follows from this that the method of correction proposed is not available, and that the discrepancies to be eliminated are not regular but incidental fluctuations. Moreover, the data of this table are uniformly smaller, scarcely $\frac{1}{2}$ or $\frac{1}{3}$ of the values referred to in § 4. They are insufficiently definite to suggest the form of the locus. New observations, Table II., were therefore needed and were made in the same way as the results given in Table I. The condenser was slightly modified, consisting in this case of a sphere of wire gauze about 1 centim. in diameter, containing a much larger piece of phosphorus than above. The larger condenser K 4 is now the standard, admitting of easier adjustment. The constancy of the ionizer is shown in fig. 4 (upper curve K 4), displaced in ordinate, for convenience.

TABLE II.—Leakage of Spherical Condensers with a medium ionized by Phosphorus. Initial potential-difference $V_0=40$ volts.

No. Radius R. Field.	Time t.	Deflexion s.	$10^5\times n_1K'$.
K 4. 6·5 cm.	$24^m\ 0^s$	7·10	97
	60	6·90	
	120	6·75	
	180	6·60	
	240	6·50	
	300	6·40	
K 3. 8·8 cm.	31^m—36^m	...	153
K 4. 6·5 cm.	38^m—43^m	...	94
K 5. 5·2 cm.	46^m—51^m	...	102
K 4. 6·5 cm.	53^m—58^m	...	103
K 6. 4·3 cm.	61^m—66^m	...	138
K 4. 6·5 cm.	68^m—73^m	...	94
K 8. 3·45 cm.	78^m—83^m	...	110
K 4. 6·5 cm.	86^m—91^m	...	103
K 10. 2·85 cm.	93^m—98^m	...	72
K 4. 6·5 cm.	100^m—105^m	...	100

In Table III. the ionizing activity of phosphorus is exhibited when the condenser, K 4, is left quite without interference. A remarkable rise and fall is as usual apparent, without discernable cause (fig. 4, lower curve). After this, the other condensers are again tested with the object of securing sufficient data to at least roughly suggest the form of the locus.

Both Tables II. and III. have been abbreviated; the results of interest admit of graphic representation, as will be explained in relation to figs. 3 and 5.

7. In view of the fact that the ionizing potency of phosphorus sometimes increases and at other times decreases, as it were incidentally and subject to arbitrary conditions* not made out, I have in fig. 5 summarized the uncorrected values of the initial currents, $\delta s/\delta t$, preferably to the corrected

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* Work still in progress has shown me that the variability of the phosphorus ionizer is related to the hygrometric conditions under which the experiments are made. Temperature is of secondary importance.

values. The former, as has been stated, usually show greater regularity than the latter, upon which the fluctuations of the standard condensers have been imposed. The diameters, 2R, are in centimetres.

TABLE III.—Leakage of Spherical Condensers with a medium ionized by Phosphorus. Initial potential-difference $V_0=40$ volts.

No. Radius R. Field.	Time t.	$10^5 \times$ $n_1 K'$.	No. Radius R. Field.	Time t.	$10^5 \times$ $n_1 K'$.
K 4. 6·5 cm.	40^m—45^m	69	K 14. 2·35 cm.	75^m—80^m	74
K 4.	46^m—51^m	109	K 14. 2·35 cm.	91^m—96^m	87
K 4.	53^m—58^m	113	K 17. 1·95 cm.	99^m—104^m	64
K 4.	58^m—63^m	103	K 20. 1·7 cm.	107^m—112	65
K 4.	64^m—69^m	99	K. 1·5 cm.	115^m—120^m	71
			K 4. 6·5 cm.	122^m—127^m	89

It appears at once that the data as a whole, though investigated with care, still fail to lend themselves for the nice discernment of the nature of the locus in a relation of current to the diameter of the condenser. The results, even of a single series, are not smooth. Indeed the exceptional positions of the currents for the standard condensers, those of K 6 (2R=8·7 cm.) being abnormally high, while those of K 4 (2R=13·1 cm.) are low, is perplexing, and has led me to suppose that some occult cause of variation has been left undiscovered. One is almost tempted to infer that each condenser behaves as an individual, a conclusion for which I am unable to discover adequate reasons. The curve, which has been put through the observations, was computed from $(dv/dt)(R+a)=A$, for reasons presently to be explained. The observations are in accord with it, in so far as they show an increase of current at an accelerated rate as diameter decreases.

8. *Working Hypothesis.*—The attempt must now be made to derive some theoretical conclusions from the experiments

detailed in the above paragraphs. As before, let n be the number of particles per cubic centim., so that n is the concentration or density of distribution of the phosphorus emanation. Let k be the "absorption" velocity of the ion, treated in the first instance as independent of the potential and of the concentration gradients. Let k' be the coefficient of decay, so that $k'n^2$ is the number of ions vanishing per cubic centim. per second. Finally, let R be the external radius of the condenser and C its effective capacity including that of the electrometer.

With regard to the electrical currents, let V be the potential at a distance r from the centre of the condenser whose external face is put to earth. Let U be the aggregate velocity of the ions in the unit electric field and e the charge of each.

In all cases the observations are made when the flux is stationary, so that $dn/dt=0$ throughout, for any shell. Moreover, as shown elsewhere, the effect of a potential gradient is but a negligible contribution to the number of ions which are absorbed by the outer surface of the condenser.

To begin with the simplest cases:—If the motion of the ion is entirely independent of $d\mathrm{V}/dr$ and n, the accumulation in an elementary shell at a distance r from the centre will be $4\pi k d(r^2 n)/dr\,.\,dr$, per second; the decay per second, $k'n^2 4\pi r^2 dr$. Hence

$$d(r^2 n)/dr=(k'/k)n^2 r^2,$$

or if A is a constant,

$$1/n=r\big((k'/k)+\mathrm{A}r\big).$$

In the absence of decay, $1/\mathrm{A}=nr^2$ so that A is the reciprocal of the concentration, n_1, at a distance 1 from the centre. If conduction were prompted solely by the ions which reach the external shell kept at $\mathrm{V}=0$, since the charge in this shell is per centim.

$$ed\mathrm{R}/(\mathrm{R}(k'/k)+\mathrm{AR})$$

and its time of discharge $d\mathrm{R}/k$,

$$\mathrm{C}d\mathrm{V}/dt=4\pi ke\mathrm{R}/\big((k'/k)+\mathrm{AR}\big).$$

In the absence of decay, $k'=0$, and

$$d\mathrm{V}/dt=4\pi ken_1/\mathrm{C},$$

where n_1 as stated holds for $r=1$. This case, in which $d\mathrm{V}/dt=ds/dt=$ const., independent of the radius of the condenser is effectually excluded by the observations given in fig. 5. If k' is not zero,

$$d\mathrm{V}/dt=(4\pi ke/\mathrm{C})\big(1/(k'/k\mathrm{R}+\mathrm{A})\big),$$

so that the current increases with R, which is not admissible. Neither of these cases, moreover, would be open to computation directly, and they are thus without immediate interest.

This may be treated in a slightly different manner, however, by supposing the number, n, just found to be correctly evaluated and then introducing the ion velocity, U, instead of the absorption velocity, k, in the usual way, and they then become suggestive. If $1/A$ is replaced by n_1, the number of ions per cubic centimetre at a distance of 1 centim. from the centre, the above concentration n at a distance r may be written,

$$n_1/n = r((k'/k)n_1(1-r)+r).$$

If decay is ignored, $n=n_1/r^2$, as is otherwise clear, is independent of k also.

Now if the electric conduction is determined by the number of ions which reach the external shell $(r=r)$,

$$-dQ/dt = -CdV/dt = 4\pi R^2 U (V/R) ne.$$

It is understood that this number is not appreciably modified by the occurrence of the field so that when decay is absent $(k'=0)$, $n=n_1/R^2$, as above deduced. Hence

$$-(dV/dt)/V = -d(\log V)/dt = 4\pi e U n_1/CR.$$

Here the first member is equivalent to $-d(\log s)/dt$, S being the deflexion of the electrometer, and is obtainable from the observations directly, $4\pi eU/C$ is a constant, n_1 expresses the waning intensity of the phosphoric source of ionization, and R is the external radius of the condenser. The equation therefore admits of being tested. The integral of the equation found for the potential gradient becomes

$$V = V_0 \epsilon^{-(4\pi e U n_1/CR)t},$$

which is compatible with the data of tables 1–3. In these tables I have therefore inserted data for

$$n_1 K = (4\pi e U/C) \cdot n_1,$$

obtained from

$$-(dV/dt)/V \cdot R$$

for each case. To facilitate computation, n_1K is left in common logarithms and written n_1k', so that

$$2{\cdot}3\, n_1 k' = n_1 K.$$

9. *Comparison of data.*—The values so found, *i. e.*,

$$n_1 K' = Rd(\log V)/dt,$$

are shown graphically in fig. 3, as ordinates in terms of R as abscissas. The curve here is apparently sinuous, due to the abnormally high values of K 6 and the abnormally low values of K4, alluded to, both of which remain unexplained. In the absence of these there would be a rise of n_1K′, of a gradual character with increasing radius. Since in n_1K, K is constant, this means that relatively more ions, n_1, are available at the larger radii of the condensers, corresponding to weaker fields, than for smaller radii and correspondingly stronger fields. But as there is no reason for excluding K 4 and K 6, and no suggestion for the occurrence of the sinuous curve obtained, n_1K must be regarded as increasing rapidly from the values for condensers of small radii, $r=2$ centim., but reaching a practically constant result after the radius 4 centim. has been surpassed. On the whole, therefore, the data so far as investigation has been possible agree with the remarks made in § 1, that the evidence in case of dilution is rather in favour of an increased number of ions and that an occurrence of decay is not manifest. This means more generally that whereas in the saturated emanation the ions are produced at the same rate at which they decay so that n is constant, in the diluted emanation at a distance from the centre $(n=n_1/r^2)$, the production is in excess of the decay and conduction relatively too great.

Another method of treating n_1K, is to refer it to strength of field. This, however, may be done more advantageously after the data of the next section, in which R is constant and V variable, have been similarly brought forward.

10. One important question as to the availability of phosphorus as an ion producer, is the intensity of its action or the number of ions produced per second in limiting cases. To make an estimate of this quality, it suffices to pass a current through the condenser for gradually increasing potential-differences between the faces, in order to ascertain to what degree the phenomenon fails to obey Ohm's law.

Results to this end were obtained with the spherical condenser, K 4, diameter 13·0 centims., the internal surface (a copper gauze bag carrying phosphorus) being about 1 centim. in diameter. Potential-differences, V, as high as 200 volts were applied, the external surface being put to earth. The current is as usual found from equidistant observations separated by an interval of 3 minutes. A few test experiments were made by repeating the earlier measurements for 21 and 63 volts, but it was thought wisest not to attempt to correct the data for the fluctuating ionizing activity of phosphorus in the lapse of time, nor to assume curvature in the initial ($t=0$, nearly) contours of the curves. It will be sufficient to report the results graphically.

The chart figure, 6, gives the relation of the current ($\delta s/\delta t$), arbitrarily in scale parts, to the potential-difference V in volts per centim., or strength of the field. The graph is obviously curved so that Ohm's law is departed from, but the curvature is small, indicating a limit as yet a great way off.

These results are now to be compared with similar data for condensers larger and smaller, the diameters chosen being $2R=6\cdot9$ centims. for K 8 and $2R=23\cdot3$ centims. for K 2, respectively.

The currents here obtained are also graphically reproduced in the chart figure 6, where the abscissas are the fields in volt./cm., and ordinates are the currents in the usual arbitrary measure. The relative conductivity of the three condensers is maintained, and the saturated states (maxima of the curves) are in all cases enormously distant. The relative curvatures of the three graphs in the same field cannot be made out.

11. *Observations for constant* V_0 *and for constant* R *compared.*—I shall now endeavour to compare the data of § 6 and § 10, by referring them either to the same radii or to the same electric fields. The latter method is preferable not only as yielding a greater range of data, but because the values of the currents, $\delta s/\delta t$, and of n_1K, have not been regarded in this light. To begin with the former, figure 7 contains the values of $\delta s/\delta t$ varying with the fields as taken from the Tables I., II., III., in which V_0/R varies by reason of varying R. These data though vague, eventually lie within the limits marked by the curves R and R′. In the same chart I have inserted the curve F, taken from fig. 6, in which V_0/R varies by reason of varying V_0. It is the curve for K 2 for which $R=11\cdot7$ is largest, and the data surest. The point of importance is clear at once: as the fields grow stronger the curve F lies quite above the curves R. If therefore high fields are produced by diminishing the radius of the condenser, the currents may be upwards of 20 per cent. or 30 per cent. too small, both because of the escape of ions around the stem and of the access of air. Indeed this state of things is not unexpected, inasmuch as the chief object of the investigation with spherical condensers was the avoidance of such losses of ions as occur in plate-condensers. Returning for additional consideration to fig. 3, it appears that the definitely low data corresponding to the radii 2 and 3 centimetres, are erroneously much too low, whence it follows that the probability of a constant n_1K is enhanced.

12. *Conclusion.*—Contrary to my expectation and in spite

of the labour spent upon them, the results for spherical condensers have not enabled me to give a decisive answer to the question at issue. The difficulty encountered and which occurs here in accentuated form, is the same which has hampered me throughout the present research, namely the elusive variability of the ionizer. Moreover, as the conditions determining it exist immediately at the surface of the active phosphorus, I do not see how they are to be put under control; merely keeping the air around the phosphorus at constant temperature, &c., is not a sufficient check on the behaviour of the surface itself.

In several respects, nevertheless, definite advances have been made. It has been shown that the best results can be reached with large spherical condensers (say 20 centims., or more, in diameter), in which changes of field are produced by applying larger potential-differences while the apparatus itself is left quite without interference.

Finally, to ascertain in how far the present experiments agree as a whole with the results for plate-condensers, where a theoretically different method is involved, it suffices to compute the value of n_1, the number of particles per cubic centim. at 1 centim. from the centre of the condenser. Since n_1K' in fig. 3 is of the order of $\cdot 0012$, or when referred to natural logarithms $n_1K = \cdot 0026$, and since $n_1K = 4\pi eU/C$, if we insert J. J. Thomson's value of $e = 2\cdot 3/10^{19}$ and put U of the order of 1 cm./sec., $C = 6\cdot 7/10^{11}$ farads, it follows that $n_1 = 6\cdot 5 \times 10^4$. The result agrees very well with the datum $n_0 = 5\cdot 5 \times 10^4$, as determined from plate-condensers by the totally different method there pursued. If in both cases $U = 1\cdot 5$ cm./sec. be assumed, the numbers will be $n_1 = 4\cdot 3 \times 10^4$ and $n_0 = 3\cdot 6 \times 10^4$, respectively. In either case the saturation is that existing at the surface of the phosphorus. They thus agree with the datum for the unsaturated emanation $n = 2 \times 10^4$, obtained by a third method from tubular condensers. Finally, all are in accord with J. J. Thomson's value for ionized air $n = 4 \times 10^4$, resulting from a method theoretically different from that of my papers.

13. *Summary.*—With the present paper I have given a brief but systematic account of the bulk of my work with the phosphorus emanation, the purpose throughout being to map out roughly the phenomena in which I am interested, as a preliminary to the more rigorous study of the subject which I am now beginning. The results as a whole are eventually to be tributary to an investigation on the colours of cloudy condensation, particularly in those regions within which Lord Rayleigh's theory must, in any case, cease to be applicable.

In endeavouring to account for the data obtained it was my endeavour to follow the established theory; but I fear that in the explanations given I have little by little made a serious departure. If I had obtained but a single coincidence the result would not have deserved record; but after finding data of a correct order of values in all the experiments, in spite of the variations of method, I have ventured to think that more than a coincidence is in question. The theory which provisionally underlies the series of papers is substantially this:—

(1) From experiments made in the absence of an electrical field I inferred that the nucleus has its own specific velocity and that this velocity is identical with the mutual velocity of the corresponding ions in the unit electrical field (volt/centim.).

(2) The nucleus produced by phosphorus is larger than the air molecule; and both for this reason and from the fact that it receives promiscuous bombardment of molecules simultaneously, its velocity is of the low order stated, being (say) less than 1/300 that of the air molecule.

(3) It is not necessary to assume that the nucleus decays or vanishes within the ionized medium; the evidence is rather in favour of a number of nuclei somewhat larger as the dilution increases.

(4) Whenever the nucleus comes in contact with a barrier (solid or liquid) it is absorbed or broken up. If the nucleus is ionized as in the case of the phosphorus emanation, the absorption is accompanied with the discharge of an electron.

(5) The observations with plate-condensers, tubular condensers, and spherical condensers are satisfied by supposing the nuclear velocity k to be independent of the concentration or degree of saturation.

(6) The nuclear velocity is independent of the potential gradient. Instead of putting U . V/R for the velocity of the ion along the potential gradient V/R, I have considered this velocity a constant, independent of V/R; while the number of nuclei capable of discharging the metallic terminal of a field varies as V/R.

(7) If for e the charge of a nucleus, J. J. Thomson's electron be taken; if $U=k$, then the number of nuclei* in the saturated phosphorus emanation agrees with J. J. Thomson's value for the number of ions in ionized air.

Brown University, Providence, U.S.A.

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* I have since found that the lower values found from tubular condensers are referable to the inherent non-saturation. By using an excess of freshly cut phosphorus, the number of ions could be nearly doubled, thus putting these results in accord with the correlative data for plate- and for spherical condensers.

VII. *The Effect of Errors in Ruling on the Appearance of a Diffraction Grating.* *By* H. S. Allen, *M.A.*, *B.Sc.**

THE effect produced by errors in ruling a diffraction-grating on the lines of the spectrum has received considerable attention. But the effect on the appearance of the grating itself has not been so carefully studied, although from its appearance useful information as to the nature of the errors may be derived.

If a spectroscope is adjusted to view a single line of the spectrum given by a plane grating, and the eyepiece of the observing-telescope is removed, the grating is seen illuminated by monochromatic light. In the absence of any errors of ruling the illumination would be perfectly uniform ; but in general the image is seen to be crossed by a number of bands parallel to the lines ruled on the grating. The bands may be more conveniently studied by focussing the observing-telescope on the surface of the grating instead of on the slit of the collimator. The object of the present paper is to explain the mode of formation of these bands.

If an absolutely perfect grating were illuminated by a parallel beam of monochromatic light, all the light going to form the spectral line of any particular order would be brought to a single focus by the objective of the telescope. If the grating were self-luminous, and at a sufficient distance from the lens, a real image of it would be formed beyond this focus. In the case we are considering, we can hardly speak of the rays forming a real image of the grating since they do not cut one another at the position of this image. However, the cone of rays emerging from the focus must be bounded by the image, and a screen placed here would be uniformly illuminated.

In the case of a grating containing two distinct rulings differing by only a small amount, the light from each portion will be brought to its own appropriate focus. Two different cases arise according to the relative positions of the wide and narrow rulings. These are shown in figs. 1 and 2. These figures are drawn for an order on the right of the central image, the observer being supposed to stand facing the grating. In fig. 1 the wide ruling is on the observer's left, the narrow ruling on his right. The light from the wide ruling is brought to a focus at F_1; that from the narrow ruling, which is diffracted at a greater angle, at F_2. The

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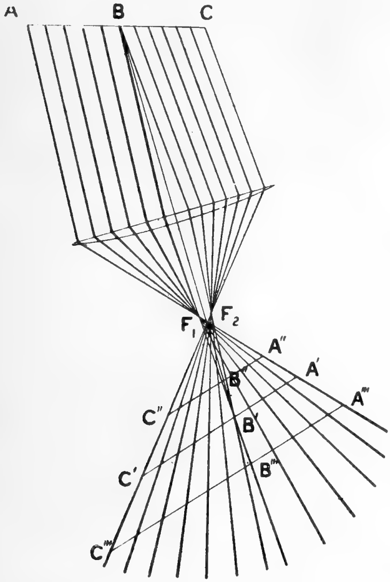
* A paper read before Section A of the British Association, Sept. 17, 1901. Communicated by the Author.

two emergent cones of light are bounded by the corresponding parts of the image of the grating. The light from the wide ruling AB is bounded by the image A′B′, that from the narrow is bounded by B′C′. Apart from interference-effects, a screen placed at A′B′C′ would be uniformly illuminated.

Fig. 1.



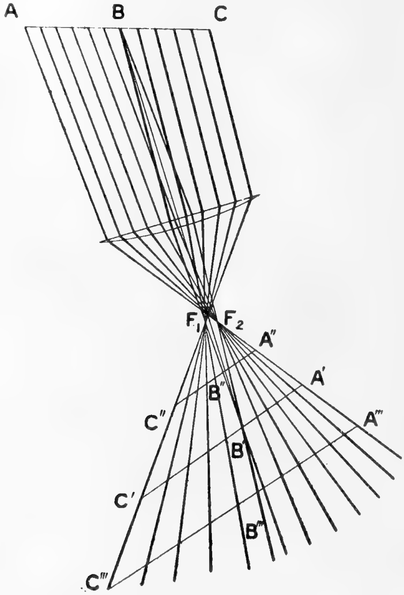
If the screen is moved nearer to the lens there will be a portion at B″ receiving light from both the cones, so that the illumination here is increased. If the screen is moved away from the lens, the part at B‴ does not receive light from either of the cones, so that a dark band is formed on the screen.

In fig. 2 (p. 94) the second case is shown in which the relative position of the rulings is reversed—the wide ruling is now on the right, the narrow one on the left. The positions of F_1 and F_2 with regard to the rulings are reversed, and it is clear at once from the figure that there will now be darkness at B″ and increased illumination at B‴.

If the figures are drawn for orders on the other side—that is the left—of the central image, the results as here expressed are exactly reversed*.

If the change in the ruling is gradual, instead of sudden, we should have to consider a series of focal points **F**, but the same line of reasoning may be followed.

Fig. 2



The conclusions may be summarized as follows:—

Orders on the right of the Central Image.

Case 1.—In passing from a wide to a narrow ruling in going from left to right.

Focus in.	Light band.
Focus out.	Dark band.

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* If light of wave-length λ falls at normal incidence on a grating ruled with N lines per unit-length and is diffracted at an angle θ,

$$\sin\theta = n\lambda N,$$
$$\delta\theta = n\lambda\delta N/\cos\theta,$$

where n is the order of the spectrum.

The distance $F_1 F_2$ is $f\delta\theta$ where f is the focal length of the lens. If the distance FB' is a, and the distance FB'' is b, the amount of overlapping at B'' is $\frac{b}{a} \cdot f\delta\theta$.

Case 2.—In passing from a narrow to a wide ruling in going from left to right.

Focus in.	Dark band.
Focus out.	Light band.

Orders on the left of the Central Image.

The results just given must be reversed.

The theoretical results so obtained were confirmed by observations on gratings with prominent errors. It had previously been noticed that when the telescope was focussed exactly on the surface of the grating the bands disappeared, by putting the telescope considerably out of focus the bands were clearly seen. Following the indications of the theory, and focussing first in and then out, it was seen that the dark bands changed to light and the light to dark. On comparing the images on the right and left of the central one, the bands were seen to be exactly reversed. The same effect was produced by the practically equivalent process of turning the plate upside down*.

In all the gratings that I have examined the principal error has been periodic in character, so that the bands are distributed over the image with a certain amount of regularity.

In conclusion I beg to express my thanks to Lord Blythswood, the resources of whose laboratory have been at my disposal in examining the gratings.

Blythswood Laboratory,
Renfrew, N.B.

VIII. *A New Specifying Method for Stress and Strain in an Elastic Solid.* *By* Lord KELVIN †.

THE method for specifying stress and strain hitherto followed by all writers on elasticity has the great disadvantage that it essentially requires the strain to be infinitely

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If we treat F_1 and F_2 as two sources of light as in Fresnel's interference experiment, the breadth of an interference-band at B″ would be $b\lambda/f\delta\theta$.

A simple calculation shows that in general this quantity is large compared with the distance within which there is overlapping, so that the interference effect is comparatively insignificant.

* I have devised an arrangement in connexion with Lord Blythswood's dividing-engine which makes it possible to cut any number of lines to the inch. Since the paper was written I have verified the results by using a plate containing two distinct rulings.

† Communicated by the Author.

small. As a notational method it has the inconvenience that the specifying elements are of two essentially different kinds (in the notation of Thomson and Tait e, f, g, simple elongations; a, b, c, shearings). Both these faults are avoided if we take the six lengths of the six edges of a tetrahedron of the solid, or what amounts to the same, though less simple, the three pairs of face-diagonals of a hexahedron *, as the specifying elements. This I have thought of for the last thirty years, but not till to-day (Dec. 16) have I seen how to make it conveniently practicable, especially for application to the generalized dynamics of a crystal.

We shall suppose the solid to be a homogeneous crystal of any possible character. Cut from it a tetrahedron ABCD of any shape and orientation. Let the three non-intersecting pairs (AB, CD), (BC, AD), (CA, BD) of its six edges be denoted by

$$(f, f'), \quad (g, g'), \quad (h, h') \quad \ldots\ldots \quad (1).$$

Parallel to the edge AB apply to the faces ADC, BDC equal and opposite pulls, P, equally distributed over them. These two balancing pulls we shall call a stress or a stress-component. Similarly, parallel to each of the five other edges apply balancing pulls on the pair of faces cutting it. Thus we have in all six stress-components parallel to the six edges of the tetrahedron, denoted as follows:—

$$(\mathrm{P}, \mathrm{P}'), \quad (\mathrm{Q}, \mathrm{Q}'), \quad (\mathrm{R}, \mathrm{R}') \quad \ldots\ldots \quad (2);$$

and we suppose that these forces, applied as they are to the surface of the solid, are balanced in virtue of the mutual forces between its particles, when its edges are of the lengths specified as in (1). Let f_0, f_0', g_0, g_0', h_0, h_0', be the values of the specifying elements in (1) when no forces are applied to the faces. Thus the differences from these values, of the six lengths shown in formula (1), represent the strain of the substance when under the stress represented by (2).

Let w be the work done when pulls upon the faces, each commencing at zero, are gradually increased to the values

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* This name, signifying a figure bounded by three pairs of parallel planes, is admitted in crystallography; but the longer and less expressive "parallelepiped" is too frequently used instead of it by mathematical writers and teachers. A hexahedron with its angles acute and obtuse is what is commonly called, both in pure mathematics and crystallography, a rhombohedron. A right-angled hexahedron is a brick, for which no Greek or other learned name is hitherto to the front in usage. A rectangular equilateral hexahedron is a cube.

shown in (2). In the course of this process we have

$$dw = \mathrm{P}df + \mathrm{P}'df' + \mathrm{Q}dg + \mathrm{Q}'dg' + \mathrm{R}dh + \mathrm{R}'dh' \quad . \; . \; . \quad (3).$$

Hence if we suppose w expressed as a function of f, f', g, g', h, h', we have

$$\frac{dw}{df} = \mathrm{P}, \quad \frac{dw}{df'} = \mathrm{P}', \quad \frac{dw}{dg} = \mathrm{Q}, \quad \frac{dw}{dg'} = \mathrm{Q}', \quad \frac{dw}{dh} = \mathrm{R}, \quad \frac{dw}{dh'} = \mathrm{R}' \quad . \; . \; . \quad (4).$$

This completes the foundation of the molar dynamics of an elastic solid of the most general possible kind according to Green's theory, expressed in terms of the new mode of specifying stresses and strains.

In a communication to the Royal Society of Edinburgh promised for Jan. 20, 1902, I hope to use with advantage this mode of specification in working out some details of the molecular dynamics of a crystal.

IX. *Some General Theorems concerning Forced Vibrations and Resonance.* By Lord RAYLEIGH, *F.R.S.*[*]

THE general equation for the small vibrations of a system whose configuration is defined by the generalized coordinates $\psi_1, \psi_2, \ldots$, may be written [†]

$$\frac{d}{dt}\frac{d\mathrm{T}}{d\dot{\psi}} + \frac{d\mathrm{F}}{d\dot{\psi}} + \frac{d\mathrm{V}}{d\psi} = \Psi, \quad . \; . \; . \quad (1)$$

where T, F, V, denoting respectively the kinetic energy, the dissipation function, and the potential energy, have the forms

$$\left.\begin{aligned} \mathrm{T} &= \tfrac{1}{2}a_{11}\dot{\psi}_1^2 + \tfrac{1}{2}a_{22}\dot{\psi}_2^2 + \ldots + a_{12}\dot{\psi}_1\dot{\psi}_2 + \ldots \\ \mathrm{F} &= \tfrac{1}{2}b_{11}\dot{\psi}_1^2 + \tfrac{1}{2}b_{22}\dot{\psi}_2^2 + \ldots + b_{12}\dot{\psi}_1\dot{\psi}_2 + \ldots \\ \mathrm{V} &= \tfrac{1}{2}c_{11}\psi_1^2 + \tfrac{1}{2}c_{22}\psi_2^2 + \ldots + c_{12}\psi_1\psi_2 + \ldots \end{aligned}\right\}, \quad . \; . \quad (2)$$

in which the coefficients a, b, c are constants.

If we substitute in (1) the values of T, F, and V, and write D for d/dt, we obtain a system of equations which may be put into the form

$$\left.\begin{aligned} e_{11}\psi_1 + e_{12}\psi_2 + e_{13}\psi_3 + \ldots &= \Psi_1 \\ e_{21}\psi_1 + e_{22}\psi_2 + e_{23}\psi_3 + \ldots &= \Psi_2 \\ e_{31}\psi_1 + e_{32}\psi_2 + e_{33}\psi_3 + \ldots &= \Psi_3 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \end{aligned}\right\}, \quad . \; . \; . \quad (3)$$

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[*] Communicated by the Author.

[†] See 'Theory of Sound,' vol. i. §§ 82, 84, 104.

where e_{rs} denotes the quadratic operator

$$e_{rs} = a_{rs}D^2 + b_{rs}D + c_{rs}. \quad . \quad . \quad . \quad . \quad (4)$$

And it is to be remarked that since

$$a_{rs} = a_{sr}, \qquad b_{rs} = b_{sr}, \qquad c_{rs} = c_{sr},$$

it follows that

$$e_{rs} = e_{sr}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If we multiply the first of equations (3) by $\dot{\psi}_1$, the second by $\dot{\psi}_2$, &c., and then add, we obtain

$$\frac{d(T+V)}{dt} + 2F = \Psi_1\dot{\psi}_1 + \Psi_2\dot{\psi}_2 + \ldots . \quad . \quad (6)$$

In this the first term represents the rate at which energy is being stored in the system; 2F is the rate of dissipation; and the two together account for the work done upon the system in time *dt* by the external forces Ψ_1, Ψ_2, &c.

In considering forced vibrations of simple type we take

$$\Psi_1 = E_1 e^{ipt}, \qquad \Psi_2 = E_2 e^{ipt}, \qquad \&c., \quad . \quad . \quad (7)$$

and assume that ψ_1, ψ_2, &c., are also proportional to e^{ipt}. The coordinates are then determined by the system of algebraic equations resulting from the substitution in (4), (3) of *ip* for D. The most general motion possible under the assumed forces would require the inclusion of *free* vibrations, but (unless $F=0$) these die out as time progresses.

By the theory of determinants the solution of equations (3) may be expressed in the form

$$\left.\begin{aligned} \nabla.\psi_1 &= \frac{d\nabla}{de_{11}}\Psi_1 + \frac{d\nabla}{de_{12}}\Psi_2 + \ldots \\ \nabla.\psi_2 &= \frac{d\nabla}{de_{12}}\Psi_1 + \frac{d\nabla}{de_{22}}\Psi_2 + \ldots \\ . \quad . \quad &. \quad . \quad . \quad . \end{aligned}\right\}, \quad . \quad . \quad . \quad (8)$$

where ∇ denotes the determinant of the symbols *e*. If there be no dissipation, ∇, or as we may write it with fuller expressiveness $\nabla(ip)$, is an even function of *ip* vanishing when *p* corresponds to one of the natural frequencies of vibration. In such a case the coordinates ψ_1, &c., in general become infinite. When there is dissipation, $\nabla(ip)$ does not vanish for any (real) value of *p*. If we write

$$\nabla(ip) = \nabla_1(ip) + ip\,\nabla_2(ip), \quad . \quad . \quad . \quad . \quad (9)$$

in which ∇_1, ∇_2 are *even* functions of ip, ∇_2 depends entirely upon the dissipation, while if the dissipation be small, ∇_1 is approximately the same as if there were none.

As it will be convenient to have a briefer notation than that of (8), we will write

$$\left.\begin{array}{l} \psi_1 = A_{11}e^{i\alpha_{11}}\Psi_1 + A_{12}e^{i\alpha_{12}}\Psi_2 + \ldots \\ \psi_2 = A_{21}e^{i\alpha_{21}}\Psi_1 + A_{22}e^{i\alpha_{22}}\Psi_2 + \ldots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \quad . \quad . \quad (10)$$

in which A, α are real and are subject to the relations

$$A_{rs} = A_{sr}, \qquad \alpha_{rs} = \alpha_{sr}. \quad . \quad . \quad . \quad . \quad (11)$$

In order to take account of the phases of the forces, we may suppose similarly that in (7)

$$E_1 = R_1 e^{i\theta_1}, \qquad E_2 = R_2 e^{i\theta_2}, \qquad \&c. \quad . \quad . \quad (12)$$

Work Done.

If we suppose that but *one* force, say Ψ_1, acts upon the system, the values of the coordinates are given by the first terms of the right-hand members of (10). The work done by the force in time dt depends upon that part of $d\psi_1/dt$ which is in the same phase with it, corresponding to the part of ψ_1 which is in quadrature with the force. Thus, taking the real parts only of the symbolic quantities, so that

$$\Psi_1 = R_1 \cos(pt + \theta_1), \qquad \psi_1 = A_{11}R_1 \cos(pt + \theta_1 + \alpha_{11}), \ . \quad (13)$$

we have as the work done (on the average) in time t

$$-pA_{11}R_1^2 \int \cos(pt + \vartheta_1) . \sin(pt + \theta_1 + \alpha_{11})\, dt,$$

or

$$-\tfrac{1}{2}pR_1^2A_{11} \sin\alpha_{11} . t. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

As was to be expected, this is independent of θ_1.

Another expression for the same quantity may be obtained by considering how this work is dissipated. From (6) we see that

$$\int \Psi_1\dot{\psi}_1 dt = 2\int F dt$$

$$= b_{11}\int \dot{\psi}_1^2 dt + b_{22}\int \dot{\psi}_2^2 dt + \ldots + 2b_{12}\int \dot{\psi}_1\dot{\psi}_2 dt + \ldots . \quad (15)$$

Taking again the real parts in (10), we have

$$\int \dot{\psi}_1^2 dt = \tfrac{1}{2}p^2R_1^2A_{11}^2 . t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\int \dot{\psi}_1\dot{\psi}_2 dt = \tfrac{1}{2}p^2R_1^2A_{11}A_{12} \cos(\alpha_{11} - \alpha_{12}) . t, \ . \quad . \quad (17)$$

$$\int \dot{\psi}_2\dot{\psi}_3 dt = \tfrac{1}{2}p^2R_1^2A_{12}A_{13} \cos(\alpha_{12} - \alpha_{13}) . t ; \qquad (18)$$

so that by (15) the work dissipated in time t is

$$\tfrac{1}{2}p^2\mathrm{R}_1{}^2t\,\{b_{11}\mathrm{A}_{11}{}^2+b_{22}\mathrm{A}_{21}{}^2+\ldots$$
$$+2b_{12}\mathrm{A}_{11}\mathrm{A}_{12}\cos(\alpha_{11}-\alpha_{12})+\ldots\}. \quad . \quad (19)$$

Equating the equivalent quantities in (14) and (19), we get

$$-p^{-1}\mathrm{A}_{11}\sin\alpha_{11}=b_{11}\mathrm{A}_{11}{}^2+b_{22}\mathrm{A}_{21}{}^2+\ldots$$
$$+2b_{12}\mathrm{A}_{11}\mathrm{A}_{12}\cos(\alpha_{11}-\alpha_{12})+\ldots \quad . \quad . \quad (20)$$

This assumes a specially simple form when F is a function of the squares only of $d\psi_1/dt$, &c., so that b_{12}, &c. vanish.

In (14) we have calculated the work done by a force Ψ_1 acting alone upon the system. If other forces act, the expression for ψ_1 will deviate from (13); but in any case we may write

$$\Psi_1=\mathrm{R}_1e^{i\theta_1}, \qquad \psi_1=r_1e^{i\phi_1}, \quad . \quad . \quad (21)$$

and the work done in unit of time by the real part of Ψ_1 on the real part of ψ_1 will be

$$-\tfrac{1}{2}p\mathrm{R}r\sin(\phi_1-\theta_1), \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and depends upon the product of the moduli and the *difference* of phases.

If ψ_1 consist of two or more parts of the form (21), the work done is to be found by addition of the terms corresponding to the various parts.

One Degree of Freedom.

The theory of the vibrations of a system of one degree of freedom, resulting from the application of a given force, is simple and well known, but it will be convenient to make a few remarks and deductions.

The equation determining ψ in terms of Ψ is

$$(-ap^2+c+ipb)\psi=\Psi\,; \quad . \quad . \quad . \quad . \quad (23)$$

so that in the notation of (10)

$$\mathrm{A}e^{i\alpha}=\frac{1}{c-ap^2+ipb}. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

As in (14), the work done by the force in unit time is

$$\frac{\tfrac{1}{2}p^2b\,\mathrm{Mod}^2\,\Psi}{(c-ap^2)^2+p^2b^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and it reaches a maximum (b and p being given) when the tuning is such that $c-ap^2=0$, that is when the natural vibrations are isoperiodic with the forced vibrations. The maximum value itself is

$$\frac{\frac{1}{2}\mathrm{Mod}^2\Psi}{b} \quad \ldots\ldots\ldots \quad (26)$$

Let us now suppose that two forces act upon the system, one of which Ψ is given, while the second Ψ' is at disposal, and let us inquire how much work can be withdrawn by Ψ'. It will probably conduce to clearness if we think of an electric circuit possessing self-induction and resistance, and closed by a condenser, so as to constitute a vibrator. In this acts a given electromotive force Ψ of given frequency. At another part of the circuit another electromotive force can be introduced, and the question is what work can be obtained at that point. Of course any work so obtained, as well as that dissipated in the system, must be introduced by the operation of the given force Ψ.

It will suffice for the moment to take Ψ such that ψ *due to it* is unity, which will happen when $\Psi = \mathrm{A}^{-1}e^{-i\alpha}$. If Ψ' be $\mathrm{R}e^{i\theta}$, the complete value of ψ is

$$\psi = 1 + \mathrm{AR}\,e^{i(\alpha+\theta)} \quad \ldots\ldots\ldots \quad (27)$$

The work done (in unit time) by Ψ' consists of two parts. That corresponding to the second term in (27) is the same as if Ψ' had acted alone and, as in (14), its value is

$$-\tfrac{1}{2}p\mathrm{R}^2\mathrm{A}\sin\alpha.$$

The work done by Ψ' upon the first part of ψ given in (27) is, as in (22),

$$-\tfrac{1}{2}p\mathrm{R}\sin(-\theta).$$

The whole work done by Ψ' is found by adding these together; and the work *withdrawn* from the system by Ψ' is the *negative* of this, or

$$\tfrac{1}{2}p\mathrm{R}^2\mathrm{A}\sin\alpha - \tfrac{1}{2}p\mathrm{R}\sin\theta. \quad \ldots\ldots \quad (28)$$

In this expression the first term is negative, and the whole is to be made a maximum by variation of R and θ. The maximum occurs when

$$\sin\theta = -1, \qquad 2\mathrm{RA}\sin\alpha = -1\,; \quad \ldots \quad (29)$$

and the maximum value itself is

$$\frac{-p}{8\mathrm{A}\sin\alpha} \quad \ldots\ldots\ldots \quad (30)$$

This corresponds to Mod $\Psi = \mathrm{A}^{-1}$; and as the work abstractable is proportional to $\mathrm{Mod}^2\,\Psi$, we have in general for the maximum

$$\frac{-p\mathrm{A}^2\mathrm{Mod}^2\,\Psi}{8\mathrm{A}\sin\alpha}.$$

Now, as we see from the values of A and α in (24), or otherwise by (20),

$$-p^{-1}\mathrm{A}\sin\alpha = b\mathrm{A}^2;$$

and thus the maximum work that can be abstracted is

$$\frac{\mathrm{Mod}^2\,\Psi}{8b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

It may at first occasion surprise that the work obtainable should be independent of a and c, upon which the behaviour of the system as a resonator depends. But the truth is that by suitable choice of Ψ' we have in effect *tuned* the system, and so reduced it to the condition of evanescent a and c—in the electrical illustration to a merely resisting circuit. Had we assumed the evanescence of a and c from the beginning, we could of course have arrived more simply at the expression (31).

In the case of maximum withdrawal of energy the complete symbolical value of ψ in (27) becomes

$$\psi = 1 + \frac{i\,e^{ia}}{2\sin\alpha} = 1 - \tfrac{1}{2} + \tfrac{1}{2}i\cot\alpha, \quad . \quad . \quad . \qquad (32)$$

the part of the complete value which is in the same phase as before being halved.

It is not difficult to recognise that the result as to the maximum work abstractable admits of further generalization. So far we have considered the case of a single degree of freedom, *e.g.*, a single electric circuit. Other degrees of freedom, *e.g.*, neighbouring electric circuits, do not affect the result, provided that the forces in them all vanish and that the only dissipation is that already considered. If in equations (3) all the quantities b except b_{11} vanish, as well as the forces Ψ_2, Ψ_3, &c., the second, third, and following equations determine *real* ratios between all the other coordinates and ψ_1, and virtually reduce the system to a single degree of freedom. The reaction of the other parts of the system will influence the force Ψ_1' required in order to abstract most work, but not the maximum value itself.

Several Degrees of Freedom.

Hitherto we have supposed that the force by means of which work is abstracted is of the same type as that which is supposed to be given and by means of which work is introduced into the system; but in the investigations which follow it will be our object to trace the effect of voluntary operation upon one coordinate ψ_1, while the system is subject to given forces Ψ_2, Ψ_3, &c., operating upon the remaining coordinates. To explain what is meant the more clearly, let us consider the simple case of two electric circuits influencing one another by induction. Each circuit may be supposed to be closed by a condenser, so as to constitute, when considered by itself, a simple vibrator. If a given periodic electromotive force (Ψ_2) act in the second circuit, the current in the first circuit would depend upon the various elements of the compound system. Let it be proposed to inquire what work can be withdrawn at the first circuit by electromotive forces (Ψ_1) there applied. For simplicity it will be supposed that the first circuit has no resistance ($b_{11}=0$).

When Ψ_2 acts alone, the ψ_1 due to it is given by

$$\psi_1 = \mathrm{A}_{12}\, e^{i\alpha_{12}}\Psi_2. \quad . \quad . \quad . \quad . \quad . \quad (33)$$

If we ascribe to Ψ_2 the value $\mathrm{A}_{12}{}^{-1}e^{-i\alpha_{12}}$, the ψ_1 due to it will be unity, and this for the present purpose is the simplest supposition to make. When a force Ψ_1 is introduced, the complete value of ψ_1 will deviate from unity, but Ψ_2 is supposed to retain throughout the above prescribed value. If $\Psi_1 = \mathrm{R}_1\, e^{i\theta_1}$, as in (12), the complete expression of ψ_1 is

$$\psi_1 = 1 + \mathrm{A}_{11}\mathrm{R}_1\, e^{i(\theta_1+\alpha_{11})}. \quad . \quad . \quad . \quad . \quad (34)$$

The work done by Ψ_1 upon this is composed of two parts. If the real components of the expressions be retained, the first is

$$-p\int \mathrm{R}_1 \cos(pt+\theta_1) \,.\, \sin pt\, dt = \tfrac{1}{2}p\mathrm{R}_1 \sin\theta_1 \,.\, t\,;$$

and the second is, as in (14),

$$-\tfrac{1}{2}p\mathrm{R}_1{}^2\mathrm{A}_{11}\sin\alpha_{11} \,.\, t,$$

a quantity necessarily positive.

Thus altogether the work done by Ψ_1 in unit time is

$$\tfrac{1}{2}p\mathrm{R}_1\sin\theta_1 - \tfrac{1}{2}p\mathrm{R}_1{}^2\mathrm{A}_{11}\sin\alpha_{11}. \quad . \quad . \quad . \quad . \quad (35)$$

The work that may be *abstracted* from the circuit is the negative of this, and it is to be made a maximum by variation of R_1 and θ_1. We must take

$$\sin\theta_1 = -1, \qquad 2\mathrm{R}_1\mathrm{A}_{11}\sin\alpha_{11} = -1\,; \quad . \quad . \quad . \quad (36)$$

and the maximum work abstractable will be

$$\tfrac{1}{4}p\mathrm{R}_1, \quad \text{or} \quad -\frac{p}{8\mathrm{A}_{11}\sin\alpha_{11}} \quad . \quad . \quad . \quad . \quad . \quad (37)$$

The symbolic expression for ψ_1 becomes at the same time from (22)

$$\psi_1 = 1 - \tfrac{1}{2} + \tfrac{1}{2}\, i \cot\alpha_{11}, \quad . \quad . \quad . \quad . \quad (38)$$

so that the part of ψ_1 in the same phase as when $\Psi_1=0$ is half as great as before.

So far as the results embodied in (37), (38), and (39) are concerned, it is a matter of indifference whether the prescribed $\psi_1=1$ when $\Psi_1=0$ is due to Ψ_2 only, or to Ψ_2 acting in conjunction with forces Ψ_3, &c., corresponding to further degrees of freedom. But for the present we will suppose that there are only two degrees of freedom.

The maximum work that can be abstracted by Ψ_1, when Ψ_2 is given, may be expressed as

$$-\frac{p\,\mathrm{Mod}^2\psi_1}{8\mathrm{A}_{11}\sin\alpha_{11}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

where ψ_1 is due to Ψ_2 acting alone. By (33)

$$\mathrm{Mod}^2\,\psi_1 = \mathrm{A}_{12}{}^2\mathrm{Mod}^2\,\Psi_2,$$

and by (20) with $b_{11}=0$, $b_{12}=0$,

$$-p^{-1}\,\mathrm{A}_{11}\sin\alpha_{11} = b_{22}\,\mathrm{A}_{21}{}^2,$$

so that the maximum work obtainable is simply

$$\frac{\mathrm{Mod}^2\,\Psi_2}{8\,b_{22}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

That it is independent not only of a_{11}, c_{11}, a_{22}, c_{22}, but also of the coefficients of mutual influence a_{12}, c_{12}, is very remarkable. To revert to the electrical example, the work abstractable in the first circuit (devoid of resistance) when a given electromotive force acts in the second, is independent of the value of the coefficient of mutual induction. If indeed this coefficient be very small, the supposition of zero resistance becomes more and more unpractical on account of the large currents which must then be supposed to flow. But the theoretical result remains true, when b_{11} is diminished without limit. In view of its independence of so many circumstances that might at first be supposed material, it may now not be surprising to note that (40) coincides with (31), that is that the work obtainable in the second circuit is the same as might have been obtained in the first where the given force itself acts.

The existence of further degrees of freedom than those corresponding to the given force Ψ_2 and the disposable force Ψ_1 makes no difference to (39). And so long as Ψ_2, Ψ_1 are the only forces in operation, we have still

$$\text{Max. work} = -\frac{p\mathrm{A}_{12}{}^2\,\mathrm{Mod}^2\Psi_2}{8\mathrm{A}_{11}\sin\alpha_{11}}. \quad . \quad . \quad . \quad . \quad . \quad (41)$$

If further all the coefficients b vanish, except b_{22}, (40) remains unaffected. If, however, we suppose that b_{33}, b_{44}, &c. are finite, while b_{11}, b_{12}, b_{13}, b_{23}, &c. still vanish, (20) gives

$$-p^{-1}\mathrm{A}_{11}\sin\alpha_{11} = b_{22}\mathrm{A}_{21}{}^2 + b_{33}\mathrm{A}_{31}{}^2 + \ldots, \quad . \quad . \quad (42)$$

and the expression for the maximum work becomes

$$\frac{\frac{1}{8}\mathrm{A}_{12}{}^2\,\mathrm{Mod}^2\Psi_2}{b_{22}\mathrm{A}_{12}{}^2 + b_{33}\mathrm{A}_{13}{}^2 + \ldots}. \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Since b_{33}, &c. are positive, the value of (43) is less than when b_{33}, &c. vanish.

The expression (43) is necessarily more complicated than (40); but a simple result may again be stated if we suppose that given forces act *successively* of the second, third, and following types, provided they be of such magnitudes that they would severally (the non-corresponding resistances vanishing) allow the same work to be abstracted by Ψ_1, that is provided

$$\frac{\mathrm{Mod}^2\Psi_2}{b_{22}} = \frac{\mathrm{Mod}^2\Psi_3}{b_{33}} = \ldots\ldots\ldots = \frac{\mathrm{Mod}^2\Psi}{b}. \quad . \quad . \quad . \quad (44)$$

On this supposition the *sum* of the energies abstractable in the various cases has the value

$$\frac{\mathrm{Mod}^2\Psi}{8b}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

of the same form as before.

In the electrical application we have to consider any number of mutually influencing circuits, of which the first is devoid of resistance. The electromotive forces acting successively in the other circuits are to be inversely as the square roots of the resistances of those circuits, *i. e.* such as would do the same amount of work on each circuit supposed to be isolated and reduced (*e. g.* by suitable adjustment of the associated condenser) to a mere resistance. The sum of all the works abstractable in the first circuit is then the same as if there were no other circuits than the first and second; or, again, as if the second circuit were isolated and it were allowed to draw work from it.

Action of Resonators.

We now abandon the idea of drawing work from the system by means of Ψ_1, and on the contrary impose the condition that Ψ_1 shall do no work, positive or negative. The effect of Ψ_1 is then equivalent to a change in the inertia a_{11}, or spring c_{11}, associated with this coordinate and the operation may be regarded as a tuning, or mistuning, of the system. If, as before, ψ_1, due to the given force Ψ_2, be unity, and $\Psi_1 = \mathrm{R}_1 e^{i\theta_1}$, the complete value of ψ_1 is that given in (34), and (35) represents the work done altogether by Ψ_1 in unit time. Equating this to zero, we get as the relation between R_1 and θ_1,

$$\mathrm{A}_{11}\mathrm{R}_1 \sin \alpha_{11} = \sin \theta_1, \quad . \quad . \quad . \quad . \quad . \quad (46)$$

and the part of ψ_1 due to Ψ_1 is

$$\frac{\sin \theta_1 \, e^{i(\theta_1+\alpha_{11})}}{\sin \alpha_{11}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

The modulus of this is a maximum when $\sin \theta_1 = \pm 1$, and the value of the maximum is cosec α_{11}. In this case (47) becomes

$$-1 + i \cot \alpha_{11}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

and the complete value of ψ_1 is

$$\psi_1 = i \cot \alpha_{11}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

in quadrature with the former value, viz., 1.

We may regard the state of things now defined as being in a sense the greatest possible disturbance of the original state of things. If the system be quite out of resonance, forces and displacements are nearly in the same phase, and α_{11} is small. The altered ψ_1 is then a large multiple of the original value.

The work done by Ψ_1 on the complete value of ψ_1 is zero by supposition; but the work done upon the part of ψ_1 *due to itself* is by (14) in unit time

$$-\tfrac{1}{2}p\mathrm{R}_1{}^2\mathrm{A}_{11} \sin \alpha_{11} = -\frac{\frac{1}{2}p}{\mathrm{A}_{11} \sin \alpha_{11}}.$$

This corresponds to the original $\psi_1 = 1$, or $\Psi_2 = \mathrm{A}_{12}{}^{-1}e^{-i\alpha_{12}}$. If the prescribed value of Ψ_2 be now left open, we have as the work in question

$$-\frac{\frac{1}{2}p\mathrm{A}_{12}{}^2 \mathrm{Mod}^2 \Psi_2}{\mathrm{A}_{11} \sin \alpha_{11}}; \quad . \quad . \quad . \quad . \quad . \quad (50)$$

and this by (20) is the same as

$$\frac{\frac{1}{2}A_{12}{}^2\,\mathrm{Mod}^2\Psi_2}{b_{22}A_{12}{}^2+b_{33}A_{13}{}^2+\dots}, \quad . \quad . \quad . \quad . \qquad (51)$$

b_{11}, b_{12}, &c. being supposed to be zero. This expression differs from (43) only as regards the numerical factor. If b_{33}, &c. also vanish, (51) becomes

$$\frac{\mathrm{Mod}^2\Psi_2}{2b_{22}} \quad \text{simply.}$$

If in (51) we introduce the suppositions of (44), we get as in (45) for the sum of all the values

$$\frac{\mathrm{Mod}^2\Psi}{2b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (52)$$

In an interesting paper entitled "An Electromagnetic Illustration of the Theory of Selective Absorption of Light in a Gas"*, Prof. Lamb has developed a general law for the maximum energy emitted by a resonator situated in a uniform medium when submitted to incident plane waves. "The rate at which energy is carried outwards by the scattered waves is, in terms of the energy-flux in the primary waves,

$$\frac{2n+1}{2\pi}\lambda^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (53)$$

where λ is the wave-length, and n is the order of the spherical harmonic component of the incident waves which is effective." Prof. Lamb remarks that the law expressed by (53) "is of a very general character, and is independent of the special nature of the conditions to be satisfied at the surface of the sphere. It presents itself in the elastic solid theory, and again in the much simpler acoustical problem where there is synchronism between plane waves of sound and a vibrating sphere on which they impinge."

The generality claimed by Lamb for (53) seemed to me to indicate a still more general theorem in the background; and it was upon this suggestion that the investigations of the preceding pages were developed. An initial difficulty, however, stood in the way. The occurrence of n, a quantity special to the spherical problem, seemed to constitute a limitation; and the further question suggested itself as to why the efficiency of the resonator should rise with increasing n. For example, why in the acoustical problem should a resonator formed by a rigid sphere, moored to a fixed point by elastic

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* Cambr. Trans. vol. xviii. p. 348 (1899).

attachments ($n=1$), be three times as effective as the simple resonator ($n=0$), for which the theory is given by my book on Sound, § 319?

The answer may be found in a slightly different presentation of the matter. In the above example the rigid sphere is supposed to be *symmetrically* moored to a fixed point, and the vibration actually assumed is in a direction parallel to that of propagation of the incident waves. Three degrees of freedom are really involved here, while the more typical case will be that in which the motion is limited to *one* direction. The efficiency of the resonator will then be proportional to the square of the cosine (μ) of the angle between the direction of vibration and that of the incident waves; and the *mean* efficiency will bear to the *maximum* efficiency ($\mu=1$) a ratio equal to that of

$$\int_0^1 \mu^2\, d\mu \;:\; \int_0^1 d\mu,$$

that is of $\frac{1}{3}$. Thus, if the vibration in the case of $n=1$ be limited to *one* direction, the *mean* efficiency of the resonator is the *same* as when $n=0$; and a similar conclusion will hold good in all cases. In this way the factor $2n+1$ is eliminated, and the statement assumes a form more nearly capable of generalization to all vibrating systems.

Now that a general theorem (52) has been demonstrated, it will be of interest to trace its application to some case of a uniform medium, for which purpose we may take the simple acoustical resonator. But this deduction is not quite a simple matter, partly on account of the extension to infinity, and also, I think, for want of a more general theory of waves in a uniform medium than any hitherto formulated. If the object be merely to obtain a result, it is far more easily attained by a special investigation from the formulæ of the Theory of Sound, on the lines indicated by Prof. Lamb. It may perhaps be well to sketch the outline of such an investigation.

The time factor e^{ikat} being suppressed, the velocity-potential ϕ of the primary waves is (§ 334) e^{ikx}, or $e^{ikr\mu}$, and the harmonic component of the nth order has the expression

$$\phi_n = (2n+1)\,\mathrm{P}_n(\mu)\,.\,\mathrm{P}_n\left(\frac{d}{d\,.\,ikr}\right)\left(\frac{\sin kr}{kr}\right), \quad . \quad . \quad (53\text{ bis})$$

while (§ 329) the corresponding expression for the divergent secondary waves is

$$\psi_n = (-1)^n k a_n \mathrm{P}_n(\mu)\,.\,\mathrm{P}_n\left(\frac{d}{d\,.\,ikr}\right)\left(\frac{\cos kr - i\sin kr}{kr}\right). \quad (54)$$

Accordingly

$$\phi_n+\psi_n=\{2n+1-(-1)^n ika_n\}\mathrm{P}_n\left(\frac{d}{d\,.\,ikr}\right)\frac{\sin kr}{kr}$$

$$+(-1)^n ka_n \mathrm{P}_n\left(\frac{d}{d\,.\,ikr}\right)\frac{\cos kr}{kr}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

Now the only condition imposed upon the appliances introduced at the surface of the sphere is that they shall do no work. The velocity is $d\phi/dr+d\psi/dr$, and the pressure, proportional to $d\phi/dt+d\psi/dt$, is in quadrature with (55). All therefore that is required is that (55) and its derivative with respect to r be in the same phase, or that the ratio of these symbolic expressions be *real*. Since P_n is a wholly odd or wholly even function, this requirement is satisfied if

$$\frac{2n+1-(-1)^n ika_n}{(-1)^n ka_n} \quad \text{be real.}$$

If A_n, which may be complex, be written $\mathrm{A}e^{i\alpha}$, we get

$$-k\mathrm{A}=-(-1)^n(2n+1)\sin\alpha. \quad . \quad . \quad . \quad (56)$$

Thus A is a maximum when

$$\sin\alpha=-(-1)^n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

and the maximum value is

$$\mathrm{A}=(2n+1)/k. \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

By (57), (58)

$$a_n=-(-1)^n i(2n+1)/k, \quad . \quad . \quad . \quad . \quad (59)$$

so that in (55)

$$2n+1-(-1)^n ika_n=0, \quad . \quad . \quad . \quad . \quad (60)$$

but $\phi_n+\psi_n$ does not itself vanish.

If the incident plane waves are regarded as due to a source at a great distance R, we have in correspondence with (53)

$$\phi=\frac{\mathrm{R}e^{-ik\mathrm{R}}}{\mathrm{R}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

with which we may compare

$$\psi_n=\frac{a_n e^{-ikr}}{r}\mathrm{P}_n(\mu). \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

The work emitted from the primary source being represented by $R^2\int_{-1}^{+1} d\mu$, that emitted, or rather diverted, by the resonator will be

$$\text{Mod}^2\, a_n \int_{-1}^{+1} P_n^{\,2}(\mu)\, d\mu.$$

Now $\int_{-1}^{+1} P_n^{\,2}(\mu) d\mu = \frac{2}{2n+1}$, and $\int_{-1}^{+1} d\mu = 2$. Also

$$\text{Mod}^2\, a_n = \frac{(2n+1)^2}{k^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

so that the ratio of works is

$$\frac{2n+1}{k^2R^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

This agrees with the result given in 'Theory of Sound' § 319 for a symmetrical resonator ($n=0$).

In order to express (64) in terms of the energy-flux (per unit area) of the primary waves at the place of the resonator, we have only to multiply (64) by the area ($4\pi R^2$) of the sphere of radius R. If we restore $2\pi/\lambda$ for k, we get as the equivalent of (64)

$$(2n+1)\,\lambda^2/\pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65^*)$$

If we limit the resonator to one definite harmonic vibration of order n and suppose that the primary waves may be incident indifferently in all directions, the *mean* of the values of (65) is λ^2/π simply, as follows from known properties of the spherical functions.

Before we can apply the general theorem (52) to an independent investigation of these results, it is necessary to consider the connexion between the formulæ for plane and spherical waves; and for this purpose it is desirable to use a method which, if not itself quite general, is of a character susceptible of generalization. If ϕ denote the velocity-potential due to a "force" $\Phi\, dV$ acting at the element of volume dV and proportional to the periodic introduction and abstraction of fluid at that place, we may write

$$\phi = B\Phi\, dV\frac{e^{-ikr}}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

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* It will be observed that (65) is the double of the value (53) above quoted.

Dec. 17.—I have since learned that Prof. Lamb's calculations for the acoustical problem have already been published. See Math. Soc. Proc. vol. xxxii. p. 11, 1900, where equation (44) is identical with (65) above. Reference may also be made to Lamb, Math. Soc. Proc. vol. xxxii. p. 120, 1900.

where $k=2\pi/\lambda$ and B is some multiplier, which may be complex. The time factor e^{ikat} is suppressed. In order to obtain plane waves we may suppose that Φ acts uniformly over the whole slice between x and $x+dx$. The effect may be calculated as in a well-known optical investigation. If $\rho^2=r^2-x^2$, the element of volume is $2\pi\rho\, d\rho\, dx$, or $2\pi r\, dr\, dx$; and for the plane waves

$$\phi=\mathrm{B}\Phi\, dx\int_x^\infty 2\pi e^{-ikr}\, dr=\mathrm{B}\,\Phi\, dx\frac{2\pi e^{-ikx}}{ik}\quad . \quad . \quad . \quad (67)$$

Here Φ acts at $x=0$, and

$$\phi_0=\frac{2\pi\mathrm{B}}{ik}\Phi\, dx, \qquad \dot{\phi}_0=2\pi a\mathrm{B}\,\Phi\, dx. \quad . \quad . \quad (68)$$

Since $\dot{\phi}_0$ must be in the same phase as Φ, it follows that B is real.

We have now to consider the work done in generating the plane waves per unit of time and per double unit area of wave-front. For this we have

$$\frac{dx}{t}\int_0^t \Phi\dot{\phi}_0\, dt=\pi a\mathrm{B}(dx)^2\,\mathrm{Mod}^2\,\phi\,; \quad . \quad . \quad . \quad (69)$$

or since by (67)

$$\mathrm{Mod}\,\phi=\frac{2\pi\mathrm{B}\, dx}{k}\,\mathrm{Mod}\,\Phi, \quad . \quad . \quad . \quad . \quad (70)$$

we get for the work propagated in *one* direction per unit of area of wave-front

$$\frac{k^2a}{8\pi\mathrm{B}}\,\mathrm{Mod}^2\,\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Reverting now to (66), we see that for divergent waves

$$\mathrm{Mod}^2\,\phi=\frac{\mathrm{B}^2}{r^2}\,\mathrm{Mod}^2\,(\Phi\, d\mathrm{V}),$$

or

$$4\pi r^2\,\mathrm{Mod}^2\,\phi=4\pi\mathrm{B}^2\,\mathrm{Mod}^2\,(\Phi\, d\mathrm{V}).$$

Accordingly by (71), since at a sufficient distance the distinction between plane and divergent waves disappears, the work emitted in unit time by a point-source $\Phi\, d\mathrm{V}$ is

$$\tfrac{1}{2}k^2a\mathrm{B}\,\mathrm{Mod}^2\,(\Phi\, d\mathrm{V}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

It may be observed that in order to preserve a better correspondence between "force" and "coordinate" a somewhat different interpretation is here put upon Φ from that adopted in 'Theory of Sound,' § 277. If we compare our present (71) with (10), § 245, we find that

$$B=\frac{1}{4\pi\rho}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

so that according to the present interpretation of Φ, (66) gives

$$\phi=\frac{1}{4\pi\rho}\Phi\, dV\,\frac{e^{-ikr}}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

whereas in the notation of § 277

$$\phi=\frac{1}{4\pi a^2}\Phi\, dV\,\frac{e^{-ikr}}{r}.$$

We are now to some extent prepared for the application of (52), but the difficulty remains that (52) deals in the first instance with a finite system subject to dissipative forces; whereas the uniform medium is infinite, and need not be supposed subject to any forces truly dissipative. There is, however, no objection to the introduction of a small dissipative force of the character supposed in the general theorem, that is, proportional everywhere to $d\phi/dt$. Under this influence plane waves are attenuated as they advance; the law of attenuation being represented by the introduction into (67) of the factor $e^{-\alpha x}$, where α is a small quantity, real and positive.

The connexion between α and b may be investigated by considering the action of the dissipative force $-b\dot{\phi}_0$ operative over a slice δx at $x=0$ in causing the attenuation. By (67) the effect at x of this force is represented by

$$\delta\phi=-\frac{2\pi Be^{-ikx}}{ik}\,b\dot{\phi}_0\,\delta x,$$

so that

$$\delta\phi/\phi=-2\pi abB\,\delta x.$$

By supposition this must be the same as $-\alpha\,\delta x$; and accordingly

$$\alpha=2\pi abB. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

If we use this result to eliminate B from (72), we get as the work emitted from a point-source

$$\frac{k^2\alpha}{4\pi b}\,\mathrm{Mod}^2\,(\Phi\, dV). \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

The formula (52) expresses the sum of all the works emitted by the resonator when submitted successively to all the various forces Ψ, subject themselves to conditions (44).

These conditions are satisfied in the present case if we identify each Ψ with the force $\Phi d\mathrm{V}$ acting over the various *equal* elements $d\mathrm{V}$ into which infinite space may be divided, the value of Φ being everywhere the same. Each point-source is regarded as the origin of plane waves which fall upon the resonator. The efficiency of the sources which lie in a given direction still depends upon the distance, the waves as they reach the resonator being attenuated by the resistance and also in the usual manner according to the law of inverse squares.

Let us compare the efficiency of the element $\Phi d\mathrm{V}$ at distance r with the efficiency of an equal element at distance unity, the value of α being so small that no perceptible attenuation due to it occurs in distance unity. The element of volume

$$d\mathrm{V}=d\sigma\,.\,d(\tfrac{1}{3}r^3), \quad . \quad . \quad . \quad . \quad . \quad (77)$$

in which for the present $d\sigma$ is kept unchanged. The efficiency of the element at distance r varies as

$$(\Phi d\mathrm{V})^2\,.\,e^{-2\alpha r}\,.\,r^{-2};$$

and

$$\Sigma(\Phi d\mathrm{V})^2\,e^{-2\alpha r}\,r^{-2}=\Phi^2 d\sigma\,d\mathrm{V}\int_0^\infty e^{-2\alpha r}\,dr=\frac{d\sigma}{2\alpha\,d\mathrm{V}}(\Phi\,d\mathrm{V})^2.$$

Hence for the sum of all the elements lying within $d\sigma$ we have

$$\frac{d\sigma}{2\alpha\,d\mathrm{V}}\times\text{efficiency of }(\Phi\,d\mathrm{V})\text{ at distance unity.}$$

This has now to be again integrated with respect to $d\sigma$. The result may be expressed by the statement that the sum of all the works emitted by the resonator is

$$\frac{4\pi}{2\alpha\,d\mathrm{V}}\times\text{mean work emitted by resonator corresponding to}$$

the various positions of the point-source on the sphere $r=1$. By the theorem (52) this sum of all the works is also expressed by

$$\frac{(\Phi\,d\mathrm{V})^2}{2b\,d\mathrm{V}},$$

or in accordance with (76) is equal to

$$\frac{2\pi}{k^2\alpha\,d\mathrm{V}}\times\text{work emitted by }(\Phi\,d\mathrm{V})\text{ itself.}$$

We see therefore that the mean work emitted by the resonator for positions of the point-sources distributed uniformly over the sphere $r=1$ is equal to the work emitted by each of the point-sources themselves divided by k^2. If the point-sources are supposed to lie at a distance r in place of unity, the divisor becomes k^2r^2 in place of k^2.

Although the above deduction may stand in need of some supplementing before it could be regarded as rigorous at all points, it suffices at any rate to show that the general theorem (52) really does include the more special cases which suggested it. In some applications, *e. g.* to an elastic solid, we should have at first to suppose the forces introduced at any element of volume dV to act in various directions, but no great complication thence arises, and the general result finally takes the same form.

Energy stored in Resonators.

In preceding investigations we have been concerned with energy emitted from a resonator. We now turn to the consideration of some general theorems relating to the energy *stored*, as it were, in the resonator when the applied forces have frequencies in the neighbourhood of the natural frequency of the resonator. And we will treat first the simple case of one degree of freedom.

As in (2) we have

$$T=\tfrac{1}{2}a\dot{\psi}^2, \qquad F=\tfrac{1}{2}b\dot{\psi}^2, \qquad V=\tfrac{1}{2}c\psi^2, \quad . \quad . \quad (78)$$

giving as the equation of vibration

$$a\ddot{\psi}+b\dot{\psi}+c\psi=\Psi=\mathrm{E}\,e^{ipt}. \quad . \quad . \quad . \quad . \quad (79)$$

The time factor being suppressed, the solution of (79) is

$$\psi=\frac{\mathrm{E}}{c-ap^2+ipb}, \quad . \quad . \quad . \quad . \quad . \quad (80)$$

whence

$$\mathrm{Mod}^2\,\psi=\frac{\mathrm{Mod}^2\,\Psi}{a^2(n^2-p^2)^2+p^2b^2}, \quad . \quad . \quad . \quad . \quad (81)$$

n, equal to $\sqrt{(c/a)}$, being the value of p corresponding to maximum resonance. If, as we suppose, b is very small, the important values of $\mathrm{Mod}^2\,\psi$ are concentrated in the neighbourhood of $p=n$, and we may substitute n for p in the term p^2b^2. Also n^2-p^2 may be identified with $2n(n-p)$. Accordingly (81) becomes

$$\mathrm{Mod}^2\,\psi=\frac{1}{n^2b^2}\,\frac{\mathrm{Mod}^2\,\Psi}{1+\dfrac{4c^2}{n^4b^2}(p-n)^2}. \quad . \quad . \quad . \quad (82)$$

We now suppose that $\text{Mod}^2\,\Psi$ is constant, while p varies over the small range in the neighbourhood of n for which alone $\text{Mod}^2\,\psi$ is sensible, and inquire as to the sum of the values of $\text{Mod}\,\psi$. Since

$$\int_{-\infty}^{+\infty}\frac{dx}{1+\alpha^2x^2}=\frac{\pi}{\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

we find

$$\int \text{Mod}^2\,\psi\,dp=\frac{\pi\,\text{Mod}^2\,\Psi}{2bc},$$

or again

$$\tfrac{1}{2}c\int \text{Mod}^2\,\psi\,dp=\frac{\pi}{4b}\,\text{Mod}^2\,\Psi. \quad . \quad . \quad . \quad . \quad (84)$$

On the left $\frac{1}{2}c\,\text{Mod}^2\,\psi$ represents the potential energy of the system at the phase of maximum displacement, which is the same as the nearly constant total energy, so that (84) gives the integral of this total energy as p passes through the value which calls out the maximum and (by supposition) very great resonance.

The most remarkable feature of (84) is perhaps that the integral is independent of a and c. Large values of these quantities will increase the energy of the system at the point where $p=n$; but on the other hand this maximum falls off more rapidly as p departs from the special value.

We pass next to the more difficult considerations which arise when the force Ψ_2 is of one kind, while the coordinate ψ_1 on which the resonance principally depends is of another. In the first instance we shall suppose that there are no other than these two degrees of freedom.

If in equation (3) we assume Ψ_1, ψ_3, ψ_4, &c. to vanish, we get

$$\frac{\psi_1}{\Psi_2}=\frac{e_{12}}{e_{12}{}^2-e_{11}e_{22}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

where e_{11}, e_{12}, e_{22} have the values given in (4) with ip substituted for D. We suppose further that $b_{12}=0$, $b_{11}=0$, so that the dissipation depends entirely on b_{22}. With these simplifications the numerator of (85) becomes

$$e_{12}=c_{12}-p^2a_{12}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

and for the denominator (taken negatively)

$$\begin{aligned}e_{11}e_{22}-e_{12}{}^2=&(c_{11}-p^2a_{11})\,(c_{22}-p^2a_{22})\\ &-(c_{12}-p^2a_{12})^2+ipb_{22}(c_{11}-p^2a_{11}). \quad . \quad . \quad . \quad (87)\end{aligned}$$

If n be one of the values of p corresponding to maximum resonance, the real part of (87) vanishes when $p=n$; so that

$$(c_{11}-n^2a_{11})(c_{22}-n^2a_{22})-(c_{12}-n^2a_{12})^2=0, \quad . \quad . \quad (88)$$

or written as a quadratic in n^2,

$$\begin{aligned}c_{11}c_{22}-c_{12}^2-n^2(a_{11}c_{22}+a_{22}c_{11}+2a_{12}c_{12})&\\ +n^4(a_{11}a_{22}-a_{12}^2)=0. \quad . \quad . \quad . \quad . \quad (89)&\end{aligned}$$

By subtraction of (89), (87) may be written

$$\begin{aligned}e_{11}e_{22}-e_{12}^2=-(p^2-n^2)(a_{11}c_{22}+a_{22}c_{11}+2a_{12}c_{12})&\\ +(p^4-n^4)(a_{11}a_{22}-a_{12}^2)+ipb_{22}(c_{11}-p^2a_{11}). \quad . \quad . \quad (90)&\end{aligned}$$

If b_{22} were zero, ψ_1 would become infinite for $p=n$. If we assume that b_{22}, while not actually zero, is still relatively very small, the values of p in the neighbourhood of n retain a preponderating importance; and we may equate p to n with exception of the factor $(p-n)$. Thus (86), (90) become

$$e_{12}=c_{12}-n^2a_{12}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

$$\begin{aligned}e_{11}e_{22}-e_{12}^2&=-2n(p-n)\{a_{11}c_{22}+a_{22}c_{11}-2a_{12}c_{12}\\ &\qquad -2n^2(a_{11}a_{22}-a_{12}^2)\}+inb_{22}(c_{11}-n^2a_{11})\\ &=-\frac{2(p-n)}{n}\{c_{11}c_{22}-c_{12}^2-n^4(a_{11}a_{22}-a_{12}^2)\}\\ &\qquad +inb_{22}(c_{11}-n^2a_{11}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)\end{aligned}$$

use being made of (89).

From (91) with use of (88),

$$\text{Mod}^2\, e_{12}=(c_{11}-n^2a_{11})(c_{22}-n^2a_{22}); \quad . \quad . \quad . \quad (93)$$

and from (92)

$$\begin{aligned}\text{Mod}^2(e_{11}e_{22}-e_{12}^2)=\frac{4(p-n)^2}{n^2}\{c_{11}c_{22}-c_{12}^2-n^4(a_{11}a_{22}-a_{12}^2)\}^2&\\ +n^2b_{22}^2(c_{11}-n^2a_{11})^2. \quad . \quad (94)&\end{aligned}$$

If we now, as for (84), carry out the integration with respect to p, Mod Ψ_2 being constant, we find from (85)

$$\frac{\int \text{Mod}^2\,\psi_1\,dp}{\text{Mod}^2\,\Psi_2}=\frac{\pi(c_{22}-n^2a_{22})}{2b_{22}\{c_{11}c_{22}-c_{12}^2-n^4(a_{11}a_{22}-a_{12}^2)\}}. \quad . \quad (95)$$

So far we have assumed merely that the compound system is in high resonance when $p=n$; but more than this is required in order to arrive at a simple result. We must further assume that the coefficients of interconnexion a_{12},

c_{12} are small (b_{12} has been already made zero), so that the resonating coordinate may vibrate with a considerable degree of independence. We are also to suppose that n corresponds to these comparatively independent vibrations, so that $c_{11}-n^2a_{11}=0$ approximately, while $c_{22}-n^2a_{22}$ is relatively large. These simplifications reduce the bracket in the denominator of (95) to

$$c_{11}c_{22}-n^4a_{11}a_{22}, \qquad \text{or to} \qquad c_{11}(c_{22}-n^2a_{22})\,;$$

whence we obtain finally

$$\frac{\frac{1}{2}c_{11}\int \mathrm{Mod}^2\,\psi_1\,dp}{\mathrm{Mod}^2\,\Psi_2}=\frac{\pi}{4b_{22}}. \quad . \quad . \quad . \quad . \quad (96)$$

In this expression, which is of the same form as (84), the numerator on the left may be considered to represent the integrated energy of the *resonator*. It must not be overlooked that the suppositions involved are to some extent antagonistic. For example, the coefficient of b_{22} in (90) is treated as constant when p varies, although $(c_{11}-n^2a_{11})$ is small. The theorem should be regarded as one applicable in the limit when b_{22} is exceedingly small.

If there be more than two degrees of freedom, the result is unaffected, provided that the forces Ψ_3, Ψ_4, &c., of the new types vanish and that the only dissipation is that represented by b_{22}. By the 3rd, 4th, &c. of (3) the new coordinates may be eliminated. In this process b_{22} is undisturbed, and everything remains as if there were only two coordinates as above.

The idea of the integration with respect to p is borrowed from a paper by Prof. Planck (*Ann. d. Phys.* i. p. 99, 1900), in which is considered the behaviour of an infinitely small electromagnetic resonator under incident plane waves. The proof of the general theorem covering Prof. Planck's case would require a process similar to that by which (51) was established. Subject to the condition

$$\frac{\mathrm{Mod}^2\,\Psi_2}{b_{22}}=\frac{\mathrm{Mod}^2\,\Psi_3}{b_{33}}=\ldots\ldots\ldots=\frac{\mathrm{Mod}^2\,\Psi}{b}, \quad . \quad . \quad (97)$$

we might expect to find, as in (52),

$$\Sigma\tfrac{1}{2}c_{11}\int \mathrm{Mod}^2\,\psi_1\,dp=\frac{\pi\,\mathrm{Mod}^2\,\Psi}{4b}. \quad . \quad . \quad . \quad (98)$$

Terling Place, Witham.
November 1901.

X. *The Ionic and Thermal Coefficients of Nitric Acid. By* V. H. Veley, *D.Sc., F.R.S., and* J. J. Manley, *Daubeny Curator, Magdalen College, Oxford* *.

In a former investigation † we have published determinations of the electric conductivity of nitric acid of different concentrations at the three temperatures 0°, 15°, and 30° C. obtained by the method of Kohlrausch with certain modifications. It has for some time been noted that the strong mineral acids do not follow Ostwald's dilution law

$$\frac{\mu^2}{(\mu_\infty - \mu)\mathrm{V}\mu_\infty} = \mathrm{K}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and though the results obtained are more in accordance with the Rudolphi-van't Hoff ‡ equation

$$\frac{\alpha^3}{(1-\alpha)^2\mathrm{V}} = \mathrm{K}, \ \alpha = \frac{\mu}{\mu_\infty}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

yet the relation can only be regarded as empirical.

It seemed, therefore, of interest to calculate from our conductivity determinations the amount of ionization $\frac{\mu}{\mu_\infty}$ at the different percentage concentrations; for though results for dilute samples have been given by Kohlrausch § and others, yet no data have been published for more concentrated samples. The value for the molecular conductivity at infinite dilution or $\mu_\infty = \mathrm{K}10^7$, according to the data of Kohlrausch is $\mathrm{K}_{18} = 342$, or $\mathrm{K}_{15} = 325{\cdot}4$ when reduced by the formula

$$\mu_{15} = \mu_{18} 342(1 + {\cdot}0162[15-18]). \quad . \quad . \quad . \quad (3)$$

In the following Table values are given: in column I. the percentage concentration, in II. the corresponding molecular dilution, in III. the values for $\mu \times 10^7$, and in IV. the values for $\frac{\mu}{\mu_\infty}$, the quotient of ionization.

It will be observed that there is an irregularity in the results for the samples from 98 per cent. to 100 per cent., which is consequent upon a reversal in the conductivity values; and in our work mentioned above attention was drawn not only to this point, but also that whereas nitric acid behaves

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* Communicated by the Authors.

† Phil. Trans. 1898, A. p. 365.

‡ *Zeits. f. physikal. Chem.* xvii. p. 385 (1895), and xviii. p. 301 (1895).

§ Wied. *Ann.* xxvi. p. 196 (1885).

as other electrolytes in possessing a positive temperature-coefficient for concentrations up to 96 per cent., yet beyond this point it behaves as a metallic conductor in possessing a negative temperature-coefficient.

TABLE I.

Percentage Proportion.	Molecular Dilution.	μ_{15} in mercury units × 10^7.	$\frac{\mu_{15}}{\mu_{15\infty}}$.
1·30	4·568	288·0	0·885
3·12	1·956	281·5	0·865
5·99	1·164	258·9	0·796
10·13	0·559	228·0	0·701
15·32	0·349	190·1	0·571
20·11	0·251	159·3	0·487
25·96	0·180	123·5	0·379
30·42	0·144	100·1	0·308
33·81	0·123	85·8	0·263
35·90	0·112	73·6	0·226
39·48	0·096	63·8	0·196
45·01	0·077	47·9	0·117
51·78	0·059	32·6	0·102
53·03	0·056	30·4	0·093
58·20	0·045	22·2	0·086
61·20	0·040	18·6	0·057
65·77	0·032	13·3	0·041
69·53	0·028	10·3	0·032
73·82	0·022	6·43	0·020
76·59	0·019	5·03	0·015
78·90	0·017	3·27	0·010
84·08	0·012	1·09	0·004
86·18	0·010	0·94	0·003
87·72	0·007	0·62	0·002
89·92	0·007	0·36	0·001
91·87	0·006	0·17	0·0005
94·32	0·004	0·07	0·0002
96·12	0·002	0·035	0·0001
98·50	0·001	0·016	0·00005
98·85	0·001	0·013	0·00004
99·27	0·001	0·017	0·00005
99·97	0	0·007	0·00002

It is evident that this phenomenon cannot be explained by the aid of the ionic dissociation theory, as the number of free ions would not exceed four per 1000. As we have observed a similar abnormality in other physical properties, and especially in the contractions as calculated from the density determinations, we consider it probable that the result is due to an incipient formation of the compound $2HNO_3 . N_2O_5$, which is prepared by dissolving nitrogen pentoxide in anhydrous nitric acid.

The values for μ for the acids of lower percentage concentration (1 per cent. to 42 per cent.) are generally somewhat lower than those given by Kohlrausch; but those for $\frac{\mu}{\mu_\infty}$ are more in accordance with those calculated by Jones * from the freezing-point determinations.

Kohlrausch and Hallwachs † have pointed out that there is an undoubted relation between the contraction, produced even in very dilute solutions of electrolytes, and ionic dissociation, since non-electrolytes do not show this phenomenon. In the present instance, though the points of alteration of direction of curvature are the same both for the contractions and the quotient of ionic dissociation, if expressed in terms of percentages, yet correlation of the two phenomena does not appear to be of a simple character.

Coefficients of Expansion.

Very few data of the coefficients of expansion of nitric acid have as yet been recorded. Kobl ‡, many years ago as a result of his density determinations, observes as follows:—"Le coéfficient de la dilatation de l'acide azotique est assez élevé. Ainsi pour l'acide azotique monohydrate, le coéfficient moyen de dilatation entre 0 degré et 15 degrés est $a = {\cdot}001263$, c'est à dire environs trois fois plus fort que celui de l'eau."

Marignac gave values for the constants a, b, and c in the equation

$$V_t = V_0(1 + at + bt^2 + ct^3) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

for a few samples; and lastly Lunge and Rey § published the corrections of densities at 15° for temperatures between 13° and 17°, in all seventeen values. From our density determinations at 4°, 14°·2, and 24°·2 corrected to a vacuum we have calculated the values for the mean cubic expansion or values of a from the equation

$$V_t = V_4(1 + at) . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

These values expressed as $\alpha \times 10^{-6}$ are given in the following table.

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* *Zeits. f. physikal. Chem.* xii. p. 623 (1893).

† *Göttingen Nachrichten*, 1893, pp. 350–357. *Cf.* Drude and Nernst, *Zeits. f. physikal. Chem.* xv. p. 79 (1894).

‡ *Ann. Chim. Phys.* [4] x. p. 140.

§ *Zeits. f. angew. Chemie*, 1891, p. 167, and 1892, p. 10.

Percentage.	$\alpha\times10^{-6}$, 4°–14°·2.	$\alpha\times10^{-6}$. 14°·2–24°·2.	Percentage.	$\alpha\times10^{-6}$. 4°–14°·2.	$\alpha\times10^{-6}$. 14°·2–24°·2.
0·625	100	158	56·29	980	950
2·732	150	241	58·32	......	950
4·595	205	246	60·42	1000	......
7·585	283	265	60·60	......	990
10·75	341	370	62·84	......	1038
13·99	391	403	65·80	1058	1043
14·95	402	446	67·85	......	1060
16·88	428	455	71·60	1054	......
18·44	476	525	75·64	1088	1076
19·32	484	......	76·55	1088	1076
23·89	540	542	78·22	1095	......
25·50	550	585	81·93	1143	1093
28·97	629	608	85·21	1143	1096
30·17	647	637	87·90	......	1096
33·55	700	672	92·24	1200	1141
38·10	770	......	94·04	1200	1141
42·33	817	790	95·62	1200	1141
44·41	847	801	96·64	1222	1172
46·56	864	817	97·33	......	1190
51·24	942	907	98·07	1250	1210
54·60	980	950	100·00	1274	1240

The value obtained for the 100-per-cent. acid for the temperature-limit 4°–14°, namely ·001274, approximates very closely to that obtained by Kolb, namely ·001263 between the limits 0°–15°; but it appears that he made an arithmetical mistake in referring to the value as about three times as great as the mean coefficient of water between the same limits; this latter, according to Rosetti's tables, is ·000047, so that the value for nitric acid is nearly thirty times as great (actually 27·1).

The above results show that as regards the mean coefficient of expansion, nitric acid of different concentrations does not behave as if it were a mere mixture of the acid and water.

The following selected values illustrate this point:—

Percentage.	Sum of Coefficients calculated at 4°–14°·2.	Coefficient found at 4°–14°·2.
10·75	180×10^{-6}	341×10^{-6}
30·17	417×10^{-6}	647×10^{-6}
51·24	676×10^{-6}	942×10^{-6}
75·64	978×10^{-6}	1088×10^{-6}
94·04	1126×10^{-6}	1200×10^{-6}

In every case the found value exceeds that calculated, though the difference decreases as the percentage-concentration increases.

Though there seems to be no simple relation between the ratios of the calculated and found coefficients of expansion, on the one hand, and the factor of ionic dissociation, on the other, yet such a general result as that given above would follow if the sums of the coefficients of expansion of the separated ions, namely, $\overset{\frown}{H}$ and $\overset{\frown}{NO}_3$, were greater than that of the undissociated molecule HNO_3. The general result is, however, probably complicated by the affinity of the acid for water and the consequent formation of either hydrates or of complex molecular structures (say x), each of which would have its particular coefficient. The net result is expressible by a general equation :

$$K=\Sigma[k(H_1)+k'(NO_3)+k''(HNO_3)+k'''(x)\ldots]. \quad . \qquad (6)$$

A further point of interest is that the values for the lower temperature-limit are for the lower percentage concentrations less than the values for the higher temperature-limit, yet for the higher percentage concentration this relation is reversed ; the reversal takes place at about 40 per cent.

If a curve be drawn in which the mean coefficients for either limits of temperature are expressed in terms of percentages, the observational results are found to lie approximately upon four straight lines—namely, firstly, from 1 to 8 per cent.; secondly, from 11 to 54 per cent.; thirdly, from 54 to 70 per cent. ; and, fourthly, from 70 to 93 per cent. ; and the remainder upon an irregular curve.

With regard to these several points, it may be noticed that the 54 per cent. corresponds to that of maximum contraction as found by Kolb and ourselves, while the irregularity from 92 to 100 per cent. has already been alluded to. The behaviour of nitric acid as regards this physical property is so far different from sulphuric acid, which has been completely investigated by Pickering*, in that, firstly, the variations are much more marked ; and, secondly, the maximum value occurs at 100 per cent., while that of sulphuric acid is at about 86 per cent., from which point the value of the coefficient decreases to 98 per cent., and from this concentration up to 100 per cent. slightly increases.

In conclusion, we trust that we may be allowed to raise a plea for the more complete investigation of the physical properties of solutions of liquids in liquids of a high degree of concentration as regards the solute. The results to be obtained will, we believe, be as full of interest as those already obtained with solutions of a high degree of dilution as regards the solvent water.

Oxford, October 1901.

* Journ. Chem. Soc. 1890, p. 114.

XI. *On the Necessity for Postulating an Æther.*

By B. Hopkinson[*].

THE meeting of the British Association generally brings into prominence the more controversial parts of science, and that just concluded was no exception to the rule. Professor Rücker's eloquent defence of the atomic theory was perhaps even more striking as an acknowledgment of a strong body of opinion for which atoms are by no means physical realities; and though he did not deal with it in any detail, the hypothesis of an æther was mentioned along with that which was more immediately the subject of the address, as a matter regarding which doubts were rife. In fact there are representatives of every shade of opinion as to the æther, from those who regard it as a wholly unnecessary conception, to those who wish to invest it with every material property that it will support.

It is not my intention to enter into the merits of this controversy, but I wish to discuss shortly the nature of the difference between the parties to it. How far is the postulate of an æther founded on metaphysical grounds, and therefore matter of dispute? how far has it what may be called a logical basis, such as all must admit?

To illustrate what is meant by these somewhat vague terms, take the phenomenon of gravitation. All we know of gravitation is that if there is matter at a place A, then another piece of matter at P distant from A will experience a force of attraction towards A. We may further analyse and express the phenomenon by attributing to every point of space a force, depending according to known laws on the nature of A, and the position of the point relative thereto, which force, however, can only be made manifest to the senses by the use of matter as a detector. Any further analysis than this leads us into the realms of speculation, and difference of opinion, hitherto out of the question, becomes possible. Thus Newton said that one who believed in the possibility of attraction between distant bodies without an intervening medium must be a fool; but J. S. Mill cheerfully avowed his sympathy with the fools. In fact the question of medium or no medium so far as gravitation is concerned may be called purely metaphysical; it is similar in kind to the question whether the gravitational force exerted by A has any existence when there is no other body to be attracted. Those

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[*] This paper is an expansion of some observations communicated to the British Association at Glasgow, 1901. Communicated by the Author.

who assert the existence of a medium do so in order to satisfy a craving for the *explanation* of gravity in terms of other forces more familar though (be it remarked) not necessarily more fundamental. For the *expression* of the laws of gravity it is not necessary to introduce any medium.

But now suppose that the force of attraction between the two bodies (supposed to be alone) were found not to depend simply on their masses and relative position, but to vary in an apparently arbitrary manner. To account for such variation and to express its laws, it would be necessary to imagine a new cause, independent of material bodies. In mathematical terminology the force of gravity would be no longer a function of the distance between the bodies only, and to express the facts mathematically we should have to introduce new independent variables—for example, the coordinates of the bodies with reference to a set of axes fixed independently of matter. Given these coordinates it might be possible, if the facts fitted, to predict the value of gravity—a prediction we could not make if the relative positions of the two bodies alone were given. There would then be what I call a logical basis for postulating a medium having parts and serving to define position independently of matter.

No such logical basis can in fact be found in the phenomenon of gravity, whose laws are entirely expressible in terms of matter only. Can any be found in the transmission of light? I think there is no doubt that the laws of all optical phenomena hitherto discovered, except one to be presently mentioned, can be adequately expressed, like that of gravitation, in terms of the state and positions of material bodies now and in the past. We know that at a point P, distant from a luminous body A, we shall by the use of appropriate instruments experience the disturbance known as light. Interference shows the disturbance to be of a periodic character, polarization that it is properly represented as a directed quantity at right-angles to AP. Its amount is $\frac{f(r-vt)}{r}$ where t is the time, r the distance AP. That is a statement of some cardinal optical laws without any mention of a medium; and there is no doubt that all known terrestrial phenomena of the kind can be dealt with in the same way. In our mathematical expressions we need to take as independent variables quantities referring to material bodies only; given the values of these quantities, the luminous effects follow determinately. There is here no logical basis for an æther. Several attempts have been made (the most notable being that of Michelson) to experience directly the effects of the æther as a disturbing cause in terrestrial experiments, and so establish such a basis, but without success. It is still open to any one to go a step further than Mill and to say that in his opinion light is an action of one body upon another, such action taking place at a distance not only in space but in time.

When we leave the earth and turn to observation of the stars, however, we find that there is a possibility of experiencing the æther as a cause. It is a consequence of the undulatory theory that aberration depends on the motion of the earth relative to the free æther, and is entirely unaffected by the motion of the star. This, if true, furnishes our logical basis. For suppose the earth and the star to be fixed, relative to one another, but in uniform motion relative to the æther. There will be aberration, the star will appear to the observer to occupy a different position from that which it would occupy if there were no motion relative to the æther, the difference depending on the amount of such motion. To one who had not thought of postulating an æther but merely concerned himself with classifying phenomena and expressing their laws mathematically, this new phenomenon would appear wholly arbitrary and inexplicable. He would in fact have to invent a new cause to explain it, and that cause would in its essence be our æther. In a way, it may be said that the æther would become manifest to his senses as a something other than matter having parts and position. We should apprehend the existence of an atmosphere in exactly the same way, and should ascribe the same properties to it, if we had to depend only on the sense of hearing for perceiving it. The fact that the noise of a railway-whistle does not always appear to proceed from the whistle, though the whistle has no motion relative to the hearer, furnishes a logical necessity for an atmosphere.

The only question that remains is this : Is there any direct evidence that the motion of the luminous body produces no effect on aberration ? For of course it will not do to take it merely as a consequence of the undulatory theory. In the ordinary observations of aberration there is no such evidence. It is quite consistent with them to suppose that the motion of the star which is uniform produces the same effect on aberration as that of the earth. We can in fact only observe the changes in aberration, such as those which depend on the periodic motion of the earth, and not its absolute amount. But there are certain stars—spectroscopic binaries—which, apparently single as seen in a telescope, are inferred to consist of two components revolving one about the other because of the periodic doubling of certain lines in their spectra. Such doubling would occur owing to the well-known Doppler effect, when the two components are moving one away from, and the other towards, the earth. The orbital velocity can be computed from the separation of the lines, and amounts in some cases to as much as 150 miles per second or many times that of the earth in its orbit. Now consider the moment when the two components are in the line joining the star to the earth. One A will be moving to the right, and the other B to the left with a velocity of 150 miles per second. If the effect of motion of the source on aberration is at all comparable with that of motion of the receiver, we should expect (when the light then leaving the star reaches the earth) to see star A displaced to the right and B to the left. The star would appear double in a telescope, and the apparent doubling would occur periodically, and alternately with the separation of the spectroscopic lines. That no such doubling is seen is very strong, if not absolutely conclusive evidence of the fact we set out to establish, that motion of the source does not affect aberration and it forms a real logical basis for postulating an æther.

XII. *A Preliminary Note on the Relation between Primary and Secondary Röntgen Radiation. By* H. S. ALLEN, *M.A., B.Sc.*[*]

WHEN the Röntgen radiation from a focus-tube falls upon the surface of certain solids, the surface becomes the source of a secondary radiation. These "Secondary Röntgen Rays" are in many respects similar to the ordinary rays which produce them. Their properties have been investigated by Sagnac[†], Perrin[‡], and Townsend[§]. Both primary and secondary rays render a gas through which they pass a conductor, but while the primary rays suffer very little absorption the secondary rays are almost entirely absorbed within a few millimetres at atmospheric pressure. The object of the present investigation is to compare the total ionization produced by the secondary rays with that which would be produced by the primary if they continued their path in the original medium. Then, supposing that the total ionization, if not equal to, is at least proportional to the

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[*] Communicated by Prof. J. J. Thomson.

[†] *Journal de Physique*, 3rd series, viii. Feb. 1899.

[‡] *Comptes Rendus*, cxxiv. p. 455.

[§] Proc. Camb. Phil. Soc. x. Part iv. 1900.

energy of the radiation, we shall obtain a measure of the ratio of the energies in the two kinds of radiation.

The experimental work involves two separate determinations. In the first place, the ratio of the total ionization caused by the secondary rays to that produced within a known space by the primary; and, secondly, the absorption experienced by the primary rays in traversing a known distance.

With the assumption just stated, the absorption of energy in a parallel beam of rays in traversing a thin layer of gas of thickness dl at a point where the intensity is I is equal to $\lambda \mathrm{I} dl$, where λ is the coefficient of absorption of the gas.

For simplicity we may consider the radiation contained in a cone of small solid angle $d\omega$, with its vertex at the anticathode. If I denote the intensity of radiation at a distance r from the vertex, we obtain the following differential equation for determining I in terms of r:

$$\left(\mathrm{I}+\frac{d\mathrm{I}}{dr}dr\right)(r+dr)^2 d\omega-\mathrm{I}r^2 d\omega=\lambda \mathrm{I} r^2 dr d\omega,$$

$$\therefore \quad \frac{d}{dr}(\mathrm{I}r^2)=\lambda \mathrm{I} r^2.$$

The solution of this equation is

$$\mathrm{I}r^2=\mathrm{I}_0 r_0^{\,2} e^{-\lambda(r-r_0)}.$$

The ionization produced by the rays in the element of volume may be expressed as $k\mathrm{I}r^2 dr d\omega$, where k is a constant.

Hence the total ionization produced within the boundary of the cone between r_0 and infinity is

$$\int_{r_0}^{\infty} k\mathrm{I}r^2 dr d\omega=\frac{k\mathrm{I}_0 r_0^{\,2} d\omega}{\lambda}.$$

The ionization actually measured is that produced in the same cone between r_0 and r_0+dr_0, and, using the same notation, may be written

$$k\mathrm{I}_0 r_0^{\,2} dr_0 d\omega.$$

Thus the total ionization may be obtained by dividing this, the observed quantity, by λdr_0.

The method employed in the first determination, that of the ionization produced by the secondary rays, was similar to that used by Perrin (*loc. cit.*), and consisted in the comparison of the rates of leak from two condensers, the leak in the first being due to primary radiation alone, that in the second to primary and secondary radiation.

The method employed in measuring the absorption of the primary rays was also a comparative one, and was practically identical with that of Rutherford[*].

As yet the determinations have only been completed in the case of a single gas—sulphuretted hydrogen—in which the coefficient of absorption is comparatively large. In this gas, calling the ionization produced by the primary rays in a distance of 2·5 cm. 100, the ionization caused by the secondary rays from a freshly polished brass plate, 10 cm. by 5 cm. in area, amounted to 11. The variation due to the absorption of 20 cm. of sulphuretted hydrogen was found to be 3·9 per cent., giving as the value of λ $1{\cdot}9 \times 10^{-3}$.

Hence the ratio of the total ionization caused by the secondary radiation from a brass plate to the total ionization for the primary radiation is equal to

$$\frac{11}{100} \times 2{\cdot}5 \times 1{\cdot}9 \times 10^{-3},$$

or $\frac{1}{1900}$ very nearly.

From this it follows on the foregoing assumptions that the total energy of the secondary radiation only amounts to about one two-thousandth part of the energy of the primary radiation.

The work here described was carried out at the Cavendish Laboratory at the suggestion of Professor J. J. Thomson, to whom I am glad to express my indebtedness.

Trinity College, Cambridge.

XIII. *The Anomalous Dispersion of Sodium Vapour.* By R. W. Wood, *Professor of Physics in the University of Wisconsin*[†].

C. E.
V. 94.

[Plates II.–IV.]

EXPERIMENTAL proof of the dispersion formulæ of Sellmeier, Helmholtz, and Ketteler has for the most part been based upon observations made upon substances in the solid state or in the state of solution, where the range of absorption extends over a considerable portion of the

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[*] Phil. Mag. [5] xliii. pp. 259–255 (1897).

[†] Read before the Royal Society June 20, 1901. We are indebted to the Royal Society for the use of the stone for the plate and to the Author for defraying part of the expense of reproducing it.

spectrum, and the molecular condition is in all probability vastly more complicated than in the gaseous state.

In liquid oxygen and in the salts of erbium and didymium we have the only examples, so far as I know, of substances other than gases which exhibit narrow absorption-bands, and in these cases the absorption is scarcely strong enough to influence the dispersion to any marked degree. A careful study of the dispersion of some substance with absorption-bands as narrow and strong as those of sodium vapour, would be of great value in aiding us to pass judgment on the various modifications of Sellmeier's original theory.

The anomalous dispersion of sodium vapour in the immediate vicinity of the D lines was first observed by Kundt, and has since been studied by Becquerel and Julius, but none of these investigators has traced the effect over a range of the spectrum greater than two or three times the distance between the lines.

In a previous paper * I have shown that a spectrum closely resembling the bright-line spectrum of incandescent sodium can be produced under certain conditions by the anomalous refraction of white light in a non-homogeneous atmosphere of sodium vapour, in precisely the manner in which Julius very ingeniously suggests that it may occur in the reversing layer of the sun, giving rise to the phenomenon of the "flash spectrum." In view of the seeming importance of the subject in its bearing on the subject of the dynamics of dispersion, and its possible connexion with solar physics, I have undertaken to determine whether sufficiently accurate quantitative data can be obtained, to make a test of the dispersion formula, as applied to sodium vapour, possible.

Though the absolute values of the refractive indices which have been found are not as satisfactory as is desirable, very interesting relative values have been obtained, and, what is of especial interest, the dispersion has been traced and measured throughout the entire range of the visible spectrum, the refractive index for all waves of higher frequency than D_2 being less than one, and for all waves of lower frequency than D_1 greater than one. In other words, I have been able to form a complete anomalous spectrum by means of sodium vapour, in which all of the colours between the extreme red and violet are present, with the exception of a range of perhaps twenty Ångström units at the D lines.

Previous investigations have been made exclusively with

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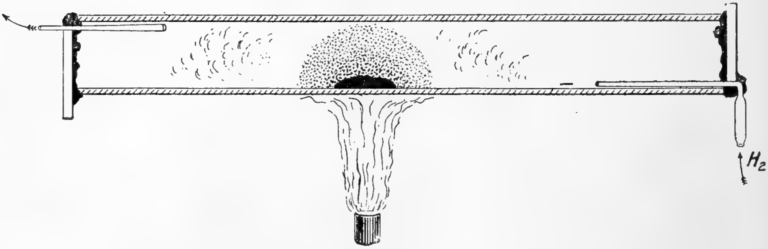
* Phil. Mag. May 1901.

sodium flames, which have been made to assume a prismatic form. In view of the great uncertainty regarding the prism angle, density of the vapour, and its molecular condition, that is whether dissociated or not, it seemed advisable to employ non-luminous vapour, obtained by heating metallic sodium in some inert gas or *in vacuo*.

The first experiments were made with a large prism of cast iron, furnished with windows of mica or thin plate glass, in which the metal was heated in an atmosphere of hydrogen. Very beautiful results were at once obtained, but certain peculiarities of the vapour's action showed that the refraction was due chiefly to the action of a non-homogeneous medium, the planes of constant density being horizontal. Great trouble was had with the windows, which soon became covered with a white deposit, which cut off most of the light. It being apparent, however, that the oblique faces played but a very small part, the effect being due almost wholly to the variable density of the vapour, it seemed best to make the most of this circumstance, and dispense with the trouble entirely by removing the glass plates to such a distance from the heated vapour that no deposit took place. The arrangement finally adopted was simply a tube of glass about 30 cm. long, provided with plate-glass ends cemented on with sealing-wax. Hydrogen, dried by passage over calcium chloride, was conducted into and out of the tube by means of two fine glass tubes arranged as shown in fig. 1.

Fig. 1.



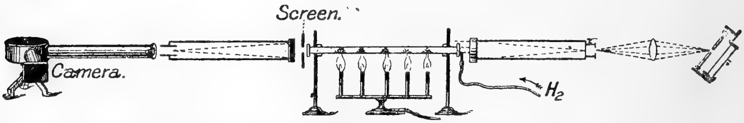
The diameter of the tubes should not be over 2 mm., and they should lie close against the sides of the large tube in order not to cut off any of the light. The most suitable diameter for the large tube is 2 cm. As the experiments to be described in this paper will, I feel sure, repay any one who takes the trouble to repeat them, and will be extremely useful in illustrating the subject of anomalous dispersion to small classes of advanced students, I shall describe in some detail the manner of preparing and using these dispersion-tubes, which I have found most advantageous. The ends of the tubes are first warmed and thickly coated with sealing-wax; one of the glass straws is then placed in position, and a small piece of plate glass, previously warmed, pressed against the wax, any crevices around the straw being closed with wax. The leading-in tube is next placed in position, and a piece of freshly-cut sodium (about 5 mm. on a side) inserted. The other window is then cemented to the tube, and the current of hydrogen started as soon as possible. Some experience is necessary properly to regulate the hydrogen stream during the experiment. When the tube is first heated much white smoke forms. If a stream corresponding to about one bubble per second is allowed to flow, the smoke will usually clear up in a few minutes and give little trouble. The tube should be heated by means of a Bunsen burner turned down low, the tip of the flame playing against the bottom of the tube. If a sodium flame is placed behind the tube the formation of the vapour can be watched, for it appears almost jet-black against the flame, though quite colourless in white light. The behaviour of the vapour is somewhat peculiar. It grows out from the sodium globule as a dark atmosphere with a sharply defined surface, which clings to the globule with great tenacity. It resembles at first a thick growth of mould more than anything else that I can think of, and a sudden gust of hydrogen scarcely moves it at all. A wire pushed up through it drags a certain amount above the free surface in much the same manner as a stick pushed up through the surface of thick molasses would do. If the tube be inverted the black cloud clings to the upper surface, behaving on the whole like a very viscous mass. It is even possible to dip some of it up on a wire.

These peculiar physical properties of the metallic vapour I have as yet only studied in a very superficial manner, and I mention them now, only because it appears to me that there is some connexion between them and the optical behaviour of the medium.

I am of the opinion that the apparent viscosity is an illusion, and that the sharply-defined surface is merely the boundary at which either condensation or chemical action (the hydrogen not being pure) is taking place. The process of dipping the vapour up on the wire might be explained by condensation on the wire followed by vaporization. A more careful study of the physical behaviour of the vapour will be made some time in the future.

The apparatus employed in the study of the dispersion of the vapour was essentially identical with that used by Becquerel. The light of an arc lamp was focussed on the horizontal slit of a collimator, after traversing which the parallel rays passed lengthwise through the dispersion-tube. A second lens brought them to a focus on the slit of a spectroscope, when the dispersion was to be studied by the method of crossed prisms, or in the focus of an eyepiece when the anomalous spectrum was to be viewed subjectively (fig. 2).

Fig. 2.



The first experiments were made by the method of crossed prisms, the spectrometer being furnished with a Rowland plane grating, which showed the sodium lines widely separated. It was at once apparent that far better results could be obtained with the dispersion-tubes than had ever been observed with prismatic flames. The curved branches of the diffraction-spectrum on each side of the D lines were perfectly sharp and steady, and the dispersion could be traced to a considerable distance up and down the spectrum. On the slit of the spectrometer appeared, instead of the white image of the horizontal slit, a most beautiful anomalous spectrum, of great brilliancy and purity. The spectrometer was at once removed and an eyepiece put in its place, when a most superb spectrum revealed itself. The general appearance is shown in the coloured plate accompanying this paper, though it is quite impossible to represent by means of pigments the sparkling brilliancy of the colours.

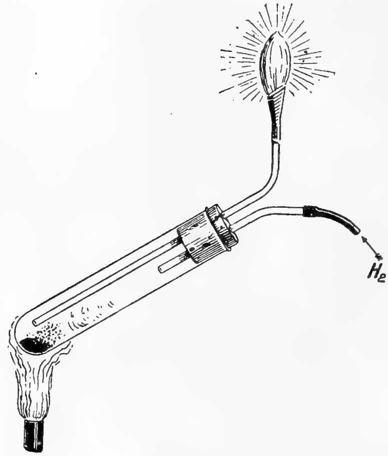
Before discussing this spectrum in detail, it will be better to take up the results of the experiments made by the method of crossed prisms. On first heating the tube, the curvature of the spectrum between the D lines as well as on each side is observed, the appearance being identical with that figured by Becquerel, but in a few seconds the vapour becomes so dense that total absorption of all the light between the lines occurs. Julius expresses the opinion that this disappearance of the light between the lines is only a result of the strong dispersion; that is, it is not absorbed but turned off to one side so that it does not enter the instrument. It appears to me, however, that this is not the case, for I have observed the same effect under conditions where lateral deflexion seems quite out of the question. As I shall show later on, the breadth of the absorption-band is sometimes twenty times the width of the spectrum comprised within the D lines.

The oppositely-curved branches adjacent to the region of absorption grow out rapidly as the tube grows hotter, the ends finally passing out of the field of the instrument. A beautiful fluted absorption appears in the red and the greenish-blue, which finally blots out a region in the blue almost entirely. Meanwhile the curvature of the spectrum increases in a most remarkable manner, and the entire red end is lifted high above the green-blue end. As the density of the vapour increases the red gradually fades away, leaving only the yellow and green and the remote blue and violet, the curvature increasing all the while. The fluted or channelled spectrum was described by Roscoe and Schuster about twenty-five years ago, but so far as I know no work has been done on it since. I have recently secured excellent photographs of it with a Rowland concave grating, from the extreme red to the violet, and find that it is much more extensive than has been supposed, for the flutings run right up to the absorption-band at the D lines on *both* sides, though they are very faint on the side of shorter wave-lengths. This spectrum will be described in a subsequent paper.

Very satisfactory photographs of the dispersed grating-spectrum were secured, some of which are reproduced. It was found impossible to maintain a sufficiently uniform density, at the low temperature, for a sufficient length of time to enable a negative to be secured showing the appearance before the light between the D lines vanished. I therefore went back to the old plan of using a prismatic flame. After some experimenting it was found that the most satisfactory flame was secured by passing hydrogen through a tube containing metallic sodium and strongly heated, and burning the gas at a flat jet piece made of platinum foil. An exceedingly dense and very uniform sodium flame is obtained in this way, which can be maintained almost indefinitely. The arrangement of the lamp is shown in fig. 3 (p. 134), the diagram requiring no description.

With this flame as a dispersing piece the photographs shown in Plate III. fig. 8 were obtained. In comparison with some of the other photographs this example is very poor, and I have included it merely to make the set complete. An exposure of about ten minutes was necessary, with an "Erythro" plate. Fig. 9 was taken with one of the dispersion-tubes, and shows what may be called the second stage. After the exposure was over, but before the plate was removed, a small sodium flame was placed before the slit of the spectrometer, and the sodium lines impressed on the plate in their proper position, to serve as a reference. For securing photographs of the dispersion resulting from still denser vapour more light was desirable, and a small prism was accordingly substituted for the grating. This prism barely resolved the sodium lines in the negatives, but gave excellent pictures of the spectrum under strong sodium dispersion. Pl. III. figs. 10 and 11 were obtained in this way. In the latter the flutings in the red and green are discernible, though not very sharp.

Fig. 3.



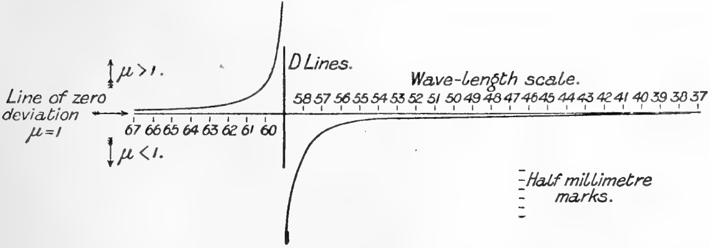
As much of the fine detail is always lost in reproduction I have marked with arrows on fig. 11 the extreme points to which the curved branches adjacent to the absorption-band can be traced in the original negative. Eye observations enable one to trace them much farther, for the tips are exceedingly faint, and the light is not very actinic.

In order to secure accurate measurements of the dispersion in different parts of the spectrum I have made use of two different methods.

The photographs were not very suitable owing to the fact that during the exposure the density of the vapour varied somewhat, producing a slight blurring of the image. Measurements of the deviation of the continuous spectrum were accordingly made with a filar micrometer, an illuminated scale giving the wave-lengths at the points where readings were taken. Several minutes were necessary, however, to get anything like a satisfactory set of readings, and changes occurred even during that short period, as was at once apparent on repeating observations. I accordingly adopted the following method.

The telescope of the spectrometer was removed and a telescope-objective of nearly two metres focus put in its place. In the focal plane of this lens a plate of glass was firmly mounted, on which a horizontal diamond scratch had been ruled. This diamond scratch was brought accurately into coincidence with the narrow continuous-spectrum image in its undeviated position. An eyepiece mounted behind the plate enabled the spectrum to be observed. When the dispersion-tube was in good condition, and the deviated branches of the spectrum appeared sharp and steady, their positions on the plate were marked with a writing diamond, the line being drawn along the middle of the spectrum, which was only about a millimetre wide on the plate. The dispersion-tube was then removed and various metals fed to the arc, the bright lines being marked on the plate with the diamond. In this way a scale of wave-lengths was secured. For the determination of refractive indices we require, however, the angular deviation of the different rays. This can be determined if we know the focal length of the lens forming the image of the spectrum on the slit of the spectrometer, and the actual deviations measured in millimetres of the images of the first slit formed by light of various wave-length at this point. To secure a record of this a glass plate, ruled with half-millimetre lines, was placed over the slit of the spectrometer, and the position of the lines on the glass plate in the focus of the large lens recorded with the writing diamond. This scale enabled the recorded deviations to be reduced to the actual deviations as they existed on the slit of the instrument.

Fig. 4.



The dispersion curve obtained in this manner, with the half-millimetre marks, is reproduced on the same scale as the original in fig. 4, the prismatic spectrum having been converted into a normal spectrum. It will be seen that for all wave-lengths shorter than those of the D lines, the refractive index is less than one, while for the rest of the spectrum it is greater than one. This is the only case that I know of in which we have a medium beautifully transparent even in considerable thicknesses, in which light travels faster than in a vacuum.

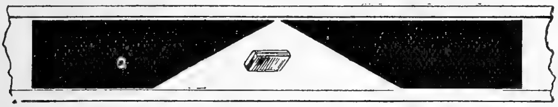
To determine the refractive indices we require the angle of the sodium prism, and here we encounter a grave difficulty, for it appears to be quite impossible to determine this directly. As I have said before, we are dealing with the optical equivalent of a prism, namely, a non-homogeneous medium, in which the vertical wave-front is retarded or accelerated in a progressively increasing amount as we pass from its upper to its lower edge. The equivalent prism appears from experiment to be bounded by concave instead of plane surfaces; in other words, the effective angle is greater near the bottom of the tube than at the top. This can be shown by screening off different levels.

The lower part of the tube acting alone gives a much greater deviation than the upper. In practice I have found it best to screen off the upper and lower portions, utilizing only the light which passes in the median plane, where the change in density appears to be quite uniform, this method having been used in preparing the dispersion curve alluded to above. The effective angle lies probably somewhere between 90° and 130°, but even this estimate is mere guesswork.

If a single observation could be made with a vapour prism of known angle, in which we could be sure that the density was uniform, quantitative values could be assigned to determinations made with the dispersion-tubes. I have accordingly tried in every way possible to obtain a prism fulfilling the required conditions. These endeavours have thus far met with only partial success, but an account of the devices tried may be of value to others working along similar lines.

Glass and mica are so quickly attacked by the sodium vapour, that the use of these substances for prism-faces seems to be out of the question. Added to this there is the difficulty of making gas-tight joints between plate-glass and metal which will stand a temperature but little below a red heat. Repeated failures to secure prisms in this way compelled me to seek some other method of giving the vapour the required form. I had observed that in glass tubes held before sodium flames the black vapour retreated before the approach of a cold obstacle. This suggested to me that it might be possible to do away with glass surfaces entirely, moulding the vapour into the required prismatic form by the proximity of cold bodies. Experiments along these lines were partially successful. Two pieces of thick-walled iron tubing, the ends of which had been cut off at an angle, were introduced into a glass tube, and the sodium placed in the clear space between the bevelled ends, as is shown in fig. 5. The ends of the tube were closed in the usual manner and the whole mounted between the collimator and telescope, in such a position that the prism formed by the sloping ends of the iron tubes stood with its refracting edge vertical. This was necessary, for in any other position the refraction due to the non-homogeneity of the vapour would have made itself felt. With a vertical slit and a vertical prism no lateral deviation could result from this cause. It was hoped that the sodium vapour formed by sudden and rapid heating of the glass tube would refuse to enter the colder iron tube, and that a prismatic form would result from the bevelled ends. The slit of the spectroscope was illuminated with monochromatic light obtained by prismatic analysis, the wave-length of which could be changed by turning the prism, which was furnished with a mirror in the manner described by Wadsworth.

Fig. 5.

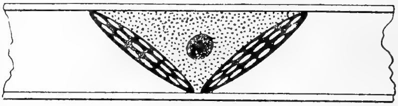


On heating the tube the black vapour was seen to spread out and fill the prismatic clear space between the iron tubes, while the lateral deviation of the image of the slit, as observed in the telescope, indicated that, to a certain extent at least, the desired prismatic form had been secured. In the extreme red the deviation was very slight, but as the spectrum was advanced across the slit by slowly turning the prism the image in the telescope moved off to one side, the deviation in this direction reaching its maximum value just before the wave-length reached that of the sodium lines. At this point the image jumped abruptly to the other side just as we should expect it to do on crossing the D lines in the spectrum, and from now on the image slowly crawled back to its undeviated position. The focal length of the telescope was 460 mm., and the maximum deviation of the rays adjacent to the D lines on the red side, as measured by an eyepiece filar micrometer, was but 1 mm., while the deviation in the opposite direction of the rays on the other side of the D lines was

1·2 mm. The angle of the prism was 130, from which data we get the following values for the refractive index of the vapour for these two wave-lengths (relative to hydrogen): $\mu=1\cdot0005$ and $\mu=0\cdot9994$.

Similar results were obtained with the device shown in fig. 6, when two elliptical pieces of perforated sheet iron were used for moulding the vapour. The images formed in this case were blurred by diffraction.

Fig. 6.



I place very little value however on these figures, for I am of the opinion that the effective angle of the sodium prism is much less than the angle between the ends of the tubes, it seeming probable that the vapour bulges out into the tubes, especially near the centre. That this is to a certain extent the case is indicated by the fact that the image of the slit is not very sharp, though this may well be caused by the varying density of the vapour. I have not yet despaired of getting a prism bounded by plates of glass, about which there can be raised no question, though the problem is a difficult one, and observations will have to be made with great rapidity.

The deviations obtained by this method are very much less than those obtained with the dispersion-tubes, indicating either that the equivalent angle is very large in the latter case (it may be nearly 180° for all we know) or that the angle of the prism formed by the iron tubes was less than the estimated value.

If we calculate the refractive indices from the data given by the curve traced with the diamond point, assuming the equivalent angle of the stratified vapour to be 130°, we get for the wave-lengths adjacent to the absorption-band the values $\mu=1\cdot0024$ and $\mu=0\cdot9969$.

The value given by Becquerel (1·0009) is intermediate between the values obtained by the two methods, but I feel sure that his flame must have been non-homogeneous, and his numerical results more or less inaccurate in consequence.

For the exhibition of the actual spectrum produced by a prism of sodium vapour, a long dispersion-tube with a battery of four or five prisms gives the best results. A small Bunsen burner should be used for each of the fragments of sodium, which should be at least 6 or 8 cm. apart. (This arrangement is shown in fig. 2, p. 132.)

The coloured drawing of the spectrum, Plate II., to which I have already alluded, was made when the spectrum was obtained in this manner. A single prism gives a very pretty anomalous spectrum, but the magnificent effect produced by the battery makes the slight amount of extra trouble well worth while. If the electric arc is employed as the source of light, extreme violet will be found to occupy the position of the undeviated image of the slit. Then comes the blue, sometimes in contact with the violet and sometimes slightly separated by a fine dark line, owing to the fact that the violet light comes from the fluted carbon band of the arc, which is separated from the blue by a comparatively dark region. Then comes a wide gap corresponding to light absorbed by the sodium vapour in the blue-green region (the channelled spectrum), and above this a beautiful flare of colour ranging from blue-green through grass-green to yellow. The red and orange portion of the spectrum is on the other side of or below the undeviated image, forming another brilliant flare of colour. It is separated from the violet by a wide dark band, due to the absorption in the vicinity of the D lines. If the density of the vapour is increased by heating the tube to a higher temperature, the red flare extends lower down, grows fainter, and finally fades away owing to the presence of the fluted absorption-bands in the red. The green and blue persist, however, becoming more widely separated, but finally the green disappears almost entirely. It is best to arrange the gas-cock so that the height of the flames can be controlled without leaving the eyepiece, for it is surprising how slight a change is necessary to completely alter the general appearance of the spectrum.

The glass tube should not be allowed to cool until the experiment is at an end, otherwise it will immediately fly to pieces as soon as the flame is again applied to it.

While glass answers very well if the experiment is to be of short duration, sheet iron is much more satisfactory. Suitable tubes can be made by any tinsmith. They should be made of thin sheet-iron, and the turned-over seam hammered until a tight joint is formed. These tubes can be heated and cooled any number of times, and can be kept in operation for an hour or two, at the end of which time the sodium is generally used up, a moss-like deposit of oxide gradually filling up the tube. These tubes can be used over and over again without deterioration, and are most satisfactory in every respect. Their only fault lies in their conductivity, the sealing-wax softening and the glass plates falling off, but this can be prevented by wrapping a strip of cloth around each end, and wetting it from time to time. I have had one tube made with water-jackets at each end, but it seems to have no especial advantage, and is more complicated. Porcelain tubes are quite satisfactory, but I prefer the iron on the whole.

By employing a tube of about 5 cm. diameter I have succeeded in projecting the anomalous spectrum, but the appearance is so very inferior to that of the phenomenon when seen subjectively, that I should never think of employing the method.

No trace of any influence of the fluted absorption-bands on the dispersion has been detected.

It is a matter of some interest to know what becomes of the energy absorbed by a sodium flame, or by non-luminous sodium vapour. Most of it is undoubtedly spent in raising the temperature of the vapour; this rise of temperature I have shown experimentally by making the dispersion-tube the chamber of a radiophone, and illuminating it with intermittent light. It was found that the radiophone sang loudly when illuminated by an intense beam of light which contained waves of the frequency of the sodium radiations, but became almost silent when these frequencies were removed.

Prismatic analysis was found to weaken the light too much: consequently a very thin film of cyanine was used which was quite opaque to the region of the D lines while transparent to the rest of the spectrum. Sunlight concentrated by a large mirror was employed for the illumination.

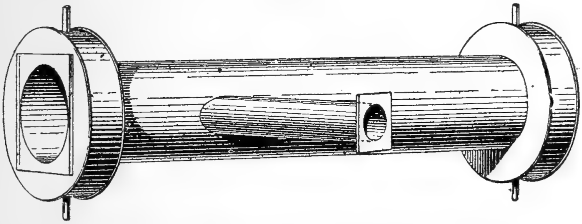
It has always appeared to me thinkable that the absorbing sodium ions, when in resonant vibration, might emit a certain amount of light laterally. Kayser mentions in his new work on spectroscopy an experiment made by Miller, in an attempt to detect this phenomenon if it existed. The possibility of such a lateral radiation is discussed by Drude in his *Lehrbuch der Optik* (p. 493), and is shown to depend on a small value of the constant r in the equation of motion of an ion vibrating under the influence of a periodic electric force. For sodium vapour, r is calculated to be something between 10 and 100. If it were equal to one we should expect a lateral radiation of light, which, says Drude, has never been observed.

While experimenting with the very intense beam of sunlight in the radiophone experiment, I observed what appeared to be a lateral radiation of a clear green light by the vapour.

My first thought was that it was light scattered by the small particles of oxide which are always floating about. If this were the case, the spectrum of the light should be that of white light after transmission through sodium vapour. An examination with the spectroscope, however, showed that this was not the case, for the spectrum consisted of a diffused red band, a very narrow bright band in the case where the D lines occur, and a diffused band in the green, in which flutings seemed to be present. The general appearance of the light reminds one of the fluorescence of uranium glass.

I am at the present time engaged in an investigation of the spectrum of this fluorescent light, but am not yet prepared to say whether the flutings in the green are discontinuities in the emission or due to the absorption of the non-luminous vapour surrounding the fluorescent vapour. A careful photographic study will be necessary before this can be decided, for which investigation I am employing an iron tube provided with copper jackets at each end, through which water circulates. (Fig. 7.) The light emitted laterally is observed through a second iron tube inserted at a right angle, and provided with a glass window. The sodium is placed at the junction of the tubes, which is then raised to a red heat with a blast-lamp.

Fig. 7.



So far as I am able to judge by passing the light causing the fluorescence through colour-screens, the wave-length of the emitted light corresponds very nearly to that of the exciting vibrations. Interposing a thin cyanine screen, which removes the light in the neighbourhood of the D lines, causes no apparent diminution in the intensity of the emitted light; but the spectroscope shows that the bright band at wave-length 5890 has disappeared. A film stained with aurantia, which removes everything below the "*b*" group, greatly diminishes the intensity of the fluorescent light, and changes its colour to red. A solution of cuprammonium of moderate strength removes from the light all of the wave-lengths which are capable of exciting the fluorescence. A more careful investigation will be necessary before deciding whether the wave-length of the emitted light is greater than that of the exciting vibrations. My impression is that the bright narrow band in the yellow is slightly on the red side of the D lines, but I do not feel absolutely sure of it. It will be interesting to see whether the fluorescence persists for an appreciable time after the light is cut off, which can be easily determined with an especially designed phosphoroscope, a matter which I expect to take up next.

A quantitative study of the absorption of the vapour is extremely desirable, though the experimental difficulties will be very great. As I have already said, the width of the band at the D lines is often from ten to fifteen times the distance between the lines. Julius has, however, called attention to the fact that we must be on our guard against attributing the absence of light in the spectrum to absorption, when the conditions are such that the rays may have been merely turned to one side by refraction. This he believes, if I read his paper rightly, is the true explanation of the widening of the D lines in the absorption-spectrum. To eliminate the possibility of this lateral deviation it is necessary, if we are working with a non-homogeneous medium, to arrange matters so that the rays of light are perpendicular to the surfaces of equal refractive index, instead of parallel, as is the case in the dispersion-tubes. One obvious method of getting round the difficulty would be to vaporize the sodium in a vacuum, thus doing away with the variations in the density, but this necessitates contact between the corrosive vapour and the glass. A better plan appeared to be to send the light through the stratified vapour in such a direction that no lateral deviation could result, a matter of some difficulty until the expedient occurred to me of using the surface of the molten metal as a reflector, the rays thus twice traversing the non-homogeneous medium in a direction perpendicular to the equi-indicial surfaces.

A photograph of the absorption-spectrum of the vapour in the red and yellow region, obtained with a 10-foot concave grating, is reproduced on Plate IV. fig. 12. The D lines were photographed just below the spectrum for comparison. The flutings in the red are well shown in this picture, which is but one of a large number that have been taken. Measurements of wave-length have not yet been made, as I feel confident of getting better plates in the near future. The fine dark lines can be found on both sides of the heavy band at the D lines, and I have found that they are present throughout the entire spectrum. An enlargement of a portion of fig. 12 is shown in fig. 13 (Pl. IV.) which gives a better idea of the appearance of the fluted bands.

On increasing the density of the vapour the red end of the spectrum finally disappears, and by employing a very powerful light in combination with the method above alluded to of reflecting the light from the mirror of molten sodium, I find that the spectrum consists of two exceedingly brilliant narrow green bands with a narrow dark line between them, and a band in the violet which is partially filled with flutings. This spectrum I have observed but once, and have given it no careful study as yet.

The flutings in the green-blue portion of the spectrum are similar in appearance to those in the red, but they make their appearance when the vapour is less dense, this part of the spectrum being blotted out entirely when the red bands are at their best.

I have not yet investigated the question of whether an increase in the thickness of the layer of vapour is the equivalent of increasing its density, but I feel quite sure that such is not the case. There remains too the question of pressure. All of the experiments that I have made thus far have been at atmospheric pressure, the variations in density being in all probability merely dilution of the metallic vapour with hydrogen.

Sodium vapour possesses another property which requires further investigation, namely, the power of emitting light in virtue of temperature alone. This was first observed by Evershed *. Iodine and a number of other substances behave in a similar manner, the temperature necessary to produce the luminous radiations being but little above a red heat. On repeating Evershed's experiment with iodine, my first impression was that anomalous dispersion might be at the bottom of it. The vapour was heated in a tube, and the purple luminosity or glow hung close to the inner wall, which was red hot. It seemed quite possible that the phenomenon might be a mirage effect, the rays of suitable wave-length coming from the red-hot wall moving in curved paths through the non-homogeneous vapour. To test this I devised a method by which this possibility was entirely eliminated, and found that the glow was more brilliant than ever. A spiral of platinum wire mounted in a small flask in which iodine was vaporized was raised to a bright red heat by an electric current. The convection current of iodine which rose through the spiral was luminous to a height of about 2 cm. above the spiral, waving from side to side in the form of a reddish-purple flame. Focussing the image of an electric arc on the spiral increased its luminosity ten-fold, without, however, altering the intensity of the glowing vapour. Probably the luminescence

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* Phil. Mag. [5] xxxix. p. 460 (May 1895).

of sodium could be shown in the same way, though the one experiment that I tried with it gave negative results.

I expect in the near future to investigate all of the questions which I have touched upon in the latter part of this paper, for it appears to me that, with proper coaxing, sodium vapour may furnish us with much information regarding molecular dynamics. The results obtained up to the present time are only a small beginning, but appear to indicate that in sodium vapour we have a medium which approaches more nearly to the ideal absorbing medium, with but a single natural period of vibration, than any substance heretofore investigated.

Of course I am speaking here only with reference to the natural vibration which appears to influence the dispersion. Strictly speaking there are two natural periods of course, which influence the velocity of the light in the medium, but when the medium is very dense the condition certainly approaches very nearly to that of a single period medium. As I have said before, the fluted absorption-bands are without influence on the dispersion, at least their influence is too slight to be detected by the methods that have been employed thus far.

XIV. *Note on Capillarity Constants of Crystal Faces.*
By HAROLD HILTON[*].

On page 520 of the 34th volume of the *Zeitschr. f. Kryst. u. Min.* G. Wulff gives a very interesting theorem ; he says:—" Nehmen wir an, dass wir ein Polyëder mit gegebenen Flächenrichtungen, welche durch die Normalen n_1, n_2, n_3. . . . bestimmt werden, gefunden haben, welches der Bedingung des Minimums der Oberflächenenergie bei gegebenem Volumen Genüge leistet. Es ist einleuchtend, dass alle Polyëder welche dem letzteren ähnlich sind und nur durch ihre Volumen sich von einander unterscheiden, ebenfalls dieser Bedingung Genüge leisten werden. Alle dieser Polyëder werden einen Krystall in verschiedenen Wachstumsstadien vorstellen, wobei es klar ist, dass der Wachstumsanfang mit dem letzten Centrum der Aehnlichkeit der ganzen erhaltenen Polyëderreihe zusammenfallen muss. Auf solche Weise kann man die Oberfläche eines dieser Polyëder folgendermassen ausdrucken : $p(n_1^2+n_2^2+n_3^3+\ldots)$, wo p eine Constant ist, und die Oberflächenenergie E bei den Capillaritätsconstanten k_1, k_2, k_3, . . . auf den verschiedenem Flächen

$$E=p(k_1n_1^2+k_2n_2^2+k_3n_3^2+\ldots)$$

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[*] Communicated by the Author.

sein wird. Das Volumen dieses Polyëders kann in folgender Form geschrieben werden:

$$V = q(n_1^3 + n_2^3 + n_3^3 + \ldots),$$

wobei q eine Constante ist.

"Da das Polyëder ein Minimum der Oberflächenenergie bei constanten Volumen besitzen muss, so müssen folgende Bedingungen erfüllt werden:

$$dE = 2p(k_1 n_1 dn_1 + k_2 n_2 dn_2 + k_3 n_3 dn_3 \ldots) = 0;$$
$$dV = 3q(n_1^2 dn_1 + n_2^2 dn_2 + n_3^2 dn_3 + \ldots) = 0,$$

dieses ist aber nur dann möglich, wenn

$$k_1 : k_2 : k_3 \ldots = n_1 : n_2 : n_3 : \ldots"$$

This proof, however, is faulty; for in writing down the expressions for E and V—which should be $(k_1 p_1 n_1^2 + k_2 p_2 n_2^2 + \ldots)$ and $\frac{1}{3}(p_1 n_1^3 + p_2 n_2^3 + \ldots)$ respectively, where $p_1, p_2, \ldots$ are constants: this, however, does not affect his argument,—he has assumed the ratios $n_1 : n_2 : n_3 \ldots$ constant, and therefore in the expressions for dE and dV, $dn_1 : dn_2 : dn_3 \ldots$ will be constant; but in deducing $k_1 : k_2 : k_3 \ldots = n_1 : n_2 : n_3 \ldots$ from $dE = dV = 0$ he tacitly assumes $dn_1, dn_2, dn_3 \ldots$ quite independent. His result, however, is correct, as the following proof shows:—

Suppose $s_1, s_2, s_3 \ldots$ the areas of the faces $\sigma_1, \sigma_2, \sigma_3 \ldots$ perpendicular to the normals through any point whose lengths are $n_1, n_2, n_3 \ldots$ respectively. We have then

$$V = \tfrac{1}{3}\Sigma ns \quad \text{and} \quad \therefore \quad dV = \tfrac{1}{3}\Sigma(nds + sdn).$$

Now if a polyhedron undergoes a small deformation, the normals to its faces remaining fixed in direction, we have $dV = \Sigma sdn$; for suppose the polyhedron to be immersed in a weightless incompressible fluid which is contained in a cylinder of cross-section A, closed at the top with a movable piston of weight W. Then the pressure at any point of the liquid is $\frac{W}{A}$.

If now the polyhedron undergoes a small deformation of the kind stated above, the work done by any surface lies between

$$\frac{W}{A}(s + ds)dn \text{ and } \frac{W}{A} sdn, \text{ and } \therefore = \frac{W}{A} sdn,$$

neglecting small quantities of the second order. If now the polyhedron's volume increases by a quantity dV, the piston is forced through a height $\frac{dV}{A}$; and therefore the work done by the surfaces of the polyhedron is $W\frac{dV}{A}$;

$$\text{and } \therefore\ W\frac{dV}{A}=\Sigma\frac{W}{A}sdn, \text{ and } \therefore\ dV=\Sigma sdn.$$

Hence

$$dV=\tfrac{1}{3}\{\Sigma(nds)+dV\} \text{ and } \therefore\ \Sigma nds=2dV.$$

Now we have relations between s_1, s_2, s_3 . . . due to the fact that they can form polyhedron faces whose normals have fixed directions. For take any point of reference O, and take planes P_1, P_2, P_3 . . . through O, parallel to σ_1, σ_2, σ_3. . . .

Consider any point A; let the perpendiculars from O on the surfaces of the polyhedron be n_1, n_2, n_3 . . ., and from A be $n_1{}^a$, $n_2{}^a$, $n_3{}^a$ . . .; and let the perpendiculars from A on P_1, P_2, P_3 . . . be a_1, a_2, a_3, a_4, a_5, . . . (considered positive or negative, as A and the corresponding polyhedron surface are on the same or different sides of the corresponding plane P). Then $\Sigma ns=\Sigma n^a s$; for each $=3V$; and

$$\therefore\ \Sigma(n-n^a)s=0, \text{ and } \therefore\ \Sigma as=0, \text{ (for } n_1-n_1{}^a=a_1, \text{ \&c.).}$$

Now a_1, a_2, a_3 . . . are sufficient to completely fix the position of the point A with reference to O, and there are linear relations of the type

$$a_4=p_4a_1+q_4a_2+r_4a_3; \quad a_5=p_5a_1+q_5a_2+r_5a_3; \quad \text{\&c.}$$

Take other points B, C, D, . . ., and let b, c, d, . . . correspond to a. Then we have $\Sigma as=0$, $\Sigma bs=0$, $\Sigma cs=0$, $\Sigma ds=0$, . . .; an indefinite number of relations, but only equivalent to 3 *independent* relations; for choosing λ, μ, ν so that

$$\lambda a_1+\mu b_1+\nu c_1=d_1; \quad \lambda a_2+\mu b_2+\nu c_2=d_2; \quad \lambda a_3+\mu b_3+\nu c_3=d_3;$$

we have

$$\begin{aligned} d_4=p_4d_1+q_4d_2+r_4d_3&=p_4(\lambda a_1+\mu b_1+\nu c_1)+q_4(\lambda a_2+\mu b_2+\nu c_2)\\ &+r_4(\lambda a_3+\mu b_3+\nu c_3)=\lambda(p_4a_1+q_4a_2+r_4a_3)+\mu(p_4b_1+q_4b_2+r_4b_3)\\ &+\nu(p_4c_1+q_4c_2+r_4c_3)=\lambda a_4+\mu b_4+\nu c_4; \end{aligned}$$

and similarly

$$d_5=\lambda a_5+\mu b_5+\nu c_5, \text{ \&c.};$$

and hence

$$\Sigma ds=\lambda\Sigma as+\mu\Sigma bs+\nu\Sigma cs.$$

We have then

$$\Sigma nds=2dV; \quad \Sigma kds=dE; \quad \text{and } \Sigma ads=\Sigma bds=\Sigma cds=0;$$

and for V constant and E a minimum we have

$$0=\Sigma nds=\Sigma kds=\Sigma ads=\Sigma bds=\Sigma cds;$$

where the quantities ds are connected *only* by these five equations.

Hence we have

$$\begin{aligned} n_1-\alpha a_1-\beta b_1-\gamma c_1-\rho k_1&=0,\\ n_2-\alpha a_2-\beta b_2-\gamma c_2-\rho k_2&=0,\\ n_3-\alpha a_3-\beta b_3-\gamma c_3-\rho k_3&=0,\\ \&c. \qquad \&c. \end{aligned}$$

where α, β, γ, ρ are (unknown) constants.

Now there can always be found a point O_1, whose distances from P_1, P_2, P_3 are

$$\alpha a_1+\beta b_1+\gamma c_1, \quad \alpha a_2+\beta b_2+\gamma c_2, \quad \alpha a_3+\beta b_3+\gamma c_3$$

respectively; the distance of O_1 from P_4 is

$$\begin{aligned} &p_4(\alpha a_1+\beta b_1+\gamma c_1)+q_4(\alpha a_2+\beta b_2+\gamma c_2)+r_4(\alpha a_3+\beta b_3+\gamma c_3)\\ &\quad=\alpha(p_4a_1+q_4a_2+r_4a_3)+\beta(p_4b_1+q_4b_2+r_4b_3)\\ &\qquad\qquad+\gamma(p_4c_1+q_4c_2+r_4c_3)=\alpha a_4+\beta b_4+\gamma c_4. \end{aligned}$$

Similarly its distance from P_5 is $\alpha a_5+\beta b_5+\gamma c_5$, &c.

Hence if the perpendiculars from O_1 on $\sigma_1, \sigma_2, \sigma_3, \ldots$ are $n_1', n_2', n_3' \ldots$, we have

$$n_1'=n_1-\alpha a_1-\beta b_1-\gamma c_1=\rho k_1, \quad n_2'=n_2-\alpha a_2-\beta b_2-\gamma c_2=\rho k_2,$$
$$n_3'=n_3-\alpha a_3-\beta b_3-\gamma c_3=\rho k_3, \text{ \&c. \&c.}$$

Hence there is a point O_1 for which

$$n_1' : n_2' : n_3' \ldots = k_1 : k_2 : k_3 \ldots,$$

which is Wulff's theorem.

It is not always easy to apply the above theorem to the case of an actual crystal. It seems therefore worth while to give an example which is of use in many cases.

Let three planes OBC, OCA, OAB meet along the lines OA, OB, OC; and let them be met by two planes ABC, A′B′C′, so that $OA=xa$, $OB=xb$, $OC=xc$; $OA'=ya'$, $OB'=yb'$, $OC'=yc'$, where a, b, c, a', b', c' are supposed known. Let the angles BOC, COA, AOB be α, β, γ, respectively.

Then the volume of the figure OABC is

$$\frac{x^3abc}{6}\sqrt{1+2\cos\alpha\,.\cos\beta\,.\cos\gamma-\cos^2\alpha-\cos^2\beta-\cos^2\gamma},$$

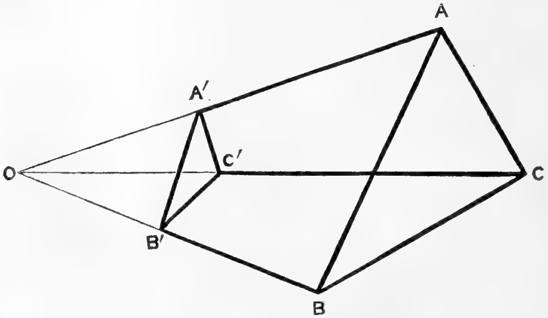
and of OA′B′C′ is

$$\frac{y^3a'b'c'}{6}\sqrt{1+2\cos\alpha\,.\cos\beta\,.\cos\gamma-\cos^2\alpha-\cos^2\beta-\cos^2\gamma};$$

the area of ABC is

$$\frac{x^2}{2}\sqrt{\Sigma(b^2c^2\sin^2\alpha - bc(b^2+c^2)\cos\alpha + 2a^2bc\cos\beta\,.\cos\gamma)};$$

the area of OBC is $\frac{x^2}{2}bc\sin\alpha$, and of OB′C′ is $\frac{y^2b'c'}{2}\sin\alpha$; and we get the areas of A′B′C′, OCA, OC′A′, OAB, OA′B′ similarly.



Suppose now that the figure ABCA′B′C′ represents a crystal which is such that its surface energy is a minimum for a given volume. Let the capillarity constants of the faces BCB′C′, CAC′A′, ABA′B′, ABC, A′B′C′ be k_1, k_2, k_3, k, and k', respectively; then the volume of the crystal is

$$\tfrac{1}{6}\sqrt{(1+2\cos\alpha\,.\cos\beta\,.\cos\gamma-\cos^2\alpha-\cos^2\beta-\cos^2\gamma)}\,(x^3abc-y^3a'b'c'),$$

and the surface energy is $P_1x^2-P_2y^2$, where

$$P_1=\tfrac{1}{2}\big(bc\,k_1\sin\alpha+ca\,k_2\sin\beta+ab\,k_3\sin\gamma$$
$$+k\sqrt{\Sigma(b^2c^2\sin^2\alpha-bc(b^2+c^2)\cos\alpha+2a^2bc\cos\beta\,.\cos\gamma)}\big),$$

$$P_2=\tfrac{1}{2}\big(b'c'k_1\sin\alpha+c'a'k_2\sin\beta+a'b'k_3\sin\gamma$$
$$-k'\sqrt{\Sigma(b'^2c'^2\sin^2\alpha-b'c'(b'^2+c'^2)\cos\alpha+2a'^2b'c'\cos\beta\,.\cos\gamma)}\big).$$

We have then at once for constant volume and surface energy a minimum, on differentiating

$$\frac{x}{y}=\frac{P_1a'b'c'}{P_2abc};\text{ with the condition }\frac{P_1{}^4a'^2b'^2c'^2}{P_2{}^3a^2b^2c^2}-P_1>0.$$

Magdalen College, Oxford.

XV. *Vapour-Pressures of Carbon Dioxide and of Ethane at Temperatures below* 0° C. *By* Professor J. P. Kuenen *and* Mr. W. G. Robson *.

ALTHOUGH from Amagat's researches the critical constants and vapour-pressures above 0° C. of carbon dioxide are accurately known, much uncertainty exists as to the pressures at temperatures below 0° C. The results obtained by Faraday, Regnault, Cailletet, and others, are not consistent; and as we wanted accurate values of the vapour-pressures of this substance—and also the vapour-pressures of ethane—in connexion with an investigation on mixtures, we resolved to undertake the determinations ourselves.

1. *The Temperatures.*—We obtained the low temperatures by using liquids boiling under reduced pressure, the most suitable liquids for our purpose being sulphur dioxide † and ammonia ‡. The liquid was collected in a Dewar vacuum-vessel of suitable shape, made by Messrs. Bender & Holbein, to our specification. Even at the lowest temperatures the liquid in this vessel boiled away quite slowly, so that as a rule there was plenty of time for taking the readings before fresh liquid had to be let into the vessel.

The vacuum vessel was closed at the top by a rubber stopper. Through the stopper passed a tube for admitting the liquid sulphur dioxide or ammonia, a tube connecting the vessel to a water-pump and open mercury-gauge, the thermometer, and the pressure-tube which contained the carbon dioxide or ethane.

In order to keep the pressure under which the liquid boiled, as read on the mercury-gauge, constant, it was found sufficient to connect the vessel with a large bottle which acted as an air-chamber, and to regulate the pressure by means of a tap and of two pinching-screws on the rubber connecting the pump with the bottle and vessel. Some gas was allowed to bubble through the liquid in order to keep it well stirred. The only source of inconstancy in the temperature was the want of purity of the boiling liquids, especially the sulphur dioxide. We had to take the precaution to read temperature and pressure as nearly as possible at the same moment. Even of the ammonia, although not submitted to a preliminary cooling, only moderate quantities were required

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* Communicated by the Authors.

† Obtained from Boake, Roberts & Co., Stratford, E.

‡ Obtained from the Scotch and Irish Oxygen Co., Glasgow.

—less than half-a-pound for a complete set of experiments lasting for two or three hours.

2 *The Pressure Apparatus.*—The substances prepared and purified by methods described in former papers [*] were inclosed in the glass tube of a Ducretet-apparatus. The part of this tube which projects outside was bent twice at right angles so as to have its top turned down. This top part passed through the stopper of the vacuum-vessel and was immersed in the low-temperature liquid.

3. *The Pressure-Gauges.*—Three gauges—(1) an open mercury-gauge reading from 1 up to 4 atmospheres; (2) a closed gas-gauge with a range of from 3 to 30 atmospheres; and (3) a closed gas-gauge reading from 25 atmospheres upwards, were used.

In a former paper [†], describing experiments in which the same gauges were used, it was stated that the two closed gauges—at that time filled with dry air—showed a difference of ·7 per cent. at 25 atmospheres. It was discovered later that some change had taken place in the high-pressure gauge, and, on taking it out of its pressure cylinder, it was found that, owing probably to the oxidation of the mercury, the volume of the air had diminished by an amount sufficient to account for the above difference. The high-pressure gauge was then filled with dry nitrogen, and its behaviour was found to be perfectly satisfactory. We would advise those who have to use a high-pressure gas-gauge to take the comparatively small trouble of filling it with atmospheric nitrogen instead of with air.

The degree of agreement between the high-pressure nitrogen-gauge and the low-pressure air-gauge may be judged from the following table:—

Low-pressure Gauge.	High-pressure Gauge.	Difference.
25·83	25·86	−·03
26·04	25·99	+·05
26·48	26·50	−·02
31·44	31·41	+·03
39·67	39·70	−·03

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[*] Phil. Mag. [5] xl. pp. 173–194; xliv. pp. 174–199.

[†] Phil. Mag. [5] xlviii. p. 180.

The differences are equally distributed between positive and negative, and are accounted for by the fact that the readings of the low-pressure gauge are somewhat inaccurate at the small volumes.

In order to deduce the pressure from a reading on a closed gas-gauge, the corrections for the deviation from Boyle's law are required. For the calculation of these corrections are available: (1) Regnault's measurements from 1 up to 25 atmospheres, and the approximate formula deduced by him from his observations; and (2) Amagat's measurements from 25 atmospheres upwards. The corrections deduced from the former are much larger than those deduced from the latter. This difference must be due to the temperature (8° C.) in Regnault's experiments having been lower than in Amagat's (20° C.).

We tried to obtain a set of consistent corrections from 1 atmosphere upwards, by reducing as well as we could Regnault's corrections to the temperature of Amagat's observations and *vice versa*. The corrections thus obtained are probably fairly accurate up to 25 atmospheres. Above this pressure, at temperatures differing from 20° C. the uncertainties become greater. It would be an extremely useful work to repeat Amagat's experiments at a lower gauge-temperature, and to extend them to pressures from 1 to 25 atmospheres. This latter part is within the scope of laboratory-research work.

In the experiments on carbon dioxide the gauges were used as above; after their completion the low-pressure gauge was filled with nitrogen, so that in the later experiments both gas-gauges contained nitrogen. But the uncertainty of the corrections remains. The gauges show complete agreement, the reading at the same pressure being 26·10 on both.

4. *The Thermometer.*—The many advantages claimed for the platinum thermometer induced us to use it in preference to a gas-thermometer or thermoelectric couple, and although we encountered many unexpected difficulties, we found it quite suitable for our purpose. It may be of interest to give here in some detail an account of our experiences with this instrument. In the first place, we are inclined to doubt the statement that when measurements are made with an ordinary post-office bridge results may be obtained which are consistent to a few hundredths of a degree up to 500° C.* Using resistance-coils and wire-bridge in the way described by Callendar†, and allowing for coil errors, wire calibration,

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* Preston, 'Heat,' 1894, p. 148.

† Phil. Mag. [5] xxxii. p. 107.

and the temperature change of coils and wire, we found that at the steam-point differences of as much as 0°·06 were possible.

(*a*) *Resistance Measurements.*—The coils and bridge-wire were of german-silver, the smallest box-coil being 0·1 ohm, and the wire having a resistance of 0·22 ohm per metre. The temperature-coefficient of the box-coils was about ·00040, and all readings were reduced to a standard temperature of 15° C. The coils were immersed in paraffin-oil, and their temperature read on a mercury thermometer divided to 0°·05. It is to be noticed that german-silver is on account of its high temperature-coefficient very unsuitable for measurements of this kind. With our apparatus an error of 0°·2 in the box temperature—an error which is quite possible when the room temperature is changing rapidly—corresponded at the steam-point to a resistance error of ·00032 ohm, *i. e.* the platinum temperature would be wrong by 0°·032. At the ice-point the same error in box temperature would cause an error in the platinum temperature of 0°·023. The expensive platinum-silver alloy would only reduce these errors to three-fifths of their values; and we would recommend, therefore, that the coils of boxes meant for use with the platinum thermometer be made of some alloy, such as manganin, whose temperature-coefficient is only ·00002, so that the above errors would be reduced to one-twentieth of their values.

(*b*) *Stem Immersion.*—The platinum thermometer is said to be free from errors due to incomplete immersion of the stem, being on this account superior to the mercury thermometer *. Our experience with regard to this is entirely in accord with that of Chree †. Defining, as he does, the "bulb" to be that part of the thermometer extending to the first mica disk—in our thermometer this is 1 cm. above the top of the spiral—we find the following errors due to insufficient immersion :—

Immersion.	Ice-point.	Steam-point.
Bulb + 1¼ cms.	+0°·11	−0°·26
„ + 3½ „	·05	—
„ + 7 „	·03	—
„ + 12 „	·00	·00

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* Griffiths, 'Nature,' vol. liii. p. 45.

† Proceedings Roy. Soc. 435. p. 28.

In our low-temperature determinations never less than the "bulb" + $1\frac{1}{2}$ cms. was immersed in the liquid sulphur dioxide or ammonia, and the bulb + 7 cms. was inside the Dewar vessel, so that $5\frac{1}{2}$ cms. of the stem was surrounded by the vapour of the boiling liquid. This insufficient immersion would account for small irregularities—up to $0^\circ{\cdot}05$—in our readings.

(*c*) *Fixed Points.*—The resistance R_0 of the platinum spiral at the ice point was found by using a bath of partly frozen distilled-water surrounded by an air-jacket on the outside of which was a freezing-mixture kept at -3° or -4°. The following are some of the readings:—

	Values of R_0.
May 10	2·8348
June 6	8
„ 10	7
„ „	8
Mean	2·8348 ohms.

The steam-point readings were:—

	Values of R_{100}.
May 10	3·8379
„	78
„	81
„	77
May 11	76
„	75
June 4	76
„ 6	76
„ 10	77
Mean	3·8377 ohms.

The platinum temperatures were calculated from the formula

$$pt = \frac{R - R_0}{R_{100} - R_0} \times 100$$

and reduced to the air-scale by means of the difference formula

$$T - pt = 1{\cdot}57\left(\frac{T}{100} - 1\right)\frac{T}{100}.$$

5. *The Results.*—Table I. contains the vapour-pressures of carbon dioxide as determined by experiment.

TABLE I.

Vapour-Pressures of Carbon Dioxide.

Temperature (Cent.).	Pressure (Atmos.).
0	34·34
− 9·95	26·04 (Low Gauge).
	25·99 (High Gauge).
−15·27	22·23
−21·67	18·32
−27·01	15·42
−32·88	12·60
−33·08	12·59
−41·02	9·46
−47·58	7·34
−54·52 *	5·45
−54·68	5·45
−56·18	5·10
−56·24	5·10 (Triple Point).
−57·80	4·743
−57·87	4·746
−61·41	4·018
−61·60	3·991
−62·39	3·823
−62·47	3·809
−63·13	3·685

Solid Carbon Dioxide.

Temperature.	Pressure.
−57·86	4·580
−61·80	3·532
−63·36	3·129
−71·41	1·683
−76·90	1·072
−78·32	1·000

These results were plotted, and Table II. gives the vapour-pressure for every five degrees as read from the curve. The pressure at 0° C. agrees with that found by Amagat † (34·3 and 34·4 atmos.) ‡.

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† *Ann. de Chim. et Phys.* [6] xxix. pp. 118, 136.

‡ Holborn (*Ann. der Physik,* vi. p. 245) finds −78·34 for the boiling-point of carbon dioxide under normal pressure.

TABLE II.

Vapour-Pressures of Carbon Dioxide (read from curve).

Temperature.	Pressure.			
	Atmospheres.		Metres of Mercury.	
0	34·3		26·1	
− 5	30·0		22·8	
−10	26·0		19·8	
−15	22·4		17·0	
−20	19·3		14·7	
−25	16·5		12·5	
−30	14·0		10·6	
−35	11·8		8·97	
−40	9·82		7·46	
−45	8·12		6·17	
−50	6·60	Solid.	5·02	Solid.
−55	5·35		4·07	
−60	4·30	3·97	3·27	3·02
−65		2·77		2·11
−70		1·88		1·43
−75		1·28		·97
−78·32		1·00		·76

On close inspection Table I. shows certain small irregularities; the reading marked by an asterisk does not get into the curve and must be affected by a small error of observation. The remaining irregularities are very small, and may be accounted for by the various uncertainties of the method: they do not, however, affect the first decimal of the values of the vapour-pressures. In addition to the accidental errors there is the systematic error due to the impurity of the gas. In the case of carbon dioxide this impurity was very small: after a change of temperature the pressure adjusted itself almost immediately to its final value, and a change of volume had a very slight effect on the pressure. At 0° C. the pressure at a large volume with only a little liquid carbon dioxide present was 34·32; and at a small volume, with most of the gas liquefied, the pressure was 34·38 atmos

Because of the scarcity of diagrams of the kind, we think it may be of interest to publish the triple-point diagram for carbon dioxide. It will be seen that the liquid may be considerably undercooled, the lowest liquid-vapour point being seven degrees below the triple-point. The triple-point was determined directly by changing the temperature until no gradual change in the amount of solid was observable, and, indirectly, by finding the point of intersection of the solid-vapour and liquid-vapour curves. The two points thus found are identical, the triple-point temperature being therefore −56°·2 C., and the pressure 5·10 atmos. Faraday found −57° C. and 5⅓ atmos., and Cailletet −54°·6 C. and 5·3 atmos. We have no direct experimental data for the liquid-solid curve, except that it is a curve rising towards the right, solid carbon dioxide being denser than liquid.

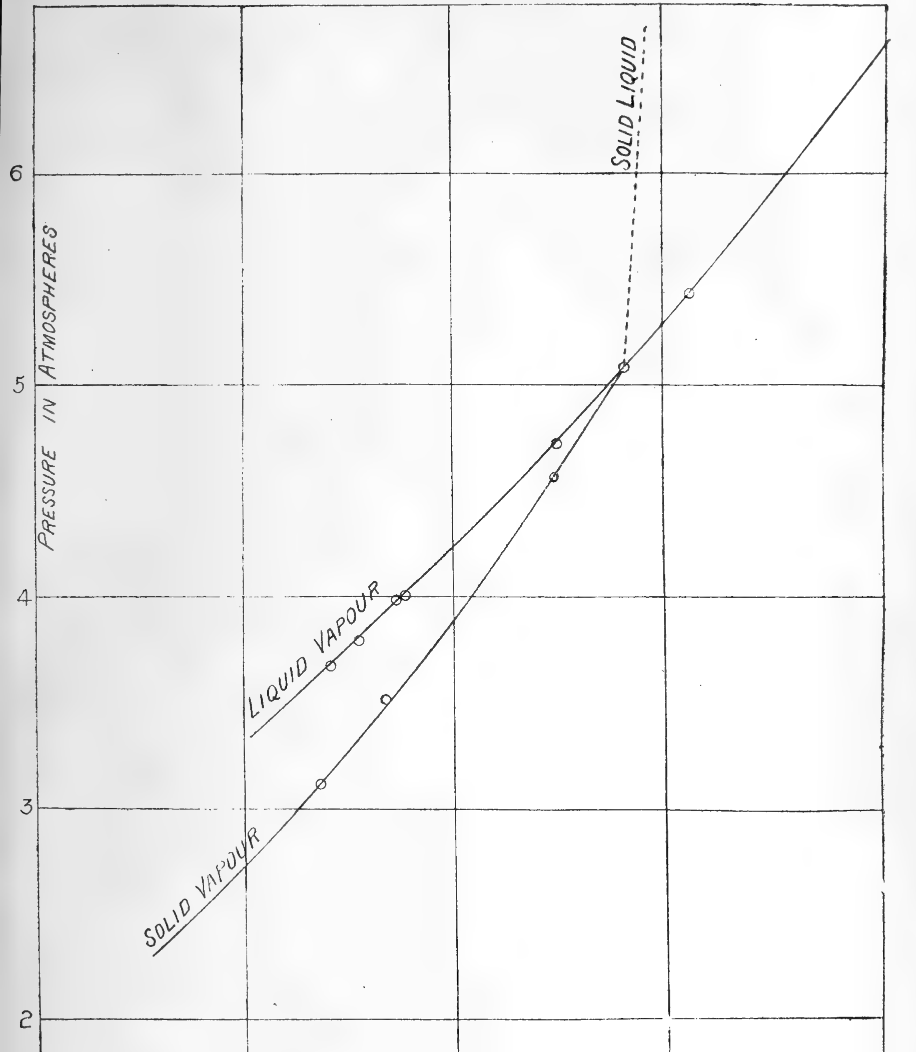
In Tables III. and IV. we give our results for ethane. We

Table III.—Vapour-Pressures of Ethane.

Temperature.	Pressure (Atmos.).
0	23·61 23·97
− 9·74	18·40 18·57
−15·06	16·08
−21·11	13·64
−26·24	11·80
−32·88	9·78
−33·24	9·53
−37·93	8·22
−43·45	6·89
−50·87	5·38
−57·84	4·21
−72·21	2·33
−78·46	1·83

Table IV.—Vapour-Pressures of Ethane (read from curve).

Temperature.	Pressure.	
	Atmospheres.	Metres of Mercury.
0	23·6	17·9
− 5	20·7	15·7
−10	18·2	13·8
−15	16·0	12·2
−20	14·0^5	10·7
−25	12·2	9·27
−30	10·5	7·98
−35	9·05	6·88
−40	7·80	5·93
−45	6·67	5·07
−50	5·60	4·26
−55	4·61	3·50
−60	3·79	2·88
−65	3·10	2·36
−70	2·58	1·96
−75	2·10	1·60


SOLID LIQUID
LIQUID VAPOUR
SOLID VAPOUR
PRESSURE IN ATMOSPHERES
6
5
4
3
2
TEMPERATURE
-70°
-65°
-60°
-55°
-50°C

could not obtain this substance, even after repeated attempts at purification, in the same condition of purity as the carbon dioxide. The observations showed this distinctly. The change of pressure with change of volume was for one sample from 23·61 to 23·97 atmos. at 0° C., and for a second sample 18·40 to 18·57 at −9°·7 C. On being heated or cooled the substance took a considerable time to attain a constant pressure; this very slow process was hastened by introducing a stirrer which was moved by a large electromagnet surrounding the Dewar vessel. We extended our observations above 0° C. up to the critical point for the sake of comparison with former results for ethane*. The agreement was satisfactory, especially if the small error of our earlier air-gauge be taken into account. Nevertheless we cannot consider the vapour-pressures as accurate within one per cent.

It had struck us that ethane, whose pressures at ordinary temperatures are much lower than those of carbon dioxide, has higher pressures at very low temperatures. It appears, however, from our results that this is entirely due to the solidification of carbon dioxide, the liquid curve for this substance, as far as we could follow it, remaining above the ethane curve. The two liquid curves do not intersect, and so far there is no evidence of association of liquid carbon dioxide. The ethane curve intersects the solid carbon-dioxide curve at −63° C.

University College, Dundee.

XVI. *A Voltameter for Small Currents.*

By R. A. LEHFELDT†.

IN the course of some experiments with mercury voltameters I had a conversation with Mr. E. C. C. Baly, during which the suggestion of a remarkably simple form of instrument arose. This I have realized in practice, as follows:—

A glass tube of from 0·5 to 1·5 mm. bore is provided with a pair of platinum electrodes, sealed in near the ends; the ends are then drawn out to a small diameter. The tube so prepared is filled—by means of the water-pump—with mercury throughout, except for one drop of mercurous-nitrate solution in the middle of its length; the ends of the tube are then sealed off. The tube is placed vertically, and the upper portion of mercury made the anode, the lower the cathode. The drop of solution creeps up the tube at a rate proportional to the current flowing, the movement being

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* Phil. Mag. [5] xliv. p. 174.

† Communicated by the Physical Society: read Nov. 8, 1901.

measured by any convenient scale, preferably one etched on the glass, with or without a micrometer.

For the successful construction of the apparatus the following points should be attended to :—The parts of the tube where the electrodes are sealed-in should should not be widened, else difficulty in filling is likely to ensue. The filling can be done by placing the tube—aslant—with one point dipping under mercury, in a dish. The mercury is covered by solution : when enough has been drawn in by the pump to form one electrode, the point is lifted momentarily so as to allow solution to flow in instead; then depressed, when more mercury enters, to form the other electrode. The filling is regulated most conveniently by a tap sealed on to the upper end of the tube, which is subsequently detached. The solution is made by weighing out about 26·3 milligrams of mercurous nitrate per c.c. of water (*i. e.* decinormal), and adding enough nitric acid to dissolve the basic salt that forms. The current-density may be anything up to 10 or 15 milliamperes per sq. cm.; more than that causes polarization. Hence a tube of 2 sq. mm. cross-section (ordinary Sprengel-pump tubing) will take 0·3 milliampere.

The electrochemical equivalent of mercurous mercury is 7·470 grams per ampere per hour, or 0·5509 c.c. Hence with a current-density of 0·010 ampere/sq. cm., the rate of movement would be 0·005509 cm. per hour, or about 1 mm. a day. The meter is therefore eminently adapted to measure long-continued currents of very small magnitude. Moreover, with proper precautions it may be shunted, as, apart from temperature changes, the resistance keeps steady.

The following is a record of such an instrument :—

Cross-section of tube 0·624 mm.: length of thread of solution about 6 mm. Micrometer used, 1 scale-division = 0·0785 mm.

	Time.		Micrometer.	
	h	m		
Oct. 18.	11	10	40·0	Started current of 1/12000 amp.
	12	15	41·0	
	13	15	42·0	Stopped.
Oct. 21.	Reset micrometer.			
	11	50	40·0	Started current of 1/12000 amp.
	15	40	43·7	
Oct. 22.	9	50	60·5	

Calculated rate of movement 1 micrometer-division in 1·067 hours; observed rate 1·073. The small discrepancy is no doubt due to errors of experiment, and could be diminished by greater care in calibration, measurement of current, and of distance moved.

XVII. *Notices respecting New Books.*

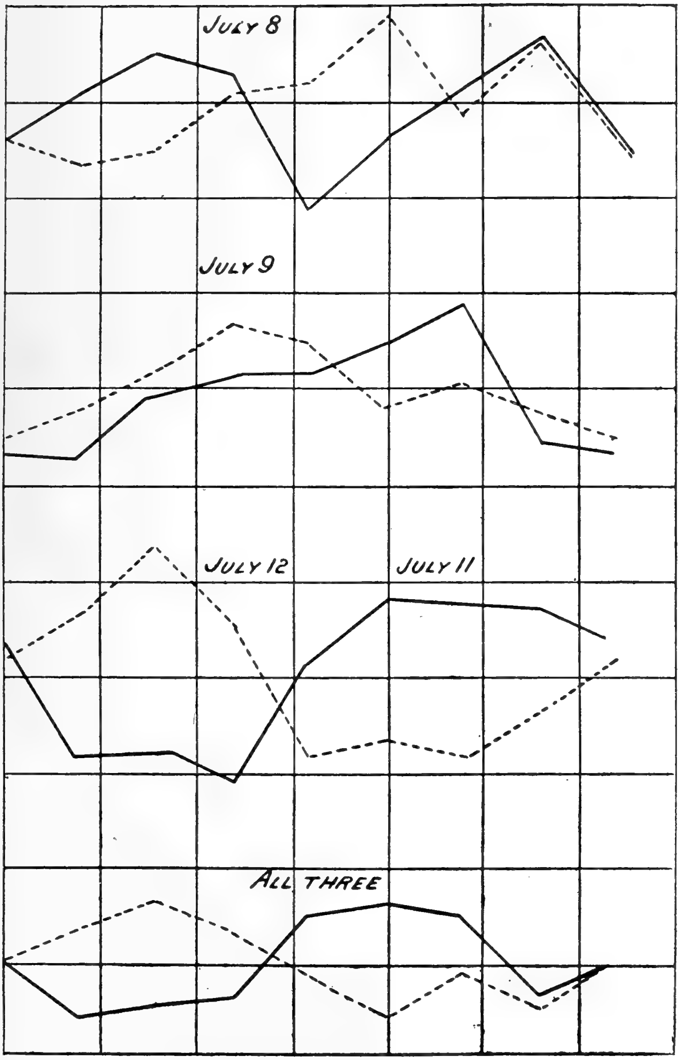
LEHRBUCH DER PHOTOCHROMIE (*Photographie der natürlichen Farben*). *Von* WILHELM ZENKER. *Neu herausgegeben von* Prof. Dr. B. SCHWALBE. Mit einem Bildniss des Verfassers. Braunschweig: F. Vieweg und Sohn, 1900. Pp. xiv+157.

IN 1810, Seebeck made the discovery that a thin layer of silver chloride when exposed to the light of the spectrum acquired colours which to some extent resembled those of the spectrum itself. This highly important and interesting discovery, published in Goethe's *Farbenlehre*, seems to have attracted very little attention at the time, and was consigned to oblivion, until John Herschel re-discovered it in 1841. Herschel carried out a number of experiments on the behaviour of silver chloride when under the influence of the light of a spectrum, and his work was continued by Edmond Becquerel, who ultimately succeeded in producing photographs in colours. Becquerel's work did not meet with the recognition which it merited, and many of his results appear to have been regarded with suspicion, if not incredulity. In Germany, however, there was at least one man who was able to appreciate to the full the practical importance of Becquerel's researches. This was Wilhelm Zenker, who in 1868 published a work on colour-photography of which the volume under review is a reprint. Zenker repeated most of Becquerel's experiments, and a photograph in colours obtained by him formed a frontispiece to his work. The present volume contains a short biographical notice of Zenker by Dr. Krech, a reprint of the 1868 edition of Zenker's *Photochromie*, and an appendix on recent developments by E. Tonn. The publishers have been unable to reproduce the original frontispiece, as in the copy of the work at their disposal the colours were so faded as to be unrecognizable.

To all interested in the subject of colour-photography, the book may be recommended as a simply and clearly written account of the consecutive steps by which the art has advanced to its present state of development.

La Série de Taylor et son prolongement analytique. Par JACQUES HADAMARD. Paris: G. Carré et C. Naud, 1901. ("Scientia" Series, No. 12.) Pp. viii + 102.

THE study of functions of a complex variable has given rise to a new and important branch of mathematics, which has been developing with phenomenal rapidity, and has already furnished the key to many interesting problems in physics. The expansion of such functions by Taylor's theorem, and the discussion of the regions within which such an expansion is valid, are problems of considerable complexity, and involve the consideration of many nice points. The little book before us is a concise account of the most recent work on this subject, and should prove extremely useful to mathematicians interested in the theory of functions. The value of the book is enhanced by the copious bibliography compiled by the author for the use of readers wishing to refer to original sources of information.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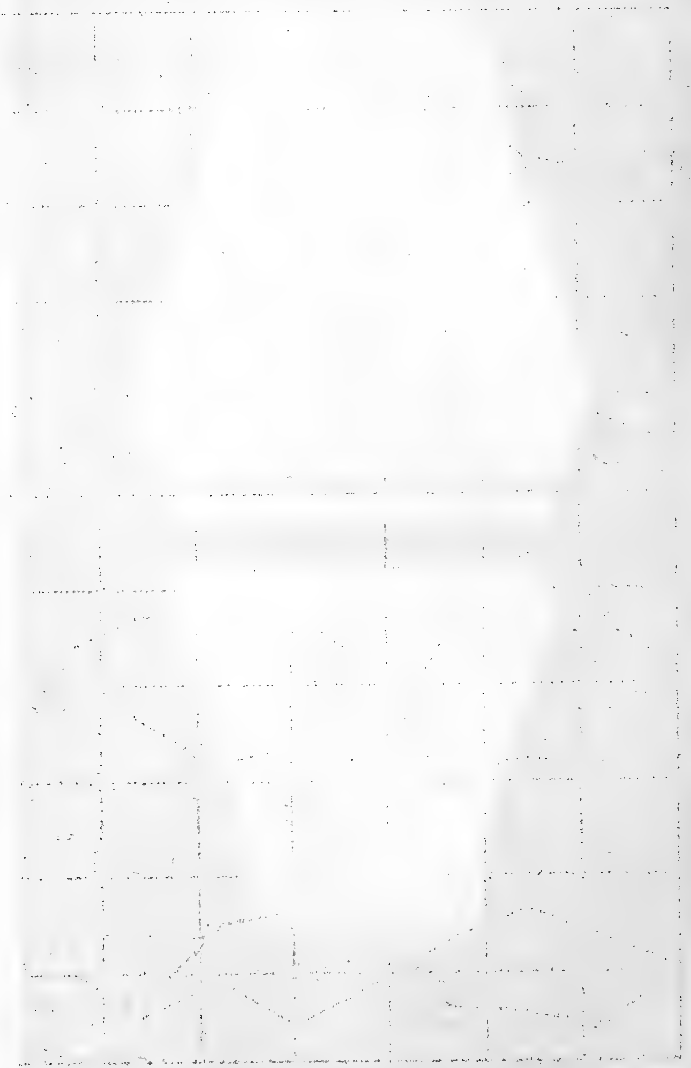
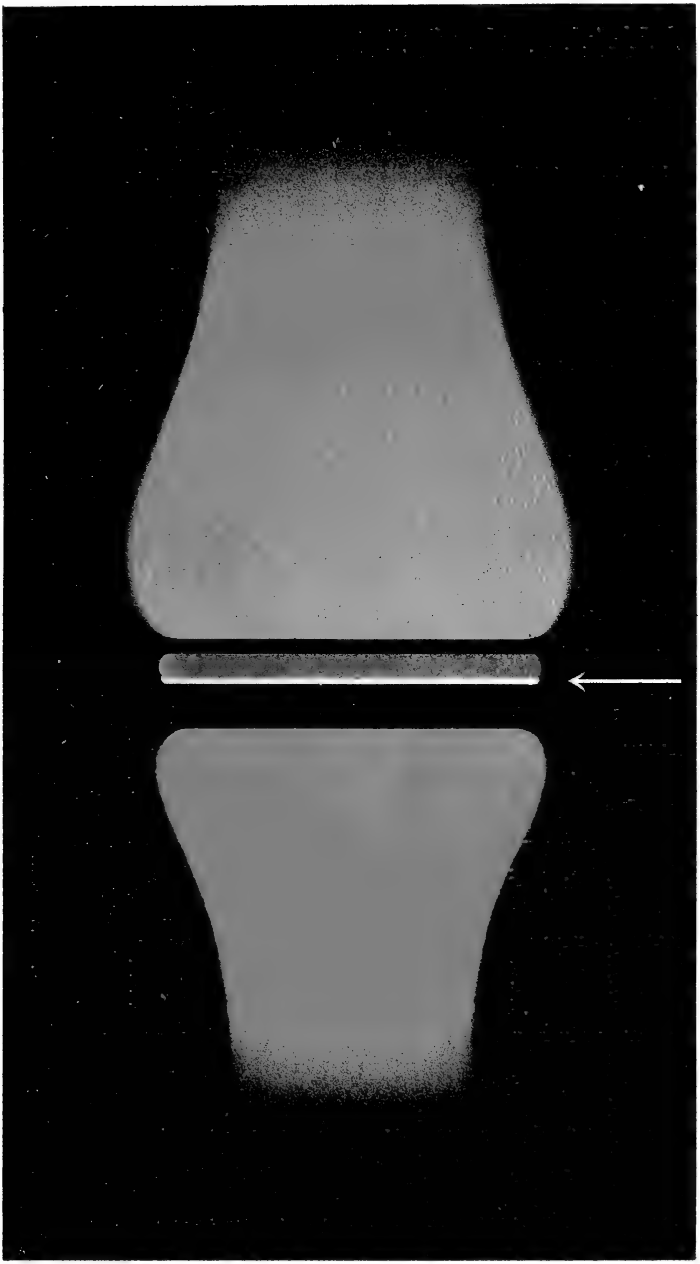




Position of
undeviated
image of slit.

West, Newman chromo.

Anomalous Spectrum produced by Prism of Sodium Vapour.

Fig. 8.

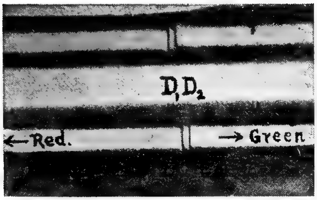


Fig. 10.

Fig. 9.

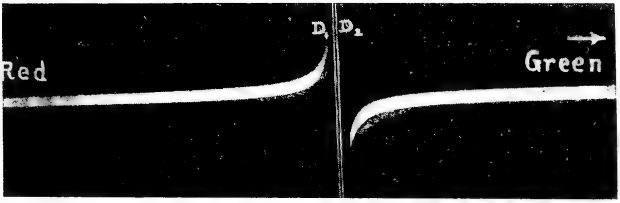


Fig. 11.

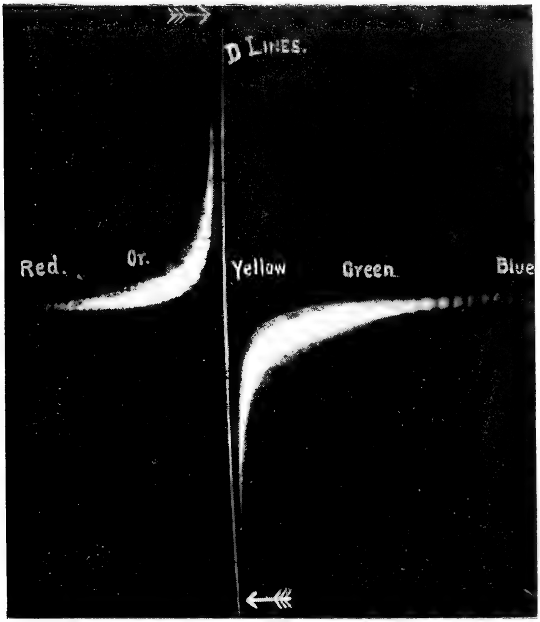


FIG. 12.

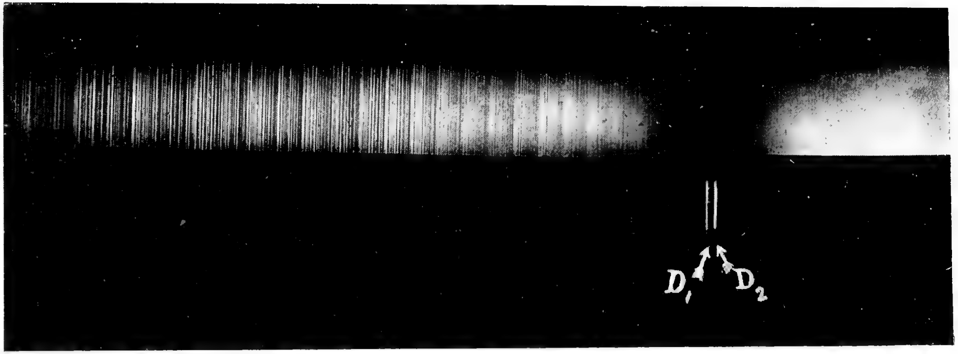
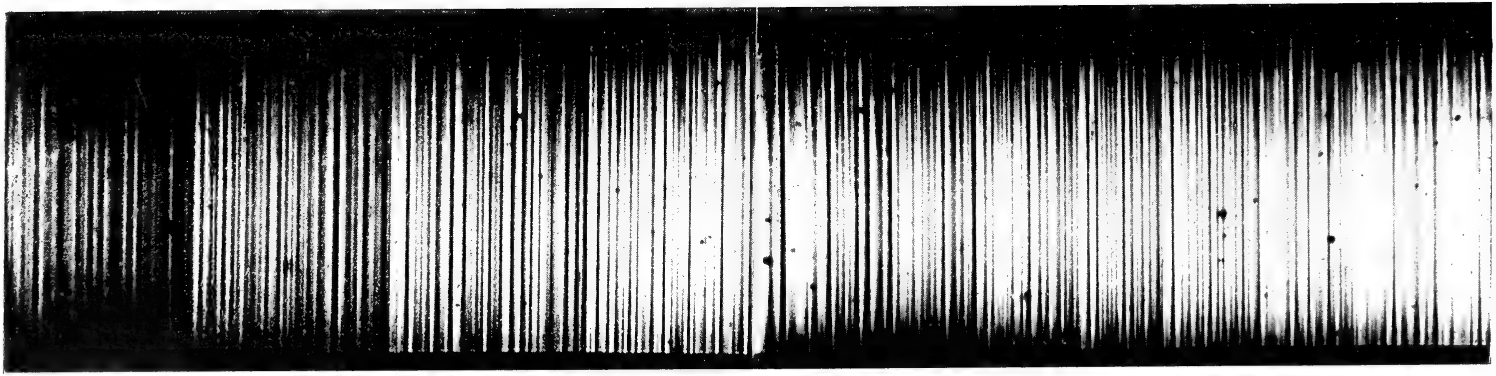


FIG. 13.



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LONDON, EDINBURGH, AND DUBLIN
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AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

FEBRUARY 1902.

XVIII. *Ionization, Ionic Velocities, and Atomic Sizes.*
By WILLIAM SUTHERLAND*.

IT is well-known that in the theory of ions as given to us by Faraday, van 't Hoff, Arrhenius, and Ostwald, there occur some gaps, which are very surprising if we consider the wonderful completeness of the theory apart from these, and also the advanced state of the kinetic theory. The gaps are:—(1) Ignorance of the cause of ionization. (2) Ignorance of the reason why Ostwald's law for the ionization of weak acids and bases does not apply to ordinary binary electrolytes. (3) Ignorance of the connexion between the ionic velocities and the other physical properties of the atoms.

The present paper is a contribution towards bridging these gaps, and consists of three parts, namely:—

1. The cause of ionization.
2. The laws of ionization in binary electrolytes.
3. Deduction of ionic velocities from the other physical properties of the atoms, and their use as a definite means of calculating the absolute sizes of the atoms.

1. *The Cause of Ionization.*

To J. J. Thomson and Nernst we owe the sagacious suggestion that ionization of a solute is caused by the high dielectric capacity of a solvent, whereby the electric force between the oppositely charged atoms or radicals is sufficiently

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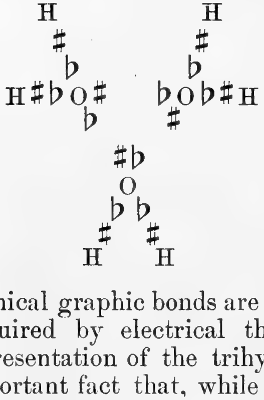
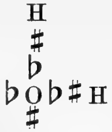
* Communicated by the Author.

reduced. Others have advanced the hypothesis that association of the molecules of the solvent is somehow the apparent cause of the dissociation of the solute molecule into ions. I think that both these suggestions can be combined in the proposition, that exceptionally high dielectric capacity, high power of ionization, and markĕd tendency to molecular association, are three effects traceable to the same cause, which I propose to call the latent or potential valency of certain atoms. Let us consider the case of water as a solvent. In a paper on the Molecular Constitution of Water (Phil. Mag. [5] vol. l. Nov. 1900), I have shown that while in steam we have H_2O, in ice we have $(H_2O)_3$, and in water a mixture of $(H_2O)_3$ and $(H_2O)_2$, varying with temperature. Now the dielectric capacity of steam is not so abnormal as that of water, being given by Lebedew (Wied. *Ann.* xliv.) as of the order 1·001 under ideal standard gas conditions, at which the index of refraction n is 1·00025. Maxwell's law $K=n^2$, or $K-1=n^2-1$, is therefore much more nearly complied with by steam than by ice and water in which K is of the order 80, while n^2 is only of the order 2. Obviously it is necessary for us to study the formula for ice or trihydrol in relation to the electrons which hold together the three H_2O constituents of its molecule, and similarly with $(H_2O)_2$. As it would be convenient to have symbols different from the operational + and − to distinguish the two sorts of electrons, I propose to use the musical types ♯ for the + electron and ♭ for the − electron, and ♮ for the neutron, which consists of the ♯ and ♭ united to form that electric doublet which gives to the æther its electric and magnetic properties. Trihydrol $(H_2O)_3$ would be represented graphically thus

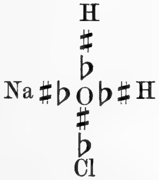


The usual chemical graphic bonds are replaced by the pairs of electrons required by electrical theory. This strictly symmetrical representation of the trihydrol molecule brings out the very important fact that, while oxygen appears as a tetrad, it is very different from the ordinary chemical tetrad, since the four electrons attached to the oxygen atom are not of the same sort, there are three $\flat$ electrons to one $\sharp$. If $(H_2O)_3$ gets dissociated it will yield three parts



$$\begin{array}{ccc} & H & \\ & \sharp & \\ & \flat & \\ \flat & O & \flat\sharp H \\ & \sharp & \end{array}$$

which could be more briefly represented by the formula $\sharp O\flat_3(\sharp H)_2$. From the graphic formula we see that in this one $\flat$ and the $\sharp$ of the oxygen atom are not in positions of equilibrium, and will tend to rush together to form a neutron $\natural$, in which the $\flat$ and $\sharp$ are much closer together than in their initial position when the dissociation of the $(H_2O)_3$ first occurred. According to this conception H_2O in steam would be represented by the formula $\natural\, O\flat_2(\sharp H)_2$. But the change from $\sharp\, O\flat_3(\sharp H)_2$ to $\natural\, O\flat_2(\sharp H)_2$ would occupy some time, during which the $\flat$ and the $\sharp$, which ultimately coalesce to form $\natural$, would be at a certain average distance apart, and would have a certain average electric moment obtained by multiplying the amount of either of the electrons by the average distance between them. This electric moment being much greater than that of the neutron confers on water its high dielectric capacity, though the special neutron in the steam molecule probably has a greater moment than that in æther. The field of electric force between $\flat$ and $\sharp$ at their average distance apart is the direct cause of electrolytic dissociation produced by water. With a solute like $\sharp NaCl\flat$ we may expect to have



$$\begin{array}{ccc} & H & \\ & \sharp & \\ & \flat & \\ Na\sharp\flat & O & \flat\sharp H \\ & \sharp & \\ & \flat & \\ & Cl & \end{array}$$

as an intermediate configuration from which the $Na\sharp$ and $Cl\flat$ are broken off, when too far apart to recombine.

We have now seen how high dielectric capacity, molecular association, and ionization in water are all consequences of the latent or potential valency of oxygen. But as this valency has the peculiar character of being associated with opposite electric charges in the atom, and gives us in $\sharp O\flat_3(\sharp H)_2$ a radical with electric charges, yet not an ion, because its charges are equal and opposite. I propose to call such a radical or atom a stion, as in static electricity it bears the same important relation to dielectric capacity as the ion to conductivity in current electricity. The neutron and the stion differ as do the electron and the ion; the neutron and electron are the molecule and atom of electricity, the stion and the ion are atoms of matter with associated electrons.

According to this line of thought those substances which fail to conform to the law $K=n^2$ fail, when K is measured electrostatically, because they contain stions, which are doublets of exceptionally large electric moment.

We spoke above of the time taken by the $\sharp$ and $\flat$ to rush together to form $\natural$, and that they do so ultimately rush together is proved by the not very abnormal dielectric capacity of steam (Lebedew, *loc. cit.*), but before they settle down to the stationary state which characterizes them in the molecule of steam, they will revolve in their relative orbits, dissipating their energy till it falls to its stationary value. But a $\sharp O\flat_3(\sharp H)_2$ group broken off from $\{\sharp O\flat_3(\sharp H_2)\}_3$ in water can enjoy only a certain average time of independent existence. Again, given a number of stions with the electric axes of their doublets uniformly distributed as to direction, and a field of electric force suddenly created where they are, it will take a certain time for the axes to adjust themselves to the directions giving maximum dielectric capacity to the region. If the field of electric force is an alternating one, and if its period of alternation is only a fraction of the time required to produce maximum dielectric capacity, we see dielectric capacity must appear to be a function of the period of alternation, with limiting values $K=2$ and $K=80$. With an order of frequency about 10^8 per second Perot (*Compt. Rend.* 1894) found $K=2{\cdot}04$, while with 10^6 alternations per second, J. Hopkinson and Wilson (Proc. Roy. Soc. lx.) found K to be less than 3. It would appear then that the time taken for the stion axes to be maximally directed in a field of electric force is of the order 10^{-6} second. This time would be an interesting subject of calculation if we knew more of the relations between electrons and atoms.

2. *The Laws of Binary Electrolytes.*

Let i be the degree of ionization of an electrolytic solution containing m gramme-equivalents of solute per litre of solution, or $1/m\ (=v)$ litres of solution per gramme-equivalent of solute, then Ostwald's well-known reasoning gives the rate of dissociation proportional to $m(1-i)$, say equal to $cm(1-i)$, and that of recombination equal to $c'm^2i^2$ with the condition for equilibrium

$$cm(1-i)=c'm^2i^2$$

$$\therefore \quad \frac{i^2}{1-i}=\frac{c}{c'}\cdot\frac{1}{m}=kv. \quad \ldots\ldots \quad (1)$$

This is the simple mass-action formula which Ostwald found to be true for over 200 weak organic acids dissolved in water, and to fail completely for ordinary binary electrolytes. It is to be noticed that the solvent is supposed to exert no effect except in so far as it influences the values of c and c'. These being determined, the processes of dissociation and combination are supposed to go on as if the solute were in a vacuum.

Recently in volume xvii. of *Zeit. f. phys. Chem.* Rudolphi has given an empirical formula for the ionization of an ordinary binary electrolyte, namely,

$$i^2/(1-i)=kv^{\frac{1}{2}}, \quad \ldots\ldots \quad (2)$$

which in vol. xviii. van't Hoff has proposed to replace by

$$i^3/(1-i)^2=kv, \quad \ldots\ldots \quad (3)$$

as perhaps a better and more easily interpreted form. In vol. xix. Storch uses $i^p/(1-i)=kv^{p-1}$, where p like k is a parameter varying from one electrolyte to another. Kohlrausch (Beibl. *Ann. d. Ph.* xxv. p. 35) finds

$$1-1/i^p=kv^{-\frac{1}{2}} \quad \ldots\ldots \quad (4)$$

to be an accurate empirical formula for dilute solutions from $m=\cdot 1$ to $m=\cdot 0001$ ($v=10$ to $v=10^4$).

All these formulæ are empirical and applicable only to dilute solutions. In them i is taken to be given by the ratio of the specific molecular conductivity of the solution at strength m to that at infinitely small strength, that is at infinite dilution. Now, no stipulation is made here about the viscosity of solutions, because the formulæ apply only to dilutions where the difference of the viscosity of the solution from that of pure water can be merged in the experimental errors in the measurement of molecular conductivity. But to satisfactorily investigate dissociation in solutions we must be free to push the investigation to far higher strengths than hitherto, in fact right up to saturation, and therefore we must take account of the effect of viscosity on conductivity. It has been argued that because an electrolyte dissolved in a stiff jelly has nearly the same conductivity as a pure aqueous solution of the same strength, viscosity can be of little importance; but it is obvious that the correct inference is that in the jelly the water is so immersed in the gelatine that it exhibits great viscosity and quasi-solidity as regards bulk movements, while to ions moving through its meshes quite freely the confined water exhibits almost only the viscosity of ordinary water. Many experiments could be mentioned to illustrate how the resistance of electrolytes is proportional to viscosity, other things being equal, but a reference to those of Manoulier (*Compt. Rend.* cxxx. p. 773) with mixtures of glycerine and water and $CuSO_4$ will suffice, as the importance of viscosity appears at once in the following elementary theoretical treatment.

Let us regard an ion as a sphere of radius a_1 moving with ionic velocity V through a solution of viscosity η, then for the frictional resistance R experienced by the sphere, if there is no slipping at its surface, we have by Stokes's formula

$$R = 6\pi V\eta a_1. \qquad \tag{5}$$

Now if the fall of electric potential is at the rate dE/dx in the direction of motion of the ♯ ion when a current is traversing the solution, then the force on each such ion is $e\,dE/dx$ where e is its charge. For steady ionic velocity V the condition is

$$e\frac{dE}{dx} = 6\pi V\eta a_1. \qquad \tag{6}$$

For the ♭ ions with velocity U we have likewise

$$-e\frac{dE}{dx} = 6\pi U\eta a_2.$$

In a c. c. of solution $10^{-3}mi$ molecules are split up into as many ions of each sort, and the current C per $cm.^2$ is

$$\begin{aligned} C &= 10^{-3}mie(V-U) \\ &= 10^{-3}mie\cdot\frac{1}{6\pi\eta}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\frac{dE}{dx}, \end{aligned} \qquad \tag{7}$$

therefore by Ohm's law and the definition of specific conductivity the latter is

$$10^{-3}mie^2\cdot\frac{1}{6\pi\eta}\cdot\left(\frac{1}{a_1}+\frac{1}{a_2}\right),$$

and λ the specific molecular conductivity is given by

$$\lambda = ie^2\cdot\frac{1}{6\pi\eta}\cdot\left(\frac{1}{a_1}+\frac{1}{a_2}\right). \qquad \tag{8}$$

If η is taken to be constant, as is usually done with dilute solutions, V and U are both constant for all dilutions, if

$dE/dx=1$; and if at infinite dilution i is assumed to become 1, then i is given by the ratio of the specific molecular conductivity at strength m to that at strength $1/\infty$. But for solutions in general we take η_0 to be the viscosity of the solvent, and η to be that of a solution of strength m, and if again we assume that i becomes 1 at infinite dilution when λ becomes λ_0, then

$$i=\frac{\lambda\eta}{\lambda_0\eta_0}.$$

Thus the ionization is found by multiplying the usual λ/λ_0 by η/η_0.

Now this simple theory must have been written down by many a physicist and found to be wanting, for it makes the ionic velocities of the different atoms at infinite dilution stand to one another inversely as their radii, a result which a brief study of data as to ionic velocities and relative atomic sizes shows to be not verified.

But we need to introduce a correction into the too simplified equation (6). The electron of the ion must be treated as if it were embedded in the atom, which has a different dielectric capacity from that of the water or other solvent.

Now if K_0 is the dielectric capacity of water, and K that of the matter of the atom of our ion, and if all the ions gathered into a single slab at right angles to the current would give it a thickness t in a distance D between the electrodes whose potentials are E_1 and E_2, then the electric force in the slab F_1 would be K_0/K times that in the water, which may be denoted by F, and

$$\begin{aligned}E_1-E_2&=F(D-t)+F_1t\\&=F(D-t+tK_0/K),\end{aligned}$$

$$\therefore\quad F_1=\frac{K_0}{K}\frac{1}{1-\frac{t}{D}\left(1-\frac{K_0}{K}\right)}\cdot\frac{E_1-E_2}{D}.$$

As $(E_1-E_2)/D$ is the same as dE/dx of our previous reasoning, we find the electric force acting on the electron of the ion to be $K_0/K\{1-t/D(1-K_0/K)\}$ times what we assumed it to be according to the ordinary too simplified method of treating ions in the theory of electrolysis. We therefore amend equation (8) to the following form:—

$$\lambda=ie^2\cdot\frac{1}{6\pi\eta}\cdot\left(\frac{1}{K_1a_1}+\frac{1}{K_2a_2}\right)\frac{K_0}{1-(1-K_0/K_1)t_1/D-(1-K_0/K_2)t_2/D}.\quad(9)$$

This introduces some interesting considerations. In the first place it shows that the dielectric capacity of each electrolytic solution must be taken into account in a complete investigation of ionization. It should be noticed that t/D is proportional to the strength of the solution. The dielectric capacity of solutions demands special investigation. But if we consider only solutions so dilute that t/D may be neglected, we find from (9) that in different solvents the specific velocity of a given ion must vary as K_0/η. Now Whetham has pointed out (Phil. Mag. [5] xxxviii., xliv.) that with water, methyl alcohol, and ethyl alcohol as solvents, the conductivities for a given electrolyte are approximately as K_0/η, one estimate of $\lambda\eta/K_0$ for the three substances in an arbitrary unit giving the relative values 1, 0·9, and 1·1, and another giving 1, 1·2, and 1·3, the conductivities standing in the ratios of 1 to ·73 and ·34. Whetham considers the dielectric capacity of the solvent to have most bearing on its ionizing power, in accordance with the suggestion of J. J. Thomson, but according to our reasoning the effect of dielectric capacity on ionization is secondary to its immediate effect on ionic velocities. But Whetham's results verify in a broad way our equation (9) when applied to very dilute solutions.

It is important to remark that this equation might appear to violate the law of the conservation of energy by seeming to make the work done in carrying a charge e from potential E_1 to E_2 in a very dilute solution to be $K_0e(E_1-E_2)/K$ instead of $e(E_1-E_2)$. But the difficulty disappears if energy $(K_0/K-1)e(E_1-E_2)$ is assumed to be taken from the dielectric. The total energy ultimately given to the dielectric is $e(E_1-E_2)$ in the form of the heat generated by the friction of the ion carrying the charge e. Evidently the dielectric has a profound role to play in ionic matters. It should be noticed that the introduction of K the dielectric capacity of the atom into the expression for ionic velocity is important, and is to be returned to in the final section.

From equation (9) with the assumption that at infinite dilution $i=1$ we can write the general value of i thus :—

$$i=\frac{\lambda\eta}{\lambda_0\eta_0}\{1-(1-K_0/K_1)t_1/D-(1-K_0/K_2)t_2/D\}. . (10$$

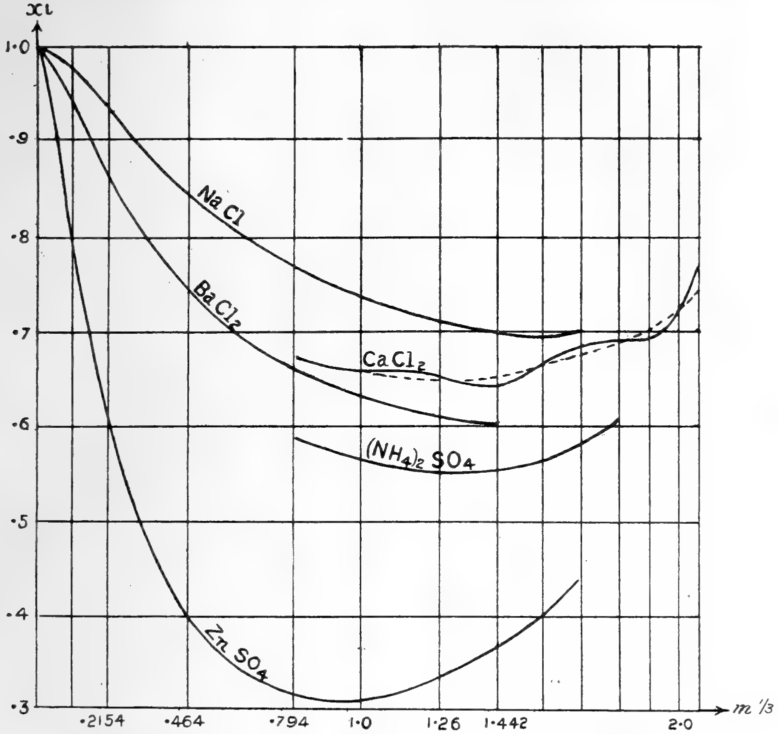
To realize the order of magnitude of the effect of dielectric capacity here, let us consider the case of an electrolytic ion having the same volume as a molecule of H_2O in a solution containing 0·1 gramme-equivalent of it per litre, then, in round numbers, we can say that we have $10^4/18=555$ molecules of H_2O to each ion, so $t/D=1/555$. For an ordinary ion we may take K as of the order 2, while K_0 is 80, and therefore $(1-K_0/K)t/D$ is of the order 1/14; for $m=\cdot01$, this expression will be 1/140, for $m=\cdot001$ it will be 1/1400. For dilute solutions it is therefore negligible, but for solutions of the order $m=\cdot01$ it cannot safely be neglected, especially in the study of formulæ where $1-i$ plays an important part, for though the error in i may be relatively small, that in $1-i$ may be relatively large. In the case of an ion for which K is larger, say 20 instead of 2, then the effect of dielectric capacity becomes less important. But this line of thought does not lead direct to the explanation of the diverse behaviour discovered by Ostwald for weak acids and ordinary binary electrolytes; because if a large value of K secures that i measured in the usual way is nearly correct for CH_3COO in acetic acid, it ought also to act in the same way for sodium acetate, and so bring this electrolyte under Ostwald's law for weak acids like acetic, whereas sodium acetate ranges itself with the ordinary binary electrolytes. It seems to me that the following is a probable explanation why Ostwald's formula applies to weak acids. Many of these acids are known to polymerize. Acetic acid is probably $(CH_3COOH)_2$, and when dissolved in water is partly dissociated into CH_3COOH, which is practically all ionized into CH_3COO and H at the dilutions of Ostwald's experiments. The conductivity measurements of Ostwald thus give him the amount of $(CH_3COOH)_2$ which has been dissociated into CH_3COOH, and his law for acetic acid is probably the expression for equilibrium between di-acetic acid and the products of its dissociation, namely, mono-acetic acid. A similar explanation will apply to all his cases of weak acids. In the case of acetates like that of sodium, we must consider that the polymerization is not so stable in the presence of water as that of acetic acid, and that in dilute solutions the di-acetate is practically all dissociated into mono-acetate, in which form the acetate is usually assumed to be present in aqueous solutions, and, finally, that with the acetates it is really the dissociation into CH_3COO, and Na or K that we investigate by means of conductivities, and not as in acetic acid the relative amounts of diacetate and monoacetate.

As it has been shown that we must have an experimental investigation of dielectric capacity in solutions before we can determine ionizations satisfactorily, I do not propose to dwell at length on the wealth of existing data, but a brief discussion of a few typical solutions is necessary. The solutions chosen are those of $NaCl$, $BaCl_2$, $CaCl_2$, $(NH_4)_2SO_4$, and $ZnSO_4$. For solutions of NaCl we have the latest determinations of Kohlrausch (*Beibl.* xxv.) for strengths from $m=\cdot 0001$ to $m=\cdot 1$ at 18° C., with which we can join on his previous determinations from $m=1$ to $m=5$. Kohlrausch estimates the specific molecular conductivity of NaCl at infinite dilution in water as 108·99 cm^{-1} ohm^{-1}. We need give then only λ/λ_0 in the following table along with η_0/η found by interpolation from Hosking's data (Phil. Mag. [5] xlix.), the viscosity of water at 18° being taken as ·01059 C.G.S.

TABLE I.—NaCl Solutions.

m	1/∞	·0001	·001	·01	·1	1	2	3	4	5
$m^{1/3}$	0·0	·0464	·10	·2154	·464	1·0	1·26	1·442	1·587	1·710
λ/λ_0	1·0	·9918	·9770	·9354	·8443	·6822	·594	·517	·450	·391
η_0/η	1·0	1·0	1·0	·9995	·995	·927	·832	·736	·645	·558
$xi=\lambda\eta/\lambda_0\eta_0$	1·0	·9918	·977	·936	·849	·736	·715	·701	·697	·700
xi calc.	1·0	·983	·964	·925	·851	·741	·712	·701	·697	·699

x is put for $1/\{1-(1-K_0/K_1)t_1/D-(1-K_0/K_2)t_2/D\}$ which for the small values of m differs so little from 1·0 that xi may be taken to give an approximate value of i for very dilute solutions. In the second row values of $m^{1/3}$ have been given, because Kohlrausch found several years ago that for dilute solutions λ/λ_0 is a linear function of $m^{1/3}$, so that xi must also be nearly a linear function of $m^{1/3}$ when m is small. But the rest of the data show that xi reaches a minimum value at nearly the strongest solution. In the graph of these data, given in the figure (p. 171) with $m^{1/3}$ as abscissa and xi as ordinate, we see that the graph instead of being a straight line for small values of $m^{1/3}$ is a curve with a point of inflexion separating two parts of small opposite curvature. Some graphs given by Whetham (Proc. Roy. Soc. 1900) for λ/λ_0 show the same effect. Kohlrausch's straight line is the simplest approximation to this part of the curve; but the whole graph looks like a parabola which at small values of $m^{1/3}$ gets distorted. The cause of the distortion is probably this. Kohlrausch pointed out that in experiments on molecular conductivity for dilute solutions it is necessary to subtract from the conductivity of the solution that of the solvent, assuming that the conductivity of the solvent is not altered by the addition of the solute. Now, for a liquid like water which contains $(H_2O)_3$ and $(H_2O)_2$ in delicately equilibrating amount, it is not safe to assume that its conductivity is not altered by the mere presence of a small amount of electrolyte. The fact that the electrolyte is dissociated makes it probable that the slight ionization and conductivity of water are altered by the presence of the electrolyte. In fact it is well known that some strong acids appear to have maximum molecular conductivities before infinite dilution is attained.



This seems to be only an exaggerated manifestation of the change of curvature shown in our graph for NaCl solutions, and to be ascribable to the same cause, namely our present inability to estimate the change in the ionization of the water brought about by the presence of the electrolytic solute. In spite of the experimental refinements in recent measurements of electrical conductivities, our knowledge of the molecular conductivities of solutions at infinite dilution is less definite than most physical chemists have assumed it to be. If we take the NaCl graph to be a parabola whose vertex is the point $m^{1/3}=1{\cdot}62$, $xi=\cdot697$, we find its equation to be

$$xi-\cdot697=\cdot1155(1{\cdot}62-m^{1/3})^2, \quad . \quad . \quad . \quad (11)$$

the values of xi given by this being entered up in Table I. as xi calc. The discrepancies between the actual and calculated values of xi are probably due to the distortion of the graph, which has been already discussed. As, theoretically, x must generally diminish with increasing values of m, because K_0/K is generally large compared to 1, it follows from the existence of a minimum value for xi that i must have a minimum value. This is important, because a formula like Ostwald's for weak acids could not give such a result. It appears that even when we can measure i, and especially $1-i$, with the requisite accuracy, we shall not be able to use Ostwald's formula for binary electrolytes.

The data for the other typical solutions are contained in Table II., the results being graphed in the figure. The conductivities are Kohlrausch's, and the viscosities derived by interpolation from Sprung's measurements (Pogg. *Ann.* clix.) and Grotrian's for $ZnSO_4$ (*ibid.* clx.). The temperature is 18° C., at which the equivalent conductivities at infinite dilution in cm.$^{-1}$ ohm^{-1} are :—

$\frac{1}{2}BaCl_2$.	$\frac{1}{2}CaCl_2$.	$\frac{1}{2}(NH_4)_2SO_4$.	$\frac{1}{2}ZnSO_4$.
123·2	118·9	133·9	117·2

Table II.

$\frac{1}{2}BaCl_2$.

m	·01	·1	1·0	3·0
$m^{1/3}$	·2154	·464	1·0	1·442
λ/λ_0	·868	·743	·567	·420
η_0/η	·999	·989	·895	·698
$xi=\lambda\eta/\lambda_0\eta_0$ ...	·869	·751	·634	·602
xi calc.	·859	·751	·630	·607

$\frac{1}{2}CaCl_2$.

m	·5	1	2	3	4	5	6	7	8	9
$m^{1/3}$	·794	1·0	1·26	1·442	1·587	1·71	1·82	1·91	2·0	2·08
λ/λ_0	·622	·566	·484	·414	·356	·298	·245	·197	·153	·116
η_0/η	·927	·863	·741	·649	·537	·437	·353	·285	·214	·150
$xi=\lambda\eta/\lambda_0\eta_0$...	·671	·656	·653	·638	·662	·682	·693	·691	·716	·776

$\frac{1}{2}(NH_4)_2SO_4$.

m	·5	1	2	3	4	5	6
λ/λ_0 ...	·557	·511	·449	·406	·368	·331	·295
η_0/η ...	·946	·902	·813	·731	·647	·568	·487
xi	·589	·566	·552	·556	·569	·583	·607

$\frac{1}{2}ZnSO_4$.

m	·00001	·01	·1	1	3	5
$m^{1/3}$	·02154	·2154	·464	1	1·442	1·71
λ/λ_0	·962	·621	·391	·226	·132	·0742
η_0/η	1·0	·997	·968	·724	·357	·168
xi	·962	·623	·404	·312	·370	·443
xi calc.	·909	·565	·402	·308	·347	·440

These tables and their graphs bring out more clearly the important fact that xi or $\lambda\eta/\lambda_0\eta_0$ passes through a minimum. The curves are parabolic, those for $BaCl_2$, $(NH_4)_2SO_4$, and $ZnSO_4$ having axes inclined to the axis of xi. The sinuosities of the $CaCl_2$ curve are smoothed out in the dotted curve. The equations for the $BaCl_2$ and $ZnSO_4$ graphs are respectively

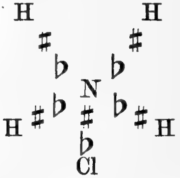
$$(\cdot969m^{1/3}-i-\cdot350)^2=7\cdot18i+\cdot824m^{1/3}-5\cdot35 \quad . \quad (12)$$

$$(\cdot969m^{1/3}-i-\cdot442)^2=3\cdot51i+\cdot403m^{1/3}-1\cdot433 \quad . \quad (13)$$

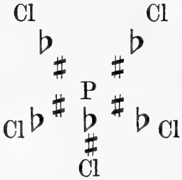
with which the values marked xi calc. in Table II. have been found. For reasons which are obvious after what has been said in discussing the NaCl graph, little importance attaches to these equations. The main point about the graphs is that they make it plain that xi attains a minimum value at a certain strength of the solution. In interpreting the form of these equations we must first consider how $m^{1/3}$ and $m^{2/3}$ come in. On general principles it is clear that they do not enter because of a direct dependence of ionization on the distance apart of the molecules of solute, which is proportional to $m^{-1/3}$. It is possible that the $m^{1/3}$ enters on account of the following train of circumstances. The solute molecules in a watery solution, such as that of NaCl, probably change a certain amount of $(H_2O)_3$ into $(H_2O)_2$, as is shown by the occurrence of shrinking on solution, and they also probably dissociate into $\sharp O\flat_2(\sharp H)_2\flat$ a number of trihydrol molecules. Let y be the average part of each second for which $\sharp O\flat_2(\sharp H)_2\flat$ is separate, $1-y$ the part for which it is combined with others, then the actions producing fresh stions do so at a rate $c(1-y)m$, while those forming trihydrol out of the stions do so at a rate $c'y^3m^3$, and for equilibrium $c(1-y)m=c'y^3m^3$, and when y is small, $y^3 \propto m^{-2}$ and y varies as $m^{-2/3}$.

Now the rate at which the stions $\sharp O\flat_2(\sharp H)_2\flat$ form the labile compound $Cl\flat\sharp O\flat_2(\sharp H)_2\flat\sharp Na$ will be proportional both to y and to m, that is to say, it varies as $m^{1/3}$, and therefore the index 1/3 enters because of the 3 in the formula for trihydrol. Our equations connecting xi and $m^{1/3}$ are therefore equations of chemical equilibrium, expressing that a rate of combination denoted by a square is proportional to a rate of dissociation denoted by a linear, whence the parabolic graphs. It would not be profitable to follow this train of thought further, until the dielectric capacities of solutions have been quantitatively studied. The complete theory of the ionization of binary electrolytes in aqueous solutions is more complicated even than it has been hitherto supposed to be.

Similar considerations to those adduced for water must apply to other ionizing liquids. Take for example liquid NH_3. The constitution of NH_4Cl suggests for it the formula



whence gaseous NH_3, like steam, has a neutron associated with each molecule, though this neutron need not have the same electric moment as the neutrons of free æther. Liquid NH_3 may consist of di- or tri-ammonia, the molecules of which on dissociation yield stions $\sharp N\flat_3(\sharp H)_3\flat$, whence the ionizing power of liquid NH_3. For PCl_5 we can imagine the constitution



in which one of the Cl atoms is quite differently attached from the other four, being ready to form $Cl\sharp\flat Cl$ and PCl_3 when PCl_5 dissociates. The easy dissociation of NH_4Cl would be explained in the same way, and probably a large part of the phenomena connected with variable valency could receive a similar electrical treatment.

3. *Ionic Velocities and Absolute Sizes of Atoms.*

For what is usually called the velocity ${}_1\lambda$ of an ion at any dilution we have from (9) the equation

$$ {}_1\lambda = ie^2 \cdot \frac{1}{6\pi\eta a_1} \cdot \frac{K_0}{K_1} \cdot \frac{1}{1-(1-K_0/K_1)t_1/D-(1-K_0/K_2)t_2/D} \quad (14) $$

At infinite dilution

$$ {}_1\lambda_0 = e^2 K_0 / 6\pi\eta a_1 K_1 . \quad . \quad . \quad . \quad . \quad . \quad (15) $$

Of the quantities entering into this last e^2, a_1 and K_1 are not capable of direct physical measurement like the others, but if K_1 were known, then since we know M_1/e from the electrochemical equivalent of the ion of mass M_1, and as $M_1 = 4\pi\rho_1 a_1{}^3$ where ρ_1 is the limiting density of the substance at absolute zero, which can be closely estimated, equation (15) would become a definite one for a_1 the atomic radius. For an ion of valency ν, e^2 must be replaced by ν^2e^2 and $_1\lambda_0$ by $\nu_1\lambda_0$ and (15) becomes

$$_1\lambda_0=\nu e^2\mathrm{K}_0/6\pi\eta a_1\mathrm{K}_1. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

We can derive values of K_1 from the relation $\mathrm{K}_1=\mathrm{N}_1{}^2$, where N_1 is the index of refraction of the stuff of the atom. In a recent paper ("The Cause of the Structure of Spectra," Phil. Mag. [6] ii.) I have given values of N_1 for a number of metals of known density, which may be taken to be the limiting density, and so the data are to hand for calculating a_1 for a number of elements. But before proceeding to absolute values we can test how the equation behaves relatively for a number of elements. As e^2, η, and K_0 are the same for all ions when we deal with infinitely dilute aqueous solutions, we must have $_1\lambda_0 a_1\mathrm{K}_1/\nu$ the same for all ions. To test this relation we gather for several atoms and radicals in Table III. the values of N_1, and of B which is the limiting volume of a gramme-atom or gramme-radical, of $_1\lambda_0$ according to Kohlrausch, and of $_1\lambda_0\mathrm{B}^{1/3}\mathrm{N}_1{}^2/\nu$ which by (16) is to be constant. The values of N_1 are derived from the refraction formula $(n-1)\mathrm{M}/\rho=(\mathrm{N}-1)\mathrm{B}$ with the values of $(n-1)\mathrm{M}/\rho$ given in the text-books as atomic refractions (see for example L. Meyer's 'Modern Chemistry'), and with values of B as given in "Further Studies on Molecular Force" (Phil. Mag. [5] xxxix.) and reproduced here. $\nu=2$ for Mg, Ca, Sr, Ba, and Zn, and $\nu=1$ for the rest.

TABLE III.

	Li.	Na.	K.	Rb.	Cs.	Mg.	Ca.	Sr.	Ba.	Zn.
$_1\lambda_0$ ...	35·5	44·4	65·3	67·3	67·8	48	53	54	57·3	47·5
B...	2	7·4	18·6	34·4	56	5·6	8·6	10·6	16·6	10·6
N...	2·9	1·65	1·44	1·41	1·24	2·25	2·21	2·28	1·95	1·96
$_1\lambda_0\mathrm{B}^{1/3}\mathrm{N}^2/\nu$	375	235	360	435	400	215	265	310	277	200

	F.	Cl.	Br.	I.	H.	OH.
$_1\lambda_0$	46·1	65·9	67·5	66·7	318	174
B	9	19	26	36	4·5	9·5
N	1·16	1·56	1·65	1·76	1·37	1·44
$_1\lambda_0\mathrm{B}^{1/3}\mathrm{N}^2/\nu$ ...	130	430	545	680	980	765

	HCOO.	$\mathrm{CH_3COO}$.	$\mathrm{C_2H_5COO}$.	$\mathrm{C_3H_7COO}$.	$\mathrm{C_4H_9COO}$.	$\mathrm{C_5H_{11}COO}$.
$_1\lambda_0$	47·2	35·4	31·8	28·3	26·5	25·3
B	24·5	42	59·5	77	94·5	112
N	1·494	1·469	1·458	1·454	1·450	1·448
$_1\lambda_0\mathrm{B}^{1/3}\mathrm{N}^2/\nu$ ..	305	265	264	255	253	256

Among the metals though ${}_1\lambda_0 B^{1/3}N^2/\nu$ varies from 200 to 400, the approach to constancy appears remarkable when it is noticed that N^2 varies from 1·53 to 8·4, and $B^{1/3}$ from 1·26 to 3·8. In the six fatty acid radicals from formic to caproic the approach to constancy is satisfactory. In the halogen atoms ${}_1\lambda_0 B^{1/3}N^2/\nu$ fails to approach constancy in a striking manner, ranging from 130 to 680. It is probable that the discrepancy here is due to the assumption that in these atoms $K=N^2$. If we remember that the halogens are heptads as well as monads, and that therefore each halogen atom contains three pairs of ♯ and ♭ as neutrons, or as doublets, giving each the possibility of acting as a tri-stion, we can see that the assumption $K=N^2$ is unsafe. The exceptional behaviour of H and OH in having such large values as 980 and 765 for ${}_1\lambda_0 B^{1/3}N^2/\nu$ is probably due to the fact that these are the ions of water itself. It is possible therefore that these two ions have their real ionic velocities largely increased by a sort of Grotthus-chain action, whereby an $\dot{H}$ or an $\grave{O}H$, instead of passing through the space of a water molecule which is in front of it, simply combines with part of the molecule and liberates the other part at the other side, so that the same effect is produced as if the ion had traversed the space of the molecule with a higher velocity than the true ionic velocity.

To proceed with our equation (16) to the calculation of a_1 absolutely, let us fix our attention on the ion of Li. We must first convert ${}_1\lambda_0$ its ionic velocity per gramme equivalent G into the velocity of an atom by dividing by $3G/4\pi\rho_1 a_1^3$, which is the number of atoms in a gramme equivalent. Again, to express ${}_1\lambda_0$ in C.G.S. units we must multiply ${}_1\lambda_0$ cm.$^{-1}$ ohm^{-1} by 10^{-9} to pass to the electromagnetic unit of resistance, and by 9×10^{20} to get the appropriate C.G.S. expression. Again, the electrochemical equivalent of hydrogen is ·0001035 in E.M. units, and therefore for Li

$$M_1/e=\cdot000725/3\times10^{10}$$

in electrostatic units. As $G/\rho_1=B$ given along with ${}_1\lambda_0$ and N in Table III., we have for Li with $K_0=80, \nu=1, \eta=\cdot01059$, and $\rho_1=3\cdot5$.

$$\frac{4\pi}{3B}a_1^3\times35\cdot5\times9\times10^{11}=(4\pi\rho_1 a_1^3)^2\frac{9\times10^{20}\times80}{\cdot000725^2\times6\pi\times\cdot01059\times2\cdot9^2 a_1}$$

$$\therefore\quad a_1^2=3\cdot8\times10^{-18},$$

$$a_1=2\times10^{-9},$$

$$\tfrac{4}{3}\pi a_1^3=7\cdot5\times10^{-27}.$$

Now Kelvin's estimate of the volume of an ordinary molecule is between 3×10^{-25} and 10^{-26} c.c. with which the value just calculated for Li, the smallest of the ions in Table III., is in agreement. The molecular radius of ordinary gases, as found by the kinetic theory from measurements of gaseous viscosity, is of the order 5×10^{-8} cm., which is about 25 times our value of a_1 just found for the Li ion. The process of calculating the size of a molecule from its observed ionic velocity might prove helpful in the case of large organic ions such as physiologists have to deal with, for the molecular weight could be estimated from the size when other recognized methods of determining it fail. Ostwald's measurements of the ionic velocities of a large number of organic radicals (Wied. *Ann.* Beibl. xiii.) might supply useful material for a preliminary study in making equation (16) available for the estimation of very large molecular weights.

Melbourne, October 1901.

XIX. *On the Variation with Temperature of the Thermoelectromotive Force, and of the Electric Resistance of Nickel, Iron, and Copper, between the Temperatures of* −200° *and* +1050°. *By* E. PHILIP HARRISON, *University College, London* [*].

THE main objects of the investigation which forms the subject of this paper are as follows :—

To trace over as wide a range as possible the change with temperature of the thermoelectromotive force and the resistance of Nickel and Iron, *using in all experiments the same specimens of metal* ; to investigate any singularities that may be present in the curves representing the change ; and to determine whether they occur at the same temperature in each of the curves. It was originally intended to investigate the magnetic properties of the same specimens, but time has not sufficed for this.

SECTION I.

THERMOELECTRIC PROPERTIES.

(1) *Results of Previous Observers.*

In his researches on thermoelectricity [†] Tait gives the results of some of his measurements for iron which extended from 0° to the melting-point. The second portion of his iron

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[*] Communicated by the Physical Society : read October 25, 1901.

[†] Proc. Roy. Soc. Edin. December 1873; 'Nature,' Rede Lecture, May 1st, 1873.

couple was either platinum or an alloy of platinum and iridium; he used mercury thermometers up to 300°, but did not correct for stem-exposure, and the experimental difficulties due to chemical action and the measurement of temperature made observations above 500° quite unreliable. His results within a moderate range of temperature (obtained with the hot junction in an oil-bath) showed that the E.M.F. temperature-curves were made up of a series of "excellent parabolas." At higher temperatures "the parabola was slightly steeper on the hotter than on the colder side." He also found that between 200° and the melting-point, two or more distinct neutral points existed.

Professors Fleming and Dewar's experiments on the same subject* extend from the temperature of boiling oxygen to 100° C. Their paper gives the experimental methods used and the numerical results, but a discussion of the latter is reserved by the authors for a further communication. The temperature of the "hot junction" in their experiments was measured by a platinum thermometer, and their results were expressed on the platinum scale.

Holborn and Day † compared a platinum-platinum-rhodium couple with the air-thermometer, and ‡ tested a number of similar couples, but were unable to obtain consistent results with oxidizable metals at high temperatures. They adopt Tait's method of representing their results by portions of parabolas for limited ranges of temperature.

Stansfield §, from observations on couples formed of platinum and platinum-iridium or platinum-rhodium alloys, found that for these couples the Peltier effect $\left(\mathrm{T}\frac{d\mathrm{E}}{dt}\right)$ approximated closely to a linear function of the temperature, but that the thermoelectric power $\left(\frac{d\mathrm{E}}{dt}\right)$ could not be so represented.

DESCRIPTION OF APPARATUS.

A. *Measurement of E.M.F.*

An ordinary potentiometer method was used and the general arrangement of apparatus is shown in fig. 1. A two-metre bridge-wire BP was connected in series with a resistance-box R and a 1 ohm rheostat O. The P.D. due to the

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* Phil. Mag. 1895, vol. xl. pp. 95–119.

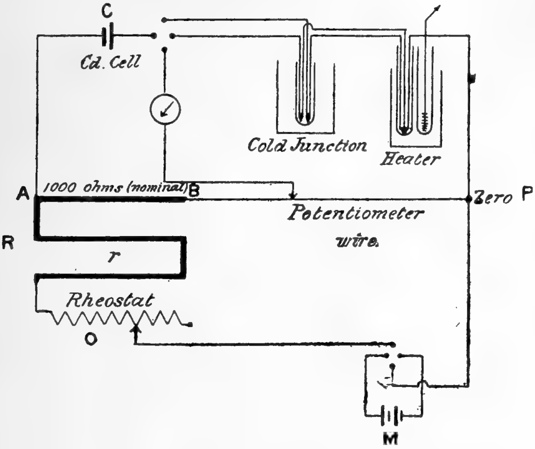
† Wiedemann's *Annalen*, August 1899.

‡ Berlin Academy, July 1900.

§ Phil. Mag. 1898, vol. xlviii.

thermo-couple was balanced against that due to two accumulators M. Before each reading of the E.M.F. of a thermocouple the standard cadmium cell C[*] was balanced on a definite resistance AB of the box by adjusting *r*, fine adjustment being made by the rheostat. Alterations in the E.M.F. of the cadmium cell with temperature were too small to affect the observations to 1 part in 1000. Since the iron-copper couple gives an E.M.F. about 10 times smaller in places than the nickel-copper, a shunt equivalent to 1/9 of the bridge-wire resistance was constructed, and arranged so that it could be inserted at will between the ends of the bridge-wire.

Fig. 1.



The bridge-wire was calibrated by the Carey Foster method. A galvanometer of the Wiedemann pattern was used with a pair of 1 ohm coils, and with the usual scale and telescope arrangement for reading deflexions. Thermoelectric effects at the slide-wire contact were avoided by always depressing the key with a glass cap. With the above arrangement of apparatus, the position of the key on the potentiometer-scale could be adjusted in the case of copper-nickel junction to the fifth of a millimetre, in the case of copper-iron junction to half a millimetre. Readings of E.M.F. of copper-nickel couples were accurate to 1·8 microvolts, while those of copper-iron couples were accurate to less than 1 microvolt at moderate temperatures.

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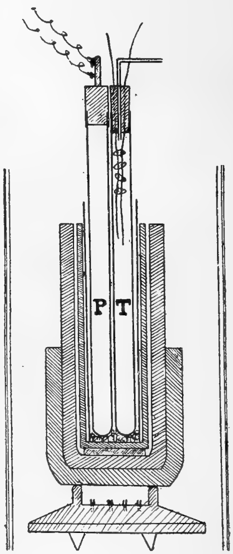
[*] E.M.F. 1019 millivolts.

B. *Measurement of Temperature.*

In all cases the hot-junction temperatures were measured by a platinum thermometer and recorded automatically by Callendar's Recorder, the advantage of this method being that simultaneous readings of E.M.F. and of temperature are obtained without the necessity for two observers. While the temperature was changing slowly (say $\frac{1}{2}^\circ$ per minute) the readings at 500° were correct to 1/10 degree. Observations were taken in all cases with rising and falling temperatures in order to make sure that the "lag" of the thermometer or recorder did not introduce appreciable errors.

Fig. 2.



The heating-apparatus (fig. 2) consisted of several concentric iron cylinders closed at one end. For temperatures below 700° this arrangement was placed on a small Fletcher burner, and the whole surrounded by a cylinder of polished tin-plate large enough to leave an air-space of two or three cms. between itself and the iron heater. The inner wall of the "tin" cylinder was lined with $\frac{1}{8}$ inch asbestos board.

For heating above 700° C. a Fletcher gas-furnace with an air draught was used, two dampers being arranged for regulating the temperature.

The hot junction formed by fusing together in a reducing blowpipe-flame the ends of the copper, nickel, and iron wires, was placed nearly at the bottom of a 12-inch porcelain tube of 9 mm. internal diameter. The platinum thermometer-tube was then placed by the side of the tube containing the couple, and the two were bound together with several layers of thick sheet-copper, extending six inches from the bottom of the tubes, which were then placed in the heater and packed round with asbestos. The top of the heater was covered with 4 or 5 layers of asbestos-paper separated from one another by air-spaces.

The other ends of the nickel and iron wires were soldered to copper leads, and the junctions placed in very small glass tubes, which together with a mercury thermometer reading to 1/10 degree C. were placed in a large test-tube full of water, the test-tube itself being also immersed in water. This arrangement formed the cold junctions, and their temperature always varied with the temperature of the room. Every observation is reduced to cold junction at 0° C. by means of a separate series of observations applied graphically. Observations at the higher temperatures were taken first in each case, so that any change that might be caused by the heating would occur at the beginning of the experiments. Confirmatory observations were made in steam, aniline vapour, and sulphur vapour. In the case of steam an ordinary hypsometer was used, the couple-tube and pyrometer being placed in it simultaneously, and the temperature of the steam calculated as well as observed by the pyrometer. The aniline was boiled in a glass beaker about 2 ft. high, provided with "tin" plates at intervals up its length to prevent convection-currents. For the sulphur point an ordinary sulphur boiling-point apparatus was used.

The effect of heating the junctions in hydrogen and in carbon was tried in every case. A large rubber bag was filled with 96°/₀ hydrogen from a cylinder. The bag communicated with one arm of a three-way tap, the other two arms of which went to the pump and couple-tube respectively. The couple-tube was washed out with hydrogen by alternately exhausting the apparatus and letting in the gas, and observations were taken at pressures slightly in excess of atmospheric pressure. The hot junctions when packed in bone-black in the porcelain tube gave consistent results up to 700°, but quite inconsistent results above that temperature. Finally, in each case observations were taken in liquid air. The junctions were put naked into the liquid with the platinum thermometer, also without its tube, beside them. Temperature of the air was recorded.

Method of avoiding Oxidation.

For observations above 500° it was necessary to protect the couples as far as possible from any gases that might chemically affect them. Owing to the difficulty of obtaining any argon at the time of the experiments, it was finally decided to use as good a vacuum as possible. The porcelain tube containing the junction was closed with a rubber cork through which a glass tube passed. The three wires were brought up between the cork and the sides of the tube, and were gently burnt into the former. Marine glue was melted and poured over the cork and top of the tube, which was then warmed till the glue ran into every hole. Finally a coating of beeswax was spread over the glue—a precaution which was found to be very necessary to the preservation of the vacuum, though as beeswax melts at a low temperature, the top of the hot-junction tube had to be kept well protected from the heat of the burner. The glass tube which passed out of the cork was then sealed directly on to a Fleuss pump. Two P_2O_5 tubes were interposed between the porcelain tube and the pump, and a mercury gauge was set up with a barometer by its side for comparison. The wires above the hot junction were in the first instance insulated by being separately wrapped in several layers of asbestos-paper as far up as the mouth of the tube. Subsequently, the wires inside the tube were insulated by little circular mica disks instead of asbestos, because the latter was found to evolve gas on heating which contaminated the couples. With mica insulation no difficulty was found in keeping the vacuum as good as that above the barometer for several days.

Regulation of the Temperature.

It was thought possible that hysteresis occurred at the singular point of the nickel curve. To investigate whether or not this was the case, it was necessary to take observations with a very slowly-rising or slowly-falling temperature. A special apparatus was constructed for regulating the gas-pressure, and for slowly turning on or off the gas-supply to the small Fletcher burner. This apparatus worked very satisfactorily, although only a small hysteresis effect was detected, which is shown by the double line at the summit of difference-curve D (Ni) (fig. 4); the upper branch was obtained with rising, the lower one with falling temperature. The difference could not be explained by temperature-lag in either the thermometer or the couple.

Method of taking the Observations.

The temperature of the hot junction being steady, or changing not more than 1 degree in five minutes, a stop-watch was started when the pen of the Recorder was crossing a horizontal "time-line." The scale of the potentiometer was adjusted by reference to the cadmium-cell, and the latter was then cut out and the Cu-Ni couple switched into the galvanometer-circuit. Time on the stop-watch at which balance occurred, and the bridge-reading at balance being noted, as soon after each observation as possible the temperature of the cold junction was read to 1/10 degree. The same observations were repeated on the copper-iron junction with the bridge-wire shunted.

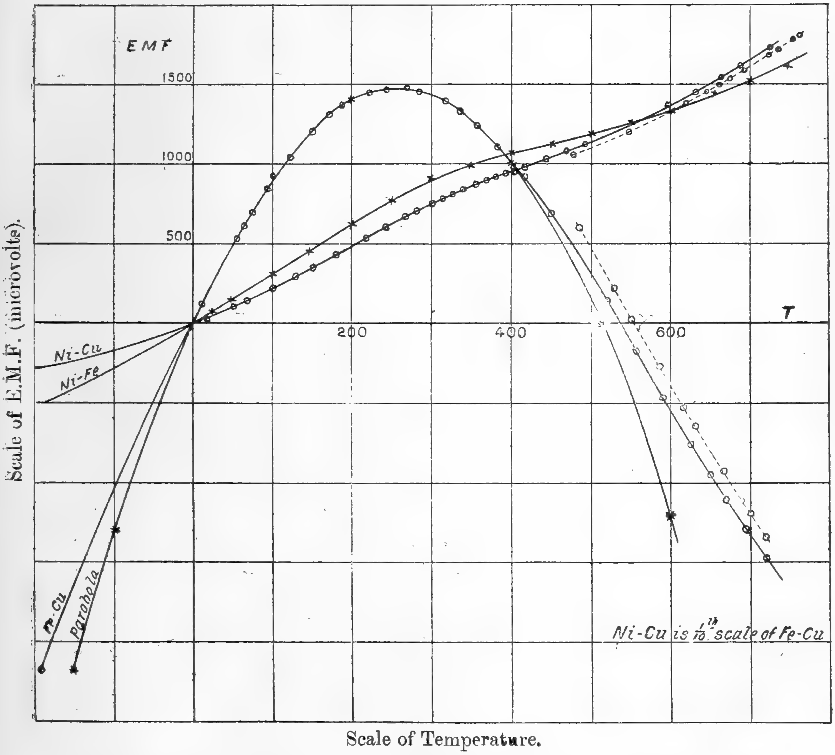
All temperatures are expressed as air-thermometer temperatures, correction from the platinum-scale being applied graphically by means of a difference-curve.

All E.M.F.'s are given in microvolts.

Discussion of Results.

The curves for variation of E.M.F. with temperature of copper-nickel and copper-iron couples might be roughly described as a straight line and a parabola respectively up to 700° C., but the differences in either case far exceed the possible errors of observation (fig. 3). In order to exhibit any peculiar points on a satisfactory scale in either case, difference-curves were constructed. For copper-nickel, differences from a straight line were plotted from the equation

Fig. 3.



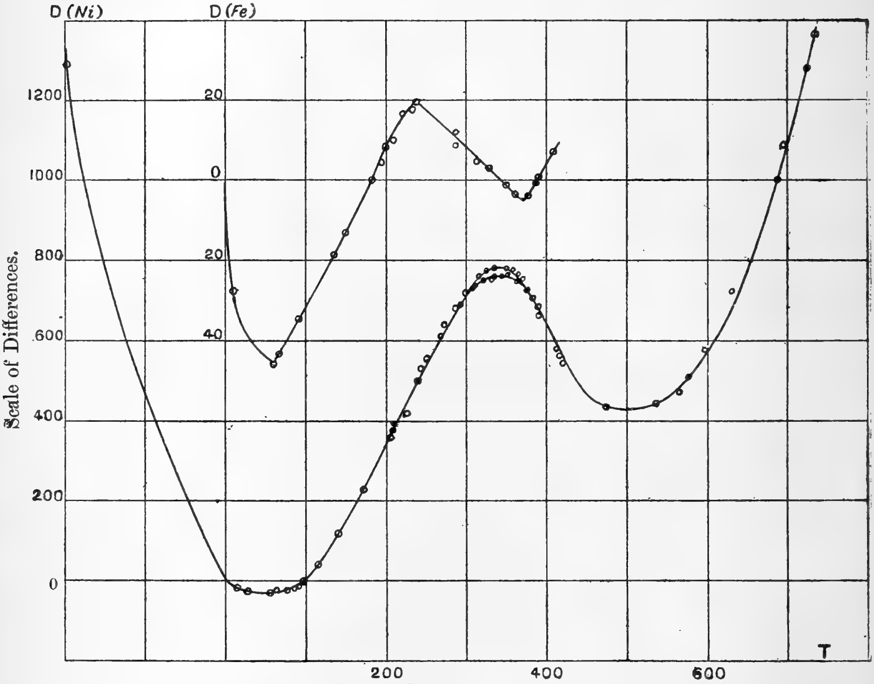
$$\text{Difference} = E_{\text{observed}} - 22t.$$

Values of D (difference) thus obtained were plotted against temperature (Curve D (Ni), fig. 4). In a similar way for the iron-copper couple a parabola

$$E_o^t = 11{\cdot}228\,t - {\cdot}021924\,t^2,$$

passing through the ice, aniline, and sulphur points, was calculated, and differences from this parabola were plotted.

Fig. 4.—Difference Curves.



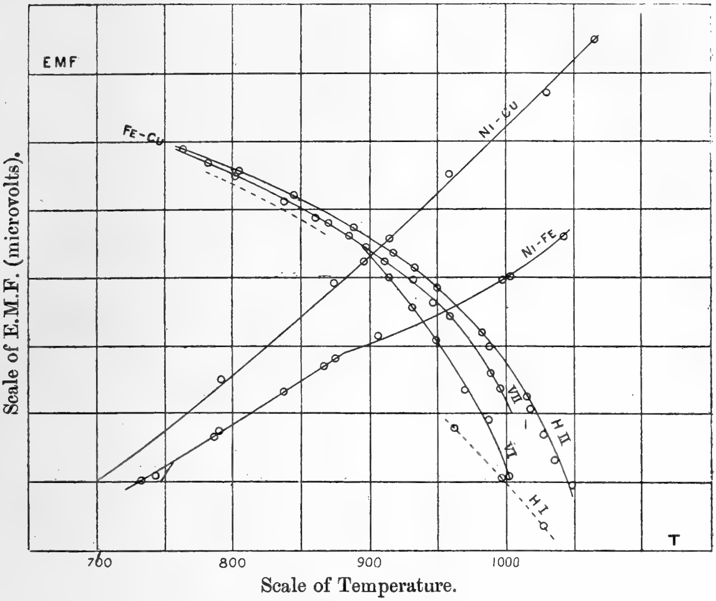
This gives Curve D (Fe), fig. 4. In this case differences are much smaller than in the case of the nickel. The parabola itself is shown in fig. 3.

From these differences it is seen that maximum variations occur in the case of Cu-Fe, (*a*) at 70°, when the thermo-electric E.M.F. is less than the corresponding parabolic ordinate by 45 microvolts; (*b*) at 230°, when the thermo-electric E.M.F. is greater than the corresponding parabolic ordinate by 20 microvolts; (*c*) at 370°, when the thermo-electric E.M.F. is again less than the corresponding parabolic ordinate by 5 microvolts. Between 0° and 200°, and between 400° and 750° there is a gradual divergence from parabolic form more strongly marked on the hot side (the reverse of Tait's observation).

The temperature of inversion, cold junction at 0°, is found to be 536° C., and is given by the intercept on the axis of temperature, between 0° and the point of section by the curve of thermoelectric power (fig. 6, p. 188).

The neutral point is at 262° C.

Fig. 5.



Above 700°, Cu-Fe E.M.F. increases almost as a linear function of the temperature till 900° is reached, when a rapid increase in the E.M.F. is noticed, with no indication of a second neutral point (fig. 5).

In the case of Cu-Ni, maximum variations occur at about 70° C. and 340° C. The divergence at 340° is very marked, and after 500° C. increases rapidly. There appears to be a small hysteresis effect at the maximum. The temperature of inversion, if there be one, does not occur within the limits of the experiment, and there is no neutral point. Above 700°, the Cu-Ni curve (fig. 5) remains almost linear up to 1050°, the limit of the experiments, while the slope of the curve is practically unchanged.

The E.M.F. curve for Ni-Fe couple (fig. 3) up to 700° C. was obtained by adding the ordinates of the copper-nickel and copper-iron curves corresponding to definite temperatures. It has no neutral point above 0°, though at the temperature of liquid air there is an indication of one. The fact that no neutral point occurs in the iron-nickel "line" is also evident from the fact that copper-nickel and copper-iron thermo-electric-power curves, when plotted on the same scale, do not intersect.

The E.M.F. curve is a nearly linear function of the temperature up to about 900°, when a decrease in the E.M.F. is noticed (fig. 5). Above 700° the curve was obtained by direct observation of the E.M.F. of a fresh nickel-iron couple, the copper being omitted in order to avoid the uncertainty due to its evaporation, which became quite appreciable at 900° C. The junction of iron and nickel was made by fusing the wires in the electric arc.

The dotted parts of the nickel and iron "lines" (fig. 3) were obtained when the couples were known to be affected by oxidation, *i. e.* when the vacuum was known to be faulty. The shift only occurred at the higher temperatures. The E.M.F., when determined on the same "oxidized" couple for lower temperatures (such as 200°), coincided with the original values. Any error in measuring temperature or E.M.F. would be constant for the copper-nickel and copper-iron "lines." The observed facts, however, show that the effect is not the same in the two cases. The vertical displacement of the nickel curve is about 425 microvolts, while at the same temperature that of the iron curve is about 120 microvolts, the horizontal shift in the first case being 14°, and in the second case 18°, in the same direction.

The effect for Cu-Fe is as if an applied E.M.F. were acting so as to produce a current in the same direction as that due to the junction. Thus the fact that the effect is an increase of ordinate above 500°, and a decrease of ordinate below 500° (in the case of Fe-Cu) is accounted for. For Cu-Ni there is always a decrease, such as might be due to an E.M.F. giving a current in the opposite direction to that due to the junction. Moreover, the two dotted bits are not absolutely parallel to the original curves, making it still more probable that the explanation of the shift is to be found in the fact that chemical action took place and altered the E.M.F.'s.

When the couples were heated in hydrogen, so as to preclude the possibility of oxidation, those points taken below 895° C. lay on the main curve, which bears out the idea that the shift in question is due to oxidation.

Above 895° other effects were observed when the couples were heated in hydrogen, which were more systematically investigated (see below).

In carbon, observations taken up to 700° agreed with the hydrogen and vacuum results. Above this temperature, carbides of iron are probably formed which give rise to the inconsistency in the results, which has already been remarked on. It may be mentioned that these observations gave values of E.M.F. which were considerably greater than those in hydrogen.

It is evident that thermoelectric couples of copper-iron and copper-nickel packed in carbon will give consistent readings of E.M.F. up to 700° or 800°, but that above this temperature the method fails.

Under every condition tried, copper, nickel, and iron wires were always extremely brittle after heating above 700°. After heating in hydrogen they were invariably quite bright, notably the copper. This last metal volatilized considerably above 900°.

The curves of thermoelectric power $\frac{dE}{dt}$, shown in fig. 6, were obtained by drawing tangents to the E.M.F. curves (figs. 3 and 5). A considerable range

$$\left(\begin{array}{c}\text{from } -200^\circ \text{ to } +100^\circ \\ \text{and from } +100^\circ \text{ to } +400^\circ\end{array}\right)$$

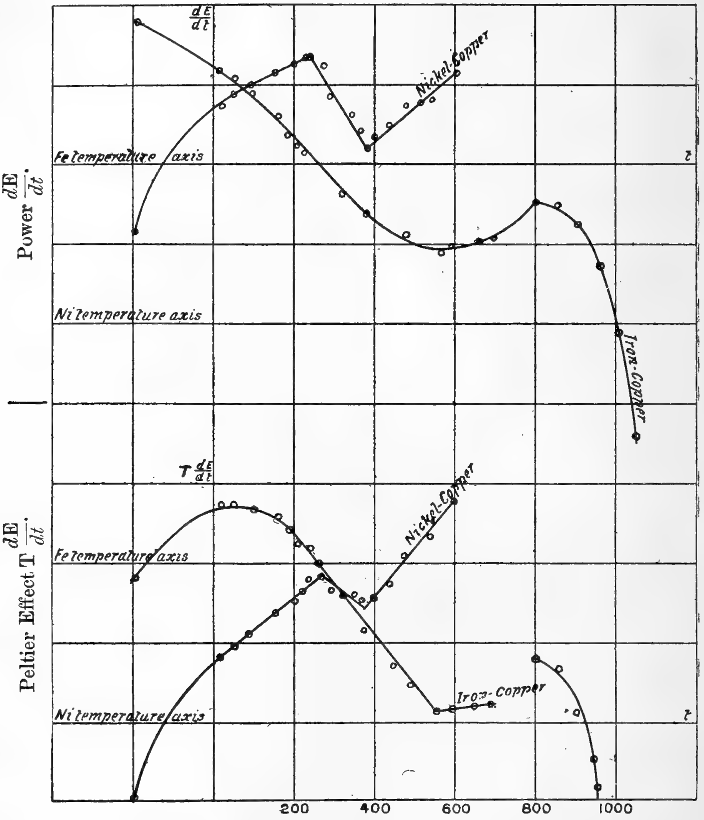
of the copper-iron curve can be represented by straight lines. Between $+400^\circ$ and $+800^\circ$, however, the curve appears to be parabolic, and cannot be built up of bits of straight lines.

Above 800° $\frac{dE}{dt}$ increases rapidly, being nearly linear between 900° and 1050°. The Cu-Ni power curve can be represented by bits of straight lines. Changes in slope occur at 240° and 380°.

The Peltier coefficients $T\frac{dE}{dt}$, obtained by multiplying the thermoelectric power by the corresponding absolute temperature, are given in the same figure (fig. 6). The curve for iron-copper is approximately parabolic between -200° and $+300^\circ$. After that it can be made up of straight lines. The break just below 800° is due to the fact that the E.M.F. curve for results above 800° did not quite join, but was approximately parallel to that for lower temperatures. For Cu-Ni the Peltier coefficient-variation can be built up of bits of parabolas, if the power-curve is strictly linear; but uncertainty in the values obtained by differentiating an

E.M.F. curve makes it better to show the Peltier coefficient-variation as bits of straight lines.

Fig. 6.



Scale of Temperature.

Changes in sign of the Thomson coefficient, or specific heat of electricity, are indicated by these changes in slope of the Peltier coefficient, and the former are remarked on by Tait, and he gives the temperatures at which they occur as just above 200°. and again just below 300°. It is also evident that the E.M.F. curve for Cu–Ni can be represented very well by bits of parabolas over the whole range.

Changes in E.M.F. on heating above 900°.

It was noticed that changes in the value of the E.M.F. for any particular temperature occurred on *continued* heating, and in order to investigate these changes observations were made with the Fe–Cu couple under the following four conditions :—

(i.) New couple in air-vacuum.
(ii.) Same couple after continued heating in air-vacuum.
(iii.) New couple in hydrogen at about 860 mm. pressure.
(iv.) Same couple after continued heating in hydrogen.

(i.) On heating a new couple rapidly to the highest temperature required, and then taking observations throughout the range as quickly as is consistent with accuracy, Curve V 1. was obtained (fig. 5).

The E.M.F increases almost as a linear function of *t* up to about 895° C., when an abrupt change of slope occurs and the E.M.F. increases more rapidly up to the limit of the experiments.

(ii.) Using the same couple after 10 or 15 hours' heating, the abrupt change of slope previously noticed no longer occurs, but as is seen by V II., the curve is continuous over the whole range so that above 895° the curve consists of two branches, the lower one obtained with a once-heated couple and the upper one with a couple that had undergone more prolonged heating, while below 895° the curve is common to the two branches, and appears to be independent of the length of time the couple has been heated. Measurements were then made with continuously rising and continuously falling temperatures to find if hysteresis occurred above 890°. Evidence was obtained of small effects of this nature, slightly greater values of E.M.F. being obtained with falling temperature, though any exact estimation of the phenomenon was made somewhat difficult by the change which was going on in the value of the E.M.F. owing to continued heating.

"Lag" errors due to a temperature which was changing too rapidly would not account for these effects, for such errors would affect the result in the opposite direction—for instance, with a too rapidly falling temperature the curve would be shifted upwards owing to "lag" in the recording of the temperature.

The variation of E.M.F. above 900° is about 20 microvolts per degree for the freshly heated couple. Probably the effects of continuous heating are going on the whole time in Curve V I.: hence the rapid, though continuous change of slope. At 895° a change, most likely chemical, occurs in the Fe, and below this temperature the couple is in a new state, and secular changes go on more slowly, if at all.

After these experiments the copper was exceedingly bright while the iron was covered with a black deposit.

(iii.) When a new couple is heated in hydrogen, Curve H I. is first obtained, no change of slope being noticeable and the values for the E.M.F. being greater than those obtained in a vacuum. This curve therefore approximates more nearly at 700° to the one obtained at lower temperatures, than does V I., the lowering of the curve under the influence of hydrogen being consistent with previous observations on the effect of this gas.

(iv.) After continued heating in hydrogen, just as in Curve V II., the values of the E.M.F. become less, and finally the curve representing them lies slightly above the values obtained *in vacuo* (Curve H II.). H II. is nearly, but not quite, parallel with V II. The wires at the end of the experiments were both perfectly clean although the iron had lost its polish. The copper was exceedingly bright.

In the case of iron heated in hydrogen, possibly a gradual reduction of impurities in the substance of the iron accounts for the change in the E.M.F. on continued heating.

[There is no break or change of slope in the thermoelectric curves for Cu-Fe at 800°, the temperature at which the resistance of Fe alters.]

In order to eliminate any possible effect of the copper, a Ni-Fe couple was used. The E.M.F. was affected by continued heating in hydrogen, in just the same way as when copper was present. Evidently the effect is due to physical or chemical changes or both going on in the iron, and the change of slope at 895° is not connected with a decomposition of copper oxide at this temperature, as was at one time thought possible.

Only the means of the observations of E.M.F. at the temperatures of steam, aniline, sulphur vapour, and liquid air are given in the table below.

The observations at other parts of the curves were much more numerous than those shown by circles in the figures, but they could not be satisfactorily represented by means of tables or empirical formulæ, on account of the singularities presented in the variations of E.M.F. The E.M.F. at any temperature can be very accurately deduced from the difference curves, *e.g.* for copper-iron at 250°

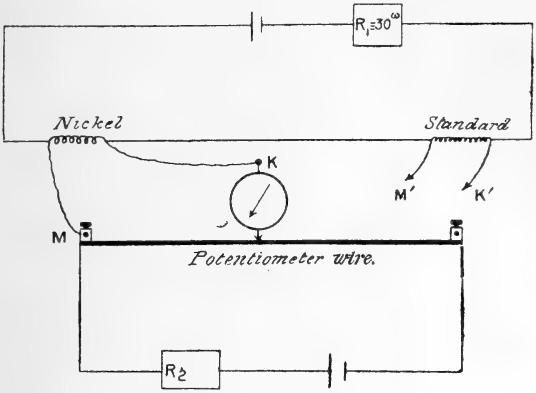
the parabolic formula gives	+1452·1	microvolts.
the reading of Curve D (Fe) gives .	+ 17·5	,,
Hence Total E.M.F. is	+1469·6	,,

SECTION II.

Resistance Experiments.

A potentiometer method was used and the general arrangement of the apparatus is shown in fig. 7.

Fig. 7.



The wire the resistance of which was to be measured was wound into a spiral, and the main current leads and potential leads of No. 28 B.W.G. copper were silver-soldered to the ends of the spiral; the other extremities of these four leads being soldered to four No. 18 copper leads which projected through the cork of the porcelain tube. The thin copper was used in order to minimise conduction of heat from the spiral. The standard coil for comparison was of manganin and, with its potential leads, was immersed in paraffin oil. It was not found necessary to apply any correction for temperature changes in the standard.

The porcelain tube containing the spiral was placed in the heater as before, with a pyrometer. Care was taken that the temperature was steady before making an observation of resistance, and in every case a reading of the standard resistance was taken before and after each reading of the resistance (Ni or Fe) required.

Discussion of Results.

The resistance of nickel increases with temperature almost parabolically up to 370°, when a change of slope occurs, and the resistance increases much less rapidly and almost as a linear function of the temperature up to 1050° (fig. 8).

A parabola which fitted the resistance observations very well up to 350° C. was calculated passing through ±100° and +300°, and was plotted from the equation

$$R_{Ni} = \cdot 10288 + \cdot 00048t - \cdot 0000007126t^2.$$

The change of slope in the nickel resistance-curve is not sudden but extends over 10° or 15°.

In the case of iron, the resistance-curve does not change its parabolic form till nearly 800°, when it becomes linear and remains so within the limits of the experiments.

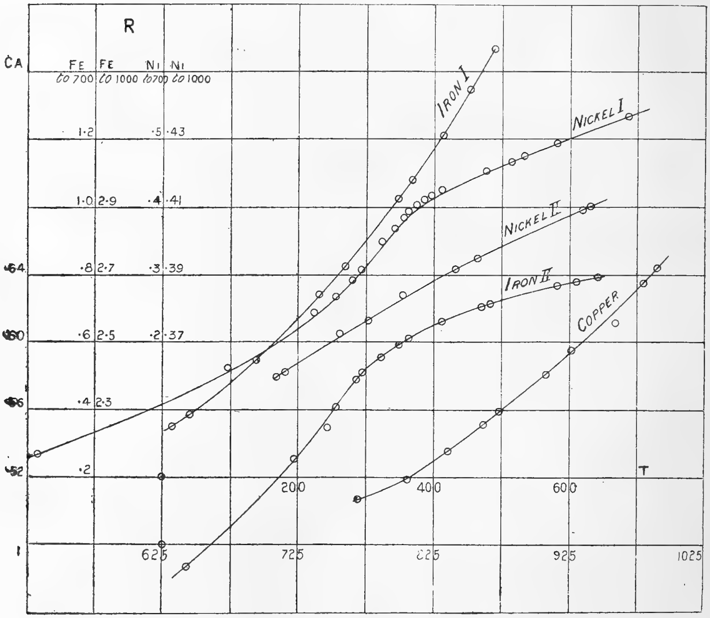
The parabola

$$R_{Fe} = \cdot 3318 + \cdot 001230t + \cdot 000002209t^2$$

fitted the iron resistance-curve up to 500°.

The shape of the nickel and iron resistance-curves are strikingly alike at the temperature of the change in each.

Fig. 8.



There was very little oxidation in these high-temperature experiments; in most cases the copper leads at their junction with the spirals appearing perfectly bright. A reddish deposit (apparently of metallic copper) was usually seen on the spiral and mica disks.

A further series of measurements was made in hydrogen.

No change whatever in the temperature at which the change of slope for iron occurred was produced by this, although the two curves were not absolutely coincident.

The change of resistance for copper was determined between 750° and 1000°. The copper wire (No. 28 B.W.G.) was wound on a mica frame and was heated in hydrogen. It is known that up to 500° or 600° the resistance of copper changes like platinum. The present experiments show that above that temperature a point of inflexion occurs, and the resistance-curve becomes convex towards the temperature axis.

Above 800° the curve is nearly linear, with another less clearly defined change of slope just below 1000°.

Numerical Results.

E.M.F. in microvolts.	Temperature (Air-Therm.).	Note.
Copper-Nickel.		
−2907	−191·2	Liquid air.
2233	99·8	Steam.
10335	445·5	Sulphur vapour.
4321	182·5	Aniline vapour.
Copper-Iron.		
−2195	−191·2	Liquid air.
1334	182·5	Aniline.
870	99·9	Steam.
684	445·5	Sulphur vapour.
98	11·2	
Iron-Nickel.		
−5111	−191·2	Liquid air.
11807	500	
3130	100	
6160	200	

Peltier Coefficients.

$T.\frac{dE}{dt}.$	Temperature.	Note.
Copper-Iron.		
+1156	−191·2	Liquid air.
Copper-Nickel.		
−7697	−191·2	Liquid air.

A Comparison with Fleming's Results for Iron, Copper, and Nickel-Copper.

	Present Experiments.	Fleming and Dewar's.
	E.M.F. in microvolts.	E.M.F. in microvolts.
Copper-Nickel.		
Liquid air.	−2907	−2380
100° C.	2233	2205
Copper-Iron.		
Liquid air.	−2191	−2589
100° C.	870	1138
Nickel-Iron.		
Liquid air.	−5111	−4969

The main object of the present research was originally to correlate the peculiar thermoelectric points with the changes of slope on the resistance-curves.

The result from this point of view is that the thermoelectric change in nickel-copper coincides approximately with the resistance change, but that no theremoelectric peculiarity exists for iron-copper at the temperature of the Fe resistance change. As is shown by the Peltier coefficients, a marked change occurs at about 500° in the case of Fe–Cu. This is approximately the temperature at which a flexure occurs in the copper resistance-curve.

It is perhaps worth noticing that if the dotted part of the Fe-Cu curve (fig. 5) (which is marked HI and represents the first few points obtained by heating a fresh couple in hydrogen) be produced, it cuts the upper and "constant" portion of the curve at a temperature just below 800°, the temperature of the resistance-changes in iron.

In conclusion I wish to express my best thanks to Professor Callendar, F.R.S., for constant help and advice during this research, which was undertaken at his suggestion.

My thanks are also due to Mr. G. M. Gibbins, of University College, for help during the first part of the thermoelectric measurements.

XX. *On a kind of Radioactivity imparted to certain Salts by Cathode Rays.* *By* J. C. McLennan, *Ph.D., Demonstrator in Physics, University of Toronto*[*].

I. *Introduction.*

It has been shown by Curie[†] that every substance, placed in the neighbourhood of radium, itself acquires a radioactivity which persists for many hours and even days after the removal of the radium. Curie[‡] has also obtained a similar but more feeble effect with the active preparation polonium.

Rutherford[§], too, has shown that thorium compounds under certain conditions possess the property of producing temporary radioactivity in all solid substances in their neighbourhood.

Up to the present time we appear to have only two recorded cases of substances becoming radioactive by the action of cathode rays.

Hofmann and Strauss[||] observed that radioactive lead sulphate prepared from the minerals uranpecherz, bröggerite, cleveite, uranglimmer, and samarskite, after losing its activity through lapse of time, was again made radioactive by exposing it to cathode rays.

Villard[¶] found that a small piece of bismuth which had been used as the anticathode of a discharge-tube produced a

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[*] Communicated by Prof. J. J. Thomson.

[†] Rapports présentés au Congrès International de Physique, Paris, tome iii. p. 108 (1900).

[‡] Curie, *Comptes Rendus*, t. cxxix. p. 714, Nov. 1899.

[§] Rutherford, Phil. Mag. vol. xlix. p. 161 (1900).

[||] *Chem. Ber.* xxxiv. pp. 8–11 & 907–13; *Beibl.* xxv. pp. 317 & 633 (1901).

[¶] Villard, *Société de Physique*, juillet 1900.

faint impression on a photographic plate. The action was very weak, and an exposure of eight days was necessary in order to make the effect visible.

This latter case appears to be the only example, hitherto recorded, of a substance normally inactive being made radioactive by exposing it to cathode rays.

In the following paper an account is given of an investigation in which several salts, showing no radioactivity as ordinarily prepared, were found to exhibit a very marked radioactivity on being gently heated after being exposed to cathode rays. With several of the salts a temperature of 100° C. sufficed to exhibit the effect, though with still higher temperatures the radioactivity was more pronounced.

The radiation emitted by these salts was found to discharge positively charged bodies, but not those negatively electrified. In no case did it appear to impart a charge to an unelectrified body.

II. *Measuring System.*

In order to study this radiation the system devised by C. T. R. Wilson* was used. A thin strip of brass, A (fig. 1), was suspended vertically in a small metallic box, B, by means of an insulating bead of sulphur, C. To this strip there was attached a gold leaf whose deflexions, observed with a micrometer microscope through a glass window in the apparatus, gave a measure of the potential of the brass strip.

The capacity of the whole measuring system was small, and at most did not exceed two centimetres.

A brass rod, D, which supported the measuring system passed through an ebonite plug, E, and was connected to one of the terminals of a battery of small storage-cells, the other terminal of the battery being joined to the case of the electroscope.

A fine steel wire which was attached at its upper end to the rod, D, extended below the sulphur bead, and was there bent into a loop surrounding the brass strip. This steel wire could be readily drawn into contact with the strip, A, by means of a small magnet, and the measuring system by this operation was raised from time to time to any desired potential.

Besides having a small capacity, this measuring system possessed the advantage of being entirely free from leakage along its supports. In practice the conducting rod, D, was maintained, throughout any measurement, at the initial potential of the gold leaf, and consequently, when the latter

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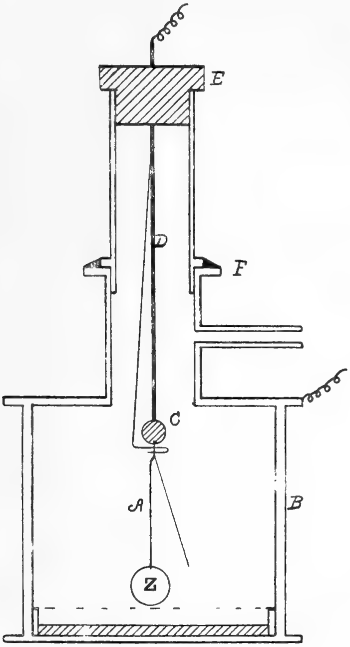
* Proc. Roy. Soc. vol. lxviii. p. 154.

indicated any loss of charge, it was evident that such loss arose from a leakage through the gas surrounding the measuring system. Any conduction across the sulphur bead could only be in such a direction as to partially counteract the loss arising from leakage through the gas.

Fig. 1.



In order to get rid of disturbances of the gold leaf by air-currents set up on applying heat to the base of the electroscope, the observations were made generally at a pressure of 15 mms. of mercury.

III. *Methods of making Salts Radioactive.*

In the initial stages of the investigation the salts to be tested for radioactivity were subjected to bombardment by cathode rays in a tube similar to that shown in fig. 2 (p. 198). This tube, while carrying but a single anode, was provided with a number of cathodes so that several salts could be simultaneously exposed.

The salts were generally exposed to the rays for half an hour, and from time to time during exposure were gently shaken in the tube so that fresh surfaces might be presented to the rays.

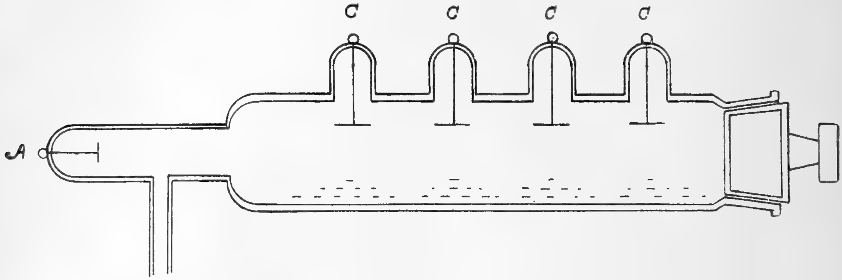


Fig. 2.

This method of exciting the salts was somewhat tedious, as it necessitated the exhaustion of the discharge-tube each time an exposure was made.

It was found, however, on trial that a radioactivity, similar and equal in intensity to that obtained with cathode rays, was imparted to the salts by simply exposing them in air at atmospheric pressure to the spark-discharge of an induction-coil or Wimshurst machine. Short thick sparks were more effective in exciting radioactivity than long thin ones; and therefore, in most cases in which this method was adopted, a leyden-jar was inserted in parallel with the terminals of the induction-coil during the exposures.

Various metals were used in turn as terminals for the coil, but no difference in their efficiency was observed.

In order to see whether the excited radioactivity was due to the presence of ultra-violet light in the spark-discharge, a quantity of one of the salts which could be made radioactive was placed in a shallow tray beneath the spark-gap and close to it. The sparks were made to pass between terminals of aluminium, and a thin quartz plate was placed between the salt and the spark-gap.

With this arrangement no radioactivity was imparted to the salt. On removing the quartz plate, however, so that nothing but air intervened between the salt and the spark-discharge, the salt became radioactive with the usual exposure.

From this experiment it seemed clear that the observed radioactivity could not be due to the action of the ultra-violet light, but was in all probability excited by a peculiar kind of radiation shown by E. Wiedemann[*] to be emitted by the spark-discharge and called by him "Entladungstrahlen."

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[*] *Zeitschrift für Electrochemie,* vol. ii. p. 159 (1895).

IV. *Radioactivity imparted to certain Salts.*

When testing a salt for radioactivity the upper part of the electroscope which carried the measuring system was detached at F, and the salt placed in a thin layer on the base of the electroscope. The measuring system was then replaced and the apparatus made air-tight with sealing-wax. The air was then exhausted to a pressure of 15 mm. of mercury, and the gold leaf was given a charge either positive or negative as desired. After this a Bunsen flame was applied to the base of the electroscope, and changes in the charge on the measuring system were observed by noting the motion of the gold leaf.

A list of the salts examined in this way, after bombardment by cathode rays, is given in Table I. With several no radioactivity was observed, but with others, such as the sulphates and sulphides of a number of the elements, a radiation was obtained which discharged the measuring system when positively electrified. In no case did this radiation discharge the gold leaf when negatively electrified. If the gold leaf was charged negatively, then on heating the salt the only effect was a slight disturbance of the leaf due, doubtless, to convection currents. If, however, the gold leaf was charged positively, no effect was observed until the salt reached a certain temperature; the leaf then began to fall, and continued falling with a rapid and regular movement for a few minutes, and then stopped even though the heating continued.

Table I.

Substance.		Column I. Gold leaf positively charged.	Column II. Gold leaf negatively charged.	Column III. Thermo-luminescence.
Manganese Sulphate.	$MnSO_4$	No leak.	No leak.	None observed.
Zinc ,,	$ZnSO_4$	,, ,,	,, ,,	Faint white.
Lead ,,	$PbSO_4$	,, ,,	,, ,,	None observed.
Beryllium ,,	$BeSO_4$	Leak (small).	,, ,,	None observed.
Calcium ,,	$CaSO_4$	Leak.	,, ,,	Very faint.
Strontium ,,	$SrSO_4$	,,	,, ,,	Faint white.
Barium ,,	$BaSO_4$	,,	,, ,,	Faint white.
Potassium ,,	K_2SO_4	,,	,, ,,	None observed.
Calcium Sulphide.	CaS	,,	,, ,,	None observed.
Strontium ,,	SrS	,,	,, ,,	None observed.
Barium ,,	BaS	,,	,, ,,	None observed.
Calcium Oxide.	CaO	No leak.	,, ,,	None observed.
Calcium Chloride.	$CaCl_2$	,, ,,	,, ,,	Faint white.
Barium ,,	$BaCl_2$	,, ,,	,, ,,	Faint white.

The sulphates of calcium, strontium, barium, and potassium were found to differ but little in their radioactive powers, while the effect obtained with beryllium sulphate was quite small. In the case of the sulphides the radioactivity was found to be only about one-fifth that of the sulphates of the same elements.

It is of interest to note in this connexion that although calcium chloride could not be made radioactive, still calcium sulphate prepared from this salt became quite radioactive on being exposed to cathode rays.

All the salts examined rapidly lost their radioactivity on being heated. From three to five minutes' exposure in the electroscope to the flame of a Bunsen burner generally sufficed to drive off completely every trace of radioactivity.

The active salts also gradually lost their radioactivity with the lapse of time without being heated. With a specimen of calcium sulphate tested twenty hours after exposure to cathode rays, the leak obtained was faint though still quite marked.

In a series of comparative measurements with active calcium sulphate, in air at pressures varying from 480 mms. down to 1 mm. of mercury, no difference either in the rate or the total amount of the positive leak was observed. At no pressure was any leak of the negative charge obtained, and the gold leaf, when initially unelectrified, did not acquire any charge from the radiation.

With the same salt (calcium sulphate) a magnetic field of 3000 C.G.S. units was found to slightly decrease the leak.

In order to obtain an approximate estimate of the lowest temperature at which the radioactivity appeared in the case of calcium sulphate, a jet of steam was directed against the base of the electroscope. The temperature so obtained was sufficient to bring on the discharge, but the rate of leak was much smaller than when the electroscope was heated with the flame of a Bunsen burner.

Each of the salts given in Table I. was carefully tested for radioactivity at the temperature of the laboratory (about 18° C.), both before being exposed to the cathode rays and after bombardment; but with the exception of potassium sulphate none of them produced a leak under these circumstances. With potassium sulphate, a small leak of positive electricity occurred if the test was made immediately after exposure to cathode rays. But this effect soon passed off, and was followed on the application of heat to the electroscope by the usual larger leak obtained with the other active salts.

None of the salts examined exhibited any radioactivity when heated without being previously exposed to cathode rays or to the spark-discharge.

As an example of the magnitude of the effects described in this paper, it may be stated that in one instance the radiation from a thin layer of active calcium sulphate, spread on the base of the electroscope over an area of twenty square centimetres, discharged from two to three electrostatic units of positive electricity.

V. *Impressed Radioactivity and Thermoluminescence.*

E. Wiedemann [*] has shown that a number of substances, when exposed to cathode rays or to the spark-discharge, acquire and possess for some time the power of becoming luminous when their temperature is raised to a point far below that at which they become luminous in their normal state. To this phenomenon he has given the name thermoluminescence.

Now the methods of exciting this phenomenon are precisely the same as those adopted in imparting radioactivity to the various salts just mentioned; and as the range of temperatures over which both phenomena appear is about the same, it seemed possible that some connexion might exist between the two effects.

To investigate this point each of the salts was carefully tested for thermoluminescence, and the results obtained are recorded in column III., Table I. A reference to this table will show that in the case of three salts only was the radioactivity accompanied by the phenomenon of thermoluminescence. In the case of the remaining salts, some exhibited thermoluminescence, but not radioactivity, while with others radioactivity was observed, but no visible thermoluminescence.

From these results it seemed clear that no connexion existed between the two phenomena; but in order to test the matter still further, some experiments were made with a number of Van't Hoff's [†] so-called solid solutions, which were known to exhibit a powerful thermoluminescence. From Table II., which contains a summary of the results, it will be seen that radioactivity was observed with but one solution, namely $CaSO_4 + 2$ p.c. $MnSO_4$. The radioactivity obtained in this case was practically of the same intensity as that obtained with the simple salt, calcium sulphate.

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[*] *Zeitschr. f. Electrochem.* vol. ii. p. 159 (1895).

[†] *Zeitschrift f. phys. Chemie*, vol. v. p. 322 (1890).

TABLE II.

Solid Solution.	Gold leaf positively charged.	Gold leaf negatively charged.	Thermo-luminescence.
$CaSO_4 + 2$ p.c. $MnSO_4$	Leak.	No leak.	Intense green.
$ZnSO_4 + 1$ p.c. $MnSO_4$	No leak.	,, ,,	Intense red.
$MgSO_4 + 1$ p.c. $MnSO_4$	,, ,,	,, ,,	Intense dark red.
$CdSO_4 + 1$ p.c. $MnSO_4$	,, ,,	,, ,,	Intense yellow.
$CaFl_2 + 2$ p.c. $MnFl_2$	,, ,,	,, ,,	Bright green.

Since, therefore, radioactivity was impressed on but one of the five solid solutions, while they all exhibited a very powerful thermoluminescence, it seems obvious that no direct connexion exists between the two phenomena.

This conclusion is also confirmed by the fact that while a temperature of 100° C., or even less, sufficed to exhibit radioactivity in connexion with most of the salts examined, a temperature considerably higher than this was found to be necessary to produce visible thermoluminescence. In fact, in a test made with the sulphates of strontium and barium the radioactivity had almost entirely disappeared before thermoluminescence became well-marked.

Moreover a difference exists in the intervals of time during which the excited salts retain the power to exhibit the two phenomena. As is well known, bodies exhibiting thermoluminescence retain this property for weeks, or even months, after exposure to cathode rays. On the other hand, the present investigation goes to show that salts rendered radioactive by cathode rays completely lose their activity in the course of one or, at most, two or three days.

VI. *Character of the Radiation.*

An experiment was made to see whether the radioactivity could be detected when the electrostatic field of the gold leaf was screened off from the salt by means of a sheet of wire gauze placed above the salt, as shown by the dotted line in fig. 1. Little difference was observed when gauze with meshes 3 millimetres in width was used, but with gauze made from wires 1 mm. in diameter, placed 1 mm. apart, the leak from the measuring system was greatly diminished.

From the experiments made so far, there appear to be but two obvious ways of explaining this peculiar radiation. One of these is to regard the effect as due to ultra-violet light produced by heating the excited salt, this light acting on the negatively-charged salt and thereby causing a leak from its surface.

To test this view, an amalgamated zinc sphere, Z (fig. 1), was attached to the lower end of the measuring system. With this arrangement, though the zinc sphere was within four millimetres of the radioactive salt, no indication of a discharge of negative electricity from the gold leaf was observed.

It seems scarcely possible that an ultra-violet radiation, capable of producing the effects observed, could be absorbed completely in passing through air at 15 mms. pressure for a distance of only four millimetres.

Probably the simpler view to take is that the salts on being heated emit a stream of negatively-charged particles or corpuscles which are drawn to the positively-charged gold leaf by the electrostatic field, and thus discharge it. If this view be correct, these negatively-charged particles must be sent out from the salt with exceedingly small velocities, otherwise we would expect the gold leaf, when initially unelectrified, to acquire at least a small negative charge.

The experiments described in this paper were conducted in the Cavendish Laboratory, and my sincere thanks are due to Professor Thomson for many valuable suggestions and much encouragement given throughout the investigation.

August 1st, 1901.

XXI. *A Comparison of various Notations employed in "Theories of Crystal-structure," and a Revision of the* 230 *Groups of Movements.* *By* Harold Hilton, *Magdalen College, Oxford*[*].

SINCE the determination of the possible groups of movements must hold an important position in any future theories of crystal structure; and since the mathematical theory of these groups has been practically brought to a completion; it seemed advisable to draw up tables comparing the different notations of the various workers on the subject, with the hope that at the same time a judgment might be formed on the correctness of the statement that the number of groups of movements which are applicable to crystallography is 230.

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[*] Communicated by the Author.

Mr. W. Barlow and Mr. G. F. H. Smith have begun a similar piece of work together for the latter reason. When they heard that I had independently finished such a comparison, they placed my work side by side with their own to some extent, and found that we agreed. They also kindly gave me some useful hints. I take this opportunity of thanking them, and also Prof. H. A. Miers, who has very kindly given me his help throughout.

The chief workers on the subject have been C. Jordan (*Annali di Matematica pura ed applicata*, Serie 2, Band ii. 1869); L. Sohncke (*Entwickelung einer Theorie der Krystallstruktur*, Leipzig, 1879); E. von Fedorow (*Zeitschr. f. Kryst. u. Min.* xxiv. 1895); A. Schoenflies (*Mathematische Annalen*, xxviii., xxix. (1887), & xxxiv. (1889), and *Krystallsysteme und Krystallstructur*, Leipzig, 1891); and W. Barlow (*Zeitschr. f. Kryst. u. Min.* xxiii. 1895). (Fedorow's work was published in Russian in 1890.)

The notations are taken from the periodicals or books just named, and compared in the tables given below.

I have myself carefully compared Barlow's notation with that in Schoenflies' book; and have also compared the two notations used by Schoenflies in his book and in the *Mathematische Annalen*. Fedorow has compared his own work with that of Barlow* and of Schoenflies†, and thus indirectly has compared Schoenflies' and Barlow's work. If there was any discrepancy between this and my own direct comparison, I have investigated the case carefully, and have given a note on the subject. The comparison of Schoenflies' work (and hence of Fedorow's and Barlow's) with that of Jordan and Sohncke is that given by Schoenflies in the *Mathematische Annalen*.

The diagrams given by Fedorow‡ are not always correct; thus *e. g.* the diagrams of 62 *s*, 63 *s*, 64 *s*, 49 *h*, 50 *h*, 91 *a*, 92 *a*, should be identical with those of 18 *s*, 20 *s*, 21 *s*, 19 *h*, 24 *h*, 29 *a*, 21 *a*, respectively, which is not always the case; in particular, as Mr. Smith has kindly pointed out to me, the diagrams of 64 *s* and 50 *h* are identical and both wrong. It would be beyond the scope of this paper to prepare fresh diagrams. I hope to do this on some future occasion.

It is as well to call attention to the fact that Fedorow altered his notation a little in *Zeitschr. Kryst. Min.* xxiv. pp. 233 & 237; that Barlow pointed out a few errors in his previous work in *Zeitschr. Kryst. Min.* xxv. pp. 86 & 87;

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* *Zeitschr. Kryst. Min.* xxiv. pp. 242–244.

† *Ibid.* xx. pp. 48–61.

‡ Tafeln V., VI., *Zeitschr. Kryst. Min.* xxiv.

and that Schoenflies did the same in *Kystallsysteme und Krystallstructur*, p. 622.

In some cases Schoenflies (in his paper in the *Math. Annalen*, xxxiv.) gave the same symbol to two groups. Where one of these groups is derived from the other by the substitution of the operation $S(\tau)$ for S, or $S(t+\tau)$ for $S(t)$, I have underlined the symbol in the case of the latter group. Similarly he gives the symbol V_3 as representing 3 distinct groups.

For convenience of printing, English capitals are substituted for German in the symbols of Schoenflies, as no error can thereby arise.

An asterisk against a symbol means that a reference is made to it in the notes following the tables.

TABLE I.

A. Schoenflies. Krystallsyst. u. Krystallstructur, 1891.	*A. Schoenflies.* Mathem. Annalen, xxviii. p. 319; xxix. p. 50 (1887); & xxxiv. p. 172 (1889).	*C. Jordan.* Annali di Matematica pura ed appl. Band ii. 1869.	*L. Sohncke.* Entwickelung einer Theorie der Krystallstruktur, 1879.	*W. Barlow.* Zeitschr. f. Krystall. u. Min. xxiii. pp. 1–63 (1894).	*E. v. Fedorow.* Zeitschr. f. Kryst. u. Min. xxiv. pp. 232–237 (1895).
C_1	no symbol given.	3	1	65	1 *s*
C_2^1	$C_1(2)$	29	2	63	3 *s*
C_2^2	$C_2(2)$	31	3	62	1 *a*
C_2^3	$C_3(2)$	33	4	64	4 *s*
V^1	V_4	91	5	56	9 *s*
V^2	V_7	abs.	6	53	4 *a*
V^3	V_8	93	12	55	7 *a*
V^4	V_9	abs.	14	54	8 *a*
V^5	V_6	abs.	* 9, 13	57	5 *a*
V^6	V_3	92	7	59	10 *s*
V^7	V_2	99	8	61	12 *s*
V^8	V_1	100	10	60	11 *s*
V^9	V_5	101	11	58	6 *a*
C_3^1	$C_1(3)$	61	17	48	38 *s*
C_3^2	$C_2(3)$	63	15	42	68 *a*
C_3^3	$C_2'(3)$	64	16	43	69 *a*
C_3^4	$C_3(3)$	62	18	51	39 *s*
D_3^1	$D_1(3)$	130	21	49	* 45 *s*
D_3^2	$D_3(3)$	abs.	25	50	* 44 *s*
D_3^3	$D_2(3)$	132	19	44	* 72 *a*
D_3^4	$D_4(3)$	abs.	23	46	* 70 *a*
D_3^5	$D_2'(3)$	133	20	45	* 73 *a*
D_3^6	$D_4'(3)$	abs.	24	47	* 71 *a*
D_3^7	$D_5(3)$	131	22	52	46 *s*

TABLE I. (*continued*).

A. Schoenflies. Krystallsyst. u. Krystall-structur, 1891.	*A. Schoenflies.* Mathem. Annalen. xxviii. p. 319; xxix. p. 50 (1887); & xxxiv. p. 172 (1889).	*C. Jordan.* Annali di Matematica pura ed appl. Band ii. 1869.	*L. Sohncke.* Entwickelung einer Theorie der Krystall-struktur, 1879.	*W. Barlow.* Zeitschr. f. Krystall. u. Min. xxiii. pp. 1–63 (1894).	***E. v. Fedorow.*** **Zeitschr. f. Kryst. u. Min. xxiv.** pp. 232–237 (1895).
C_4^1	$C_1(4)$	54	30	34	22 *s*
C_4^2	$C_2(4)$	55	26	26	30 *a*
C_4^3	$C_3(4)$	56	29	29	33 *a*
C_4^4	$C_2'(4)$	57	27	27	31 *a*
C_4^5	$C_5(4)$	58	31	38	23 *s*
C_4^6	$C_4(4)$	59	28	28	32 *a*
D_4^1	$D_1(4)$	116	36	39	30 *s*
D_4^2	$D_2(4)$	124	41	40	43 *a*
D_4^3	$D_3(4)$	117	32	30	* 44 *a*
D_4^4	$D_4(4)$	125	38	32	* 48 *a*
D_4^5	$D_5(4)$	118	35	36	47 *a*
D_4^6	$D_6(4)$	126	40	37	50 *a*
D_4^7	$D_3'(4)$	119	33	31	* 45 *a*
D_4^8	$D_4'(4)$	127	39	33	* 49 *a*
D_4^9	$D_7(4)$	120	37	41	31 *s*
D_4^{10}	$D_8(4)$	121	34	35	46 *a*
C_6^1	$C_1(6)$	47	47	23	49 *s*
C_6^2	$C_2(6)$	48	42	14	74 *a*
C_6^3	$C_2'(6)$	52	43	15	75 *a*
C_6^4	$C_3(6)$	49	44	16	76 *a*
C_6^5	$C_3'(6)$	51	45	17	77 *a*
C_6^6	$C_4(6)$	50	46	20	78 *a*
D_6^1	$D_1(6)$	108	53	25	54 *s*
D_6^2	$D_2(6)$	109	48	18	82 *a*
D_6^3	$D_2'(6)$	113	49	19	83 *a*
D_6^4	$D_3(6)$	110	50	21	84 *a*
D_6^5	$D_3'(6)$	112	51	22	85 *a*
D_6^6	$D_4(6)$	111	52	24	86 *a*
T^1	T_3	150	54	7	59 *s*
T^2	T_2	164	55	6	61 *s*
T^3	T_1	151	56	10	60 *s*
T^4	T_5	155	58	1	89 *a*
T^5	T_4	156	57	2	90 *a*
O^1	O_4	166	59	12	68 *s*
O^2	O_5	169	64	11	98 *a*
O^3	O_2	168	60	8	70 *s*
O^4	O_3	172	63	9	97 *a*
O^5	O_1	167	61	13	69 *s*
O^6	O_7	170	65	3	94 *a*
O^7	O_7'	171	66	4	95 *a*
O^8	O_6	abs.	62	5	96 *a*

The work of drawing up the tables has led me to the conclusion that the number of possible groups of movements which contain 3 independent translations which are not infinitesimal, is really 230.

NOTES.—Sohncke's groups 9 and 13 are identical, as Schoenflies pointed out.

Fedorow gives* 44 $s=D_3^1$, 45 $s=D_3^2$, 70 $a=D_3^3$; 71 $a=D_3^5$, 72 $a=D_3^4$, 73 $a=D_3^6$, in his table of his own and Schoenflies' results. That this is wrong appears from his own words on p. 228 of the *Zeitschr. f. Kryst. u. Min.* xxiv. Similarly he has wrongly compared 55 $s=D_{3d}^1$, 56 $s=D_{3d}^3$, 45 $h=D_{3d}^2$ 46 $h=D_{3d}^4$, 47 $s=D_{3h}^1$, 48 $s=D_{3h}^3$, 42 $h=D_{3h}^2$, 43 $h=D_{3h}^4$.

Again, he gives in the same place 44 $a=D_4^4$, 45 $a=D_4^8$, 48 $a=D_4^3$, 49 $a=D_4^7$; which is incorrect as appears from p. 229 of the *Zeitschr. f. Kryst. u. Min.* xxiv., and from Table V. at the end of the same volume.

In Barlow's table† there is a small error; there he says that 58 B_1, 58 B_2, 58 B_3 are of type 62 (3 of Sohncke), so specialized that its axes lie as those of the type 58 or 60 (11 or 10 of Sohncke); now these two last groups, which are V^9 and V^8 of Schoenflies, belong to Schoenflies' Raumgitter Γ_v''' and therefore all their 3 series of binary axes are of type C_2^3, not C_2^2 (3 of Sohncke). For "62 (3 of Sohncke)" should be read "64 (4 of Sohncke)."

V_h^{27} appears as "V_h^{24}" on p. 51 of the *Zeitschr. f. Kryst. u. Min.* xx.—obviously a misprint. The corresponding group (58 a_2) of Barlow is not given quite correctly by him. He gives as the position of the centre "On a line midway between two neighbouring screw-axes of different sorts in its point of intersection with a rotation-axis." This would make his types 58 a_1 and 58 a_2 identical, and the same as Schoenflies' V_h^{28}; the last phrase should run "midway between its points of intersection with a rotation axis."

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* *Zeitschr. Kryst. Min.* xx. (1892) p. 56.

† *Ibid.* xxiii. (1895) p. 56.

TABLE II.

A. Schoenflies.		*W. Barlow.*	*E. von Fedorow.*	*A. Schoenflies.*		*W. Barlow.*	*E. von Fedorow.*	*A. Schoenflies.*		*W. Barlow.*	*F. von Fedorow.*
C_i	No symbol given.	$65\,a_1$	$2\,s$	C_{2v}^{18}	$c_3^s\,(2)$	$61\,B_1$	$17\,s$	V_h^{24}	v_2^i	$61\,a_2$	$24\,h$
C_s^1	No symbol given.	$65\,B_1$	$5\,s$	C_{2v}^{19}	$c_3^{m_1}(2)$	$61\,B_2$	$16\,h$	V_h^{25}	v_1^h	$60\,a_1$	$20\,s$
C_s^2	Abs.	$65\,B_2$	$1\,h$	C_{2v}^{20}	$c_3^{d_1}(2)$	$*\,58\,B_1$	$16\,s$	V_h^{26}	v_1^m	$60\,a_2$	$23\,h$
C_s^3	No symbol given.	$65\,B_3$	$6\,s$	C_{2v}^{21}	$c_3^{d_1}(2)$	$*\,58\,B_2$	$15\,h$	V_h^{27}	v_5^i	$*\,58\,a_2$	$21\,a$
C_s^4	Abs.	$65\,B_4$	$2\,h$	C_{2v}^{22}	$c_3^d\,(2)$	$*\,58\,B_3$	$14\,h$	V_h^{28}	v_5^h	$58\,a_1$	$20\,a$
C_{2h}^1	$c_1^h\,(2)$	$63\,a_1$	$7\,s$	V_h^1	v_4^h	$56\,a_1$	$18\,s$	C_{3i}^1	$c_1^i\,(3)$	$48\,a_1$	$51\,s$
C_{2h}^2	$c_2^h\,(2)$	$62\,a_1$	$2\,a$	V_h^2	v_4^i	$56\,a_3$	$19\,h$	C_{3i}^2	$c_3^i\,(3)$	$51\,a_1$	$52\,s$
C_{2h}^3	$c_3^h\,(2)$	$64\,a_1$	$8\,s$	V_h^3	v_4^m	$56\,a_2$	$17\,h$	C_{3v}^1	$c_1^s\,(3)$	$48\,b_3$	$41\,s$
C_{2h}^4	$c_1^i\,(2)$	$63\,a_2$	$3\,h$	V_h^4	$v_4^{i_1}$	$56\,a_4$	$18\,h$	C_{3v}^2	$c_1^a\,(3)$	$48\,b_1$	$40\,s$
C_{2h}^5	$c_2^i\,(2)$	$62\,a_2$	$3\,a$	V_h^5	v_7^s	$53\,a_1$	$14\,a$	C_{3v}^3	$c_1^s\,(3)$	$48\,b_4$	$40\,h$
C_{2h}^6	$c_3^i\,(2)$	$64\,a_2$	$4\,h$	V_h^6	$v_7^{i_1}$	$53\,a_4$	$17\,a$	C_{3v}^4	$c_1^a\,(3)$	$48\,b_2$	$39\,h$
C_{2v}^1	$c_1^s\,(2)$	$56\,B_1$	$13\,s$	V_h^7	v_7^{m1}	$53\,a_2$	$15\,a$	C_{3v}^5	$c_3^a\,(3)$	$51\,b_1$	$42\,s$
C_{2v}^2	$c_2^s\,(2)$	$53\,B_1$	$9\,a$	V_h^8	v_7^i	$53\,a_3$	$16\,a$	C_{3v}^6	$c_3^a\,(3)$	$51\,b_2$	$41\,h$

C^3_{2v}	c^s_1 (2)	56 B_2	5 *h*	V^9_h	v^h_8	55 a_3	22 *a*	D^1_{3d}	D^a_1 (3)	49 a_1	* 56 *s*
C^4_{2v}	c^m_1 (2)	56 B_3	6 *h*	V^{10}_h	$\mathrm{v}^{i_1}_8$	55 a_6	27 *a*	D^2_{3d}	D^i_1 (3)	49 a_2	* 46 *h*
C^5_{2v}	c^m_2 (2)	53 B_3	11 *a*	V^{11}_h	$\mathrm{v}^{m_1}_8$	55 a_2	23 *a*	D^3_{3d}	D^s_3 (3)	50 a_1	* 55 *s*
C^6_{2v}	c^m_1 (2)	56 B_4	7 *h*	V^{12}_h	v^m_8	55 a_4	25 *a*	D^4_{3d}	D^i_3 (3)	50 a_2	* 45 *h*
C^7_{2v}	c^m_2 (2)	53 B_2	10 *a*	V^{13}_h	v^s_8	55 a_1	24 *a*	D^5_{3d}	D^a_5 (3)	52 a_1	57 *s*
C^8_{2v}	$\mathrm{c}^{m_1}_1$ (2)	56 B_5	9 *h*	V^{14}_h	v^i_8	55 a_5	26 *a*	D^6_{3d}	D^i_5 (3)	52 a_2	47 *h*
C^9_{2v}	$\mathrm{c}^{m_1}_2$ (2)	53 B_4	12 *a*	V^{15}_h	v^i_9	54 a_1	29 *a*	S^1_4	c^q_1 (2)	63 *c*	26 *s*
C^{10}_{2v}	$\mathrm{c}^{m_1}_1$ (2)	56 B_6	8 *h*	V^{16}_h	v^h_9	54 a_2	28 *a*	S^2_4	c^q_3 (2)	64 *c*	27 *s*
C^{11}_{2v}	c^d_1 (2)	59 B_1	14 *s*	V^{17}_h	v^s_6	57 a_1	18 *a*	C^1_{4v}	c^s_1 (4)	34 b_1	24 *s*
C^{12}_{2v}	c^d_2 (2)	57 B_1	13 *a*	V^{18}_h	$\mathrm{v}^{m_1}_6$	57 a_2	19 *a*	C^2_{4v}	c^d_1 (4)	34 b_3	26 *h*
C^{13}_{2v}	c^d_1 (2)	59 B_2	10 *h*	V^{19}_h	v^s_3	59 a_1	19 *s*	C^3_{4v}	c^s_3 (4)	29 b_1	* 37 *a*
C^{14}_{2v}	$\mathrm{c}^{s_1}_3$ (2)	59 B_3	15 *s*	V^{20}_h	v^m_3	59 a_2	20 *h*	C^4_{4v}	c^d_3 (4)	29 b_3	* 38 *a*
C^{15}_{2v}	$\mathrm{c}^{s_1}_3$ (2)	59 B_4	11 *h*	V^{21}_h	$\mathrm{v}^{m_1}_3$	59 a_3	21 *h*	C^5_{4v}	c^s_1 (4)	34 b_2	25 *h*
C^{16}_{2v}	c^m_3 (2)	59 B_5	12 *h*	V^{22}_h	v^i_3	59 a_4	22 *h*	C^6_{4v}	c^d_1 (4)	34 b_4	27 *h*
C^{17}_{2v}	c^m_3 (2)	59 B_6	13 *h*	V^{23}_h	v^h_2	61 a_1	21 *s*	C^7_{4v}	c^a_3 (4)	29 b_2	* 36 *a*

TABLE II. (*continued*).

A. Schoenflies.		*W. Barlow.*	*E. von Fedorow.*	*A. Schoenflies.*		*W. Barlow.*	*E. von Fedorow.*	*A. Schoenflies.*		*W. Barlow.*	*E. von Fedorow.*
C_{4v}^{8}	C_{3}^{d} (4)	$29\,b_4$	$39\,a$	D_{4h}^{5}	D_{2}^{h} (4)	$40\,a_1$	$54\,a$	D_{6h}^{1}	D_{1}^{h} (6)	$25\,a_1$	$58\,s$
C_{4v}^{9}	C_{5}^{s} (4)	$38\,b_1$	$25\,s$	D_{4h}^{6}	D_{2}^{m} (4)	$40\,a_2$	$56\,a$	D_{6h}^{2}	D_{1}^{m} (6)	$25\,a_2$	$48\,h$
C_{4v}^{10}	C_{5}^{b} (4)	$38\,b_2$	$28\,h$	D_{4h}^{7}	D_{2}^{s} (4)	$40\,a_3$	* $55\,a$	D_{6h}^{3}	D_{4}^{h} (6)	$24\,a_2$	$87\,a$
C_{4v}^{11}	C_{4}^{d} (4)	$28\,b_1$	$34\,a$	D_{4h}^{8}	D_{2}^{i} (4)	$40\,a_4$	* $57\,a$	D_{6h}^{4}	D_{4}^{m} (6)	$24\,a_1$	$88\,a$
C_{4v}^{12}	C_{4}^{d} (4)	$28\,b_2$	$35\,a$	D_{4h}^{9}	D_{5}^{h} (4)	$36\,a_2$	$60\,a$	T_{h}^{1}	T_{3}^{h}	$7\,a_1$	$62\,s$
C_{4h}^{1}	C_{1}^{h} (4)	$34\,a_1$	$28\,s$	D_{4h}^{10}	D_{5}^{m} (4)	$36\,a_1$	$61\,a$	T_{h}^{2}	T_{3}^{i}	$7\,a_2$	$49\,h$
C_{4h}^{2}	C_{3}^{h} (4)	$29\,a_1$	$41\,a$	D_{4h}^{11}	D_{5}^{i} (4)	$36\,a_4$	$63\,a$	T_{h}^{3}	T_{2}^{h}	$6\,a_1$	$64\,s$
C_{4h}^{3}	C_{1}^{i} (4)	$34\,a_2$	$29\,h$	D_{4h}^{12}	D_{5}^{d} (4)	$36\,a_3$	$62\,a$	T_{h}^{4}	T_{2}^{i}	$6\,a_2$	$50\,h$
C_{4h}^{4}	C_{3}^{i} (4)	$29\,a_2$	$42\,a$	D_{4h}^{13}	D_{6}^{h} (4)	$37\,a_2$	* $66\,a$	T_{h}^{5}	T_{1}^{h}	$10\,a_1$	$63\,s$
C_{4h}^{5}	C_{5}^{h} (4)	$38\,a_1$	$29\,s$	D_{4h}^{14}	D_{6}^{m} (4)	$37\,a_1$	* $64\,a$	T_{h}^{6}	T_{5}^{i}	$1\,a_1$	$91\,a$
C_{4h}^{6}	C_{4}^{i} (4)	$28\,a_1$	$40\,a$	D_{4h}^{15}	D_{6}^{a} (4)	$37\,a_4$	$67\,a$	T_{h}^{7}	T_{4}^{i}	$2\,a_1$	$92\,a$
V_{d}^{1}	V_{4}^{d}	$56\,\beta_1$	$32\,s$	D_{4h}^{16}	D_{6}^{s} (4)	$37\,a_3$	* $65\,a$	T_{d}^{1}	T_{3}^{d}	$7\,b_1$	$65\,s$
V_{d}^{2}	V_{4}^{q}	$56\,\beta_2$	$30\,h$	D_{4h}^{17}	D_{7}^{h} (4)	$41\,a_1$	$37\,s$	T_{d}^{2}	T_{2}^{d}	$6\,b_1$	$67\,s$

V_d^3	v_8^d	$55\,\beta_1$	$52\,a$	D_{4h}^{18}	$\mathrm{D}m_7$ (4)	$41\,a_2$	$38\,h$	T_d^3	T_1^d	$10\,b_1$	$66\,s$
V_d^4	v_8^q	$55\,\beta_2$	$53\,a$	D_{4h}^{19}	D_8^d (4)	$35\,a_1$	$58\,a$	T_d^4	T_3^q	$7\,b_2$	$51\,h$
V_d^5	v_3^d	$59\,\beta_1$	$33\,s$	D_{4h}^{20}	D_8^i (4)	$35\,a_2$	$59\,a$	T_d^5	T_2^q	$6\,b_2$	$52\,h$
V_d^6	v_3^q	$59\,\beta_2$	$31\,h$	C_{3h}^1	c_1^h (3)	$48\,b_5$	$43\,s$	T_d^6	Abs.	$2\,b_1$	$93\,a$
$*V_d^7$	$v_3^q\ a$	$59\,\beta_3$	$32\,h$	D_{3h}^1	D_1^h (3)	$49\,b_1$	$*\ 48\,s$	O_h^1	o_4^h	$12\,a_1$	$71\,s$
$*V_d^8$	$v_3^q\ \beta$	$59\,\beta_4$	$33\,h$	D_{3h}^2	D_1^m (3)	$49\,b_2$	$*\ 43\,h$	O_h^2	o_4^i	$12\,a_2$	$53\,h$
V_d^9	v_2^d	$61\,\beta_1$	$35\,s$	D_{3h}^3	D_3^h (3)	$50\,b_1$	$*\ 47\,s$	O_h^3	o_5^h	$11\,a_1$	$102\,a$
V_d^{10}	v_2^q	$61\,\beta_2$	$34\,h$	D_{3h}^4	D_3^m (3)	$50\,b_2$	$*\ 42\,s$	O_h^4	o_5^d	$11\,a_2$	$103\,a$
V_d^{11}	v_1^d	$60\,\beta_1$	$34\,s$	C_{6v}^1	c_1^a (6)	$23\,b_1$	$50\,s$	O_h^5	o_2^h	$8\,a_1$	$*\ 73\,s$
V_d^{12}	v_5^q	$58\,\beta_1$	$51\,a$	C_{6v}^2	c_1^a (6)	$23\,b_2$	$44\,h$	O_h^6	$o_2^{m'}$	$8\,a_2$	$54\,h$
D_{4h}^1	D_1^h (4)	$39\,a_1$	$36\,s$	C_{6v}^3	c_4^a (6)	$20\,b_1$	$80\,a$	O_h^7	o_3^d	$*\ 9\,a_1$	$100\,a$
D_{4h}^2	D_1^m (4)	$39\,a_2$	$35\,h$	C_{6v}^4	c_4^d (6)	$20\,b_2$	$79\,a$	O_h^8	o_3^i	$9\,a_2$	$101\,a$
D_{4h}^3	D_1^d (4)	$39\,a_3$	$*\ 36\,h$	C_{6h}^1	c_1^h (6)	$23\,a_1$	$53\,s$	O_h^9	o_1^h	$13\,a_1$	$*\ 72\,s$
D_{4h}^4	D_1^i (4)	$39\,a_4$	$*\ 37\,h$	C_{6h}^2	c_4^h (6)	$20\,a_1$	$81\,a$	O_h^{10}	o_6^i	$5\,a_1$	$99\,a$

Schoenflies does not appear to give his groups V_d^7 and V_d^8 quite correctly*; he puts the plane of the operations $S_d(\tau_d)$, $S_d(\tau_d+\tau_z)$ in the same position as the operations S_d and $S_d(\tau_z)$ of V_d^5 and V_d^6; but in that case the operations referred to would not bring the axes of V^6 into self-coincidence, as is necessary. Barlow gives the position of the plane quite correctly on p. 54, *Zeitschr. Kryst. u. Min.* xxiii.; and this remark agrees with Schoenflies' own treatment of these groups in the *Math. Annalen*, xxxiv.

Fedorow's comparison of his groups 36 *h*, 37 *h*, 55 *a*, 57 *a*, 60 *a*, 61 *a*, 65 *a*, 66 *a* with Barlow's types seems to be incorrect, judging by the Tables V. and VI. given at the end of the 24th volume of the *Zeitschr. f. Kryst. u. Min.*; as is also his comparison of 36 *a*, 37 *a*, 38 *a*, 64 *a*, 65 *a*, 66 *a* with the corresponding groups of Schoenflies.

Fedorow compares 72 *s*, 73 *s* with O^9 and O^5; an obvious misprint for O_h^9 and O_h^5.

Fedorow's remark on Barlow's type 9 a_1 (*Zeitschr. Kryst. u. Min.* xxiv. p. 244) seems not to be accurate. The position of the inversion centre given by Barlow agrees entirely with that given by Schoenflies, and involves symmetry planes parallel, not to the surface planes, but to the diagonal planes of the cube.

XXII. *On the Electrical Conductivities produced in Air by the Motion of Negative Ions.* By P. J. KIRKBY, *M.A., Fellow of New College, Oxford*†.

IN the February number of the Philosophical Magazine for 1901 Prof. Townsend described in detail a series of experiments by which he demonstrated the remarkable fact that negative ions produced in a gas are capable of disintegrating other molecules with which they collide, and thus generate other ions by collision. As this adds one more to the few known methods by which ions are produced, the discovery opens up a new field of research. A series of experiments were given (J. S. Townsend and P. J. Kirkby) in the June number of the Phil. Mag. 1901, which proved that the same results which apply to air apply also to hydrogen and carbonic acid gas. These researches exhibited the extraordinary difference in the behaviour of positive and negative ions. It seemed therefore desirable further to emphasize this difference,

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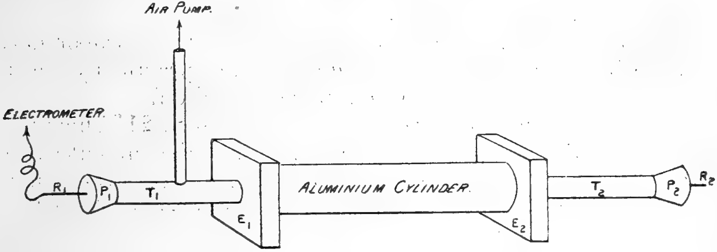
* *Krystallsysteme und Krystallstructur*, p. 497.

† Communicated by Prof. J. S. Townsend.

and for this purpose the following experiments were undertaken.

1. The form of the apparatus is shown in the accompanying figure. It consisted of two coaxial cylinders. The outer one was of aluminium in order that the Röntgen rays, which were used to ionize the air within, might pass freely through it. The internal diameter was 41·5 mm. A copper wire formed the inner cylinder: its diameter was ·206 mm. The ends of the external cylinder were cemented into grooves turned in pieces of ebonite E_1, E_2, as shown in fig. 1. Brass tubes, T_1, T_2, coaxial with the aluminium cylinder projected from the ebonite pieces E_1, E_2, and carried the ebonite plugs P_1, P_2, into the centres of which metallic rods R_1, R_2 were fastened. The copper wire was stretched between these rods. The brass tubes T_1, T_2, which were permanently earthed, formed a divided insulation which effectually prevented any leakage between the cylinders when the outer cylinder was charged to a high potential. The space within the aluminium cylinder was exhausted to the required amount by means of the thin vertical tube shown in the diagram, which was soldered into the brass tube T_1. The pressure was determined by a Macleod gauge which was in permanent connexion with the aluminium cylinder.

Fig. 1.



The Röntgen rays passed through a slit in a lead-covered box which contained the Crookes tube and the Ruhmkorf coil. The aluminium cylinder was placed over the slit.

2. The method of conducting the experiments was similar to that adopted in the paper already referred to. One pair of quadrants of the electrometer was permanently connected to the copper wire. The other pair was permanently to earth. The potential of the aluminium cylinder was raised to any required amount by connecting it to one terminal of a battery of small Leclanché cells, the other terminal being to earth. The voltage used ranged in general from 37 to 370.

The cells throughout these experiments underwent no perceptible variation.

Before each observation the quadrants in connexion with the copper wire were put to earth and then insulated. The Röntgen rays were then turned on for a certain time (generally 10 seconds), and the electrometer-deflexion was observed.

The copper wire which formed the inner cylinder was never raised to a potential materially higher than zero when compared with the potential of the aluminium cylinder. And so, the intensity of the field of force between the cylinders did not vary materially during an experiment. The ions formed by the Röntgen rays playing upon the attenuated air within the apparatus were driven along the lines of force which radiated out from the wire.

It was found that the electrometer-reading was proportional to the time, showing that the current was practically steady and ceased the instant the rays were turned off.

3. The observations given in the table on p. 215 show that when the aluminium cylinder was negative, the current varied greatly with the electric intensity. Thus, on some occasions, on multiplying the intensity between the cylinders by 10 the current increased twenty- or even thirtyfold. On the other hand, if the aluminium cylinder was positive, the current remained nearly of the same order of magnitude, although the electric force between the cylinders was great enough almost to produce sparking.

At the head of each double column will be found the pressure of the air measured in millimetres of mercury by the Macleod gauge. The left of each pair of columns gives the reading when the external, *i. e.* the aluminium, cylinder was at a negative potential, so that the negative ions were driven away from it onto the copper wire and the positive ions attracted to it away from the wire. The right of each pair of columns gives the reading when the force was reversed. The numbers are simply the electrometer-readings without being reduced. Those which are given under each pressure are proportional to the quantities of electricity received by the copper wire during an observation, the potential of the aluminium cylinder (positive or negative as the case may be) for that observation being given in the first column of all. The electrometer-reading for a Clark cell was nearly always less than 80, so that though the potential of the wire departed from zero during an experiment, yet in comparison with the potential of the aluminium cylinder, it is easy to see that this departure may be neglected. The bulb which produced the rays varied from day to day, so that no relation exists between the

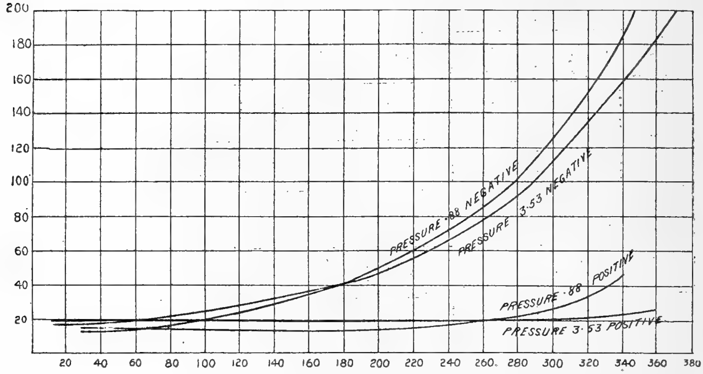
	A.		B.		C.		D.		E.		F.		G.		H.		I.		J.		K.	
Pressure in millims.	6·8.		3·53.		2·14.		1·74.		1·2.		·953.		·88.		·673.		·32.		·247.		·142.	
Voltage of Aluminium Cylinder.	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+
8·8	......	......	17	18·5	......	......	3·3	4·4														
17·7	......	......	18	18·3	......	......	3·5	4·5														
37	45	48·5	17·2	19·5	15·8	17	5	5·5	8·8	10	7·7	8	12·5	13·7	3	3	5	4·1	5·6	4·2	4	2·7
74	......	......	20	20	17·7	17·3	5·6	......	10·8	11·1	11	8·3	15·2	......	3	2·5	......	5·1				
111	......	......	25	18·5	22·3	......	......	......	14·2	......	13·7	......	20·6	......	......	......	10	6				
148	......	......	32	20	30	......	......	......	20	......	21·6	7·8	31	......	......	......	14·8	......	17·1			
185	67·6	......	40	18·5	......	......	12·1	6	......	......	......	......	41·8	......	10·1	......	22·8	......	22·4	......	17·8	
222	83	......	56	18·3	50	19·3	21·2	5·7	38	......	44	10·5	60	15	15·3	......	31·5	......	34·2	......	25·6	
259	105	......	75·5	20·5	......	......	23·9	5·7	......	......	......	......	83·1	18	22·7	6	46·6	......	49·3	11	35·3	
296	138	......	103·5	19·5	97	20·6	35	7·0	69·5	......	97·7	11·5	118·5	24·2	34·2	11	70	18	72	16	51·3	8
333	180	50	150	21·5	......	......	58·3	7·5	101	......	......	......	172·5	37·2	52·7	......	118	......	108	......	82·7	9
370	225	......	202	23·8	187	30·3	80·7	8·0	148·5	55·7	250	18·7	251·5	......	85	......	......	......	189	......	156	12

numbers which belong to different pressures. Nevertheless, only those sets of observations are presented during which the bulb remained constant within the limits of other experimental errors.

4. It thus appears that there is a remarkable difference in the behaviour of positive and negative ions when moving in a small pressure under electric force.

The curves here given (fig. 2) serve as an illustration of the columns B and G. The abscissæ measure the potentials of the aluminium cylinder whether positive or negative. The ordinates of the four curves are simply proportional to the numbers in the four columns respectively of B and G. Each pair of curves exhibits how, as the intensity of the field between the cylinders is increased, the negative current rapidly separates itself in magnitude from the corresponding positive current.

Fig. 2.



It was to give prominence to this curious distinction between the positive and negative electrolytic currents in a gas under low pressure, that coaxial cylinders were used. And to demonstrate it the more emphatically, a thin wire was used in order that there might be a great electric intensity in the field of force.

5. Owing to the constants of the apparatus, it does not appear possible to exhibit any exact or detailed agreement between all the numbers given in the table above and Professor Townsend's theory; but the tests which have suggested themselves show, as far as they go, a satisfactory agreement.

Let a, b be the radii of the internal and external cylinders. Let N_0 be the number of ions of one kind generated per second in unit volume by the Röntgen rays. Let C_r be the steady current inwards of negative ions across a coaxial cylinder of unit length and of radius r.

Then C_r increases as r diminishes. Let α be the number of ions generated by a single ion in moving through 1 centimetre.

Then

$$-dC_r = C_r\alpha\, dr + 2\pi r\, dr\, N_0 e.$$

e being the charge on an ion.

The integral of this equation is

$$C_r = 2\pi N_0 e \int_r^b \rho\, d\rho\, e^{\int_r^\rho \alpha dr}$$

since $C_b = 0$.

Therefore the quantity of negative electricity received per second by the wire is

$$C = 2\pi N_0 e \int_a^\rho \rho\, d\rho\, e^{\int_a^\rho \alpha d\rho}.$$

If $\alpha = 0$ we have

$$C_0 = \pi N e(b^2 - a^2).$$

Therefore

$$C = \frac{2C_0}{b^2 - a^2}\int_a^b r\, dr\, e^{\int_a^r \alpha dr}.$$

6. C_0 may be approximately determined by observation. The tables show that for small forces between the cylinders the positive current is greater than the corresponding negative current. This arises from the secondary radiation which originates from the copper wire. Thus it has been shown by P. Curie and G. Sagnac (*Comptes Rendus*, vol. cxxx. 9th April, 1900), that a metal like copper which gives out a large secondary radiation loses negative electricity under the action of the Röntgen rays. In the present case, the secondary radiation and the loss of negative electricity were small on account of the small dimensions of the wire. Nevertheless, it was sufficient to indicate that voltage beyond which the potential of the aluminium cylinder cannot be lowered without causing the negative current sensibly to exceed C_0. For example, to consider column B, it is clear that when the potential of the cylinder was −8·8, −17·7, −37, there was no detectable increase in the negative current, the negative being so far less than the corresponding positive current, and that the differences in both currents up to this point are due

to errors of observation. But when the potential was lowered to -74 the negative current began perceptibly to increase; for then it had begun to exceed the positive current which is represented too highly by 20. Therefore the voltage of the cylinder beyond which the negative current ceases to be practically C_0 lies between -37 and -74. And, similarly, it may be seen about at what potential of the cylinder the negative current begins to increase for the other pressures of the tables.

7. Unfortunately, at the low pressures given the initial ionization was so small that it would have been useless to employ a voltage numerically lower than -37 with the view of finding C_0. To bring C_0 within the range of accurate observation the aluminium cylinder would require to be larger. On the other hand, it was too large to admit of an evaluation of the integral $\int_a^b r\,dr\,e^{\int_a^r \alpha\,dr}$ in the expression given above for C, though this would be possible under certain more restricted circumstances, by using the approximate form for α, which is given later.

8. Professor Townsend gave α/p both by means of a curve determined experimentally and by means of a series of the form

$$\sum_0^\infty \beta_n e^{-\gamma_n p/X},$$

the coefficients of which he partly determined (Phil. Mag. Feb. 1901). But his numbers were found to be quite inadequate to the present case where the circumstances are different from his, and they lead to a series which scarcely converges at all. In addition to this, though, since $X \propto \frac{1}{r}$, $\int \alpha\,dr$ can be immediately found, yet the next step, if it were possible, would involve great labour. Nevertheless it is possible, without effecting this integration, to show to some extent that the figures of the tables are in agreement with Professor Townsend's results.

9. The force between the cylinders at the distance r from their common axis is

$$\frac{V}{r\log\frac{b}{a}},$$

which, since $a = \cdot0103$ cm., $b = 2\cdot08$ cm., is equal to $\frac{V}{5\cdot3 \times r}$, where V is the potential of the outer cylinder, the wire being approximately at zero. The strong field of force lies close to the inner axis, the intensity at the wire being fifty times the intensity at points where $r=5$ mm. Thus, for moderate values of V the region where ions are generated by collisions is a small cylindrical volume inclosing the wire. Let the radius of this multiplying space be R. Let I negative ions entering this cylindrical space from regions external to it be increased by multiplication to xI ions when the wire is reached. Those ions which are generated by the Röntgen rays within the cylinder R will be multiplied by the process of collision to a lesser extent. Therefore the whole negative current will be

$$C_- = x\left\{C_0 - \frac{R^2-a^2}{b^2-a^2}C_0\right\} + y\frac{R^2-a^2}{b^2-a^2}C_0, \quad . \quad . \quad (1)$$

where $y<x$.

When the force is reversed, the only ions which undergo multiplication are those which start within the cylinder R, so that the corresponding positive current will be

$$C_+ = C_0 - \frac{R^2-a^2}{b^2-a^2}C_0 + z\frac{R^2-a^2}{b^2-a^2}C_0, \quad . \quad . \quad (2)$$

in which z is plainly $<y$.

Now, since $x>y>z>1$, it follows that the negative current always exceeds the positive, the secondary radiation being neglected. And if R^2 is small compared with b^2, the negative current will be about x times the positive.

10. As an illustration we may suppose in particular that R is 1 mm., *i. e.* about twice the mean free path of a negative ion in air at a pressure of 1 mm. Let the cylinder R be divided into two parts by means of a coaxial cylinder of radius $\frac{1}{2}$ R$=\cdot 5$ mm. Then if 100 ions are uniformly distributed within the cylinder R, 25 will *q. p.* fall within the inner cylinder R/2, and 75 between the two cylinders. Further, let it be supposed that every ion executes its mean path R/2, and at the end of it generates another negative ion by collision. Then, when the negative ions are driven to the wire, the original 100 will be increased to $25+2\times 75$ on reaching it. Therefore $y=1\cdot 75$. Now let the force be reversed. Then the same 100 ions will produce a total flux of $2\times 25+75$ ions across the cylinder R. Therefore $z=1\cdot 25$.

Thus $$y=1\cdot 4\times z.$$

11. To apply these results to the tables above, it is to be observed that α/p is negligible unless

$$\frac{X}{p}>30 \text{ (Curve, Phil. Mag. Feb. 1901, p. 212)},$$

and that then it is at first extremely small.

Therefore, to determine R, one has

$$X = \frac{V}{R \log \frac{b}{a}} = 30 \times p. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In the case of column A we may take $C_0 = 47$. The negative current corresponding to the voltage 333 is 180.

Therefore by equation (1)

$$\frac{C_-}{C_0} = \frac{180}{47} = x - (x - y)\frac{R^2 - a^2}{b^2 - a^2}. \quad . \quad . \quad . \quad (4)$$

From (3) since $\log \frac{b}{a} = 5{\cdot}3$, $V = 333$, $p = 6{\cdot}8$, we get

$$R = \frac{333}{30 \times 6{\cdot}8 \times 5{\cdot}3} = {\cdot}31 \text{ cm.}$$

and therefore,

$$\frac{R^2 - a^2}{b^2 - a^2} = \frac{{\cdot}093}{4{\cdot}3} = {\cdot}0216,$$

for $a = {\cdot}0103$ cm., $b = 2{\cdot}08$ cm. Therefore (4) gives

$$x - (x - y) \times {\cdot}0216 = 3{\cdot}83.$$

Now $x > y > 1$, so that x lies between 3·83 and 3·89. Therefore by equation (2) the positive current cannot exceed

$$47\{1 + 2{\cdot}89 \times {\cdot}0216\} \text{ or } 50.$$

If we take z to be equal to half the lower limit just found, which is more or less justified by the illustration given above, we should get for the positive current 48. This agrees very well with the observed value 50.

We may apply the same theory to the columns B, C, D for the voltage 370. Simple calculations give :—

	C_0.	R.	$\frac{R^2 - a^2}{b^2 - a^2}$.	Limits of x.	Values of $C+$	
					calculated.	observed.
B	18	·66	·1	11·4, 12·6	23·5	23·8
C	16	1·083	·272	11·6, 15·5	36·8	30·3
D	3·8	1·32	·405	21·2, 35	18·6	8·0

It will thus be seen that these calculations, rough as they are, are capable of giving accurate agreement with observed quantities so long as R is small. But as the multiplying space is expanded on diminishing the pressure, the limits assigned by the method to the positive current far exceed its actual value.

12. There is, however, another simple test which may be mentioned, and which serves to define the voltage of the outer cylinder for which the negative current just begins perceptibly to increase owing to the ionizing effects of collisions.

Referring to the curve for air (Phil. Mag. Feb. 1901, p. 213) connecting α_1 with X_1, it is easy to show that from $X_1=440$ and onwards the equation

$$\alpha_1=20\,e^{-\frac{440}{X_1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

represents the curve with accuracy. It is necessary, before proceeding, to give a physical interpretation of this equation.

Out of n free paths which a molecule of gas traverses

$$ne^{-m}$$

exceed m times the mean path. Now if one supposes that every ion starts from rest and that every ion which has fallen through more than U volts after its previous collision has sufficient velocity to produce another pair of ions by its impact, then every ionizing impact must terminate a path greater than $\frac{U}{X}$, where X is the force in the neighbourhood of the collision. Let L be the mean free path of a negative ion for a pressure of 1 mm. Then $\frac{L}{p}$ is its mean path for the pressure p millimetres. Therefore in traversing a centimetre it makes $\frac{p}{L}$ collisions and of these

$$\frac{p}{L}\times e^{-\frac{U}{X}/\frac{L}{p}}$$

will be ionizing collisions.

Now if $L=\frac{1}{20}$ cm., the previous equation becomes

$$20\,pe^{-\frac{20U}{X_1}} \text{ where } X_1\equiv X/p.$$

The *statistical* meaning of the equation (5)

$$\alpha_1=20e^{-\frac{440}{X_1}}$$

is therefore this: when X_1 exceeds 440, *i. e.* when X exceeds

$440 \times p$, we may suppose ·05 to be the mean free path of a negative ion, and that whenever its free path constitutes a fall through more than 22 volts, the molecular impact which terminates the free path breaks up the molecule struck into two other ions. This is, of course, only a statistical interpretation. Some collisions which terminate a greater fall than one of 22 volts are ineffective. Others which terminate a lesser fall than this may be effective.

13. When $X < 440$ the above equation fails to represent the curve. The value of α_1

$$20\, e^{-440/X_1}$$

is then too small. The smallest value of X_1 which gives an appreciable value of α_1 is 40. The curve may then be represented for a small range of X_1 by

$$\alpha_1 = 20\, e^{-\frac{220}{X_1}}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now $\frac{1}{20}$ cm. is about the true free path of a negative ion in a pressure of 1 mm., so that this equation will admit of a similar statistical interpretation to that given to the equation

$$\alpha_1 = 20\, e^{-\frac{440}{X_1}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

provided the field of force is constant and its intensity equal to $40\,p$. In the case of the equation (1) just written the only condition is that X_1 must exceed 440 and not vary too rapidly.

14. To apply these results it is to be remarked that in the theory of gases only $100\, e^{-3}$, *i. e.* only about 5 paths in 100, exceed three times the mean free path L. Let V be the potential of the aluminium cylinder when the current shows a tendency to increase along with the force, the increase being slightly less than 5 per cent.

Then if equation (1) held for all values of X_1 we should have

$$\int_a^{a+3L} \frac{V}{r \log \frac{b}{a}}\, dr = 22 \text{ volts},$$

and if equation (2) held universally

$$\int_a^{a+3L} \frac{V}{r \log \frac{b}{a}}\, dr = 11 \text{ volts}.$$

Therefore

$$V \log\left(1+\frac{3L}{a}\right) \Big/ \log\frac{b}{a}$$

lies between 11 and 22.

The limiting voltage of the aluminium cylinder at which the increase of the negative current should be just measurable therefore falls between

$$\frac{22 \log\frac{b}{a}}{\log\left(1+\frac{3L}{a}\right)} \quad \text{and} \quad \frac{11 \log\frac{b}{a}}{\log\left(1+\frac{3L}{a}\right)}.$$

These limits calculated for the pressures in the table are respectively:

for	A	102	and	51
	B	70		35
	C	57		28
	D	52		26
	E	45		22
	F	42		21
	G	40		20
	H	37		18
	I	31		15
	J	28		14
	K	25		12

It will be seen that where the limiting voltages can be detected in the tables they fall between these limits.

15. This is another test which is too rough to be conclusive. Yet if the mean ionizing fall of an ion were taken to be one which fell outside the limits (11, 22) determined above, conspicuous discrepancies would arise between the numbers of the tables and those furnished by theory. And, as a matter of fact, it is often difficult to fix the limiting voltage of the cylinder by observation more accurately than the calculated numbers determine it.

16. It has been shown that the observations presented in this paper are, so far as it has been found practicable to submit them to analysis, in accord with Professor Townsend's Theory of Collisions. And to this extent they afford additional evidence of its truth. Thus it appears necessary to suppose that negative ions, when moving with adequate velocity, disintegrate other molecules upon colliding with them. This at least must be the case when the pressure is

small. On the other hand, the positive ions do not appear to have the same effect.

17. Professor J. J. Thomson (Phil. Mag. Dec. 1899) deduces from experiments with ultra-violet light 3×10^{-26} of a gramme as the mass of a negative ion. Professor Townsend (Phil. Trans. 1899) finds $4{\cdot}5 \times 10^{-24}$ grammes as the weight of a molecule of hydrogen, a determination which makes $6{\cdot}6 \times 10^{-23}$ the corresponding weight for air. This is more than three times the number given by Meyer (Kinetic Theory of Gases, art. 121). Thus, if we assume that the negative ions produced by the Röntgen rays are identical with those obtained by means of ultra-violet light, the molecules of air are 2200 times the size of the negative ion according to these numbers. If the positive ion is what remains of the molecule after the negative is detached from it, it follows that the positive ion does not materially differ in mass from the ordinary molecule of air. In any case it appears evident that it far exceeds in magnitude the negative ion. Therefore the mean path of the latter exceeds that of the former. In fact, Prof. Townsend (Phil. Mag. Feb. 1901) finds about ·47 mm. as the path of the negative ion in 1 mm. pressure of air. The corresponding path of the positive ion assumed to be equal to that of the molecule of air, is about ·094 of a millimetre. This must be multiplied by $\sqrt{2}$ when the ion is moving under electric force, its velocity far exceeding then the velocity of agitation of air. Its free path then becomes ·13 mm. If e is the charge on a positive or negative ion, the energy acquired between successive impacts in a field of electric force, X at any point, is $e\int X ds$, the integral being taken along the free path. Thus there will be a marked difference, so far as energy is concerned, between the impacts of the positive and negative ions. It appears, however, from observations of sparking points during these experiments, that this difference of energy before impact does not suffice to account for the difference of the effect of impact. But this is, of course, only to be expected.

18. It is proved above that the negative current is always greater than the positive when only negative ions generate others by collision. Nevertheless, observation seems to show that sparking occurs sooner when the aluminium cylinder is positive than when it is negative. Moreover, it is possible to find for this cylinder a potential which upsets the stability of the gas, so that a current starts across to the wire either spontaneously or when the slightest amount of ionization is supplied by the Röntgen rays. This unstable condition

appears also to be reached sooner when the outer cylinder is positive than when it is negative. It follows that as the potential of the cylinder is numerically raised, a voltage is reached at which the positive current overtakes the negative; an event which, however, only happens when the sparking point is nearly reached.

19. These phenomena possibly indicate that when X/p attains a certain magnitude, positive ions under certain circumstances of impact are capable of generating others by collision, and such an hypothesis would easily account for all the facts, including sparking. But experiments will be undertaken in order, if possible, to throw light on this subject. Another series of experiments are being undertaken to exhibit the difference of behaviour between positively and negatively charged ions generated from, and moving in, hydrogen and carbonic acid gas.

I must take this opportunity of thanking both Professor Townsend, who suggested this research, for his many suggestions, and Professor Turner for his kindness in providing a laboratory, doubtless at great personal inconvenience, in the University Observatory, Oxford, where the experiments were carried out.

XXIII. *On Irreversible Processes and* Planck's *Theory in relation thereto.*

By S. H. BURBURY, *F.R.S.*[*]

PLANCK'S theory is developed in his treatise "Ueber irreversible Strahlungsvorgänge" (*Berlin. Sitzungsberichte*, 1897, pp. 441–480), and is modified in subsequent papers in detail only. It will be found to depend on some well-known algebraic theorems, and on a certain law of interchange of energy between the parts of a material system.

1. If u, v be two quantities necessarily positive, and which vary subject to the condition that $u+v=2c$, a constant, the least possible value of $u \log u + v \log v$ is when $u=v$, and the greatest is when either $u=0$ or $v=0$. For we may write $u=c+x$, $v=c-x$, x being positive and less than c. Then

$$\frac{d}{dx}(u \log u + v \log v) = \frac{d}{dx}(\overline{c+x} \log \overline{c+x} + \overline{c-x} \log \overline{c-x})$$

$$= \log \frac{c+x}{c-x};$$

and as this is necessarily positive, the proposition is proved.

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[*] Communicated by the Author.

Corollary—If u', v' lie between u and v, $u'\log u'+v'\log v' < u\log u+v\log v$.

A System of Exchanges.

2. Let u, u' be two essentially positive quantities, for example the energies of two material systems. In time dt let u emit, and u' receive, the energy $ku\,dt$, and let u' emit, and u receive, the energy $k'u'\,dt$, k and k' being constants.

Then
$$\frac{du}{dt}=k'u'-ku,\quad \frac{du'}{dt}=ku-k'u',$$

and (*a*) let $S=u(\log(qu)-1)+u'(\log(q'u')-1)$

$$\frac{dS}{dt}=\log(qu)\frac{du}{dt}+\log(q'u')\frac{du'}{dt}$$
$$=(k'u'-ku)\log\frac{qu}{q'u'}.$$

If then, k, k' being given by the physical conditions, q, q' be so chosen that $q/q'=k/k'$, $\frac{dS}{dt}$ is necessarily negative.

S may be put in other forms. For instance,

(*b*) let
$$S=qu\log(qu)-(1+qu)\log(1+qu)+q'u'\log(q'u')-(1+q'u')\log(1+q'u')$$
$$\frac{dS}{dt}=(k'u'-ku)\log\frac{qu}{q'u'}\frac{1+q'u'}{1+qu},$$
which is again negative if $q/q'=k/k'$.

The Physical Application of this Theorem.

3. A material system is divided into n parts, $a_1\ a_2\ \dots\ a_n$, and we assume that the energy, U, which the system possesses can be localized, so that at any instant $a_1\dots a_n$ possess respectively the energies $u_1\dots u_n$, which vary with the time subject to the condition that $u_1+\dots+u_n=\Sigma u=U$.

For the law of this time variation, let us assume that every a emits in time dt a quantity of energy proportional to the corresponding u, that is a_1 emits k_1u_1dt, &c.

By conservation of energy, what a_1 emits is received by some one or more of the other parts $a_2\dots a_n$. Let us then assume that $k_{12}\,u_1\,dt$ is received by a_2

$k_{13}\,u_1\,dt$ „ a_3,

and so on, where $k_{12}+k_{13}+\dots+k_{1n}=k_1$.

In the same way let a_1 receive from a_2 in time dt the energy $k_{21}\,u_2\,dt$,

from a_3 $k_{31}\,u_3\,dt$, &c.

Then
$$\frac{du_1}{dt} = k_{21}u_2 + k_{31}u_3 + \dots + k_{n1}u_n$$
$$- k_{12}u_1 - k_{13}u_1 - \dots - k_{1n}u_1.$$

Similarly
$$\frac{du_2}{dt} = k_{12}u_1 + k_{32}u_3 + \dots + k_{n2}u_n$$
$$- k_{21}u_2 - k_{23}u_2 - \dots - k_{2n}u_2$$
and so on.

4. Let us now assign to $u_1 \dots u_n$ respectively the constant positive coefficients $q_1 \dots q_n$, and form the function

$$S = \Sigma(u \log(qu) - u) = \Sigma u \log \frac{qu}{\epsilon}, \text{ if } \epsilon = 2{\cdot}7182818.$$

Then —

$$\frac{dS}{dt} = \Sigma \log(qu) \frac{du}{dt}$$
$$= \log(q_1u_1)(k_{21}u_2 - k_{12}u_1) + \log(q_2u_2)(k_{12}u_1 - k_{21}u_2)$$
$$+ \log(q_3u_3)(k_{13}u_1 - k_{31}u_3) + \&c.$$
$$+ \log(q_1u_1)(k_{31}u_3 - k_{13}u_1) + \log(q_2u_2)(k_{32}u_3 - k_{23}u_2)$$
$$+ \log(q_3u_3)(k_{23}u_2 - k_{32}u_3) + \&c.$$
$$+ \log(q_1u_1)(k_{41}u_4 - k_{14}u_1) + \log(q_2u_2)(k_{42}u_4 - k_{24}u_2)$$
$$+ \log(q_3u_3)(k_{43}u_4 - k_{34}u_3) + \&c.$$
$$+ \&c.$$

$$= \log \frac{q_1u_1}{q_2u_2}(k_{21}u_2 - k_{12}u_1) + \log \frac{q_1u_1}{q_3u_3}(k_{31}u_3 - k_{13}u_1) + \&c.$$

$$+ \log \frac{q_2u_2}{q_3u_3}(k_{32}u_3 - k_{23}u_2) + \&c.$$

and this is necessarily negative, provided that constants $q_1 \dots q_n$ exist such that $k_{12}/k_{21} = q_1/q_2$, $k_{13}/k_{31} = q_1/q_3$, &c., and generally $k_{pr}/k_{rp} = q_p/q_r$.

If therefore the exchanges of energy between the parts of our system take place according to the above law, the function S necessarily diminishes with the time, and we may call it the entropy of the system. Further, when S has reached its minimum, we have:

(1) By conservation of energy $\Sigma \frac{du}{dt} = 0$, and (2) by conservation of entropy $\Sigma \log(qu) \frac{du}{dt} = 0$, and therefore $\log qu$ is an absolute constant, having the same value for every u.

Any function whose time-variation has always the same sign until a certain state is reached, and is then zero, may be called an entropy function. It is not necessary that it should have the logarithmic form. For instance, let u, v be two functions of t which satisfy at every point in a given space the conditions

$$\frac{du}{dt}=kv, \quad \text{and} \quad \frac{dv}{dt}=-k'u$$

where k, k' are positive constants. These correspond to circular functions. Then if $S=\iiint uv\,dx\,dy\,dz$ throughout the space in question,

$$\frac{dS}{dt}=\iiint(kv^2-k'u^2)dx\,dy\,dz,$$

and has always the same sign until on average of the whole space $k\overline{v^2}=k'\overline{u^2}$. Further, when this state is reached, $\frac{dS}{dt}=0$, and $\frac{d^2S}{dt^2}=-4kk'\iiint uv\,dx\,dy\,dz$, and S retains its minimum value if $\overline{uv}=0$.

Planck's Theory.

5. A vacuum space is traversed by an arbitrary system of electromagnetic waves. In this space is a linear electric resonator, or Dipol, whose proper period of vibration corresponds to a wave-length very great in comparison with its own linear dimensions. And it is assumed that its oscillations are damped only by radiation of energy into the surrounding space, and not in any degree by ohmic resistance or other internal dissipative process. Let $f(t)$ denote the moment at time t of the resonator, Z the component at time t in the direction of the resonator of the intensity of the electric field at the point where the resonator is, both f and Z being expressed in absolute electrostatic measure. Then the oscillation of the resonator is given by the equation

$$\frac{d^2f}{dt^2}+2\sigma\nu_0\frac{df}{dt}+4\pi^2\nu_0^2f=\frac{3c^3\sigma}{4\pi^2\nu_0}Z, \quad . \quad . \quad . \quad (1)$$

in which c is the velocity of light *in vacuo*, ν_0 the number of oscillations of the resonator per unit of time, supposing it uninfluenced by any other bodies, and σ is the damping, or logarithmic decrement of the amplitude of its oscillations. It is essential to the theory that σ, and also $\sigma\nu_0$, be very small.

6. The vibrations Z in the surrounding medium may consist of waves of all periods. But expressing Z in a series of the form

$$Z=\int_0^\infty d\nu\,C_\nu\cos(2\pi\nu t-\theta_\nu),$$

where C_ν and θ_ν are functions of ν, and C_ν is always positive, Planck obtains (p. 443) for the solution of (1)

$$f(t) = \frac{3c^3}{16\pi^3\nu_0^2}\int_0^\infty d\nu \frac{C_\nu}{\nu} \sin\gamma_\nu \cos(2\pi\nu t - \theta_\nu - \gamma_\nu),$$

in which

$$\cot\gamma_\nu = \frac{\nu_0^2 - \nu^2}{\sigma\nu_0\nu}\pi.$$

Now σ is assumed very small. Therefore $\sin\gamma_\nu$ is negligible for all values of ν except values nearly equal to ν_0. Whence also we may write $\cot\gamma_\nu = \frac{\nu_0 - \nu}{\sigma\nu_0}2\pi$. Planck concludes that only those waves which have a period very nearly equal to that of the resonator affect it or are affected by it.

Henceforward we will use ν without the suffix to denote the frequency of the resonator.

7. Planck now distinguishes between rapidly and slowly varying quantities. Z is a *rapidly varying* quantity, and on the average of a time τ, which though very short contains many complete periods $\frac{1}{\nu}$, the mean value of Z is zero. But the important thing is not its mean value but the mean value of its square. And $\overline{Z^2}$, if taken on average for two different periods of time, each equal to τ, will generally vary. It belongs to the class of *slowly varying* quantities. And Planck now defines $\overline{Z^2}$ to be the *intensity of the exciting oscillation.* In like manner the mean energy of the resonator is understood to be the mean taken over an interval of time τ many times greater than the period of the resonator, and other quantities are treated in like manner. The assumption in Planck's theory is fundamental, that we may use these mean values, p. 445 & p. 457.

8. Let U be the energy of the resonator. Then by known formulæ

$$U = \tfrac{1}{2}Kf^2 + \tfrac{1}{2}L\left(\frac{df}{dt}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which

$$K = \frac{16\pi^4\nu^3}{3c^3\sigma}, \quad L = \frac{4\pi^2\nu}{3c^3\sigma} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and, σ being very small, $Kf^2 = L\left(\frac{df}{dt}\right)^2$ on average.

Further, Planck obtains, analysing Z,

$$\frac{dU}{dt} + 2\sigma\nu U = \frac{3c^3\sigma}{16\pi^2\nu}I_\nu, \text{ (22) p. 455,}$$

in which I_ν is the intensity of the vibrations of the surrounding æther whose period is the same as that of the resonator. And $2\sigma\nu U$ is the whole energy emitted per unit of time by the resonator having mean energy U. This equation, (22) in Planck's notation, is the fundamental equation.

9. Now suppose such a resonator fixed in space, and electromagnetic waves to fall upon it, coming from all directions, and consider these waves at the small distance r from O the centre of the resonator. Taking the axis of the resonator for polar axis, let θ, ϕ be the usual angular coordinates, so that $\sin\theta d\theta d\phi$ defines the solid angle $d\Omega$ at O in direction $\theta\phi$.

Let us consider these waves divided into separate waves, each having front $r^2 d\Omega$. Let $\mathbf{K}$ denote the *intensity* of the vibrations of the wave. Consider an element of area $ds = r^2 d\Omega$ on the surface of the r sphere, and an element $d\sigma$ at the centre perpendicular to the radius to ds. Then the energy which in time dt passes from ds to $d\sigma$ is $dt\,\frac{ds\,d\sigma}{r^2}\mathbf{K}$ (p. 456), that is $dt\,d\sigma . d\Omega\,\mathbf{K}$. It follows that the energy per unit of volume at the centre due to the wave is the last expression divided by $c\,dt\,d\sigma$, that is $\frac{\mathbf{K}}{c}\,d\Omega$, and the whole energy per unit of volume at the centre is, if $\mathbf{K}$ is constant for all positions of $d\Omega$, $\frac{4\pi\mathbf{K}}{c}$.

10. The vibrations are in the plane of the wave. But the polarization may have any direction in that plane. There are then, continues Planck, in that plane two mutually perpendicular directions in which the vibrations have intensity respectively greater and less than in any other. These are called the *principal directions*, and the intensities of vibration in them, which shall be denoted by $\mathbf{K}$, $\mathbf{K}'$, are the *principal intensities.*

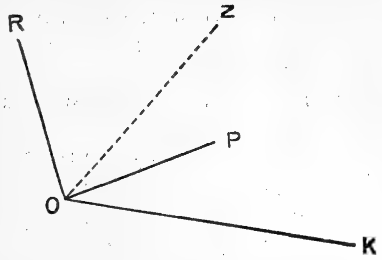
We might, however, resolve the vibrations in any other two mutually perpendicular directions in the plane of the wave, making with the principal directions the angles ω and $\frac{\pi}{\omega} + \omega$ respectively. If $\mathbf{K}_1$, $\mathbf{K}_2$ denote the intensities of these resolved vibrations respectively,

$$\mathbf{K}_1 = \mathbf{K}\sin^2\omega + \mathbf{K}'\cos^2\omega,$$
$$\mathbf{K}_2 = \mathbf{K}\cos^2\omega + \mathbf{K}'\sin^2\omega,$$

and $\mathbf{K}_1 + \mathbf{K}_2 = \mathbf{K} + \mathbf{K}'$ whatever ω may be; also $\mathbf{K}_1$ and $\mathbf{K}_2$ lie between $\mathbf{K}$ and $\mathbf{K}'$.

Let now the plane of the paper be that of the wave, and let the dotted line Oz be the direction of the resonator axis which makes the angle θ with the normal.

Let OP be the projection of Oz on the plane, OR a line in



the plane perpendicular to OP, and therefore to Oz. Also let OK denote the principal direction corresponding to the intensity **K**.

Then the angle $\mathrm{KOP}=\omega$

and $\quad \mathrm{PO}z=\frac{\pi}{2}-\theta.$

$\mathbf{K}_1$ is the intensity of vibration in OR,

$\mathbf{K}_2$ " " " OP.

The electric force due to the vibrations $\mathbf{K}_2$ in OP is perpendicular to OP, that is in OR, and is therefore perpendicular to Oz. Therefore the vibrations $\mathbf{K}_2$ have no electric force in Oz. They do not affect, and are not affected by, the resonator. If $\partial\mathbf{K}$ denotes for any **K** the change in it due to incidence on the resonator, $\partial\mathbf{K}_2=0$.

On the other hand the vibrations in OR, whose intensity is $\mathbf{K}_1$, have electric force in OP, which makes the angle $\frac{\pi}{2}-\theta$ with the axis. Therefore a part of the energy of these vibrations is absorbed by the resonator, and lost to the wave. And this part is in time dt, $\frac{3c^2\sigma}{4\pi\nu}\mathbf{K}_1\sin^2\theta\, dt\, d\Omega$ (p. 461, equation (36)).

It should be noted that

$$\frac{3c^2\sigma}{4\pi\nu}=\frac{3}{4\pi}\frac{c^2}{\nu^2}\sigma\nu$$

$$=\frac{3}{4\pi}\lambda^2\sigma\nu$$

where λ is the wave-length. Now $\sigma\nu$ has been assumed to be very small. Therefore $\frac{3c^2\sigma}{4\pi\nu}$ is a quantity whose square may be neglected.

The result stated above for the absorbed energy involves a relation between the magnitude of the elementary area $r^2 d\Omega$, that is the section of the wave which we are considering, and the cross section of the resonator. For the quantity of energy absorbed by the resonator will depend on the cross section, so therefore must $d\Omega$. I understand Planck to state this, p. 461.

11. Again, the energy emitted by the resonator per unit of time is $2\sigma\nu\mathrm{U}$ (art. 8). It is emitted uniformly in all directions, making with the axis angles between θ and $\theta+d\theta$. But the energy emitted in such directions is proportional to $\sin^2\theta$. Therefore the energy emitted in any direction $d\Omega$ is per unit of time

$$\frac{2\sigma\nu\mathrm{U}\sin^2\theta\, d\Omega}{\int\sin^2\theta\, d\Omega}=\frac{3\sigma\nu\mathrm{U}}{4\pi}\sin^2\theta\, d\Omega,$$

because $\int\sin^2\theta\, d\Omega=\frac{8\pi}{3}$.

12. The energy thus emitted by the resonator is absorbed by the waves of the same period $\frac{1}{\nu}$. It follows that if $f\mathbf{K}_1 d\Omega\, dt$ denotes the energy of the wave of period $\frac{1}{\nu}$ which passes the elementary area $r^2 d\Omega$ in time dt,

$$\frac{d}{dt}(f\mathbf{K}_1)=f\frac{d\mathbf{K}_1}{dt}=-\frac{3c^2\sigma}{4\pi\nu}\mathbf{K}_1\sin^2\theta+\frac{3\sigma\nu}{4\pi}\mathrm{U}\sin^2\theta$$

$$=\frac{3c^2\sigma}{4\pi\nu}\left(\frac{\nu^2\mathrm{U}}{c^2}-\mathbf{K}_1\right)\sin^2\theta.$$

But
$$\frac{d\mathrm{U}}{dt}=\frac{3c^2\sigma}{4\pi\nu}\left(\mathbf{K}_1-\frac{\nu^2\mathrm{U}}{c^2}\right)\sin^2\theta.$$

Therefore
$$\frac{d(f\mathbf{K}_1)}{dt}+\frac{d\mathrm{U}}{dt}=0$$

expresses the conservation of energy.

13. The conditions of the transfer of energy between resonator and wave are precisely the conditions necessary for application of the theorem of art. 3. Each system emits to the other per unit of time an amount of energy proportional to the energy which the emitting system has for the time being. Further, the ratio k_{12}/k_{21} of art. 2 is here c^2/ν^2. We expect then to find that an entropy function exists.

The Electromagnetic Entropy.

14. According to Planck's definition, the entropy, S, of the resonator whose energy is U, is

$$S = -\frac{U}{a\nu}\left(\log\frac{U}{b\nu}-1\right)$$

where a and b are constants. It is independent of the entropy of the incident waves.

Again, if s denote the entropy per unit of volume of a monochromatic wave $d\Omega$ of intensity **K**, we have by analogy to art. 9, $s = d\Omega\,\frac{L}{c}$, where

$$L = -\frac{\mathbf{K}}{a\nu}\left(\log\frac{c^2\mathbf{K}}{b\nu^3}-1\right),$$

which has a determinate value at every point. The actual entropy of any given volume of the wave throughout which **K** is constant is, or is proportional to, the last expression multiplied by the given volume.

15. Up to this point I have closely followed Planck. I now deviate somewhat from his method. Let $f\mathbf{K}d\Omega dt$ denote as before the energy of the wave of intensity **K** which passes the elementary area $r^2 d\Omega$ in time dt. Then f is proportional to the volume of the wave in question. And let us define the entropy of that same wave as follows:

$$s = -\frac{1}{a\nu}f\mathbf{K}\left(\log\frac{c^2\mathbf{K}}{b\nu^3}-1\right)d\Omega\,dt,$$

whence $$\frac{ds}{dt} = -\frac{1}{a\nu}\log\frac{c^2\mathbf{K}}{b\nu^3}\,\frac{d(f\mathbf{K})}{dt}\,d\Omega.$$

Then we have for an *unpolarized wave*

$$\frac{d}{dt}(S+s) = -\frac{1}{a\nu}\log\frac{U}{b\nu}\,\frac{dU}{dt} - \frac{1}{a\nu}\log\frac{c^2\mathbf{K}}{b\nu^3}\,\frac{d(f\mathbf{K})}{dt}.$$

But

$$\frac{dU}{dt} = \frac{3c^2\sigma}{4\pi\nu}\left(\mathbf{K}-\frac{\nu^2 U}{c^2}\right)\sin^2\theta,$$

and

$$\frac{d(f\mathbf{K})}{dt} = \frac{3c^2\sigma}{4\pi\nu}\left(\frac{\nu^2 U}{c^2}-\mathbf{K}\right)\sin^2\theta.$$

Therefore

$$\frac{d(S+s)}{dt} = -\frac{1}{a\nu}\,\frac{3c^2\sigma}{4\pi\nu}\sin^2\theta\left(\frac{\nu^2 U}{c^2}-\mathbf{K}\right)\log\frac{c^2\mathbf{K}}{\nu^2 U},$$

which, owing to the minus sign prefixed, is necessarily positive if not zero, and then only zero when $\mathbf{K}=\frac{\nu^2 \mathrm{U}}{c^2}$ for every wave and resonator.

A Polarized Wave.

16. We cannot in calculating entropy, as we did in calculating energy, use $\mathbf{K}_1$ and $\mathbf{K}_2$ instead of the principal intensities $\mathbf{K}$, $\mathbf{K}'$. For $\mathbf{K}_1$ and $\mathbf{K}_2$ lie between $\mathbf{K}$ and $\mathbf{K}'$.

Therefore by art. 1

$$-\frac{\mathbf{K}_1}{a\nu}\left(\log\frac{c^2\mathbf{K}_1}{b\nu^3}-1\right)-\frac{\mathbf{K}_2}{a\nu}\left(\log\frac{c^2\mathbf{K}_2}{b\nu^3}-1\right)$$

is greater than

$$-\frac{\mathbf{K}}{a\nu}\left(\log\frac{c^2\mathbf{K}}{b\nu^3}-1\right)-\frac{\mathbf{K}'}{a\nu}\left(\log\frac{c^2\mathbf{K}'}{b\nu^3}-1\right).$$

Let us then define

$$s=-\frac{1}{a\nu}f\mathbf{K}\left(\log\frac{c^2\mathbf{K}}{b\nu^3}-1\right)-\frac{1}{a\nu}f\mathbf{K}'\left(\log\frac{c^2\mathbf{K}'}{b\nu^3}-1\right)$$

to be the *actual entropy* of the wave above defined before incidence. And let

$$s_1=-\frac{1}{a\nu}f\mathbf{K}_1\left(\log\frac{c^2\mathbf{K}_1}{b\nu^3}-1\right)-\frac{1}{a\nu}f\mathbf{K}_2\left(\log\frac{c^2\mathbf{K}_2}{b\nu^3}-1\right)$$

denote the *hypothetical entropy* of the wave before incidence. It is what the entropy would be if $\mathbf{K}_1$ and $\mathbf{K}_2$ were the principal intensities.

After incidence let s become s', and s_1 become s_1'. And similarly the entropy of the resonator shall be S before and S′ after, incidence.

Now, as we have seen, $\frac{d}{dt}(f\mathbf{K}_2)=0$. But

$$\frac{d}{dt}(f\mathbf{K}_1)=-\frac{3c^2\sigma}{4\pi\nu}\left(\mathbf{K}_1-\frac{\nu^2\mathrm{U}}{c^2}\right)\sin^2\theta.$$

We will, following Planck, denote by $\mathbf{K}_3$ what $\mathbf{K}_1$ becomes after incidence. We have then

$$s_1'-s_1=-\frac{1}{a\nu}\log\frac{c^2\mathbf{K}_1}{b\nu_3}\frac{d(f\mathbf{K}_1)}{dt}dt$$

$$=+\frac{1}{a\nu}\log\frac{c^2\mathbf{K}_1}{b\nu^3}\frac{3c^2\sigma}{4\pi\nu}\left(\mathbf{K}_1-\frac{\nu^2\mathrm{U}}{c^2}\right)\sin^2\theta\,dt.$$

Also

$$S'-S=-\frac{1}{a\nu}\log\frac{U}{b\nu}\frac{dU}{dt}dt$$

$$=+\frac{1}{a\nu}\log\frac{U}{b\nu}\frac{3c^2\sigma}{4\pi\nu}\left(\frac{\nu^2U}{c^2}-\mathbf{K}_1\right)\sin^2\theta\,dt.$$

Therefore

$$s_1'+S'-(s_1+S)=-\frac{1}{a\nu}\frac{3c^2\sigma}{4\pi\nu}\sin^2\theta\left(\frac{\nu^2U}{c^2}-\mathbf{K}_1\right)\log\frac{c^2\mathbf{K}_1}{\nu^2U}dt,$$

which is necessarily positive owing to the minus sign prefixed.

But $s_1>s$ by art. 1.

Therefore, *a fortiori*, $s_1'+S'>s+S$; or the entropy of the resonator plus the hypothetical entropy of the wave after incidence is greater than the sum of the actual entropies of wave and resonator before incidence.

17. But also $s_1'>s'$ by art. 1.

It does not therefore yet follow that $s'+S'>s+S$, which is the proposition we wish to prove.

To make the proof complete at this point it is necessary to assume that $\mathbf{K}_2$ and $\mathbf{K}_3$ are after incidence the principal intensities; and therefore $s_1'=s'$. And this Planck asserts to be the case (p. 468).

He does not, however, explain the grounds for this assertion. The principal directions before incidence are independent of the direction Oz (see figure) of the resonator axis. They cannot therefore generally be OP, OR, which are determined by the direction Oz. Therefore generally $\mathbf{K}_2$ in OP is neither the greatest nor the least intensity. By the incidence on the resonator the intensity $\mathbf{K}_1$ in OR is altered by a very small quantity proportional to $\frac{3c^2\sigma}{4\pi\nu}$. This cannot generally have the effect of making $\mathbf{K}_2$ either the greatest or the least intensity.

18. The proof can, however, as seems to me, be completed as follows:—For any given wave-front, all directions of the axis Oz of the resonator are equally probable. Therefore on average of all waves and resonators with the same ν,

$$(s_1-s)=(s_1'-s').$$

And now

$$s'+S'-(s+S)=s_1'+S'-(s_1+S)+(s_1-s)-(s_1'-s').$$
$$=s_1'+S'-(s_1+S),\text{ on average.}$$

And this we have proved to be positive.

It follows that, on average of all waves and resonators having the same ν, the entropy must increase until a state is reached, the stationary state, in which

$$\mathbf{K}_1=\mathbf{K}_2=\mathbf{K}=\mathbf{K}'=\frac{\nu^2 U}{c^2}.$$

Another difficulty that I find is as follows:—Assuming that $\mathbf{K}_2$ and $\mathbf{K}_3$ are the principal intensities after incidence, the argument proceeds in Planck's own words as follows:—"wird der Resonator von einem irgendwie polarisirten Strahlenbündel getroffen, dessen Energiestrahlung die Hauptintensitäten $\mathbf{K}$ und $\mathbf{K}'$, und dessen Entropiestrahlung daher die Intensität $L+L'$ besitzt. Dieses Strahlenbündel lässt in der Zeit dt die Entropie

$$\text{“}\ (L+L')dt\frac{3c^2\sigma}{4\pi\nu}\,d\Omega,$$

auf den Resonator fallen, und dadurch wird auf dieser Seite der nämliche Entropiebetrag dem Felde entzogen. Auf der anderen Seite geht vom Resonator ein in bestimmter Weise polarisirtes Strahlenbündel aus dessen Energiestrahlung die *Hauptintensitäten* $\mathbf{K}_2$ und $\mathbf{K}_3$, und dessen Entropiestrahlung daher die entsprechende Intensitäten L_2 und L_3 besitzt. Dadurch wird dem umgehenden Felde die Entropie

$$\text{“}\ (L_2+L_3)dt\frac{3c^2\sigma}{4\pi\nu}\,d\Omega,$$

zugeführt. Im ganzen beträgt also die in der Zeit dt eingetretene Entropieänderung des den Resonator umgebenden Feldes

$$\text{“}\ dt\frac{3c^2\sigma}{4\pi\nu}\int(L_2+L_3-L-L')\,d\Omega. \quad (46)\text{”}$$

And to this has to be added the change of entropy of the resonator itself. And Planck thence proves that the sum of the entropies is increased.

The difficulty in this reasoning is in the introduction of the factor $\frac{3c^2\sigma}{4\pi\nu}$. If ∂ denotes for any function the change due to incidence on the resonator, or the difference between the values of the function before and after incidence, then the quantity in brackets in the expression (46) is equivalent (on Planck's assumption that $\mathbf{K}_2$ and $\mathbf{K}_3$ are principal intensities) to $\partial(L+L')$, that is to

$$-\frac{1}{a\nu}\left\{\log\frac{c^2\mathbf{K}}{b\nu^3}\,\partial\mathbf{K}+\log\frac{c^2\mathbf{K}'}{b\nu^3}\,\partial\mathbf{K}'\right\}.$$

Further, from the equations

$$\mathbf{K}_1 = \mathbf{K}\sin^2\omega + \mathbf{K}'\cos^2\omega,$$
$$\mathbf{K}_2 = \mathbf{K}\cos^2\omega + \mathbf{K}'\sin^2\omega,$$

$\mathbf{K}$ and $\mathbf{K}'$ may be expressed as linear functions of $\mathbf{K}_1$ and $\mathbf{K}_2$. And since $\partial\mathbf{K}_2=0$, and $\partial\mathbf{K}_1$ is proportional to $\frac{3c^2\sigma}{4\pi\nu}$ by (36), it follows that $\partial\mathbf{K}$ and $\partial\mathbf{K}'$, and therefore also $\partial(\mathrm{L}+\mathrm{L}')$, and therefore also the quantity in brackets in (46), is proportional to $\frac{3c^2\sigma}{4\pi\nu}$. The factor $\frac{3c^2\sigma}{4\pi\nu}$ is then not wanted outside of the bracket, and has no right to be there.

It is true that since L/c denotes entropy per unit of volume, the quantity in brackets requires some factor to make it express an actual amount of entropy. But such factor must be independent of σ, and therefore cannot be $\frac{3c^2\sigma}{4\pi\nu}$. The above are the difficulties which I have found. Very probably they are capable of explanation.

On Planck's Theory in relation to the Distribution of Energy in the Spectrum.

19. Planck has given no account of interchanges of energy between systems of different vibration periods. His method is in fact based on the assumption or proof (art. 6) that waves of different period from that of a resonator pass the resonator unaffected, so that no interchange of energy takes place. This, however, is not quite rigorous. If the difference of periods, though not zero, be very small, some very small interchange of energy between the wave and the resonator will consistently with the equation of p. 443 take place. We may admit, therefore, that if all values of ν are represented, ν varying continuously between different systems, energy will pass indirectly from systems of period $\frac{1}{\nu}$ to systems of period $\frac{1}{\nu'}$, though the rate of interchange will be much slower than between systems of the same period.

20. Now Planck does not investigate the law of these slow interchanges. He assumes that an entropy function exists for them, and that it is precisely the same function (but with variable ν) which has been defined above for systems having the same period. That may be true, but it cannot, I think, be accepted as an axiom. It seems to me that this branch of the subject requires further elucidation.

21. According to Planck, if the entropy has the general form $pU \log \frac{qU}{\epsilon}$ for the resonator, and $p\mathbf{K} \log \frac{q'\mathbf{K}}{\epsilon}$ for the wave, we have

$$p = -\frac{1}{a\nu}, \quad q = \frac{1}{b\nu}, \quad q' = \frac{c^2}{\nu^2}\frac{1}{b\nu},$$

where ν is now a variable. As he himself explains, the constants p, q, q' are not determined by the analysis (which appears to determine the ratio q'/q but no more), but are chosen to make the results of analysis agree with those of experiment.

22. For any infinitely small "virtual" change of state the variation of the total entropy must in the stationary state vanish. The "virtual" change of state supposed is that a small quantity of energy proportional to $\partial \mathbf{K}_1$ is added to one system of period $\frac{1}{\nu_1}$, and another small quantity proportional to $\partial \mathbf{K}_2$ is added to another system of period $\frac{1}{\nu_2}$, *cæteris paribus*. Now a "virtual" change—if there is any virtue in the term—means a change consistent with the conditions of our system. In this supposition, therefore, we have definitely thrown overboard the restriction that energy can only be interchanged between systems of the same period.

However, given that the above change takes place, we have by conservation of energy

$$\partial \mathbf{K}_1 + \partial \mathbf{K}_2 = 0.$$

And by conservation of entropy in the stationary state,

$$p_1 \log (q_1 \mathbf{K}_1) \partial \mathbf{K}_1 + p_2 \log (q_2 \mathbf{K}_2) \partial \mathbf{K}_2 = 0.$$

From which it follows that in the stationary state $p \log (q\mathbf{K})$ is an absolute constant, independent of ν.

Assume

$$p \log (q\mathbf{K}) = -\frac{1}{\theta},$$

whence

$$\mathbf{K} = \frac{1}{q} \epsilon^{-\frac{1}{p\theta}}.$$

Instead of ν Planck now introduces the new variable λ, the wave-length, by the formula

$$\lambda = \frac{c}{\nu},$$

$$d\nu = -\frac{c}{\lambda^2} d\lambda.$$

And thence using his own form for p and q, namely $p = \frac{1}{a\nu}$, $q = \frac{1}{b\nu^3}$, he obtains for the intensity E_λ, of the wave-length λ,

$$E_\lambda = \frac{2bc^2}{\lambda^5}\epsilon^{-\frac{ac}{\lambda\theta}},$$

which is Wien's law. And this at the date of Planck's paper now under discussion was considered to be sufficiently accurate.

23. Later researches of Beckmann, Pringsheim, and others throw doubt on the accuracy of Wien's law. It becomes then necessary, in order to make the results of Planck's analysis agree with experiments, either to vary the constants p, q, q', or else to vary the form of the function S. And this Planck does in his paper "Ueber irreversible Strahlungsvorgänge" (*Preuss. Akad. Wiss. Berlin, Sitzungsberichte*, xxv. pp. 544–555, May 9, 1901). He there, without altering the general theory as developed in the former treatise, assumes for the entropy of the resonator

$$S = k\left\{\left(1+\frac{U}{h\nu}\right)\log\left(1+\frac{U}{h\nu}\right) - \frac{U}{h\nu}\log\frac{U}{h\nu}\right\},$$

where k and h are constants, and obtains results in accordance with the later experiments.

Is Planck's a true irreversible process?

24. According to the theorems of art. 2, every interchange of energy between the parts of a material system, if such interchanges are made in accordance with the law there assumed, is an irreversible process, just as, and for the same reason that, interchanges of heat between hot and cold bodies are irreversible. Now according to Planck the interchanges of energy between different parts of his system do take place according to the law assumed in art. 2. The process, if it exists in nature, is therefore an irreversible process.

But the existence of such a machine as Planck's resonator cannot be proved by experiment. For any actual electric vibration such as he supposes must be accompanied by dissipation of energy. It may be that æther, the universal solvent of mathematical difficulties, supplies us with the instrument required. But if not, Planck's process is non-existent. And can a non-existent process be irreversible?

25. It seems to me that if at any instant all the velocities in Planck's problem were reversed, the system would *not* retrace

its course, executing in the reverse direction all the changes through which it has passed. In this respect Planck's problem differs *toto cœlo* from that of Boltzmann's H theorem, where we have a process, consisting of a number of separately reversible processes, supposed to be irreversible in the aggregate.

XXIV. *Note on a Paper by* Prof. Fleming, F.R.S., *and* Mr. Ashton, *entitled "On a Model which Imitates the Behaviour of Dielectrics." By* JOHN BUCHANAN, *D.Sc. (Lond.)*[*].

UNDER the above title there has appeared in the Phil. Mag. for August 1901 a description of a very ingenious model invented by the authors of the paper. The diagrams, obtained by help of the model, which illustrate the paper are exceedingly interesting and suggestive.

There are some points in the theory of the action of the model, and of the behaviour of the dielectric in a condenser, which may be deemed worth attention.

1. The action of the model clearly depends on the viscosity of a liquid. The diagrams show by their form the very interesting fact that the motion of the pencil which traced them approximated closely to what may be expressed by the term "motion of a viscous fluid by diffusion"[†].

In other words, the displacement curves obtained from the model, and their derived velocity curves, are of the same form as the graphs of certain solutions of Fourier's well-known equation

$$\frac{dv}{dt}=\mathrm{K}\frac{d^2v}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For comparison with figs. I. & III. (Pl. V.) respectively, of Prof. Fleming and Mr. Ashton's paper, I give here figs. 1 and 2, which are the respective graphs of equations (2) and (3) below. In these solutions of (1) v denotes the displacement, t the time, x the distance from the origin at which exists the motion under consideration, and K denotes the "diffusivity."

Without the guidance afforded by some such theory as is here given, it would be quite impossible to discover from experiment such a law of displacement as is expressed by (2) for instance, or of velocity as expressed by (3). Thus, the

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[*] Communicated by the Physical Society: read November 8, 1901.

[†] *Cf.* Lord Kelvin's Math. and Phys. Papers, vol. iii. art. xcviii. p. 433.

right-hand member of (2) expresses that when $t=0$ a sudden displacement C (=" charge ") is impressed on the model at the point $x=0$. This displacement is maintained until a time $t=T$ has elapsed, when release (=" discharge ") is allowed to take place.

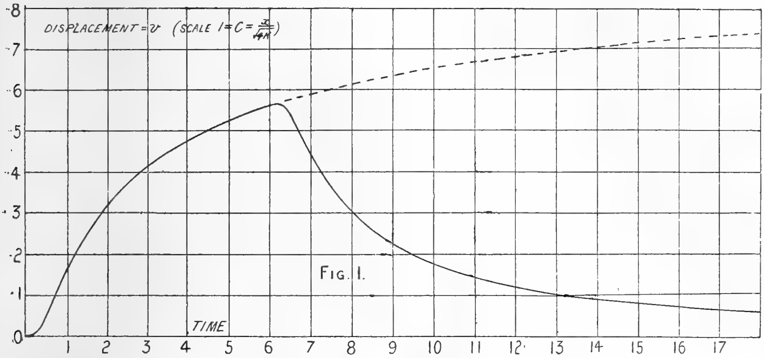

FIG. 1.

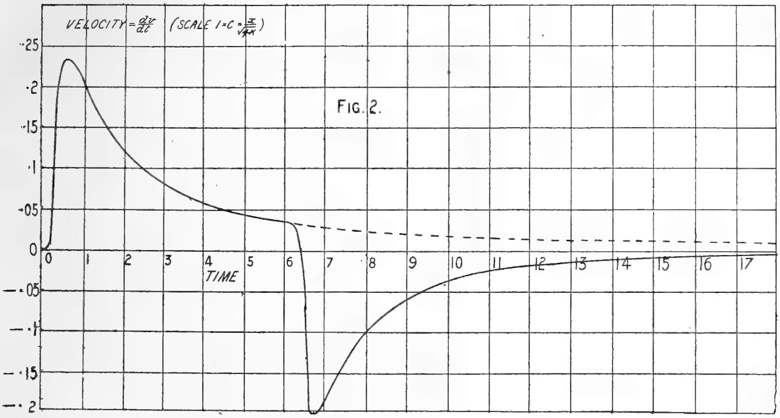

FIG. 2.

As an example of a more complex set of operations, for comparison with fig. v, Pl. V. *loc. cit.*, I give here figs. 3 & 4 (p. 242).

The former is the graph of (4) below, the latter is the derived velocity-curve.

We can from these examples see how, by solutions of (1), we can express the effect of any given set of conditions as regards " charge " or " discharge."

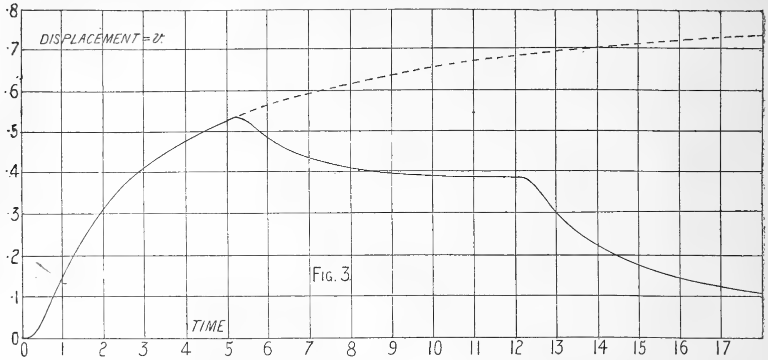

FIG. 3.

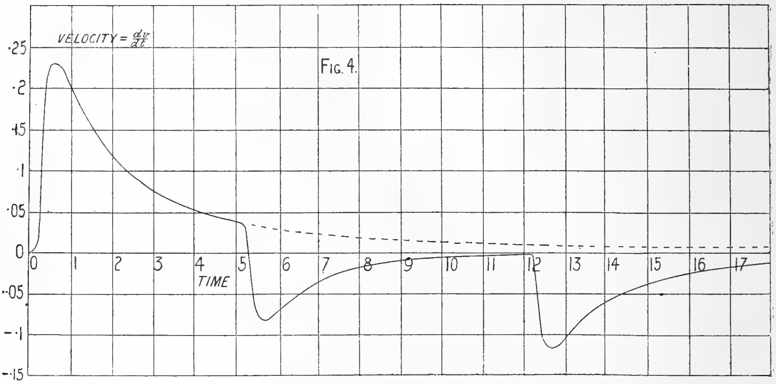

FIG. 4.

The solutions of (1) referred to above are :—

$$v=C\left(1-\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4Kt}}} dz\epsilon^{-z^2}\right)-C\left(1-\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{2K(t-T)}}} dz\epsilon^{-z^2}\right), \quad (2)$$

$$\frac{dv}{dt}=C\,\frac{x}{\sqrt{4\pi K}}\left(\frac{\epsilon^{-\frac{x^2}{4Kt}}}{t^{\frac{3}{2}}}-\frac{\epsilon^{-\frac{x^2}{4K(t-T)}}}{(t-T)^{\frac{3}{2}}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also,

$$v=C\left(1-\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4Kt}}} dz\epsilon^{-z^2}\right)-\frac{C}{2}\left(1-\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4K(t-T)}}} dz\epsilon^{-z^2}\right)$$

$$-\frac{C}{2}\left(1-\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4K(t-T')}}} dz\epsilon^{-z^2}\right). \quad . \quad . \quad (4)$$

2. The reasons why the model gives curves of the same

form as the curves of "charge" and "discharge" of a condenser appear fairly obvious.

Lord Kelvin showed long ago in his paper "On the Theory of the Electric Telegraph"[*], that the potential and the current at any point in the wire of a cable can be expressed by appropriate solutions of (1) above. In precisely the same manner, by use of solutions of (1) we can treat the question of the diffusion of electricity into or out of the dielectric of a condenser. That such a dielectric has usually an enormous resistance per centim. compared with a centim. length of the wire of a cable, does not alter the fact that the law, according to which electric diffusion goes on in both, is the same. Resistance merely affects the magnitude of the proportion of distance, or of time, at which corresponding states of potential are reached in the dielectric of the condenser and in the wire of the cable. In the case of a distance the magnitude of this proportion may be of the order 10^{-10}, in the case of a time it may be of the order 10^{20}.

The whole argument, then, may be summarized in the statement that the motion of the model, and the diffusion of electricity in the dielectric of a condenser, are subject to one and the same mathematical law expressed by equation (1) above.

In conclusion, I would take the liberty of suggesting to the inventors of the model to obtain "hysteresis" diagrams by cyclical loading of the springs. I feel sure that, when published, such diagrams would also prove extremely instructive.

Gordon's College, Aberdeen,
Sept. 1901.

XXV. *Notes on Gas-Thermometry.*—II. *By* Dr. P. CHAPPUIS, *Attaché au Bureau International des Poids et Mesures, Sèvres*[†].

MESSRS. Holborn and Day have published recently in a research on the air-thermometer[‡] the results of a new determination of the expansion of Berlin porcelain between 0° and 1000°.

These experiments are of considerable importance as

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[*] Math. and Phys. Papers, vol. ii. art. lxxiii.

[†] Communicated by the Physical Society: read November 22, 1901.

[‡] *Annalen der Phys. und Chem.* 4th series, vol. ii. 1900, p. 505.

bearing on the reduction of temperature measurements, in which reservoirs of Berlin porcelain have been employed. I drew attention in a former note * to the fact that part of the divergence found between the results of Messrs. Callendar and Griffiths and of Harker and myself for the boiling-point of sulphur may be attributed to the uncertainty in the values assumed for the expansion of porcelain. I propose now to examine in what way our results would be modified by the introduction of the value for the expansion deduced from the experiments of Messrs. Holborn and Day. I would like at the same time to make clear and correct one or two errors in the note previously mentioned, to which Dr. Chree has been kind enough to draw my attention.

1. Messrs. Holborn and Day have measured the expansion of a rod of unglazed porcelain about 483 mm. in length by the method of the comparator. Their observations are represented in a fairly satisfactory manner by the expression

$$L_t - L_0 = \{2954t + 1{\cdot}125t^2\}10^{-9}$$

between the limits of temperature +250° and 625°. This function diverges considerably from the observations at 750° and at 875° to again agree better at 1000°. Below 250° it gives values which are too high.

In fact the expansion of Berlin porcelain cannot be represented exactly by a two-term function throughout an interval of temperature exceeding a few hundred degrees.

It follows therefore that an expression for the expansion deduced empirically cannot be applied outside the limits of the experiments without running the risk of committing serious errors †.

The following table contains the values of the mean coefficient of expansion between 0° and T° deduced from our measurements, those of Messrs. Holborn and Wien in 1892 ‡, and the recent ones of Messrs. Holborn and Day.

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* Phil. Mag. [5] vol. l. p. 436 (Oct. 1900).

† The case is not the same with platinum, whose expansion measured by Messrs. Holborn and Day by the same method between 0° and 1000° has been found to be $(8889t + 1{\cdot}274t^2)\,10^{-9}$. M. Benoit had previously obtained between 0° and 100° in the Fizeau apparatus (*Trav. et Mém. du Bur. Inter.* t. vi. p. 190) the almost identical expression

$$(8901t + 1{\cdot}21t^2)10^{-9}.$$

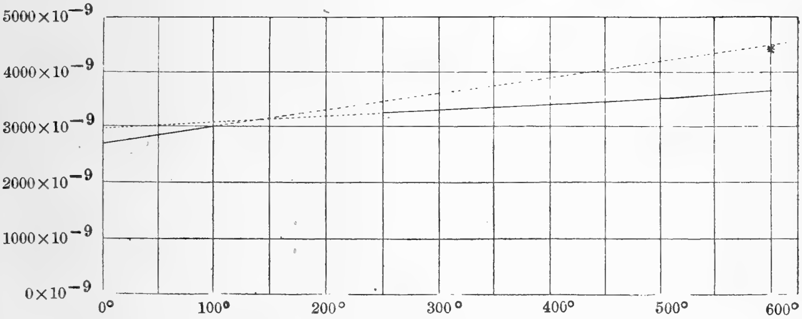
Here extrapolation would have led to no serious inaccuracies.

‡ Holborn & Wien, *Ann. der Phys. und Chem.* Bd. xlvii. p. 121 (1892).

T.	Harker & Chappuis, 1898.	Holborn & Wien, 1892.	Holborn & Day, 1900.
0°	2690×10^{-9}		2954×10^{-9}
100°	2989 „		3066 „
200°	(3288) „		3179 „
300°	(3587) „		3291 „
400°	(3886) „		3404 „
500°	(4185) „		3516 „
600°	(4484) „	4400×10^{-9}	3629 „

At the time of our experiments the absence of precise data for the expansion of porcelain compelled us to deduce the values for the higher temperatures by extrapolation of the expression determined between the limits 0° and 100°. The values we thus obtained seemed to be confirmed by Messrs. Holborn and Wien, but are not in agreement with the new experiments of Messrs. Holborn and Day.

The following diagram represents the mean coefficients as given by our determination and that of Messrs. Holborn and Day.



Since the formula of Messrs. Holborn and Day gives too high values for temperatures below 250° the curve should approach ours in this region. It is evident from the diagram that there is therefore no incompatibility between their results and ours, and that they may be considered as approximate values of the same function representing the expansion of Berlin porcelain.

Although there may be doubts as to the identity of the material subjected to experiments in the two cases, and even as to the invariability of the properties of a single specimen, when subjected on several occasions to a lengthy annealing process, it seemed to me interesting to recalculate some of our observations on the boiling-point of sulphur, assuming for the mean coefficient of expansion of the reservoir the value given by the experiments of Messrs. Holborn and Day for the temperature 445°. I have kept in this calculation the coefficient obtained by me for nitrogen between 0° and 100°.

It follows from the introduction of the new values, that the boiling-point of sulphur deduced from our experiments with a porcelain-reservoir thermometer would be lowered 0°·5 C. from 445°·2 to 444°·7. This number is very close to that obtained by Messrs. Callendar and Griffiths.

It is possible that the measurements in which we employed a reservoir of "verre dur," and which also gave us a value for the boiling-point close to 445°·2, may be affected by a similar error, but the data at present available for the expansion of this glass at high temperatures do not now permit of the error, if any, being calculated.

I think there would be some interest in redetermining the boiling-point of sulphur, using a bulb of platinum-iridium whose expansion follows a regular and better known law.

2. In the note mentioned above (Phil. Mag. [5] l. p. 431) I gave the values of the coefficient $\frac{1}{P_0}\frac{dP}{dt}$ for nitrogen for an initial pressure P_0 of one metre, at different temperatures comprised between 0° and 100°. Examination of the table of these values shows that the coefficient of nitrogen gradually diminishes and reaches near 75° a limiting value equal to

$$\alpha_{\text{lim}} = 0·003\,673\,80.$$

It may be assumed that starting from this temperature the gas is in the perfect state.

Direct observations of the constants of the nitrogen thermometer having given

$$P_0 = 1·000\,000 \text{ m.} \quad \text{and} \quad P_{100} = 1·367\,466 \text{ m.},$$

we may deduce the initial pressure P_0', which should have been observed had the nitrogen retained down to 0° the properties of a perfect gas; that is to say, if the pressure had continued to vary from 100° downwards at the rate of 0·003 673 80 metre per degree.

We should then have

$$P_0' = P_{100} - 0·003\,673\,80 \times 100 = 1·000\,086 \text{ m.}$$

The thermometer supposed perfect would therefore have at 0° the pressure $P_0' = 1·000086$ m. and at 100° $P_{100} = 1·367466$ m., whence

$$\alpha_1 = \frac{P_{100} - P_0'}{100\, P_0'} = 0\cdot003\ 673\ 48.$$

The values for α_1 and P_0' here found should be substituted for those given in my previous communication ($p_0 = 1\cdot000\ 063$ and $\alpha_1 = 0\cdot003\ 663\ 80$).

The divergences between the uncorrected nitrogen scale and the theoretical scale, whose constants have just been calculated and which represents the normal scale of temperatures, are proportional to the temperature measured from 100° and have the following values:—

At	100°	0°·000
	200°	0·023
	300°	0·047
	400°	0·070

The difference between these values and those given previously is too small to be of appreciable practical importance, and our previous conclusions remain unaltered *.

Sèvres, Jan. 1901.

XXVI. *On Asymmetry of the Zeeman Effect.* By George W. Walker, *M.A., A.R.C.Sc., Fellow of Trinity College, Cambridge* †.

In his valuable papers on the Zeeman Phenomenon, Professor W. Voigt ‡ predicted an asymmetry of the normal triplet in the sense that the new component lying towards the violet end should be at a greater distance from the central component than the new component lying towards

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* I take this opportunity of correcting a few mistakes in my previous note, *loc. cit.*

p. 433, line 23. $\frac{1}{P_0}\frac{dP}{dt}$ instead of $\frac{1}{P}\frac{dP}{dt}$.

p. 438, line 7. $v_t = v_0(1 + (10275t + 3\cdot24t^2)10^{-9})$.

p. 438, line 10. $v_t = v_0(1 + (9715\cdot6t + 4\cdot43t^2)10^{-9})$.

p. 438, line 14. $v_t = v_0(1 + (9781\cdot4t + 4\cdot276t^2)10^{-9})$.

p. 440, line 23. Read—"the thick part of the tube expands *more* than the thinner part," instead of "*less* than etc."

p. 441, Table, 3rd column, 1st line. Read −0·17 instead of −0·27.

p. 442, Table, 2nd column, 4th line. Read +0·081 instead of +0·031.

† Communicated by the Physical Society: read October 25, 1901.

‡ *Annalen der Physik und Chemie*, lxvii. 1899, p. 345; *ibid.* i. 1900, p. 376.

the red end of the spectrum. Professor Zeeman has verified this and is endeavouring to measure the amount, which is excessively small.

Professor Voigt finds that

$$\delta = -\xi \pm \sqrt{c^2 R^2 + \xi^2},$$

where τ_0 and τ are the undisturbed and disturbed periods

$$\vartheta = \frac{\tau}{2\pi}, \qquad \vartheta_0 = \frac{\tau_0}{2\pi}, \qquad \delta = \vartheta - \vartheta_0,$$

$$\xi = \epsilon_1 \vartheta_0 ;$$

R = strength of the magnetic field.

The quantities ϵ_1 and c are constants depending on the system which produces the fundamental line. ξ is supposed very small, and upon it the asymmetry depends; for if $\xi = 0$ we get

$$\delta = \pm c R,$$

which represents the ordinary Zeeman effect. If ξ is retained we see that the asymmetry will be most marked in a weak magnetic field.

I find that asymmetry may be accounted for as a second order term arising from the magnetic field, and will now obtain the result.

Let us take as our representative molecule producing radiation, a system consisting of two atoms equally and oppositely charged. Let the charge be e and the effective masses m_1 and m_2 respectively. In order to avoid difficulties about the law of force between the two atoms we shall consider the motion as a disturbed circular orbit, so that we may write the equations of motion as

$$m_1 \ddot{x}_1 + a^2(x_1 - x_2) = e\mathrm{H}\dot{y}_1, \qquad m_2 \ddot{x}_2 - a^2(x_1 - x_2) = -e\mathrm{H}\dot{y}_2,$$

$$m_1 \ddot{y}_1 + a^2(y_1 - y_2) = -e\mathrm{H}\dot{x}_1, \qquad m_2 \ddot{y}_2 - a^2(y_1 - y_2) = +e\mathrm{H}\dot{x}_2,$$

$$m_1 \ddot{z}_1 + a^2(z_1 - z_2) = 0, \qquad m_2 \ddot{z}_2 - a^2(z_1 - z_2) = 0,$$

where $x_1 y_1 z_1$, $x_2 y_2 z_2$ are the coordinates of the centres of the two atoms and H is the strength of magnetic field supposed uniform and parallel to the z axis.

In general a in these equations may differ slightly from the undisturbed value, but for the present purpose this does not matter.

As first integrals of the equations we get

$$m_1\dot{x}_1+m_2\dot{x}_2=(m_1+m_2)u+e\mathrm{H}(y_1-y_2),$$
$$m_1\dot{y}_1+m_2\dot{y}_2=(m_1+m_2)v-e\mathrm{H}(x_1-x_2),$$
$$m_1\dot{z}_1+m_2\dot{z}_2=(m_1+m_2)w,$$

where *u*, *v*, *w* are constants of integration.

Using these equations and putting

$$p_0^2=a^2\left(\frac{1}{m_1}+\frac{1}{m_2}\right),$$

$$x_1-x_2=\xi, \qquad y_1-y_2=\eta, \qquad z_1-z_2=\zeta,$$

we get

$$\ddot{\xi}+p_0^2\xi+\frac{e^2\mathrm{H}^2}{m_1m_2}\xi=\frac{e\mathrm{H}(m_2-m_1)}{m_1m_2}\dot{\eta}+\frac{e\mathrm{H}(m_1+m_2)}{m_1m_2}v,$$
$$\ddot{\eta}+p_0^2\eta+\frac{e^2\mathrm{H}^2}{m_1m_2}\eta=-\frac{e\mathrm{H}(m_2-m_1)}{m_1m_2}\dot{\xi}-\frac{e\mathrm{H}(m_1+m_2)}{m_1m_2}u,$$
$$\ddot{\zeta}+p_0^2\zeta \qquad =0.$$

The terms in *u* and *v* have no influence on the periodic parts of ξ and η. Hence putting ξ, η, ζ proportional to ϵ^{i} we get for the ζ vibration

$$p=p_0,$$

and for the ξ and η vibrations the roots of p given by

$$p^2-p_0^2-\frac{e^2\mathrm{H}^2}{m_1m_2}=\pm\frac{e\mathrm{H}(m_2-m_1)}{m_1m_2}p\,;$$

and hence

$$p=\pm\frac{1}{2}\frac{e\mathrm{H}(m_1-m_2)}{m_1m_2}+p_0\sqrt{1+\frac{e^2\mathrm{H}^2(m_1+m_2)^2}{4p_0^2m_1^2m_2^2}}.$$

Neglecting squares of

$$\frac{e^2\mathrm{H}^2(m_1+m_2)^2}{4p_0^2m_1^2m_2^2}$$

we get

$$p=p_0\pm\frac{1}{2}\frac{e\mathrm{H}(m_1-m_2)}{m_1m_2}+\frac{1}{8}\frac{e^2\mathrm{H}^2(m_1+m_2)^2}{p_0m_1^2m_2^2}.$$

Putting

$$p=\frac{1}{\vartheta}, \qquad p_0=\frac{1}{\vartheta_0}, \qquad \delta=\vartheta-\vartheta_0,$$

we get

$$\delta=\pm\frac{1}{2}\frac{e\mathrm{H}(m_2-m_1)}{m_1m_2}\vartheta_0^2-\frac{1}{8}\frac{e^2\mathrm{H}^2(m_1+m_2)^2}{m_1^2m_2^2}\vartheta_0^3.$$

The new lines looking at right angles to the field are therefore:—

The central line $\vartheta=\vartheta_0$ polarized perpendicular to the direction of the lines of force,

and the lateral components

$$\delta+\vartheta_0=\vartheta_0 \pm \frac{1}{2}\frac{e\mathrm{H}(m_2-m_1)}{m_1 m_2}\vartheta_0^2-\frac{1}{8}\frac{e^2\mathrm{H}^2(m_1+m_2)^2}{m_1^2 m_2^2}\vartheta_0^3 . \quad . \quad (\mathrm{A})$$

polarized perpendicularly to the central line.

The first term represents the ordinary Zeeman effect. For comparison I rewrite Voigt's formula

$$\delta=-\xi\pm\sqrt{c^2\mathrm{R}^2+\xi^2},$$

or approximately

$$\delta=\pm c\mathrm{R}-\xi\pm\frac{1}{2}\frac{\xi^2}{c\mathrm{R}}\cdot\cdot\cdot\cdot$$

The asymmetry is therefore in the same sense in the two theories, viz., the lateral component towards the violet having the greater displacement. They differ, however, in one important respect. On the present theory the asymmetry is greater the greater the strength of the magnetic field, whereas on Voigt's theory it is more marked the smaller the field.

The term

$$\pm\frac{1}{2}\frac{e\mathrm{H}(m_2-m_1)}{m_1 m_2}\vartheta_0^2$$

in this theory is of course the same as $\pm c\mathrm{R}$ in Voigt's theory.

Let us next compare the effect in different parts of the spectrum.

On Voigt's theory the difference of displacement of the two components is

$$2\xi \quad \text{or} \quad 2\epsilon_1\vartheta_0,$$

and the fraction of the separation is

$$\frac{2\xi}{2c\mathrm{R}} \quad \text{or} \quad \frac{\epsilon_1\vartheta_0}{c\mathrm{R}}: \quad \text{that is} \quad \propto\frac{1}{\vartheta_0\times\mathrm{H}}.$$

On the present view the difference of displacement is

$$\frac{1}{4}\frac{e^2\mathrm{H}\,(m_1+m_2)^2}{m_1^2 m_2^2}\vartheta_0^3,$$

and the fraction of the separation is

$$\frac{1}{4}\frac{e\mathrm{H}(m_1+m_2)^2}{m_1m_2(m_1-m_2)}\vartheta_0.$$

On both views, then, the actual difference of displacement is greater, other things being equal, at the red end. But the fraction which the asymmetry is of the separation would be greater at the violet end of the spectrum on Voigt's theory, and greater at the red end on the present theory. This has an important bearing on the experimental detection of the effect.

We may further consider the probable numerical values.

The quantity ϵ_1 is connected with the molecule in such a way that the dielectric constant

$$\mathrm{K}=1+\Sigma\epsilon_1,$$

the summation referring to all the molecules: ϵ_1 is thus a quantity of order about 10^{-25}, and if $\vartheta_0=4\times10^{-16}$,

$$\xi \text{ is of order } 4\times10^{-41}.$$

If, as a variety of experimental work indicates, m_2 is very small compared with m_1, the formula (A) becomes

$$\delta=\mp\frac{1}{2}\frac{e\mathrm{H}}{m_2}\vartheta_0^{\,2}-\frac{1}{8}\frac{e^2\mathrm{H}^2}{m_2^{\,2}}\vartheta_0^{\,3}.$$

For a field of 10^4 C.G.S. units we may take $\dfrac{e\mathrm{H}}{m_2}$ about 10^{11};

therefore $\dfrac{1}{2}\dfrac{e\mathrm{H}}{m_2}\vartheta_0^{\,2}$ is about 8×10^{-21},

and $\dfrac{1}{8}\dfrac{e^2\mathrm{H}^2}{m_2^{\,2}}\vartheta_0^{\,3}$ is about 8×10^{-26}.

Thus ξ appears to be very small compared with

$$\frac{1}{8}\frac{e^2\mathrm{H}^2}{m_2^{\,2}}\vartheta_0^{\,3},$$

and the latter term might just come within measurable amount by increasing the field.

One more point in connexion with the proposed view may be noted, which is that it provides an explanation of why a line may not be resolvable. If $m_1=m_2$ there would be no doubling of the line, but only a small shift towards the violet. In this case the system is dynamically symmetrical.

It is almost unnecessary to remark that the simple system selected for discussion is merely illustrative of the characteristic features of the problem.

XXVII. *Notices respecting New Books.*

Expédition Antarctique Belge. Résultats du Voyage du S.Y. 'Belgica' en 1897–1898–1899 ...... *Aurores Australes par* HENRYK ARCTOWSKI. Anvers, 1901. 64 pp. quarto & 2 plates.

Expédition Norvégienne de 1899–1900 *pour l'étude des Aurores boréales. Résultats des recherches magnétiques par* KR. BIRKELAND. Christiania, 1901. 81 pp. large 8vo & 12 plates.

THESE two works possess a special interest in view of the expeditions which have recently set out, or are now in contemplation, having the Antarctic for their objective. A study of their contents will give the reader an idea of the nature of some of the more important problems with which these expeditions have to deal, and hence of the special training and wide scientific knowledge desirable in the observers.

Mr. Arctowski was one of the observers on the 'Belgica.' Of the 62 auroras he deals with, most were observed by himself, and all, with one exception, during the seven months March to September 1898. During the observations in 1898 the latitude of the 'Belgica' varied from 69° 51′ to 71° 36′ S., the longitude from 83° 35′ to 92° 21′ W. (of Greenwich). Particulars are usually given of the times of appearance and disappearance of each aurora, as well as of the more important changes observed. The more characteristic forms—arcs, bands, curtains, ribbons, &c.—are illustrated freely in the text, and in the two fine plates at the end of the volume. The descriptions occupy the larger part of the work. Mr. Arctowski concludes, p. 50, that the visual phenomena were very similar to those described by Nordenskiöld during the voyage of the 'Vega' in 1878–79. As he remarks, this is all the more significant from the fact that the times of the two voyages occupied similar positions in the sun-spot cycle, and that the distance of the 'Belgica' from the assumed position of the south magnetic pole was similar to that of the 'Vega' from the north magnetic pole.

Mr. Arctowski investigates, pp. 53–56, the diurnal variation:—(1) treating all the auroras; (2) including only those where the meteorological conditions were throughout favourable for visibility; (3) allowing weights according to the luminous intensity (estimated apparently by the number of special glass plates required to render the aurora invisible, see footnote to p. 8). The time found for the maximum varied between 9 and 11 P.M., according to the method of treatment. The great majority of the auroras were observed between 8 P.M. and 2 A.M. The results are very similar to those obtained at the polar station, Jan Mayen, in 1882–3. As concerns the annual variation, Mr. Arctowski found very decided maxima in March and September, with a much smaller maximum in July, and two nearly equal minima in May and August. The existence of maxima near the equinoxes has been generally observed in the northern hemisphere.

During a single appearance of an auroral arc there was usually a movement from the horizon towards the zenith and back again —sometimes several such alternations—the maximum altitude attained varying considerably even at the same season. As between one month and another Mr. Arctowski remarks, p. 63, on a distinct tendency for the altitude to be greater at the equinoxes and less at midwinter. The azimuth of the crown of the arc, though usually not very remote from the magnetic meridian, varied considerably. On p. 35, fig. 25 illustrates an interesting case in which the lower of two simultaneously existing arcs appeared distinctly elliptical, whilst the crowns of the two arcs differed appreciably in azimuth.

A number of other points are dealt with, on which the observations were somewhat few to throw much fresh light, and references are given to many recent authorities. In a few cases—*e.g.*, in that of the supposed 26-day period dealt with on p. 60—Mr. Arctowski is perhaps a little too ready to treat as facts what are more safely regarded as hypotheses; but this is a matter of opinion.

Mr. Arctowski's volume is the work of an observer who has mainly in view the description of phenomena, and not the attainment of a complete physical explanation. In many respects Prof. Birkeland's volume stands in marked contrast. The occasion for its appearance was an expedition, partly at the expense of the Norwegian Government, which occupied a station on the mountain of Haldde, near Bossekop, in the north of Norway, during the winter of 1899–1900. In addition to Prof. Birkeland, the expedition comprised two physical assistants—one killed by an avalanche—and a telegraph engineer. It had some of the newest patterns of instruments, including self-recording magnetographs of the Eschenhagen pattern, with arrangements for rotating the drum carrying the photographic paper in 24 hours, as usual, or in 2 hours. It had also an electrograph, with similar slow or rapid drum rotation, in which radium took the place of the ordinary water-dropper as "collector."

The book gives only a general preliminary summary of the observations—more especially of the magnetic ones—and a good deal of it is devoted to Prof. Birkeland's theoretical deductions and to laboratory experiments supporting his views. From the observational standpoint, the most interesting part is that dealing with the magnetograph records. By pre-arrangement, quick rotation of the drum carrying the photographic paper was simultaneously adopted on a number of days at Haldde and Potsdam, and Prof. Birkeland was subsequently allowed free access to all the Potsdam records. Time telegraph-signals were received at Haldde from Christiania twice a week, and special attention was given to recording exact times on the magnetic curves. In the case of the declination, 44 small perturbations were identified in the Haldde and Potsdam curves. In no case did the apparent difference in time exceed 18 seconds, and in 38 of the 44 cases the difference did not exceed 10 seconds. The mean apparent difference was 1 second. As Prof. Birkeland does not guarantee accuracy to nearer than 10 seconds, practical coincidence in time seems well established. The importance of this result is increased by the very considerable distance—about 2000 kilometres—between the two stations. Another interesting feature was the occurrence of well-marked magnetic waves, similar to those to whose elucidation so much has been done by Prof. Eschenhagen, whose recent death is a serious loss to Terrestrial Magnetism.

At first, it may be remembered, Prof. Eschenhagen's observations pointed to the conclusion that these waves were limited to one or two approximately constant periods, one notably of 30 seconds. Prof. Birkeland, however, gives, pp. 8 & 9, a complete list of all the "waves" recorded by the quick-moving apparatus at Potsdam from May 1898 to May 1900, and this shows periods of all lengths from 7·8 to 125·0 seconds. At Haldde the average periodic time would seem to be longer, for in the wave series observed there in March 1900 the periods varied from 62 to 130 seconds. Waves with such periods have been of course copiously recorded by Kew pattern magnetographs during the last 40 years; but with drums run at the usual rate, only the general features can be traced.

Whilst arriving, as we have seen, at the conclusion that distinctive small magnetic movements occurred simultaneously at Haldde and Potsdam, Prof. Birkeland concludes, p. 14, from a comparison of the more notable large disturbances observed at Haldde with those recorded on the same days at a number of observatories (including Pawlowsk, Copenhagen, Parc St. Maur, Greenwich, and Toronto), that there was not absolute identity in the times of occurrence, but that on the contrary there was a distinct retardation of phase in westerly as compared to easterly stations. The magnetic curves on which this conclusion is based appear in Plates III. to VII. If confirmed as a general fact, this would be very important, but meantime it should be borne in mind that no such phenomenon seems to have been detected by either W. G. Adams or Ellis, who made independent and more comprehensive investigations of this kind.

Pp. 15 to 38 are devoted mainly to the *cause* of magnetic disturbances, which Prof. Birkeland believes to be mainly, if not exclusively, electrical currents in the upper atmosphere. On this hypothesis he investigates the position, direction, and to some extent the intensity of the currents to which may be ascribed the magnetic disturbances observed at Haldde, and also those recorded at various stations during the international polar year

1882–3. The results are shown graphically in Plates II., VIII., IX., & X.

The remainder of the volume is mainly devoted to a description of experiments made with cathode rays and magnetic fields in high vacua, to a discussion of recent physical work by Goldstein, J. J. Thomson and others, and to the enunciation of Birkeland's own theory of the proximate cause of magnetic storms and aurora. He believes in the emission from the sun—notably at times of maximum sun-spot frequency—of cathode rays. When these reach the earth's rarefied atmosphere they come under the influence of the earth's magnetic field, and tend to coincide with the direction of the lines of magnetic force. For aurora the primary requisite is local intensity of current; for big magnetic disturbance a large total current is required, but it need not always be sufficiently concentrated to occasion much aurora.

The laboratory experiments consist mainly in the production of phenomena resembling aurora. Judging by the descriptions, by fig. 6 p. 57, and by Plates XI. and XII., the resemblance between the laboratory phenomena and auroral bands, ribbons, and "merry dancers" must be striking.

That aurora is an electrical manifestation, and that it is usually concurrent with magnetic storms and earth currents, is now generally believed; but it would be prudent as yet to regard with reserve any of the theoretical speculations advanced to account for the interconnection. A theory may be in general accord with a number of facts discussed by its founder, and yet be found wanting when tested over another range of phenomena by independent critics. This may be the fate in store for Prof. Birkeland's theory; but in any case it should serve a useful purpose in suggesting lines of research and focussing attention on crucial points.

By the older type of magnetic observer, the physicist, with his laboratory experiments and proclivities to theory, is apt to be regarded as a relative of the chamber philosopher, who aims at the immediate regeneration of Society. There is sometimes a good deal to be said for this view; but at the present time another side of the picture may be more profitably regarded, at least in this country. We have in England various magnetic observatories, some of which devote a good deal of time to recording magnetic results and dealing with them in a routine way; but there is no adequate provision for securing the existence on their staffs or the co-operation of men possessing wide theoretical knowledge, with the brain-power and the leisure necessary to secure the best results. The volume before us seems evidence that the Norwegian Government and People take a sympathetic view in such matters, and it is to be hoped that they have in contemplation the continuance of the Haldde observations during the forthcoming "Antarctic Year."

C. CHREE.

Die Partiellen Differential-Gleichungen der Mathematischen Physik. Nach Riemann's Vorlesungen in vierter Auflage neu bearbeitet von HEINRICH WEBER, Professor der Mathematik an der Universität Strassburg. Zweiter Band. Mit eingedruckten Abbildungen. Braunschweig: F. Vieweg und Sohn. 1901. Pp. xii+527.

IT is not very long since we had occasion to notice the first volume of this highly important work on mathematical physics. The second volume has now made its appearance, and fully maintains the high standard of excellence of its predecessor. It is divided into five sections. Section I. is purely mathematical, and contains a succinct account of that portion of the theory of linear differential equations which is of importance in connexion with physical applications. Section II. deals with the theory of heat conduction. Section III. is devoted to the theory of elasticity, and includes a brief general account of elasticity, statical elastic problems, and the vibrations of strings and plates. Section IV. is on electrical oscillations, and deals with electromagnetic waves, the propagation of current along a cable, and the reflexion of electromagnetic waves. The concluding Section V. is devoted to Hydrodynamics.

To the student of mathematical physics the completed work will be one of exceptional value and interest.

XXVIII. *Intelligence and Miscellaneous Articles.*

THE MICHELSON-MORLEY EXPERIMENT.

Dunheved, Endcliffe Crescent,
Sheffield
January 9, 1902.

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

WITH reference to my remark in the second paragraph on page 38 of the last number of the Philosophical Magazine, I learn from Prof. Morley that although the observations were taken in reverse order, the results were printed in direct. As he remarks, the curves would therefore show either that the maximum drift took place about 3 P.M., or that they were due to instrumental errors.

Whatever may be the cause, it may be noted that if, as the latest authorities say, Vega is the apex of the sun's way, and the sun's proper motion is 15 miles per second, then on July 8, the resultant velocity of the earth would give at Cleveland, Ohio (lat. 42° N.), a maximum drift at about 7.45 P.M.

I am, Yours faithfully,
W. M. HICKS.

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

MARCH 1902.

XXIX. *Aepinus Atomized.* *By* Lord KELVIN*.

§ 1. ACCORDING to the well-known doctrine of Aepinus, commonly referred to as the one-fluid theory of electricity, positive and negative electrifications consist in excess above, and deficiency below, a natural quantum of a fluid, called the electric fluid, permeating among the atoms of ponderable matter. Portions of matter void of the electric fluid repel one another; portions of the electric fluid repel one another; portions of the electric fluid and of void matter attract one another.

§ 2. My suggestion is that the Aepinus' fluid consists of exceedingly minute equal and similar atoms, which I call electrions †, much smaller than the atoms of ponderable matter; and that they permeate freely through the spaces occupied by these greater atoms and also freely through space not occupied

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* Communicated by the Author. From the Jubilee Volume presented to Prof. Bosscha in November 1901.

† I ventured to suggest this name in a short article published in 'Nature,' May 27, 1897, in which, after a slight reference to an old idea of a "one-fluid theory of electricity" with *resinous electricity as the electric fluid*, the following expression of my views at that time occurs:—"I prefer "to consider an atomic theory of electricity foreseen as worthy of thought "by Faraday and Clerk Maxwell, very definitely proposed by Helmholtz "in his last lecture to the Royal Institution, and largely accepted by "present-day theoretical workers and teachers. Indeed, Faraday's law "of electrochemical equivalence seems to necessitate something atomic "in electricity, and to justify Johnstone Stoney's word *electron*. The "older, and at present even more popular, name *ion* given sixty years "ago by Faraday, suggests a convenient modification of it, *electrion*, to "denote an atom of resinous electricity. And now, adopting the

by them. As in Aepinus' theory we must have repulsions between the electrions; and repulsions between the atoms independently of the electrions: and attractions between electrions and atoms without electrions. For brevity, in future by atom I shall mean an atom of ponderable matter, whether it has any electrions within it or not.

§ 3. In virtue of the discovery and experimental proof by Cavendish and Coulomb of the law of inverse square of distance for both electric attractions and repulsions, we may now suppose that the atoms, which I assume to be all of them spherical, repel other atoms outside them with forces inversely as the squares of distances between centres; and that the same is true of electrions, which no doubt occupy finite spaces, although at present we are dealing with them as if they were mere mathematical points, endowed with the property of electric attraction and repulsion. We must now also assume that every atom attracts every electrion outside it with a force inversely as the square of the distance between centres.

§ 4. My assumption that the electrions freely permeate the space occupied by the atoms requires a knowledge of the law of the force experienced by an electrion within an atom. As a tentative hypothesis, I assume for simplicity that the attraction experienced by an electrion approaching an atom varies exactly according to the inverse square of the distance from the centre, as long as the electrion is outside; has no abrupt change when the electrion enters the atom; and decreases to zero simply as the distance from the centre when the electrion, approaching the centre, is within the spherical boundary of the atom. This is just as it would be if the electric virtue of the atom were due to uniform distribution through the atom of an ideal electric substance of which each infinitely small part repels infinitely small portions of

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" essentials of Aepinus' theory, and dealing with it according to the " doctrine of Father Boscovich, each atom of ponderable matter is an " electron of vitreous electricity; which, with a neutralizing electrion of " resinous electricity close to it, produces a resulting force on every " distant electron and electrion which varies inversely as the cube of the " distance, and is in the direction determined according to the well-known " requisite application of the parallelogram of forces." It will be seen that I had not then thought of the hypothesis suggested in the present communication, that while electrions permeate freely through all space, whether occupied only by ether or occupied also by the volumes of finite spheres constituting the atoms of ponderable matter, each electrion in the interior of an atom of ponderable matter experiences electric force towards the centre of the atom, just as if the atom contained within it, fixed relatively to itself, a uniform distribution of ideal electric matter.

the ideal substance in other atoms, and attracts electrions, according to the inverse square of the distance. But we cannot make the corresponding supposition for the mutual force between two *overlapping* atoms ; because we must keep ourselves free to add a repulsion or attraction according to any law of force, that we may find convenient for the explanation of electric, elastic, and chemical properties of matter.

§ 5. The neutralizing quantum of electrions for any atom or group of atoms has exactly the same quantity of electricity of one kind as the atom or group of atoms has of electricity of the opposite kind. The quantum for any single atom may be one or two or three or any integral number, and need not be the same for all atoms. The designations monelectrionic, dielectrionic, trielectrionic, tetraelectrionic, polyelectrionic, &c., will accordingly be convenient. It is possible that the differences of quality of the atoms of different substances may be partially due to the quantum-numbers of their electrions being different ; but it is possible that the differences of quality are to be wholly explained in merely Boscovichian fashion by differences in the laws of force between the atoms, and may not imply any differences in the numbers of electrions constituting their quantums.

§ 6. Another possibility to be kept in view is that the neutralizing quantum for an atom may not be any integral number of electrions. Thus for example the molecule of a diatomic gas, oxygen, or nitrogen, or hydrogen, or chlorine, might conceivably have three electrions or some odd number of electrions for its quantum so that the single atoms, O, N, H, Cl, if they could exist separately, must be either vitreously or resinously electrified and cannot be neutral.

§ 7. The present usage of the designations, positive and negative, for the two modes of electrification originated no doubt with the use of glass globes or cylinders in ordinary electric machines giving vitreous electricity to the insulated prime conductor, and resinous electricity to the not always insulated rubber. Thus Aepinus and his followers regarded the prime conductors of their machines as giving the true electric fluid, and leaving a deficiency of it in the rubbers to be supplied from the earth. It is curious, in Beccaria's account of his observations made about 1760 at Garzegna in Piedmont on atmospheric electricity, to read of "The mild excessive electricity of the air in fair weather." This in modern usage would be called mild positive electricity. The meaning of either expression, stated in non-hypothetical language, is, the mild vitreous electricity of the air in fair weather.

§ 8. In the mathematical theory of electricity in equilibrium, it is a matter of perfect indifference which of the opposite electric manifestations we call positive and which negative. But the great differences in the disruptive and luminous effects, when the forces are too strong for electric equilibrium, presented by the two modes of electrification, which have been known from the earliest times of electric science, show physical properties not touched by the mathematical theory. And Varley's comparatively recent discovery* of the molecular torrent of resinously electrified particles from the "kathode" or resinous electrode in apparatus for the transmission of electricity through vacuum or highly rarefied air, gives strong reason for believing that the mobile electricity of Aepinus' theory is resinous, and not vitreous as he accidentally made it. I shall therefore assume that our electrions act as extremely minute particles of *resinously* electrified matter; that a void atom acts simply as a little globe of atomic substance, possessing as an essential quality vitreous electricity uniformly distributed through it or through a smaller concentric globe; and that ordinary ponderable matter, not electrified, consists of a vast assemblage of atoms, not void, but having within the portions of space which they occupy just enough of electrions to annul electric force for all places of which the distance from the nearest atom is large in comparison with the diameter of an atom, or molecular cluster of atoms.

§ 9. This condition respecting distance would, because of the inverse square of the distance law for the forces, be unnecessary and the electric force would be rigorously null throughout all space outside the atoms, if every atom had only a single electrion at its centre, provided that the electric quantities of the opposite electricities (reckoned according to the old definition of mathematical electrostatics) are equal in the atom and in the electrion. But even if every neutralized separate atom contains just one electrion in stable equilibrium at its centre, it is obvious that, when two atoms overlap so far that the centre of one of them is within the spherical boundary of the other, the previous equilibrium of the two electrions is upset, and they must find positions of equilibrium elsewhere than at the centres. Thus in fig. 1 each electrion is at the centre of its atom, and is attracted and repelled with equal forces by the neighbouring atom and electrion at *its* centre. In fig. 2, if E and E′ were at the centres C, C′, of the two atoms, E would be repelled by E′ more than it would be

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* Proc. Roy. Soc. vol. xix. pp. 239, 240 (1871).

attracted by the atom A′. Hence both electrions being supposed free, E will move to the right; and because of its diminished repulsion on E′, E′ will follow it in the same direction. The equations of equilibrium of the two are easily written down, not so easily solved without some slight arithmetical artifice. The solution is correctly shown in fig. 2, for the case in which one radius is three times the other, and the distance between the centres is 2·7 times the smaller radius *. The investigation in the footnote shows that if the atoms are brought a little nearer, the equilibrium becomes unstable; and we may infer that both electrions jump to the right, E′ to settle at a point within the atom A on the left-hand side of its centre; and E outside A′, to settle at a point still within A. If, lastly, we bring the centres closer and closer together till they coincide, E comes again within A′, and the two electrions settle, as shown in fig. 3, at distances on the two sides of the common centre, each equal to

Fig. 1.

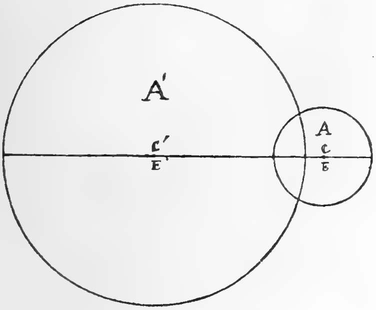
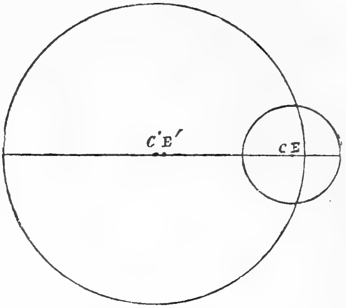


Fig. 2.



Radii 3 and 1.
C′C=2·7. C′E′=·1458. CE=·0462.

$$\frac{1}{2}\sqrt[3]{\frac{2}{\frac{1}{a^3}+\frac{1}{a'^3}}}$$

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* Calling e the quantity of electricity, vitreous or resinous, in each atom or electrion; ζ the distance between the centres of the atoms; a, a' the radii of the two atoms; x, x' the displacements of the electrions from the centres; X, X′ the forces experienced by the electrions; we have

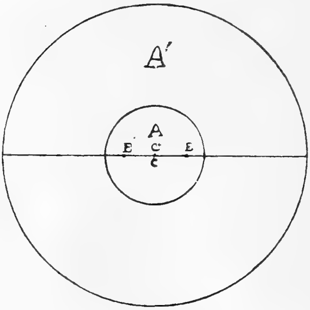
$$X=e^2\left[-\frac{x}{a^3}+\frac{1}{(\zeta+x-x')^2}-\frac{\zeta+x}{a'^3}\right];$$

$$X'=e^2\left[-\frac{x'}{a'^3}+\frac{1}{(\zeta-x')^2}-\frac{1}{(\zeta+x-x')^2}\right].$$

Each of these being equated to zero for equilibrium gives us two equations which for the case $\alpha' = 3\alpha$ is

$$\frac{1}{2}\alpha\sqrt[3]{\frac{2.27}{28}} = \cdot 622\,\alpha.$$

Fig. 3.



E'C=CE=·622.

§ 10. Mutual action of this kind might probably be presented in such binary combinations as O_2, N_2, H_2, Cl_2, CO, SO, NaCl (dry common salt) if each single atom, O, N, H, Cl, C*, S, Na†, had just one electrion for its neutralizing

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which are not easily dealt with by frontal attack for the determination of two unknown quantities x, x'; but which may be solved by a method of successive approximations, as follows:—Let $x_0, x_1, \ldots x_i, x'_0, x'_1, \ldots x_i'$, be successive approximations to the values of x and x', and take

$$x_{i+1} = \frac{1}{\frac{1}{a^3} + \frac{1}{a'^3}}\left(\frac{1}{D_i^2} - \frac{\zeta}{a'^3}\right); \quad x'_{i+1} = a'^3\left\{\frac{1}{(\zeta - x'_i)^2} - \frac{1}{(\zeta + x_{i+1} - x'_i)^2}\right\};$$

where $D^2_i = (\zeta + x_i - x'_i)^2$. As an example, take $a=1$, $a'=3$. To find solutions for gradual approach between centres, take successively $\zeta = 2\cdot 9$, $2\cdot 8$, $2\cdot 7$, $2\cdot 6$. Begin with $x_0 = 0$, $x'_0 = 0$, we find $x_4 = \cdot 01243$, $x'_4 = \cdot 0297$, and the same values for x_5, and x'_5. Take next $\zeta = 2\cdot 8$, $x_0 = \cdot 01243$, $x'_0 = \cdot 0297$; we find $x_4 = x_5 = \cdot 0269$, $x'_4 = x'_5 = \cdot 0702$. Thus we have the solution for the second distance between centres. Next take $\zeta = 2\cdot 7$, $x_0 = \cdot 0269$, $x'_0 = \cdot 0702$; we find $x_6 = x_7 = \cdot 0462$, $x'_6 = x'_7 = \cdot 1458$. Working similarly for $\zeta = 2\cdot 6$, we do not find convergence, and we infer that a position of unstable equilibrium is reached by the electrions for some value of ζ between $2\cdot 7$ and $2\cdot 6$.

* The complexity of the hydrocarbons and the Van't Hoff and Le Bel doctrine of the asymmetric results (chirality) produced by the quadrivalence of carbon makes it probable that the carbon atom takes at least four electrions to neutralize it electrically.

† The fact that sodium, solid or liquid, is a metallic conductor of electricity makes it probable that the sodium atom, as all other metallic elements, takes a large number of electrions to neutralize it (see below, § 30).

quantum. If the combination is so close that the centres coincide, the two electrions will rest stably at equal distances on the two sides of the common centre as at the end of § 9. I see at present no reason for considering it excessively improbable that this may be the case for SO, or for any other binary combinations of *two atoms of different quality* for neither of which there is reason to believe that its neutralizing quantum is not exactly one electrion. But for the binary combinations of two atoms of identical quality which the chemists have discovered in diatomic gases (O_2, N_2, &c.) there must, over and above the electric repulsion of the two similar electric globes, be a strong atomic repulsion preventing stable equilibrium with coincident centres, however strongly the atoms may be drawn together by the attractions of a pair of mutually repellent electrions within them; because without such a repulsion the two similar atoms would become one, which no possible action in nature could split into two.

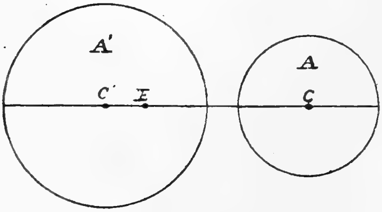
§ 11. Returning to fig. 3, let us pull the two atoms gradually asunder from the concentric position to which we had brought them. It is easily seen that the electrions will both remain within the smaller atom A, slightly disturbed from equality of distance on the two sides of its centre by attractions towards the centre of A′; and that when A′ is infinitely distant they will settle at distances each equal to $\frac{1}{2}\alpha\sqrt[3]{2} = \cdot 62996\,\alpha$ on the two sides of the centre of A. If, instead of two monelectrionic atoms, we deal with two polyelectrionic atoms as in § 9, we find after separation the number of electrions in the smaller atom increased and in the larger decreased; and this with much smaller difference of magnitude than the three to one of diameters which we had for our monelectrionic atoms of § 9. This is a very remarkable conclusion, pointing to what is probably the true explanation of the first known of the electric properties of matter; attractions and repulsions produced by rubbed amber. Two ideal solids consisting of assemblages of monelectrionic atoms of largely different sizes would certainly, when pressed and rubbed together and separated, show the properties of oppositely electrified bodies; and the preponderance of the electrionic quality would be in the assemblage of which the atoms are the smaller. Assuming as we do that the electricity of the electrions is of the resinous kind, we say that after pressing and rubbing together and separating the two assemblages, the assemblage of the smaller atoms is resinously electrified and the assemblage of the larger atoms is vitreously electrified. This is probably the true explanation of the old-known fact that ground glass is resinous relatively to polished glass. The process of polishing might be expected to smooth down the smaller atoms, and to leave the larger atoms more effective in the surface.

§ 12. It probably contains also the principle of the explanation of Erskine Murray's* experimental discovery that surfaces of metals, well cleaned by rubbing with glass-paper or emery-paper, become more positive or less negative in the Volta contact electricity scale by being burnished with a smooth round hard steel burnisher. Thus a zinc plate brightened by rubbing on glass-paper rose by ·23 volt by repeated burnishing with a hard steel burnisher, and fell again by the same difference when rubbed again with glass-paper. Copper plates showed differences of about the same amount and in the same direction when similarly treated. Between highly burnished zinc and emery-cleaned copper, Murray found a Volta-difference of 1·13 volts, which is, I believe, considerably greater than the greatest previously found Volta-difference between pure metallic surfaces of zinc and copper.

§ 13. To further illustrate the tendency (§ 9) of the smaller atom to take electrions from the larger, consider two atoms: A′, of radius a', the greater, having an electrion in it to begin with; and A, radius a, the smaller, void.

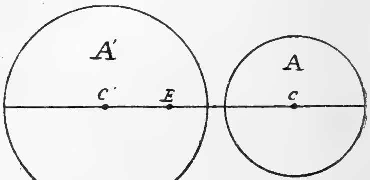
By ideal forces applied to the atoms while the electrion is free let them approach gradually from a very great distance apart. The attraction of A draws the electrion from the centre of A′; at first very slightly, but farther and farther as the distance between the atoms is diminished. What will be the position of the electrion when the distance between the centres is, as in fig. 4, $2a'$? Without calculation we see that the electrion would be in equilibrium if placed at the point

Fig. 4.



$a'=1.$ C′C=2. C′E=·38.

Fig. 5.



C′C=1·89. C′E=0·63.

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* "On Contact Electricity of Metals," Proc. Roy. Soc. vol. lxiii. p. 113 (1898). See also Lord Kelvin, "Contact Electricity of Metals," Phil. Mag. vol. xlvi. pp. 96–98 (1898).

in which the surface of A′ is cut by the line of centres; but the equilibrium would be obviously unstable, and a simple calculation* shows that the stable position actually taken by the electrion is $\cdot 38\,\alpha'$ from C′, when the distance between the centres is $2\alpha'$ (fig. 4). If the distance between the centres is now diminished from $2\alpha'$ to $1\cdot 89\alpha'$ (α being now supposed to be anything less than $\cdot 89\alpha'$) the electrion comes gradually to distance $\cdot 63\alpha'$ from C′ (fig. 5); its equilibrium there becomes unstable; and it jumps out of A′ towards A (like a cork jumping out of a bottle). It will shoot through A (A′ and A being held fixed); and after several oscillations to and fro, perhaps† ten or twenty, if it has only quasi inertia due to condensation or rarefaction ‡ produced by it in ether; or perhaps many times more if it has intrinsic inertia of its own; it will settle, with decreasing range of excursions, sensibly to rest within A, attracted somewhat from the centre by A′. If, lastly, A′ and A be drawn asunder to their original great distance, the electrion will not regain its original position in A′, but will come to the centre of A and rest there. Here then we have another illustration of the tendency found in § 9, of the smaller atom to take electrions from the larger.

§ 14. In preventing the two atoms from rushing together by holding them against the attractive force of the electrion, we shall have gained more work during the approach than we afterwards spent on the separation; and we have now

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* Denoting by ζ the distance between the centres, and by X the force on E when its distance from C′ is x', we have

$$X = e^2\left[\frac{1}{(\zeta - x')^2} - \frac{x'}{a'^3}\right].$$

Hence for equilibrium $\dfrac{1}{(\zeta - x')^2} = \dfrac{x'}{a'^3}$. This is a cubic for x' of which the proper root (the smallest root) for the case $\zeta = 2a'$ is $\cdot 38\,a'$. The formula for X has a minimum value when $\zeta - x' = a'\sqrt[3]{2}$, which makes

$$X = \frac{e^2}{a'^2}\left[\frac{3}{2}\sqrt[3]{2} - \frac{\zeta}{a'}\right].$$

Hence the value of x' for equilibrium coincides with the value of X, a minimum, and the equilibrium becomes unstable, when ζ is diminished to $\dfrac{3\sqrt[3]{2}}{2}a' = 1\cdot 890\,a'$. For this, the value of x' is $\dfrac{\sqrt[3]{2}}{2}a' = \cdot 63\,a'$.

† "On the Production of Wave Motion in an Elastic Solid," Phil. Mag. Oct. 1899, § 44.

‡ "On the Motion of Ponderable Matter through Space Occupied by Ether," Phil. Mag. Aug. 1900, §§ 15, 17.

left the system deprived of the further amount of energy carried away by ethereal waves into space.

§ 15. The system in its final state with the electrion at the centre of the smaller atom has less potential energy in it than it had at the beginning (when the electrion was at the centre of A′), by a difference equal to the excess of the work which we gained during the approach above that which we spent on the final separation of A′ and A, plus the amount carried away by the ethereal waves. All these items except the last are easily calculated from the algebra of the footnote on § 13; and thus we find how much is our loss of energy by the ethereal waves.

§ 16. Very interesting statical problems are presented to us by consideration of the equilibrium of two or more electrions within one atom, whether a polyelectrionic atom with its saturating number, or an atom of any electric strength with any number of electrions up to the greatest number that it can hold. To help to clear our ideas, first remark that if the number of electrions is infinite, that is to say if we go back to Aepinus' electric fluid, but assume it to permeate freely through an atom of any shape whatever and having any arbitrarily given distribution of electricity of the opposite kind fixed within it, the greatest quantity of fluid which it can take is exactly equal to its own, and lodges with density equal to its own in every part. Hence if the atom is spherical, and of equal electric density throughout as we have supposed it, and if its neutralizing quantum of electrions is a very large number, their configuration of equilibrium will be an assemblage of more and more nearly uniform density from surface to centre, the greater the number. Any Bravais homogeneous assemblage whatever would be very nearly in equilibrium if all the electrions in a surface-layer of thickness a hundred times the shortest distance from electrion to electrion were held fixed; but the equilibrium would be unstable except in certain cases. It may seem probable that it is stable if the homogeneous assemblage is of the species which I have called* equilateral, being that in which each electrion with any two of its twelve next neighbours forms an equilateral triangle. If now all the electrions in the surface-layer are left perfectly free, a slight rearrangement among themselves and still slighter among the neighbouring electrions in the interior will bring the whole multitude (of thousands or millions) to equilibrium. The subject is of

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* "Molecular Tactics of a Crystal," § 4, being the Second Robert Boyle Lecture, delivered before the Oxford University Junior Scientific Club, May 16, 1893 (Clarendon Press, Oxford).

extreme interest, geometrical, dynamical, and physical, but cannot be pursued further at present.

§ 17. To guide our ideas respecting the stable equilibrium of moderate numbers of electrions within an atom, remark first that for any number of electrions there may be equilibrium with all the electrions on one spherical surface concentric with the atom. To prove this, discard for a moment the atom and imagine the electrions, whatever their number, to be attached to ends of equal inextensible strings of which the other ends are fixed to one point C. Every string will be stretched in virtue of the mutual repulsions of the electrions; and there will be a configuration or configurations of equilibrium with the electrions on a spherical surface. Whatever their number there is essentially at least one configuration of stable equilibrium. Remark also that there is always a configuration of equilibrium in which all the strings are in one plane, and the electrions are equally spaced round one great circle of the sphere. This is the sole configuration for two electrions or for three electrions; but for any number exceeding three it is easily proved to be unstable, and is therefore not the sole configuration of equilibrium. For four electrions it is easily seen that, besides the unstable equilibrium in one plane, there is only the stable configuration, and in this the four electrions are at the four corners of an equilateral tetrahedron.

§ 18. For five electrions we have clearly stable equilibrium with three of them in one plane through C, and the other two at the ends of the diameter perpendicular to this plane. There is also at least one other configuration of equilibrium: this we see by imagining four of the electrions constrained to remain in a freely movable plane, which gives stable equilibrium with this plane at some distance from the centre and the fifth electrion at the far end of the diameter perpendicular to it. And similarly for any greater number of electrions, we find a configuration of equilibrium by imagining all but one of them to be constrained to remain in a freely movable plane. But it is not easy, without calculation, to see, at all events for the case of only five electrions, whether that equilibrium would be stable if the constraint of all of them but one to one plane is annulled. For numbers greater than five it seems certain that that equilibrium is unstable.

§ 19. For six we have a configuration of stable equilibrium with the electrions at the six corners of a regular octahedron; for eight at the corners of a cube. For ten, as for any even number, we should have two configurations of equilibrium (both certainly unstable for large numbers) with two halves of the number in two planes at equal distances on the two sides of the centre. For twelve we have a configuration of stable equilibrium with the electrions at positions of the twelve nearest neighbours to C in an equilateral homogeneous assemblage of points*; for twenty at the twenty corners of a pentagonal dodecahedron. All these configurations of § 19, except those described for ten electrions, are stable if, as we are now supposing, the electrions are constrained to a spherical surface on which they are free to move.

§ 20. Except the cases of § 18, the forces with which the strings are stretched are the same for all the electrions of each case. Hence if we now discard the strings and place the electrions in an atom on a spherical surface concentric with it, its attraction on the electrions towards the centre takes the place of the tension of the string, provided it is of the proper amount. But it does not secure, as did the strings, against instability relatively to radial displacements, different for the different electrions. To secure the proper amount of the radial force the condition is $\frac{ie^2r}{\alpha^3}=\mathrm{T}$; where i denotes the number of electrions; e the electric quantity on each (and therefore, § 8, ie the electric quantity of vitreous electricity in the atom); r denotes the radius of the spherical surface on which the electrions lie; α the radius of the atom; and T the tension of the string in the arrangement of § 17. We have generally $\mathrm{T}=q\frac{e^2}{r^2}$ where q is a numeric depending on the number and configuration of the electrions found in each case by geometry. Hence we have $\frac{r}{\alpha}=\sqrt[3]{\frac{q}{i}}$ for the ratio of the radius of the smaller sphere on which the electrions lie to the radius of the atom. For example, take the case of eight electrions at the eight corners of a cube. T is the resultant of seven repulsions, and we easily find

$$q=\frac{3}{4}\left(\sqrt{3}+\sqrt{\frac{3}{2}}+\frac{1}{3}\right) \text{ and finally } \frac{r}{\alpha}=\cdot 6756.$$

Dealing similarly with the cases of two, three, four, and six electrions, we have the following table of values of $\left(\frac{r}{\alpha}\right)^3$ and $\frac{r}{\alpha}$; to which is added a last column showing values of

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* "Molecular Tactics of a Crystal," § 4.

$$i^2\frac{3\alpha^2-r^2}{2\alpha^2}-\Sigma\frac{\alpha}{\mathrm{D}},$$

being $\frac{\alpha}{e^2}$ of the work required to remove the electrions to infinite distance.

Number of Electrions.	Configuration.	$\left(\frac{r}{a}\right)^3$.	$\frac{r}{a}$.	$\frac{\alpha}{e^2}\times$ work required to remove the electrions to infinite distance, $=w$.
2	At the ends of a diameter	$\frac{1}{8}$	·5000	4·500
3	At the corners of an equilateral triangle	$\frac{1}{3\sqrt{3}}$	·5774	9·000
4	At the corners of a square	$\frac{\sqrt{2}}{8}+\frac{1}{16}$	·6208	14·750
4	At the corners of an equilateral tetrahedron	$\frac{3}{16}\sqrt{\frac{3}{2}}$	·6124	15·000
6	At the corners of an equilateral octahedron	$\frac{1+4\sqrt{2}}{24}$	·6522	33·335
8	At the corners of a cube	$\frac{3}{32}\left(\sqrt{3}+\sqrt{\frac{3}{2}}+\frac{1}{3}\right)$	·6756	52·180

§ 21. In the configurations thus expressed the equilibrium is certainly stable for the cases of two, three, and four electrions. It seems to me, without calculation, also probably stable for the case of six, and possibly even for the case of eight. For the case of twenty at the corners of a pentagonal dodecahedron the equilibrium is probably not stable: and even for the cases of twelve electrions and ten electrions, the equilibrium in the configurations described in §§ 18, 19 may probably be unstable, when, as now, we have the attraction of the atom towards the centre instead of the inextensible strings.

§ 22. In fact when the number of electrions exceeds four, we must think of the tendency to be crowded out of one spherical surface, which with very large numbers gives a tendency to uniform distribution throughout the volume of the atom as described in § 16 above. Thus, in the case of five electrions, § 18 shows a configuration of equilibrium in which the two electrions lying in one diameter are, by the mutual repulsions, pushed very slightly further from the centre than are the three in the equatorial plane. In this case the equilibrium is clearly stable. Another obvious configuration, also stable, of five electrions within an atom is one at the centre, and four on a concentric spherical surface at the corners, of a tetrahedron. From any case of any number of electrions all on one spherical surface, we may pass to another configuration with one more electrion placed at the centre and the proper proportionate increase in the electric strength of the atom. Thus from the cases described in § 19, we may pass to configurations of equilibrium for seven, nine, eleven, thirteen, and twenty-one electrions. All these cases, with questions of stability or instability and of the different amounts of work required to pluck all the electrions out of the atom and remove them to infinite distances, present most interesting subjects for not difficult mathematical work; and I regret not being able to pursue them at present.

§ 23. Consider now the electric properties of a real body, gaseous, liquid, or solid, constituted by an assemblage of atoms with their electrions. It follows immediately from our hypothesis, that in a monatomic gas or in any sufficiently sparse assemblage of single atoms, fixed or moving, Faraday's "*conducting power for lines of electric force,*" *or what is now commonly called the specific electro-inductive capacity, or the electro-inductive permeability, exceeds unity by three times the ratio of the sum of the volumes of the atoms to the whole volume of space occupied by the assemblage,* whether the atoms be monelectrionic or polyelectrionic, and however much the electrion, or group of electrions, within each atom is set to vibrate or rotate with each collision, according to the kinetic theory of gases. To prove this, consider, in a uniform field of electrostatic force of intensity F, a single atom of radius α, and, at rest within it, a group of i electrions in stable equilibrium. The action of F produces simply displacements of the electrions relatively to the atom, equal and in parallel lines, with therefore no change of shape and no rotation; and, x denoting the amount of this displacement, the equation for the equilibrium of each electrion is $\frac{iex}{\alpha^3}=\mathrm{F}$. This gives $iex=\alpha^3\mathrm{F}$ for the electric moment of the electrostatic polarization induced in the atom by F. In passing, remark that $\alpha^3\mathrm{F}$ is also equal to the electric moment of the polarization produced in an insulated unelectrified metal globe of radius α, when brought into an electrostatic field of intensity F: and conclude that the electric inductive capacity of a uniformly dense assemblage of fixed metallic globules, so sparse that their mutual influence is negligible, is the same as that of an equal and similar assemblage of our hypothetical atoms, whatever be the number of electrions in each, not necessarily the same in all. Hence our hypothetical atom realizes perfectly for sparse assemblages Faraday's suggestion of "small globular conductors, as shot" to explain the electro-polarization which he discovered in solid and liquid insulators. (Experimental Researches, § 1679.)

§ 24. Denoting now by N the number of atoms per unit volume we find $NV\alpha^3F$ as the electric moment of any sparse enough assemblage of uniform density occupying volume V in a uniform electric field of intensity F. Hence $N\alpha^3$ is what (following the analogy of electromagnetic nomenclature) we may call the electro-inductive susceptibility* of the assemblage; being the electric moment per unit bulk induced by an electric field of unit intensity. Denoting this by μ, and the electro-inductive permeability by ω, we have (Electrostatics and Magnetism, § 629 (14))

$$\omega=1+4\pi\mu=1+3\left(N\frac{4\pi\alpha^3}{3}\right)$$

which proves the proposition stated at the commencement of § 23.

§ 25. To include vibrating and rotating groups of electrions in the demonstration, it is only necessary to remark that the time-average of any component of the displacement of the centre of inertia of the group relatively to the centre of the atom will, under the influence of F, be the same as if the assemblage were at rest in stable equilibrium.

§ 26. The consideration of liquids consisting of closely packed mobile assemblages of atoms or groups of atoms with their electrions, forming compound molecules, as in liquid argon or helium (monatomic), nitrogen, oxygen, &c. (diatomic), or pure water, or water with salts or other chemical substances dissolved in it, or liquids of various complex chemical constitutions, cannot be entered on in the present communication, further than to remark that the suppositions we have made regarding forces, electric and other, between electrions and atoms, seem to open the way to a very definite detailed dynamics of electrolysis, of chemical affinity, and of heat of chemical combination. Estimates of the actual magnitudes concerned (the number of molecules per cubic centimetre of a gas, the mass in grammes of an atom of any substance, the diameters of the atoms, the absolute value of the electric quantity in an electrion, the effective mass or

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* Suggested in my 'Electrostatics and Magnetism,' §§ 628, 629.

inertia of an electrion) seem to show that the intermolecular electric forces are more than amply great enough to account for heat of chemical combination, and every mechanical action manifested in chemical interactions of all kinds. We might be tempted to assume that all chemical action is electric, and that all varieties of chemical substance are to be explained by the numbers of the electrions required to neutralize an atom or a set of atoms (§ 6 above); but we can feel no satisfaction in this idea when we consider the great and wild variety of quality and affinities manifested by the different substances or the different "chemical elements"; and as we are assuming the electrions to be all alike, we must fall back on Father Boscovich, and require him to explain the difference of quality of different chemical substances by different laws of force between the different atoms.

§ 27. Consider lastly a solid; that is to say, an assemblage in which the atoms have no relative motions, except through ranges small in comparison with the shortest distances between their centres*. The first thing that we remark is that every solid would, at zero of absolute temperature (that is to say all its atoms and electrions at rest), be a perfect insulator of electricity under the influence of electric forces, moderate enough not to pluck electrions out of the atoms in which they rest stably when there is no disturbing force. The limiting value of F here indicated for perfect insulation, I shall for brevity call the disruptional force or disruptional intensity. It is clear that this disruptional force is smaller the greater the number of electrions within an atom.

§ 28. The electro-inductive permeability of a solid at zero temperature is calculable by the static dynamics of § 24, modified by taking into account forces on the electrions of one atom due to the attractions of neighbouring atoms and the repulsion of their electrions. Without much calculation it is easy to see that generally the excess of the electro-inductive permeability above unity will be much greater than three times the sum of the volumes of the electric atoms per unit volume of space, which we found in § 24 for the electro-inductive permeability of an assemblage of single atoms, sparse enough to produce no disturbance by mutual actions. Also without much calculation, it is easy to see that now the induced electric moment will not be in simple proportion to

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* I need scarcely say that it is only for simplicity in the text that we conveniently ignore Roberts-Austen's admirable discovery of the inter-diffusion of solid gold and solid lead, found after a piece of one metal is allowed to rest on a piece of the other for several weeks, months, or years.

F, the intensity of the electric field, as it was rigorously for a single atom through the whole range up to the disruptional value of F; but will tend to increase more than in simple proportion to the value of F; though for small practical values of F the law of simple proportion is still very nearly fulfilled.

§ 29. Raise the temperature now to anything under that at which the solid would melt. This sets the electrions to performing wildly irregular vibrations and rotations, so that some of them will occasionally be shot out of their atoms. Each electrion thus shot out will quickly either fall back into the atom from which it has been ejected, or will find its way into another atom. If the body be in an electric field F, a considerable proportion of the electrions which are shot out will find their way into other atoms in the direction in which they are pulled by F; that is to say, the body which was an infinitely perfect insulator at zero absolute temperature has now some degree of electric conductivity, which is greater the higher the temperature. There can be no doubt that this is a matter-of-fact explanation of the electric conductivity, which so nearly perfect an insulator as the flint glass of my quadrant electrometer at atmospheric temperature shows, when heated to far below its melting-point, (according to Prof. T. Gray* $\cdot 98.10^{-24}$ at 60° cent.; $4\cdot 9.10^{-24}$ at 100°; 8300.10^{-24} at 200° cent.); and of the enormous increase of electric conductivity of rare earths at rising temperatures above 800° C., so admirably taken advantage of by Professor Nernst in his now celebrated electric lamp.

§ 30. If the hypotheses suggested in the present communication are true, the electric conductivity of metals must be explained in the same way as that of glass, guttapercha, vulcanite, Nernst filament, &c., with only this difference, that the metallic atom must be so crowded with electrions that some of them are always being spilt out of each atom by the intermolecular and electrionic thermal motions, not only at ordinary atmospheric temperatures, and higher, but even at temperatures of less than 16° centigrade above the absolute zero of temperature. I say 16° because in Dewar's Bakerian Lecture to the Royal Society of London, June 13, 1901, "The Nadir of Temperature," we find that platinum, gold, silver, copper, and iron have exceedingly high electric conductivity at the temperature of liquid hydrogen boiling under 30 mms. of mercury, which must be something between $20^\circ\cdot 5$, the boiling-point of hydrogen at 760 mms. pressure,

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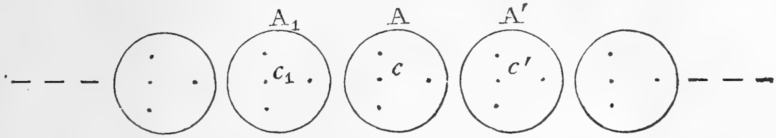
* Proc. Roy. Soc. Jan. 12, 1882.

and 16°, the temperature of melting solid hydrogen, both determined by Dewar with his helium thermometer. There is no difficulty in believing that the electrions in each of the metallic atoms are so numerous that, though they rest in stable equilibrium within the atoms, closely packed to constitute the solid metal at 0° absolute, and may move about within the atom with their wildly irregular thermal motions at 1° of absolute temperature, they may between 1° and 2° begin to spill from atom to atom. Thus, like glass or a Nernst filament below 300° absolute, a metal may be an almost perfect insulator of electricity below 1° absolute: may, like glass at 333° absolute, show very notable conductivity at 2° absolute : and, like glass at 473° absolute as compared with glass at 333° absolute, may show 8000 times as much electric conductivity at 2°·8 as at 2°. And, like the Nernst filament at 1800° or 2000° absolute, our hypothetical metal may at 6° absolute show high conductivity, comparable with that of lead or copper at ordinary temperatures. The electric conductivity in the Nernst filament goes on increasing as the temperature rises till the filament melts or evaporates. Nevertheless it is quite conceivable that in our hypothetical metal with rising temperature from 2° to 16° absolute the electric conductivity may come to a maximum and decrease with further rise of temperature up to and beyond ordinary atmospheric temperatures. In fact, while some extent of thermal motions is necessary for electric conductivity, too much of these motions must mar the freedom with which an electrion can thread its way through the crowd of atoms to perform the function of electric conduction. It seems certain that this is the matter-of-fact explanation of the diminution of electric conductivity in metals with rise of temperature.

§ 31. Regretting much not to be able (for want of time) to include estimates of absolute magnitudes in the present communication, I end it with applications of our hypothesis to the pyro-electricity and piezo-electricity of crystals. A crystal is a homogeneous assemblage of bodies. Conversely, a homogeneous assemblage of bodies is not a crystal if the distance between centres of nearest neighbours is a centimetre or more ; it is a crystal if the distance between nearest neighbours is 10^{-8} of a cm. or less. Pyro-electricity and piezo-electricity are developments of vitreous and resinous electric forces such as would result from vitreous and resinous electrification on different parts of the surface of a crystal, produced respectively by change of temperature and by stress due to balancing forces applied to the surfaces.

§ 32. To see how such properties can or must exist in crystals composed of our hypothetical atoms with electrions, consider first merely a row of equal tetraelectrionic atoms in a straight line, each having its quantum of four electrions within it. Fig. 6 shows a configuration of stable equilibrium of the electrions not, however, truly to scale. The sets of three dots indicate trios of electrions at the corners of equilateral triangles, the middle dot in each row being alternately on the far side and the near side of the plane of the paper, which contains the centres of the atoms and the remaining electrion of each four. Let C_1, C, C′, be the centres of the atom A_1, A, A′. An easy calculation shows that the quartet of electrions within A, regarded for the moment as a group of four material points rigidly connected, is attracted to the left with a less force by A_1 than to the right by A′ (in making the calculation remember that A_1 attracts all the electrions within A as if it were a quantity e of vitreous electricity collected at C_1, and similarly in respect to A′). There are corresponding smaller differences between the opposite attractions of the more and more remote atoms on the two sides of A. Let δ denote the excess of the sum of the rightwards of these attractions above the leftwards. The geometrical centre of the electrions within A is displaced rightwards to a distance, l, from C equal to $\frac{\alpha^3\delta}{16e^2}$.

Fig. 6.



§ 33. Imagine now a crystal or a solid of any shape built up of parallel rows of atoms such as those of § 32. The amount of the displacing force on each quartet of electrions will be somewhat altered by mutual action between the rows, but the general character of the result will be the same; and we see that throughout the solid, except in a thin superficial layer of perhaps five or ten atoms deep, the whole interior is in a state of homogeneous electric polarization, of which the electric moment per unit of volume is $4eNl$; where N is the number of atoms per unit volume, and l is the displacement of the geometrical centre of each quartet from the centre of its atom. This is the interior molecular condition of a di-polar pyro-electric crystal, which I described in 1860[*] as probably accounting for their known pyro-electric quality,

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[*] 'Collected Mathematical and Physical Papers,' vol. i. p. 315.

and as in accordance with the free electro-polarities of fractured surfaces of tourmaline discovered by Canton*. If a crystal, which we may imagine as given with the electrions wholly undisturbed from their positions according to § 32, is dipped in water and then allowed to dry, electrions would by this process be removed from one part of its surface and distributed over the remainder so as to wholly annul its external manifestation of electric quality. If now either by change of temperature or by mechanical stress the distances between the atoms are altered, the interior electro-polarization becomes necessarily altered ; and the masking superficial electrification got by the dipping in water and drying will now not exactly annul the electrostatic force in the air around the solid. If at the altered temperature or under the supposed stress the solid is again dipped in water and dried, the external electric force will be again annulled. Thus is explained the pyro-electricity of tourmaline discovered by Aepinus.

§ 34. But a merely di-polar electric crystal with its single axis presents to us only a small, and the very simplest, part of the whole subject of electro-crystallography. In boracite, a crystal of the cubic class, Haüy found in the four diagonals of the cube, or the perpendiculars to the four faces of the regular octahedron, four di-polar axes : the crystal on being irregularly heated or cooled showed as it were opposite electricities on the surfaces in the neighbourhood of opposite pairs of corners of the cube, or around the centres of the opposite pairs of triangular faces of the octahedron. His discoveries allow us to conclude that in general the electric æolotropy of crystals is octopolar with four axes, not merely dipolar as in the old-known electricity of the tourmaline. The intensities of the electric virtue are generally different for the four axes, and the directions of the axes are in general unsymmetrically oriented for crystals of the unsymmetrical classes. For crystals of the optically uniaxal class, one of the electro-polar axes must generally coincide with the optic axis, and the other three may be perpendicular to it. The intensities of the electro-polar virtue are essentially equal for these three

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* Wiedemann (*Die Lehre von der Elektricität*, Second Edition, 1894, vol. ii. § 378) mentions an experiment without fully describing it by which a null result, seemingly at variance with Canton's experimental discovery and condemnatory of my suggested theory, was found. Interesting experiments might be made by pressing together and reseparating fractured surfaces of tourmaline, or by pressing and rubbing polished surfaces together and separating them. It would be very difficult to get trustworthy results by breakages, because it would be almost impossible to avoid irregular electrifications by the appliances used for making the breakage. The mode of electric measurement followed in the experiment referred to by Wiedemann is not described.

axes: it may be null for each of them: it may be null or of any value for the so-called optic axis. Haüy found geometrical differences in respect to crystalline facets at the two ends of a tourmaline; and between the opposite corners of cubes, as leucite, which possess electropolarity. There are no such differences, there are only the chiral differences, between the two ends of a quartz crystal (hexagonal prism with hexagonal pyramids at the two ends) but there are differences (visible or invisible) between the opposite edges of the hexagonal prism. The electropolar virtue is null for the axis of the prism, and is proved to exist between the opposite edges by the beautiful piezo-electric discovery of the brothers Curie, according to which a thin flat bar, cut with its faces and its length perpendicular to two parallel faces of the hexagonal prism and its breadth parallel to the edges of the prism, shows opposite electricities on its two faces, when stretched by forces pulling its ends. This proves the three electropolar axes to bisect the 120° angles between the consecutive plane faces of the prism.

§ 35. For the present let us think only of the octopolar electric æolotropy discovered by Haüy in the cubic class of crystals. The quartet of electrions at the four corners of a tetrahedron presents itself readily as possessing intrinsically the symmetrical octo-polar quality which is realized in the natural crystal. If we imagine an assemblage of atoms in simple cubic order, each containing an equilateral quartet of electrions, all similarly oriented with their four faces perpendicular to the four diagonals of each structural cube, we have exactly the required æolotropy; but the equilibrium of the electrions all similarly oriented would probably be unstable; and we must look to a less simple assemblage in order to have stability with similar orientation of all the electrionic quartets.

§ 36. This, I believe, we have in the doubled equilateral homogeneous assemblage of points described in § 69 of my paper on "Molecular Constitution of Matter," republished from the Transactions of the Royal Society of Edinburgh for 1889 in volume iii. of my 'Collected Mathematical and Physical Papers' (p. 426); which may be described as follows for an assemblage of equal and similar globes:—Beginning with an equilateral homogeneous assemblage of points, A, make another similar assemblage of points, B, by placing a B in the centre of each of the similarly oriented quartets of the assemblage of A's. It will be found that every A is at the centre of an *oppositely* oriented quartet of the B's. To understand this, let A_1, A_2, A_3, A_4 be an equilateral quartet of the A's; and imagine A_2, A_3, A_4 placed on a horizontal glass plate* with A_1 above it. Let B_1 be at the centre of A_1, A_2, A_3, A_4, and let B_1, B_2, B_3, B_4 be a quartet of the B's similarly oriented to A_1, A_2, A_3, A_4. We see that B_2, B_3, B_4 lie below the glass plate, and that the quartet B_1, B_2, B_3, B_4 has none of the A's at its centre. But the vertically opposite quartet B_1, B'_2, B'_3, B'_4 contains A_1 within it; and it is oppositely oriented to the quartet A_1, A_2, A_3, A_4. Thus we see that, while the half of all the quartets of A's which are oriented oppositely to A_1, A_2, A_3, A_4 are void of B's, the half of the quartets of B's oppositely oriented to A_1, A_2, A_3, A_4 have each an A within it, while the other half of the quartets of the B's are all void of A's.

§ 37. Now let all the A's and all the B's be centres of equal and similar spherical atoms, each containing a quartet of electrions. The electrions will be in stable equilibrium under the influence of their own mutual repulsions and the attractions of the atoms, if they are placed as equilateral quartets of proper magnitude, concentric with the atoms, and oriented all as any one quartet of the A's or B's. To see that this is true, confine attention first to the five atoms A_1, A_2, A_3, A_4, B_1. If the electrions within A_1, A_2, A_3, A_4 are all held similarly oriented to the quartet of the centres of these atoms, the quartet of electrions within B_1 must obviously be similarly oriented to the other quartets of electrions. If, again, these be held oriented oppositely to the quartet of the atoms, the stable configuration of the electrions within B_1 will still be similar to the orientation of the quartets within A_1, A_2, A_3, A_4, though opposite to the orientation of the centres of these atoms. If, when the quartets of electrions are all thus similarly oriented either way, the quartet within B_1 is turned to reverse orientation, this will cause all the others to turn and settle in stable equilibrium according to this reversed orientation. Applying the same consideration to every atom of the assemblage and its four nearest neighbours, we have proof of the proposition asserted at the commencement of the present section. It is most interesting to remark that if, in a vast homogeneous assemblage of the kind with which we are dealing, the orientation of any one of the quartets of electrions be reversed and held reversed, all the others will follow and settle in stable equilibrium in the reversed orientation.

§ 38. This double homogeneous assemblage of tetra-electrionic atoms seems to be absolutely the simplest† molecular structure in which Haüy's octo-polar electric quality can

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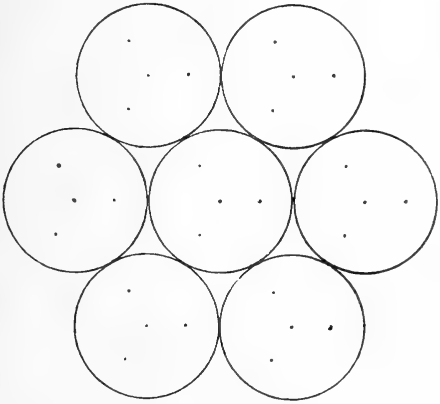
* Parallel glass plates are useful auxiliaries for graphic construction and illustrative models in the molecular theory of crystals.

† Not the simplest. See § 40 below.

exist. To see that it has octo-polar electric quality, consider an octahedron built up according to it. The faces of this octahedron, taken in proper order, will have, next to them, alternately points and triangular faces of the electrionic quartets within the atoms. This itself is the kind of electric æolotropy which constitutes octo-polar quality. Time prevents entering fully at present on any dynamical investigation of static or kinetic results.

§ 39. [Added Oct. 23.] Since what precedes was written, I have seen the explanation of a difficulty which had prevented me from finding what was wanted for octo-polar electric æolotropy in a homogeneous assemblage of single atoms. I now find (§ 40 below) that quartets of electrions will rest stably in equilibrium, under the influence of the mutual repulsion between electrion and electrion and attraction between atom and electrion, in an equilateral homogeneous assemblage in the configuration indicated in fig. 7. The quartets of electrions are supposed to have their edges parallel to the six lines of symmetry of the assemblage. The plane of the paper is supposed to be that of the centres of the seven atoms. The central point in each circle represents a simple electrion which is at distance r, according to the notation of § 20 above, from the plane of the paper on the near side; and therefore the other three are at the corners of an equilateral triangle at distance $\frac{1}{3}r$ on the far side to make the electric centre of gravity of the quartet coincide with the centre of its atom. The radius of circle on which these three lie is $\frac{2\sqrt{2}}{3}r$ or $\cdot 94\,r$. The diagram is drawn correctly to scale

Fig. 7.



according to the value $\cdot 612\alpha$ given for r in the table of § 20, on the supposition that the circles shown in the diagram represent the electric spheres of the atoms in contact.

§ 40. Imagine now the electrions of each quartet to be rigidly connected with one another and given freedom only to rotate about an axis perpendicular to the plane of the paper. To all of them apply torques; turning the central quartet of the diagram slowly and keeping all the others at rest. It is clear that the first 60° of turning brings the central quartet to a position of unstable equilibrium, and 60° more to a position of stable equilibrium corresponding to the first position, which we now see was stable when the others are all held fixed. We are now judging simply from the mutual actions between our central quartet and the six shown around it in the diagram; but it may be easily proved that our judgment is not vitiated by the mutual action between the central quartet and all around it in space, including the six in the diagram. Similarly we see that any one quartet of the assemblage, free to turn round an axis perpendicular to the plane of the paper while all the others are fixed, is in stable equilibrium when oriented as are those shown in the diagram. And similarly again we see the same conclusion in respect to three other diagrams in the three other planes parallel to the faces of the tetrahedrons or corresponding octahedrons of the assemblage. Hence we conclude that if the axial constraints are all removed, and the quartets left perfectly free, every one of them rests in stable equilibrium when oriented either as one set or as the other set of equilateral tetrahedronal quartets of the assemblage. It is interesting to remark that if, after we turned the central quartet through 60°, we had held it in that position and left all the others free to rotate, rotational vibrations would have spread out among them from the centre; and, after losing in waves spreading through ether outside the assemblage the energy which we gave by our torque acting on the central quartet, they would come to stable equilibrium with every one of them turned 60° in one direction or the other from its primitive position, and oriented as the central quartet in the position in which we held it.

§ 41. We have thus found that an equilateral homogeneous assemblage of atoms each having four electrions within it, arranges these electrions in equilateral quartets all oriented in one or other of two ways. The assemblage of atoms and electrions thus produced is essentially octo-polar. Of the two elementary structural tetrahedrons, of the two orientations, one will have every one of its electrionic quartets pointing towards, the other from, its faces. The elementary structural octahedron has four of its faces pointed towards, and four pointed from, by its electrionic quartets. This is essentially a dynamically octo-polar* assemblage; and it supplies us with a perfect explanation of the piezo-electric quality to be inferred from the brothers Curie's experimental discovery, and Voigt's mathematical theory.

§ 42. Look at the diagram in § 39; and remember that it indicates a vast homogeneous assemblage consisting of a vast number of parallel plane layers of atoms on each side of the plane of the paper, in which seven atoms are shown. The quartets of electrions were described as all similarly oriented, and each of them equilateral, and having its geometrical centre at the centre of its atom; conditions all necessary for stable equilibrium.

§ 43. Let now the assemblage of atoms be homogeneously stretched from the plane on both sides to any extent, small or great, without any component motions of the centres of the atoms parallel to the planes of the layers. First let the stretch be very great; great enough to leave undisturbed by the other layers the layer for which the centres of atoms are, and the geometrical centres of the quartets were, in the plane of the paper. The geometrical centres of the quartets are not now in the plane of the paper. The single electrions on the near side seen in the diagram over the centres of the circles are drawn towards the plane of the paper; the equilateral triangles on the far side are also drawn nearer to the paper; and the equilateral triangles are enlarged in each atom by the attractions of the surrounding atoms. The contrary inward movements of the single atoms on one side of the plane, and of the triplets on the other side, cannot in general be in the proportion of three to one. Hence the geometrical centres of gravity of the quartets are now displaced perpendicularly to the plane of the paper to far side or near side; I cannot tell which without calculation. The calculation is easy but essentially requires much labour; involving as it does the determination of three unknowns, the length of each side of the equilateral triangle seen in the diagram, the distance of each of its corners from the electrion on the near side of the paper, and the displacement of the geometrical centre of

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* The octo-polar *pyro-electricity*, which is supposed to have been proved by Haüy's experiment, must have been due to something æolotropic in the heating. Uniform heating throughout a regular cube or octahedron could not give opposite electric manifestations in the four pairs of alternate corners of the cube, or alternate faces of the octahedron. Nevertheless the irregular finding of electric octo-polarity by Haüy is a splendid discovery; of which we only now know the true and full significance, through the experimental and mathematical labours of the brothers Curie, of Friedel, and of Voigt.

gravity of the four to one side or other of the plane. Each one of the three equations involves summations of infinite convergent series, expressing force-components due to all the atoms surrounding any chosen one in the plane. A method of approximation on the same general plan as that of the footnote to § 9 above would give a practicable method of calculation.

§ 44. Return to § 42, and consider the diagram as representing a crystal in its natural unstressed condition, consisting of a vast train of assemblages of atoms with centres in the plane of the paper, and in parallel planes on each side of it. We now see that the forces experienced by the electrions of one quartet from all the surrounding atoms in the plane of the paper would, if uncompensated, displace the geometrical centre of gravity of the quartet to one side or other of the plane of the paper, and we infer that the forces experienced from all the atoms on the two sides of this plane give this compensation to keep the centre of gravity of the quartet in the plane. Stretch now the assemblage to any degree equally in all directions. The quartets remain equilateral with their centres of gravity in the plane of the paper and parallel planes. Lastly stretch it farther equally in all directions parallel to the plane of the paper, with no component motion perpendicular to this plane. This last stretching diminishes the influence of all the atoms whose centres are in the plane of the paper tending to displace the centres of gravity of their electrions in one direction from this plane; and therefore leaves all the atoms out of this plane to predominate, and to cause a definite calculable displacement of the centres of gravity of all the quartets in the contrary direction to the former.

§ 45. To realize the operations of § 44, cut a thin hexagonal plate from the middle between two opposite corners of a cubic crystal, or parallel faces of an octahedron. Fix clamps to the six edges of this plate, and apply forces pulling their pairs equally in contrary directions. The whole material of the plate becomes electro-polar with electric moment per unit bulk equal to $4\mathrm{N}ex$; of which the measurable result is uniform electrostatical potentials* in vacuous ether close to the two sides of the plate, differing by $4\pi \, . \, 4\mathrm{N}ext$, where t denotes the thickness of the plate, x the calculated displacement of the centre of gravity of each quartet from the centres of the atoms parallel to the two faces of the plate, e the electric mass of an electrion, and N the number of atoms per cubic

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* See my 'Electrostatics and Magnetism,' § 512, cor. 3.

centimetre of the substance. This, for a crystal of the cubic class, is, in Voigt's mathematical theory, the analogue to the electric effect discovered in quartz by the brothers Curie, and measured by aid of thin metal foils attached to the two faces of the plate and metallically connected to the two principal electrodes of an electrometer.

XXX. *The Rate of Recombination of Ions in Gases under Different Pressures.* By R. K. McClung, *M.A., formerly Demonstrator of Physics, McGill University, Montreal*; 1851 *Exhibition Scholar, McGill University, Trinity College, Cambridge* *.

ANY gas which has been exposed to Röntgen rays, retains the ionization thus produced for a short time after the source of radiation has been removed. The negative and positive ions take an appreciable time to recombine with one another; and it is necessary in some cases to take into consideration the rate at which these ions disappear by recombination. This question of the rate of recombination of ions has been investigated by Prof. Rutherford (Phil. Mag. Nov. 1897) for air and some other gases at atmospheric pressure. In that paper he points out that if, when the rate of production becomes equal to the rate of recombination, the rays be then stopped, the rate at which the conducting particles decay is given by the equation

$$\frac{dn}{dt} = -\alpha n^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where n is the number of ions per c.c. in the gas at any time t after the rays have ceased, and α is a constant for any given gas at ordinary atmospheric pressure. If N be the maximum number of ions per c.c. when a steady state has been reached before the cessation of the rays, then the above equation becomes by integration

$$\frac{1}{n} - \frac{1}{N} = \alpha t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Object of Investigation.

At the suggestion of Professor Rutherford, the present research was undertaken in order to ascertain the conditions which exist in regard to the recombination of the ions in gases under pressures other than atmospheric pressure, and to find the relation between the rates of recombination at different pressures. The objects of the present investigation have been therefore :—

(1) To determine whether the law of recombination, as set

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* Communicated by Prof. E. Rutherford.

forth in equation (1), holds true for air under pressures above and below an atmosphere.

(2) If it does hold true, to find the relation which the quantity α at any given pressure bears to that at any other pressure.

(3) To make an absolute determination of α.

This quantity α may be called the coefficient of recombination.

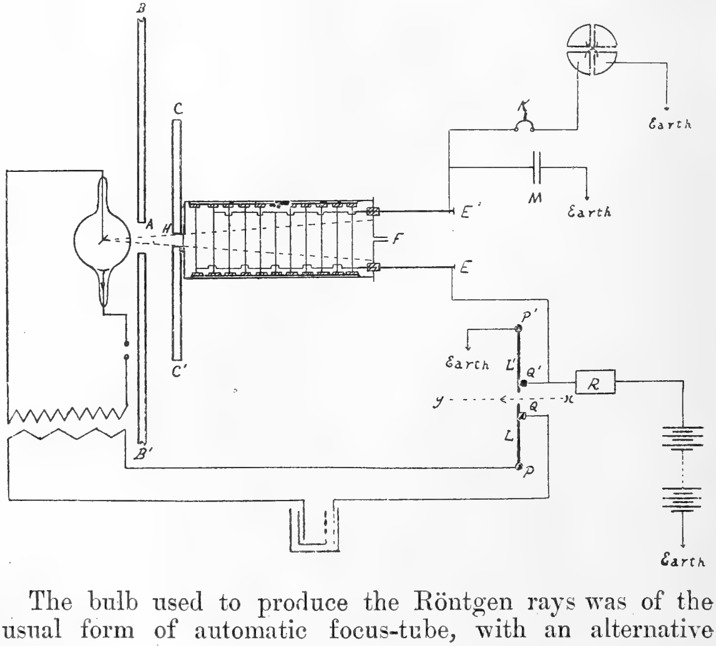
For this purpose the method employed by Professor Rutherford in his previous investigation was adopted in principle, but with an apparatus differing somewhat in detail.

In order to investigate the law, as given by equation (2), it is necessary to be able to measure the maximum number of ions in a given volume of the gas when a steady state has been reached, and also to measure the number remaining at the end of any given interval of time after the rays have ceased. For this purpose, then, the following arrangement of apparatus was adopted.

Description of Apparatus.

The complete arrangement of the several parts and connexions is shown in the diagram (fig. 1).

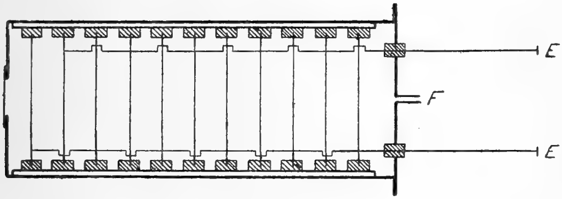
Fig. 1.



The bulb used to produce the Röntgen rays was of the usual form of automatic focus-tube, with an alternative spark-gap connexion, and was driven by a large Ruhmkorff induction-coil. The coil was run by the current from the 110-volt circuit, with a Wehnelt interrupter of slow frequency in series. In order to intensify the discharge in the focus-tube, a spark-gap of about a centimetre and a half was placed directly in series with the secondary of the coil and the electrodes of the tube.

The induction-coil and *x*-ray bulb were completely inclosed in a small metal room connected to earth and the rays issued through a small orifice A in a thick lead plate B B′. A short distance in front of this was placed another thick lead plate C C′. The rays passed through a small rectangular orifice H in this plate and then into a brass cylinder through an aluminium window. This cylinder, which is shown more in detail in fig. 2, contained the electrodes between which the leak was measured.

Fig. 2.



The electrodes employed consisted of extremely thin aluminium-foil, stretched on narrow zinc rings, the rings being for the purpose of support. These rings were supported on ebonite blocks, which were rigidly fastened to a metal frame, which could easily be removed from the cylinder. These thin plates were eleven in number, and were placed at distances of about two centimetres apart. Six of them, namely the two end ones and the four alternate ones of the remainder, were all connected together and also to the terminal E, while the other five alternate plates were connected to E′.

This arrangement of electrodes was adopted in order to secure a large volume of gas to be ionized, and thus obtain a larger effect than could be obtained otherwise, while at the same time the electrodes were close enough together to secure a saturation-current with a reasonable voltage. All the ions could thus reach the electrodes in a very short time. This precaution was essential as it was necessary that all the ions should reach the electrodes before disappearing by recombination. An arrangement consisting of a guard-ring was tried, but the effect produced was so small that it was not suitable for measurement, and so was abandoned.

Thin aluminium-foil was used in the first place because, being so very thin, it would not absorb the rays to any appreciable extent, and, secondly, because the secondary radiation produced at the surface of metals is small in the case of aluminium. The actual thickness of the foil used was only ·00036 of a centimetre. Before finally adopting the aluminium a set of electrodes consisting of paper covered with powdered graphite and then soaked with dilute acid so as to make them conductors, was tried. These gave very satisfactory results while they lasted; but they were not durable, and so had to be abandoned also. The final arrangement of the aluminium plates seemed to be the most satisfactory, and was therefore adopted. The terminals connected to the electrodes passed through ebonite corks in the end of the cylinder to secure insulation.

The cylinder was made so that the air might be exhausted from it through the opening F, and it was also made strong enough to stand considerable internal pressure.

In order to measure the maximum number of ions in a given volume of the gas when a steady state had been reached, and also the number remaining at the end of any given interval of time, it was necessary to be able to apply the electromotive force to the electrode E at the same instant that the rays ceased, and also at any given time after the cessation of the rays. For this purpose the following method was used. L and L′ were two contact-breakers, pivoted at P and P′ and making contact with Q and Q′ respectively. The primary circuit of the induction-coil was closed through the contact-maker L. The lever L′ was connected to earth, while Q′ was connected to the electrode E and also, through a very large resistance R, to one pole of a battery of small accumulators, the other pole of the battery being connected to earth. The resistance R which was used was a carbon one of about a megohm.

Now since the resistance between Q′ and E was extremely small compared with R, then, if the lever L′ were closed against Q′, the plates connected with E would be practically at zero potential, since L′ is connected to earth. But as soon as the contact was broken between L′ and Q′, then E would be instantly charged up to the potential of the battery. The contacts at Q and Q′ were broken by means of a heavy pendulum which swung along the path xy. If the levers were placed directly opposite to each other, the rays could be stopped and the electromotive force applied to E at the same instant; while if they were separated from each other along the path $x\ y$, the rays could be stopped and the E.M.F. applied at a known interval afterward. This interval could be varied by varying the distance between the levers L and L′.

The time taken by the pendulum to pass from one contact-breaker to the other was determined by calibrating the pendulum. This was done by attaching a smoked-glass plate to the pendulum and obtaining a trace on this plate made by a vibrating tuning-fork of known frequency. By this means the time taken by the pendulum to pass from one point to another could be calculated. The electrode E′ was connected, through a suitable key at K, to one pair of quadrants of the electrometer, while the other pair was connected to earth.

Method of Observation.

The following was the method of observation for determining the values of N and n in equation (2). The induction-coil circuit was closed and the rays allowed to pass into the cylinder for about ten seconds, to insure the ionization reaching a maximum. This point was tested by running the bulb for a longer time before applying the E.M.F., and it was found that about ten seconds was quite sufficient for the ionization to reach a maximum. The pendulum was then released, which stopped the rays and applied the E.M.F. to E by breaking the contacts. During all this time both pairs of quadrants of the electrometer were to earth. In order to avoid the violent throw of the electrometer-needle, which would occur by induction if the pair of quadrants connected to E′ were insulated before the E.M.F. was applied to E, the connexion between the electrometer and E′ was broken by the key K until after the lever L′ was replaced against Q′, which was done immediately after the passage of the pendulum. After replacing L′, the charge in the plates connected with E′, due to the ions which passed across on the application of the E.M.F. to E, was thrown into the electrometer by closing the contact K. The deflexion of the electrometer-needle should then be proportional to the number of ions in the gas between the electrodes at the instant of the application of the E.M.F. to E, if the E.M.F. is large enough to produce saturation. The potential-gradient employed in most of the observations was about 110 volts per centimetre.

One of the great difficulties encountered in this investigation was the want of constancy in the intensity of the radiation given out by the bulb. To obviate this to some extent when taking a series of observations, the bulb was run at regular intervals as far as possible. Besides that, instead of depending on a single observation in any one case, several observations were made, and then the mean of these taken. All through the investigation this system of averages has been adopted.

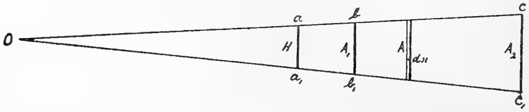
Theoretical Considerations.

Before proceeding to the actual results obtained by experiment, some theoretical questions must be considered.

Suppose that the number of ions per c.c. which are being produced at any given point in the gas per second is q, and suppose n equals the actual number per c.c. in the gas at the same point, then it has been shown (J. J. Thomson and E. Rutherford, Phil. Mag. Nov. 1896) that when a steady state has been reached, that is, when the rate of production is equal to the rate of recombination,

$$q = \alpha n^2. \qquad (3)$$

Fig. 3.



In fig. 3 let O represent the source of the rays, and let O C C_1 represent the cone of rays passing through the rectangular orifice H (fig. 1) and thence into the cylinder. Let $b\,b_1\,c_1\,c$ represent the part of the cone included between the two extreme plates within the cylinder. Suppose H, A_1, and A_2 to represent the cross-sectional areas of the cone at the orifice and the two plates, respectively. Let h and d represent the distances of $a\,a_1$ and $b\,b_1$ respectively from O, while l is the distance between the plates $b\,b_1$ and $c\,c_1$.

Let q_0 = the number of ions per c.c. produced per second at the plate $b\,b_1$.

Let Q = total number produced per second in the volume between $b\,b_1$ and $c\,c_1$;

$$\therefore \quad Q = \int q\,dV,$$

where q = the number of ions per c.c. at a distance x from O, and $dV = A\,dx$, the cross-sectional area at the distance x being A.

$$A = \frac{x^2}{d^2} A_1;$$

and since the number of ions per c.c. varies inversely as the square of the distance from the source, neglecting the absorption of the rays by the air,

$$q=\frac{d^2}{x^2}q_0.$$

This absorption of the air may be neglected without causing any serious error; for it has been found (E. Rutherford and R. K. McClung, Phil. Trans. Roy. Soc. vol. cxcvi. 1901) that only about 3 per cent. of strong radiation was absorbed in passing through a metre of air at atmospheric pressure. The absorption was shown, by the same authors, to be proportional to the pressure of the gas. In the present case, the distance between the two plates $b\,b_1$ and $c\,c_1$ was only 20 centimetres; so the absorption would be very small, and therefore negligible.

Therefore, supplying these values in the above equation, we get

$$\begin{aligned}Q&=\int_d^{d+l}\frac{d^2}{x^2}q_0\cdot\frac{x^2}{d^2}A_1dx\\&=q_0A_1l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)\end{aligned}$$

Again, let N = total number of ions in the same volume at the cessation of the rays;

$$\therefore \quad N=\int ndV,$$

where n = residual number per c.c. at the distance x from O.

$$\begin{aligned}\therefore \quad N&=\int\sqrt{\frac{q}{\alpha}}\,dV, \quad \text{since } q=\alpha n^2\\&=\frac{1}{\sqrt{\alpha}}\int_d^{d+l}\sqrt{\frac{d^2}{x^2}q_0\cdot\frac{x^2}{d^2}}A_1dx\\&=lA_1\sqrt{\frac{q_0}{\alpha}}\times\left(1+\frac{l}{2d}\right); \quad . \quad . \quad . \quad . \quad (5)\end{aligned}$$

$$\therefore \quad \alpha=l^2A_1{}^2\left(1+\frac{l}{2d}\right)^2\frac{q_0}{N^2}.$$

But $q_0=\dfrac{Q}{A_1l}$ from equation (4);

$$\therefore \quad \alpha=lA_1\left(1+\frac{l}{2d}\right)^2\frac{Q}{N^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This equation gives the value of the constant α in absolute measure, since the quantities A_1, l, and d can be measured and Q and N determined. This equation may be written

$$\alpha = \mathrm{K}\frac{\mathrm{Q}}{\mathrm{N}^2}, \quad \ldots \ldots \quad (7)$$

where K is equal to the constant $l\mathrm{A}_1\left(1+\frac{l}{2d}\right)^2$, which depends entirely upon the dimensions and position of the apparatus.

The value obtained for α will be discussed later in the paper.

The theory of recombination may be considered also in regard to the time taken by the maximum number of ions to fall to half their value.

Let n_0=maximum number of ions per c.c. at a point distant x from O (fig. 3).

Let n_t=number per c.c. at the end of a time t after the rays have ceased.

Then by equation (2)

$$\frac{1}{n_t}-\frac{1}{n_0}=\alpha t.$$

The decrease in time t per c.c.$=n_0-n_t$.

The decrease in a thin cross-sectional area A, of a thickness dx, would be $\mathrm{A}(n_0-n_t)dx$.

$$\frac{1}{n_t}=\frac{1}{n_0}+\alpha t;$$

$$\therefore \quad n_0-n_t=\frac{n_0^2\alpha t}{1+n_0\alpha t}.$$

Also

$$\alpha n_0^2 = q = \frac{q_0 d^2}{x^2};$$

$$\therefore \quad n_0^2=\frac{q_0}{\alpha}\cdot\frac{d^2}{x^2}.$$

Therefore the decrease in the cross-sectional area A in time t is equal to

$$\frac{\mathrm{A}_1 x^2}{d^2}\left(\frac{q_0 d^2}{\alpha x^2}\cdot\frac{\alpha t}{1+\alpha t\frac{d}{x}\sqrt{\frac{q_0}{\alpha}}}\right)dx$$

$$=\frac{\mathrm{A}_1 q_0 t x}{x+td\sqrt{q_0\alpha}}\,dx$$

$$=\mathrm{A}_1 q_0 t\left(1-\frac{td\sqrt{q_0\alpha}}{x+td\sqrt{q_0\alpha}}\right)dx;$$

Therefore the total decrease in the number of ions in the volume bb_1c_1c in the time t

$$=\int_d^{d+l} A_1q_0t\left(1-\frac{td\sqrt{q_0\alpha}}{x+td\sqrt{q_0\alpha}}\right)dx$$

$$=A_1q_0t\left\{l-td\sqrt{q_0\alpha}\log_\epsilon\left(\frac{d+l+td\sqrt{q_0\alpha}}{d+td\sqrt{q_0\alpha}}\right)\right\}.$$

If we let T be the time that it takes the number of ions to fall to half their value, then the total decrease in the time T

$$=A_1q_0\mathrm{T}\left\{l-\mathrm{T}d\sqrt{q_0\alpha}\log_\epsilon\left(\frac{d+l+\mathrm{T}d\sqrt{q_0\alpha}}{d+\mathrm{T}d\sqrt{q_0\alpha}}\right)\right\}.$$

But the maximum number of ions is equal to

$$lA_1\left(1+\frac{l}{2d}\right)\sqrt{\frac{q_0}{\alpha}} \text{ by equation (4);}$$

$$\therefore\quad A_1q_0\mathrm{T}\left\{l-\mathrm{T}d\sqrt{q_0\alpha}\log_\epsilon\left(\frac{d+l+\mathrm{T}d\sqrt{q_0\alpha}}{d+\mathrm{T}d\sqrt{q_0\alpha}}\right)\right\}$$

$$=\tfrac{1}{2}lA_1\left(1+\frac{l}{2d}\right)\sqrt{\frac{q_0}{\alpha}}.$$

This equation for obtaining the value of T is incapable of an exact solution. An approximate solution may, however, be obtained as follows:—

$$A_1q_0\mathrm{T}\left\{l-\mathrm{B}\log_\epsilon\left(\frac{d+l+\mathrm{B}}{d+\mathrm{B}}\right)\right\}=\tfrac{1}{2}lA_1\left(1+\frac{l}{2d}\right)\sqrt{\frac{q_0}{\alpha}},$$

where $\mathrm{B}=\mathrm{T}d\sqrt{q_0\alpha}$,

$$\therefore\ A_1q_0\mathrm{T}\left\{l-\mathrm{B}\log_\epsilon\left(1+\frac{l}{d+\mathrm{B}}\right)\right\}=\tfrac{1}{2}lA_1\left(1+\frac{l}{2d}\right)\sqrt{\frac{q_0}{\alpha}};$$

$\therefore$ by expanding $\log_\epsilon\left(1+\dfrac{l}{d+\mathrm{B}}\right)$,

$$A_1q_0\mathrm{T}\left\{l-\mathrm{B}\frac{l}{d+\mathrm{B}}\right\}=\tfrac{1}{2}lA_1\left(1+\frac{l}{2d}\right)\sqrt{\frac{q_0}{\alpha}}$$

as a first approximation, provided that l is small compared with $d+\mathrm{B}$.

It is found by supplying the experimental values of the quantities involved in the expression $\dfrac{l}{d+\mathrm{B}}$, that the second term in the expansion of the logarithm amounts to approximately ·125 ; and therefore the above expression is a fair value for the first approximation.

$$\therefore\quad A_1 q_0 T l\left(\frac{d}{B+d}\right)=\tfrac{1}{2} l A_1\left(1+\frac{l}{2d}\right)\sqrt{\frac{q_0}{\alpha}};$$

$$\therefore\quad T=\frac{1}{2}\cdot\frac{1}{\sqrt{q_0\alpha}}\left(1+\frac{l}{2d}\right)\frac{B+d}{d};$$

$$=\frac{1}{2\sqrt{q_0\alpha}}\left(1+\frac{l}{2d}\right)\frac{Td\sqrt{q_0\alpha}+d}{d}$$

$$=\frac{1}{2}\left(1+\frac{l}{2d}\right)\left(T+\frac{1}{\sqrt{q_0\alpha}}\right);$$

$$=\frac{2d+l}{4d\sqrt{q_0\alpha}}\,\frac{4d}{2d-l};$$

$$=\frac{1}{\sqrt{q_0\alpha}}\cdot\frac{2d+l}{2d-l}$$

$$=\frac{2d}{2d+l}\cdot\frac{N}{Q}\cdot\frac{2d+l}{2d-l}\text{ by equations (4) and (6);}$$

$$=\frac{2d}{2d-l}\cdot\frac{N}{Q}. \quad \ldots\ldots\ldots\ldots \quad (8)$$

This gives a fairly close approximation for the time taken for the number of ions in the given volume in the cylinder to fall to half of their maximum value.

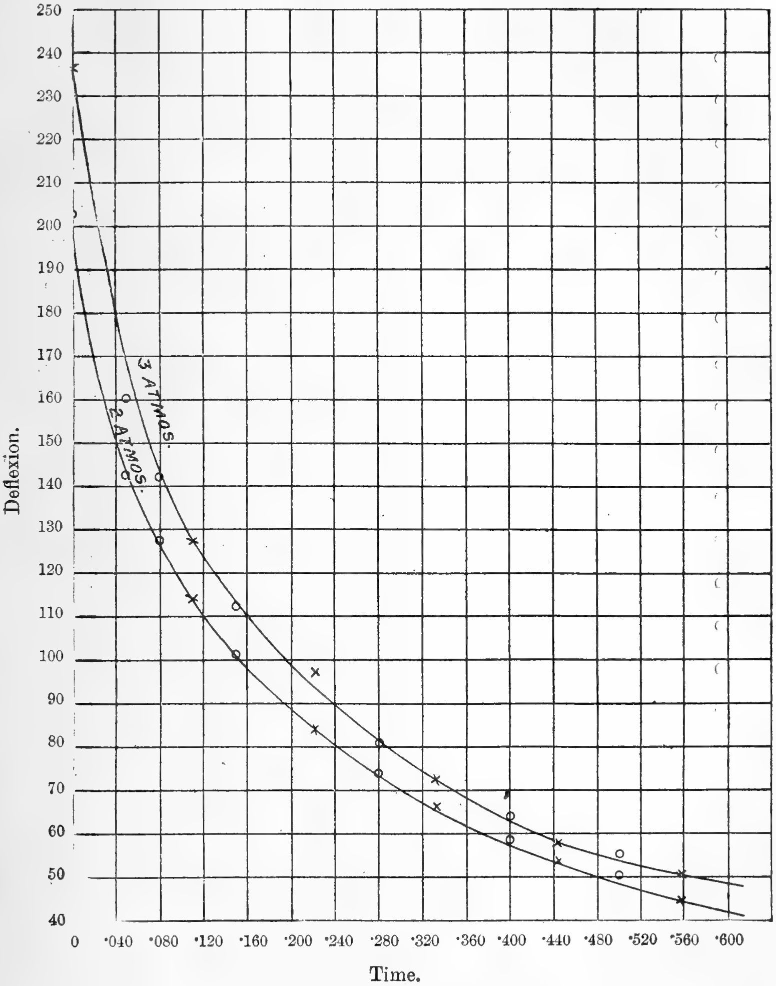
Having discussed the mathematical theory, we are now in a position to give some experimental results in support of the above theory.

Experimental Proof of the Law of Recombination.

We have endeavoured to establish the foregoing theory at the various pressures by obtaining experimental curves showing the relation between the number of ions and the corresponding time-interval after the cessation of the rays and then comparing them with the theoretical values, as deduced from equation (2). Curves were obtained for different pressures, varying from three atmospheres to ·125 of an atmosphere. A considerable number of such experimental curves were obtained, some of which are shown in figs. 4, 5, 6, and 7.

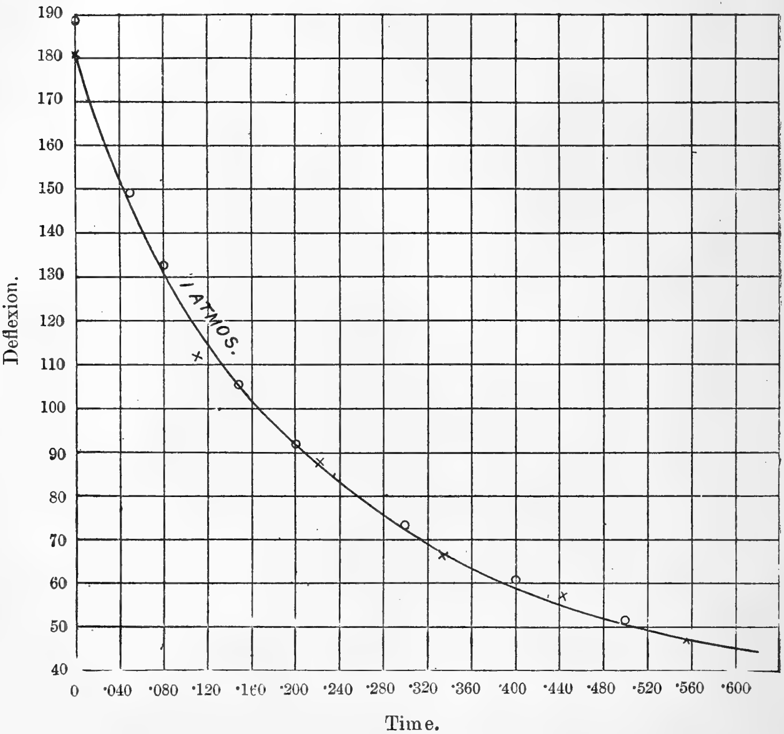
In these curves, the ordinates represent deflexions, which are proportional to n, while the abscissæ represent the corresponding times in seconds. The experimental points in each curve are marked with a cross, while the theoretical points deduced from equation (2) are marked with a circle. Only the experimental curve is drawn in each case in order to avoid confusion by having the two curves so close together. The theoretical points are obtained by choosing two experimental

Fig 4.



points on the curve with which to determine α and N, and having obtained these, then the different values of n are calculated from the formula for arbitrarily chosen values of t. On referring to the diagrams it is seen that the experimental and theoretical curves correspond very closely with each other. Any slight disagreement is well within the limits of experimental error, especially taking into consideration the fact that the intensity of the rays may vary slightly during the course of taking the observations necessary for one such curve.

Fig. 5.



The agreement is quite close enough to show the accuracy of the theory. The experimental investigation therefore supports the theory, and we see that the same law of recombination holds true at different pressures, at least for the range of pressures investigated.

It was not easy to obtain curves for pressures much below ·125 of an atmosphere, for at the very low pressures the effects obtained by this method were so small that they could not be relied upon with any great degree of certainty. However, there are no indications that the law does not hold below this pressure.

Fig. 6.

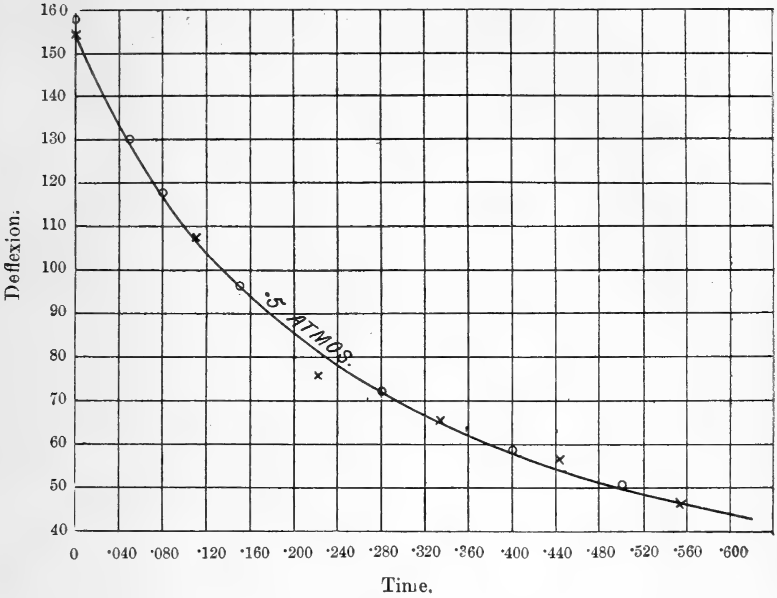
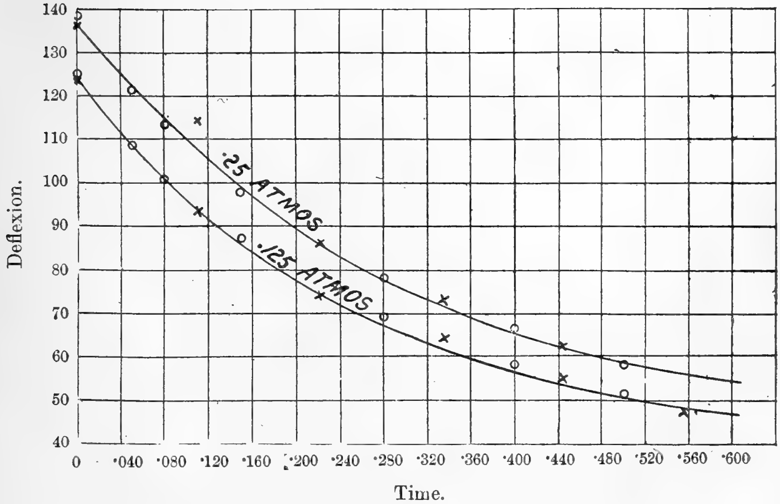

Deflexion.
160
150
140
130
120
110
100
90
80
70
60
50
40
·5 ATMOS.
0
·040
·080
·120
·160
·200
·240
·280
·320
·360
·400
·440
·480
·520
·560
·600
Time.

Fig. 7.


Deflexion.
140
130
120
110
100
90
80
70
60
50
40
·25 ATMOS
·125 ATMOS
0
·040
·080
·120
·160
·200
·240
·280
·320
·360
·400
·440
·480
·520
·560
·600
Time.

From these curves we see, what we should naturally expect, that the rate at which the ions recombine is much greater at the high pressures than at the low ones, owing, of course, to the fact that at the high pressures there are many more ions produced in a given volume of the gas, and therefore the chance of their coming in contact with one another is much greater than at low pressures.

If we consider the theory from the point of view of the time taken for the maximum number of ions to fall to half their value, we get a further confirmation of it from experiment. By determining the ratio of N to Q at each pressure and substituting these values, along with the values of d and l in centimetres in equation (8), we obtain the value of T in seconds in each case. Comparing these calculated values with the values of T obtained directly from the curves, we find the agreement fairly close, considering the fact that formula (8) is only an approximate formula, and also that a slight variation in the intensity of the rays would cause a corresponding deviation in the shape of the curve, and therefore a slight alteration in the value of T obtained from it. A comparison of these values of T is given for several different pressures in Table I.

TABLE I.

Pressure.	Deflexion for N.	Deflexion for Q.	Value of $\frac{2d}{2d-l}$.	Calculated value of T.	Value of T obtained from curve.
3 atmos.	236	143	1·66	·114	·137
2 "	196	112	1·66	·122	·161
1 "	181	51	1·66	·247	·203
$\frac{1}{2}$ "	154	41	1·66	·259	·249
$\frac{1}{4}$ "	136	26	1·66	·360	·368

The agreement between theory and experiment is close enough to show that the theory is quite valid no matter from which point of view it may be considered.

We see, therefore, that the relation $\frac{dn}{dt} = -\alpha n^2$ gives actually the rate at which the ions recombine in air, not only at atmospheric pressure, but also for air at pressures other than atmospheric, at least between the limits already indicated.

Although it has not been practicable to take observations beyond these limits, yet we think the same law would, in all probability, hold true at other pressures as well.

The Relation of α to Pressure.

The next question to be considered is the relation which the value of the quantity α, at any given pressure, bears to that at any other pressure: in other words, to decide whether the coefficient of recombination is a constant for all pressures, or whether it varies when the pressure varies.

In order to get over the difficulty of any variation in the intensity of the radiation which might occur in the course of taking observations at different pressures, the following method was adopted:—

Selecting two given pressures at a time, sets of observations were taken alternately for the two pressures, and then taking the mean value of the observations for one pressure, I compared them with the mean for the other. By this means it was possible to compare the value of α at various pressures with that at atmospheric pressure.

Suppose α_1 is the value of the coefficient for air at atmospheric pressure, and α_2 that any other given pressure, then

$$\alpha_1 = K\frac{Q_1}{N_1^2} \text{ and } \alpha_2 = K\frac{Q_2}{N_2^2},$$

where Q_1, N_1, and Q_2, N_2 are the corresponding values of Q and N;

$$\therefore \frac{\alpha_1}{\alpha_2} = \frac{Q_1}{Q_2} \cdot \frac{N_2^2}{N_1^2}.$$

By comparing the values of α by this formula, it was found that the coefficient is not only a constant for any one given pressure, but has the same value at different pressures. In other words, the coefficient of recombination is independent of the pressure. The value of α at several pressures, as compared with the value at atmospheric pressure, is given in Table II. (p. 298).

In the table, α_1 is the value of α at atmospheric pressure, while α_2 is the value at the other pressure compared with it.

The numbers given in the last column of Table II. do not show an absolute constancy in the value of α at the different pressures, but the difference from unity, in each case, is well within the limits of experimental error, considering the various sources of error which may come into the results. Since N is squared, any small error which might occur would be intensified by squaring.

TABLE II.

Pressure.	Deflexion for N.	Deflexion for Q.	$\frac{\alpha_1}{\alpha_2}$.
1 atmos. 3 ,,	121·0 193·0	34·4 79·7	1·094
1 ,, 2·5 ,,	67·9 103·2	16·6 35·5	1·083
1 ,, 2 ,,	70·4 96·0	15·2 26·5	1·058
1 ,, 1·5 ,,	128·6 148·7	38·2 49·4	1·028
1 ,, ·5 ,,	110 0 93·5	29·3 20 7	1·018
1 ,, ·25 ,,	70·0 43·5	23·5 9·0	1·011
1 ,, ·125 ,,	61·9 29·9	22·7 5·1	1·029

The set of comparisons above were not all made at the same time, but several of the observations were made on different days, extending over a period of about two weeks, and the comparison in each case was made with air at atmospheric pressure, so they do not represent a single set of values.

We have thus arrived at the fact that not only does the same law of recombination of the ions in air hold at various pressures, but the value of the coefficient of recombination is independent of the pressure of the air in which the ionization takes place. If the value of α is therefore determined for air at atmospheric pressure, it is determined for air at other pressures as well.

Determination of the Absolute Value of α.

We now proceed to determine the absolute value for this coefficient of recombination. Since the value of α is the same for different pressures, it is only necessary to determine it for a single given pressure, so atmospheric pressure has been selected for the determination given below. A considerable number of absolute determinations at other pressures have been made as well, which it is not necessary to reproduce here.

On referring to equation (7), it will be seen that it is necessary to make an absolute determination of the three quantities K, Q, and N, in order to determine α. K is a constant depending entirely on the dimensions and position of the apparatus. Q is the total number of ions produced per second within the volume of the cone of rays included between the two extreme electrodes, and is proportional to the deflexion of the electrometer-needle per second. N is the total number of ions in the same volume at the instant that the rays cease, and is proportional to the steady deflexion of the electrometer-needle.

From formula (6)

$$K = lA_1\left(1 + \frac{l}{2d}\right)^2,$$

where the quantities l, A_1, and d represent the dimensions as described in fig. 3.

$$l = 20 \text{ cms.}$$

$$d = 25{\cdot}07 \text{ cms.}$$

$$A_1 = \frac{d^2}{h^2} \times H,$$

where H = area of the orifice in the lead plate through which the rays passed before entering the cylinder, and h its distance from the source.

$$\therefore \quad A_1 = \frac{25{\cdot}07^2}{19{\cdot}07^2} \times 7{\cdot}14$$

$$= 12{\cdot}28 \text{ sq. cms.}$$

$$\therefore \quad K = 20 \times 12{\cdot}28 \left(1 + \frac{20}{50{\cdot}14}\right)^2$$

$$= 478{\cdot}9.$$

If d_1 = deflexion of the electrometer-needle per second due to the passage of the ions to the electrodes ;

S = deflexion due to a standard Clark cell; and

c_1 = capacity in microfarads of electrometer, connexions, &c.,

then the quantity of electricity which passes between the electrodes per second is equal to

$$\frac{d_1}{S} \times \frac{1{\cdot}434}{1} \times \frac{c_1}{10^6} \times 3 \times 10^9 \text{ electrostatic units.}$$

But quantity of electricity = number of ions × charge on an ion. Q=number of ions, and let ϵ = charge on an ion :

$$\therefore \quad \frac{d_1}{S} \times \frac{1{\cdot}434}{1} \times \frac{c_1}{10^6} \times 3 \times 10^9 = Q \times \epsilon.$$

$$\therefore \quad Q = \frac{d_1}{S} \times \frac{1434}{1} \times \frac{c_1}{\epsilon} \times 3.$$

The value of N may be determined by the same formula if we substitute d_2 for the deflexion corresponding to N and c_2 for the capacity in the circuit. Substituting the mean value of the quantities experimentally observed, viz.:—

$d_1 = 24{\cdot}07$ divisions of scale,

$S = 93{\cdot}2$ " "

$c_1 = {\cdot}00380$ m.f.,

we obtain

$$Q = \frac{24{\cdot}07}{93{\cdot}2} \times \frac{1434}{1} \times \frac{{\cdot}0038}{\epsilon} \times 3$$

$$= \frac{422}{\epsilon \times 10^2}.$$

Also

$d_2 = 104{\cdot}5$ divisions of scale,

$c_2 = {\cdot}00016$ m.f.

$$\therefore \quad N = \frac{104{\cdot}5}{93{\cdot}2} \times \frac{1434}{1} \times \frac{{\cdot}00016}{\epsilon} \times 3$$

$$= \frac{7722}{\epsilon \times 10^4},$$

$$\therefore \quad \frac{Q}{N^2} = 7{\cdot}076 \times \epsilon;$$

$$\therefore \quad \alpha = 478{\cdot}9 \times 7{\cdot}076\,\epsilon$$

$$= 3384\,\epsilon.$$

This value of α represents the mean value of a number of determinations.

If we substitute the value of ϵ obtained by Prof. J. J. Thomson, namely $6{\cdot}5 \times 10^{-10}$, then α becomes $2{\cdot}20 \times 10^{-6}$, which gives the absolute value of the coefficient.

This value of the coefficient of recombination agrees very closely with that obtained by Prof. Townsend for gases at atmospheric pressure (Phil. Trans. vol. cxciii. 1899) by an entirely different method. The method which he used was that of sending a current of ionized air along a tube containing an insulated electrode connected to an electrometer. By taking the deflexion for the electrode at different positions along the tube, he was able to determine the rate at which the number of ions diminished in consequence of recombination. In that paper he gives the value as $3420\,\epsilon$. The agreement between the two determinations is remarkably close considering the diversity of methods and apparatus employed by Prof. Townsend and myself.

The value of α which has been obtained by this determination for air at atmospheric pressure may be considered as the value of the coefficient of recombination for air at all other pressures, at least between the limits of ·125 and 3 atmospheres, since it has been shown in a former part of this paper that α is independent of the pressure of the air.

It might be worth while to mention briefly the method which was employed to determine the electric capacities in the circuit, as it is rather an unusual one.

To determine the value of N, the capacity in the circuit was simply that of the plate electrodes, the connexions, and electrometer. To determine Q, however, it was necessary to introduce a considerably larger capacity into the circuit, as the rate of movement of the electrometer-needle was too rapid without it. For this purpose, an air-condenser was connected in parallel, as shown at M in fig. 1. It was necessary therefore to determine the capacity both of this condenser and of the electrometer, together with the connexions. To do this, the following method was used.

If the air between two insulated parallel plates be ionized by means of uranium oxide, and if one of the plates be charged while the other is connected to the electrometer; then the deflexion of the electrometer-needle per second is inversely proportional to the capacity in the circuit.

Two parallel zinc plates were therefore arranged horizontally in a closed metal box. The inclosing box was connected to earth, while the two plates were carefully insulated. On the lower plate a thin layer of uranium oxide was placed. The lower plate was connected to the battery, while the upper one was connected in parallel with the electrometer circuit. This of course introduced the additional capacity of the two parallel plates. The ratio of the capacity of these plates to that of the rest of the electrometer circuit was, however, determined by the method of mixtures.

The rate of leak was taken with a condenser of known capacity in the place of M, fig. 1. Then the known capacity was removed and the air condenser, to be determined, introduced, and the rate of leak taken again. A third observation was then taken without any condenser in the circuit at all.

Now by comparing the last observation with that obtained when the condenser of known capacity was in circuit, the capacity of the electrometer, connexions, and plate-electrodes was determined. Then, having obtained this capacity, that of the air condenser was determined by a comparison of the deflexions obtained in the second and third observations.

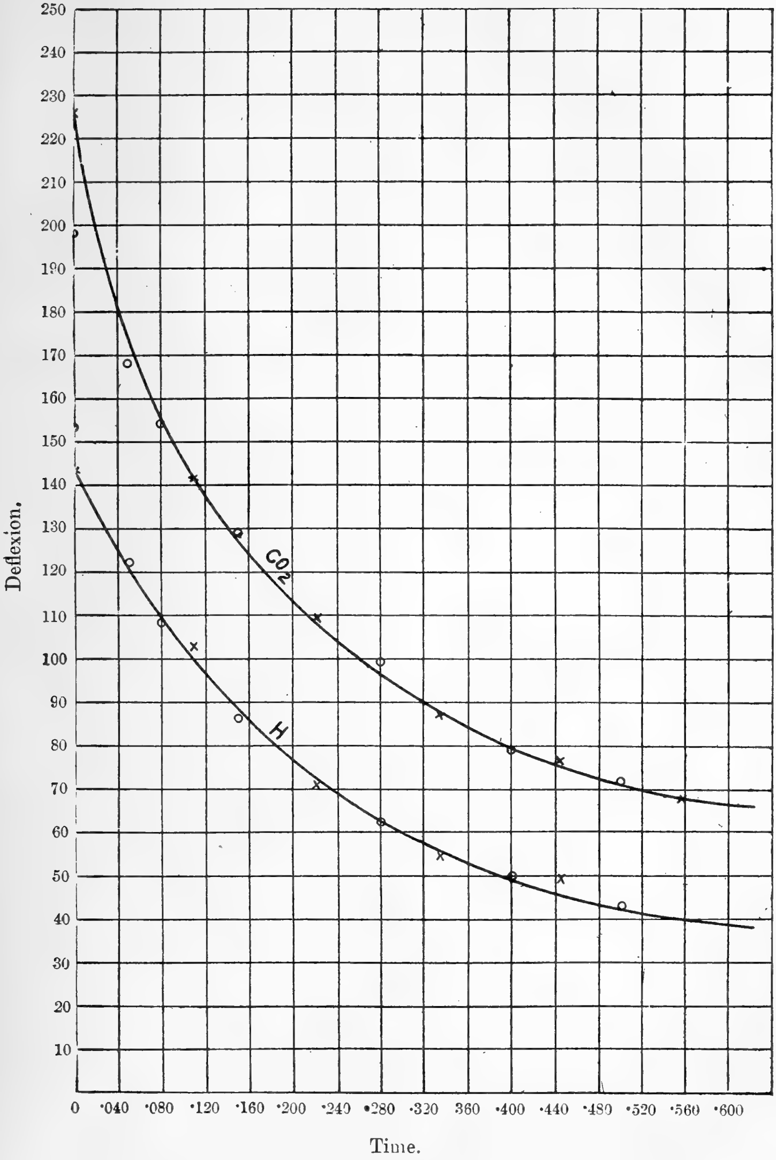
This method of determining capacities was found to be much more satisfactory than the method of mixtures. The capacity of the electrometer, connexions, &c., was found to be ·00016 of a microfarad, while that of the air condenser was ·00364 of a microfarad.

Consideration of other Gases.

This question of recombination of ions was investigated also in the case of carbonic acid gas and hydrogen gas. The same method was employed as for air. The recombination of the ions in these gases under different pressures could not, however, be satisfactorily investigated, as the supply of gas was limited in each case and no means of filling the cylinder with these gases, under increased pressure, was at our disposal. The results given here, therefore, for carbon dioxide and hydrogen are for these gases under a pressure just slightly above atmospheric pressure. The pressure was a few millimetres above that of the atmosphere, in consequence of the pressure of a short column of the acid, from which the gas was generated in each case, in the generating apparatus. In the case of both hydrogen and carbon dioxide, it was tested whether the ions in these gases followed the same law of recombination as was found to hold in the case of air. The law was found to be the same in these gases as in air. A comparison of the experimental and theoretical curves is shown for carbon dioxide and hydrogen in fig. 8.

The agreement between theory and experiment in these two gases is therefore seen to be as close as is the case with air. The indications are then that this law of recombination of the ions is an invariable one. It has been shown conclusively by this research, that the rate at which ions recombine is governed by this law in the cases of air, carbon dioxide, and hydrogen.

Fig. 8.


250
240
230
220
210
200
190
180
170
160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
Deflexion.
CO_2
H
0
·040
·080
·120
·160
·200
·240
·280
·320
360
·400
·440
·480
·520
·560
·600
Time.

The absolute value of the coefficient α for these two gases was also determined. Carbon dioxide presented a little difficulty, as the determinations made on it did not keep perfectly constant from day to day. A considerable number of determinations were made, however, extending over several days and the mean of these values taken. This value of α was found to be $3492 \times \epsilon$, which on substituting the value of ϵ becomes $2{\cdot}27 \times 10^{-6}$. This is practically the same as the value of the coefficient obtained for air.

The corresponding determination for hydrogen resulted in a value only about 87 per cent. of the value for air. The actual result for hydrogen was $2938 \times \epsilon$, which becomes $1{\cdot}91 \times 10^{-6}$ on substituting the value of ϵ. From this it appears that, although the coefficient of recombination is a constant for any gas, it has not necessarily the same value for all gases.

Referring again to Prof. Townsend's paper (Phil. Trans. Roy. Soc. vol. cxciii. 1899) we find that these results are again in close agreement with his. The result for hydrogen given above is also the mean result of several determinations.

Removal of Nuclei.

Throughout this investigation, care was taken to thoroughly dry the gas and to remove dust particles from it before allowing it to pass into the cylinder in which the measurements were made on it. For this purpose, the air which was used in the cylinder was first passed slowly through an iron tube (about two feet in length) filled with calcium chloride, to remove the moisture, and then through a bulb containing cotton-wool, to remove any dust particles.

When hydrogen and carbon dioxide were used, the gas was passed through a series of drying-bottles containing small pieces of pumice-stone moistened with strong sulphuric acid, and thence through the cotton-wool as before. These precautions were taken of course on account of the fact, which has been previously investigated (E. Rutherford, Phil. Mag. Nov. 1897), that when any nuclei are present in the gas, the ions collect about these nuclei and become neutralized. The number of ions decreases therefore much more rapidly when any nuclei are present in the gas than it does if the gas is quite free from these particles. As the object was to determine the rate of decrease in the number of ions due simply to the recombination with one another, the precaution was therefore taken to remove all foreign particles from the gas as far as possible.

Summary of Results.

The results obtained from this research may be summarized as follows:—

(1) The rate at which the ions recombine in ionized air is determined by the same law, namely, $\frac{dn}{dt} = -\alpha n^2$, no matter what the pressure of the gas may be, at least between the limits of ·125 of an atmosphere and 3 atmospheres.

(2) The coefficient of recombination for the ions in air is independent of the pressure of the air for the same range of pressures as above.

(3) The absolute value of this coefficient has been determined and has been found to be $3384\,\epsilon$, or on substituting the value of the charge on an ion it becomes $2{\cdot}20 \times 10^{-6}$.

(4) Hydrogen and carbon dioxide were also examined, and the same law of recombination was found to hold true as for air.

(5) On determining the coefficient for carbon dioxide, it proved to be practically the same as for air, while the coefficient for hydrogen is only about 87 per cent. of the value for air.

In conclusion, I wish to express my sincerest thanks to Prof. Rutherford, at whose suggestion this research was undertaken, and who has so kindly rendered me the most valuable assistance by his advice and suggestions throughout the whole investigation.

Macdonald Physical Laboratory,
McGill College, Montreal,
July 4, 1901.

XXXI. *Condensation of the Vapours of Organic Liquids in Presence of Dust-free Air.* By F. G. Donnan, *M.A., Ph.D.*[*]

[Plate V.]

§ 1. *Introduction.*

THE question why some liquids act as ionizing solvents while others do not, is one of fundamental importance in the theory of electrolytic dissociation, yet, although various suggestions have been made, no satisfactory explanation of these differences has hitherto been given. The experiments

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[*] Communicated by the Author.

to be described below were made in order to test a certain view of the matter, which may be briefly sketched as follows. In a solution in which the solute is more or less ionized, one might suppose the ions to be surrounded by clusters of solvent-molecules which had, so to speak, "condensed" round them and opposed an obstacle to their recombination. Now one might regard this state of things as being caused by some sort of "specific attraction" between the solvent-molecules and electricity, *i.e.* the electrons or electrical charges which are associated with the ions. Were this the case, one might expect this specific attraction to manifest itself in other ways. For example, if electrical nuclei of any sort were present in, or were produced by any means in, air which was saturated with the vapour of an ionizing liquid, then it would be just possible that the specific attraction referred to above might help to produce condensation of the vapour round these nuclei under suitable conditions; *i.e.* if the vapour were supersaturated by a sudden adiabatic expansion. If the liquid in question did not act as an ionizing solvent, it would be natural to expect that the condensation just alluded to would only occur when the vapour entered the really unstable (labile) region, or at any rate would only be produced by a much higher degree of supersaturation.

The very interesting experiments of C. T. R. Wilson * on the condensation of water-vapour in the presence of dust-free air and other gases, have rendered it extremely probable that this condensation occurs round electrical nuclei of some sort. It was found that an adiabatic expansion produced no condensation in dust-free air saturated with aqueous vapour until the expansion attained to a certain value, viz., 1·25. This expansion (ratio) produces a small shower of fine rain, and expansions ranging from the above value up to 1·38 produce small fine showers of not much greater density. At an expansion of 1·38, however, the phenomenon changes in character, a more or less dense fog being then produced. Now air or any other gas exposed to Röntgen radiation becomes a conductor; and the experiments of Prof. J. J. Thomson and his fellow-investigators have made it practically certain that this conduction is of a convective nature, due to electrically-charged nuclei produced in the gas. The importance of this in the present connexion lies in the fact that Wilson's experiments showed that air which was saturated with water-vapour and exposed to Röntgen radiation gives,

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* Phil. Trans. A. 1897, vol. clxxxix. pp. 265–307; A. 1899, vol. cxciii. pp. 289–308.

when adiabatically expanded, no condensation before the expansion 1·25, and a dense fog instead of a fine rain at that point. The natural conclusion is that the rain-like condensation for expansions ranging from 1·25 to 1·38 is due to condensation round a comparatively small number of electrical nuclei either existing in the gas or produced by the sudden expansion. Here, then, is a case of the condensation of the vapour of a liquid of unique ionizing power round electrical nuclei. It becomes a matter of interest to compare with water-vapour the vapours of other liquids whose ionizing powers are either much smaller or practically zero, and the following experiments were carried out for that purpose.

§ 2. *Apparatus.*

The apparatus employed was, so far as concerns the method used to obtain adiabatic expansion, essentially the same as the apparatus described by Wilson in his later experiments. A drawing of it is shown on Plate V. *a* is a glass tube floating on mercury and fitting pretty accurately into the cylinder (*b*). The end (*e*) of the glass rod (*c*) is rounded into a knob and covered with rubber; it is pressed, by the elasticity of the piece of rubber-tubing (*d*), firmly against the turned-out end of the tube (*f*), into which the rubber-covered knob accurately fits. *f* connects with the vacuous space (*g*). This was a large strong bottle which could be readily evacuated by means of a Fleuss vacuum-pump. On pulling the glass rod (*c*), connexion is made between the interior of *a* and *g*, the result of which is that the piston-tube (*a*) is suddenly and sharply brought down against the smooth flat rubber stopper (*h*), and a practically adiabatic expansion produced in the cloud-vessel (*i*), where the formation of the rain or fog is observed. The lower end of *a* must be very carefully ground flat so that it fits perfectly tightly against the upper surface of the rubber stopper. The liquid to be experimented on is introduced into *i* through *k*, which can be closed by a ground-glass cap rendered secure against entrance of air by an external mercury seal. The cloud-vessel is connected by capillary tubing with the manometer (*l*). The mercury in the manometer was always brought before an expansion to a fixed mark just below the capillary joint by means of the fine adjustment (hinge and screw) *n*. In this way the dead space at *m* was rendered negligibly small. During the expansion the tap (*p*) was closed, and then, when the expanded air and vapour had recovered its thermal and saturation equilibrium, *p* was opened and the mercury brought to the same fixed mark. The ratio of the pressures, after allowance has been made for the pressure of the saturated vapour at the temperature of the experiment, gives the volume-expansion ratio. An etched scale on *b* enables one to repeat any given expansion as often as necessary, and serves as a fiducial scale generally. A layer of mercury protects the rubber stopper (*r*) from the action of the liquid in *i*. The wired rubber connexion (*s*) enables the cloud-vessel to be readily detached and cleaned. The experiments were made in a dark room, and the cloud-vessel (*i*) was illuminated as recommended by Wilson*.

§ 3. *Liquids used.*

Water.—Ordinary distilled water.

Ethyl alcohol.—Dried by lime first, and then by repeated distillation from small quantities of sodium. B.-Pt. constant.

Methyl alcohol.—Similarly treated. (Kahlbaum's best quality.)

Benzene, Chlorobenzene, Carbon tetrachloride, Carbon disulphide.—From Kahlbaum. They were dried by P_2O_5, and fractionated to constant B.-Pt. These B.-Pt.'s agreed with the values taken by interpolation from Ramsay and Young's results for the corresponding pressures. The carbon disulphide was purified by shaking with mercury before distillation.

The data as regards the vapour-pressures of the organic liquids employed were obtained by calculation from Ramsay and Young's results.

§ 4. *Results.*

Substance.	Rain-point Expansion.	Fog-point Expansion.	Temperature of Experiments.
Water	1·29	1·42	20·5–20·8
Methyl alcohol	1·32	1·42	16·8
Ethyl alcohol	1·20	1·25	17·2–17·5
Carbon tetrachloride	1·89	Not observed.	16·5
Carbon disulphide	1·05	1·08	16·5
Chlorobenzene	1·48	1·60	16·3
Benzene	1·53	1·78	18–18·5

The results for water are not quite the same as the values given by Wilson; but, as Wilson has pointed out himself, the values obtained depend somewhat on the form of apparatus used. Doubtless in the present case the connecting neck between the cloud-vessel and the expansion-cylinder would tend to make the measured expansion greater than the

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* For the methods of obtaining dust-free air and other particulars of manipulation, reference must be made to Wilson's papers.

effectively adiabatic expansion in (*i*). For comparative purposes, however, this source of uncertainty is of no account. As pointed out by Wilson and confirmed by myself, differences of two or three degrees are practically without influence on the results, so that the differences recorded in the last column do not materially affect the comparison.

A more important point is the fact that, owing to differences in the vapour-pressures of the liquids, a proper comparison would require the calculation of the degree of supersaturation in each case, not simply the expansion-ratio. This calculation seems to me to be very uncertain, as it requires the assumption that the supersaturated vapour obeys the simple gas laws. By comparing liquids of such very different volatilities as say carbon tetrachloride, benzene, and chlorobenzene, this difficulty can be more or less obviated so far as a *qualitative* comparison is concerned.

Without going into fine distinctions, the above results on a cursory examination seem to justify the conclusion that condensation takes place with greater difficulty in the case of non-ionizing solvents. The case of carbon disulphide is, however, very peculiar. An extremely small degree of expansion appeared to produce condensation in this case, and scarcely any region of increasing expansion and rain-like condensation appeared to exist. And, so far as I know, it does not act as an ionizing solvent. This cannot be due to its greater volatility, as a comparison with carbon tetrachloride shows*.

On the other hand, experiments with acetone, whose results I have not included in the table, as the vapour-pressure data for pure acetone do not appear to be known with sufficient certainty, showed conclusively that it required an extremely high expansion-ratio to produce even rain-like condensation, and yet acetone may be classed with ethyl and methyl alcohols as a solvent of moderate ionizing power.

Under these circumstances I do not feel entitled to draw any certain conclusion from the above experiments. They seemed, however, to be of sufficient interest to warrant publication, owing to the importance which attaches at the present time to the new theories which are being developed in connexion with the question of electrical nuclei in gases.

In conclusion I wish to point out that the term "fog-point" is somewhat of a misnomer in the case of the organic liquids enumerated above. I found it always possible to

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* Light is known to induce chemical change in carbon disulphide. It is just possible that this chemical change may be preceded by electrical disturbances in the molecule, giving rise to ions or free electrons.

measure with considerable accuracy a point at which the previously obtained very sparse rain-like condensations passed rapidly into very much denser condensations; these, however, I could scarcly term fogs, as their duration was never very great.

My very best thanks are due to Professor Ramsay, at whose suggestion these experiments were begun and whose advice and assistance were constantly at my disposal.

Chemical Laboratory, University College, London,
April 1901.

XXXII. *On Circular Filaments or Circular Magnetic Shells equivalent to Circular Coils, and on the equivalent Radius of a Coil.* *By* Professor THOMAS R. LYLE*.

[Plate VI.]

1. THE magnetic potential of a coil, with current C, at any point in its axis is given to a high order of approximation by

$$V=2\pi nC\left(1-\frac{x}{\rho}+\frac{\eta^2 x}{24\rho^3}+\frac{\xi^2-\eta^2}{8\rho^5}a^2x\right) \quad . \quad . \quad \text{I.}$$

if the dimensions of the cross section are small compared with the radius of the coil and the winding is uniform;

where a = mean radius,
ξ = axial breadth,
η = radial depth,
x = distance of point on the axis from the centre of the coil,
and $\rho^2=a^2+x^2$.

(See Maxwell, sect. 700.)

Expanding in ascending powers of x/a,

$$\begin{aligned}V=2\pi nC\Bigg\{1-\frac{x}{a}\left(1.1-\frac{3\xi^2-2\eta^2}{24a^2}\right)+\frac{x^3}{2a^3}\left[1.2-\frac{3(5\xi^2-4\eta^2)}{24a^2}\right]\\ -\frac{x^5}{2.4.a^5}\left[1.3-\frac{3.5(7\xi^2-6\eta^2)}{24a^2}\right]\\ +\frac{x^7}{2.4.6.a^7}\left[1.3.5-\frac{3.5.7(9\xi^2-8\eta^2)}{24a^2}\right]\\ -\,\&c. \qquad \&c.\Bigg\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{II.}\end{aligned}$$

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* Communicated by the Physical Society: read December 13, 1901.

2. The potential V' of a circular filament concentric and coaxal with the coil of radius r and with current nC flowing is given by

$$V' = 2\pi nC\left\{1 - \frac{x}{\sqrt{x^2+r^2}}\right\},$$

which expanded in ascending powers of x/r gives

$$V' = 2\pi nC\left\{1 - \frac{x}{r} + \frac{x^3}{2r^3} - \frac{1.3.x^5}{2.4.r^5} + \frac{1.3.5.x^7}{2.4.6.r^7} - \&c.\right\}.$$

3. The two potentials V in (1) and V′ in (2) will be identical provided

$$\frac{1}{r} = \frac{1}{a}\left\{1 - \frac{3\xi^2 - 2\eta^2}{24a^2}\right\},$$

$$\frac{1}{r^3} = \frac{1}{a^3}\left\{1 - \frac{3(5\xi^2 - 4\eta^2)}{24a^2}\right\},$$

$$\frac{1}{r^5} = \frac{1}{a^5}\left\{1 - \frac{5(7\xi^2 - 6\eta^2)}{24a^2}\right\},$$

$$\&c. \qquad \&c.$$

which equations can all be satisfied to the order of approximation adopted if

$$3\xi^2 - 2\eta^2 = 5\xi^2 - 4\eta^2 = 7\xi^2 - 6\eta^2 = \&c.,$$

that is if $\xi^2 = \eta^2$

and

$$\frac{1}{r} = \frac{1}{a}\left\{1 - \frac{\eta^2}{24a^2}\right\}$$

or

$$r = a\left(1 + \frac{\eta^2}{24a^2}\right).$$

4. If we expand the potentials in ascending powers of a/x the same result is arrived at, hence the potentials of a circular coil of square cross section (η, η), mean radius a, with n turns and carrying a current C, and of a circular filament of radius $a\left(1+\frac{\eta^2}{24a^2}\right)$ carrying a current nC, lying in the median plane of, and coaxal with the coil, are identical at all points on the common axis, and hence by Legendre's theorem identical at all points of space without the coil.

This particular filament, therefore, is equivalent to the coil and can replace it, and the radius of this filament I shall call the *equivalent radius of the coil*, and a coil of square cross-section I shall call a *single-shell coil.*

5. In the construction of coils there is no reason why with ordinary care a very approximately square section could not be attained, and then the above theorem would, as I will explain, simplify the theory of galvanometers and electrodynamometers, as well as the determination of coil-constants, coefficients of mutual induction, and current-balance constants.

6. A coil of n turns the axial breadth ξ of whose section is greater than its radial depth η, of mean radius a and carrying a current C, can be replaced by two equal filaments coaxal with the coil, each carrying a current $\frac{1}{2}n\mathrm{C}$, whose radii are $a\left(1+\frac{\eta^2}{24a^2}\right)$, and which are placed at equal distances β on either side of the median plane of the coil, where

$$\beta^2=\frac{\xi^2-\eta^2}{12}.$$

For:

The potential of the two filaments as specified above, at any point on the common axis distant x from their median plane, is

$$\mathrm{V}_2=2\pi n\frac{\mathrm{C}}{2}\left\{1-\frac{x-\beta}{\rho_1}+1-\frac{x+\beta}{\rho_2}\right\}$$

where

$$\rho_1^2=r^2+(x-\beta)^2, \qquad \rho_2^2=r^2+(x+\beta)^2,$$

r being the radius of either filament. Expanding by Taylor's theorem

$$\mathrm{V}_2=2\pi n\mathrm{C}\left\{1-\frac{x}{\rho}+\frac{\beta^2}{1.2}\cdot\frac{3r^2x}{\rho^5}\right\}.$$

$$\text{(where } \rho^2=x^2+r^2\text{)}$$

$$\mathrm{V}_2=2\pi n\mathrm{C}\left\{1-\frac{x}{r}\left[1-\frac{3\beta^2}{2r^2}\right]+\frac{x^3}{2r^3}\left[1-\frac{3.5.\beta^2}{2r^2}\right]\right.$$
$$\left.-\frac{x^5}{2.4.r^5}\left[1.3.-\frac{3.5.7.\beta^2}{2r^2}\right]+\&\text{c.}\right\},$$

and in order that this may be identical with the axial potential of the coil given in section 1, the following equations have to be satisfied:—

$$\frac{1}{r}\left\{1-\frac{3\beta^2}{2r^2}\right\}=\frac{1}{a}\left\{1-\frac{3\xi^2-2\eta^2}{24a^2}\right\},$$

$$\frac{1}{r^3}\left\{1-\frac{3\,.\,5\,.\,\beta^2}{2r^2}\right\}=\frac{1}{a^3}\left\{1-\frac{3(5\xi^2-4\eta^2)}{24a^2}\right\},$$

$$\frac{1}{r^5}\left\{1-\frac{5\,.\,7\,.\,\beta^2}{2r^2}\right\}=\frac{1}{a^5}\left\{1-\frac{5(7\xi^2-6\eta^2)}{24a^2}\right\},$$

&c. &c.

It is easily seen that this series of equations will be satisfied to the order of approximation adopted if

$$\frac{1}{r}=\frac{1}{a}\left(1-\frac{\eta^2}{24a^2}\right)$$

or

$$r=a\left(1+\frac{\eta^2}{24a^2}\right)$$

and

$$\beta^2=\frac{\xi^2-\eta^2}{12}.$$

If we expand the potentials in ascending powers of a/x for the portion of the axis where x is $> a$ the same result will be arrived at; hence the potentials of the coil and of the double filament are equal at all points in space external to the coil, and in all calculations (except those bearing on self-induction) the coil can be replaced by this double-shell or filament.

A coil of this type (*i.e.* in which $\xi > \eta$) I shall call a *thick double-shell coil*; the radius of either of the two replacing filaments I shall call the *equivalent radius* of this coil; and the distance 2β between the two filaments I shall call the *equivalent breadth* of the coil.

It is obvious now that the coil will be completely specified when its type, its equivalent radius, and its equivalent breadth are given; that all magnetic actions of the coil on systems external to itself will be identical with those of the equivalent filaments; and that all magnetic action on the coil from external systems will be identical with the action of these systems on the filaments, the order of approximation being up to the fourth power of the dimensions of the section of the coil divided by its radius.

7. A coil of n turns, the axial breadth ξ of whose cross section is less than its radial depth η, of mean radius a, and carrying a current C, can be replaced by two concentric and coplanar filaments coaxal with and lying in the median plane of the coil, each carrying a current $\frac{1}{2}n\mathrm{C}$, and whose radii are $r+\delta$ and $r-\delta$ respectively, where

$$r=a\left(1+\frac{\xi^2}{24a^2}\right),$$

$$\delta^2=\frac{\eta^2-\xi^2}{12}.$$

It is unnecessary to give the proof of this, as it follows easily on the same lines as that of the last theorem. A coil of this type might be called a *thin double-shell coil*, while r as defined above I shall call its *equivalent radius* and 2δ its *equivalent depth*.

8. In the case of a coil whose cross section is so large that the fourth power of its dimensions divided by its radius cannot be neglected, it is easy to imagine it divided up into portions whose dimensions are small enough for the above theorems to apply, and then to determine the system of filaments that will replace each of these portions. A few extra filaments adds nothing to the difficulty and little to the time required to calculate a coefficient of mutual induction or a current-balance constant, seeing that tables are available by means of which either of these quantities referring to any two coaxal filaments can be quickly determined.

9. The principles embodied in the preceding sections may be established in a totally different way, which is interesting enough to record here.

Its application to a thick double-shell coil ($\xi > \eta$) will be sufficient to explain the method.

If the position of any turn of the coil be defined by x the distance of its centre from a fixed point on the axis, and y the radius of the turn, then N the magnetic flux passing through this turn due to any system whatever of magnets or currents will be a function of x and y.

Let us specify a small portion of the coil considered by rectangular coordinates p, q referred to axes through the centre of the section parallel to the axis of the coil and its radius respectively, then the current round an element dp, dq of the coil will be $\frac{n\mathrm{C}}{\xi\eta}\,dp\,.\,dq$.

ξ and η having the same meaning as in previous sections.

The mutual energy of this filament and the external magnetic field will be

$$\frac{n\mathrm{C}}{\xi\eta}\left[\mathrm{N}_0+p\left(\frac{d\mathrm{N}}{dx}\right)_0+q\left(\frac{d\mathrm{N}}{dy}\right)_0+\frac{1}{1\,.\,2}\left\{p^2\frac{d^2\mathrm{N}}{dx^2}+2pq\frac{d^2\mathrm{N}}{dxdy}+q^2\frac{d^2\mathrm{N}}{dy^2}\right\}\right]dp\,.\,dq,$$

where N_0 is the value of N at the centre of the section.

Integrating between the limits $(+\frac{1}{2}\xi,-\frac{1}{2}\xi)$ $(+\frac{1}{2}\eta,-\frac{1}{2}\eta)$ we get the total mutual energy

$$=n\mathrm{C}\left\{\mathrm{N}_0+\frac{1}{24}\left(\xi^2\frac{d^2\mathrm{N}_0}{dx^2}+\eta^2\frac{d^2\mathrm{N}_0}{dy^2}\right)+\ldots\right\}.$$

Again, if we have two equal filaments whose p and q coordinates are (α,β), and $(-\alpha,\beta)$ respectively, and if a current $\frac{1}{2}n\mathrm{C}$ circulates in each, the mutual energy of the external system and the filaments will be

$$\frac{1}{2}n\mathrm{C}\left\{\mathrm{N}_0+\alpha\frac{d\mathrm{N}_0}{dx}+\beta\frac{d\mathrm{N}_0}{dy}+\frac{1}{1\,.\,2}\left[\alpha^2\frac{d^2\mathrm{N}_0}{dx^2}+2\alpha\beta\frac{d^2\mathrm{N}_0}{dxdy}+\beta^2\frac{d^2\mathrm{N}_0}{dy^2}\right]\right\}$$

$$+\frac{1}{2}n\mathrm{C}\left\{\mathrm{N}_0-\alpha\frac{d\mathrm{N}_0}{dx}+\beta\frac{d\mathrm{N}_0}{dy}+\frac{1}{1\,.\,2}\left[\alpha^2\frac{d^2\mathrm{N}_0}{dx^2}-2\alpha\beta\frac{d^2\mathrm{N}_0}{dxdy}+\beta^2\frac{d^2\mathrm{N}_0}{dy^2}\right]\right\}$$

$$=n\mathrm{C}\left\{\mathrm{N}_0+\beta\frac{d\mathrm{N}_0}{dy}+\frac{1}{2}\left[\alpha^2\frac{d^2\mathrm{N}_0}{dx^2}+\beta^2\frac{d^2\mathrm{N}_0}{dy^2}\right]+\ldots\right\};$$

but

$$\frac{1}{y}\frac{d\mathrm{N}}{dy}=\frac{d^2\mathrm{N}}{dx^2}+\frac{d^2\mathrm{N}}{dy^2}$$

(See Maxwell, section 703);

then the energy of the two filaments reduces to

$$n\mathrm{C}\left\{\mathrm{N}_0+(\beta y_0+\tfrac{1}{2}\alpha^2)\frac{d^2\mathrm{N}_0}{dx^2}+(\beta y_0+\tfrac{1}{2}\beta^2)\frac{d^2\mathrm{N}_0}{dy^2}+\&\mathrm{c.}\right\};$$

and in order that this may be identical with the expression for the energy of the coil

$$\beta y_0+\tfrac{1}{2}\alpha^2=\frac{\xi^2}{24}\quad\text{and}\quad\beta y_0+\tfrac{1}{2}\beta^2=\frac{\eta^2}{24}.$$

Solving the second equation we have to the order of approximation adopted

$$\beta=\frac{\eta^2}{24y_0},$$

and then from the first

$$\alpha^2 = \frac{\xi^2 - \eta^2}{12};$$

but the radius of the replacing filaments being $y_0 + \beta$ is

$$= y_0\left(1 + \frac{\eta^2}{24y_0{}^2}\right),$$

and their distance apart is 2α where $\alpha^2 = \frac{\xi^2 - \eta^2}{12}$ as before.

As an illustration of this method of treating the problem, the following case may be considered:—

Let the external field be uniform and the axis of the coil parallel to the lines of force of the field.

The mutual energy is $C\Sigma N$, where N is the magnetic flux through any one turn.

But $\quad C\Sigma N = (n\,.\,C\,.\,H) \times$ mean area of the coil

$$= nCH \times \pi\left(a^2 + \frac{\eta^2}{12}\right)$$

as is easily shown by integration: a being the mean radius as before.

In this case the equivalence of the filaments replacing a thick double shell or a single shell to their corresponding coils is obvious.

For a thin double shell the mean area of the two filaments is

$$\pi(r^2 + \delta^2)$$

$$= \pi\left\{a^2 + \frac{\xi^2}{12} + \frac{\eta^2 - \xi^2}{12}\right\}$$

$$= \pi\left\{a^2 + \frac{\eta^2}{12}\right\},$$

and this is again equal to the mean area of the corresponding coil.

Applications of the foregoing Principles.

10. The constant of a single-shell galvanometer-coil corrected for cross section is $2n\pi/r$ where r is its equivalent radius; while that of a thick double-shell coil is

$$\frac{2n\pi}{r}\left(1 - \frac{3}{2}\frac{\beta^2}{r^2}\right),$$

where r is its equivalent radius and 2β its equivalent breadth, and that of a thin double-shell coil is

$$\frac{2n\pi r}{r^2-\delta^2} \quad \text{or} \quad \frac{2n\pi}{r}\left(1+\frac{\delta^2}{r^2}\right)$$

where r is its equivalent radius and 2δ its equivalent depth.

11. The magnetic force H at any point on the axis of the three types of coils distant x from their centres is given by

$$\mathrm{H}=2\pi n\mathrm{C}\frac{r^2}{\rho^3} \quad \text{for a single shell,}$$

$$\mathrm{H}=2\pi n\mathrm{C}\frac{r^2}{\rho^3}\left\{1+\frac{3}{2}\cdot\frac{4x^2-r^2}{\rho^5}\beta^2\right\} \text{ for a thick double shell,}$$

and

$$\mathrm{H}=2\pi n\mathrm{C}\left\{\frac{r^2}{\rho^3}+\frac{1}{2}\cdot\frac{2x^4-11x^2r^2+2r^4}{\rho^7}\delta^2\right\} \text{ for a thin double shell,}$$

where $\rho^2=r^2+x^2$.

It will be noticed that in the expression for the axial H of a thick double-shell coil the second term, depending on the equivalent breadth, will disappear when $x=r/2$: hence a Helmholtz galvanometer made of equal coils of either the first or second type which are placed at *half their equivalent radii* apart is an ideal instrument of its kind, as the corrections both for the cross sections of its coils and for the length of its needle disappear from its constant.

12. In practice the equivalent radius of a coil may be determined in one or other of three ways.

a. By measurement of the mean radius and cross section and substituting the values so obtained in the expressions given for the equivalent radii of the three types of coils in §§ 4, 6, and 7.

b. By comparison by Bosscha's method with a standard coil, preferably a single-shell one, whose equivalent radius has been carefully determined by method *a*.

c. By a second electrical method which I will presently describe.

13. It is important to note that the comparison of two single-shell coils by Bosscha's method gives directly the ratio of their equivalent radii, no corrections having to be added if the length of the small needle at the common centre of the two coils be neglected. (This latter correction is, however, larger than, I think, many people imagine and should be applied in most cases, taking five-sixths of the actual length of the small magnet for the distance between its poles *.)

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* See W. Hallock and F. Kohlrausch, Wied. *Ann.* xxii. p. 411, or abstract in Phil. Mag. [5] vol. xviii. p. 390, "On the Distance apart of the Poles of a Magnet."

Thus if R_1 and R_2 be the resistances of the parallel branches in which the two coils r_1, n_1 and r_2, n_2 are included when the needle at their common centre is unaffected, then

$$\frac{r_1}{r_2}=\frac{n_1R_2}{n_2R_1};$$

and if the distance 2λ between the poles of the small magnet be taken into account (see Maxwell, § 711),

$$\frac{r_1}{r_2}=\frac{n_1R_2}{n_2R_1}\left\{1+\tfrac{3}{4}\lambda^2\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)\right\},$$

where for r_1 and r_2 on the right side the mean radii a_1 and a_2 got by approximate measurement may be substituted.

In general we have

$$\frac{R_1}{R_2}=\frac{G_1}{G_2}\left\{1+\tfrac{3}{4}\lambda^2\left(\frac{1}{a_1^2}-\frac{1}{a_2^2}\right)\right\}$$

where R_1, R_2 are the resistances of the two parallel branches in which the coils lie, and G_1, G_2 their galvanometer constants given in terms of equivalent radius, breadth or depth in § 10.

14. The proposed electrical method of measuring the equivalent radius of a coil will be easily understood from the following. If, with the apparatus used in Bosscha's method for determining the ratio of the equivalent radii of two coils, some arrangement be made by means of which the smaller coil can slide to either side of the larger one, still remaining coaxal with and parallel to it, then if the former be moved a distance x to either side, and if the resistances be readjusted so that the magnet at the centre of the large coil is not affected, we have

$$\frac{n_1}{r_1R_1'}=\frac{n_2r_2^2}{R_2'(r_2^2+x^2)^{\frac{3}{2}}}, \quad . \quad . \quad . \quad . \quad . \qquad \text{(I.)}$$

if the coils be single-shell ones, where R_1' and R_2' are the new resistances in the parallel branches. If with the same resistances R_1' and R_2' in the branches, balance is obtained with the small coil first at one side and then at the other, the distance between the two positions of the small coil will be $2x$; and as the ratio of r_1 to r_2 determined by the method in the last section is known and given by

$$\frac{r_1}{r_2}=\frac{n_1R_2}{n_2R_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(II.)}$$

we can from equations I. and II. determine both r_1 and r_2 in terms of $2x$ and the two ratios of resistances.

The most sensitive position in which to place the small coil is that at which dH/dx is a maximum, that is when the small coil is at a distance from the needle (and the plane of the large coil) equal to one half of its equivalent radius.

For this position, equation I. becomes

$$\frac{n_1}{r_1 R_1'} = \frac{n_2}{r_2 R_2'} \frac{8}{5\sqrt{5}};$$

so that

$$\frac{R_1'}{R_2'} = \frac{5\sqrt{5}}{8} \frac{R_1}{R_2},$$

R_1 and R_2 being the resistances in Bosscha's comparison method.

Hence we arrange the resistances R_1' and R_2' so that their ratio is $5\sqrt{5}/8$ times the resistance ratio in Bosscha's method, adjust the small coil accurately first on the east and then on the west of the large coil so that the magnet is unaffected; the distance between these two positions is the equivalent radius of the small coil. Four determinations of r_2 can be made by varying the relative aspects of the faces of the two coils, and increased sensitiveness can, if desired, be obtained by using a compensating magnetic system to partly neutralize the earth's field. This is, however, rarely necessary as the method is exceedingly sensitive. It would be an advantage, however, when extreme accuracy is required, as one could then work with smaller currents, and so minimize the variation of the resistances due to heating.

15. Similar treatment can obviously be applied to the large coil. Thus when the coplanar resistance ratio is now changed to R_1''/R_2'' so that $\frac{R_1''}{R_2''} = \frac{8}{5\sqrt{5}} \frac{R_1}{R_2}$, and the large coil moved to equilibrium positions east and west of the small coil and the needle, the distance between these two positions of the large coil will be equal to its equivalent radius. This determination can then be used to check the one made by moving the small coil.

16. Plate VI. shows a diagrammatic section of the apparatus I have used for the above purpose.

A is a strong wooden base on levelling-screws. B is a strong flat bar which can slide parallel to the length of the base, and can be clamped in any position to a slow-motion screw *a* of small pitch and having a small range. To B is attached an upright piece C, one of whose faces is very carefully planed and set perpendicular to the upper surface of B and to the line of motion of B. To C the larger coil is to be attached by means of a clamp as in the diagram. A pair of uprights F and another pair F′ rise from the base, each pair being joined near the top by a cross-piece. In the upper sides of these cross-pieces rectangular slots are cut in which a strong bar D can slide parallel to the other slide B. D can be clamped in any position to the slow-motion screw *b* which is of fine pitch and has a short range. To the lower side of D is fixed a piece E, one of whose faces is carefully planed and set perpendicular to D. To this piece the smaller coil is to be attached. Resting on the upper ends of the four pillars F and F′ is a brass table fitted with short levelling-screws for carrying the magnetometer. The telescope and scale are set up on the left side, and to the opposite ends of the beams B and D are attached (as in the diagram) carefully ruled glass millimetre-scales S and S′ by means of which, with the assistance of two micrometer-microscopes, the motion of B and of D and so of either coil in the axial direction can be accurately measured.

There is little difficulty in setting up and centering the coils, but I have found the following method convenient:—The magnetometer being replaceable in a given position, the axial line through the centre of the magnet is determined by directing a horizontal telescope whose line of sight is parallel to D to the centre of the magnet. A small scriber which takes the place of the magnetometer is now adjusted so that the image of its point coincides with the cross wires of the telescope. Across one face of each coil a fine wire is stretched along a diameter by means of a centre square and the middle points of the diameters marked. The coils are now clamped in position so that the middle points of these diametral wires coincide with the point of the scriber.

17. In measuring the equivalent radii of the coils we must first determine the ratio of the resistances that are included in the parallel arms containing the coils when their actions on the needle balance each other, the needle being at the common centre of the coils. The adjustment of the coils so that the needle is at their common electrical centre is of importance, and is managed as follows. First adjust by eye, then arrange the resistances so that with currents flowing the needle is not affected; now, by means of the slow-motion screws attached to each slide, the coils can in turn be moved axially until they both have maximum magnetic effect on the needle. When this has been effected in the usual way for adjusting for a maximum or minimum, we know that the centre of the needle lies at the common electrical centre of the two coils.

The resistance ratio of the branches will now require slight readjustment, and then its value R_1/R_2 is determined as Lord Rayleigh[*] and Mrs. Sidgwick have done, by means of a specially arranged Wheatstone's bridge. The resistances in the branches are then changed to R_1' and R_2',

where
$$\frac{R_1'}{R_2'}=\frac{5\sqrt{5}}{8}\frac{R_1}{R_2},$$

if both the coils are single-shell ones, and the small coil moved first to one side then to the other to positions in which with currents flowing the needle is not affected. The distance between these two positions is the equivalent radius of the small coil.

The small coil having been replaced at the centre, the resistances in the branches are again changed to R_1'' and R_2'', where

$$\frac{R_1''}{R_2''}=\frac{8}{5\sqrt{5}}\frac{R_1}{R_2};$$

and the large coil is now moved, first to one side and then to the other, to positions of equilibrium. The distance between these two positions will be the equivalent radius of the large coil.

As the ratio R_1/R_2 determined in the first part of this process gives us (see § 13) the ratio of the equivalent radii of the two coils,

$$\frac{r_1}{r_2}=\frac{n_1R_2}{n_2R_1},$$

this value of the ratio r_1/r_2 can be used as a check on the ratio of the values of r_1 and r_2 determined separately.

18. If both coils be thick double-shell ones the second resistance-ratio, that is the one (R_1'/R_2') used in measuring r_2, must be adjusted so that

$$\frac{R_1'}{R_2'}=\frac{5\sqrt{5}}{8}\left(1-\frac{3}{2}\frac{\beta_2^2}{a_2^2}\right)\frac{R_1}{R_2};$$

and the third resistance-ratio R_1''/R_2'', *i. e.* the one used in measuring r_1, must be adjusted so that

$$\frac{R_1''}{R_2''}=\frac{8}{5\sqrt{5}}\left(1-\frac{3}{2}\frac{\beta_1^2}{a_1^2}\right)\frac{R_1}{R_2},$$

where $2\beta_1$ and $2\beta_2$ are the *equivalent* breadths of the two coils and a_1 and a_2 their approximate mean radii.

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[*] Phil. Trans. 1884.

For any combination of two coils of different types, the factors that have to be applied to the coplanar resistance-ratio to get the second and third ratios can easily be deduced from the expressions given in §§ 10, 11.

18*a*. The length of the small needle in the magnetometer should be known, and the factors for getting the second and third resistance-ratios from R_1/R_2 should be corrected for it. The correction to be applied to each factor will be the same for all kinds of coils.

When the coils and needle are coplanar there is equilibrium, provided

$$\frac{2\pi n_1}{R_1 r_1}\left(1+\frac{3}{4}\frac{\lambda^2}{r_1^2}\right)=\frac{2\pi n_2}{R_2 r_2}\left(1+\frac{3}{4}\frac{\lambda^2}{r_2^2}\right),$$

where 2λ is the distance between the poles of the needle. (See Maxwell, section 711.)

When the small coil is displaced along the axis a distance = half its radius, there will be equilibrium provided the resistances are changed to R_1' and R_2' so that

$$\frac{2\pi n_1}{R_1' r_1}\left(1+\frac{3}{4}\frac{\lambda^2}{r_1^2}\right)=\frac{2\pi n_2}{R_2' r_2}\frac{8}{5\sqrt{5}};$$

for as the needle is now in the Gaugain position with respect to the small coil, its length disappears from the expression for the torque exerted by the small coil on the needle.

Hence from the two equations above we get

$$\frac{R_1'}{R_2'}=\frac{R_1}{R_2}\left(1+\frac{3}{4}\frac{\lambda^2}{r_2^2}\right)\frac{5\sqrt{5}}{8}.$$

Similarly it may be shown for the other factor that

$$\frac{R_1''}{R_2''}=\frac{R_1}{R_2}\left(1+\frac{3}{4}\frac{\lambda^2}{r_1^2}\right)\frac{8}{5\sqrt{5}};$$

so that the factors in § 18 would become

$$\frac{5\sqrt{5}}{8}\left(1-\frac{3}{2}\frac{\beta_2^2}{a_2^2}\right)\left(1+\frac{3}{4}\frac{\lambda^2}{a_2^2}\right),$$

and

$$\frac{8}{5\sqrt{5}}\left(1-\frac{3}{2}\frac{\beta_1^2}{a_1^2}\right)\left(1+\frac{3}{4}\frac{\lambda^2}{a_1^2}\right);$$

and similarly in other cases.

19. The practical application of the above plan of measuring the equivalent radii of coils offers no difficulty to a

careful experimenter. The method is accurate and sensitive, its accuracy depending (1) on the accuracy of the comparison and adjustment of resistances; (2) on the slide-apparatus being well enough made to give very nearly true axial motion to the coils with a minimum of friction; (3) on the delicacy of motion given by the slow-motion screws to the sliding-bars.

The accurate measurement of the displacement of a coil offers no difficulty.

In regard to (1), the arrangement used by Lord Rayleigh and Mrs. Sidgwick to compare the mean radii of the coils of their current-balance leaves nothing to be desired. The apparatus I have described is the one I have used, and is not necessarily of the most suitable design. The slow-motion attachments to the sliding-bars are not altogether a success, and will have to be modified. The motion realized was not delicate enough and was rather jerky, so that I could rarely adjust the displaced coil in an exact equilibrium position. The necessity for a sensitive slow motion will appear from the following.

Two ordinary galvanometer-coils were being measured, one, the larger, of 36 turns and resistance nearly 1 ohm, the other of 30 turns and resistance a little over half an ohm. The battery-power used was three secondary cells with a resistance of 20 ohms included in the circuit. When the small coil was very near the equilibrium position ($\frac{1}{2}r$ from the large coil), a motion of ·016 cm. of the small coil caused a motion of the needle represented by 14 millimetres on a scale 40 cm. from the mirror. The little magnet was swinging in the earth's field slightly neutralized so that it made one swing in 2 seconds. Thus 1 mm. motion of the scale in the telescope corresponded very nearly to a motion of one-hundredth of a millimetre of the coil, and at a distance of 40 cm. from the mirror a half-millimetre scale could be used with comfort, and each half millimetre divided into tenths.

In order to indicate the degree of accuracy obtainable, as well as the necessity of having the motion of the slide-bars as frictionless as possible, I will quote the following results for the equivalent radius of a small galvanometer-coil obtained with a rough preliminary apparatus.

In this apparatus the slow-motion screw could only apply a thrust to the slide-bar at the end near the observer, the bar being drawn back by hand, when it was necessary to do so.

The two equilibrium positions of the small coil are indicated by A and B, A being on the side of the magnetometer furthest from the observer.

1. Coil adjusted at A so as to give a balance, then drawn to other side of magnetometer and adjusted similarly at B. Distance A to B=8·7280 cm
2. Similarly, but pushing the coil from B to A position. Distance B to A=8·7258 cm.
3. Same as 1. ,, A to B=8·7278 cm.
4. Same as 2. ,, B to A=8·7255 cm.

It will be seen that there is a nearly constant difference between the results obtained when the slide-bar is drawn towards the observer from the A to the B position, and when it is thrust from the observer from the B to the A position. There was, I found, considerable friction between the slide-bar and the far upright which fully accounted for this discrepancy, as the force of a few pounds was sufficient to compress that part of the bar (it was not a very thick one) between the coil and the far upright, in which it slid, by an amount equal to the discrepancy.

20. From the theory established in the early sections of this paper, it is obvious that the determination of the mutual inductance of any two coaxal circular coils is reduced to that of coaxal circular filaments; and in this method no terms are neglected of order lower in small quantities than the fourth power of the ratio of the cross section to the radius.

Hence by using Maxwell's tables (see Maxwell, Chap. xiv. Appendix) for $\log \frac{M}{4\pi\sqrt{r_1 r_2}}$ as a function of γ, where

$$\sin^2\gamma = \frac{4r_1 r_2}{(r_1+r_2)^2+x^2},$$

r_1 and r_2 being the radii of the filaments, and x the distance between their planes, we can quickly and accurately determine the mutual induction of two coils.

21. For single-shell coils of equivalent radii r_1 and r_2 and distance x apart the process is very expeditious, for M, determined as in § 20, has only to be multiplied by $n_1 n_2$ to get the mutual inductance of the coils.

22. To determine the mutual inductance of two double-shell coaxal coils, we replace the coils by their equivalent filaments, determine the M's as in § 20 for the four pairs got by combining a filament of one coil with a filament of the other, and multiply the mean of these by $n_1 n_2$.

For, the mutual inductance of two coils being their mutual energy when unit current is circulating in each, and as unit current in a double-shell coil is replaced by a current of $\frac{1}{2}n$ in each of its equivalent filaments, then, if $\overset{\frown}{ab}$ be Maxwell's M for two coaxal filaments a and b, and if $a\,a'$ be the pair equivalent to one coil and $b\,b'$ be the pair equivalent to the other, the mutual energy of currents of $\frac{1}{2}n$ circulating in a and a' with respect to currents $\frac{1}{2}n'$ circulating in b and b' will be

$$\tfrac{1}{4}nn'\{\overset{\frown}{ab}+\overset{\frown}{ab'}+\overset{\frown}{a'b}+\overset{\frown}{a'b'}\}$$

$=nn'\times$ mean of the four M's.

Thus four determinations of an M are required in the case of double-shell coils, but this does not involve four times the labour of a single determination if the four calculations are worked concurrently, as corresponding numbers in the four will be found on the same or consecutive pages of a logarithm book.

23. As an example I will give the leading figures in the calculation of the mutual inductance of Rowland's coils A and B used in his determination of the ohm.

Coil A...... $a_1=13{\cdot}710,\ \xi_1={\cdot}84,\ \eta_1={\cdot}90,\ n_1=154.$
Coil B...... $a_2=13{\cdot}690,\ \xi_2={\cdot}84,\ \eta_2={\cdot}90,\ n_2=154.$
Distance apart $=6{\cdot}534.$

So both coils are thin double-shell ones and are specified in the notation used by

$$\text{A}\begin{cases} r_1=a_1\left(1+\dfrac{\xi_1^{\,2}}{24a_1^{\,2}}\right)=13{\cdot}7121, \\[2ex] \delta_1^{\,2}=\dfrac{\eta_1^{\,2}-\xi_1^{\,2}}{12}={\cdot}0087,\ \delta_1={\cdot}0933. \end{cases}$$

$$\text{B}\begin{cases} r_2=a_2\left(1+\dfrac{\xi_2^{\,2}}{24a_2^{\,2}}\right)=13{\cdot}6921, \\[2ex] \delta_2=\delta_1={\cdot}0933. \end{cases}$$

Hence the radii of the filaments equivalent to A being

$r_1+\delta_1$ and $r_1-\delta_1$ are

13·8054 and 13·6188

and to B are

13·7854 and 13·5988,

so we have to determine the four M's where

	Radii.			Distance.
1...	13·8054	and	13·7854	6·534
2...	13·8054	,,	13·5988	6·534
3...	13·6188	,,	13·7854	6·534
4...	13·6188	,,	13·5988	6·534

From which we get (see Maxwell, chap. xiv.)

$$\gamma_1=76^\circ\,40'\frac{182}{300},\quad \gamma_2=76^\circ\,34'\frac{288}{302},\quad \gamma_3=76^\circ\,35'\frac{29}{302},\quad \gamma_4=76^\circ\,30'\frac{12}{303}.$$

$$\begin{array}{ll} M_1=161{\cdot}2805 & \\ M_2=159{\cdot}1388 & \\ M_3=159{\cdot}1642 & \text{Mean } M=159{\cdot}1857. \\ M_4=157{\cdot}1594 & \end{array}$$

$$\begin{aligned}\text{Mutual Inductance} &= n_1 n_2 M \\ &= 3775250.\end{aligned}$$

Rowland's value for the same pair of coils was

3775500.

24. In determining the constant of a current-balance of the type used by Lord Rayleigh and Mrs. Sidgwick in their determination of the electrochemical equivalent of silver, and of which the coils should preferably be single-shell ones, we should first determine the equivalent radii of the two large coils and of the suspended coil by the method I have explained above, which involves in the first instance determining the ratios of these equivalent radii.

It is possible also to realize an accuracy, if due care be taken, that is comparable with what can be obtained in measuring a straight line by the well-known methods.

We can now imagine the coils replaced by their equivalent filaments and calculate for each pair of attracting or repelling filaments the value of the parameter γ where

$$\sin^2\gamma=\frac{4r_1r_2}{(r_1+r_2)^2+x^2}\quad\text{as before.}$$

From Lord Rayleigh's Table* we get the value of

$$\log\left(\frac{\sqrt{r_1r_2}}{x}\frac{dM}{dx}\right)$$

corresponding to each value of γ and thence $\frac{dM}{dx}$ for the

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* Phil. Trans. 1884.

pair of filaments considered. The total force between the four mutually attracting or repelling pairs of filaments is

$=\frac{1}{4}n_1n_2C^2\times$ sum of the four values of $\frac{dM}{dx}$.

25. As an example I give the leading figures in the calculation of the constant of the current-balance used by Lord Rayleigh and Mrs. Sidgwick.

Particulars of the large coils :—

$$a_1=a_1'=24{\cdot}81016$$

$$\xi_1=1{\cdot}5,\quad \eta_1=1{\cdot}29.$$

Number of turns on each $=n_1=225$.

It is a thick double-shell coil and its equivalent radius r_1

$$\text{being}\quad =a_1\left(1+\frac{\eta^2}{24a_1^2}\right),$$

$$\text{is}\quad =24{\cdot}81295.$$

Its equivalent breadth 2β is given by

$$\beta_1^2=\frac{\xi_1^2-\eta_1^2}{12},$$

hence

$$\beta_1^2={\cdot}0490 \text{ and } \beta_1={\cdot}221.$$

The distance between the median planes of the two large coils was 25 cms.

Particulars of the suspended coil :—

$$a_2 \text{ approximately}=10{\cdot}25,$$

$$\xi_2=1{\cdot}3843,\quad \eta_2={\cdot}9690,$$

Number of turns $(n_2)=242$.

It is also a thick double-shell coil whose equivalent radius (r_2) will be determined by Bosscha's method and whose equivalent breadth $2\beta_2$ is given by

$$\beta_2^2=\frac{\xi_2^2-\eta_2^2}{12},$$

hence

$$\beta_2^2={\cdot}0814 \text{ and } \beta_2={\cdot}285.$$

It was found that when balance was obtained in Bosscha's comparison method with resistances R_1 and R_2 in the parallel arms containing either of the large coils and the small coil respectively, that the mean value of

$$\frac{\mathrm{R}_2}{\mathrm{R}_1} \text{ was} = 2{\cdot}60070.$$

Hence (see §§ 10 and 13) as the ratio of the equivalent radii

$$= \frac{r_1}{r_2} = \frac{n_1\mathrm{R}_2}{n_2\mathrm{R}_1}\left\{1 - \frac{3}{2}\left(\frac{\beta_1{}^2}{a_1{}^2} - \frac{\beta_2{}^2}{a_2{}^2}\right)\right\}$$

we get

$$\frac{r_1}{r_2} = \frac{225}{242} \times 2{\cdot}60070 \times 1{\cdot}001044,$$

or
$$r_2 = \frac{24{\cdot}81295}{2{\cdot}42053} = \mathbf{10{\cdot}25104}.$$

The force between the suspended and one of the fixed coils is now equal to the sum of the four forces between two coaxal circular filaments of radii 24·81295 and 10·25104 [each small filament carrying current $= \frac{1}{2}n_2\mathrm{C}$ and each large filament carrying current $\frac{1}{2}n_1\mathrm{C}$] at the four different distances

$$x + \beta_1 + \beta_2, \quad x - \beta_1 - \beta_2,$$
$$x + \beta_1 - \beta_2, \quad x - \beta_1 + \beta_2,$$

that is 13·006, 11·994, 12·436, 12·564.

The parameter γ for each of these pairs will be

58° 31′·74, 59° 23′·88, 59° 1′·29, 58° 54′·69,

which give respectively for $\frac{d\mathrm{M}}{dx}$ the values

1·032198, 1·055872, 1·046662, and 1·043648

whose mean value is

1·044595.

Lord Rayleigh's value for the same was

1·044627.

26. If, in the above calculation, I made allowance for the length of the small magnet used in comparing the equivalent radii of the coils, there would be a very appreciable difference in the result even if the needle were very short. I do not know

the length of the needle actually used by Lord Rayleigh [*], but if it was half a centimetre long or 4 mm. between its poles M would be

$$1{\cdot}045154 \quad \text{instead of}$$

$$1{\cdot}044595,$$

that is the correction would be 54 in 100,000. For other lengths the correction can be got as it varies as the square of the distance between the poles of the needle. Thus, if the pole distance were only 1 mm., the correction would be more than 3 parts in 100,000, and if it were 3 mm. (a fairly short needle) the correction would be 3 parts in 10,000, which would affect the measurement of current to the extent of $1\frac{1}{2}$ parts in 10,000.

27. In the case of a standard of mutual inductance of which the coils are very approximately single-shell ones whose equivalent radii have been accurately determined by the electrical method, it might be worth while taking account of the effect of change of temperature on the value of M.

If the distance-piece between the two coils be of the same material as that on which the coils are wound, and whose coefficient of linear expansion α is nearly equal to that of the copper wire in the coils, then at once

$$M_t = M_0(1+\alpha t),$$

seeing that M is a homogeneous function of r_1, r_2, and x of one dimension.

For a current-balance in which the two large coils are wound on the same frame, and the suspended coil wound on a ring of the same material whose coefficient of expansion is nearly equal to that of copper, the temperature-coefficient of its constant vanishes seeing that $\frac{dM}{dx}$ is a homogeneous function of $r_1 r_2$, and x of no dimensions.

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[*] In the discussion on the paper, when read before the Physical Society, Lord Rayleigh pointed out that the length of the needle was one-tenth of an inch, and the error due to the neglect of the length was less than one part in ten thousand.

XXXIII. *The Velocity of Light.* *By* A. A. MICHELSON[*].

THE fact that the velocity of light is so far beyond the conception of the human intellect, coupled with the extraordinary accuracy with which it may be measured, makes this determination one of the most fascinating problems that fall to the lot of the investigator.

The experiment of measuring the velocity of light was originally devised by Arago, as a means of furnishing a crucial test between the two great rival theories of light. History has recorded the triumph of the undulatory theory, as the result of the classic experiments of Foucault and Fizeau.

It was found in the course of these and subsequent experiments that the experimental determination of the absolute value (V) of the velocity of light was capable of so high a degree of accuracy that instead of obtaining V by observations of the eclipses of Jupiter's satellites, or the aberration of the fixed stars, together with the astronomically calculated value of the sun's distance (with results of relatively small accuracy)—the process is exactly reversed; so that the experimental value of V, together with observed values of the light equation, or the constant of aberration, furnishes a far more accurate value of the sun's distance than can be obtained by rare and costly expeditions for observing the transit of Venus. This appears quite clearly if we contrast the numbers given by Professor Harkness[†], which give the solar parallax as obtained from astronomical data[‡],

$$8{\cdot}78 \pm {\cdot}05,$$

with the results given by Professor Todd of the same quantity, as determined by the combination of velocity of light with the light equation or with the constant of aberration.

In an article on "Solar Parallax from the Velocity of Light"[§], Professor D. P. Todd gives the following *résumé* of the determinations of the solar parallax from L, the light equation, and from α, the constant of aberration:—

"The elements of sensible uncertainty considered are:

"(1) Uncertainty in the determination of terrestrial velocity of light. . . . I am disposed to think that the limit

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[*] Communicated by the Author.

[†] The numbers here given are not Professor Harkness's estimates, but only a rather rough guess from the data he presents, which I think are nevertheless of about the right order of magnitude.

[‡] Am. Journ. Sci. 1881.

[§] Am. Journ. Sci. vol. xix. p. 59.

of uncertainty of the velocity of light concluded above (299,920 kilometres) may be fairly taken at 70 kilometres.

"(2) Uncertainty in the coefficient of the light equation from observations of the satellites of Jupiter. . . . The amount of uncertainty is probably not far from one second of time*.

"(3) Uncertainty in the constant of sidereal aberration. I conceive that a variation of 0·025 in this well-determined constant† will not be regarded far from the limit of uncertainty. . . .

"(4) Uncertainty in the relation of the absolute terrestrial velocity to the velocity in space. . . . The impossibility of an experimental determination of this relation renders the assumption of identity necessary.

"In conclusion, then, all the experimental determinations of the velocity of light hitherto made give, when combined with astronomical constants, the mean equatorial horizontal parallax of the sun,

$$8''\cdot808 \pm 0\cdot006.$$

The corresponding mean radius of the terrestrial orbit is

149,345,000 kilometres."

Roughly speaking, it appears that the velocity-of-light method of obtaining the solar parallax is about ten times as accurate as the astronomical method.

It also appears that in the last-named method the order of accuracy in the determination of the astronomical factors is about one one-thousandth part.

The undulatory theory does not specify the nature of the undulations, and it is no argument against it that the special mode of motion assumed by Fresnel (vibrations of an elastic solid, which so elegantly explains all the known phenomena of light) has met with some serious objections. If, according to Maxwell, these undulations are electromagnetic changes, these objections no longer apply, and this electromagnetic theory of light is now universally accepted‡.

In accordance with Maxwell's theory, the ratio of any electric or magnetic unit measured electrostatically to the

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* This is taken from the observations of Delambre, 493·2, weight 1, and of Glasenapp, 500·84±1·02, weight 2.

† Professor Todd accepts Struve's value 2''·0445.

‡ While the electromagnetic theory gives a satisfactory explanation of light and goes a long way toward the explanation of the mechanism of radiation, yet it seems not at all unlikely that a return to something analogous to the now discarded elastic solid theory may be necessary to explain electromagnetic phenomena.

same unit measured electromagnetically, the ratio usually designated as Maxwell's v, must be equal to the velocity of light. The verification of this relation would be a powerful argument (were any such needed) in favour of the theory; and in fact the value of v agrees as well with V as do the separate accepted values of v among themselves.

M. Abraham, in a report to the International Congress of Physics, gives the following *résumé* of these determinations:

Himstedt	$3{\cdot}0057 \times 10^{10}$
Rosa	$3{\cdot}0000 \times 10^{10}$
J. J. Thomson	$2{\cdot}9960 \times 10^{10}$
H. Abraham	$2{\cdot}9913 \times 10^{10}$
Pellat	$3{\cdot}0092 \times 10^{10}$
Hurmuzescu	$3{\cdot}0010 \times 10^{10}$
Perot and Fabry	$2{\cdot}9973 \times 10^{10}$

M. Abraham accepts the mean value

$$v = 3{\cdot}0001 \times 10^{1}$$

which he regards as probably correct to the thousandth part, and concludes with the following significant remark:

"Etant donné l'intérêt qui s'attache à la détermination de la vitesse v, il paraît désirable que de nouvelles expériences soient entreprises. La précision des anciennes mesures peut être dépassés : toutes les méthodes s'y prêtent. Il y a encore à réduire quelques corrections trôp incertaines; il y a à simplifier quelques mesures auxiliaires trôp complexes, et par ce nouvel effort on pourra sans aucun doute, apporter dans la mesure de v une précision, supérieure à celle aujourd'hui acquise pour la vitesse de la lumière."

Again, Blondlot and Gutton, in the same report, give the following *résumé* of some of the best determinations of the velocity of Hertzian waves:

Blondlot	302,200	km./sec.
Blondlot	296,400	,, ,,
	298,000	,, ,,
Trowbridge and Duane	300,300	,, ,,
MacLean	299,110	,, ,,
Saunders	298,200	,, ,,
	299,700	,, ,,

Messrs. Blondlot and Gutton do not make any estimate of the most probable value, but doubtless the mean of the preceding results,

$$299{,}130 \pm 1000$$

cannot be far from the truth. The authors conclude as follows:

"Ils (ces valeurs) sont sensiblements égaux au rapport des unités électromagnétique et électrostatique de quantité d'électricité, comme la théorie de Maxwell l'indique, et aussi à la vitesse de la lumière. Il y a un intérêt capital a rechercher si cette égalite des trois nombres est seulement approximative, ou si elle offre un caractère absolu; on devra donc s'efforcer, dans les déterminations ultérieures de la vitesse de propagation des ondes électromagnétiques, non seulement d'atteindre la plus grande exactitude possible, mais aussi de déterminer le degré d'approximation des nombres obtenus."

It would appear, then, that there is but little doubt that in the near future both these determinations will be made with at least the same high order of accuracy as obtains in the measurement of the velocity of light. In this case, any possible difference in the resulting values would not cast any doubt upon the electromagnetic theory, but would doubtless be traceable to the enormous difference in the conditions determined by light-waves on the one hand, and electric oscillations or static charges on the other. On the contrary, such a difference might almost certainly be predicted, and would probably throw much light on the structure and mode of action of dielectrics.

Having reviewed in some detail the order of accuracy attained in the astronomical elements which are to be combined with V, and the electromagnetic results which are to be compared with V, let us consider the actual order of accuracy obtained in the measurement of V itself.

Concerning this, as in the cases just reviewed, a considerable diversity of opinion will prevail concerning the relative weight which should be attributed to the various determinations.

The following table is taken from Professor Newcomb's report:

(1) Foucault, 1862	298,000
(2) Cornu (1), 1874	298,500
(3) Cornu (2), 1878	300,400
(4) Cornu (2) as discussed by Listing	299,990
(5) Young and Forbes, 1880–81	301,382
(6) Michelson (1)	299,910
(7) Michelson (2)	299,853
(8) Newcomb (selected results)	299,860
(9) Newcomb (all observations)	299,810

There is no doubt that the weights which should be assigned to these results vary enormously. Still, in view of the limited number, and keeping in mind the criticism of M. Cornu, which will be presently referrred to, it may be of interest to find the mean of these nine values, giving them equal weight. The result is

$$V=299{,}664,$$

with an average difference from the mean of 600 kilometres.

But the results (1), (2), (6) are admitted to be only first approximations, and (5) is undoubtedly affected by some serious constant error; so that a much closer approximation to the true value will be obtained by combining (3) or (4), the mean of (6) and (7), and (9).

The first combination gives:

Cornu	300,400 ± 300
Michelson	299,882 ± 60
Newcomb	299,810 ± 60

Giving these equal weight we find

$$V=300{,}030,$$

with an average difference from the mean of about 250.

If, however, we take Listing's value of Cornu's results, we get:

Cornu	299,990
Michelson	299,882
Newcomb	299,810

Giving for the mean

$$V=299{,}890,$$

with an average difference from the mean of about 60.

In a paper presented to the International Congress in 1900, M. Cornu expresses serious doubts concerning the order of accuracy of the results obtained by means of the revolving mirror; and, giving equal weights to the two methods regardless of the number of determinations, and ignoring Listing's discussion of his own results, gives as his estimate of the most probable value of the velocity of light

$$V=300{,}130,$$

with average difference from the mean of 270.

The criticism of M. Cornu embraces the following points:

1. Can the ordinary laws of reflexion be applied to the case of a mirror in rapid rotation?

2. Can the ordinary laws of reflexion be applied to the case of a light-ray whose axis moves with a velocity comparable with the velocity of light?

3. Does the dragging along (*entrainement*) of the light waves in the air-vortex close to the mirror affect the displacement?

These criticisms have been carefully considered by H. A. Lorentz*, who states that "Quoiqu'elles (quelques considérations . . .) laissent encore beaucoup à désirer au point de vue de la rigueur, elles me semblent bien propres à dissiper les doutes que je viens de rappeler."

It seems to me that M. Lorentz has satisfactorily answered M. Cornu's questions; and my own estimate of the most probable value of V agrees with the second result given on p. 334, namely,

$$V = 299{,}890 \pm 60.$$

Nevertheless an experimental determination which should more or less completely avoid the necessity for the consideration of these questions, or at least materially diminish any error which could possibly arise, even if they were valid; and which, in any case, would furnish one more independent result to be combined with the very few reliable ones already obtained,—would surely be welcome.

The preceding exposition makes it clear that the measurements and observations to be combined with or to be compared with the velocity of light, namely:

(1) Astronomical observations of L,
(2) Electromagnetic ratio v,
(3) Velocity of Hertzian oscillations,

are all capable of measurement with an order of accuracy which even now approaches, if does not equal, that of the measurement of the velocity of light.

This, it seems to me, is a sufficient excuse—if any were needed—for once more attacking the problem.

The following plan suggested itself during the experiments upon the "Relative Motion of the Earth and the Luminiferous Æther"†.

The essential feature is the combination of a grating with a revolving mirror, which combination acts as a toothed wheel, the grating-space representing the distance between the teeth, the radius being the distance from the revolving mirror to grating.

It was proposed to utilize this combination in an attempt to solve the problem of the "relative motion" by measuring the velocity of light *in one direction*, that is, without returning the light to the source. This was before the celebrated work

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* "Sur la méthode du miroir tournant pour la détermination de la vitesse de la lumière." Arch. Néer. des Sciences, sér. ii. t. vi. p. 303.

† Am. Journ. of Science, vol. xxxiv. Nov. 1887.

of Hertz showed that the electrical impulses (which were to be used to establish the required phase relation between the two revolving mirrors) would be affected in the same way as would the light-waves themselves.

(It may also be noted in this connexion that the method proposed in the same article, and illustrated by fig. 1, is also not sound.)

It may also be worth mentioning that some preliminary experiments, made about two years ago, have shown that it is not entirely impossible to employ a mechanical method of keeping the two revolving mirrors in a constant phase relation. For instance, it was shown that the vibrations of a tuning-fork could be transmitted over a mile of piano-wire with a diminution of amplitude of less than one half.

The plan proposed is virtually a combination of the methods of Foucault and Fizeau. The essential feature of the observation of eclipses corresponding to the latter method, while the production of the eclipses is brought about by a revolving beam of light as in the former method.

Figures 1 & 2, in which the lettering is the same, will illustrate the essential features, subject to such minor modifications as experiment may suggest.

Fig. 1.

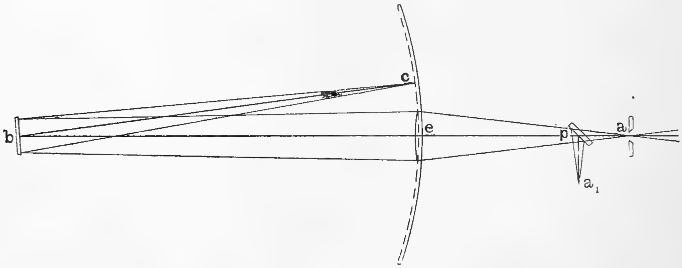
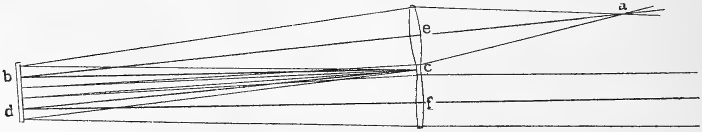


Fig. 2.



The light starts from a slit at *a*, passes through a lightly silvered glass plate P and a lens *e*, and falls upon the upper half of the revolving mirror *b*. Thence it proceeds to the grating *c*, upon the surface of which it forms an image of the slit. It is thence reflected to the lower half *d* of the revolving mirror, which reflects the beam through the lens f to the distant mirror upon the surface of which the second image of the slit is formed. The light then retraces its course and returns to its source at a, part being reflected to a_1 for convenience of observation by the eyepiece.

The limit of closeness of the grating-space is determined by the aperture of the revolving mirror viewed from c.

If the "radius" bc is 3 metres and the revolving mirror is 6 cm. wide, this angle will be 0·02 ; and the breadth of the diffraction image at c will be of the order $\lambda/{\cdot}02$, or, say, 1/40 mm. The grating-space should therefore be at least 1/20 mm., and probably better 0·1 mm. If the number of revolutions is 250, and the distance to the fixed mirror be 3 kilometres, the displacement of the first image over the grating surface will be 18 cm., corresponding to 1800 eclipses. There need be no difficulty in counting the order of the eclipse observed, if the speed is gradually increased to its final value. The fractions could probably be observed correctly to something like 2 per cent., so that this element of the computation for V could be measured to something like one part in a hundred thousand.

This same or even a higher order of accuracy may be obtained in the measurement of all the other elements. Previous experiments have shown that the speed of the mirror may be obtained, by means of a rated tuning-fork, to within one in one hundred thousand.

The measurement of the distance may be made directly on a base line especially prepared for such work to within one in two hundred thousand.

Finally, the grating may be calibrated to an order of accuracy depending on the angle subtended by the diffraction-fringes, *i. e.*, $1/40 \times 1/3000$, or less than one in 100,000.

It seems not unreasonable to hope that with proper care and patience the value of this great fundamental constant of Nature may be found to within five kilometres or less.

It will be noted that in the form of experiment here proposed, the more serious of the difficulties pointed out by M. Cornu no longer exist, or are very much diminished; while the possible accuracy is greatly increased. It may be hoped, therefore, that the result of this combination of the methods of Foucault and Fizeau will be to reconcile the differences which thus far seem to exist between the results of the work of their respective followers.

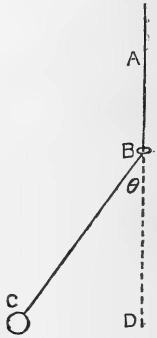
XXXIV. *On the Pressure of Vibrations.*

By Lord Rayleigh, *F.R.S.*[*]

THE importance of the consequences deduced by Boltzmann and W. Wien from the doctrine of the pressure of radiation has naturally drawn increased attention to this subject. That æthereal vibrations must exercise a pressure upon a perfectly conducting, and therefore perfectly reflecting, boundary was Maxwell's deduction from his general equations of the electromagnetic field; and the existence of the pressure of light has lately been confirmed experimentally by Lebedew. It seemed to me that it would be of interest to inquire whether other kinds of vibration exercise a pressure, and if possible to frame a general theory of the action.

We are at once confronted with a difference between the conditions to be dealt with in the case of æthereal vibrations and, for example, the vibrations of air. When a plate of polished silver advances against waves of light, the waves indeed are reflected, but the medium itself must be supposed capable of penetrating the plate; whereas in the corresponding case of aerial vibrations the air as well as the vibrations are compressed by the advancing wall. In other cases, however, a closer parallelism may be established. Thus the transverse vibrations of a stretched string, or wire, may be supposed to be limited by a small ring constrained to remain upon the equilibrium line of the string, but capable of sliding freely upon it. In this arrangement the string passes but the vibrations are compressed, when the ring moves inwards.

Fig. 1.



We will commence with the very simple problem of a pendulum in which a mass C is suspended by a string. B is a ring constrained to the vertical line AD and capable of moving along it; BC$=l$, and θ denotes the angle between BC and AD at any time t. If B is held at rest, BC is an ordinary pendulum, and it is supposed to be executing small vibrations; so that $\theta=\Theta\cos nt$, where $n^2=g/l$. The tension of the string is approximately W, the weight of the bob; and the force tending to push B upwards is at time t W$(1-\cos\theta)$. Now this expression is closely related to the potential energy of the pendulum, for which

$$V=Wl(1-\cos\theta).$$

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[*] Communicated by the Author.

The mean upward force upon B is accordingly equal to the mean value of $V \div l$; or since the mean value of V is half the constant total energy E of the system, we conclude that the mean force (L), driving B upwards, is measured by $\frac{1}{2} E/l$.

From the equation

$$L = \tfrac{1}{2} E/l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

it is easy to deduce the effect of a *slow* motion upwards of the ring. The work obtained at B must be at the expense of the energy of the system, so that

$$dE = -L\,dl = -\tfrac{1}{2} E\,dl/l.$$

By integration

$$E = E_1 l^{-\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where E_1 denotes the energy corresponding to $l=1$. From (2) we see that by withdrawing the ring B until l is infinitely great, the whole of the energy of vibration may be abstracted in the form of work done by B, and this by a uniform motion in which no regard is paid to the momentary phase of the vibration.

The argument is nearly the same for the case of a stretched string vibrating transversely in one plane. The string itself may be supposed to be unlimited, while the vibrations are confined by two rings of which one may be fixed and one movable.

If the origin of x be at one end of a string of length l, the transverse displacement may be expressed by

$$y = \phi_1 \frac{\sin \pi x}{l} + \phi_2 \sin \frac{2\pi x}{l} + \ldots, \quad . \quad . \quad . \quad (3)$$

$$\dot{y} = \dot{\phi}_1 \frac{\sin \pi x}{l} + \dot{\phi}_2 \sin \frac{2\pi x}{l} + \ldots, \quad . \quad . \quad . \quad (4)$$

where $\phi_1, \phi_2, \ldots$ are coefficients depending upon the time. For the kinetic and potential energies we have respectively ('Theory of Sound,' § 128)

$$T = \tfrac{1}{4}\rho l \sum_{s=1}^{s=\infty} \dot{\phi}_s^{\,2}, \qquad V = \tfrac{1}{4} W l \sum_{s=1}^{s=\infty} \frac{s^2\pi^2}{l^2} \phi_s^{\,2}, \quad . \quad . \quad (5)$$

in which W represents the constant tension and ρ the longitudinal density of the string. For each kind of ϕ the sums of T and V remain constant during the vibration; and the same is of course true of the totals given in (5).

From (3)

$$\frac{dy}{dx} = \frac{\pi}{l}\left(\phi_1 \cos \frac{\pi x}{l} + 2\phi_2 \cos \frac{2\pi x}{l} + \ldots\right),$$

so that when $x=l$

$$\frac{dy}{dx}=\frac{\pi}{l}(-\phi_1+2\phi_2-3\phi_3+\ldots).$$

Accordingly the force tending to drive out the ring at $x=l$ is at time t

$$\tfrac{1}{2}\mathrm{W}\,.\,\frac{\pi^2}{l^2}(-\phi_1+2\phi_2-3\phi_3+\ldots)^2,$$

or in the mean taken over a long interval,

$$\tfrac{1}{2}\mathrm{W}\,.\,\mathrm{Mean}\;\Sigma\frac{s^2\pi^2}{l^2}\phi_s^{\,2}.$$

Comparing with (5), we see that the mean force L has the value $2l\times$ mean V; or since mean V = mean T = $\frac{1}{2}$E, E denoting the constant total energy,

$$\mathrm{L}=\mathrm{E}/l. \qquad (6)$$

The force driving out the ring is thus numerically equal to the *longitudinal density of the energy.*

This result may readily be extended to cases where the vibrations are not limited to one plane; and indeed the case in which the plane of the string uniformly revolves is especially simple in that T and V are then constant with respect to time.

If the ring is allowed to move out slowly, we have

$$d\mathrm{E}=-\mathrm{L}\,dl=-\mathrm{E}\,dl/l,$$

or on integration

$$\mathrm{E}=\mathrm{E}_1 l^{-1}, \qquad (7)$$

analogous to (5), though different from it in the power of l involved. If l increase without limit, the whole energy of the vibrations may be abstracted in the form of work done on the ring.

We will now pass on to consider the case of air in a cylinder, vibrating in one dimension and supposed to obey Boyle's law according to which $p=a^2\rho$. By the general hydrodynamical equation ('Theory of Sound,' § 253 *a*),

$$\varpi=\int\frac{dp}{\rho}=-\frac{d\phi}{dt}-\tfrac{1}{2}\mathrm{U}^2, \qquad (8)$$

where ϕ denotes the velocity-potential and U the resultant velocity at any point; so that in the present case, if we integrate over a long interval of time,

$$a^2\int\log p\,dt+\tfrac{1}{2}\int\mathrm{U}^2\,dt \qquad (9)$$

retains a constant value over the length of the cylinder. If p_0 denote the pressure that would prevail throughout, had there been no vibrations, $p-p_0$ is small and we may replace (9) by

$$a^2\int\left\{\frac{p-p_0}{p_0}-\tfrac{1}{2}\frac{(p-p_0)^2}{p_0^{\,2}}\right\}dt+\tfrac{1}{2}\int \mathrm{U}^2\,dt\ .\ \ .\ \ .\quad (10)$$

The expression (10) has accordingly the same value at the piston as for the mean of the whole column of length l. Now for the mean of the whole column

$$\int(p-p_0)\,dx=0;$$

and thus if p_1 denote the value of p at the piston where $x=l$,

$$a^2\int\left\{\frac{p_1-p_0}{p_0}-\tfrac{1}{2}\frac{(p_1-p_0)^2}{p_0^{\,2}}\right\}dt$$

$$=-\frac{a^2}{2l}\iint\frac{(p-p_0)^2}{p_0^{\,2}}\,dx\,dt+\frac{1}{2l}\iint \mathrm{U}^2\,dx\,dt\ .\ \ .\quad (11)$$

It is not difficult to prove that the right-hand member of (11) vanishes. Thus, expressing the motion in terms of ϕ, suppose that

$$\phi=\cos\frac{s\pi x}{l}\cos\frac{s\pi at}{l}\ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\quad (12)$$

Then

$$p-p_0=\rho_0\,d\phi/dt,\qquad \mathrm{U}=d\phi/dx;$$

and since $p_0=a^2\rho_0$, we get

$$\frac{1}{2l}\iint\left\{\left(\frac{d\phi}{dx}\right)^2-\frac{1}{a^2}\left(\frac{d\phi}{dt}\right)^2\right\}dx\,dt,$$

and this vanishes by (12). Accordingly

$$\int(p_1-p_0)\,dt=\int\frac{(p_1-p_0)^2}{2p_0}\,dt\ .\ \ .\ \ .\ \ .\ \ .\quad (13)$$

Again by (12)

$$\int\left(\frac{d\phi}{dt}\right)_l^2 dt=\frac{2}{l}\iint\left(\frac{d\phi}{dt}\right)^2 dx\,dt,$$

so that

$$\int(p_1-p_0)\,dt=\frac{1}{p_0 l}\iint(p_1-p_0)^2\,dx\,dt\ =\ \frac{\rho_0}{l}\iint \mathrm{U}^2\,dx\,dt.$$

Now $\rho_0\iint \mathrm{U}^2\,dx\,dt$ represents twice the mean total kinetic energy of the vibrations or, what is the same, the constant total energy E. Thus if L denote the mean additional force due to the vibrations and tending to push the piston out,

$$L = E\,l^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

As in the case of the string, the total force is measured by the longitudinal density of the total energy; or, if we prefer so to express it, the additional *pressure* is measured by the volume-density of the energy.

In the last problem, as well as in that of the string, the vibrations are in one dimension. In the case of air there is no difficulty in the extension to two or three dimensions. Thus, if aerial vibrations be distributed equally in all directions, the pressure due to them coincides with *one-third* of the volume-density of the energy. In the case of the string, where the vibrations are transverse, we cannot find an analogue in three dimensions; but a membrane with a flexible and extensible boundary capable of slipping along the surface, provides for two dimensions. If the vibrations be equally distributed in the plane, the force outwards per unit length of contour will be measured by one-half of the superficial density of the total energy.

A more general treatment of the question may be effected by means of Lagrange's theory. If l be one of the coordinates fixing the configuration of a system, the corresponding equation is

$$\frac{d}{dt}\left(\frac{dT}{dl'}\right) - \frac{dT}{dl} + \frac{dV}{dl} = L, \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

where T and V denote as usual the expressions for the kinetic and potential energies. On integration over a time t_1

$$\int \frac{L\,dt}{t_1} = \frac{1}{t_1}\left[\frac{dT}{dl'}\right] + \frac{1}{t_1}\int\left(\frac{dV}{dl} - \frac{dT}{dl}\right)dt.$$

If dT/dl' remain finite throughout, and if the range of integration be sufficiently extended, the integrated term disappears, and we get

$$\int \frac{L\,dt}{t_1} = \frac{1}{t_1}\int\left(\frac{dV}{dt} - \frac{dT}{dl}\right)dt. \quad . \quad . \quad . \qquad (16)$$

On the right hand of (16) the differentiations are partial, the coordinates other than l and all the velocities being supposed constant.

We will apply our equation (16) in the first place to the simple pendulum of fig. 1, l denoting the length of the vibrating portion of the string BC. If x, y be the horizontal and vertical coordinates of C,

$$x=l\sin\theta,\qquad y=l-l\cos\theta;$$

and accordingly if the mass of C be taken to be unity,

$$\mathrm{T}=\tfrac{1}{2}l'^2(2-2\cos\theta)+l'\theta'\,.\,l\sin\theta+\tfrac{1}{2}\theta'^2l^2, \quad . \quad . \quad (17)$$

l', θ' denoting dl/dt, $d\theta/dt$. Also

$$\mathrm{V}=gl\,(1-\cos\theta). \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

From (17), (18)

$$\frac{d\mathrm{V}}{dl}=g(1-\cos\theta),\qquad \frac{d\mathrm{T}}{dl}=l'\theta'\sin\theta+\theta'^2l. \quad . \quad . \quad (19)$$

These expressions are general; but for our present purpose it will suffice if we suppose that l' is zero, that is that the ring is held at rest. Accordingly

$$\frac{d\mathrm{V}}{dl}=\frac{\mathrm{V}}{l},\qquad \frac{d\mathrm{T}}{dl}=\frac{2\mathrm{T}}{l},$$

and (16) gives

$$\int\frac{\mathrm{L}\,dt}{t_1}=\frac{1}{t_1}\int\frac{\mathrm{V}-2\mathrm{T}}{l}\,dt. \quad . \quad . \quad . \quad . \quad (20)$$

On the right hand of (20) we find the mean values of V and and of T. But these mean values are equal. In fact

$$\int\mathrm{V}dt=\int\mathrm{T}\,dt=\tfrac{1}{2}\mathrm{E}t_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

if E denote the total energy. Hence, if L now denote the mean value,

$$\mathrm{L}=-\tfrac{1}{2}\mathrm{E}/l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

the negative sign denoting that the mean force necessary to hold the ring at rest must be applied in the direction which tends to diminish l, i. e. downwards. In former equations (1), (6), (14), L had the reverse sign.

We will now consider more generally the case of one dimension, using a method that will apply equally whether for example the vibrating body be a stretched string, or a rod vibrating flexurally. All that we postulate is homogeneity of constitution, so that what can be said about any part of the length can be said equally about any other part. In applying Lagrange's method the coordinates are l the length of the vibrating portion, and ϕ_1, ϕ_2, &c. defining, as in (3), the displacement from equilibrium during the vibrations. As functions of l, we suppose that

$$\mathrm{V}\propto l^m,\qquad \mathrm{T}\propto l^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Thus, if L be the force corresponding to l, we get by (16)

$$\int \frac{\mathrm{L}\,dt}{t_1} = \frac{1}{t_1}\int\left(\frac{m\mathrm{V}}{l} - \frac{n\mathrm{T}}{l}\right)dt,$$

in which

$$\int \mathrm{V}\,dt = \int \mathrm{T}\,dt = \tfrac{1}{2}\mathrm{E}\,.\,t_1,$$

E representing as before the constant total energy. Accordingly, L now representing the mean value,

$$\mathrm{L} = \frac{(m-n)\mathrm{E}}{2l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

In the case of a medium, like a stretched string, propagating waves of all lengths with the same velocity, $m=-1$, $n=1$, and $\mathrm{L}=-\mathrm{E}/l$, as was found before.

In the application to a rod vibrating flexurally, $m=-3$, $n=1$, so that

$$\mathrm{L} = -2\mathrm{E}/l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

If $m=n$, L vanishes. This occurs in the case of the line of disconnected pendulums considered by Reynolds in illustration of the theory of the group velocity *, and the circumstance suggests that L represents the tendency of a group of waves to spread. This conjecture is easily verified. If in conformity with (13) we suppose that

$$\mathrm{V} = \mathrm{V}_0\, l^m \phi_1^2, \qquad \mathrm{T} = \mathrm{T}_0\, l^n \dot{\phi}_1^2,$$

and also that

$$\phi_1 = \sin\frac{2\pi t}{\tau}, \qquad \dot{\phi}_1 = \frac{2\pi}{\tau}\cos\frac{2\pi t}{\tau},$$

τ being the period of the vibration represented by the coordinate ϕ_1, we obtain, remembering that the sum of T and V must remain constant,

$$\mathrm{V}_0\, l^m = \mathrm{T}_0\, l^n\,.\,4\pi/\tau^2.$$

This gives the relation between τ and l. Now v, the wave-velocity, is proportional to l/τ; so that

$$v \propto l^{1-\frac{1}{2}n+\frac{1}{2}m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

Thus, if u denote the group-velocity, we have by the general theory

$$u/v = \tfrac{1}{2}n - \tfrac{1}{2}m\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

and in terms of u and v by (24)

$$\mathrm{L} = -\frac{u\mathrm{E}}{vl}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

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* See Proc. Math. Soc. ix. p. 21 (1877); Scientific Papers, i. p. 322. Also Theory of Sound, vol. i. Appendix.

Boltzmann's theory is founded upon the application of Carnot's cycle to the radiation inclosed within movable reflecting walls. If the pressure (p) of a body be regarded as a function of the volume v *, and the absolute temperature θ, the general equation deduced from the second law of thermodynamics is

$$\frac{dp}{d\log\theta}=\text{M}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where M dv represents the heat that must be communicated while the volume alters by dv and $d\theta=0$. In the application of (29) to radiation we have evidently

$$\text{M}=\text{U}+p, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where U denotes the density of the energy—a function of θ only. Hence †

$$\frac{dp}{d\log\theta}=\text{U}+p. \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

If further, as for radiation and for aerial vibrations,

$$p=\tfrac{1}{3}\text{U}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

it follows at once that

$$d\log\text{U}=4\,d\log\theta.$$

whence

$$\text{U}\propto\theta^4, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

the well-known law of Stefan. It may be observed that the existence of a pressure is demanded by (31), independently of (32).

If we generalize (32) by taking

$$p=\frac{1}{n}\text{U}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where n is some numerical quantity, we obtain as the generalization of (33)

$$\text{U}\propto\theta^{n+1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

It is an interesting question whether any analogue of the second law of thermodynamics can be found in the general theory of the pressure of vibrations, whether for example the energy of the vibrations of a stretched string is partially unavailable in the absence of appliances for distinguishing *phases*. It might appear at first sight that the conclusion already given, as to the possibility of recovering the whole energy by mere retreat of the inclosing ring, was a proof to

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* Now with an altered meaning.

† Compare Lorentz, Amsterdam Proceedings, Ap. 1901.

the contrary. This argument, however, will not appear conclusive, if we remember that a like proposition is true for the energy of a gas confined adiabatically under a piston. The residual energy of the molecules may be made as small as we please, but the completion of the cycle by pushing the piston back will restore the molecular energy unless we can first abolish the infinitesimal residue remaining after expansion, and this can only be done with the aid of a body at the absolute zero of temperature. It would appear that we may find an analogue for temperature, so far as the vibrations of *one* system are concerned; but, so far as I can see, the analogy breaks down when we attempt a general theory.

XXXV. *On the Physical Peculiarities of Solutions of Gases in Liquids.* *By* J. Alfred Wanklyn, *Corresponding Member of the Royal Bavarian Academy of Sciences*[*].

THOSE who are acquainted with Bunsen's methods of manipulating gases, and especially those chemists who have enjoyed the privilege of Bunsen's personal instruction, will be familiar with Bunsen's zealous care in so using his "Absorptiometer" that every drop of the liquid solvent should come into repeated contact with the gas to be dissolved.

Bunsen's arrangements presuppose that only those particles of the solvent brought into actual contact with the gas become charged with the gas, and that actual passage of the gas from one stratum of liquid to an adjacent stratum either does not take place at all, or is exceedingly slow.

The question may be fairly asked:—Given a quantity of gas in contact with the surface of pure water in a state of absolute quiescence, will that gas penetrate below the surface of the water except with a degree of slowness calling to mind the slow passage of the less diffusive salts which do not traverse a space of 100 millimetres in a fortnight? An experiment on the action of carbonic acid has been made in my laboratory, which I will now describe.

First of all there was the very simple observation that carbonic acid, confined in a tube over mercury, is capable of being absorbed by distilled water kept at rest, that is to say, without being subjected to the shaking up which is usually resorted to in order to bring about such absorptions. The rate of absorption was also noted, and found to be about one

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[*] Communicated by the Author.

cubic centimetre per hour, the absorption-tube being of such a diameter that a length of 3·75 millimetres corresponded to a capacity of one cubic centimetre.

Of course, after the lapse of time, the water in the tube would become more or less charged with the gas, and then the rate of absorption would diminish.

This observation would seem to have disposed of the question; but the experiment about to be described puts a different face forward.

A graduated tube in the mercurial trough was charged with a given volume of carbonic acid superposed over a known volume of water saturated with carbonic acid: and the whole arrangement was left at rest for two days, and then a careful reading was performed. The volume of the CO_2 being found to be 52·39 c. c. at 0° centigrade and 760 mm. pressure (dry).

The volume of the layer of water saturated with CO_2 was 19 c. c.

By means of a very delicate pipette with suitable delivery tube, 2 c. c. of strong solution of caustic potash (25 grammes KHO in 100 c. c. of the solution) was passed up through the mercury into the tube, without disturbing the 19 c. c. of water saturated with CO_2.

Under these conditions, the natural result was looked for that the CO_2 in the water would have passed into the potash-solution, and that the water would have taken up the CO_2 which rested upon the water. Nothing of the kind came to pass. The volume of the CO_2 in the tube remained almost absolutely constant for 48 hours; the first reading (20 hours after the addition of the potash) being 52·05 c. c.: the second reading (about 44 hours after the addition of the potash) being 52·19 c. c.: the third reading 51·33 c. c. about 70 hours after the potash.

Of course it will be understood that after the lapse of 70 hours a little of the caustic potash on the bottom layer would have had time to reach the surface of the aqueous liquid by liquid diffusion, as was shown in Graham's memorable investigation of forty years ago.

The experiment was brought to a conclusion by shaking up the contents of the tube—whereupon the potash was enabled to absorb the CO_2, which consequently disappeared in the course of a few minutes, leaving behind a very small residue of atmospheric air.

The author considers that this experiment with potash establishes the fact that isolated molecules of CO_2 do not travel amongst the molecules of H_2O, but that the single molecule CO_2 attaches itself to many of the molecules of H_2O, and forms a big compound molecule.

It has been shown by direct experiments, several years ago, that a solution of carbonic acid in water has a greater specific gravity than pure water. When, therefore, CO_2 comes into contact with the upper surface of a column of water, a liquid is produced which is specifically heavier than pure water; and the heavier liquid sinks, and so a current is set up, and in that manner water may become saturated with carbonic acid without any shaking up.

Carbonic acid is more than thirty times as soluble as oxygen in water; and the penetration of oxygen into water kept at rest must be very much slower than the penetration of carbonic acid into water. There can be no doubt that a layer of still water of four inches in thickness would protect organic matter from the oxidizing action of the oxygen of the atmosphere for more than a month—a fact which has an important bearing on the sanitary questions of the present time.

The Laboratory, New Malden, Surrey,
5th February, 1902.

XXXVI. *Arrangement of Bands in the First Group of the Positive Band-Spectrum of Nitrogen.* By CLIVE CUTHBERTSON[*].

I SHALL attempt to show that the bands which make up the first group ($\lambda=6622$ to $\lambda=5000$) in the positive band-spectrum of nitrogen can be arranged in series which conform to a law similar to that given by Deslandres for the second group, which lies between $\lambda=5000$ and $\lambda=2820$.

The results of Deslandres are published in the *Comptes Rendus*, vols. ciii., civ., and cvi. In the first of these papers he states that the bright lines in the band $\lambda=391$ of the negative band-spectrum of nitrogen are distributed according to the law that "the intervals between one line and the next are approximately in arithmetical progression"; and he adds that he has verified this law in the case of several banded spectra. In the second paper he announces that the same law applies to the frequencies of the heads of the members of a group of bands in a certain number of spectra. This was first discovered by him in the case of the second group of

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[*] Communicated by the Author.

bands of nitrogen ($\lambda=500$ to $\lambda=280$). The following are his figures :—

N. observés (Series I.).	N. observés (Ser. II.).	N. observés (Ser. III.).	N. observés (Ser. IV.).	N. calculés (Ser. IV.).	N. observés (Ser. V.).	Intervalles calculés.
...	...	...	183251	183241	201045	
						14136
...	...	...	197360	197377	215179	
						14428
...	...	193162	211792	211805	229578	
						14719
...	188372	207770	226517	226524	244279	
						15010
...	203388	222787	241518	241526	259316	
						15302
198776	218685	238080	256837	256828	274664	
						15593
214352	234256	253714	272444	272430	290247	
						15885
230247	250117	269587	288337	288315	306077	
						16176
246374	266350	285791	304492	304492	322247	
						16468
262872	282831	302230	320946	320961	338707	
						16759
279636	299569	318990	337723	337721	355479	
						17051
296664	316628	336020	354772	354773	...	

The measurements are partly by Hasselberg, partly by Lecoq de Boisbaudran, and partly by Deslandres himself. The results are sufficiently good to justify his claim to have established the law, though in many instances the intervals depart widely enough from the figures given in column 7. Deslandres states that he has verified this law on the first positive group of nitrogen (λ 700 to λ 500) "with at least three series," and on other spectra. In 1890 Prof. J. S. Ames[*] published a fresh set of measurements of the wave-numbers of the middle line in the head of the bands of the second group of the spectrum of nitrogen, and added an amended table of the series picked out by Deslandres. The following are his figures, with the addition of the first differences between the wave-numbers (p. 350).

The regularity of the first differences seems to prove beyond reasonable doubt that we have here the expression of a law which governs the bands of the second group.

So far as I am aware, however, the bands of the first group have never been reduced to order in the same way.

[*] Phil. Mag. [5] vol. xxx. p. 48.

Prof. J. S. Ames's arrangement of the Second Group.

I.	II.	III.	IV.	V.
				2010·05
				141·23
				2151·28
				144·25
			2117·48	2295·53
			147·17	147·07
		2077·32	2264·65	2442·60
		150·05	150·11	150·16
	2033·72	2227·37	2414·76	2592·76
	152·69	153·10	153·13	152·98
	2186·41	2380·47	2567·89	2745·74
	155·98	155·96	155·90	156·01
2143·00	2342·39	2536·43	2723·79	2901·75
158·81	158·86	158·88	158·82	158·69
2301·81	2501·25	2695·31	2882·61	3060·44
161·85	161·76	161·71	161·71	161·42
2463·66	2663·01	2857·02	3044·32	3221·86
164·56	164·64	164·68	164·51	164·52
2628·22	2827·65	3021·70	3208·83	3386·38
167·54	167·62	167·38	167·38	167·09
2795·76	2995·27	3189·08	3376·21	3553·47
170·54	170·39	170·34	170·27	
2966·30	3165·66	3359·42	3546·48	

This first group consists of about 36 bands, the first lines of which respectively have the following frequencies given to them by Dr. Hasselberg (Watts, 'Index of Spectra,' p. 213).

15096		17080		18726	
	182		148		116
15278		17228		18842	
	180		147		113
15458		17375		18955	
	179		144		112
15637		17519		19067	
	178		146		111
15815		17665		19178	
	176		143		108
15991		17808		19286	
	174		143		109
16165		17951		19395	
	154		142		107
16319		* 18093		19502	
	155		139		105
16474		* 18232		19607	
	153		141		118
16627		18373		19725	
	152		118		
16779		18491			
	152		120		
16931		18611			
	149		115		

* Ångström and Thalén selected 18129 and 18251 as heads of bands in place of these two lines. But there is overlapping here, and the lines given above make the most regular series.

The differences between these numbers form three rough arithmetical progressions, which are shown divided by lines; and these I believe to be the three series of which Deslandres speaks. But the divergencies from strict regularity are too large to be accidental, and the breadth of the bands is only about one-tenth of that of the bands of the second group. It seemed, however, improbable that the law which is so abundantly evident in the second group should not be represented by some similar rule in the adjoining portion of the spectrum.

I searched, therefore, for such a law, and found that the frequencies of the heads of bands could be arranged in the following order on the model of Deslandres' series (see p. 352).

The figures, thus distributed, fall into thirteen series, in some of which only two members are present, and the first differences decrease in arithmetical progression from 153·1 to 112·9.

This arrangement differs so much from that of the second group made by Deslandres that its reality may appear open to doubt. The number of bands is so great, and the differences between the frequencies of the heads of consecutive bands so regular, that it might be surmised that almost any law could be "fudged" out of the figures. But the following reasons tend to show that this arrangement is a true one, and not accidental:—

1. It accounts for every one of the bands mapped by Ångström and Thalén, and all, with two exceptions (16319 and 16474) conform to the law with very considerable accuracy. Only one (18373) is used twice.

2. In addition, it accounts for many of the subsidiary strong lines interspersed in some bands, and, in particular, for the overlapping which occurs at about $\frac{1}{\lambda}=18000$ (*t*, *u*, and *v* of Ångström and Thalén). Thus, the strong lines at 18093, 18232, and 19782 become the heads of bands, and the curious line at 19872 is accounted for. So also the line at 21225 is seen to be the head of a band in the Series IV.

3. The distribution was arrived at by tabulating differences between the frequencies of every pair of bands in the group. On examining the table it was seen that the numbers lying in a certain line transversely across the page were connected in the manner shown above. No other series, so far as I can discover, can be picked out of the table so as to give anything like the same regularity.

4. The series die out after the thirteenth. And it is remarkable that the last member of that series, 19502, is consecutive to the last member of the first series, 19607. The chances against this occurring in an accidental arrangement are high.

Hasselberg's Measurements*.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.	XIII.
15096												
153·1												
16627	15278											
150·2	150·1											
18129	16779	15458										
147·8	147·2	147·3										
19607	18251	16931	15637									
		144·2	144·3									
		18373	17080	15815								
		140·9	141·1	141·3								
		19782	18491	17228	15991							
		138·6	138·2	138·3	138·4							
		21168	19872	18611	17375	16165						
			135·3		135·1	135·4						
			21225 ?		18726	17519	16319					
						132·3	134·6					
						18842	17665	16474				
							129	133·4				
							18955	17808				
								125·9				
								19067	17951			
									122·7			
									19178	18093		
										119·3		
										19286	18232	
											116·3	
											19395	18373
												112·9
												19502

* In this table the decimal point is shown in the differences, so as to facilitate comparison with the figures in Prof. Ames's table.

5. The arithmetical series which connects the frequencies in the first group of bands is not merely similar to, but identical with, that found by Deslandres for the second group, and it begins almost exactly where the other leaves off.

If these facts are sufficient to show that the arrangement given above is a real one, there are several points of interest to be noticed in it.

(1) The true breadth of a band seems to be the interval of frequency between the strongest line in the head and the corresponding line in the head of the next consecutive band *of that series* (not the next numerically consecutive band, *e.g.* 15096 to 16627, and not 15096 to 15278); for it is only by classifying the bands on this system that we discover the law of their distribution.

(2) As a rule the visible portion of the band is confined to the first eighth or tenth of its breadth.

(3) The breadth of the bands in each series of the first group decreases as the frequency increases, while in the second group, of which the law was found by Deslandres, the breadth increases as the frequency increases. This fact, as interesting as it was unexpected, seems to offer a clue to the cause of the banded spectrum.

(4) The arrangement of the bright lines in a band of the first group differs slightly from that of those in a band of the second group. In the former the lines are comparatively few and the three lines forming the head are widely separated. In the latter the lines are numerous and run in triplets, and the three lines forming the head are close together. As the bands also differ in the circumstance mentioned above (3), it is possible that these two results are both due to the same cause.

(5) The bands in each series seem to be related like the notes in a scale, while the series may be compared to the same scale in different keys.

XXXVII. *On the Effect of a Transverse Magnetic Field on Metallic Resistance.*

Cavendish Laboratory, Cambridge,
February 12, 1902.

To the Editors of the Philosophical Magazine.

GENTLEMEN,

IN 'Communications from the Physical Laboratory of Leiden,' No. 72, Dr. E. van Everdingen gives an investigation of the effect of magnetic force on the resistance of metals, and comes to the conclusion that the theory that the electric current is carried by charged particles which move freely through the metal requires the resistance to be *diminished* by a transverse magnetic force and not increased.

In a Report presented to the International Congress of Physics at Paris, 1900, I had previously come to the opposite conclusion, namely, that the resistance on this theory ought to be increased. I wish in this letter to give the reasons for adhering to my original result. Dr. van Everdingen's results are based on the assumption that the corpuscles which carry the current behave like a perfect gas ; *that the collisions which the corpuscles make with the molecules through which they move are similar to those which take place between hard elastic bodies*; and thirdly, *that the corpuscle between two collisions is free from any force except that due to the external field producing the current.*

The second and third of these assumptions seem to me extremely unlikely ; but before discussing them I will explain how they are required in the calculation given by Dr. van Everdingen. In the paper alluded to I showed that if X is the external electric force, H the magnetic force, e the charge, m the mass of a corpuscle, U the velocity of the corpuscle parallel to x at the beginning of the free path, t the free time, *i. e.* the time between two collisions, then the mean velocity of an ion parallel to the axis of x during the free time is

$$\frac{1}{2}\frac{\mathrm{X}et}{m}\left(1-\frac{1}{12}\frac{\mathrm{H}^2e^2}{m^2}t^2\right)+\mathrm{U}\left(1-\frac{1}{6}\frac{\mathrm{H}^2e^2}{m^2}t^2\right). \quad . \quad . \quad (1)$$

Thus the current parallel to x is equal to

$$\Sigma\,\tfrac{1}{2}n\frac{\mathrm{X}et}{m}\left(1-\frac{1}{12}\frac{\mathrm{H}^2e^2}{m^2}t^2\right)+\Sigma n\mathrm{U}\left(1-\frac{1}{6}\frac{\mathrm{H}^2e^2}{m^2}t^2\right),$$

where n is the number of corpuscles having the free time t. The effect of the magnetic field is represented by the terms

$$-\frac{1}{24}\frac{\mathrm{H}^2e^3}{m^3}\Sigma n\mathrm{X}t^3-\frac{1}{6}\frac{\mathrm{H}^2e^2}{m^2}\Sigma n\,\mathrm{U}t^2. \quad . \quad . \quad . \quad (2)$$

The first of these represents a decrease in the current, and therefore an increase in the resistance ; it is only the second term that we need consider. In the paper referred to I took zero as the most probable value of this term. Dr. van Everdingen, on the contrary, makes it have a large positive value, large enough to swamp the negative term and change the sign of this effect. The way in which this result is arrived at is substantially as follows. If we consider two corpuscles, one projected in the direction of the current and the other with equal velocity in the opposite direction, since the one moving along the current has its velocity increased while that moving against the current has its velocity diminished by the external electric field; thus if the *free path* is unaltered, t the free time for the one moving against the current and for which U is negative will be greater than the time for the one moving down stream and for which U is positive, because the average velocity in the first case is less than that in the second. If we assume that the free path does not depend upon the velocity, and that the corpuscle is not acted on by any forces except those due to the external electric field, we can easily show that if $2\delta t$ is the excess of the free time for the ion projected against the current over that for the ion projected along it, then

$$\frac{2\delta t}{t}=\frac{\frac{1}{2}\mathrm{X}\frac{e}{m}t}{\mathrm{U}}. \quad \ldots\ldots \quad (3)$$

The second term in (2) is proportional to the value of δt, and with the value of δt given by equation (3) is greater than the first, indicating that the resistance is diminished by the magnetic field. Equation (3) depends, however, entirely upon the assumption that the mean free path is independent of the velocity; *i. e.*, that the collision between a corpuscle and a molecule is analogous to that between two hard elastic bodies which do not exert any force upon each other until they are in contact. If we remember that the corpuscles are highly charged with electricity and always within very short distance of the molecules, it seems extremely improbable that the collision can be of the type of that between hard elastic spheres. It would seem much more reasonable to adopt a view of the collisions analogous to that used by Maxwell when he investigated the properties of a gas in which the force between the molecules varied inversely as the fifth power of the distance. On this view the collision consists of a deflexion of the path of the corpuscle due to the force exerted on it by a molecule near to which it passes. If we take this view, then equation (3) no longer holds: the equation which replaces it depends upon the law of force between the corpuscle and the molecule. If, for example, the law is inversely as the fifth power of the distance, $\delta t=0$ and the second term in (2) vanishes ; if the law of force varies less rapidly with the distance than the inverse fifth power, δt is negative and the second and first terms of (2) are of the same sign : thus the increase in resistance is greater than that calculated by the first term alone. I have calculated that it is not until the force varies more rapidly than the inverse ninth power of the distance that the second term overpowers the first. These considerations show that we cannot calculate the second term in (2) unless we know the law of force; but that there is no reason why it should reverse the effect of the first term : in fact, the probability is the other way. I think, however, whatever be the law of force, the second term will be small compared with the first: for take the most favourable case for the second term, *i. e.* when the collision is between hard elastic bodies—in this case (3) is obtained on the supposition that the only force acting on the system during its free path is that due to the external electric field. Now the corpuscles are highly charged and are always within distances of less than 10^{-7} cm. of the molecules of the metal: it is almost certain, therefore, that the local forces exerted on the corpuscle by the surrounding molecules are enormously greater than those exerted by the external electric field, and that at the end of its free path the corpuscle rushes into or past the molecule with which it is colliding with a velocity very large compared with that with which it started. Now when the velocity is changing during the free path it is easy to prove that $\delta t/t$ is less than that given by equation (3) in the proportion of the initial to the final velocity of the corpuscle. As this proportion is small, we see that $\delta t/t$ will be very small compared with the value given by (3), and therefore that the effect of the second term is very small compared with that calculated by Dr. E. van Everdingen.

The conclusion at which we arrive is thus that, while the value of the second term in (3) is uncertain, depending, as it does, upon the law of force between the molecule and the corpuscle, this uncertainty is not important, as the effect of it is small compared with that of the first term in (3) which gives an increased resistance in a magnetic field.

I am, Gentlemen,
Yours very truly,
J. J. THOMSON.

XXXVIII. *Notices respecting New Books.*

Original Papers by the late John Hopkinson, D.Sc., F.R.S. Vol. I. *Technical Papers.* Pp. lxii+294. Vol. II. *Scientific Papers.* Pp. vii+393. Edited, with a memoir, by B. Hopkinson, B.Sc. Cambridge: At the University Press, 1901.

OF the many names associated with the development of electrical engineering during the last quarter of the nineteenth century, that of John Hopkinson claims a position of undisputed and honoured pre-eminence. Gifted with remarkable ability and originality, he stood head and shoulders above his fellow-workers as the man of genius by whose unwearied efforts were largely built up those general scientific principles on which is based the practice of modern electrotechnology.

The term "magnetic circuit" will ever be associated with the name of John Hopkinson. No less remarkable than his technical work are his researches in pure science. John Hopkinson's position is unique in that he was the first to convince an incredulous school of engineers that a thorough training in pure science does not disqualify a man from grappling with the problems of practical engineering.

In view of the importance and variety of Hopkinson's contributions to both pure science and electrotechnology, and of his restless mental activity, the events of his outward life strike one as being singularly calm and peaceful. If we except his tragical death, there seems to be nothing in his career which could arouse the interest of the general public, or appeal to the student of human nature. Life for him seems to have spelt success. At no stage in his brilliant progress to the head of his profession does he appear to have met with serious disappointment, or to have known its bitterness. Adverse circumstances appeared to have no power in the face of such brilliant endowments as those which nature had bestowed on John Hopkinson. It seemed as though everything he laid his hand to was bound to succeed.

It is only fitting that his papers should be published in collected form by the Cambridge University Press. A most interesting memoir is contributed by Mr. B. Hopkinson.

This is not the place to attempt any detailed review of John Hopkinson's work. The sterling value and permanent quality of that work are well known to all students of physical science, and his 'Original Papers' will find their place among the classical works in this branch of science.

The publishers deserve great credit for the beautiful "get-up" of these two volumes, the first of which is embellished with two portraits of John Hopkinson, at the ages of twenty-three and forty-eight respectively.

A Handbook for the Electrical Laboratory and Testing-Room. By J. A. Fleming, *M.A., D.Sc., F.R.S.* Volume I. Pp. viii+538. London: *The Electrician* Printing and Publishing Company, Limited.

Dr. Fleming is so well known as a writer of books on electrotechnology, and his success in this line has been so great, that one naturally expects either novelty of matter or novelty of treatment in anything bearing his name. A perusal of the volume before us has not disappointed our expectations. The book fills a distinct gap, and lacks none of that vigour and freshness of treatment which have established the author's reputation as a writer of text-books.

The introductory chapter deals with the equipment of an electrical laboratory, and contains a detailed description of the various standards and instruments employed in electrical measurements. The author recommends the use of incandescent lamps for galvanometers, a sharp image of the filament being obtained on the scale. It appears to us that this method is never so good for exact *deflectional* work as the older one in which a brightly illuminated spot is used with a dark line across it. The former method, is, in fact, not suited to a really dark room, as there is then no illumination of the scale, and it becomes difficult if not impossible to take readings. The importance of a good galvanometer mirror is, perhaps, not sufficiently insisted on. It is wonderful how many manufacturers spoil their highest class instruments by providing them with mirrors which render it impossible to obtain a good definition of the "spot" on the scale. Another frequent source of annoyance is the extremely poor optical quality of the glass used for the windows of galvanometer cases. Any kind of mirror or glass seems, by many makers, to be thought good enough for instruments of the highest finish and delicacy. We are glad to notice the author's strong condemnation of all the usual forms of tachometers and speed-indicators, and have often wondered how people could be induced to squander money over these costly but useless instruments. The extremely sensitive hydraulic speed-indicator described by the author should prove very useful. On p. 103, in the description of the cadmium cell, "melted paraffin" should read "paraffin oil." On p. 127, we came across the following sentence (foot of page): "the logarithm of one excursion bears a nearly constant ratio to the logarithm of the next one. This ratio is called the *logarithmic decrement* of the galvanometer." This must evidently have been written in a hurry, and requires an obvious correction.

Chapter II. contains a very full account of various methods of measuring resistance. In describing Price's Guard-wire method, p. 290, the author says, "The guard wire short-circuits the galvanometer." Surely this statement is, to say the least of it, extremely misleading. At the end of this chapter, we find a very useful collection of tables of resistivity.

Chapter III. deals with the measurement of current. The author passes in review the electrolytic and absolute methods of measurement, giving full directions and useful hints in connexion with each method. On p. 342 there is an obvious *erratum*, the electrochemical equivalent of silver being given once as 0·001182, and again as 0·001193, instead of ·0011182 and ·0011193 respectively. This chapter also contains an account of wave-form measurements, frequency- and phase-meters.

Electromotive-force measurements form the subject of Chapter IV. The use of a Kelvin balance in this connexion is carefully explained, and is followed by an account of potentiometer methods, several forms of this instrument being described in detail. The calibration of a high-tension voltmeter is next dealt with, and descriptions are given of some standard forms of laboratory and switchboard voltmeters.

The concluding Chapter V. deals with the important subject of power measurement. We are probably not far wrong in asserting that the wattmeter is an instrument of which more utterly useless specimens have been offered to a confiding electrical public than of any other instrument. The author gives a thorough discussion of the precautions which must be observed in constructing and using this important instrument.

The value of the work under review is largely increased by the numerous references to original sources of information. We hope that the author will soon be able to bring out Vol. II., and thus complete a work which will earn him the gratitude of all interested in electrical engineering measurements.

Annuaire du Bureau des Longitudes. Avec des Notices Scientifiques. Paris: Gauthier-Villars, 1902, Pp. iv+850. Prix 1 fr. 50 c.

THIS well-known and handy little annual contains, as usual, a mass of useful information for the physicist, chemist, astronomer, and engineer. Among the articles specially contributed for this year must be mentioned the following:—*Polyphase Currents*, by M. A. Cornu; *Wireless Telegraphy*, by M. H. Poincaré; and *The Decimal Division of the Circumference*, by M. Guyou.

XXXIX. *Intelligence and Miscellaneous Articles.*

ON THE FLUORESCENCE AND ABSORPTION-SPECTRUM OF SODIUM VAPOUR. BY R. W. WOOD, PROFESSOR OF EXPERIMENTAL PHYSICS, JOHNS HOPKINS UNIVERSITY.

IN my paper on the anomalous dispersion of sodium vapour, which was communicated to the Royal Society last June, and which appeared in the January number of the Philosophical Magazine as well as in the Society's Proceedings, I mentioned having observed an emission of light by sodium vapour illuminated by a powerful beam of light. At the time of writing this paper I was unaware that the phenomenon had already been observed by Wiedemann and Schmidt, whose paper I have since discovered.

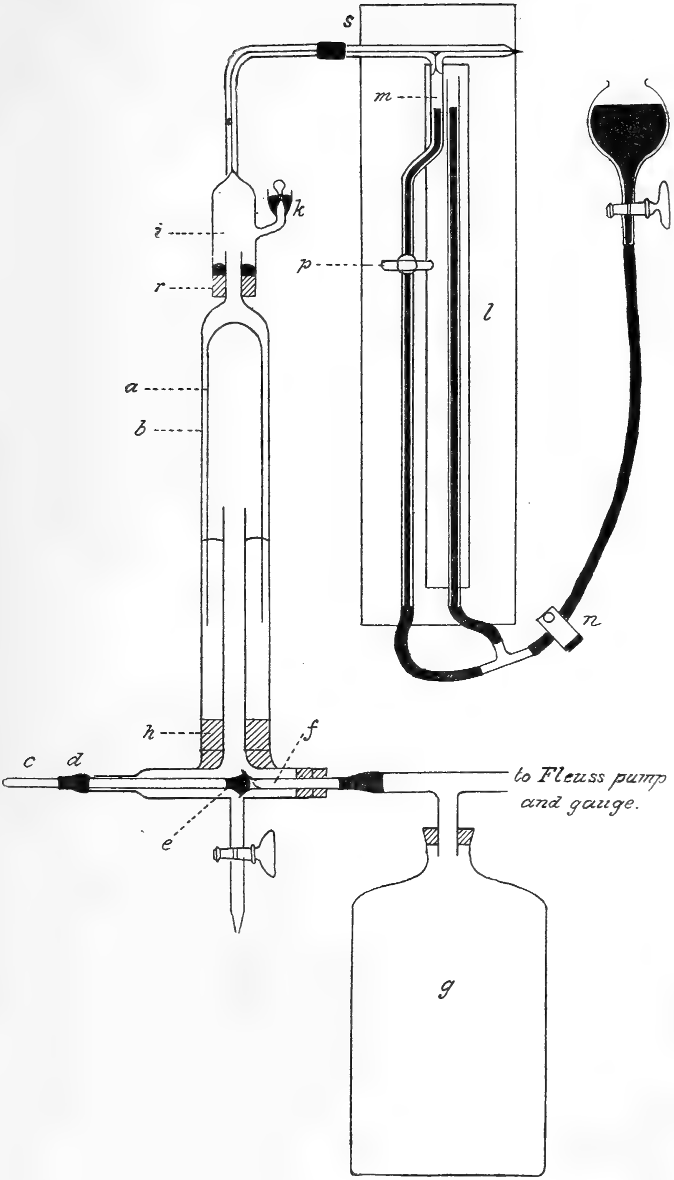
A paragraph in Drude's 'Optics,' which I cited in the original paper, led me to infer that such a behaviour of the vapour had never been observed, though it had been repeatedly sought for; and I therefore took no trouble to go over the literature on the subject.

Wiedemann and Schmidt worked with exhausted glass bulbs, which I find, on repeating their experiments, soon become opaque to light, owing to the liberation of silicon by the action of the sodium vapour on the glass. One of my students is at the present time engaged in an investigation of the phenomenon with an improved apparatus, which permits of a much more careful study of the spectrum than can be made with glass bulbs. We are also making an exhaustive study of the absorption of the vapour over a wide range of density and pressure. I have found, as I said in my previous paper, that when the vapour is exceedingly dense the transmitted light is of a deep violet colour, the spectrum consisting of a rather narrow bright band in the yellow-green and a broad band in the violet, the rest of the spectrum being absorbed, the transition from the channelled spectrum first observed by Roscoe and Schuster being a gradual increase in the extent and intensity of the flutings, photographs of which are shown in my last paper. As the vapour becomes still more dense, I find that a narrow and very black line appears almost exactly in the centre of the yellowish-green band. My first impression was that this corresponded in position to one of the well-known emission-lines of the element; but I now find that it is apparently a new absorption-line, its wave-length being very nearly 552.

As I have already shown, the channelled absorption extends throughout the entire range of visible spectrum. The behaviour in the ultra-violet and infra-red we are now examining. I am in hopes of detecting photographically anomalous dispersion at the infra-red lines which appear in the emission-spectrum. This is a matter of considerable importance, for they are near enough to the visible spectrum to modify the dispersion-curve which I have already given.

It is also important to determine whether the complicated channelled spectrum depends on the density of the vapour or its amount; in other words, whether the thousands of free periods are present in the sodium atom or molecule as it ordinarily occurs in flames or vapour of small density, or whether molecular aggregates form under the conditions under which it is observed. I propose to investigate this question by means of very long absorption-tubes heated to a rather low temperature.

Baltimore, January 28, 1902.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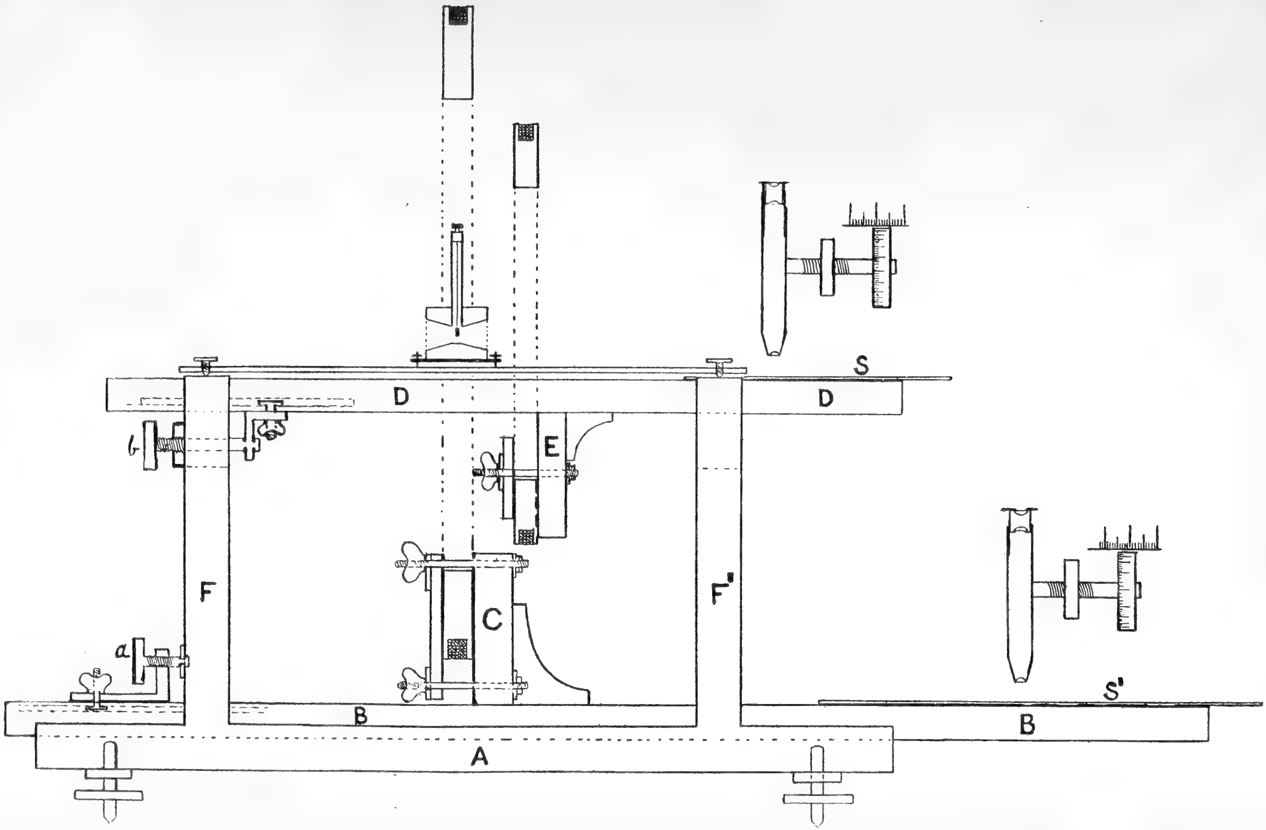
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THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
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[SIXTH SERIES.]

APRIL 1902.

XL. *Some Experiments upon the Relations between Æther, Matter, and Electricity.* *By* NORMAN E. GILBERT*.

IN the following paper are described a number of experiments, undertaken at the suggestion of Professor H. A. Rowland, upon the relations between æther, matter, and electricity. The theories advanced in the paper as suggestive of the experiments are, to the best of the writer's ability, statements of Professor Rowland's ideas. Unfortunately, the sudden illness and death of Professor Rowland prevented him from reviewing the paper.

Since the development of the wave theory of light by Huygens in 1678, the most important problem which has confronted the physicist has been the determination of the nature and the properties of the medium which we must imagine to fill all space for the propagation of the waves which give rise to the sensation of light. Numerous æthers have been postulated, each with properties which might account for the phenomenon under consideration, but none of which has made any claim to universal application. Green has developed extensively the elastic solid theory and has even made estimates of the density and rigidity of the æther. The recent development of the electromagnetic theory of light and the location of the electromagnetic energy in the æther, have demanded properties entirely different from any which could be furnished by a rigid elastic solid, and new æthers have been postulated accordingly. Faraday's discovery of the rotation

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* Communicated by Prof. J. S. Ames.

of the plane of polarization of light in a magnetic field suggested that the particles of matter, or the æther in connexion with them, must be in rotation. As the result of the theories proposed by Ampère and Weber, and developed by Maxwell, modern theories of magnetism are based on some kind of rotary or vortical motion in the æther; and if a piece of iron is magnetized, we imagine that the molecules, or something about them, rotate also. Maxwell* has tried to detect the presence of any such rotation in an electromagnet. With a kind of gyroscope he showed that, if it exists, the angular momentum must be small compared with any quantities which we can measure. An attempt was made, at the suggestion of Professor Rowland, by Mr. Paul McJunkin and the author, to determine within what limits it is possible to say that there is no frictional or viscous resistance in the æther connected with such rotation.

Frictional or Viscous Resistance in the Æther.—The existence of permanent magnets shows that any retardation due to any kind of resistance must be very slight. In the case of an electromagnet, any energy used in overcoming such resistance, if it exists, must be derived from the exciting current, and the disappearance of such energy will produce an apparent resistance added to that of the wire. An attempt was therefore made to determine whether a wire carrying a current had the same electrical resistance when producing a magnetic field that it had when not producing it.

The experiment consisted in winding two coils of wire together on an iron core and determining whether the resistance was the same in two cases:—

(1) When the current was so passed through the coils that both produced a field in the same direction;

(2) When the current was so passed that the fields produced counterbalanced each other.

The great difficulty in the experiment lay in the necessity of measuring the resistance of a coil in which a comparatively large current was flowing. In order to overcome the effect of changes in resistance due to changes in temperature, two coils were wound, as nearly as possible identical, and these double coils were used for the four arms of a Wheatstone's bridge so that the temperature would rise in all four arms equally. Each coil consisted of about 2500 turns of doubled No. 30 copper wire, the whole inclosed in an iron case, boiled in wax for five hours and cooled in a vacuum. The insulation resistance was then about eleven megohms. Iron cores were used, and it was found that the cases effectually protected the

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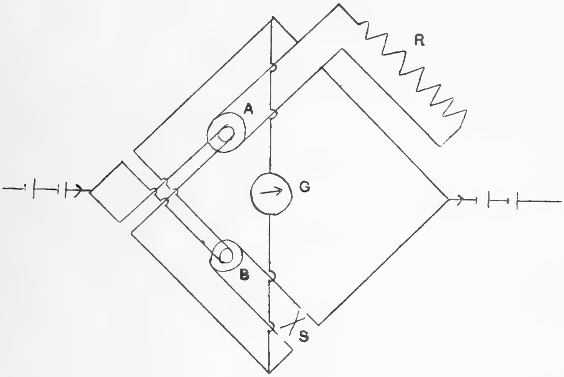
* Maxwell, Elect. and Magn., art. 575.

coils against sudden changes in temperature due to air-currents, while at the same time they served as yokes to the magnets. A current of one-tenth ampere was used, which insured a high state of magnetization in the iron when the two coils were in series, giving 5000 turns.

The coils were connected in the bridge in such a way that the two coils in one case formed the opposite arms of the bridge, as shown in the diagram.

Fig. 1.



A and B are the two cases and G the galvanometer. At S was a reversing switch by which the current in one of the coils could be reversed. This changed the field which might affect two opposite arms of the bridge, and thus doubled the effect on the galvanometer. Another switch might have been inserted in the other pair of arms, and thus the effect have been again doubled; but errors due to the switches would also have been doubled and no advantage gained. The reversing switch was carefully constructed with large copper rods fitting into mercury cups made of copper. The contact of the copper on copper was so good that thermal effects were probably inappreciable. However, at best, the inaccuracies of the switch limited the accuracy of the experiment.

The fine adjustments were made by resistance-boxes shunted around one of the coils. About 15,000 ohms in this shunt balanced the bridge. A change of one ohm in the shunt gave a deflexion of two millimetres, and indicated a change in the resistance of the arm of 5/100,000 ohm. The whole resistance being over 100 ohms, this would give a determination of one part in 2,000,000 or, since the deflexion is doubled, one part in 4,000,000 for each arm.

The final readings were made on the night of May 29, 1899, when the galvanometer was reasonably steady. The switch S was so placed that the currents in the two coils in the case B maintained magnetic fields in the same direction and the bridge was balanced by means of the shunt R. The direction of flow of the current in one of the coils in B was then reversed, causing the two fields to oppose each other, and the bridge was balanced again. The current was then reversed again and the bridge balanced once more. From fifteen to thirty seconds were required to balance the bridge between readings. Thirty readings were taken in this way, the mean of which showed the shunt resistance to be 3·4 ohms less when the magnetic field was maintained in B. The shunt was so placed (see fig 1) that this indicates a smaller resistance, in the coils in B, by one part in 1,200,000 when maintaining a field than when not maintaining a field.

The above result is not in the direction to indicate that any energy is used in maintaining the field. The difficulty may lie in the fact that the galvanometer, though used at night, was unsteady at best, the probable error of the mean being 1·9 ohms; or it may be due to leakage. The resistance of the coils was 100 ohms, while the insulation resistance was 11,000,000 ohms. If the leakage is symmetrical along the doubled wire, it will not affect the galvanometer upon reversing the current in one coil, but the assumption that it is symmetrical may not be justified.

Relative motion of Æther and Matter.—Another important question to be decided by experiment concerns the relative motion of æther and matter. Does the æther immediately surrounding a mass of matter move with the matter in its excursions through space, or does it allow the matter to pass unopposed? Experiments upon this subject give discordant results. If, as the above experiment seems to show, the æther offers no frictional or viscous resistance, we might expect it to remain stationary, allowing the free passage of matter through it. The phenomenon of aberration and the shift of spectrum lines in accordance with Doppler's principle are explained on this hypothesis, as well as the fact that no comet has yet shown any acceleration due to æthereal resistance. The elaborate experiments of Oliver Lodge*, in which he passed a beam of light several times around the space between two rapidly rotating disks, failed to show any difference in the velocity of the beams passing round in opposite directions.

On the other hand, the fact that a vibrating molecule can

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* Phil. Trans. vol. clxxxiv. p. 727 (1893).

set up vibrations in the æther may indicate that there is some sort of frictional connexion between the molecule and the æther. Fizeau [*] has shown that when two beams of light, passing in opposite directions through a tube of water, are made to interfere, the fringes are displaced when the water is set in motion, indicating that the æther is carried with the water. His measurements indicated that the water moves a little faster than the æther. This experiment has been repeated and the results confirmed by Michelson and Morley [†]. Again, Michelson and Morley [‡] have made an elaborate series of experiments with their interferometer, and have been able to detect no displacement of the fringes as the instrument was rotated through different angles with respect to the direction of motion of the earth, thus seeming to show that the æther moves with the earth. Sutherland [§] tried to explain this by showing that the displacement, while perfectly possible, would be too small to be detected by the adjustment of their experiment; but Lodge [||] dismisses this explanation and suggests, with FitzGerald [¶] and Lorentz [**], that the molecules of the instrument and of the heavy stone slab on which it rests may bear such a relation to the æther that they are compressed along the line of motion and so distort the instrument just enough to balance the effect of the relative motion. Whether or not this explanation is valid, it remains for future experiments to decide.

Another method by which this problem may be attacked is to study the nature of electricity and its relation to matter. Maxwell [††] made two interesting experiments along this line. One of these, in which he showed that a coil of wire, carrying a current, had no angular momentum due to the current, has already been mentioned. In the other, he showed that electricity has no linear momentum, since a delicately suspended coil showed no tendency to rotate in its own plane when a current was started or stopped in it. It was while discussing these experiments that Professor Rowland suggested the experiment which has since been carried out. The experiment consisted in trying to detect a current generated in a wire wound on the periphery of a wheel in such a way that it would move in the direction of its length when the

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[*] *Ann. de Chimie et Physique*, tome lvii. p. 385 (1859).

[†] Am. Journ. of Sc. (3) vol. xxxi. p. 377 (1886).

[‡] Phil. Mag. xxiv. p. 449 (1887); Am. Journ. of Sc. (3) p. 475 (1897).

[§] Phil. Mag. xlv. p. 23 (1898).

[||] Phil. Mag. xlvi. p. 343 (1898).

[¶] Public Lectures in Trinity College, Dublin.

[**] *Versuch einer Theorie der Electrischen Korpern.*

[††] Elec. and Mag. vol. ii. arts. 574, 575.

wheel was rotated. Connexion was made to a delicate galvanometer by bringing out the ends of the coil of wire at the centre of the axis, one on either side of the wheel.

Several lines of thought might lead us to expect to find a current in such a circuit when the wheel was rotated. If we consider an electric current to be a continuous giving way of the æther under the action of the electric stresses, which, in a dielectric, give rise to electric displacement, then we may think of a moving stream of æther as constituting an electric current, and we might expect that a wire, moving relative to the æther, would have a current generated in it. Moreover, we are familiar with several phenomena in which the behaviour of positive electricity is different from that of negative, such as the discharge of negative, but not of positive, by ultra-violet light, and the complete dissimilarity between the phenomena at the electrodes of a Crookes' tube when a discharge is passing through the tube. In practically all of these cases the positive electricity has been shown to be more sluggish in its action than the negative. J. J. Thomson * has imagined that ordinary metallic conduction may be only a kind of electrolytic action, in which case we might expect the positive electricity to move more slowly or to lag behind the negative in a moving conductor. Such a lag would constitute the current for which we are looking.

But the consideration which, above all others, has led us to look for such a current is the fact that it would give us at once a simple explanation of the cause of the earth's magnetism. If we may assume that a moving body has generated in it, by virtue of its motion, an electromotive force, then a rotating body like the earth, which presents a closed circuit, will contain a current capable of producing a magnetic field identical with that which a constant current, produced by an E.M.F. equal to Σ E.M.F. around the circuit, would produce in a stationary earth. Dr. Schmidt †, by an extension of Gauss' harmonic analysis, has recently arrived at the conclusion that 97·5 p. cent. of the terrestrial magnetism is due to causes within the earth, while Schuster ‡ believes that not more than 5 p. cent. can possibly be attributed to outside causes. This fraction is so small that in the present investigation it may be neglected entirely.

The earliest explanations of this phenomenon, which depend upon the existence of permanent magnetism within the earth,

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* Recent Researches, Par. 34. *Rapports présentés au Congrès International de Physique* (Paris, 1900), p. 138.

† *Ciel et Terre,* Dec. 16, 1900.

‡ B. A. Report, 1898, p. 745.

apparently fail, since no known substance can retain its magnetism at the high temperatures which exist in the interior. In 1879 Ayrton and Perry [*] advanced a theory which depends upon Rowland's experimental proof that a moving electric charge acts magnetically like a current. This theory assumes the presence of a large negative charge of static electricity placed upon the surface of the earth and rotating with it. Rowland [†] disproved this theory, by showing that a surface-density great enough to account for terrestrial magnetism would involve a repulsive force sufficient to tear away articles on the earth's surface. To overcome this difficulty Sutherland [‡] has assumed that an equal positive charge is concentrated at the centre which will confine the field to the interior of the earth. To keep these charges apart an insulation resistance is necessary which will stand a fall of potential of 2×10^8 volts per centimetre. The high temperature within the earth would destroy the insulating power of most substances which are known at the surface, so that Mr. Sutherland is forced to assume that pressure will counteract this effect and restore the insulating power. Until we have more evidence on this point it seems that this assumption is hardly warranted. As far back as 1825 Barlow suggested that magnetic polarity might be induced by mere rotation of matter, and tried the experiment with iron spheres, but found no effect except that due to hysteresis. In recent years, however, the belief has been gaining ground that terrestial magnetism is due to the rotation of the earth. Schuster [§], in his presidential address before Section A of the British Association, 1892, asks the question: "Is every large rotating mass a magnet?"; while Lord Kelvin [||] "finds it unimaginable but that terrestial magnetism is due to the greatness and the rotation of the earth;" and Professor Rowland has frequently suggested such a cause in his lectures.

If now we may assume that matter, by virtue of its motion, has induced in it an electromotive force in the direction opposite to that of its motion, currents will be set up within the volume of the earth, flowing in cylindrical sheets about the axis of the earth, and producing a magnetic field at any point on the surface. Assuming the electromotive force in

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[*] Phil. Mag. vii. p. 401 (1879); Proc. Phys. Soc. of London, iii. p. 57 (1880).

[†] Phil. Mag. viii. p. 102 (1879); Proc. Phys. Soc. of London, iii. p. 93 (1880).

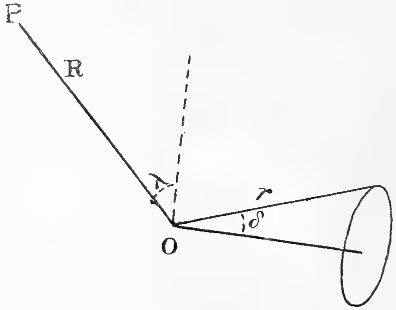
[‡] 'Terrestial Magnetism and Atmospheric Electricity,' June 1900.

[§] British Association Reports, 1892, p. 634.

[||] Popular Addresses, vol. ii. p. 511.

any elementary circuit to be proportional to the linear velocity and to the length of the circuit, we may find an expression for the magnetic intensity at a point on the surface of the earth due to such currents in the interior.

Fig. 2.



The magnetic potential at a point P (fig. 2) due to a current in an elementary circuit is equal to the value of the current multiplied by that of the solid angle subtended at P by the circuit. Let O represent the centre of the earth and r the distance of the circumference of the circuit from the origin, δ the angle subtended at O by the circuit, R the distance of P from the origin and λ the latitude of P. Then the expression for the solid angle at P is

$$\omega = 2\pi\left[\frac{r}{\mathrm{R}}\mathrm{P}_1(\theta) - \frac{r^2}{\mathrm{R}^2}\mathrm{P}_2(\theta)\mathrm{P}_1\left(\frac{\pi}{2}+\lambda\right) + \frac{r^3}{\mathrm{R}^3}\mathrm{P}_3(\theta)\,\mathrm{P}_2\left(\frac{\pi}{2}+\lambda\right) - \dots\right]$$

$$-2\pi\cos\theta\left[\frac{r}{\mathrm{R}} - \frac{r^2}{\mathrm{R}^2}\mathrm{P}_1(\theta)\mathrm{P}_1\left(\frac{\pi}{2}+\lambda\right) + \frac{r^3}{\mathrm{R}^3}\mathrm{P}_2(\theta)\,\mathrm{P}_2\left(\frac{\pi}{2}+\lambda\right) - \dots\right],$$

where P_1 ( ), P_2 ( ), &c. represent zonal harmonics.

To find the current, due to the rotation of the earth, in an elementary circuit whose centre is on the axis and whose plane is perpendicular to the axis of rotation,—

let v be the linear velocity of a point on the circumference,
T be the period of rotation,
K be the E.M.F. generated in one cm. moving at rate of 1 cm. per sec. in direction of its own length,
E be E.M.F. generated in one circle,
C be the current generated in one circle,
ρ be the specific resistance, *i.e.* the resistance of 1 c.c.,
m be the magnetic potential at P due to one circle,
μ be the magnetic potential at P due to whole earth.

Then (fig. 2),

$$v=2\pi r\sin\theta\,.\,\frac{1}{\mathrm{T}}=\frac{2\pi r}{\mathrm{T}}\sin\theta.$$

$$\mathrm{E}=\mathrm{K}\frac{2\pi r}{\mathrm{T}}\sin\theta\,.\,2\pi r\sin\theta.$$

$$\mathrm{C}=\mathrm{K}\frac{2\pi r\sin\theta\,.\,2\pi r\sin\theta}{\mathrm{T}\rho\,.\,\dfrac{2\pi r\sin\theta}{r\,d\theta\,dr}}=\frac{2\pi\mathrm{K}r^2}{\rho\mathrm{T}}\sin\theta\,d\theta\,dr.$$

Whence

$$m=\mathrm{C}\omega=\frac{4\pi^2\mathrm{K}}{\mathrm{R}\rho\mathrm{T}}\Big[r(\mathrm{P}_1(\theta)-\cos\theta)-\frac{r^2}{\mathrm{R}}(\mathrm{P}_2(\theta)$$

$$-\mathrm{P}_1(\theta)\cos\theta)\mathrm{P}_1\left(\frac{\pi}{2}+\lambda\right)+\dots\Big]r^2\sin\theta\,d\theta\,dr.$$

Integrating this from $\theta=0$ to $\theta=\pi$ and from $r=0$ to $r=\mathrm{A}$, the radius of the earth, we have

$$\mu=-\frac{8}{15}\frac{\pi^2\mathrm{K}\mathrm{A}^5}{\rho\mathrm{T}\mathrm{R}^2}\sin\lambda.$$

If $\mathrm{F}_v=$ the vertical component and F_h the horizontal component of the magnetic intensity at a point, we have

$$\left.\begin{aligned}\mathrm{F}_v&=-\frac{d\mu}{d\mathrm{R}}=\frac{16}{15}\frac{\pi^2\mathrm{K}}{\rho\mathrm{T}}\mathrm{A}^2\sin\lambda\\ \mathrm{F}_h&=-\frac{d\mu}{d(\mathrm{R}\lambda)}=-\frac{1}{\mathrm{R}}\frac{d\mu}{d\lambda}=\frac{8}{15}\frac{\pi^2\mathrm{K}}{\rho\mathrm{T}}\mathrm{A}^2\cos\lambda\end{aligned}\right\}\quad.\qquad(1)$$

when R=A, whence

$$\left.\begin{aligned}\frac{\mathrm{F}_v}{\sin\lambda}&=\frac{16}{15}\frac{\pi^2\mathrm{K}}{\rho\mathrm{T}}\mathrm{A}^2=2\mathrm{C}\\ \frac{\mathrm{F}_h}{\cos\lambda}&=\frac{8}{15}\frac{\pi^2\mathrm{K}}{\rho\mathrm{T}}\mathrm{A}^2=\mathrm{C}\end{aligned}\right\}\quad.\ .\ .\ .\ .\ .\qquad(2)$$

where C is a constant defined by the above equation.

Also

$$\frac{\mathrm{K}}{\rho}=\frac{15}{8}\frac{\mathrm{TC}}{\pi^2\mathrm{A}^2}\quad.\ .\ .\ .\ .\ .\ .\ .\qquad(3)$$

The present investigation requires a knowledge of only the order of magnitude of the values of $\mathrm{F}_v/\sin\lambda$ and $\mathrm{F}_h/\cos\lambda$. To determine these ten observatories, representing all sections of the earth, were chosen from the tables given by Bigelow in Johnson's Cyclopædia, vol. v. p. 469. In each case the value of the latitude was reduced to magnetic co-ordinates, the position of the magnetic pole being taken from the same article. The average of the values of the constant C derived from the two components of intensity are

$$C = F_h/\cos\lambda = \cdot 316, \qquad 2C = F_v/\sin\lambda = \cdot 614\,;$$

and these agree as well as could be expected in view of the irregularity in distribution of the magnetic elements owing, possibly, to unsymmetrical permeability and conductivity in the interior and particularly to local causes of disturbance. From the two series of determinations we obtain a mean value $C = \cdot 311$. Substituting this in equation (3) with $T = 86{,}400$ sec. and $A = 6336.10^5$ cm. we have $K/\rho = 1276.10^{-17}$ ($\log = 6\cdot 10585 - 20$).

It now remains to determine by experiment whether such a value of K exists as shall give to ρ a reasonable value.

The galvanometer used for this test was one which could be made extremely sensitive, more sensitive, in fact, than it was possible to use in Baltimore. The laboratory has trolley-lines on two sides, and a railway tunnel, in which heavy electric locomotives are used, passes nearly underneath. As a result, there were only one hundred minutes daily, divided into three periods between 1.30 and 4 A.M., when the instrument could be expected to be usable, and occasionally entire nights passed without the possibility of obtaining any satisfactory readings. Owing to this cause the progress of the work has been slow and the results at best are unsatisfactory.

The magnetic system of the galvanometer consisted of two sets of magnets, each containing three small magnets about one eighth of an inch long, mounted on a fine glass thread at a distance apart of about 1 inch. Midway between the two sets of magnets was placed a very small mirror. The inertia of the whole system was reduced to a minimum. The suspension was a quartz fibre. Frequent attempts were made, by testing the sensibility in both directions on the scale, to determine whether there was any appreciable torsion in the fibre, but none was detected which was comparable with outside irregularities, though the system sometimes turned through several turns while being made astatic. A magnetic shield consisting of three concentric cylinders of soft iron was used during a part of the time, but even with this the galvanometer could be used only during the three quiet periods. It was found advisable to demagnetize the shield occasionally either by heating to a red heat or by placing around it a coil carrying an alternating current, and then slowly reducing the strength of the current. By "sensibility" of the galvanometer is meant the current required to give one millimetre deflexion when the resistance of the galvanometer (for coils in series) was 50 ohms and the scale was distant one metre. The test-current was derived from a dry cell of 1·4 volts E.M.F. cut down by shunts of 10/1400 and 100/10,000 and then passed through 10,000 ohms in series with the galvanometer. The coil on the experiment wheel was always in series when the sensibility was tested, the resistance being negligible in comparison with the 10,000 ohms. The testing system was kept connected so that it could be used at frequent intervals during the progress of the readings and while the wheel was running. The sensibility during the last and best of the readings was kept at 10^{-10}. At this sensibility the galvanometer was "dead-beat" and the time required for a single throw, or one-fourth of a complete period, was about fifteen seconds. The sensibility could be increased beyond this by further cutting down the field. Occasionally 10^{-11} and on one trial $4·5 \times 10^{-12}$ was reached, but the time required for a single throw was increased to two or even three minutes; and the time demanded for a complete reading seldom elapsed without a variation in the thermal current or some magnetic disturbance from outside,

At these high sensibilities it is interesting to note that the throw due to a small instantaneous induction-current depended directly upon the current, but was practically independent of the sensibility of the galvanometer, showing that the inertia of the system was negligible compared with the damping.

The first attempt to discover an electromotive force due to the longitudinal motion of a wire was made with a coil of rectangular cross section, of No. 30 copper wire, wound in a slot cut in the side of the rim of a wheel. The rim was slit radially to avoid currents in the wheel itself. The mean radius of the coil was 6·25 cms. and the speed was 70 turns per second. The length of wire was about 42,000 cms., with a resistance of 146 ohms. This wheel was rotated in both directions at a time when a deflexion of one millimetre might have been detected; but no reversible deflexion of this amount was obtained, though some irregular deflexions occurred, which were due, doubtless, to slight variations in the magnetic field through the coil. The continuity of the circuit was frequently tested by moving a magnet near the coil and noting the throw due to the induced current. The sensibility of the galvanometer was $1·6 \times 10^{-9}$ and the resistance of the circuit 200 ohms, so that an E.M.F. of $3·2 \times 10^{-7}$ volts in the circuit, corresponding to $K = 3 \times 10^{-15}$ volts (K being the hypothetical E.M.F. generated in one centimetre of wire moving with unit velocity) would probably have been detected, though this is by no means certain.

The form of the above coil was not satisfactory. It was inclosed on three sides by brass, and the turns on the interior were so shielded by those on the exterior that comparatively few would be exposed directly to the action of the æther if the æther were dragged along as a viscous fluid would be. A new wheel was therefore built whose periphery was a cylinder 4·2 cms. broad and 7·3 cms. radius. On this was wound in a single layer 175 turns of No. 33 copper wire giving a resistance of 57 ohms. A test similar to the above was made with this wheel with the result that no reversible E.M.F. as large as $K=1{\cdot}2\times 10^{-14}$ volts was detected. The practical result of these two tests was to show that any E.M.F. which might exist could be detected only by averaging a long series of readings, and this was next undertaken.

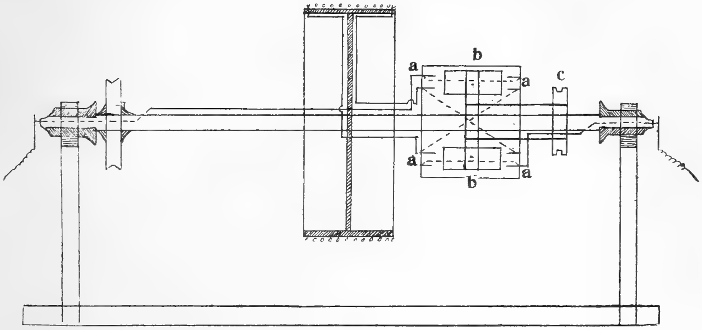
The sources of difficulty in making these measurements were numerous. In order to insure smooth running at the high speeds attained, the most delicate balancing of the wheel was necessary. The wheel was mounted on a long slender steel shaft which was sufficiently flexible to allow rotation about a true principal axis when a high speed was once reached ; but trouble was experienced in passing the point at which the speed of the wheel equalled the period of vibration of the wheel on the shaft. Here vibration became so excessive that the motor was sometimes unable to increase the speed beyond that point. The difficulty was overcome by allowing the cast iron base of the machine to stand unclamped on cotton or on several thicknesses of cardboard, and by most careful balancing. Another source of trouble lay in the thermal currents in the circuit caused by the heating of the copper-silver junctions by the heat flowing away from the bearings when the wheel was running. The terminals of the coil were led out through the end of the shaft in order to avoid as far as possible any friction and consequent heating at the brushes, and to reduce to a minimum any alternating currents due to motion of conductors in the earth's field. After running for some minutes the temperature would become approximately steady ; but at best the variations in the thermal E.M.F. at the junctions of the wire were much greater than the E.M.F. for which we were looking. These thermal currents were reduced to a minimum by using the same wire, so far as possible, for all connexions ; but this could not be done where the circuit passed from the tip of the axis to the brush, for it was necessary to use silver to insure good contact. Silver wires were used for several inches in both directions from the brushes. The contact at the brushes was another source of difficulty. The lead wires were led side by side from the rim to the axis of the wheel, but still a slight alternating current existed sufficient to give a throw of about five millimetres on the galvanometer scale when the wheel was turned quickly through a half-turn. This would give no difficulty at high speeds unless the brush were thrown off periodically so as to act as a commutator. This appeared sometimes to occur when the silver tip dug a small cavity in the plane surface of the spring bearing against it. This was overcome by frequently smoothing off the plane surface with a file. This difficulty was so great with copper contacts that it was necessary to use silver, as was mentioned above. Another difficulty lay in the fact that the inertia was so great that, though all joints were carefully soldered, earth-connexions and breaks were frequent at the joints or in the wire itself. These breaks were frequently not complete, but simply gave a variable contact and were extremely hard to locate without removing all connexions and replacing with new. This was particularly true among the complicated connexions of the reversing switch in the later experiments.

The wheel was run by a belt from a one-sixth horse-power electric motor. It was found necessary to keep the frame of the machine in electrical connexion with the gas fixture, in order to avoid violent throws of the galvanometer-needle due to static electricity. The speed first used was 85 turns per second. Later a larger pulley was used giving 125 turns, and this was sometimes increased to 150 by shifting the brushes on the motor at each reversal. If this was done after a high speed was reached, sparking was not excessive. In the early experiments the motor magnets changed the zero of the galvanometer by about two centimetres. It was impossible to detect any unsteadiness due to the motor when running ; but in the later experiments it was moved to the farther side of the room where it had no perceptible effect on the galvanometer.

In taking the first extended set of readings, the wheel was run in one direction while five deflexions were taken. Then the motor was reversed by reversing the current in the armature, so as to affect the galvanometer as little as possible, and five more deflexions were recorded, and so on. A reading was thrown out if there was any indication that it had been affected by outside influences. The wire was so wound on the wheel, that positive electricity lagging behind would flow to the galvanometer through the end of the axis away from the pulley when the wheel was rotating in the positive direction. To determine in which direction this would deflect the galvanometer, a small test battery was used. This consisted of two copper wires to one end of each of which brass plates were soldered ; to the other end of one wire a zinc plate was soldered ; the brass plates were separated by a piece of paper and inserted between the silver spring and the tip in the axle ; on the zinc plate was placed a moistened piece of paper, and this was touched by the copper wire which was connected to the brass plate nearer the galvanometer. Thus the current from this battery passed through the galvanometer in the same direction as a positive current from the wheel. This direction was such, during the first readings, as to give a "negative" deflexion, and these readings, while very variable, were uniformly in the direction looked for. The sensibility of the galvanometer was 3×10^{-9}, length of wire 8000 cms., velocity 3900 cms. per second, and resistance of circuit 100 ohms. The mean deflexion was ·335 mm. in each direction, which would give $K = 3 \times 10^{-15}$ volts. Substituting this in equation (3) we find $\rho = {\cdot}34$ ohm, a resistance about 3000 times that of mercury. Such a conductivity is easily possible considering the high temperature existing in the interior of the earth ; and at this stage the experiment appeared most promising.

The experiment was next varied by crossing the lead wires on the wheel so as to bring them out at opposite ends of the shaft, with the result that the readings failed to reverse properly. The galvanometer terminals were also reversed occasionally, the readings being consistent in every case. The wires were reversed on the wheel several times, and the readings were so inconsistent that a reversing switch was devised which was placed directly on the shaft and could be reversed while the wheel was in motion. This device removed the large variations in thermal currents due to stopping the wheel, and eliminated the effects of all electromotive forces in the circuit except those on the wheel itself, only these being commuted. The switch with its connexions is shown in fig. 3. The wires from the wheel and from the silver tips in the shaft end in copper springs *a a*. Contact is made between these springs by means of two copper plates *b b*, mounted on a fibre collar which can be moved along the shaft by a small rod placed in the groove in the movable collar *c*. The copper springs were properly bent and were controlled by adjusting-screws, so that good contact was assured. The complete switch shown plane in the figure was cylindrical and was only 3 cms. in diameter, so that the various parts were kept as nearly as possible at the same temperature. The reversing-rod was touching the collar *c* only during the instant of reversal. The switch moved easily, but was held firmly in position at the end of the throw by the pressure of the copper springs. The connexions of the switch were such that the positive terminal from the wheel was connected to that end of the shaft towards which the switch was thrown. This was verified by passing a current through the machine and noting the deflexion of a small compass held above the wheel. This test was also applied when the wheel was rotating, as an assurance that everything was in order. Another test frequently applied was to pass a current through a delicate milliammeter in series with the wheel. The slighest change or unsteadiness in the needle when the machine was started was an indication of trouble.

Fig. 3.



If any friction exists between the æther and the moving wire, and if there is any viscosity within the æther itself, it is probable that the motion of the wheel would produce convection-currents which would greatly reduce the amount of relative motion between the wire and the æther in immediate contact with it. To overcome this as far as possible a copper ring or shield was cast and placed around the wheel. The width of the shield was the same as that of the wheel, the internal diameter 3 mms. greater than the diameter of the wheel, and the thickness of copper was 2 cms. The weight was 12 lbs. To eliminate any possible effect due to the earth's magnetic field, the wheel was rotated alternately in and at right angles to the plane of the magnetic meridian, but no effect due to this cause was detected.

Table I. contains the results of 920 readings by this method.

Each reading consists of a zero, a deflexion given by throwing the reversing switch, and another zero given by throwing the reversing switch back. In the first row, 11 is the sum of 40 readings to the west on the galvanometer-scale, and 55 is the sum of the corresponding readings to the east; the sixth column gives the sums of differences in the various sets; the last two columns give the connexions of the wires. The galvanometer circuit was kept closed, thus eliminating thermal currents which may have appeared in the earlier readings. The galvanometer terminals were reversed by a single mercury switch. Care was taken to avoid touching any part of the circuit during a series of readings. The sensibility of the galvanometer was 10^{-9}. The signs have been so adjusted in the columns of differences, that positive differences always mean deflexions in the direction looked for. The wheel during this series was wound with 8000 cms. of No. 36 copper wire, giving a resistance of 180 ohms in the circuit. The speed was 6000 cms. per second. The average deflexion of ·04 mm. then corresponds to $K = 15 \times 10^{-17}$ volt or to $\rho = \cdot 012$ ohm, a resistance only 130 times as great as that of mercury. In the series recorded in Table II., taken eight months later with a galvanometer ten times more sensitive, a deflexion of ·175 mm. in the opposite direction was obtained, and this is about sufficient to balance the last. It seems certain, then, that a resistance as small as 130 times the resistance of mercury would be required for the interior of the earth if terrestrial magnetism were due to this cause. Schuster* has shown that, to account for the earth-currents induced by the currents in the upper atmosphere, which are undoubtedly the primary cause of the short-period variations, the average specific resistance of the earth must be of the order of magnitude $\rho = 1 \cdot 23 \times 10^{13}$ C.G.S. units, a value 1,000,000 times as great as that derived above.

It is evident from this comparison that the largest value obtained for the hypothetical electromotive force is far too small to account for the magnetism of the earth. The second series of readings gives an average which just balances the first, reducing the final average to zero. The result of the entire research is then purely negative. If any such cause for terrestrial magnetism exists, it has not been detected by this experiment.

As has been mentioned, an attempt was made in some of the experiments to reduce possible convection-currents in the æther by placing a heavy copper shield around the wheel.

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* Phil. Trans. of Royal Soc. 1889, p. 496.

TABLE I.

		ROTATION +				ROTATION —					
1900.	No. of readings.	W.	E.	Diff.		W.	E.	Diff.		Wires on Wheels.	Galv.-terms.
Switch West.—No Shield.											
Feb. 3 ...	40	11·0	55·0	+44·0		20·5	30·5	−10·0		X	+
3 ...	20	10·5	4·5	− 6·0		15·5	10·0	+ 5·5		X	−
7 ...	30	14·5	2·5	+12·0		9·0	3·0	− 6·0		X	−
7 ...	30	7·5	6·5	+ 1·0	+51·0	3·0	5·0	+ 2·0	− 8·5	X	+
Copper Shield on.											
Mar. 15...	30	7·0	23·5	+16·5		4·5	7·5	− 3·0		X	+
17...	30	5·5	21·0	−15·5	+ 1·0	3·5	17·5	+14·0	+11·0	=	+
No Shield.											
Mar. 20...	30	7·0	10·0	+ 3·0		12·0	6·5	+ 5·5		=	−
22...	30	5·0	12·0	− 7·0		4·0	17·5	+13·5		=	+
22...	30	7·5	12·5	+ 5·0	+ 1·0	17·0	4·0	+13·0	+32·0	=	−
	270				+53·0				+34·5		
Switch South.—Copper Shield on.											
		S.	N.			S.	N.				
Mar. 24...	30	17·0	0·5	−16·5		9·0	29·0	−20·0		=	−
31...	20	15·0	0	−15·0		5·0	6·5	− 1·5		=	−
Apr. 3...	30	15·5	3·0	−12·5		7·0	8·5	− 1·5		=	−
3...	30	8·5	4·0	− 4·5		2·5	15·5	−13·0		=	+
	110				−48·5				−36·0		
Switch North.—Copper Shield on.											
		N.	S.			N.	S.				
Apr. 7...	20	0·0	25·0	−25·0		5·0	28·0	+23·0		=	+
11...	30	6·5	14·5	− 8·0		2·0	26·0	+24·0		=	+
11...	30	38·0	1·5	+36·5		24·0	5·5	−18·5		=	−
	80				+3·5				+28·5		
	460				) +8·0				) +27·0		
					·02				·06		

TABLE II.

		ROTATION +				ROTATION −				
1900.	No. of Readings.	N.	S.	Diff.		N.	S.	Diff.		Galv.-terms.
					Switch North.					
Nov. 23...	20	61·0	122·0	− 61·0		13·0	73·0	+ 60·0		+
25...	20	40·0	42·0	− 2·0		36·0	48·0	+ 12·0		+
25...	20	148·0	2·0	−146·0	−209·0	115·0	18·0	+ 97·0	+169·0	−
	60									
					Switch South.					
		S.	N.			S.	N.			
Nov. 27...	20	12·0	110·0	− 98·0		6·0	142·0	+136·0		+
28...	20	20·0	7·0	− 13·0		37·0	5·0	+ 32·0		−
28...	20	11·0	10·0	+ 1·0	−110·0	8·0	16·0	+ 8·0	+176·0	+
	60									
					Switch West.					
		W.	E.			W.	E.			
Dec. 1...	20	16·0	63·0	− 47·0		21·0	32·0	+ 11·0		+
1...	20	55·0	10·0	− 45·0	− 92·0	38·0	20·0	+ 18·0	+ 29·0	−
	40									
					Switch East.					
		E.	W.			E.	W.			
Dec. 2...	20	23·0	20·0	− 3·0		9·0	22·0	− 13·0		−
2...	20	3·0	40·0	− 37·0	− 40·0	5·0	23·0	+ 18·0	+ 5·0	+
	40				) −450·0				) +379·0	
	200				− 2·25				+ 1·90	

Another attempt to accomplish the same thing was made by placing an electric charge on an insulated brass shield surrounding the wheel. Professor Rowland's Berlin convection experiment proves that a moving charge is accompanied by the æther, thus producing the magnetic phenomena. Conversely, a charge held at rest should hold the surrounding æther at rest. The shield, with the wire on the wheel, formed a thin cylindrical condenser whose dimensions were

$l=4{\cdot}3$ cms. = length.
$b=8{\cdot}0$ cms. = radius of shield.
$a=7{\cdot}3$ cms. = radius of wheel.
$d=0{\cdot}7$ cm. $=b-a=$ distance between plates.
$S=197$ sq. cms. = area of wheel.

Thus the capacity was

$$C=\frac{l}{2\log b/a}=23{\cdot}4 \quad \text{or} \quad C=\frac{S}{4\pi d}=22{\cdot}4 \text{ C.G.S.}$$

The potential V was 5000 volts or 16 C.G.S. units, giving for the surface-density

$$\rho=\frac{CV}{S}=1{\cdot}9 \text{ C.G.S. units.}$$

In making these readings, one point of the galvanometer circuit was connected to earth while the galvanometer frame, magnetic shield, frame of wheel, and one pole of the electric machine were also earthed. The other pole of the electric machine was connected to the shield. A battery of six one-gallon leyden-jars was used to prevent rapid changes in potential. The wheel was wound with 8000 cms. of No. 33 copper wire, single silk-covered, giving a resistance of 70 ohms. Owing to some difficulty which it was found impossible to locate or eliminate, the silver brushes were removed and the silver tips in the shaft were replaced by fine copper wires passing out along the line of the shaft and entering small horizontal glass tubes filled with mercury. The galvanometer terminals entered the other end of the tubes. After several hours of running the wheel, the large unsteady deflexions almost entirely disappeared.

There was an air-gap in the shield which prevented the electricity from flowing around on the shield when the wheel was running. If now the stationary charge on the shield holds stationary the induced charge on the wheel, and if the charge holds the æther, then the wire must move with reference to the æther in its immediate neighbourhood. This experiment was tried with negative result, three times at intervals of several months. The final series of readings from the last trial, taken when the sensibility of the galvanometer was 10^{-10}, is given below :—

Jan.	No. of readings.	Rotation	charge.	Diff. of readings.	Average.
20......	10	+	+	+32	+3·2
20......	10	+	−	+36	+3·6
21......	10	−	+	− 2	− ·2
21......	10	−	−	+17	+1·7

The last set was taken within 30 minutes of an accident which necessitated the soldering of two joints on the wheel in the galvanometer circuit, and therefore cannot be compared with the one above taken on the same night. If, however, we average these readings, assuming that the direction of the readings should reverse both with the charge and with the direction of rotation, we obtain a mean of ·35 mm. or ·175 mm. in each direction, which is identical in magnitude with that obtained under similar circumstances with no charge.

In conclusion I take pleasure in acknowledging my obligation to Professor Rowland, who suggested the work and who followed its progress with much interest ; to Professor Ames for his kindness in many ways ; to Mr. Harold Pender for his assistance in obtaining the later readings, and to Professor L. A. Bauer for valuable criticisms and references.

Johns Hopkins University,
Feb. 1, 1901.

XLI. *A new Hygrometric Method.*

By E. B. H. Wade, *M.A.*[*]

(1) *THE object of this note* is to describe a new hygrometric method, in which a thermometer is wetted not with water but with sulphuric acid of suitable strength.

(2) *It is claimed* that the method is (*a*) theoretically sound, (*b*) cheap and easy to manipulate, (*c*) specially suited for the study of ordinary wet bulbs, (*d*) independent of ventilation.

(3) *Theory of the method.*—Let t = temperature of dry bulb, t' = the same for wet bulb, θ = the same for bulb wet with acid (acid bulb) ; f = tension of vapour in air, f' = maximum tension of water-vapour at t', ϕ = maximum tension for acid at θ. If

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[*] Communicated by the Physical Society: read December 13, 1901. Published by permission of the Under Secretary of State, Public Works Ministry.

$t=\theta$, $f=\phi$, and ϕ is known from Regnault's work. Hence f' follows. In studying wet bulbs by this method assume further that over a small range of conditions

$$f=f'-c(t-t')=\phi-k(t-\theta),$$

and from two such comparisons obtain c and k, or from a large number treated by the method of least squares.

The range of conditions over which it is admissible to assume that c and k are constant must be found by experiment, but it must be admissible over some range. Special attention is drawn to this device for finding c and k, since it does not involve an appeal to any method supposed to be standard.

(4) *Manipulation.*—To apply this theory to the study of wet bulbs, the bulb of a thermometer is wrapped in linen secured by a rubber band just above the bulb. A little higher the stem passes through a rubber stopper, which closes the mouth of a test-tube containing a little acid which wets the rag. When this thermometer reads about the same as the dry bulb, withdraw the test-tube exposing the rag to air. Between the fourth and sixth minutes after this, read repeatedly the acid thermometer and the wet and dry bulbs, and take the respective means. Rinse the linen repeatedly with the stock of acid and replace the test-tube. A very little experience enables one (from a reading of the wet and dry bulbs at 8 A.M.) to select a solution which will be appropriate to the humidities met with during most of the morning and again in the evening. A point is made of the simplicity of the method.

(5) *Illustrative results.*—The results below are to illustrate the method and to show that k may be taken constant between limits $t-\theta=\pm 2°$ and in very different ventilations. Table I. gives results with the apparatus described and an ordinary psychrometer. Light winds prevailed, but an exact record was not kept. The series is preliminary : k appears constant, but the question of ventilation remains. To test this the ordinary psychrometer was replaced by Assmann's well-known construction. The high artificial ventilation of this instrument is thought to render it independent of changes in the natural ventilation. The acid bulb was still exposed to the natural ventilation which was estimated on a scale whose unit is about 6 kilometres per hour. The agreement of columns f_1 f_2 in Table II. is held to show the acid bulb's independence of ventilation. k is practically the same as in Table I. over range $t-\theta=\pm 2°$. *The value of* c *was taken as unknown.*

TABLE I.

D. Dry bulb Fuess No. 2035, Reichsanstalt certificate No. $\frac{12050}{1899}$.
W. Wet bulb Fuess No. 2043, corrected by No. 2035.
A. Acid bulb Fuess No. 2063, corrected by No. 2035.
} Bulbs spherical 9 mm. diam.
f_1. Tension by formula $f'-c(t-t')$.
f_2. ,, ,, $\phi-k(t-\theta)$.

D.	W.	A.	f_1.	f_2.	Remarks.
25·9	16·1	24·3	8·58	8·53	Acid 46·20 °/o $c=\cdot52$ $k=\cdot92$
24·7	16·0	24·1	9·29	9·25	
23·4	15·6	23·4	9·11	9·30	
22·6	15·4	23·0	9·26	9·37	
21·6	15·4	22·5	9·78	9·63	
22·8	15·8	23·0	9·70	9·22	
23·3	16·1	23·6	9·86	9·78	
22·8	15·8	23·2	9·70	9·47	
21·9	15·6	21·5	9·77	10·21	Acid 41·50 °/o $c=\cdot54$ $k=\cdot91$
20·6	15·6	20·8	10·47	10·33	
19·2	15·2	20·2	10·68	10·67	
22·4	16·0	21·6	10·05	9·93	
23·2	16·0	21·8	9·61	9·51	
19·9	15·9	21·3	11·34	11·63	
18·6	12·1	17·2	7·00	6·83	
17·4	11·7	16·8	7·13	7·45	
16·4	11·4	16·0	7·33	7·16	
15·6	10·6	15·2	6·81	6·81	
15·0	9·6	14·0	5·90	5·71	
19·1	14·5	18·2	9·97	10·11	Acid 33·45 °/o $c=\cdot50$ $k=\cdot91$
17·8	13·8	17·0	9·73	9·40	
15·4	12·2	15·5	8·97	9·32	
13·2	10·8	13·6	8·44	8·51	
10·6	9·0	11·4	7·75	7·74	
14·4	11·3	14·3	8·41	8·44	
17·2	10·9	14·6	6·55	6·33	

TABLE II.

D = Air temperature in Renou screen for mean of thermometers
Fuess 2035 Reichsanstalt certificate No. 12050 } agreeing
and Fuess 1143 ,, ,, ,, 12207 } closely.
W = Wet bulb temperature by Fuess No. 1144 corrected by No. 2035, certificate No. 12207.
A = Acid bulb as in Table I. $c = \cdot4$ and $k = \cdot93$.
f_1 = Tension of vapour in air from formula $f_1 = f_W - \cdot4(D-W)$.
f_2 = ,, ,, ,, ,, $f_2 = f_A - \cdot93(D-A)$.
Constants c and k deduced from all the observations.
Acid A 43·75°/o. B 52·13°/o.

D.	W.	A.	f_1.	f_2.	Diff.	Acid.	V.
16·8	13·8	19·3	10·5	10·5	0·0	A	1
16·7	13·3	19·1	10·0	10·2	−0·2	A	1–2
17·2	13·6	19·3	10·2	10·1	+0·1	A	1–2
18·4	14·5	20·3	10·7	10·5	+0·2	A	1–2
19·7	15·6	23·9	11·7	11·3	+0·4	B	0
18·4	14·0	20·0	10·1	10·0	+0·1	A	0
24·0	19·5	26·2	15·0	14·6	+0·4	A	1
19·3	14·6	20·8	10·5	10·3	+0·2	A	1

D.	W.	A.	f_1.	f_2.	Diff.	Acid.	V.
20·0	15·2	21·4	10·9	10·4	+0·4	A	4–5
20·0	15·7	22·0	11·6	11·5	+0·1	A	0
20·9	15·2	21·9	10·5	10·5	0·0	A	0
22·5	16·7	23·8	11·8	11·9	−0·1	A	5
21·9	15·9	22·6	11·0	10·7	+0·3	A	4
22·2	16·1	22·9	11·2	10·9	+0·3	A	3
22·7	16·4	23·3	11·4	11·0	+0·4	A	4
22·6	16·2	25·6	11·1	11·0	+0·1	B	0
21·6	14·8	21·6	9·8	9·4	+0·4	A	0
23·9	17·0	24·1	11·6	11·1	+0·5	A	4
22·9	16·0	23·2	10·7	10·6	+0·1	A	3
27·1	20·2	28·2	14·8	15·0	−0·2	A	3
26·3	19·4	27·2	13·9	14·2	−0·3	A	2
21·3	14·3	21·3	9·3	9·2	+0·1	A	3
23·5	16·5	23·8	11·1	11·0	+0·1	A	2
24·0	16·8	24·1	11·3	11·0	+0·3	A	3
21·7	14·4	21·4	9·3	9·0	+0·3	A	0
21·2	14·0	23·3	9·0	9·2	−0·2	B	0
21·4	13·8	23·2	8·7	8·8	−0·2	B	2
27·0	19·4	27·6	13·7	13·7	0·0	A	5
23·2	15·6	25·3	10·2	10·2	0·0	B	2
24·9	17·1	27·6	11·4	11·7	−0·3	B	1
23·6	15·9	23·2	10·3	9·9	+0·4	A	4
27·2	19·2	27·2	13·3	13·2	+0·1	A	4
24·0	15·9	26·2	10·2	10·5	−0·3	B	2
22·7	14·6	22·0	9·2	8·9	+0·3	A	2–4
24·9	16·2	23·9	10·2	9·8	+0·4	A	0
22·1	13·3	22·9	7·9	7·7	+0·2	B	3
25·5	16·7	27·5	10·6	11·1	−0·5	B	2
25·9	17·0	27·5	10·8	10·7	+0·1	A	1
27·2	18·2	26·3	11·9	11·7	+0·2	A	5
24·5	15·4	23·3	9·4	9·3	+0·1	A	0
23·4	14·3	22·2	8·5	8·8	−0·3	A	0
23·3	14·0	21·6	8·2	7·8	+0·4	A	0
23·0	13·7	21·4	8·0	7·7	+0·3	A	2
26·4	17·1	28·2	10·8	11·3	−0·5	B	1
25·4	16·0	24·0	9·7	9·6	+0·1	A	2
26·1	16·5	27·2	10·1	10·2	−0·1	B	2
23·3	13·7	21·5	7·9	7·6	+0·3	A	3
25·3	15·6	26·3	9·3	9·4	−0·1	B	1
25·6	15·9	23·8	9·5	9·1	+0·4	A	0
25·5	15·7	23·8	9·4	9·1	+0·3	A	1
23·6	13·6	21·7	7·6	7·6	0·0	A	1
24·6	14·3	22·2	8·0	7·5	+0·5	A	2
26·1	15·6	26·7	9·0	9·4	−0·4	B	2
27·3	16·5	27·7	9·7	9·7	0·0	A	2
25·0	14·1	22·2	7·6	7·1	+0·5	A	4
25·0	14·1	22·2	7·6	7·1	+0·5	A	0
24·4	13·4	24·2	7·0	7·4	−0·4	A	0
26·9	15·8	27·2	8·9	9·3	−0·4	A	3
27·1	15·7	27·1	8·7	9·0	−0·3	A	3
25·5	14·3	22·8	7·6	7·6	0·0	B	2
25·7	14·3	23·2	7·5	8·0	−0·5	A	0
26·6	14·6	23·2	7·6	7·1	+0·5	A	3
27·1	14·9	24·0	7·7	8·0	−0·3	A	2
26·3	14·2	25·2	7·2	7·0	+0·2	B	4
28·4	16·2	28·1	8·8	9·2	−0·4	B	2
25·9	13·3	24·8	6·4	6·8	−0·4	B	4
29·3	16·5	28·1	8·8	8·4	+0·4	A	0
25·4	12·5	24·0	5·7	6·1	−0·4	A	3
24·8	11·8	23·2	5·1	5·6	−0·5	A	1
29·1	16·1	28·2	8·4	8·8	−0·4	A	3

Table III. deals further with ventilation. A wet bulb and an acid bulb were mounted at opposite ends of a diameter of a whirling machine, in a room where the air was very calm and of even temperature. They were read at rest and again after 2 minutes' whirling at about 6 metres per sec. The result shows an almost perfect *immunity towards ventilation* except in the cases *where* D − A *exceeds* ± 3° (see Table III.). The observers who kindly assisted were kept entirely unaware of the effect to be looked for.

TABLE III.

D = Temperature by dry bulb thermometer.
W_1 = ,, wet ,, before } whirling.
W_2 = ,, ,, ,, after
A_1 = ,, acid ,, before } whirling.
A_2 = ,, ,, ,, after
All bulbs 0·9 centimetre diameter.

Strengths of acid adjusted to give very different values of D − A for similar values of D − W.

D.	W_1.	W_2.	A_1.	A_2.	$D-A_1$.	A_1-A_2.	W_1-W_2.	Observer.
24·0	15·8	15·0	20·7	20·6	3·3	+0·1	+0·8	M.K.
23·8	15·8	14·8	21·2	21·0	2·6	+0·2	+1·0	,,
23·8	16·0	15·3	18·6	18·2	5·2	+0·4	+0·7	,,
23·8	15·8	15·0	22·8	22·8	1·0	0	+0·8	,,
23·7	16·1	15·3	23·7	23·8	0	−0·1	+0·8	,,
23·8	16·2	15·3	26·2	26·1	2·6	+0·1	+0·9	,,
23·7	16·2	15·3	29·6	29·2	5·9	+0·4	+0·9	,,
23·6	16·0	15·3	27·5	28·0	−3·9	−0·5	+0·7	,,
24·4	16·2	15·4	27·0	27·2	−2·6	−0·2	+0·8	,,
23·6	15·8	15·0	27·6	27·6	−4·0	0	+0·8	W.
23·6	15·8	15·0	27·2	27·3	−2·6	−0·1	+0·8	,,
23·4	16·0	15·2	26·0	26·0	−2·6	0	+0·8	,,
23·4	15·6	14·8	25·4	25·6	−2·0	−0·2	+0·8	,,
23·4	15·8	15·0	25·2	25·4	−1·8	−0·2	+0·8	,,
23·4	15·6	14·6	25·2	25·2	−2·2	0	+1·0	,,
23·4	15·8	15·0	24·8	24·9	−1·4	−0·1	+0·8	,,
23·2	15·8	15·1	24·4	24·6	−1·2	−0·2	+0·7	,,
23·0	16·0	14·8	24·4	24·5	−1·4	−0·1	+1·2	,,
23·4	15·7	14·8	24·4	24·4	−1·0	0	+0·9	M.S.
21·4	15·6	15·1	20·6	20·6	+0·8	0	+0·5	W.
22·2	15·0	14·4	19·0	19·0	+3·2	0	+0·6	,,
22·2	14·9	14·2	19·9	19·9	+2·3	0	+0·7	M.K.
22·6	15·4	14·8	20·8	20·8	+1·8	0	+0·6	,,
22·8	15·6	14·9	18·5	18·1	+4·3	+0·4	+0·7	,,

(6) *Applicability of the method.*—The method is recommended as a reference method to those who wish to find the constants for their wet and dry bulb. They may proceed exactly as described above, preferably taking *c* and *k* as both unknown. The method is also available as a substitute

for the wet bulb. It cannot compete in simplicity with the wet bulb in its simplest form, but it can compete with the various modifications (whirled thermometers, &c.).

For the range studied the constant ·93 can be used for k, and it is hoped that this will be tested for other climates and below the freezing-point. A small error in k is of very little importance, owing to the small value of $t-\theta$.

(7) *Conclusion.*—The method has been shown simple; the elementary theory is thought rigorous; the extension to small thermal effects is justified in numerous experiments; k has turned out to be very nearly ·93 over as wide a range as was open to study; c is unexpectedly low. The last point requires some discussion. According to a mass of published dew-point observations, a value 0·5 for c was expected in Table II., and the writer thinks it superfluous to make additional dew-point observations. The low value 0·4 found for c, though opposed somewhat to the dew-point results, is favoured by Maxwell's theory[*]. To make $c = {\cdot}5$ in high ventilation is to make his ratio $\frac{K}{D}$ (*loc. cit.*) practically unity, but he considers it probably ·77 and "certainly less than unity." His estimate is supported by the experiments of Graetz, Winkelmann, and of Kundt and Warburg, so that a considerably lower value of c is suggested. *It is not, however, the object of this note to determine the value of* c. It is the *method* with its theoretical and practical claims which is put forward. A wider basis was rendered impossible from want of leisure.

Abbasia Observatory, Egypt.
August 22, 1901.

XLII. *Air-Pressures used in playing Brass Instruments.* By Edwin H. Barton, *D.Sc., F.R.S.E., Senior Lecturer in Physics at University College, Nottingham, and* S. C. Laws, *B.Sc., "1851 Exhibition" Science Research Scholar*[†].

[Plates VII. & VIII.]

It is well known that in playing upon the "brass" or "wood-wind" instruments of the orchestra the particular note, at any instant desired, is produced by the simultaneous use of the mechanism of the instrument and the corresponding "embouchure," through which air at a suitable pressure is driven by the performer. The mechanism adjusts the

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[*] Art. "Diffusion," *Encyc. Britt.*

[†] Communicated by the Physical Society: read December 13, 1901.

length of the tubing in use so that the note in question is one of those with which the instrument is now in resonance. The correct embouchure and air-pressure pick out, as it were, from the notes then possible to the instrument, the particular one sought, and cause it to sound.

Some of the problems here involved are chiefly of a musical character.

But the pressure of air which solicits the desired note, although several others are possible with the same fingering, and the variations of this pressure with the pitch, loudness, and other circumstances form a problem belonging rather to the domain of physics, and one which is attacked in the experiments here described.

Some previous work on the air-pressures required for various instruments has been done by Dr. W. H. Stone (see p. 171, 'Elementary Lessons on Sound,' Macmillans, 1891), who gives maximum and minimum pressures for the oboe, clarinet, bassoon, horn, cornet, trumpet, euphonium, and bombardon. No details, however, are there given of the pressures for each note of the scale, nor of the variations of the pressure with loudness.

The present work deals with three brass instruments, viz., the tenor trombone, the trumpet, and the cornet. The results of the experiments on these are sufficiently consistent to support certain general conclusions which are summarized at the close of the paper.

Experimental Method.—Following the hint given by Dr. Stone, the pressures were taken by a water-manometer connected to the performer's mouth by an indiarubber tube terminating in a glass nozzle which could be held by the side teeth. After a little practice, the instrument could be played without the insertion of this side nozzle producing any serious annoyance. The open end of the manometer was provided with a plug of cork and cotton-wool which just damped out the oscillations of the water columns in it without interfering with their prompt assumption of the correct levels when a note was sounded. To arrest the columns immediately in these positions and retain them there until the readings were taken, a pinch-cock was used on the indiarubber pipe connecting the manometer with the player's mouth. Thus the *modus operandi* was as follows:—A series of notes and their loudness being decided upon and indicated in the note-book and the player and recorder in position, the recorder calls for a certain note, at the same time opening the pinch-cock. The player sounds the note and sustains it till it is judged to be of the desired intensity; the pinch-cock is then released, thus immediately closing and so retaining the water columns in place. These are read and recorded and then the next note is called for.

The manometer-scales were graduated to centimetres and estimated to quarters of a division.

A mercury manometer was tried for some of the higher pressures exceeding a metre of water, but was quickly abandoned as being far less convenient.

Lines of Investigation.—It was decided to find how the air-pressure required to sound the different notes varied with (1) the pitch of the note, (2) its loudness, (3) the fingering or other manipulation of the instrument, (4) on the instrument itself. In each of these cases of inquiry as to the effect of any one variable cause, the other variables were kept constant so far as circumstances permitted. Thus, to test how the pressure depended upon pitch, a scale was played with the loudness maintained as nearly as possible the same. The fingering was, however, necessarily changed to produce the consecutive notes of the scale. Again, for dependence of pressure on loudness, the same note was played with various intensities, then another note with various intensities, and so on. Thirdly, where the same note was obtainable with two fingerings, the sound was obtained of the given pitch and with the same intensity, first with the usual and then with the alternate fingering.

The Scale on Brass Instruments.—In order to render subsequent references quite clear, it seems desirable to explain here how the scale is obtained both with slide and valve instruments. In the slide instruments, when the slide is in the first position, *i. e.* closed, the minimum length of tubing is in use, and the only notes obtainable are those with which that length of tubing is in resonance. These form the harmonic series, the relative frequencies of the notes being 1, 2, 3, 4, 5, 6, &c.; the fundamental or prime is termed by musicians the "pedal" and is rarely used. By drawing the slide out to the second position, sufficient length of tubing is introduced to put the instrument in resonance with notes a semitone lower. The other positions to the seventh inclusive lower the notes of the instrument by a tone, a tone and a half, &c. to three tones respectively. The use of the slide has thus three effects: (1) it furnishes the complete chromatic scale in the largest interval which occurs between the notes in use natural to the instrument with the slide closed, viz., between the second and third harmonics, whose interval is a fifth ; (2) and consequently it more than bridges all the other intervals (fourth, major third, &c.) which occur in the higher part of the harmonic series, thus giving alternate methods of playing various notes ; (3) it extends the scale downwards.

In valve instruments the same results are usually approximately obtained by the use of three valves. The first valve, when depressed, adds sufficient tubing to lower the notes by a whole tone, the second valve a semitone only, and the third about a tone and a half.

These facts are exhibited collectively in Table I. for an instrument tuned to B♭ like the Cornet and Tenor Trombone. The lower notes are at the bottom of the table. Those in round brackets indicate alternative fingerings. The numerals under some of the notes give approximate frequencies for the trombone in complete vibrations per second. For the cornet, all these require doubling.

TABLE I.—The Scale on Slide and Valve Instruments.

Positions of Slide on Trombone.	1st.	2nd.	3rd.	4th.	5th.	6th.	7th.
Valves depressed on Cornet.	None.	2nd.	1st.	1st & 2nd.	1st & 3rd.	2nd & 3rd.	1st, 2nd, & 3rd.
Number of partial tone or "Harmonic" elicited by suitable "embouchure" and air-pressure.	Names of Notes.						
8	B♭ 480	A		G			
7	This general	harmo ly used	nic is	not			
6	F 360	E		(D) 300			
5	D 300		C		(B♭) 240		
4	B♭ 240	A		G		(F) 180	
3	F 180	E		D		C	B
2	B♭ 120	A		G		F	E
1	B♭ 60	This is not	prime genera	note, cal lly used.	led the	"pedal,"	

Experimental Results. *Tenor Trombone.*—The instrument used for these experiments is in B♭ by Millereau & Co. of Paris, and was throughout played by E. H. Barton with a Higham mouthpiece.

The mean results of scales and exercises over two octaves tried on it at different intensities are given in Table II., and graphically exhibited in Plate VII. The ordinates of the curves are the manometer pressures in centimetres of water, and the abscissæ represent the intervals in cents, the intonation being assumed in equal temperament.

Both in the table and on the Plate the C in square brackets is the middle C between the staves, and is of frequency about 269 complete vibrations per second.

Table II.—Pressures for Scales on Trombone.

Notes	F	G	A	B♭	C	D	E	F	G	A	B♮	[C]	D	E	F
Positions of Slide.	6	4	2	1	6	4	2	1	4	2	1	3	1	2	1
f. Mean of 4 sets.	18	26	32	37	40	47	53	58	65	73	80	88	108	117	122
m. Mean of 11 sets.	17	23	27	31	34	42	51	52	57	63·7	64·2	71	76	81	90
p. Mean of 4 sets.	13	17	22	27	27·5	35	42	39	44	48	45·5	52	59	65	74

The results of experiments on given notes sounded in close succession but with different intensities are given in Table III. (p. 390).

The experiments as to change, if any, of pressure with change of fingering are detailed in Table IV. (p. 390).

Cornet.—These experiments were made with a Higham Cornet played by a professional trumpeter and cornettist, Mr. E. C. Pickerill of Nottingham. The results for scales over two octaves soft, medium, and loud, each played once only, are shown by the three curves on Plate VIII. marked respectively *p*, *m*, and *f*. The C in square brackets denotes the middle C between the staves. The abscissæ and ordinates are as in Plate VII.

TABLE III.—Pressures for given Notes, Loud and Soft.

Notes	F	B♭	F	B♭	D
Approximate frequencies.	90	120	180	240	300
Positions of Slide.	6	1	1	1	1
pp.	13	24*	27	50	73·5
p.	18	34·5	50	57	83
m.	19·5	36·5†	55·5	63	88·5
f.	23	40	67·5	82	106
Single crescendo notes.		*pp* 20 to to *f* 38	*p* 50 to to *ff* 78	*p* 61 to to *ff* 101	

* This pressure was obtained with a small aperture between the lips. With a larger aperture, the note still being *pp*, the pressure fell to 15 !

† The pedal note an octave lower than this was produced *m* with a pressure of 7 cm.

TABLE IV.—Pressures for same Notes in Alternative Positions.

Notes		F		B♭		D	
Approximate frequencies.		180		240		300	
Positions of Slide.		1	6	1	5	1	4
No. of "Harmonic" used		3	4	4	5	5	6
1st Series (1897).	*p*	50·5	48	72	70	72	67
	mf	50	50	75	75	98	92
	f	73	68	94	92	119	117
	ff	80	83	113	110		
2nd Series (1901)	*m*	55	50	67·5	68·5	72	72

The experiments on the cornet as to different intensities of given notes are detailed in Table V.

TABLE V.—Pressures for given Notes, Loud and Soft.

Notes ……		A	E	A	C♯	E
Approximate frequencies.		226	339	452	568	678
		Pressures.				
Intensities.	*f*	26·75	34·25	44	52·75	66
	m	22·5	26	31·5	38·75	41·5
	p	12	15·5	20·5	30·25	34

Trumpet.—For the experiments on the Trumpet a Besson trumpet in F was used, played by Mr. E. C. Pickerill with a mouthpiece by H. Keats & Son. The results of once playing the scale in F through two octaves at medium intensity is shown on Plate VIII. by the dotted curve marked TT.

Comparison of Trumpet and Cornet.—Table VI. gives the result of a comparison of trumpet and cornet as to the pressures for given notes of different intensities.

TABLE VI.—Trumpet and Cornet compared.

Notes ……		B♭	F	B♭	D	F
Approximate frequencies.		240	360	480	600	720
Trumpet.	*f*	25·75	43·75	45·5	48·75	68
	m	23·25	28	34·5	41·5	57·25
	p	16	18·5	26·75	28·75	42
Cornet.	*f*	26·5	45	47·25	55·75	75·75
	m	18	28·5	32·75	41	51·75
	p	13	24	22·5	32·25	39·25

Conclusion.—In reviewing the results of the experiments it is clear that the following general inferences may be drawn:—

1. Other things being equal, the louder the note the greater the pressure, as was to be expected. See Tables III., IV., V., and VI.

2. Again, all else being retained the same, the higher the pitch of the note played on a given instrument the greater the air-pressure used. See Tables III., V., and VI.

In the scales (Table II. and Pls. VII. & VIII.) this is sometimes apparently violated, not only on the trombone played by the writer but also in the professional playing of the trumpet and cornet. This seems to be due to a slight unintentional difference in intensity or in the manner of producing the notes. (See first footnote to Table III.) In the higher parts of the scale it is hard to avoid increasing the loudness. This may account for the upward turn of the curves at this part.

3. In the preliminary experiments, on plotting the frequency of the notes of the scale as abscissæ and the pressures as ordinates fairly smooth curves were obtained convex upwards. When, however, quantities proportional to the logarithms of the frequencies are taken for abscissæ, the curves become almost straight lines. See Pls. VII. & VIII., in which the abscissæ are Mr. Ellis's logarithmic cents, 100 to the equally-tempered semitone, represented by the large squares, and the intonation is assumed to be in equal temperament.

4. Now if the logarithms of the frequencies be taken to measure pitches of notes, then the difference of logarithms measures the ratios of frequencies, *i. e.* measures the intervals between them. And, as is well known, this is the only way which admits of simple addition of numbers representing the component intervals giving the number representing the resultant interval. Thus the logarithm of the frequency is in a certain very real and important sense the best physical measure of the pitch of a note. Hence, adopting this method, and noting that the curves in Pls. VII. & VIII. are nearly straight lines, we may say that the air-pressure required to sound any note with given intensity is approximately proportional to its pitch defined logarithmically.

5. Where alternative positions or fingerings are used for the same note the pressures are practically the same, see Table IV., the results in which were confirmed also by Mr. Pickerill with his instruments. In other words, it is almost indifferent as regards pressure required, whether a note is played on a given instrument as the third harmonic

of a short tube or the fourth of a longer one, &c., &c. This is contrary to what is implied by some writers.

6. The pressures for identical notes on trumpet and cornet are almost the same for any given intensity, but very much less than those for the same notes on the trombone. See Table VI. and Plate VIII. (*cf.* with Table III. and Plate VII.).

7. The pressures used for loud low notes may exceed those for soft high notes, in some cases even where the notes are an octave apart. See Tables III., IV., V., and VI., also Pls. VII. & VIII.

8. The present results as to maximum and minimum pressures are in accord with Dr. Stone's, so far as comparison is possible.

Our thanks are due to Mr. Pickerill for kindly placing his services at our disposal.

University College, Nottingham,
August 1901.

XLIII. *The Induction Coil.* *By* Prof. JOHN TROWBRIDGE *.

LORD RAYLEIGH, in a recent article (Philosophical Magazine, Dec. 1901), has shown the significance of the electrostatic capacity between the primary and the secondary of the induction-coil, and has proved that in certain cases a condenser in the primary is of no advantage, and in certain cases may be prejudicial.

The performance of a powerful coil devised by Mr. Heinze, of Boston, U.S., for me, operated by his liquid interrupter, justifies the points raised by Lord Rayleigh; and, on account of the importance of the induction-coil in wireless telegraphy, merits I think wider knowledge than it has received.

The dimensions of the coil are as follows:—

Length of iron of primary coil, 48 inches.
Diameter of " " " $3\frac{1}{2}$ "
Wound with one layer of no. 6 copper wire.

The secondary coil consists of 140 coils separated from each other by plates of glass one eighth of an inch thick. Each of these coils is wound with, between 400 and 500 turns, fine wire. The entire resistance of the secondary coil is 246,600 ohms.

The secondary is merely inclosed in a box. By unscrewing the cover of this box the sectional coils can be inspected, and any sectional coil can be removed if its insulation breaks down; for the coils are not imbedded in paraffin. It was

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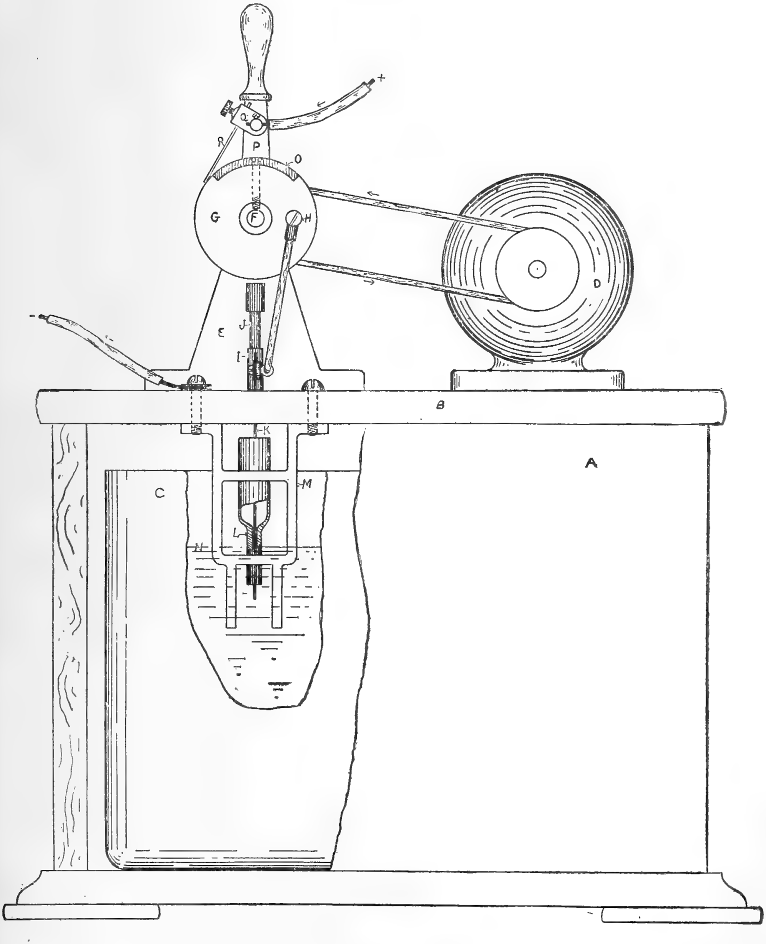
* Communicated by the Author.

found necessary to use glass at least one eighth of an inch thick between the sectional coils, on account of the great electric density developed. Mr. Heinze's experience in this respect corroborates my own. In employing a storage-battery of twenty thousand cells with an electromotive force of forty thousand volts to charge Franklin plates, I have found it necessary to also employ plates of glass one eighth of an inch thick. The great surface-density developed by a powerful rush of current into such condensers to raise them to a difference of potential of even five or ten thousand volts, makes it unsafe to employ glass of less thickness than one eighth of an inch.

When this coil is excited by five storage-cells and is provided with a condenser in the primary, it gives, under the effect of an ordinary vibrating break, a fifteen-inch spark. When, however, the condenser is removed, and the coil is operated by a liquid interrupter shown in fig. 1, the length of spark is increased to thirty inches and is of astonishing body.

The interrupter is contained in a wooden box A (fig. 1), the cover of which swings on hinges and supports all the mechanism. By lifting the cover, the parts can be easily examined. The front of the box consists of a door which, when open, exposes a glass jar C containing acidulated water, through which the mechanism can be seen. A small motor on the top of the box drives the oscillating mechanism, which consists of a base-casting E having a bearing and a shaft F, upon one end of which is a crank-disk G, and at the other end a pulley for receiving its power from the motor. By the rotation of disk G and crank-pin H, an up and down movement is imparted to a brass sleeve I, which moves up and down upon a vertical spindle J secured to the base-casting E. This sleeve I has adjustably secured to it a platinum wire K, which projects below the cover B, and moves in and out through a capillary tube closely fitting the wire. This glass tube L is held in a lead electrode M, so constructed as to partly surround and to come very close to the platinum wire when the latter projects through the capillary tube. The electrode M, glass tube L, and platinum wire K are immersed in the acidulated solution, so that the end of the capillary tube is about an inch below the top of the liquid N. The liquid is a twenty per cent. solution of sulphuric acid. The crank-disk G is made of slate, and has upon its periphery a 90-degree segment of brass O which is connected to the shaft F. The segment O bears a certain relation in position to brass sleeve I and depth of platinum wire K. There is also a movable brush-holder yoke P, which has secured to it, but insulated from it, a brush-holder Q and brush R, which rest on disk and segment O during one-fourth of its revolution.

Fig. 1.



The electric current flows as follows:—The positive wire from the mains (I use a one-hundred-and-ten volt circuit) connects with the brush-holder P, and the current flows through the brush R to segment O, to shaft F, to base-casting E, to vertical spindle J, to brass sleeve I, to platinum wire K; and when the wire is immersed in the liquid N, the current flows through it to the lead electrode M, to one end of the primary of the coil, and out by the other end to the negative wire of the mains. By the movement of the brush one can regulate the amount of current passing through the primary.

The number of interruptions per minute can be changed from sixty to five thousand by varying the speed of the motor. The regulating handle of the motor will change the spark-length of the coil from zero to thirty inches and will also vary the quantity of the discharge.

The momentary rush of current through the primary coil is probably very great, although the ammeter in the primary circuit indicates not more than 5 or 10 amperes. I have found it impossible to produce a spark in air longer than seven feet by means of a storage-battery and a Planté transformer. I imagine that a large coil built on the above principle and actuated by very powerful commercial currents could produce sparks much longer than seven feet. Dynamos are now built of a size far exceeding what was once thought possible. The needs of wireless telegraphy may demand a corresponding increase in the size of induction-coils; and the coil of the type I have described, without a condenser and with a liquid break, seems to be the type for the future, if powerful discharges are desired. The mechanical break and the primary condenser must be relegated to the historical physical cabinet.

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XLIV. *A suspected case of the Electrical Resonance of Minute Metal Particles for Light-waves. A New Type of Absorption. By* R. W. Wood, *Professor of Experimental Physics in the Johns Hopkins University*[*].

CERTAIN experiments on which I have been engaged of late have led me to believe that I have found a new type of light absorption, which it may be possible to refer to the electrical resonance of small metallic particles for waves of light. The experiments of Garbasso and Aschkinass have shown that a plate of glass covered with uniformly arranged strips of tinfoil of equal size, which serve as resonators, shows the phenomenon of selective transmission and reflexion for electromagnetic waves of different wave-lengths. In other words, a plate of this description exhibits the electrical analogy of surface-colour. I have succeeded in producing

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[*] Communicated by the Physical Society: read March 14, 1902.

metallic deposits on glass which the microscope shows to be made up of particles smaller than the wave-lengths of light, which by transmitted light exhibit colours quite as brilliant as those produced by aniline dyes. I have sought in every way to explain these colours, by all the well-known principles of interference and diffraction, and at the present am forced to accept the hypothesis suggested in the title of this paper, which I have chosen not without some misgivings.

In the present paper I shall content myself with giving a full account of the experimental data which I have thus far obtained, postponing for the present anything like a definite conclusion.

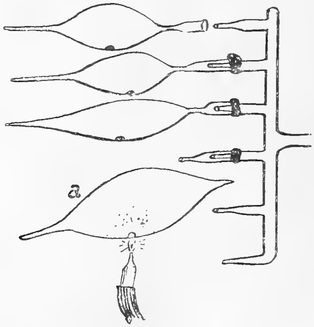
The metallic deposits are obtained by heating small fragments of the alkali metals in glass bulbs, thoroughly exhausted and hermetically sealed. Only the small portion of the bulb on which the metal particle lies is heated, leaving the remainder, where the condensation is to take place, quite cold. The metallic film which condenses on the wall, when viewed by transmitted light, shows colours of excessive brilliancy, as brilliant in fact as films strongly stained with aniline dyes. On first obtaining these films, I was of the opinion that these colours could be attributed in some way to interference, but more careful consideration showed that many difficulties were present. In the first place, the transmitted colours, in the case of thin plates, are never of any considerable intensity, being diluted with a large excess of white light. Moreover, the extraordinarily high absorption-coefficient of metallic sodium makes it impossible for us to apply the theory of the colours of thin plates, except on the assumption that the thickness of the film is exceedingly small in comparison with the wave-length, and that there is a relative change of phase at the two surfaces, which varies with the wave-length. This seemed on the whole not very probable, for, as far as experiments go, indications are that the phase-change in metallic reflexion is nearly independent of the wave-length. Moreover, metallic films obtained in other ways, as by chemical or cathodic deposition, do not show these colours. Thin films of metal show a more or less marked colour by transmitted light, but the colour is fixed for any definite metal, and, except in the case of gold and silver, not very pronounced. The sodium films, on the contrary, may be deep purple, blue, apple-green, or red according to conditions. I noticed at the outset that where the deposit was somewhat dense in the immediate vicinity of the sodium drop, the outside of the bulb, seen by reflected light, exhibited halo effects, similar to those shown by certain minerals. This pearly or silky lustre, which is difficult to describe, I at once recognized as a diffraction effect due to the scattering of light by small reflecting particles. Examination under the microscope showed this to be the case, the sodium having deposited on the glass in the form of small isolated drops. It at once occurred to me that the condition might be the same in the coloured films, the particles here being small enough to respond to the light-waves, in the same manner as the resonators of Garbasso and Aschkinass. I examined some of the more transparent of the coloured films, under a one-twelfth inch oil-immersion objective, and found that my suspicions were verified, the films being made up of particles just barely visible as individual grains lying close together. I was unable to satisfy myself that the size varied with the colour of the film, and am of the opinion that if the resonance theory is the true one, the free period depends more on the proximity of the particles than on their actual size. The diameter of the particles, as measured with the micrometer, varies from about ·0003 to ·0002 mm. In other words, they are of about the size of the smallest micrococci described by the bacteriologists.

It occurred to me at first that the colours might be similar to those shown by mixed plates, which are explained in the same way as the coloured central images seen in certain transparent diffraction-gratings. This would, however, necessitate the presence of minute transparent particles of a thickness sufficient to give a retardation of $\lambda/2$. The formation of such particles could be conceived of as resulting from the oxidation of the sodium particles by a layer of absorbed air on the glass. It is therefore necessary to show at the outset that the cause of the colour is the presence of metallic sodium. That this is the case I feel sure, for if the slightest trace of air be allowed to enter the bulb, the colour, no matter how intense, instantly vanishes, the glass becoming as clear as before the formation of the film. The oxidation process could be made to take place slowly, by drawing out one end of the bulb into a long fine capillary, with a bore of about ·01 mm., and breaking off the end after the coloured film had been formed.

In certain cases it was observed that the colour changed before disappearing entirely. I also found that the colour of the film could be permanently changed by carefully heating the glass to a temperature just below that necessary to evaporate the film. Local heating to a slightly higher temperature drives off the film entirely, leaving the glass perfectly clear. By means of a small flame, it is possible to drive a patch of colour all over the inside of a bulb, and it is in this way that some of the best films are prepared. It was next found that cooling the films by the application of ice to the exterior of the bulb produced most extraordinary changes of colour. Pale green films, almost transparent, on being cooled ten or fifteen degrees, changed to a violet as deep as that shown by dense cobalt-glass. Pink films were changed to deep blue-green, while films originally deep blue became transparent. These effects were most puzzling at first, but I finally succeeded in finding out the cause, as I shall show later on. At the beginning of the work it was almost impossible to predict what sort of a film would be obtained in any experiment. Sometimes, on first heating the sodium, a film of a deep claret colour would be obtained; once formed, it could be driven from one part of the bulb to another without any great change of colour. In another experiment, made under exactly the same conditions, so far as I could see, it was impossible to obtain anything but a deep blue. I prepared and experimented with fully fifty bulbs, without being able to arrive at any definite conclusion in regard to the conditions necessary for producing a definite result. It was finally found however, by blowing connected bulbs, that if the sodium was thoroughly boiled in one of them, and while still molten was shaken into an adjoining bulb, the surface of which was of course free from any deposit, on heating the globule in the fresh bulb, only blue films could be produced. This made it appear as if the claret-coloured film were produced by something volatile in the sodium which was driven off by the first heating. It was necessary, if possible, to find out what this was, for these red films were found to be more interesting than the blue, and it was desirable to be able to produce them at will. As a result of many experiments, I had formed a vague notion that bits of sodium cut from certain lumps in the bottle produced the red films on the first heating, while portions taken from other lumps did not. It seemed not unlikely that the sodium might contain potassium as an impurity, which might be the agent concerned. I accordingly prepared a set of bulbs containing potassium, and immediately found that magnificent reds and purples could be obtained at will; and I felt firmly convinced that the anomalies shown by sodium were due to potassium impurities. Similar coloured films could be obtained with lithium, though, owing to its higher boiling-point, bulbs of bohemian glass are required. Letting this serve as an introduction, it will be well to take up the experiments somewhat in detail.

As the optical behaviour of the films is extremely interesting, regardless of the ultimate cause of the colour, I shall describe in some detail the manner of preparing and filling the bulbs. It is in the first place a great economy of time if a number are exhausted at once. To accomplish this I have fitted the mercurial pump with a glass tube having half-a-dozen lateral branches drawn down as shown in fig. 1. The bulbs are blown of the form shown in the same figure, and a piece of sodium or potassium is cut up under ligroin into blocks about 3 mm. on each edge. These are introduced into the bulbs as quickly as possible after wiping off the fluid, and the stems of the bulbs drawn down to a small bore for subsequent sealing.

Fig. 1.



They are quickly fastened to the branch tubes and exhausted. I find it a good plan to heat the metal until it fuses while the bulb is still on the pump; the whole bulb may be warmed by a Bunsen flame to drive off absorbed air. If the exhaustion is carried down to the point where the mercury begins to hammer, it is generally sufficient. The bulbs are now sealed off from the pump, and may be put away for future use, or experimented with at once. A burner should be made by drawing out a glass tube, which will give a pointed flame about half a centimetre high, and the tip of the flame allowed to play against the spot on the bulb where the metal lies (fig. 1, *a*). Sometimes the whole bulb will suddenly flash a deep violet or blue, and sometimes the film will develop more slowly. A chain of three or four bulbs may be made, the sodium heated in one, and the clean molten metal shaken into the others, drops of various sizes sticking to the bulbs. Coloured films can then be formed by heating these clean drops in the bulbs. This shows that the oxide present in the first lump has nothing to do with the production of the colour.

I shall now take up in order and in some detail the appearances of the films by reflected and transmitted light, the spectral analysis of the light, the polarization effects, and the very remarkable temperature changes.

If the metal is heated at one end of a rather long tapering bulb, the colour is most intense near the metal and gradually fades away to nothing at the other end of the bulb. If the bulb is placed in strong sunlight with a black background, it is seen that in some places where the deposit is too slight to show much colour by transmission the light is scattered or diffused, and this diffused light is coloured. The claret-coloured or purple film, where the deposit is slight, scatters a green light, the surface appearing as if fluorescent. Now the spectrum of the transmitted light in these purple films has a heavy absorption-band in the yellow-green, consequently the scattered light is the complementary colour of the transmitted. If the film is greenish blue, the scattered light is reddish. The microscope shows that in these deposits, which have the power of scattering light, the individual particles are rather widely separated, that is, the distance between them is large in comparison to their diameters. I shall speak of this light as the "fluorescent" light, to distinguish it from the light scattered by the large particles, which is not coloured except by diffraction. This light, which gives the bulbs a slightly silky lustre where the particles are large, I shall call diffused light.

Returning now to the fluorescent light: in the case of the purple films, the appearance of a bulb in strong light is very much as if certain portions of its interior surface had been painted over with a solution of fluorescein. No trace of regular reflexion is shown by these films, except of course the reflexion due to the glass. The particles are so far apart that they apparently act as independent sources, the interference necessary for rectilinear propagation not being present. If the incident light is polarized, the fluorescent light is also polarized, which is not the case for ordinary diffuse reflexion. On the resonance theory, we may regard this fluorescent light as the energy radiated from the resonators, as a result of their forced vibrations. Passing now to a part of the film where the colour of the transmitted light is deeper, we find that there is no longer any trace of this fluorescent light. The colour absent in the transmitted light is now regularly reflected, the particles being so close together that interference, as imagined by Huygens, takes place. In strong sunlight, this portion of the film shows neither the fluorescent nor the diffused (diffracted) light. Except at the spot where the regularly reflected image of the source of light appears, it is quite dark like the clear glass. Still farther down the bulb where the particles are very large, we get ordinary scattered light, and the film has the silky lustre to which I have alluded.

Summing up the results thus far, we find :—

Coarse particles diffract or scatter light, and give the bulb a silky lustre.

Minute particles very close together, regularly reflect those wave-lengths absent in the transmitted light, but give no scattered light.

Minute particles far apart, diffuse light of the same wave-lengths as those which are, to some extent, absent in transmitted light. This type of scattered light, which for convenience I have called fluorescent, is exhibited by deposits which are too scant to show any trace of colour when viewed by transmitted light.

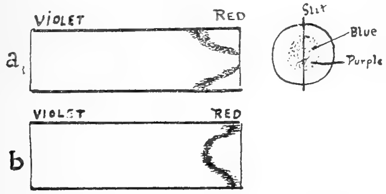
Spectrum of the Transmitted Light.

An examination of the transmitted light with a small direct-vision spectroscope showed that in some cases the films completely refused transmission of certain wave-lengths. The claret-coloured films had an absorption-band in the yellow, which was exceedingly black and quite narrow, its width rather less than one tenth of the easily visible spectrum. Any theory accounting for the colour of this film must explain the complete extinction of certain wave-lengths. In the case of the blue films, the absorption-band was in the red, but in this case it invariably appeared more diffused. The green films apparently absorbed, more or less completely, the blue and red ends of the spectrum.

The changes of colour which accompany change of temperature are of two distinct types : a permanent change usually the result of local heating, and a temporary change, the result of cooling.

I was much puzzled by the fact that the change in colour was in the same direction in both cases ; the region of absorption appearing to shift in the direction of greater wave-lengths. Purple-red films, with an absorption-band in the yellow, changed to blue, the change being due to the shifting of the band into the red. The effect can be best observed by touching the outside of the bulb with a hot glass rod. A transparent spot then appears where the metal has entirely evaporated, surrounded by a blue ring on a purple field. Placing the slit of the spectroscope along a diameter of the ring, the spectrum appears as in fig. 2, *a*. With potassium films, I have sometimes obtained permanent changes in the opposite direction ; a violet-blue film becoming pink when heated, the spectrum appearing as in fig. 2, *b*. I do not feel sure that I have determined the exact cause of the permanent colour-changes, due to heating. It doubtless depends on the partial evaporation of the metallic particles, and we must bear in mind that the period of vibration depends, not only on their size, but on the distance between them ; both of which quantities will be affected by evaporation. The matter is probably further complicated by the fact that, in some cases, I have had films made up of particles of two different sizes. In this case it is not difficult to imagine that the smaller ones are completely vaporized, while the larger ones, which previously were of too great size to produce colour, are brought down to dimensions small enough to enable them to resonate. This might explain the change from purple to blue. Experimental evidence is, however, rather against this hypothesis; and for the present I prefer to leave the cause of the permanent change in colour an open question.

Fig. 2.



The changes produced by cooling are much more startling, and of greater interest, for the cause has been discovered. We may summarize these effects something as follows :—Pink and purple films become blue when cooled, the original colour returning as soon as they are warmed up again. A pale green film becomes deep violet, while in some cases I have found that films, originally blue, become almost perfectly transparent when cooled, the blue colour returning subsequently. These effects are best shown by touching the outside of the bulb with a small piece of ice. At first sight, it seems as if the only possible effect of cooling would be a slight reduction of the size of the particles, and a contraction of the supporting wall. That this should be sufficient to affect the period of their vibration was almost unthinkable. If it were so, a further lowering of temperature should produce further changes, but I found that it was quite immaterial whether the bulb was touched with ice, or with a mixture of solid carbonic acid and ether ; the resulting colour being the same in both cases. I have frequently obtained films so sensitive, that merely moistening the outside of the bulb and blowing upon it was sufficient to produce the change. On wiping it, or on stroking the bulb with the finger, the original colour would return. This was strongly suggestive of a molecular change, and for a while I was inclined to regard it as analogous to the well-known case of iodide of mercury. The final solution occurred to me in the following way. I had been for some time endeavouring to introduce into the bulb some fluid of high dielectric constant, to determine its effect on the colour. All fluids tried apparently contained sufficient oxygen to instantly bleach the films. The introduction of such a fluid should, it seemed to me, increase the capacity, slow down the period, and shift the absorption-band towards the red, or perhaps out of the visible spectrum entirely. Films had often been obtained which showed absolutely no trace of colour-change on cooling, and it occurred to me that possibly traces of hydrocarbon vapour, from the ligroin in which the metal had been kept, might be condensed on the wall by the application of a cold body. I first prepared some films from sodium and potassium which had been heated for some time *in vacuo*, in order to remove all traces of the hydrocarbon. These films were invariably unresponsive, to either ice or solid carbonic acid. I next made a double bulb, and formed a film in the usual way in one of the bulbs. This film proved to be exceedingly sensitive to the application of ice, changing from apple-green to the colour of dense cobalt-glass. I now packed the other bulb in carbon dioxide and ether, in order to condense in it all of the suspected hydrocarbon vapour. It was then found that the film was no longer sensitive to the application of cold. To make absolutely sure that the true cause had been found, I cooled a small spot on a pink film to a low temperature by means of a small piece of carbon dioxide, which was left on the bulb for some time. The spot became almost colourless, the absorption having passed entirely out of the spectrum. On examining the spot quickly under the microscope, I had no trouble in seeing minute globules of a transparent liquid, sufficient quantity having condensed to form droplets. These grew rapidly smaller as the glass warmed, and the instant they disappeared, or perhaps a fraction of a second afterwards, the field of the instrument became first bluish, then pink, in consequence of the entrance of the absorption-band into the spectrum again.

There is a temperature effect of a different nature, which at first seems wholly inexplicable, but which I now think I understand. On one or two occasions I have had potassium films, originally deep violet, become almost transparent on the application of cold, and remain so *permanently*. What was still more remarkable, the appearance of the spot viewed in reflected light indicated clearly that the size of the particles had been increased, the spot showing the silky lustre to which I have already alluded, while the film surrounding it scattered no light at all. I had noticed that these films were apparently made up of two sets of particles—one set large enough to be seen under a low power, the other only coming into view under the highest power obtainable. After the temperature change, the smaller ones seemed to have disappeared entirely; and I am now of the opinion that the condensation of the fluid film enabled them in some way to gather about and adhere to the larger particles, building up aggregates large enough to diffuse light.

Changes produced by Oxidation.

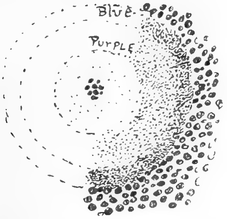
If the tip of one of the bulbs is cut off, the entrance of the air causes the coloured film to vanish like a flash. In some instances, I was of the impression that I detected a momentary change of colour before the film disappeared. To lengthen the process, I adopted the expedient of drawing the end of the bulb out into a long fine capillary, with a bore less than one one-hundredth of a millimetre. In this bulb a film of a deep pink colour was formed, and on cutting off the tip of the capillary the colour changed to blue, and the film vanished almost immediately. The small amount of air necessary to efface the films is indicated by the fact that at the end of an hour there was still a fairly good vacuum in the bulb, notwithstanding the fact that the end of the capillary had been open all the while. If the mouth of the capillary is fused immediately after the oxidation of the film, a new coloured film can be formed by heating the metal, the oxygen having been removed by the first film. This process can be repeated a number of times, showing that the presence of the oxide on the glass does not interfere with the deposition of the coloured film. If the process of oxidation be made very slow, by employing a very long capillary, and the process watched under the microscope, the black particles slowly become dim, and finally fade away. I am of the opinion that the microscope merely shows us the diffraction-disk due to the opaque particle. This would become dimmer as the size of the particle is reduced, without any apparent change of size. Examination with the spectroscope shows us that, during the process of oxidation, the absorption-band sometimes moves out of the spectrum through the red end, and sometimes merely fades away without any motion. I have tried filling the bulbs with carefully dried hydrogen gas, after the film had been formed, but the traces of oxygen present in the gas were sufficient to bleach the films. By employing two adjoining bulbs, each lined with a coloured film, and cooling one of these in carbonic acid and ether, I was enabled to fill the bulbs with hydrogen at atmospheric pressure without destroying the coloured film in the cooled bulb. The low temperature prevented chemical action, and the oxygen was wholly removed by the film in the warmer bulb. On warming the bulb again, the colours persisted, showing that no change was produced by the presence of pure hydrogen.

I find that it is even possible to fill a bulb with air at atmospheric pressure without destroying the colour, if the bulb is thoroughly cooled with the carbon dioxide and ether, the chemical action not taking place at this low temperature. These experiments were tried to test a certain theory of the cause of the colours which occurred to me one day, which it is not necessary to go into as it was found to be worthless. The extreme sensitiveness of these coloured films to minute traces of oxygen makes it seem probable that they will prove useful in the study of the flow of gases through small apertures and capillaries. I have noticed that if the capillary is exceedingly fine, the film, instead of disappearing all at once, gradually melts away, beginning at the end of the bulb near the capillary. Sometimes it is half-a-minute before the colour at the other end of the bulb bleaches out. It is as if a fluid were slowly rising in the bulb and dissolving the film. This seems rather surprising, in view of the ideas which we hold about the diffusion of gases in highly exhausted receivers.

Relation between the Colour of the Film and the Size and Distribution of the Particles.

One peculiarity of the coloured films is that the colour produced under a given condition seems to depend in some way on the condition of the glass surface on which the film is deposited. Circular pink spots are frequently found on a blue field, while sometimes circular spots quite devoid of colour are observed. The centre of the circle is usually occupied by a dark speck, which I believe to be a small particle of the metal thrown up when the globule is heated. This speck, strangely enough, seems to modify the nature of the surface of the glass over a circular area surrounding it, so that a film of a different colour forms on it, or in certain cases no film at all forms. One of these spots which I studied under the microscope proved interesting. The black speck in the centre was found to be made up of a number of coarse particles of the metal lying close together. Surrounding this was a transparent ring, the particles here being very minute and very widely separated. Next came a ring of pinkish purple (absorption-band about at the D lines), while outside of this came a ring of deep blue (absorption-band in the red). The surrounding field was practically colourless, being made up of large particles similar to those at the centre of the ring (fig. 3). In the blue ring the particles were distinctly closer together than in the pink one; a circumstance which I made sure of by having a number of other observers express an opinion on the matter. If now the particles are packed closer together in the blue film, we should expect the capacity to be increased, the same as by the introduction of the medium of high dielectric constant. The period would consequently become slower, and the absorption-band would lie further along towards the red, which is exactly what was observed. In some instances, I have been of the opinion that the particles in the blue films are larger than those in the pink ones, but of this I am not sure.

Fig. 3.



Behaviour of the Films with Polarized Light.

One of the first experiments which I tried with the deposits which were made up of particles large enough to scatter light, was their polarizing power. The angle of complete polarization in the case of non-metallic particles is 90°, as is well-known. J. J. Thomson (Recent Researches) has calculated the scattering effect of metallic spheres on light-waves, and has found that we should expect complete polarization on a line making an angle of 120° with the direction of the incident light. I am not aware that any experiments have been made to test this calculation. The results which I obtained gave a value somewhat larger than this, the mean being about 135°. It is difficult, however, to get very accurate results with films deposited on spherical surfaces.

Very interesting results were obtained as soon as the coloured films were examined by polarized light. To enumerate all of the different effects which I have recorded in my note-book would require too much space, and I shall only mention two or three typical effects. The arrangement of the apparatus was as follows. The light of a Welsbach burner passed through a Nicol prism mounted in an opaque screen in such a way that it could be turned through a right angle, then through the coloured film at oblique incidence, and then into the direct-vision spectroscope. It was found in the case of one of the purple films, which had a strong absorption-band in the yellow-green, that this band was only present when the electric vector was parallel to the plane of incidence. When the electric vector was parallel to the surface the spectrum appeared continuous. In every case the colour was much more pronounced when the electric vector had a component normal to the surface. On turning the nicol from one position to the other, no motion of the absorption-band could be detected in this film, the change consisting in a gradual fading away. In other films, however, I have sometimes found that the absorption-band shifts its position when the nicol is turned. For example, in the case of a film which showed an absorption-band a little below the D lines with the electric vector perpendicular to the plane of incidence, on turning the nicol the band moved along into the red and became broader and fainter, the effect being similar to that produced by cooling the film.

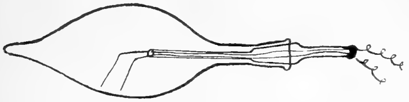
When the films are viewed between two nicols more complicated effects are obtained, which I shall omit for the present as they have not yet been carefully studied.

Until someone has calculated the behaviour of a sheet of resonators at oblique incidence with reference to waves polarized in and at right angles to the plane of incidence, it will be impossible to say whether the polarization experiments are in accordance with the resonance hypothesis or not. It seems to me, however, that the results obtained with polarized light will be the means of eventually determining the exact cause of the colours, or at least in deciding in favour of, or against, the theory of electrical resonance.

Electrical Conductivity of the Films.

The coloured films are apparently non-conducting. Inasmuch as it was difficult to make sure that the films were deposited up to and against the "sealed-in" electrodes, a bulb was fitted with a ground joint through which the wires passed as shown in fig. 4. After the film had been depoisted the joint was turned, causing the tips of the wire to be drawn over the film. With a sensitive galvanometer in circuit with the bulb, 75 volts gave no deflexion when the wires were drawn over the film, though they scraped the glass clean of the deposit. No change was produced in the appearance of the film between the wires with a potential-difference of 700 volts obtained from a storage-battery. With higher voltages a gas-discharge commenced, and the colour changed in the neighbourhood of the electrodes, owing to the heat. I have tried the effect of illuminating the films with the light of an arc brought close up to the bulb, but there is absolutely no trace of conductivity, even under these conditions. This is true even for deposits so dense, that under the microscope the particles appear to be in contact with, and even piled up on top of one another. A possible explanation of the absence of conductivity may be, that the granules consist of a metal core, surrounded by a non-conducting film of some sort, possibly oxide. I have noticed that bulbs, in which films have been repeatedly formed and driven off by the application of a flame, finally reach a state in which it is no longer possible to form coloured deposits, but I have not yet determined whether this is due to a modification of the surface of the glass, the removal of the oxygen, or some other cause. At the present time, I am studying the effects of the introduction of a small amount of liquid ligroin, which can be accomplished without impairing the vacuum, or destroying the coloured film, in the following way:—One end of the bulb is drawn out into a thin-walled tube with a bore of about 1 mm. After a satisfactory coloured film has been formed, a swinging motion of the arm drives the drop of molten metal down into the small tube, sealing it hermetically. The tip of the tube is then cut off under ligroin, which completely fills the tube, as far up as the plug. By means of a small flame, it is possible to close the mouth of the tube, the empty portion containing practically nothing but hydrocarbon vapour. On carefully warming the metal plug to the fusing-point, the hydrocarbon is discharged into the bulb, passing through the shell of liquid metal which adheres to the walls of the tube and removes from the

Fig. 4.



fluid any impurities which are detrimental to the film. In this way, I have been able to introduce ligroin in quantities sufficient to thoroughly wet the inside of the bulb, and have obtained in this way colours of even greater brilliancy than before.

Conclusion.

At the present stage of the work, it seems to me that it is impossible to decide either in favour of, or against, the theory of resonance. Most of the changes produced by varying the conditions seem to be what we should expect. The idea of electrical resonance has, at all events, proved useful as a working hypothesis, for it led directly to the discovery of the cause of the changes produced by low temperatures. While I have obtained coloured films from sodium, potassium, and lithium, I have been unable to obtain them from thalium and cadmium, which give granular deposits very similar in other respects. I cannot help feeling that I shall eventually find some other explanation than the one ventured in the title of this paper, though I have been unable to conceive of any arrangement of media which will show colours of this nature by interference. At all events, it appears to me that we have colour phenomena quite unlike any that are already known, and which are of some interest, regardless of the ultimate explanation which may be given them.

Baltimore, February 1, 1902.

XLV. *Aggregates of Minors of an Axisymmetric Determinant.*

By Thomas Muir, *LL.D., C.M.G., F.R.S.*[*]

1. FOR the purpose of stating Kronecker's theorem regarding certain aggregates of minors of an axisymmetric determinant of even order, there does not seem to be a more appropriate notation than that introduced in the Philosophical Magazine[†] in 1884, which consists in giving a typical term of the aggregate in question and adding some distinguishing mark to the letters or figures appearing therein whose variation gives rise to the other terms. Thus, the aggregate

$$\begin{vmatrix} 1234 \\ 5678 \end{vmatrix} - \begin{vmatrix} 1235 \\ 4678 \end{vmatrix} + \begin{vmatrix} 1236 \\ 4578 \end{vmatrix} - \begin{vmatrix} 1237 \\ 4568 \end{vmatrix} + \begin{vmatrix} 1238 \\ 4567 \end{vmatrix},$$

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[*] Communicated by the Author.

[†] The article is entitled "An Overlooked Discoverer in the Theory of Determinants," and appears in vol. xviii. pp. 416–427.

which according to Kronecker vanishes when the parent determinant $\begin{vmatrix} 12345678 \\ 12345678 \end{vmatrix}$ is axisymmetric, may be written

$$\Sigma \begin{vmatrix} 123\overline{4} \\ \underline{5678} \end{vmatrix},$$

it being understood that in each term under the sign of summation the row-numbers are taken in natural order and the column-numbers in order, and that the sign-factor of the term is $(-1)^v$ where v is the number of inverted-pairs in the single line of numbers formed by writing the column-numbers immediately after the row-numbers.

2. Kronecker's theorem was originally published in 1882, and since that date considerable additional light has been thrown on it, including such light as comes from generalization. The first generalization was pointed out in 1897 * and the second in 1901 †. The latter, due to Professor Metzler, is noteworthy because of its width; and on this and other grounds deserves some special attention such as the present paper aims at giving to it. In effect, Professor Metzler says that not only $\Sigma \begin{vmatrix} 123\overline{4} \\ \underline{5678} \end{vmatrix}$ vanishes in the circumstances referred to, but also $\Sigma \begin{vmatrix} 12\overline{34} \\ \underline{5678} \end{vmatrix}$ and $\Sigma \begin{vmatrix} 1\overline{234} \\ \underline{5678} \end{vmatrix}$; in other words, that Kronecker need not have confined the proposition to the case where $n-1$ of the row-numbers are invariable, because all that is necessary is that the number of such invariable row-numbers shall be greater than 0 and less than n. The mode of investigation is that employed in my second proof of Kronecker's theorem, viz., expansion of each member of the aggregate in terms of products of complementary minors and re-condensation resulting from the union of such of these products as have a factor in common.

3. The main point calling for notice in the new generalization is the manner in which the vanishing of the aggregates comes about. A slight examination of a special case makes clear that in this respect there is an essential difference between the new and the old, and that further investigation is

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* Transactions Roy. Soc. Edinburgh, vol. xxxix. p. 226.

† Transactions American Math. Soc. vol. ii. pp. 395–403.

therefore called for. Let us consider the case of $\Sigma \begin{vmatrix} 1\overline{234} \\ \underline{5678} \end{vmatrix}$. In less compact notation it is

$$\begin{gathered}
\begin{vmatrix} 1234 \\ 5678 \end{vmatrix} - \begin{vmatrix} 1235 \\ 4678 \end{vmatrix} + \begin{vmatrix} 1236 \\ 4578 \end{vmatrix} - \begin{vmatrix} 1237 \\ 4568 \end{vmatrix} + \begin{vmatrix} 1238 \\ 4567 \end{vmatrix} \\
+ \begin{vmatrix} 1245 \\ 3678 \end{vmatrix} - \begin{vmatrix} 1246 \\ 3578 \end{vmatrix} + \begin{vmatrix} 1247 \\ 3568 \end{vmatrix} - \begin{vmatrix} 1248 \\ 3567 \end{vmatrix} + \begin{vmatrix} 1256 \\ 3478 \end{vmatrix} \\
- \begin{vmatrix} 1257 \\ 3468 \end{vmatrix} + \begin{vmatrix} 1258 \\ 3467 \end{vmatrix} + \begin{vmatrix} 1267 \\ 3458 \end{vmatrix} - \begin{vmatrix} 1268 \\ 3457 \end{vmatrix} + \begin{vmatrix} 1278 \\ 3456 \end{vmatrix} \\
- \begin{vmatrix} 1345 \\ 2678 \end{vmatrix} + \begin{vmatrix} 1346 \\ 2578 \end{vmatrix} - \begin{vmatrix} 1347 \\ 2568 \end{vmatrix} + \begin{vmatrix} 1348 \\ 2567 \end{vmatrix} - \begin{vmatrix} 1356 \\ 2478 \end{vmatrix} \\
+ \begin{vmatrix} 1357 \\ 2468 \end{vmatrix} - \begin{vmatrix} 1358 \\ 2467 \end{vmatrix} - \begin{vmatrix} 1367 \\ 2458 \end{vmatrix} + \begin{vmatrix} 1368 \\ 2457 \end{vmatrix} - \begin{vmatrix} 1378 \\ 2456 \end{vmatrix} \\
+ \begin{vmatrix} 1456 \\ 2378 \end{vmatrix} - \begin{vmatrix} 1457 \\ 2368 \end{vmatrix} + \begin{vmatrix} 1458 \\ 2367 \end{vmatrix} + \begin{vmatrix} 1467 \\ 2358 \end{vmatrix} - \begin{vmatrix} 1468 \\ 2357 \end{vmatrix} \\
+ \begin{vmatrix} 1478 \\ 2356 \end{vmatrix} - \begin{vmatrix} 1567 \\ 2348 \end{vmatrix} + \begin{vmatrix} 1568 \\ 2347 \end{vmatrix} - \begin{vmatrix} 1578 \\ 2346 \end{vmatrix} + \begin{vmatrix} 1678 \\ 2345 \end{vmatrix},
\end{gathered}$$

the number of terms being of course $C_{7.3}$, *i. e.* 35. This may be viewed as consisting of four sub-aggregates, A, B, C, D, viz. A occupying the first line, B the second and third lines, C the fourth and fifth lines, and D the sixth and seventh lines. Of these the first is seen to be a Kronecker aggregate. The second B consists of two parts which cancel each other; for the sum of the first four terms being equal to $\begin{vmatrix} 1243 \\ 5678 \end{vmatrix}$ diminished by a Kronecker aggregate is equal to $-\begin{vmatrix} 1234 \\ 5678 \end{vmatrix}$, and the sum of the last six terms can be shown to be $\begin{vmatrix} 1234 \\ 5678 \end{vmatrix}$. Similarly, the sum of the first four terms of C cancels the sum of the last six terms: and the sum of the first six terms of D cancels the sum of the last four terms. The new aggregate, $\Sigma \begin{vmatrix} 1\overline{234} \\ 5678 \end{vmatrix}$, is thus seen to be the sum of four zeros; and it will be found that all the others have a like constitution.

4. The four sub-aggregates in the preceding are easily seen to be

$$\Sigma \begin{vmatrix} 123\overline{4} \\ \underline{5678} \end{vmatrix}, \quad \Sigma \begin{vmatrix} 12\overline{45} \\ 3\underline{678} \end{vmatrix}, \quad \Sigma \begin{vmatrix} 13\overline{45} \\ 2\underline{678} \end{vmatrix}, \quad \Sigma \begin{vmatrix} 1\overline{456} \\ 23\underline{78} \end{vmatrix}.$$

The last three are not essentially different; so that to

Kronecker's fundamental aggregate $\Sigma\left|\begin{matrix}123\overline{4}\\ \underline{5678}\end{matrix}\right|$ there has to be added one other, $\Sigma\left|\begin{matrix}12\overline{45}\\ 3\underline{678}\end{matrix}\right|$, the common characteristic of the two being the invariability of the line-numbers 1, 2, 3.

The vanishing of this new aggregate, $\Sigma\left|\begin{matrix}12\overline{45}\\ 3\underline{678}\end{matrix}\right|$, is established by separating the four terms of it which have 4 as a row-number from the six terms which have 4 as a column-number, and showing that the one group cancels the other. Thus

$$\begin{aligned}\Sigma\left|\begin{matrix}12\overline{45}\\ 3\underline{678}\end{matrix}\right| &= \Sigma\left|\begin{matrix}124\overline{5}\\ 3\underline{678}\end{matrix}\right| + \Sigma\left|\begin{matrix}12\overline{56}\\ 34\underline{78}\end{matrix}\right|,\\ &= \left|\begin{matrix}1243\\ 5678\end{matrix}\right| + \left|\begin{matrix}1234\\ 5678\end{matrix}\right|,\\ &= 0.\end{aligned}$$

The penultimate step here presents two expressions for $\left|\begin{matrix}1234\\ 5678\end{matrix}\right|$. The second of these is new; the first is not, since the vanishing of Kronecker's aggregate $\Sigma\left|\begin{matrix}123\overline{4}\\ \underline{5678}\end{matrix}\right|$ may also be viewed as a statement that

$$\left|\begin{matrix}1234\\ 5678\end{matrix}\right| = \Sigma\left|\begin{matrix}123\overline{5}\\ 4\underline{678}\end{matrix}\right|,$$

where, as is often convenient in dealing with sub-aggregates, the typical term on the right is reckoned positive and the others determined from it. We are thus led to a series of identities

$$\left|\begin{matrix}1234\\ 5678\end{matrix}\right| = \Sigma\left|\begin{matrix}123\overline{5}\\ 4\underline{678}\end{matrix}\right| = \Sigma\left|\begin{matrix}12\overline{56}\\ 34\underline{78}\end{matrix}\right| = \Sigma\left|\begin{matrix}1\overline{567}\\ 234\underline{8}\end{matrix}\right|.$$

5. If in the case of five-line minors we denote the fundamental Kronecker aggregate,

$$\Sigma\left|\begin{matrix}1234\overline{5}\\ \underline{6789\tau}\end{matrix}\right| \text{ by K;}$$

the Metzler aggregates

$$\Sigma\left|\begin{matrix}123\overline{45}\\ \underline{6789\tau}\end{matrix}\right|,\quad \Sigma\left|\begin{matrix}12\overline{345}\\ \underline{6789\tau}\end{matrix}\right|,\quad \Sigma\left|\begin{matrix}1\overline{2345}\\ \underline{6789\tau}\end{matrix}\right|, \text{ by } M_1,\ M_2,\ M_3;$$

and the still more recent aggregates

$$\Sigma\begin{vmatrix}123\overline{56}\\ 4\underline{789\tau}\end{vmatrix},\quad \Sigma\begin{vmatrix}12\overline{567}\\ 34\underline{89\tau}\end{vmatrix},\quad \Sigma\begin{vmatrix}12\overline{456}\\ 3\underline{789\tau}\end{vmatrix},\quad \Sigma\begin{vmatrix}1\overline{3456}\\ 2\underline{789\tau}\end{vmatrix}$$

by

$$\mu_1,\qquad \mu_2,\qquad \mu_3,\qquad \mu_4;$$

the following partition of the 126 terms of $\Sigma\begin{vmatrix}1\overline{2345}\\ \underline{6789\tau}\end{vmatrix}$ will make clear the various relationships involved.

$$M_3=\begin{cases} M_2=\begin{cases} M_1=\begin{cases} K & \text{(6 terms)},\\ \mu_1 & \text{(15 terms)},\end{cases}\\ \mu_3\;=\begin{cases} \mu_1' & \text{(15 terms)},\\ \mu_2 & \text{(20 terms)},\end{cases}\end{cases}\\ \mu_4\;=\begin{cases} \mu_3'=\begin{cases} \mu_1'' & \text{(15 terms)},\\ \mu_2' & \text{(20 terms)},\end{cases}\\ \mu_3''=\begin{cases} \mu_2'' & \text{(20 terms)},\\ \mu_1''' & \text{(15 terms)}.\end{cases}\end{cases}\end{cases}$$

Of course for perfect completeness we should have started with the 252 terms of the aggregate $\Sigma\begin{vmatrix}\overline{12345}\\ \underline{6789\tau}\end{vmatrix}$, dividing it into $\Sigma\begin{vmatrix}1\overline{2345}\\ \underline{6789\tau}\end{vmatrix}$ and $\Sigma\begin{vmatrix}\overline{23456}\\ 1\underline{789\tau}\end{vmatrix}$,—that is to say, into M_3 and M_3'; but nothing would thereby have been gained, as the difference between the one half and the other is made to disappear by row-and-column interchange.

The fundamental aggregates in the collection are seen to be K, μ_1, μ_2; that is to say,

$$\Sigma\begin{vmatrix}1234\overline{5}\\ 6789\tau\end{vmatrix},\quad \Sigma\begin{vmatrix}123\overline{56}\\ 4\underline{789\tau}\end{vmatrix},\quad \Sigma\begin{vmatrix}12\overline{567}\\ 34\underline{89\tau}\end{vmatrix},$$

the others on the extreme right being variants of two of these and therefore indicated by the same letters with one or more dashes. The common characteristic of the three is the invariability of four line-numbers, all the possible distributions of four such numbers among rows and columns being

$$4+0,\qquad 3+1,\qquad 2+2.$$

6. The proof that μ_1 and μ_2 vanish on the imposition of axisymmetry is accomplished by continuing in each case the process of fission by which they themselves arose, and then

showing that the two components differ only in sign. Thus the fifteen-termed aggregate

$$\mu_1', \textit{ i. e. } \Sigma\begin{vmatrix} 124\overline{56} \\ 3\underline{789\tau} \end{vmatrix},$$

$$= \Sigma\begin{vmatrix} 1245\overline{6} \\ 3\underline{789}\tau \end{vmatrix} + \Sigma\begin{vmatrix} 124\overline{67} \\ 35\underline{89\tau} \end{vmatrix},$$

$$= \begin{vmatrix} 12453 \\ 6789\tau \end{vmatrix} + \begin{vmatrix} 12435 \\ 6789\tau \end{vmatrix},$$

$$=0:$$

and the twenty-termed aggregate

$$\mu_2'', \textit{ i. e. } \Sigma\begin{vmatrix} 14\overline{567} \\ 23\underline{89\tau} \end{vmatrix},$$

$$= \Sigma\begin{vmatrix} 145\overline{67} \\ 23\underline{89\tau} \end{vmatrix} - \Sigma\begin{vmatrix} 146\overline{78} \\ 235\underline{9\tau} \end{vmatrix},$$

$$= \begin{vmatrix} 14523 \\ 6789\tau \end{vmatrix} - \begin{vmatrix} 14235 \\ 6789\tau \end{vmatrix},$$

$$=0.$$

The 35-termed aggregate μ_3 and the 70-termed aggregate μ_4 are seen to resemble the M's in their constitution, the one being the sum of two zeros and the other the sum of four.

7. Finally, let it be noted that two lines of generalization are herein marked out. The first is indicated in the series of identities—

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \Sigma\begin{vmatrix} \alpha & \overline{\gamma} \\ \beta & \underline{\delta} \end{vmatrix};$$

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \zeta \end{vmatrix} = \Sigma\begin{vmatrix} \alpha & \beta & \overline{\delta} \\ \gamma & \underline{\epsilon} & \underline{\zeta} \end{vmatrix} = -\Sigma\begin{vmatrix} \alpha & \overline{\delta} & \overline{\epsilon} \\ \beta & \gamma & \underline{\zeta} \end{vmatrix};$$

$$\begin{vmatrix} \alpha & \beta & \gamma & \delta \\ \epsilon & \zeta & \eta & \theta \end{vmatrix} = \Sigma\begin{vmatrix} \alpha & \beta & \gamma & \overline{\epsilon} \\ \delta & \underline{\zeta} & \underline{\eta} & \underline{\theta} \end{vmatrix} = \Sigma\begin{vmatrix} \alpha & \beta & \overline{\epsilon} & \overline{\zeta} \\ \gamma & \delta & \underline{\eta} & \underline{\theta} \end{vmatrix} = \Sigma\begin{vmatrix} \alpha & \overline{\epsilon} & \overline{\zeta} & \overline{\eta} \\ \beta & \gamma & \delta & \underline{\theta} \end{vmatrix};$$

$$\begin{vmatrix} \alpha & \beta & \gamma & \delta & \epsilon \\ \zeta & \eta & \theta & \iota & \kappa \end{vmatrix} = \Sigma\begin{vmatrix} \alpha & \beta & \gamma & \delta & \overline{\zeta} \\ \epsilon & \underline{\eta} & \underline{\theta} & \underline{\iota} & \underline{\kappa} \end{vmatrix} = -\Sigma\begin{vmatrix} \alpha & \beta & \gamma & \overline{\zeta} & \overline{\eta} \\ \delta & \epsilon & \underline{\theta} & \underline{\iota} & \underline{\kappa} \end{vmatrix} = \Sigma\begin{vmatrix} \alpha & \beta & \overline{\zeta} & \overline{\eta} & \overline{\theta} \\ \gamma & \delta & \epsilon & \underline{\iota} & \underline{\kappa} \end{vmatrix} = \dots$$

. . . .

where the opening identity in each line is Kronecker's. The second is seen in the series,

$$0=\Sigma\begin{vmatrix}\alpha & \overline{\beta}\\ \underline{\gamma} & \underline{\delta}\end{vmatrix},$$

$$0=\Sigma\begin{vmatrix}\alpha & \beta & \overline{\gamma}\\ \underline{\delta} & \underline{\epsilon} & \underline{\zeta}\end{vmatrix}=\Sigma\begin{vmatrix}\alpha & \overline{\gamma} & \overline{\delta}\\ \beta & \underline{\epsilon} & \underline{\zeta}\end{vmatrix},$$

$$0=\Sigma\begin{vmatrix}\alpha & \beta & \gamma & \overline{\delta}\\ \underline{\epsilon} & \underline{\zeta} & \underline{\eta} & \underline{\theta}\end{vmatrix}=\Sigma\begin{vmatrix}\alpha & \beta & \overline{\delta} & \overline{\epsilon}\\ \gamma & \underline{\zeta} & \underline{\eta} & \underline{\theta}\end{vmatrix}=\Sigma\begin{vmatrix}\alpha & \overline{\delta} & \overline{\epsilon} & \overline{\zeta}\\ \beta & \gamma & \underline{\eta} & \underline{\theta}\end{vmatrix},$$

$$0=\Sigma\begin{vmatrix}\alpha & \beta & \gamma & \delta & \overline{\epsilon}\\ \underline{\zeta} & \underline{\eta} & \underline{\theta} & \underline{\iota} & \underline{\kappa}\end{vmatrix}=\Sigma\begin{vmatrix}\alpha & \beta & \gamma & \overline{\epsilon} & \overline{\zeta}\\ \delta & \underline{\eta} & \underline{\theta} & \underline{\iota} & \underline{\kappa}\end{vmatrix}=\Sigma\begin{vmatrix}\alpha & \beta & \overline{\epsilon} & \overline{\zeta} & \overline{\eta}\\ \gamma & \delta & \underline{\theta} & \underline{\iota} & \underline{\kappa}\end{vmatrix}=\dots$$

. . . .

where again on the extreme left the same Kronecker identities appear.

Capetown, South Africa,
3rd February, 1902.

XLVI. *On the Question of Hydrogen in the Atmosphere.* *By* Lord Rayleigh, *F.R.S.*[*]

It will be remembered that M. Armand Gautier, as the result of very elaborate investigations, was led to the conclusion that air, even from the Atlantic, contains by volume nearly 2 parts in 10,000 of free hydrogen. The presence of so much hydrogen, nearly two thirds of the carbonic acid which plays such an important part, is of interest in connexion with theories pointing to the escape of light constituents from the planetary atmospheres. Besides the free hydrogen, M. Gautier found in the air of woods and towns considerable quantities of hydrocarbons yielding CO_2 when led over hot copper oxide.

Spectroscopic Evidence.

In the Philosophical Magazine for Jan. 1901, I described some observations upon the spectrum of sparks taken in dried air at atmospheric pressure, which seemed "to leave a minimum of room for the hydrogen found by M. Gautier." Subsequently (April 1901), these experiments were repeated with confirmatory results. The spectra, taken from platinum points, of pure country air and the same to which $\frac{2}{10,000}$ of

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[*] Communicated by the Author.

hydrogen had been added were certainly and easily distinguished by the visibility of the C-line. An improvement was afterwards effected by the substitution of aluminium points for platinum. A strong preliminary heating reduced the C-line with a stream of pure dried air to the least yet seen, only just continuously visible, and contrasting strongly with the result of substituting the air to which the two parts in 10,000 of hydrogen had been added.

To air from outside one thousandth part of hydrogen was introduced and allowed time to mix thoroughly. Excess of chlorine was then added, and after a while the whole was exposed to strong sunshine, after which the superfluous chlorine was removed by alkali. Tested in the spectroscope, this sample showed only about the same signs of hydrogen as the pure air, indicating that the added hydrogen had effectively been removed—a result which somewhat surprised me.

As there now appeared to be a margin for further discrimination, three samples were prepared, the first pure air, the second air to which was added $\frac{1}{10,000}$ of hydrogen, and the third air with addition of $\frac{2}{10,000}$ of hydrogen. In the spectroscope the three were just certainly distinguishable, showing C in the right order. The chlorine-treated mixture showed about the same as the pure air. On repetition with fresh samples these results were confirmed.

In my former note I mentioned that nitrous oxide and oxygen showed the C-line as much as, if not more than, air. I cannot say whether this result is inevitable, but the gases were prepared with ordinary care. In the more recent repetition N_2O showed the C-line about the same as the air to which $\frac{1}{10,000}$ H_2 had been added. Oxygen, prepared from permanganate, showed C *much* more than does pure air; and there was not much change when oxygen from mixed chlorates of potash and soda was substituted.

The impurity in the oxygen, if it be an impurity *, does not appear to be easily removed. The visibility of C was not perceptibly diminished by passage of either kind of oxygen over hot copper oxide. On the other hand the air containing $\frac{2}{10,000}$ of added hydrogen was reduced by the same treatment to equality with pure air. Possibly the impurity is a hydrocarbon not readily burnt.

Neither by treatment with chlorine could oxygen from either source be freed from the property of exhibiting the C-line.

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* It is possible that traces of hydrogen, derived from the electrodes or from the glass, show more in oxygen than in air.

The spectroscopic evidence here set forth is certainly far from suggesting that air, previously to any addition, already contains 2 parts in 10,000 of free hydrogen. The passage from 2 to 3 parts in 10,000 might possibly produce the observed change of visibility which followed the introduction of one ten-thousandth of hydrogen; and the behaviour with chlorine and hot copper oxide is not absolutely inconsistent with the initial hydrogen. But the reconciliation seems to involve coincidences of little *à priori* probability.

Determinations by Combustion.

In M. Gautier's experiments large volumes of dried air were passed through tubes containing copper oxide heated in a specially constructed furnace, the water formed being collected in suitable phosphoric tubes and accurately weighed. In unsystematic experiments the source of the water so collected might be doubtful, but it is explained that the apparatus was tested with pure dry *oxygen*, and that under these conditions the phosphoric tube showed no increase of weight exceeding ·1 mg. The work was evidently very careful and thorough; and the impression left upon the mind of the reader is that the case is completely made out. Indeed, had I been acquainted with the details, as set forth in *Ann. d. Chimie*, t. xxii. Jan. 1901, at an earlier stage, I should probably have attempted no experiments of my own. It so happens, however, that I had already begun some work, which has since been further extended, and which has yielded results that I find rather embarassing, and am even tempted to suppress. For the conclusion to which these determinations would lead me is that the hydrogen in country air is but a small fraction, perhaps not more than one eighth part, of that given by M. Gautier. Although I am well aware that my experience in these matters is much inferior to his, and that I may be in error, I think it proper that some record should be made of the experiments, which were carefully conducted with the assistance of Mr. Gordon and many times repeated.

The quantity of air upon which I operated was almost uniformly 10 litres, much less than was used by M. Gautier. A glass aspirating bottle, originally filled with water, was discharged upon the lawn, so that the water was replaced by fresh *country* air. During an experiment the air was driven forward, at the rate of about $1\frac{1}{4}$ litres per hour, by water entering below. After traversing a bubbler charged with alkali, it was desiccated first by passing over a surface of sulphuric acid, and subsequently by phosphoric anhydride. Next followed the hot copper oxide, contained in a hard glass tube and heated by an ordinary combustion-furnace. Next in order followed the **U**-tube charged with phosphoric anhydride whose increase of weight was to indicate the absorption of water, formed in or derived from the furnace-tube. The **U**-tube was protected upon the further (down-stream) side by other phosphoric tubes. It was provided with glass taps and was connected on either side by short pieces of thick rubber of which but little was exposed to the passing air. The counterpoise in the balance was a similar closed phosphoric tube of very nearly the same volume, and allowance was made for the pressure and temperature of the air included in the working-tube at the moment when the taps were closed.

Two parts in 10,000 of free hydrogen, *i. e.* 2 c.c., yield on combustion the same volume of water-vapour, and of this the weight would be 1·5 mg. to be collected in the phosphoric tube. Any water, due to hydrocarbons originally present in the air and oxidized in the furnace-tube, would be additional to the above 1·5 mg.

The earlier experiments, executed at the end of 1900 and beginning of 1901, gave results which I found it difficult to interpret. The gain of weight from the passage of 10 litres of fresh air was about ·4 mg., that is, far too little; and, what was even more surprising, this gain was not diminished when the air after passage was collected and used over and over again. The gain appeared to have nothing to do with hydrogen originally present in the air, being maintained, for example, when a single litre of air was passed round and round eight or nine times. Neither did the substitution of oxygen for air make any important difference. Subsequently it was found that the gain was scarcely diminished when the furnace remained cold during the passage of the air.

Warned by M. Gautier, I was prepared for a possible gain of weight due to retention of oxygen; but this gain ought to be additional, and should not mask the difference between air containing and not containing free hydrogen. Faulty manipulation might be expected to entail an excessive rather than a defective gain; and the only cause to which I could attribute the non-appearance of the hydrogen was a failure of the copper oxide to do its work. The sample which I had employed was of the kind sold as granulated. M. Gautier himself found a considerable length of copper oxide necessary to complete the action. The question is, of course, not one of length merely, but rather of the *time* during which the travelling gas remains in close proximity to the oxidizing agent. Taking into account the slower rate of passage in litres per hour, it would seem that my arrangement had the advantage in this respect.

An attempt was made to improve the phosphoric anhydride by a preliminary heating for many hours to 260° in a current of dry air, somewhat as recommended by M. Gautier, but the results were not appreciably altered. When one remembers the experiments of Baker, from which it appears that, if all is thoroughly dry, heated phosphorus does not combine with oxygen, it is difficult to feel confidence in this process.

It was certain that the sample of phosphoric anhydride hitherto employed in this work was inferior. When a tube of it which had been used for some time (in other work) and had become gummy at the ends, was strongly heated over a spirit flame, occasional flashes could be seen in the dark. Treatment in the cold with *ozonized* air seemed to effect an improvement; but experiments in this direction were not pursued to a definite conclusion in consequence of the discovery that when another sample of phosphoric anhydride was substituted for that hitherto in use the anomaly disappeared. Thus in four trials where 10 litres of air were passed without a furnace, the gains were

Nov. 26, 1901	—·00016
,, 29, ,,	—·00003
,, 30, ,,	+·00014
Dec. 2, ,,	+·00006
Mean	·00000

thus on the whole no gain of weight. The errors would appear somewhat to exceed ·1 mg., but it may be noted that in this and following tables the real error is liable to appear exaggerated. If in consequence of an error of weighing, or of allowance for weight of included air, or of a varied condition of the outer surface of the tube, a recorded gain is too high, the next is likely to appear too low.

In the operations which followed, the furnace-tube was charged with copper oxide prepared *in situ* by oxidation of small pieces of thin copper foil with which the tube was packed. Examination once or twice after a breakage showed that the oxidation was not complete, but no measurements were taken until there was no appreciable further absorption of oxygen. After the copper oxide has been exposed to the air of the room, many hours' heating to redness in a current of dry air are required to remove the adherent moisture.

The results, referring in each case to 10 litres of fresh *country* air, are as follows, the weights being in gms. :—

Dec. 6, 1901	+·00023
„ 9, „	+·00042
„ 11, „	+·00010
„ 13, „	+·00014
„ 14, „	+·00025
„ 15, „	−·00001
„ 17, „	+·00031
„ 18, „	+·00028
„ 20, „	+·00025
„ 24, „	+·00016
Mean	+·00021

It will be seen that the mean water collected is only about one seventh of that corresponding to the complete combustion of the hydrogen, according to M. Gautier's estimate of the amount.

As has already been suggested, a defective gain of weight can hardly be explained by faulty manipulation. The important question is as to the efficiency of the copper oxide. Did my furnace-tube allow the main part of the free hydrogen to pass unburnt? The question is one that can hardly be answered directly, but I may say that variations of temperature (within moderate limits) did not influence the result.

What it is possible to examine satisfactorily is the effect of small additions of hydrogen to the air as collected. In my later experiments the added hydrogen was only 1 c.c., that is, $\frac{1}{10,000}$ by volume, or half the quantity originally present according to M. Gautier. The hydrogen was first diluted in a gas pipette with about 100 c.c. of air and allowed time to diffuse. The 10 litre aspirating-bottle being initially full of water, the diluted hydrogen was introduced at the top, and was followed by 10 litres of air from the open, after which the mixture stood over night, precautions which had been found sufficient to ensure a complete mixture in the spectroscopic work. The results were :

Dec. 19, 1901.	+·00091
„ 23, „	+·00084
„ 27, „	+·00103
Mean	+·00093

The additional gain is thus ·00072, very nearly the full amount (·00075) corresponding to the 1 c. c. of added hydrogen. We may say then that the copper oxide was competent to account for a small *addition* of hydrogen to air.

Following a suggestion from the spectroscopic experiments, I have examined the effect of treatment with *chlorine*. To 10 litres of air 10 c. c. (in one case 5 c. c.) of hydrogen were added, followed by excess of chlorine and exposure to sunshine. The excess of chlorine having been removed by alkali, the air was desiccated and passed through the furnace as usual. The resulting gains of weight in the phosphoric tube were :

Dec. 31, 1901.	+·00011
Jan. 2, 1902.	+·00050
„ 6, „	+·00044
„ 8, „	+·00030
Mean	+·00034

According to these figures, the treatment with chlorine after hydrogen left the gains somewhat larger than in the case of pure air ; but the sunshine, especially on Jan. 2, was feeble, and the difference of ·13 mg. may be the consequence of incomplete insolation. In any case the added hydrogen was *very nearly* removed by the chlorine.

In two experiments 10 litres of pure air were subjected to the chlorine treatment, and after removal of excess were mixed with 1 c. c. of hydrogen. The gains of weight recorded in the phosphoric tubes were

Jan. 1, 1902	+·00082
„ 7, „	+·00069
Mean	+·00075

The difference between this and ·00093 would suggest that perhaps some original hydrogen had been removed by chlorine, but the amount is very small.

I shall best fulfil my intention if I refrain from attempting to sum up the whole of the evidence. So far as my own work is concerned, the natural inference from it would be that the free hydrogen in country air does not exceed $\frac{1}{30,000}$ of the volume. If I may make a suggestion, it would be in favour of working with some such quantity as 10 litres, collected in glass bottles, and of comparisons between fresh air and air already passed once or twice through the furnace-tube.

February 1902.

XLVII. *On a Law of Molecular Attraction.*

By J. W. MELLOR, *D.Sc.* (*N.Z.*)*.

THE term $\frac{a}{v^2}$ which occurs in the well-known van der Waals-Budde equation of state,

$$\left(p+\frac{a}{v^2}\right)(v-b)=\mathrm{R}\theta,$$

was added by van der Waals in order to provide for the diminished outward pressure p of the gas caused by molecular attractions. a is a constant.

Let us assume van der Waals' correction to be an approximate representation of the magnitude of the molecular attraction.

The total work of expansion W′ is, therefore,

$$\mathrm{W}'=\int\left(p+\frac{a}{v^2}\right)dv=\int p\,.\,dv+\int\frac{a}{v^2}\,dv.$$

The first term $\int p\,.\,dv$ represents the *external* work of expansion, while the second term gives the *internal* work of expansion. That is, the work W done against molecular force when a gas expands from a volume v_1 to a volume v_2, is

$$\mathrm{W}=\int_{v_1}^{v_2}\frac{a}{v^2}dv=a\left(\frac{1}{v_1}-\frac{1}{v_2}\right). \quad . \quad . \quad . \quad . \quad (1)$$

It is apparent that this expression represents the internal work done when a liquid expands into a very great volume of its vapour, and it is a simple matter to evaluate the constant a when the latent heat of vaporization† of the given substance is known. v_1 then represents the volume of the liquid, v_2 the volume of the vapour.

For the sake of fixing our ideas, consider a volume of gas containing two molecules at a distance r apart. Since r is linear, the volume of the gas will vary as r^3. Let

$$v=cr^3,$$

where c is the constant of variation. From (1), therefore,

$$\mathrm{W}=\frac{a}{c}\left(\frac{1}{r_1{}^3}-\frac{1}{r_2{}^3}\right),$$

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[*] Communicated by the Author.

[†] See van 't Hoff's *Vorlesungen*, iii. p. 55 (1900).

which may be more compactly written

$$W = k\int_{r_1}^{r_2} \frac{1}{r^4}\, dr, \quad \ldots\ldots\ldots \quad (2)$$

where k has been written in place of a/c.

If F denotes the force of attraction between the two molecules, the "internal" work dW done against molecular attraction when the molecules are separated a distance dr, is

$$dW = F\,.\,dr, \quad \ldots\ldots\ldots \quad (3)$$

and if the molecules at a distance r_1 apart are separated to a distance r_2,

$$W = \int_{r_1}^{r_2} F\,.\,dr. \quad \ldots\ldots\ldots \quad (4)$$

From (2) and (4),

$$\int_{r_1}^{r_2} F\,.\,dr = \int_{r_1}^{r_2} \frac{k}{r^4}\, dr.$$

This identical equation in dr allows us to write

$$F = \frac{k}{r^4}; \text{ or } F \propto \frac{1}{r^4}.$$

That is to say, the force of attraction between two molecules is inversely as the fourth power of the distance between them (Sutherland's law).

Van der Waals' assumption that the force of attraction between two molecules is inversely proportional to the square of the volume occupied by the gas, and Sutherland's law of the inverse fourth*, are, therefore, two modes of stating the same hypothesis.

I am not aware that this simple relation has been previously pointed out.

The Owens College, Manchester.

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* Sutherland, Phil. Mag. [5] xxii. p. 81 (1886).

XLVIII. *An Experimental Investigation into the 'Skin'-effect in Electrical Oscillators.* *By* C. A. Chant[*].

1. *Introductory and Theoretical.*

THE first explicit reference to the fact that, when a conductor is subjected to a periodic electromotive force, the current is not uniformly distributed over a cross-section of the conductor is to be found in Art. 690 of Maxwell's 'Electricity and Magnetism.' Upon obtaining the equation connecting the impressed electromotive force with the effective electromotive force and the inductive electromotive force, he introduces terms which "express the correction of this value [of the inductive electromotive force] arising from the fact that the current is not of uniform strength at different distances from the axis of the wire. The actual system of currents has a greater degree of freedom than the hypothetical system in which the current is constrained to be of uniform strength throughout the section. Hence the electromotive force required to produce a rapid change in the strength of the current is somewhat less than it would be on this hypothesis."

It is quite certain, however, that Maxwell did not foresee the great interest and importance which the subject was destined soon to develop.

In a series of papers written between 1884 and 1887, Heaviside [†] dealt with the entire question of the propagation of electric currents into conductors and of magnetization into cores when produced by a periodic electromotive force. He was one of the first to insist that the action should be considered as entering the conductor from the surrounding dielectric. He compares the transmission of the effect into the metal to the transmission of motion into the inner portions of liquid in a cylindrical vessel when the vessel is given a rotatory vibration about its axis.

Especial attention was drawn to the subject by Hughes [‡], who treated the question experimentally.

In 1886 Rayleigh [§] published his well-known paper in which he obtained expressions for the resistance and self-induction of a straight conductor carrying a periodic current.

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[*] Communicated by Prof. J. Trowbridge.

[†] Electrical Papers, vol. i. pp. 353, 429; vol. ii. p. 168.

[‡] Jour. Soc. Tel. Engineers, Jan. 28, 1886.

[§] "On the Resistance and Self-induction of Straight Conductors," Phil. Mag., May 1886, p. 382; Scientific Papers, vol. ii. p. 486.

For very rapid oscillations the resistance

$$R' = \sqrt{(\tfrac{1}{2}pl\mu R)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which l is the length of the conductor, μ its magnetic permeability, R its resistance to steady currents, and $p=2\pi n$, where n is the frequency.

In 1890 Stefan*, in a paper on electric oscillations in straight conductors, also obtained formulas for the resistance and self-induction. With very high frequency his expression for the resistance is

$$R' = R\pi a\sqrt{\frac{n\mu}{\sigma}}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where a is the radius of the conductor and σ its specific resistance. This formula is equivalent to that given by Rayleigh. He remarks that for very great frequencies metallic conductors act much as though without resistance, but electrolytes behave very differently on account of their very high resistance. He finds that for a cylindrical copper conductor 1 cm. in diameter, with a frequency of 50 millions, the current-density at a depth of 0·004 cm. is only 1/100 of that at the surface; while for a tube of equal size of carbon disulphide the current-density at the centre is but 0·8 per cent. lower than at the surface,—in other words, the current is practically uniform.

If, now, the action enters the conductor from the surrounding dielectric and is prevented from penetrating very far by the rapidity of the oscillations, it is evident that very thin layers of metal should be sufficient to ward off electrical undulations, either by absorption or reflexion.

In a paper published in 1889, Hertz † described experiments made to find out how thick a metallic film was needed to screen from his rapid oscillations. Tinfoil, Dutch metal, and gilt paper acted perfectly. The thickness of the metal on the latter he estimated at 1/20 mm., though it was probably much less than that amount. Chemically deposited silver failed when the film was so thin as not to be opaque to light. The thickness of the film he places at less than 1/1000 mm. It was probably not 1/10 of that thickness and, moreover, hardly continuous metal. He remarks that the action of the waves scarcely penetrates farther into the wire than does the light which is reflected from its surface. Similar experiments on the screening effect of extremely thin metal leaf are given by Lodge and others.

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* Wied. *Ann.* xli. p. 400 (1890).

† 'Electric Waves,' p. 160.

A calculation of the superficial shell effective in the reflexion of Hertzian oscillations is given by Poincaré *, who finds the thickness at which the effect is $\frac{1}{e}$ of its amount at the surface,

for frequency, $n=50 \times 10^6$, thickness $=0{\cdot}002$ cm.;
„ „ $n=500 \times 10^6$, thickness $=0{\cdot}0006$ cm.

This estimate is probably too high.

J. J. Thomson has treated the 'skin'-effect with considerable fulness. In a note appended by him to Art. 690 of the third edition of Maxwell †, he obtains as the resistance per unit length of the conductor

$$R'=\left(\frac{\mu p \sigma}{2\pi a^2}\right)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

where the symbols have the meanings given above. This, again, is the same as the values obtained by Rayleigh and Stefan.

We can obtain the relative current-densities at different depths below the surface for any given frequency in the following way. Stefan ‡ has shown that if w be the component of the current in the direction of the axis of z, and if it does not vary with z, the equation

$$\frac{dw}{dt}=\frac{\sigma}{4\pi\mu}\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

must be satisfied.

When the depth to which the action penetrates is small, the effect of curvature of the surface may usually be neglected, in which case (4) may be replaced by

$$\frac{dw}{dt}=\frac{\sigma}{4\pi\mu}\frac{d^2w}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

This is Fourier's well-known equation of diffusion which Lord Kelvin § has shown to be applicable to the motion of a viscous fluid, of closed electric currents within a homogeneous conductor, of heat, of substances in solution, of electric potential in the conductor of a submarine cable; and, indeed, to every case of diffusion in which the substance concerned is in the same condition at all points of any one plane parallel to a given plane.

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* *Oscillations Électriques*, p. 246 & fol.

† See also 'Recent Researches,' p. 246 & fol.

‡ *Sitzungsberichte der Wiener Akad. der Wiss.* xcv. 2 Abth. p. 917 (1887).

§ Report of British Assoc. 1888, p. 571.

Suppose, now, the periodic current at the surface to be represented by

$$w = \mathrm{I} \sin pt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

We have to solve (5) subject to the conditions

$$w = \mathrm{I} \sin pt, \qquad \text{when } x = 0, \quad . \quad . \quad . \quad . \quad (7)$$

$$w = 0, \qquad\qquad ,, \quad t = 0. \quad . \quad . \quad . \quad (8)$$

The solution is

$$w = \frac{2}{\sqrt{\pi}} \mathrm{I} \left[\sin pt \int_{\frac{x}{2\kappa\sqrt{t}}}^{\infty} e^{-\beta^2} \cos \frac{px^2}{4\kappa^2\beta^2} d\beta - \cos pt \int_{\frac{x}{2\kappa\sqrt{t}}}^{\infty} e^{-\beta^2} \sin \frac{px^2}{4\kappa^2\beta^2} d\beta \right], \quad . \quad (9)$$

wherein $\kappa^2 = \sigma/4\pi\mu$.

As t increases the condition of affairs approaches a 'permanent' state, and then (9) reduces to *

$$w = \mathrm{I}e^{-x\sqrt{\frac{2\pi\mu p}{\sigma}}} \sin\left(pt - x\sqrt{\frac{2\pi\mu p}{\sigma}}\right). \quad . \quad (10)$$

At the surface, *i. e.* when $x = 0$, the maximum value of the current is I. It becomes $\frac{1}{e}$ of this value at a depth

$$x = \sqrt{\frac{\sigma}{2\pi\mu p}} = \frac{1}{2\pi}\sqrt{\frac{\sigma}{\mu n}}. \quad . \quad . \quad . \quad (11)$$

This depth J. J. Thomson † and Poincaré ‡ take as the thickness of the 'skin.' The difference in phase between the current at the surface and that at this depth is easily obtained from (10), and is

1 (radian) = 57°·3.

For high frequencies this thickness becomes exceedingly small, and an object of the present investigation was to see whether an oscillator behaved differently when the metal constituting it was thinner than this 'skin'; and if so, what was the critical thickness in any particular case.

In the experiments to be described, oscillators were used with frequencies approximately 375, 825, 2000, 3200 millions per second, respectively. Substituting these values for n in the above value (11) for x; and taking $\mu = 1$, $\sigma = 1600$, approximately, for copper or gold, 13,500 for platinum, and

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* See Byerly's 'Fourier's Series and Spherical Harmonics,' Art 51.

† Recent Researches, pp. 260, 281.

‡ *Oscillations Électriques,* p. 252.

4,770,000 for electric-light carbon, we obtain the following table:—

TABLE I.

Frequency n.	Thickness of 'Skin.'		
	Copper.	Platinum.	Carbon.
Millions.	cm.	cm.	cm.
375	0·00033	0·00092	0·01749
825	0·00022		
2000	0·00014		
3200	0·00011		

We can obtain an approximate value for the thickness of the 'skin' in another way. Suppose the conductor to be a circular cylinder. From Stefan's formula (2)

$$\frac{R'}{R} = \pi a \sqrt{\frac{n\mu}{\sigma}};$$

and assuming that, as with steady currents, the resistance is inversely proportional to the area of the section used by the current, the oscillatory current must occupy the R/R'th part of the section. Since this portion is a thin layer next the surface we have

$$2\pi a x = \frac{R}{R'} \times \pi a^2,$$

$$= \pi a \sqrt{\frac{n\mu}{\sigma}} \times \pi a^2,$$

and $$x = \frac{1}{2\pi}\sqrt{\frac{\sigma}{n\mu}};$$

which turns out to be precisely the value we obtained before (11).

Hertz[*] stated that, as far as he could observe, the nature of the metal out of which his *resonator* was formed had no influence upon the phenomena, but experiments by Bjerknes[†] did not confirm this conclusion. He found the efficiency of the metals copper, brass, silver, platinum, nickel, iron, to be in the order in which they are here named. By depositing

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[*] 'Electric Waves,' p. 45.

[†] Wied. *Ann.* xlviii. p. 592 (1893).

electrolytically on the surface of a metal forming a resonator a shell of another metal, he determined what thickness he must add in order that the resonator should behave as though formed of the second metal. He used an electrometer method, and his results are shown in the following table :—

TABLE II.

	Original Reading.	Final Reading.	Thickness of Layer.
Copper on Iron	13·4	105	0·012 mm.
Zinc on Iron............	13·4	71	0·016 „
Zinc on Copper	100	80	0·012 „
Nickel on Copper......	100	40	0·010 „
Cobalt on Copper......	100	10	0·003 „
Iron on Copper	100	10	0·003 „

The wave-length used by Bjerknes was 420 cms. and the frequency, therefore, about 70 millions per second.

I have not been able to find any record of experiments on the thickness of the effective portion of the metal of an *oscillator* except by Preece and Righi.

The former, in describing experiments made in conjunction with Marconi on signalling without wires, says * :—

"The distance at which effects are produced with such rapid oscillations depends chiefly on the energy of the discharge that passes. A six-inch-spark coil sufficed through 1, 2, 3, up to 4 miles, but for greater distances we have used a more powerful one—one emitting sparks 20 inches long. It may be pointed out that this distance increases with the diameter of the spheres and it is nearly doubled by making the spheres solid instead hollow."

He used the regular Righi oscillator with 10 cm. spheres sparking in vaseline oil, from which, he says, there were emitted waves 120 cms. in length, with a frequency of 250,000,000 per second. How this wave-length was determined is not given.

Righi states † :—

"Finally I made the perhaps unlooked-for observation that an oscillator composed of hollow spheres is less effective than one such of massive spheres. For example, when the

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* Proc. Royal Inst. vol. xv. p. 472 (1897).

† Righi, *Optik der elektrischen Schwingungen*, pp. 14, 15.

oscillator had massive spheres of about 4 cms. diameter the action on a resonator first ceased at about 11 metres; while, on the other hand, it ceased at about 6 metres when these spheres were replaced by hollow ones of equal external diameter and about 1·5 mm. of wall-thickness. Two others with yet decidedly thinner walls gave still smaller effects. The knobs connected to the machine * might, however, quite as well be hollow as solid, and also, within certain limits, might be of less or greater diameter without the efficiency of the oscillator being thereby affected."

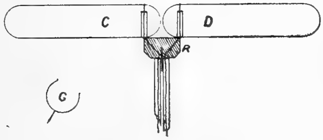
As just remarked, these are the only references to the subject that I have found. Indeed, considering the importance of the oscillating doublet, it is somewhat surprising that so little has been done in the direction of a satisfactory theory of it.

2. *The Oscillators.*

The oscillators were of two types, cylindrical and spherical respectively. The cylinders were circular in section with hemispherical ends; and were 2·5 cms. in diameter and 12·5 cms. long. For exciting the oscillator an induction-coil capable of giving a 12·5 cm. spark was used. It was placed in a metal-lined box and connexions were made to it as follows.

The cylinders C, D (fig. 1) were placed along the focal line of a parabolic reflector M, made of sheet zinc, being held in position by wooden supports hollowed to receive them. To the ends of the wires L, L were soldered strips r, r of brush copper about 5 mm. wide bent into circular form as shown in G (fig. 1). The ends of the cylinders were slipped into these copper holders, and a rubber band placed about the holder drew it closely to the cylinder, thus ensuring good and ample contact. The wires were then led through a wooden block R (as shown in fig. 2, p. 432), into which they were securely wedged, and then through small glass tubes firmly held together by rubber tape; and were finally joined

Fig. 1.



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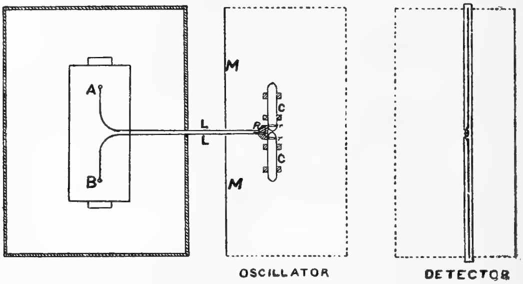
* Influence electrical machine.

to A and B, the terminals of the secondary of the coil. The glass tubes passed through openings in the mirror M and the box.

Fig. 2.



By this arrangement a pair of cylinders could be removed and another pair substituted for them while the connexions remained precisely the same.

The spark-gap was always the same for the cylinders, being about 0·7 mm. long.

The spherical oscillators were of the Righi pattern. The spheres C, D (fig. 3) were held in circular disks of hard rubber, in which openings had been turned to receive them, by melting paraffin about them. The smaller knobs K, K were on the ends of jointed rods R, R which were held in hard rubber rods N, N rigidly fastened to the disks carrying the spheres. By this means the knobs could be adjusted to the spheres, and would remain so when the spheres were adjusted to each other. To allow this latter to be done, one disk was held to the horizontal support beneath by a screw passing up through the support into it, while the other could be slid along the support by a screw. The entire arrangement is shown in fig. 4. The spheres were of three sizes.

Fig. 3.

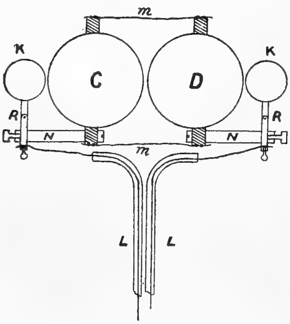
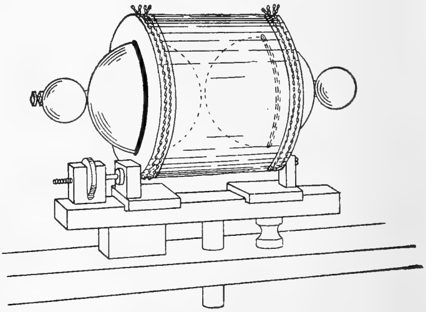


Fig. 4.



With those 2·5 cms. and 4 cms. in diameter, knobs K, K of diameter 19 mm. were used. When the spheres had a diameter of 10 cms., the knobs had a diameter of 37 mm.

The membrane *m*, *m* was first softened by soaking for a few minutes in water and glycerine, and was then securely bound to the circumference of the hard rubber disks by cords which rested in grooves made to receive them. The membrane was such as is used as a container for the familiar 'Bologna sausage,' and was obtained at a large packing house. It made a perfectly oil-tight vessel, the largest oscillator being in action for hours without a single drop of oil escaping.

The oil used was described as *white liquid vaseline*

When in operation the oscillator was placed in the parabolic reflector on a suitable support, seen in fig. 7, and was connected to the coil by wires L, L passing through tubes as in the case of the cylinders.

The interrupter was that used by G. W. Pierce with his Radio-micrometer *. It consisted of a platinum-tipped rod which, by means of a fan-motor, was rapidly plunged into and withdrawn from a cup of mercury, the surface of which was kept clean by a stream of water continually flowing over it. In series with the interrupter was a pendulum contact-maker. The metallic rod of the pendulum carried a platinum wire soldered to its lower end, which, as the pendulum swung to and fro, dipped into a mercury trough. By raising or lowering the pendulum the time during which it was immersed in the mercury was shortened or lengthened. This was so adjusted that whilst the pendulum made circuit during a single semi-oscillation, the motor-break made about 6 interruptions. In addition an ordinary key was in circuit. This was pressed down while the pendulum made any desired number of vibrations. For most of the work the key was depressed for 10 swings of the pendulum, that is, for about 60 interruptions of the coil. Thus in the primary circuit the current traversed in succession the primary of the induction-coil, the motor-break, the pendulum rod into the mercury and the circuit-key. The battery consisted of ten storage-cells arranged in two banks in multiple, *i. e.* giving 10 volts of E.M.F.

The parabolic mirrors, both for oscillator and receiver, were 75 cms. high, 60 cms. wide, and had a focal length of 12·5 cms. At first reflectors of but 5 cms. focal length were

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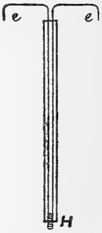
* American Jour. of Science, vol. ix. p. 252 (1900).

tried, but they were entirely useless, the reflected waves apparently destroying completely the direct waves *.

3. *The Receiver.*

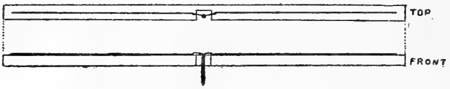
As a receiver I used the magnetic detector devised by Rutherford † and described by him in 1896. A few inches of iron wire 0·14 mm. in diameter were dipped in melted paraffin and, after cooling, the wire was cut into about 20 pieces. Holding these together, about 90 turns of fine (No. 40) insulated copper wire were wound upon about 1 cm. of their length; and after the protruding ends of the iron wire had been cut off and the fine wire had been soldered to heavier copper wires, the helix with its iron core was mounted in the end of a small glass tube, the connecting wires being drawn through the other end of the glass tube. The helix was kept in place by simply dipping the tube in melted paraffin and then allowing it to cool. The entire arrangement is shown (half size) in fig. 5. The helix is seen at H and the ends of the wire are shown at *e*, *e*, curved as they were used.

Fig. 5.



On a wooden rod about 65 cms. long two wires (fig. 6) were fastened with soft wax, their near-together ends terminating in small mercury cups. The bar was cut out at its central portion so that the detector could hang vertically with its ends in the mercury cups. Thus the wings became electrically a portion of the wire of the helix. The wooden bar was mounted in a parabolic reflector so that the wire lay along a focal line (fig. 7).

Fig. 6.



The method of using was as follows. The iron of the detector was first magnetized to saturation by being thrust into a helix in which a current was made to circulate. The detector was then hung so as to connect with its wings, and on placing it in the field of an electric oscillator, the surgings

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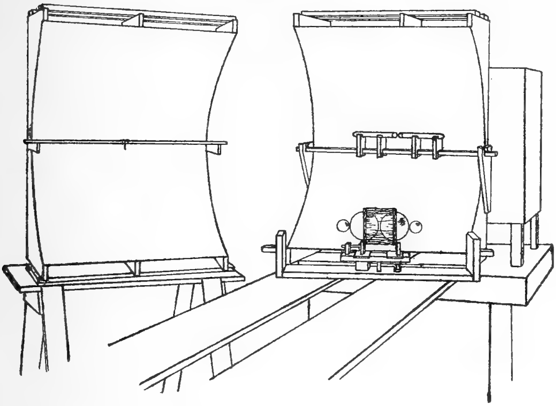
* I had forgotten Hertz's experience. 'See Electric Waves,' pp. 172, 175.

† Phil. Trans. A 1897, vol. clxxxix. pp. 1–24.

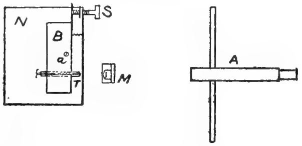
gathered up by the wings destroyed a portion of the magnetization of the iron. The change in the magnetization was observed in a magnetometer.

Fig. 7.



A section of the magnetometer is shown at M in fig. 8. It was of the plainest construction, consisting simply of a mirror 10 mm. in diameter with two small steel magnets fastened on the back by paraffin, and suspended by a silk fibre about 12 cms. long. Behind it was a block B, in which a small glass tube, closed at one end, was wedged. The inner diameter of this tube was approximately equal to the outer diameter of the detector-tube, so that the latter moved snugly into the former, and, when pushed in as far as it would go, was in a definite position which could easily be recovered.

Fig. 8.



The block B was fastened on the upper face of a board N by a screw *a* around which it could be given a slow motion by a screw S, thus easily making the detector approach or recede from the magnetometer by a small amount. The telescope and scale A were about 120 cms. from the magnetometer.

The position of the magnetometer was adjusted so that when the detector, fully saturated, was pushed into the tube T, the magnetometer deflexion was 40 cms., the scale being numbered so that this read zero. After partial demagnetization through the electrical oscillations, it was replaced in its position behind the magnetometer and the alteration in reading observed directly.

It was found that the detector could be removed and replaced with no observable difference in the reading; and successive magnetizations to saturation produced the same deflexion.

Rutherford * showed that the demagnetization produced by rapid oscillations is confined to the outer layers of the magnet. The diffusion of the magnetic force from the dielectric into a magnetizable substance is, indeed, similar to the passage of electrical action into a conductor. Hence for rapidly alternating currents the magnetic action is confined to the outer layers, the more rapid the alternation the thinner the layer affected. Hence the reason for using fine wire and insulating the pieces from each other. By so doing more surface is presented to the dielectric and the effect is greater. In some of his detectors Rutherford used iron wire 0·07 mm. in diameter (*i. e.* only one half the diameter of mine), with 160 turns of fine wire on the helix, and much longer wings. Thus he was able to obtain noticeable effects at a distance of over half a mile from the oscillator, which consisted of large plates. My detector, however, was sensitive enough for my purpose.

In the report † on Hertzian Oscillations which Righi presented to the International Congress of Physicists in Paris in 1900, he enumerated 21 kinds of apparatus for indicating the presence of electric oscillations. Of these the magnetic detector is one of the easiest to manage and possibly the simplest for quantitative comparisons, though I know of few investigations in which it has been used ‡. Its sensitiveness increases as the wings are lengthened, and the greatest difficulty I experienced arose from the fact that it is specially sensitive to the surgings in the connecting wires of the oscillator. An illustration may be interesting.

A wire 3 metres long was stretched from one terminal of

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* *Loc. cit.*

† *Rapports présentés au Congrès International de Physique réuni à Paris en* 1900, Tome ii. p. 301. (Paris, 1900.)

‡ In the June (1901) Phil. Mag. is a paper by C. G. Barkla, in which are described experiments on the relative velocities of electromagnetic disturbances along wires of different diameters, in which the magnetic detector was used as an indicator.

the secondary of an induction-coil across to the side of the room, and sparks were made to pass between the ordinary pointed terminals of the coil. The coil was in the basement of the laboratory. A fully magnetized detector, with wings each about 90 cms. long arranged parallel to the wire running out from the coil, was placed in position by the magnetometer, giving the usual deflexion of 40 cms. The magnetometer was in a room on the next floor above, and was distant about 35 metres, with brick walls, numberless wires and other pieces of metal between; and yet, as soon as sparks began to pass at the coil, the magnetometer deflexion was reduced by 2 cms.

The detector was also used to determine the wave-length of the oscillators used. The wave-length for the cylinders was found to be approximately 80 cms.; that for the 10 cm. spheres approximately 38·5 cms., *i. e.* 3·85 times the diameter. An account of these experiments will be published later. Using these results the frequencies given in Table I. were obtained.

4. *Experiments on the 'Skin'-effect.*

During the study of the 'skin'-effect cylindrical oscillators with the dimensions given above were made as follows:—

No. 1. Solid brass.

No. 2. Solid Norway iron.

No. 3 & No. 4. Copper 1·5 mm. and 0·8 mm. thick respectively. The mantle was of copper tubing and the hemispherical ends were spun and then soldered on.

No. 5. Solid electric-light carbon.

No. 6. Sheet platinum, over a wooden form. The mantle was 0·0013 cm. thick; the hemispherical ends, which were soldered on, were 0·015 cm. thick.

No. 7. Sheet silver. Mantle, 0·002 cm.; ends, 0·014 cm.

No. 8. Tin-foil, 0·0025 cm. thick, on a wooden form.

No. 9. Gold-leaf laid upon a wooden form. According to manufacturer, the leaf was about 1/220,000 inch, or 0·0000114 cm. thick.

No. 10. Silver-leaf laid upon wood. Thickness about 0·00003 cm.

Upon cylinders covered with gold-leaf, copper was deposited electrolytically, the thickness of the copper being in No. 11, No. 12, No. 13, No. 14, No. 15, respectively, 0·0001, 0·00015, 0·0002, 0·0003, 0·0005 cm.

No. 16. Silver deposited chemically on glass blown to the proper shape. The thickness, obtained by weighing, was approximately 0·000013 cm.

In order to have the ends between which the sparks were to pass as nearly alike as possible, these ends were faced with sheet platinum of a thickness 0·003 cm. By means of a die disks of 1 cm. diameter were punched from the sheet. In the case of cylinders No. 1 to No. 4, these disks, after being given the proper spherical curvature, were soldered directly on the cylinder, after which the surplus solder was removed and the platinum polished by a buffing-wheel.

In the case of No. 5, copper was first deposited on the end, and a cap with curvature rather greater than that of the end face was cemented on with shellac. To do this, a small piece of shellac was placed on the end, and the cap placed in position over it. The shellac was melted by holding a hot wire on the cap, which was then pressed by the finger firmly against the end until the shellac hardened. In this way excellent contact was made all about the circumference of the cap.

No. 6 had excellent platinum faces already, and No. 7 was left without them. On all the rest caps were fastened as on No. 5, *i.e.*, directly cemented with shellac.

Later on, some of the caps were secured in place by holding them firmly against the end of the cylinder and then depositing copper on the edge of the cap and on the cylinder where it rested. This certainly insured perfect electrical connexion, and is superior to the cementing method, being more durable, but the results obtained were not noticeably better.

By referring to Table I. it will be seen that for every frequency mentioned there, gold and silver leaf are decidedly thinner than the 'skin' as there given; while in the case of the lowest frequency (that of the cylinders), beside the gold and the silver leaf, the metallic portion of cylinders numbered 11, 12, 13, 14, and 16 was thinner than the calculated 'skin.' One would expect, therefore, that if the efficiency of the oscillator depends on the thickness of the metal constituting it, that dependence would manifest itself in the use of these cylinders; but extended and careful observation failed to detect any evidence whatsoever of such an effect.

The manner of experimenting was as follows. The cylinders were placed in position in the zinc reflector, and the connexions and sparking-distance carefully adjusted. Then the magnetized detector was hung in the other reflector, which was placed directly before the first reflector with a certain distance (usually 30 cms. at first) between the focal lines of the two. The key was then depressed while the pendulum made 10 swings, after which the detector was placed in its pocket behind the magnetometer and the change noted in the scale-reading. The detector was then magnetized again, and the same work performed with the mirrors at a greater distance apart. This was continued until readings at a sufficient number of distances had been secured. Then another pair of cylinders was taken and a similar set of readings obtained.

Rutherford reported that in his experiments no difference could be detected whether the first semi-oscillation in the receiver tended to magnetize the core or the reverse, but in my experiments this was very evident. For instance, on putting the detector in place at 30 cms. from the oscillator, a scale-reading of 57 mm. was obtained. Next, after remagnetizing, the detector was hung up so that the face presented to the oscillator was the reverse of that in the former case. In other words, if the detector were now raised out of its mercury cups, turned through 180° about its axis and then dropped into its cups again, its first position would have been recovered. With the detector in this second position, a deflexion of but 37 mm. was obtained. The oscillator is very dead-beat, and with the detector in the first position, the first (and greatest) pulse received by it was in the direction to demagnetize the core ; in the second position this first pulse was in the opposite direction in the helix, and the total resulting demagnetization was not so great. This difference in reading gradually diminished as the detector was withdrawn from the oscillator, and at about 60 cms. it disappeared. In Table III. is given a series of readings at distances between 30 and 90 cms., the two positions of the detector being denoted by 1 and 2.

TABLE III.

Distance of Oscillator from Detector in cms.	Oscillator ; position of detector ; and scale-reading in mm.							
	No. 1.		No. 9.		No. 11.		No. 12.	
	1	2	1	2	1	2	1	2
30	50	25	57	39	50	30	63	48
40	30	22	47	31	34	24	50	41
50	27	26	35	35	30	24	44	38
60	27	25	36	32	29	32	44	43
70	23	22	28	28	25	29	...	...
80	22	20	31	25	27	26	35	35
90	20	20	29	26	25	27	...	...

It would naturally be suspected that this reversal effect was due to the direct action of the coil or the connecting wires; but on removing the cylinders and substituting therefor small knobs, no effect was obtained. Indeed, the wings of the detector were made at right angles to the wires leading from the coil to the oscillator in order to avoid such action. Moreover, if it were due to the direct action, the effect would have been more pronounced with the spherical oscillators, but such was not the case.

The readings in the table show irregularity in the action of the oscillators, though I believe it is not so great as in many experiments with electric waves, a fact due largely to the constancy of the interrupter used.

In Table IV. readings obtained with various cylinders are exhibited. Each reading is the mean of the two readings obtained, at the distance indicated, by the detector in its two positions. An examination of the results will lead to the conclusion that, for frequencies as high as those here used, the effectiveness of an oscillator is not impaired by using a thin shell instead of a solid.

As stated above, the platinum caps were soldered to the iron and brass cylinders, and were then given an excellent polish. The caps on the gold-leaf and other delicate shells could not be polished in this way. They were simply rubbed clean with chamois. One would naturally expect, therefore, that the solid cylinders would perform best, but such was not the case. Indeed the brass and iron oscillators were the hardest to adjust and seemed about the *least* effective; and the gold-leaf acted so continuously better than the solid, that I began to think its thinness was a determining factor. The average of the readings for gold-leaf given in the table is decidedly better than for the brass oscillator, and numerous readings accentuated this result. Afterwards, however, the platinum shell (No. 6) was constructed, and proved the most reliable and efficient of all. Now this shell was thicker than the calculated ‘skin’ (Table I.), and further, on wrapping two layers of heavy tinfoil (thickness 0·005 cm.) about the mantle and binding it closely there, no alteration whatever was made in its efficiency. The proper conclusion, therefore, seemed to be that, within the limits of the experiments, the thickness of the metal had no effect.

If one could obtain a thin continuous carbon shell possibly more decisive experiments might be made, as the ‘skin’ for carbon is comparatively thick, but such was not available.

The magnetic nature of the iron made no difference whatever; iron and brass were indistinguishable in their behaviour. Silver did not appear equal to platinum as a spark-surface,

TABLE IV.

Distance of Oscillator from Detector in cms.	No. of Oscillator; scale-reading in mm.													
	No. 1.	No. 2.	No. 5.	No. 6.	No. 7.	No. 8.	No. 9.	No. 10.	No. 11.	No. 12.	No. 13.	No. 14.	No. 15.	No. 16.
30	37 47 49	46 26	47 49	46 69 54	37 45	39	39 58 47	52	40 42	48	49 55	46	57 45	44 34
40	22 26 29	40	28	38 51	...	33	30 39 40	40	28 33	34	37 46	43	45	38
50	26 30 30	28 17	28 30	37 44 35	...	32	26 35 38	35	27 32	30	47 41	37	41 29	37
60	20 26 28	26	32	39 47	35	30	19 34	33	30 31	32	41 43	40	40 26	27 27
70	18 22	...	25	...	...	32	19 28	28	27	28	31	26	36	29
80	16 21 20	21 12	25	33 35	22	30	18 38 29	26	27 23	25	26 35	31	30 22	...
90	20	...	24	...	...	29	27	25	26	21	23	27	29	...
100	18 22	...	18 23	25 26	...	26	23 24	21	26	19	21	22	28	...
120	13 15	17 10	12	22 23	...	20	14 23	19	22 19	15	18 26	19	25 15	...
140	11 10	...	... 15	19 18	...	15	11 17	14	17	13	15	15	17	...
160	12 10	12 8	8	12 17	...	17	14 21	14	14 12	13	12 19	18	17 10	...
180	12	...	7	...	...	12	11	11	12	9	12	12	14	...
200	8	5	6 11	11 12	...	11	10	10	11 12	6	10	12	10	...

As the experiments with cylindrical shells failed to exhibit any trace of the 'skin'-effect, it was decided to try with spheres, and Righi oscillators with spheres 2·5, 4, and 10 cms. respectively in diameter were constructed. By these, solid brass spheres were compared with gold-leaf spherical shells of the same diameter. The latter were made by covering with gold-leaf accurately turned and beautifully polished lignum-vitæ spheres of the required sizes.

In the oscillators composed of solid spheres the brass surface was used as sparking faces. Observation showed that, in air, newly polished brass is at least equal and possibly superior to sheet platinum, and the oil dielectric prevents the rapid deterioration of the surface.

For the gold-leaf shells platinum faces were necessary. Two were fastened on each sphere, at the extremities of a diameter, by electro-deposition of copper.

The results obtained with the three oscillators are given in Tables VI. and VII., which explain themselves. With the solid spheres the first reading was usually the highest, but with the shells there were usually two or three indifferent ones to start with. This was observed with platinum faces generally. The readings recorded are: for the solid spheres, the first 8 taken; for the 4 cm. shells, the first 8; for the 2·5 cm. shells, the last 8 of a series of 14; and for the 10 cm. shells, all the satisfactory readings obtained.

Table V.—Spheres 2·5 cms. in diameter.

Distance of Oscillator from Detector in cms.	Detector position; and scale-readings in mm.								Mean.
	I.		II.		III.		IV.		
	1	2	1	2	1	2	1	2	
	1. Solid.								
30	40	44	41	40	46	39	46	35	41·4
50	27	23	24	23	18	20	25	25	23·1
100	15	13	12	8	11	10	12	13	11·8
140	10	7	9	7	8	8	11	7	8·4
200	7	7	7	4	7	4	10	5	6·4
250	7	5	6	5	5	7	8	6	6·1
300	7	6	5	4	4	5	4	5	5·0
	2. Shell.								
30	39	40	46	37	40	41	50	45	42·3
50	18	22	20	16	22	24	27	28	22·1
100	10	12	10	12	13	13	16	15	12·6
140	8	7	8	7	8	7	12	12	8·6
200	6	4	6	5	5	5	15	9	6·9
250	6	6	6	4	7	5	10	11	6·9
300	5	7	5	7	4	3	10	8	6·1

TABLE VI.—Spheres 4 cms. in diameter.

Distance of Oscillator from Detector in cms.	Detector position; and scale-readings in mm.								Mean.
	I.		II.		III.		IV.		
	1	2	1	2	1	2	1	2	
	1. Solid.								
30	73	64	...	60	65	56	68	50	62·3
50	33	...	46	40	30	43	32	30	36·3
100	26	28	20	18	16	17	14	10	18·6
140	18	15	12	10	10	10	7	10	11·5
200	8	8	...	9	...	...	5	4	6·8
250	8	9	9	7	7	6	4	...	7·1
300	6	5	7	5	5	5	3	2	4·8
	2. Shell.								
30	70	43	74	60	66	48	60	51	59·0
50	32	31	40	36	28	33	33	33	33·3
100	15	14	15	16	14	15	18	16	15·4
140	7	9	15	11	10	13	11	11	10·9
200	10	9	10	6	5	9	7	7	7·9
250	8	10	8	8	6	...	6	8	7·7
300	7	7	12	7	...	...	5	4	7·0

TABLE VII.—Spheres 10 cms. in diameter.

Distance of Oscillator from Detector in cms.	Detector position; and scale-readings in mm.								Mean.
	I.		II.		III.		IV.		
	1	2	1	2	1	2	1	2	
	1. Solid.								
50	97	117	120	80	133	78	98	95	102·3
100	81	90	88	104	78	100	66	72	84·9
200	55	57	52	43	47	40	39	45	47·3
300	34	34	37	35	38	35	29	27	33·6
500	13	14	10	10	12	10	10	9	11·0
600	9	11	8	7	6	8	6	7	7·8
700	7	4	6	5	5	3	5	8	5·4
	2. Shell.								
50	110	92	113	107	136	115	120	85	109·5
100	101	85	73	89	76	80	93	89	85·8
200	45	45	49	44	...	...	...	...	45·8
300	30	34	32	34	...	31	...	...	32·2
500	15	12	...	...	...	...	...	...	13·5
600	7	10	...	...	...	...	...	...	8·5
700	5	6	...	...	...	...	...	...	5·5

The 4 cm. spheres seem to show evidence, at the distance of 30 cms., of the reversal effect mentioned above, but it disappeared before a distance of 50 cms. was reached. With the other spheres no trace of the phenomenon can be found.

No difficulty at all was experienced in handling the smaller spheres, but the larger ones gave some trouble. But a single reading had been secured with the first pair of 10 cm. shells before one of them failed. Veins appeared in the gold-leaf encircling one platinum cap. On replacing this sphere by a new one, only three or four readings were secured before the other sphere of the first pair failed in the same way. This was removed and a new one put in its place, and with the pair thus formed the readings in Table VII. were obtained.

If curves be drawn with the mean readings of Tables V., VI., and VII. as ordinates and the distances from the oscillator as abscissæ, the curve for a shell will be found to practically coincide with that for the solid sphere of the same diameter. The experiments, therefore, indicate that, both in the case of the cylindrical and of the spherical doublets, the excessively thin gold-leaf shells were quite as efficient as the solid metal bodies.

This investigation was made in the Jefferson Physical Laboratory, Harvard University; and while under especial obligation to Professor Trowbridge for his never-failing consideration and encouragement, I would express my gratitude to every member of his staff for innumerable kindnesses.

Jefferson Physical Laboratory,
Harvard University, U.S.

XLIX. *A New Specifying Method for Stress and Strain in an Elastic Solid.* *By* Lord KELVIN *.

THE method for specifying stress and strain hitherto followed by all writers on elasticity has the great disadvantage that it essentially requires the strain to be infinitely small. As a notational method it has the inconvenience that the specifying elements are of two essentially different kinds (in the notation of Thomson and Tait *e, f, g*, simple elongations; *a, b, c*, shearings). Both these faults are avoided if we take the six lengths of the six edges of a tetrahedron of the solid, or, what amounts to the same, though less simple, the three pairs of face-diagonals of a

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* Communicated by the Author. From the Proceedings of the Royal Society of Edinburgh of Jan. 20, of which a preliminary notice was published in the Phil. Mag. for Jan. 1902.

hexahedron*, as the specifying elements. This I have thought of for the last thirty years, but not till a few weeks ago have I seen how to make it conveniently practicable, especially for application to the generalized dynamics of a crystal.

§ 1. We shall suppose the solid to be a homogeneous crystal of any possible character. Cut from it a tetrahedron ABCD of any shape and orientation. Let the three non-intersecting pairs (AB, CD), (BC, AD), (CA, BD) of its six edges be denoted by

$$(3p, 3p'), \quad (3q, 3q'), \quad (3r, 3r') \quad . \quad . \quad . \quad (1).$$

This notation gives

$$(p, p'), \quad (q, q'), \quad (r, r') \quad . \quad . \quad . \quad . \quad (2)$$

for the six edges of a tetrahedron, similar to ABCD, formed by taking for its corners $(\alpha, \beta, \gamma, \delta)$ the centres of gravity † of the four triangular faces BCD, CDA, DAB, ABC respectively, so that we have $p=\alpha\beta$, $q=\beta\gamma$, $r=\gamma\alpha$, $p'=\gamma\delta$, $q'=\alpha\delta$, $r'=\beta\delta$. Consider now, in advance, the amounts of work done by the six pairs of balancing forces constituting the six stress-components described in § 2, when the strain-components vary; for example, the balancing pulls P, parallel to AB, when $\alpha\beta$ increases from p to $p+dp$, all the other five lengths q, r, p', q', r' remaining constant. For the reckoning of work we may suppose the opposite forces, P, to be applied at α and β, instead of being equably distributed over the faces ADC, BDC. Hence the work which they do is Pdp; and other five pairs of balancing pulls, Q, R, P′, Q′, R′, do no work.

§ 2. Parallel to the edge AB apply to the faces ADC, BDC equal and opposite pulls, P, equally distributed over them. These two balancing pulls we shall call a stress or a stress-component. Similarly, parallel to each of the five other edges apply balancing pulls on the pair of faces cutting it. Thus we have in all six stress-components parallel to the

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* This name, signifying a figure bounded by three pairs of parallel planes, is admitted in crystallography; but the longer and less expressive "parallelepiped" is too frequently used instead of it by mathematical writers and teachers. A hexahedron, with its angles acute and obtuse, is what is commonly called, both in pure mathematics and crystallography, a rhombohedron. A right-angled hexahedron is a brick, for which no Greek or other learned name is hitherto to the front in usage. A rectangular equilateral hexahedron is a cube.

† For brevity I shall henceforth call the centre of gravity of a triangle, or of a tetrahedron, simply *its centre*.

six edges of the tetrahedron, denoted as follows :—

$$(\mathrm{P}, \mathrm{P}') \quad (\mathrm{Q}, \mathrm{Q}') \quad (\mathrm{R}, \mathrm{R}') \quad . \quad . \quad . \quad . \quad (3);$$

and we suppose that these forces, applied as they are to the four faces of the solid, are balanced in virtue of the mutual forces between its particles, when its edges are of the lengths specified as in (1). Let p_0, p_0', q_0, q_0', r_0, r_0' be the values of the specifying elements when no forces are applied to the faces. Thus the differences from these values, of the six lengths shown in formula (2), represent the strain of the substance when under the stress represented by (3).

Let w be the work done when pulls upon the faces, each commencing at zero, are gradually increased to the values shown in (3). In the course of this process we have

$$dw = \mathrm{P}dp + \mathrm{P}'dp' + \mathrm{Q}dq + \mathrm{Q}'dq' + \mathrm{R}dr + \mathrm{R}'dr' \quad . \quad . \quad (4).$$

§ 3. Hence if we suppose w expressed as a function of p, p', q, q', r, r', we have

$$\frac{dw}{dp} = \mathrm{P}, \quad \frac{dw}{dp'} = \mathrm{P}', \quad \frac{dw}{dq} = \mathrm{Q}, \quad \frac{dw}{dq'} = \mathrm{Q}', \quad \frac{dw}{dr} = \mathrm{R}, \quad \frac{dw}{dr'} = \mathrm{R}' \quad (4).$$

This completes the foundation of the molar dynamics of an elastic solid of the most general possible kind according to Green's theory, expressed in terms of the new mode of specifying stresses and strains, without restriction to infinitely small strains.

§ 4. To understand thoroughly the state of strain specified by (1) or (2), let the tetrahedron of reference, $\mathrm{A}_0\mathrm{B}_0\mathrm{C}_0\mathrm{D}_0$, for the condition of zero strain and stress, be equilateral (that is to say, according to the notation of § 2 (1) let $\frac{1}{3}$ of each edge $= p_0 = q_0 = r_0 = p_0' = q_0' = r_0'$). In $\mathrm{A}_0\mathrm{B}_0\mathrm{C}_0\mathrm{D}_0$ inscribe a spherical surface touching each of the six edges. Its centre must be at K_0, the centre of the tetrahedron ; and the points of contact must be the middle points of the edges. Alter the solid by homogeneous strain*, to the condition (p, q, r, p', q', r') in which $\mathrm{A}_0\mathrm{B}_0\mathrm{C}_0\mathrm{D}_0$ becomes ABCD. The inscribed spherical surface becomes an ellipsoid having its centre at K, the centre of ABCD, and touching its six edges at their middle points †. This ellipsoid shows

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* Thomson and Tait's 'Natural Philosophy,' § 155 ; 'Elements,' § 136.

† Thus we have an interesting theorem in the geometry of the tetrahedron :—If an ellipsoid touching the edges of a tetrahedron has its centre at the centre of the tetrahedion, the points of contact are at the middles of the edges.

fully and clearly the state of strain specified by p, q, r, p', q', r'. It is what is called the "strain ellipsoid"[*].

§ 5. Two ways of finding the ellipsoid touching the six edges of a tetrahedron are obvious. (1) Through AB and CD draw planes respectively parallel to CD and AB; and deal similarly with the two other pairs of non-intersecting edges. The three pairs of parallel planes thus found, constitute a hexahedron which contains the required ellipsoid touching the six faces at their centres; or (2) draw AK, BK, CK, DK, and produce to equal distances KA′, KB′, KC′, KD′ beyond K. We thus find four points, A′, B′, C′, D′, which, with A, B, C, D, are the eight corners of the hexahedron which we found by construction (1). A circumscribed hexahedron being thus given, the principal axes of the ellipsoid, and their orientation, are found by the solution of a cubic equation.

§ 6. Another way of finding the strain-ellipsoid, which is in some respects simpler, and which has the advantage that in its construction it does not take us outside the boundary of our fundamental tetrahedron, is as follows:—In the equilateral tetrahedron $A_0B_0C_0D_0$ describe, from its centre K_0, a spherical surface touching any three of its faces. It touches these faces at their centres; and it also touches the fourth face, and at its centre. Hence, if we solve the determinate, one-solutional, problem *to draw an ellipsoid touching at their centres any three of the four faces of any tetrahedron* ABCD, *and having its centre at* K, this ellipsoid touches at its centre the fourth face of the tetrahedron; and it is the strain-ellipsoid for the homogeneous strain by which an equilateral tetrahedron of solid is altered to the figure ABCD.

§ 7. To bring our new method of specifying strain and stress into relation with the ordinary method for infinitesimal strains and the corresponding stresses:—Let λ denote the length of each edge of the equilateral tetrahedron of reference, $A_0B_0C_0D_0$; and let h be the edge of the cube of which A_0, B_0, C_0, D_0 are four corners (this cube being the hexahedron found by applying either of the constructions of § 5 to the tetrahedron $A_0B_0C_0D_0$). The twelve face-diagonals of this cube are each equal to λ, and therefore $\lambda = h\sqrt{2}$. Let now the cube be infinitesimally strained so that its edges become $h(1+e)$, $h(1+f)$, $h(1+g)$; and so that the angles in its three pairs of faces are altered from right angles to acute and obtuse angles differing respectively by a, b, c from right angles. This is the strain (e, f, g, a, b, c) in the notation of

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[*] Thomson and Tait's 'Natural Philosophy,' § 160; 'Elements, § 141.

Thomson and Tait referred to in the introductory paragraph above. By the infinitesimal geometry of the affair, we easily find the corresponding alterations of the face-diagonals, which according to our present notation are $(p-1)\lambda$, $(p'-1)\lambda$, $(q-1)\lambda$, etc., and thus we have as follows :—

$$\left.\begin{aligned} p-1&=\tfrac{1}{2}(f+g+a)\\ p'-1&=\tfrac{1}{2}(f+g-a)\\ q-1&=\tfrac{1}{2}(g+e+b)\\ q'-1&=\tfrac{1}{2}(g+e-b)\\ r-1&=\tfrac{1}{2}(e+f+c)\\ r'-1&=\tfrac{1}{2}(e+f-c) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

for the relation between the two specifications of any infinitesimal strain. Adding these, and denoting $e+f+g$ by s, we find

$$p+p'+q+q'+r+r'-6=2s \quad . \quad . \quad . \quad (6).$$

And solving for a, b, c, e, f, g, in terms of p, q, r, p', q', r', we have

$$\left.\begin{aligned} &a=p-p' \;; &&b=q-q' \;; &&c=r-r' : \\ &e=s-p-p'+2 \;; &&f=s-q-q'+2 \;; &&g=s-r-r'+2 \end{aligned}\right\} \; (7).$$

§ 8. The work required to produce an infinitesimal strain e, f, g, a, b, c, in a homogeneous solid of cubic crystalline symmetry is expressed by the following formula :—

$$2w=\mathfrak{A}(e^2+f^2+g^2)+2\mathfrak{B}(fg+ge+ef)+n(a^2+b^2+c^2). \quad (8).$$

This may be conveniently modified by putting

$$k=\tfrac{1}{3}(\mathfrak{A}+2\mathfrak{B}) \;;\; n_1=\tfrac{1}{2}(\mathfrak{A}-\mathfrak{B}) \quad . \quad . \quad . \quad . \quad (9),$$

where k denotes the bulk modulus and n_1, n the two rigidity-moduluses. With this notation (8) becomes

$$2w=k(e+f+g)^2+\tfrac{2}{3}n_1[(f-g)^2+(g-e)^2+(e-f)^2]+n(a^2+b^2+c^2) \quad (10).$$

The rigidity relative to shearings parallel to the pairs of planes of the cube, or, which is the same thing, changes of the angles of the corners of the square faces from right angles to acute or obtuse angles, is n_1. The rigidity relative to changes of the angles between the diagonals of the faces from right angles to acute or obtuse angles is n. The compressibility modulus is k. Using now (7) in (10) we have

$$\begin{aligned} 2w=ks^2&+\tfrac{2}{3}n_1[(q+q'-r-r')^2+(r+r'-p-p')^2+(p+p'-q-q')^2] \\ &+n[(p-p')^2+(q-q')^2+(r-r')^2] \quad . \quad . \quad . \quad . \quad . \quad . \quad (11). \end{aligned}$$

L. *Multiple Transmission Fixed-Arm Spectroscopes.* *By* W. Cassie, *M.A., Professor of Physics in the Royal Holloway College*[*].

VARIOUS forms of spectroscope have been devised to diminish the number of prisms required for great dispersion by using each prism more than once. Contrivances for reflecting the beam of light back through a prism or train of prisms a second time by means of a mirror or a reflecting prism have been used by many observers. An instrument has been devised by Prof. Wadsworth in which the beam is sent six times through one prism by the use of seven mirrors.

The spectroscopes described in the present paper secure many transmissions of the beam through one or two prisms by the use of a principle which will be best explained by describing the instruments themselves.

I.

The first form of instrument I have to describe, which is in some respects the most satisfactory of the first three, is made with two half-prisms each silvered on one side. Figs. 1 and 2 (p. 450) show the essentials of the instrument, fig. 1 in plan and fig. 2 in elevation.

A is the collimator, B the telescope. *a b c* and *d e f* are two equal half-prisms silvered on the faces *a c* and *d f* and placed with their faces vertical. The beam of light, after emerging from the collimator A, passes under the prism *a b c* and strikes the prism *d e f* near the lower edge of the face *d e*. This prism is so placed that the beam entering at *d e* is reflected almost normally at the silvered face *d f*, and comes out again at the face *d e*. Now this beam is not horizontal in direction, but is slightly inclined upwards. The consequence is that the beam returning after emergence from *d e* can be caused to strike the prism *a b c* on the lower part of the face *a b*. This prism also is placed so that the beam is reflected almost normally at the silvered face *a c*, and after emerging from *a b* again strikes *d e* at a place higher than at first; and so travels backwards and forwards between the two prisms, gradually climbing upwards until at last the beam from *a b* passes over the top of the prism *d e f* and enters the telescope B. Each reflexion through a half-prism is equivalent to transmission with minimum deviation through a prism whose refracting angle is twice that of the half-prism. So that this arrangement is equivalent to a train of as many prisms as there are

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[*] Communicated by the Physical Society: read November 22, 1901.

reflexions of the beam. The loss of light can be reduced to a minimum by choosing the angles of the prisms so that the light is incident at the angle of polarization.

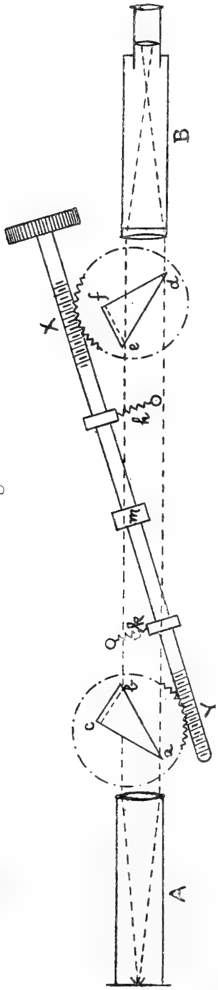


Fig. 1.

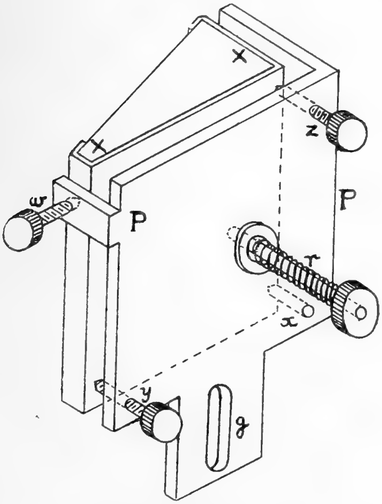
Fig. 2.

The adjustment for light of different refrangibilities is secured by turning the prisms in opposite directions so that they are always inclined to the beam of light at equal angles.

This is done by supporting the prisms on disks with cogged edges, represented by the circles surrounding them in fig. 1, and having a rod X Y with tangent-screws pressed against these disks by springs *h*, *k*. The rod X Y can be turned about the point *m* so as to remove the tangent-screws from the disks for a quick motion of the prisms. The prisms are attached to the disks by mountings which admit of their accurate adjustment. When the instrument has been adjusted for light of one refrangibility, it is adjusted for all refrangibilities; and the whole spectrum can be made to cross the field of view by turning the head of the tangent-screws.

The mounting of a prism is shown in fig. 3. The prism is fixed in a brass cell X which is attached by springs and screws to the upright plate P. From the back of the prism-cell a rod *r* passes freely through a hole in the plate P, and is surrounded by a spiral spring which presses the cell against the fixed point *x* (which enters a slight depression in the back of the cell) and the points of the screws *y* and *z*. The cell is pressed against the point of the screw *w* by a spring on the opposite side. The plate P is supported with freedom for adjustment by a bolt and nut passing through the slot *g*.

Fig. 3.



The breadth of the beam which can be used is limited horizontally only by the breadth of the face of the prism; but vertically the beam must be narrow if a number of reflexions are required. Thus the resolving power of the instrument depending, as it does, on the horizontal breadth of the beam, is not affected by this restriction. The resolving power is that of a train of as many complete prisms as there are reflexions in the half-prisms.

The beam does not traverse the prisms exactly in a principal plane. But the smallness of this obliquity and the narrowness of the beam in a vertical direction make any such effect of small consequence. Moreover, the obliquity can be indefinitely diminished by increasing the distance between the prisms.

II.

Another form of instrument is illustrated in plan in fig. 4 and in elevation in fig. 5. In this case one prism and two plane mirrors are used.

A is the collimator and B the telescope. C is the prism mounted on a levelling-table which rests on and turns with the arm C*f*. *d* is a plane mirror fixed above the object-glass of the collimator and facing towards the prism. *e* is a plane mirror supported on the extremity of the arm C *e*, and also facing towards the prism. The arm C *e* is free to turn round the centre of the table of the instrument. To the arm C*f* is attached a screw carrying a block Z, which is free to move along the arm, and is linked to a pin X fixed on the table of the instrument and to a pin Y on the arm C *e*. To use the prism with minimum deviation Z X and Z Y are made equal, and the prism and the mirror *e* are moved simultaneously by turning the screw.

The beam of light emerging from the collimator A passes under the mirror *d* and is deviated by the prism. The mirror *e* is so placed that the light strikes it near its lower edge and goes back through the prism. The beam being inclined slightly upwards, now strikes the mirror *d*, and so is reflected backwards and forwards between the mirrors traversing the prism at each journey and gradually climbing upwards until finally it passes from the mirror *d* over the top of the prism into the telescope B. If the prism and mirrors are once properly adjusted for minimum deviation for light of one refrangibility, they are brought in adjustment into the position for any other refrangibility simply by turning the head of the screw C*f*.

III.

A third form of instrument which may be convenient when it is desired to have the reversed beam in a separate position

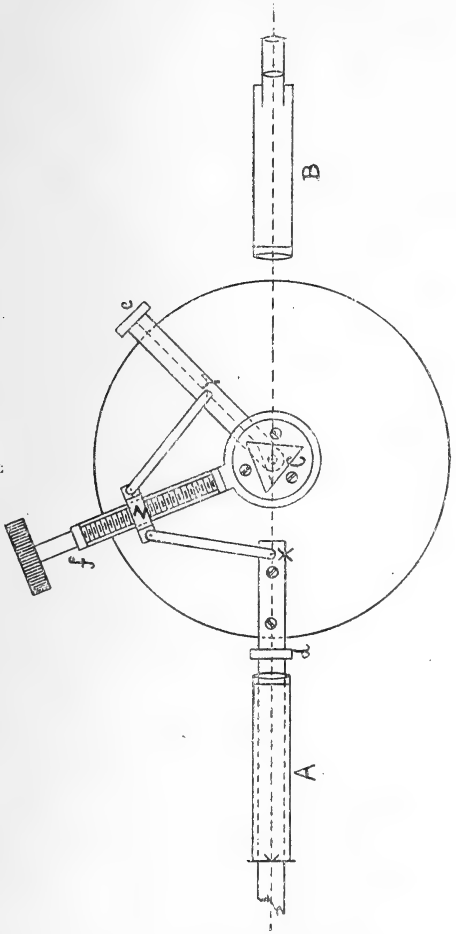


Fig. 4.

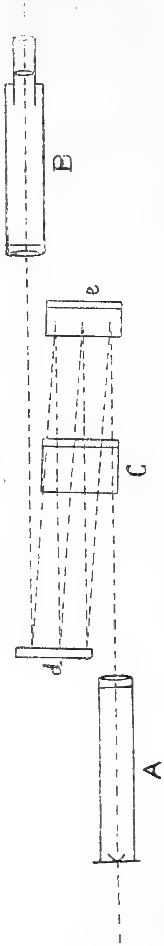


Fig. 5.

from the direct one, is shown in plan in fig. 6 and in elevation in fig. 7 (p. 454).

A and B are the collimator and telescope as before. The refracting prisms are *c d e* and *f g h*. *k* is a right-angled reflecting prism above the object-glass of the collimator. The beam from the collimator inclined slightly upwards as before is refracted at the face *c d* of one prism, is totally reflected at the face *d e*, and emerging at the face *e c* enters

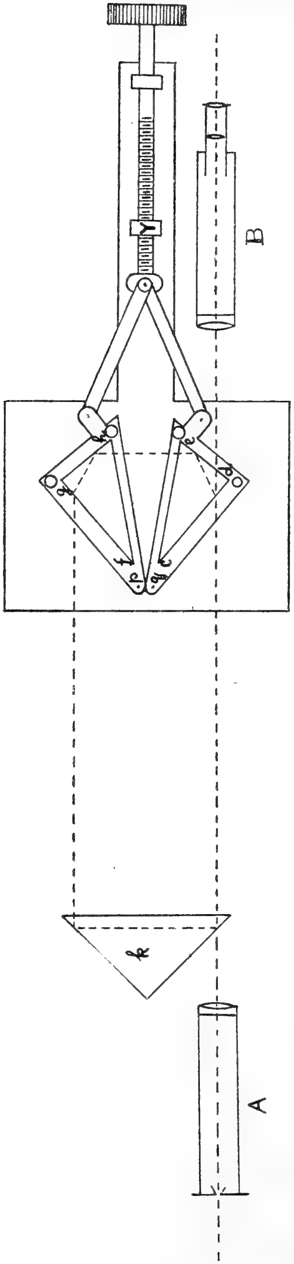


Fig. 6.

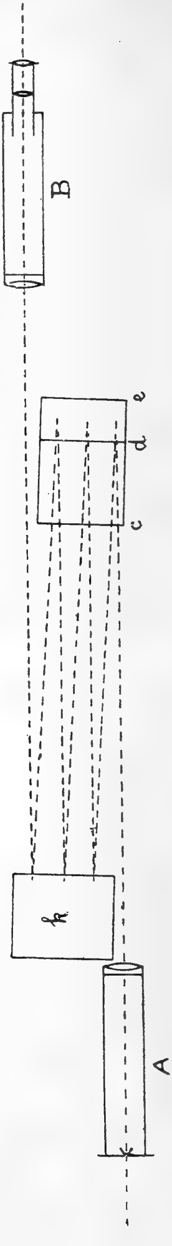
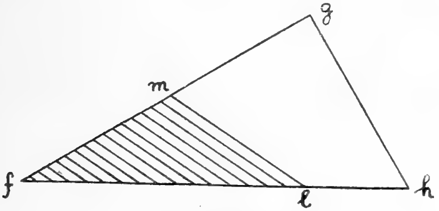


Fig. 7.

the face *h f* of the next prism, is totally reflected at the face *g h*, and refracted out at the face *f g*. The angles and positions of the prisms are made such that the beam emerging from *f g* is parallel to that entering at *c d* (except for a slight inclination upwards). After leaving the two refracting prisms the beam enters *k*, which returns it after two reflexions to the face *c d* in a direction parallel to that in which it first entered that face. Thus the beam circulates round the system of three prisms, gradually rising until it finally passes over the top of the prism *c d e* into the telescope B. For adjustment of the refracting prisms they are mounted on levelling-tables which are free to turn about centres at *p* and *q*, and are linked as shown in fig. 6 to a screw working in a fixed block Y, so that by turning the screw the angle between the faces *c e* and *f h* can be altered while the positions of the prisms are kept symmetrical with respect to the central line of the system. Thus when the instrument is adjusted for one wave-length, the whole spectrum can be brought into view by simply turning the adjusting screws.

It is not necessary that the refracting prisms be complete, because a ray entering near *c* strikes the face *e c* after refraction, and is for present purposes lost; consequently the portion shaded in fig. 8 may be cut from the prisms at the corners *c* and *f*. If *h g m l* is the actual form of prism used, it is necessary to cut it so that the beam in passing between the refracting prisms meets the faces *c e* and *f h* as nearly as possible at normal incidence; because if, for instance, the prism were made triangular of the form *h m g*, the refraction at the face *m h* would produce dispersion opposite to that due to *m g*.

Fig. 8.



This instrument is much inferior to the others in resolving power. To get the internal reflexions in the refracting prisms total, it is necessary to make the refractions at a large obliquity. That gives, no doubt, considerable dispersion. But the difference between the longest and shortest paths of rays through the refracting prisms is comparatively small.

For flint-glass the angles at d and g do not differ much from 90°; so that if $de = a = gh$ and the angles at e and h are about 60°, the difference of the paths for the pair of prisms is about $\frac{a}{2}$. Whereas the same two prisms with the corners at c and f not cut off, used as reflecting half-prisms in the arrangement first described, would give a difference of path about $4a$ for rays traversing each once. That is to say, the resolving power of such an instrument of type I. would be about eight times that of one of type III.

Other combinations involving similar principles will readily suggest themselves. But the three described are sufficient to illustrate the method. The number of transmissions may by increased by lengthening the prisms or mirrors. The length of slit that can be used may be increased in the same way. In the instruments I have had made for trial the prisms and mirrors were 5 centimetres high and gave very good results. They are simple in construction, and the adjustments are easy compared with those of a train of prisms. Moreover, by a few turns of an adjusting-screw one can change the number of transmissions, and with them the degree of dispersion, step by step through a wide range.

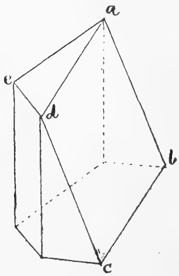
IV.

[*Addition made Dec.* 6, 1901, *after the reading of the paper.*]

Another arrangement in which the reflectors are totally reflecting prisms is shown in figs. 10 and 11.

The back of the half-prism instead of being silvered is formed into a right-angled totally-reflecting prism with its edge at right angles to the refracting edge of the half-prism, as shown in fig. 9. The beam is reflected back to this combined half-prism and reflector by two right-angled prisms fh and lm, as shown in figs. 10 and 11. The prism fh is exactly

Fig. 9.



half the breadth of the refracting prism, and is exactly opposite to the full upper half of it. The prism $l\,m$ is of the same size as $f\,h$, but has its lower edge at a lower level than the bottom of the refracting prism, so that a space is left between the reflecting prisms. A beam of light from the collimator passes from C between the reflecting prisms, is refracted and reflected at the half-prism, and so passes to and fro, taking the course indicated by the arrows in figs. 10 and 11, and finally passes from the reflector $l\,m$ under the refracting prism into the observing-telescope at T. The result in dispersing and resolving power is equivalent to passage through a train of as many complete prisms as there are to-and-fro journeys of the beam.

Fig. 10.

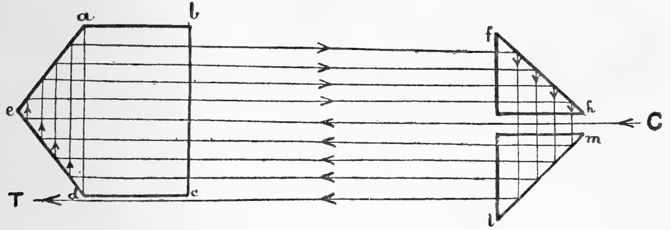
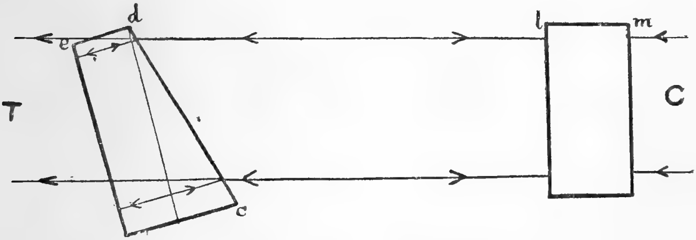


Fig. 11.



This arrangement is better than I. and II., because less light is lost by reflexion; and it is better than III. because it gives full resolving power. Moreover, the beam passes through the refracting prism in a principal plane, so that all the prisms may be placed quite close together, and the apparatus thus made more compact. The number of transmissions possible depends upon the width of the opening left between the reflecting prisms $f\,h$ and $l\,m$. A very large number of transmissions may be used if this opening is made small. Another possible form would be to make the prisms $f\,h$ and $l\,m$ also refract, by giving them the form produced by cutting the prism of fig. 9 by a plane through e at right angles to the face $a\,b\,c\,d$.

LI. *Notices respecting New Books.*

Die Theorie der Beobachtungsfehler und die Methode der kleinsten Quadrate mit ihrer Anwendung auf die Geodäsie und die Wassermessungen. Von Otto Koll. Mit in den Text gedruckten Figuren. Zweite Auflage. Berlin: Julius Springer, 1901. Pp. xii+323+31.

THIS is in every way an admirable text-book. The somewhat difficult and generally neglected subject of the theory of observations and the method of least squares is here handled with a lucidity of exposition and a wealth of illustration which compel the reader's attention. The treatment is a thoroughly practical one, the main purpose of the book being to train the reader in the intelligent application of the theory of probabilities to the results of measurements connected with surveying operations and hydraulic engineering. The book is divided into two parts, Part I. dealing with the theory of errors of observation; and Part II. with the method of least squares. A collection of the various working formulæ forms an appendix of 31 pages to the book. A noticeable feature is the liberal use of fully worked out numerical examples, which should prove extremely useful to the student. A further point of interest which may be referred to, is that the working formulæ have been thrown into a shape which renders them convenient to use in connection with a calculating machine, the author expressing his belief that tables of logarithms will be displaced by such machines.

LII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from vol. ii. p. 660.]

November 6th, 1901.—J. J. H. Teall, Esq., M.A., V.P.R.S., President, in the Chair.

THE following communications were read:—

1. 'Note on a Submerged and Glaciated Rock-Valley recently exposed to view in Carmarthenshire.' By Thomas Codrington, Esq., M.Inst.C.E., F.G.S.

This valley was brought to light in building a bridge across the River Towy at Dryslwyn, 9 miles above Carmarthen to which the tide now flows. At the bridge, the valley is narrowed to about half a mile. Near the water-edge the rock sloped down gradually to 23 feet below summer water-level, and was glaciated in large furrows a foot or more across, and striated blocks of grit rested upon it. About 60 feet farther out into the river, rock was not met with till depths of from 34 to 42 feet below summer level were reached, and the rock-surface was found to be sloping towards the south at an angle of from 28° to 18° with a vertical line; it was followed down to between 45 and 56 feet below summer water-level. Scratched stones were again met with in the clay near the rock. The glaciated surface on the northern bank is only 25 feet above sea-level; and the rock-surface is sloping down at a precipitous angle at 8 feet below sea-level at a distance of 18 miles from the mouth of the river.

2. 'On the Clarke Collection of Fossil Plants from New South Wales.' By Edward Alexander Newell Arber, Esq., B.A.

3. 'On an Altered Siliceous Sinter from Builth (Brecknockshire).' By Frank Rutley, Esq., F.G.S.

A rock-specimen, given to the author many years ago by the late H. W. Bristow, forms the main subject of this paper. It shows no trace of original sand-grains; it is compact, and with a fracture platy to conchoidal: small splinters of it can be fused on their edges to a white, frothy glass. Under the microscope the rock is decidedly tufaceous, containing small fragments chiefly of pumice, less often of crystals which are apparently epidote. In the slides of this rock and in some of the siliceous sinters from New Zealand, used for comparison, there are small patches of a brown substance which may possibly be of organic origin; in connexion with it, Prof. Weed's discovery of algous growths in some of the New Zealand sinters is mentioned. A specimen of hard breccia, also from the vicinity of Builth, is described. The cement of this rock is also possibly siliceous sinter, as well as some of the fragments, which latter show faint evidence of the inclusion of little shreds of pumice.

November 20th.—J. J. H. Teall, Esq., M.A., V.P.R.S., President, in the Chair.

The following communications were read:—

1. 'Notes on the Genus *Lichas*.' By Frederick Richard Cowper Reed, Esq., M.A., F.G.S.

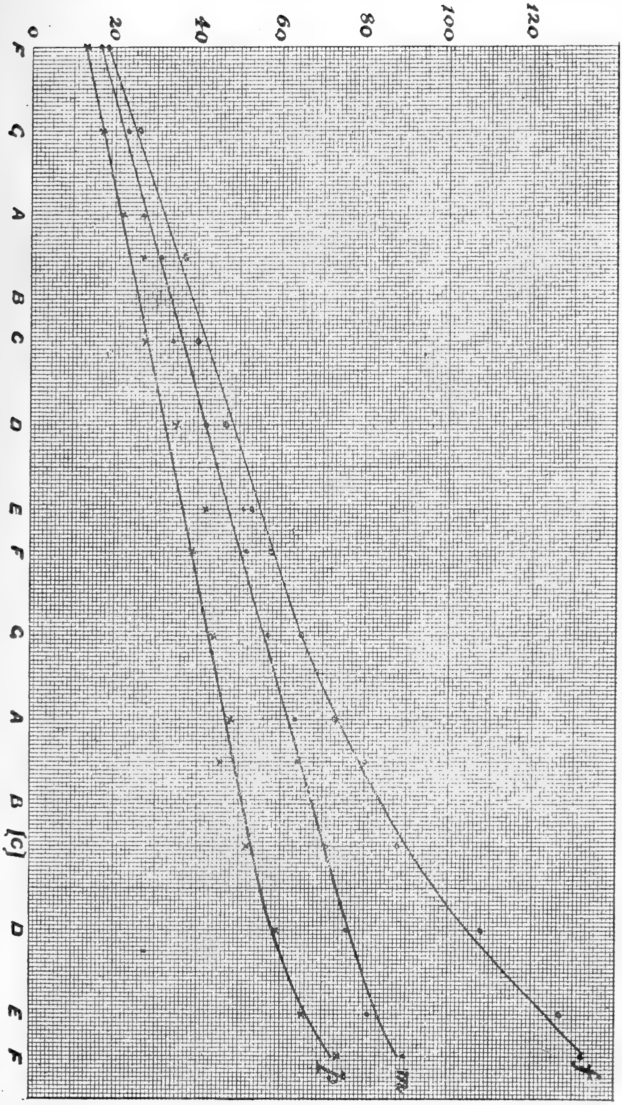
2. 'Some Remarks on the Meteorological Conditions of the Pleistocene Epoch.' By Dr. Nils Ekholm, Meteorologiska Central-Anstalten, Stockholm.

The opinion of the author on this subject differs in some important respects from that of Mr. Harmer. He considers the subject under two heads: (1) What are the meteorological conditions necessary and sufficient to produce a permanent ice-sheet such as that of the Great Ice-Age? (2) What will be the influence of such glaciation on the meteorological conditions, especially on the cyclones and anticyclones, of the ice-covered land and on its neighbourhood? The snow-line does not correspond with the mean annual isotherm of 32°, for Verchojansk in Siberia is not glaciated, whereas the southern point of Greenland is. The former has a winter anticyclone, while the latter is traversed by the central or northern part of cyclones during the whole year. The area of Pleistocene glaciation in America and Europe coincides with the areas now traversed by the most regularly-frequented storm-tracks. There seems to have been about the same difference between the mean annual temperatures of Europe and North America in the Great

Ice-Age as now, and it is generally agreed that a lowering of the present snow-line by 1000 metres would give rise to a similar Ice-Age. The hypothesis that a glaciation of North America would raise the temperature of Europe, and *vice versa,* seems to the author physically untenable. The positions and movements of anticyclones are not generally ruled by the ground temperature in our latitudes: they are in most cases eddies formed by the air-circulation in general, and in this the greater area and receipt of heat by the Equatorial regions must always be a preponderating factor. The author considers that the influence of the Glacial Period on atmospheric circulation would probably be similar to that of a cold winter nowadays. The cyclones would be gradually deviated into a more and more southerly track, while an anticyclone would be formed in the north, not, however, a stationary one, but travelling like a cyclone, only more slowly and irregularly. The summer must have been cold and stormy, with frequent fogs, somewhat like that of Cape Horn or Kerguelen Island at the present day. The author considers that Mr. Harmer underrates the effect of insolation and overrates that of the winds. 'The temperature of the summer only is essential for the phenomenon of glaciation.'

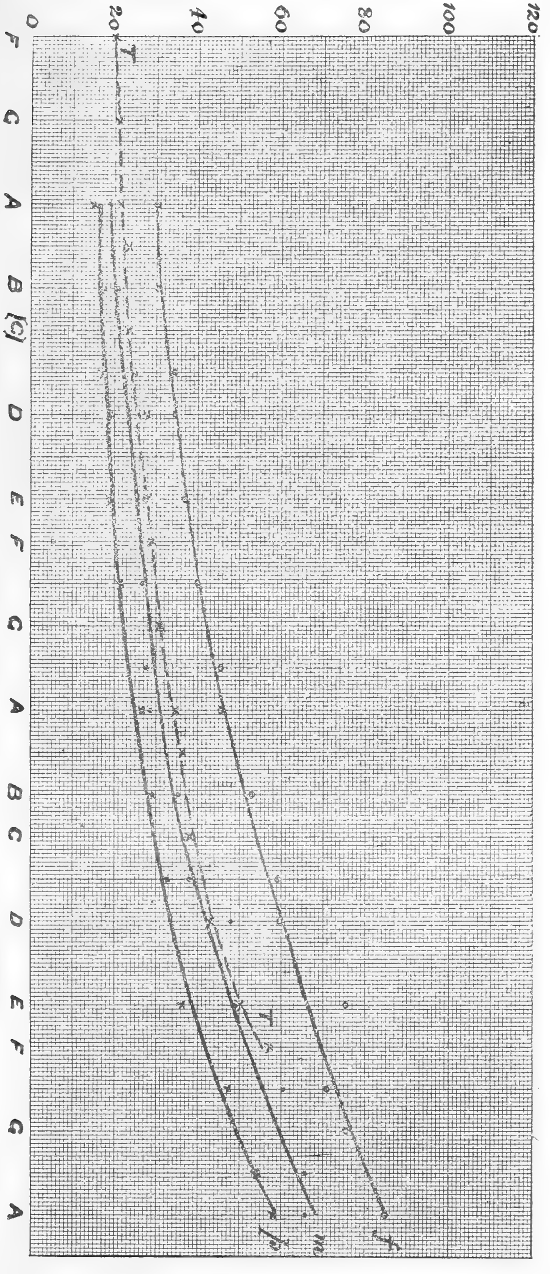
3. 'On the Origin of certain Concretions in the Lower Coal-Measures.' By H. B. Stocks, Esq., F.I.C., F.C.S.

In certain of the Lower Coal-Measures of Lancashire and Yorkshire and in the 'hard-bed coal,' peculiar concretions known as 'coal-balls' occur, which have a considerable interest because they contain well-preserved plant-remains. The author's analysis shows that they consist mainly of calcium carbonate and iron-pyrites, in varying proportions. Carbonate of lime appears to have been introduced by osmosis through the cell-walls; and that it was introduced in small quantity and under exceptional circumstances appears to be proved by the comparative rarity of the concretions and their presence in this seam of coal only. During the decay of the vegetable matter of which coal is formed, in contact probably also with animal matter, some of the organic matter would pass into solution in water, causing the absorption of the oxygen in solution; the result of this is that further decay would take place under anaërobic conditions. This, occurring in water containing sulphates, would give rise to sulphuretted hydrogen and mud blackened by the presence of ferrous sulphide, while carbonates would also be produced. Experiments were tried (1) on the precipitation of carbonate of lime under varying conditions (in presence of organic matter, etc.); (2) on the action of salts of lime and of iron on wood; and (3) on the action of bacteria on solutions containing calcium sulphate in solution and ferric oxide in the deposit. In the first series carbonate of lime was deposited in spheres; in the second it was found that iron-salts are preservatives but lime-salts are not; and in the third, black mud largely consisting of ferrous sulphide was produced, while the calcium sulphate was converted into carbonate. It is considered that these experiments explain the origin of the 'coal-balls.'



PRESSURES FOR TROMBONE.

Phil. Mag. Ser. 6. Vol. 3. Pl. VIII.



PRESSURES FOR CORNET AND TRUMPET.

THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[SIXTH SERIES.]

MAY 1902.

LIII. *A Recording Manometer for High-Pressure Explosions.* *By* J. E. Petavel, *Harling Fellow of the Owens College, Manchester*[*].

[Plates IX. & X.]

Introduction.

THE study of the phenomena which take place during an explosion is of considerable importance both from a theoretical and practical point of view. An exact knowledge of the variation of pressure will provide information on the rate of cooling of gases under high pressures, on the variation of specific heat at high temperatures, and on the question of dissociation.

From a practical point of view such information is needed in the design of modern artillery, and in all problems connected with ballistics. It enables the civil engineer to select the explosive suitable for the purpose to be attained. It guides the mechanical engineer in the design of the heat-engine.

Brief Historical Account.

It is impossible within the scope of the present paper to give more than an enumeration of the principal devices which have been used for the measurement of explosive pressures. The first systematic experiments were carried out by Robins

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[*] Communicated by the Author. Read before the Manchester Literary and Philosophical Society on November 26th, 1901.

in 1743, Hutton in 1778, and Count Rumford in 1797. Rumford's method consisted in determining the minimum weight required to prevent a valve from lifting under the pressure of the explosion, and was used, with slight variations, by a long succession of investigators, the best known experiments being those carried out in 1867 by Bunsen.

The relative strength of solid explosives was in these early days estimated by the two following methods:—

I. Firing them in a closed chamber in the centre of a block of lead, the permanent increase of volume of the chamber being taken as a measure of the explosive force.

II. By measuring the angle through which a heavy pendulum was moved when acted upon by the explosion.

The Rodman gauge, invented in 1859, was improved and modified some ten years later by the Committee on Explosives, the work resulting in the instrument known as the Crusher-gauge, which since then has been almost exclusively used. A piston works in a steel cylinder screwed into the explosion-chamber. One end of the piston is exposed to the pressure of the explosion, the other rests on a short copper cylinder. The explosion crushes the copper by an amount which bears a known relation to the maximum pressure attained.

In 1875, Noble and Abel measured the explosive pressure of gunpowder by determining the rate of acceleration of a projectile.

In recent years many new instruments have been devised. Le Chatelier and Mallard have used a modification of the Bourdon gauge. Vieille has used a piston controlled by a stiff spring. Noble and others have obtained records from an instrument not unlike an ordinary steam-engine indicator, the initial compression of the spring being, however, regulated to nearly correspond to the maximum explosive pressure.

Many useful results have been obtained by causing the piston of the ordinary crusher-gauge to inscribe its rate of motion on a revolving cylinder.

Finally, the chemical phenomena which occur during the firing of the charge, and the rate of explosion of several gaseous mixtures, have been most carefully studied by Berthelot in France and H. B. Dixon in this country.

Theoretical Consideration.

The all important condition for a gauge destined to record the rise and fall of pressure caused by an explosion is that its time-period should be as small as possible. If "A" represent the force required to produce unit deflexion of the vibrating system, W the weight of the moving parts, the time-period will be $=2\pi\sqrt{\frac{W}{Ag}}$.

We have therefore two variables at our disposal, namely, the weight of the moving parts and the controlling force. The former must be made a minimum, the latter a maximum.

In most instruments where a short period is desirable the strains to which the parts are subjected are very small, and the desired result is obtained by decreasing the size of all moving parts, and using, wherever possible, materials of low density. This method is employed in the case of all oscillographs, telegraph-recorders, phonograph-receivers, galvanometers, &c.

In the present case, the instrument having to withstand pressures of ten or twenty thousand pounds per square inch applied with extreme suddenness, strength becomes a condition of vital importance; and steel is the only material which will withstand the strain. We cannot therefore use materials of small density, neither can we reduce the dimensions of the moving parts below a certain limit.

It is thus evident that we must have recourse to the second variable factor to secure the short time-period which is necessary. As we have seen above, the controlling force brought into play per unit length of motion must be as great as possible. In other words, we must use the stiffest spring we can obtain.

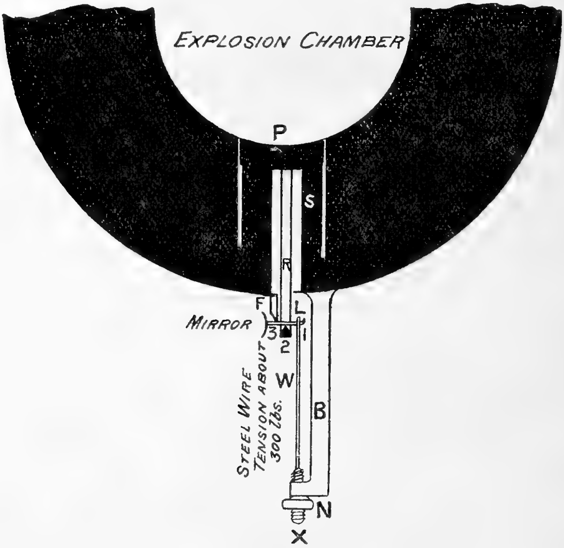
The stiffness of a spring will vary with the material of which it is made and its shape, increasing for a given size as the shape approaches more nearly to that of a solid bar subjected to longitudinal strain. This bar can be made as short as may be desired, and in theory the time-period of the system is only limited by the density of the material and by its modulus of elasticity.

Design of the Recorder.

In practice, however, the travel of the moving parts cannot be indefinitely decreased, for the deflexions must remain of such dimensions as to be accurately measurable. The following figure is a theoretical representation of a recorder designed on the principles we have just established.

A cylindrical groove is cut half through the walls of the inclosure. The upper part P of the cylinder thus obtained represents the piston of our indicator, and the lower portion S the spring. Under the pressure of the explosion the piston P will be forced outwards a certain small amount corresponding to the elastic compression of the material of which the spring is made. This motion is transmitted to the exterior by the rod R.

Fig. 1.—Diagrammatic Representation of the Recording Manometer.



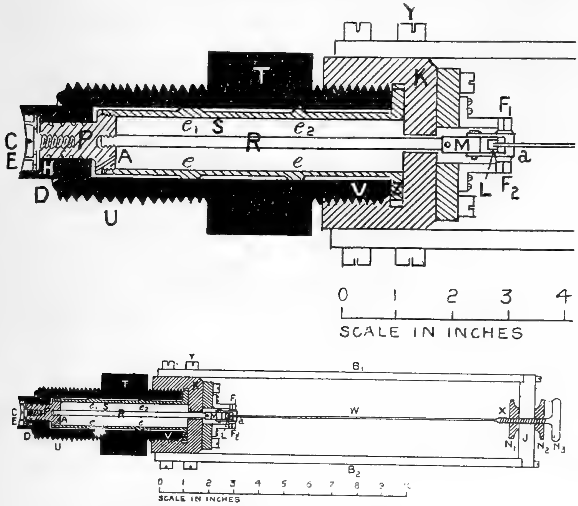
The lever L supporting the mirror rests on the fulcrum F at "3"; it is kept against the knife-edge "2" of R by the tension of the wire W. The wire W is of considerable length, and it is stretched to near its limit of elasticity. The lever L can therefore follow the small advance of the rod R without greatly diminishing the tension of the wire W.

The mirror focusses a point source of light on to a rapidly revolving cylinder, thus recording on a magnified scale the motion of the piston P. It is not impossible that an indicator of this type would work in practice; but the deflexion of the mirror, and therefore the scale of the records obtained, would be much too small. To increase the deflexions three modifications are necessary: the spring S must be made longer, the ratio of its cross-sectional area to that of the piston must be decreased, and the knife-edges "2" and "3" brought closer together.

In fig. 2 the design of the actual instrument is given, the lettering being the same as in the previous figure.

By means of the thread U the gauge screws into the explosion-chamber, C being flush with the inside surface; an air-tight joint is formed by the ring D pressing against a flat steel ledge*. The end of the gauge from D to E is a good fit in the walls of the explosion-chamber, and the joint is thus protected from the direct effect of the explosion.

Fig. 2.—Sectional Plan of the Recording Manometer.



The spring S, about 5 inches in length, is tubular in shape. To prevent any buckling it is made to closely fit the cylinder in which it is contained at two places, e_1 and e_2. The spring is fixed at the outer end Z, being held in place by the nut K; at the inner end it is free and supports the piston P. The ordinary U-leather is replaced by a leather washer attached to the piston by the screw C and to the fixed part of the gauge by the rim E. The end of the piston projects about an hundredth of an inch above the rim H, and it can therefore move back without straining the leather.

The mirror (not visible in the figure) is carried by the lever L. This lever is so designed that the knife-edges 1, 2, and 3 (see fig. 1) are in the same plane, it being at the same time possible to bring the knife-edges 2 and 3 within one-hundredth of an inch of each other should so great amplification be found necessary. Up to the present, however, the distance has not been decreased below $\frac{1}{16}$ inch, the scale obtained with this distance being found satisfactory.

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* In the case of apparatus designed for gases under high pressures all joints should be made directly metal to metal, no packing being used. A joint thus made, if properly designed, is and remains absolutely air-tight. It can be made or broken in an instant, and as many times as may be required.

With regard to the chronograph there is but little to be said: the films on which the deflexions are photographically recorded are wound on a drum which is kept in rapid rotation by an electric motor; the usual devices being used to regulate and measure the speed. The drum is inclosed in a light-tight box; a long narrow slit (about $\frac{1}{32}$ inch in width) runs the entire length of the box parallel to the axis of rotation. One of the filaments of an incandescent lamp is focussed by the mirror on to this slit, forming a fine straight line perpendicular to the axis of rotation and to the slit. The sharp point of light thus formed on the film moves from right to left as the pressure increases. To secure the quality and intensity of light which is necessary, the lamp is run at twice its normal voltage at the moment of the explosion.

To avoid the blurring of the zero-line the light is cut off an instant later, and the zero marked in when the products of the explosion have had ample time to cool to atmospheric temperature.

The gauge is calibrated by hydraulic pressure, and the results are checked by comparison with the values of the maximum pressure obtained by the statical method described at last year's Meeting of the British Association *.

For each explosive mixture two records are taken, one at a high speed giving the rise of the pressure, the other at a low speed giving the rate of cooling.

Records obtained.

Two typical records are given in Plates IX. and X.†; they refer to a mixture of air and coal-gas fired at an initial pressure of about eleven hundred pounds per square inch. Oxygen was in excess, the residual gases containing about 3 per cent. of it.

In tracing out diagrams like the above, no part of the instrument except the light frame carrying the mirror moves more than one or two thousandths of an inch. In spite, therefore, of the suddenness of the rise in pressure the velocity of the moving parts remains small and the usual trouble due to inertia does not arise. Pl. IX. is an illustration of this fact; 0·055 of a second after ignition the pressure is rising at the rate of over a million pounds per square inch per second, none the less the curve turns sharply at nearly a right angle

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* See Proceedings of the Brit. Assoc., Bradford, 1900, p. 655.

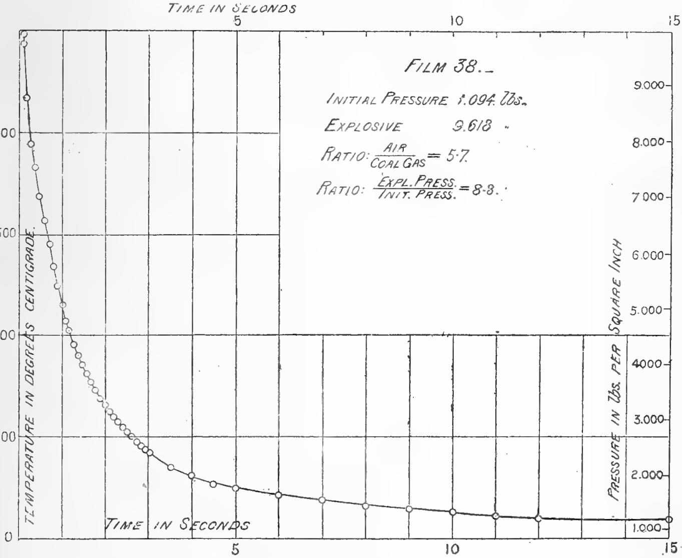
† The original films can be measured to about one thousandth of an inch; the points given in Plates IX. and X. have lost somewhat in sharpness, but are still sufficiently good to give a fair idea of the records

without any sign of vibration. The necessary data with regard to Pls. IX. and X. will be found in Tables I. and II.

It is intended to obtain similar curves covering a range of pressure up to 20,000 lbs. per square inch, and referring to mixtures from the most explosive to the non-explosive. The work will be divided into four parts: namely, I. Oxygen and hydrogen; II. Coal-gas and air; III. Other gaseous explosive mixtures; IV. Solid and liquid explosives.

With regard to coal-gas and air, barely half the work has as yet been finished; it is therefore too soon to draw any general conclusions. A few points, however, in the records here given deserve attention.

Fig. 3.—Coal-gas and Air. Rate of Cooling.



1. The time required to reach the maximum pressure, namely 0·058 second, is not far from that which would be required with the same mixture at atmospheric pressure.

2. The ratio of explosive to initial pressure has been increased. At or near atmospheric pressure the ratio for this mixture would be about 7; in the present case it is 8·6. This fact is due to three causes which work simultaneously, namely:—(A) The departure of gases from Boyle's law; (B) The relative decrease of thermal loss during the time occupied by the combustion; (C) The increase in the absolute temperature at which dissociation would take place.

3. The rate of cooling has greatly decreased.

The quantity of heat dissipated per unit of cooling surface increases with the temperature interval and with the pressure of the gas, but not at the same rate as the latter *. The heat developed, on the other hand, is simply proportional to the pressure.

By increasing the pressure from 1 to 70 atmospheres we increase the heat generated in a given volume 70 times, but we do not increase the rate at which heat is dissipated in anything like the same ratio. The increase of efficiency which, in the case of gas-engines, has always been connected with the use of high initial pressures, is mainly due to this cause. It is also to some extent due to the higher temperature obtained and to the smaller dimensions of the moving parts.

One more point deserves attention. It will be noticed that 0·05 second after firing the rate of rise of pressure suddenly increases, and becomes over nine times as fast as before. For the less explosive mixtures this change in curvature does not occur, the curve of rise of pressure being similar to the cooling-curve, only of course much steeper. It is worthy of note that the change of curvature occurs when the gas is at a mean temperature about equal to that at which spontaneous ignition would take place. A similar result would therefore be obtained if we heated the gases by the combustion of a certain portion of them until the entire bulk was at the "flash-point"; the combination would then take place simultaneously throughout the entire mass, resulting in an almost instantaneous rise to the maximum temperature and pressure.

Explosives may be divided into three classes—instantaneous, medium, slow,—according to the speed at which the maximum pressure is attained. The first class comprises fulminate of mercury, nitro-glycerine, &c.: the duration of explosion being equal to or less than the natural period of vibration of solid bodies, the term pressure hardly applies, the phenomenon being not static, but kinetic, of the nature of an impact rather than a pressure. If the ignition takes place in a closed vessel, a certain pressure will follow the combustion; but this pressure is no measure of the strains exerted. The second class would contain such explosives as ordinary gunpowder, cordite, and mixtures of hydrogen or coal-gas with oxygen or air. The time of rise varying according to circumstances from 0·0001 to 0·1 second. The more diluted mixtures of hydrogen and of coal-gas take their place in the third category with speeds from 0·1 second to 2 seconds.

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* See Phil. Trans. of Royal Soc. vol. cxcvii. p. 229 (1901).

It is advisable to reserve the term "explosion" for the second of the above classes, denoting by the word "detonation" a reaction which occupies less than one ten-thousandth of a second, and by the word "combustion" one which requires more than one tenth of a second.

TABLE I.—*Rise of Pressure.*

See Plate IX. (Film 41, explosion 181.)

Spherical inclosure 4 in. diameter. Capacity 551·91 c.c. =0·0195 cub. ft.

Temperature of inclosure before firing 18° C.
" " after products of combustion had cooled 24° C.

Initial pressure	77·28 atmospheres =1136 lbs. per sq. inch.
Ratio: Air to Coal-gas ..	6·0
Maximum explosive pressure	646·2 atmospheres. =9508 lbs. per sq. inch.
Ratio: maximum explosive pressure to initial pressure	8·4.
Residual pressure	63·8 atmospheres =937 lbs. per sq. inch.

Analysis of Residue.

Carbon dioxide................	9·8	per cent.
Oxygen	3·0	"
Nitrogen	87·2	"
	100·0	

Time in seconds.	Reading in millimetres.	Absolute Pressure in lbs. per sq. in.	Time in seconds.	Reading in millimetres.	Absolute Pressure in lbs. per sq. in.
0·000	......	1136	0·058	21·99	9508
0·010	0·77	1237	0·060	21·91	9477
0·020	1·57	1549	0·062	21·91	9477
0·030	3·06	2130	0·064	21·67	9385
0·040	5·03	2898	0·066	21·63	9369
0·042	5·21	2968	0·068	21·60	9357
0·044	5·84	3204	0·070	21·56	9340
0·046	6·38	3424	0·080	21·22	9209
0·048	6·86	3612	0·090	21·07	9149
0·050	7·47	3850	0·100	20·87	9071
0·052	10·80	5147	0·150	19·73	8628
0·054	15·92	7143	0·200	18·61	8192
0·056	19·83	8667			

TABLE II.—*Fall of Pressure.*

See Pl. X. and fig. 3. (Film 38, explosion 175.)

Spherical inclosure 4 in. diameter. Capacity 551·91 c.c. =0·0195 cub. ft.

Temperature of inclosure before firing 21° C.
" " after products of combustion had cooled 27° C.

Initial pressure	74·38 atmospheres =1094 lbs. per sq. inch.
Ratio: Air to Coal-gas ..	5·71.
Maximum explosive pressure	654·2 atmospheres =9618 lbs, per sq. inch.
Maximum temperature......	2483° C.
Ratio: maximum explosive pressure to initial pressure	8·8.
Residual pressure	58·98 atmospheres =867 lbs. per sq. inch.

Analysis of Residue.

Carbon dioxide	11·2	per cent.
Oxygen	2·0	"
Nitrogen	86·8	"
	100·0	

Time in seconds.	Reading in millims.	Absolute Pressure in lbs. per sq. in.	Temperature of Gas in degrees cent.*	Time in seconds.	Reading in millims.	Absolute Pressure in lbs. per sq. in.	Temperature of Gas in degrees cent.*
0·08	23·54	9618	2483	2·2	5·68	2979	581
0·1	23·19	9487	2445	2·3	5·54	2927	566
0·2	20·55	8506	2164	2·4	5·32	2845	542
0·3	18·62	7789	1957	2·5	5·13	2774	522
0·4	16·95	7167	1781	2·6	5·00	2726	508
0·5	15·83	6751	1661	2·7	4·83	2663	490
0·6	14·78	6361	1548	2·8	4·68	2607	474
0·7	13·79	5993	1445	2·9	4·51	2544	456
0·8	12·82	5632	1341	3·0	4·35	2484	439
0·9	11·92	5298	1245	3·5	3·82	2287	382
1·0	11·08	4986	1156	4·0	3·33	2094	327
1·1	10·38	4726	1081	4·5	2·96	1968	291
1·2	9·72	4481	1011	5·0	2·72	1878	265
1·3	9·20	4287	955	6·0	2·47	1785	239
1·4	8·68	4094	900	7·0	2·16	1670	205
1·5	8·16	3901	845	8·0	1·95	1592	183
1·6	7·68	3722	793	9·0	1·74	1514	161
1·7	7·30	3581	753	10·0	1·54	1439	139
1·8	6·86	3418	706	11·0	1·43	1399	128
1·9	6·54	3299	672	12·0	1·32	1358	106
2·0	6·23	3183	639	15·0	1·15	1295	98
2·1	5·95	3079	609				

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* The temperatures in this column are calculated in the usual manner from the pressure allowing for the quantity of water-vapour formed.

Before closing it should be stated that the present research has been greatly facilitated by funds awarded by the Government Grant Committee of the Royal Society. This opportunity is taken to thank Professor Schuster for so kindly placing the ample resources of his laboratory at my disposal; my thanks are also due to the Committee of the Davy-Faraday Research Laboratory for the loan of certain parts of the apparatus. Lastly, it is only just to state that the satisfactory results obtained with the apparatus described are attributed to the care and skilled workmanship of Mr. Charles W. Cook.

LIV. *On Flutings in a Sound-Wave and the Forces due to a Flux of a Viscous Fluid around Spheres. By* S. R. Cook, *M.S., A.M., Former Fellow in Physics, University of Nebraska; Professor of Physics in Washburn College, Kansas*[*].

ACOUSTIC attractions and repulsions have been carefully studied and much of the accompanying phenomena explained by Messrs. Guyot[†], Schellbach[‡], Guthrie[§], and Lord Kelvin[||]. Attractions and repulsions with the sound-wave have been more recently discussed by Walter Koenig[¶].

In his discussion Koenig makes use of the hydrodynamical equations for a perfect fluid, and from them deduces formulæ which represent forces brought into action by the flux of a fluid around spheres. He finds that the flux of a perfect fluid around spheres produces a force of repulsion parallel, and attraction perpendicular to the stream-lines; also a movement of rotation tending to move an ellipsoid or disk into a plane transverse to the stream-lines. These forces are considered by Koenig sufficient to explain the formation of laminæ and flutings in the sound-wave.

Although Koenig's explanation has been generally accepted[**], yet it seemed to the writer that frictional forces must be present, and that opposite forces were necessary to maintain the particles forming the laminæ in their respective positions.

The purpose of this paper was to make a careful study

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[*] Read before the Denver Meeting of the American Association for the Advancement of Science. Communicated by Prof. D. B. Brace.

[†] *Presse Scientifique*, 1861, t. iii. p. 130.

[‡] Pogg. *Ann.* vol. cxxxix. st. 4, p. 670.

[§] Phil. Mag. [4] xli. 1871, p. 405.

[||] Phil. Mag. [4] xli. 1871, p. 423.

[¶] Wied. *Ann.* t. xlii. pp. 353, 549; t. xliii. p. 43 (1891).

[**] Lord Rayleigh, Theory of Sound, vol. ii. p. 46.

of the formation of laminæ by different fluting materials, with different media under varying conditions, for the purpose of making manifest other forces than those of a perfect fluid, and also to experimentally verify the presence of the forces theoretically deduced from hydrodynamical equations for a perfect fluid.

The apparatus were modified forms of the Kundt sound-apparatus. The vibrating brass bar was 277·7 cm. in length, with a frequency of 1287·5 per second and an amplitude of vibration, with ordinary bowing, of 0·2 mm.; the bar being clamped for its first overtone.

In the stationary sound-wave, as that of the organ-pipe or Kundt tube, are to be found two distinct phenomena:—1st, The separation of the tube into nodes and antinodes; 2nd, the formation in the antinodes or ventral segments of flutings and laminæ. Two theories have been advanced to explain the formation of these laminæ:—1st, They are the nodes of very high overtones produced in the vibrating system; 2nd, they are produced by forces originated by the flux of the medium.

I. *As Nodes of High Overtones.*

The rapid motion of the particles in the laminæ, the movements of these laminæ, the breaking up of old and the forming of new laminæ, make it highly improbable that they are produced by overtones. However, in order that this might be more firmly established, a tone was produced by a siren in front of a tube closed at one end by a moving piston. With the proper adjustment of the vibrating air-column, laminæ of magnesium carbonate could easily be produced.

II. *Laminæ due to Hydrodynamical Forces.*

Koenig's solution of the hydrodynamical equations for a perfect fluid seem to give a more rational explanation of the problem. The formulæ obtained by Koenig for the forces between two spheres, whose radii are R and R_1 and whose distance apart is r_0, were

$$X = -\frac{\frac{3}{2}\pi\rho R^3 R_1{}^3 W_0{}^2}{r_0{}^4} \sin\theta\,(1-5\cos^2\theta), \quad . \quad . \quad (1)$$

$$Z = -\frac{\frac{3}{2}\pi\rho R^3 R_1{}^3 W_0{}^2}{r_0{}^4} \cos\theta\,(3-5\cos^2\theta); \quad . \quad . \quad (2)$$

W_0 being the velocity of the stream, ρ the density of the medium, and θ the angle which the line of centres of the spheres makes with the direction of the stream. If the particle is in the form of an ellipsoid or disk, Koenig finds a moment M tending to diminish the angle θ,

$$M = \tfrac{4}{3}\rho R_2^3 W_0^2 \sin 2\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where R_2 is the radius of the disk.

When $\theta = \frac{n\pi}{2}$, n being an integer,

$$X = -\frac{3}{2}\frac{\pi\rho R^3 R_1^3 W_0^2}{r_0^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

When $\theta = n\pi$

$$Z = \frac{3\pi\rho R^3 R_1^3 W_0^2}{r_0^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This gives repulsion parallel and attraction perpendicular to the stream-lines.

In the equations, the factors susceptible to variation are the density of the medium ρ, the size and shape of the particles used for fluting material, R, R_1, and R_2, and the stream velocity W_0.

I shall first allow ρ and W_0 to remain constant and vary the fluting material.

1. *A study of various fluting materials with air as a medium.*—The ordinary form of the Kundt sound-apparatus was used ; the medium being maintained at normal pressure. Flutings were obtained with the following materials :—magnesium carbonate, lycopodium, amorphous silica, cork dust, magnesium, anthracene, ammonium chloride (as fumes), sand, aluminium, brass and iron filings, timothy blue grass, red top, and petunia seed.

After a careful microscopical examination of these materials, it was noticed that as the radii of the particles or disks increased, the mass of the particles remaining the same, the more readily flutings formed ; that the more nearly spherical the particles were the more easily they swept into the nodes ; and that the greater the radii of the particles the farther apart the laminæ.

2. *A study of various fluting material with carbon dioxide and chlorine as media.*—With carbon dioxide as medium flutings were obtained in all the substances used with air, and in addition flutings were obtained in filings of coin silver. It was noted that flutings were obtained more easily in carbon dioxide than in air. This was clearly shown by the fact that flutings with coin silver could not be obtained in air. With chlorine as medium, in addition to the above-mentioned substances, flutings were obtained with platinum filings. It is to be noted that in the last two experiments the density ρ

has been increased, and the forces X and Z and the moment M have also been proportionally increased.

We may, however, use the same fluting material so that R, R_1, R_2 will remain constant, and vary the density ρ.

3. *The ratio of the density of air to the density of carbon dioxide and hydrogen for minimum disturbance.*—In order to determine this ratio, it is quite essential that not only the fluting material be exactly the same for the two gases, but also that the stream-velocity W_0 be the same. This was accomplished by constructing a double-tube Kundt apparatus, one tube at each end of the vibrating brass bar, arranged with manometers and drying trains, and so placed that the tubes were parallel and close to each other. One of these was filled with air; the other with the gas to be compared. The pressure was then adjusted for minimum disturbance, *i. e.* the pressure at which the fluting material could just be made to form striæ. The manometers were then read and the observation repeated with gases interchanged.

The results obtained for different fluting material with air and carbon dioxide are given in the following table:—

Table I.

Substance.	T.	N.	P_1.	P_2.	ρ_1.	ρ_2.	ρ_2/ρ_1.
Magnesium carbonate.	°21	9	97·1	21·7	·0001258	·0000514	·42
	18	20	124·6	52·8	·0001982	·0001278	·64
Amorphous silica.	19	8	336·8	137·5	·0005376	·0003335	·62
	18	11	340·5	142·2	·0005415	·0003438	·63
Aluminium filings.	20	10	400·2	180·5	·0006360	·000434	·68
	22	10	335·0	140·0	·000533	·000338	·64
Lycopodium ...	21	10	223·3	114·7	·0008552	·000275	·77
Anthracene ...	17	10	213·9	117·5	·0003401	·0002840	·83
Sand	20	10	641·0	343·5	·0010200	·0008360	·82
	21	10	627·3	337·3	·0009980	·0008153	·82

T equals the temperature; N, the number of determinations in each experiment; P_1, P_2, ρ_1, ρ_2, equal the respective pressures and densities of air and carbon dioxide for minimum disturbance.

It will be seen by the last column that the ratio of the density of carbon dioxide to air is always less than unity. The determination of the ratio of air to hydrogen required great precaution as the slightest admixture of air would vitiate the results. With the apparatus used the minimum disturbance was obtained at a pressure for hydrogen of 1694 mm. of mercury, and for air at a pressure of 20 mm. of mercury. The corresponding densities were ·0001868 and

·0000318, the ratio of the density of hydrogen to air being 5·85, that of air to hydrogen ·17.

In these experiments, since R, R_1, R_2, r_0, and W_0 have been made constant for the two gases, if X_1, Z_1, and M_1 are the forces of minimum disturbance in air; X_2, Z_2, M_2 for the carbon dioxide, then

$$\frac{X_1}{X_2}=\frac{Z_1}{Z_2}=\frac{M_1}{M_2}=\frac{\rho_1}{\rho_2}; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

where ρ_1 and ρ_2 are the densities of air and carbon dioxide respectively. And since at minimum disturbance X_1, Z_1, M_1 must be equal respectively to X_2, Z_2, M_2, then, if there are no forces other than the forces of a perfect fluid, the ratio ρ_1/ρ_2 should be unity; and since this is not the case, the supposition is that there are other forces which do not vary directly as the density, and since the internal friction of fluids, within the limits of these experiments, is independent of the density, it is not improbable that these forces partake of the nature of frictional forces.

For any gas the coefficient of internal friction $R=k\rho c\lambda$ * and for any other gas of equal density $R_1=k\rho c_1\lambda_1$.

Taking the ratios

$$\frac{R}{R_1}=\frac{c\lambda}{c_1\lambda_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Hence, if there are forces other than the perfect fluid forces proportional to the coefficient of internal friction, they must also be proportional to the product of the velocity and the mean free path. This led to the study of the influence of the density on flutings and the ratio of the mean free path to the distance apart of the laminæ.

4. *Ratio of the mean free path of the molecule to the distance apart of the laminæ.*—A Kundt tube was so arranged that the pressure could be reduced to any desired amount and accurately determined. The following table obtains for amorphous silica and air.

TABLE II.

Density.	Molecular Velocity C.	Mean free path $=\lambda$.	Distance apart of flutings $=d$.	d/λ.
·000180	502	6114×10^{-8}	·72	118000
·000228	506	4748×10^{-8}	·50	105000
·000360	504	3077×10^{-8}	·32	104000
·000695	502	1557×10^{-8}	·23	146000
·001152	501	940×10^{-8}	·15	159000

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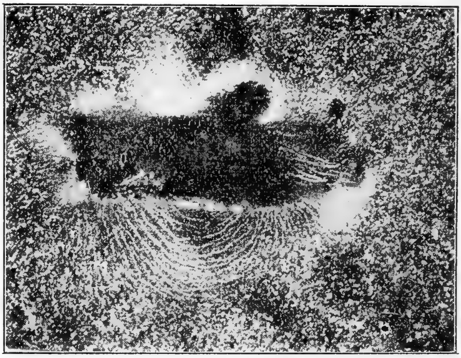
* Boltzmann, *Gas Theorie*, t. i. p. 81.

The last column of this table indicates that for a given medium and fluting material there may exist a direct ratio between the distance apart of the flutings and the mean free path of the molecule.

Brief studies on flutings are here presented on account of their peculiar interest in acoustic attractions and repulsions.

5. *Flutings produced by a direct sound-wave.*—A plane glass plate was substituted for the Kundt tube. The vibrating cork was cut by a horizontal plane and the plate was placed about one millimetre below. When the bar was clamped for its fourth overtone flutings of amorphous silica were produced on the plate. These flutings were elliptical in form, having for their common axis the line between the vibrating cork and the glass plate (fig. 1). Flutings were also produced when the bar was clamped perpendicularly and the glass plate placed within a centimetre of the cork. These flutings were concentric circles around the cork (fig. 2).

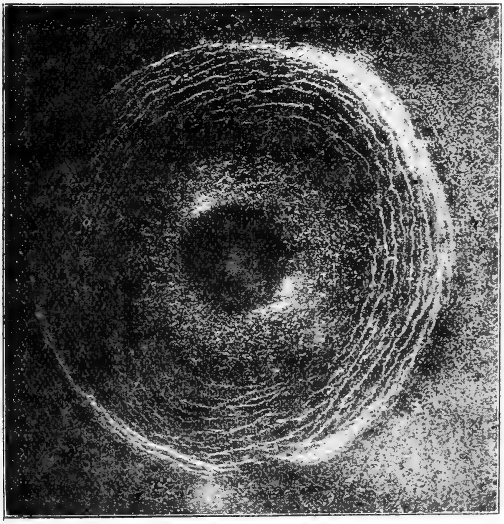
Fig. 1.



6. *The vibration of laminæ.*—A mica disk was threaded on a fine wire stretched through the centre of the Kundt tube. A cork was fastened to one prong of a large tuning-fork—frequency 32 per second—and placed in the end of the Kundt tube. The cork was thus made to vibrate in unison with the prong of the tuning-fork. The mica disk, though in no way connected to the cork, vibrated in unison with it. The amplitude of the vibration of the cork was from one to two centimetres. That of the mica disk was less.

This experiment suggests that the laminæ might vibrate in unison with the vibrating bar. In order to examine this, a glass tube was split longitudinally, the edges were ground smooth, and the opening was overlaid with cover-glasses sealed around to make them air-tight. A microscope was focussed on the amorphous silica or other substance used for flutings. When the sound-wave was established, the particles of amorphous silica could be seen quickly gathering into thin laminæ which would either sweep out of the field of view or move about in the field with irregular motion.

Fig. 2.



As the number of vibrations of the bar was 1287·5 per second, and the amplitude of vibration for ordinary bowing 0·2 mm., the only means by which a vibration of the laminæ could have been detected would have been by the broadening of the laminæ, and this, when under the microscope, was such a heterogeneous mass of moving particles that no vibration could be detected. The study became very interesting, however, in watching the motion of the individual particles. A larger particle would move towards a smaller particle until within a certain distance, when the small particle would suddenly move away. Again, a particle might be swept up close to another until it seemed to be within another sphere of action, and be drawn to it. The two would then take a position with their line of centres transverse to the stream-lines. Particles lying in a transverse position at some distance apart would be attracted, always assuming the laminal condition. It was noted, however, that these particles, with their spheres of attraction and repulsion.

appeared never to come into actual contact, and it was distinctly evident that they were not in actual contact in the laminæ.

If only the three forces deduced from the equations of a perfect fluid are to be considered, the particles should at least be in contact transversely. After a careful study of these phenomena it was decided, if possible, to obtain spheres small enough and light enough to flute.

III. *Forces due to a Flux of a Viscous Fluid around Spheres.*

The three forces deduced from the theory of a perfect fluid have already been discussed, and the experiments have pointed to other forces. The presence of these new forces will be more clearly manifest in the following experiments with spheres in a viscous fluid.

The spheres used were obtained by blowing out from a glass tube melted sealing-wax, and allowing the hot spray to filter through the air into a vessel of water. In this manner a large number of variously-sized spheres could be quickly and easily obtained. The spheres, when placed under a microscope, were found to be very perfect in form, with smooth surfaces, and having a diameter of from 0·1 to 2·0 mm. When placed in the resonance-tube of the Kundt apparatus, spheres whose diameter was not more than 1 mm. readily formed flutings. A careful microscopic study was then made of attractions and repulsions of these spheres in the centre of the ventral segment of a stationary sound-wave, and the following observations were determined :—

1. *Observations on a Concave Surface.*

(1) Spheres whose line of centres was parallel to the stream-lines and whose distance apart was greater than between one and two tenths of a millimetre repelled.

(2) Spheres whose line of centres was transverse to the stream-lines and whose distance apart was greater than about half the diameter of the spheres attracted.

(3) Spheres whose line of centres was transverse to the stream-lines and whose surfaces, owing to the concavity of the tube, were in contact when there was no sound-wave present, did not remain in contact when the sound-wave obtained but were driven by some repellant force farther up the side of the tube.

(4) Spheres whose line of centres was parallel with the stream-lines and whose distance was less than half their

diameter rotated as a single body until their line of centres was transverse to the stream-lines.

The roughness and shape of the bottom of the tube interfered somewhat with the best observation. To overcome this difficulty a plane piece of plate-glass was fringed with paper, to prevent the spheres from rolling off, and placed in the tube. The spheres resting on the surface of the glass readily formed flutings across its plane surface, but by slightly tilting the glass surface the spheres could be made to roll from side to side where flutings would be formed projecting somewhat up the incline. With this arrangement of the apparatus the following observations were determined:—

2. *Observations on a Plane Surface.*

(1) Observations (1) and (2) of the previous section were verified.

(2) Two spheres with line of centres transverse to stream-lines were attracted and then repelled to a distance of about half their diameter, and remained thus while moving forward and back on the glass.

(3) Three spheres situated as in (2) acted in all respects like two spheres.

(4) The surface of the glass plate was inclined at an angle of about 20° to the horizontal. One sphere was resting against the side of the tube, another was in contact with it, resting on the inclined plate. Their line of centres was transverse to the stream-lines. When the sound-wave was present the second sphere was repelled to a distance of about half its diameter up the incline from the first, and maintained that position during the duration of the sound-wave, then returned to its original position. Three spheres in line acted in a similar manner.

(5) Glass plate as in (4). First sphere fixed to upper side of plane surface. The two spheres in relatively the same position as in (4). When the sound-wave ceased the second sphere would start to roll down the inclined surface from the first sphere, but as soon as the sound-wave was again started this sphere was attracted up the inclined plane to within a distance of about half its diameter, where it was maintained during the maintenance of the sound-wave.

Although the sealing-wax spheres seemed to move with perfect freedom on the surface of the plate-glass, and there seemed to be nothing that could possibly vitiate the results obtained, yet the question of possible surface-conditions arose, and it was decided to suspend these same spheres in the centre of the tube.

3. *Observations with Suspended Spheres.*

A small hole was blown at the centre of a ventral segment in the Kundt resonance-tube, and a small cork was fitted flush with the inner surface of the tube. To this cork two spheres were suspended on the end of fine spider-threads about two centimetres in length, and at any desired distance apart.

Two spheres, whose diameter was about 0·9 mm., thus suspended, with their line of centres parallel to the stream-lines, were repelled, but with their line of centres perpendicular to the stream-lines they were attracted. When the line of centres was at an angle of about 45°, or less, they would rotate their line of centres perpendicular to the stream-lines and approach each other.

The sealing-wax spheres were replaced by glass spheres of the same size. The results were the same, but not so pronounced.

These observations were repeated with other sealing-wax spheres, whose diameters were from ·5 mm. to ·7 mm., their distance apart being from ·1 mm. to ·8 mm. The same results were observed.

Spheres of from ·2 mm. to ·3 mm. were then suspended in like manner with similar results, except that the forces were more pronounced. With these small spheres suspended within half of their diameter apart, it was observed that the suspensions approached each other when the spheres were in contact.

In order to eliminate this attraction of the fibres, one of the spheres was mounted on the end of a fine glass fibre, which was brought in from the side of the Kundt resonance-tube, and the other was mounted on a single fibre-suspension as before.

When the two spheres were brought within a distance of about half their own diameter, with their lines of centres perpendicular to the stream-lines, the suspended sphere was repelled from the sphere mounted on the glass fibre. When their line of centres was parallel to the stream-lines, and their distance apart less than half their diameter, the suspended sphere approached the fixed sphere and moved around until their line of centres was perpendicular to the stream-lines.

These observations were verified by other sets of spheres in other ventral segments.

4. *Influence of Surface on the Forces between two Spheres.*

In order to determine whether or not a surface would influence the forces between two spheres, a cover-glass was mounted on the end of a glass rod, and was made to approach the spheres from below. It was found that when the cover-glass was very close to the spheres, the attraction parallel and repulsion transverse to the stream-lines was increased.

IV. *Discussion of the Data.*

The data here presented have experimentally verified the presence of the forces deduced from the theory of a perfect fluid, and make manifest other forces. The presence of these new forces, which are opposite in direction to the perfect fluid forces, were indicated in the experiments with different gases, but in the observations with spheres the effect of these forces has not only been seen, but the relative magnitude of the two forces has been roughly measured, so that the function of the distance apart of the spheres varies more rapidly than the perfect fluid function.

The experiments with different gases, whose coefficient of viscosity varies, the increase of these forces on the approach of a plane, and the fact that they are opposite in direction to the perfect fluid forces, indicate that these forces are due to internal friction or viscosity of the fluid used as medium.

V. *Formation of Laminæ.*

As has already been stated, Koenig considered the two forces, and the moment deduced from the theory of a perfect fluid, sufficient to explain the formation of the laminæ produced in the ventral segment of a Kundt's resonance-tube. If no other than the attractive force existed between particles whose line of centres is perpendicular to the stream-lines, the particles would, in the transverse plane, only be in stable equilibrium when they are in actual contact. But this condition is not in accord with the most careful observation both of particles and spheres. If, however, there is a repulsive force which is only effective when the particles are not more than one half their diameter apart, there will be a position in the transverse plane in which two particles are in stable equilibrium, and this position is the position in which the particles will be found in the laminæ.

And if in the direction parallel to the stream-lines we consider only a repulsive force, all particles having their lines of centres parallel to the stream-lines would be repelled from each other regardless of their distance apart; but this does not agree with observation for particles or spheres lying within about ·3 mm. apart. Particles thus situated act as a single system, and are rotated by the moment M into the plane perpendicular to the stream-lines.

The attractive force parallel to the stream-lines, in conjunction with the moment of rotation, causes the particles which are within the sphere of action to form in laminæ transverse to the stream-lines; and the attraction and repulsion perpendicular to the stream-lines maintain these particles in the laminæ. If, however, there are particles in the plane parallel to the stream-lines, and not within the sphere of action of sufficient intensity, they will be repelled to some greater distance, and may come within the sphere of action of some other particles, and with them form other laminæ.

VI. *Summary.*

1.—The flutings are not confined to the ventral segment of a stationary sound-wave, but the conditions for the formation of laminæ are found wherever there is a to and fro flux of a viscous fluid around solid particles.

2.—The ease with which flutings may be formed depends upon the size, shape, and specific gravity of the fluting-material, and also upon the molecular weight, density, and viscosity of the medium.

3.—A perfect fluid contains forces which are essential but are not sufficient to produce laminæ in the form in which they exist.

4.—The experiments here noted show that there is another and opposite force both parallel and perpendicular to the stream-lines.

5.—The attractive and repulsive forces are increased by the presence of a surface in close proximity to the solid particles.

6.—These new forces are most probably due to the viscosity of the medium.

7.—The new forces, in conjunction with the forces of the perfect fluid, make possible a complete and clear explanation of laminæ and flutings produced in sound-waves.

My sincere thanks are due to Dr. D. B. Brace for many valuable suggestions and for the interest manifested in the results.

Physical Laboratory, University of Nebraska,
Lincoln, Nebraska, Aug. 22, 1901.

LV. *On Focal Lines, and Anchor-Ring Wave-Fronts.*

By Prof. J. D. Everett, *F.R.S.*[*]

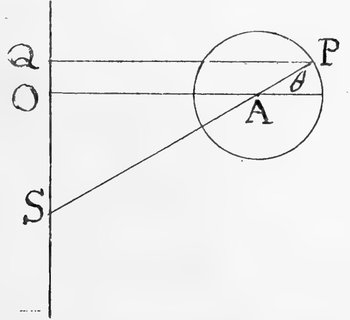
WHEN a small cone of rays is obliquely incident on a spherical reflecting or refracting surface, the rays after reflection or refraction no longer compose a true cone. Instead of meeting in a point they form a narrow neck; and this neck is flattened in two places called the primary and secondary foci, the planes of flattening being at right angles to each other. Optical writers give the name *focal lines* to the sections of the pencil made at these two places by planes perpendicular to the axis of the pencil; but it would be more appropriate to give the name to the sections which most nearly resemble lines, whatever angle they may make with the axis of the pencil.

Clearness of conception, in intricate matters, is greatly aided by sharply defined illustration; and I wish to call attention to a case (which appears to have been hitherto overlooked) in which all the rays, even of a large pencil, pass accurately through two definite lines: one of these lines being a circular arc cutting the pencil at right angles; and the other being a straight line, which may have any inclination to the axis of the pencil.

The case is that in which the wave-front in one of its positions is a *tore* (or anchor-ring).

A tore may be defined as the surface generated by the revolution of a circle round a fixed straight line in its plane; this line we shall refer to as the *axis of revolution*. A tore has also what may be called a *circular axis*—the circle generated by the motion of the centre of the revolving circle. In the figure QOS is the axis of revolution, O the centre of the tore, and OA the radius of the circular axis. P is any

Fig. 1.



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[*] Communicated by the Physical Society: read February 28, 1902.

point on the revolving circle, PQ a perpendicular on the axis of revolution, and θ the inclination of the straight line PAS to PQ or AO.

The two focal lines are always at the centres of curvature of two mutually rectangular normal sections of the wave-front, one being the section of greatest and the other of least radius of curvature. For an element of the wave-front at P one of these sections is the circle shown in our diagram, and PA is its radius of curvature. The other is the section of the tore made by a plane through PAS perpendicular to the plane of the diagram. To find its radius of curvature note that QP is the radius of a circular section made by a plane perpendicular to the axis QS; hence, by Meunier's theorem, the required radius of curvature is PQ $\sec\theta$, that is PS. The two focal lines are accordingly at A and S. A is called the *primary*, and S the *secondary* focus. If we make P travel round the circle shown in the figure the primary focus remains fixed at A, and the secondary focus travels along the axis QS, its distance from O being OA $\sec\theta$, which runs from zero to infinity in both directions. On the other hand, if we make P revolve round the axis QS the secondary focus remains fixed at S, and the primary focus generates the circular axis of the tore.

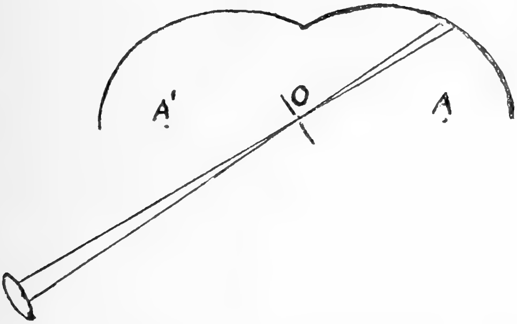
For a circular element of the wave-front, of small diameter d, having P for its centre, the primary focal line will be an arc of the circular axis, of length $\frac{\mathrm{SA}}{\mathrm{SP}}d$; and the secondary focal line will be a portion of the axis of revolution, of length $\frac{\mathrm{AS}}{\mathrm{AP}}d \,.\sec\theta$. If, instead of regarding this absolutely sharp line as the focal line, we follow the usual convention, and adopt, as the secondary line, the section of the pencil by a plane at S perpendicular to PS, its two ends will be blurred, so that it will resemble a figure of 8, and its length will be $\frac{\mathrm{AS}}{\mathrm{AP}}d$.

At any point T between A and S the breadth of the pencil in the plane of the diagram is $\frac{\mathrm{AT}}{\mathrm{AP}}d$, and its breadth perpendicular to the plane of the diagram is $\frac{\mathrm{ST}}{\mathrm{SP}}d$. The ratio of the latter breadth to the former is $\frac{\mathrm{ST}}{\mathrm{AT}}$ multiplied by the constant $\frac{\mathrm{AP}}{\mathrm{SP}}$, and is unity when SA is divided internally and externally in the same ratio.

Toric wave-fronts can be produced by constructing a concave reflector, of the form generated by making an ellipse, or any portion of an ellipse, revolve through any convenient angle round an ordinate erected at one focus O of the ellipse; and employing it to reflect rays diverging from a small source of light at this focus. The length of path is the same for all once-reflected rays from O to the circle traced by the other focus A, and will, therefore, be the same for all such rays from O to a tore having this circular axis.

Figs. 2, 3, 4 represent an experimental illustration. Fig. 2 contains an axial section of a reflecting surface of revolution; the section being made up of portions of two equal ellipses of eccentricity $\frac{1}{2}$. O and A, A′ are points corresponding to O and A in fig. 1. A point source at O is provided by means of an opaque disk pierced with a small hole at O ; the disk being carried by an axis in its own plane, coinciding with a diameter of the bounding circle of the reflector, as shown in figs. 3, 4 (p. 486), which represent the reflector supported on a stand, with its bounding plane vertical, the axis of rotation of the disk being also vertical. By rotation round this axis the disk can be made to face a source of light at its own level, at any obliquity to the bounding plane. In fig. 2 a beam is shown focussed on the hole by a lens. The beam may be furnished by a lantern, or by a bright lamp-flame properly screened. Widening the angle of the pencil given by the lens lengthens and at the same time time brightens the focal lines.

Fig. 2.



The primary line will be a circle having A A′ for diameter, and will be always real. The secondary line will coincide with the axis of revolution, and will be real or virtual according to the position of the area of incidence. For incidence at the extremity of the ordinate through A, the reflected ray is parallel to the axis, and the secondary line goes off to infinity. Both lines when real can be shown on a translucent screen. The secondary line, when virtual, can be seen by looking into the reflector. The complete circle of the primary can be obtained at once by making the axis of the incident beam (of wide angle) coincide with the axis of the reflector, so that the area of incidence surrounds the central boss.

Fig. 3.

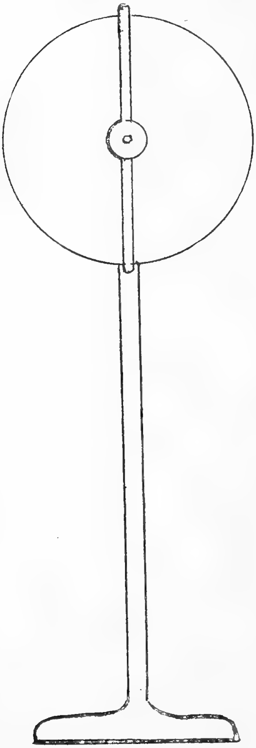
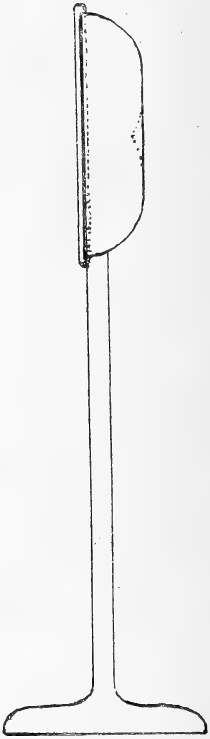


Fig. 4.



An iris diaphragm (as used by microscopists) might with advantage be substituted for the pierced disk.

Postscript.—Since reading the paper I have found that Maxwell, in a paper entitled "On the Cyclide," has investigated the general form of the wave-surface "when one or both of the so-called focal lines is really a line," and has indicated the anchor-ring as a particular case. ('Collected Papers,' vol. ii. pp. 144 & 151.)

LVI. *The Temperature Variations of the Specific Molecular Conductivity and of the Fluidity of Sodium Chloride Solutions.* *By* Prof. THOMAS R. LYLE *and* Mr. RICHARD HOSKING, *University of Melbourne*[*].

[Plates XI. & XII.]

THE statement has often been made that the temperature variations of electrical conductivity and fluidity (the inverse of viscosity) are identical for a given electrolytic solution. In order partly to test the accuracy of the statement in a particular case, and generally to obtain a set of accurate values for the two quantities mentioned, the experiments which yielded the following results were undertaken by us.

Previously one of us had obtained values of the Viscosity of Sodium Chloride solutions over a considerable range of temperature (Phil. Mag. March 1900, p. 274); but the effect of change of density and temperature on the viscosity is so great that it was determined that portions of the solution to be experimented upon should be run into suitable instruments fixed in the same bath, and that the observations necessary for calculating the density, fluidity, and electrical resistance should be made at the same time, the temperature being kept constant until all the readings had been taken for that particular temperature.

From the results so obtained, we are able, by interpolation, to compare the electric conductivity and the fluidity over the range of temperature between 0° C. and 100° C. of solutions of constant concentration; that is, we compare the conductivity and the fluidity at different temperatures of such solutions as contain, say, 1 gram molecule per litre at each of these temperatures. This we are enabled to do by means of the values of the density at the different temperatures of any particular solution experimented with, which we determine simultaneously with its electric resistance and viscosity.

The bath was a large beaker of water, contained in the upper compartment of an iron jacket provided at the front and back with glass windows, and inclosed at the top by a thick wooden cover. The two compartments were separated by a plate of asbestos on which the beaker rested, and the lower one contained the gas-jet used for heating the bath and the very small controlling flame which was used for keeping the bath from cooling while readings were being taken. The cover was pierced with the holes necessary for holding the

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[*] Communicated by the Authors.

various instruments in position in the bath, namely: the glischrometer, dilatometer, electrolytic cell, thermometer, and siphon. The stirring was managed by a swiftly rotating propeller. The thermometers were standardized before being used. They were graduated in tenths, and their readings were observed through a cathetometer. The sample of sodium chloride used in the solutions was specially purified for the purpose by David Avery, Esq., M.Sc., Instructor in Chemistry at the Working Men's College, Melbourne.

Figures 1, 2, and 3 are drawings of the glischrometer, dilatometer, and electrolytic cell used. The formula used in determining the viscosity was

$$\eta = \mathrm{A}\rho\mathrm{T} - \frac{\mathrm{B}}{\mathrm{T}}\rho,$$

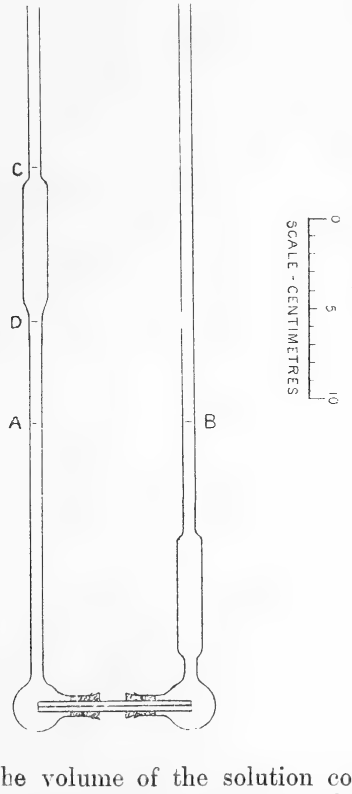
where A and B are constants, ρ is the density and T the transpiration time of the solution. The constants A and B were obtained from known values of η supplied from the original curves at two temperatures for which T had been observed with the new glischrometer, the liquid used being water. The values obtained were

$$\mathrm{A} = 0{\cdot}00002834, \quad \mathrm{B} = 0{\cdot}0360\,;$$

and the new determinations of the viscosity of water were identical with the old ones up to 70° C. Above that there were very slight variations. With the new form of glischrometer the liquid flowed through the capillary-tube under a pressure due to the difference of level of its two surfaces only. No external pressure was used. Moreover, measurements could only be made with the liquid flowing in the one direction through the capillary-tube, so that we could not eliminate errors due to this tube not being quite horizontal. For these reasons the new values obtained as absolute values are not quite so reliable as the original ones. However, we took all precautions to get as accurate values as possible. The amount of solution before each reading was adjusted so that the two surfaces were at the marks A and B in the same horizontal line. The solution was then forced up into the left limb until its surface was just above C. The time of flow when the meniscus passed from C to D gave the transpiration time (T), usually from 2 to 6 minutes.

The cylindrical bulb of the dilatometer (figure 2) was about 5 c.c. in capacity, and the remaining portion had an internal diameter of 2 mm. approximately. Between A and C were graduations etched in the glass. The graduations above B in the other limb were much further apart, and were only used for setting the surface of the solution at a definite level in that limb, while readings in the other limb were being made. Sufficient of the solution was first run in to fill the bulb and tubes just above A and B. Its weight was obtained, and the pressure of a screw-clip on an air-reservoir attached to D was so adjusted that the level was exactly at the mark B. The reading in A C was then noted and the temperature and the volume of the solution could be calculated, and the density at that temperature deduced. At higher temperatures, the mark II. in B D was used instead of mark I., and at still higher temperatures, marks III. and IV. The actual values in scale-divisions of these intervals had been previously calculated, and the coefficient of expansion of the glass in the dilatometer was also determined. The volume between B and A was also found, and the absolute value of the scale-divisions in A C.

Fig. 1.—Glischrometer.



The volume of solution in the dilatometer at any temperature t° C. was given by

$$V_t = (5{\cdot}030 + n \times {\cdot}001296)\,(1 + 0{\cdot}0000287t)\ \text{. c.c.,}$$

where n was the scale-reading. With this instrument, values for the density of water were obtained agreeing with the true values to within ·01 per cent. between temperatures 0° and 90° C. which was sufficiently accurate for our purpose.

Fig. 2.
Dilatometer.

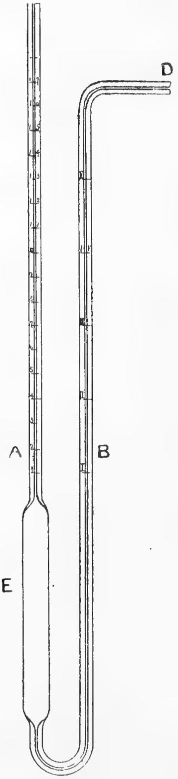
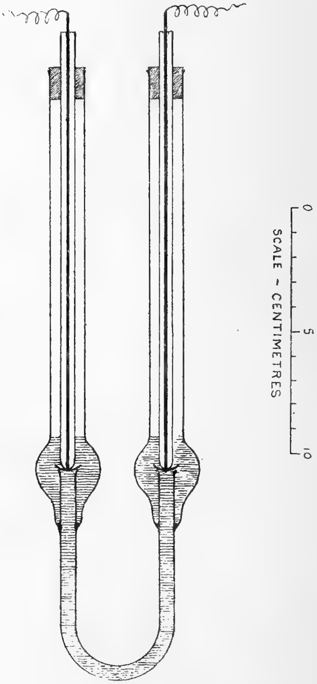


Fig. 3.
Electrolytic Cell.



The electrolytic cell was constructed wholly of glass (fig. 3), and the electrodes could always be fixed at exactly the same place, for when they were in position they rested on the open ends of the tube containing the solution, these ends being in the centre of pear-shaped bulbs which connected with wide tubes passing through the cover of the bath, and supported by it. Sufficient solution was used to fill the cell to the top of the bulbs. The wires leading to the electrodes passed down narrow glass tubes, and were sealed through the ends to the platinum electrodes which were cup-shaped and had been coated with platinum-black. The tubes were kept in position by passing through corks which fitted into the open ends of the wide tubes.

When all the instruments had been throroughly cleaned, and had been fixed in position in the bath, the solution, which had been made up by weight according to Bender's (Wied. *Ann.* xxii.) values for the density, was distributed among the three instruments in the proper proportions. After any excess had been removed, and the dilatometer had been weighed and replaced, the experiments on that solution were begun at 0° and the readings were taken at intervals of 10° C. up to a degree or two below the boiling-point of water. Everything was then allowed to cool, and final readings were taken at the temperture of the room in order to see whether there was any appreciable variation in any of the quantities observed on account of evaporation. During all this time the instruments remained untouched and unaltered in position. The electrical resistances were measured by a modification of Kohlrausch's method.

The alternating current was derived from a small magneto-alternator driven by a water-motor, and a Wheatstone's bridge was connected up in the ordinary way. A telephone, however, was not used as an indicator, but a reflecting D'Arsonval galvanometer; the alternating current in this portion of the circuit having been transformed into a direct one by passing from the bridge to a commutator attached to the axle of the alternator and rotating with it. The galvanometer connexions were in this way reversed as often as the direction of the current from the alternator changed, and so the usual deflexion of the galvanometer could be readily detected, and the resistances adjusted to give no deflexion.

Owing to the shape of the vessel containing the electrolyte, relative measurements of the resistance were obtained. These were, however, standardized by using Kohlrausch's recent values (Journal of Chemical Society, February 1900) for the specific molecular conductivity of 1·0 gramme-molecule and of 0·5 gramme-molecule solutions of sodium chloride at 18° C.

The strengths of the solutions used were very approximately 0·1, 0·2, 0·5, 1·0, 2·0, and 4·0 gramme-molecule equivalents per litre at 15° C., the exact strengths at the various temperatures being determined as the experiments proceeded. In the accompanying Table, specimen observations are given, with the results deduced from them.

Specimen Observations.

Fluidity Data.				Density Data.				Conductivity Data.	
Strength of Soln. *n*.	Temp. °C.	Time (secs.).	Viscosity (Calc.).	Temp. °C.	Weight of Soln. (gms.)	Scale-Reading.	Density (Calc.).	Temp. °C.	Resistance (Legal Ohms.)
0·1	10·96	452·0	·01278	10·90	5·0660	9·9	1·0043	11·00	17760
0·1	90·05	124·8	·00315	90·05	,,	137·4	0·97022	90·05	5250
0·2	20·85	351·2	·00992	20·90	5·0881	16·0	1·0069	20·80	7507
0·2	80·50	137·4	·00356	80·50	,,	114·8	0·9802	80·45	3160
0·5	70·52	158·2	·00424	70·53	5·1521	97·7	0·9972	70·70	1570
0·5	35·72	265·6	·00749	35·74	,,	35·5	1 0140	35·70	2500
1·0	41·05	246·0	·00704	41·10	5·255	44·4	1·0317	41·00	1261
1·0	60·95	182·4	·00508	61·00	,,	79·7	1·0220	61·00	951
2·0	51·20	223·4	·00656	51·20	5·446	63·8	1·0636	51·15	643
4·0	0·15	809·6	·02656	0·15	5·8423	5·0	1·1600	0·15	1087
4·0	98·35	156·0	·00463	98·30	,,	183·3	1·1060	98·70	247

Curves were drawn to represent the changes in the density, viscosity, and electrical resistance of the different solutions with change of temperature, and the values obtained from these curves for every 10° C. are contained in Tables I., II., and III. These curves, and others obtained from interpolated values, were plotted with extreme care on very large sheets of squared paper in order to keep the required degree of accuracy throughout.

In the experiments, the extreme temperatures of the bath were 0°·15 C. and 98°·70 C., and the transpiration times varied between 809 secs. and 118 secs.

Table I.—Density of NaCl Solutions.

Temp.	Water.	NaCl Solutions.					
		$n_{15}=0·1$	0·2	0·5	1·0	2·0	4·0
0°C.	0·9999	1·0049	1·0095	1·0220	1·0442	1·0841	1·1600
10	0·9997	1·0043	1·0087	1·0210	1·0421	1·0809	1·1554
15	0·9992	1·0036	1·0080	1·0201	1·0408	1·0792	1·1530
18	0·9986	1·0032	1·0075	1·0195	1 0401	1·0782	1·1517
20	0·9983	1·0028	1·0071	1·0190	1·0394	1·0774	1·1507
30	0·9958	1·0002	1·0044	1·0160	1·0361	1·0734	1·1457
40	0·9923	0·9966	1·0005	1·0123	1·0320	1·0690	1·1405
50	0·9882	0 9924	0·9962	1·0079	1·0275	1·0643	1·1351
60	0·9834	0·9876	0·9915	1·0029	1·0224	1·0588	1·1295
70	0·9779	0·9821	0·9862	0·9975	1·0170	1·0533	1·1236
80	0·9719	0·9763	0·9805	0·9916	1·0111	1·0474	1·1176
90	0·9656	0·9700	0·9742	0·9854	1·0049	1·0414	1·1113
100	0·9586	0·9632	0·9675	0·9792	0·9983	1·0353	1·1049

Table II.—Viscosity of NaCl Solutions.

Temp.	Water.	NaCl Solutions.					
	$n_{15}=0·0$	$n_{15}=0·1$	0·2	0·5	1·0	2·0	4·0
0°C.	·01794	·01805	·01805	·01820	·01851	·02039	·02676
10	·01309	·01312	·01312	·01342	·01392	·01506	·02014
15	·01143	·01147	·01147	·01177	·01223	·01328	·01762
18	·01060	·01063	·01063	·01092	·01135	·01237	·01636
20	·01009	·01013	·01013	·01041	·01082	·01182	·01557
30	·00802	·00805	·00805	·00837	·00869	·00956	·01254
40	·00657	·00657	·00657	·00685	·00715	·00788	·01034
50	·00553	·00553	·00553	·00579	·00599	·00668	·00873
60	·00472	·00472	·00472	·00493	·00515	·00578	·00746
70	·00407	·00407	·00407	·00427	·00447	·00502	·00647
80	·00358	·00358	·00358	·00375	·00394	·00442	·00569
90	·00316	·00316	·00316	·00333	·00349	·00397	·00506
100	·00285	·00285	·00285	·00300	·00314	·00355	·00456

Table III.—Resistance between the Electrodes of NaCl Solutions.

Temp.	NaCl Solutions.					
	$n_{15}=0·1$	0·2	0·5	1·0	2·0	4·0
0° C.	24000	12500	5300	2970	1785	1097
10	18170	9670	4110	2325	1316	864
15	16120	8580	3670	2062	1177	777
18	15070	8000	3450	1925	1108	729
20	14420	7625	3300	1841	1065	701
30	11720	6250	2700	1512	893	575
40	9850	5270	2300	1279	762	485
50	8420	4550	2000	1103	653	419
60	7330	3955	1762	963	562	367
70	6500	3525	1581	853	492	329
80	5810	3155	1425	768	440	297
90	5250	2850	1280	702	403	270
100	4730	2560	1154	649	375	250

From Table I. the Molecular Concentrations of the solutions were obtained by using the formula

$$N_t = N_{15} \times \frac{\rho_t}{\rho_{15}} = \frac{N}{\rho} \cdot \rho_t,$$

where N_{15}=number of gramme-molecules per litre at 15° C.,
ρ_{15}= density of solution at 15° C.,
and t is the temperature.

Also ρ is the density at 15° C. given by Bender (Wied. *Ann.* xxii.) for solutions containing N gramme-molecules per litre at that temperature. The results are collected in Table IV.

Table V. contains the Fluidities, the reciprocals of the Viscosities, and Table VI. the Specific Molecular Conductivities. To obtain the latter the formula $C_t = \frac{x(1-gt)}{N_t R_t}$ was employed; where x was the constant of the cell used (1427·0), g was the coefficient of linear expansion of the glass, N_t was the concentration and R_t the resistance of the solution used at the temperature $t°$ C.

TABLE IV.—Molecular Concentrations.
NaCl Solutions.

Temp.	ρ_{15}=1·0035	1·0075	1·0200	1·0400	1·0788	1·1522
	n_{15}=0·1	0·2	0·5	1·0	2·0	4·0
0° C.	·10014	·20040	·50095	1·0042	2·0100	4·0275
10	·10008	·20024	·5005	1·0021	2·0040	4·012
15	·10001	·20010	·5000	1·0008	2·0007	4·003
18	·09997	·20000	·4998	1·0001	1·9990	3·999
20	·09993	·19993	·4995	0·9994	1·9974	3·995
30	·09966	·19940	·4980	·9962	1·9900	3·978
40	·09932	·19861	·4962	·9923	1·9817	3·960
50	·09890	·19780	·4941	·9879	1·9730	3·941
60	·09842	·19683	·4916	·9832	1·9630	3·922
70	·09788	·19580	·4890	·9778	1·9528	3·902
80	·09728	·19470	·4861	·9722	1·9419	3·881
90	·09666	·19340	·4831	·9663	1·9308	3·859
100	·09598	·19206	·4800	·9099	1·9194	3·836

TABLE V.—Fluidity of NaCl Solutions.

Temp.	Water.	NaCl Solutions.					
		n_{15}=0·1	0·2	0·5	1·0	2·0	4·0
0° C.	55·75	55·40	55·40	54·95	54·03	49·05	37·37
10	76·40	76·22	76·22	74·52	71·84	66·40	49·65
15	87·49	87·19	87·19	84·97	81·78	75·30	56·75
18	94·34	94·08	94·08	91·58	88·11	80·85	61·13
20	99·11	98·72	98·72	96·06	92·43	84·61	64·23
30	124·7	124·2	124·2	119·5	115·1	104·6	79·74
40	152·2	152·2	152·2	146·0	139·9	126·9	96·72
50	180·8	180·8	180·8	172·7	167·0	149·7	114·5
60	211·9	211·9	211·9	202·8	194·2	173·0	134·1
70	245·7	245·7	245·7	234·2	223·7	199·3	154·6
80	279·4	279·4	279·4	266·7	253·8	226·3	175·8
90	316·5	316·5	316·5	300·3	286·5	252·0	197·6
100	350·9	350·9	350·9	333·3	318·5	281·7	219·3

TABLE VI.—Specific Molecular Conductivities. NaCl Solutions.

Temp.	NaCl Solutions.					
	$n_{15}=0{\cdot}1$	0·2	0·5	1·0	2·0	4·0
0° C.	59·38	56·97	53·75	47·85	39·78	32·30
10	78·48	73·70	69·37	61·25	54·11	41·36
15	88·54	83·12	77·78	69·16	60·60	45·88
18	94·76	89·20	82·76	74·06	64·43	48·95
20	98·88	93·60	86·56	77·55	67·10	50·95
30	122·15	114·47	106·10	94·71	80·28	62·37
40	145·8	136·3	125·0	112·4	94·47	74·28
50	171·3	158·5	144·3	130·9	110·7	86·36
60	197·7	183·2	164·6	150·6	129·3	99·06
70	224·2	206·6	184·5	171·0	148·5	111·1
80	252·3	232·2	205·9	191·0	166·9	123·7
90	281·0	258·7	230·6	210·2	183·3	133·4
100	314·1	290·0	257·4	228·8	198·1	148·7

Using now the figures contained in Tables IV., V., and VI., two sets of isothermal curves were drawn, one set having fluidity as ordinates and absolute concentration as abscissæ, and the other set having specific molecular conductivity as ordinates and absolute concentration as abscissæ.

The fluidity-concentration isothermals are, with a slight deviation for weak solutions near the freezing- and boiling-points, straight lines meeting very approximately in a point on the axis of abscissæ for which the concentration c is 10·74.

Hence these isothermals can be expressed by the formula

$$F=\lambda\,(10{\cdot}74-c),$$

where λ is a function of the temperature only.

The actual deviations of the experimental results from this empirical formula only occur, as already mentioned, in the case of weak solutions near 0° C. and 100° C., F being slightly greater near 100° C. and being slightly less near 0° C. than the formula indicates.

The sp. mol. conductivity-concentration isothermals were not so simple. For concentrations greater than 1 gr.-mol. they were approximately straight lines apparently meeting at the same point on the axis of abscissæ as the fluidity-concentration isothermals, but sufficient experiments have not been carried out to properly test this. For concentrations less than 1 gr.-mol. these isothermals curve up, showing as they ought an accelerated increase in sp.-mol. conductivity as the solution becomes more dilute.

If, however, we admit that both sets of isothermals meet the axis of abscissæ at a point corresponding to a concentration of 10·74 gr.-mol. per litre, it would indicate obviously that such a solution, if physically possible, would at all temperatures have zero fluidity and zero sp.-mol. conductivity.

From these isothermal curves values were obtained at different temperatures of the fluidity and the sp.-mol. conductivity for solutions of constant concentration throughout, namely, for 0·1, 0·2, 0·5, 1·0, 2·0, 3·0, 4·0 gr.-mol. per litre. From the same curves also, by extrapolation, we obtained the values of the fluidity and the sp.-mol. conductivity at infinite dilution for all temperatures between 0° C. and 100° C.

These results are collected in Tables VII. and VIII., and plotted on Plate XI. The curves so arrived at are remarkable in that they indicate that for solutions of the strengths experimented with, both the fluidity and the sp.-mol. conductivity vanish at a temperature of $-35^{\circ}{\cdot}5$ C.

It is obvious at once from these constant-concentration curves that the temperature-coefficients of fluidity and sp.-mol. conductivity are not identical, though the two sets of curves possess considerable similarity.

The values of the ratio of the sp.-mol. conductivity (C) to fluidity (F) were now calculated for solutions of constant concentration at different temperatures, and the results are collected in Table IX., and plotted in Plate XII., where the values of C/F are abscissæ with the corresponding temperatures as ordinates.

Table VII.—Fluidity (F) at Constant Concentration.

Temp.	Infinite Dilution.	$n=0{\cdot}1$	$n=0{\cdot}2$	$n=0{\cdot}5$	$n=1{\cdot}0$	$n=2{\cdot}0$	$n=3{\cdot}0$	$n=4{\cdot}0$
0° C.	55·7	55·4	55·2	55·0	54·0	49·4	43·9	37·7
10	76·4	76·3	76·0	74·6	71·8	66·4	58·8	49·9
18	94·35	94·0	93·2	91·6	88·1	80·8	71·4	61·1
20	99·1	98·6	98·0	96·0	92·4	84·6	74·4	64·2
30	124·7	123·2	122·5	120·0	115·0	104·5	92·1	79·1
40	152·2	151·0	149·8	146·1	140·0	126·7	112·0	96·2
50	180·8	179·4	178·0	173·9	166·4	149·3	131·4	113·7
60	211·9	210·0	208·0	202·0	192·0	172·6	152·8	132·9
70	245·7	243·2	241·0	234·0	222·4	198·9	175·7	152·2
80	279·4	276·8	274·0	266·1	252·8	226·0	192·2	172·6
90	316·5	313·3	310·0	300·0	283·9	251·7	221·7	193·4
100	350·9	347·1	343·4	332·5	314·2	278·4	245·4	217·0

TABLE VIII.—Specific Molecular Conductivity (C) at Constant Concentration.

Temp.	Infinite Dilution.	$n=0·1$	$n=0·2$	$n=0·5$	$n=1·0$	$n=2·0$	$n=3·0$	$n=4·0$
0° C.	66·2	59·4	55·0	50·5	47·9	42·4	37·4	32·65
10	88·0	78·5	73·7	66·6	61·25	54·0	47·0	41·4
18	107·5	94·75	89·2	80·9	74·06	64·4	56·1	48·9
20	112·5	98·9	93·6	84·6	77·55	67·1	58·1	50·9
30	140·0	122·1	114·5	104·1	94·7	80·2	70·0	62·0
40	166·0	145·8	135·4	123·9	112·2	94·2	82·4	74·0
50	194·0	171·3	158·5	143·4	130·5	110·0	96·1	85·9
60	225·0	197·7	183·2	164·0	150·1	128·8	111·7	98·1
70	252·0	224·2	206·3	184·0	170·1	147·2	127·1	109·6
80	283·0	251·7	231·8	205·3	190·0	165·8	142·9	121·5
90	311·0	279·4	257·0	229·3	209·4	181·9	155·4	133·0
100	350·0	312·6	288·5	253·0	227·6	196·0	168·7	144·9

TABLE IX.—Ratio $\frac{C}{F}$ at Constant Concentration.

Temp.	Infinite Dilution.	$n=0·1$	$n=0·2$	$n=0·5$	$n=1·0$	$n=2·0$	$n=3·0$	$n=4·0$
0° C.	1·189	1·072	0·996	0·918	0·887	0·858	0·852	0·866
10	1·152	1·029	·970	·893	·853	·813	·799	·830
18	1·139	1·008	·957	·883	·841	·797	·786	·800
20	1·135	1·003	·955	·881	·839	·793	·782	·793
30	1·123	·991	·935	·867	·823	·768	·760	·784
40	1·091	·966	·904	·848	·801	·744	·736	·769
50	1·073	·955	·890	·825	·784	·737	·731	·755
60	1·062	·942	·881	·812	·781	·746	·731	·738
70	1·026	·922	·856	.786	·765	·740	·723	·720
80	1·013	·909	·846	·772	·752	·734	·717	·704
90	·983	·892	·829	·764	·738	·723	·701	·688
100	·998	·901	·840	·761	·724	·704	·687	·668

The plots come out as parallel straight lines to a very close approximation, which shows that the ratio C/F for solutions of constant concentration decreases proportionally to rise of temperature, and that for solutions of all strengths the temperature-coefficient involved is the same. Its value may be taken as ·00174; hence we may express the relations shown on Plate XII. by the formula

$$\frac{C}{F}=u\,(1-·00174\,t),$$

where u depends only on the concentration and t is the temperature.

For infinite dilution $u=1{\cdot}177$, and for solutions of different strengths n we get the following values of u:—

n	0·1	0·2	0·5	1·0	2·0	3·0	4·0
u	1·048	·982	·906	·869	·827	·816	·825

If these values of u be substituted in the above formula for C/F, and the latter quantity calculated for different states of temperature and concentration, it will be found that the values so obtained agree very closely with those contained in Table X. The average difference is 1 per cent., and the largest deviations occur for the temperatures 0° C. and 100° C.

The degree of dissociation at different temperatures for solutions of constant concentration is given in Table X.; and it is interesting to note that for the stronger solutions at least there is a well-marked diminution of the degree of dissociation with rise of temperature.

Table X.—Degree of Dissociation at Different Temperatures of Solutions of Constant Concentration.

Temp.	NaCl Solutions.						
	$n=0{\cdot}1$	$n=0{\cdot}2$	$n=0{\cdot}5$	$n=1{\cdot}0$	$n=2$	$n=3$	$n=4$
0° C.	·897	·830	·763	·723	·640	·565	·493
10	·892	·838	·756	·696	·614	·535	·470
18	·882	·830	·753	·689	·599	·522	·454
20	·879	·832	·752	·689	·596	·517	·451
30	·872	·818	·744	·677	·573	·501	·443
40	·876	·813	·744	·674	·566	·495	·444
50	·882	·816	·738	·672	·566	·495	·442
60	·879	·814	·729	·669	·573	·496	·437
70	·890	·820	·731	·675	·585	·505	·435
80	·889	·819	·725	·671	·586	·505	·429
90	·899	·826	·737	·673	·585	·500	·428
100	·893	·824	·723	·650	·560	·482	·414

LVII. *On the Physical Peculiarities of Solutions of Gases in Liquids. No. 2. By* J. Alfred Wanklyn, *Corresponding Member of the Royal Bavarian Academy of Sciences*[*].

THE solubility of ammonia in water at ordinary pressures and temperatures is seven hundred times as great as the solubility of carbonic acid. There is, therefore, great interest in ascertaining whether a corresponding retardation of the act of solution of the ammonia is possible. The following experiments answer this question in the affirmative.

The same tube as was employed in the experiments upon

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[*] Communicated by the Author.

carbonic acid was taken. The tube is closed at one end and open at the other end. Its diameter (inside measure) is about 19 millimetres, its length 490 millimetres.

The desired conditions—an atmosphere of gaseous ammonia resting upon a perfectly saturated ammoniacal solution superposed upon a layer of pure water—were realized by the following procedure :—

The tube, its closed end downwards, was nearly filled with mercury, viz., to about 8 c. c. from the top, and that volume of strong liquor ammoniæ was poured into the tube so as to fill it. Then the tube was closed with the thumb, reversed and introduced into the mercurial trough in the well-known manner. The strong solution of ammonia, being in that manner confined at reduced pressure over mercury, boiled vigorously and evolved an atmosphere of ammonia resting upon a perfectly saturated aqueous solution of ammonia. The evolution of ammonia having ceased, a reading of the volume of ammonia, together with the temperature and pressure, was performed. Then a measured quantity of distilled water was passed up, using a delicate pipette and carefully avoiding any disturbance of the liquid. The operation having been successfully accomplished, the layer of distilled water underneath the strong ammonia was quite visible, and remained visibly distinct for a long time. The following readings were made :—

	Observed volume of NH_3: cubic centims.	Temperature centigrade.	Pressure: millimetres.	Volume of NH_3 corrected at 0° and 760 mm. (dry).
A…………	46·5	12°	495·5	29·04 c. c.
B…………	43·5	12°	501·7	27·50 „
C…………	40·0	12°	490·2	24·70 „
D…………	39·0	12°·5	485·0	23·80 „

The volume of ammonia solution was 8·0 c. c., and A is the volume of gaseous ammonia before the introduction of the distilled water.

The volume of the distilled water was 5·1 c. c. B is the volume of gaseous ammonia remaining after the introduction of the distilled water.

C is a second reading of the gaseous ammonia, one hour having elapsed since the reading B.

D is a later reading after the lapse of another hour.

Finally the tube was gently shaken, and immediately the gaseous ammonia vanished leaving only a very small air-bubble.

The experiment was repeated, the initial volume of solution of ammonia being 7·5 c. c., and the volume of distilled water 5·5 c. c.

The initial volume of gaseous ammonia (at 0° and 760 mm. dry) was 38·2 c. c. The volume after addition of the water was 34·8 c. c., which three hours standing reduced to 23·1 c.c.

After the lapse of a further ten hours the volume fell to 7·13 c. c., which instantly disappeared on slight shaking.

The experiments are being continued, and the results are far-reaching.

The Laboratory, New Malden, Surrey,
25th March, 1902.

LVIII. *On the Magnetism induced in Iron by Rapidly Oscillating Current-fields.* *By* W. MANSERGH VARLEY, *B.Sc., Ph.D.*, 1851 *Exhibition Scholar*[*].

1. AS long ago as 1827 Savary[†] noticed that a steel needle could be magnetized by putting it in the neighbourhood of a wire through which a leyden-jar was discharged, and some years later similar experiments led Henry[‡] to suspect the oscillatory character of the leyden-jar discharge. Still later, however, both Feddersen[§] and Hertz[||] expressed the opinion that iron is not magnetized by rapidly oscillating fields: but Thomson[¶], Trowbridge[**], Klemenčič[††], Marchand[‡‡], and Hemsalech[§§] have shown by various methods that when iron is placed in such an oscillating field, there is an increase in the energy absorbed such as can only be accounted for on the supposition that the iron is rapidly magnetized and demagnetized. St. John[||||] also has compared the self-inductions of similar iron and copper circuits,

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[*] Communicated by Prof. J. J. Thomson.

[†] F. Savary, Pogg. *Ann.* x. p. 73 (1827), and *Ann. de Chim. et de Phys.* xxxiv. p. 5 (1827).

[‡] Jos. Henry, Lodge's 'Modern Views of Electricity,' p. 370 (1899), or Fleming's 'Alternate Current Transformer,' vol. i. p. 223.

[§] W. Feddersen, Pogg. *Ann.* cviii. p. 497 (1859).

[||] H. Hertz, Wied. *Ann.* xxxiv. p. 558 (1888).

[¶] J. J. Thomson, Phil. Mag. xxxii. p. 445 (1891).

[**] J. Trowbridge, Phil. Mag. xxxii. p. 504 (1891).

[††] I. Klemenčič, *Wien. Ber.* ciii. p. 205 (1894); civ. p. 724 (1895); cvii. p. 330 (1898).

[‡‡] E. W. Marchand, 'Nature,' lxii. p. 413 (1900).

[§§] Hemsalech, *Comptes Rendus*, cxxxii. p. 917 (1901).

[||||] C. E. St. John, Phil. Mag. xxxviii. p. 425 (1894).

and his results show that the permeability of the iron is still very great with rapid oscillations.

Since the very early experiments first mentioned, none have been described showing directly that iron is magnetized in a rapidly alternating field, or showing how the induced magnetization depends on the frequency of oscillation. The present research was undertaken to show, by a direct method, how the magnetization induced in iron by such oscillatory fields depends on the strength and frequency of the field, and on the diameters of the wire used.

The relations between these quantities will be most clearly displayed by considering the alteration in the magnetization produced by (1) altering the frequency while keeping the magnetizing field constant, (2) altering the magnetizing field while keeping the frequency constant; or if B represents the maximum induced magnetism, n the frequency of oscillation, H the maximum intensity of the magnetizing field; (1) $\left(\frac{\partial B}{\partial n}\right)_{H=\text{const.}}$ and (2) $\left(\frac{\partial B}{\partial H}\right)_{n=\text{const.}}$.

2. When a leyden-jar of capacity C is discharged through a circuit containing self-induction L and resistance R, the oscillatory current, i, at any time t is given by*

$$i=\frac{2Q}{\sqrt{4LC-R^2C^2}}\cdot e^{-\frac{Rt}{2L}}\cdot\sin\left(\frac{\sqrt{4LC-R^2C^2}}{2LC}\cdot t\right),$$

where Q is the quantity of electricity in the condenser at time $t=0$.

If, as is the case in all the following experiments, $\frac{R^2C}{4L}$ is very small compared with unity, we can write the equation in the usual form

$$i=\frac{Q}{\sqrt{LC}}\cdot e^{-\frac{Rt}{2L}}\cdot\sin\frac{t}{\sqrt{LC}},$$

which shows that i goes through harmonic oscillations of frequency $n=\frac{1}{2\pi\sqrt{LC}}$ and of decreasing amplitude. For the first and greatest amplitude $\frac{di}{dt}=0$, whence

$$t=\sqrt{LC}\tan^{-1}\left(\frac{2}{R}\sqrt{\frac{L}{C}}\right),$$

or practically, since $\frac{R^2C}{4L}$ is very small,

$$t=\frac{\pi\sqrt{LC}}{2},$$

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* See Bedell & Crehore's 'Alternating Currents,' p. 107.

and we obtain for the first amplitude

$$i_{\text{max.}} = \frac{Q}{\sqrt{LC}} \cdot e^{-\frac{\pi}{2}\frac{R}{2}\sqrt{\frac{C}{L}}}$$

which, for the same reason, further simplifies to

$$i_{\text{max.}} = \frac{Q}{\sqrt{LC}}.$$

Now $Q=VC$, where V is the potential-difference between the plates immediately before the discharge, so that

$$i_{\text{max.}} = V \cdot \sqrt{\frac{C}{L}}.$$

We can utilize the two equations

$$\text{(I.)}\ n = \frac{1}{2\pi\sqrt{LC}}, \qquad \text{(II.)}\ i_{\text{max.}} = V\sqrt{\frac{C}{L}} = kH$$

(where H is the maximum field inside a solenoid included in the discharge circuit) for the purpose of the present problem.

For if we wish to investigate the dependence of the magnetization (B) on the frequency (n) alone, we need only alter L and C, keeping the ratio $\frac{C}{L}$ unchanged; while to investigate the dependence on the field-strength, the frequency remaining constant, we need only alter L and C, keeping the product L . C the same.

In practice it is not possible, nor indeed necessary, to fulfil these conditions exactly, but they are so nearly fulfilled in the experiments described, that only the simplest interpolation is necessary to reduce the measurements back to the ideal case.

In order to determine the magnitude of the maximum current ($i_{\text{max.}}$) and the magnetization (B) induced in the iron by it, use was made of the deflexion of cathode-rays after the method of Braun*. The experiments were so arranged that at one time the deflexion was due only to the maximum of the magnetizing current, and at another time due only to the magnetization induced by the same in the iron. The frequency was calculated from the self-induction and capacity in circuit. A more detailed description of the method of experiment is given in the following paragraph.

3. The oscillatory current was produced by allowing the condensers C, consisting of leyden-jars specially fitted up so as to give good internal and external contact, to discharge

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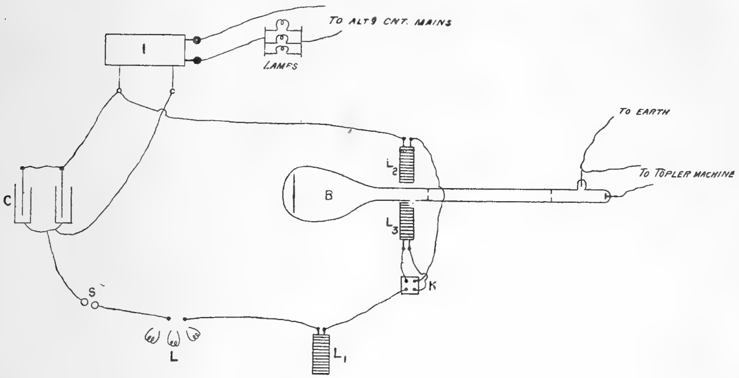
* F. Braun, Wied. *Ann.* lx. p. 552 (1897).

through a circuit containing a micrometer spark-gap S, and various solenoids and coils, L_1, L_2, L_3, L. The jars were charged by connecting to the secondary terminals of an induction-coil, I, through the primary of which either an alternating or direct current was passed. The three solenoids L_1, L_2, and L_3 were made exactly alike, and each had the same number of turns. Two such systems of solenoids were wound, each coil of the first system having 645 turns, and of the second system 140 turns, the insulation of the D.C.C. wire being in both cases made as perfect as possible by being drawn through melted wax, rubber strips being also placed between adjoining rows of turns. Various other coils, L, were similarly made, but of such dimensions that the self-inductions could be accurately calculated by means of Maxwell's formula.

Fig. 1.



The Braun's tube was one with two diaphragms, and having the improvements suggested by Zenneck *, including a glass screen (instead of mica) coated with $CaWO_4$ as phosphorescent material. The tube was excited by a twenty-plate Töpler influence-machine driven at a very constant rate by a small water-motor. A damp thread was used as conductor between the Töpler machine and the anode of the Braun's tube, which was connected to earth in order to eliminate the surface charge otherwise present on the surface of the tube in the neighbourhood of the diaphragms, and to give increased steadiness to the phosphorescent spot. Two of the three similar solenoids (viz., L_2 and L_3, see sketch) were placed symmetrically with respect to the Braun's tube, and their positions

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* J. Zenneck, Wied. *Ann.* lxix. p. 844 (1899).

kept unaltered throughout a whole series of measurements. The direction of the current through one of the coils (L_3) could be reversed by means of the commutator (K), so that the coils acted in the same or in opposite senses on the cathode-rays. A few less important precautions must also be taken in arranging the apparatus.

A scale cut in tinfoil, on glass, strongly illumined from behind, was projected on to the screen of the Braun's tube, the lenses being arranged so as to give exactly millimetre divisions on the screen, this arrangement enabling the deflexions to be read with the utmost accuracy and ease in the perfectly dark room.

In order to eliminate the electrostatic effect of the solenoids acting as small capacities, the tube was surrounded by a layer of tinfoil in the neighbourhood of the solenoids.

4. The iron wires investigated were placed as evenly as possible in glass tubes of such a diameter as to slide easily in and out of the cores of the solenoids. Each wire was perfectly insulated from the others in the same bundle by being coated with paraffin or shellac. The particulars of the wires used in four of the bundles are given in the following table, where N is the number of wires in the bundle, d the diameter of the wire, σ the total cross section of the wires in the bundle, and S the whole surface of the wire divided by the length. The length of each wire was 9·0 cms.

TABLE I.

Bundle.	N.	d, cms.	σ, cm^2.	S, cms.
A	20	0·100	0·157	6·28
B	37	0·071	0·147	8·25
C	70	0·0525	0·152	11·54
D	409	0·0221	0·157	28·4

The iron wires in each of these four bundles were from the same kind of iron, and kindly placed at our disposal by Messrs. Siemens & Halske. As the figures show, the cross section in each of the four bundles was nearly the same, and it was found, on placing the bundles in the same direct-current field, that the induced magnetization in each was also very nearly equal, thus affording a suitable series to test the influence of the diameter on the magnetization in an oscillating electrical field.

5. As stated in § 2, we wish to determine (1) the deflexion due to the maximum of the magnetizing current, and (2) that due to the magnetization induced by the same in the bundle of iron wires under consideration. For this purpose, the two coils L_2 and L_3 were first connected up so as to act in the same sense on the cathode-rays, and the iron was placed in the distant solenoid L_1. The deflexion obtained on closing the primary circuit of the induction-coil, after connecting up the required self-induction and capacity in the secondary circuit, was that due to the maximum of the magnetizing current—which for shortness we shall call simply the "current." The direction of the current through L_3 was now reversed, so that the cathode-rays were undeflected and the spot in its position of rest, the iron brought from the solenoid L_1 and placed in L_2. The deflexion now obtained was due only to the magnetization induced in the iron by the current just determined. The experiments were repeated after suitably altering the values of the self-induction and capacity in circuit.

In order that the "current" and frequency may remain unaltered, the iron must occupy the same relative position in the solenoids L_1 and L_2, and must in fact be placed in the same positions throughout a series of measurements.

The value of the "current" was determined by calibrating the tube by means of direct current deflexions, the currents being measured directly by a Weston ammeter. The magnetizations induced in the iron by the oscillatory currents were similarly compared with those induced by direct currents of the same strength. Measurements showed that when the self-induction and spark-gap were kept constant, the deflexions due to the oscillatory currents were proportional to the square root of the capacity in circuit, proving that for the frequencies used the cathode-rays possessed no detectable inertia. In order to be able to vary the frequency within wider limits, two sets of coils were used, the coils of each set having respectively 645 and 140 turns. It is easy to see that by the substitution of one set of solenoids for the other in the secondary circuit, other self-inductions being absent, and the capacity and sparking potential remaining the same, the magnetizing force inside the solenoids will remain practically unaltered, as also the "current" deflexions, thus affording an excellent means of obtaining wide variations in the frequency of oscillation for our purpose. The more exact comparison or reduction of the measurements was carried out in the following manner:—

For the first systems of solenoids let $H_0' = a' i'$, and

$\delta'=b'i'$, whence $H_0'=c'\delta'$; and similarly for the second system $H_0''=c''\delta''$, whence

$$\frac{H_0'}{H_0''}=\frac{1}{c}\,\frac{\delta'}{\delta''},$$

H_0', H_0'' being the fields in the cores of the solenoids in the two cases, δ_0', δ_0'' the deflexions due to the currents alone, and a', b', c', c'', and c constants.

A certain specimen of iron was placed in each of the two solenoids in turn and magnetized by direct current to the same intensity in the two cases, all deflexions being noticed. Now we have $H_0'=H_0''$; and the relation of the current deflexions $\frac{\delta'}{\delta''}=c$ gives the reduction factor, by means of which it is possible to reduce the deflexions obtained with the second system of coils to the same unit as those obtained with the first system. All measurements were reduced to those of the 645 turn system, the reduction factor by which the current deflexion in the case of the 140 turn system had to be multiplied being found as a mean of 12 measurements to be 1·46.

6. The capacities of the leyden-jars used were determined by comparison with a standard high resistance by the tuning-fork method, modified so as to be suitable for measuring accurately capacities as low as 1000 cms. The capacities of the several jars used were found to be 1640, 2430, 2270 cms. &c. These measurements were repeated after an interval of several months, and identical values obtained.

The coils used as standard or known self-inductions had the dimensions given in the following table, the self-inductions being calculated from Maxwell's formula *, or rather Stefan's form † of the same.

TABLE II.

Coil.	Mean radius, cms.	Breadth, cms.	Thickness, cms.	No. of turns.	Self-induction, cms.
I. ...	11·9	1·8	2·0	140	$8{\cdot}0\times10^{6}$
II. ...	12·0	3·5	2·0	250	22×10^{6}
III. ...	10·85	2·3	1·7	47	$0{\cdot}77\times10^{6}$

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* C. Maxwell, Phil. Trans. clv. p. 508 (1865); see also Lord Rayleigh, Proc. Roy. Soc. xxxii. p. 117 (1881).

† J. Stefan, Wied. *Ann.* xxii. p. 112 (1884).

Rayleigh's method was employed for determining the largest self-induction (137×10^6 cms.) used.

The other self-inductions, including those of the sets of solenoids, were too small to enable Rayleigh's method to be applied, and were not of a form to permit the use of Maxwell's formula; so advantage was taken of the deflexion of the cathode-rays by the oscillatory currents produced by the leyden-jar discharges, to compare them with the standard self-inductions given above. As we have already seen, the deflexion due to the oscillatory current is proportional to $V\sqrt{\frac{C}{L}}$; if now the sparking distance and capacity are kept constant, the deflexion is inversely proportional to the square root of L.

To determine, for example, the self-induction of the three coils of the first (or 645 turn) system of solenoids, placed in series, by comparison with the standard coils I. and II. :—

Let α be the deflexion when only the three solenoids are in the secondary circuit, no other self-induction being in circuit *;

β the deflexion when in addition the standard coil I. is in circuit, and γ the deflexion when this is replaced by the standard coil II.

Then

$$L = \frac{8 \times 10^6}{\left(\frac{\alpha}{\beta}\right)^2 - 1} = \frac{22 \times 10^6}{\left(\frac{\alpha}{\gamma}\right)^2 - 1}.$$

The following table gives the deflexions actually observed.

TABLE III.

α.	β.	γ.	L in cms. as determined with	
			Standard Coil I.	Standard Coil II.
66	50	38	$11{\cdot}2 \times 10^6$	$11{\cdot}2 \times 10^6$
65	49	38	10·5 „	11·4 „
63	48	37	11·1 „	11·6 „
61	47	37	11·7 „	12·8 „
63	48	37	11·1 „	11·6 „
70	54	42	11·8 „	12·4 „
70	53	41	10·8 „	11·5 „
47	37	28	13·0 „	12·0 „
51	40	30	12·8 „	11·6 „
Mean values for L			$11{\cdot}5 \times 10^6$	$11{\cdot}8 \times 10^6$

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* *Note.* Two of the three solenoids were used to deflect the rays—see sketch.

Mean value for $L = 11{\cdot}65 \times 10^6$ cms., or nearly 12×10^6 cms. Similar determinations gave the self-induction of the three 140 turn solenoids in series as $0{\cdot}45 \times 10^6$ cms., the coil III. being used as the standard for comparison.

One other example will serve to show the application of this method to determining a rather larger self-induction.

Let α be the deflexion when the three 645 turn solenoids, whose self-induction we have just determined, and the standard coils I. and II. are in circuit, and β the deflexion when the standard coils are replaced by the unknown self-induction L. Then

$$L = \left\{ \left(\frac{\alpha}{\beta}\right)^2 . 42 \times 10^6 - 12 \times 10^6 \right\} \text{cms.}$$

TABLE IV.

Self-induction, cms.	Capacity, cms.	n.	δ_H.	δ_M for the bundles of iron			
				A	B	C	D
$0{\cdot}55 \times 10^6$	2430	130700	23	7	10·5	16	23
0·55 ,,	3810	104300	27	10·5	14·5	20	30
0·55 ,,	4760	93300	32	12·5	17	22	35
0·55 ,,	6080	82600	37·5	14·5	19	25·5	39
0·55 ,,	8510	69800	44	16	21	28·5	44
1·32 ,,	4760	60300	22·5	11	15	18·5	26
1·32 ,,	7030	49600	26	13	17	22	32
1·32 ,,	8510	45000	29	14·5	19	25·5	35·5
1·32 ,,	10090	41400	37·5	19	23·5	28	41
1·32 ,,	13150	36200	43·5	21	26·5	34	48
16 ,,	1480	31000	36	19	23·5	29·5	41
24 ,,	1480	25300	36	20	24·5	31	42
24 ,,	1480	25300	29	17·5	22	28·5	36
38 ,,	1480	20100	17	13·5	17	21	25
46 ,,	1480	18300	26	18	22	28·5	35
46 ,,	2430	14300	34	23	28·5	34·5	44
24 ,,	3060	17600	47	26·5	32	40	53
46 ,,	3060	12700	37	28	34	40	47
96 ,,	3060	8820	29	23·5	30	36	42
153 ,,	3060	7000	22	21	25	29	33
46 ,,	4640	10300	44	30	38	46	55
96 ,,	4640	7150	30	28	34·5	40	44
153 ,,	4640	5540	27	26·5	32	37	40
263 ,,	4640	4320	21	23·5	27	31	33
96 ,,	6220	6200	44	36	42	52	55·5
153 ,,	6220	4900	35	32	38	44	47
263 ,,	6220	3730	27	28	33	37	40
153 ,,	8550	4200	44	42	47·5	55	56
Direct Current.							
0·40 amp.		0	18	26	27	27·5	27
0·58 ,,		0	26	39	40	42	41
0·88 ,,		0	38	50	52	54	52·5
1·32 ,,		0	55	59	61	...	61·5

The value of L for the coil thus determined was found as a mean of 9 measurements to be $79{\cdot}5 \pm 0{\cdot}4 \times 10^6$, or nearly 80×10^6 cms. The increase of self-induction of the circuit due to the introduction of iron into one of the equal solenoids was also determined in an exactly similar manner, and for different frequencies of oscillation.

7. The full results of a series of observations taken with the bundles of iron A, B, C, and D over the maximum range of frequencies obtainable with the coils at our disposal are given in Table IV. (p. 508). In this table δ_H represents the deflexion due to the "current" alone, which is proportional to the magnetizing force, δ_M represents the corresponding deflexion due to the magnetization induced in the iron, and n the number of oscillations per sec.

From these figures the following Table V. was derived, to show the alteration of the magnetization with the periodicity for various maximum magnetizing forces (H_0). The values

TABLE V.

δ_H.		H_0.	n.	δ_M for the bundles							
				A		B		C		D	
44		82 C.G.S. units.	69800	16		21		28·5		44	
44	43·5		36200	21		26·5		34		48	
44	47		17600	25	26·5	30·5	32	38	40	51·5	53
44			10300	30		38		46		55	
44			6200	36		42		52		55·5	
44			4200	42		47		55		56	
44			0	54		56		58		56·5	
36	37·5	66 C.G.S. units.	82600	14	14·5	18	19	24	25·5	38	39
36	37·5		41400	18	19	22	23·5	26·5	28	40	41
36			31000	19		23·5		30		41	
36			25300	20		24·5		31		42	
36	37		12700	27	28	33	34	40	40	46	47
36	35		4900	33	32	39	38	45	44	48	47
36			0	48		50		52		51	
26	27	47 C.G.S. units.	104300	10	10·5	13·5	14·5	18·5	20	28	30
26			49600	13		17		22		32	
26			18300	18		22		28·5		35	
26	27		5540	25	26·5	30	32	35	37	38	40
26	27		3730	27	28	32	33	36	37	39	40
26			0	29		40		42		41	
23		41 C.G.S. units.	130700	7		10·5		16		23	
23	22·5		60300	11		15		18·5		26	
23	22		7000	22	21	26	25	30	29	33	33
23			0	34		35		36		35	

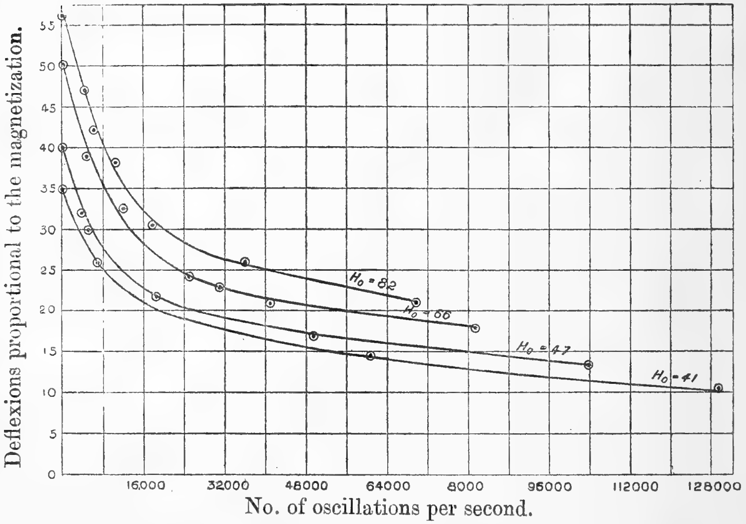
of the magnetizing force are calculated from the ordinary solenoid formula

$$H = \frac{4\pi n i}{10}.$$

Where interpolation has been necessary the figures from Table IV. are given in small type for comparison.

The curves I. were constructed from the figures of this table to show graphically the relation between the decrease in the magnetization with the increase in the frequency for

Fig. 2.—Curves I.



the bundle B of iron for different magnetizing forces (H_0); and the curves II. to show the same for different diameters of wire with the same magnetizing force ($H_0 = 66$ C.G.S. units).

A third set of curves (III.) is also given, which show the relation between the magnetization and the magnetizing force for another specimen of iron for direct currents and for oscillatory currents of frequencies 10,000 and 20,000 per second.

For the most frequent oscillations worked with, the magnetization was found to be proportional to the surface of iron in the case of the bundles A, B, and C, though not in the case of D, as may be seen by comparing the figures for the highest frequencies given in Table IV. with those given in the last column of Table I. This shows that with the most rapid oscillations employed, the magnetizing force did not penetrate to the centre of the thickest wires used, but did penetrate to the middle of the thinnest wires.

Fig. 3.—Curves II.

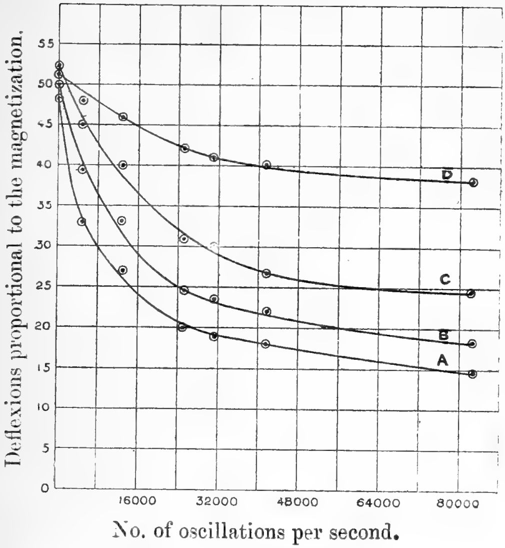
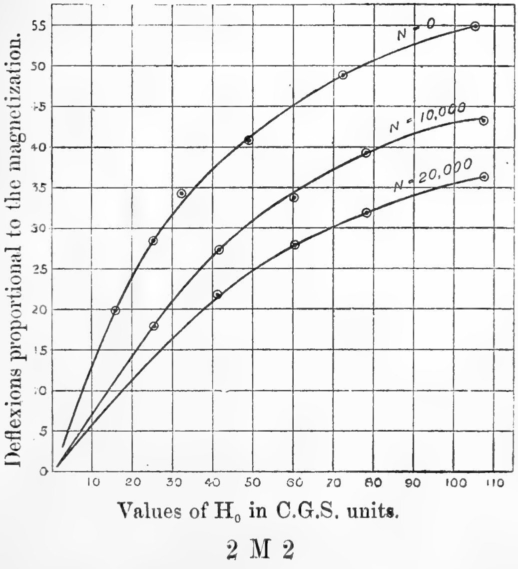


Fig. 4.—Curves III.



The theory of the experiment does not admit of any accurate calculation of the permeability (μ) from the results, as the depth in the wire at which the magnetizing force has sunk to an insignificant fraction of its value at the surface itself involves μ. An attempt was, however, made to work out an approximate value for μ from the measurements obtained; and the mean of some ten values gave $\mu=110$ for a certain specimen of soft iron for about 10,000 oscillations per second. which is just about the value obtained by Klemenčič under similar conditions by a totally different method, but it is much smaller than that got by St. John for much more rapid oscillations.

In conclusion, may I offer my best thanks to Prof. F. Braun of Strassburg, in whose laboratory the experiments were carried out, and to Prof. M. Cantor, for their many kindnesses during the progress of the research.

Cavendish Laboratory, Cambridge,
March 1901.

LIX. *The Rôle of Water Vapour in Gaseous Conduction.*
By Professor PERCIVAL LEWIS[*].

IN a recent number of this Magazine[†] Professor Trowbridge expresses his conviction that pure gases, hydrogen in particular, are perfect insulators, and that the presence of a trace of water-vapour or oxygen is necessary to produce the dissociation or other conditions upon which conduction depends. When we consider the more or less analogous phenomena of electrolytic conduction in solutions, this view does not seem improbable; but some facts may make us hesitate to accept the conclusion that water-vapour is so essential an agent in discharges through gases.

In the first place, some of the reasons given by Professor Trowbridge for believing pure hydrogen to be an insulator do not seem to me conclusive. He cites the fact, discovered by Schumann[‡], that hydrogen at atmospheric pressure seems perfectly transparent to ultra-violet radiation, as demonstrating that pure hydrogen is an insulator under all conditions. It may be that hydrogen has a finite but very small absorption-coefficient, no more noticeable within the limits of a laboratory than the absorption of carbon dioxide for visible radiation under the same conditions; but whether this be so or not does

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[*] Communicated by the Author.

[†] Trowbridge, Phil. Mag. [6] ii. p. 379 (1901).

[‡] Schumann, *Ann. der Physik* (1901).

not seem material, for we know that all fairly dry dust-free gases are insulators under ordinary conditions. It is still to be shown that pure hydrogen under electric stress in a vacuum-tube is an insulator.

Another reason assigned by Professor Trowbridge for holding hydrogen to be an insulator, is that when occluded by platinum or palladium the resistance of these metals is increased. It must be remembered, however, that alloys usually have a greater specific resistance than that of either of their constituents. Silver added to platinum gives an alloy with more than twice the specific resistance of platinum, yet no one can claim that silver is an insulator. Nor is the condition of hydrogen when occluded, or when liquefied, the same as that in vacuum-tubes.

In the second place, it seems that the complete removal of water-vapour from vacuum-tubes is perhaps possible, and that still the hydrogen or other gases contained therein may transmit the current and give their characteristic spectra. All who have worked with vacuum-tubes know how pertinacious water-vapour is; but some experiments by Deslandres [*] and by the present writer [†] indicate that it may be effectively removed by the prolonged heating of metallic sodium in the tube. Deslandres states that only by such treatment could he cause the characteristic water-vapour lines to disappear from the spectrum of nitrogen, and it also causes the disappearance of that group of bands in the extreme ultra-violet spectrum of nitrogen attributed by Deslandres to an oxide of nitrogen, while the remaining part of the nitrogen spectrum was as bright as or brighter than before.

E. Wiedemann [‡] heated metallic sodium in vacuum-tubes containing hydrogen or nitrogen. As the vapour-density of the sodium increased, its spectrum grew in intensity at the expense of that of the other gas; but on cooling the tube, the hydrogen or nitrogen spectrum reassumed its original appearance.

In order to again test the matter, the writer recently placed a quantity of metallic sodium in a vacuum-tube containing hydrogen. The sodium was repeatedly heated to drive off its more volatile impurities, and the tube pumped out and filled with fresh electrolytically prepared hydrogen. After the tube had been well covered with sodium "dew," it was closed. For two weeks it was frequently heated, and the spectrum examined by a pocket spectroscope. After heating, or in parts of

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[*] Deslandres, *Ann. de Chim. et de Phys.* [6] xv. p. 46 (1888).

[†] Lewis, *Ann. der Physik* [4] ii. p. 465 (1900).

[‡] E. Wiedemann, Wied. *Ann.* v. p. 517 (1878).

the capillary where sodium was most abundant, the hydrogen spectra, both simple and compound, were weak; on cooling the tube, they both increased in intensity and showed no signs of approaching dissolution. The tube was finally accidentally broken, and on being immediately examined it was found that there were large patches of sodium with clean metallic lustre. Whatever other impurities might have been there, it is difficult to see how any free water-vapour or oxygen could have been present. Moreover, as the hydrogen spectrum had its full intensity at the first flash through the tube, it seems improbable that the current, first passing through the sodium-vapour, could have set free oxygen from any of its compounds present, thus enabling hydrogen to take subsequent part in the conduction.

Some experiments by Warburg[*] appear to have some bearing on this matter. He found that in a hydrogen tube containing some pure electrolytically prepared metallic sodium, the cathode fall of potential had a minimum value of 168 volts. With measurable traces of oxygen present the fall was 240 volts, gradually falling to a minimum of about 200 volts as the oxygen, after combination with the hydrogen, was removed by the ordinary drying-tubes. Only after the more nearly perfect drying produced by metallic sodium was the lower minimum reached. This indicates that, so far as the processes at the cathode are concerned (and here the current meets with the greatest opposition), the current can pass with more facility through perfectly dry hydrogen than through that containing traces of oxygen or water-vapour. In the case of nitrogen, Warburg found that small traces of these impurities lowered the cathode fall, but in dry nitrogen it had a definite upper limit. The writer has used one of Professor Warburg's nitrogen tubes, and although the gas had been in contact with sodium for about ten years, the current passed with ease and gave the ordinary spectrum of nitrogen.

It certainly seems impossible to eliminate all traces of impurity from vacuum-tubes, and it may be that one or more such impurities may take a very active and essential part in the processes which transform a gas into a conductor; but if, under the conditions referred to, comparatively small electromotive forces can drive a current through hydrogen or nitrogen, it seems at least doubtful whether water-vapour is necessary to gaseous conduction.

University of California, Berkeley,
January 1902.

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[*] Warburg, Wied. *Ann.* xl. p. 1 (1890).

LX. *On the Temperature Variation of the Electrical Resistances of Pure Metals, and Allied Matters.* By W. Williams, *B.Sc., Municipal Technical College, Swansea*[*].

MANY of the physical properties of metals are periodic functions of their atomic weights. This is especially true of the valency and atomic volume. In a lesser degree it is also true of the melting-point and electric resistance. Now, the periodic variations of these four quantities for most of the metals can be very simply related to one another. For if σ is the specific resistance, at 0° C., m the maximum chemical valency, V the atomic volume, and T the absolute melting-point, to a first approximation $\sigma = \frac{\mathrm{K}m\mathrm{V}}{\mathrm{T}}$, where K is a quantity which appears to have different values for different *groups* of metals. Dr. Fleming has already pointed out that the metals of large specific resistance are those having large atomic volumes and large valencies. For silver, copper, magnesium, zinc, cadmium, thallium, lead, mercury (in the solid state), and probably calcium, $\mathrm{K} = 1{\cdot}72 \times 10^2$ if σ is measured in microhms for a centimetre cube, and V is the quotient of the atomic weight by the density. For gold, indium, tin, and aluminium, however, K has only half this value. For sodium and potassium also its value appears to be half that given above; while for arsenic, antimony, and bismuth it is about double.

In the table that follows, σ is the value calculated for the specific resistance at 0° C. from $\frac{\mathrm{K}m\mathrm{V}}{\mathrm{T}}$, while σ' is the value obtained by observation. The maximum chemical valency is as given by the Periodic Law, except in the case of gold and copper, where the values 3 and 2 are taken as being more consistent with the chemical characteristics of the metals.

	m.	V.	T.	σ.	σ'.	Diff.
Silver	1	10·2	1173	1·5	1·5	0 p. cent.
Copper	2	7·1	1333	1·82	1·7	7 ,,
Magnesium	2	13·9	1023	4·6	4·1	12 ,,
Zinc	2	9·1	676	4·65	5·8	20 ,,
Cadmium	2	12·9	593	7·5	7·5	0 ,,
Thallium	3	17·2	561	15·9	17·6	9·6 ,,
Lead	4	18·1	605	20·6	20·0	3 ,,

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[*] Communicated by the Physical Society: read March 14, 1902.

The resistance of a metal is approximately proportional to the absolute temperature. Hence $\sigma = \frac{K}{273} \cdot \frac{mV\theta}{T}$, where θ is the absolute temperature, so that the specific resistance in the solid state at the temperature of fusion is $\frac{KmV}{273}$. Now, the resistance of pure mercury in the solid state has been very carefully investigated by Dr. Fleming (Proc. Royal Society, vol. lx. No. 359, p. 79). According to his results, the process of fusion appears to set in at $-50°$ C., and is not properly completed until $-36°$ C. is reached, the value of σ' at $-50°$ C. just before fusion begins being 21. By calculation, taking m to be 2, and $V=14{\cdot}1$, we get 17·8. Again, at $-95°$ C. ($-100°$ C. on the *pt* scale), the calculated value is 14·4, Dr. Fleming's value being 15·5.

The melting-point of calcium is not definitely given. The metal, however, is said to melt at "red heat," and will therefore have a melting-point between 500° C. and 600° C., so that T is about 853. Hence, m being 2 and V 25·4, $\sigma=10{\cdot}6$, Matthiessen's value being 7·5 *.

Taken as a whole, the accordance between σ and σ' is not unsatisfactory, being about the same as the accordance observed in the case of the law of Dulong and Petit as to atomic heat. Thus, taking 6·3 for the constant of that law, the value is 15 per cent. above it in the case of calcium, and 20 per cent. below it in the case of sulphur. Also, considering the wide variations that occur in the values of m, V, T, and σ, the accordance cannot be regarded as a matter of accident, but must be taken as an indication of a definite physical law.

Certain metals, however, fail to conform with the relation for σ although they are closely related, both physically and chemically, to those that do. Thus gold, indium, tin, and aluminium fail, although the first is allied to copper and silver, the second to thallium, and the third to lead. Thus:—

	m.	V.	T.	σ.	σ'.
Gold	3	10·1	1310	3·99	2·0
Indium	3	15·3	449	17·64	8·4
Tin	4	16·3	503	22·05	10·5
Aluminium	3	10·6	923	5·94	2·9

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* The melting-point of calcium is probably underestimated, 800° C., or 1073 absolute being perhaps more nearly correct.

The metals of the alkali and arsenic groups also fail. Thus:—

	m.	V.	T.	σ.	σ'.
Potassium	1	45·4	335	23·4	8·4
Sodium	1	23·7	369	11·1	5·1
Arsenic	5	13·2	773	14·8	35·1
Antimony	5	17·9	710	21·7	43·1*
Bismuth	5	21·1	538	33·9	108·0

According to Pictet, if α is the coefficient of thermal expansibility, $V^{\frac{1}{3}}\alpha T$ is a constant for metals. Calling this constant c, we get the following for its values in the case of the metals discussed above:—

	V.	*a.*	T.	*c.*
Silver	10·2	·0000194	1173	·0486
Copper	7·1	·0000170	1333	·0436
Gold	10·1	·0000147	1310	·0417
Magnesium	13·9	·0000270	1023	·0664
Zinc	9·1	·0000298	676	·0421
Cadmium	12·9	·0000316	593	·0440
Tin	16·3	·0000230	503	·0293
Thallium	17·2	·0000302	561	·0437
Aluminium	10·6	·0000222	923	·0450
Indium	15·3	·0000417	449	·0465
Lead	18·1	·0000290	605	·0461

Excluding magnesium and tin, the probable value of the constant is ·045.

The alkali metals appear to constitute a separate group having a different value for c, thus,

	V.	*a.*	T.	*c.*
Sodium	23·7	·000072	369	·0763
Potassium	45·4	·000083	335	·0992

The mean value for c being ·088, or about double its value for copper, &c. The arsenic metals, or at least bismuth and antimony, also appear to constitute a separate group, having $c=$ ·021 or half its value for copper, &c. Thus,

	V.	*a.*	T.	*c.*
Antimony	17·9	·0000116	710	·0215
Bismuth	21·1	·0000137	538	·0204

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* For antimony, $\sigma'=$ 45·8 according to Lorenz, and 40·3 according to Berget.

On account of the irregularities of the metal, the result in the case of arsenic cannot be taken as definite *.

Now, the groups that are thus characterized by different values for c are also characterized by corresponding values for K. Thus, where $c = {\cdot}045$, K is $1{\cdot}72 \times 10^2$. Where c is approximately $(\frac{1}{2} \times {\cdot}045)$, K is approximately $2 \times 1{\cdot}72 \times 10^2$, and where c is approximately $(2 \times {\cdot}045)$, K is approximately $(\frac{1}{2} \times 1{\cdot}72 \times 10^2)$. This suggests that the relation for σ can be rendered more comprehensive if written $\sigma = \frac{\mathrm{A}}{273} \cdot \frac{m\mathrm{V}\theta}{c\mathrm{T}}$, where A is a constant. Thus modified, the relation gives the following values for σ:—

	Observed.	Calculated.
Silver	1·5	1·5
Copper	1·7	2·0
Magnesium	4·1	3·4
Sodium	5·1	7·1
Zinc	5·8	5·36
Cadmium	7·5	7·95
Potassium	8·4	11·5
Thallium	17·6	17·7
Lead	20·0	21·7
Arsenic	35·1	34·3
Antimony	43·1	50·2
Bismuth	108·0	80·1

Except in the case of sodium, the order in which the metals stand is the same as regards the observed and calculated values of σ; and although the differences are great in the case of some of the metals, the values are throughout of the proper order of magnitude. The modified relation fails to hold, however, for gold, indium, tin, and aluminium, so that these metals, as before, are exceptional. Whether it holds for calcium and mercury cannot be ascertained until the value of c can be obtained for those metals. If they follow Pictet's law, so that c lies between ·04 and ·05, they will conform with the relation in its modified form to the same degree of approximation as they do in the case of the original one.

If m, in the relation for σ, is to be the maximum valency according to the Periodic Law, the relation will approximately hold for platinum and palladium. Thus:

	V.	m.	T.	c.	σ.	σ'.	Diff.
Palladium	9·2	8	1775	·0411	8·44	10·2	17 p. cent.
Platinum	9·1	8	2050	·0379	7·82	9·0	13 ,,

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* The metal sublimes when heated and has a definite melting-point only under pressure.

The metals of the platinum and palladium groups have a valency of 4 in their sesqui- and tetra-chlorides. Ruthenium and osmium, however, form potassium compounds analogous to chromates, manganates, and ferrates, in which the valency must be 6. In addition, they form tetroxides RuO_4 and OsO_4, indicating a valency of 8 in accordance with the Periodic Law, a valency that also accords with the potassium compounds K_2PtCl_6, and K_2PdCl_6. It therefore appears that a maximum valency of 8 is not incompatible with the chemical characteristics of the metals.

The metals of the Iron Group do not conform with the relation for σ, a matter to be fully dealt with on a future occasion.

According to the law of Dulong and Petit, s is approximately $\frac{6{\cdot}3}{w}$, where s is the specific heat and w the atomic weight. Also, $V=\frac{w}{d}$, where d is the density. Hence $V=\frac{6{\cdot}3}{sd}$. Again, if α is the coefficient of thermal expansibility, supposed to be constant, and if $\lambda=V^{\frac{1}{3}}\alpha\theta$ and $c=V^{\frac{1}{3}}\alpha T$, we get

$$\sigma=\frac{A}{273}\cdot\frac{mV\theta}{cT}=\frac{Bm\lambda}{c^2sd}=\frac{Bm\lambda}{c^2n\rho}=\frac{Bm\lambda\theta}{c^2nh}.$$

where B is a constant, n the average number of atoms per unit volume, ρ the atomic specific heat, and h or $(\rho\theta)$ the average thermal energy of an atom at temperature θ. Now, V is proportional to the average space assigned to an atom at the absolute zero. Hence λ or $V^{\frac{1}{3}}\alpha\theta$ will be proportional to the average relative displacement of the centres of two neighbouring atoms between absolute zero and the temperature θ, which we may call the average atomic displacement at temperature θ.

If α is constant, so that λ for any metal is proportional to θ, and if ρ is also constant, α must be proportional to θ, so that $\frac{\sigma}{\sigma_0}=\frac{\theta}{273}=1+\epsilon t$, where σ and σ_0 are the values at t° C. and 0° C. respectively, and $\epsilon={\cdot}003665$. According to this, the temperature resistance-coefficients of all pure metals must be the same, namely $\frac{1}{273}$, and the resistance must vanish at the absolute zero. The investigations of Fleming and Dewar, however, have shown that while the resistance tends to vanish at the absolute zero, the temperature-coefficient is usually higher than $\frac{1}{273}$, its average value being about ·004. Now we know that α and ρ are not constant, but slowly increase with rise of temperature. Hence the view suggests itself that the deviation of the temperature-coefficients from the simple value $\frac{1}{273}$ may be due to the temperature-variations of α and ρ.

Since the relation $\sigma=\sigma_0(1+\epsilon t)$ is true to a first approximation for all pure metals, we can put

$$\frac{\sigma}{\sigma_0}=\frac{\theta}{273}\phi(\theta)=(1+\epsilon t)(1+\gamma t+\eta t^2+\ldots),$$

which suggests that temperature enters into the relation for σ in two ways essentially different, namely:

(1) Directly, as an essential factor, and represented above by $\frac{\theta}{273}$; and

(2) Indirectly, and represented above by $\phi(\theta)$, as the temperature-variation of one or more of the other essential factors.

We have then to inquire whether these essential factors can be α and ρ, and whether $\phi(\theta)$ can be built up from their temperature-variations.

According to Prof. J. J. Thomson ('Nature,' May 10th, 1900), the specific resistance of a metal on the electronic theory is given by

$$\sigma=\frac{2\mu u^2}{qe^2l^2},$$

where q is the average number of electrons dissociated in unit volume per unit time, μ is the mass of an electron, e its electrical charge, u its average velocity of translation, and l the length of its mean free path. Hence, since μ, e, and l are constant as regards temperature, the temperature-variation of σ must depend upon the temperature-variations of u and q. On this matter, Prof. J. J. Thomson makes the following observation:—"If we suppose that u, the mean velocity of translation of the corpuscles, varies with the temperature in the same way as the velocity of translation of the molecules of a gas, μu^2 would be proportional to the absolute temperature, and the specific resistance would, considered as a function of the absolute temperature θ, vary as $\frac{\theta}{q}$; if q, the amount of ionization, increases as the temperature increases, the resistance will vary more slowly than the absolute temperature; if q diminishes as the temperature increases, the resistance will vary more rapidly than the temperature." Hence, according to Prof. J. J. Thomson, in the temperature relation $\frac{\sigma}{\sigma_0}=\frac{\theta}{273}\phi(\theta)$, the factor $\frac{\theta}{273}$ enters in the determination of the average velocity of translation of the moving electrons, while $\phi(\theta)$ enters as the temperature-variation of the electronization-coefficient q.

But $\sigma \propto \frac{m\lambda\theta}{c^2nh}$. Hence $\frac{\sigma}{\sigma_0} = \frac{\theta}{273} \cdot \frac{\lambda h_0}{\lambda_0 h}$, so that $\phi(\theta) = \frac{\lambda h_0}{\lambda_0 h}$, and therefore, since according to Prof. J. J. Thomson $\phi(\theta) = \frac{q_0}{q}$, we must have $q \propto \frac{h}{\lambda}$. This result, taken in conjunction with $\sigma \propto \frac{m\lambda\theta}{c^2nh}$, suggests that $q \propto \frac{nh}{m\lambda}$. That is, that for the metals that conform with the relation for σ, the electronization per unit volume increases with the average thermal energy of the atom and the number of atoms per unit volume, and decreases with the average relative displacements of the atoms and their maximum chemical valency.

The investigations of Fleming and Dewar have shown that $\phi(\theta)$ is not a linear function of θ. If, however, we confine ourselves to a limited range, say from 0° C. to 100° C., we may consider it to be approximately linear, and so considerably simplify the matter. In that case we get $\sigma = \sigma_0(1+\epsilon t)(1+\gamma t) = \sigma_0(1+at+bt^2)$, where $a = (\epsilon+\gamma)$, and $b = \epsilon\gamma$. As might be expected, this temperature-factor fails to hold both at very high and at very low temperatures, but may be taken to hold with a considerable degree of accuracy between 0° and 100°, the only range within which the temperature-variations of α and ρ can be said to be fairly accurately known.

Let α and s be linear functions of the temperature, which is sufficiently accurate for our present purpose, and let $\alpha = \alpha_0(1+\beta t)$ and $s = s_0(1+\delta t)$, where α and s are the values at $t°$ C., α_0 and s_0 at 0° C., and β and δ the temperature-variations. Then

$$\lambda = \mathrm{V}^{\frac{1}{3}}\alpha_0\left[\left(1-\frac{273\beta}{2}\right)+\tfrac{1}{2}\beta t\right](273+t),$$

$$h = \rho_0\left[\left(1-\frac{273\delta}{2}\right)+\tfrac{1}{2}\delta t\right](273+t)\ ;$$

and therefore,

$$\phi(\theta) = \frac{h_0\lambda}{h\lambda_0} = \frac{1+\frac{1}{2}\beta' t}{1+\frac{1}{2}\delta' t}$$

$$= 1+\tfrac{1}{2}(\beta'-\delta')t - \tfrac{1}{4}\delta'(\beta'-\delta')t^2 + \&c.,$$

where $\beta' = \dfrac{\beta}{1-\dfrac{273\beta}{2}}$, and $\delta' = \dfrac{\delta}{1-\dfrac{273\delta}{2}}$.

Thus $\gamma = \frac{1}{2}(\beta' - \delta')$, $\eta = -\frac{1}{4}\delta'(\beta' - \delta')$, &c., and in the approximate temperature-factor $\frac{\sigma}{\sigma_0} = 1 + at + bt^2$, which holds between 0° and 100°, $a = \epsilon + \frac{1}{2}(\beta' - \delta')$ and $b = \frac{1}{2}\epsilon(\beta' - \delta')$.

This result, however, cannot at present be subjected to any reliable and decisive test. For the accurate investigation of the temperature-variation of α over wide ranges of temperature has not been made for pure metals, while, though various observers have investigated the temperature-variations of s, there is a considerable want of accordance between the results obtained. The accurate determination of both thermal expansibility and specific heat are matters of very great difficulty, trifling differences in the values of either quantity producing very serious differences in the values of the temperature-variations deduced from them. Thus, according to Naccari, the specific heat of copper at 100° C. is ·09422, and ·09634 at 200° C. Assuming the latter value to be only $\frac{1}{2}$ per cent. too high, the value at 200° C. would be ·09586. In the former case the temperature-variation is ·0000212, in the latter ·000016; so that a difference of only $\frac{1}{2}$ per cent. per 100° in the value of s produces a difference of 30 per cent. in the value of the temperature-variation. Now, apart altogether from experimental difficulties, which are certainly very considerable, differences greater than this can arise in the values of s at the same temperature of different specimens of the same metal from differences in mechanical qualities alone. Unless, therefore, the values of σ, α, and s are accurately determined *for the same specimen and over a considerable range of temperature*, we cannot expect a close and definite relation to exist between their temperature-variations. Hence, as the experimental information has to be collected from different observers for different specimens, probably in different mechanical conditions, we must be satisfied if the relation $\gamma = \frac{1}{2}(\beta' - \delta')$ yields values of the proper order of magnitude, differing from those obtained experimentally by no more than those values for the same metals according to different observers differ from one another.

In spite, however, of the imperfections of the experimental data available to test the matter, enough appears to be known to show that $\phi(\theta)$, for many of the metals, is *mainly* determined by β and δ, that the probable form of the function is $\frac{1 + \frac{1}{2}\beta' t}{1 + \frac{1}{2}\delta' t}$, and that $\gamma = \frac{1}{2}(\beta' - \delta')$, as the following results will show.

In selecting experimental data respecting β and δ, the results of Fizeau by the interferometer method have been taken for α between 0° and 100°; these being taken because they depend upon measurements susceptible of more accurate determination than those of other methods.

As regards specific heats, the results of Naccari (1888) have principally been followed.

Naccari's values for s are as follows—the values at 0° C. being calculated from the values given at 100° C., 200° C., and 300° C.:—

	0°.	100°.	200°.	300°.	δ.
Copper	·0921	·09422	·09634	·09846	·0002302
Silver........................	·05449	·05663	·05877	·06091	·0003927
Cadmium	·0546	·0570	·0594	·0617	·0004385
Lead	·02972	·03108	·03244	·03380	·0004575
Zinc	·0906	·0951	·0996	·1040	·0004967
Aluminium	·2116	·2211	·2306	·2409	·0004489

Naccari's results for copper are not in accord with those of Bedè and Trowbridge. Thus, according to Bedè, the mean value of s between 15° and 100° is ·09331; ·09483 between 16° and 172°; and ·09680 between 17° and 247°. Hence, the average value of δ is ·000516. Again, according to Trowbridge ('Science Abstracts,' Dec. 1898), the mean value of s between $-181°$ and 13° is ·0868, and ·094 between 23° and 100°. Hence the average value of δ is ·000546.

Naccari gives no results in the case of tin. Lorenz gives the following: s at $0° = ·05368$; at $50° = ·05534$; at $75° = ·05643$. Hence the average value of δ is ·00064.

Naccari's results for aluminium are confirmed by those of Richards. Thus, according to Richards, the mean value of s between 0° and 100° is ·2270; ·2370 between 0° and 300°; and ·2520 between 0° and 600°. Hence the average value of δ is ·000463.

Fizeau's values for α are as follows—α_{40} denoting the value at 40° C., and α_{50} that at 50° C.:—

	α_{40}. ·0000	α_{50}. ·0000	β.
Copper	1678	1698	·001251
Silver........................	1921	1936	·000806
Cadmium	3069	3102	·001124
Lead	2924	2948	·000848
Zinc	2918	2915	
Aluminium	2313	2336	·001036
Tin............................	2234	2269	·001671

According to Fizeau, zinc behaves in an anomalous manner as regards its thermal expansibility, the value of α being greater at 40° than at 50° C. This, however, is not in accord with the observations of Matthiessen, according to whose results for zinc, $l=l_0[1+\cdot00002741t+\cdot0000000234t^2]$, where l and l_0 are the lengths of the same bar at t° and 0° respectively. Hence $\beta=\cdot00172$. For most of the metals, however, for which both Matthiessen and Fizeau give values, Fizeau's values for β are about half those deduced from Matthiessen's observations. Thus:

	Matthiessen.	Fizeau.	Ratio.
Copper	·0025	·00125	2·0
Silver	·00149	·000806	1·85
Cadmium	·00354	·001124	3·15
Tin	·00259	·001671	1·55
Lead	·000543	·000848	·64
Palladium	·00184	·00088	2·09
Gold	·00165	·000567	2·91
Platinum	·000823	·000923	·89
		Mean =	1·89

Fizeau's values for copper and aluminium are confirmed by those of Dulong and Petit, and Mr. Tutton. Thus, according to Dulong and Petit, the mean value of α for copper between 0° and 100° C is ·00001718, and ·00001883 between 0° and 300°. Hence $\beta=\cdot00101$. Again, according to Mr. Tutton (Proc. Royal Society, lxiii. p. 208), the value of α for pure aluminium is $10^{-8}[2204+2\cdot12\,t]$, so that $\beta=\cdot000962$. Hence, perhaps, until more decisive determinations are made in the case of pure zinc, by Mr. Tutton's improved interferometer method for instance, and as the molecular constitution of the metal is simple like that of copper and silver, we may take for β half the value deduced from Matthiessen's observations as in the case of most of the metals tabulated above.

From the values of β and δ given above, we get the following results:—

	β'.	δ'.	$\frac{1}{2}(\beta'-\delta')$.	a.	b.
	·00	·000	·000	·00	·00000
Copper	1509	590	460	4125	167
Silver	0906	393	245	3910	090
Tin	2163	700	731	4396	270
Cadmium	1327	468	431	4096	158
Lead	0959	488	236	3901	087
Zinc	0966	533	217	3882	080
Aluminium	1206	478	364	4029	133

For magnesium β appears to be abnormally high. Thus, according to Fizeau, the value of α at 40° is ·00002694, and ·00002762 at 50°, so that $\beta = {\cdot}00281$. We must therefore conclude, either that magnesium does not conform with the relation for γ, or that Fizeau's value for β is not characteristic of the metal itself, but of a particular specimen only. For the remaining metals, with the exception of those afterwards to be mentioned, data cannot be obtained respecting δ and β.

According to the temperature relation $\frac{\sigma}{\sigma_0} = 1 + [\epsilon + \frac{1}{2}(\beta' - \delta')t] + \frac{1}{2}\epsilon(\beta' - \delta')t^2$, since in the case of the metals tabulated above δ' is less than β', the locus of u or $\frac{\sigma}{\sigma_0}$ must be concave upwards, and the value of $\frac{du}{dt}$ must increase as the temperature rises. This is in accord with the results of observation, as the following table will show, where $\frac{du}{dt}$ and its mean value between 0° and 100° are tabulated. For comparison, the values deduced from Fleming and Dewar's observations,

		$\frac{du}{dt}$.	Mean.
Tin	T	·00440 + ·0000054 t	·00467
	F	·00425 + ·0000040 t	·00445
	B	·00403 + ·0000117 t	·00461
Copper	T	·00413 + ·0000033 t	·00429
	F	·00412 + ·0000033 t	·00428
	E	·00412 + ·0000060 t	·00442
Cadmium	T	·00410 + ·0000032 t	·00425
	F	·00405 + ·0000028 t	·00419
	V	·00402 + ·0000018 t	·00412
Lead	T	·00390 + ·0000017 t	·00399
	F	·00399 + ·0000024 t	·00411
	B	·00395 + ·0000028 t	·00409
Zinc	T	·00388 + ·0000016 t	·00396
	F	·00395 + ·0000020 t	·00405
Silver	T	·00391 + ·0000018 t	·00400
	F	·00391 + ·0000018 t	·00400
	B	·00397 + ·0000014 t	·00404
Aluminium	T	·00403 + ·0000027 t	·00416
	F	·00408 + ·0000030 t	·00423
	B	·00388 + ·0000026 t	·00401

denoted by F, those given by Benoit denoted by B, and those given by Vicentini and Omodei denoted by V, have also been inserted. The values calculated from Fleming and Dewar's observations have been obtained by applying to them the approximate relation $\frac{\sigma}{\sigma_0}=(1+\epsilon t)(1+\gamma t)$ between 0° and 100°.

T denotes the values obtained from thermal data, δ and β, while E, in the case of copper, denotes the values given by Elmore.

In the case of the metals tabulated above, it will be found that not only is γ approximately $\frac{1}{2}(\beta'-\delta')$ and a approximately $\epsilon+\frac{1}{2}(\beta'-\delta')$, but that γ is also approximately $\frac{1}{3}\beta$ or $\frac{1}{3}\beta'$, and a approximately $\epsilon+\frac{1}{3}\beta$, or $\epsilon+\frac{1}{3}\beta'$, as the table opposite (p. 527) will show.

In the table the column headed γ contains the values calculated for that constant from Fleming and Dewar's observations; the first column headed Diff. indicates the percentage difference of $\frac{3\gamma}{\beta}$ from its mean value 1·046, while the second is the percentage difference of $\frac{3\gamma}{\beta'}$ from its mean value ·896.

From the table it will be seen that the order in which the metals stand as regards the values of γ is, with the exception of aluminium, the same as that as regards the values of $\frac{1}{3}\beta$ and $\frac{1}{3}\beta'$; that the ratios $\frac{3\gamma}{\beta}$ and $\frac{3\gamma}{\beta'}$ are approximately constant; and that the values deduced for a from γ, namely $\epsilon+\gamma$, are practically the same as those deduced from β and β', namely $\epsilon+\frac{1}{3}\beta$, and $\epsilon+\frac{1}{3}\beta'$.

Hence, since for these metals γ is approximately $\frac{1}{2}(\beta'-\delta')$, and also approximately $\frac{1}{3}\beta$ or $\frac{1}{3}\beta'$, it follows that δ' must be approximately $\frac{1}{3}\beta'$, and δ approximately $\frac{1}{3}\beta$. That is, that the temperature-variation of the specific heat must depend upon that of the thermal expansibility, or, more correctly perhaps, that *the temperature-variation of the specific heat and that of the thermal expansibility must depend upon the temperature-variation of the same physical property of the metal.* This result, again, on account of uncertainties respecting the experimental data bearing upon it, cannot be accepted without considerable doubt and hesitation. I have only ventured to put it forward because it appears to have enough probability in its favour, and to be sufficiently interesting and suggestive to deserve a fuller consideration, and to have a special investigation into the experimental facts bearing upon it made by competent observers.

Metal.	γ. ·000	$\frac{1}{3}\beta$. ·000	$\frac{1}{3}\beta'$. ·000	$\frac{3\gamma}{\beta}$.	Diff. Per cent.	$\frac{3\gamma}{\beta'}$.	Diff. Per cent.	$\epsilon+\gamma$. ·00	$\epsilon+\frac{1}{3}\beta$. ·00	$\epsilon+\frac{1}{3}\beta'$. ·00
Tin	538	557	721	·97	7·3	·75	16·3	4203	4222	4386
Copper	450	417	503	1·08	3·2	·9	·5	4115	4082	4168
Cadmium ..	384	375	443	1·02	2·5	·87	2·8	4049	4040	4108
Lead	326	283	320	1·15	10·0	1·02	13·8	3991	3948	3985
Zinc	289	283	320	1·02	2·5	·9	·5	3954	3948	3985
Silver	245	269	302	·89	14·5	·81	9·6	3910	3934	3967
Aluminium ..	412	345	403	1·19	13·8	1·02	13·8	4077	4010	4070

If ϵ is the average increase of s per unit rise of temperature, and if $\delta = \frac{1}{3}\beta$, as suggested above, then, approximately, $\epsilon = \frac{1}{3}(s_0\beta) = \frac{2 \cdot 1\beta}{w}$, where w is the atomic weight. The following are some of the results obtained for ϵ from this relation:—

	β. ·00	w.	ϵ (*calc.*) ·0000	ϵ (*obs.*) ·0000	
Aluminium	1036	27	800	950	Naccari.
				1000	Richards.
Copper	1251	63·3	410	460	Bède.
				490	Trowbridge.
Tin	1671	119	295	341	Lorenz.
Cadmium	1124	112	210	240	Naccari.
Palladium	0880	106	175	210	Violle.
Silver	0806	108	160	210	Naccari.
Iridium	1198	192·5	131	151	Violle.
Platinum	0923	195	100	112	Violle.

The relation does not hold for the metals of the Iron group. Also, for many of the metals experimental results cannot be obtained to test the matter, while for metals like lead, magnesium, and zinc, the results available are not conclusive.

According to the relation $\sigma \propto \frac{\theta}{q}$, since θ is constant during fusion, the change of resistance that accompanies fusion must depend upon the change in the value of q*. Hence, if σ_s is the value of σ in the solid state just before fusion has begun, and σ_e its value in the liquid state just when fusion is completed, and if q_s and q_e be the corresponding values of q, $\frac{\sigma_e}{\sigma_s} = \frac{q_s}{q_e}$. For copper, silver, magnesium, zinc, cadmium, calcium, thallium, lead, and mercury, $q \propto \frac{nh}{m\lambda}$. If we assume that h in this relation is to stand for the *internal*, and not for the *total* energy of the atom, it will be the same in the solid and liquid states at the temperature of fusion. Hence, if λ_e, λ_s be the values of λ corresponding to σ_e, σ_s, $\frac{\sigma_e}{\sigma_s} = \frac{\lambda_e}{\lambda_s}$†.

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* Which implies that the average velocity of translation u of the moving electrons is not affected by the act of fusion, but remains unaltered like the temperature.

† This assumption implies that the electronization per unit volume, so far as it depends upon temperature, is determined:—

(1) By the kinetic energy of an atom considered as a system of electrons in orbital motion,—this being unaffected by fusion, since the temperature of the atom is not altered.

(2) By the average distance apart of the atomic centres,—which is directly affected by fusion.

The ratio $\frac{\lambda_e}{\lambda_s}$ can be calculated from the increase of volume that accompanies the fusion. Thus, if d is the average distance between the centres of two neighbouring atoms at the absolute zero, $d+\lambda_e$ and $d+\lambda_s$ will be its values in the liquid and solid states respectively at the temperature of fusion. Also, if ρ_e and ρ_s be the corresponding densities, and if $\left(\frac{\rho_s}{\rho_e}\right)^{\frac{1}{3}}$ or $\frac{d+\lambda_e}{d+\lambda_s}$ be denoted by x, we get

$$\lambda_e=(d+\lambda_s)\,x-d.$$

Now the only cases where reliable information respecting ρ_s, ρ_e, σ_s, and σ_e can at present be obtained, are those of lead, cadmium, tin, and zinc, and in the case of these metals, since the melting-point is low, the average value of α between 0° and 100° is probably not far from its value between absolute zero and the melting-point T, so that very approximately $\lambda_s=d\alpha\mathrm{T}$. Hence

$$\frac{\lambda_e}{\lambda_s}=\frac{(1+\alpha\mathrm{T})x-1}{\alpha\mathrm{T}},$$

the ratio being thus independent of the actual value of d. In order, however, to compare the value of λ_e with the constant of Pictet's law, namely $\mathrm{V}^{\frac{1}{3}}\alpha\mathrm{T}$, we shall put $d=\mathrm{V}^{\frac{1}{3}}$.

The increase of volume that accompanies fusion has been specially investigated for several metals by Vicentini and Omodei, their results being taken in the table that follows except in the case of zinc.

	ρ_s.	ρ_e.	λ_e.	.	$\frac{\lambda_e}{\lambda_s}$.	$\frac{\sigma_e}{\sigma_s}$.	Diff.
Lead	11·005	10·645	·076	·046	1·66	1·9	+12 p. cent.
Cadmium ...	8·366	7·989	·085	·044	1·93	2·3	+16 ,,
Tin	7·183	6·988	·054	·029	1·86	2·1	+12 ,,
Zinc	......	6·480	·087	·042	2·07	2·0	− 4 ,,

Vassura's values for $\frac{\sigma_e}{\sigma_s}$ have been taken in the case of cadmium and tin, and De la Rive's in the case of lead and zinc.

Omadei gives no values for ρ_s and ρ_e in the case of zinc. According to Roberts-Austen, however, the density of zinc just when fusion is completed is 6·480, from which value $\frac{\lambda_e}{\lambda}$ can be calculated.

Taking the results in the table as experimentally valid, it is seen that to a rough approximation $\frac{\sigma_e}{\sigma_s}=\frac{\lambda_e}{\lambda_s}$. The determination of $\frac{\lambda_e}{\lambda_s}$, however, is a matter of difficulty, the results obtained above from $\frac{\rho_s}{\rho_e}$ being undoubtedly imperfect and unreliable. Thus, Omadei's value for the percentage increase of volume of thallium in the act of fusion is 4·32, whereas, according to Toepler, it ought to be 3·22. The former of these values gives $\lambda_e=\cdot 081$ and $\frac{\lambda_e}{\lambda_s}=1\cdot 84$, whereas the latter gives $\lambda_e=\cdot 072$, and $\frac{\lambda_e}{\lambda_s}=1\cdot 64$. Moreover, it is not probable that the increase of resistance that accompanies fusion is determined by the increase in λ *alone*, though possibly λ may be the most important factor concerned in the matter. At any rate it is worthy of note that, whereas in the case of metals which expand on melting the specific resistance is greater in the liquid than in the solid state at the temperature of fusion, in the case of bismuth and antimony, which contract on melting, it is less.

Platinum and palladium differ from the metals already discussed, as regards their temperature-coefficients, in the fact that for them the value of a is less than $\frac{1}{273}$, and the locus of σ concave towards the axis of temperature, whereas, in the case of the other and apparently simpler metals, a is greater than $\frac{1}{273}$ and the locus of σ convex towards that axis. This anomaly indicates that for platinum and palladium, the temperature-variation of the electronization-coefficient q cannot be accounted for by the temperature-variations of λ and h, as in the case of the simpler metals. Some other factor has to be superposed upon these in order to convert into an increase the decrease that would arise in q from them alone. The thermoelectric curves of platinum and palladium are also exceptional as compared to those of the simpler metals. For whereas the curves, as given by Fleming and Dewar, have a positive curvature for the simpler metals, for platinum and palladium the curvature is decidedly negative. These exceptional characteristics are probably closely related, since on the electronic theory both the electrical resistance and the thermoelectric properties of a metal depend upon the electronization-coefficient q.

As might be expected from their ferro-magnetic properties, iron, nickel, and cobalt, as regards their temperature-coefficients, constitute an exceptional and highly characteristic group, the discussion of which must be reserved for a future occasion.

The results discussed in the paper may be summarized thus :—

(1) A relation $\sigma = \frac{\mathrm{K}m\mathrm{V}\theta}{c\mathrm{T}}$ which roughly holds for

Silver,	Platinum,
Copper,	Palladium,
Magnesium,	Thallium,
Sodium,	Lead,
Zinc,	Mercury,
Cadmium,	Arsenic,
Calcium,	Antimony,
Potassium,	Bismuth ;

but fails for gold, indium, tin, and aluminium. It also fails, as we should expect, for iron, nickel, and cobalt—while in the case of the remaining metals data cannot be obtained to test the matter.

(2) A temperature factor

$$\frac{\sigma}{\sigma_0} = (1+\epsilon t)\left[1+\frac{1}{2}(\beta'-\delta')\,t+\ldots\right]$$

obtained from the relation $q \propto \frac{nh}{m\lambda}$, approximately true for tin, copper, cadmium, lead, zinc, silver, and aluminium, but not for platinum and palladium, and of course the metals of the Iron group.

(3) A specific heat relation $\epsilon = \frac{2{\cdot}1\beta}{w}$, approximately true for

Aluminium,
Copper,
Tin,
Cadmium,
Palladium,
Silver,
Iridium,
Platinum ;

but not, of course, for the metals of the Iron group. The results are uncertain in the case of lead, magnesium, and zinc; while in the case of the remaining metals, the relation cannot at present be tested.

(4) A relation $\frac{\sigma_e}{\sigma_s}=\frac{\lambda_e}{\lambda_s}$ at the temperature of fusion, roughly true for lead, cadmium, tin, and zinc, the only metals for which, at present, it can be tested.

The results, however, in many respects are incomplete and unsatisfactory, as of course is to be expected from the imperfect character of their experimental basis. Perhaps, however, the paper—in spite of its imperfections, may prove useful as a statement of what may be done to correlate expansion-coefficients, specific heats, and electrical resistances, and that it may do good by calling attention to the matter, and stimulating experienced observers to take up its systematic investigation.

LXI. *The Magnetic Elements at the Cape of Good Hope from* 1605 *to* 1900. *By* J. C. BEATTIE, *D.Sc., and* J. T. MORRISON, *M.A., B.Sc.*[*]

§ 1. THE first recorded observation of declination at the Cape was made in 1605, the first measurement of intensity about 1841, the first observation of inclination or dip in 1751. Other observations have been made at irregular intervals till 1840. In 1841 a detachment of R.A. began observations at the Royal Observatory, Capetown, and carried them on till 1846. This observational work was continued at the permanent magnetical observatory established in connexion with the Royal Observatory of the Cape of Good Hope; the magnetic work seems to have been discontinued after 1869.

In recent years observations have been taken by the magnetic observers attached to various expeditions. In addition to these, yearly observations are taken at the Royal Observatory, Capetown, by Messrs. Beattie and Morrison with a set of field-instruments (Kew pattern) obtained by Sir David Gill in 1894.

§ 2. *Inclination.*

The following is a list of the determinations of inclination at the Cape.

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[*] Communicated by the Authors. Abstract of paper communicated to the South African Philosophical Society in 1901.

Date.	Observers.	Dip. ° '	Authorities.	Remarks.
1751.	La Caille	−43 0	From Sabine's 'Magnetical and Meteorological Observations at the Cape of Good Hope,' Vol. I. Magnetism.	
1770.	Ekeberg..............	−44 25		
1774.	Bayley	−45 37		
1774.	Ekeberg..............	−44 29		
1775.	Wales	−45 19		
1775.	Abercrombie.........	−46 21		
1776.	Bayley	−46 31		
1780.	Bayley	−46 46		
1791.	Vancouver............	−48 30		
1792.	Dentrecasteaux......	−47 25		
1818.	Freycinet	−50 47		
1836.	Fitzroy	−52 35		
1839.	Du Petit Thouars.	−53 06		
1840.	Ross	−53 08		
1841.	R.A. Detachment...	−53 09		The results for 1841 to 1846 inclusive are the mean of observations taken as a rule twice weekly, four hours before and four hours after noon.
1842.	R.A. ,, ...	−53 12		
1843.	R.A. ,, ...	−53 19		
1844.	R.A. ,, ...	−53 36		
1845.	R.A. ,, ...	−53 31		
1846.	R.A. ,, ...	−53 33		
1847.	....	−53 41	Admiralty hydrographic records.	
1848.	....	−53 47		
1849.	....	−53 52		
1850.	....	−53 58		
Feb. 6, 1851.	Sir Thomas Maclear.	−54 02	Sabine.	
1852.	....	−54 04	Dip-book Royal Observ. C. of G. Hope.	The results for 1852–1857 inclusive are the mean of observations taken twice weekly, morning and afternoon.
1853.	....	−54 09		
1854.	....	−54 19·6		
1855.	....	−54 24·5		
1856..	....	−54 23·9		
Jan. 1857.	....	−54 23·0		
Oct. 1857.	....	−54 36·4	Reise der Novara.	
March 1858.	....	−54 29·3	Dip-book R. O., C. of G. Hope.	
Aug. 1871.	Stone	−55 45·4		
Sept. 1871.	Stone	−55 34·9		
Nov. 1873.	....	−55 56·3	'Challenger' Report, Narrative ii..	
Sept. 30, 1874.	....	−56 6·0	Voyage of the 'Gazelle.'	
Jan. 31, 1890.	Preston	−57 15·2	U.S. Coast and Geodetic Survey, Bull. 23.	
Dec. 29, 1894.	Combe	−57 50·0	R. O. Records, C. of G. Hope.	With Dip Circle 9, by Dover.
Jan. 11, 1895.	Finlay	−57 52·0		
Oct. 23, 1897.	Finlay	−58 07·0		
Oct. 2, 1899.	Beattie & Morrison.	−58 31·0	R. O. Records, C. of G. Hope.	
Aug. 19, 1900.	Beattie & Morrison.	−58 38·9		
Dec. 1900.	Beattie & Morrison.	−58 39·7		
Dec. 1900.	Beattie & Morrison.	−58 41·1	....	With Dip Circle 142, by Dover.

From the above results the secular variation of inclination is as follows :—

Period.	Secular Variation.
1751 to 1840	6'·94 annual increase of south inclination.
1841 ,, 1846	5·45 ,, ,, ,,
1843 ,, 1854	5·10 ,, ,, ,,
1854 ,, 1873	5·20 ,, ,, ,,
1873 ,, 1890	4·90 ,, ,, ,,
1890 ,, 1900	7·80 ,, ,, ,,

§ 3. *Declination.*

The following list contains all the records of declination determination at the Cape of Good Hope which have come under our observation.

Date.	Observers.	Declination.	Authorities.	Remarks.
		° ′		
1605.	Davis	0 30 E. of N.	Taken from Sabine's 'Magnetical and Meteorological Observations at the Cape of Good Hope.' Vol. I. Magnetism.	
1609.	Keeling	0 12 W. of N.		
1614.	Pring	1 30 ,,		
1614.	Daunton	1 45 ,,		
1622.		2 0 ,,		
1675.		8 0 ,,		
1675.	Leydeker	8 28 ,,		
1691.		11 0 ,,		
1721.	Mathews	16 25 ,,		
1724.	Mathews	16 23 ,,		
1751.	La Caille	19 15 ,,		
1753.	La Caille	19 00 ,,		
1768.	Wallis	19 30 ,,		
1768.	Carteret	19 30 ,,		
1770.	Ekeberg	19 10 ,,		
1771.	Cook	20 30 ,,		
1772.	Wales	20 26 ,,		
1774.	Ekeberg	21 39 ,,		
1774.	Bayley	21 36 ,,		
1775.	Wales	21 14 ,,		
1780.	Cook	22 16 ,,		
1783.	Lodberg	22 23 ,,		
1788.	Bligh	24 4 ,,		
1791.	Vancouver	25 40 ,,		
1792.	Dentrecasteaux	24 31 ,,		
1818.	Freycinet	26 31 ,,		
1836.	Fitzroy	28 30 ,,		
1839.	Du Petit-Thouars	29 9 ,,		
1841.	R.A. Detachment	29 0·2 ,,		From April 1841 to July 1846 inclusive the declination was observed hourly. The declination given for each year is the mean of all the observations for that year.
1842.	R.A. ,,	29 5·0 ,,		
1843.	R.A. ,,	29 6·0 ,,		
1844.	R.A. ,,	29 6·2 ,,		
1845.	R.A. ,,	29 7·4 ,,		
1846.	R.A. ,,	29 9·2 ,,		From Sept. 1846 to Aug. 1850 the declination was observed five times daily. The declination given for the year is the mean of all the observations for that year.
1847.		29 12·4 ,,		
1848.		29 14·0 ,,		
1849.		29 16·4 ,,		
1850.		29 18·8 ,,		
1851.		29 20·9 ,,	R. Obs. C. of G. Hope Magnetic Records.	From Sept. 1850 to Mar. 1852 the declination was observed five times daily.
1852.		29 22·9 ,,		
Oct. 11, 1857.		29 34·4 ,,	Reise der 'Novara.'	
1860.		29 41·8 ,,	Magnetic Records, R. Obs. C. of G. Hope.	From October 1860 to January 1869 the declination was observed twice daily.
1861.		29 44·8 ,,		
1862.		29 50·3 ,,		
1863.		29 52·1 ,,		
1864.		29 53·9 ,,		
1865.		30 0·1 ,,		
1866.		30 2·0 ,,		
1867.		30 1·7 ,,		
1868.		30 1·9 ,,		
Jan. 1869.		30 1·5 ,,		
Nov. 1873.		30 4·0 ,,	'Challenger' Reports, Narrative, vol. ii.	
Jan. 1890.	Preston	29 36·0 ,,	U.S. Coast and Geodetic Survey, Bull. 23.	
Jan. 1895.	Finlay	29 18·0 ,,	Magnetic Records, R. Obs. C. of G. Hope.	
Nov. 1897.	Finlay	29 2·0 ,,		
Dec. 1900.	Beattie & Morrison	28 53·0 ,,		

From the above results the secular variation of declination is as follows :—

Period.	Secular Variation.
1605 to 1839	7·56 increase of westerly declination.
1841 „ 1850	1·30 „ „ „
1843 „ 1866	2·30 „ „ „
1866 „ 1869	Very nearly constant.
1870 „ 1890	2·80 decrease of westerly declination.
1890 „ 1900	3·91 „ „ „

§ 4. The observations for intensity are fewer. The first recorded trustworthy observation was made in 1843. So far as is known to us, the results of all observations made since that date are contained in the following list. The results are given in C.G.S. units.

Date.	Observer.	Horizontal Intensity.	Total Intensity.	Authorities.
1843.	R.A. Detachment ...	·2089	·3498	Sabine, 'Magnetical and Meteorological Observations at Cape of Good Hope,' vol. i.
1844.	„ „ ...	·2069	·3470	
1845.	„ „ ...	·2082	·3495	
1846.	Smalley	·2080		
1847.	„	·2077		
1848.	„	·2072		
1850.	„	·2066		
1852.	Maclear	·2059	·3506	Magnetic Records, R. Obs. Cape of Good Hope.
1853.	„	·2056	·3511	
1854.	„	·2050	·3516	
1855.	„	·2048	·3517	
1856.	„	·2044	·3511	
1857,	„	·2041	·3507	
Sept. 1857.	....	·2056	·3684	Reise der 'Gazelle.'
1873·9.	....	·1989	·3551	'Challenger' Report, Narrative, vol. ii.
1890·1.	Preston	·1916	·3542	U.S. Coast and Geodetic Survey, Bull. 23.
1895·05.	Finlay	·1900	·3572	Magnetic Records, R. Obs. Cape of Good Hope.
1897·8.	Finlay	·18835	·3566	
1901·0.	Beattie & Morrison.	·1851	·3559	

Secular Variation.

	Horizontal Intensity.	Total Intensity.
1843 to 1855	·00035 annual decrease.	·00016 annual increase.
1855 „ 1901	·00043 „ „	·00009 „ „
1890–1 „ 1901	·00059 „ „	?

Capetown, February 1902.

LXII. *Determination of Inversion Temperature of Kelvin Effect in Hydrogen.* *By* K. Olszewski [*].

In 1854, Kelvin and Joule showed that during an irreversible expansion hydrogen behaves differently from all other gases :—while air in expanding from a higher to a lower pressure without the performance of external work is

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[*] Communicated by the Author.

cooled by an amount proportional to the difference of the two pressures, hydrogen under like circumstances becomes heated. As is well known, the Kelvin-Joule effect has been applied by Hampson and Linde to the production of liquid air on a large scale. So far as the liquefaction of hydrogen is concerned, one might have expected, in view of the peculiar behaviour of hydrogen just mentioned, that the apparatus of Hampson and Linde would not be directly applicable for this purpose; this supposition was verified by experiment. There appeared, however, to be no doubt that the apparatus of Linde, and still more so that of Hampson, could, with suitable modifications, be used for liquefying hydrogen *: it being merely a question of cooling the gas, before expansion, below the temperature corresponding to the inversion of the Kelvin effect, after which it could be liquefied by expansion.

As a matter of fact, this gas was successfully liquefied by means of apparatus constructed on the above lines first by Dewar, and then by Travers †. These investigators used liquid air for cooling the hydrogen, the air boiling under diminished pressure, whereby the apparatus was cooled to about −200°. The question arose as to whether so strong a cooling is an indispensable condition for the liquefaction of hydrogen by means of irreversible expansion, or whether a more moderate amount of cooling would suffice.

This question could be readily answered without the necessity of undertaking a series of experiments at different temperatures, provided the temperature of inversion of the Kelvin effect were known for hydrogen.

This, however, had not been experimentally determined. Theoretically it had been deduced by Witkowski ‡ in two ways. First, by assuming the thermodynamic coincidence of the inversion temperatures for hydrogen and for air—which led to a value of about −46°; secondly, by making use of an empirical formula given by Rose-Innes §. This investigator proposed the following formula for determining the cooling in the case of the Joule-Kelvin effect:

$$e=\frac{\alpha}{T}-\beta,$$

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* Professor Kammerling Onnes (Communications of the Leyden Laboratory, no. 23, p. 16, 1896) pointed out the possibility of liquefying hydrogen in Linde's apparatus, and stated the conditions, based upon thermodynamical similarity, under which liquefaction can occur.

† Phil. Mag. [6] i. April 1901.

‡ *Rozprawy* of the Cracow Academy, Section of Math. and Natural Sciences, xxv. p. 247 (1898).

§ Phil. Mag. [5] xlv. p. 228 (1898).

where α and β, calculated from the experimental data of Joule and Kelvin, amounted to 64·1 and 0·331 respectively for hydrogen. Assuming that there is no cooling, we obviously get by means of the above formula the temperature of inversion, which in this case amounts to 193°·7 on the absolute scale, *i. e.* to −79°·3 C.

In view of the considerable difference between the values obtained by the above two methods, an experimental determination of this temperature appeared desirable.

Description of the Apparatus.

The hydrogen used in these experiments was prepared by acting on commercial zinc with pure dilute sulphuric acid. In order to purify the gas, it was led through wash-bottles containing solutions of caustic soda and potassium permanganate, and lastly through a vessel filled with pieces of pumice-stone soaked in mercuric chloride. The gas was collected in a large zinc gasometer of 1200 litres capacity, and finally compressed, by means of a Whitehead compressor, into a steel cylinder of 13 litres capacity under a pressure of about 180 atmospheres. In the interior of this cylinder was placed a long tube constructed of wire netting and filled with sticks of caustic potash.

In preparing and compressing the hydrogen, care was taken to remove the air from all parts of the apparatus. The hydrogen could therefore be regarded as pure, in so far as it was possible, in working on a large scale, to avoid its contamination with small traces of air.

The essential part of the apparatus is shown in the figure (p. 538).

The steel cylinder containing the hydrogen under high pressure is connected, by means of a copper tube, to a metallic manometer, and also to the copper tube *a*, the continuation of which is coiled to form a worm *b*, and is closed by a valve *c*. This valve is secured to the cover *n n* by means of an asbestos-packed stuffing-box and gland *o*. By means of the hand-wheel *d* the valve may be opened, when the hydrogen escapes into the box *h h*, which is made of sheet brass and stuffed with chamois leather, and expands to atmospheric pressure, escaping through the pipe *pi*. In the same box is contained a resistance-thermometer *e*, a full description of which is given in my paper on the determination of the critical temperature and the boiling-point of hydrogen *. The terminals *f* and *g* serve to connect the thermometer with the Wheatstone's

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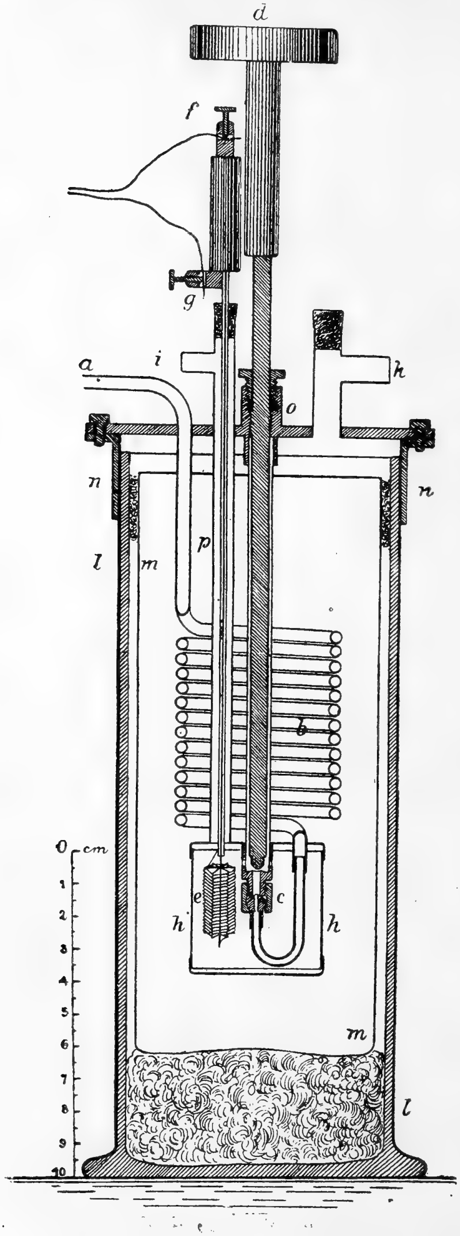
* *Rozprawy* of the Cracow Academy, xxix. p. 404 (1895); also Phil. Mag. [5] xl. p. 202 (1895).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bridge. The whole of this apparatus is, by means of the metallic ring *n n*, cemented to a thick-walled glass cylinder *l l*, inside which there is a thin-walled beaker *m m*, which serves to receive the cooling agent.

Liquid air, liquid ethylene, and a mixture of solid carbonic acid and ether were used as cooling agents. The air and ethylene were poured in through the upper opening of the T-shaped pipe *k*, to a level above that of the worm *b*. The opening *k* serves as escape-tube for the gases of the cooling substance, or—the upper opening having been closed—for connecting the interior of the apparatus with an air-pump for the purpose of lowering the pressure and temperature. In the experiments with solid carbon dioxide, this substance was introduced into the apparatus before fixing the cover *n n*. For measuring the pressure of the gases coming from the cooling substance, a mercury manometer was used, which was also in connexion with the pipe *k*, but which is not shown in the figure.

Description of the Experiments.

In the experiments with liquid air as the cooling substance, the initial temperature was about $-190°$, and the pressure of the hydrogen before expansion about 170 atmospheres. The expansion took place slowly and lasted from 4 to 5 seconds. The cooling was considerable: the galvanometer-deflexion amounted to about 200 mm. scale-divisions. In consequence of this, it became necessary to use a less powerful cooling agent; for this purpose liquid ethylene appeared suitable. In the experiments with this substance the initial temperature was about $-103°$, the initial pressure about 150 atmospheres. In this case also the temperature fell during expansion, but to a much smaller extent than when liquid air was employed: the galvanometer-throw amounting to about 30 mm on the scale. Thus it appeared that the temperature of liquid ethylene was too low; but for the production of higher temperatures this gas is unsuitable, hence in the third series of experiments a mixture of solid carbonic acid and ether was employed. The initial temperature was $-78°$, the initial pressure about 117 atmospheres. In this series of experiments 25 expansions were observed, during which the pressure in the steel cylinder fell from 117 to 110 atmospheres; *i. e.*, during a single expansion the pressure of the hydrogen was altered by about $\frac{1}{4}$ of an atmosphere, the volume of the expanded gas (measured under atmospheric pressure) amounting to about $3\frac{1}{4}$ litres. This point is specially mentioned because in some previous experiments, not recorded here, in which the fall of pressure during a single expansion amounted to several atmospheres, the cooling was much stronger; in these experiments a small cylinder, of 0·6 litre capacity, which served as a reservoir for the gas to be employed during each expansion, was used in addition to the large cylinder. With such an arrangement, there is a cooling due to the performance of external work as well as a cooling due to an irreversible expansion.

At a temperature of −78° the hydrogen in expanding became slightly heated, causing a galvanometer-throw of about 3 mm. in a direction opposed to that formerly obtained. By slow pumping and the resultant lowering of temperature, the galvanometer-throws were gradually reduced, until finally at −80°·5 no deflexion was noticeable. A further lowering of temperature produced cooling, and at −83° this gave a throw of 5 mm. in the opposite direction.

Results.

From the above experiments it follows that the temperature of inversion of the Kelvin effect for hydrogen amounts to −80°·5. This number agrees fairly well with the value deduced by Witkowski from the equation of Rose-Innes (−79°·3). This agreement between the two values makes it interesting to calculate the critical temperature of hydrogen from the assumed thermodynamic similarity of the critical temperatures of air and hydrogen, and the inversion temperature of air (deduced by Witkowski from Rose-Innes's equation). If we assume the critical temperature of air to be 133° on the absolute scale (= −140° C.), the inversion temperature of hydrogen to be 192°·5 absolute (= −80°·5 C.), that of air 633° absolute (= +360°C.), then we obtain for the critical temperature of hydrogen 40°·4 absolute, *i.e.* −232°·6C. This temperature differs from that found by me experimentally * (−234°·5 C.) by only 1°·9.

From the above experiments we may draw the conclusion that in order to liquefy hydrogen by means of the Kelvin effect, it is not absolutely necessary to cool it below −200°, but that with a thermally well-insulated apparatus the temperature of about −100°, which is easily obtained by using solid carbonic acid and ether, is already sufficient. There is, however, no doubt that a stronger cooling by means of liquid air helps to accelerate the liquefaction of hydrogen.

Cracow, Chemical Institute of the Jagellon University.

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* *Rozprawy* of the Cracow Academy; also Phil. Mag. [5] xl. p. 202 (1895).

LXIII. *On* MacCullagh *and* Stokes's *Elliptic Analyser, and other Applications of a Geometrical Representation of the State of Polarization of a Stream of Light. By* JAMES WALKER, *M.A.*[*]

1. IN 1843 MacCullagh[†] showed that an imperfectly adjusted Fresnel's rhomb can be employed within certain limits for the investigation of a stream of elliptically-polarized light, and in the same paper says:—"In making experiments on elliptically-polarized light, a plate of mica, or any other doubly-refracting crystal, placed perpendicular to the ray, may be used instead of Fresnel's rhomb . . . The two cases are precisely similar; and if it is necessary not to neglect the errors of the rhomb, it is certainly not less necessary to take into account those which may arise from a want of accuracy in the thickness of the plate, considering how difficult it is to make the thickness correspond exactly to the particular ray which we wish to observe."

These remarks seem to have escaped attention; and Sir G. G. Stokes[‡], in 1851, suggested the use of an imperfect quarter-wave plate combined with a Nicol's prism as a New Elliptic Analyser, giving formulæ for its use that are substantially the same as those published by MacCullagh.

In both cases the formulæ are stated without proof, but this is of course easily supplied. The reason for referring to the subject in the present paper is that the method of proof herein employed affords a good illustration of the advantages of a geometrical representation of the state of polarization of a stream of light that Poincaré[§] has employed for explaining Mallard's theory of Rotary Polarization, without using it, as far as I am aware, for the purpose of obtaining numerical results.

2. It may perhaps be as well to recall the main features of this geometrical representation.

Taking the axis of z in the direction of propagation, a stream of polarized light may be represented by its components polarized in planes parallel respectively to the axes of x and y with the polarization-vectors

$$\xi=\bar{a}e^{\iota nt}, \quad \eta=\bar{b}e^{\iota nt},$$

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[*] Communicated by the Author.

[†] Proc. R. I. Acad. ii. p. 384 (1843); Collected Works, pp. 238–242.

[‡] B. A. Report for 1851, Part ii. p. 14; Collected Works, iii. p. 197.

[§] *Théorie Math. de la Lumière*, ii. ch. xii.

wherein $\bar{a}$ and $\bar{b}$ are in general complex, and their ratio

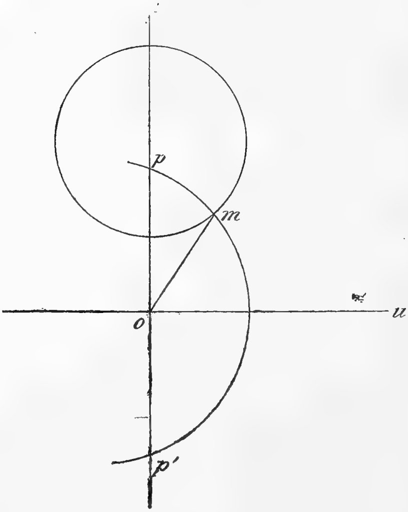
$$\bar{b}/\bar{a}=(b/a)e^{\iota\Delta'}=(b/a)\cos\Delta'+\iota(b/a)\sin\Delta'=u+v\iota, \text{ say,}$$

a and b being the amplitudes of the components, and Δ' the acceleration of phase of the second relatively to that of the first.

This ratio defines the form and orientation of the elliptic vibration of the extremity of the polarization-vector of the stream; and we may therefore represent the state of polarization by a point on a plane, for which the abscissa is u and the ordinate is v, the length of the radius-vector to the representative point giving the ratio of the amplitudes and the angle that it makes with the axis of abscissæ being the difference of phase. Since the polarization is right- or left-handed according as Δ' lies between 0 and π or between π and 2π, the vibrations in the stream will be right- or left-handed, according as the representative point is above or below the axis of u.

When the point is on the axis of u, the stream is plane-polarized in an azimuth $\tan^{-1}u$ with respect to the plane of xz; if the point be on the axis of v, the difference of phase is $\pi/2$, and the planes of maximum and minimum polarization are parallel to the axes of x and y. Points p, p' on the axis of ordinates at unit distance from the origin represent circular polarization.

Fig. 1.



Now if θ be the angle that the planes of maximum and minimum polarization make with the coordinate axes, $\tan\beta$ be the ratio of the axes of the elliptic vibrations and $\tan\sigma = b/a$, we have

$$\tan 2\theta = \cos\Delta' \,.\, \tan 2\sigma, \quad \sin 2\beta = \sin\Delta' \,.\, \sin 2\sigma,$$

which give

$$u^2+v^2+2\cot 2\theta \,.\, u-1=0, \quad u^2+v^2-2\operatorname{cosec} 2\beta \,.\, v+1=0.$$

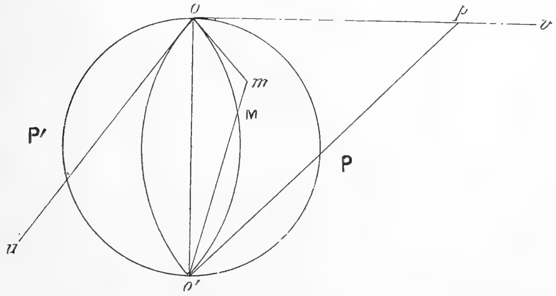
Thus if θ be constant, the points representing the different states of polarization lie on a circle through p and p', and if the ratio of the axes of the elliptic vibrations be constant, the points corresponding to different orientations of the axes are on a circle cutting the first circle orthogonally.

Any point is the intersection of a circle of the one system with a circle of the second system: the distance from the origin of the point, in which the circle of the first (θ) system cuts the axis of u, is the tangent of the angle that the plane of maximum or of minimum polarization makes with the axis of x, according as the representative point is within or without the circle of radius equal to unity with its centre at the origin.

3. Let us now transform this representation by a stereographic projection.

Describe a sphere of unit diameter touching the plane of uv at the origin of the coordinates, and let the points of the plane be projected on the surface of this sphere by joining them to O′, the extremity of the sphere through O.

Fig. 2.



Then the axes of u and v project into great circles at right angles to one another, the former of which may be called the equator; the points p, p' become the poles; the circles (θ)

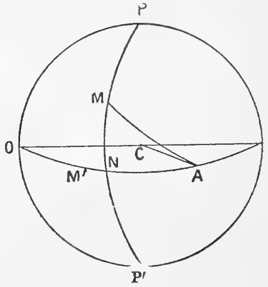
will be represented by the meridians 2θ, and the circles (β) by the parallels of latitude 2β; and if $\tan\sigma$ be the length of the radius-vector Om and the angle $m\text{O}u=\Delta'$, the point m will be projected into M where the arc $\text{OM}=2\sigma$ and the angle $\text{MOO}'=\Delta'$.

Thus any point on the sphere will represent the state of polarization of a stream of light, the azimuth of its plane of maximum polarization being half the longitude, and the ratio of the axes of the elliptic vibrations being the tangent of half the latitude of the point; and the polarization is right- or left-handed according as the point is in the Northern or Southern Hemisphere.

It is now easy to represent the effect on the polarization produced by the passage of the stream through a crystalline plate.

Let α be the angle between the axis of x and the plane of polarization of the most retarded stream in the plate, Δ the relative retardation of phase introduced by the plate; then the effect of the transmission is represented by three successive rotations:—(1) round the polar axis PP′ through an angle -2α, (2) round the diameter OO′ in a left-handed direction through the angle Δ, (3) round the polar axis through an angle 2α; and these three operations are clearly equivalent to a single rotation in a left-handed direction through an angle Δ round an equatorial diameter A′A, where the arc $\text{OA}=2\alpha$.

Fig. 3.



If the resulting polarization be plane, this rotation must bring the representative point into the plane of the equator to the point M′, say, and the arc AM′ is twice the angle σ that the resulting plane of polarization makes with that of the most retarded stream in the plate.

4. The application to the elliptic analyser follows at once.

Let the point M, representative of the polarization of the stream to be investigated, be determined by its longitude $ON=2\theta$ and its latitude $NM=2\beta$, then if the arc $NA=2\phi$, the spherical triangle ANM, in which $AM=AM'=2\sigma$, gives

$$\sin 2\phi=\tan 2\beta \cot \Delta, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\cos 2\sigma=\cos 2\beta \cos 2\phi, \quad . \quad . \quad . \quad . \quad (2)$$

$$\cos \Delta=\tan 2\phi \cot 2\sigma. \quad . \quad . \quad . \quad . \quad (3)$$

Whence it follows that there are two possible positions of the axis, A′A and B′B, such that $NA+NB=\pi$, and that the values of σ corresponding to these positions are complementary to one another. If the polarization be right-handed, A will lie within or without the arc ON, according as Δ is greater or less than $\pi/2$; the reverse being the case if the polarization be left-handed.

If then χ_1 and $90+\chi_2$ be the azimuths, measured from a fixed plane of reference in a direction from right to left, of the plane of polarization of the most retarded stream in the plate of the analyser, when the emergent light is plane-polarized, and θ be the azimuth of the plane of maximum or of minimum polarization of the primitive stream, we have

$$\chi_1=\theta\pm\phi, \ \chi_2=\theta\mp\phi, \text{ and } \theta=(\chi_1+\chi_2)/2, \ \phi=(\chi_1\sim\chi_2)/2. \quad (4)$$

Again, if σ_1, σ_2 be the azimuths of the plane of polarization of the stream emerging from the plate in its first and second positions, measured from right to left from a plane of reference fixed in the plate, $(\sigma_1+\sigma_2)/2$ gives a direction inclined at an angle $\pm\pi/4$ to the plane of polarization of the most retarded stream in the plate and $\sigma_2\sim\sigma_1=\pi/2\pm2\sigma$, whence

$$\cos 2\beta=\sin(\sigma_2\sim\sigma_1)\sec(\chi_2\sim\chi_1). \quad . \quad . \quad . \quad (5)$$

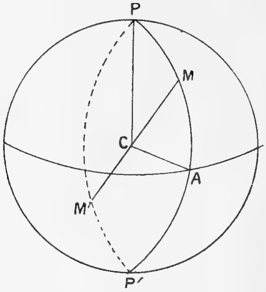
Further, it is easy to see that $\tan(\sigma_2-\sigma_1)$ and $\tan(\chi_2-\chi_1)$ have the same or opposite signs according as Δ is less or greater than $\pi/2$, whence

$$\cos_2 \Delta=\tan(\sigma_2-\sigma_1)\tan(\chi_2-\chi_1). \quad . \quad . \quad . \quad (6)$$

To complete the specification of the state of polarization of the primitive stream, we require to know the azimuths of the resulting plane of polarization measured from the plane of polarization of the most retarded stream in the plate of the analyser, which in the case of a mica plate is the plane perpendicular to that of the optic axes, and in selenite plates is the plane perpendicular to the first mean line. If σ_1' and σ_2' be those azimuths measured in a left-handed direction, the stream is right- or left-handed, according as σ_1', σ_2' are greater or less than $\pi/2$, and the angle θ determines the plane of maximum or of minimum polarization according as $\sin\sigma_2'$ is greater or less than $\sin\sigma_1'$.

5. As a second illustration of the method, we may take Gouy's explanation* of streams of permanent type in active crystals, according to which they are to be ascribed to the superposition of the effects of ordinary double refraction and of an independent rotary power of the crystal.

Fig. 4.



If this be so, the result of normal passage of the stream through unit thickness of a plate of an active crystal is given by a rotation through an angle $\kappa(\mu_1-\mu_2)$ round the axis CA corresponding to the principal section of the plate, C being the centre of sphere, together with a rotation 2ρ round the polar axis CP, where $\kappa=2\pi/\lambda$, μ_1 and μ_2 are the refractive indices of streams polarized in and perpendicularly to the principal section of the plate, supposed devoid of rotary power.

Regarding these rotations as small and neglecting small quantities of the second order, the resultant rotation is represented by the sum of the vectors obtained by taking along CA, CP lengths proportional to the rotations about these axes, and is therefore a rotation $\sqrt{\kappa^2(\mu_1-\mu_2)^2+4\rho^2}$ round the axis CM, where $\tan \text{AM}=2\rho/\{\kappa(\mu_1-\mu_2)\}$.

The point M and the diametrically opposite point M' represent the polarizations of streams of permanent type, and we see that these streams are oppositely polarized with their planes of maximum polarization respectively in and perpendicular to the principal section, and that the ratio $\tan\beta$ of the axes of the elliptic vibrations of the ends of their polari-

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* *J. de Phys.* (2) iv. p. 149 (1885).

zation-vectors is given by

$$\tan 2\beta = 2\rho / \{\kappa(\mu_1 - \mu_2)\}.$$

In the case of quartz and other positive crystals, $\mu_2 > \mu_1$, and the stream with its plane of maximum polarization in the principal section of the plate, is left- or right-handed, according as ρ is positive or negative, that is according as the crystal is left- or right-handed.

In traversing unit thickness the phase of the vibrations in the right-handed stream is retarded relatively to that of the left-handed vibrations by an amount

$$\kappa(\mu' - \mu'') = \pm \sqrt{\kappa^2(\mu_1 - \mu_2)^2 + 4\rho^2},$$

the upper or lower sign being taken, according as ρ is positive or negative; and in order to obtain the actual retardations of phase $\kappa\mu'$ and $\kappa\mu''$ of the streams we require to know the value of $\kappa(\mu' + \mu'')$.

In order to determine this *, let a stream of permanent type be replaced by its components polarized in planes parallel and perpendicular to the principal section with the polarization-vectors

$$\xi = \mathrm{A}e^{\iota nt}, \quad \eta = \mathrm{B}e^{\iota nt},$$

and suppose that after traversing unit thickness these become

$$\xi' = \mathrm{A}'e^{\iota nt}, \quad \eta' = \mathrm{B}'e^{\iota nt};$$

then we have

$$\mathrm{A}' = \alpha\mathrm{A} + \beta\mathrm{B}, \quad \mathrm{B}' = \gamma\mathrm{A} + \delta\mathrm{B},$$

where α, β, γ, δ are constants depending upon the nature of the plate.

But the stream being of permanent type, we have

$$\mathrm{A}'/\mathrm{A} = \mathrm{B}'/\mathrm{B} = e^{-\iota\kappa x},$$

where κx is the retardation of phase; hence

$$(\alpha - e^{-\iota\kappa x})\mathrm{A} + \beta\mathrm{B} = 0, \quad \gamma\mathrm{A} + (\delta - e^{-\iota\kappa x})\mathrm{B} = 0,$$

and

$$\begin{vmatrix} \alpha - e^{-\iota\kappa x}, & \beta \\ \gamma, & \delta - e^{-\iota\kappa x} \end{vmatrix} = 0.$$

The roots of this equation give the values of $e^{-\iota\kappa\mu'}$, $e^{-\iota\kappa\mu''}$, and their product is

$$\alpha\delta - \beta\gamma = e^{-\iota\kappa(\mu' + \mu'')}.$$

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* Poincaré, *loc. cit.* p. 299.

But according to Gouy's hypothesis,

$$\alpha/\cos\rho = -\beta/\sin\rho = e^{-\iota\kappa\mu_1}, \quad \gamma/\sin\rho = \delta/\cos\rho = e^{-\iota\kappa\mu_2},$$

whence

$$\alpha\delta - \beta\gamma = e^{-\iota\kappa(\mu_1+\mu_2)},$$

and

$$\mu' + \mu'' = \mu_1 + \mu_2.$$

6. As a final example, let us consider the passage of a stream of polarized light through a closed symmetrical packet of plates, the case considered qualitatively by Poincaré.

The effect of the packet on the polarization of the stream is given by the resultant of successive rotations about axes CA_1, CA_2 in the plane of the equator through an angle δ, where C is the centre of the sphere, δ the relative retardation of phase introduced by each plate, and $A_1A_2 = A_2A_3 = \ldots = 2\pi/p$, p being the number of plates in the packet.

If then Δ_n denote a rotation δ about the axis CA_n, and S_p a rotation $2\pi/p$ round the polar axis CP, the combined rotation is

$$\Delta_1 . \Delta_2 . \Delta_3 \ldots \Delta_p = \Delta_1 . S_{-p}\Delta_1 S_p . S_{-2p}\Delta_1 S_{2p} \ldots . S_{-(p-1)p}\Delta_1 S_{(p-1)p},$$

but

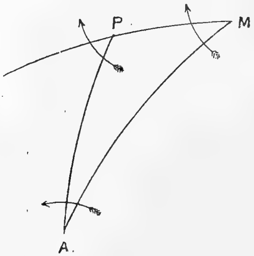
$$S_{(p-1)p} . S_p = S_{p.p} = S_{2\pi} = 1 \quad \text{or} \quad S_{(p-1)p} = S_{-p},$$

whence

$$\Delta_1 . \Delta_2 . \Delta_3 \ldots \Delta_p = (\Delta_1 S_{-p})^p;$$

or the effect of the p successive rotations is the same as p times the resultant of the rotations Δ_1 and S_{-p}.

Fig. 5.



To determine this resultant, we must draw through A_1 a great circle making an angle $\delta/2$ with A_1P in a direction opposite to the rotation round CA_1, and through P a great circle making with PA_1 an angle π/p in the same direction as the rotation round CP; then if these circles intersect in the point M, the resultant of the two successive rotations is a rotation round CM through an angle equal to $2A_1MP$.

Now since $A_1P = \pi/2$, we have

$$\tan PM = \tan \frac{\delta}{2} \operatorname{cosec} \frac{\pi}{p} \quad \text{or} \quad PM = \frac{\delta}{2} \operatorname{cosec} \frac{\pi}{p},$$

if δ be small: also

$$\cos A_1MP = \cos \frac{\delta}{2} \cos \frac{\pi}{p},$$

whence, if $A_1MP = \pi/p + \omega$, we have, neglecting ω^2,

$$\omega = \frac{\delta^2}{8} \cot \frac{\pi}{p},$$

and the resultant rotation $(\Delta_1 S_{-p})^p$ is a rotation round CM through an angle

$$\left\{ \frac{2\pi}{p} + \frac{\delta^2}{4} \cot \frac{\pi}{p} \right\} \times p = 2\pi + \frac{p\delta^2}{4} \cot \frac{\pi}{p}.$$

Hence, if δ be very small, the effect of the packet is very nearly to move the representative point along a parallel of latitude to a meridian differing from the original meridian by an angle

$$2R = \frac{p\delta^2}{4} \cot \frac{\pi}{p},$$

the rotation being in a direction opposite to that in which the angle of combination of the plates is measured.

Thus, in traversing the packet, the plane of maximum polarization of the stream is turned through an angle R, the form of the vibrations of the extremity of the polarization-vector remaining unchanged.

LXIV. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 460.]

December 4th, 1902.—J. J. H. Teall, Esq., M.A., V.P.R.S., President, in the Chair.

THE following communications were read:—

1. 'On a New Genus belonging to the Leperditiadæ, from the Cambrian Shales of Malvern.' By Prof. Theodore Thomas Groom, M.A., D.Sc., F.G.S.

2. 'The Sequence of the Cambrian and Associated Beds of the Malvern Hills.' By Prof. Theodore Thomas Groom, M.A., D.Sc., F.G.S. With an Appendix on the Brachiopoda by Charles Alfred Matley, Esq., B.Sc., F.G.S.

The series, exclusive of some 600 feet of igneous rocks, may be estimated at between 2500 and 3000 feet, and consists of the following members, tabulated in descending order:—

4. The Bronsil Shales, 1000 feet thick; grey shales containing *Dictyonema* and many Tremadoc brachiopods and trilobites.
3. The White-leaved Oak Shales; black shales, including:
    - (*b*) The zone of *Peltura scarabæoides, Sphærophthalmus alatus, Ctenopyge pecten, Ct. bisulcata, Agnostus trisectus*; 500 feet.
    - (*a*) The zone containing *Kutorgina pusilla, Protospongia fenestrata*, a new variety of *Acrotreta*, and a new genus of the Leperditiadæ; 30 feet.
2. The Hollybush Sandstone, comprising:
    - (*b*) Massive Sandstone probably not less than 1000 feet thick, and containing *Kutorgina Phillipsi, Orthotheca fistula, Scolecoderma antiquissima*, and new species of *Hyolithus.*
    - (*a*) Flaggy and Shaly Beds, not less than 75 feet thick; chiefly flaggy and shaly glauconitic sandstones, with *Kutorgina Phillipsi, Scolecoderma antiquissima, Hyolithus*, etc.
1. The Malvern Quartzite,[1] consisting chiefly of grey quartzites and conglomerates, rarely glauconitic; probably at least several hundred feet thick; containing *Kutorgina Phillipsi, Hyolithus primævus*, and a new species of *Obolella.*

The last rock, though now separated by faults from the older Malvern Series, contains angular fragments both of Uriconian and Malvernian type. It is correlated with the Wrekin Quartzite and with the lower divisions of the Hartshill Quartzite. The Flaggy and Shaly Beds appear to correspond with the *Olenellus*-beds and the zone of *Paradoxides Groomi* in Shropshire. The bulk of the Hollybush Sandstone probably represents the greater part of the Paradoxidian of other localities, and may in part correspond with the Purley Beds of Nuneaton.

The lower division of the White-leaved Oak Shales may represent the Swedish zone with *Beyrichia Angelini*, and perhaps the Festiniog Beds of North Wales. It is, however, more probable that it represents the uppermost portion of the Paradoxidian. The greater part of these shales, however, belongs to the zone of *Sphærophthalmus alatus*; but it is possible that other zones, both immediately above and immediately below, may be represented in the district. The middle part of the Bronsil Shales has yielded Asaphids and Olenids in association with *Dictyonema*, and may be correlated with the Tremadoc Beds which yield the *Euloma-Niobe* fauna. The author

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[1] [This was originally termed by the Author 'Hollybush Quartzite.']

prefers to endorse the Continental view of these rocks, and to group the Tremadoc Series with the Ordovician, with the reservation that the *Dictyonema*-shales of Europe should be regarded as belonging to the Tremadoc, the base-line of the Ordovician being drawn immediately below these shales.

The paper contains an account of the Hyolithidæ and trilobites of these rocks. Three new species of *Hyolithus* are named and described in full, and four in outline, while a revision of Holl's species *H. fistula* is given. Notes are also given, by Mr. Philip Lake and the author, on *Agnostus trisectus*, *Cheirurus Frederici*, and other trilobites, and a name is given to certain cylindrical bodies which appear to be the eggs or excreta of some animal.

In the Appendix on the Cambrian Brachiopoda of the Malvern Hills, after making a few brief remarks as to our present knowledge of these fossils, Mr. C. A. Matley proceeds to describe a new species of *Obolella* and a new variety of *Acrotreta Sabrinæ*. Species of *Lingulella*, *Lingula*, and *Acrotreta* are described, and a revision is given of *Obolella (?) Salteri*, *Lingulella Nicholsoni*, *Acrotreta Sabrinæ*, *Linnarssonia Belti*, and *Kutorgina cingulata* vars. *Phillipsi* and *pusilla*.

December 18th.—J. J. H. Teall, Esq., M.A., V.P.R.S., President, in the Chair.

The following communications were read:—

1. 'Coal and Petroleum-Deposits in European Turkey.' By Lieut.-Colonel Thomas English, F.G.S.

In this paper an account is given of the formations which include some recently discovered coal-seams and naphtha-bearing sands of Tertiary age in the little visited stretch of country lying to the north of the Gulf of Xeros in the Mediterranean, and of the western portion of the Sea of Marmora. The geological map which accompanies the paper is founded on the topography of that compiled by the Russian military staff. The lowest beds seen are soft brown limestones, with nummulites of Lutetian age, overlaid by blue shales. These, again, are covered conformably by brownish-grey calcareous sandstones, with subordinate beds of clay and shale, and occasional interruptions of basalt and rhyolite, which are folded into a well-marked syncline and anticline. In these sandstones there are outcrops of several seams of coal, one being 3½ feet thick. It is a bituminous, hard, non-coking steam-coal, with a sandstone-floor and hard clay-roof containing impressions of dicotyledonous leaves. Associated with the coal is a layer of brecciated rhyolite, which may have had some influence in converting lignite into true coal. The section of a trial-boring, striking the coal at 122 feet from the surface, is given; and also that of a boring 225 feet deep immediately below the coal. Apparently overlying the sandstones are palagonite-tuffs, shales, and hard limestones of Miocene age. The Eocene sandstones are occasionally overlaid by soft sandy strata, probably Pliocene, with which naphtha-bearing beds are interstratified. Naphtha in quantity, under considerable gas-pressure, has been obtained from borings, a section of one of which is given, 300 feet deep. The strata are much disturbed and are nearly vertical in places. A stony clay having no apparent connexion with the present drainage-system, and containing scratched, striated, and facetted boulders, spreads unconformably over all the formations previously mentioned. At the Hora lighthouse a well-marked 'raised beach' occurs at 130 feet above present sea-level, sometimes cemented into a concrete, in which occur *Dreissena polymorpha* and a *Neritina*, probably *N. danubialis.* The fresh water within the Sea of Marmora must at this epoch have stood at a height sufficient to collect the drainage of the whole of South-eastern Europe and Western Asia, an area of at least 2,000,000 square miles, into a freshwater sea, from which the volume of water discharged (even with the present small rainfall) would be not less than twice that of Niagara. The Dardanelles channel is itself a gorge cut back through soft horizontal Miocene strata, with every appearance of rapid erosion by falling water.

2. 'On the Geological and Physical Development of Dominica; with Notes on Martinique, St. Lucia, St. Vincent, and the Grenadines.' By Prof. Joseph William Winthrop Spencer, Ph.D., M.A., F.G.S.

These islands form a continuation of the volcanic chain extending from Guadeloupe, though separated one from the other by embayments in the submarine plateau, reaching to depths of more than 6000 feet, within the line connecting the shores of the islands. These submarine valleys head in cirques, like the amphitheatres which occur on the slopes descending from high plateaux. From the ends of the cirques, valley-like channels can be traced landward on the submerged plateaux, or can be found to cross them in order to join like features on the other side. The cols between the opposite valleys vary in depth from about 2000 to 3600 feet, except that between the Grenadines and the Trinidad banks, where the divide may not be more than 750 feet below the surface of the sea, and one south of St. Vincent (less than 1300 feet). Some of the submarine channels have remarkable tributaries. The drowned valleys, like those about the islands to the north, assume two very different forms—those with broad undulating outlines, such as characterize the features produced during the long Miocene-Pliocene period of erosion, when the surfaces of the land were at or near the base-level of erosion, and other types where very deep valleys and gorges incise the more rounded features of the drowned plateau, which in the early Pleistocene epoch thus appears to have stood for a limited time at an altitude of 6000 or 7000 feet, as shown within the limits of the Antillean mass (and still higher from evidence beyond). There are no coastal plains, strictly speaking; only to a very limited extent are the islands surrounded by shelves submerged to a depth of less than 200 feet. But the Grenadine banks are extensive. One or two outlying remnants of the Antillean plateau occur south-east of Dominica, and another about 60 miles east of Martinique, all of which may be fragments of the old coastal plains.

All the islands are underlaid by old Tertiary or pre-Tertiary igneous rocks, as in Guadeloupe. Such, where exposed, are found to be very much decayed. Elsewhere, they are covered by tuffs with only angular breccia. Upon such surfaces, denuded, rest other tuffs derived from older deposits, containing waterworn pebbles, in lines of bedding. These last may be the equivalent of the Tertiary tuffs and limestones of Guadeloupe. Upon their eroded surface rests a gravel-formation, which itself has largely been washed away. In the hollows of its surface is found another formation composed of coral-limestone, containing a fauna which still survives, with one or two possibly extinct forms. The deposit occupies a position similar to one in Guadeloupe and another in St. Kitts. Its surface is also greatly eroded, and then covered with another stratified sand-and-gravel accumulation. The surface of the slopes is often covered by a loam, which is, in part at least, a land-formation. The Pleistocene Period is thus seen to have been one of changing physical conditions. The older Tertiary history must be inferred from that of the neighbouring islands.

Lavas may be seen underlying the gravels, and accordingly we find that the volcanic activity was renewed, after a very long Tertiary quiescence, in the Pleistocene Period.

The plains underlaid by the beds of old tuffs have been so raised up as to give rise to sloping terraces dipping outward from the late volcanic centres, showing that their elevation has been due to local uplifts, and not to regional movements, and also that to the igneous centres alone are confined the volcanic uplifts, which do not extend to such remnants of coastal plains as are found in the Antillean region. The recent terraces, which are not deformed, are small, and perhaps do not rise to more than 70 feet above the sea. Youthful cañons are being formed near the mouths of the streams, showing the recent re-elevation of the land.

3. 'On the Geological and Physical Development of Barbados, with Notes on Trinidad.' By Prof. Joseph William Winthrop Spencer, Ph.D., M.A., F.G.S.

Barbados, over 100 miles east of the main chain of islands, is a remnant of the dismembered and sunken Antillean plateau, with the embayment in it, west of the island, reaching to a depth of over 7000 feet. But the drowned Barbados ridge extends far, both to the south and to the north of the island, and is connected by another ridge with the Martinique mass. Its hydrographic features are best understood when studying the question of the general relationship of the islands.

Trinidad is part of the South American continent, being on the subcoastal shelf which extends much farther seaward.

The Scotland Sands and the deep Oceanic Series of Barbados have been comprehensively studied by others. The Oceanic Series is here shown to be no newer than the Eocene Period, and the Scotland Series is supposed to date back to the beginning of the Tertiary or even to an older epoch. The 'Raised Coral-Reefs' are differentiated into three formations, all of white limestone or marl. The oldest beds are all tilted to considerable angles. They contain an Oligocene fauna. Their surfaces are eroded into rounded forms, showing that the topographic feature was completed at a low elevation above the base-level of erosion. As no other Tertiary formation occurs until the close of the Pliocene, it is inferred that the region was a land-surface throughout the long Miocene-Pliocene Period. This limestone is the equivalent of the Antigua Formation. In the hollows of its surface occur the remains of a mechanical limestone (the Ragged Point Series), which has been mostly carried away by subsequent denudation. It lies in a horizontal position. Newer still is the Bath Series, or raised coral-reefs, with a fauna still living. However, it is an old Pleistocene formation, and its surface is very much denuded. Some of the newer fossiliferous beds may represent even another distinct Pleistocene epoch. The greatest elevation and denudation appears to have been in the early Pleistocene days. The terraces occur in Barbados at much higher elevations than in the other islands, as this district does not appear to have undergone so great an amount of recent subsidence as that which carried down the surfaces, with little cañons and channels, below sea-level, adjacent to the other islands. The local features of Barbados extend our knowledge of the history of the whole chain of islands, besides agreeing with the phenomena found elsewhere.

Trinidad has more continental features than the other islands. Its surface-topography has been found to owe its origin to the erosion-features of the Miocene-Pliocene Period, which have been covered by only thin mantles as in Barbados, so that its life-history falls into harmony with that of the other islands. In its older beds it has the deep oceanic oozes as in Barbados. No volcanic phenomena have been added to the features of these islands.

This paper is the last of a descriptive series, but a clearer understanding would be obtained by bringing together the results, showing the variations and developments in different localities.

LXV. *Intelligence and Miscellaneous Articles.*

THE MICHELSON-MORLEY EXPERIMENT.

Physical Laboratory, Univ. of Nebraska.
February 28, 1902.

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

IN the paper of Dr. Hicks, which appeared in the January number, discussing the Michelson-Morley experiment, there is introduced a term in the expression for the displacement of the bands, which it seems to me is at variance with the commonly accepted views concerning the effect of the drift of the æther on optical phenomena.

He gives the expression $D\frac{\xi^2}{\Lambda}c\cos 2\alpha + D'\xi\cos\alpha$ for the displacement of the bands, and interprets the term $D'\xi\cos\alpha$ by saying that if a photographic plate be exposed where interference is taking place, an image of the fringes as they exist at the plane of the photographic plate will be printed on the plate, but if an image of the fringes be projected on the retina of the eye, by focussing the eye directly or with a telescope on a position in space, the fringes on the retina will be displaced relatively to those in the plane on which the eye was focussed. This displacement he calls the $\cos\alpha$ effect. The distance between the eye and the plane of the fringes in space he calls D', and α is approximately equal to the angle which the wave-fronts make with the direction of the drift.

Dr. Hicks, therefore, evidently believes in the objective reality of the $\cos\alpha$ effect, and several times in his paper he speaks of the necessity of correcting for or eliminating it. Now a first order effect of ξ has been looked for by a number of investigators, notably Mascart and Ketteler, who were unable to observe any such effect. These considerations, taken with the fact that the mathematical investigations of Lorentz and others lead to the conclusion that there should be no such effect observable, would make it unlikely that the phenomenon to which Dr. Hicks calls attention is real. If it were, one might determine ξ very simply by measuring the displacement observable at different distances from the interferometer. It is admissible that there is a displacement of the fringes relatively to a fixed point in the æther, and that this displacement is proportional to $D'\xi\cos\alpha$, but there is an effect due to the motion of the observer, exactly equal and of opposite sign, which nullifies it.

Dr. Hicks gives the following more complete expression, where X is the distance of the central band from the point of intersection of the planes of the mirrors, and $d\sin\alpha$ is the aberration term, d being linear with ξ.

$$X = a + b\cos 2\alpha + c\cos\alpha - d\sin\alpha.$$

Aberration is usually given as proportional to the sine of the angle between the ray and the direction of motion of the earth, but in this case a refers to the wave-front and not to the ray, so that cos a is the function that should be used, and it can easily be shown that the numerical value of the aberration is equal to $D'\xi \cos a$. The last two terms of the equation, therefore, cancel one another, and the equation reduces to the usual well-known form.

William B. Cartmel,
Fellow in Physics,
University of Nebraska,
Lincoln, Nebr.

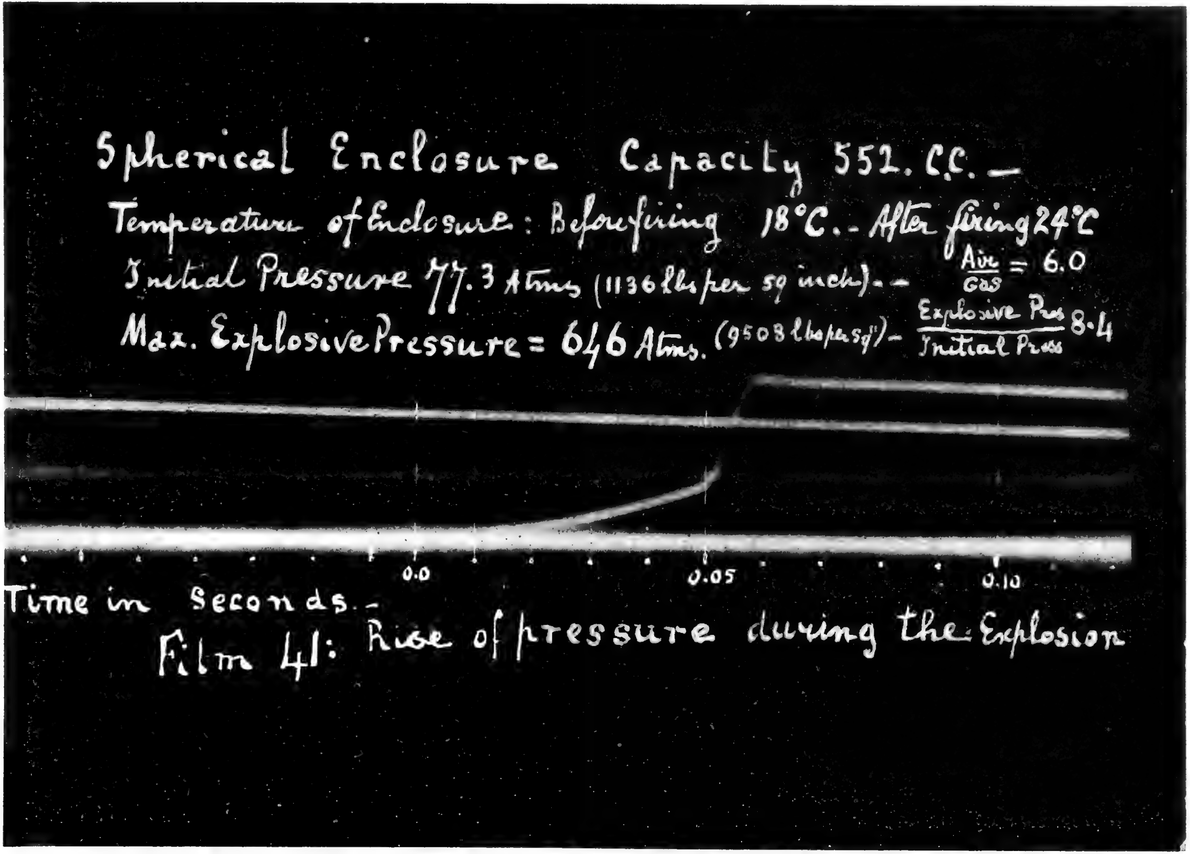
University College, Sheffield,
March 20, 1902.

Mr. Cartmel seems to suppose that I have attempted to give an explanation of a term in the expression for the position of the central fringe, whereas I have introduced the term because, for the reasons given in §§ 4, 15, it appeared to be called for. I do not see that Mr. Cartmel has met that argument. I may say that before the paper was published I had a good deal of discussion with Dr. Larmor on this very point. His argument was based on the minimum path for relative rays, and he suggested that the effect might be counteracted by the action of the drift in altering the refraction through the lenses of the instrument. This is the only way I can see out of the difficulty; but even then this action could only be produced between the object-glass and the image, whereas the effect itself depends principally on the distance of the object-glass from the place observed.

The correction of cos a for sin a in the aberration effect was noticed, but overlooked in the final revise. I cannot, however, follow Mr. Cartmel in his statement that it can be shown that it will numerically cancel the former effect. In fact, if so the matter may be looked at from the opposite point of view as completely annulling any aberration effect—which would be a curious result in itself—besides verifying the reality of the effect he calls in question.

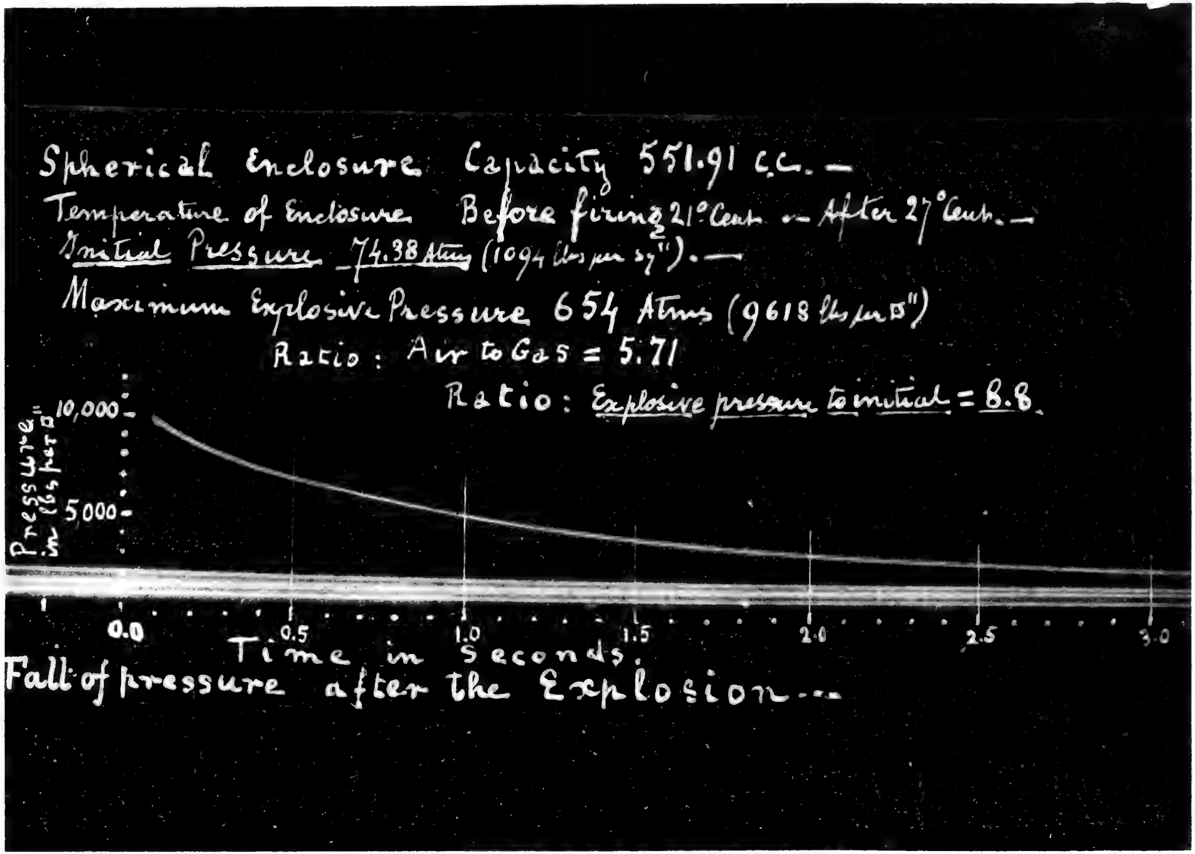
The aberration displacement at the focal plane of the instrument would be $F\xi \cos a$, whereas that due to what I have called the cos a effect would be $(d+F)\xi \cos a$, where F is the focal length of the object-glass and d its distance from the observed fringe. As a matter of fact, if the curves of the observations published be plotted for a complete revolution of the instrument, they show evidence of both a cos a and a cos 2 a effect. But too much stress should not be laid on this. W. M. Hicks.

EXPLOSION OF A MIXTURE OF COAL GAS AND AIR.

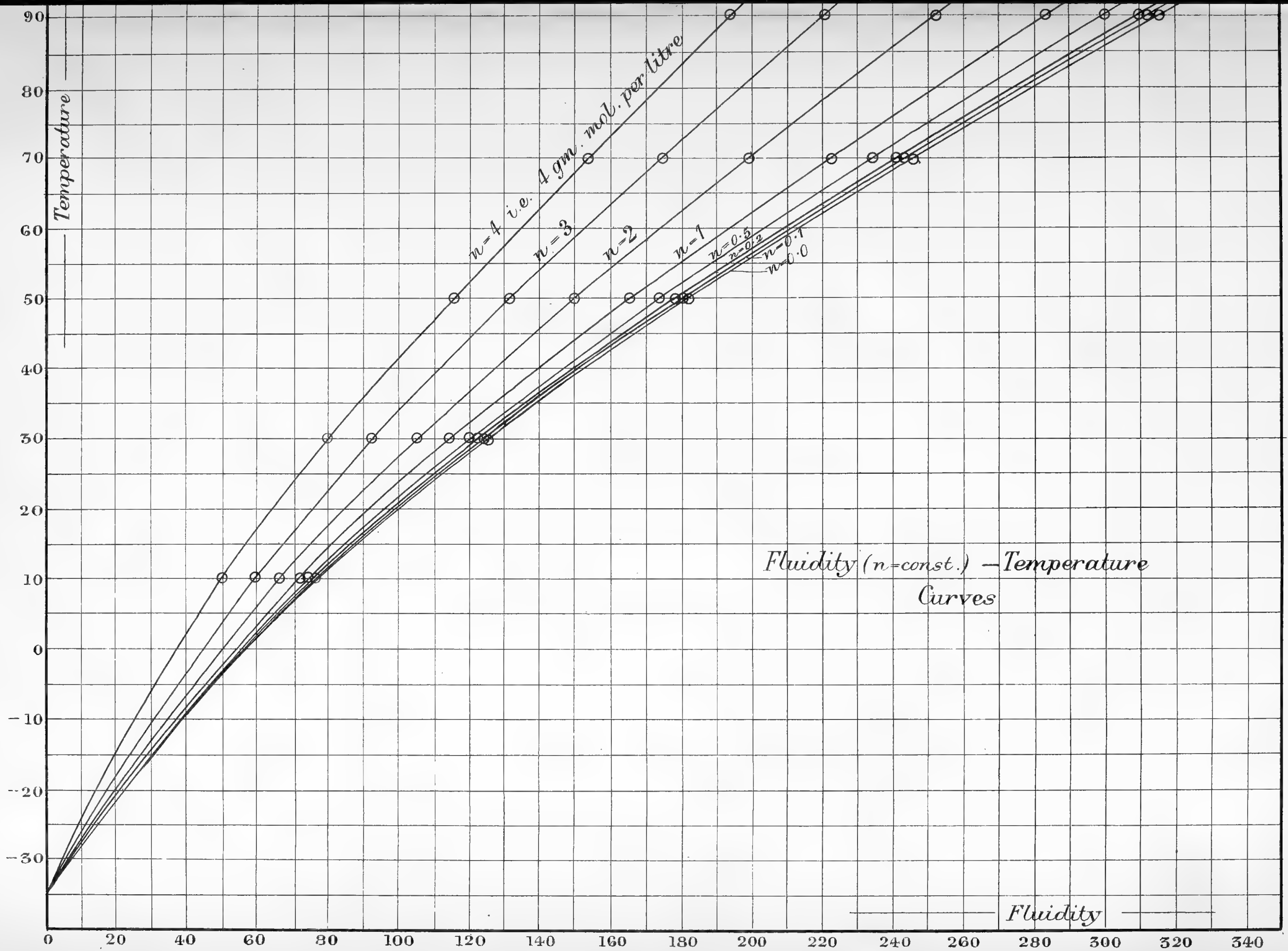


I. RISE OF PRESSURE.

EXPLOSION OF A MIXTURE OF COAL GAS AND AIR.



II. FALL OF PRESSURE.



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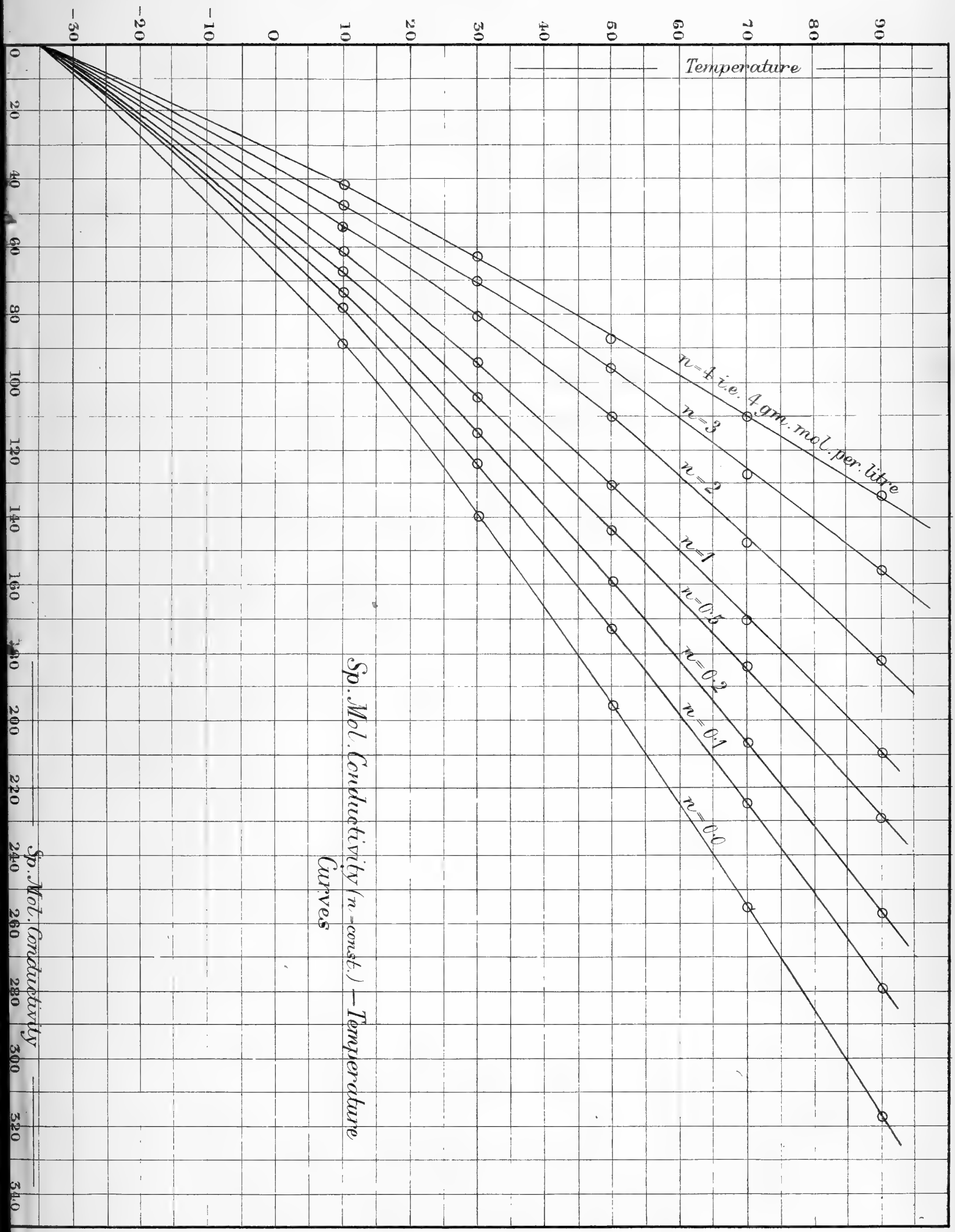
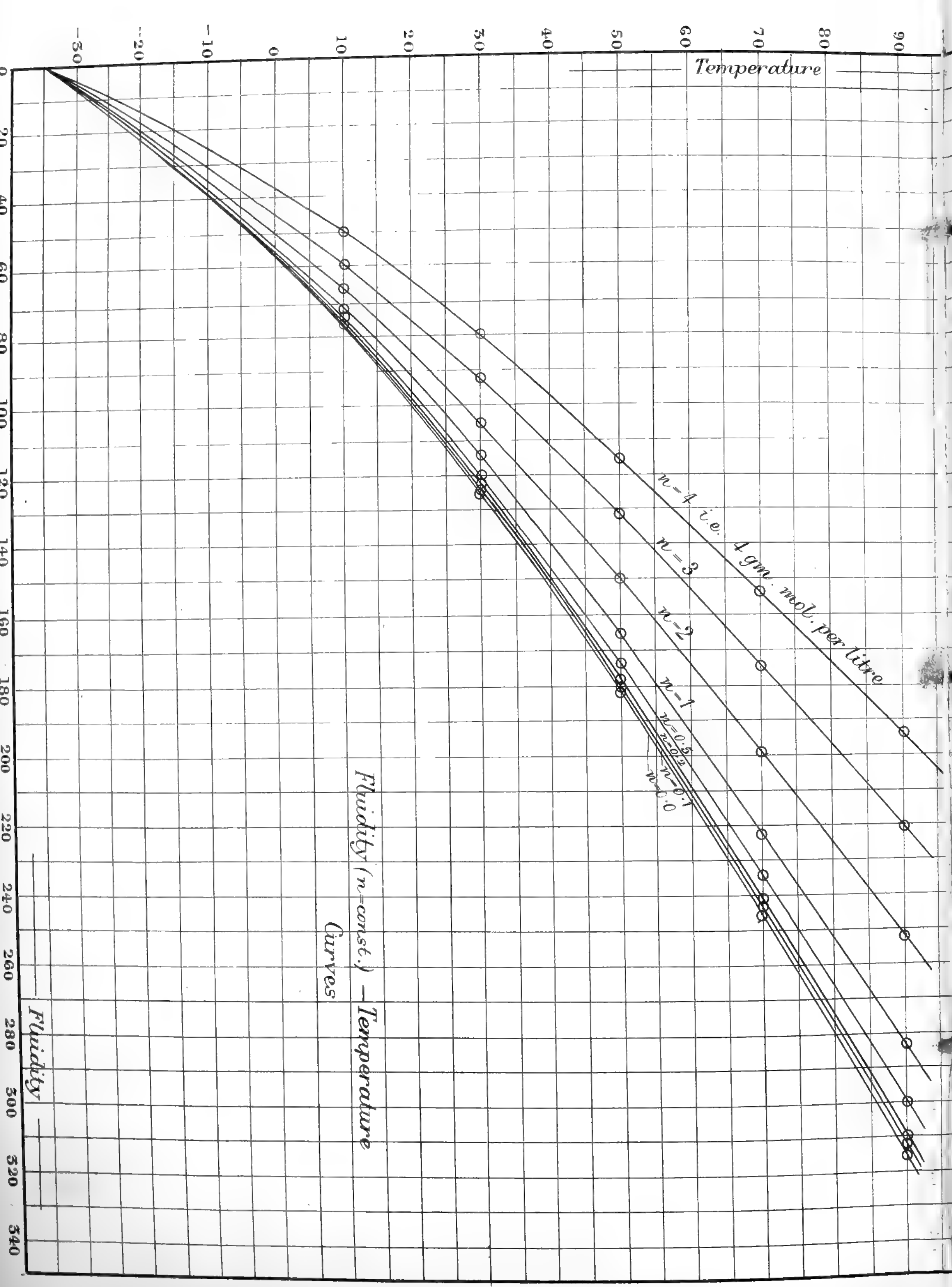
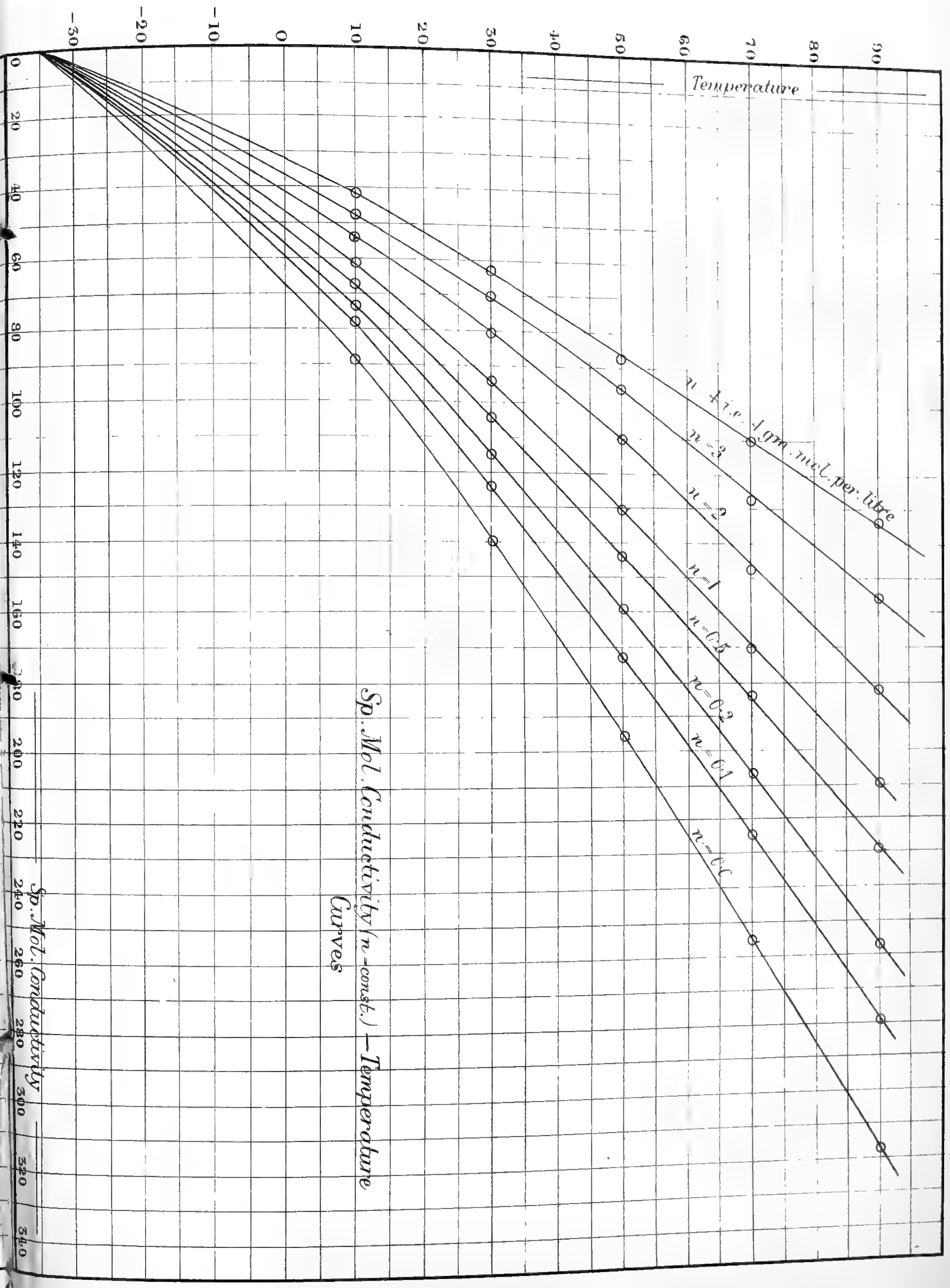

Temperature
-30
-20
-10
0
10
20
30
40
50
60
70
80
90
Sp. Mol. Conductivity
0
20
40
60
80
100
120
140
160
180
200
220
240
260
280
300
320
340
n=4 i.e. 4 gm. mol. per. litre
n=3
n=2
n=1
n=0.5
n=0.2
n=0.1
n=0.0
Sp. Mol. Conductivity (n = const.) — Temperature Curves



Fluidity (n=const.) — Temperature Curves



Sp. Mol. Conductivity (n=const.) — Temperature Curves

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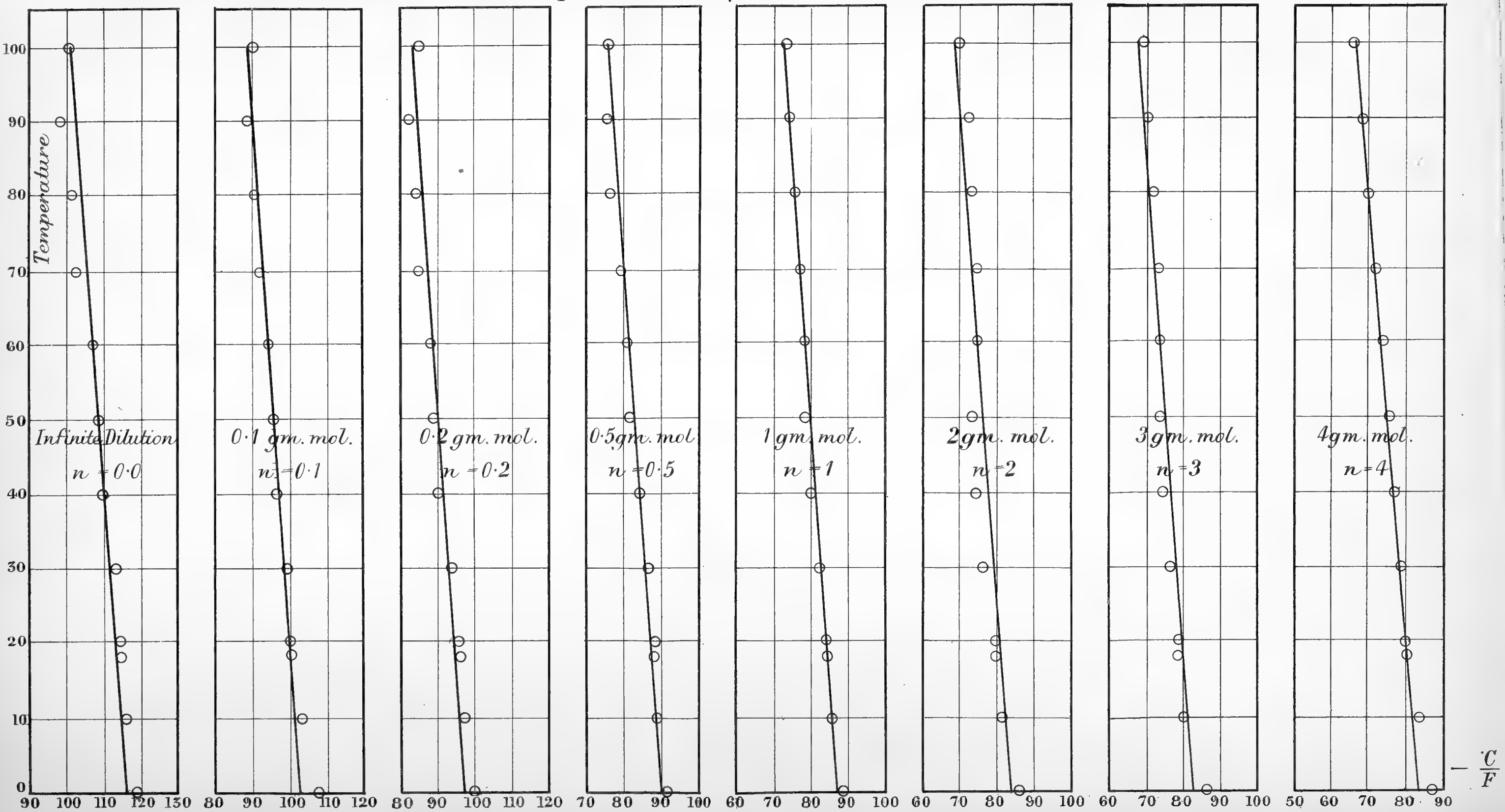
The Temperature variations of the Specific Molecular Conductivity and the Fluidity of Sodium Chloride Solutions.

$\frac{C}{F}$(n = const.)–Temperature Curves



Mintern Bros. lith.

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

JUNE 1902.

LXVI. *The Conductivity produced in Gases by the aid of Ultra-Violet Light.* *By* JOHN S. TOWNSEND, *M.A., Wykeham Professor of Physics, Oxford**.

1. IN some previous papers on the electrical conductivity of gases it has been proved that a number of complicated phenomena can be explained by the theory of the genesis of ions by collision. In those experiments which were designed to test the theory the conductivity was started in the gas by Röntgen rays. (J. S. Townsend, Phil. Mag. Feb. 1901; J. S. Townsend & P. J. Kirkby, Phil. Mag. June 1901; P. J. Kirkby, Feb. 1902.)

The electric forces which were used were such that the potential-differences between the electrodes were not sufficient of themselves to produce or maintain a continuous discharge. A number of interesting results obtained by Stoletow † with ultra-violet light can also be explained by the same theory, but as his experiments are not arranged so as to give an accurate test of the theory, and were made with air alone, it was considered desirable to make some new experiments with air and other gases.

It is easy to devise experiments so as to obtain a simple test of the theory of the genesis of ions by collision, when the conductivity is started by means of ultra-violet light. In the experiments which are here described the conductivities

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* Communicated by the Author.

† Stoletow, *Journal de Physique*, ser. 2, vol. ix. 1890.

were measured between two parallel plates at different distances apart. The light fell on the negative electrode, which was a zinc plate. The light has the effect of setting free negative ions from the surface of the metal on which it falls. These ions, which I shall call the z ions, as they move through the gas under an electric force produce others by collisions with molecules. The increase of conductivity obtained by separating the plates is thus explained. The experiments also lead to the conclusion that the ions generated in the gas have exactly the same properties as the z ions. In order to show this, the conductivity was determined for different distances between the plates, the electric force, X, and the pressure, p, of the gas being kept constant. The number of ions, α, generated by one of the z ions in moving through a centimetre of gas is a function of X and p, so that α remains constant in the experiments with different distances between the plates. If the negative ions generated in the gas have the same properties as the z ions, the conductivity will increase by the factor $\epsilon^{\alpha x}$ when the distance between the plates is increased by the length x. The total number of ions n reaching the positive plate will be given by the formula

$$n = n_0 \, \epsilon^{\alpha d},$$

n_0 being the number of ions set free from the zinc plate, and d the distance between the plates. It was found that this formula gives the connexion between the values of n and d over a large range of forces and pressures.

2. The apparatus used in these experiments is shown in figures 1 and 2.

The conductivity took place between the two parallel plates A and B, figure 1. The plate A was of quartz silvered on the inside so as to obtain a conducting surface. The silvering was ruled with a series of fine parallel lines to allow some of the light to pass through the plate and fall on the surface of the lower plate B, which was of zinc. The plate A rested on a brass cap C, and the junction between the plate and the brass was made air-tight. The short ebonite tube E was screwed to the flat part of the cap on which the quartz plate rested, and a metal disk F was fixed to the lower end of the ebonite. This disk was thus in rigid connexion with the quartz plate A and parallel to it. The distance between the plates A and B could be adjusted by means of the screw S, which worked in a half-millimetre thread cut in the plate F. The wide glass tube D which contained the apparatus was ground flat at one end, and the brass cap was cemented to it. The other end was joined to a narrower tube about 33 inches long, which dipped into a vessel containing mercury.

Some of the ebonite tube E was cut away so as to allow the gas a free passage from the space between the plates to the other parts of the apparatus.

Fig. 1.

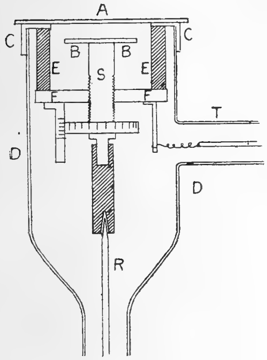
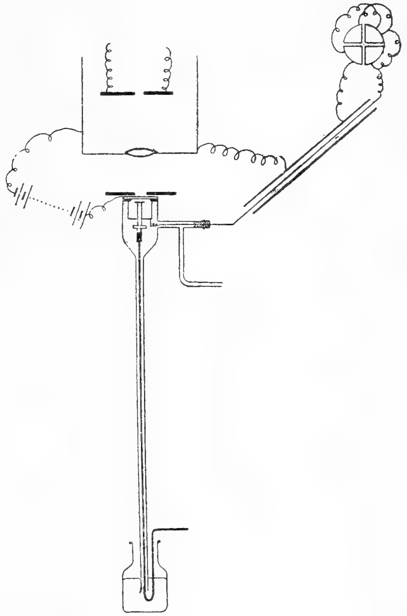


Fig. 2.



A short piece of ebonite was fixed to the screw S, and the rod R, coming through the mercury column below D, fitted into a notch cut in the ebonite. The distance between the plates A and B could be altered, without changing the pressure, by turning the end of the rod R, which was bent as shown in figure 2.

The apparatus was connected to a Toepler pump and a McLeod gauge. A small variation of pressure could be detected by means of the gauge, and it was found that the pressure only altered by about a five-hundredth of a millimetre during a set of observations.

When experimenting with hydrogen and carbonic acid, a fresh supply of gas was introduced into the apparatus before each set of experiments, so that the small leakage could not have produced any appreciable effect in the results.

Metallic connexion was made with the zinc plate by a rod passing through the side-tube T. The rod entered the apparatus through an ebonite plug in the end of the side-tube, and was thus insulated from the glass tube.

The zinc plate was in connexion with one pair of quadrants of an electrometer, the other pair of quadrants being to earth. The wire leading from the apparatus to the insulated quadrants passed along the centre of an earth-connected brass tube to prevent induction-effects. The silvered plate A was connected to the positive terminal of a battery of small Leclanché cells, and the other terminal was put to earth. When it was required to make experiments with a constant electric force, the number of cells employed was proportional to the distance between the plates. The cells were set up in sets of 24; each set had exactly the same potential, 35 volts.

The plate B connected to the electrometer never deviated much from zero potential, so that during an experiment the electric force between the plates was practically constant and equal to the potential of the battery divided by the distance between the plates.

Strips of tinfoil were wrapped round the ebonite tube E (about halfway between the brass cap and the disk E), and connected to earth through a second side-tube (not shown in the figure). This arrangement prevented any leakage of electricity over the surface of the ebonite from the plate A to the apparatus connected with the electrometer.

The ultra-violet light was produced by the spark of a leyden-jar discharge taking place between two aluminium points. An induction-coil was used for charging the leyden-jar. The whole apparatus for producing the light was surrounded by a metal screen connected to earth, so that the sparking should have no direct effect on the electrometer.

The spark-gap was at the focus of a quartz lens placed in an aperture of the screen vertically above the parallel plates A and B. A lead screen with a circular hole one centimetre in diameter was placed between the lens and the quartz plate, in order to confine the light to the central portion of the zinc plate. The latter was 2·4 centimetres in diameter, so that when the light fell upon it the conductivity took place only in the space where the field of force was uniform.

The apparatus was tested at various times, and it was found that no electrometer deflexion was produced by the sparking when the aperture in the lead screen was closed.

3. The observations were made in the usual manner. The coil which charged the leyden-jar was turned on for a fixed time, and the electrometer deflexion was observed. Several

observations were thus made with different distances between the plates, the electric force and the pressure being kept constant. These experiments were repeated alternately, and the mean of the observations was taken. As a rule the observations did not vary by more than 4 or 5 per cent., so that considerable accuracy could be obtained.

It is necessary to apply a small correction to the electrometer readings owing to the increase of capacity which occurs when the plate B is moved up towards A. The effect of moving B from a position 5 millimetres from A to 1 millimetre from A was to increase the capacity of the apparatus connected to the quadrants (including the quadrants) in the ratio of 82 : 80. The observations at the latter distance were increased by $2\frac{1}{2}$ per cent., and intermediate observations by a smaller factor, in order that the observations should be proportional to the current between the plates.

In most cases three conductivities, n_1, n_2, and n_3, were found for three distances, d_1, d_2, and d_3, between the plates, the electric force and pressure being constant. The distances were such that $d_2-d_1=d_3-d_2$.

According to the formula, $n=n_0\epsilon^{\alpha d}$, the corresponding conductivities satisfy the condition $\frac{n_2}{n_1}=\frac{n_3}{n_2}$ when the increase in conductivity arises from the action of the negative ions. By adopting this method of experimenting it is possible to guard against using forces which are too high, and which give rise to conductivities so large that they cannot be attributed to this process of ionization alone.

In the ordinary discharge arising from a high potential-difference between the electrodes, there must be at least two processes of ionization at work. The genesis of ions by the motion of negative ions alone cannot explain the phenomenon of a continuous discharge. It is evident, therefore, that some other cause is at work, and these experiments show that it begins to take effect at potentials much lower than the potential necessary to produce a discharge. With a suitable pressure, it was found that the ratios of the conductivities corresponding to equal increments of the distance *d* continually rose as the distance between the plates was increased, although the potential-difference between the plates was less than 300 volts. With hydrogen this effect is obtained with potentials less than 200 volts.

I have as yet made only a very incomplete examination of this second process of ionization, and in the following tables only a few examples of it are given.

The tables correspond to the different pressures. The force X is given in volts per centimetre. The conductivities n_1 n_2, n_3, &c. are the electrometer deflexions obtained in 10 seconds, and corrected for capacity as explained, the suffix denoting the distance between the plates in millimetres. The ratio of each conductivity to the preceding is given in the column R. It is not necessary to have the same intensity of light for the experiments at different pressures and forces, so that in the tables of results the numbers in the vertical columns do not accurately correspond to equal intensities of light.

TABLE I.—Air. Pressure 6 mms.

X.	n_1.	n_3.	n_5.	R.
350	16·9	19·2	22·2	1·14, 1·14
525	20·7	37·5	69	1·81, 1·84
700	28·2	97·5	345	3·47, 3·53

TABLE II.—Air. Pressure 2·5 mms.

X.	n_1.	n_3.	n_5.	R.
350	23	47·5	97·2	2·06, 2·05
525	29·7	126	547	4·24, 4·31
700	37·3	312	...	8·56 —

TABLE III.—Air. Pressure 1·04 mms.

X.	n_1.	n_3.	n_5.	R.
262	27·6	57	121	2·06, 2·12
350	31·5	86	241	2·73, 2·80
438	40	140	518	3·5, 3·7

TABLE IV.—Air. Pressure ·375 mm.

X.	n_1.	n_3.	n_5.	R.
262	34	63	126	1·83, 2·00
350	35·3	74	168	2·1, 2·26
438	42	99	231	2·35, 2·34
525	44·5	113	280	2·53, 2·48

TABLE V.—Air. Pressure ·148 mm.

X.	n_1.	n_5.	R.
262	35·8	73	2·04
350	38·4	83	2·16
525	41	92	2·25
700	42	99	2·35

TABLE VI.—Hydrogen. Pressure 67 mms.

X.	n_1.	n_2.	R.
1750	60·2	84	1·39

TABLE VII.—Hydrogen. Pressure 30 mms.

X.	n_1.	n_2.	n_3.	R.
1050	70·2	101·5	144	1·44, 1·43
1400	111	275	...	2·48
1750	208	990	...	4·75

Table VIII.—Hydrogen. Pressure 12 mms.

X.	n_1.	n_3.	n_5.	R.
350	46	...	64	1·39
525	61	112	200	1·83, 1·79
612	72·3	179	450	2·47, 2·51

Table IX.—Hydrogen. Pressure 5·06 mms.

<table>
<tr><th>X.</th><th>n_1.</th><th>n_3.</th><th>n_5.</th><th>R.</th></tr>
<tr><td>262</td><td>57·5</td><td>86</td><td>134</td><td>1·49, 1·55</td></tr>
<tr><td>350</td><td>67·5</td><td>145</td><td>312</td><td>2·15, 2·15</td></tr>
<tr><td>525</td><td>97</td><td>424</td><td>1800</td><td>4·37, 4·22</td></tr>
<tr><td>700</td><td>134</td><td>1120</td><td>...</td><td>8·35 —</td></tr>
<tr><td></td><td colspan="2">n_1.</td><td>n_2.</td><td></td></tr>
<tr><td>875</td><td colspan="2">187</td><td>721</td><td>3·82</td></tr>
</table>

Table X.—Hydrogen. Pressure 1·82 mms.

<table>
<tr><th>X.</th><th colspan="2">n_1.</th><th colspan="2">n_3.</th><th colspan="2">n_5.</th><th>R.</th></tr>
<tr><td>262</td><td colspan="2">62·5</td><td colspan="2">128</td><td colspan="2">276</td><td>2·05, 2·16</td></tr>
<tr><td>350</td><td colspan="2">74</td><td colspan="2">194</td><td colspan="2">546</td><td>2·62, 2·80</td></tr>
<tr><td></td><td>n_1.</td><td colspan="2">n_2.</td><td colspan="2">n_3.</td><td>n_4.</td><td></td></tr>
<tr><td>525</td><td>97</td><td colspan="2">187</td><td colspan="2">359</td><td>875</td><td>1·93, 1·93, 2·44</td></tr>
</table>

Table XI.—Hydrogen. Pressure ·82 mm.

X.	n_1.	n_3.	n_5.	R.
262	62·5	112	204	1·79, 1·82
350	66·5	131	272	1·97, 2·07
525	67·5	144	405	2·14, 2·81

Table XII.—Hydrogen. Pressure ·385 mm.

X.	n_1.	n_5.	R.
262	55·4	112·5	2·03
350	57·5	123·5	2·15
525	66·6	147	2·21
700	71·8	158	2·21

Table XIII.—Carbonic Acid. Pressure 44 mms.

X.	n_1.	n_2.	R.
1750	70·3	89	1·26

Table XIV.—Carbonic Acid. Pressure 20·5 mms.

X.	n_1.	n_2.	n_3.	R.
1050	69·5	92	122	1·32, 1·33
1750	161	540	...	3·35

TABLE XV.—Carbonic Acid. Pressure 8·75 mms.

X.	n_1.	n_3.	n_5.	R.
350	49	...	65	1·33
700	82·5	199	480	2·41, 2·41
	n_1.	n_2.		
1050	144	407		2·82

TABLE XVI.—Carbonic Acid. Pressure 3·9 mms.

X.	n_1.	n_3.	n_5.	R.
262	64	...	107	1·67
350	74·5	124	210	1·67, 1·70
525	97	305	993	3·15, 3·24
	n_1.	n_2.	n_3.	
700	139	356	882	2·55, 2·49
1050	278	1400	...	5·04 —

TABLE XVII.—Carbonic Acid. Pressure 1·375 mms.

X.	n_1.	n_3.	n_5.	R.
262	65·5	132	266	2·01, 2·02
350	81	230	646	2·84, 2·81
	n_1.	n_2.	n_3.	
525	110	264	598	2·40, 2·26
700	152	450	1380	2·96, 3·07
1050	242	1080	...	4·46 —

TABLE XVIII.—Carbonic Acid. Pressure ·606 mm.

X.	n_1.	n_3.	n_5.	R.
262	88	199	462	2·26, 2·32
350	97·5	282	853	2·90, 3·03
525	122	518	2270	4·24, 4·38
700	157	840	—	5·35 —

TABLE XIX.—Carbonic Acid. Pressure ·321 mm.

X.	n_1.	n_3.	n_5.	R.
262	85	171	355	2·02, 2·07
350	95	217	508	2·28, 2·33
522	114	300	860	2·63, 2·86
	n_1.	n_2.	n_3.	
700	126	221	378	1·76, 1·71
1050	152	268	...	1·77 —

4. In these experiments the greatest potential-difference between the plates was 350 volts. An appreciable increase of conductivity can be obtained with this voltage with gases at several centimetres' pressure by having short distances between the plates. This can be seen from Tables VI., VII., and XIII., the conductivities having been determined with the plates at 1 and 2 millimetres apart.

The equality of the two ratios $\frac{n_3}{n_1}$ and $\frac{n_5}{n_3}$ (given in the last column of each table) shows that the conductivity is proportional to $\epsilon^{\alpha \times d}$, d being the distance between the plates, and α a constant depending on the force and pressure. When the force is constant and the pressure lowered the value of α at first increases as the pressure diminishes, then reaches a maximum, and afterwards diminishes. It will be shown that the pressure for which α is a maximum is proportional to X.

In certain cases the ratios $\frac{n_3}{n_1}$ and $\frac{n_5}{n_3}$ are not equal, the latter being much the larger of the two. This effect can be obtained with air at a millimetre pressure when the maximum potential-difference between the plates is less than 300 volts. The phenomenon occurs at lower voltages in hydrogen, as is shown by the experiments given in Tables X. and XI., in which a force of 52·5 volts per millimetre was used. With this force the conductivity of hydrogen at 1·82 mms. pressure increases by the factor 1·93 when the plates are separated from 1 to 2 or from 2 to 3 millimetres apart; a further separation of the plates from 3 to 4 millimetres increases the conductivity by the factor 2·44. This process continues until eventually a continuous discharge is obtained without the aid of the ultra-violet light. With the same force (52·5 volts per millimetre) a similar effect is produced at ·82 millimetre, the ratios $\frac{n_3}{n_1}$ and $\frac{n_5}{n_3}$ being 2·14 and 2·81 respectively. When the pressure is reduced to ·385 millimetre or increased to 5·06 millimetres, the effect disappears for distances up to 5 millimetres between the plates. I hope to make a more complete examination of this effect, as it may give an explanation of some of the complicated phenomena which occur in the vacuum-tube discharge.

For the present I shall confine the discussion to the effects produced by the motion of the negative ions through the gas.

5. The results of the experiments can be easily represented in a simple form. The ratios R were used to determine the values of α corresponding to the different forces and pressures, the quantity α is the number of ions which a negative ion produces in one centimetre of the gas at pressure p under an electric force X. In the calculations I have not included those values of $\frac{n_5}{n_3}$ which are greater than $\frac{n_3}{n_1}$ by an amount which cannot be attributed to experimental errors. In other cases the mean of the two ratios was equated to $\epsilon^{\alpha \times \cdot 2}$ in order to determine α, ·2 centimetre being the increase in the distance between the plates.

As I have already pointed out (Phil. Mag. February 1901), the connexion between α, X, and p should be of the form $\frac{\alpha}{p} = f\left(\frac{X}{p}\right)$. The simplest method of testing this property of the three variables is to take a number of values of $\frac{\alpha}{p}$ and $\frac{X}{p}$ determined over large ranges, and see whether the points whose coordinates are $\frac{\alpha}{p}$ and $\frac{X}{p}$ lie on a curve. All the values of $\frac{\alpha}{p}$ (derived from the experimental determinations) are marked on the accompanying diagrams, p being measured in millimetres of mercury, and X in volts per centimetre. It is evident that a curve runs through the set of points belonging to each gas. Each point is numbered in order to show from which table of observations the value of α was calculated. The range of values of $\frac{X}{p}$ and $\frac{\alpha}{p}$ are so large that it was found necessary to have two diagrams on different scales. From these curves it is possible to obtain the value of α for any force and any pressure.

6. The properties of the curves are interesting in many ways, and are in accordance with what we should expect from simple considerations.

When an ion travels through a gas under an electric force, it makes a number of collisions with the molecules which is proportional to the pressure. If the velocity of the ion is sufficiently great, the effect of a collision will be to produce two new ions. The free paths of the ion between the collisions are of various lengths which are inversely proportional to the pressure. The velocity acquired in a path of length x is proportional to $\sqrt{X \cdot x}$, so that the velocities of the ions must depend on the quantity $\frac{X}{p}$. The value of $\frac{X}{p}$ must be large in order that the ion may acquire a velocity along the shorter paths which will be sufficiently great to produce new ions on collision. When this effect is obtained further increases in X cannot give rise to larger values of α, as new ions will be produced at every collision. The maximum value of $\frac{\alpha}{p}$ represents the total number of collisions that an ion can make in going through one centimetre of a gas at a millimetre pressure.

The curves show that $\frac{\alpha}{p}$ approaches a maximum as $\frac{X}{p}$ increases, and, as we should expect, this value is larger for carbonic acid than for air, and larger for air than for hydrogen.

The pressure at which α is a maximum for a given force X is obtained by differentiating the equation $a=pf\left(\frac{X}{p}\right)$ with respect to p; we thus obtain

$$f\left(\frac{X}{p}\right)-\frac{X}{p}\cdot f'\left(\frac{X}{p}\right)=0,$$

so that p is proportional to X.

Fig. 3.

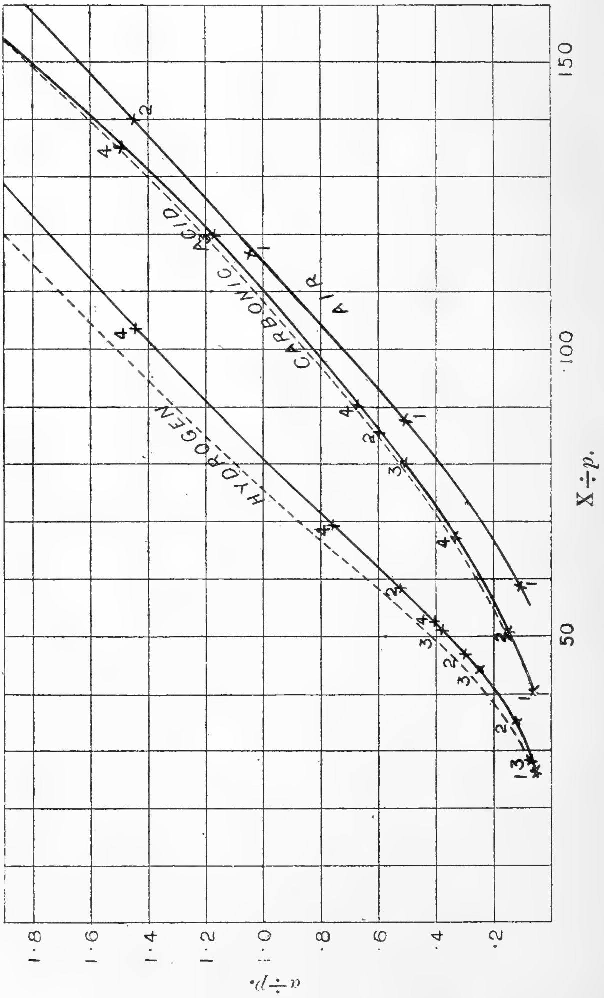
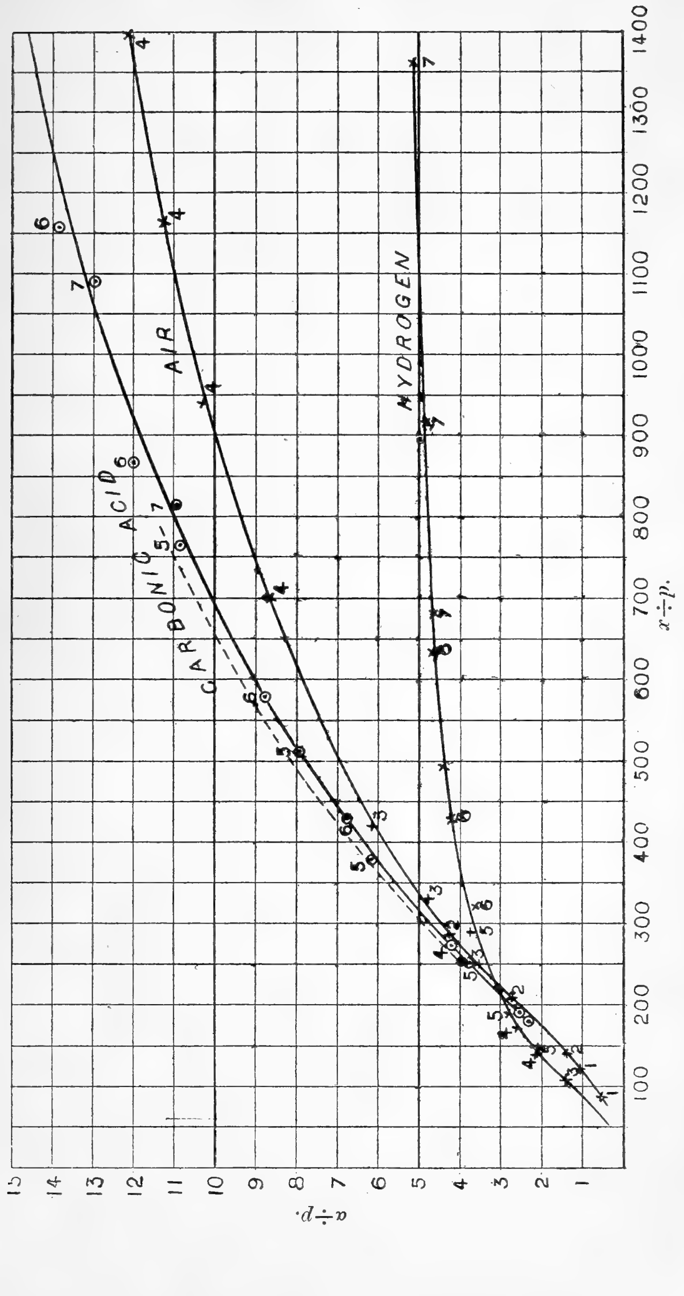


Fig. 4.


CARBONIC ACID
AIR
HYDROGEN
100
200
300
400
500
600
700
800
900
1000
1100
1200
1300
1400
$x \div p$.
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
$\alpha \div p$.

The required value of $\frac{X}{p}$ is found by drawing a tangent from the origin to the curve. The values of $\frac{X}{p}$ for which α is a maximum are thus found to be approximately:—350, 160, 380 for air, hydrogen, and carbonic acid respectively.

Stoletow (*loc. cit.*) obtained this result for air experimentally. He found that when the force was constant and the pressure variable, the conductivity reached a maximum for a certain critical pressure. He also found that this pressure was proportional to the force and independent of the distance between the plates. When the force is expressed in volts per centimetre, the critical pressure, according to Stoletow's experiments, is given by the equation:

$$p=\frac{X}{380},$$

which is practically the same result that I obtained by drawing a tangent to the curve. The exact value of $\frac{X}{p}$ which gives the critical pressure cannot be found very accurately by either of these methods, so that we cannot expect the two numbers to agree very accurately.

We can, however, fix the maximum value of α with greater precision. It can be easily seen from the theory that the maximum value of α for a given force X is got from the slope of the tangent which passes through the origin.

The coordinates of the points of contact of the tangents to the curves for air, hydrogen, and carbonic acid are respectively:

$$\frac{\alpha}{p}=5{\cdot}2, \quad 2{\cdot}35, \quad 6{\cdot}1$$

$$\frac{X}{p}=350, \quad 160, \quad 380.$$

Hence when X is given and the pressure altered so as to obtain the maximum conductivity, the value of α will be

$$\frac{5{\cdot}2\,X}{350}, \quad \frac{2{\cdot}35\,X}{160}, \quad \text{and} \quad \frac{6{\cdot}1\,X}{380}$$

for the three gases respectively.

It is remarkable that these three numbers are nearly equal.

7. If the negative ions with which we have been dealing

are the same as those produced in a gas by Röntgen rays, the values of α should be the same under similar conditions of force and pressure. A comparison of the results shows that this is the case. There are, however, differences between the curves which are given in this paper and those which have been already published in the papers to which I have referred. The values of α as determined by the experiments with Röntgen rays are in some cases much too large. The greatest differences between the results arise with the larger values of $\frac{X}{p}$, and are principally due to the fact that when experimenting with Röntgen rays the forces at the lower pressures were so large that other sources of ionization came into play. As I have pointed out, it is easy to guard against this error when experimenting with ultra-violet light; but at the time the earlier experiments were being made I did not realize that the potential must be kept more than 100 volts below the sparking-point. Thus, in the experiments with hydrogen (Phil. Mag. June 1901) at 1·77 mm. pressure, the plates being 5·5 millimetres apart, potential-differences up to 220 volts were used; and at ·84 mm. the potential was taken up to 250 volts. Referring to Tables X. and XI. of this paper it will be seen that, with these pressures, potential-differences greater than 175 volts give rise to increases in conductivity which cannot be attributed to the action of negative ions. The values of α deduced from those experiments are therefore too high, and we cannot expect to find a good agreement between the present results and the determinations of α made with Röntgen rays for the larger values of $\frac{X}{p}$.

The curve corresponding to the smaller values of $\frac{X}{p}$ determined by Röntgen rays for hydrogen lies above the curve determined by ultra-violet light, in the position of the dotted line in the diagram.

The corresponding curve for air would lie between the air and carbonic-acid curves given in the diagram. In order not to complicate the figure this line has not been inserted. The potentials used with air were also too high in some of the experiments, so that the values of α are from 10 to 20 per cent. larger than those found with ultra-violet light for values of $\frac{X}{p}$ greater than 300.

With carbonic acid larger forces may be used than with air or hydrogen without producing conductivities which do not arise from the collisions of the negative ions. The curve for this gas obtained with Röntgen rays practically coincides with that obtained with ultra-violet light over a large range of forces. It lies slightly above the latter curve.

We cannot expect a closer resemblance between the two sets of curves as there are other effects, besides the one I have mentioned, which make it difficult to obtain very accurate determinations of the quantity α with Röntgen rays. The initial distribution of ions is not uniform between the plates, and the secondary radiation carries a charge which introduces complications. The larger values of $\frac{X}{p}$ can only be obtained from experiments at low pressures, and in these cases the initial ionization is small and of the same order as the charge carried by the secondary radiation. This method is therefore not suited for the determination of the larger values of $\frac{\alpha}{p}$. It ought to be easy to determine these numbers with ultra-violet light, and I hope to continue the experiments with air and carbonic acid, so as to obtain the maximum values of α for these gases.

The curve which I have given for hydrogen practically reaches the maximum value of $\frac{\alpha}{p}$, which is about 5·2.

8. The results obtained with the smaller values of $\frac{X}{p}$ are sufficient to lead to the conclusion that the negative ions generated in a gas by the motion of the zinc ions set free by ultra-violet light are identical with the ions produced by Röntgen rays in the same gas. The experiments with ultra-violet light show that the negative ions produced in any gas by the motion of the ions from the zinc plate are identical with the zinc ions themselves. Consequently the negative ions produced by Röntgen rays in any of these gases are exactly the same as the ions set free by ultra-violet light from a zinc plate.

We are also led to conclude from the experiments, that these negative ions are small compared with the molecules of any of the gases. A molecule of a gas makes more than four times the number of collisions that a negative ion makes in going through the same distance of the gas at a given pressure. Also, the property which they possess of generating new ions does not belong to the positive ions under similar conditions of force and pressure. These properties of the negative ion show that it must be small both as regards mass and linear dimensions compared with a molecule of hydrogen.

9. The size of the negative ion, when acted on by small forces, may be estimated from the determination of the rate of diffusion or the velocity under an electromotive force. In determining the latter quality in air, Rutherford* made experiments over a range of pressures extending from atmospheric pressure to 34 mms. At the lowest pressure a force of about 50 volts a centimetre was used, so that the value of $\frac{X}{p}$ was less than 2. The results of Rutherford's investigations showed that the negative ions are the same size over this range of pressures, and are large compared with the molecules.

On the other hand, it can be seen from the experiments described in this paper, that the ions in hydrogen at 67 millimetres pressure are of small mass while under a force of 1750 volts per centimetre, since they produce others by collisions. From the theory and the experiments which showed that $\alpha = pf\left(\frac{X}{p}\right)$, it is to be supposed that the negative ions would remain of small mass and produce others by collision in a gas at any pressure provided that the electric force is sufficiently strong. This immediately follows from the fact that the velocity at collision depends on the value of $\frac{X}{p}$ (see section 6).

It would appear from these investigations, that when ions are generated in a gas the negative ions are of a definite size independent of the gas and small compared with the molecules of hydrogen. When the ratio $\frac{X}{p}$ is small the positive and negative ions quickly acquire the properties of masses which are large compared with molecules, an effect which has usually been attributed to the formation of groups of molecules round the ions. If the force acting on the gas is increased, and the ratio $\frac{X}{p}$ exceeds a certain limit, some of the collisions between the ions and the molecules will be sufficiently violent to prevent the formation of a group of molecules round the ions. A further increase in the force causes the negative ions to collide with sufficient velocity to

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* E. Rutherford, Cambridge Phil. Soc. vol. ix. pt. viii. (1898).

disintegrate the molecule into positive and negative ions. The apparent masses of the ions therefore depend on the ratio of the electric force to the pressure.

10. The results of these experiments throw some light on the constitution of molecules of gases. We are led to conclude that:—It is possible to detach a particle from the molecule of a gas which is small, as regards mass and linear dimensions, compared with the molecule of hydrogen, also the particles produced from molecules of different gases are identically the same.

The mass of the negative ion coming from a zinc plate was previously shown by Professor Thomson[*] to be small compared with the molecule of hydrogen. The method which he adopted did not involve any of the principles underlying the present investigations.

A considerable number of phenomena connected with the electric discharge in gases may be explained in a general way by taking into consideration the physical properties of these negative ions. Thus some of the effects of variation of pressure, electric force, and distance between the plates can be accounted for. Also the high conductivity of gases under rapidly alternating forces may be due to the fact that the negative ions traverse a long distance before they are discharged by the electrodes. There are, however, many phenomena for which these physical properties supply no explanations: such as the appearance, at the electrodes, of the constituents of compound gases.

The experiments with ultra-violet light show that in carbonic acid the conductivity may arise from the genesis of small negative ions. I am at present continuing the researches with other gases and vapours, so as to obtain some additional evidence on this point, as it is to be expected that similar phenomena may occur with other compound gases.

LXVII. *On Vector Differentials.*

By Frank Lauren Hitchcock[†].

1. IN studying physical quantities we are led to make a distinction between those which have by their very nature a direction in space, and those which, on the other hand, may be thought of as mere numbers. Directed quantities are conveniently called *vectors*, and non-directed ones *scalars*.

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[*] J. J. Thomson, Phil. Mag. December 1900.

[†] Communicated by the Author.

The mathematical connexion between these two kinds of magnitudes is extremely intimate: if we have any scalar function continuously distributed through a portion of space, there is a vector function immediately derivable from it by the operator ∇, which derived vector was called by Maxwell the space-variation of the original scalar.

The object of the present paper is to study briefly the differentiation of vectors, a subject inseparably bound up with the quaternion operators ∇ and ϕ. I shall assume that the reader has some slight acquaintance with the calculus of Hamilton, and shall occasionally refer to Tait's 'Quaternions,' 3rd edition, 1890.

2. From the definition

$$d(\mathrm{F}q)=\underset{n=\infty}{\mathrm{Lim}}\, n\left\{\mathrm{F}\left(q+\frac{dq}{n}\right)-\mathrm{F}q\right\}$$

follows the very general proposition that a differential is a *linear* function: both q and $\mathrm{F}q$ are, in general, quaternions; but one or both habitually "degenerate" into vectors or scalars. In any case $d(\mathrm{F}q)$ is linear in dq.

It follows that if P is any scalar function of a point ρ, then $d\mathrm{P}$ is linear in $d\rho$.

Now every possible scalar term linear in $d\rho$ may, by very elementary transformations, be put in the form $\mathrm{S}\lambda d\rho$, where λ is of course a vector function of ρ. If there are several such terms we may *assume* that $\Sigma\lambda=-\nabla\mathrm{P}$, where the minus sign is introduced in order that our results may agree with Hamilton's original definition of ∇. Therefore

$$d\mathrm{P}=-\mathrm{S}d\rho\nabla\mathrm{P}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is a fundamental equation.

From this, remembering that $d\rho$ may, like any other vector, be thought of as the product of a scalar and a unit-vector, we have

$$\frac{d\mathrm{P}}{dh_{\epsilon}}=-\mathrm{S}\epsilon\nabla\mathrm{P}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

by writing $d\rho=\epsilon dh$ and then dividing both sides by dh. Here $\frac{d}{dh}$ may be thought of as an operator. It signifies differentiation with regard to any direction whatever in space, and ϵ is the corresponding unit-vector, either a constant or a function of ρ.

We have also

$$\begin{aligned} d\mathrm{P} &= -\mathrm{S}d\rho\nabla\mathrm{P} \\ &= +\mathrm{S}d\rho(i\mathrm{S}i\nabla\mathrm{P}+j\mathrm{S}j\nabla\mathrm{P}+k\mathrm{S}k\nabla\mathrm{P}) \\ &= -\mathrm{S}d\rho\left(i\frac{d\mathrm{P}}{dx}+j\frac{d\mathrm{P}}{dy}+k\frac{d\mathrm{P}}{dz}\right), \text{ by (2)}, \\ &= \frac{d\mathrm{P}}{dx}dx+\frac{d\mathrm{P}}{dy}dy+\frac{d\mathrm{P}}{dz}dz, \end{aligned}$$

because $dx=-\mathrm{S}id\rho$, &c.

From (1) it appears that if $d\mathrm{P}$ be given, $\nabla\mathrm{P}$ can be written by inspection.

3. Taking next σ any vector function of ρ, we have

$$d\sigma=\phi d\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where ϕ is a linear and vector function. And, directly,

$$\frac{d\sigma}{dh_\epsilon}=\phi\epsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For a fascinating account of the various types of these functions, see the last chapter of Kelland and Tait's 'Introduction to Quaternions.' The function ϕ is there considered as a homogeneous strain, and it seems convenient so to speak of it, even in those cases where it could not exist in a physical sense; for example, when the sum of the roots of the strain-cubic is zero.

To show that $\nabla\sigma$ may be written by inspection when $d\sigma$ is given, we may put

$$q=i\phi i+j\phi j+k\phi k\,;$$

and, if ϕ consists of several terms, we may consider each of them as a separate linear and vector function, call them ϕ_1, ϕ_2, &c.; to these will correspond $q_1, q_2, \ldots$, whose sum, since q is linear in ϕ, must give the q of the whole function ϕ.

But $\nabla\sigma=q$, by (4); thus we can write down $\nabla\sigma$ if we know the part of q contributed by each term of ϕ.

Taking special cases, a term of the form $\beta\mathrm{S}\alpha\lambda$, which we may call ϕ_1, and where λ is any vector whatever, gives

$$\begin{aligned} q_1 &= i\beta\mathrm{S}\alpha i+j\beta\mathrm{S}\alpha j+k\beta\mathrm{S}\alpha k \\ &= -\alpha\beta\,; \end{aligned}$$

and in a similar manner the forms $\mathrm{V}\alpha\lambda$, $\mathrm{V}\alpha\lambda\beta$, and $g\lambda$ give, in order, 2α, $\mathrm{S}\alpha\beta$, and $-3g$. Any other terms that may occur are to be treated in this way, and the sum of the results taken.

We have thus the means of finding the effect of ∇ on any function, scalar or vector, by merely differentiating it.

4. The following useful formulæ will be familiar to students of Tait:—

$$\nabla(\mathrm{FP}) = \frac{d(\mathrm{FP})}{d\mathrm{P}} \nabla \mathrm{P},$$

and

$$\nabla(\mathrm{PP}_1) = \mathrm{P}_1 \nabla \mathrm{P} + \mathrm{P} \nabla \mathrm{P}_1,$$

in which the order is not important, and also

$$\nabla(\mathrm{P}\sigma) = \nabla \mathrm{P} \,.\, \sigma + \mathrm{P} \nabla \sigma,$$

where the order is vital. Here P and P_1 are scalars, FP is a scalar function of P, and σ is a vector.

To find the effect of ∇ on the product of any two vectors σ and τ we may adopt the notation $d\sigma = \phi d\rho$ and $d\tau = \theta d\rho$; whence

$$\begin{aligned} d(\sigma\tau) &= d\sigma \,.\, \tau + \sigma \,.\, d\tau \\ &= \phi d\rho \,.\, \tau + \sigma \,.\, \theta d\rho. \end{aligned}$$

From the scalar part of this differential we have

$$d\mathrm{S}\sigma\tau = \mathrm{S} d\rho (\phi' \tau + \theta' \sigma),$$

whence by (1),

$$\nabla \mathrm{S}\sigma\tau = -\phi'\tau - \theta'\sigma; \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

and from the vector part,

$$d\mathrm{V}\sigma\tau = \mathrm{V}\sigma\theta d\rho - \mathrm{V}\tau\phi d\rho,$$

each term of which, by the last article, contributes its portion of $\nabla \mathrm{V}\sigma\tau$. If we take $\phi_1 = \mathrm{V}\sigma\theta$, we have

$$\begin{aligned} q_1 &= i\mathrm{V}\sigma\theta i + j\mathrm{V}\sigma\theta j + k\mathrm{V}\sigma\theta k \\ &= -\mathrm{S} \,.\, \sigma\nabla\tau - \sigma \mathrm{S}\nabla\tau - \theta\sigma, \text{ (Tait, §§ 89, 90)} \end{aligned}$$

by the ordinary transformations.

Similarly the part corresponding to $-\mathrm{V}\tau\phi d\rho$ is

$$+\mathrm{S} \,.\, \tau\nabla\sigma + \tau \mathrm{S}\nabla\sigma + \phi\tau:$$

by adding the vector parts of these two quaternions we have

$$\mathrm{V}\nabla\mathrm{V}\sigma\tau = \tau \mathrm{S}\nabla\sigma - \sigma \mathrm{S}\nabla\tau + \phi\tau - \theta\sigma, \quad . \quad . \quad . \quad (6)$$

and by adding the scalar parts,

$$\mathrm{S}\nabla(\sigma\tau) = \mathrm{S} \,.\, \tau\nabla\sigma - \mathrm{S} \,.\, \sigma\nabla\tau; \quad . \quad . \quad . \quad . \qquad (7)$$

we have thus the three parts of $\nabla(\sigma\tau)$. Combining them gives

$$\nabla(\sigma\tau)=\tau S\nabla\sigma+(\phi-\phi')\tau+S.\tau\nabla\sigma$$
$$-\sigma S\nabla\tau-(\theta+\theta')\sigma-S.\sigma\nabla\tau;$$

but we have, identically,

$$V.V\nabla\sigma.\tau=(\phi-\phi')\tau, \quad . \quad . \quad . \quad . \quad (8)$$

by Tait, § 186; accordingly the first three terms of $\nabla(\sigma\tau)$ reduce to $\nabla\sigma.\tau$, and the last three, similarly, to $-\sigma.\nabla\tau-2\theta\sigma$; whence, finally,

$$\nabla(\sigma\tau)=\nabla\sigma.\tau-\sigma\nabla\tau-2\theta\sigma. \quad . \quad . \quad . \quad . \quad (9)$$

It may be noticed that $-\theta\sigma$ is the same as $S\sigma\nabla.\tau$.

If r and q are any two quaternion functions of ρ we have

$$\nabla(qr)=\nabla q.r-q\nabla r+2Sq.\nabla r+2S(Vq\nabla)r, \quad . \quad (10)$$

which follows on combining (9) with $\nabla(P\sigma)$, &c., and which the reader may verify with ease.

5. It is convenient to classify vectors by the effect of ∇ upon them: if $V\nabla\sigma$ vanishes, σ is derivable from a scalar potential and its distribution is *irrotational*; if $S\nabla\sigma$ vanishes, σ is derivable from a vector potential, and its distribution is *solenoidal*; while if both these conditions are fulfilled at once, so that $\nabla\sigma=0$, then the distribution is *Laplacean*. These distinctions are of fundamental importance in Physics.

There are also vectors which, though they do not directly satisfy the equation $V\nabla\sigma=0$, yet do so when multiplied by a variable scalar. Hamilton and Tait showed that we then have $S\sigma\nabla\sigma=0$. The simplest example is a unit-vector normal to a series of surfaces, and capable, therefore, of being written $U\nabla P$.

Taking the two vectors ∇P and $U\nabla P$, we shall adopt the notation:

$$dU\nabla P=d\nu=\chi d\rho,$$
$$d\nabla P \;=d(t\nu)=\psi d\rho;$$

the operators ψ and χ are then vector differentials, functions of ρ, and always linear in $d\rho$: their properties appear to be of considerable interest.

If α and β are any two *constant* unit-vectors, we shall have

$$d\left(\frac{dP}{dh_\alpha}\right)=-dS\alpha\nabla P, \text{ by (2)},$$
$$=-S\alpha d\nabla P, \text{ because } \alpha \text{ is constant},$$
$$=-S\alpha\psi d\rho;$$

whence by putting βdh_β for $d\rho$,

$$\frac{d^2\mathrm{P}}{dh_\beta dh_\alpha} = -\mathrm{S}\alpha\psi\beta = -\mathrm{S}\beta\psi\alpha, \quad . \quad . \quad . \quad . \quad (11)$$

where α and β are perfectly interchangeable, because either of them is any constant unit-vector whatever.

One consequence is that

$$\nabla\frac{d\mathrm{P}}{dh_\alpha} = \psi\alpha = \frac{d}{dh_\alpha}\nabla\mathrm{P}, \quad . \quad . \quad . \quad . \quad (12)$$

which may be extended to a vector by the usual method (Tait, § 149). Thus the operators ∇ and $\frac{d}{dh}$ are commutative, provided the direction h is constant. A single case of the same kind will presently be exhibited where the direction of differentiation is *not* constant.

6. The function χ, found by differentiating $\mathrm{U}\nabla\mathrm{P}$, or ν, owes most of its peculiarities to the fact that the differential of a unit-vector is always at right angles to the unit-vector itself (Tait, § 140, (2)); this is expressed by the equation

$$\mathrm{S}\nu\chi\epsilon = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where ϵ is any direction whatever. Thus the strain χ turns every vector into the tangent plane to the surface $\mathrm{P} = \text{const.}$

If we form the strain-cubic in the usual manner we find that the absolute term vanishes, so that

$$\chi(\chi^2 - m_2\chi + m_1) = 0$$

for any direction whatever. Thus the cubic has a zero root; for another way of finding it we have, λ and μ being any vectors whatever,

$$\chi\mathrm{V}\chi'\lambda\chi'\mu = 0, \text{ Tait, § 157, (2).}$$

By interchanging χ and χ' in this last equation we obtain

$$\chi'\nu = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

for $\mathrm{V}\chi\lambda\chi\mu$ is parallel to ν, by (13). It appears from these results that that direction for which $\chi = 0$ is at right angles to the plane into which χ' turns every vector; and *vice versâ*.

Whence, by taking a special case of (8),

$$(\chi - \chi')\nu = \mathrm{V}(\mathrm{V}\nabla\nu)\nu,$$

the left side reduces to $\chi\nu$, that is, $\frac{d\nu}{dn}$; and remembering that ν satisfies the equation $\mathrm{S}\nu\nabla\nu = 0$, we see that ν, $\chi\nu$, and

$V\nabla\nu$ are mutually at right angles, while $T\chi\nu = TV\nabla\nu$. These facts are expressed by the equation

$$\nu\chi\nu = V\nabla\nu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

Writing, as above, $\frac{d}{dn}$ for differentiation along the normal to the surface $P = \text{const.}$ we shall have

$$\begin{aligned} d\left(\frac{dP}{dn}\right) &= -d(S\nu\nabla P) \\ &= -S\nu\psi d\rho - S(t\nu)\chi d\rho\,; \end{aligned}$$

the last term vanishes by (13), and ψ is self-conjugate by (11), hence

$$\nabla\frac{dP}{dn} = \psi\nu = \frac{d}{dn}\nabla P, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

an equation which should be compared with (5) and with (12), from the former of which it may be deduced by applying (14).

7. We are now able to examine the criterion that the vector ν shall, besides being derivable from a scalar potential by means of a scalar factor, be derivable from one particular scalar potential which shall satisfy Laplace's equation; to find, in other words, the condition that a scalar t can be found such that $\nabla(t\nu) = \nabla^2 P = 0$.

Remembering that $\frac{dP}{dn} = t$, we shall have

$$\begin{aligned} \psi d\rho &= d(t\nu) \\ &= \nu dt + t d\nu \\ &= -\nu S d\rho\nabla t + t\chi d\rho \\ &= -\nu S d\rho\frac{d(t\nu)}{dn} + t\chi d\rho, \text{ by (16)}, \\ &= -\frac{dt}{dn}\nu S\nu d\rho - t\nu S\chi\nu d\rho + t\chi d\rho \\ &= -\frac{dt}{dn}\nu S\nu d\rho - t\chi\nu S\nu d\rho + t\chi' d\rho, \end{aligned}$$

where the last step follows because ψ is self-conjugate.

By inspection of this result it is evident that, upon any vector in the tangent plane, the strain ψ has the same effect as χ', with the sole difference that ψ allongates the vector by the factor t. There are important geometrical applications of this fact, some of which will be found in the examples at the end of this paper. But we are now concerned to get an expression for $\nabla(t\nu)$. It is proved above that

$$\psi d\rho = -\frac{dt}{dn}\nu S\nu d\rho - t\nu S\chi\nu d\rho + t\chi d\rho,$$

where, by Art. 3, the first two terms give $-\frac{dt}{dn}$ and $-t\nu\chi\nu$, and the last term gives $t\nabla\nu$. Thus if P satisfies Laplace's equation, then

$$-\frac{dt}{dn} - t\nu\chi\nu + t\nabla\nu = 0;$$

the vector part gives an independent proof of (15); the scalar part is

$$\frac{dt}{dn} = tS\nabla\nu,$$

and since it has already been proved that, in general,

$$\nabla t = \frac{d(t\nu)}{dn} = \nu\frac{dt}{dn} + t\chi\nu,$$

we have, provided P satisfies Laplace's equation,

$$\nabla t = t(\nu S\nabla\nu + \chi\nu). \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The vector $\nu S\nabla\nu + \chi\nu$ may be written $\nabla\nu\,.\,\nu$; and because $\nabla^2 t$ is a scalar,

$$\begin{aligned} V\nabla(t\nabla\nu\,.\,\nu) &= 0 \\ &= V\,.\,\nabla t(\nabla\nu\,.\,\nu) + tV\nabla(\nabla\nu\,.\,\nu); \end{aligned}$$

which reduces at once to

$$V\nabla(\nabla\nu\,.\,\nu) = 0;$$

from (10), putting $\nabla\nu$ for q and ν for r and taking vectors,

$$V(\nabla^2\nu)\nu - V(\nabla\nu)^2 + 2S\nabla\nu\,.\,V\nabla\nu - 2\chi V\nabla\nu = 0,$$

where the second and third terms destroy each other, so that finally

$$V\nu\nabla^2\nu + 2\chi V\nabla\nu = 0, \quad . \quad . \quad . \quad . \quad (18\,a)$$

which is the required condition.

The same essential fact is expressed by saying that $\nabla\nu\,.\,\nu$ must be integrable *without* a factor, or that there must exist a scalar—call it u—such that

$$u = \nabla^{-1}(\nabla\nu\,.\,\nu). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

8. To examine the properties of $\chi V\nabla\nu$, we may write, as a

special case of (8),

$$(\chi-\chi')V\nabla\nu=0,$$

which means that χ and its conjugate have the same effect on $V\nabla\nu$. But it was shown in Art. 6 that χ turns every vector into a certain plane, and χ' turns every vector into another plane; hence $\chi V\nabla\nu$ lies along the line of intersection of these two planes.

If ν' be a unit-vector such that $\chi\nu'=0$, it follows that

$$\chi V\nabla\nu=xV\nu\nu';$$

to determine the unknown scalar x, take ϵ and η two unit-vectors such that $\chi\epsilon=g\epsilon$ and $\chi\eta=g_1\eta$; it may be easily shown that ϵ, η, and ν will then form a rectangular system (see Ex. 2); and they may be taken so that $\epsilon\eta=\nu$. It is then legitimate to write $\frac{d\nu}{dn}$ in the following form (Tait, § 176):

$$S\nu\nu'\,.\,\chi\nu=-g\epsilon S\epsilon\nu'-g_1\eta S\eta\nu';$$

operate by $V\nu$,

$$S\nu\nu'V\nabla\nu=-g\eta S\epsilon\nu'+g_1\epsilon S\eta\nu',$$

and by using again the same form of χ,

$$-\chi V\nabla\nu=\frac{gg_1}{S\nu\nu'}\{-\epsilon S\eta\nu'+\eta S\epsilon\nu'\}$$

$$=\frac{m_1}{S\nu'\nu}V\nu'\nu,$$

where m_1 is the coefficient of χ in the strain-cubic. Thus if l be the angle between ν and ν', the tensor of $\chi V\nabla\nu$ is $m_1\tan l$.

9. If, further, σ be any vector in the tangent plane, so that at all points $S\sigma\nu=0$, then by (8)

$$(\phi-\phi')\nu+V\nu V\nabla\sigma=0;$$

here $\phi\nu$ may be written $\frac{d\sigma}{dn}$; by (5) we obtain

$$\nabla S\sigma\nu=0=-\chi'\sigma-\phi'\nu;$$

the values of $\phi\nu$ and $\phi'\nu$ give by substituting,—

$$\frac{d}{dn}+\chi'+V\nu V\nabla=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

provided the operand be at right angles to ν.

Operating on $V\nabla\nu$ and substituting the result in (18 *a*) gives

$$V\,.\,\nu\nabla S\nabla\nu=\left(\frac{d}{dn}-\chi\right)V\nabla\nu. \quad . \quad . \quad . \quad (18\,b)$$

Again, by using the value of $V\nabla\nu$ from (15),

$$\frac{d}{dn}V\nabla\nu=V\nu\frac{d^2\nu}{dn^2},$$

and this, combined with the result of the last article, gives

$$V\,.\,\nu\nabla S\nabla\nu=V\nu\left(\frac{d^2\nu}{dn^2}-m_1\nu' S^{-1}\nu'\nu\right). \quad . \quad . \quad (18\,c)$$

One other transformation is obtained from the $\chi\nu$ of the last article by putting $\omega\nu'=-g_1\eta S\eta\nu'-g\epsilon S\epsilon\nu'$, so that the component of ν' at right angles to ν is $S\nu'\nu\,.\,\omega^{-1}\chi\nu$, and this gives, by substituting in (18 *c*),—

$$V\nu\left(\frac{d^2\nu}{dn^2}-m_1\omega^{-1}\frac{d\nu}{dn}-\nabla S\nabla\nu\right)=0\,; \quad . \quad . \quad (18\,d)$$

that is, the vector in parentheses is normal to the surface $P=\text{const.}$ Here it is noteworthy that both the vector $V\nu\nabla S\nabla\nu$ and the linear and vector function $m_1\omega^{-1}$ are numerically determinate all over a given surface $P=P_0$. Thus (18) shows the character of ν, provided $\nabla^2P=0$, in the immediate neighbourhood of the given surface.

If ν be so given as to satisfy (18), P may be written $\nabla^{-1}(t\nu)$, and is determined by (19), since $u=\log t$ by (17).

Examples.

1. Give in terms of χ the curvature of a normal section of the surface $P=\text{const.}$ (Tait, § 350, where ν is the $t\nu$ of this paper.)

2. Show that two of the roots of the cubic in χ correspond to the sections of greatest and least curvature.

3. If ν' correspond to the other root, show that if ν, ν' and $\chi\nu$ are coplanar, $\chi^2\nu$ is parallel to $\chi\nu$. Of what class of surfaces is this a property?

4. Show that if P is a homogeneous function of x, y, and z, any straight line through the origin cuts the surfaces denoted by P at a constant angle.

5. Show that if P is a homogeneous spherical harmonic, $\nabla^{-1}(\nabla\nu\,.\,\nu)=\text{const.}$ is the equation of a cone.

6. For what class of surfaces *may* ν' lie in the tangent plane?

7. For what class of surfaces is χ self-conjugate? (Tait, § 332).

8. Discuss the pure and the rotational parts of the strain χ.

9. Prove the identities:—

$$(a) \quad \chi^2\nu = \mathrm{V}\,.\,\nu\chi\mathrm{V}\nabla\nu - \mathrm{S}\nabla\nu\,.\,\chi\nu,$$

$$(b) \quad \chi'\chi\nu = \chi^2\nu - \nu(\chi\nu)^2.$$

10. Discuss the pure strain $\chi + \chi'$.

11. Interpret $\mathrm{TV}\nabla\nu$ and $\frac{d}{dn}\mathrm{UV}\nabla\nu$. (Tait, §§ 299, 300.)

12. Show by (11) of Art. 5 that ∇^2 is the negative of Laplace's operator.

13. Show that $\frac{d}{dn}$ and $\frac{d}{dh}$ are commutative when applied to P, h being parallel to $\mathrm{V}\nabla\nu$.

14. With everything as in Art. 4, prove

$$\nabla(\sigma\tau) = \nabla\sigma\,.\,\tau - \nabla\tau\,.\,\sigma - 2\theta'\sigma.$$

15. Show by (7) that $\mathrm{V}\nabla\chi\nu$ lies in the tangent plane.

16. Use (6) of the same article to find $\mathrm{V}\nabla(\nu\mathrm{S}\nabla\nu + \chi\nu)$.

LXVIII. *Some Experiments on Electric Waves in Short Wire Systems, and on the Specific Inductive Capacity of a Specimen of Glass.* *By* J. A. Pollock, *Professor of Physics, and* O. U. Vonwiller, *Deas-Thomson Scholar in Physics in the University of Sydney*[*].

THE experiments described in the following paper include observations of the waves along free wires, and also of the vibrations in the two systems formed when the wires are bridged at various points. In the former case it is shown, that when the electrical vibrations of the wire system are forced, they are in that mode whose free period is near to that of the vibrator oscillation, and therefore the distance between the nodes along the wires does not vary continuously with change of the period of the condenser discharge In the other case an explanation is found which accounts for the varying heights of the maxima deflexions observed when the bridge is moved along the wires. A method for finding the specific inductive capacity of solid dielectrics with

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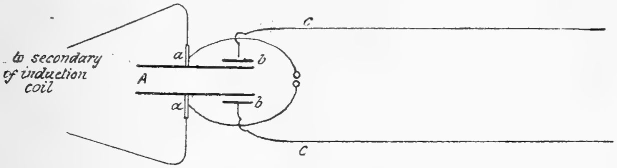
[*] Communicated by the Authors.

Hertzian waves is developed, which is more sensitive than others which have hitherto been used.

Apparatus.

The arrangement of the apparatus used in these experiments is shown in the following figure.

Fig. 1.



A is a condenser made of two circular brass plates 30 cms. in diameter and 0·853 cm. thick, the inner surfaces being scraped plane. One plate is mounted on a stand which can be moved by a screw. This plate can thus be set at any desired distance from the other plate, the two being always strictly parallel. The plates are supported by stout hard rubber tubes in which are placed the brass lugs *aa*. Attached to these lugs are two brass wires 0·3 cm. thick, each 30 cms. long, bent to form the discharge circuit as shown in the figure, the sparks taking place between aluminium spheres 1·1 cms. in diameter. These wires are in a horizontal plane; 1·8 cms. from their free ends they are bent vertically downwards, the aluminium spheres dipping into a large basin of paraffin oil. It is found essential for uniformity in the character of the spark to keep the oil well stirred. To enable the capacity of the condenser to be changed without altering the length of the spark-gap the discharge wire attached to the movable plate is cut, and connexion remade through a little pool of mercury. The capacity of the condenser circuit can thus be altered without changing any other condition; the alteration in the inductance of the circuit due to the slight change in its length through the pool of mercury being negligible.

Attached to the condenser-plates, but carefully insulated from them, are two smaller plates, *b b*, 8·8 cms. in diameter, 0·31 cm. thick, the distance between them and the condenser-plates being 0·75 cm. To these plates are attached brass wires, *c c*, 0·04 cm. in diameter, which are carried 30 cms. apart in a horizontal plane to distances varying from 7 to 40 metres, their ends being attached by small loops to insulated supports.

Across these wires is placed a light wood slider, the wires *c c* passing through small glass tubes attached to it. The slider carries a small piece of hard rubber provided with two mercury cups. From each mercury cup runs a short piece of wire, which ends in a single loop encircling the glass tube at the end of the slider.

The magnetic detector invented by Rutherford, and described by him (Phil. Trans. clxxxix. (1897) p. 8), is used to determine the characteristics of the waves along the wires, the ends of the little solenoid of the detector being attached to the mercury cups of the slider. The slider is in fact similar to that used by Rubens (Wied. *Ann.* xlii. p. 154, 1891), Rutherford's detector being used instead of the bolometer-wire. In some cases, observations have been taken with the slider in various positions along the wires ; in others, the slider was kept near the ends of the wires, and a wire bridge of negligible resistance placed across them at various points along their length. In both cases the steel wires of the detector are magnetized to saturation, the detector then placed in an invariable position behind a reflecting magnetometer, and the reading taken. The solenoid of the detector is then attached to the slider wires, the condenser discharged once, and the reading of the magnetometer again taken. The difference between these readings is in all cases called the deflexion.

Calculation of the Constants of the Condenser Circuit.

In the following discussion, the wave-length in air of the condenser radiation is required; it is therefore necessary to calculate the capacity and inductance of the circuit. The capacity was measured by the tuning-fork method, that of the tuning-fork apparatus and connexions being carefully eliminated.

The results of these experiments show that the capacity of the condenser-plates alone, in air, when separated by 1·837 cms. is 42·4 electrostatic units. The discharge wires and knobs add 3·1 units. The plates *b b* add nothing to the capacity. When these small plates were attached to the wires *c c*, the measurements being made with the apparatus *in situ*, the measured capacity was increased by 5 to 7 units, according to the length of the wires. It is doubtful if this wire capacity affects the period of the condenser oscillation, and it has been left out of account. The capacity of the condenser-plates alone, as calculated by Kirchhoff's formula (*Abhandl.* p. 113), is 30·62 + 9·48 (edge correction) = 40·10. The difference between this value and that found experimentally (42·4) is due to the presence on the outer sides of the condenser-plates, but near to their edges, of eight small brass screws employed in holding hard rubber brackets which were used for adjusting the plates parallel to one another. These screws were subsequently dispensed with.

In some of the experiments, a circular glass disk, 1·837 cms. thick and 17·2 cms. in diameter, was mounted between the condenser-plates, coaxial with them, the plates being in contact with the glass on either side. The specific inductive capacity of the glass as determined later is 7·8. The formula for calculating the capacity of the condenser under these circumstances is given in the discussion of the specific inductive capacity measurements. To this calculated capacity must be added the small corrections for screws and discharge circuit as found previously. In other experiments, a glass plate of the same thickness, but 35·5 cms. in diameter, was placed between the condenser-plates, the plates touching the glass as before. The glass plate being larger in diameter than the brass plates, it is impossible to evaluate the edge correction to the capacity. All that need be said, however, is that the capacity is greater than $\frac{\mathrm{K}r^2}{4a}$ + the edge correction in air, K being the specific inductive capacity of the glass 7·8, r the radius of the brass plates and a the distance between them.

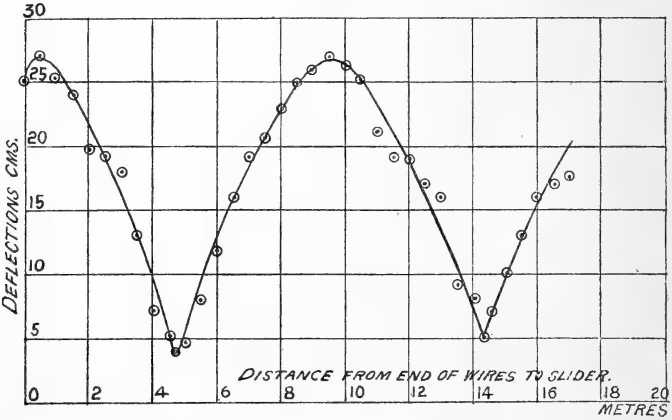
The formula which has been used for calculating the inductance of the discharge circuit is $\mathrm{L}=2l\left(\log_e\frac{8l}{\pi d}-2\right)$, l being the length of the circuit assumed circular, and d the diameter of the wires. The circuit is made up of a pair of thick brass wires 0·3 cm. in diameter and of total length 60 cms., and a pair of stout brass lugs connected to the condenser-plates, of total length 4·5 cms. and 1·2 cms. in diameter, into which the discharge-wires are fastened. The inductance has been calculated on the assumption that the total length of the circuit is 64·5 cms., the extra diameter of the lugs being neglected.

The wave-length in air of the condenser radiation has been found by the formula $\lambda=2\pi\sqrt{\mathrm{L.C}}$, C being in electrostatic units. From a resonance experiment, to be described later, the calculated wave-length is found to be about 5 per cent. too large, and other evidence points to the same conclusion. The calculated values have therefore been reduced by this amount. For the present purpose, only approximate values are required.

Experiments without a Bridge across the Wires.

In experiments with waves along short wires where no bridge is used, the system of longitudinal currents is determined by the condition that the currents must vanish at the ends of the wires. All modes of vibration having an integer number of nodes along the wires are therefore possible. In these experiments various lengths of wires have been used, the position of the nodes being determined to within 5 cms. With more observations, the position could have been found to within 1 cm., but this accuracy was considered unnecessary. Successive values of the internode in any one experiment usually agree to within 2 per cent. of the mean value. Fig. 2 is the plot of the observations with wires 17·5 inches long, the ordinates being deflexions in centimetres, and the abscissæ distances of the slider along the wires in metres, measured from their free ends.

Fig. 2.



The results of the experiments are given in Table I. In calculating the distance between the nodes for various modes of vibration, 1·6 m. has been added to the length of the wires, an amount which will be justified later.

In such experiments, when the vibrations of the electrical condition of the wire system are forced, it may be said generally that that mode of vibration occurs whose free period is near to that of the condenser oscillation. The distance between the nodes on the wires depends on the length of the wires, and does not alter continuously with change in the condenser capacity. It is therefore impossible to determine the ratio of the condenser capacities in two such experiments

TABLE I.

Wire-length, in metres.	Wave-length in Air of Condenser Radiation, in metres.	Observed Internode × 2, in metres.	Calculated internode × 2 for various number of nodes in wire-length + 1·6 metres.				
			Nodes.				
			2.	3.	4.	5.	6.
7·00	9·5	8·7	8·60	5·73			
8·98	9·5	7·0	10·58	7·05	5·29		
9·47	9·5	7·2	11·07	7·38	5·53		
10·05	9·5	7·7	11·65	7·77	5·82		
10·05	Slightly>22	11·4	11·65	7·77	5·82		
12·05	9·5	9·2	13·62	9·08	6·81		
15·00	9·5	8·8	...	11·07	8·30	6·64	
17·50	Slightly>22	19·0	19·10	12·73			
20·00	9·5	8·4	21·60	14·40	10·80	8·64	7·20
20·00	Slightly>22	21·2	21·60	14·40	10·80	8·64	7·20
21·00	Slightly>22	21·6	22·60	15·07			
41·50	Slightly>22	21·0	...	28·73	21·55	17·24	

with the same length of wire, from the observed distance between the nodes; and Thomson's value of the specific inductive capacity of glass, 2·7, as deduced from experiments described in Proc. Roy. Soc. xlvi. p. 292 (1889), must be considered quite erroneous.

Experiments with the Wires Bridged.

When the detector is placed near the ends of the wires and a bridge moved along them, maxima deflexions occur as shown by Lecher (Wied. *Ann.* xli. p. 850, 1890) and Rubens (Wied. *Ann.* xlii. p. 154, 1891), when the lengths of the two circuits into which the wires are divided by the bridge are in the ratio of small odd numbers. Small amounts have to be added to the measured lengths of the two circuits, on account of change of phase by reflexion. Having determined the positions of the bridge corresponding to maxima deflexions, assuming that these positions depend only on the lengths of the two circuits, the lengths to be added may be determined in the well-known way. Let M be the total length of one of the wires, a the distance of the bridge from their free ends for any maximum, l the length of the bridge, c the amount to be added to the free ends of the wires, C that to the other ends. The measured lengths of the two circuits are

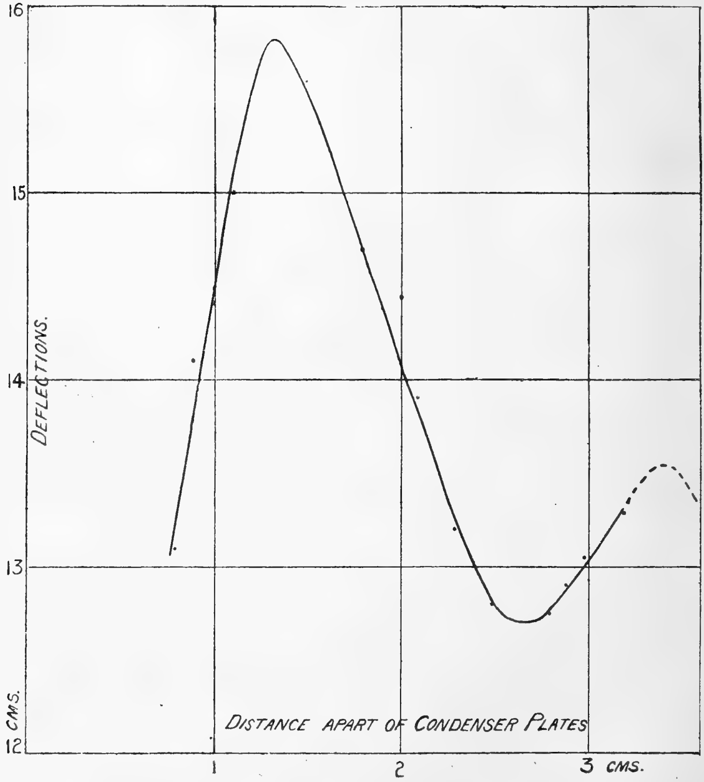
$$2(\mathrm{M}-a)+l, \quad 2a+l;$$

let the ratio of these be nearly equal to m/n where m and n

are small odd numbers, then the ratio of $2(M-a+C)+l$ to $2(a+c)+l$ is to be equal to m/n. Ten such equations have been formed from the results of experiments with various lengths of wires. The main values of C and c found from these equations are $C=1{\cdot}52$ metres, $c=0{\cdot}11$ metre. $C+c=1{\cdot}63$, the value which has been used in Table I. These corrections apply to all the experiments.

With the bridge in a fixed position, the magnitude of the readings obtained with the detector depend on the period of the oscillation of the condenser circuit, and also on the length of the spark-gap. Keeping the latter constant, the curve in fig. 3, freely drawn through the observations, gives the relations between the deflexions as ordinates and the distance between the condenser-plates as abscissæ, both measured in centimetres. The wires in this experiment were 13·62 metres long, uncorrected, and the bridge was placed at 2·33 m. from their free ends.

Fig. 3.



Calling that part of the wire system near the condenser the primary, and the part remote the secondary circuit, in the case shown in fig. 3 the corrected length of the primary circuit is 25·92 m., that of the secondary being 5·18 m. The ratio of the lengths of the primary and secondary circuits being 5/1, it is necessary for complete resonance between them that there should be five nodes in the primary circuit, or any odd multiple of this number. As the condenser-plates are separated the amplitude of the vibration in the secondary circuit, as shown by the deflexions, increases as the period of the condenser oscillation approaches more nearly that of the free period of some mode of vibration of the primary circuit. From the figure, it appears that there is complete resonance between the condenser and primary circuits when the condenser-plates are separated by 1·33 cms. At this point the calculated wave-length in air of the condenser radiation is 11·0 m., while if it is considered that the length of the primary circuit is $\frac{5}{2\lambda}$, λ comes out 10·4 m. The calculated values may therefore be considered about 5 per cent. too high.

As the condenser-plates are further separated the deflexion rapidly decreases. A second maximum occurs when the wave-length of the condenser radiation is equal to 2/7 of the length of the primary circuit; a condition which could not be quite reached with the present apparatus. The second maximum is shown in the figure by the dotted line. It is less than the first one, for although at this point there is complete resonance between the condenser and primary circuits, the conditions are not right for complete resonance between the primary and secondary circuits, there being 7 nodes in the primary circuit, the ratio of the lengths of the two circuits being as before 5/1.

In general, when the bridge is in a fixed position the amplitude of the vibration in the secondary circuit will depend on that in the primary circuit. Let the ratio of the lengths of the two circuits be m/n, m and n being small odd numbers. If P is the length of the primary circuit and λ the wave-length of the condenser radiation, the amplitude of the vibration in the primary circuit will be a maximum when

$$\mathrm{P}=\frac{2k+1}{2}\lambda,$$

where k is an integer. So the amplitude of the vibration in the secondary circuit (directly measured by the detector deflexions) will be a maximum as the condenser capacity is altered, whenever λ is such as to be equal to $2P/2k+1$. Of these maxima that one will be the greatest corresponding to $2k+1=m$, or any odd multiple of m.

When the bridge is moved along the wires, it is not right to assume that maxima deflexions of a detector in the secondary circuit occur when the wire system is divided exactly in the ratio of small odd numbers. For suppose the bridge in such a position that it exactly divides the system in such a ratio, and consider that the length of the primary circuit is such that a certain mode of its vibration has a free period very near to that of the oscillation of the condenser circuit. A small movement of the bridge certainly makes the resonance less perfect between the primary and secondary circuits, but may increase the amplitude of the vibration in the primary circuit. The amplitude of the vibration in the secondary circuit may, on the whole, become greater. This effect would be more marked in those cases where the length of the primary is small compared with that of the secondary circuit.

If the detector is placed in the primary instead of in the secondary circuit, a similar curve to the first part of the one shown in fig. 3 gives the relation between deflexion and distance apart of condenser-plates as the period of the condenser oscillation decreases from a value greater to a value less than that of some mode of vibration of the primary circuit.

Experiments have been made with the following lengths of wires :—8·98, 9·47, 10·05, 10·50, 12·00, 18·85 metres. Observations have in general only been taken for successive positions of the bridge differing by 0·5 metre, so no doubt some fine detail is not included. At the time the experiments were made, the object was merely to locate the position of the bridge giving the principal maxima.

Fig. 4.

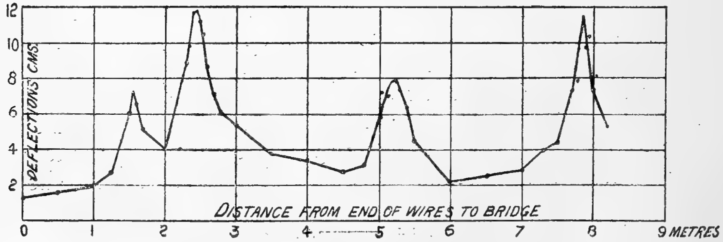


Fig. 4 gives the plot of the results obtained with wires 8·98 metres long, the ordinates being deflexions in cms., and the abscissæ lengths measured along the wires from their free ends to the bridge.

The results in three cases are given in Table II. It is not considered necessary to give more than these, as the range in the lengths of the wires used is not sufficient to alter the general character of the observations. The lengths of the circuits given in the table are those when maxima deflexions occurred, corrected as previously mentioned, the bridge being 0·30 metre long. Under the heading of Length of Primary × 2 ÷ *k* are given the wave-lengths of some possible modes of vibration of the primary circuit.

TABLE II.

Length of primary circuit. *m.*	Length of secondary circuit *n.*	*m/n.*	Wave-length of condenser radiation.	Length of primary × 2 ÷ *k*				Character of Maximum.
				k. 1	3	5	7	
II *a.* Wire-length 8·98 m.								
18·20	3·62	5/1	15·2	36·40	12·13	7·28	...	Low.
16·40	5·42	3/1	...	32·80	10·93	6·56	...	High.
10·90	10·92	1/1	...	21·80	7·27	...	...	Low.
5·50	16·32	1/3	...	11·00	3·67	...	...	High.
II *b.* Wire-length 10·05 m.								
19·94	4·02	5/1	9·5	39·88	13·29	7·98	...	High.
17·97	5·99	3/1	...	35·94	11·98	7·15	...	None.
15·34	8·62	—	...	30·68	10·23	6·14	...	Low.
12·04	11·92	1/1	...	24·08	8·02	4·82	...	Medium.
9·44	14·52	—	...	18·88	6·29	3·78	...	Low.
II *c.* Wire-length 18·85 m.								
35·85	5·72	7/1	15·2	71·68	23·89	14·33	10·24	Medium.
34·64	6·92	5/1	...	69·28	23·09	13·86	9·90	Highest.
31·04	10·52	3/1	...	62·08	20·69	12·42	8·87	Low.
26·04	15·52	5/3	...	52·08	17·36	10·41	7·44	High.
21·04	20·52	1/1	...	42·08	14·03	8·42	...	High.
15·74	25·82	3/5	...	31·48	10·49	6·29	...	High.

Rubens (Wied. *Ann.* xlii. p. 154, 1891) has shown that the electrical vibration in the secondary circuit when the deflexion is a maximum, is in that mode of vibration required for resonance with the vibration in the primary circuit. Assuming that that mode of vibration of the primary circuit occurs whose free period is near to that of the condenser oscillation, a condition which has been shown to hold for unbridged wires, the results contained in Table II. may, with one or two exceptions, be explained in the following manner:—In II *a* for the first maximum that mode of vibration of the primary circuit having three nodes would no doubt occur, the wave-length in this case 12·13 being nearer to the wave-length of the condenser radiation 15·2, than that of any other possible mode of vibration. Although the lengths of the primary and secondary circuits are as 5/1, there is not perfect resonance, for this requires that there should be at least five nodes in the primary circuit. The maximum is therefore low. For the second maximum, the same mode of vibration of the primary circuit having a wave-length 10·93 is more out of tune with the condenser circuit than before, but the condition for resonance between the primary and secondary circuits is now perfect, the lengths being as 3/1. The more perfect resonance between the primary and secondary circuits more than compensates for the lower amplitude of vibration in the primary circuit, and the maximum is higher than the previous one.

In II *b* for the first maximum the number of nodes in the primary circuit would be five, and as the lengths of the two circuits are as 5/1, the maximum is high. There is no maximum in this case when the lengths of the circuits are in the ratio of 3/1, the reason apparently being that the vibration in the primary circuit is ill-defined on account of the wave-length of the condenser radiation 9·5 being midway between those of two of the possible modes of vibration of the primary circuit, 11·98 and 7·15. Such a condition is illustrated in the case of Melde's well-known experiments on the vibrations of strings when the tension is midway between the tensions required for two successive modes of vibration. A similar explanation accounts for the third maximum being low in the case of the experiments with wires 8·98 long, the results of which are given in the first division II *a* of the Table.

In II *b* the second maximum, which is a low one, occurs for a position of the bridge which makes the lengths of the circuits nearly in the ratio of 5 to 9. In this case the primary circuit with three nodes is very nearly in tune with the condenser circuit, and the amplitude of the vibration in it must be considerable; the conditions for perfect resonance between the primary and secondary circuits are not, however, satisfied. The character of the next maximum is one of the exceptions to this general explanation. The primary circuit with three nodes is not more out of tune with the condenser circuit than it was with five nodes in the case of the first maximum.

It might certainly have been expected that the third maximum would be as high as the first one; it is, however, less. The fourth maximum occurs for a position of the bridge which does not make the lengths of the circuits in the ratio of small odd numbers. It would appear here that a comparatively strong vibration of twice the period of the condenser oscillation is excited in the primary circuit, giving a stronger impulse to the secondary circuit than in the case of adjacent positions of the bridge, the wave-length of the condenser radiation being 9·5, and that of the primary circuit 18·88.

Another exception occurs in II *c*. The fifth maximum should be higher than the second, it is a little less. For the fifth maximum the conditions are favourable for perfect resonance between the primary and secondary circuits, the ratio of their lengths being as 1 to 1. The mode of vibration of the primary circuit having three nodes gives a wave-length 14·03 rather nearer to that of the condenser radiation 15·2 than 13·86, the wave-length of the mode of vibration having five nodes in the case of the second maximum. With these two exceptions, the explanation seems to account for all the results which have been obtained. The question of the relative heights of the maxima was not considered when the observations were taken, so no special efforts were made to keep the character of the spark absolutely unaltered.

The explanation depends on the assumption that the vibrations in the primary circuit are forced and not free vibrations of the system. A similar explanation probably holds in the case of the beautiful curves given by Rubens (Wied. *Ann.* xlii. p. 154, 1891). If the vibrations of the primary circuit in Rubens's experiments are free and not forced by the vibrator, it is difficult to see why any other mode of vibration should occur than the fundamental one. If the primary circuit always vibrates in its fundamental mode, those positions of the bridge which divide the wires into circuits requiring this mode for perfect resonance between them should give higher maxima than other positions. This is not the case. On the other hand, J. J. Thomson's argument in discussing these experiments ('Recent Researches,' § 388), showing that the vibrations in the wires which are detected by the bolometer cannot be forced by the vibrator, is not sound. Thomson assumes that if the vibrations are forced, no capacity being at the ends of the wires and no bridge placed across them, the places where the electromotive intensity at right angles to the wire vanishes are an odd number of quarter-wave-lengths of the condenser radiation from the ends of the wires. This is incorrect, as shown by the results of the experiments described in the early part of this paper.

The Specific Inductive Capacity of a Specimen of Glass.

It has been shown above that, with the bridge in a fixed position, the magnitude of the readings obtained with the detector depend on the period of the oscillation of the condenser circuit. Keeping the inductance of the circuit constant, under certain conditions, the deflexions alter rapidly with change of capacity. The proper relation between the period of the condenser discharge and the length of the wires for most rapid variation of deflexion with respect to change of capacity, is readily found for any set of circumstances. The conditions must be such as to give a relation between deflexion and distance apart of condenser-plates similar to that represented by the steep sides of the curve in fig. 3.

Advantage has been taken of this fact of rapid change of deflexion with respect to change of capacity, to determine the specific inductive capacity of a piece of glass.

All the specimens of glass which have been used are pieces of "thick polished plate glass" made by the Compagnie de Floreffe, Floreffe, Belgium. They were cut at the same time from one sheet. The indices of refraction at 15° C., as determined from observations of the angles of minimum deviation with a prism of the material, are given in the following table:—

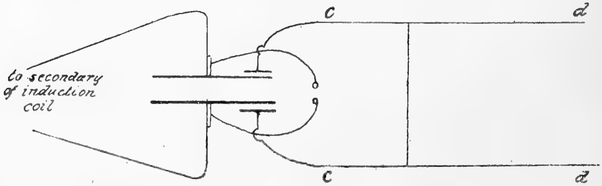
B 1·5283, C 1·5296, D 1·5320, E 1·5355,
b_1 1·5361, F 1·5386, G 1·5446.

The density of the glass at 23° C. is 2·54.

Care has been taken in every case to have the glass clean and dry.

The scheme of the arrangement by which the specific inductive capacity of the glass has been determined is shown in fig. 5.

Fig. 5.



Each of the wires *c d* was 4·94 metres total length, 4·675 m. straight, and 0·265 m. bent round at condenser end. The detector was placed in the primary circuit 30 cms. from *c c*, the bridge being 1·10 m. from the same line. Preliminary trials had shown that the capacity of the condenser with its plates separated by 0·6 cm., air being the dielectric, was approximately equal to its capacity when the glass was between the plates. The position of the bridge was so chosen, by trial, that a slight change in the distance between the condenser-plates, when the distance was about 0·6 cm., resulted in the greatest change in the deflexion given by the detector. The sensitiveness was such that a change of 0·1 cm. in the distance between the plates caused a change of about 3 cms. in the observed deflexion. An equally sensitive arrangement can be made with the detector in the secondary, instead of in the primary circuit.

It must be remembered that any change in the spark produces a considerable change in the deflexions, so great precautions have to be taken to keep the character of the spark constant during a set of observations. This was effected in the present instance by not allowing the spark-gap to alter, by only taking single sparks for each observation, and so minimizing the deterioration of the surface of the spark-knobs, and by keeping the large bath of paraffin oil through which the sparks passed well stirred.

Great care has been exercised to keep the plates parallel to one another, and the distances in each case have been very carefully measured.

The procedure consists in finding a mean deflexion with the glass between the plates, then with the glass removed obtaining mean deflexions for two or three distances with air.

The following tables give the observations which have been made, a being the distance between the condenser-plates. The diameter of the glass disk is 17·2 cms. and its thickness 1·837 cms.

Table III.

	Glass between plates. $a=1·837$ cm. Defls., cms.	Air between plates. $a=0·554$ cm. Defls., cms.	Air between plates. $a=0·578$ cm. Defls., cms.	Air between plates. $a=0·602$ cm. Defls., cms.
Dec. 18th, 1901.	17·3	...	16·7	18·0
	17·4	16·8	17·4	17·4
	17·6	16·9	18·2	17·4
	17·1	16·6	17·0	
	17·4	...	17·4	
Mean	17·36	16·77	17·34	17·60

$a=0·578$ cms. with air gives the same deflexion as $a=1·837$ cms. with glass between the plates. Temp. 27°·8–28°·5 C.

TABLE IV.

	Glass between plates. $a=1{\cdot}837$ cm. Defls., cms.	Air between plates. $a=0{\cdot}498$ cm. Defls., cm.	Air between plates. $a=0{\cdot}584$ cm. Defls., cms.
Dec. 19th, 1901.	16·6	13·5	15·5
	14·9	14·7	15·7
	15·8	13·1	16·8
	15·7		16·4
	16·3		15·0
	15·5		
Mean	15·80	13·77	15·88

$a=0{\cdot}581$ cm. with air gives the same deflexion as
$a=1{\cdot}837$ cms. with glass between the plates. Temp. 25·0 C.

TABLE V.

	Glass between plates. $a=1{\cdot}837$ cm. Defls., cms.	Air between plates. $a=0{\cdot}572$ cm. Defls., cms.	Air between plates $a=0{\cdot}585$ cm. Defls., cms.	Air between plates. $a=0{\cdot}592$ cm. Defls., cm.
Dec. 19th, 1901.	14·7			
	13·9	13·1	15·0	14·4
	14·2	15·1	15·2	15·5
	13·5	15·1	14·5	14·5
	13·3	13·5	13·6	
	14·5	13·3	14·3	
	13·7	14·3		
	14·6			
Mean	14·05	14·07	14·52	14·80

$a=0{\cdot}572$ cm. with air gives the same deflexion as
$a=1{\cdot}837$ cms. with glass between the plates. Temp. 25°·0 C.

Calculation of the Specific Inductive Capacity.

If in two successive experiments the deflexions are the same, it is assumed that the capacities in the condenser circuits are equal. The condenser circuit has been carefully arranged so that when the distance between the plates is altered, the only capacity change is that of the plates. In these experiments there are no attachments to the plates except the central lugs on their outer surfaces. Whatever may be the capacity of these lugs, it may be considered not to change when the distance between the plates alters by a small amount. The total capacity in the condenser circuit is, for the present purpose, a matter of no concern. The change in the capacity is of importance alone, and this may be accurately determined by an application of Kirchhoff's formula (*Abhandl.* p. 113).

Let C be that part of the capacity of the condenser circuit which does not change when the distance between the plates is altered. With air between the plates the capacity of the circuit is:—

$$\frac{r^2}{4a}+\frac{r}{4\pi a}\left[a+a\log_e\frac{16\pi r(a+d)}{a^2}+d\log_e\frac{a+d}{d}\right]+\mathrm{C},$$

r being the radius of the plates, d their thickness, and a the distance between them.

Professor Bragg, to whom we are indebted for much help in this matter, pointed out to us that under certain conditions the capacity of a circular plate condenser with a slab of material between the plates can be accurately calculated. If, for instance, a cylindrical disk of glass considerably smaller in diameter than the condenser-plates is placed coaxially with them, the plates touching the glass on either side, the edge of the glass disk may be considered everywhere parallel to the tubes of force touching it, and no disturbance of the external field takes place. This condition has been carefully kept in view in these experiments. With the glass between the condenser-plates the capacity of the circuit is therefore:—

$$\frac{\mathrm{K}r_1^2}{4a_1}+\frac{r^2-r_1^2}{4a_1}+\frac{r}{4\pi a_1}\left[a_1+a_1\log_e\frac{16\pi r(a_1+d)}{a_1^2}+d\log_e\frac{a_1+d}{d}\right]+\mathrm{C},$$

K being the specific inductive capacity of the glass, r_1 the radius of the glass disk, and a_1 its thickness (=the distance between the plates), the other symbols being as just mentioned.

If a value of a can be found which makes the capacity in the two cases equal, equating the two expressions enables the value of K to be calculated. Calculating K with the value of a, given in Tables III., IV., and V., the results are:—

7·85
7·80
7·95

Mean ...... 7·87 at 26°·0 C.

The specific inductive capacity of the same disk of glass has been measured by an absolute electrometer method with alternating potential-differences having a frequency of about 50 per second, the details of which are given later. The results so obtained for K are:—

7·73
7·81
7·75
7·87

Mean .... 7·97 at 25°·5 C.

The evidence afforded by this investigation is not sufficient to show that there is any difference between the specific inductive capacity of the piece of glass used when acted on by alternating electric forces with a frequency of 20 million per sec., and that found by the aid of forces having a frequency of only 50 per sec.

Lecher, in determining the specific inductive capacity of various dielectrics (Wied. *Ann.* xlii. p. 150, 1891), joins the ends of the secondary wires to condenser-plates. Placing a slab of the dielectric between the plates, he finds a position of the bridge such that the two circuits into which it divides the wire system are in resonance. This position depends on the capacity of the end condenser. With air between the condenser-plates, he then determines for various capacities the corresponding positions of the bridge, and by interpolation finds the capacity which would give the same position of the bridge as in the former case. This capacity is then equal to that of the condenser with the material under investigation as the dielectric. Cohn and Heerwagen (Wied. *Ann.* xliii. p. 343, 1891), and Thomson (Recent Researches, p. 340) give the formula applicable to Lecher's end circuit. The formula may be written:—

$$\cot\frac{2\pi s}{\lambda} = \frac{2\pi}{\lambda}\frac{C}{c},$$

s being the length of the wire circuit of the secondary system when resonance obtains, λ the wave-length of the vibration in the primary system, C the capacity of the secondary condenser, and c the capacity per unit length of the secondary wires, the capacities being in electrostatic units. The secondary wires may be considered parallel, so Lord Rayleigh's value for c may be adopted (Phil. Mag. xliv. p. 199, 1897). Differentiating the equations and substituting values appropriate to the apparatus used in the present re-research, the change of C necessary to maintain resonance for a given change in the position of the bridge may be readily found. An error of 1 cm. in the position of the bridge for resonance between the two circuits with the glass between the condenser-plates, would produce a considerable difference in the calculated value of K. The method is not so sensitive as the one which has been used in the present investigation.

Many observers, in determining specific inductive capacities, have used slabs of dielectrics in fields not rigorously uniform without taking that fact into consideration, and have neglected altogether the edge correction to the capacity of the condensers. The values of K given by such experimenters may be very much in error.

Northrup (Phil. Mag. xxxix. p. 78, 1895) describes a method for comparing the values of the specific inductive capacities of a substance under slowly and rapidly changing fields. His apparatus, a modified form of Gordon's five-plate balance (Phil. Trans. p. 427, 1879, and Hopkinson, Phil. Trans. clxxii. pt. ii., 1881), may be capable of showing that the specific inductive capacity of a material is not greatly different in slowly and in rapidly varying fields; but his method is quite inadequate for determining the actual value of the constant. He neglects altogether the edge correction to the capacity of the condensers formed by the small plates and the middle one. As the glass is much larger than the small plates, and as one of these is not always in contact with the glass plate, the capacity of the condenser of which the glass forms part of the dielectric would be very difficult to estimate. A balance similar to that described by Northrup has been used in the present investigation, a Rutherford detector being adopted instead of the spark method for determining when the small plates were at the same potential. Following Northrup's method exactly, the specific inductive capacity of a plate of glass, cut at the same time and from the same sheet as the disk used in the experiments described previously, comes out 10, being greater than the true value, as one would expect from the incorrect formula used. An improved method at once suggests itself—to have the glass plate less in diameter than the small plates, and in contact with one of them and the middle plate. It is doubtful, however, if the method is worth considering, on account of the difficulty of calculating the disturbance in the field caused by the small plates.

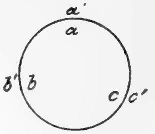
Blondlot (*Comptes Rendus*, May 11th, 1891, p. 1058) compares the specific inductive capacity of a piece of glass with that of sulphur in a rapidly changing field. The way in which the value of K is calculated is not shown, but from the value given it appears to have been assumed that the formula for plates and dielectric infinite in extent, the dielectric slabs completely filling the space between the plates, is applicable to the method. This is very far from being the case, and the value of K, 2·9, deduced from the experiments must be taken as incorrect.

Details of Absolute Electrometer Method.

The apparatus consists of a brass ring and a circular brass plate, each 1 cm. thick and 23 cms. in external diameter. They are placed one above the other, being separated by three hard-rubber distance-pieces of any required length. The ring forms a guard-ring to a movable plate attached to the arm of a balance. The diameter of the swinging plate is 8·95 cms., that of the aperture in the ring being 9·05 cms. The lower plate is provided with three levelling-screws, which enable the final adjustment for parallelism to be made with great accuracy. The opposed surfaces of the plates are carefully scraped plane. The plates are webbed across their outer surfaces, being purposely made very massive to avoid any uncertainty in the distance measurements due to bending.

Distance measurements.—The distance measurements are made by a specially constructed calipers which rests on the lower plate, and which may be placed to measure the distance between the plates at any point. The lever is moved by a micrometer-screw outside the plates. The head of the screw is 3 cms. in diameter, and is divided into 25 divisions, the value of each division being 0·023 mm. The arrester of the balance is altered so that when the beam is resting on its knife-edge, the mass in the pan being greater than that necessary to counterbalance the swinging plate, the beam can be tilted until the pointer is exactly at the middle mark of the scale. When the pointer is in this position, distance measurements are taken at three equidistant points near the edge of the swinging plate. A full set of the most discordant readings is given here to show the accuracy attained. The readings are in terms of divisions of the micrometer-head, the revolution of the screw being the same in all. Only mean readings are given to one decimal place; in no case did individual readings differ by more than 0·7 div.

Fig. 6.



In figure 6, *a*, *b*, *c* represent the three equidistant points on the swinging plate at which measurements are made, *a′*, *b′*, *c′* being three adjacent points on the guard-ring; the readings are as follows:—

	$a'=3·5$	$b'=6·7$	$c'=6·0$
Before weighing......	$a=5·9$	$b=5·8$	$c=5·9$
After weighing	$b=5·2$	$b=4·9$	$c=5·4$

The mean distance reading is taken as 5·51.

The lever calipers measuring the distance between the jaws of a Brown & Sharp vernier calipers set at 30·00 mm., reading = 6·00.

Distance between plates = 30·00 mm. + equivalent of (6·00 − 5·51) divs. The accuracy of the distance measurements is much greater than that of the determinations of the electrostatic pull. The distance between the plates must be known to 0·05 mm. if the value of K is to be determined to 1 p. cent. The distance is really known to a value well within this limit.

Electrical details.—The lower plate of the electrometer carefully insulated is attached to the insulated terminal of a large induction-coil, the other terminal and the upper plate of the electrometer being earthed. In parallel with the secondary of the induction-coil is a battery of twelve large leyden-jars. The primary of the coil is fed from the alternating-current side of a Westinghouse converter, run as a motor on the direct-current side with current from storage-cells. The frequency of the alternations is about 50 per second. The primary circuit includes a variable resistance the last part of which is a straight manganin wire with a sliding mercury-cup contact. The difference of potential between the two plates is indicated by a Kelvin electrostatic voltmeter. Every precaution is taken to screen the balance from electrical influence, and to prevent the glass plate becoming electrified.

A weighing is made in the following manner. With the beam resting on its knife-edge, the mass in the pan being greater than that required to balance the swinging plate, the pointer is brought up to the middle mark of the scale, the pointer and scale being observed through a telescope. The primary circuit is closed and the variable resistance altered until the pointer of the balance swings over, the reading of the voltmeter being taken by a second observer at the moment when this occurs. Two such experiments have to be made, one with the glass disk between the plates and one with air, the masses being adjusted until the upsetting of the balance in the two cases occurs for the same reading of the voltmeter. The whole uncertainty in the method is caused by fluctuations in the value of the potential. Up to the present, with a difference of potential between the plates of 5000 volts, it has not been found possible to determine the mass, whose weight counterbalances the electrostatic pull, to less than one milligram. The masses have, however, been determined with certainty to this amount. The mass which counterbalances that of the swinging plate, the resting-point being the middle division of the scale, is readily determined to a tenth of a milligram.

The mean observations are given in Table VI., the value of K being calculated from the well-known absolute electrometer formula. The thickness of the glass is in all cases 1·837 cms. Each determination involved a complete relevelling and adjustment of the apparatus ; it is merely a coincidence that the distance measures gave the same result on two days.

TABLE VI.

1901.	Dielectric between plates.	Pot. Diff. volts.	Extra mass in pan when balance upset.	Mean distance between plates.	K.	Temp. C.
Dec. 20th.	Glass.	5000	0·307 grm.	3·001 cms.	7·73	26·6
„	Air.	„	0·067	„		
Dec. 21st.	Glass.	5000	0·3084	2·999	7·81	25·2
„	Air.	„	0·0669	„		
Dec. 23rd.	Glass.	5000	0·308	2·999	7·75	25·0
„	Air.	„	0·067	„		
„	Glass.	5000	0·308	2·999	7·78	25·2
„	Air.	„	0·127	2·172		
				Mean	7·79	25·5

During this investigation we have received much help, which we gratefully acknowledge, from Professor McAulay, who took the most active interest in our work while Acting Professor of Mathematics at Sydney, and from Professor Bragg. We are indebted to Mr. A. Boyd and Mr. P. L. Weston, who while students in the laboratory took the preliminary observations in the earlier part of the work, and also to Mr. H. W. Myers, Mr. T. P. Tivey, and Miss F. Martin for considerable assistance. To Mr. Cook, the laboratory assistant, our best thanks are given for the great care with which the apparatus has been made.

LXIX. *The Absorption, Dispersion, and Surface-Colour of Selenium. By* R. W. Wood, *Professor of Experimental Physics in the Johns Hopkins University*[*].

THE dispersion curves of substances with absorption-bands lying wholly within the visible or infra-red spectrum have been carefully examined within the past few years, in connexion with the modern theory of dispersion. To the best of my knowledge little or nothing has been done with substances in which the absorption begins in the visible spectrum and extends into the remote ultra-violet. Of these media amorphous selenium is a type, strong absorption beginning not far from the D lines, and increasing continuously with decreasing wave-length until, in the ultra-violet, the extinction coefficient has as high a value as in the case of metals. Another substance which I have found worthy of investigation is nitroso-dimethyl-aniline. This substance is most remarkable in its behaviour. It crystallizes in green laminæ which melt at 85 degrees and can be formed into fluid prisms between glass plates. These prisms must be kept fluid by an air-bath, as on solidification they become opaque. A prism of four or five degrees gives a most astonishing dispersion when an incandescent lamp is viewed through it. It is quite transparent up to the greenish-blue, and gives a spectrum *twelve times as long as a quartz prism of equal angle.* Pressed into a thin film, it seems to be nearly as transparent as glass of a very pale canary-yellow colour; but if the transmitted light be examined with a spectroscope, the blue and violet end of the spectrum is cut off almost as sharply as if by an opaque screen in the eyepiece. This indicates an exceedingly steep extinction curve; and the course of the dispersion curve within this region will prove most interesting. I am at the present time engaged on the investigation of the absorption and dispersion of this substance, and further discussion will be postponed for the present.

Selenium I have already investigated with the assistance of Mr. A. H. Pfund, one of my students at the University of Wisconsin, to whom I am indebted for a large amount of very faithful work.

The only determinations that I have been able to find of the dispersion of this substance were made by Sirks[†] by the method of the colours of thin plates; a method involving considerable error, as is apparent from a comparison of his curve

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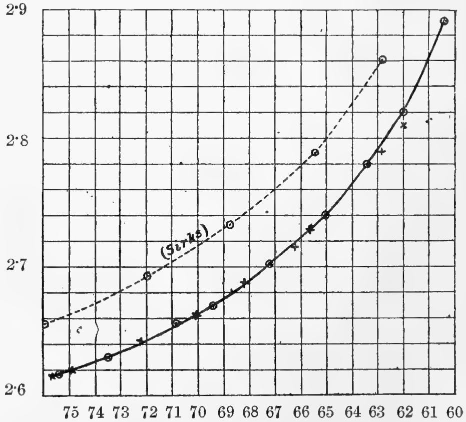
[*] Communicated by the Physical Society: read February 28, 1902.

[†] Sirks, Dispersion des Selens, Pogg. *Ann.* cxliii. p. 429.

with the curve obtained by direct spectrometer measurements with selenium prisms.

These prisms were made in the same manner as the cyanine prisms which I have previously described (Phil. Mag. i. p. 624, 1901). The substance is much more transparent than cyanine, and larger angles can consequently be used. Four or five degrees is about the maximum angle which can be employed to advantage. A strip of thin German plate-glass about 1 cm. wide and 2 cm. long is about right. A narrow strip of cardboard 1·5 mm. thick is pasted along the narrow edge, and a small piece of selenium fused on the opposite edge by holding the plate over a small flame. The other plate must be heated at the same time to about the same temperature, and the two clamped in a small vice in the manner described for cyanine prisms. It is necessary to heat the plate until the substance is quite fluid, which is considerably above the temperature at which it becomes pasty. After the medium has solidified, one of the plates can be removed by a blow from a small hammer, if the prism is required for measurements with the spectrometer, as the angle can then be easily determined by the reflexion of light from the oblique surfaces. If the prism is to be used for demonstration only, it is better to leave the plate on. One of the prisms crossed with a diffraction-grating shows the dispersion curve very well when an arc-light is viewed through the combination.

Determinations were made of the dispersion by means of three selected prisms, the results of which are shown below.



The results obtained by Sirks are plotted for comparison. The values obtained with the different prisms agree very well except at the edge of the absorption-band, where the chance of error is much greater owing to the broadening of the image of the spectrometer-slit by diffraction, the light only passing through a very narrow strip of the prism bordering the refracting edge.

The spectrometer was illuminated with monochromatic light obtained by prismatic analysis, the wave-length being determined for each observation with a glass grating of 14,000 lines. I found it quite impossible to construct prisms of small enough angle to make determinations below wave-length 61 possible. With one very acute prism I fancied that I detected evidences of a decrease of refractive index after passing this point, and accordingly determined to follow the curve further down into the spectrum by means of the interferometer, in the same manner as has already been done in the case of cyanine.

Beautifully uniform films of selenium were obtained on plates of plane parallel glass, by means of a flat selenium cathode in a high vacuum. This cathode was made by rubbing a stick of selenium over a hot aluminium plate, the device being that used by Longden in obtaining films showing Newton's rings.

Displacements of the interferometer fringes by the films were obtained in the manner described in previous papers on the dispersion of cyanine and carbon; a film covering one half of each of the two interferometer paths giving double the displacement of a single film without any increased loss of light. The displaced fringes were photographed by monochromatic light of known wave-length, the photographs being subsequently measured with a filar micrometer.

In the photographic work some trouble was had with superfluous light which, being superposed on the image of the fringe system, lessened the contrast between the maxima and minima. Some of this light comes from the glass surfaces and some from the selenium films. Mr. Pfund finally got rid of this light by the ingenious device of placing a convex lens between the interferometer and the photographic plate, which brought the light to two point-foci less than a millimetre apart. A screen with a pin-hole held back the superfluous light, which was collected at one focus, allowing the light which formed the fringes to pass. It was also found that better results were obtained with unsilvered interferometer-plates.

When working in the red and orange comparatively thick films could be used, but in the blue and violet only exceedingly thin ones allowed any light to pass through.

The apparatus as arranged for photographing the fringes and measuring the wave-length of the monochromatic light employed, is shown in fig. 2. The light of an arc-lamp A falls on a mirror cemented to the back of a prism which stands on a revolving graduated circle. The collimator objective T projects a spectrum on the slit S, which can be shifted by turning the circle which carries the prism. The monochromatic light which leaves the slit, after passage through a collimating-lens L, falls on the interferometer. Plane parallel plates carrying the selenium films are shown at *b*. To get rid of the multiple

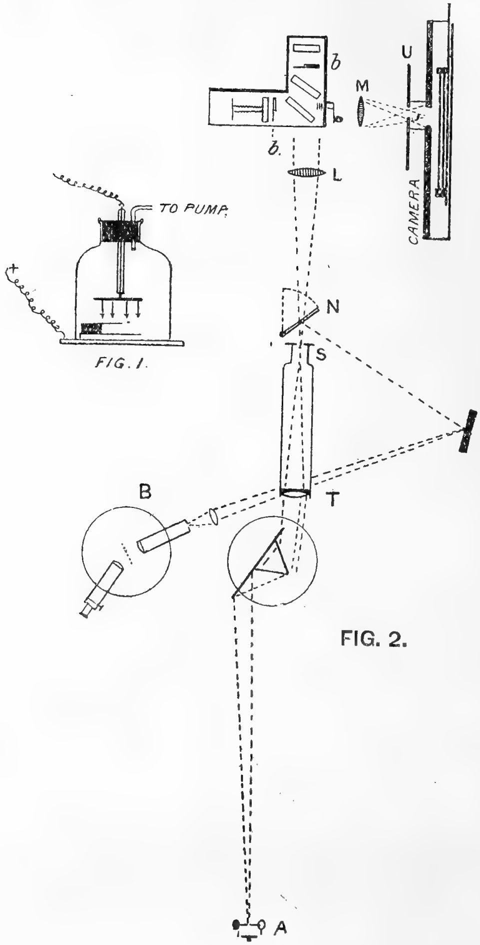


FIG. 1.

FIG. 2.

images, the lens M is arranged so as to focus the light on the perforated screen U *.

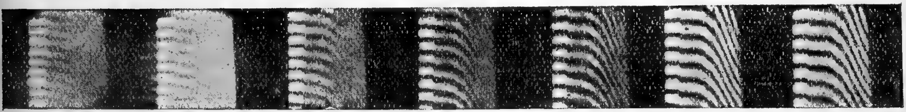
No other lens is used, the light falling directly on the photographic plate after passage through the pin-hole.

The camera is simply a long narrow box, the plate being carried on a slide, so that eight exposures can be made one after the other on a single plate, measuring 12 × 2 cms.

A small mirror N, hinged in front of the slit, enables us to divert the light from the interferometer and throw it into the grating spectrometer S, by means of which the wave-length can be at once determined. By means of this arrangement it was possible to work with considerable rapidity. After each exposure, the small mirror was turned in front of the slit, and the cross-hairs of the spectrometer set on the diffracted image. The prism was then turned through a small angle and a second exposure given, the circle of the spectrometer being read while the exposure was in progress. The longest exposures in the red were only five or six minutes in duration.

After making a number of measurements in different parts of the spectrum by means of films of different thicknesses, it occurred to me that if a wedge-shaped film were employed, curved fringes would be obtained which would allow the displacement for any wave-length to be measured for the maximum thickness capable of transmitting the light. Films of this nature were obtained by shielding the plate during the deposition with a strip of mica mounted a centimetre or so above its surface, as shown in fig. 1. The drift of the selenium under this shield was very regular, and excellent wedges were obtained which showed straight interference-bands when viewed in reflected light. A series of photographs obtained with one of these prismatic deposits is reproduced in fig. 3. It will be seen from the nature of the displacement of the fringes that the film is of nearly uniform thickness for a short distance, and then becomes wedge-shaped, the fringes plunging down suddenly.

Fig. 3.



Another advantage of the wedge-shaped films is that the

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* By an error of the engraver the aperture in U is shown much too large. Only one of the two convergent beams passes through it.

displaced fringe can be identified with the undisplaced. In working with films in the red and orange, where the displacement is equal to the width of several fringes, the only method of identifying the fringes is to make visual observations with white light at the centre of the system. With the wedge-shaped film there is no difficulty in telling which fringes belong together, for the same fringe can be traced clear across the plate, and the total displacement determined. The chance of making an error of an entire fringe-width in estimating the displacement by visual observations with white light is very great, owing to the great dissimilarity in the appearance of the chromatic fringes seen through clear glass and selenium.

The photographs of the curved fringes were measured by placing them in contact with a glass plate ruled with parallel diamond scratches, one of the scratches being placed in coincidence with the line marking the thin edge of the film. The displacement was measured as before with a filar micrometer, setting the cross-hair first on the undisplaced fringe, and then on the point at which the corresponding fringe intersected the scratch nearest the end of the oblique system.

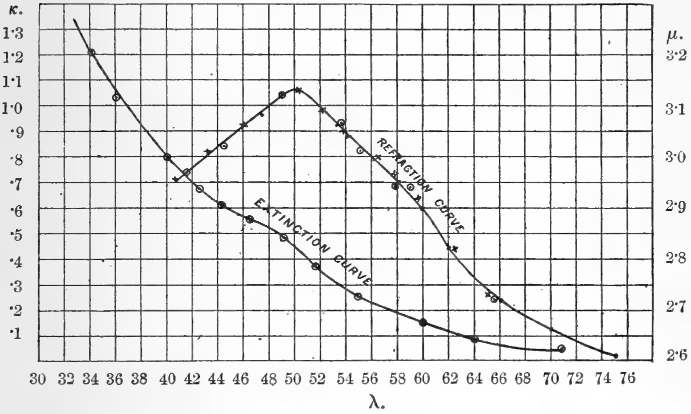
The values for the refractive indices obtained in the red by means of the prisms were taken as a basis for the calculation of the indices in the rest of the spectrum from the interferometer measurements. If we find the displacement for wave-length λ_r to be n_r (n being measured in terms of fringe-width), and know the ref. index μ_r for this wave-length, we can determine the refractive index μ_g for wave-length λ_g (in the green) by measuring the displacement n_g for the same thickness, from the formula

$$\mu_g=1+(\mu_r-1)\frac{n_g\lambda_g}{n_r\lambda_r}$$

Determinations were made in this way down to wave-length ·0004, beyond which point it was impossible to go owing to the powerful absorption, which made the fringes too faint to measure. It was only by employing exceedingly thin films and making the measurements with the greatest care that consistent results were obtained in the violet. The reason for this will be apparent when the subject of absorption is taken up, for a film which in red light is so transparent that its presence cannot be detected, in the violet and ultra-violet absorbs nearly as strongly as a metal film of the same thickness.

The dispersion-curve obtained by means of the interferometer is given on fig. 4, points determined with uniform films being indicated by circles, while points determined with wedge-shaped films are denoted by crosses. The turning point of the curve is at wave-length ·0005, where the refractive index reaches the value 3·13, which, so far as I know, is the highest value ever found for any substance.

Fig. 4.



The continued rise of the dispersion-curve after entrance into the region of the absorption-band is worthy of notice, for usually the curve falls abruptly as soon as the band is entered. As the matter is of considerable theoretical importance it seemed best to check, if possible, the data furnished by the interferometer by means of very acute prisms. Examination of the wedge-shaped plate by means of sodium light, showed that a strip about 8 mm. wide was suitable for the purpose, the interference-fringes being straight, parallel, and equidistant. The angle was found from the fringes to be very nearly 22 seconds.

The deviations were observed by means of a large spectrometer, which I have recently set up for quantitative measurements of the dispersion of sodium vapour. The slit is illuminated by means of the monochromatic illuminator made by Fuess. The telescope has a focal length of 1·5 metres, the eyepiece being furnished with a filar micrometer. The graduated screw of the monochromatic illuminator, by means of which the wave-length of the light on the slit is changed, is operated from the eye-end of the telescope by means of a long steel rod, while the graduations are read by means of the finder of the telescope, which points towards a mirror in which a portion of the illuminator is seen reflected.

This arrangement is extremely convenient, for the observer can run rapidly through a series of observations without leaving his seat at the eyepiece, a matter of great importance when working with sodium prisms. The plane parallel glass plate with the selenium wedge was covered by a piece of black paper furnished with two apertures, one exposing the prism, the other a portion of the clear glass. The illuminator was set for red light, and the prism placed in front of the telescope: a small screen enabled either aperture to be covered. The image of the slit was slightly broadened by diffraction, but not enough to interfere with accurate setting of the cross-hairs. On changing from clear glass to the selenium wedge, a displacement equal to three times the width of the band was observed. If both apertures were opened simultaneously, both the direct and displaced images appeared in the field, the images being furrowed, however, by the fine interference minima due to two apertures. Though it is quite possible that these minima would furnish the means of a more accurate setting of the cross-hairs on the centre of the band, they were not made use of in the present case, the apertures being uncovered in succession. This seemed advisable in view of the uncertainty regarding the effect of the change of wave-length on their location, for it must be remembered that when working with the strongly absorbed blue light, the amplitude falls off rapidly across the selenium-covered aperture, its effective width contracting with decrease of wave-length. The angular deviation for the red was measured, and found to agree fairly well with the observations made with prisms of larger angle. The selenium aperture being exposed, the cross-hair was set on the displaced image and the wave-length decreased. The image immediately moved off the hair, which was made to follow it, and a continued increase in the deviation was noted as far down the spectrum as wave-length ·00055, beyond which point the image became too faint to observe.

I was disappointed not to be able to follow the curve to its turning point in the blue, and accordingly tried another method.

The wedge-shaped film was observed by the light of lithium, sodium, and thallium flames in succession, and the positions of the dark bands recorded by needle scratches. The eighth band for lithium light coincided with the tenth band for sodium light, which agrees well with the observed values of the refractive index. With thallium light only three bands next to the thin edge could be seen, owing to the strong absorption, but their position indicated an increase of refractive index, the third thallium dark band coinciding with the second lithium band. The first dark band moved nearer to the thin edge with decreasing wave-length. A screen consisting of a glass cell filled with a solution of cuprammonium and a sheet of dense cobalt glass was now placed in front of a Welsbach light. This gave a deep blue light free from red. In this case the first dark band was the only one that appeared, but it was distinctly nearer the edge than in the case of the thallium light, which shows that the wave-length of the blue light in the selenium is considerably less than the wave-length of the thallium light, which would not be the case if the refractive index for the blue was considerably less than for the green. In this last case the band was too ill-defined to admit of an accurate measurement of its position.

A preliminary examination of the light transmitted by a very thin film squeezed out between two plates of quartz, by means of a quartz spectrograph, showed that there was no return of transparency in the ultra-violet, at least up to wave-length ·00028 or thereabouts. To determine if possible the position of the centre of the band, which gives us λ_m in the dispersion formula, a series of photometric measurements were made of the visible and ultra-violet light transmitted by a thin film deposited *in vacuo* by means of the cathode discharge. In the visible spectrum measurements were made by placing the glass plate, partly covered by the film, in contact with the double slit of a spectrometer, the line terminating the film being brought into coincidence with the junction of the slits. The thickness of the film was determined by means of fringe displacements, making use of the data obtained with the selenium prisms.

Owing to the difficulty of obtaining accurate data in the blue and violet by the visual method, and to enable us to carry on a study of the absorption in the ultra-violet, a photographic method was adopted, the details of which were worked out largely by Mr. Pfund.

The double slit was removed and a single slit put in its place, which was opened to a width of 0·5 mm. The ocular slit of the spectrophotometer was also opened to the same width. A short distance behind the ocular slit, the photographic plate was mounted with the selenium film covering the upper portion. A part of the plate was therefore exposed to light coming through clear glass, and part to light which had passed through the film of selenium. This method was found to give better results than a previous method which we tried, which consisted in substituting a photographic plate for the eyepiece of the instrument, and using the Vierordt slit as in the visual method. To determine the ratio of the intensities we made use of a photographic wedge, made by giving to a long narrow photographic plate an exposure decreasing uniformly in duration. To this strip a graduated scale was attached, and it was arranged to slide in a frame behind a narrow vertical aperture, against which the plate which had been exposed in the instrument, developed and dried, could be placed immediately below the wedge, which was then moved along until the spot of equal density was found. This spot could be determined to within 1 mm., the total length of the wedge being about 10 cms. In this way very good ideas of the relative intensities could be obtained, assuming Roscoe's rule governing the blackening of a plate to be true, as it probably is within the limits comprised in our experiments.

Below is given a table showing the values of $\frac{I}{I'}$, the ratio of the intensities, obtained by the two methods for a film the thickness of which was 0·0001 mm., and the calculated values of the extinction coefficient κ, defined by the equation

$$\frac{I}{I'}=e^{-\frac{4\pi\kappa d}{\lambda}},$$

where d is the thickness of the film and e the base of the natural logarithms.

	$\frac{I}{I'}$ (Visual).	$\frac{I}{I'}$ (Photographic).	κ.
760	1·05	1·04	·0234
710	1·10	1·07	·0456
640	1·20	1·18	·0879
589	1·40	1·39	·155
550	1·80	1·80	·254
515	2·40	2·43	·358
490	3·40	3·44	·478
466	4·40	4·40	·571
442	6·20	6·15	·598
425	7·50	7·53	·677
415	9·50	9·46	·736
400	9·50	12 00	·785

The values of κ plotted as ordinates with wave-lengths as abscissæ give the extinction curve, which is reproduced on fig. 4 with the dispersion curve.

To carry the work into the ultra-violet we abandoned the spectrophotometer with its glass prism and lenses, and made use of a short-focus concave grating with 14,000 lines to the inch. This grating was sent to me by Mr. Thorp of Manchester. It was made by mounting one of his celluloid replicas of a Rowland grating on a concave lens, the celluloid film being subsequently silvered. It was only intended as a device for illustrating the action of the concave grating*, but I found that if its aperture were reduced somewhat it gave surprisingly sharp lines.

A film of selenium was deposited on a thin plate of quartz, and mounted in front of the slit of the grating outfit, the lower half of the slit being covered with clear quartz, the upper half with quartz and selenium.

The slit was illuminated by a spark-discharge between zinc terminals, and the spectrum photographed. The absorbing action of the selenium manifested itself as a more or less complete obscuration of the upper half of the spectrum. By giving the selenium-covered portion of the slit a very long exposure, and the clear portion a short one, any given part of the spectrum could be balanced, the ratio of the times giving us an approximate value of $\frac{I}{I'}$.

We exposed a number of plates, and fully established the fact that the turning-point of the extinction curve, if it exists at all, lies beyond wave-length ·00022. In the case of one plate the clear portion of the slit was exposed for 30 seconds, the selenium-covered portion for 62 minutes. The resulting photograph showed lines down to wave-length ·00022 for the clear part of the slit, while the selenium-covered part only recorded itself down to ·00028, the lines being, however, very much fainter than the corresponding ones in the other half of the spectrum. The lines were of about the same intensity at wave-length ·00033, for which we get, from the ratio of the times of exposure, the value 1/124 for I/I′. This gives us the value 1·255 for κ. The thickness of the film was the same as in the previous experiments. Still longer exposures would have to be given to balance the two halves of the spectrum for shorter waves than these, from which it is evident that the extinction curve is still rising rapidly at wave-length 0·00022, where the coefficient has as high a value as in the case of the metals.

The extinction curve as figured is not to be regarded as

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* I have mounted this grating on a light wooden frame designed according to the well-known Rowland device, which can be placed on the lecture-table and used for class illustration. The slit is illuminated with sun or arc light and the spectra received on a strip of ground glass. It is useful for showing how the spectra and the camera move relatively to each other.

very accurate, as no precautions were taken to eliminate the reduction in the intensity of the light due to the reflexion at the surface. In determining curves of absorption it is always better to use two films of different thickness. The errors in the case of selenium may be considerable on account of the high refractive index, and I expect to redetermine the curve in the near future. In the present preliminary work the object was merely to determine whether there was a return to partial transparency in the ultra-violet region, which question appears to be answered in the negative, although a possible turning-point in the curve may be masked by the high reflexion coefficient. It appears to me to be extremely doubtful that there is but a single absorption-band in the present case, the more probable condition being a series of overlapping bands. We are in fact forced to this conclusion if we attempt to apply the dispersion formula to the results. Though we are unable to determine experimentally the value of λ_m, the wave-length corresponding to that at the centre of the absorption-band, we can calculate a value for it from three determinations of the refractive index in the region of comparative transparency. Writing the dispersion formula in the form*

$$n^2 = m + \frac{m'\lambda^2}{\lambda^2 - \lambda_m{}^2}$$

we can calculate λ_m from

$$\lambda_m{}^2 = \frac{\lambda_3{}^2(\lambda_1{}^2 - \lambda_2{}^2) - \lambda_2{}^2(\lambda_1{}^2 - \lambda_3{}^2)\mathrm{C}}{(\lambda_1{}^2 - \lambda_2{}^2) - (\lambda_1{}^2 - \lambda_3{}^2)\mathrm{C}},$$

in which

$$\mathrm{C} = \frac{n_2{}^2 - n_1{}^2}{n_3{}^2 - n_1{}^2},$$

where n_1, n_2, and n_3 are the values of the refractive index for wave-lengths λ_1, λ_2, and λ_3. Applying this formula to the values determined by the prism method we find λ_m to be 0·00056, which we may assume to be not very far from the centre of the first band in the series. The true position of the centre of the first member of the series cannot be exactly determined, owing to the effects of the free periods of higher frequency, but the value given above can be considered a fair approximation. Ketteler has given a method for decomposing a complex absorption-band into its constituents; but as the process is very laborious and somewhat arbitrary, it has not

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* "Dispersion of Ultra-Violet Rays," F. F. Martens, *Annalen der Physik*, No. 11, p. 612 (1901).

seemed worth while to attempt to apply it until a more accurate curve has been determined.

If we take into account the first member of the series only, and apply the results obtained with the prisms to the formula

$$n^2 = b^2 + \frac{M_1}{\lambda^2 - \lambda_m^{\ 2}},$$

from which the other formula may be derived by letting $b^2 = m + m'$ and $M_1 = m'\lambda_m^{\ 2}$, in which b^2 is the dielectric constant, we find b^2 to be very nearly 6·0. This latter form of the formula is the one generally used. If we use the original formula and determine m' and m from the expressions

$$m' = \frac{(n_3^{\ 2} - n_1^{\ 2})(\lambda_1^{\ 2} - \lambda_m^{\ 2})(\lambda_3^{\ 2} - \lambda_m^{\ 2})}{\lambda_m^{\ 2}(\lambda_1^{\ 2} - \lambda_3^{\ 2})},$$

$$m = n_2^{\ 2} - \frac{m'\lambda_2^{\ 2}}{\lambda_2^{\ 2} - \lambda_m^{\ 2}},$$

we find $m = 5$ and $m' = 1{\cdot}02$

or $b^2 = m + m' = 6{\cdot}02.$

If m in the original formula turns out to be greater than unity, it indicates that other absorption-bands must be taken into account. The dielectric constant of selenium has been measured by Romich and Nowack, the value found being 10·2.

The dielectric constant of a substance according to the electromagnetic dispersion theory is the dielectric constant of the æther, plus the dielectric constants of the various ions whose action on the æther waves gives rise to the phenomena of absorption and dispersion. Until the behaviour of selenium in the infra-red region has been investigated, it will be impossible to say whether the high value of the dielectric constant is due wholly, or only in part, to the ions whose free periods correspond in frequency to the green, blue, and ultra-violet rays. The infra-red work is already under way, and until it is completed further discussion of the results given in the present paper is impossible*.

The very high values of κ in the ultra-violet led me to look for traces of selective reflexion in this region. The light of an arc-lamp was reflected successively from six surfaces of selenium deposited on glass. The image of the crater after

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* Since writing this paper, I have had the opportunity of examining the infra-red transmission up to wave-length 2·3 μ with Prof. Mendenhall's very sensitive bolometer at the University of Wisconsin. No absorption-bands were found up to this point.

the sixth reflexion was very faint, but so far as the eye could judge without trace of colour. To determine whether or not ultra-violet light was present in excess, the light from the sixth surface was received upon a photographic plate one half of which was covered with a plate of glass which would absorb everything below wave-length 0·00034. The two halves of the plate were equally blackened, indicating that light of shorter wave-length than the above value was not present in any great excess.

Metals, however, have a much weaker reflecting power for ultra-violet radiations than for waves in the visible spectrum, and the failure to obtain traces of ultra-violet "Rest-strahlen" is doubtless due to related causes.

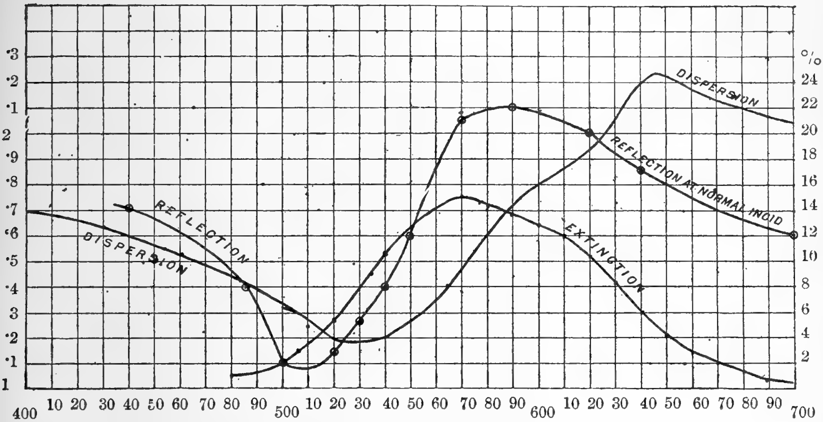
At the time when these experiments were tried the question of the possible detection of "Rest-strahlen" in the ultra-violet had never been discussed so far as I know. A paper has since appeared by Martens (*loc. cit.*) in which experiments are described which appear to prove that selective reflexion in the ultra-violet actually exists in certain circumstances.

We must be on our guard, however, against the error often made in assuming that the rays most strongly absorbed are identical with those most strongly reflected. This statement appears in so many of the text-books that it is well to call attention to the fallacy. Selective reflexion depends quite as much upon the refractive index of the medium as on the absorption. The index has a high value on the red side of the absorption-band, and a low value on the blue side; consequently the strongest reflexion will be for those rays on the red side of the centre of the absorption-band; that is to say, the maximum of the reflexion curve is shifted in the direction of the longer wave-lengths with respect to the absorption curve. The minimum of the reflexion curve lies on the blue side of the absorption-band, and in some cases may be equal to zero. The whole thing is of course contained in the formula for reflexion from absorbing media, which states that the percentage reflected,

$$R=\frac{n^2(1+\kappa^2)+1-2n}{n^2(1+\kappa^2)+1+2n}.$$

I have calculated from Pflüger's values of n and κ the reflexion curve which represents the surface colour of cyanine at normal incidence (fig. 5), which illustrates well the fact that the intensity of the reflected light in different parts of the spectrum is in no way a measure of the absorption. In point of fact, it is easily seen that the characteristic plum-colour of cyanine films is due rather to the almost complete *absence* of reflexion of the green rays than to strong metallic reflexion of any particular colour.

Fig. 5.—Curve illustrating Surface-colour of Cyanine.



In the case of selenium, if we apply the reflexion formula to the data that have been obtained thus far, we obtain a curve which indicates that the reflexion increases rapidly with decreasing wave-length. As I have said before, multiple reflexions from selenium surfaces give no trace of colour, which suggests that errors may exist in either the refraction or extinction curve. I have already pointed out where the source of trouble may lie in the case of the latter curve. In regard to the refraction-curve, it may be considered fairly accurate up to the point where the prism determinations stopped. In the case of films the thickness of which is less than the wave-length of light, I do not yet feel sure that the displacement of interference-fringes gives us a true measure of the refractive index. I hope some time to make a rigorous test of this point, for so far as I know it has never been settled.

The fact that the dispersion-curve of cyanine determined by the interferometer agreed so well with the curve obtained by means of a thin prism, does not prove conclusively that the method is not open to objection; for the same errors might occur in both cases: in other words, the deviation produced by a prism, the base of which was only a wave-length or two thick, might be affected by phase changes at the surfaces.

Selenium shows a strong surface-colour if the angle of incidence is in the neighbourhood of the polarizing angle, and the reflected light is examined through a nicol turned so as to extinguish as much as possible of the reflected light. The colour ranges from a greenish white through blue to deep violet as the angle of incidence increases.

The elliptical polarization of light reflected from selenium and the determination of the absorption and dispersion curves by the katoptric method is now in progress, as well as an investigation of the behaviour of the substance in the infra-red. It will be interesting to compare the results obtained in this way with the results obtained by the dioptric method.

The refraction and extinction curves for selenium, obtained by the katoptric method, are given in Ketteler's 'Theoretical Optics,' p. 552; but it is not stated whether the experiments were made with the glassy or metallic modification. The refraction curve in the red and orange region is obviously wrong, if it is intended to represent the dispersion of the glassy modification, as can be seen by comparing it with the curve obtained with the prisms.

LXX. *The Thermal Properties of Carbon Dioxide and of Ethane.* *By* J. P. Kuenen *and* W. G. Robson[*].

I. Carbon Dioxide.

OUR measurements of the vapour-pressures of carbon dioxide communicated in a recent paper[†] make it possible to discuss the thermal properties of this substance within wider limits than could be done hitherto. By our results, together with those of Amagat[‡], the vapour-pressures of the liquid are known between the critical temperature and −65° C., and the vapour-pressures of the solid from the triple point downwards. The densities of the coexisting liquid and vapour above 0° C. have been determined by Amagat; and below 0° C. there is a recent determination of the liquid densities down to −60° C. by Behn[§]. Behn has also measured the density of the solid at the boiling-point. Mathias[||] has determined the latent heat of evaporation above 0° C. up to the critical point, while Behn[¶] obtained the latent heat of sublimation at the boiling-point −78° C.

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[*] Communicated by the Authors.

[†] Kuenen and Robson, Phil. Mag. [6] iii. p. 149.

[‡] Amagat, *Ann. Phys. Chimie* [6] xxix. p. 136.

[§] Behn, *Ann. der Physik*, iii. p. 733.

[||] Mathias, *Thèses à la Faculté des Sc. de Paris,* No. 687, 1890.

[¶] Behn, *Ann. der Physik*, i. p. 270.

The above are, so far as we are able to judge, the most trustworthy data for carbon dioxide. We must here refer to some researches which escaped our notice until after our former paper was published. Vapour-pressures of solid carbon dioxide at pressures below and just above one atmosphere were measured by du Bois and Wills[*]. Although their value for the boiling-point is probably less accurate than ours, which is in complete agreement with Holborn's value, their results are of importance for determining the direction of the vapour-pressure curve at and below the boiling-point: and for that purpose they have been used in this paper. Vapour-pressures and triple-point constants were determined by Villard and Jarry[†]. As determined by them, the triple point is at −56°·7 C. and 5·1 atmospheres (our values are −56°·24 C. and 5·10 atmos.); the boiling-point they find to be −79° C., our result being −78°·32 C. The slight difference in the temperatures may be due to their having used a toluene thermometer. Their pressures were measured on a metal gauge. Lastly, we must mention a research by Tammann[‡] on the fusion-curve of carbon dioxide. Tammann's results will be compared with ours later in this paper.

Law of Corresponding States.—We have first of all ascertained how far the substance obeys the law of corresponding states. Owing to Young's researches[§], the comparison can be made with much greater facility now than at the time when the law was first enunciated by van der Waals[||].

The following table (p. 624) contains the reduced boiling-points, liquid volumes, and vapour volumes at the same corresponding pressures as used by Young, deduced from the researches enumerated above. (Amagat=A, Behn=B, and Kuenen-Robson=K.R.)

In comparing the second, third, and fourth columns of the table with the first, second, and third tables in Young's paper, it is found that there is no systematic divergence from the other substances. The temperatures are approximately the means of those for carbon tetrachloride and stannic chloride: the liquid volumes agree with those of stannic chloride, and are 1 or 1½ per cent. below the volumes of carbon tetrachloride; while the vapour volumes are between ½ and 1 per cent. below those of carbon tetrachloride. Within the

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[*] du Bois and Wills, *Verhandl. der D. Phys. Gesellschaft*, i. p. 168.

[†] Villard and Jarry, *Compt. Rend.* cxx. p. 1413.

[‡] Tammann, Wied. *Ann.* lxviii. p. 569.

[§] Young, Phil. Mag. [5] xxxvii. p. 1, &c.

[||] van der Waals, *Die Continuität &c.* 1899, p. 142.

same limits as other non-associating substances *carbon dioxide obeys the law of corresponding states*, and is thus a normal substance.

Table I.

Corresponding States. Carbon Dioxide.

Reduced Pressure.	Reduced Temperature.	Reduced Liq. Volume.	Reduced Vap. Volume.
·04423	·681 K.R.	·383 B.	
·08846	·731 ,,	·401 ,,	
·1474	·774 ,,	·420 ,,	
·2064	·806 ,,	·435 ,,	
·2949	·843 ,,	·457 ,,	
·4423	·889 ,,	·495 ,,	
·5899	·926 A.	·535 ,, ·537 A.	3·61 A.
·7372	·956 ,,	·590 ,,	2·64 ,,
·8257	·972 ,,	·633 ,,	2·16 ,,
·8846	·982 ,,	·671 ,,	1·86 ,,
·9436	·991 ,,	·730 ,,	1·57 ,,
1·0000	1·000	1·000	1·00

Law of Density Diameter.—Owing to the absence from the table of vapour volumes below 0° C., the diameter law of Cailletet and Mathias cannot be tested completely below 0° C. It is well known that Amagat's results for temperatures above 0° C. agree with this law. It is easy to show that the law does not hold accurately below 0° C., and in what direction the deviation lies. We have only to produce the diameter as deduced from Amagat's numbers down towards low temperatures: it is found that the mean density thus obtained at −60° C. is less than half the value of the liquid density, and this would lead to a negative value for the vapour density. We are thus driven to the conclusion that *the diameter is not a straight line and is slightly convex towards the temperature axis.*

This result for carbon dioxide agrees with recent results obtained by Young[*] with regard to the same law. Young finds that, with the exception of normal pentane for which the diameter is a straight line, all substances show slight deviations on one side or the other, according to whether the ratio of the critical density to the theoretical density is smaller or larger than the ratio for normal pentane—3·765. For carbon dioxide this ratio is 3·61, and is thus smaller than any of the numbers in Young's table; the group into which it falls

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[*] Phil. Mag. [5] l. p. 291.

contains carbon tetrachloride and stannic chloride, to which substances we have already found it to bear a close resemblance, and the deviations from the diameter law are also of the same sense as for the other substances in the group. Carbon dioxide thus confirms Young's generalization on this head.

Volume of Saturated Vapour below 0° C.—By the law of corresponding states we may now deduce the volumes of the vapour below 0° C. with a fair degree of accuracy. It is unlikely that the results thus obtained should be more than a few per cent. wrong. The following table contains the results at the reduced pressures occurring in Young's tables corresponding to temperatures below 0° C. (Amagat's result for 0° C. has been added.) The volume of the saturated vapour should be slightly smaller than the volume calculated by the gaseous laws from the data at 0° C. and one atmosphere. This appears to be the case. For −65°·7 C. the volume so calculated is 120 as compared with 113, at −37°·5 the figures are 41 and 35·9, and at −2°·3 the figures are 15·65 and 11·21.

Table II.

Reduced Pressure.	Temp.	Vapour Volume.	Liquid Volume.	$\frac{dp}{dt}$.	Latent Heat.	
					Calc.	Mathias.
	°					
·02241	......	215	—	—	—	—
·04482	−65·7	113	·83	·165	95·3	83·6
·08846	−50·4	58·6	·86	·272	84·7	80·4
·1474	−37·5	35·9	·90	·390	78·0	76·5
·2064	−27·8	25·6	·94	·490	72·0	72·7
·2949	−16·4	17·56	·98	·648	68·4	67·2
·4423	− 2·3	11·21	1·07	·877	58·4	58·2
—	0	10·42	1·09	·907	56·0	56·5
·5899	+ 8·9	7·77	1·16	1·078	48·8	48·6

Latent Heat of Evaporation.—Having obtained the volume of saturated vapour, we may now calculate the latent heat of evaporation of the liquid by the formula

$$\mathrm{L} = (v_v - v_l)\mathrm{T}\frac{dp}{d\mathrm{T}}.$$

The value of the liquid volume is taken from Behn's tables, and $\frac{dp}{d\mathrm{T}}$ we determined directly from our vapour-pressure curve. In this manner the values contained in Table II. were obtained.

There are no direct measurements of this quantity below 0° C. with which our results might be compared. Our result at 0° C. agrees with the number obtained by Chappuis [*], which is contained in Mathias's table [†]. The comparison above 0° between calculated and observed values was made by Mathias himself and by Tsuruta [‡], to whose paper we had unfortunately no access. Mathias also gives a formula proposed by Cailletet and himself from which the latent heat may be calculated:—

$$\lambda^2 = 117{\cdot}303(31-t) - {\cdot}466(31-t)^2.$$

Above 0° this formula agrees very well with experiment; at temperatures below 0° the differences between our results and the calculated values ultimately increase in magnitude, as shown by the last column of Table II. More particularly there is no evidence of a maximum in the latent heat as given by the quadratic formula. Our results seem to show that, starting from the critical temperature, the latent heat first rises quickly, then more slowly, and at very low temperatures more quickly again.

Latent Heat at the Triple Point.—By interpolation in Table II. we are now able to obtain the value of the latent heat of evaporation at the triple point. Our result is 86·1 calories.

Table III.

Triple Point of Carbon Dioxide.

Temp.	Pressure.	V_v.	V_l.	V_s.	L_{lv}.	L_{sv}.	L_{sl}.
−56°·24	5·10	74	·850	·676	86·1	129·9	43·8
				$\frac{dp}{dt}$ =	·224	·337	47·9

In order to obtain the latent heat of sublimation of the solid at this point the volume of the solid is required. Behn [§] finds for the density at −78° the value 1·53. Assuming that the solid expands about half as much as the liquid, the density at the triple point must be about 1·48. For the latent heat of sublimation we now obtain 129·9 calories.

By subtraction the latent heat of fusion is found to be 43·8 calories. From this figure we may now deduce $\frac{dp}{dt}$ for

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[*] Chappuis, *Ann. de Chim. et Phys.* [6] xv. p. 514.

[†] Mathias, *l. c.* p. 57.

[‡] Tsuruta, *Journal der Phys.* [3] ii. p. 272.

[§] Behn, *Ann. der Physik,* iii. p. 733.

the third line which passes through the triple point, the solid-vapour line or fusion curve. The result is that $\frac{dp}{dt}=47\cdot9$ atmospheres per degree centigrade. The dotted line which we drew in the triple-point diagram contained in our former paper * is thus seen to be much less steep than it ought to have been: our only object at the time was to show clearly which way it slopes. According to Tammann's measurements referred to above, the fusion curve rises with a slight curvature, concave towards the p-axis, starting at the triple point with a rise of 50 atmospheres per degree, which is in fair agreement with the result of our calculation.

It is impossible to make similar calculations with regard to the transformation on the solid-vapour curve owing to our ignorance of the volume of the vapour. Neither the law of corresponding states nor the diameter law is applicable to this curve. An approximate result may be obtained by using the gaseous laws and assuming the volume to be slightly smaller than that thus obtained. At the boiling-point the theoretical gaseous volume is 362. It appeared above that the deviation from the gaseous laws was diminishing as the temperature fell along the liquid-vapour line: it is thus unlikely that at the boiling-point the actual volume should be more than 2 or 3 per cent. smaller than the calculated value. If we assume 355 as the volume we cannot be very far wrong. Calculating the latent heat of sublimation in the same way as before, we find at the boiling-point $-78^\circ\cdot32$, 121 calories. This result is not in good agreement with the number found experimentally by Behn, $-142\cdot4$ †. It is improbable that the calculated value is at fault; it cannot be assumed that the gas volume as calculated is too small, and the only other experimental quantity in the calculation is the slope of the vapour-pressure curve at the boiling-point. We have adopted for this quantity the number obtained by du Bois and Wills ‡, our own readings not being numerous enough near that point to determine the slope with sufficient accuracy. Our curve, as far as it went, gave a number agreeing with that obtained by du Bois and Wills. We must therefore conclude that Behn's value is too large, although we cannot say why it should be so.

The latent heat at $-78^\circ\cdot32$ being 121 and at $-56^\circ\cdot24$

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* Kuenen and Robson, Phil. Mag. [6] iii. p. 149.

† Behn, *Ann. der Phys.* i. p. 272. Favre and Silbermann (*Ann. de Chimie et de Phys.* [3] xxxvii. p. 470) found 142·2 calories.

‡ du Bois and Wills, *l. c.*

130 calories, it appears that the latent heat of sublimation increases as the temperature rises. This phenomenon is very rare for liquids, and in that case may be looked upon as due to association of liquid molecules, *e. g.* in the case of alcohol *; in any case there is nothing strange in the fact of the latent heat of a solid rising with the temperature.

II. ETHANE.

From our results for ethane contained in the same paper we have calculated the figures in Table IV. For easy comparison we have added the corresponding temperatures for carbon tetrachloride and the percentage differences between that substance and ethane. All the reduced temperatures are lower than for carbon tetrachloride, and are thus lower than for any other substance in Young's tables. The differences increase extremely regularly as the temperature falls. As far as accidental errors are concerned, our estimate of the uncertainty of our measurements given in our former paper (less than 1 per cent.) was thus evidently too great. But there is a systematic difference which rises to 3 per cent. at

TABLE IV.

Corresponding States. Ethane.

Critical Temperature ... 305°·05.
Critical Pressure......... 48·43 atmospheres.
Critical Volume .... 4·84 c.c. per gramme.

Reduced Pressure.	Reduced Temperature.	CCl_4.	Difference per cent.
·04423	·6501	·6728	3
·08846	·7082	·7251	2·5
·1474	·7543	·7697	2
·2064	·7909	·8025	1·5
·2949	·8313	·8411	1·2
·4423	·8830	·8889	·7
·5899	·9205	·9260	·6
·7372	·9530	·9565	·4
·8257	·9700	·9725	·3
·8846	·9807	·9822	·15
·9436	·9910	·9917	·1
1·0000	1·0000	1·0000	—

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* Tsuruta (Phil. Mag. [5] xxxv. p. 435) finds a maximum for the latent heat of hydrochloric acid, which is probably a normal substance.

the lowest temperature, and is of opposite sign to the deviations for abnormal associating substances. We cannot say with certainty whether these deviations are characteristic of pure ethane. The ethane, although submitted to repeated fractional distillation, was not so pure as the carbon dioxide; and it is quite possible that the deviations in question are to be accounted for by the presence of impurity. At the temperatures above 0° C. the whole mass of the substance was immersed in the temperature-bath, but below 0° C. an approximately constant part of the tube was cooled in the Dewar vessel. Because of this complexity in the experimental conditions, it is impossible to account for the above differences in detail, or to estimate the actual error which the impurity must have produced at the various temperatures, if we assume them to be due to that cause.

We have no data for the liquid and vapour densities of ethane sufficiently accurate to test the diameter law, and to find by extrapolation the densities at the lower temperatures which are required for the calculation of the latent heat.

The critical volume as determined directly by experiment was ·0065 of the gas volume at 0° C. and one atmosphere. From this we calculate for the critical volume 4·84 c.c. per gram. The ratio of the critical density to the density calculated by the gaseous laws we find to be 3·55. The uncertainty of these figures we estimate at about 2 per cent. If we consult Young's tables referred to before *, we find that, judging by the position which the other normal hydrocarbons take up in it, the ratio for ethane should be less than for pentane, 3·765, as is actually the case. The ratios for propane and butane are not yet accurately known; but those for hexane, heptane, &c. increase regularly the higher we go up the series.

In order to obtain approximate values for the latent heat of evaporation of ethane, we have assumed that the corresponding volumes of vapour and liquid are the same as for carbon tetrachloride, considering that this substance resembles ethane more than any other contained in Young's tables.

In comparing the volumes thus obtained with the volumes calculated by the gaseous laws, we find the latter slightly larger, as they should be, with the exception of the volume at the lowest temperature −74°·7, the volumes being 253 and 256. The latter number is thus seen to be a few per cent. too large.

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* Young, Phil. Mag. [5] l. p. 296.

Table V. contains the values of the latent heats thus found. It will be noticed that the change of this quantity with temperature, except at the lowest temperature, follows, on the whole, the same law as for carbon dioxide. From the critical point downwards it first rises quickly, then more slowly, and ultimately begins to rise more quickly again, the last value being, however, smaller than the last but one. The irregularities are doubtless due to the irregularities in the value of $\frac{dp}{dt}$. When we compare with each other the slopes as read from the curves for carbon dioxide and ethane at corresponding temperatures, we find that the ratio of the slopes varies from 1·59 to 1·74, the average being 1·65. Irregularities of an amount up to 5 per cent. are thus accounted for. It need not be explained how difficult it is, even by means of a flexible rule, to draw a perfectly satisfactory curve through a comparatively small number of experimental points, and again how easily the slope of the curve, especially near its end, may be slightly over- or under-estimated. Our chief object was to obtain a rough idea of the value of the latent heat of ethane, for the sake of comparison with carbon dioxide and mixtures of the two substances.

TABLE V.

Latent Heat of Ethane.

Temperature.	Vapour Volume.	Liquid Volume.	$\frac{dp}{dt}$.	L.
° −74·7	256	1·9	·091	111
−57·0	133	2·0	·169	116
−42·9	81·1	2·1	·224	99
−31·75	57·8	2·1	·301	98
−19·4	39·6	2·2	·390	90
− 3·65	25·3	2·4	·548	82
+ 7·8	17·8	2·60	·678	70
+17·7	12·87	2·86	·772	54·5
+22·9	10·50	3·05	·855	45·7
+26·17	9·10	3·22		
+29·3	7·84	3·45		
+32·05	4·84	4·84		

University College, Dundee.

LXXI. *The Thermal Expansion of Porcelain.*
By A. E. Tutton, *B.Sc., F.R.S.**

THE thermal expansion of porcelain has formed the subject of several investigations during the last three years, and has become a subject of considerable importance in consequence of the employment of porcelain of the Bayeux and Berlin varieties for the reservoirs of air-thermometers.

The present investigation has been carried out with a specimen of Bayeux porcelain, which was kindly placed at the author's disposal by Prof. Callendar. It is of especial interest as being cut from the same tube which was employed (1 metre length of it) by Bedford in an investigation, by the method of Callendar, of the expansion of this variety of porcelain between 0° and 830° (Phil. Mag. vol. xlix. p. 90); and of which same tube also a piece was used by Chappuis in a series of determinations, by the Fizeau method, for the interval 0° to 83°. It was the fact of there being some discrepancy between the results of these observers that induced Prof. Callendar to suggest to the author the carrying out of an independent series of determinations, with the delicate interference-dilatometer described by the author to the Royal Society in the year 1898 (Phil. Trans. A. vol. cxci. p. 313, and vol. cxcii. p. 455).

The interference-dilatometer in question possesses all the advantages over the original Fizeau apparatus of the Bureau International des Poids et Mesures, Sèvres, of that devised by Abbe and elaborated by Pulfrich (*Zeitschr. für Instrumentenkunde,* 1893, p. 365); together with further improvements on the Abbe form. Two of these improvements are of considerable importance, namely, the separation of the observing part of the apparatus from the expansion-chamber, and its removal to an adequate distance from the heated atmosphere above the bath in which the latter is heated; and the measurement of the temperature of the interference tripod, and the substance under investigation which it carries, by means of a thermometer bent just above the cylindrical bulb and so arranged that the latter lies on the platinum-iridium tripod table, in tied contact with one of the screws supporting the glass cover-disk, and only a millimetre distant from the substance itself. For the author has proved by direct experiment that a thermometer merely hung alongside in the bath does not register the temperature of the tripod, which at the higher limit of 120° may be as much as 3° lower. The chief

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* Communicated by the Physical Society: read March 14, 1902.

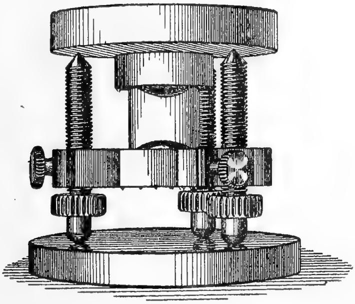
advantages over the Fizeau apparatus are briefly: (1) The employment of a micrometric method of measuring the position and width of the interference-bands; (2) the use of autocollimation; (3) the employment of C hydrogen light instead of sodium light, as being more truly monochromatic and not subject to secondary interference; and (4) an arrangement of the thermal chamber which readily permits an extension of the range to an upper limit of 120°.

The author's method also renders the observer independent of the nature of the surface of the substance investigated, as to whether it is polishable or not. This is attained by the use of the aluminium compensator, a relatively thick disk of aluminium laid on the top of the substance. It is thus the upper surface of this disk, and not that of the substance, which, together with the under surface of the glass cover-disk, reflects the interfering red hydrogen light. As aluminium expands about 2·6 times as much as platinum-iridium, it is easy to choose a disk, out of a series prepared for the purpose, which will roughly compensate for the expansion of the lengths of the platinum-iridium screws, which project above the table of the same alloy on which the substance rests. By so doing, approximately the whole expansion of the substance is recorded by the movement of the interference-bands, affording at once an idea of its relative expansibility, or in the case of contraction indicating the fact. The author's value (quoted later) for the expansion of aluminium agrees fairly well with that given by Fizeau, and it has been proved that any error introduced by the use of this third substance cannot exceed ±0·02 of an interference-band, an amount far less than the differences observed between the numbers of bands afforded by successive determinations under identical conditions. The aluminium compensator is simply invaluable when substances incapable of good polish are being investigated, and "biscuit" porcelain is eminently such a case; and there is the further advantage in the case of a piece of tube, of which the section is only an annulus, that there is a large field of bands afforded instead of a mere ring showing parts of bands. The polish taken by aluminium is pre-eminently suitable, as the intensity of the reflected light is about the same as that from the cover-glass. The compensators keep unimpaired if stored in a desiccator. As some misapprehension has arisen as to the use of the compensator, the author wishes to emphasize the fact that no accession of accuracy is claimed for the method, and, as has just been shown, it may introduce ±0·02 of error; but as this latter is absolutely negligible, the other advantages referred to are such as to make its use in the specified cases most convenient and valuable. Given, however, a substance capable of taking a good polish, and whose thermal deformation is approximately known, the compensator is both unnecessary and inadvisable. The reflecting surface of the aluminium disk used in these determinations was minutely concave, affording bands slightly curved. This is an advantage, as it enables the observer to know at once whether expansion or contraction is occurring, according as the bands move from or to their centre of curvature.

The specimen of Bedford's tube used by Chappuis was 15 mm. long, and was furnished at its two ends with plane surfaces. The one placed uppermost was polished as far as possible, and the fringes of sodium light obtained with this annulus were used for his determinations.

The author had an ample amount of Bedford's tube to furnish three specimens for investigation. They were cut in lengths somewhat over 12 mm. and ground with truly plane ends by Messrs. Troughton and Simms, the makers of the dilatometer. It was not found possible to get a satisfactory polish; and this has been shown to be unnecessary. Portions of the two ends were then ground out somewhat, at three places in each case, leaving three equidistant small patches of the original truly plane end, so that the specimens might stand with three-point contact on the truly plane polished surface of the platinum-iridium table (that side of the table being employed which was not furnished with projecting points). Likewise the aluminium disk lay, when put into position, with three-point contact on the upper porcelain blunt points. The diameter of the pieces of porcelain was about 17 mm., and the walls were about 3 mm. thick and fairly uniform, the hole being as nearly as possible central.

Fig. 1.



The arrangement of the tripod and its contents will be clear from the accompanying illustration (fig. 1).

Three independent determinations were carried out with each piece of porcelain, each occupying seven to eight hours. The method followed was precisely that described in the memoir on the "Thermal Deformation of the Crystallized Sulphates of Potassium, Rubidium, and Cæsium" (Phil. Trans. 1899, A. vol. cxcii. p. 467).

Only the very slightest signs of inequality of expansion were observed, as indicated by a just perceptible amount of rotation of the bands, nothing like so much as is described by Chappuis for the fringes of the annular rim, and usually it was only temporary, the original position being regained before the close of each half of the determination, after constancy at the two higher limiting temperatures in the neighbourhood of 60° and 120° was attained.

The results of the determinations, together with the necessary data, are set forth in the tables on pp. 636, 637.

L_{t_1} represents the length of the specimen of porcelain, as measured at the ordinary temperature by the author's thickness-measurer (Phil. Trans. A, vol. cxci. p. 337, 1898). It is the length at the central axis, immediately under the minute silvered ring on the under side of the glass cover-disk, the centre of which is the point of reference for the micrometric measurement of the initial and final positions of the bands for each interval of temperature.

l_a is the thickness of the aluminium disk at the centre of reference.

l represents the mean length of the three platinum-iridium screws, or vertical distance between silver ring and surface of platinum-iridium table.

d is the thickness of the air-layer at the centre of reference.

t_1 is the initial temperature; in every case it was the temperature of the room (as determined by the thermometer in contact with the tripod) at the moment of commencing the determination, about 6.30 A.M. in mid-winter, after the apparatus had been left 16–17 hours since completing on the previous day either adjustment or a determination. It was considered more satisfactory to start with a lower limit of this constant character, than to attempt to approach nearer zero by artificial cooling, as is the method at the Bureau International. t_2 is the limiting temperature of the first interval, and t_3 that of the second interval. $t_2 - t_1$ is considered the first interval, $t_3 - t_1$ the second.

b_1, b_2, and b_3 are the corresponding barometric pressures, required for computing the correction to the observed number of bands.

f_2 is the observed number of bands for the first interval, and f_3 for the second.

The small correction which follows is that which has to be applied for the alteration of the refractive index of air due to change of temperature and pressure. It is fully discussed on p. 349 of the memoir (*loc. cit.*) on the dilatometer. The formula is

$$f' = f + d(t_2 - t_1) \cdot \frac{b_1}{760} \cdot \frac{1}{1+\alpha t_1} \cdot \frac{1}{1+\alpha t_2} \cdot 2\frac{(n-1)\alpha}{\lambda}$$

$$- d(b_2 - b_1) \cdot \frac{1}{1+\alpha t_2} \cdot 2\frac{n-1}{760\,\lambda};$$

where f represents the observed and f' the corrected number of bands, α the coefficient of expansion of air, and n the refractive index of air for the wave-length λ of the light employed. f_2' and f_3' are consequently the corrected numbers of bands for the two intervals.

Then follows the actual amount of diminution of thickness of the air-layer, as given by the product $f'\lambda/2$, where $\lambda/2$ is 0·0003281 mm.

The expansion of the platinum-iridium screws and of the aluminium compensator are next given, calculated from the formulæ (*vide* Phil. Trans. A, vol. cxci. pp. 353 & 356, for determinations of these coefficients):—

For the screws, $l\left[10^{-9}\left(8600 + 4{\cdot}56\frac{t_1+t_2}{2}\right)\right](t_2 - t_1)$,

" aluminium, $l_a\left[10^{-8}\left(2204 + 2{\cdot}12\frac{t_1+t_2}{2}\right)\right](t_2 - t_1)$,

for the first interval, and substituting t_3 for t_2 for the second interval.

The slight excess of expansion on the part of the aluminium over that of the screws, acts like the expanding porcelain in diminishing the thickness of the air-layer, and has therefore to be deducted from the total observed diminution; the resulting differences represent the amounts of expansion of the porcelain for the two intervals. Then follow the calculations of the two constants a and b of the coefficient of expansion of porcelain. For the discussion of the formulæ employed, and the mode of deriving them, the memoir on the dilatometer should be consulted (*loc. cit.* p. 352).

Expansion of Bayeux Porcelain.—Specimen 1.

L_{t_1}=12·259 mm. l_a=8·187 mm. l=20·907 mm. d=0·461 mm.

		Determination 1.	Determination 2.	Determination 3.
Temperatures ...	t_1	6°·8	10°·0	11°·9
	t_2	63°·4	64°·7	64°·6
	t_3	120°·8	122°·2	120°·9
Bar. pressures ...	b_1	761·2 mm.	753·7 mm.	755·2 mm.
	b_2	760·9	753·1	756·1
	b_3	760·6	752·5	757·0
No. of bands ...	f_2	7·38	6·97	6·83
	Corrn....	−0·08	−0·08	−0·07
	f_2'	7·30	6·89	6·76
	f_3	17·42	17·14	16·72
	Corrn....	−0·14	−0·14	−0·13
	f_3'	17·28	17·00	16·59
Dim. of thickness of air-layer ...	$f_2'\lambda/2$...	0·0023951 mm.	0·0022606 mm.	0·0022180 mm.
	$f_3'\lambda/2$...	0·0056696	0·0055778	0·0054433
Exp. of Pt.-Ir.	1st int. ...	0·0103662	0·0100297	0·0096676
	2nd int. ...	0·0211900	0·0208805	0·0202885
Exp. of Al. ...	1st int. ...	0·0105577	0·0102247	0·0098592
	2nd int....	0·0218328	0·0215323	0·0209247
Excess exp. Al.	1st int. ...	−0·0001915	−0·0001950	−0·0001916
	2nd int. ...	−0·0006428	−0·0006518	−0·0006362
Expansion of Porcelain. ...	$L_{t_2}-L_t$ ...	0·0022036	0·0020656	0·0020264
	$L_{t_3}-L_{t_1}$...	0·0050268	0·0049260	0·0048071
	θ ...	0·000 032 618	0·000 029 784	0·000 030 777
	ϕ ...	0·000 000 089 94	0·000 000 106 81	0·000 000 100 34
	L_0 ...	12·2588 mm.	12·2587 mm.	12·2586 mm.
	a ...	0·000 002 661	0·000 002 430	0·000 002 511
	b ...	0·000 000 007 34	0·000 000 008 71	0·000 000 008 19

Mean values .... a=0·000 002 534. b=0·000 000 008 08.

Expansion of Bayeux Porcelain.—Specimen 2.

L_{t_1}=12·334 mm. l_a=8·187 mm. l=20·645 mm. d=0·124 mm.

		Determination 1.	Determination 2.	Determination 3.
Temperatures ...	t_1	5°·5	7°·2	5°·7
	t_2	61°·9	62°·2	63°·1
	t_3	120°·7	121°·7	123°·2
Bar. pressures ...	b_1	771·1 mm.	768·5 mm.	769·3 mm.
	b_2	770·8	768·5	769·2
	b_3	770·5	768·5	769·2
No. of bands ...	f_2	7·34	7·30	7·38
	Corrn....	− 0·02	− 0·02	− 0·02
	f_2'......	7·32	7·28	7·36
	f_3	17·52	17·81	17·97
	Corrn....	− 0·04	− 0·04	− 0·04
	f_3'......	17·48	17·77	17·93

Specimen 2 (*continued*).

		Determination 1.	Determination 2.	Determination 3.
Dim. of thickness of air-layer ...	$f_2'\lambda/2$...	0·0024017 mm.	0·0023886 mm.	0·0024148 mm.
	$f_3'\lambda/2$...	0·0057352	0·0058304	0·0058829
Exp. of Pt.-Ir.	1st int. ...	0·0101925	0·0099444	0·0103770
	2nd int....	0·0211373	0·0210237	0·0215744
Exp. of Al. ...	1st int. ...	0·0105065	0·0102556	0·0106997
	2nd int....	0·0220486	0·0219412	0·0225159
Excess exp. Al.	1st int. ...	−0·0003140	−0·0003112	−0·0003227
	2nd int....	−0·0009113	−0·0009175	−0·0009415
Expansion of Porcelain ...	Lt_2-Lt_1...	0·0020877	0·0020774	0·0020921
	Lt_3-Lt_1...	0·0048239	0·0049129	0·0049414
	θ ...	0·000 031 447	0·000 031 780	0·000 030 029
	ϕ ...	0·000 000 082 63	0·000 000 086 32	0·000 000 093 29
	L_0...	12·3338 mm.	12·3338 mm.	12·3338 mm.
	a ...	0·000 002 550	0·000 002 577	0·000 002 435
	b ...	0·000 000 006 70	0·000 000 007 00	0·000 000 007 56

Mean values .... $a=0{\cdot}000\ 002\ 521$. $b=0{\cdot}000\ 000\ 007\ 09$.

Expansion of Bayeux Porcelain.—Specimen 3.

$Lt_1=12{\cdot}208$ mm. $l_a=8{\cdot}187$ mm. $l=20{\cdot}549$ mm. $d=0{\cdot}154$ mm.

		Determination 1.	Determination 2.	Determination 3.
Temperatures ...	t_1	8°·4	10°·4	7°·5
	t_2	62°·2	63°·8	61°·7
	t_3	122°·8	125°·5	123°·0
Bar. pressures ...	b_1	764·9 mm.	764·2 mm.	762·9 mm.
	b_2	764·7	764·3	762·8
	b_3	764·5	764·5	762·7
No. of bands ...	f_2	7·20	7·03	7·22
	Corrn....	−0·03	−0·03	−0·03
	f_2'	7·17	7·00	7·19
	f_3	17·90	17·96	18·12
	Corrn....	−0·05	−0·05	−0·05
	f_3'	17·85	17·91	18·07
Dim. of thickness of air-layer ...	$f_2'\lambda/2$...	0·0023525 mm.	0·0022967 mm.	0·0023591 mm.
	$f_3'\lambda/2$...	0·0058567	0·0058764	0·0059288
Exp. of Pt.-Ir.	1st int. ...	0·0096855	0·0096225	0·0097542
	2nd int....	0·0209204	0·0210737	0·0211174
Exp. of Al. ...	1st int. ...	0·0100371	0·0099795	0·0101056
	2nd int....	0·0219452	0·0221269	0·0221483
Excess exp. Al.	1st int. ...	−0·0003516	−0·0003570	−0·0003514
	2nd int....	−0·0010248	−0·0010532	−0·0010309
Expansion of Porcelain ...	Lt_2-Lt_1...	0·0020009	0·0019397	0·0020077
	Lt_3-Lt_1...	0·0048319	0·0048232	0·0048979
	θ ...	0·000 031 315	0·000 029 613	0·000 030 987
	ϕ ...	0·000 000 083 25	0·000 000 090 45	0·000 000 087 50
	L_0...	12·2077 mm.	12·2077 mm.	12·2078 mm.
	a ...	0·000 002 565	0·000 002 426	0·000 002 538
	b ...	0·000 000 006 82	0·000 000 007 41	0·000 000 007 17

Mean values......... $a=0{\cdot}000\ 002\ 510$. $b=0{\cdot}000\ 000\ 007\ 13$.

It will be sufficient here merely to state the formulæ, which are as follows:—

$$\theta = \frac{(t_1+t_3)(L_{t_2}-L_{t_1})}{(t_2-t_1)(t_3-t_2)} - \frac{(t_1+t_2)(L_{t_3}-L_{t_1})}{(t_3-t_1)(t_3-t_2)},$$

$$\phi = \frac{L_{t_3}-L_{t_1}}{(t_3-t_1)(t_3-t_2)} - \frac{L_{t_2}-L_{t_1}}{(t_2-t_1)(t_3-t_2)},$$

$$L_0 = L_{t_1} - \theta t_1 - \phi t_1^2,$$

$$a = \frac{\theta}{L_0}, \qquad b = \frac{\phi}{L_0}.$$

The results for the three pieces of porcelain tube are compared in the following table, and the final mean result for the linear expansion of Bayeux porcelain is given at the foot.

	a.	*b.*
Specimen 1.	0·000 002 534	0·000 000 008 08
„ 2.	2 521	7 09
„ 3.	2 510	7 13

Final mean values: $a = 0{\cdot}000\,002\,522$, $b = 0{\cdot}000\,000\,007\,43$.

This investigation, therefore, affords as the mean coefficient of linear expansion, $a+bt$, of Bayeux porcelain, between 0° and t°, within the limits of 0° and 120°, the value

$$10^{-9}(2522 + 7{\cdot}43\,t).$$

That is,

$$L_t = L_0[1 + 10^{-9}(2522\,t + 7{\cdot}43\,t^2)].$$

The true coefficient, α, of linear expansion at t°, or the mean coefficient between any two temperatures (within the limits of 0° and 120°) whose mean is t, is as follows:—

$$\alpha = a + 2bt = 0{\cdot}000\,002\,522 + 0{\cdot}000\,000\,014\,86\,t\,;$$

or $$10^{-9}(2522 + 14{\cdot}86\,t).$$

The mean coefficient of the cubical expansion between 0° and t°, for the same limits of 0° and 120°, derived from this investigation, is:

$$a + bt = 0{\cdot}000\,007\,566 + 0{\cdot}000\,000\,022\,29\,t\,;$$

or $$10^{-9}(7566 + 22{\cdot}29\,t).$$

That is

$$V_t = V_0[1 + 10^{-9}(7566\,t + 22{\cdot}29\,t^2)].$$

The actual coefficient of cubical expansion, α, at any temperature t, within the limits 0° to 120°, and also the mean coefficient of cubical expansion between any two temperatures (within the same specified limits) whose mean is t, is expressed by

$$\alpha = a + 2bt = 10^{-9}(7566 + 44{\cdot}58t).$$

The value obtained by Bedford for the linear expansion for the interval 0° to 600° was as follows:—

$$L_t = L_0[1 + (3425t + 1{\cdot}07t^2)10^{-9}],$$

and $$\alpha = a + 2bt = (3425 + 2{\cdot}14t)10^{-9}.$$

Beyond 600° the formula does not accurately reproduce the observations, which were carried out as far as 830°.

For the cubical expansion Bedford gives:

$$V_t = V_0[1 + (10275t + 3{\cdot}24t^2)10^{-9}],$$

and $$\alpha = a + 2bt = (10275 + 6{\cdot}48t)10^{-9}.$$

The result obtained by Chappuis for the linear expansion between 0° and 83° was the following:—

$$L_t = L_0[1 + (2824{\cdot}1t + 6{\cdot}17t^2)10^{-9}],$$

$$\alpha = a + 2bt = (2824{\cdot}1 + 12{\cdot}34t)10^{-9}.$$

And for the cubical expansion:

$$V_t = V_0[1 + (8472{\cdot}4t + 18{\cdot}53t^2)10^{-9}],$$

$$\alpha = a + 2bt = (8472{\cdot}4 + 37{\cdot}06t)10^{-9}.$$

The considerable difference thus presented between the results of Bedford and Chappuis is most striking with regard to the increment of the coefficient, the constant b in the linear expansion being six times greater according to Chappuis than according to Bedford. The constant a is at the same time reduced.

The results now presented by the author agree tolerably well with those of Chappuis, but emphasize this difference, the constant b being seven times larger than according to Bedford, and the constant a slightly smaller than according to Chappuis.

In the following table are set forth the actual coefficients of linear expansion at particular temperatures, for every 10° up to the limit of the determinations, deduced from the observations of the author and of Chappuis respectively. The actual lengths when $L_0 = 1$ are also graphically expressed by the two curves in fig. 2 (the dotted one representing the results of Chappuis), for which degrees of temperature are taken as abscissæ and the lengths as ordinates.

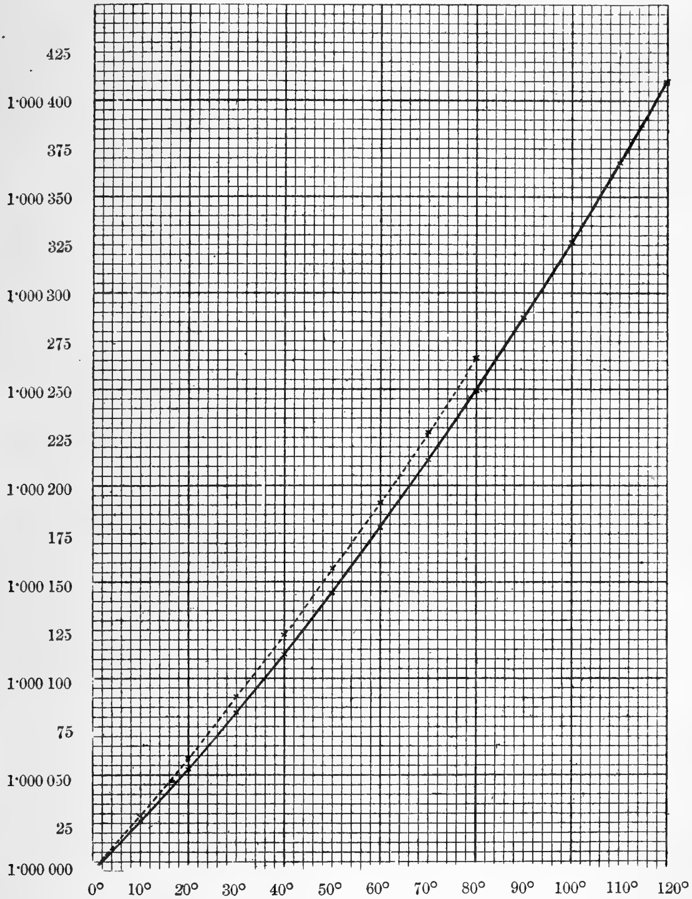
Table showing Values of $\alpha = a + 2bt$ for various Temperatures.

Temperature.	Tutton.	Chappuis.
0°	0·000 002 522	0·000 002 824
10	2 671	2 947
20	2 819	3 071
30	2 967	3 194
40	3 116	3 317
50	3 265	3 441
60	3 414	3 564
70	3 562	3 688
80	3 711	3 811
90	3 859	
100	4 008	
110	4 157	
120	4 305	

The value for 120°, calculated from the formula of Chappuis, is 0·000 004 305, exactly the same as the author's value.

Chappuis offered, towards an explanation of the difference between Bedford's results and his own, the observation that some rotation of the fringes occurred, indicating inequality of expansion and curving of the tube; and further, that the hole through the tube was considerably eccentric, so that opposite parts of the wall presented variations in thickness, and the effect of a glaze differing in expansibility from the "biscuit" might be to slightly deform the tube on heating. In the author's opinion this is not the cause of the discrepancy, for in the author's experiments the amount of rotation of the interference-bands was negligible, and the hole was central. The results now presented indicate rather that the discrepancy is due to a fundamental real difference, dependent on the interval of temperature for which the determinations were made. They indicate clearly, in the author's opinion, that the character of the expansion between 0° and 100° is substantially that exhibited by the results of Chappuis for the interval of 0° to 83°, those results, however, not going quite far enough in their difference from those of Bedford; also, that the considerable difference from the latter observer's values is owing to the large interval included by Bedford's, 0° to 830°. In other words, presuming the accuracy of Bedford's results (and there is every reason to do so, as they are confirmed by the results of other observers with Berlin porcelain), it appears proved that the increment per degree ($\Delta\alpha/\Delta t = 2b$), of the coefficient of expansion, in the case of porcelain is not a constant quantity, but one which is much larger between 0° and 100° than at the higher temperatures explored by Bedford. Two entirely independent series of

Fig. 2.—Lengths at $t°$ when $L_{0°}=1$.



determinations, carried out by the highly accurate interference method, and employing the quite different forms of apparatus of Fizeau and the author, involving also the use of light of entirely different wave-lengths, have now conclusively settled the rate of expansion for the short interval of 0° to 100°, and, indeed, if the author's higher limit is included, for the interval 0° to 120°. The temperature of 120° herein attained, is the maximum which can with safety be employed in an interference-dilatometer.

The conclusion now arrived at with respect to Bayeux porcelain is very similar to that derived from a research by Holborn and Day (*Ann. der Phys. und Chem.* vol ii. p. 505, 1900), on the expansion of Berlin porcelain between 0° and 1000°. A rod of unglazed porcelain 483 mm. long was employed, and the method of the comparator used in making the determinations of its expansion. It was found that between 250° and 625° the results are fairly reproduced by the expression

$$l_t = l_0[1 + (2954t + 1{\cdot}125t^2)10^{-9}].$$

But below 250° the values afforded by the formula are too high. Again, between 700° and 900° the determinations are not accurately reproduced by the formula; but agreement becomes better again at 1000°.

Prof. Callendar, in discussing the relative results of Chappuis and Bedford (Proceedings of the Physical Society, Nov. 22nd, 1901), stated that "Bedford's results agreed very fairly, allowing for difference of material, with Holborn and Day's from 200° to 600°, and that both differed from those of Chappuis between 0° and 80° when extrapolated in a precisely similar manner. It was quite possible, as he (Prof. Callendar) had previously suggested, that the expansion of porcelain between 0° and 100° was anomalous. It appeared certain that some anomaly in the expansion at 800° was indicated in the experiments of Bedford, and also in those of Holborn and Day."

The supposition of Prof. Callendar, as to an anomalous expansion between 0° and 100° in the case of porcelain, appears to be well founded, the anomaly consisting in a very much larger increment for this interval than for the higher interval 200° to 600°. The author fully agrees with Dr. Chappuis that the thermal expansion of the porcelain employed for air-thermometers, whether of the Bayeux or Berlin varieties, cannot be represented by a two-term function throughout an interval of temperature exceeding a very few hundred degrees. On no account can such an expression be safely employed outside the interval for which it was actually determined, extrapolation leading to grave errors in the case of porcelain.

LXXII. *On the Change of the Electrical Resistance of Metals when placed in a Magnetic Field.* *By* J. Patterson, *B.A.* (*Toronto*), 1851 *Exhibition Science Scholar, Emmanuel College, Cambridge* [*].

THE change of the electrical resistance of metals when placed in a magnetic field was first observed in iron and nickel by Lord Kelvin [†] in 1856. He found that the electrical resistance of these metals was increased when magnetized longitudinally, and diminished when magnetized transversely to the direction of the current.

Tomlinson [‡] showed that bismuth had its resistance increased by a field parallel to the direction of the current; while Righi [§] observed that its resistance was also increased when the field was transverse to the direction of the current.

Goldhammer [||] made more accurate experiments on bismuth and on the magnetic metals iron, cobalt, and nickel, He also found that antimony and tellurium exhibited this phenomenon.

Recently Lenard [¶] made very careful experiments on bismuth with both steady and alternating currents. He found that the change of resistance was different when the current was alternating from what it was when steady, and also that the increase of resistance for a longitudinal field was less than for a transverse one. He obtained a small change in german-silver and copper, but in this case it was the same whether the current was steady or alternating.

This phenomenon has recently been explained by Prof. J. J. Thomson [**] on the corpuscular theory of electric conduction in metals. According to this theory, the electric currents in the metals are carried by negatively-charged corpuscles moving with a definite velocity under an applied E.M.F. These corpuscles are considered to act as a perfect gas: consequently they have a mean free path and mean velocity and exert a definite pressure.

He has shown that if the corpuscles be moving under an electric force, and then a transverse magnetic force be applied, the particles move in a cycloid. Their path is thus increased, and consequently the resistance of the conductor is changed.

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[*] Communicated by Prof. J. J. Thomson, F.R.S.

[†] Math. & Physical Papers, vol. ii. p. 307.

[‡] Phil. Trans. 1883, p. 1.

[§] *Journal de Physique*, ser. 2, tome iii. p. 355 (1884).

[||] Wied. *Annalen*, vol. xxxi. p. 360 (1887).

[¶] Wied. *Annalen*, vol. xxxix. p. 619 (1890).

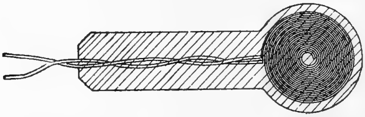
[**] *Rapports présentés au Congrès International de Physique*, iii. p. 138, Paris, 1900.

If the corpuscles be acted upon by a magnetic force in the same direction as the electric force, they describe spirals about the lines of magnetic force, and consequently the resistance is, in this case, also changed.

It was suggested by Prof. Thomson that I should make a series of experiments on some of the non-magnetic metals, to see whether they also showed a change of resistance in the transverse magnetic field, and, if so, to measure its magnitude, and from these measurements to obtain the mean free path, mean velocity, and pressure of the corpuscles.

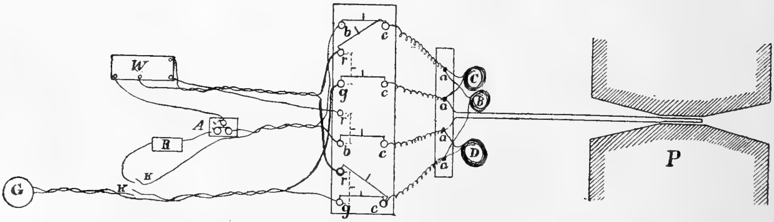
The metals, which were in the form of wires, double silk-covered, were wound non-inductively in flat spirals. These spirals were about 15 mm. diameter and 1 mm. thick; they were wound between two plates, and then immersed for a few minutes in hot paraffin-wax. When cold, one of the plates was carefully removed and a thin strip of mica put over the spiral and fastened with paraffin-wax. The other side was then treated in a similar manner. The ends were soldered to two stout pieces of copper wire twisted together. Fig. 1 shows the spiral when complete.

Fig. 1.



Two similar spirals were made of the same metal, each having, as nearly as possible, the same resistance, so that the second would act as a compensating resistance. It was hoped that by this arrangement greater constancy would be secured than if another resistance was used. Two other non-inductive resistances, nearly equal to the others, were made usually of german-silver. These four resistances were joined together in the form of a Wheatstone bridge. The general arrangement is shown in fig. 2.

Fig. 2.



All the junctions (*a*, fig. 2) were soldered and mounted on ebonite. For convenience, these junctions were connected to mercury cups (*c*, fig. 2) with copper wire. The bridge was balanced by placing one of the arms, either B or C as the experiment required, in multiple arc with a large variable resistance W.

The current was furnished by a single Leclanché cell with a variable resistance R in its circuit. This allowed the current in the bridge to be adjusted so as not to produce appreciable heating.

A D'Arsonval galvanometer, G, was used as it was not affected by a magnet. It had a resistance of 95 ohms and, by using a telescope and scale, it gave a deflexion of 1 mm. for $2{\cdot}4 \times 10^{-8}$ ampere, or a deflexion of 1 mm. on the scale would correspond to a change of resistance in one arm of the bridge to, at most, one part in 50,000.

There was an extra wire from r' to the variable resistance W, and also a mercury key in the battery-circuit at A. This arrangement permitted the resistance W to be used as an ordinary Wheatstone bridge by simply changing the mercury keys to the dotted position as shown in figure.

All the circuits in the Wheatstone bridge were carefully insulated and supported on paraffin blocks, so that there could be no leaking to the galvanometer from extraneous sources.

The spiral and the other resistances forming the bridge and the connecting wires to the mercury cups were well covered with paraffin-wax, so that they could be immersed in a water-bath.

To produce the magnetic field, the large ring-magnet of the Du Bois type in the Cavendish Laboratory was used. The poles, P (fig. 2), were about 2 mm. apart. It gave a very concentrated field, so that the other arms of the bridge, which were about 20 cm. away, were not affected by the magnet. The fields were measured with a carefully calibrated bismuth spiral for known currents through the magnet: these fields were taken as constant throughout the experiment for that position of the poles.

The poles of the magnet were surrounded by a water-bath, which was kept at the temperature of the room. The spiral and the resistances were all immersed in this bath.

The first experiments were made with german-silver wire 0·004 inch diameter. The resistances of the two spirals were 44·035 and 44·06 ohms respectively. The other two, which were also of german-silver, had 42·41 and 43·58 ohms respectively. With this arrangement, no change could be detected with a field of 26,600 lines per sq. cm.

Lenard * obtained a slight change of resistance for german-silver in a magnetic field. He does not state, however, the composition of the metal he used. The specimen used in this experiment contained about 12 per cent. of nickel; and it is most probable that the different results are due to different compositions of the alloy.

A series of experiments were next made with pure copper in the form of white-silk covered wire, hard drawn, ·002 inch diameter. The two spirals had resistances of 24·04 and 22·92 ohms. The other two, of german-silver, had 24·12 and 22·95 ohms respectively.

It was exceedingly difficult to obtain a balance on the bridge for any length of time. This was probably due to heating of the resistances, for as they were insulated with paraffin, the heat would not be conducted away very rapidly. Besides, one of the spirals being between two large pole-pieces of the magnet, the heat in all probability was conducted away from that spiral at a different rate than from the other. The galvanometer, however, always indicated an increase of resistance when the magnetic field was on, but when the field was put off and the bridge again tested the original balance had altered generally in the direction which indicated an increase of resistance in the compensating spiral. Any change in the resistance of the coils due to heating would therefore tend to diminish the effect due to the magnetic field rather than increase it. This shows that the increase in resistance of the test-coil must be due to the magnetic field. The compensating spiral was replaced by a german-silver resistance of 22·92 ohms. This gave much more satisfactory results, but there was a slow change from time to time.

To obtain a reading the bridge was first balanced, then the magnetic field put on and the bridge again balanced. The field was put off, and the bridge tested to see if the original balance had altered. If it had changed a little, the experiment was repeated until the same balance was obtained after as before the field was put on. It required but a few seconds to make a complete observation.

If A is the resistance of the test spiral, B, C, D the resistances in the other arms of the bridge, C being in multiple arc with the variable resistance R, then for equilibrium we must have

$$\frac{D}{B} = \frac{A}{\frac{CR}{C+R}}$$

$$= \frac{A}{\sigma},$$

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* Wied. *Ann.* vol. xxxix. p. 619 (1890).

where

$$\sigma = \frac{\mathrm{CR}}{\mathrm{C}+\mathrm{R}}.$$

If, now, when the magnetic field is put on, A is increased by $\partial\mathrm{A}$, and consequently σ by $\partial\sigma$ in order that equilibrium may be preserved, then

$$\frac{\mathrm{A}+\partial\mathrm{A}}{\sigma+\partial\sigma} = \frac{\mathrm{D}}{\mathrm{B}} = \frac{\mathrm{A}}{\sigma},$$

or

$$\frac{\partial\mathrm{A}}{\mathrm{A}} = \frac{\partial\sigma}{\sigma}.$$

Hence the value of the change of resistance due to the magnetic field is given by $\frac{\partial\sigma}{\sigma}$.

Table I. gives the results obtained for the first set of observations.

TABLE I.—Copper Wire.

H= Magnetic field.	R=Resistance in multiple arc to balance bridge.		$\sigma = \frac{24.12\times \mathrm{R}}{24.12+\mathrm{R}}$.		$\partial\sigma$= increase of resist. due to field.	$\frac{\partial\sigma}{\sigma}10^4$.	Mean value of $\frac{\partial\sigma}{\sigma}10^4$.	$\frac{\partial\sigma}{\sigma}\frac{10^{13}}{\mathrm{H}^2}$.
	Field off.	Field on.	Field off.	Field on.				
26500	2175	2215	23·8555	23·8602	·0047	1·97		
26500	2240	2285	23·8631	23·8681	·0050	2·09	2·01	29
26500	2300	2345	23·8697	23·8744	·0047	1·97		
21300	2210	2240	23·8596	23·8631	·0035	1·46		
21300	2370	2405	23·8770	23·8806	·0036	1·50	1·50	33
21300	2300	2335	23·8697	23·8734	·0037	1·55		

To test whether the apparent change in the resistance was real or due to induction effects or the magnetizing current, the galvanometer-circuit was closed, while the battery-circuit remained open, but in this case the deflexion of the galvanometer, if any, was the same whether the field was on or off. Reversing the current in the magnet did not affect the result. As the four resistances of the Wheatstone bridge formed a closed circuit, the change might be due to a sudden heating of the resistances by any induced currents at the make or break of the magnetizing circuit. To prevent this, the bridge was arranged so that the circuit could be left open when the field was put on or off. This also had no

TABLE II.—Copper Wire.

Position of Poles.	Current in amperes in magnet.	H = magnetic field.	R = Resistance in multiple arc to balance bridge.		$\sigma = \frac{24.12 \times R}{24.12 + R}$.		$\partial\sigma$ = increase of resistance due to field.	$\frac{\partial\sigma}{\sigma}10^4$.	Mean value of $\frac{\partial\sigma}{\sigma}10^4$.	$\frac{\partial\sigma}{\sigma}\frac{10^{13}}{H^2}$.
			Field off.	Field on.	Field off.	Field on.				
3rd ……	20	28000	2400	2445	23·8800	23·8844	·0044	1·84	1·80	23
,, ……	20	28000	2185	2220	23·9116	23·9158	·0042	1·76		
1st ……	20	26000	2250	2290	23·8642	23·8686	·0044	1·84	1·85	27
,, ……	20	26000	2830	2895	23·9162	23·9207	·0045	1·88		
,, ……	20	26000	2880	2945	23·9197	23·9241	·0044	1·84		
3rd ……	12	25850	2400	2440	23·8800	23·8839	·0039	1·63	1·63	24
1st ……	12	24200	2545	2585	23·8936	23·8971	·0035	1·47	1·55	26
,, ……	12	24200	2810	2865	23·9147	23·9186	·0039	1·63		
2nd ……	20	22400	1760	1777	23·7939	23·7970	·0031	1·30	1·18	24
,, ……	20	22400	1900	1918	23·8176	23·8204	·0028	1·17		
,, ……	20	22400	2235	2260	23·8625	23·8653	·0028	1·17		
,, ……	20	22400	2310	2337	23·8708	23·8736	·0028	1·17		
,, ……	20	22400	2550	2580	23·8940	23·8966	·0026	1·09		
3rd ……	6	20500	1716	1730	23·7857	23·7880	·0023	0·96	1·07	25
,, ……	6	20500	2445	2475	23·8844	23·8872	·0028	1·17		
2nd ……	12	20400	1695	1709	23·7816	23·7843	·0027	1·13	0·99	24
,, ……	12	20400	2255	2275	23·8647	23·8670	·0023	0·96		
,, ……	12	20400	2525	2550	23·8918	23·8940	·0022	0·92		
,, ……	12	20400	1734	1748	23·7891	23·7917	·0026	1·09		
,, ……	12	20400	2370	2392	23·8770	23·8792	·0022	0·92		
,, ……	12	20400	2570	2595	23·8959	23·8971	·0022	0·92		
1st ……	6	18400	2655	2680	23·9028	23·9049	·0021	0·87	0·89	26
,, ……	6	18400	2780	2810	23·9125	23·9147	·0022	0·92		
2nd ……	6	12700	2280	2290	23·8675	23·8686	·0011	0·46	0·45	28
,, ……	6	12700	2400	2410	23·8800	23·8810	·0010	0·42		
,, ……	6	12700	2530	2542	23·8922	23·8933	·0011	0·46		

effect on the result. Moreover, had it been due to the above cause, the resistance should have been greater after the field was put off, owing to the induced current at break.

As a final test a series of observations were made with the poles of the magnet at different distances apart, so that different fields were obtained with the same current in the magnetizing circuit. The values obtained are given in Table II.

These results show that when the field is constant the change of resistance for this field is independent of the current in the magnetizing circuit. There are considerable differences, it is true, in the magnitude of the change of resistance for the same field. It will be seen, however, in the column giving the increase of resistance due to the field, that these differences occur in the fourth place of decimals and this figure is only approximate.

These results confirm those obtained by Lenard* for copper. He, however, does not give the magnitude of the change which he obtained, except stating that it was very small.

It was hoped that by surrounding the poles of the magnet with a water-jacket through which the water could be kept flowing, the resistance would be more constant. It was found, however, that much more satisfactory results were obtained when the water, instead of circulating, was allowed to stand in the jacket.

Another spiral of the same copper wire was made with a resistance of 58·68 ohms. The other three resistances were made of "Eureka" wire, and had 59·6, 58·2, 58·48 ohms respectively. With this spiral another series of observations were made, the results of which are given in Table III.

TABLE III.—Copper Wire.

H= Magnetic field.	R=Resistance in multiple arc to balance bridge.		$\sigma=\frac{59.6\times R}{59.6+R}$.		$\partial\sigma=$ increase of resist. due to field.	$\frac{\partial\sigma}{\sigma}10^4$.	Mean value of $\frac{\partial\sigma}{\sigma}10^4$.	$\frac{\partial\sigma}{\sigma}\frac{10^{13}}{H^2}$.
	Field off.	Field on.	Field off.	Field on.				
27650	2090	2076	57·9475	57·9367	·0108	1·86	1·96	26
27650	2080	2065	57·9398	57·9281	·0117	2·02		
27650	2090	2075	57·9475	57 9339	·0116	2·00		
25850	2072·5	2060	57·9340	57·9241	·0099	1·70	1·67	25
25850	2062	2050	57·9257	57·9162	·0095	1·64		

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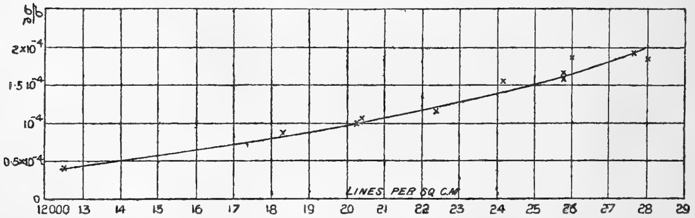
* Wied. *Ann.* vol. xxxix. p. 619 (1890).

In these observations the bridge was balanced by placing B (fig. 2) in multiple arc with the variable resistance; then, in order that the relation $A\sigma = CD$ may hold when A is increased, σ must be decreased. Hence the increase in A appears in the table as a decrease in σ.

The average of all the results given in Tables I., II., and III. are plotted in fig. 3, where the abscissæ represent the field and the ordinates the change of resistance.

Fig. 3.



If the change of resistance due to the magnetic field varied as the square of the resistance, then $\frac{\partial\sigma}{\sigma}\frac{1}{H^2}$ should be a constant. The values obtained for $\frac{\partial\sigma}{\sigma}\frac{10^{13}}{H^2}$ would indicate that this is so.

Experiments were also made with "Eureka" wire, which contains iron and has practically no temperature-coefficient. The spiral had a resistance of 94·87 ohms, and the other three resistances, also of "Eureka" wire, were each of about 95·21 ohms. In this case a decrease of resistance of 0·75 in 10,000 for a transverse field of 26,900 lines was obtained. As the composition of the wire was not known, these experiments were not pursued further. They are of interest, however, in that they show that iron in its alloys still retains the property of decreasing its resistance when placed in a transverse magnetic field.

To measure the change of resistance in mercury, a spiral was made of capillary glass tubing and filled with mercury.

The resistance was about 2·39 ohms. In this case the deflexions of the galvanometer were observed, and the change calculated from the formulæ

$$z = (a\alpha - b\beta)\frac{E}{\Delta},$$

$$= a((\alpha + \delta\alpha) - b\beta)\frac{E}{\Delta},$$

$$= a\delta\alpha\frac{E}{\Delta},$$

where

$$\Delta = BG(a+b+\alpha+\beta) + B(b+\alpha)(a+\beta) + G(a+b)(\alpha+\beta)$$
$$+ ab\alpha + ab\beta + a\alpha\beta + b\alpha\beta,$$

where z is the current through the galvanometer, E the E.M.F. of the battery: B the resistance of the battery circuit and G that of the galvanometer; a, b, α, β the resistances of the arms of the bridge. The results obtained are given in Table IV.

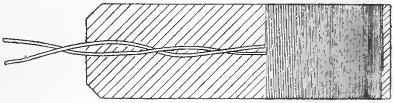
Table IV.—Mercury.

Magnetic field.	σ=resistance of spiral. No field.	Defl. of Galvanometer. Field on.	$\partial\sigma$=increase of resistance due to field.	$\frac{\partial\sigma}{\sigma} 10^4$.
24900	2·3990	4·1 mm.	·00076	3·18
22600	2·3990	3·6 „	·00067	2·80
22300	2·3990	3·8 „	·00070	2·92
18100	2·3774	2·9 „	·00054	2·27
14400	2·3990	2·0 „	·00037	1·54

For the remainder of the experiments, the sensitiveness of the galvanometer was changed from $2{\cdot}4 \times 10^{-8}$ to $1{\cdot}9 \times 10^{-9}$ ampere per scale-division. This made the period of the galvanometer much longer, but it allowed the current in the bridge to be decreased very much. Usually 500 ohms were inserted in the battery circuit.

Instead of winding the wire to be tested in a spiral, it was found much better and more convenient to wind it on strips of mica about $\frac{1}{10}$ mm. thick, 13 mm. wide, and 14 mm. long (fig. 4).

Fig. 4.



The small fraction of the wire that was parallel to the direction of the lines of force would not amount to 1 per cent. of the total, and in these experiments was negligible. No water-bath was required with this arrangement.

Experiments were made with gold, platinum, silver, zinc, tin, cadmium, and carbon. The gold, platinum, and silver were pure, and the wires were double white-silk covered, 0·0025 inch diameter. The tin was practically pure; the zinc contained about 1 per cent. lead and a trace of iron; the cadmium was about 99½ per cent. pure and had a little zinc and a trace of iron. The zinc and cadmium were tested spectroscopically for bismuth, but no trace could be detected. The wires were in the latter set 0·004 inch diameter, and double white-silk covered. The carbon was obtained by carefully removing the filament from an 8-volt incandescent lamp so that the connexions were not broken. It was then mounted on mica and imbedded in paraffin. A large number of observations were made with each metal, and the average values are given in Table V.

Table V.

Magnetic field.	Cadmium. $\frac{\partial\sigma}{\sigma}10^4$.	Zinc. $\frac{\partial\sigma}{\sigma}10^4$.	Gold. $\frac{\partial\sigma}{\sigma}10^4$.	Platinum. $\frac{\partial\sigma}{\sigma}10^4$.	Tin. $\frac{\partial\sigma}{\sigma}10^4$.	Silver. $\frac{\partial\sigma}{\sigma}10^4$.	Carbon. $\frac{\partial\sigma}{\sigma}10^4$.
29200	21·16	6·35	3·02	·44	1·87	2·02	
28600	20·86	5·98	2·93	·44	1·84	1·76	
26200	...	5·72	...	...	...	1·86	
25500	18·0	5·31	...	...	1·52		
25300	17·36	5·09	2·39	·33	1·47	1·46	
24400	...	...	...	...	...	...	2·73
21400	13·25	4·11	...	...	1·24	1·54	
21100	...	...	...	...	...	...	2·42
20400	12·32	3·77	...	...	1·09	1·22	
19800	...	...	...	...	...	...	1·49
18400	10·51	3·47	...	...	...	·96	
17900	...	...	...	...	...	...	1·18
17300	9·28	3·40	1·18	·27	·74	·68	

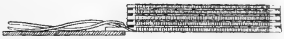
The average values of $\frac{\partial\sigma}{\sigma}\frac{10^{13}}{H^2}$ in order of magnitude obtained from above tables are

Cd 282. Zn 87. Hg 54.
C 44. Au 37. Ag 26.
Cu 26. Sn 23. Pt 6.

These numbers thus represent the relative values of $\frac{\partial\sigma}{\sigma}$ for the same magnetic field.

To measure the change of resistance where the current is parallel to the lines of force, the wire was wound on thin strips of mica 15 mm. long and 3 or 4 mm. wide. These strips were placed one on top of the other so as to form a small rectangular parallelepiped. Magnified views in plan and elevation are shown in figs. 4 and 5.

Fig. 5.



This resistance could be placed so that the current was either parallel or perpendicular to the lines of force. A series of experiments were made on german-silver, copper, silver, tin, and zinc. The german-silver gave no change for a field of 19,800 lines. A number of observations were made on each of the others, and the average values obtained are given in Table VI.

TABLE VI.

Metal.	Magnetic field.	Resistance of wire in field.	$\partial\sigma$ = increase due to field.	$\frac{\partial\sigma}{\sigma} 10^4$.
Silver	22700	10·7189	·00105	·98
	20300	10·7159	·00085	·79
Tin..............	22700	25·5704	·0024	·93
	20300	25·5672	·0019	·74
Zinc	22700	19·0671	·0057	3·09
	20300	19·0671	·0047	2·46
Copper	23500	...	...	1·80
	20800	...	...	1·54

The same piece of copper wire was used to measure the change of resistance in both transverse and longitudinal fields. The values obtained for the change of resistance in the same field appeared to be slightly greater when the field was transverse than when longitudinal.

In Table V. the values for silver appear somewhat discordant as are also those for copper given in Tables I., II., & III. These two metals, however, are the best conductors, and considerable more difficulty was experienced in measuring the change of resistance due to the magnetic field in these metals than in the others.

The difficulties connected with measuring such small changes are too great to expect very accurate results: hence not much more can be claimed for the numbers given in this paper than that they represent the order of magnitude of the change of resistance in the magnetic field.

Prof. Thomson * has shown that if

n be the number of corpuscles in unit volume,
e the charge on a corpuscle and m its mass,
u_0 the velocity acquired under a unit field,
λ the mean free path of the corpuscle,
c the initial velocity, which is about $7{\cdot}6 \times 10^6$ at 27 °C.,
t the time a corpuscle takes to complete its mean free path,
q the number of corpuscles produced per sec. in the steady state,
H the transverse magnetic field;

then the conductivity is $\frac{1}{\sigma} = neu_0$. Also if $\partial\sigma$ is the change of resistance due to the magnetic field,

$$\frac{\partial\sigma}{\sigma} = \frac{1}{3}H^2u_0^{\,2}.$$

Using the values for the change in the resistance of platinum given in Table V. and substituting in above equation,

then $$\cdot 44 \times 10^{-4} = \frac{1}{3}(29200)^2u_0^{\,2}$$

or $$u_0 = 3{\cdot}9 \times 10^{-7}.$$

Also $$u_0 = \frac{1}{2}\frac{e}{m}\frac{\lambda}{c},$$

or $$\lambda = 2u_0\frac{m}{e}c,$$

$$= 2 \times 3{\cdot}9 \times 10^{-7} \times 10^{-7} \times 7{\cdot}6 \times 10^6,$$

$$= 5{\cdot}9 \times 10^{-7} \text{ for platinum.}$$

Since

$$neu_0 = \frac{1}{\sigma},$$

$$ne = \frac{1}{\sigma u_0},$$

$$= \frac{1}{9100 \times 3{\cdot}9 \times 10^{-7}} \text{ in the case of platinum,}$$

$$= 280 \text{ about.}$$

If N is the number of molecules of hydrogen in 1 c.c.

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* J. J. Thomson, *Rapports présentés au Congrès International de Physique*, iii. p. 138, Paris, 1900.

under atmospheric pressure and at 0° C., and e the charge on an ion,

$$Ne=0{\cdot}4,$$

and therefore $$\frac{ne}{Ne} = 700 \text{ about};$$

that is, the pressure exerted by the corpuscles in the case of platinum is about 700 atmospheres.

Again, the charge on a corpuscle is 6×10^{-10} electrostatic units, and this gives

$$\begin{aligned} n &= \frac{280}{6\times10^{-10}}\times3\times10^{10}, \\ &= 1{\cdot}4\times10^{22}. \end{aligned}$$

Richardson * has obtained for n the value $1{\cdot}3\times10^{21}$ at a temperature of 1542° absolute, from the negative radiation of a hot wire. The agreement of this with the above is fairly close considering the difficulties in connexion with the experiment, and the wide difference of temperature at which the results were obtained.

Since $t=\frac{\lambda}{c}$ this becomes for platinum 8×10^{-14}, and from this we obtain for $q=\frac{n}{\lambda/c}$ the value $1{\cdot}8\times10^{35}$.

The results for the other metals are tabulated in the following table, those for platinum being also given to make the table complete.

Metal.	u_0.	λ.	ne in atmos.	n.	$\frac{\lambda}{c}$.	q.
Platinum ...	$3{\cdot}9\times10^{-7}$	$5{\cdot}9\times10^{-7}$	700	$1{\cdot}4\times10^{22}$	8×10^{-14}	$1{\cdot}8\times10^{35}$
Gold	$1{\cdot}04\times10^{-6}$	$1{\cdot}6\times10^{-6}$	1115	$2{\cdot}2\times10^{22}$	$2{\cdot}1\times10^{-13}$	$1{\cdot}0\times10^{35}$
Tin	$8{\cdot}3\times10^{-7}$	$1{\cdot}3\times10^{-6}$	225	$4{\cdot}5\times10^{21}$	$1{\cdot}7\times10^{-13}$	$2{\cdot}6\times10^{34}$
Silver	$8{\cdot}5\times10^{-7}$	$1{\cdot}3\times10^{-6}$	1840	$3{\cdot}6\times10^{22}$	$1{\cdot}7\times10^{-13}$	$2{\cdot}1\times10^{35}$
Copper	$8{\cdot}8\times10^{-7}$	$1{\cdot}34\times10^{-6}$	1730	$3{\cdot}4\times10^{22}$	$1{\cdot}8\times10^{-13}$	$1{\cdot}9\times10^{35}$
Zinc	$1{\cdot}5\times10^{-6}$	$2{\cdot}3\times10^{-6}$	293	$5{\cdot}8\times10^{21}$	3×10^{-13}	$1{\cdot}9\times10^{34}$
Cadmium ...	$2{\cdot}7\times10^{-6}$	$4{\cdot}1\times10^{-6}$	136	$2{\cdot}7\times10^{21}$	$5{\cdot}4\times10^{-13}$	5×10^{33}
Mercury......	$1{\cdot}2\times10^{-6}$	$1{\cdot}82\times10^{-6}$	21	$4{\cdot}3\times10^{20}$	$2{\cdot}4\times10^{-13}$	$1{\cdot}8\times10^{33}$
Carbon	$1{\cdot}18\times10^{-6}$	$1{\cdot}79\times10^{-6}$	0·54	$1{\cdot}08\times10^{19}$	$2{\cdot}35\times10^{-13}$	$4{\cdot}6\times10^{31}$

In order that the corpuscular pressure may be in equilibrium at the surface of the metal, the pressure tending to drive out

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* Proc. Camb. Phil. Soc. vol. xi. pt. iv. p. 286 (1901).

the corpuscles must be equal to the force tending to keep them in. Taking the axis of x as perpendicular to the surface of the metal, then at any point x if p is the pressure, X the force, and n the number of corpuscles per unit volume,

$$-\frac{dp}{dx}=\mathrm{X}ne.$$

But p is proportional to the absolute temperature, and the number of corpuscles therefore

$$-c\theta\frac{dn}{dx}=\mathrm{X}ne,$$

or
$$-\frac{c\theta}{e}[\log n]=\int \mathrm{X}dx,$$
$$=\mathrm{W};$$

where W is the work required to take the corpuscle from the inside to the outside of the metal.

On the assumption that the work is all electrical, W would be the difference of potential, and in the case of two metals in contact it would be equal to V the contact-difference of potential. If n_1 and n_2 are the number of corpuscles per unit volume in each metal, then on the above assumption we have

$$\mathrm{V}=\frac{c\theta}{e}\log\frac{n_1}{n_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{A})$$

Taking Maxwell's value of 2×10^{19} for the number of molecules per c.c. of a gas at standard temperature and pressure, we have

$$\frac{c\theta}{e}=0{\cdot}027 \text{ volt.}$$

Substituting the values of n obtained for the different metals in (A), the values tabulated below are obtained, on making the assumption stated above, for the contact-difference of potential in volts :

Cu–Pt.	Cu–Au.	Cu–Ag.	Cu–Cd.	Cu–Zn.	Cu–Sn.	Cu–C.
0·0239	0·0117	–0·0015	0·0684	0·0478	0·0546	0·2176

In conclusion I wish to acknowledge my indebtedness to Prof. Thomson for his kindly interest and valuable suggestions throughout the course of the investigation.

Cavendish Laboratory,
March 14th, 1902.

LXXIII. *Notices respecting New Bokos.*

Lehrbuch der Potentialtheorie. II. *Allgemeine Theorie des logarithmischen Potentials und der Potentialfunktionen in der Ebene.* Von Dr. ARTHUR KORN. Mit 58 in den Text gedruckten Figuren. Berlin, 1901. Ferd. Dümmlers Verlagsbuchhandlung. Pp. x + 366.

THIS volume forms Part II. of the author's Treatise on the Theory of Potential, but may be regarded as an entirely self-contained work, since no knowledge of the results arrived at in the first portion of the complete treatise is assumed. The theory of logarithmic potential enters largely into many two-dimensional problems of modern mathematical physics, and the author is to be congratulated on the clear exposition which he has given us of this important subject. The work is divided into six sections, and is preceded by a short introduction on line and surface integrals. In Section I. the author defines and classifies the various types of logarithmic potential, considers their general properties, and then discusses the special characteristics of point, line, and surface potentials. In Section II. the expansion of log $1/r$ is considered, and is followed by an account of the expansion of functions generally in terms of circular functions. Sections III., IV., and V. deal with the theory of potential functions generally, of their representation by means of line-integrals, their turning-values, and the solution of Dirichlet's problem in two dimensions. The concluding Section VI. is devoted to an account of conformal representation and to Green's function. A useful bibliographical appendix and table of *errata* are given at the end of the book.

Die Heterogenen Gleichgewichte vom Standpunkte der Phasenlehre. Von Dr. H. W. BAKHUIS ROOZEBOOM, *Professor an der Universität Amsterdam.* Erstes Heft: Die Phasenlehre—Systeme aus Einer Komponente. Mit 54 eingedruckten Abbildungen. Braunschweig: F. Vieweg und Sohn. 1901. Pp. xiv + 222.

THE domain of physical chemistry is one in which Dutch physicists appear particularly to excel; and the history of this branch of science is very largely associated with the names of Dutch investigators. It is therefore only natural to expect that a treatise on an important branch of physical chemistry should come from the pen of a Dutch author. The book under review forms the first part of a systematic treatise on the equilibrium of heterogeneous systems, regarded from the point of view of Willard Gibbs's Phase Rule, which is adopted as the basis of classification in various cases of chemical and physical equilibrium.

After a very careful and lucid explanation of the Phase Rule, the author proceeds to the consideration of systems of a single component, taking the cases of equilibrium between a liquid and a gaseous phase, a solid and a gaseous one, and a solid and liquid one. The "triple point"—solid-liquid-gas—is next discussed.

The remainder of the book is devoted to the consideration of more complicated cases of systems of a single component.

In his treatment of the subject, the author avoids, as far as possible, the use of higher mathematics and the more difficult parts of thermodynamics, his primary aim being to furnish chemists with a clear account of the present state of the theory of equilibria.

The treatment of systems of two components is promised in Part II., and of systems of three or more components in Part III.

Outlines of Electrochemistry. *By* HARRY C. JONES, *Associate-Professor of Physical Chemistry in the Johns Hopkins University*, New York: D. Van Nostrand Company. 1901. Pp. vi+106.

WE know of no other book which contains, within so short a compass, so clear an account of the leading ideas of modern electrochemistry. The author is an enthusiast, and unmistakable evidences of this appear on almost every page of the book, which forms delightful reading. No attempt is made to present the reader with an exhaustive and systematic account of the subject, but the method followed will, we venture to think, furnish him with a much clearer conception of modern electrochemical theories than could be derived from the study of more elaborate treatises. The subjects dealt with comprise osmotic pressure, electrolytic dissociation, electrolysis and theories of electrolysis, conductivity of solutions, and calculation of electromotive-forces of cells. It is very unfortunate that the author should use the terms "tension" and "solution-tension" for "pressure" and solution-pressures. The absurdity of using "tension" in this sense has repeatedly been pointed out; yet chemists are great sinners in this respect. Again, *why* should electrochemists arbitrarily seize upon the much-abused "π" and make it do duty for "P.D."? Why not use symbols commonly employed by electricians? On p. 83, foot of second column, the printer has made havoc of the author's meaning. Apart from these and one or two other minor blemishes, we have little fault to find with the book, and can heartily commend it for its general excellence.

Halbmonatliches Litteraturverzeichniss der Fortschritte der Physik im Jahre 1902. Dargestellt von der Deutschen Physikalischen Gesellschaft. Redigirt von KARL SCHEEL und RICHARD ASSMANN. 1 Jahrg. Nos. 1 und 2. Braunschweig: F. Vieweg und Sohn.

IN order to enable its members to keep in touch with contemporary scientific literature, the Deutsche Physikalische Gesellschaft has decided to issue a classified fortnightly register of all articles dealing with physical subjects (including meteorology) in various languages. We have before us the first two numbers of this new publication, which for purposes of reference should prove very important. Considerable care appears to have been bestowed on the classification of the articles under suitable headings.

LXXIV. *Intelligence and Miscellaneous Articles.*

THE PRODUCTION OF A MAGNETIC FIELD BY A FLIGHT OF CHARGED PARTICLES. BY PROF. R. W. WOOD.

IN the various modifications of Rowland's experiment on electric convection, centrifugal force places an upper limit to the attainable velocity of the moving charge. It has seemed to me that if the revolving disk could be abandoned, and a flight of charged particles substituted for it, much higher velocities could be obtained. A number of experiments have been planned, which I am at the present time carrying out in collaboration with Dr. Harold Pender. The first one has already yielded positive results, a deflexion of a magnetic needle having been obtained by driving a cloud of charged particles of solid carbon dioxide through a long glass tube placed under the needle. The particles become charged in the same manner as the steam-jet in the hydroelectric machine of Armstrong, the potential being so high that sparks over a centimetre long can be drawn from the iron cylinder in which the liquefied gas is contained. While we have obtained scale-deflexions as high as 10 centimetres with the tube close to the needle, it has been found better to work with the tube about 2·5 cms. below the needle, in which case the deflexion is about 2 cms. Reversing the direction of the jet invariably reverses the direction of the deflexion. At the present time we are employing two glass tubes side by side in a larger tube, the space between being filled with paraffin. The tubes are about 1·5 metre long, each one being permanently attached to a cylinder of the gas. If the needle is deflected by a stream flowing through one tube, it can be brought nearly back to its original position by setting the other tube in action. The particles appear to carry positive charges, which is the case in the hydroelectric machine, and the direction of the deflexion is the same as when an electric current flows from the carbon to the zinc in the direction of the gas-jet. As yet no attempt has been made to determine experimentally the velocity of the moving charge. About a pound of gas escapes in 20 seconds through the tube, which has a bore of about 2·5 mm. Taking the temperature of the gas as $-80°$, and assuming that one half of it is solidified (doubtless much too high an estimate), the calculated velocity is not far from 2000 feet per second, while the maximum attainable rim-speed in the case of a revolving disk is less than 300 feet. To calculate the velocity it will be necessary to know not only the amount of gas solidified, but also the pressure-gradient in the tube. In the above calculation the pressure was taken as that of the atmosphere, which is obviously not the case, though the error introduced may be partially offset by the large estimate of the amount of solid gas present.

The experiments have been in progress for but a few days, and only this very brief preliminary communication is possible at the present time.

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