THE MEASUREMENT

OF

GENERAL EXCHANGE-VALUE

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BY

CORREA MOYLAN WALSH

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ERRATA.

Page 3, line 4 in notes, before *Del Commercio* insert *Elementi di economia pubblica*
Page 3, line 8 in notes, *for* later *read* latter
Page 9, line 2, *for* had *read* played
Page 23, line 19, *for* outer *read* other
Page 108, line 20, *before* the second *insert* at
Page 136, line 2 in note, *before* value *insert* the
Page 160, line 10, *for* variation *read* variations
Page 198, line 6 in notes, *for* some *read* same
Page 202, below, twice, *for* $\frac{P_2}{P_1}$ *read* $\frac{P_3}{P_2}$
Page 271, in the upper equation, for $1 + \beta_2'$ read $1 - \beta_2'$
Page 317, line 3 in note, *for* argument *read* agreement
Page 318, line 10 in note, *for* Sect. II. *read* Sect. III.
Page 370, in the heading, for SUMS read MASSES
Page 409, in the formula, for $x_2 + x_2$ read $x_1 + x_2$
Page 462, lines 23 and 24, for p read p'
Page 522, line 2, in the denominator, for n_2 read n_1
Page 530, line 26, for *variation* read *variations*
Page 535, line 4, *for* substracted *read* subtracted
Page 552, for $\sqrt{x_1x_2}$ and $\sqrt{y_1y_2}$ read $\sqrt{a_1a_2}$ and $\sqrt{\beta_1\beta_2}$
Page 557, line 21, *for* '53 *read* '52

L'algebra non essendo che un metodo preciso e speditissimo di ragionare sulle quantità, non è alla sola geometria od alle altre scienze matematiche che si possa applicare, ma si può ad essa sottoporre tutto ciò che in qualche modo può crescere o diminuire, tutto ciò che ha relazioni paragonabili tra di loro. Quindi anche le scienze politiche possono fino ad un certo segno ammetterla.

BECCARIA, 1765.

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CHAPTER I.

THE NATURE OF EXCHANGE-VALUE.

I.

§ 1. "Value" is an ambiguous term, in the common use of which may be detected several meanings that deserve to be distinguished by special epithets. Applying the term to species of material things, we often have in mind their usefulness or utility, as when we speak of water being very valuable for mankind; and therefore what we refer to should be called *use-value.* Or we may be thinking of the particular things in a species, and whether the species be important or not as such, yet because we highly prize or esteem the particular things for their rarity, we attach also value to them, and this is something which should be characterized as *esteem-value.* Again, we sometimes refer merely to the fact that the things are produced or procured only by labor, and we value them because they are endeared to us by past effort and cannot be replaced except at the cost of more effort; and then we should call the quality we are thinking of, *cost-value.* Lastly, we may be considering only the fact that things once possessed may be exchanged for other things, whereby a thing useless in itself to its owner, which he does not esteem for itself, and which perhaps has come to him gratuitously, may procure for him a useful and needed object, and save him the trouble of special effort to produce that object,—which attribute in things is properly called their *exchange-value*.

Having the first meaning in mind, we say a thing is more or is less valuable according as its class is more or is less useful as a whole—its "total utility" is greater or smaller—the whole class is less or more dispensable—our first want for it is greater or smaller. Having in mind the second, we say that an individual thing, along with its mates, is more or is less valuable according as it is more or is less prized by people in general, which is generally according as their uses for its class are greater or smaller and as the abundance of its class proportionally to their numbers is smaller or greater, or according to the magnitude of what has been called its "final utility"—its usefulness when satisfying the last want which its class is abundant enough to satisfy. Thus when we say "water is more valuable than diamonds," we are comparing the species water with the species diamond; when we say "diamonds are more valuable than water," we are comparing individual diamonds with equally large individual drops of water. Having in mind the third meaning, we say a thing is more or is less valuable according as there is greater or smaller difficulty for people to produce it by finding it or by making it. Having in mind the last, we say a thing is more or is less valuable according as it procures in exchange, or purchases, more or less of other things.

§ 2. A term rarely has several meanings without some one idea running through them all. In the case of "value" the underlying idea is that everything which we pronounce "valuable" is an object of desire. Everything desired, however, is useful to us. But some species of physically useful things are so abundant that we are not aware of our desire for them until we stop to imagine ourselves deprived of them. Then we recognise their value—their use-value. For other things we know our desire because in their absence we feel the want of them. Thus the more useful and the more absent are they, the more we desire them, provided they are not so entirely absent as to leave us unacquainted with their utility; and when we possess them, the more we prize, esteem, and value them, for fear of losing them. That we desire things which cost us effort to produce, is shown by the fact that we spend effort to procure them. And anything is desirable to anybody, however useless in itself to him, that serves the useful purpose of providing him with other useful things, and so saves him the labor otherwise necessary to obtain them.

By reason of this common essence, "value" may be defined as the *valor*—might, power—in things by which, in general, or under certain circumstances, they are rendered objects of desire. Things are objects of desire in four different ways. They are objects of desire because of their utility alone; wherefore, using them, we assign to them use-value. They are objects of desire because of their utility and rarity; wherefore, esteeming them and holding fast to them, we assign to them esteem-value. They are objects of desire because of their utility and the difficulty of making or replacing them; wherefore, laboring to produce them, we assign to them cost-value. They are objects of desire because of their special utility in providing us with other desired objects; wherefore, accepting or laboring to get even what we otherwise do not want in order by exchange to get what we do want, we assign to them exchange-value. Apart from the common reference to desirableness, the four meanings of "value" are very distinct, so that they represent *four kinds or species* of value.[1] The distinctness of these is also shown by the fact that the same thing, compared with others, may possess different degrees of use-value, of esteem-value, of cost-value, and of exchange-value.

§ 3. The second and the last kind of value were distinguished by Turgot, who called the former "valeur estimative" and the latter "valeur échangeable."[2] The first and the last

[1] Beccaria said that originally "value" meant "having force, habitude, ability to fulfil a purpose," and this "absolute value" later became "relative and venal" and meant "the power which everything has of being exchanged with all others." *Del commercio*, written about 1769, ed. Custodi, Vol. I., p. 339. These two senses are not specifically distinct, but the latter is one species under the former as genus. The other species were not noticed.

[2] *Valeurs et monnaies*, about 1750, ed. Daire, pp. 82, 83. He also called the later "valeur appréciative," because we generally estimate the exchange-value of things by appraising, or setting prices on, them, p. 87.

were distinguished by Adam Smith, who called the first "value in use" and the last "value in exchange" or, obviously borrowing from Turgot, "exchangeable value."[3] But the latter term Adam Smith did not confine to the meaning of exchange-value. He divided "value in exchange" into two sub-species, which he called "nominal value" and "real value" or even "real exchangeable value." By the former of these he meant merely one instance of exchange-value, namely a thing's exchange-value in one special other thing, money,—the reverse of its "nominal price," the quantity of money we must pay to purchase it; and by the latter he meant the quantity of labor the thing will purchase,—the reverse of its "real price," the quantity of labor it costs to acquire the thing.[4] Thus what is here called esteem-value was vaguely, and by a means of measuring it, rather than by its nature, referred to as "real value"; and what is here called cost-value was given an entirely different appellation, "real price."[5] While Adam Smith's first and incomplete dichotomy of "value" has been widely accepted, his distinction between "real value" and "real price" has hardly even been noticed.[6] By Ricardo the term "real value" was applied to cost-value, when he spoke of it as "the quantity of labor and capital"—itself, in his view, a product of labor—"employed in producing" the thing,[7]—in which sense he also indifferently employed the term "natural price."[8] This kind of value he never differentiated from esteem-value, although all

[3] [illegible] MacCulloch's ed. [illegible]

[4] [illegible]

[5] [illegible] The idea of it was [illegible] would have value to an [illegible] and had spoken of [illegible] labor as the price he [illegible] mere figure of speech was [illegible]

[6] [illegible] or would [illegible] value" in the [illegible] what we give or [illegible] between these terms, [illegible] labor as an ex- [illegible] Adam Smith's doctrines.

[7] [illegible]

[8] [illegible]

his investigations into the causes of variations in what he called "relative value" hit upon the causes of variations in the relative esteem-values of things. Thus what is here called exchange-value he often referred to by the term "relative value"; and he also occasionally spoke of it as "nominal value," namely as value "in coats, hats, money, or corn," *i. e.*, in various objects named.[9] Like Adam Smith, he does not expressly tell us that he is subsuming these values under "exchangeable value"; yet he too must do so, as he made his first division of "value" likewise only into "value in use" and "value in exchange,"[10] and it is apparent they cannot be subsumed under "value in use," which he identified with "riches."[11] Moreover there are passages in which he uses "exchangeable value" as he elsewhere uses "real value,"[12] while it is evident that "exchangeable value" must mean "relative value" when he defines it as "the power of purchasing [other things] possessed by any one commodity."[13]

This error of confounding at times two, or even three, distinct kinds of value together under a term applicable only to one of them, along with that of assigning principal importance, implied in the term "real," to the very one which does not deserve to come under that term, has run through almost all the so-called "classic" political economy, notwithstanding that this school has constantly claimed exchange-value to be the chief topic it dealt with, and has rarely treated of "value" except

[9] *Ibid.*, p. 32. "Value" was sometimes used by Ricardo also in the sense of what is in this work called general exchange-value, pp. 293, 401; but he also denied this use, p. 171, and ignored it, p. 13.—As for "nominal value," this term was best defined by a contemporary economist: "The nominal value of a commodity is strictly speaking its value in any one commodity named; but as the precious metals are on almost all occasions the commodity named, or intended to be named, the nominal value of a commodity, when no object is specifically referred to, is always understood to mean its value in exchange for the precious metals," Malthus, *Principles of political economy*, 2d ed., 1836, p. 54. This narrow sense is the one above seen to have been used by Adam Smith. For it "money-value" is a shorter term, and free from all ambiguity. Even in its wider sense the term "nominal value" is not satisfactory.

[10] *Works*, p. 9.

[11] *Ibid.*, pp. 169–173, against J. B. Say, who had sometimes identified "real value" with "value in use."

[12] *Ibid.*, pp. 172, 377.

[13] *Ibid.*, p. 49.

under the heading of Exchange or Distribution, and never under the heading of Production, which would seem to be the proper place for treating of cost-value, nor under the heading of Consumption, which would seem to be the proper place for treating of esteem-value. Hence it has been the cause of untold amount of confusion of thought, and of wasted effort to get straightened out. Recognition of the distinction between all four kinds, and of the at least coördinate importance of the last kind, has not yet come into general consciousness, although a beginning has been made. It was not till a little over twenty years ago that three of the four kinds were distinguished. Then Jevons separated from each other use-value, esteem-value, and exchange-value, under the titles of "value in use," "esteem, or urgency of desire," and "purchasing power, or ratio of exchange."[14] It is high time that all four should be distinguished, and their distinctness observed.

It would be out of place in this work to pursue the distinction further. Of all the four kinds, the last is the only one which has always been treated of by economists, and, as just observed in another form, many of them have asserted that their science is specially and even wholly concerned with exchange-value. In this position may be some exaggeration, especially when we sever from exchange-value the other kinds of value which they unconsciously associated with it.[15] It is proper, however, that special treatises should be confined to this one particular kind of value, and it is the whole and sole subject of the present work.

[14] *Theory of political economy*, 2d ed., 1879, pp. 85, 87, 3d ed., 1888, pp. 78, 81. These terms are not in the 1st ed. published in 1871. But in the 1st edition he had really made the same distinctions in thought. At the same time Walras was working out the laws of esteem-value in very much the same way, publishing them in the 1st edition of his *Éléments d'économie politique pure*, 1874. While Jevons generally used the term "value" confined to the meaning of exchange-value, Walras has employed it mostly in the sense of esteem-value, occasionally using the term "exchangeable value" when treating of exchange-value, and frequently employing long and tedious phrases, from lack of short and clear-cut terms whereby to distinguish the kinds of value he had in mind, and sometimes falling into confusion in consequence of dropping the long phrases.

[15] As political economy does deal with all the kinds of value, a better technical name for it than Whateley's "catallactics" would be "timiotology," although perhaps Hearn's "plutology" is still better.

II.

§ 1. Exchange-value is a relative quality in material things. A material thing has exchange-value, as it has weight, only because of other material things to which it relates in a particular way. This is its exchanging for them, or its ability to exchange for them. Gravity is the power in a thing by which it attracts other things toward itself and is attracted toward other things by a similar power in them. Exchange-value is the power[1] in a thing by which it procures for its owner other things, which procure it for their owners by a similar power in them. As we cannot conceive of the gravity of one thing alone, without reference to other things, we cannot conceive of the exchange-value of one thing alone, without reference to other things.[2] Furthermore, in the case of exchange-value, we know that for its existence is required something else, namely the men who make exchanges.[3] But it is likewise believed that for the existence of weight, or attraction, in material things, there is required some other thing as its cause, which has been variously placed. Now just as we can conceive of weight or attraction without

[1] Cf. Beccaria and Ricardo above. McCulloch speaks of "exchangeable value being the power which a commodity has of exchanging for other commodities," *Principles of political economy*, 1825, p. 213 (repeated in his ed. of the *Wealth of nations*, p. 439). Courcelle Seneuil: "The value of a commodity is the force or power of exchange of this commodity," *Traité d'économie politique*, 1858, Vol. I., p. 256. And Walras speaks of a relation establishing itself between appropriated things such that "each one acquires, as a special property, a faculty of exchanging for each of the others in definite proportions," *Éléments*, 1st ed., pp. 25–26; and on p. 48 he applies this "property" to "exchangeable value."

[2] Yet Bourguin says that exchange-value is not a property, a quality, an attribute, of things—there is no intrinsic value, that is, something which we can conceive of in an isolated body as a quality inherent in it independently of all other things, adding "length and weight can be so conceived in a body, apart from any relation, from any comparison with another thing: they are therefore intrinsic qualities," B. 132, pp. 22–23; cf. p. 268. This is curious. An isolated body could not have weight, and we cannot conceive of its having weight, *e. g.*, being unsupported, in which direction would it fall? As for length, we can conceive of an isolated body having length provided we conceive of it having many distinct parts, themselves not isolated. Length is properly the distance between such parts, and distance is not an intrinsic quality inherent in any isolated thing.

[3] The existence of the men who use, prize, and produce things, is also necessary for the attaching of the other kinds of "value" to things. But any material thing can have use-value, esteem-value, or cost-value, without reference to any other material thing.

bringing into the question consideration of that other thing, so we can conceive of exchange-value, or of exchanges, without bringing into the question consideration of the men who make them.[4] At all events in our present limited inquiry it is unnecessary to investigate the relationship between valuable things and their owners, or the motives by which these are actuated in exchanging them in certain quantities. The owners are agents, and their motives are determining reasons, for the making of exchanges, and consequently for attaching exchange-value to things. In seeking to compare and measure the exchange-values already formed and put into things, comformably to qualities already under given conditions found existing in the things, we no more need to know the causes of exchange-value than we do to know the cause of gravitation. Indeed psychology holds somewhat the same relation to objective or formal economics (the study of the *phenomena* of exchanges and the laws of their relations) as theology to physics. We may make use of psychology in some branches of economics, and get back to causes not manifested in the phenomena themselves. This means that we can go further in economics than we can in physics. But for merely measuring the relations of exchange-

[4] J. B. Clark: "The inaccuracy of the term purchasing power, often used as synonymous with value in exchange, consists mainly in its implying a power in the commodity itself to effect a purchase. Such power resides in men, not in things," *The philosophy of wealth*, 1886, p. 88. To be sure, the power of effecting exchanges (the *causa agendi*) resides only in men; but without possessing some material thing men have no power to purchase anything (although they may have power to produce or to earn something). More fully described, exchange-value is the power in things to be taken in exchange for other things (the *causa fiendi*). It is a power in things by means of which their possessors have power of effecting an exchange. A derrick has no power "to effect the lifting" of anything; yet it has a power by means of which men can effect the lifting of things. It would seem as if F. A. Walker tried to avoid Clark's objection when he defined "value" as "the power which an article confers upon its possessor . . . of commanding, in exchange for itself," other things, *Political economy*, 1887, pp. 5, 81. This would place value in the possessor, which is absurd. The same idea is expressed by Bourguin, who, however, immediately appended the old opinion that "value" resides in the commodity, saying that it is "the commodity's exchangeability for another commodity," B. 132, p. 20. It is interesting to note that, without thinking of this question, Adam Smith described "value in exchange" as "the power of purchasing other goods, which the possession of that object [one having utility] conveys," *op. cit.*, p. 13. This means that neither the object nor the possessor, but the possession has exchange-value. To conform to this position in our speech would only be to use much pleonasm.

value, as manifested in actual exchanges, we might be disembodied spirits investigating the laws of a world in which we had no part and in which we could not go behind the scenes.

§ 2. The assertion that exchange-value is a quality *of* material things, or *in* material things, is often denied on the ground that exchange-value is only an estimation which men set upon things and so is only in our minds—only subjective, not at all objective. To maintain this denial is merely to employ metaphysics in the wrong place. For on this line of reasoning there are no qualities in things, since there is no quality said to be in things known to us but it requires our presence for its existence as we know it. We are, however, permitted by metaphysics to speak of qualities *of* or *in* material things, on the ground that the material things themselves are in us in the same sense in which it is said the qualities we assign to them are in us. We may, then, be permitted to continue using the same popular phraseology in all cases, and to speak of the exchange-value (or other values) *of* things so long as we speak of the weight, size, hardness, color, etc., *of* things. It is perfectly correct to say that exchange-value is something in our minds. But it is also correct, as we all really believe, to say that exchange-value is something in things, and it is not correct to say that exchange-value is only in our minds. Exchange-value is not merely subjective; it is also objective. We believe there is something in material things (whether these be in us or not) that, along with our own constitution, is the, or a, cause of our desiring them and behaving toward them as we do; and it is this something in them to which we refer when we think of their value, or, in particular, of their exchange-value.[5]

[5] H. D. Macleod: "Value is not a quality of an object . . . it is an affection of the mind. The sole origin, form, or cause of value is human desire," *Theory of credit*, 1893, Vol. I., p. 200. This omits to say what is the cause of our desire for things. Surely our desire for things is not wholly independent of the things themselves. W. L. Trenholm: "It is obvious that it cannot be any special quality in the thing desired which gives value to it, but that the value comes wholly from unsatisfied desire," *The people's money*, 1893, p. 226. This ignores the connection between the "special quality" and the desire.—C. Menger would also deny the existence of exchange-value in things, affirming that it is not a "real phenomenon," not only because it cannot exist in an isolated body, but because it is "the thought-of cause of the existing various exchange relations and of their

This whole question, however, though important in metaphysics, is of no consequence in economics;[6] for we should continue to think of, and to investigate the relations of, exchange-value in precisely the same manner whether metaphysicians decided there is exchange-value in things or whether they decided it is only in our minds. The difficulties which confront us in the metrology of exchange-value do not arise from its subjectivity, if it be only subjective, and would exist the same if it be also objective. They arise solely from its extreme variability, whatever be the cause of this. But it has been necessary to point this out here, on account of the frequency with which the opinion is advanced that we can find no invariable standard of exchange-value because of its being only subjective *unlike* weight and other qualities which we have succeeded in measuring with tolerable exactness.[7]

§ 3. A thing which has exchange-value has exchange-value in relation to other particular things, or, by combining these under the same terms, to other particular kinds of things. Individuals of one kind exchanging in certain proportions for individuals of this or of that or of any other kind, it is said the former kind has a certain exchange-value in this kind, in

variations, which cause is in our thoughts ascribed to the commodity in question," art. *Geld* in the Handwörterbuch der Staatswissenschaften, Jena, Vol. III., 1892, p. 740. But at the same time he allows it to be an *Austauschmöglichkeit* belonging to things. Then why not also a *Tauschkraft?* And if we define exchange-value as such, it would belong to things.—The denial is made also on the ground that exchange-value is only a relation. So Denis, B. 100, p. 171. The error of this will be seen presently.

[6] Of course to complain, with H. Dabos, *La théorie de la valeur*, Journal des Économistes, March, 1888, p. 406, that exchange-value is not a physical property of things, like their color, density, porosity, etc., is to go out of one's way in search of trouble, since economists should leave physical properties to physicists and concern themselves with the economic properties of things.

[7] *E. g.*, J. P. Smith, B. 7, p. 39.—T. Martello says that speaking of a fixed unit of value is like speaking of a fixed unit of love, *La moneta e gli errori che corrono intorno ad essa*, 1883, p. 101.—It is only a step to say that, because value is only in us, there is no such thing as the value of a material object. This step has been taken even by Walras in the very work in which he started out by describing exchange-value as a property acquired by things under certain circumstances, whence he concluded that the term "franc" cannot refer to the value of a piece of metal, there being none, but only to the piece of metal itself, *Éléments*, p. [illegible], 2d ed., p. [illegible]. But this has not prevented him from later writing a work on a method of regulating variations in the "value of money," B. [illegible]. A similar position is adopted by Bourguin, B. 132, p. 38.

that kind, and so on,—a contracted expression for "in relation to the other kind." The exchange-value of one thing in another, or of one kind of thing in another kind of thing, may be called a *particular exchange-value* of the thing, or of the kind of thing; so that we may speak either of the particular exchange-value of one thing in another thing or of the particular exchange-value of one kind of thing in another kind of thing.[8]

Such a particular exchange-value of one thing in another, or of one kind in another, is not the other thing itself, or the definite quantity of the other kind of thing, procurable in exchange, as has often been carelessly said.[9] Nor is it simply a relation, or ratio, between the two things.[10] Indeed it is difficult to see what meaning there is in such a definition of exchange-value. If a bushel of wheat exchanges for two bushels of barley, is the exchange-value of wheat the ratio *two?* We commonly say in this case that the exchange-value of wheat is *twice* that of barley; but this is only a statement of comparison between the exchange-values of equal quantities of wheat and of barley. It does not pretend to tell us what the nature is of the exchange-value of either of these things.[11] It is plain that,

[8] This is another term for "nominal value" as above defined by Malthus.

[9] *E. g.*, Adam Smith, as already observed in Note 6 in Sect. I.; also J. B. Say, *Traité d'économie politique*, 5th ed., 1826, Vol. II., p. 156; J. S. Mill, *Principles of political economy*, 1848, ed. 1878, Vol. I., p. 588. Macleod, *op. cit.*, Vol. I., pp. 112–113, 170, 172.—The error has been pointed out by Jevons, *op. cit.*, 1st ed., pp. 81–82, 3d ed., p. 78.

[10] *E. g.*, "Value in exchange is the relation of one object to some other or others in exchange," Malthus, *op. cit.*, p. 50, and similarly again, p. 61.—Beccaria had said: "Value indicates the proportion of one quantity with another," *op. cit.*, Vol. II., p. 8. There is a great difference between these expressions.

[11] Thus Jevons himself was in error when he added: "Value in exchange is nothing but a ratio, and the term should not be used in any other sense," *loc. cit.* All that he had a right to say was that the ideas of the other kinds of value should be excluded from the idea of this kind of value. He had himself previously declared of this kind of value: "Value is a vague expression for potency in purchasing other commodities," B. 22, p. 20; and he continued to use the term in this sense, cf. *Investigations*, p. 358. What absurdity this position may lead to is well illustrated in the following: "Value . . . is a relation between certain things to which men attribute value," Trenholm, *op. cit.*, p. 246.—Walras has gone to a peculiar extreme also here. He says: "Strictly speaking, there are no values, there are only relations of values," *Éléments*, 1st ed., p. 189 (again in B. 71, p. 4). (Perhaps the meaning intended to be conveyed in this sentence is that exchange-value is a relation between esteem-values.)—Carelessness in trebly describing value as (1) a *ratio*, (2) the *quantity* of another or other things, (3) purchasing

as already shown of exchange-value in general, the exchange-value of one thing in another is the power in the one thing of exchanging for the other—a power which necessarily presupposes certain relations of quantity between the things exchanged, but is itself very different from those relations. Thus the exchange-value of one thing in another is neither the other thing nor the relation between the two; but it is the power in the one which can exist only in connection with a similar power in the other, and can be estimated, as we shall see, by the relation between the quantities exchanged.

§ 4. A thing has many such particular exchange-values—as many, in fact, as there are kinds of things with which it can exchange. Now several, many, or all of these particular exchange-values, as they exist together, may be combined into a single concept, and so provide us with the idea of the thing's exchange-value simply so called.[12] This idea of simple exchange-value, like that of the gravity of the heavenly bodies, is difficult to grasp at first. Yet it is a necessary idea, which we all do inevitably form with various degrees of definiteness and accuracy.[13] It is at any given time and place a single exchange-

power, is also shown by J. L. Laughlin, *Facts about money*, Chicago, 1895, pp. 75–76, 147, 192, and Parsons, B. 136, pp. 81–82.

[12] J. L. Shadwell: "The human mind can only compare two things at once, and when it is said that a commodity has a certain power of purchasing all other commodities, the words, though they may be pronounced, written, and printed, do not really present any idea to the mind. The power of gold to purchase silver is a definite idea, and so is its power to purchase copper; but the power of gold to purchase silver and copper means nothing at all," *System of political economy*, London, 1877, p. 93. The only reason offered why we cannot strike an average is that "we have no standard by which to measure the objective importance of different articles," p. 95. This is a difficulty, to be sure, but one by no means insuperable,—and one which has long been discussed, it forming one of the definite problems in our subject (here to be treated in Chapter IV.). Few persons would admit in their own case the impotency here claimed for every human mind.

[13] Bourguin asserts that a thing has no "value in general," but only "particular values" (by "value" always meaning "exchange-value," B. 132, p. 3). His only reason seems to be, because exchange-value is not a property inherent in things, p. 22. Cf. p. 135, where he says: "The [purchasing] power of money is only a word; it designates, not a quality, an intrinsic value, but an *ensemble* of relations which have nothing in common, not being equations between magnitudes of the same kind." But if the particular exchange-values are not intrinsic, neither would the general exchange-value be intrinsic, and no reason is shown why there is no non-intrinsic general exchange-value except the statement that the relations (*i. e.*, the particular exchange-values) have nothing in common. But this is not so,

value somehow made up of many particular exchange-values. A thing has many exchange-values in other things separately; it has one exchange-value in other things collectively. Or rather the more correct statement is that when we do reach this idea of a thing's simple exchange-value we must view it as the thing's only real exchange-value,[14] and regard the particular exchange-values as merely this one and the same exchange-value in its various relations to the similarly single and simple exchange-values of other particular things. One of the tasks of political economy is to explicate and render more intelligible this conception of a thing's exchange-value in other things, or in all other things. To contribute to the accomplishment of this task is one of the objects of these pages. It may be premised that we shall find an exchange-value of a thing *in all things*, including itself,—an idea never yet distinguished from that of the exchange-value of a thing in all *other* things. These two ideas have some points of contact, and both may lay claim to the title of a thing's *exchange-value* simply, or its *general exchange-value.*[15] It is plain that we have no right to speak simply of a thing's exchange-value, if we have in mind only its exchange-value in some other thing, or in a few other things. Speaking of its exchange-value simply, we should be using language wrongly unless we referred to its exchange-value in all, or in all other things—or at least in all others to which we practically can refer,—that is, to its one simple exchange-value as measured by comparison with the simple exchange-values of all the other things.[16]

as the particular exchange-values of a thing have in common powers of exchanging for certain quantities of other things, and these powers are magnitudes of the same kind. As well say the attraction of the earth for the moon is different in kind from the attraction of the earth for the sun. Bourguin elsewhere says he will use the term "the value of a thing," in the singular, meaning the *ensemble* of all its particular exchange-values, only for convenience, p. 23 (and he will attempt to measure variations in this *ensemble*, pp. 138–139). But we may be sure that when a term is found to be convenient, it expresses an idea or concept.

[14] But of course not its only "real value."

[15] This term was much used by J. S. Mill. In the same sense "general value" was used by Hallam, *View of the state of Europe in the Middle Ages*, 1816, Chapt. IX., Part II.

[16] Naturally in speaking of exchange-value simply we do not mean something without relation to any other things. Neglect of this has led Macleod into curious

III.

§ 1. Exchange-value is quantitative. A certain quantity of one thing exchanges for a certain quantity of another thing. If it exchanges for more of the other, its exchange-value in the other is proportionally greater. If it exchanges for less of the other, its exchange-value in the other is proportionally smaller. If it exchanges for the same quantity of the other, its exchange-value in the other is the same. Thus the exchange-value of one thing in another may have different degrees of intensity. This being so of its exchange-value in any one other, it is so of its exchange-value in every other, and consequently of its exchange-value in all other things, or of its general exchange-value. And so with the exchange-values of everything.

Such variations in exchanges, and consequently in exchange-values, be it said in passing, may occur at different times and places. Consequently to speak of a thing's exchange-value in another thing, or in general, is to refer to its exchange-value at a given time and place. Exchange-value is something temporal and local. In omitting explicit declarations, we generally imply that we are dealing with exchange-value at the same place and are referring to changes happening in time.

§ 2. From the above comparisons it results that, *the quantity of one thing being assumed, its exchange-value in another is proportional to the quantity of the other it exchanges for* (Proposition I.). Consequently we can measure the exchange-value of one thing in another at two dates (or places) by the relative quantities of the other it exchanges for at the two dates (or places).

error. He says: What is wanted by economists who seek an invariable standard of value is "something by which they can at once decide whether gold is of more value in A. D. 30, in A. D. 1588, or in A. D. 1893; in Italy, in England, or in China; without reference to anything else," *Theory of credit*, Vol. I., p. 212. And so he had long before said: "As no single body can be a standard of distance or equality [without reference to others], so no single object can possibly be a standard of value" without reference to others, *Theory and practice of banking*, 1875, Vol. I., p. 16, and similarly again, p. 77. Of course the exchange-value of a single body *with* reference to others may be a standard of exchange-value, just as the length of a single body—a certain distance between its extremities—compared with others, may be a standard of length. (But some of the economists referred to treated of cost-value, and so took account only of cost of production.)

These quantities of the other, we must remember, do not constitute the exchange-value of the one in that other. The exchange-value of the one in the other is its power of acquiring that other for its owner. The magnitude of this power over the other is manifested by the effect it accomplishes in exchanging for that other, that is, by the quantity of that other thing it acquires.[1] Here we are treating merely of the power of the one thing over the other, or its exchange-value in the other, and therefore need pay no attention to the power of the other, or to its exchange-value. But if we were comparing the exchange-value of the one with the exchange-value of the other, or trying to measure how much exchange-value the former is manifesting when it is exchanging for the latter—by "exchange-value" here meaning exchange-value simply, or general exchange-value,—we should have to take into consideration also the general exchange-value of the latter.

It is plain that *at any one place individual things exactly alike physically will always exchange for one another indifferently, or have the same exchange-value in their own class; and also will exchange for the same quantities of other things, or have the same exchange-value in other things: that is, all the individuals in a homogeneous class have the same exchange-value* (Proposition II.). Like things are not generally exchanged, because there is generally no object in making such an exchange. But such an exchange is possible, and sometimes occurs. Dealers who store wheat together, probably get back different wheat, and so have exchanged wheat for wheat. Also in the case of money, we frequently exchange, say, ten dollars in one piece for ten dollars in two or more pieces. It is evident that when such exchanges are made people do not exchange more for less of the same thing. Now a bushel of wheat and any other bushel of wheat having the power of exchanging for a bushel of wheat, every bushel of wheat has the same exchange-value in wheat (all of the same quality). And when one kind of things is exchanged for another kind, it is indifferent which of many like individuals is given or received. Therefore, each of these in the one class

[1] Cf. Courcelle Seneuil, *op. cit.*, Vol. I., p. 243.

having the power of exchanging for the same thing in the other, they all have the same exchange-value in that other kind of thing. And for the same reason they have the same exchange-value in any and every other kind of things, consequently in all other kinds of things together, that is, in exchange-value simply, or in general exchange-value. It may be that the different individuals which we include under a class with the same name are generally not exactly alike, and perhaps they never are exactly alike—it has been maintained that no two grains of wheat are ever exactly alike;—but they are often nearly enough alike to pass in practice for alike. Of course what is here said refers to materials in the same form only; for the form itself has utility. A cubic foot of wood in a lumber yard has not the same exchange-value as a similar cubic foot of wood in a building.

It is plain also that, *the exchange-value of a quantity of one thing in another being given, the power of acquiring quantities of the latter by means of the former is proportional to the quantity of the former employed in exchanging for the latter* (Proposition III.). For example, if one bushel of wheat has the power of exchanging for two bushels of barley, two bushels of wheat have double this power, that is, they have the power of exchanging for four bushels of barley; for, according to the preceding proposition, the second bushel of wheat has the same power as the first. This proposition does not mean that if ten bushels of wheat have the power of exchanging for a diamond of a certain size, twenty bushels of wheat have the power of exchanging for a diamond of twice that size. It means only that they have the power of exchanging for two such diamonds. It also does not mean that if one bushel of wheat actually exchanges for two bushels of barley, any quantity of wheat might have been exchanged for twice the same quantity of barley; for this would require that the exchange-values should remain the same whatever be the quantities offered in the market. It means only that such proportional exchanges can be made while the exchange-values do remain the same.

§ 3. Again, *of two kinds of things the quantities which exchange for each other are equivalent in the sense that the exchange-*

value of the one in the other is equal to the exchange-value of the latter in the former (Proposition IV.). This looks as if it were an expletive proposition, like those which have preceded, springing from the meaning of the terms employed, or like saying that the distance of the sun from the earth is equal to the distance of the earth from the sun. Yet it may be objected that, the power of wheat to acquire barley being measured by the quantity of the barley, and the power of barley to acquire wheat being measured by the quantity of the wheat, as the quantity of barley and the quantity of wheat are two distinct and generally unequal things, the equality of the two exchange-values is not shown by a comparison of these quantities. If a proof be demanded, however, a proof is forthcoming. We have seen that a bushel of wheat has the power of acquiring a bushel of wheat. But, according to our supposition, two bushels of barley have the power of acquiring a bushel of wheat. Therefore two bushels of barley have the same exchange-value in wheat as one bushel of wheat. And similarly a bushel of wheat has the same exchange-value in barley as two bushels of barley. Therefore, having the same exchange-value both in barley and in wheat, a bushel of wheat has the same exchange-value in barley as the two bushels of barley have in wheat.[2]

To say that the quantities of things which exchange for each other are equivalent in the sense of having the same exchange-value simply, that is, the same general exchange-value, needs still further proof.

Now, *of two kinds of things the quantities which exchange for each other exchange for the same quantity of any other kind of things, and therefore have the same exchange-value in that other kind of things* (Proposition V.). The first part of this proposition is the law of stable equilibrium in an open and free market, which equilibrium must exist on the average in the long run. It is possible for fluctuations to occur at times, but then forces are set at work to restore the equilibrium. For instance, while

[2] The case is comparable with weights. We should never know by the balancing of two bodies in opposite scales that they are of equal weight except by alternating them, or by employing "double weighing."

one bushel of wheat exchanges for two bushels of barley and for three bushels of oats, if it should happen that some one is willing to give up four bushels of oats for two bushels of barley, immediately those who have wheat and want oats would exchange their wheat for barley and it for this man's oats, and those who have barley and want wheat would exchange their barley for this man's oats and them for wheat. This man's supply of oats would then be soon exhausted, and he would retire from the market. The condition which renders useless such roundabout exchanges (which in money exchanges between different places are called *arbitrages*)—a condition which those roundabout exchanges themselves tend to produce,—exists in our example when three bushels of oats exchange for two bushels of barley. Then, representing one bushel of each by A, B and C, and equivalence by the sign $\asymp$, we have between the kinds

$$\begin{array}{ccc} & & 2\,B \\ & \asymp & \\ 1\,A & & \asymp \\ & \asymp & \\ & & 3\,C \end{array}$$

of things the interrelation here depicted. That any two of these quantities have the same exchange-value in the third is apparent.

This being so of *any* two things in *any* third, it must be true that the quantities of everything which exchange for each other have the same exchange-value in all things beside themselves. But as their own exchange-values are equal each in the other, these may be added, and we have: *Of everything the quantities which exchange for each other have the same exchange-value in all other things and in all things, that is, the same general exchange-value, or the same exchange-value simply* (Proposition VI.); which is what we wished to prove.

It follows also that *all the many particular exchange-values of one thing in other things singly are singly equal to the thing's general exchange-value, and to one another* (Proposition VII.). The exchange-value of wheat in barley is, for instance, its power of acquiring two bushels of barley, and its exchange-value in oats, its power of acquiring three bushels of oats; but the two bushels of barley and the three bushels of oats have the same general exchange-value, therefore the one bushel of wheat is acquiring the same exchange-value, to which its own is equal, whether it be exchanged for two bushels of barley or for three bushels of oats. It is evident, moreover, that when a bushel of wheat is used to acquire barley, it is manifesting its whole exchange-value, but only in relation to barley; and when it is exchanged for oats, it is manifesting its whole exchange-value, but only in relation to oats. Therefore it is manifesting the same exchange-value in both cases, but in different relations. The two particular exchange-values, as we have already noticed, are the same as the general exchange-value, are equal to it, and consequently are equal to each other. And so of all the particular exchange-values of any one thing.

§ 4. To measure even the particular exchange-value of one thing in another is not an easy task; for at the same place during the same period of time the same quantity of the same kind of thing may be exchanged for various quantities of another kind of thing. It is sometimes said that the exchange-value of one thing in another is determined by an actual exchange. This is not so, as it often happens that in an actual exchange the one party rejoices over a good bargain and the other is worried lest he have made a bad bargain—*i. e.*, the one thinks he has got more, the other less, than the thing given was worth. We also ascribe exchange-value to things never exchanged, and especially we want to know the proper exchange-value of a thing which we are going to part with, before we effect its exchange for anything else. We estimate exchange-values rather by the general run of exchanges of similar other things. The particular exchange-value of one kind of thing in another kind of thing is not an affair of a single exchange, but of many. The single exchanges may fluctuate around an average, which is what we call the exchange-value of the thing (the class) during the period in question. When certain kinds of things are habitually exchanged in large quantities, the fluctuations in short intervals of time are not apt to large, so that in these cases it is tolerably easy to determine ir particular exchange-values within narrow limits of error.

§ 5. If the particular exchange-value of one thing in another sometimes difficult to estimate, it is always difficult to estite its general exchange-value; for this includes not only the ermination of its many particular exchange-values, some of ich are sure to be troublesome, but also the combination of se into a whole. This problem, however, may perhaps ad- : of a satisfactory theoretical solution, whereupon the diffities will reside only in the practical details.

The labor of finding the particular exchange-values of things uld be almost infinite if we were to undertake to find all the ticular exchange-values of all things. For as one thing has articular exchange-value in another thing, this has a parılar exchange-value in it, so that there are two relations of ticular exchange-value between every two things; and, refore, among a hundred kinds of things, each one of the ndred having a particular exchange-value in every one of the er ninety and nine, there would be 100 × 99 = 9,900 parılar exchange-value relations, and among two hundred there uld be 200 × 199 = 39,800, and so on in rapidly increasing gression. Through this maze of interminable interrelations have been enabled to make our way, as is well known, by the ention of money, which, as is said, serves as the "common nominator" for all other things. By means of prices, money s as a perfectly satisfactory measure of the exchange-values other things in the same place at the same time. Now money ne is brought into direct relationship with every other exngeable thing, the relations between these others being interdiated through their relations to money. Hereby our atten- is confined to the particular exchange-values of all other ngs in money, as indicated by their prices, or, which is the ne thing inverted, to the particular exchange-values of money all other things singly. Consequently it is primarily only general exchange-value of money in all other things collvely that we are concerned with measuring; for after suring it and finding its constancy or variation at different times, or in different places, we can measure the constancy or variation of any other thing in general exchange-value by its known constancy or variation in relation to money. It is not an uncommon opinion that it is easier to estimate the variation in exchange-value, simply spoken, of any commodity than it is to estimate the variation in the exchange-value of money, on the ground that in the former case we only have to notice the variation of its price, while in the latter we have to consider variations of all prices. But the variation of a commodity's price only tells the variation of the commodity's particular exchange-value in money, and gives us no information concerning the variation or constancy of the commodity's exchange-value simply so called, or general exchange-value, until we know the variation or constancy of the general exchange-value of money. Thus the former calculation really presupposes the latter. And as a matter of fact, not only is it easier to measure the constancy or variation of the general exchange-value of money than of anything else, but money is the only thing of which the general exchange-value can be measured by us directly; for we should never be able to find all the particular exchange-values of any other thing without taking account of its and of the other things' prices, or exchange-values in money.

§ 6. Because exchange-value is quantitative, to conceive of exchange-value involves measurement. The measurement may be rough or exact, but measurement there must be. And we make measurement not only of particular exchange-values, but also of general exchange-value—especially of money's general exchange-value. Everybody has some notion of "money's worth," and some opinion as to whether through a course of years this worth or exchange-value has remained constant or varied, or whether it is greater in one place than in another. Only this notion is generally very badly formed and left vague and unprecise, wherefore the opinion is generally weak and irresolute, or inclines in favor of constancy merely through lack of proof of variation. Political economy, if it be a science, cannot avoid the duty of attempting to rid this notion of its vagueness[3] and to provide a method of measuring the general exchange-value of money with theoretical precision as a model to be realized as closely as possible in practice.[4]

IV.

§ 1. In measuring quantities we must bear in mind that we are not concerned with the causes of their constancy or of their variations. In measuring from year to year the weight and tallness of a boy, we have nothing to do with the causes that make him grow. To measure variations, and to explain them by pointing out their causes, are two distinct operations.[1] And the former is the primary; for we can be scientifically prepared for investigating the cause of variations only after measuring the variations with scientific precision. When direct measurement of things we desire to measure is not feasible, we frequently measure them, as less apparent causes, by their more apparent effects, if we can eliminate all other causes, as in the familiar example of heat, by expansion. But less apparent effects we rarely attempt to measure by their apparent causes, since these effects generally escape our control and we cannot know whether they are operated upon only by the causes employed to measure them by. And equally apparent effects it would be useless to measure by their equally apparent causes, as we can measure them directly. To attempt, then, to measure apparent effects by their less apparent causes, would be the height of folly. Now the variations of exchange-values are more apparent than

[3] J. S. Mill spoke of "the necessary indefiniteness of the idea of general exchange-value," *op. cit.*, Vol. II., p. 102. Of course until it is made definite, this idea will remain indefinite. But what shows the necessity of its remaining indefinite? And why should the task of clarifying it be shirked by any economist?

[4] J. B. Say said we cannot "measure" exchange-value at different times and places, but we can only "appraise" it, or form "approximative valuations," because of the absence of an invariable measure—*i. e.*, because we cannot make an absolutely exact measurement, *op. cit.*, Vol. II., pp. 85, 93. This distinction is too hard and fast, since an inexact measurement is a measurement, and if we took his statement literally, we should have no measures, as none is absolutely invariable. We ought at least to aim at more than appraising; we ought to aim at measuring.

[1] Cf. Jevons, B. 22, pp. 21, 59; Nicholson, B. 94, pp. 304-305.

their causes; for, difficult though it be to measure accurately a variation in the general exchange-value of money, it is less difficult than to assign all the exact causes which have produced it (although it may be easier to adduce many possible causes in a general way than actually to measure the precise variation). Therefore it is especially absurd in political economy to say that in attempting to measure exchange-value we must pay attention to its causes. On the contrary, we should be especially careful to drop all consideration of its causes.

§ 2. It is necessary to state this obvious truth because of the prevalence of the opposite opinion in this one subject alone among all subjects of metrology. Thus, for example, a writer has recently asserted: "To measure the variations of money there is required not only detailed knowledge of prices, but also of the causes which produce the variations."[2] And another has said that this measurement is impossible because of the impossibility of knowing all these causes.[3] It would be difficult to match such assertions with similar assertions in any outer branch of science.[4] Yet opinions of this sort in political economy have no less an authority than that of Ricardo.[5]

The reason for this error is twofold. It lies in the confusion between cost-value and exchange-value, and in the doctrine that "value," including exchange-value, is determined by the labor-cost of production. But it is only cost-value which is proportionate to labor-cost; and even if exchange-values were determined by the relative labor-costs, it would not follow that the exchange-value of a thing can be measured by its labor-cost, but only by its labor-cost compared with the labor-costs of other things,[6] which comparison would be more difficult

[2] Nitti, *La misura delle variazioni di valore della moneta*, 1895. (Quoted from the *Economic Journal*, Vol. V., p. 260.) Cf. V. Pareto, *Cours d'économie politique*, Lausanne 1896, Vol. I., p. 266.

[3] Martello, *op. cit.*, p. 333.

[4] On the contrary, *e. g.*, Whewell speaks of "an important maxim of inductive science, that we must first obtain the *measure* and ascertain the *laws* of phenomena, before we endeavor to discover their *causes*," *Philosophy of the inductive sciences*, 1847, Vol. II., p. 240.

[5] *Works*, pp. 400–401.

[6] As long ago clearly pointed out by R. Torrens, *Essay on the produciton of wealth*, London 1821, pp. 49, 56, and by S. Bailey, *Critical dissertation on the nature, measure, and causes of value*, London 1825, pp. 6–11, 17–18, and *Letter to a political economist*, 1826, pp. 53–54.

than the comparison between their actual ratios of exchange, and so would be worthless even if that doctrine were universally true, and is especially worthless since that doctrine is not universally true (and, in fact, never even pretended to be).

§ 3. Another objection is a variation upon the same theme. This is that the measurement of the general exchange-value of money, however successfully made, is useless, because it gives no information about the causes of the variations.[7] It may seem strange that any economists could raise such an objection; for in no subject does a measurement disclose causes, and in nothing else is mensuration reproached for its inability to do so. The explanation is that what these economists want is really a measurement of cost-value, and so they are dissatisfied with what turns out to be a measurement of exchange-value alone, which being called simply a measurement of "value," may have seemed to them to give promise of being a measurement also of cost-value.[8] What they want is a measurement of the variations in the cost-values of commodities and in the cost-value of money (gold)—or rather, a measurement of the variations in money as a standard of cost-value. Or even sometimes they want such a measurement of money as a standard of esteem-value; for some of them recognize that gold is a semi-monopolized product, with "value" enhanced by rarity

[7] So D. A. Wells, *Recent economic changes*, 1889, p. 121. Cf. Malthus, *op. cit.*, p. 120, and McCulloch, *Political economy*, p. 214, (Note to *Wealth of nations*, pp. 439–440).

[8] Malthus offers a peculiar example of confusion. He wrote: "The exchangeable value of a commodity can only be proportioned to its general power of purchasing [general exchange-value] so long as the commodities with which it is exchanged continue to be obtained with the same facility," *op. cit.*, pp. 58–59. Thus although he expressly uses the term "exchangeable value," he distinguishes it from purchasing power or exchange-value proper, and identifies it with cost-value. He does so still more plainly when he amplifies the term into "intrinsic value in exchange," p. 60. Yet the idea of exchange-value always attaches to these terms, being embodied in them. The fundamental fault, which runs through all his long disquisition on the measure of value, and prevents it from reaching a satisfactory conclusion, is the fact that he is seeking the impossible—a single measure both of exchange-value and of cost-value.—Ricardo had the same idea as Malthus when he wrote: "Why should . . . all commodities together be the standard, when such a standard is itself subject to fluctuations in value? *Works*, p.166, *i. e.*, in cost-value (and in esteem-value). But of course their fluctuation in other kinds of value is not a reason why they should not be the standard of exchange-value if they be found not to fluctuate in this.

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Währungsfrage, 1895, p. 18. Probably this was the
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to the measurement of exchange-value as being mis-
. Fiamingo, *The measure of the value of money accord-*
s, Journal of Economics, Chicago, Dec. 1898, pp. 74-
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same direction, but the one more. Here it has actually been said, with much reiteration, that without regard to other things, it is sufficient for us only to know, or it is necessary for us to know, on which side the cause of the change lay, and then we should know which of the two has changed—in "value," still appearing to mean exchange-value, now of some absolute sort (except that we know that these writers associate even the term "value in exchange" with value of other kinds).[1] The cause in this matter has generally been sought in the labor of producing the article (though it might equally well be sought in the quantity of the article forthcoming, relatively to the number of people desiring it); and so it is said that if we knew that this labor (or this quantity) has changed in the case of one of the articles only, we should know which of them alone has changed in "value"—supposedly meaning exchange-value. But if so, the statement is wrong. For, under the supposition, we know nothing about other things; and so if we knew only of such a change in the labor-cost (or in the quantity) of one of the articles, we should only know that its *cost-value* (or *esteem-value*) has changed, not that its *exchange-value* has changed—by "exchange-value" properly meaning only exchange-value simply so called, that is, exchange-value in all other things; for the rest are unknown, or left out of account. But as regards its exchange-value in the one other thing, we know that this has changed exactly as we know that the other's exchange-value in it has changed, learning both from actual exchanges; and we know these changes just as well whether we know the changes in their costs of production (or quantities), or not. In considering the mutual exchange-values of two things alone, we are abstracting all other things; and now the only change we have a right to consider is the change between them, which involves both a rise and a fall.

§ 2. In thus putting aside all other things and abandoning the use of them as a common measure, we can measure the exchange-value in other things of each of the two objects only by

[1] *E. g.*, J. E. Cairnes, *Some leading principles of political economy*, 1874, pp. 13–14. So even Walras, B. 71, pp. 5–6.—Shadwell disposes of this question by pronouncing it "puerile," *op. cit.*, p. 103.

the other. As we employ a different measure for each, the rise and the fall are not found to be in the same proportion. Evidently if A, after being equivalent to B, rises by one-half, or by 50 per cent., in exchange-value in B, so that A exchanges for $1\frac{1}{2}$B, then, $1\frac{1}{2}$B exchanging for 1A, 1B exchanges for $\frac{2}{3}$A, wherefore B has fallen by one-third, or by $33\frac{1}{3}$ per cent., in exchange-value in A.

The relationship just exemplified may be formulated and generalized as follows. Here and throughout these pages whenever percentage is represented in algebraic formulæ by the sign p or the like, this refers to the percentage on 1, or to percentage expressed in hundredths.[2] Now, then, if A, after being equivalent to B, rises p per cent. in exchange-value in B, so that it commands $(1+p)$B, and has risen to $(1+p)$ times its former exchange-value in B, then B, now exchanging for $\frac{1}{1+p}$A, has fallen by $1-\frac{1}{1+p}$ or $\frac{p}{1+p}$ per cent. to $\frac{1}{1+p}$ of its former exchange-value in A. Reversely if A, after being equivalent to B, falls p' per cent. to $(1-p')$B, B rises by $\frac{1}{1-p'}-1$ or $\frac{p'}{1-p'}$ per cent. to $\frac{1}{1-p'}$ times its former exchange-value in A. Thus the exchange-value of A in B and of B in A being in each instance representable at first by 1, we find that unity is the geometric mean between the later exchange-value of A in B and the later exchange-value of B in A; for

$$1+p : 1 :: 1 : \frac{1}{1+p},$$

and

$$1-p' : 1 :: 1 : \frac{1}{1-p'}.$$

This relationship may be stated thus: *When of two equivalents the one rises or falls in exchange-value in the other, the other falls or rises*

[2] If the popular form be desired, p referring to integral figures, all the formulæ which follow may be maintained by substituting 100 wherever 1 occurs or is understood. *E. g.*, here 100 A, after being equivalent to 100 B, rising p per cent., command $(100+p)$ B, etc. And always the expression "$(1+p)$ times" must be changed to "$\frac{100+p}{100}$ times." It will be seen that the method adopted is the simpler.

in exchange-value in it so that their subsequent mutual exchange-values are reciprocals of each other, the quantities in which they first exchange being taken as units. Or, again, more briefly, *the rise of the one measured in the other is the inverse of the fall of the latter measured in the former; and reversely* (Proposition IX.).

We may, of course, in our supposition replace B by M, representing a sum of money, say one unit. The same rule holds, but may be differently worded. Price is the expression of the exchange-value in money of the article priced.[3] If A rises in price from 1.00 to 1.50, its exchange-value in M rises in direct proportion, and the exchange-value of M in it falls in inverse proportion, M now purchasing only so much of A as .66⅔M previously purchased. The rise of A in price by 50 per cent., is a depreciation of money in A by 33⅓ per cent. Reversely if the price of A falls from 1.00 to .50, or by 50 per cent., the exchange-value of M in A has risen by 100 per cent., M now purchasing 2A. *The price being originally one unit, the subsequent price and the subsequent exchange-value of money in the article priced (expressed in its original exchange-value in it) are reciprocals of each other* (Proposition X.).

§ 3. In these reciprocal changes it is possible to affirm that, in spite of the difference in the proportions, which arises from the differences in the measures used, *the rise of the one thing in the other and the corresponding fall of the latter in the former are equal* (Proposition XI.). A rise of A to double its former exchange-value in B, for instance, and the fall of B to half its former exchange-value in A, are equal changes. For the exchange-value of A in B rises from the power possessed by 1A of purchasing 1B to that of purchasing 2B; and if it should fall again from this power to that of purchasing 1B, this fall would

[3] Economists have not been careful in their use of the term "price," defining it indifferently in two distinct ways: (1) as the sum of money given (or asked) in exchange for the thing, (2) as the value of the thing in money—and sometimes the same writer has given both, *e. g.*, J. S. Mill, *op. cit.*, Vol. I., p. 538; J. Garnier, *Traité d'économie politique*, 1848, 9th ed., 1889, p. 293; Macleod, *Elements of political economy*, 1858, p. 39 and *Theory of credit*, Vol. I., p. 176. The first is more conformable to popular usage, and is the best. Price is not the exchange-value of the thing in money, but, being the sum of money itself, it expresses or measures the exchange-value of the thing in money.

evidently be equal to that rise. In the original variation the exchange-value of B in A falls from the power possessed by 2B of purchasing 2A to that of purchasing 1A. This fall is evidently equal to the last mentioned fall of A, and consequently to that rise of A. In effect, a rise from 1 to 2 and a fall from 2 to 1 are equal to each other.

§ 4. *Whenever such mutual changes take place between two articles relatively to each other, their exchange-values in all things beside themselves vary in the same proportions relatively to each other* (Proposition XII.). When A and B are equivalent to each other, we have seen that they possess the same exchange-value in all other things, whatever this exchange-value may be. Then if A rises 50 per cent. in its particular exchange-value in B, it is evident that, in relation to all the other things which are outside them both, A's exchange-value in all those other things has risen 50 per cent. above B's exchange-value in all those other things, whatever this exchange-value of B may be; and inversely B's exchange-value in all those other things has fallen 33⅓ per cent. below A's exchange-value in all those other things, whatever this exchange-value of A may be.

What has just been said is *not* to be said of the exchange-value of A in all other things, which would include B, because B has changed relatively to A, and so may be a disturbing factor; nor of the exchange-value of B in all other things, which would include A, because A has changed relatively to B, and so may be a disturbing factor; wherefore the relationship between these exchange-values is slightly different. But the disturbance caused by B's presence in A's general exchange-value is counteracted if A also is included; similarly in the case of B's general exchange-value, if B's presence is also included, wherefore the above relationship is restored. That is, *when mutual changes take place between two things, relatively to each other, their exchange-values in all things vary in the same proportions relatively to each other* (Proposition XIII.).[4] Another

[4] In practice, of course, the variation in their exchange-values in all *other* things will be almost the same as their variations in *all* things,—as will be shown [illegible] This Proposition, then, is *approximately* true of the "general purchasing [illegible] two things.

demonstration of this proposition will be forthcoming when we have discovered how to measure exchange-value in all things.

It follows, of course, that *if two things (classes) remain unchanged relatively to each other, their exchange-values, both in all the common other things and in all things, vary, or remain constant, alike* (Proposition XIV.); but not necessarily do their exchange-values in all other things vary alike.

Naturally *when mutual changes take place between two things, some change must take place in one or in both relatively to the common other things* (or relatively to all things, and also to all other things)[5] (Proposition XV.). The exchange-values of the two things in all the common things (or in all things) are yoked together so that, the variation of the one in the other being given, and also the variation (or constancy) of either in all the others (or in all things), the variation (or constancy) of the other in all the others (or in all things) may be easily calculated. In general the possible changes may be classified into *five principal types:* relatively to the other things (1) A may have risen, while B has remained unchanged, (2) A may have remained unchanged, while B has fallen, (3) A may have risen and B fallen, (4) both A and B may have risen, but A more, (5) both A and B may have fallen, but B more. Mere knowledge that A has risen in B, with or without the percentage being known, conveys no knowledge as to which of these possible combinations of changes relatively to the other things has been effected; but it does convey knowledge that one or another of them must have been. Which of them it is, or exactly what the changes are relatively to all the common other things (or to

[5] This Proposition, therefore, is true of the general purchasing power of two things. Yet the following statement has been made:—"Whilst gold has not risen in purchasing power, and silver has not declined in purchasing power, the relative value of silver to gold has declined to less than half," A. Ellissen, *The errors and fallacies of bimetallism*, London 1895, p. 12. The author, however, is conscious of inconsistency, for he adds: "In mathematics such a problem would amount to a preposterous absurdity; but in the case before us it is only too true." It is tenable only by an equivocation, the author having treated "purchasing power" like "value" in general, and then not having distinguished between its different species. He means that silver has not fallen in purchasing power proper and gold has not risen in cost-value (or esteem-value)—perfectly simple and unconnected statements, not deserving to be so expressed as to involve contradiction.

all things) must be sought—not by trying to find the causes, or changes in the costs of production (or in the quantities) of either or both, or of all the other things,[6] but—by direct measurement either of A's or of B's exchange-value in all the other things (or in all things), or of both, which measurement must take account of all the particular exchange-values of either or each in all the other things singly, and thereby also of the particular exchange-values of all the other things in either or each of these two.

II.

§ 1. Let us now pay attention to the first and the second, as the simplest, of the above five possibilities when the other things are also taken into consideration. Let us suppose that, A rising or falling in exchange-value in B, B retains its exchange-value unchanged relatively to all the other things, and that these other things all remain without change relatively to one another and to B. Now, then, if A rises in exchange-value in B, it rises also in exchange-value in every one of the other things, that is, in every other thing beside itself, in the same proportion; and, B *ipso facto* falling in exchange-value in A, so does every one of the other things fall in exchange-value in A, and in the same proportion as B, that is, in inverse proportion to A's rise in B. Thus if A rises 50 per cent. in B, it rises 50 per cent. in C, in D, and so on throughout all the other things; and as B falls 33⅓ per cent. in A, so do C and D and the others, each singly, fall 33⅓ per cent. in A. And the reverse is the case if A falls in B. Then, like B, all the others rise singly in A in inverse proportion to A's fall in B. Thus we have the general law: *A variation of one thing in exchange-value in the same proportion relatively to every other thing involves an opposite variation in inverse proportion of every one of the others relatively to it* (Proposition XVI.).

[6] These are subjects that must be investigated in order to measure variations of cost-value, or of esteem-value. Naturally, different kinds of value must be measured in different ways, and the ways required for measuring variations of other kinds of value are not the ways required for measuring variations of exchange-value.

When A thus varies, say rising or falling 50 per cent., in relation to every one of the other things, all these remaining unchanged amongst themselves, as they must do under the first supposition, it is evident that A has varied in the same proportion, rising or falling 50 per cent., in exchange-value in all these other things together. In other words, a variation in the particular exchange-values of any one thing in the same proportion in all other things singly is a variation in its general exchange-value in the same proportion in all other things collectively. Or more briefly, *when all the particular exchange-values of a thing in other things vary alike, their common variation is the variation of its general exchange-value in all other things* (Proposition XVII.).

It may be noticed that absolutely all the particular exchange-values of a thing cannot vary; for its particular exchange-value in itself (or in its mates within its class) never varies (according to Proposition II.). Hence the need of confining the statement to the particular exchange-values in other (*i. e.*, in other kinds of) things.

§ 2. But when B and C and D and every other thing have, say, fallen by 33⅓ per cent. in exchange-value in A, it is equally plain that B and C and D and every other thing have *not* fallen so much in exchange-value in all other things; for they have not fallen relatively to one another. All A's particular exchange-values in other things have risen in the same proportion, namely by 50 per cent., and therefore A's general exchange-value in all other things has risen in that proportion. But only one particular exchange-value of B, of C, of D, etc., has fallen, while all the other particular exchange-values of B, of C, of D, etc., have remained unchanged; therefore their totals of particular exchange-values, that is, their general exchange-values in all other things, have been less altered. Of course the reverse takes place in the case of a fall of A. If A falls in exchange-value in every other thing in the same proportion, it falls in exchange-value in all other things in that proportion; and every one of the other things rises in exchange-value in it alone in inverse proportion without rising in anything else, and therefore every one of them rises in exchange-value in all other things in a proportion smaller than the inverse. Thus we have the general law: *A variation of any one thing in exchange-value in the same proportion relatively to all other things involves an opposite variation of every one of the others in exchange-value in all other things to a smaller extent than the inverse proportion* (Proposition XVIII.).

And evidently, *in such cases, the extent of the opposite variations will be smaller and smaller the more numerous are the other things* (Proposition XIX.). We perceive this by supposing at first the existence of only three articles and then extending the number. Let A, as before, be the article which rises in exchange-value in every other single thing, say by 50 per cent. Then, as all the other things are in the same box, having the same exchange-values, we need to follow the fortunes of only one of them, say B. Now with only three articles,[1] B was at first equivalent to A and to C, and after the rise of A it is equivalent to $\frac{2}{3}$A and to C; with four articles, it was equivalent to A, to C and to D, and later is equivalent to $\frac{2}{3}$A, to C and to D; with five, it was equivalent to A, to C, to D and to E, and later is equivalent to $\frac{2}{3}$A, to C, to D and to E; and so on. It is plain that the larger the number of other things, the larger is the number of its particular exchange-values which B retains unchanged, and the comparatively smaller and less appreciable becomes the loss of its exchange-value in A—or smaller and less appreciable the loss of its purchasing power over A compared with its purchasing power over the other things which remain unchanged;—and consequently, the whole being enlarged, the smaller is the fixed loss in one member relatively to the remainder. With very many articles, a small change in A relatively to the rest, may be a practically inappreciable change in B and the others relatively to all other things as a whole.[2] But theoretically, in any such case, how-

[1] We must begin with three in order to get two others, so as to make the plural. If, however, we started with only two things, having only one other for "all the others," the opposite variation would be exactly in the inverse proportion, this condition reducing to the condition in Propositions IX. and X. This is, for really other things, the unattainable limit of greatness in the opposite variation.

[2] Of course in all this passage the sizes of the other articles are supposed to be

ever small the change of A, there is *some* change in B and the others relatively to all other things as a whole.[3] The reverse is equally plain if we suppose A to fall.

§ 3. What is said of the correlation of exchange-values in the case of things in general, is of course true when one of them is money. To reinterpret these relations in terms of prices, is a simple matter. (1) If money rises in exchange-value in every other article by p per cent., it rises in exchange-value in all other things by p per cent. But its rise in exchange-value in every other article by p per cent. is the same as a fall of every one of these in price by $\frac{p}{1+p}$ per cent. Therefore a uniform fall of price by $\frac{p}{1+p}$ per cent. is a rise of money in exchange-value in all other things by p per cent., that is, in inverse proportion. And while every article has fallen by $\frac{p}{1+p}$ per cent. in price, that is also, by $\frac{p}{1+p}$ per cent. in exchange-value in money, every article has fallen to a much smaller extent in exchange-value in all other things.[4] And similarly in the case of a fall of money or a rise of prices. Yet we some-

given. If we increase their number simply by breaking them up into parts, the extent of the opposite change in their exchange-values is not thereby affected. The sizes of articles, or classes, will be treated of later.

[3] If it be theoretically permissible to suppose an infinite number of equally larger other things, then the opposite change, no matter how much A or any finite number of things finitely rise in every one of the common other things, the opposite variation of every one of these in exchange-value in all other things is, even theoretically, infinitely small, or nil, that is, there is no variation in their exchange-values in all other things. This is the unattainable limit of smallness in the opposite variation.

[4] It is a mistake to say that the "general value" of these articles has not changed, as said by J. Prince-Smith, *Valeur et monnaie*, Journal des Économistes Dec. 1853, p. 373. But if we purposely ignore the exchange-value of money on the ground that money is only an intermediary in exchanges of goods, it is true to say that the general exchange-value of these articles in all other, or in all, articles except money, has not changed. We should remember, however, that money is not an intermediary merely in contemporaneous exchanges, but that frequently a long interval passes between the time when we get money for our property and the time when we get other property for our money, so that the intervening state of the exchange-value of money is very important. Still the importance seems to be *sui generis*.

times hear such faulty expressions as this: If, prices rising by 50 per cent., money depreciates by 33⅓ per cent., commodities have appreciated by 50 per cent.; or conversely. It is true that commodities, each singly, have appreciated by 50 per cent. in money; but to say simply that commodities have appreciated 50 per cent. means that they have risen 50 per cent. in exchange-value simply, which is not true.[5] Again (2) if money rises by p per cent. in exchange-value in one article alone while retaining unaltered its exchange-value in every other article, this is only another way of saying that the price of that one article has fallen by $\frac{p}{1+p}$ per cent., while the prices of all other articles has remained unchanged. Then the exchange-value of that one article in every other thing, and in all other things has fallen by $\frac{p}{1+p}$ per cent.; and the exchange-value of every other article, as of money, in that one thing, has risen by p per cent., but the exchange-value of money, and of every other article, in all other things, though it has risen somewhat,[6] has by no means risen to that extent.

III.

§ 1. A corollary from the preceding laws is this: *There cannot be a variation in the exchange-value of one thing alone* (Proposition XX.).[1] There can, of course, be a rise or a fall in exchange-value of one thing alone, since one thing may rise without anything else rising, and one thing may fall without anything else falling. But the moment one thing alone so rises or falls in exchange-value, all the others inversely fall or rise.

[5] Another similarly faulty expression is that in the case of a rise of prices (values have risen), and conversely. It is only values in money, or values expressed in money, that have changed exactly as the prices. Of course on the stock market, where the only values thought of are money-values, this expression is natural.

[6] As correctly said by Prince-Smith, *op. cit.*, p. 373, and by Jevons, B. 15, p. [illegible]

[1] See Walras, *Éléments*, 1st ed., pp. 167-168, 2d ed., p. 456. This is very different from what is referred to in Note 1 in Sect. I. of this chapter. Here he is speaking of exchange-value; there what he had in mind was esteem-value.

Hence it is a solecism to speak simply of a change of one thing alone in exchange-value. There may be a change of one price alone; but as we have just seen, this is not a change in the exchange-value of the one article alone. Furthermore there may be a change in the use-value, in the cost-value, and perhaps even in the esteem-value, of one thing alone. But here is only one more reason for distinguishing use-value and cost-value and esteem-value from exchange-value. Likewise it is conceivable that a cause may act upon one thing alone to make its price change, or its use-value change, or its cost-value or esteem-value change, without affecting the price, use-value, cost-value or perhaps (although this is questionable) even the esteem-value, of anything else. It is not possible, however, for a cause to act upon one thing alone so as to render it of greater or smaller exchange-value, since this cause will also act upon all other things to render them of smaller or greater exchange-value in this thing and thereby, to some extent, in all things.[2]

§ 2. What is here said shows the wrongness of a doctrine sometimes taught concerning "value," supposedly in the sense of exchange-value. This is that when a sensible rise or fall takes place in the value or price of one thing alone (or in the value of money, and therefore in all prices in the opposite direction), the rest remaining tolerably steady relatively to one another, there is reason for holding that the change is altogether in the one thing and not at all in the others—that is, that the one thing alone has changed in "value," and the others have remained unchanged in "value." The reason assigned is that it is easier to explain these phenomena by supposing one cause affecting this one thing than by supposing causes affecting the other things, which causes would have to be as numerous as the other things; or, in other words, that the hypothesis invoking

[2] Hence there is impossibility in the assumption made by H. Fawcett (following J. S. Mill, *op. cit.*, Vol. I., pp. 539–540) in his *Manual of political economy*, 1863, 6th ed., 1883, p. 314, at the commencement of his inquiry into "the causes which regulate the price of commodities," namely that a variation of price "is always supposed to be produced by something which affects the value of the commodity and not the value of precious metals"; for by "value" he meant exchange-value. Fawcett has been followed by A. S. Bolles, *Chapters in political economy*, 1874, p. 56.

one cause is far the simpler and more probable—just as the Copernican theory is simpler and more probable than the Ptolemaic.[3]

The statement made in this reason assigned for the doctrine in question is in itself not incorrect. In default of data for finding the cause or causes directly, or in our impatience at the delay required for such a direct investigation, it is easier for us to suppose, and we are inclined to believe, that there is only one cause affecting, or, as is sometimes carelessly said, residing in, the one thing whose exchange-value in all others is most prominently changing.[4] But to take this view of the matter as a reason for supposing a change in exchange-value only in the one thing, is absolutely incorrect. We can—it being supposed all along that we know the particular variations—calculate with mathematical precision, when we have fully developed the theory of exchange-value, almost the exact variation in the exchange-value (in all other things, and also in all things) of the one thing and almost the exact variations in the (similar) exchange-values of all the other things. There is no occasion for employing the law of probabilities at all.

The explanation of this doctrine itself—and of the use of the law of probabilities with regard to the causes as a reason for applying the same to the changes of "value"—is the retention in the idea of "value" of the ideas of cost-value and of esteem-value. In the case of cost-value, and perhaps also of esteem-value, it would be possible for the change to reside only in the cost-value, or in the esteem-value, of the one thing; and accordingly here, in default of knowledge to the contrary, it is more probable, it is a more likely hypothesis, that the change has taken place in the cost-value, or in the esteem-value, of the one thing alone. This is because, at least in the case of cost-value, this value is wholly dependent upon the labor required for producing the thing, and so is measured by that

[3] Ricardo, *Works*, p. 13; A. Cournot, *Recherches sur les principes mathématiques de la théorie de la richesse*, 1838, pp. 18–19; the latter followed (with addition of the simile) by C. Gide, *Principes d'économie politique*, 1884, pp. 81–83.

[4] Thus this hypothesis in regard to the cause, but not in regard to the change in value, was used by Jevons, B. 22, pp. 19–20, B. 24, p. 156.

labor, that is, is measured by a measurement of its cause. Now if it should really happen that one thing alone rose or fell in cost-value, the relation of its cost-value to the cost-value of all the other things would have varied, and, if it rose compared with them, they would have fallen compared with it; and if, as is sometimes the case, the exchange-value of the thing varies with its cost-value, its exchange-value would rise in other things and theirs would fall in it (and consequently in exchange-value simply). Here then we should be reaching a right conclusion by means of the law of probabilities if we supposed the change of cost-value to be solely in the one thing; but we should be violating the assumed conditions if we concluded either that the relative cost-value of the thing compared with the others has alone changed and not also their relative cost-values compared with its, or that the exchange-value of the one thing (meaning its exchange-value simply, *i. e.*, in all other things, or in all things, or compared with the exchange-values of the other things) has alone changed and not also the exchange-values of the other things (meaning their exchange-values simply, *i. e.*, in this thing and in all other or in all things, or compared with this thing's exchange-value). The very data assumed require these correlative changes both in relative cost-values and in exchange-values—for all exchange-value is relative.[5]

IV.

§ 1. Another corollary from the above-given laws is the already mentioned distinction between exchange-value in all

[5] There is, therefore, no need of speaking of "relative exchange-value." In fact, the very term "exchange" in this compound term has the meaning of "relative," so that "exchange-value" is similar to "relative value." Sometimes exchange-value is relative cost-value; always it is relative esteem-value. In the course of time a thing retains the same exchange-value if it retains the same relative esteem-value, that is, whether its esteem-value vary or not, if it varies or not just as the esteem-values of all other things on the average vary or not. Or in the course of time its exchange-value varies if its relative esteem-value varies, that is, if its esteem-value varies compared with the average of the esteem-values of all other things, for instance rising (1) if it rises more than they do, or (2) if it rises while they are stationary, or (3) if it rises while they fall, or (4) if it is stationary while they fall, or (5) if it falls less then they do—in the same five typical relations we have already noticed in another connection.

other things and exchange-value in *all* things. For the law that if one thing alone rises in exchange-value in every other thing to the same extent, it rises exactly to that extent in exchange-value in all other things, is perfectly evident. And also the law is equally plain that every one of those other things has sunk somewhat in exchange-value in all other things. Putting these two laws together, we see that when a thing rises a certain percentage in exchange-value in all other things, as it rises to that extent in things that have themselves sunk somewhat in exchange-value in all other things, it has not risen quite that percentage in exchange-value after all, but to a somewhat lesser extent. The performance is comparable with the rise of a body above a plane which is depressed by the force raising the body; in which case the measurement of the rise of the body in relation to the plane gives a greater result than the true rise. Similarly in the reverse case of the fall of a body by which the plane from which it falls is repelled upward, the measurement of the fall of the body from the plane also gives too great a fall. So a fall of one thing in exchange-value in all other things, being a fall which raises their exchange-values somewhat, is not so great a fall really as it appears to be, judged by comparison with them only.

The explanation of the difference is obvious. In treating of the exchange-value of one thing in all other things we are using all *other* things as the standard; but then in treating of the exchange-value of any one of the other things also in all other things we are using a different standard, for this standard excludes the latter thing which was included in the former standard, and includes the former thing, which was excluded from that standard. These two standards, then, are shifting standards; but together they include *all* things. Thus when we perceive that a thing which has risen a certain extent in exchange-value in all other things has not risen quite so high in exchange-value, by this exchange-value we mean an exchange-value measured in a standard in which not only all the other things are included, but also the thing itself of which the altered exchange-value is being measured. Thus this exchange-value is exchange-value, not in all other things, but in all things. And it is the superior and final exchange-value, the one more than any other appearing to deserve the single term exchange-value. We might even be tempted to call it "absolute exchange-value," except that there are metaphysical and logical objections to the use of this term. These objections could not hold against calling it "universal exchange-value." But perhaps it is as well not to give it any name.

The simple term, then, exchange-value, or general exchange-value, is slightly ambiguous. The same ambiguity extends to the related terms, appreciation and depreciation, and the like. Thus a thing which appreciates a certain percentage in exchange-value in all other things, does not appreciate quite so much in exchange-value in all things. Yet almost all writers on the subject have measured appreciation and depreciation, or in general any variation in exchange-value, in the former way only. Thus it is commonly asserted that if all other things remain unchanged amongst themselves, including money, the change of one thing relatively to any one of them, and so its change in price, exactly measures its change in general exchange-value;[1] or again that if all prices change alike (or on an average) to a certain extent, money has changed to the inverse extent in exchange-value simply.[2] Such statements are only half true. They are true of the variations in exchange-value in all other things; but they are not true of the variations in exchange-value in all things.

§ 2. The differences between these two kinds of general exchange-value and the laws common to them both, may be briefly surveyed. It is evident that *a thing's exchange-value in all things always varies less than its exchange-value in all other things* (Proposition XXI.). For, given the deviation of a thing's ex-

[1] *E. g.*, J. S. Mill, *op. cit.*, Vol. I., pp. 539–540.

[2] So Prince-Smith, *op. cit.*, p. 373. J. S. Mill says that, *caeteribus paribus*, prices vary directly, the exchange-value of money inversely, with the quantity of money, *op. cit.*, Vol. II., pp. 16–17. It is usual, after getting a method or formula for measuring the variation of the average of prices, simply to invert this as the measure of appreciation or depreciation of money; so Levasseur, B. 18, p. 195; Jevons, B. 22, pp. 53–54, 58; Drobisch, B. 29, p. 39, B. 30, p. 149; Lehr, B. 68, p. 40; Nicholson, B. 94, pp. 317–318; Lindsay, B. 114, p. 26.

change-value in all other things, as its exchange-value in itself never varies, when this unchanged particular exchange-value of itself in itself is added to the standard by which its deviation is measured, it must have the effect of lessening the deviation somewhat. But it cannot annul the deviation, since there is some particular variation as before, which must still be counted. Therefore *the one general exchange-value of a thing cannot vary without the other varying also* (Proposition XXII.). And, of course, the lessening just shown cannot deflect the deviation in the opposite direction; wherefore *the two kinds of general exchange-value always deviate from constancy in the same direction* (Proposition XXIII.). Further, as we have seen that when one thing alone rises in exchange-value in all other things, the others fall somewhat, and less the more numerous they are, and conversely, so we now see that there are two rises of the one thing and consequently also two falls of the other things, and conversely. It is now plain that in both cases, *the divergence between the two deviations of the two kinds of general exchange-value is smaller the more numerous are the articles* (Proposition XXIV.). For the divergence is determined by the difference between the whole number of things minus one thing and the whole number of things compared with the whole number of things, and this comparative difference is smaller the larger the whole number of things. Thus, for example, if we know the variation of a thing in exchange-value in all *other* things (as when conditions satisfy Proposition XVII.), then we know that its variation in exchange-value in *all* things is almost as great, if the other things are very numerous (or very much larger than the one thing, or than its class). Similarly, *given the number of (equal) articles, the divergence between the two deviations is always in the same proportion* (Proposition XXV.). For here the comparative difference is the same in all the measurements. We shall later find formulæ from which, in some cases, the exact proportions can be derived, if desired, and a very simple proposition expressing them.

§ 3. Upon these laws of variation and divergence follows a law of constancy and identity. It is plain that if there is no variation of a thing's exchange-value in all other things, there can be no variation of the thing's exchange-value in all things, since nothing is added in the latter case to make a variation not existing in the former. Nor, reversely, if there is no variation of a thing's exchange-value in all things, can there be a variation of the thing's exchange-value in all other things, because, if an element of variation existed in the latter case, it would have existed in the former also. In other words, *constancy of the one kind of general exchange-value possessed by anything is constancy of the other* (Proposition XXVI.). That is, again, the two kinds of general exchange-value *coincide* and are *identical* when either of them is constant. Another demonstration of this theorem may be offered as follows: If changes in a thing's particular exchange-values occur so that its exchange-value in all other things rises, its exchange-value in all things rises, but less; and if such changes occur so that its exchange-value in all other things falls, its exchange-value in all things falls, but less; therefore if we suppose these changes on each side to be smaller and smaller until the change in the thing's exchange-value in all other things is so small as to be inappreciable, its exchange-value in all things, always being even smaller, must be still less appreciable, and when the former becomes zero, the latter must also be zero.

On account of this coincidence of the two kinds of general exchange-value at rest, we are justified in subsuming them under the same term; and whenever we have occasion to speak of constant general exchange-value, or simply of constant exchange-value, there is no need of distinguishing between the two kinds, which separately exist only when there is variation.[3]

[3] Beginning at a given period, we see that if there is a variation of a thing's general exchange-value, this splits up into two variations from the original coincident positions of the two kinds of general exchange-value; but if there had been constancy of the thing's general exchange-value, there would have been no such splitting, and the two kinds would not have manifested their distinctness. Now when such a variation and such a splitting have occurred, if we start over again from the second period as a new starting point, the same phenomena may repeat themselves; that is, if from this starting point there is constancy, the general exchange-value remains undivided, but if there is variation, it divides again into two. And so on from any later period as a new starting point. In every case the two kinds of general exchange-value at the second period, com-

V.

§1. Constancy of general exchange-value may happen under two different conditions of things. The first is the simplest and most obvious. It is evident that if all the particular exchange-values of a thing in all other things remain constant, its general exchange-value in all other things and, since its exchange-value in itself must remain constant, its general exchange-value in all things remain constant. This condition takes place when there are no changes whatsoever between the particular exchange-values of any thing in any other amongst all things; so that the general exchange-value not only of the thing in question, but of everything else, is without change. Thus, *if all things retain the same exchange-values relatively to one another, the general exchange-value of any one thing is constant* (Proposition XXVII.).

The second condition is not quite so obvious, but is equally demonstrable. For evidently it is possible, when one thing changes alike in exchange-value in all others except one, for all the rest beside these two to remain unchanged in exchange-value in all other things, and consequently in all things, provided that the other one thing change in respect to them in a manner to compensate for the change of the first one. And consequently, any two compensating for each other, or several compensating for one or for several, or many for many, it is possible for one thing to retain the same general exchange-value, in spite of changes in any two or in any higher number of others up to all others.

§ 2. Here we meet with an error which would be fatal in our subject. A few economists have asserted that the general exchange-value of anything can remain unchanged only under the first condition, and as that condition is never fulfilled in the actual world except over the briefest of intervals, they have denied even the *possibility* of any one thing, or in especial, money, retaining the same general exchange-value through the

pared with the first, may have remained the same or have divided; but in every case the two kinds at the first period, to be compared with the second, are taken as identical.

course of time[1]—and one writer has even rejected the expression "stable value" as a collocation of words without meaning, like "triangular square."[2] The same position has been taken by Roscher concerning general purchasing power[3]—with better reason, as we shall see, for one sense in which this term is used. Now if this view were correct in regard to exchange-value, that is, if the constancy of general exchange-value under any changes of particular exchange-values is even theoretically impossible, or inconceivable, it would be absurd to speak of a greater or less variation of a thing's general exchange-value through the course of time, since this expression always has reference to its variation from a supposed constant position, and if this has no possible existence, the magnitude of the variation has no possible existence—nay, the variation itself could not possibly exist, and the general exchange-value of a thing at one time would be wholly incommensurable with its general exchange-value at another time, as indeed has been maintained by some extremists.[4] Then not only to measure a thing's constancy in general exchange-value would be an idle endeavor, but also to measure its variation in general exchange-value would be meaningless. And the statement frequently made by the very persons who deny the possibility of constancy of general exchange-value, or the possibility of measuring it, that the desirable feature in money is that it should vary as little as possible in exchange-value, would be an absurdity.[5]

[1] Galiani, *Della moneta*, 1750, ed., Custodi, Vol. II., p. 11; McCulloch, *Political economy*, p. 213 (Note to *Wealth of nations*, p. 439); Prince-Smith, *op. cit.*, p. 373; Macleod, *Theory of banking*, Vol. I., p. 69, *Theory of credit*, Vol. I., p. 176; Walras, *Élements*, 1st ed., pp. 167–168, 185, 2d ed., p. 456; Martello, *op. cit.*, pp. 402–403.—Rossi seems to have had the same idea when he said there can be no measure of value because as a standard it should be invariable and as a value it must be variable, *Cours d'économie politique*, 1840, Vol. I., p. 150.

[2] F. Ferrara, Introduction to Martello's work, p. CXLIX.

[3] B. 32, § 127.

[4] *E. g.*, A. Held maintained that we cannot properly say that in a financial crisis the exchange-value of money is greater than it was, but only greater than it would have been had no crisis arisen, *Noch eimmal über den Preis des Geldes. Ein Beitrag zur Münzfrage*, Jahrbücher für Nationaloekonomie und Statistik, Jena 1871, p. 328. He might as well say that when a yard-stick expands under the influence of heat, it is not longer than it was, but only longer than it would have been had it not been heated. How it is we can compare a value with what it would have been, but not with what it was, is not explained.

[5] Among the writers mentioned in Notes 1 and 2 only Ferrara and Martello

§ 3. But the reasoning by which is maintained the denial of the possibility of a constant exchange-value through variations of particular exchange-values, is the plainest of fallacies. It is based on our Proposition XX. The exchange-value of a thing is affected and altered if any *one* other thing varies in exchange-value. Therefore, it is concluded, its exchange-value must be still more affected and altered, if two other things, if three, if many, vary in exchange-value—that is, if *any* other thing or things vary in exchange-value. Now it is plain that the greater the number of other things that vary in exchange-value, the more the exchange-value of the thing in question is *affected;* but as soon as it is affected by *two* or more other influences, the possibility arises that it may be affected in opposite directions, so that the *alteration* of its general exchange-value, instead of being increased, may be diminished, and furthermore, that it may be reduced to zero by the opposite influences exactly counterbalancing each other.

This may be rigorously demonstrated as follows. Suppose at first

$$M = A = B = C = \ldots\ldots,$$

and later A rises in exchange-value in all the others, so that

$$M = \frac{1}{x}A = B = C = \ldots\ldots;$$

then M has fallen somewhat in general exchange-value. Again, instead of this rise of A, suppose B falls, so that

$$M = A = yB = C = \ldots\ldots,$$

(in each of these expressions x and y being greater than unity); then M has risen somewhat in general exchange-value. Now it is evident that if these two changes take place together, so that

$$M = \frac{1}{x}A = yB = C = \ldots\ldots,$$

are guilty of this inconsistency. [illegible] not only to measure general exchange-value, but also to regulate the general exchange-value of money with a view to its constancy. [illegible] who have, for various reasons, denied that we can measure exchange-value, or that money is a measure of it, have fallen into similar inconsistency.

then the general exchange-value of M is influenced to a fall by the one change and by the other to a rise, and it can be that, the amount of the first change being fixed, the influence of the second change may outweigh its influence, conducing to a fall, or the influence of the second change may be insufficient, permitting a rise. It is also evident, in accordance with the principle of continuity, that between these higher and lower influences of the variable second change, compared with the fixed influence of the first, there must be a point at which that influence exactly counterbalances this influence, so that, the two influences neutralizing each other, or having the same effect as if neither existed, the general exchange-value of M remains unchanged—just as two opposite and equal forces will keep a body at rest.

Now the rise of A in exchange-value in all other things is the inverse fall of M's particular exchange-value in A, and the fall of B in exchange-value in all other things is the inverse rise of M's particular exchange-value in B. Therefore, *the general exchange-value of anything may remain unchanged in spite of a variation in one of its particular exchange-values (or an inverted variation in the general exchange-value of one other thing), if this be compensated by an opposite variation in another of its particular exchange-values (or an opposite variation in the general exchange-value of another thing)* (Proposition XXVIII.).

Furthermore, it is obviously not necessary for the counterbalancing to be done by a variation of one other thing alone. *The compensation for the variation of one thing may be effected by variations, in the opposite direction, of many others, or even of all the others* (Proposition XXIX.). If done by more than one other, it is, however, plain that the variations of the others individually must be smaller than the variation of only one other, because their combined influence constitutes the compensation. If, for instance, the rise of A were already compensated by the fall of B, then a fall of C would disturb the balance and itself require to be compensated either by the rise of something else or by a still higher rise of A. Thus a great rise of A may be compensated by two lesser falls, equal or unequal to each other, of B and C; consequently, in general, any rise of one thing, by any number of falls of other things, and conversely. And, as the compensating influence must be distributed over all the other changes, it is plain that, the sizes of the things (or classes) being given, *the greater the number of the compensating variations the smaller must they be individually or on an average* (Proposition XXX.).

Also it is plain that, *a certain variation in one direction being compensated by one or many variations in the opposite direction, it is possible for still another variation in either direction to be compensated by still another or other variations in the direction opposite to it, and so on indefinitely, all at the same time*, the one thing in question remaining unchanged in general exchange-value so long as all variations in one direction are compensated, in pairs, or in combination, by variations in the opposite direction (Proposition XXXI.).

§ 4. Now when the one thing we have in mind—M in the above example—remains constant in general exchange-value under changes in two or more other things, it is plain that all the other things that have not varied relatively to it—the C, etc., in the above example—have also remained constant in general exchange-value (in accordance with Proposition XIV.). Consequently, *the constancy of any one thing in general exchange-value is unaffected by the number of other things likewise constant in general exchange-value*, and in any calculation it ought to be indifferent whether we include them or not (Proposition XXXII.). Evidently it is only things that vary in general exchange-value that can exert influence to cause a variation in the general exchange-value of anything that would otherwise be constant.[6]

But with regard to variations a distinction is to be drawn. *The things which have varied in certain proportions relatively to the thing or things remaining constant have varied in those same proportions in their general exchange-value*, not in all other things, but *in all things* (or in all but any or all of the constant

[6] Of course if money has varied in general exchange-value, the number of things that are constant in price affects the condition of anything otherwise constant, because these things have varied in general exchange-value with money.

things) (Proposition XXXIII.). Thus, while the variation of anything in general exchange-value in all *other* things is affected by the number of other things constant in general exchange-value, *the variation of anything in general exchange-value in all things* (*or in at least all the variable things, including itself*) *is not affected by the number of other things remaining constant in general exchange-value* (Proposition XXXIV.). The latter part of this proposition is apparent, since the other things that are constant in general exchange-value do not affect the general exchange-value of the one thing above used, by comparison with which the variation of the one thing may be measured, and this variation may be measured as well by comparison with any one of the constant things as by comparison with the one first used. The distinction, then, is needed in order to carry out Proposition XXIV., which shows that the relation between the two variations in general exchange-value of anything is affected by the number of other things; for then the number of other things must affect the two variations differently, and if the number of other things that are constant in general exchange-value does not affect the variation of any one thing in general exchange-value in all things, it must affect the variation in general exchange-value in all other things. More particularly, the distinction is evident, because in the calculation by which the constancy of the constant things is determined has entered the variation of the exchange-value of the thing in question that has varied, but this variation is excluded from the calculation by which its variation in exchange-value in all other things is determined. It is plain that only if we found that money, for instance, remained constant in exchange-value in all other things but one, this one being left out of the calculation, would a variation of this one relatively to money (indicated by its price-variation) measure the variation of its exchange-value in all other things. And what would not affect the exchange-value in all other things of a thing that varies is, not something remaining constant in exchange-value in all things, but something remaining constant in exchange-value in all but that one thing, that one thing being left out of

4

the calculation of the constant thing's constancy just as it is left out of the calculation of its own variation in exchange-value in all other things. In exchange-value in all things everything is measured by the same standard, and so has the same variation whether it be directly measured in relation to all things or whether it be measured in relation to something whose position in relation to all things has already been determined. But in exchange-value in all other things the standards are different in every measurement, and therefore the results do not fit and dove-tail into one another as they do in the other case.

The last Proposition may be altered into this: *If in measuring the exchange-value of money in all things (including itself) we find it has varied to a certain extent, then it ought to be indifferent in the calculation whether we include or exclude anything whose price has varied inversely* (Proposition XXXV.). In other words, for instance, if in measuring money we find that the general level of prices, including its own, which never varies, has risen so as to be r times what it was before, wherefore the exchange-value in all things of money has fallen by $\frac{1}{r}$, then anything whose price has risen so as to be r times what it was before ought to be indifferent in the calculation by which the result for money was obtained; for that thing has remained constant in exchange-value in all things. Furthermore also, *If in measuring the exchange-value of money in all other things (i. e., in all commodities) we find it has varied to a certain extent, then it ought to be indifferent in this calculation whether we include or exclude any commodity whose price has varied inversely* (Proposition XXXVI.), since this commodity retains constant exchange-value, not in all things, but *in all commodities*, and so is constant in exchange-value in all things other to money, and the addition or subtraction of such a commodity to or from the standard by which the variation of money is measured does not alter the relation of that standard to money.[7] Thus the standard of all com-

[7] Here if $r = 1$, indicating constancy, both these Propositions fall directly under Proposition XXXII.

modities (excluding money) may take the place of the standard of all things (including money).

§5. We may also perceive the following relations, provided we suppose that the things be the same (or similar and in equal quantities) at all the periods compared. Then, *when any one thing has remained constant through compensatory variations of others, all the others collectively have remained constant* (Proposition XXXVII.). For as the changes in the other things have compensated the relations of this one thing to all other things, they have compensated the relations of all the other things to this one thing, and as this one thing is constant, they have compensated their relations to something constant, and therefore themselves, collectively, are constant. Also, according to Proposition XVIII., if the one thing had risen or fallen in exchange-value in every other thing alike, it would have depressed or raised all the others individually somewhat (and therefore collectively as many times more); therefore if it neither rises nor falls in every other thing, it cannot influence them either to a rise or to a fall, and they, individually (and collectively), must remain constant. Here, however, we are supposing the one thing to change relatively to the others individually, but not collectively; yet it is evident that, although the others change relatively to it individually, they cannot do so collectively. It follows that *all things taken together, namely all things permanent over all the periods compared, must be constant in general exchange-value when any one is* (Proposition XXXVIII.). And even if no one thing has remained constant, yet as any one by rising depresses all the others, and each of the others less in proportion to their numbers (according to Proposition XIX.), and reversely by falling, it is plain that *all things collectively, provided they be the same (or similar and in equal quantities) at all the periods compared, must be constant in general exchange-value* (Proposition XXXIX.). This principle will call for further elucidation later, when the need of the proviso will be proved, and it will be shown in which kind of general exchange-value, when there are variations, the proposition is true.

§ 6. To return to the earlier principles about the possibility of compensation, it follows as a *résumé* of them that *it is possible for the general exchange-values of two or any greater number of things up to all but one to vary, and yet for the general exchange-values of all the others, that is, of any number of things from all but two down to one, to remain constant* (Proposition XL.). Of course, for this result to happen, the variations must be in certain proportions, which it is our object to discover. As yet we only know the general possibility—and not even that this possibility must necessarily always exist, but that it may exist. We know also an impossibility, which is, that if any one thing alone rises or falls, it is impossible for anything to retain the same general exchange-value, since the counterweight is lacking.

All this might seem to apply also to the case of general purchasing power. It can be so if this term be identified with exchange-value in all other things. But the term has not always been used in this, its proper, sense; and then what is here proved of general exchange-value cannot be proved of what is meant by "general purchasing power." This subject will be discussed later.

The statement that one thing is remaining constant in general exchange-value while other things are varying in their general exchange-values, means also, as we have seen, that this thing's particular exchange-values in those varying things are varying and are compensating for one another. This statement may be true of money—of which, in fact, we proved it;—and as the particular exchange-values of money are inversely measured by prices, it follows that a change of price, or many changes of prices in one direction, may be compensated by one or many changes of prices in the opposite direction, with the resultant that money remains constant in general exchange-value, the changes of prices neutralizing one another and yielding the same combined influence as if none of them had taken place. Therefore, although we have seen that a change of a single price cannot directly indicate the extent of the thing's variation in exchange-value in all things, it is possible for a change of one price to do so if there is at least one other change of price in the opposite direction to the proper extent for counterbalancing; or a change of any prices may do so, provided they are all together such as to leave the exchange-value of money unchanged. And although we have seen that a change of all prices alike does not indicate an exactly inverse change of money in exchange-value in *all* things, yet such a change of all prices but one may do so if that one changes in the same direction still more to the proper extent for counterbalancing in all others the opposite change of money in them. We are, however, more interested in the former case of money being stable in general exchange-value, which permits particular changes of prices directly to indicate the true variation in exchange-value in all things, or in all commodities, of the things whose prices change. This can be, if all the changes of prices counterbalance one another, in pairs or otherwise. We shall later see that such counterbalancing is not wholly left to accident—which if it were, the chances might be as small of its ever happening as of the proverbial letters falling out of a bag and composing the Iliad. There is no connection between those letters: there is close connection between all exchange-values.

CHAPTER III.

ON THE MEASUREMENT OF GENERAL EXCHANGE-VALUE.

I.

§ 1. The correlations reviewed in the preceding chapter—the variation of other things in exchange-value consequent upon a variation of any one, and the need of compensation in the variations of other things in order to keep any one thing constant—seem to point to a peculiarity in exchange-value such as to separate it off from other quantities that are subjects of mensuration. In other subjects of mensuration we consider ourselves justified in thinking we have some things possessing fixed quantities of the attribute to be measured, by comparison with which we can measure the constancy or variation of the things of which the constancy is not known or the variation is suspected. Thus in measuring lengths at different periods of time we think we can take the circumference of the earth as a fixed and unchanging length, or even as such the length of any carefully preserved stick at the same temperature, or, more abstrusely, the length of a pendulum swinging in a second at a given spot. But in the measurement of exchange-value at different periods we find that not even all other things always possess a fixed exchange-value, much less any selected one from amongst them. It is as if we lived in a world of gases and were seeking to measure length. In such a world, in which a change in size of one thing would involve a change of all others, perhaps we might succeed in the attempt, perhaps not. In regard to exchange-value, however, the case is not hopeless. In fact, the way out of the difficulty has already been indicated. The other things, taken collectively, are constant in certain circumstances, and in others their collective variation will be measurable when the variation in them of the one thing in question is measured.

The most striking difference is this. In other matters we think we have some fixed quantity of the attribute to be measured, in small and convenient compass, permanently held for us by some material thing, which fixed quantity of the attribute in question can serve as a small unit whereby we may measure other larger or smaller quantities of the same attribute in other things. In matters of exchange we find not even the appearance of any fixed quantity of exchange-value being permanently held for us by any material thing, so that we are not justified even in assuming any fixed quantity to be attached to any material thing we know of; or, in other words, we have no naturally provided material on which we can lay out a small unit of exchange-value serviceable for measuring at different periods (or places) other exchange-values larger or smaller than it. Thus other measurable things at least seem to admit of our forming absolutely fixed small units, by comparison with which we seem to be able to speak of the constancy or variation of absolute quantities of the attribute possessed by other things. And consequently people have come to think of our possessing absolute measures in those subjects. But in exchanges we have no even apparently fixed small unit, so that we seem to be without power of measuring the variations of anything in absolute exchange-value. Exchange-values are recognized as only relative quantities; or exchange-value is viewed as only relatively quantitative. And so exchange-value has been distinguished as merely relative from other quantities, in which it is supposed we have something absolute.

§ 2. Hence it has been the burden of the song of many economists, who sometimes appear to find pleasure in the thought, that the measurement of general exchange-value through the course of time or in distant places is impossible because nature has not provided us with any material permanently and everywhere possessing the same general exchange-value. That nature has not rendered us this service is a fact, although we have seen error in the view that this omission is due to the essential impossibility of anything having fixed exchange-value. Others, again, have despaired of measuring general exchange-value on the ground, which we have seen to be insufficient as well as incorrect, that it is not an attribute in things like weight or color. Still others have denied that we can measure exchange-value at all, for the reason that it is a relation, and that we cannot measure relations.[1] None of these is a good reason for despairing.

The only real difference so far advanced is in the mistaken attitude which people have adopted. In other subjects of mensuration we have nothing more absolute than what we have in the mensuration of exchange-value itself. The differences which exist are differences of degree, and not of kind.

II.

§ 1. That exchange-value is a relative quantity, is not a peculiarity at all. All quantities are relatively quantitative. When a branch two feet long grows to be three feet long, it has changed from twice to thrice the length of a foot-stick. But

[1] F. A. Walker: "Value is a relation. Relations may be expressed, but not measured. You cannot measure the relation of a mile to a furlong; you express it as 8 to 1," *Money*, 1877, p. 288 (repeated in *Money, Trade and Industry*, 1879, pp. [illegible]). The last statement is merely like saying that we cannot measure, but only express, the relation of a dollar to a dime. But this does not prevent us from measuring other exchange-values by dollars or dimes just as we measure other distances by miles or furlongs.—Macleod: "An invariable standard of value . . . is in itself absolutely impossible by the very nature of things. Because value is a ratio: and a single quantity cannot be the measure of a ratio. A measure of length or capacity is a single quantity: and can measure other single quantities . . . But value is a ratio, or a relation: and it is utterly impossible in the very nature of things that a single quantity can measure a ratio, or a relation. It is impossible to say that $a : b :: x$," *Theory of credit*, Vol. I., p. 213. Length is no more, and no less, a single quantity than exchange-value. As exchange value involves a relation between things in exchange, so length involves a relation between points in space. Neither is a mere relation, the one being the distance between two points, the other a power in one thing of purchasing another. Measurement is only of quantitative relations. In measuring length we say that the length of the thing in question is to a foot (the length of certain sticks) as $x : 1$. In measuring exchange-value we say that the exchange-value of the thing in question is to a dollar (the exchange-value of certain bits of metal) as $x : 1$. The processes are exactly alike. We measure one relation by another relation.—Laughlin: "Since value is a relation there never can be an absolutely perfect measure of it. . . . Length is positive, and not a relative thing," *op. cit.*, p. 100. In reality it is only because value, and length also, are relative (quantitative), that they can be measured at all.

the foot-stick itself has changed from a half to a third of the length of the branch. Only here we say, it is the branch, and not the foot-stick, that has changed in length. We do so, as already remarked, because we have other things in mind, and the foot-stick is the one which keeps its length unchanged relatively to them. Of course it cannot be supposed that we know it is the branch which grows because it is living, and the foot-stick is constant because it is dead, wood. For we must know which sticks grow and which are constant, before we can know the distinction between living and dead wood. And we are not endowed with instinctive or innate knowledge as to which it is among things which change relatively to one another that is constant and which it is that is varying. We acquire this knowledge only by comparison with many things, —and comparison is the primary act in measuring. Thus even the mensuration of length is a relatively quantitative affair. We measure the foot-stick by other things before we pronounce it a good measure for measuring other things.[1] There is, then, some standard in this mensuration superior to the material instrument we use as our measure. And so in other subjects of mensuration. What is the nature of this standard?

§2. In measuring some attributes or qualities, we employ a standard foreign to them, which we assume to be a fixed standard, so far as they are concerned. We do so because any variation that might occur in it would be independent of variations in them, and would be the subject of another measurement. Therefore the variations in them, measured in this standard, are regarded as absolute variations, over against their variations in comparison with one another, as relative variations. Yet this view of the absoluteness of the variations measured by this standard, does depend upon the further question as to whether this standard can be found to be really fixed, and so leads us back to a further measurement. Thus in the general subject of value, the cost-value of a thing is to be measured by the quantity of labor required to produce the thing, and its esteem-

[1] T. Mannequin: "We do not measure length in the meter, nor capacity in the liter," *Le problème monétaire*, 1879, p. 28. Of course while using a meter-stick we do not stop to measure it. But we confide in the results of previous measurements.

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… It is the relation be-
… measured and the
… only constancy or
… of some part rel-
… y in that most
… angles. Here
… circle, of which
… an angle, we
… is a part. And
… same relation to
… anything, we

are really comparing it with a larger whole—if not the whole extended universe, the whole earth, or at least the whole of things we see around us. Thus when a branch and a foot-stick change in length relatively to each other, we perceive that the branch changes in length relatively to the earth, and we do not perceive any change in the foot-stick relatively to the earth. To be sure, the foot-stick has changed relatively to one other thing. But we know of other things becoming smaller at the same time, in comparison with which the foot-stick has become larger, and, among the infinitude of things with infinitely different lengths, we are unable to perceive any change in the length of the foot-stick (at the same temperature) compared with the whole of things. If we did (as we do, in fact, at different temperatures), we should have to say that the foot-stick had changed in length.

§ 4. The fixity is not necessarily in the whole itself independently of its parts, nor in the parts independently of the whole. The only fixity we can have knowledge of is in the relation between the parts and the whole.

It is not necessarily in the whole itself; for our universe might expand or contract in size without our knowing it, so long as all particular things, including our bodies, kept the same relationship to the whole, as we perceive it,—and who knows, or who cares, whether the universe of material things is one day half as large as another in so-called absolute size? Such a change would, after all, have to be a change in relation to something else than our material universe—say, another material universe, or something called absolute space. Then if such a change actually did take place, what we should desire for our measure of length would be, not absolute permanence—permanence in relation with that extra-mundane thing,—but relative permanence, in relation to our own universe or world. What happens to our universe and its parts in comparison with something else beside it, in no wise concerns us, and we do not care to measure this relation. Moreover, even if we did find such a change, we could not think of it as being more in our universe than in that other thing, except by comparison with still another thing; and so on without end.

Nor is the fixity necessarily in the parts. In one physical theory our material universe is conceived to be composed of extended atoms, which do not touch one another. Now suppose that all atoms are approaching toward or receding from one another, that is, that the distances between the atoms are enlarging or decreasing compared with the distances through the atoms. Such a change, say of contraction, may be stated in two (among five) typical ways: the one, that the atoms are permanent in size, and the intervening spaces are decreasing; the other, that the intervening spaces (between the centers) are permanent, and the atoms are enlarging—in each case relatively to something else outside our universe. By the physicist the former of these interpretations is the more likely to be adopted, because it fits in better with his preconceptions, which generally do not take into account what is meant by permanence in size—namely, permanence in relation to some other size. Yet, if we could actually see this change going on, we should be as likely to say that the atoms are growing larger as to say that the universe is growing smaller. Practically, however, so long as we remain out of touch with these smallest things, if the things which are the smallest for us, and all the things which we see and feel, including our bodies, grew smaller compared with the atoms in them in exactly the same proportion (or on the average, allowing for relative changes in some things, such as go on anyhow), so that they all (with these exceptions) retained the same relations to the whole, we should want our measure of length, at least for all practical purposes, to decrease with the rest, so as to retain the same relation to the whole as it had before, the sameness of this relation being our standard, although then our measure of length would decrease in size relatively to the atoms, which are, according to the hypothesis, outside our visible and tangible world. Then, practically for us, measured by the standard which we use, it would be the atoms that are growing in size.[2]

§ 5. In a similar manner, when one thing, say a branch,

[2] Probably the measurement of length by the pendulum would indicate a change. But then we would be at a loss as to whether the change were in our usual mensuration of length, or in our mensuration of time.

changes in length before our eyes in comparison with other things, it would be possible to assume that the branch has retained the same size and all the rest of our universe has grown smaller,—that is, relatively to something else beside the universe. But if we adopted this view,—and if it were true,—we should still want our measure of length to go with the rest of our universe, instead of remaining constant with this one thing in its mystical connection with something else about which we know nothing beyond its agreement with this thing. We, therefore, adopt the opposite course, and think, not really of our universe as unchanging, but of the unchanging measure of length as the one which remains permanent in relation to our universe, and as the branch is changing in relation to our universe and to this measure, we think of the branch as changing in length. It is not that we bring in any theory of probabilities, and argue that it is more probable that the branch changes and that the universe does not. What we see is that the branch *does* change relatively to the whole, and to other things which are not seen to change relatively to the whole. While it is doing so, the universe and many things in it do change equally much relatively to it, but we do not see that the universe does change at all relatively to the things, the foot-sticks, which we have chosen for our measures. Hence, we assign the change to the branch, and not to the foot-stick or to the universe. What we measure is a fact, not a probability.

What is here said of length, is true also of other subjects of simple mensuration. So, for instance, in the case of mass of matter. A body has permanent mass, if it always has the same quantity in relation to the whole quantity in our universe. This is our ultimate conception of permanence of mass; for we know nothing about any other permanence of the mass of the whole universe itself, or of any of its parts, which permanence, if it existed, would have to be in relation to something else. If new matter were injected into all things in the same proportion, we might not perceive any change, and if somehow we did learn of it, we should be glad that our measures of mass were increased along with the rest. Similarly in the case of force, in its simplest forms.

But if any of these hypothetical and miraculous changes took place in a part, say a half, of our world, especially if scattered about, we should be at a loss where to place it (whether to conceive of it as an increase in the one half, or as a decrease in the other), and should distribute it over the whole; and then our measure would be some size, or some weight, which has the same relation to the whole as our old measures had.

§ 6. In all these subjects we are somewhat careless of the widest standard theoretically possible, and content ourselves with the most practicable and the most prominent. Thus our ordinary measures are measured, not by their relation to the whole material universe, but by their relation to the earth, or even to particular regions on the earth, and have even been measured with more especial reference to the sizes of our bodies. If, therefore, such changes as just supposed were to take place in the outer stellar regions, we should probably pay no attention to them as regards our measures of length or of mass; for the relation between our measures and our standard for them, the earth, would be unchanged.[3] Again, if something were miraculously increased on our earth, we should probably think it had received matter from the outside, and not alter our conceptions. But if we all woke up some fine morning and found half the things with which we are familiar become larger compared with the other half, and this other half therefore smaller compared with those, then, even if all our measuring sticks happened to remain alike and were constant with the one or the other of these sets, we should have to adapt our measure of length to some average of all these changes; for to say that our measuring sticks had remained constant, would be to take the half of things with which they continue to agree as the constant ones, although we should have no more

[3] At all events in our measurements of things on the earth. For in astronomy we have to make use of other bodies beside the earth in our measurement of distance. It is conceivable that the astronomical and the geographical miles could vary. If the world were still molten and were contracting, we being salamanders that live in fire, we might find this to be the case. Perhaps we could find it even now if our measurements were sufficiently accurate.

right to do this than to take the other half as the constant ones. What we should desire for our standard of length would be some portion of the lengthened things, and some addition to the shortened things, that has the same relation to the whole as our standard measures had before.

§ 7. To be sure, such variations are not found to take place in the lengths and in the weights of things on the face of the earth,—which is why we should regard them as miraculous if such changes should suddenly begin to take place. When we open our eyes every day, what we see contains many things—mountains, plains, rocks, buildings and land-marks of thousands of kinds and descriptions,—that do not appear to alter, do appear to remain the same, in their relative positions and in their relative distances or lengths. And many things, when they are left to themselves, appear, when we weigh them, to have the same weight (relatively to many other similar things) always. Upon these we hit for our standard. Thus the foot-stick, or the pound weight, which we use for measuring, is one which we find not to vary in relation to other things—to be one among the things that do not apparently vary relatively to one another. Yet, although this constancy of so many things relatively to one another obscures, it does not alter, the principle of mensuration. It is only because the foot-stick (at the same temperature) is not perceived to vary relatively to the whole of things, that we are justified in thinking of it as a constant measure. We are not justified in thinking so simply because there are some things which keep constant relations of length amongst themselves. There must be enough of these to form a whole, in which not only our measures, but all the things in which we are interested are included. And in this whole must be included also the things which do vary. And if these things which do vary are not to affect the measure by its variations relatively to them, it must be because of compensations in their variations. For instance, if all the things that vary in length varied only by growing larger relatively to the things that are constant amongst themselves, which then are always becoming smaller relatively to those other things, then (supposing those other things to be sufficiently important to interest us) we should not consider the things stationary amongst themselves to be a good standard by which to measure the others. We should want to include the others in the whole, and the measure would have to increase partly with the increase of the others. We are saved from this need only by the fact that the things which do vary in length relatively to the things stationary amongst themselves, become both larger and smaller in about the same proportions. Trees which grow, also decay and fall or are chopped to bits, and disappear, as others appear. Animals increase, and die away. Clouds form, and fade. Heat expands, and cold contracts. It is only because our foot-sticks, which remain steady in relation to the things that do not change amongst themselves, remain steady also in relation to all these increasing and decreasing things, on the whole, to all appearance, that they are suitable measures of length.

§ 8. Now the exchange-value of everything forms part, with the exchange-value of everything else, of a whole. By comparison with this whole we ought to be able to reach the same kind of fixity in this subject as in other quantitative matters. To be sure, we are here left without the help of things that remain constant relatively to one another. In the economic world there are no mountains, plains, rocks, buildings and other land-marks that retain the same exchange-values relatively to one another. We have only things that change relatively to one another. Yet this difference is only a difference of degree, and not of kind. The relative fixity in length of land-marks *inter se* is a help, and a great help, but not a requirement, in the mensuration of length. The standard is the relation of a part to the whole—or to a practicable whole. This standard we can have in the measurement of exchange-value, as well as of other attributes. The principle of simple mensuration is the same in all cases. In measuring exchange-value what we need is to form a proper conception of the whole of exchange-values. This done, we shall be able to compare with it the exchange-value of anything, in order to find whether this exchange-value is constant or varying in relation to the whole, and how much varying. If it be constant, not only the whole including it, but the whole of all the rest of things, will be constant relatively to it. If it rise, the latter whole will fall ; and reversely ; diverging from the fall or rise of the whole including the thing in a definite proportion. It will, then, be sufficient to be able to compare the exchange-value of any one thing with the exchange-value of all the others.

If we succeed in this, the absence of any particular thing that naturally keeps its relation to the whole permanently unchanged will in no wise prevent us from reaching satisfactory results. We shall have a *method* of measuring exchange-value, in the place of a measuring instrument. Instead of having a bit of material which keeps the unit tolerably constant, we shall be able to lay out a unit of exchange-value upon materials that vary in exchange-value. In doing so, however, we shall do only what is done in all other matters of mensuration when the greatest possible accuracy is desired. The metal used as bearer of the unit of length does not remain constant under changes of temperature, and the engineer must allow for its variation, often using as a foot a length which is not the length of his foot-stick. But more than this. The possibility exists that, although nature does not provide us with a material unvarying in exchange-value, we shall be able to make such a thing for ourselves, with tolerable exactness, and so be in possession of a dollar-bit of metal or paper comparable with a foot-stick of wood or metal.

That exchange-value is not something absolute, is, therefore, no objection against our being able to measure it with as great precision as we can attain to in other subjects of mensuration, since in no subject of mensuration is the attribute itself, which we measure, anything absolute.[4] And what we have of fixity,

[4] J. Garnier: "Unfortunately, all value being *essentially variable*, it follows that there cannot be an *invariable unit* of value, and that we cannot estimate the *absolute* magnitude of the value of things, but only their *relative and comparative* magnitude. . . . The value of this sum of money [by which we attempt to measure the value of a house] is not value existing by itself, abstraction being made of all comparison, and we can form an idea of it only by comparing it with all the things we can obtain in exchange for it [including the house itself]. . . . It follows therefore from the internal nature of value that the search after a (mathe-

or of absoluteness in this sense, in other measurements, we have also in the case of exchange-value, so far as its relativity is concerned—namely, fixity of the relation between a whole and its parts.[5]

§ 9. It is strange how great is the fondness for absoluteness, and how great the dislike of relativity. In subjects wholly relative, where much was once carelessly thought to be absolute, people are fain to retain the absolute in some nook or corner, letting it in by the back-door after driving it out in front. The eminent mathematician and economist, Cournot, drew the distinction that we cannot have an absolute exchange-value, but we can have an absolute variation of exchange-value.[6]

Such a distinction is due to an ambiguity in the term "absolute." The literal meaning of "absolute" is "without relation to anything else." It has acquired the meaning of "really fixed or permanent," or "really invariable in its relations." Besides which, the adverbial form is often used in a merely intensive sense, as when we say "absolutely all," meaning "all without any exception whatsoever." Plainly it is only the second, the acquired, sense that has any importance in matters of mensuration. Nothing—not even the universe, or "absolutely" all things—can be absolutely variable or absolutely constant in size or in any other quantitative attribute, in the original sense.

matically) *exact standard or meter of value is impossible*," *op. cit.*, p. 290 (the italics in the original). This passage finds fault with the mensuration of value for being able to be no more than it ought to be, a measurement of the relative or comparative magnitude of values. It also makes the mistake of implying that the meter, the unit of length, is not only a mathematically exact, but also an absolute, standard, existing by itself, with abstraction of all comparison. Thus the fault found with the mensuration of exchange-value is that it is not what the mensuration of length also is not.

[5] We have seen that esteem-value is measured by esteem, and cost-value by labor-cost. We may notice, in passing, that esteem and cost are to be measured in the same way as above described. We have a certain esteem for all the things we possess. If the things we possess become more numerous, the size of each compared with the whole decreases. So our esteem for each decreases. And with it each thing's esteem-value. Reversely if our possessions grow less. Then every single one becomes larger in relation to the whole; consequently it grows in esteem, and in esteem-value. Similarly with cost-value. If an hour's work comes to produce more things, each of these becomes smaller in relation to the whole product: it falls in labor-cost, and in cost-value.

[6] *Op. cit.*, p. 22.

But, in the acquired sense, a thing can be absolutely permanent as well as absolutely variable in relation to the standard adopted—and if a thing be found absolutely permanent, say in size, its size might just as well be spoken of as absolute—in this secondary sense. Exactly so with exchange-value. Cournot himself distinguished "absolute variation" from "relative variation" only because when two things change in exchange-value relatively to each other, we can conceive of the change as being wholly in the one.[7] He apparently failed to see that in this case we are merely comparing each of the things with all things, and the change is wholly in the one only because this one is changing relatively to all other things, while the other is not changing relatively to all other things. The variation of the one, then, is as relative as the permanence of the other. And if the variation of the one can be called absolute, because of the comparison with the true standard, so can the permanence of the other. And if we can say a thing can be absolutely permanent in exchange-value, we can equally well say it is permanent in absolute exchange-value. All this use of the term "absolute," however, should be avoided when dealing with quantities, quantities being wholly relative. For to use it in the acquired sense involves the risk of importing into it also the literal sense of "without relation."

III.

§ 1. Even the distinction between two kinds of general exchange-value—its division into exchange-value in all other things and exchange-value in all things—is not peculiar to our subject. Thus in our extended universe if something in changing in its size does not affect the size of the whole, this distinction exists; for if, for example, the thing were reduced to half its size in relation to the whole, it would be slightly more than halved in relation to the other things. Or again, if in changing in its size it affects the size of the whole by so much, then, if for example it were halved compared with what the whole used to be, it would be halved compared with all other things,

[7] *Op. cit.*, p. 18.

but not quite halved compared with all things. The distinction may be more plainly seen in the case of motion, which yields a closer analogy with our subject—already used in a partial way.

In a finitely extended universe of points, the whole of which we cannot conceive to move, since there is nothing in relation to which it could move, if all others of its points retain their positions unchanged relatively to one another, the apparent motion of a single point a certain distance (compared, say, with the diameter of the whole) will be its real motion. This is its motion relatively to all *other* points. Again, in a finitely extended material universe, in which its parts have attractions so that for the whole there is a center of gravity, which we cannot conceive to move (except in relation to other parts of the universe), since its motion would represent the combined motion of the whole, and we cannot conceive of the whole as moving, for the same reason as before; then, all other bodies retaining their positions unchanged relatively to one another, the motion of a single body a certain distance through the others in a certain direction would displace the center of gravity in the whole a slight distance in the same direction, and therefore, this center being conceived as unmoved, it would cause the rest of the universe to move a slight distance in the opposite direction (relatively to its center of gravity), and its own apparent motion (relatively to all the others) will not be its real motion (in relation to the center of gravity of the whole), which will be slightly less in the same direction. This is obviously its motion relatively to *all* things, including itself. Thus in the former case, or in the latter also if we conceive of the motion only as compared with all the other things, that motion is comparable with the rise or fall of the exchange-value of a single exchangeable thing in all *other* things; and in the latter case, what we regard as the body's real motion is comparable with the rise or fall of the exchange-value of a single exchangeable thing in *all* things. In both the subjects, when our attention is called to the distinction, we regard the latter point of view as the truer one—the one alone suitable for all measurements,—but the former as also a possible one. In the case of motion a body cannot move without pushing or pulling something else in the opposite direction, and if that something else retains the same position relatively to all other things, all these other things must have been shoved in the opposite direction, though only to a very slight extent, so that the motion of the body in question is not so far in reality as it is in relation to them, since they are moving slightly in the opposite direction. And so with the movement of an exchangeable thing upward or downward in exchange-value relatively to all other things: it pushes them somewhat in the opposite direction, wherefore its motion relatively to them is greater than it really is, as they are moving in the opposite direction.[1] For convenience, however, especially as the opposite motion of all the other things is infinitesimally small (though less so in economics than in physics), we can, if we choose, regard all the *other* things, or even only those of them which remain unchanged relatively to one another, as our standard, and measure not merely, as we do, the motion of a body by its relation only to all or even to some other things, but also the variation of a thing's exchange-value merely by its relation to the *other* things.

§ 2. When we have chosen which method we shall adopt, and what shall be our standard, there is of course no occasion for employing in our measurements the law of probabilities—as was asserted also in this connection by Cournot.[2] We do not say: it is more probable that all the other things have remained stationary than that this one has stood still and they moved; or, it is more probable that all things have together

[1] If the moving body pushed something else equally heavy equally far in the opposite direction, or several things appropriately lesser distances, the motion of this body (no longer the only one moving compared with the rest) would be compensated, so that its apparent motion compared with all the common other things which have remained unchanged amongst themselves would be its real motion compared with all things (including itself), while those common other things would remain unchanged also in their real positions. The similarity with the case of exchange-value as exposited in Proposition XXXIII. is manifest. There is only one difference, which gives greater simplicity to the case of exchange-value. The motion of bodies in space may be in three dimensions; the motion of things in exchange-value can be only in one dimension—hence always only in either the same or opposite directions. In exchange-value there is no parallelogram of forces, except as this is reduced to a straight line.

[2] *Op. cit.*, pp. 15–16. Similarly Bourguin, B. 132, p. 24.

remained stationary, wherefore both this and the others have moved relatively to the whole. But having adopted our point of view, we simply measure, as best we can, what we see happening before us. And our point of view itself in these matters we adopt, not by any use of the law of probabilities, but because the myriad interrelations which do not change, or which do not change on the average, make more impression on us than the particular ones which do change.

IV.

§ 1. Having found these points of resemblance between the mensuration of exchange-value and the mensuration of other ultimate quantitative attributes, let us turn to a difference that is of considerable moment.

The ultimate standard, consisting of a relation between the parts and the whole composed of them, would seem to demand, for its perfection, that the whole should be the same, or exactly similar, whole at both the periods, or at both the places, between which the comparison is instituted. This requirement is observed in the mensuration which we have already noticed as being the most perfect, namely the mensuration of angles. For here the circles with which we compare angles are so exactly similar that we do not hesitate to pronounce them the same, and even speak of all circles as being only one circle. And in physical matters the requirement seems to be satisfied; for our physical world appears every day to be made up of things so exactly alike that we consider them to be the same things, and although there are some new formations and destructions of old things, yet a little induction teaches that the matter in these, or the ultimate bodies composing them, are constantly the same. But in economics the state of things is very different. From age to age, from century to century, even from year to year and from week to week, the economic world is a different world, composed of many things at one time which do not exist at another. For our economic world is only a part of the whole material world, and may draw not only new bodies, but new matter from it, and return old things to it. In the economic world there is creation and annihilation. If particular things, when consumed, were always replaced by similar things, and nothing new were produced, we should always have exactly similar worlds, which is all we want.[1] But not only the particular things appear in different quantities, constituting classes of different sizes, but wholly new classes come into existence at times, and some old ones pass out, or qualities become better or worse, really constituting different classes. And this is not all. At the same time the economic world in one locality is different from the economic world in another.

Here we have what probably constitutes a difference in kind between the subjects of mensuration in economics and in the physical sciences. It may be that there is no creation or annihilation in the material universe, which therefore is the same whole always; wherefore the economic world differs from that whole in kind. And this difference in kind would seem to go over into our mensuration of exchange-value compared with our mensuration of physical attributes, such as length and weight.

§ 2. Yet compared with our practical measurement of physical attributes this generic distinction does not exist in entirety; for we never use the relation between the parts and the whole universe as our standard, but only the relation of the parts to some lesser whole within the larger whole, and this lesser whole, being only a part of the larger whole, may receive new matter from the outside or yield up old matter to the outside. For instance, we measure the weight of bodies by their relation to the earth below them; but through volcanoes the earth below them sends forth matter to the air above them, and from outer space it receives meteorites, so that the whole with which the weight of bodies is compared is a variable one. These changes

[1] For as it is not the other things themselves which constitute a thing's particular exchange-values, but these are its power of purchasing the other things, it is indifferent what the other things are, provided they be alike. The difficulty does not lie in the distinctness of the things, but in their differences. The principle is broad. We do not *know* that the physical things we see every day are the same; but this ignorance is no source of trouble to the physicist.

are relatively so small that their influence is imperceptible to us, and we neglect them. It is worth enquiring, however, what we should do if they were large enough to provoke attention.

It is well known that the meter is supposed to be a definite portion of the circumference of the earth. Now if a small planet, or comet, were to collide with the earth and unite itself to it, perceptibly enlarging our world, and some of us should survive the catastrophe, it is probable that, as already remarked, we should regard the same meter to be, not the same proportion to the new earth, but the same proportion to that part of the new earth which alone constituted the old earth. Or if the planet in colliding with our earth should scoop away some of it, and carry it off into space, we should want our meter to remain the same proportionately to the smaller earth as it was before to this same part of the old earth. In other words, we should disregard accretions and subtractions, and use for our constant whole with which we compare the parts only a whole which exists at both periods compared—a whole common to both the periods.

§ 3. In economic matters it is easy to imitate this procedure. There are even two ways of doing so. The one is this. In measuring the exchange-value of money at two periods when the material constitution of the economic world has varied, we might merely take, in every class of goods, the largest amount of it which exists at both periods. Thus if one class has grown larger, we should cut off what has been added at the second period, and take into account only the quantity of the first period. Or if another class has diminished, we should disregard the surplus which existed at the first period, and take only the quantity of the second period. The world so reduced would be a world which exists at both the periods. And we might do the same if we were comparing two economic worlds at the same time, but in different places.

The other way is this. Instead of taking what is common to both periods in every class separately, we might take what is common to both the periods in all the classes together as a whole. Thus in measuring the exchange-value of money in commodities we should compare the total mass-quantities of goods a given sum of money will purchase at each period in the proportions in which the total sum of money is found to have been spent on the goods at each period. Or, reversely, in comparing the exchange-values, collectively, of all commodities in money, we should compare the total sums of money a given mass-quantity of goods will command at each period, this mass-quantity being composed at each period of classes in the proportions in which the actual mass-quantity of goods is found to have been composed at each period. In order to carry out this method, we shall need to find what constitutes sameness in a mass-quantity of goods at different periods, or in different places.

The second of these methods is conformable to the analogy of physical mensurations. In physical matters our standard whole does not have to be composed of classes of things individually the same or similar at both the periods compared. This is plainer in a more complex case. Suppose it should miraculously (as we should say) happen that some classes of physical things should be enlarged by creation of new bodies and others diminished by annihilation of old ones. Then we should want to eliminate only the excess in the total of the one period over the other. What we want in our standard whole is that it should at all periods be composed of the same amount of substance or material. Now in economic things the substance or material is utility—or the importance we attach to things. Hence, we want our whole at each period to contain the same amount of utility, or importance; and it is indifferent how this is distributed in the various classes of things.

It would seem, then, that even this difficulty, which threatened to be disastrous, may be overcome. The analogy of the mensuration of exchange-value with other kinds of mensuration may again be made perfect. The only peculiarity that remains in the mensuration of exchange-value is that here the wholes given us by nature are not the same at both periods, so that we have the task of reducing them; while in physical subjects the material world seems to be given to us by nature as a constant—or at least as a sufficiently close approximation to a constant. The economic world may be admitted to be different in kind from the physical; yet economic mensuration is the same as the physical.

A certain defect must further exist in measuring the general exchange-value of anything. This is the impossibility of taking into account all the things which possess exchange-value even in ordinarily small economic worlds during even very short periods. There is a necessary confinement of our attention to the exchange-value of money in the most prominent classes of staple commodities. But, again, this defect is not peculiar to the mensuration of exchange-value; for in all mensuration we omit notice of a major part of the universe, and generally our standards are relationships to lesser wholes within the complete whole.

§ 4. It happens, however, that we do not need so much precision in the mensuration of exchange-value as we do in the mensuration of many other quantitative attributes. We need precision in any measurement only to the extent necessary to prevent the discovery of misfits in any subsequent combinations. Thus if a surveyor, measuring round a field, reckons that his two last points are a certain distance apart, and then finds that they are not that distance apart, his first measurements have not been conducted with the accuracy desirable, and corrections must be made. But in the measurement of the exchange-value of money over several periods there are few opportunities for the exercise of correction more sure than the original calculation, if conducted on right principles and with care. At least this is the condition to-day; for there is no knowing what degrees of accuracy may in future be obtainable, and therefore be desired. All that we need at present to strive after is to find the proper method whereby we may rectify the carelessly made calculations, or guesses, which everybody is apt to make—some asserting that money has appreciated, much or little, others that it has depreciated, others again that it has not varied at all,—and to approximate as nearly as possible to the truth, thereby reaching a result which nobody can question.[2]

[2] J. B. Say made the statement that in economics the problem of finding a constant exchange-value is like the problem, in geometry, of squaring the circle, in that both are unattainable with perfect exactness, *op. cit.*, Vol. II., p. 89, cf. *Cours complet d'économie politique pratique*, 2d ed., p. 181. This statement has often been repeated as if it expressed the hopelessness of all attempts to measure exchange-value. Yet if we could attain to anything like the approximation to exactness in measuring exchange-value between two periods—or in measuring anything else—that we can reach in measuring the ratio of the circumference to the diameter, we should have very good reason to congratulate ourselves.—Macleod has altered the simile by saying the search after an invariable standard of value is like the search after the philosopher's stone or perpetual motion, *Elements of political economy*, 1858, p. 171. It is strange that what is not only a legitimate but a necessary problem in economic science should by an economist be likened to things which never were objects of science, but only of cupidity.

CHAPTER IV.

SELECTION AND ARRANGEMENT OF PARTICULAR EXCHANGE-VALUES.

I.

§ 1. In order to compare the general exchange-value of any one thing (generally money) at different periods, a preliminary labor is that of obtaining an expression for its general exchange-value at each period separately. To do this involves two distinct operations. The one is to select and properly arrange the thing's particular exchange-values. The other is to combine these into the thing's one general exchange-value, which they compose. The latter has been the subject of more dispute than the former, which, though also a subject of many discordant opinions, has not received the attention it deserves, and is by no means so easy a problem as it has been taken for. It is the subject which naturally calls for attention first, although it may not admit of complete solution independently of the other.

To obtain with complete theoretical exactness the exchange-value of money—or, to be precise, of a certain sum of money, say the money-unit—at any place during any period—a week, a month, a year,—we ought to take account of every individual thing which has been exchanged at that place during that period, and of its price or exchange-value in money, which will give the money-unit's exchange-value in it, whether it was actually exchanged for money or not. To do this is impossible; and so our practical measurement of general exchange-value during any period is subject to curtailment. The whole which we can employ can only be a part of the total of exchangeable things. But our efforts must be directed at making this practicable whole as large as possible; for, as has been said by an eminent investigator in this subject, "the result is more accurate, the greater the number of the data, and the smaller the number of omitted articles." [1]

§ 2. The curtailment must begin by leaving out of account things which appear only as individuals—such as race horses, paintings, antiques and the like. These individual things must be omitted not only because their number is legion, but because each one is exchanged only occasionally, and so would not appear in all the successive periods, and none can stand for another. Moreover their omission is only a small loss, as, in spite of their great numbers, their sales all told form but a small part of the immense quantity of all sales.

The majority of exchangeable things fall into classes, under generic or specific names, in which all individuals at the same time and place have the same price, or different prices according to different qualities, which form sub-classes; and there is always a succession of similar individuals appearing in the market during every period. These classes we can employ instead of the individual things themselves, and so our labor is already enormously reduced. For when one of the classes, representing the individuals in it, is said to vary in exchange-value or in price, we at once know that all the individuals in it have so varied. The quoted price of a bushel of wheat is not the price of a particular bushel of wheat, but of any bushel of wheat (of the same quality) at the same time and place.

All classes of things, however, do not equally well represent the individuals in them; for in some classes the individuals vary infinitely in quality or in size or in many attributes that go to make them valuable. Thus all complex products, such as machines, buildings, ships, railroads, are too variegated in their individuals to make the logical subsumption of these into classes of much importance for the object we have in view. Therefore these things, as also fancy breeds of animals, precious stones, and most articles of luxury, must be neglected as being little

[1] Edgeworth, B. 60, p. 197.

more than collections of untractable individuals. The omission of complex products is of little consequence, because their exchange-values generally vary in somewhat the same way as do the materials of which they are made or the simple products which they help to make. Land also is such a heterogeneous class, embracing lots and fields and forest districts of infinitely various exchange-values according to situation and natural fertility. The omission of these from an attempt to measure the general exchange-value of money in all exchangeable things is of considerable moment. But if we are seeking rather to establish for the exchange-value of money a standard composed of products, land would not belong to this. Stocks or shares in railroads and industrial companies are not to be counted, partly because the prices, or money costs, of these are not to be counted for the reason just given, and partly because their prices are dependent upon the profits, which are greatly dependent upon the prices of the services or products, already counted. For a similar reason the money cost of transportation of goods ought not to be counted, because it is a factor in the price of goods, and so is already counted in them. But the money cost of transportation of persons, or of travel, ought to be counted. Bonds are not to be counted because their prices depend upon the general rate of interest on the one hand and on the other upon the particular credit of each company, both of which factors have nothing to do with the make-up of the exchange-value of money, although they are both affected by variations in the exchange-value of money, which therefore needs to be measured independently of them.

The classes to be counted are, then, to be confined to so-called raw products and to those things which have been called fungible, namely things sufficiently alike, every one in its own class, to replace one another with indifference on the part of their owners, which are things that can be meted out by weight or other measures, or by the piece. These things include not only raw products—grain, cattle, metals, etc.—but also many manufactured products in a medium stage—steel, flour, yarn, cloth and a few finished products—rope, some simple tools, bread, and even ready-made clothing and shoes, which are now turned out in large quantities in almost uniform grades. All of these classes, which continuously provide a succession of very nearly similar individuals definitely measured and priced, are to be sought for and included. Perhaps two or three hundred such classes may be found, which number is not too large to be dealt with by a board of trained statisticians, collecting data and making calculations in the course of present time.

§ 3. In trying to discover the ups and downs of the exchange-value of money in times past, it is impossible to find continuous price-lists of so many classes, or information about the quantities of the individual things composing them; so that here we must confine our researches to a few. The few then must be chosen as samples. Now serviceable for samples are: (1) only the most important or staple products, (2) only those things whose prices are independent of each other, (3) only those whose prices are not subject to special causes of fluctuation; and, furthermore, (4) some of these articles should not appear twice in the lists, once as a raw material, and again in the manufactured article made of it, while others appear only once. Some economists have even advocated this "exclusive" method for present times. There is no good reason for doing so. This method is only a makeshift, the best we can do in reviewing past times. We can do better for present times, and should, therefore, seek to do so.[1] Some of the early writers were content to measure the variations in the exchange-value of money by comparing it only with one other article, generally wheat, or the most commonly used food product.[2] The line of progress has been to widen the range of the comparison more and more, until it reaches the utmost limit practicable. Now, it should be noticed,

[1] Slightly different is the position of those who advocate a "multiple standard" for contracts, or for regulating the exchange-value of paper money, composed of twenty, of a hundred, or of any other small number of classes, selected for their importance or from other motives. Their position is, not that this procedure yields the best measure of the exchange-value of money, but that it forms a much better standard than the standard resting on one or two metals. Still, while they are about it, they might as well have the largest "multiple standard" possible.

[2] So Locke, Adam Smith, Condillac, Say, D. Stewart, Storch, Cibrario, and others.

the canons suitable for the "exclusive" method have little or no application for the broad "inclusive" method.[3]

§ 4. In all cases only wholesale prices are to be employed. This is because they are the only ones practicable. Objection is often made that it is retail prices which give the exchange-value of money to the consumers. This is not altogether true, as the economic "consumer" is often the purchaser in wholesale quantities, as the manufacturer, who consumes raw materials; and also many large institutions, hotels, governments, etc., mostly get their stores at wholesale prices. Although on some occasions retail prices do not follow the variations of wholesale prices, and may even vary slightly while those are stable, it is probable that the true average of retail prices follows the ascertained average of wholesale prices much more closely than the attempt to reckon the average of retail prices would succeed in following the true average.[4]

II.

§ 1. We have seen that when a class of commodities varies in price, all the individuals in it do so. Now if one class is larger than another, when they both vary in price, the variation of the larger class represents a variation of more individuals, and the variation of money in exchange-value in that class is a variation of it in more individuals, than is its variation in the other class. The particular exchange-value of money in a larger class is not necessarily a higher or greater exchange-value, but it is, so to speak, a wider or larger exchange-value, than its particular exchange-value in a smaller class. Therefore the variation of money in the larger class should count for more, or, which is the same thing, the variation in price of the larger

[3] The distinction is not observed in the following statements: "No article should be scheduled twice in different stages of manufacture," J. B. Martin, *Gold* [illegible], in the Journal of the British Association for the Advancement of [illegible] for the year 1888, p. 626; "We might count the wool instead of the things made of it, (or of course we ought not to count both)," Marshall, B. 93, p. 373.

[4] At all events, if retail prices are used, they must be used in a measurement by themselves. Wholesale and retail prices must never be mixed in one table. [illegible], B. [illegible], pp. 110 111, 114, 116.

class should count for more. The making allowance for the sizes of the classes, which consists in assigning to each class its proper importance or weight in the calculation of the general variation of prices has been called "weighting," and the size assigned to each class has been called its "weight." These not very well chosen terms have become consecrated by usage.[1] We should notice that the idea of *weight* attaching to all classes refers to their influence upon the result in our calculations. The relative weights of the classes in our calculations are not relative weights of the classes *per se*. They are dependent upon the relative *sizes* of the classes *per se*.

§ 2. The prices of the classes of things are quoted in the market lists on mass-units of various kinds and magnitudes, differently in different places and times, as it has been found convenient for merchants to bargain for (*e. g.*, iron by the ton and copper by the pound). The quoted prices of some things are therefore very much higher than those of others, without reference either to the preciousness or to the importance of the classes. If these prices be taken simply as they occur and be combined in a certain way, the operation, as will be shown later, is virtually that of weighting the classes according to the accidental height of their quoted prices at the first period, in a comparison of two or more periods. Thus the weighting here is purely accidental and haphazard, without any principle or reason for assigning more importance to one class than to another, except the chance of mercantile customs, which have grown up without reference to this subject. Such a method of calculating general exchange-value and its variations was employed—of course without knowledge of what they were doing—

[1] The terms have long been in use in the reduction of observations, especially in astronomy. They appear to have been introduced into economics by Jevons, who in his important first work on our subject wrote: "It must be confessed that the exact mode in which preponderance of rising or falling prices ought to be determined is involved in doubt. Ought we to take all commodities on an equal footing in the determination? Ought we to give most weight to those which are least intrinsically variable in value? [Cf. Malthus, above in Chapt. I., Sect. I., Note 8.] Ought we to give additional weight to articles according to their importance, and the total quantities bought and sold? The question, when fully opened, seems to be one that no writer has attempted to decide—nor can I attempt to decide it." B. 22, p. 21.

by one of the earliest investigators in this subject in the eighteenth century, the French financier, Dutot, and in the century just elapsed even by two prominent economists.[2] It was avoided in the middle and toward the end of the prior century by two other investigators, the Italian writer on monetary matters, Carli, and the English physicist, Sir George Shuckburgh Evelyn, the latter of whom introduced the practice of reducing all prices at one period, taken as basis, to 100 (*i. e.*, 100 money-units), whatever be the quantities of goods which are sold at this price in each class, or whatever be the number of times this money's worth of goods of each class are sold during the periods compared. Here, though it is usual to say there is no weighting, there really is weighting, since all the classes are treated as if they were of the same size—there is *even weighting*, which is only next worst after the earlier haphazard weighting. It is, in fact, impossible to avoid assigning one weight or another to the classes reviewed ; wherefore it is plain that the proper thing for us to do is to distribute the weighting with the greatest care possible.

§ 3. Not infrequently it has been asserted that it is shown by experience not to be worth the trouble to assign proper weighting to classes, as the so-called "unweighted" calculations (really with even weighting) yield results very little divergent —so it is claimed—from those reached with proper uneven weighting. Whether it is worth the trouble to be careful, even though the divergence were small, would seem to depend somewhat upon the question who is to take the trouble; for what would excuse a voluntary investigator surveying past events for a merely historical purpose would not excuse a board of official statisticians employed by the State for the practical purpose of providing a guide for contracts or for the regulation of money in the present course of time. But, although the general trend upward or downward in a series of years may be somewhat similar in the two methods, the particular results are often very divergent, as shown by the very calculations upon

[2] See Appendix C, I. As an example of the curious combinations this method, or want of method, may lead to, it may be noticed that in one of his calculations Levasseur allowed 1750 times greater influence to tin than to cotton !

which the other opinion has been founded. Thus in the comparison given by Mr. Palgrave of the *Economist* series of "unweighted" index-numbers and the "weighted" index-numbers calculated upon the same prices, we find the following contrasts:

Year	Evenly weighted	Unevenly weighted
1880	87	89
1881	81	93
1882	83	87
1883	79	88 [3]

Here the calculated movements of general prices go in exactly opposite directions in every sequence of years. Between the first and the second years, for instance, the *Economist* figure falls 7 per cent., and the "corrected" figure rises 4½ per cent., —a difference of 12 per cent. Divergences of this sort are to be seen in every case where in a series of periods the same prices have been treated in both ways for comparison. By the same argument, therefore, by which it has been attempted to show the needlessness of uneven weighting, the need of it is proved.[4]

To assign the "weights" with perfect precision would involve a great amount of labor—principally in discovering the relative sizes of the classes; for the mere introduction of the ascertained weights in the calculations does not much increase the labor. But to assign uneven weighting with approximation to the relative sizes, either over a long series of years or for every period separately, would not require much additional trouble; and even a rough procedure of this sort would yield results far superior to those yielded by even weighting. It is especially absurd to refrain from using roughly reckoned uneven weighting on the ground that it is not accurate, and instead to use even weighting, which is much more inaccurate.

[3] B. 77, pp. 329–330.

[4] We shall, however, later find that both the methods used in the above calculations are very defective, and especially so in the matter of affording comparison between any two years neither of which is the basic year, so that much of their divergences may be due to error even in the "corrected" figures. What difference would, then, exist in practice between the true method and the simple method is still an unknown quantity. But we shall later also find that in the proper way of forming a series of index numbers a slight error in every calculation may, in a wrong method, accumulate before long into large error. Therefore every contrivance to secure accuracy is imperative.

§ 4. Still, although few of the practical investigators have actually employed anything but even weighting, they have almost always recognized the theoretical need of allowing for the relative importance of the different classes ever since this need was first pointed out, near the commencement of the century just ended, by Arthur Young.[5] But the method of measuring the sizes of the classes has been the subject of diverse views, and even the reasons offered for the most commonly adopted method have not been quite satisfactory.

Arthur Young advised simply that the classes should be weighted according to their importance. Some early critics of the plan of judging the value of money "by its relation to the mass of commodities" objected that the importance of articles is different to different persons, and therefore there could be no one standard of this sort.[6] Joseph Lowe yielded so much to this criticism as to recommend for different ranks of society different weightings, and consequently so many different standards; but again he wanted one standard to be formed with weighting according to the importance of the classes for the whole community, as indicated by the total "values" consumed.[7] That the classes should be weighted according to their relative total money-values, has become the prevalent doctrine, being frequently re-invented as if by instinct,[8] and generally with the same explanation. One of the best forms in which

[5] Young did so in opposition to Evelyn's method, which he condemned as manifestly wrong in counting the articles as equally important, B. 6, pp. 68, 70. There have, however, been relapses. Thus Porter, in B. 11, totally ignored uneven weighting, although his attention had been called to it by Tooke, who thereupon declared his table to be misleading because it allowed equal influence in the result to unimportant as to important articles, *Evidence* before the Select Committee on Banks of Issue, 1840, q. 3615. Then a period followed in which even weighting was employed with little notice of the need of anything else, until Drobisch demolished Laspeyres' method for this neglect, B. 30, p. 145, cf. B. 29, pp. 32-33. Laspeyres admitted the theoretical need of uneven weighting, B. 26, p. 304; and since then attention has been paid to it by almost all writers.

[6] Ricardo, *Works*, p. 401; Malthus, *op. cit.*, pp. 119-120. So also Von Jacob, the German translator of Lowe, according to S. D. Horton, *Silver and gold*, Cincinnati, 1876, 2d ed., 1895, p. 39.

[7] B. 8, Appendix, pp. 93-99.

[8] Tooke, *loc. cit.*; Giffen, B. 45, q. 709; Sidgwick, B. 56, p. 66; Palgrave, B. 77, p. 344; Sauerbeck, B. 79, p. 595; Marshall, B. 93, p. 372; Wasserab, B. 105, pp. 87, 89, 94 ff.; Westergaard, B. 110, p. 220; Fonda, B. 127, p. 160.

this explanation has been presented is that the weighting then follows the importance of the articles to the "average consumer."[9] The standard of exchange-value now becomes what has aptly been called a "standard of desiderata."[10]

To this view the objection has sometimes been made that it is too "subjective." An "objective" criterion has been invoked by saying that "more weight should be assigned to those commodities which, being circulated in greater quantities, make greater demand on the currency."[11] Here the measurement of the exchange-value of money is viewed in a peculiar manner. For no one would dream of measuring the exchange-value of any commodity by weighting the other articles in which its particular exchange-values vary merely by the quantities of them actually exchanged for that commodity. In order that one thing have exchange-value in another it is not necessary that the two should be actually exchanged for each other; and the consideration by which the size of the other thing's class is to be judged cannot possibly be confined to the quantity of it exchanged for this thing. There is nothing peculiar here in the case of money. We are not measuring the demand for money, but the general exchange-value of money. This doctrine, therefore, is not so good as the one it seeks to supersede.

In regard to the question of objectivity and subjectivity, it may be noticed that these terms are used in two senses. One of them is that "objective" refers to what is universal to all

[9] Marshall, B. 93, p. 372.—The "average consumer," of course, is the whole community, and not a few samples of "normal families." The method of ascertaining the "average consumer" by means of family budgets (especially if confined to those of day laborers, as often done—and advised by Pomeroy, B. 135, p. 332) is certainly not so accurate as the method seeking the total volume of goods in trade. The investigation of family budgets, introduced by Eden and elaborated by Engel, is very important for sociological studies, but hardly interests us here. The statistician who has used them most in our department is Falkner (see especially B. 111, pp. XL–LV), who defends them in B. 112, pp. 63–64 and B. 113, pp. 269–270. The two methods are discussed by Taussig, B. 121, pp. 24–25.

[10] Horton, *op. cit.*, p. 35; *The silver pound*, 1887, pp. 3–4. That such a standard is one of the objects of these enquiries has been asserted by the British Association Committee, *First Report*, B. 99, p. 249 n.

[11] Edgeworth, B. 66, p. 139. This would lead to a slight change in the weighting, as perceived by Foxwell, who, according to Edgeworth, B. 63, p. 135, wanted weighting to be assigned according to the values of the classes multiplied by the numbers of times the articles in them are sold before being consumed.

men, in distinction from what is peculiar to individual men as "subjective." In this sense the above measurement of the sizes of the classes according to their importance for the whole community is objective. Subjective would be the many standards, one for each person according to the importance of the classes to him individually. Still we desire a measurement, and a reason for it, which are "objective" in the other sense of being out of dependence upon persons. The case is somewhat like the measurement of heat. Heat is subjective, regarded as a feeling in men, though it be in all men. Its variations are also subjective as being differently felt by different persons. But we measure heat without regard to the susceptibilities of individual persons. And more still: we measure it without regard to its universal influence upon all men. A being who never had the sensation of heat, could measure heat. Could not a being who knows not the idea of importance rightly measure out the proper weighting to be assigned to the classes of exchangeable articles?

§ 5. A few economists have held that weighting should be according to the relative masses that are consumed of the different classes, all the masses being measured by the same common mass-unit, a unit of weight being generally preferred.[12] This position claims to be the truly objective. It has the fault that it may be objective in two distinct ways, neither of which has any superiority over the other. For if we measured the masses of the classes by a common measure of capacity, the relative sizes of the classes would turn out differently. It may be replied that it is weight which gives the mass proper, or quantity of matter, contained in commodities. The counter-reply is that capacity is the measure of volume, or size in space; and there is no reason why the quantity of an inscrutable thing called matter should be more considered, in economics, than the quantity of visible space which the objects occupy. We compare

[12] So Drobisch, B. 29, pp. 30, 35, B. 30, pp. 148, 153; Lehr, B. 68, p. 11; Lindsay, B. 114, p. 26; and apparently the British Association Committee, *First Report*, B. 99, pp. 249–250: and Edgeworth, B. 59, p. 264. But statements on this matter are often unprecise, the ambiguous term "quantities" being used. We shall also see that in some cases weighting which appears to be by masses is really by total money-values.

some exchangeable objects by volume or bulk, and consider it an advantage that they be heavy. We compare other exchangeable objects by weight, and consider it an advantage that they be large (and therefore light). We never compare all things either by weight alone or by bulk alone. And furthermore, some exchangeable objects, such as land, have neither weight nor volume, while other things, such as railroad tickets, no one would think of sizing according to their weight or volume; and some gases, which now are commercial articles, and sold by capacity at a certain pressure, it would be difficult to bring into line with solid or liquid things, since the pressure chosen is arbitrary. But to this position a more direct objection will be noticed presently.

III.

§ 1. Such is a brief history of the question of weighting brought down to the present day. It is obvious that the question has not been thoroughly discussed. Even the nature of weighting in general has rarely been understood, the term itself being misleading; and the special difficulties concerning weighting in our subject have never been pointed out, wherefore they have never been overcome.

We shall see that another question which naturally arises in our subject is a question of averaging the variations of exchange-values. Now weighting is a question connected with averaging, and though even weighting is the more common in simple theoretical problems, in practical problems occasion for uneven weighting is always likely to present itself. The question of weighting must be treated first, because weighting of some sort is a prerequisite in all averaging.

The nature of weighting may be illustrated by a simple example. We may suppose that two proud fathers, each with three sons, dispute as to which has the taller sons, and proceed to measure and to average them. The one finds a different figure for the tallness of each son. He simply adds up the three figures, and divides by three. He has used even weighting, allowing equal importance in the calculation to each measurement. The other, let us suppose, finds two of his sons to be equally tall and the other differently tall. He then has only two different figures. The one represents the tallness of two sons, the other the tallness of one son. If he adds these two figures and divides by two, the result would be wrong. If he notes down the figures for every son separately, although the same figure occurs twice, and adds these three figures and divides by three, he would get the right result, and be doing exactly what the other did—he would be using even weighting, with three figures. But if he employs only the two distinct figures, but takes the one representing the two measurements twice, by multiplying it by two, and adds this doubled figure to the other single figure, and divides by three, he would be using uneven weighting; for he would be allowing twice as much influence upon the result to the one figure as to the other. The result is the same as if he employed even weighting with addition of three figures. In this simple example the difference in the labor of performing either of these operations is not great. But in more complex matters it may amount to a great deal. The operation of first multiplying all the similar figures by the number of times they occur and adding their products to one another and to the figures that occur only once, and dividing by the total number of repeated and single figures, is a simpler one than that of adding all the individual figures. Thus uneven weighting, though appearing to be more difficult than even weighting, is really a means of simplifying and abridging the calculation.

Weighting, then, consists in allowing for the number of individuals which possess a common attribute in certain quantities, which are being averaged. Even weighting is employed when every individual is measured, and its measurement, whether the same or different from others, is separately employed in the calculation. Uneven weighting is employed when equal measurements are treated as one, for convenience merely, and yet allowance is made for the number of times they are repeated. Or uneven weighting is to be employed, again, when we make only a few observations on different classes of individuals, and know (or assume) that the other individuals are like those measured; for then the quantity of the attribute possessed by each individual in the class is really to be repeated the number of times it is possessed by an individual, that is, by the number of individuals in the class. Only if the numbers of individuals in all the classes happen to be the same, can even weighting be properly employed.

§ 2. Now in most subjects of averaging, the nature of the individuals in each class is directly given, wherefore the finding of their numbers is not difficult—at least theoretically speaking. Thus, in the above example, the three sons in each set were the individuals in question. Or in the wider case of measuring the average tallness of the population of a country, after getting the average tallness for different districts, we should weight the calculation of the general average of these by the numbers of persons in each district. This is because, by the nature of the problem, every full-grown person, be he prince or pauper, is an individual equally important with every other as a factor in determining the result we are seeking.

In the subject of averaging exchange-values the difficulty which confronts us is apparent. The individuals with which we are dealing in every class of commodities whose prices are reported, are not directly given; for, as already remarked, it is purely accidental what the actual individuals are which have prices recorded against them. The price of wheat is now in America generally reported in bushels. In England it used to be reported in quarters. If these two systems be applied to the same mass of wheat, the former would report the price of eight times as many individual portions of wheat as the latter, the portions referred to in the latter being eight times as large as those in the former. It is obvious that the number of times the price of a bushel of wheat is repeated has no more right to be the "weight" of the wheat-price in our calculations than has the number of times the price of a quarter of wheat is repeated, or reversely, or than the number of any other accidentally chosen mass-unit of wheat. And so with the prices, and the numbers of individuals which bear them, in all other fungible articles. It is, therefore, a problem what are the individuals we are dealing with. This question must be answered before we can find the sizes of the classes, which sizes depend upon the numbers of individuals there are in each class.

§ 3. Here a certain position above reviewed seems to offer aid. This is the position that we ought always to take the prices of the same mass-unit, preferably one of weight, applied to all commodities, and to judge the sizes of the classes by the numbers of this mass-unit they contain. According to this doctrine, the individual in all classes is the same mass-quantity, or the same space-quantity, and the relative sizes of the classes are according to their relative total weights or volumes. The former idea being adopted, the "weights" of the classes in our calculations are to be literally according to the weights of the classes. This is, evidently, a very convenient doctrine.

Against it some objections have already been urged. But the principal one has not yet been noticed. This is that weighting so determined would be according to a different attribute in the things from the one we are measuring. It would be like weighting the calculation of the average tallness of the three sons, above cited, by the weight or girth of each of the boys. It is plain that the relative weight or volume of two classes of commodities does not indicate the relative importance of these two classes. We have, however, agreed not to judge the sizes of classes by their importance. But it is equally plain that the relative weight, or volume, of two classes does not indicate their relative sizes from the point of view in which we have the classes under consideration. Their relative weights indicate their sizes judged by weight. Their relative volumes indicate their sizes judged by volume. What indicates their sizes judged as things possessing exchange-value? Evidently only the relative amounts of the exchange-values of their whole quantities.[1]

[1] Of course in the above examples we would not weight the figures according to the tallness of the boys, or according to the total tallness of the people in the different districts. This is because in those examples the individuals are given, being discrete wholes. The proper comparison is with averaging in the case of things with uniform substance that are found in different sizes. In averaging,

To prove this, we must review the nature of the subject for which we are seeking the weighting.

§ 4. What we are seeking ultimately is the combined result on the general exchange-value of something when we know the variations in its particular exchange-values. We have seen that when a particular exchange-value of the thing rises to a certain height, this rise may be compensated by another of its particular exchange-values falling to a certain extent, although we have not yet found the proportion between these balancing variations. But we have also seen that if two of its other particular exchange-values fall instead of only one, these would each have to fall to a less extent to counterbalance the one rise (in Proposition XXX.). We want, then, to know what constitutes one particular exchange-value, and what constitutes two particular exchange-values—that is, two particular exchange-values that are twice as large (or wide, so to speak) as the one. Thus far we have merely treated of a particular exchange-value as one if it be merely an exchange-value viewed in relation to one other exchange-value, without regard to the size of either of the two classes. This we now find to be not quite adequate. We find the need of another distinction. Let us keep the term "particular-value" in its previous usage, and introduce for what we now want the term "individual exchange-value." A particular exchange-value of a thing is its exchange-value in one other class of things; an individual exchange-value is its exchange-value in one other thing. An individual exchange-value relates to a particular exchange-value very much as a particular exchange-value relates to general exchange-value; for as many particular exchange-values compose one general (or generic) exchange-value, so many individual exchange-values compose one particular (or specific) exchange-value. Our question then is, What constitutes one individual exchange-value, whose variation is to be counterbalanced by an opposite variation in some other individual exchange-value, and by a lesser variation in other two individual exchange-values?

for instance, the yields of different fields (in which the superficies is the element of importance) in order to find the average yield of a country, we should count every field, not as one individual, but according to the relative extent of its superficies.

Now when one thing has a certain exchange-value in another thing, it is equal in exchange-value to the quantity of that other thing by which its exchange-value in it is measured. Hence if the thing 1A is equivalent to b B and to c C (these capital letters referring to mass-units, and the small letters to numbers), we see that when we speak of A's rise in exchange-value in B being compensated by its fall in exchange-value in C, we have in mind its rise from equivalence to b B and its fall from equivalence to c C, wherefore these are the two individual exchange-values we are comparing with each other at that period. Therefore also the two individual things in the classes B and C with which we are comparing 1A are b B and c C—its equivalents at the period in question. Similarly if 1A be equivalent to c D, the economic individual in the class D with which we are comparing 1A is c D. Suppose the rise of 1A to equivalence with $b'b$ B is compensated by the fall of 1A to equivalence with $\frac{1}{c'}$ c C (b' and c' being certain figures larger than unity). Then if A falls in D also, the balance is disturbed. Therefore, for the balance to be maintained, when A falls in C and in D equally, it must fall in each less than by $\frac{1}{c'}$. Now suppose the individuals c C and c D are exactly alike, so that we put them in the same class and refer to them by the same term, say Z. Then when A falls in Z, it falls both in C and in D equally. Therefore, to counterbalance its rise in B, it must fall less in Z than it would have to fall in C alone or in D alone.[2] It is obvious that the class Z is larger than, and just twice as large as, the class B, at the period in question, because it embraces twice as many individuals as the class B, since it embraces two, namely c C and c D, while B embraces only one, b B. And so if B and Z are variously large

[2] Proposition XXX., it should be noticed, may apply both to other classes and to other individuals. It may mean that when the compensatory changes are in a greater number of (equal) classes, they must be smaller, and when they are in a greater number of (equal) individuals (in the same class), they must be smaller. The reason is that in the former case as in the latter the compensation is distributed over more individual things. It is plainly indifferent whether the individuals, their number being given, be all in one class or in many classes. The important factor is, not the number of classes, but the number of individuals.

classes, their relative sizes will be determined by the numbers of *b* B's and of *c* Z's they contain—that is, the numbers of equivalents to 1A. Generalizing still more, we see that it is the number of equivalents to 1A in any class, which constitutes its relative size, these equivalents to 1A being the *economic individuals* in them corresponding to the posited economic individual 1A. And, of course, whatever be the quantity 1A, a large or a small mass-unit, the proportions of the numbers of equivalents to 1A in the other classes, and consequently of their sizes, will be the same relatively to one another and to the class A. Hence if instead of 1A we use 1M—a money-unit, say one dollar,—then the relative sizes of all the classes of commodities are proportioned to the "dollar's worths" they contain, which are indicated by the total money-values of these classes.

All this reasoning is obviously independent of any consideration of the mass-quantities of *b* B and of *c* C, supposed to be equivalent to 1A, or of the mass-units in which they are expressed. If 1A is one pound of wheat, and this happened to be equivalent to one pound of butter as *b* B and to one pound of wool as *c* C, this would in no wise improve the reasoning, which would be the same if the one pound of wheat happened to be equivalent to two pounds of butter and to seven and three-quarter pounds of wool. On the other hand, if this last happens to be the case, it is plain that one pound of butter and one pound of wool have no claim to be considered economic individuals compared with one pound of wheat. As masses, one pound of butter and one pound of wool are individuals compared with one pound of wheat, because equal to it in weight. But we are not dealing with masses. We are dealing with exchange-values. And the exchange-value of one pound of wheat being taken as our given individual, it is only the quantities of butter and of wool whose exchange values are equal to this exchange-value, at the period in question, that are individuals compared with it, in our economic point of view.

Or if we compare classes with money, the state of things is still plainer. Suppose copper is worth ten cents a pound and iron one cent a pound. It is evident that when we compare a variation of the exchange-value of one dollar in copper with a variation of the exchange-value of one dollar in iron, we are comparing its variation relatively to ten pounds of copper and to a hundred pounds of iron. And if during a certain period the masses of copper and of iron with which we are dealing are ten million pounds of the former and fifty million pounds of the latter, the relative sizes of these classes—the relative numbers of individual exchange-values of 1M in each of these classes, that vary when its price-quotation varies—is, at the period in question, not as ten for copper to fifty for iron (or 1 to 5), but as one hundred for copper to fifty for iron (or 2 to 1).

It will be noticed that this explanation does away with dependence upon the idea of importance. The sizes of the classes are not measured by their importance to the consumers. They are measured by the numbers of equivalents they contain, these equivalents being the economic individuals, and the sizes of the classes naturally being according to the numbers of individuals they contain. It is true that the importance of the classes is measured in exactly the same way, so that the relative sizes of the classes go hand in hand with their relative importance—and we may continue to speak of their sizes being according to their importance. But it is not directly by their importance that the measurement of their sizes is made. It is made by the measurement of the above equivalent individuals they contain. This measurement is just as objective as the measurement by the number of equiponderant individuals they contain.

§ 5. It should be noticed, further, that the size of the class whose exchange-value is being measured is of no consequence. and therefore need not be considered, except only in the one case when we are seeking to measure a variation of its exchange-value in *all* things. For if we are seeking to measure its variation in exchange-value in all *other* things, or if we find that it has remained constant in exchange-value in all other things (and consequently in all things), we have to consider only the relative sizes of the *other* classes, the size of the class in question having no influence in the result. Thus in measuring the variations in the exchange-value of money, we are dispensed from enquiring about the size of the class money itself, so long as we confine ourselves to measuring its variations in general exchange-value in all other things. To this we shall confine ourselves for the greater part of our researches, and shall treat of the more difficult problem of measuring variations in the general exchange-value of money in all things only at the end of our work, where, too, we shall consider whether such a measurement is needed or not.

IV:

§ 1. The number of individuals or equivalents in each class during a given period is, of course, not to be measured by the number of them in existence at any one moment. For articles are variously durable, and the stocks on hand at any moment do not represent the relative quantities in which they are ordinarily used. These quantities can only be ascertained by taking all the quantities brought into trade during a period sufficiently long to cover all the ups and downs of the stocks—at least a year as being the shortest natural cycle. If land be one of the things taken into account, not its total money-value, nor even the total money-value sold during a year, but its annual rental is the item to be compared with the total money-values of the commodity classes.[1]

§ 2. It is a question whether the total money-values of commodity classes to be considered are those of the quantities annually produced or those of the quantities annually consumed; for in every country importation and exportation disturb the equality between production and consumption in the case of many articles. Perhaps it would be best to count both—that is, in any one case either the total product or the total consumption according as the one or the other is the greater. Thus in the United States we should count all the wheat produced and all the tin consumed, and in England all the wheat consumed and all the tin produced. There is overlapping here; but this is right, because the English and the American economic worlds

[1] For a similar reason it is only the variations of rent, and not the variations of the price of land, that are to be counted.

do overlap. This double counting, which takes place in the two measurements, could be avoided by making the measurement of the exchange-value of money in the two countries together, if that be possible; but then there would be similar overlapping with other countries, which could be avoided again only by making the measurement for the whole world at large—for then we should have to count only the total quantities produced *or* consumed, without counting exports and imports twice.[2] Of course, the total money-values must not be confined to the total money-values sold. A bushel of wheat consumed by a farmer is as valuable as a bushel of wheat which he sells.

§ 3. Articles that pass through various stages and appear in each under a special name—as hemp, rope; wheat, flour, bread; wool, yarn, cloth, clothing,—should not be counted as a distinct quantity in each stage. If that were done, as some articles appear in more forms than others, the sizes of these would be unduly magnified. One method of treating these things would be to group together all the various stages, and to assign to this group a size according to the total value of its highest forms (or the highest counted). Another would be to confine attention to two stages only. But in the second stage there might be several branches, according to the kinds of things the raw material is made into.

§ 4. To find with accuracy the total money-value and the average price of any commodity class during a given period is a statistical task of considerable difficulty; but the theory of the operation is simple. Of all the prices reported of the same kind of article the average to be drawn is the arithmetic;[3] and the prices should be weighted according to the relative mass-quantities that were sold at them. For example, if there is sold twice as much cloth in September as in July, the price in Sep-

[2] Some statisticians have counted only these overlappings, and have weighted the classes according to their relative sizes in the country's exports and imports. If this weighting be employed merely in an attempt to measure the volume of foreign trade, as done by Giffen and De Foville, less objection can be found. But if it be employed in a measurement of the general exchange-value of money, it is a very defective kind of weighting. Its defect has been noticed by Nasse, B. 104, p. 332.

[3] Jevons, B. 15, pp. 41, 43; Walras, B. 61, p. 6.

tember should be given twice as much weight as the price in July.[4]

V.

§ 1. As yet we have examined the sizes of the classes only at a single period. But in any enquiry into the variation of the general exchange-value of anything we are concerned with at least two periods. Between the two periods not only the particular exchange-values may have varied, but also the sizes of the classes. And the sizes of the classes vary partly because of the very variations in the particular exchange-values whose common variation we are attempting to measure. The difficulty in the subject of weighting, in the case of averaging exchange-values, is only commencing.

The older writers, from Arthur Young on, who touched upon weighting, recommended merely a rough weighting according to the importance of the classes as vaguely measured, by their relative total money-values, or by masses, over a number of years, although a few, using custom-house returns, actually did use varying weighting, without understanding its nature.[1] The need of paying attention more minutely to the weighting of each period compared was first pointed out about thirty years ago by the German philosopher and mathematician, Drobisch. Drobisch was the originator of the idea that we should weight the classes according to their actual physical weights, and he confined his

[4] E. Segnitz gave this formula for finding one average price,

$$P = \frac{pq + p'q' + p''q'' + \cdots}{q + q' + q'' + \cdots},$$

in which $p\,p'\,p''$ represent the prices at which respectively $q\,q'\,q''$ quantities of the article sold during the period in question, *Ueber die Berechnung der sogenannten Mittel sowie deren Anwendung in der Statistik und anderen Erfahrungswissenschaften*, Jahrbücher für Nationaloekonomie und Statistik, 1870, Band XIV., p. 184. E. Heitz, though not approving of it himself, says this method is employed in the statistical bureaus of several German states (Würtemberg, Bavaria, Gotha, and partly in Prussia and Hannover), *Ueber die Methoden bei Erhebung der Preisen*, in the same Jahrbücher, 1876, Band XXVII., pp. 347–351. This use of uneven weighting has been recommended also by Marshall, B. 93, p. 374, and the British Association Committee, *First Report*, B. 99, pp. 250–251.—Weighting according to the length of time each price lasts was recommended by Beaujon, B. 95, p. 116.

[1] See Appendix C IV § 2 (2).

attention to differences at the two periods between the mass-quantities of the classes. Considering the single case of a comparison between two contiguous periods, he raised the following questions:—Shall we employ in our calculations the mass-quantities of both the periods separately? or only the mass-quantities of one of the periods? and in the latter case shall it be those of the first period, or those of the second? or, again, shall we combine these two systems, and employ a single one that is a mean between them? Between these questions Drobisch decided in favor of the first course suggested. In doing so he originated what may be called a method of *double weighting*. This consists in drawing an average of the prices at each period separately, and at each period on the mass-quantities of that period; and in then comparing these averages. At each period compared the mass-quantities may be measured in different ways. The way Drobisch chose was, as we have seen, to measure them by the number in every class of a common mass-unit. Upon the method of double weighting he decided on the ground that the mass-quantities of neither of the two periods is preferable the one to the other,[2] and that there is no better reason for the mean between the two,[3] apparently considering that the course he adopted was the only one not exposed to objections. Drobisch has found but few followers in the use of double weighting, though some involuntary ones are among them.[4] His position was immediately attacked by Professor Laspeyres, for a reason which will be noticed later—a reason, however, which is valid, not against his use of double weighting, but against his use of a common weight-unit; and a little later by Paasche, on the ground that we must use the mass-quantities of only one period so as to get the variation in the sums of money needed to purchase the same mass-quantities at both the periods.[5] And a similar reason has been given more recently by Professor Falkner, namely that "we must compare like with like,"[6]—although,

[2] B. 29, p. 39.
[3] B. 31, pp. 423–425.
[4] See Appendix C V.
[5] B. 33, pp. 171–173.
[6] B. 112, p. 63.

if the like does not exist at both the periods, the comparison may appear somewhat forced. Laspeyres recommended using the mass-quantities of the first period, Paasche those of the second.[7] The mean between these two positions may be either by making two calculations, the one on the mass-quantities of the first period, the other on those of the second, and then taking the (arithmetic) mean between the two results; or by making a single calculation with mass-quantities that are the (arithmetic) mean between those of the two periods. The last has sometimes been recommended, though only half-heartedly.[8] Thus every obvious position seems to have been occupied. Amidst all this diversity of opinion Professor Sidgwick has asserted that as we cannot decide in favor of any of these methods, the use of the mass-quantities of the one period being as good as of the other, and the mean between them having no "practical significance," and as the answers obtained by each method may differ, we can therefore make no one authoritative measurement of the variation of money's general exchange-value.[9] And the same criticism has recently been reaffirmed by Dr. Wicksell, who says that the problem is insolvable unless it happens that the same result is yielded by using the mass-quantities of each period separately, the use of the mean between the mass-quantities at the two periods having nothing but a "conventional meaning." [10]

The truth is, however, that these are not the only positions that present themselves, and the subject has hitherto been treated most inadequately. Like Drobisch, the writers who have noticed this question have considered only the difference at the two periods between the mass-quantities. But it is strange that persons who have not followed Drobisch in adopting mass-quantities as the test of weighting, and who have even asserted, many of them, that weighting should be according to the total exchange-values of the classes, when they come to discuss the question concerning divergent weights at the two periods, should confine

[7] Pasche has been followed by v. d. Borght and Conrad, and without dependence by Mulhall, by Sauerbeck at times, and, following the latter, by Powers. See Appendix C IV § 2 (2).

[8] See Appendix C IV § 2 (3).

[9] B. 56, pp. 67–68.

[10] B. 139, pp. 8–9.

their attention to differences that may exist between the mass-quantities at the two periods.[11] Changes in the mass-quantities, and changes in the weights, of the classes are two distinct things, and the former are of importance only as they affect the latter. Since no one has so much as noticed the question in its proper aspect, we are left to our own devices, and must investigate the subject *ab ovo*.

§ 2. There are three variables: (1) the exchange-values, or prices, of given mass-units of many classes of commodities; (2) the total mass-quantities of these classes; (3) their total exchange-values, or money-values. The first are the ones we are trying to average. The second and third are factors in the weighting of the classes in our averaging of their exchange-values or prices, or of the variations of these. Yet the third itself is dependent upon the first two, being their product. As we are now examining only the factors that enter into the weighting of the classes, we are not concerned with the first variations except so far as they affect the third.

With the two variables that enter into weighting we have, then, two distinct elemental cases, and a third composed of these two. The first, and simplest, is when the total sums spent on every class remain constant over both the periods, so that the relative sizes of the classes remain the same at both the periods, the variations of the prices being counteracted by inverse variations of the mass-quantities. The second is when the mass-quantities in every class remain constant over both the periods, so that the sizes of the classes vary from period to period directly as the prices. The third is when both the mass-quantities and the total sums spent on them vary from period to period.

§ 3. I. In this simple case the relative importance, or the relative sizes, of the classes remain constant at each of the periods, whether the exchange-value of money be the same at both the periods or not. It would therefore seem natural to treat the classes as having, in our averaging of their price-variations, the relative sizes that they have at each of the periods.

[11] Giffen and Palgrave are exceptions. The former used weighting according to the money-values of a single period, the latter according to those of every later period. But each has done so without discussing the question.

Thus, for example, if at the first period the total money-value of the class A is twice as great as the total money-value of the class B, and if at the second period each of these classes has the same total money-value as before,—or even if they have both varied in their total money-values, but in the same proportion, so that the class A still has twice the money-value (and consequently twice the exchange-value, as may be deduced from Proposition VII.) as the class B,—we should look upon the class A as twice as large, or as containing twice as many economic individuals, as the class B.

Yet each of the economic individuals in these classes would consist at each period of a different mass-quantity of the article. When this is perceived, we may hesitate to adopt this position. We may best discuss this question, however, when we are examining the more general and more complex cases where both the total exchange-values and the total mass-quantities are variable.

§ 4. II. The next simplest case is when it happens that the mass-quantities of the various classes remain constant over both the periods, and their prices varying, their total money-values are different at each period. It is plain that this suppositional case escapes all the questions that have hitherto been raised on this subject, above reviewed; and yet the question of weighting is still before us. For the sizes of the classes are different at each of the periods directly according to the variations in their relative prices. Hence we have alternative positions like those already noticed, except that here there is no opportunity for double weighting. Shall we use the weighting of only one of the periods? and then shall it be that of the first or that of the second period? or shall we use a mean between the two systems? And there is still another possible position. Shall we use only the smaller total money-value that occurs at either period as the weight of the class for both the periods together, eliminating what is in excess at one of the periods?

Plainly there is no reason why in general we should choose the weighting of one of the periods in preference to that of the other. And there is good reason for rejecting each of them—

in general, unless later discussion of the averages shows that the one fits one of the averages and the other another. For if one class happens to be more important, or larger, than another at the first period, and at the second, through a fall in the price of the former and a rise in the price of the latter, is less important, or smaller, it would be absurd to take either of these conditions as alone representative of the proper relationship between these classes. Or if two classes are equally large in total exchange-value at the first period and the one becomes larger and the other smaller at the second period, and if two other classes, at the first period the one larger and the other smaller, become equal in size at the second period, it is plain that we have no better right to treat the classes in the one of these pairs as equal in size than to treat the classes in the other as such; but it is impossible to do so with both, if we are to use the weighting of only one period—all which shows that the classes in neither of these pairs are to be treated as equal, yet not so unequal as in each they are at one of the periods.[12]

§ 5. Therefore we must use either a mean between the weighting of the two periods, or the weights partly of the one and partly of the other, picking out the smaller weight of a class at either period. The reason for entertaining such a position as the last is obtained from the preceding Chapter. We have seen a desideratum to be that we should have the same or a similar whole at both periods, in which to compare the relationship between a part and the whole of which it is a part. An economic individual being an exchange-value quantum rather than a mass quantum, it may be maintained that we ought to eliminate all those individuals which exist at one period alone, which can be done by taking the total exchange-value that is the smaller at either period. The mass-quantities being supposed in all cases constant over both periods, if the class A rises in exchange-

[12] It may be objected that it is only the individuals at the first period that vary, the individuals at the second period being merely a result of the variation of the former. But really the variation is *from* the individuals at the first period *to* the individuals at the second period; and the number of individuals that have varied is not the number existing at the first period any more than it is the number existing at the second.

value between the first and the second period, its total exchange-value increases, and only that which it had at the first period should be considered; and if the class B falls in exchange-value, it contained at the first period more economic individuals than at the second, and so only those existing at the second period should be considered.

A practical objection to this position is that the economic individuals at each period should be the same, that is, have the same exchange-value, or be equivalents to a money-unit having the same exchange-value at both the periods; but whether the money-unit has the same exchange-value at the second period as at the first, is the question to be determined. Merely to compare the total money-value of the classes at the two periods is not to compare their total exchange-values; yet that is all we can do till we have found the constancy or variation of the exchange-value of money. It is plain that if the exchange-value of money be falling, *i. e.*, if prices in general be rising, the smaller total money-values might all, or most of them, occur at the first period, and then in selecting them for our weights, we should be taking merely, or mostly, the weights of the first period; or if the exchange-value of money be rising, *i. e.*, if prices in general be falling, the weights might be mostly those of the second period. Or between one country and another, where, the mass-quantities being the same, the levels of prices are considerably different, we might be really taking the weights almost altogether from those of one of the countries alone, namely the one in which the level of prices is the lower. It is perhaps possible that we might approximate to the ultimate result by several stages of approach. Assuming at first that money is stable, we should take the total money-value as representing the total exchange-values of the classes, and, choosing the smaller ones at either period, should use these as the weights of the classes; then if the result showed constancy in the exchange-value of money, we should rest content; but if it showed a variation, we should take this as a means of correcting the estimate of the total exchange-values of the classes at the second period compared with what they were at the first, and repeat the proc-

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classes, their relative sizes will be determined by the numbers of b B's and of c Z's they contain—that is, the numbers of equivalents to 1 A. Generalizing still more, we see that it is the number of equivalents to 1A in any class, which constitutes its relative size, these equivalents to 1A being the *economic individuals* in them corresponding to the posited economic individual 1A. And, of course, whatever be the quantity 1A, a large or a small mass-unit, the proportions of the numbers of equivalents to 1A in the other classes, and consequently of their sizes, will be the same relatively to one another and to the class A. Hence if instead of 1A we use 1M—a money-unit, say one dollar,—then the relative sizes of all the classes of commodities are proportioned to the "dollar's worths" they contain, which are indicated by the total money-values of these classes.

All this reasoning is obviously independent of any consideration of the mass-quantities of *b* B and of *c* C, supposed to be equivalent to 1A, or of the mass-units in which they are expressed. If 1A is one pound of wheat, and this happened to be equivalent to one pound of butter as *b* B and to one pound of wool as *c* C, this would in no wise improve the reasoning, which would be the same if the one pound of wheat happened to be equivalent to two pounds of butter and to seven and three-quarter pounds of wool. On the other hand, if this last happens to be the case, it is plain that one pound of butter and one pound of wool have no claim to be considered economic individuals compared with one pound of wheat. As masses, one pound of butter and one pound of wool are individuals compared with one pound of wheat, because equal to it in weight. But we are not dealing with masses. We are dealing with exchange-values. And the exchange-value of one pound of wheat being taken as our given individual, it is only the quantities of butter and of wool whose exchange values are equal to this exchange-value, at the period in question, that are individuals compared with it, in our economic point of view.

Or if we compare classes with money, the state of things is still plainer. Suppose copper is worth ten cents a pound and iron one cent a pound. It is evident that when we compare a the second period let the classs A be r_2 times as large as the class B and s_2 times as large as the class C. Then at this period the class B is $\frac{s_2}{r_2}$ times as large as the class C. Now if the arithmetic mean be used in averaging these relations in order to get the size relationship for both the periods together, we should have to say that A is $\frac{1}{2}(r_1 + r_2)$ times as large as B and $\frac{1}{2}(s_1 + s_2)$ times as large as C. Then if we draw the relationship between B and C from these relations, we should have to say that B is $\frac{s_1 + s_2}{r_1 + r_2}$ times as large as C. But if we draw the relationship between B and C in the same way from the first separately given relations between B and C, we should have to say that B is $\frac{1}{2}\left(\frac{s_1}{r_1} + \frac{s_2}{r_2}\right)$ times as large as C. But this expression is no the same as the preceding, and is equal to it only in special ci cumstances $\left(\text{namely if } r_1 = r_2, \text{ or if } \frac{s_1}{r_1} = \frac{s_2}{r_2}\right)$. Therefore t employment of the arithmetic mean involves an inconsisten which shows it to be absurd. The geometric mean is f from this inconsistency. This mean being employed, the relationships for both the periods together are to be expre by saying that A is $\sqrt{r_1 \cdot r_2}$ times as large as B and $\sqrt{}$ times as large as C, wherefore, using these relations, we say that B is $\frac{\sqrt{s_1 \cdot s_2}}{\sqrt{r_1 \cdot r_2}}$ times as large as C. And if we use first separately given relations between B and C, we should to say that B is $\sqrt{\frac{s_1}{r_1} \cdot \frac{s_2}{r_2}}$ times as large as C; which expr

... preceding only in form, and is universally the same as if we compared every class with every other in the way described. This may be shown as follows. The total mass-quantity of the class A, which is supposed to be the same at both the periods, may be represented by x, and the total mass-quantity of the class B by y (x and y standing for certain numbers of certain mass-units of A and B, what these mass-units are being indifferent). And the price (of the mass-unit before used for the quantity) of A may be represented by α_1 at the first period and by α_2 at the second; and the price (of the mass-unit before used for the quantity) of B may be represented by β_1 at the first period and by β_2 at the second. Then the weight of the class A is $x\alpha_1$ for the first period, and for the second $x\alpha_2$; and the weight of the class B is $y\beta_1$ for the first period, and for the second $y\beta_2$. Let $x\alpha_1 = r_1 \cdot y\beta_1$, and $x\alpha_2 = r_2 \cdot y\beta_2$. Then by the above method, the weight of the class A for both the periods is $\sqrt{r_1 \cdot r_2}$ times the weight of the class B. Now the geometric mean of the weights at both periods of the class B is $\sqrt{y\beta_1 \cdot y\beta_2} = y\sqrt{\beta_1\beta_2}$, and that of the two weights of the class A is $\sqrt{r_1 y\beta_1 \cdot r_2 y\beta_2} = y\sqrt{\beta_1\beta_2 \cdot r_1 r_2}$. The relationship between these is

$$\frac{\text{weight of A for both periods}}{\text{weight of B for both periods}} = \frac{y\sqrt{\beta_1\beta_2 \cdot r_1 r_2}}{y\sqrt{\beta_1\beta_2}} = \sqrt{r_1 r_2},$$

the same as before. What is here demonstrated of two classes, may be demonstrated in the same way of three classes, of four classes, and of any number of classes.

It is plain also that this method is unaffected by the possible variations in the exchange-value of money, or the general level of prices. For, statically considered, the variation of the exchange-value of money affects all prices alike. Then, at the second period, if the prices of the mass-unit of A is $t\alpha_2$ instead of α_2, the price of the mass-unit of B will be $t\beta_2$ instead of β_2, and so on. This t will appear as $\sqrt{t}$ in the weight of every class for both the periods together; and so will not alter the relationship between these weights, disappearing in the comparisons just as $\sqrt{\beta_1\beta_2}$ has done. Hence even weighting is here really carried out in our averaging of the weights of every class at both periods, or in two countries.

§ 7. We may illustrate this position by an example, which will not only render it clearer, but will also disclose certain important details. Suppose at both periods we have seven pounds of A and seven pounds of B (or any greater quantities in the same proportion), and that at the first period each of these seven pounds is valued at 100 dollars, but that at the second period the money-value of the seven pounds of A is 196 dollars, and the money-value of the seven pounds of B is 4 dollars. (Such extreme variations are purposely chosen, because they bring out the distinction more clearly.) Then, for both the periods together the class A is $\sqrt{\frac{19600}{400}} = 7$ times larger than the class B, —which relationship is also shown by the fact that $\sqrt{19600} = 140$ and $\sqrt{400} = 20$, and $\frac{140}{20} = 7$. Then it is proper to say that over these two periods together the class A contains seven economic individuals, and the class B one economic individual; which means that the economic individual in A consists of one pound of A and the economic individual in B consists of seven pounds of B. Now the money-value, or price, of the economic individual in A, namely one pound of A, is $14.2857 at the first period, and the second $28.00, while the money-value, or price, of the economic individual in B, namely seven pounds of B, is $100.00 at the first period, and at the second $4.00. Thus the individual in A is at the first period one seventh as large, and at the second seven times as large, as the individual in B; for $14.2857 = \frac{100}{7}$, and $28 = 4 \times 7$. And the geometric mean price of these individuals at both the periods is the same, being in each case $20.00; for $\sqrt{14.2857 \times 28} = \sqrt{100 \times 4} = 20$.

These relations are universal, as may be demonstrated in the following simple manner. Using the same symbols as above, the total money-values of the class A are $x\alpha_1$ at the first period and $x\alpha_2$ at the second, and those of the class B $y\beta_1$ and $y\beta_2$ at the two periods respectively. Therefore, the class A is $\frac{x}{y}\sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}}$

times larger (or smaller) than the class B, or, in other words, if the class B be regarded as containing one economic individual, being taken, so to speak, as the unit-class, the class B contains $\frac{x}{y}\sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}}$ individuals. Then the money-value, or price, of the individual in the class A is $\frac{x\alpha_1}{\frac{x}{y}\sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}}} = \sqrt{\frac{\alpha_1}{\alpha_2}} \cdot y\sqrt{\beta_1\beta_2}$ at the first period, and at the second $\frac{x\alpha_2}{\frac{x}{y}\sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}}} = \sqrt{\frac{\alpha_2}{\alpha_1}} \cdot y\sqrt{\beta_1\beta_2}$; while the money-value, or price, of the individual in the class B is at the first period $y\beta_1$ and at the second $y\beta_2$. Now $\sqrt{\frac{\alpha_1}{\alpha_2}} \cdot y\sqrt{\beta_1\beta_2} : y\beta_1 :: y\beta_2 : \sqrt{\frac{\alpha_2}{\alpha_1}} \cdot y\sqrt{\beta_1\beta_2}$; and the geometric mean of the prices of the individual in A at both the periods and the geometric mean of the prices of the individual in B at both the periods are both $y\sqrt{\beta_1\beta_2}$. What is so demonstrated of two classes, can be demonstrated of any number of classes.

A consequence of these relations is that if we suppose the prices of these individuals in A and in B, etc., to vary at a uniform rate (that is, at the same percentage, compounding), their prices will pass the geometric mean, the same in all cases, at the same moment, which is the timal half-way point, and they will then be actually of the same size or importance (in exchange-value), and at any other moment their sizes will be above and below this in corresponding proportions. These, however, are imaginary or ideal relationships, as we cannot expect prices to vary uniformly, and if the periods be contiguous, the price-variations are really within each period, and not between them. Still they are the relationships we should demand when we conceive of the problem in its ideal conditions.

We have, then, four reasons for preferring the geometric mean between the total money-values at the two periods, namely, to recapitulate: (1) the geometric mean avoids an inconsistency into which the arithmetic mean falls; (2) it alone[13] is unaffected by variations or differences in the exchange-value of money; (3) according to it alone the economic individuals in the classes have inverted relationships at each of the periods, and (4) coincide in size or importance at the timal half-way point, if their price-variations are supposed to be at a uniform rate.

So conceived, the economic individual in every class, when the mass-quantities are constant over both periods, is composed at both periods of a constant mass-quantity, which varies in importance or economic size (exchange-value), in such a way that at the one period it is as much more or less important than every other as at the other period it is less or more important than every other, so that altogether it has the same importance as every other. Naturally the sizes of the classes are according to the numbers of such economic individuals they contain.

§ 8. III. Lastly we have to consider the cases, probably the only actual ones, when both the mass-quantities and the total exchange-values of the classes vary from the one period to the other. Here there are only three, or at most four, main positions that claim our serious attention. The first is that we should take the smaller total exchange-value of every class at either period as its weight in the comparison of the two periods. The second is like unto this, from the point of view of the mass-quantities, namely that we should take only the smaller mass-quantity of every class at either period, and then treat these quantities as we would treat them were they the only ones at both the periods, taking for the weight of every class the geometric mean between the total money-values of these reduced mass-quantities at the prices of each of the periods. The third is that we should treat the total money-values of the total mass-quantities at both periods in the same way we have just recommended for treating the total money-values of the mass-quantities that are constant over both periods, namely by taking for

[13] Consideration of the harmonic mean is here omitted because no claim has ever been put forward for it, and no reason is apparent in its favor. It involves the same inconsistencies and difficulties, inverted, as the arithmetic.

the weight of every class the geometric mean between the class's full total money-values at both periods. Among the several suggestions above reviewed for treating the divergent mass-quantities only one has any good claim for consideration. We cannot use the mass-quantities of each period separately, in the system of double weighting as employed by Drobisch, because that perverts the problem, as we shall find in the next chapter. There may be other systems of double weighting, as notably one invented by Professor Lehr; but we may more conveniently examine these later. And at all events, so long as we consider the question of averaging price-variations, instead of measuring variations of price-averages—a distinction which will be made plainer in the next Chapter,—we shall need to choose a method of single weighting; and the single system itself that is adopted will be the basis for any double weighting we may later find proper to adopt. To take the mass-quantities of either period alone is absurd, as neither is of more importance than the other. To make one calculation on the mass-quantities of the one period and another on those of the other, and then to take the (arithmetic) mean between the two results, is an arbitrary proceeding. The only position hitherto recommended that deserves any attention is that of taking the (arithmetic) means between the mass-quantities at the two periods, or, which is the same thing, the sums of the mass-quantities at both the periods, and treating these as if they were mass-quantities constant over both the periods.[14] And yet a question will arise as to whether the geometric mean is not better here also.

The first of these methods is very much like the method above discussed of taking the smaller total money-value at either period when the mass-quantities are the same at both the periods. It shares the defects of that method, and adds to them another. This is that if we happen to be comparing a prosperous period with a dull period, or a large country with a small country, the smaller total money-values may be mostly, and even all, those of the dull period, or of the small country, even though the ex-

[14] This and the preceding method probably yield very nearly the same results. There is one condition in which their results always agree exactly. This is indicated in Appendix C IV § 2 (3).

change-value of money may be the same at both the periods, or in both the countries, and the total money-values correctly represent the total exchange-values. For this added defect, however, a correction is possible, which will be pointed out presently.

The second has more to recommend it. We are seeking to measure the exchange-value of money in relation to other classes of things, all which compose a whole. If all these classes of things consist of the same masses at both periods, the material whole, or world, which they compose, is the same, or similar, at both periods. And that the material world should be the same, or similar, at both periods, in spite of, and underneath, the variations in the exchange-values of the things in it, has the appearance of being a postulate of simple mensuration. Now by eliminating all the surplus mass-quantities that exist at either period alone, we pick out all those mass-quantities which are common to both the periods, so that a material world composed of these mass-quantities is actually the same, or similar, at both periods.[15] At all events the selection of these mass-quantities seems to provide an answer to the objection advanced by Professor Sidgwick and by Dr. Wicksell. There is also another argument, apparently in favor of this position, that may be derived from a more general position, which here deserves to be noticed.

§ 9. When new classes of commodities appear upon the scene in the course of our comparisons, the appearance of these in no wise affects the exchange-value of the others. At first, to take a simple example, our money-unit might have purchased so much of one kind of thing, or so much of another; that it should later be able to purchase the same amounts of these things and so much of a new kind of thing, widens the range of its exchanges, but does not increase its power over other things. The change is extensive, not intensive. Or, viewed in another way, at first the money-unit could purchase half as much of the first

[15] If this be the right position, then the treatment above recommended of the special cases when the total money-values of every class happen to remain constant, or to vary in the same proportion, has to be abandoned. For then, some of the prices being supposed to have varied, the total mass-quantities must have varied, and therefore, according to the present method, only the smaller of these total mass-quantities, and their reduced total money-values, must be taken into consideration.

and half as much of the second kind of things; and later it will purchase a third of the first and a third of the second and a third of the new article. Here what is gained in extension, is lost in intension. A new class of goods adds a new particular exchange-value to the general exchange-value of everything else; but this new exchange-value is only equal to each of the old ones, and equal to the general exchange-value (according to Proposition VII.), and therefore makes no change in any of those magnitudes. A variation in anything's general exchange-value can be brought about only by a variation in one or more of its already existent particular exchange-values; and such a variation can take place only relatively to a class of objects existing at both the periods. A particular exchange-value relatively to a class which exists only at one of the periods compared cannot have varied. Consequently *a class of things is not to be counted in the one period. when it is not counted in the other.*[16] And in a series of comparisons, if a new class reaches sufficient importance to deserve to be counted at a certain period, in comparing this period with the preceding, in which it is not counted, it must still be neglected; and it is to be counted only in the comparison of the next period with this period, and thereafter.[17] And in comparing with the present a period in the distant past we must leave out the classes which have come into existence since and those which have passed out of existence, comparing only those parts of each world which are common to both. Here, however, there is much necessary imperfection in the comparison, because some classes may have passed out of existence only recently, and others may have come into existence soon after the first period, and these may have varied in the intervening periods.[18]

[16] Thus Fauveau said he assumed only the same species of things at both the periods, B. 54, p. 355.—This principle *may* be violated in Drobisch's method. Indifference to its violation is shown by Lindsay, B. 114, p. 26. The need of observing it is treated as a defect by Nicholson, B. 94, ff. 313–315.

[17] Attention was called to this problem by Sidgwick, who thought it one of great difficulty, B. 56, p. 68. It was solved by Marshall, B. 93, p. 373 n. The solution was adopted by the British Association Committee, *First Report*, B. 99, p. 250.

[18] Hence in such cases true comparisons can be made only by passing through all the intervening periods. Where this is impossible, a true comparison is impossible.

The knowledge gained by this investigation into the want of influence of new classes it may be well now to state in propositional form. From this investigation (and really as a corollary to Proposition VII.) we perceive that *the height of the general exchange-value of anything is in no wise determined by the number of its particular exchange-values, but solely by their* (common) *height* (Proposition XLI.). And so, *there being no variation of exchange-values (or of prices), the appearance of a new class of things, with any exchange-value whatsoever, has no influence upon the height of the exchange-value of anything already existing*, nor can it have any such influence until it has existed over at least two periods, thus having time to remain constant or to vary; *nor can the disappearance of an old class have any such influence, no matter what was its exchange-value at the time when it disappeared* (Proposition XLII.).

Now what is thus shown to be true of the appearance or disappearance of a class, it might be argued, is also true of the appearance or disappearance of individual (physical or material) things in old classes. It might be argued that just as we must count only the classes of things which are common to both the worlds compared, so in each class we must count only the (material) individuals which are common to them both. The reason why the classes which appear only at one of the periods cannot be counted is that such classes have neither varied nor remained constant in exchange-value and money has neither varied nor remained constant in exchange-value in them. The same reason might seem to apply to (material) individuals which are present at only one of the periods. These have neither varied nor remained constant in exchange-value, nor has money either varied or remained constant in exchange-value in them. Therefore these single-period individuals, it might seem, should be neglected in our calculations.

§ 10. What has above been said of classes is true of (material) individuals as follows. *The height of the general exchange-value of anything is not determined by the number of its individual exchange-values, but solely by their* (*common*) *height* (Proposition XLIII.). Hence, if there are no variations of prices, or of the exchange-values of things in money, and of money's inverse exchange-values in all other things, so that the general exchange-value of money is constant (according to Proposition XXVII.), then the coming into existence of new individuals, or the passing out of existence of old ones, with the constant prices of the other individuals in the same classes, does not affect the height of any particular exchange-value of money, and consequently not the height of its general exchange-value.[19] In other words, *a change in the size of any classes, without a variation in any exchange-values (or prices), does not cause a variation in anything's general exchange-value* (Proposition XLIV.).[20] This means that *when there are no variations of exchange-values (or of prices), the weighting of the classes is indifferent.* Hence it is indifferent, in this case, whether we neglect the surplus individuals or not. It may be added that also *if all prices vary in the same proportion, changes in the sizes of any classes do not affect the variation in the exchange-value of money, which is the inverse of this common price-variation* (Proposition XLV.);[21] and consequently *the weighting of the classes is indifferent also in this case.*

But when there is irregular variation of some exchange-values, or prices, between two periods, then the weighting is very important. And now also it is important whether any of the classes[22] has augmented or diminished or remained constant in size. That allowance should be made for such changes in total exchange-value (or in total money-value) size, we have already

[19] For at a given moment final utilities are, to the people at large, according to prices. At a given moment, then, prices being given, if people spend their money more on one class than on another, or reversely, this only shows that they are choosing the final utilities in a different manner, not that they are getting more final utility by spending their money in one way than in another. Therefore at two moments, or periods, all prices still being the same, if people actually do spend their money differently, this shows nothing more or less than was shown in the previous supposition.

[20] This Proposition flows directly from Proposition XXVII., which is unconditional.

[21] This Proposition flows directly from Proposition XVII., which is unconditional. Compare also Propositions XXXV. and XXXVI., from which a similar corollary may be drawn concerning the weighting, in all cases, of any class whose price varies inversely to the exchange-value of money—either in all things or in all other things, according to the measurement that is being made.

[22] Except as indicated in the preceding note.

conceded. Then why should not allowance be made also for changes in material size? The economic individual may change in exchange-value size between the two periods. Then why should not the economic individual change also in material size? It is true that it is only the lesser mass-quantity of any class existing at either period, or the mass-quantity common to both periods, that has varied or remained constant in exchange-value. Yet money has varied or remained constant in exchange-value in a changing mass-quantity of this class, just as it has varied or remained constant in exchange-value in a changing exchange-value quantity. When a class is wholly absent from the one period, though present at the other, we have no price-quotation of it at all at the one period, and so it can have no variation or constancy in its price or exchange-value.[23] But when merely a certain material quantity of a class is absent from the one period, yet, as its similar mates are present at both periods, we do know the variation or constancy of money in exchange-value in this changing class of things.

§ 11. A practical objection also exists against taking only the lesser mass-quantities at either period, like that brought against taking only the lesser total money-values of the classes at either period. This is that if we are comparing a prosperous period with a dull period, or a large country with a small country, the mass-quantities might be only those of the small country—which would be treating the small country as if its commodities were more important than those of the large country, —or only those of the dull period, which might in one case be the earlier, and in another the later period, so that in the former case we should be using the mass-quantities of the first period in the comparison of the two periods, and in the later the mass-quantities of the second period, the only reason for the difference in the choice of the period being that more importance is attached to the conditions in the dull period than to those in the prosperous period,—which is absurd.

Still a correction of this defect in this method—and at the

[23] For surely we cannot supply what would have been its price had it existed, as suggested by Nicholson, B. 94, p. 314.

same time of the similar defect in the first-noticed method—could be made in the following manner. Finding the total money-expenditure at each of the periods, or in each of the countries, reduce the expenditure in the one to the expenditure in the other, and reduce all the mass-quantities in the same proportion; then take the smaller of these mass-quantities at either period, or country, and operate as before—or, in the other method, take the smaller of the total money-values so reduced. This correction, however, generally needs a further correction. For it involves that the total money-expenditures so compared shall be of the same exchange-value; and, therefore, it can be safely employed only when the exchange-value of money is the same at both the periods, or in both the countries. When this is not the case, or not known to be the case, as always when we are attempting to measure the exchange-value of money, the only further correction can be by an uncertain method of approach, like that previously noticed. For perhaps the truth may be reached by taking the first result of this method as representing the variation in the exchange-value of money, and then again reducing the total money-expenditures to the same total exchange-value expenditure on the assumption of this variation being correct, and repeating the operation until the result tallies with the last assumption.

Another of the above suggested methods, namely that of taking the arithmetic mean between the mass-quantities at each period, or, which is the same thing, the total mass-quantities over both the periods, has no theoretical reason in its favor; for it is not the total mass-quantities during both the periods, or the halves of these, that have varied from the one period to the other, since more than half of them may be within one of the periods.

Moreover this method stands in need of the same sort of correction as the preceding. For between a small country and a large country, or between a dull period and a prosperous period, the arithmetic mean of the mass-quantities would be more influenced by the conditions in the large country, or in the prosperous period—the influence here being on the opposite side from what it was on in the preceding cases. The correction is to be made in the same way as before, namely by reducing the mass-quantities, before averaging them, in the same proportion, so that their grand total exchange-values—not merely money-values—shall be the same at each period; all which again involves the need of approach through many assumptions and repetitions, so that, from being a simple and convenient method, it becomes a very laborious one.

A simpler means of avoiding at once the difficulty arising from differences in sizes of the countries or in prosperity of the periods and that arising from variation in the exchange-value of money, consists in eschewing the arithmetic mean of the mass-quantities and substituting the geometric.

It may be, however, that in spite of its theoretical insufficiency, the method of taking merely the arithmetic means of the mass-quantities, uncorrected, may in ordinary cases yield results so near the truth that it may be preferable to any of the other methods thus far examined, and on account of its greater convenience may be even preferred to the truer methods which we shall later discover. But the practical merits of these methods it is impossible to examine here. We shall examine them later in connection with the averages that are to be drawn of the price variations.

§ 13. The suggested method above set down as the third still remains for examination. This is simply to draw the geometric mean of the full weights of every class at both periods, that is, of the total money-values (which indicate the relative total exchange-values) of the classes at both the periods, without having any more concern for the changes in the mass-quantities of the classes than for the changes in their total exchange-values. In the very first case considered, where the total money-value of every class happens to be constant over both periods, or to change in the same proportion, so that the relative sizes of the classes remain unchanged, we have seen that the most natural treatment is to take these relative money-values as the weights of the classes. Then, in the simplest cases, the economic individual is a constant money-value at both periods, but is a changing mass-quantity. In the second general case, where the mass-quantities in all the classes happen to be constant over both periods, the economic individual we have unhesitatingly found to be a constant mass-quantity at both periods, but a changing money-value (or exchange-value). Neither of these cases is likely ever to occur in actuality. Now, when both the total money-values and the total mass-quantities in every class are various at both the periods, it seems most natural, and not improper, to combine both these conceptions, and to have for our economic individual one changing both in money-value (or exchange-value) and in mass-quantity.[24] As before, the nature of this individual may be best understood by aid of an example.

Suppose at the first period we have 100 pounds of A at $1.00 apiece and 100 pounds of B at $1.00 apiece, and at the second period 50 pounds of A at $1.96 apiece and 1,000 pounds of B at $0.04 apiece. The total money-values of the class A are $100 at the first period and $98 at the second, and those of the class B $100 and $40 respectively. The relative sizes of these classes, therefore, according to this method of measurement, are $\sqrt{9800} = 98.995$ for the class A, and $\sqrt{4000} = 63.246$ for the class B. Or if the latter be taken as one unit, the weight for A is $\frac{98.995}{63.246} = 1.565\left(= \sqrt{\frac{9800}{4000}} = \sqrt{2.45}\right)$. Therefore, the individual in the class A consists at the first period of $\frac{100}{1.565} =$ 63.89 pounds, worth $63.89, and at the second of $\frac{50}{1.565} =$ 31.945 pounds, worth $62.61; and the individual in the class B consists at the first period of 100 pounds, worth $100, and at the second of 1000 pounds, worth $40. But $\frac{100}{63.89} = \frac{62.61}{40}$ $= 1.565$, that is, the individual in B is 1.565 times more valuable than the individual in A at the first period, and at the second period the individual in A is 1.565 times more valuable than the individual in B, although the individual in A has contracted by

[24] If this method be admitted, both those methods are to be retained, and employed when the conditions are met. This method is comprehensive, and includes both those.

half, and the individual in B has grown tenfold, in mass-size. Thus in spite of the changes in their mass-sizes the individuals, so obtained, in the classes have inverse relations of money-value (and consequently of exchange-value) at each of the two periods, and therefore are equally valuable, or equivalent, over the two periods together. Also $\sqrt{63.89 \times 62.61} = \sqrt{100 \times 40} = 63.246$, that is, the geometric mean of the prices of the individual in A is the same as the geometric mean of the prices of the individual in B, which means that at the half-way moment between the two periods, the prices being supposed to vary at a uniform rate, the prices of the two individuals are the same, so that then these two individuals are actually equivalent.

That these two relationships are universal, can be demonstrated as easily as before in the case when the mass-quantities were the same at both periods (see above, § 7). We have only to distinguish x into x_1 and x_2, and y into y_1 and y_2, and then the price of the individual in the class A becomes $\sqrt{\frac{x_1 a_1}{x_2 a_2}} \cdot \sqrt{y_1 y_2 \beta_1 \beta_2}$ at the first period and $\sqrt{\frac{x_2 a_2}{x_1 a_1}} \cdot \sqrt{y_1 y_2 \beta_1 \beta_2}$ at the second, and everything works out as before.

Variations in the exchange-value of money have no influence to derange the weights here, just as they have none in the cases where the mass-quantities are the same at both periods, every weight being affected in the same proportion here as there. Nor does a difference in the sizes of the countries, or in the prosperity of the periods, have any such deranging influence. The geometric mean with even weighting being employed, equal weight is really attached to the weights of each period. There is no need of correction. This method of weighting, then, is not only the best, but even the simplest.

At all events this method seems to give us the true conception of the economic individual in a comparison between two periods. That the variations of the prices of the classes, between two periods, should be weighted according to the relative numbers of such individuals the classes contain over the two periods, would seem to be proper. Or if we average prices at each period separately, preparatory to comparing the averages, it would seem proper to make use of the numbers of these individuals the classes contain at each period, determined with reference to the other period with which it is compared.

§ 14. The subject has to be left in this somewhat unsatisfactory state until we have examined the question of the averages in which these systems of weighting are to be employed. One system of weighting may perhaps be found suitable for one kind of averaging, and not for another. And one kind of weighting with one kind of averaging may be found to yield exactly, or very nearly, the same results as another kind of weighting with another kind of averaging. Hence our choice may be more narrowed than it is at present.

So far we have been attempting, without complete success as yet, to reach the theoretically exact position. To attain to this position itself would, however, be of little service, unless we have very exact data to apply the theoretically correct method to. Hence the theoretically correct method of weighting will be serviceable only for measuring the course of the exchange-value of money during present time, as we advance into the future. For reviewing what is already past, it is hopeless to expect to find data sufficient to justify us in making use of any but a rough and ready weighting, the same for many consecutive periods together, the finer shades of difference from year to year being untraceable. This weighting, the same for many periods, must either be some general average of the relative importance of the classes over these periods, or be according to some general average of the mass-quantities in the classes over these periods. Which of these methods is the better, and of what sort the average ought to be, we must postpone examining till we have reviewed the subject of weighting in connection with the subject of averaging the price variations.

VI.

§ 1. One more item remains, about which there has been dispute. Many economists have maintained that among the prices of things which are to be counted in measuring the exchange-value of money we should include also the "price of labor," namely, wages and salaries.[1]

No opinion could be more erroneous.

§ 2. In the first place labor has no exchange-value. Labor is not a possessible thing: it does not pass from one owner to another; it is not exchanged for anything else. What! it may be said, does not the employer pay money for labor, and does not the laborer get money for his labor? By no means. The essence of the contract between a manufacturer and his employees is that the former shall put materials and machines in the hands of the latter and shall take the products which they shall make —he buys from them the improvements they make, they sell to him those improvements.[2] Even in domestic service what the employer pays for is the product of labor:—the charwoman is

[1] Wages were included in the lists, with the prices of commodities, for calculating variations in the "value" of money, by Dutot, Evelyn and Young. Adam Smith found the "value" of silver at different epochs by considering the sums needed to purchase "the same quantity of labor *and* commodities," *op. cit.*, p. 101. (His inexactness is shown by his making the same measurement by "the quantity of labor which any particular quantity of them [gold and silver] can purchase or command, *or* the quantity of other goods which it will exchange for," p. 14.) McCulloch: to be constant in "exchangeable value" a thing must "at all times exchange for, or purchase, the same quantity of all other commodities and labor," *Political economy*, p. 213 (and in Note to *Wealth of nations*, p. 439). Roscher gives a truly German reason for assigning "an important place" in the lists to daily wages: "The desire to exert influence upon other men and to be prominent socially is a very universal one; and of its attainability there is no better sign than the power of disposal over many days of labor," B. 32, § 129. That wages should be counted along with commodities has been of late asserted by Martin, *op. cit.*, p. 626; Nasse, *Die Währungsfrage in Deutschland*, Preussische Jahrbücher, 1885, p. 313; Giffen, B. 44, pp. 127, 128, B. 45, q. 780; Newcomb, B. 76, p. 212; Wasserab, B. 105, p. 75; G. P. Osborne, *Principles of economics*, Cincinnati 1893, p. 332; Lindsay, B. 114, p. 332; G. H. Dick, *International bullion money*, London 1894, p. 3; Wiebe, B. 124, pp. 168–169; Edgeworth, B. 65, p. 386; A. M. Hyde, *Gold, labor and commodities as standards of value*, Journal of Political Economy, Chicago, Dec. 1897, p. 97; Parsons, B. 136, pp. IV., 97, 115, 128.

[2] The difficulty in carrying out the ordinary opinion is well shown by James Mill in his *Elements of political economy* (2d ed., 1824). He starts out by saying; "The laborer who receives wages sells his labor. . . . The manufacturer who pays these wages buys the labor," p. 21; and later says that when a capitalist provides raw materials and tools, and the laborer works up the product, this "belongs to the laborer and capitalist together," but that, "when the share of the commodity which belongs to the laborer has been all received in the shape of wages [paid in advance of the sale of the product], the commodity itself belongs to the capitalist, he having in reality, bought the share of the laborer and paid for it in advance," pp. 40–41. On p. 90 he makes both these statements!

paid for clean windows and floors made out of dirty windows and floors, the cook for cooked food in the place of raw food, the waiter for food produced on the table from food produced in the kitchen, the coachman for a moving carriage instead of a stationary one.[3] Employers do not want labor—they would pay much more if they could be served, like Psyche in Cupid's bower, with the hands of invisible spirits. What they want, and what they pay for, is something which they cannot get without paying some one to produce it for them.[4] And what laborers give in return for their hire is not the labor which nobody wants, but the products of that labor.[5]

§ 3. But this is not the principal reason, although the principal reason flows from this. Labor and material things being uninterchangeable, the wages of labor and the prices of commodities are categorically distinct. To put them in the same list is to try to mix oil and water. Wages belong to another list.

[3] Some writers have admitted that the wages of "productive labor" (wages paid for the production of goods that are to be sold) are not to be counted because they are a factor entering into the prices of the goods produced, and therefore, being already counted in those, they would be counted twice; but claim that the wages of so-called "unproductive labor" (of domestics, etc., whose products are not sold) ought to be counted:—Edgeworth, B. 59, p. 266 n.; Marshall, B. 93, p. 372; Nicholson, B. 94, p. 324; the British Association Committee, *First Report*, B. 99, p. 249; H. J. Davenport, *Outlines of economic theory*, New York 1896, p. 227; Wicksell, B. 139, p. 18. Here a distinction is to be drawn. The wages of domestics, etc., contribute only to the retail prices we pay for what we consume. Therefore if retail prices are being used in our lists, these wages would properly belong there. But retail prices are usually excluded, for reasons already explained. Hence the wages even of domestics should be excluded.

[4] So Fonda, B. 127, pp. 14-15, and Davenport, *op. cit.*, p. 53. Cf. Aristotle, who said we should have no need of workmen if shuttles moved themselves, *Politics*, I., 2, 5.—On the other hand, that labor is the only thing we pay for has been asserted by W. D. Wilson, *First principles of political economy*, Philadelphia 1882, p. 100.

[5] The products of labor for which we exchange money may be immaterial things—dramatic scenes, music, etc. (pleasureable sensations and thoughts), or states and conditions (safety, health, etc.)—in short *services*, if by this term be meant, not actions, but the effects of actions. We do not pay musicians for playing, but for music; nor lawyers for pleading, but for the results of their pleading,—often paying, however, for a chance of getting what we do not get. Now immaterial products ought, theoretically, to be counted in the lists—we might count theatrical tickets, fees, etc. The sole reason why they are to be excluded is the practical one already applied to many material products, that their qualities are so diverse, the individual products under the same names so various, that they do not form homogeneous classes. Yet one of them, transportation of persons, we have seen to deserve to be counted.

Labor is the cost at which the majority of mankind procure the commodities whose prices go into the list drawn up for measuring the exchange-value of money. According as is the labor which it costs a man to produce a certain thing, so is the cost-value of the thing to him. According to the average labor which it costs the producing part of mankind to produce things of a certain class, so is the average cost-value of that class of things in general. The cost-value of a thing is not necessarily according to the labor merely of producing the thing physically; for things are carried to places where they are not so produced, and the economic production—offering in the market—of them there includes the labor of transportatation. The majority of mankind, however, do not produce but a small part of the things they consume, and so are more interested in the cost of *procuring* things, which is according to the cost of producing other things, material or immaterial, and the rate at which these can be exchanged for those. Here the idea of cost-value in the narrow sense of cost of production proper, passes over into a wider sense of cost of acquisition, in which it becomes identical in magnitude with esteem-value. If a man gets a dollar a day for what he produces, whether in wages or otherwise, then anything the price of which is a dollar costs him a day's work. Such a thing has not to him the exchange-value of a day's work; for he does not exchange the day's work for it, and if he does not get it, through not working for it, he has lost his day's work as well as the thing—he has idleness without the thing in place of the thing with work, so that if there is any exchange, it is exchange of idleness for the thing, and the thing should be said to have the exchange-value, not of a day's work, but of a day's idleness. Nothing, however, is reached along this line of reasoning. There is properly no exchange, but there is cost.[6] There is comparison

[6] The conception of Turgot and Adam Smith that labor is the real *price* we pay for what we get is a misuse of terms, as it confounds "price" with "cost." Price, in a wide and in itself not very proper sense, is the thing which one man gives to another in exchange for what the other gives to him. Cost is what we give up, without anybody else getting it. If a man who owns a ton of coal gives it to another in exchange for a ton of iron, the ton of coal is, in this wide sense, the price of the ton of iron (and the exchange-value of the ton of iron in coal is measured by this amount). If a man burns up a ton of coal in smelting a ton of iron,

of the pleasure in procuring the thing with the displeasure of doing the work. This comparison is the essence of esteem-value, which permits a certain amount of cost-value to things, and no more. This feature in esteem-value shows itself again in the comparison of the pleasure of possessing one thing with the displeasure of abandoning the possession and use of another, or of renouncing possession and use of other things also procurable in the place of the thing obtained; which comparison regulates the exchange-values of things.

§ 4. Now it is desirable that we should be able to measure the cost-value of things in both the narrow and the wide sense. Industrial progress consists in cheapening the cost-value of things, and it is well to know whether progress is being made, and how rapidly. But the trouble of measuring the cost-value of everything separately might be saved by measuring the cost-value of money, provided this be its cost-value in the wide sense, identical with esteem-value. For if we should find, for instance, that this cost-value of money has been constant, then we should know that, as the price of anything varies, the cost-value, at least in the wide sense, of that thing directly varies; and, if we have measured also the general exchange-value of money by means of an average of prices, then, as the average price of all things varies, so does the cost-value of all things on the average directly vary. Or if we should find that the exchange-value of money in all other things has been constant, but that its cost-value has fallen, then we should know that a steady price of anything means a fall in its cost-value to just that extent, a fall of price a still greater fall in cost-value, or a rise of price either a lesser fall in cost-value or a stationary cost-value or a rise in cost-value, according to the proportions between the fall of money in cost-value and the rise of the price; and we should know that all other things have on the average fallen in cost-value to the same extent as money. Or we might find

the ton of coal is one of the costs in producing the ton of iron. Labor is another one of the costs of producing the ton of iron; and as labor was expended in producing the ton of coal, labor is an ultimate cost of production (and cost-value is measured by labor). Labor cannot be the price of anything, because one man cannot give it to another.

that the exchange-value and the cost-value of money have both changed, the one rising and the other falling, or both together rising or falling, only the one more and the other less. There are, in fact, all the five typical possibilities we have noticed in another case between the variations of two independent quantities. But whichever of these changes take place, when we know their proportions, it is easy to calculate out all the particular variations concerning which knowledge is desired. Therefore the measurement of the cost-value of money is also desirable.

§ 5. Now the cost-value of anything, in the wide sense, as determined by the labor cost of acquisition, and identical in magnitude with esteem-value, is measured, not by the average labor-cost of producing the thing in question, but by the average labor-cost of procuring it. In the case of paper money, it is evident that we have no interest in the cost of producing it; and even in the case of metallic money, our interest in the cost of producing it is small, as regards the subject before us. We are interested in the labor-cost of procuring it. Evidently such a thing as the general labor-cost of procuring money is to be measured by the average labor-cost to the average man of procuring money. This may be found by finding the total money-earnings of a country in a given time, say an hour, and dividing it by the number of workers, with allowance for the number of hours a day each one works.[2]

Here incidentally it deserves of notice that in measuring the cost of procuring money, we should not confine our calculation to the consideration of wages, or of wages and salaries. To confine attention to wages, and especially to the wages of the cheapest and

[2] There was a dispute in the early part of the nineteenth century as to whether the measure of "value" is the quantity of labor needed to produce the thing or whether it is the quantity of labor the thing will command in exchange. Adam Smith had advanced both positions, and sides were taken by his followers, Ricardo assuming the former tenet, while Malthus defended the latter, each having partisans. Now it is plain that the measure of *cost-value* proper is the average quantity of labor needed to produce the thing; and it would seem as if the measure of cost-value in the wide sense, or *esteem-value*, may be taken to be the average quantity of labor the thing will command. Thus both those positions were correct and incorrect. They were correct each of one kind of "value," and incorrect of the other kinds of "value." They were both incorrect of exchange-value.

commonest sort of laborers, or to agricultural laborers, as recommended by some writers,[3] would be like confining our attention in the attempt to measure the exchange-value of money to the price of wheat or other food, as performed by some of the earlier workers in this field. Not only all wages and all salaries, but all profits (apart from rent and income, if these be already counted in the gross income of those who pay them), should be included in the examination, and some means of averaging them should be found. The common word for wages, salaries, and profits, is earnings. The general money-earnings of the hour's work of all the working part of the community should be somehow found. Perhaps some earnings, as immeasurable in practice, would have to be omitted, just as in measuring the exchange-value of money we have to omit the prices of some things. But the effort should be made to include as many as possible.

§ 6. Over against this measurement of the cost-value of money, we have curiosity to know the cost-value in the same sense, or esteem-value, of all commodities in general. We might be tempted to measure this in the same way, by finding the total product of a country in a given time and dividing by the number of producers, or of consumers. But to this there is objection, in that the total product would have to be measured all by weight or all by capacity, and when a change takes place between two periods a different result would be reached according as the one or the other of these measures were used—a difficulty we have already noticed in another connection. For this lumping together of a total mass product would mix up materials of different qualities and fineness, and from period to period a change in the mere total mass would disregard possible improvements or retrogressions in the qualities[4]—a difficulty we shall meet again. Instead, we should have to measure the vari-

[3] Harris, *Essay upon money and coins*, 1757, Part I., p. 13; Malthus, *op. cit.*, pp. 96, 112, 116; Shadwell, *op. cit.*, pp. 202–203, and in the Journal of the British Association for the Advancement of Science, 1883, p. 626; T. I. Pollard, *Gold and silver weighed in the balance: a measure of their value*, Calcutta 1886, p. 75.

[4] What we want really to find is the total quantity of pleasure procurable from our productive activity. But, in equal quantities, coarse goods do not yield so much pleasure as fine goods.

ation in the cost of production of every kind of article separately, by finding its total product in a given time and dividing by the number of its producers, at the two periods compared; and then somehow, by some method of averaging never yet investigated, combine the variations in the costs of production of the different kinds in one variation (or constancy) of the cost of production of all things. This is a very complex method. It is the only one possible for measuring general cost-value in the narrow and proper sense. But for the wider sense, in which cost-value becomes almost identical with esteem-value, another simpler method is also possible.

This is to find the esteem-value of money alone, and to find the exchange-value of money alone; for by comparing the results of these two measurements we can get the result desired. We thus see that we have need of two distinct operations, which are supplementary to each other, and together help us to reach a final result in regard to the esteem-value of commodities in general.

§ 7. Now if, instead of these two separate operations, we should attempt to perform a single operation by including earnings in the same measurement with the prices of commodities, we should form a hodge-podge that has no meaning. Its results would indicate neither the exchange-value nor the esteem-value of money, and as it is undertaken only with a view to measuring the "value" of money, it would mean nothing, there being no economic value apart from the four kinds we have analyzed out, nor any value compounded of any two or more of these. For instance, take the case which is believed to have been going on for the past twenty odd years. Prices in general have been falling, and at the same time money earnings in general have been rising. A whole school of modern writers, reviving views which triumphed during a similar period in the early part of the century, have concluded from this state of things that the "value" of money has been about stationary.[5] This result

[5] *E. g.*, "Gold prices fell only 19 per cent. from 1873 to 1891. . . . Wages, in gold, rose more than 14 per cent. from 1873 to 1891. . . . The advance in wages since 1873 so nearly offsets the decline in prices that when fairly tested by both prices and wages the value of gold in 1873 and 1891 was practically the same," B.

would be obtained, in fact, if we put money earnings, which have risen, and the prices of commodities, which have fallen, in the same table, and, attaching equal importance or weight to each, drew an average between them all—on the supposition that the rise of the former has been about equal to the fall of the latter. And yet it is plain that the fall of the prices of commodities means a rise of the exchange-value of money in commodities; and the rise of people's earnings in money does not mean a fall of the exchange-value of money in labor, there being no such exchange-value, money not being exchanged for labor; but it means a fall in the cost-value or esteem-value of money. Then what "value" is it of money that has remained stationary? Surely there is no value that is a mean between exchange-value and cost-value (or esteem-value).

Again, if the prices of commodities should for a time remain constant on an average, and if also the average money-earnings of workers remain stationary, the same result would be obtained from the mixed method of measuring the "value" of money—namely that the "value" of money has remained unchanged. Yet, in this case, both the exchange-value and the cost-value or esteem-value of money have remained unchanged. Thus the mixed method will give the same result in regard to the "value" of money, even though the separate measurements of the exchange-value and of the cost-value or esteem-value of money give different results. The separate methods, however, give us another result not indicated by the mixed method. In the last suppositional case it is evident that the cost-value and esteem-value of commodities in general have remained stationary. But in the first actual example it is plain, from the separate measurements, that the cost-value and esteem-value of commodities in general have fallen even more than the cost-value and esteem-value of money. For a fall of the esteem-value of money with a rise of its exchange-value, means that men are obtaining money more easily and that their money is purchasing for them more commodities, that therefore commodities are being

W. Holt, *Interest and appreciation*, Sound Currency (Reform Club), New York, Nov. 15, 1898, p. 368.

obtained still more easily than money, and hence their cost-value and esteem-value have been falling still more rapidly than those values of money. The influence of falling prices and of falling esteem-value of money is cumulative upon the esteem-value of commodities—in our estimates of those things. But the mixed method, which indicates merely that an anomalous "value" of money is stationary as well in the one case as in the other, does not distinguish, does not inform us, whether commodities are falling in cost-value, or esteem-value, or whether they are stationary. And still another exactly opposite example is conceivable, in which money-earnings might fall and prices rise, in which therefore the cost-value and esteem-value of money are rising and the exchange-value of money is falling, where the mixed method of measuring might also indicate constancy in the "value" of money. Thus on three very different occasions the mixed method would give the same answer, although on two of these occasions the exchange-value and the cost-value (or esteem-value) of money are acting in diametrically opposite ways, and in the other they are standing still in the mean. Also this method might give the same answer no matter how much the exchange-value of money is falling, provided the esteem-value of money is rising sufficiently to counter-balance, or *vice versâ*. The utter worthlessness of such a method, which mixes up distinct things, is apparent.

If such a mixed method is worthless when we carry out the one-half of it thoroughly, by including in the list all earnings, it is *à fortiori* worthless if we execute this half of it imperfectly by including in the lists only wages, and still more if only some wages.

§ 8. There has really been gross confusion of thought in the recommendation of this method—an extension to a whole of what belongs only to a part. Considering only the case of persons who have fixed incomes and who spend some of it in buying commodities and some of it in hiring servants, economists have seen that if the prices of commodities fall and the wages of servants rise, there is some tendency here toward compensation and balancing. In attempting to measure the purchasing power of these incomes, therefore, it would be proper to take account of the wages of servants along with the prices of goods. But then the prices of goods to be used, in order to observe parallelism, should be retail prices. And investigation should be made as to how much, on the average, is spent on goods and how much in wages, in order to weight goods and wages accordingly —probably with the result of finding the weight of wages to be relatively small, wherefore, for the counterbalancing, the rise of wages would have to be considerably greater than the fall of prices. In doing this, however, all that is accomplished is the construction of a special standard for a part of the community. Then, forgetting the limitation of this procedure, and wanting to use wholesale prices, and to form a standard for the whole community of producers (including the servants) and consumers, and therefore to weight labor as equally important with the products of labor, some economists have assumed that a compensation which exists for a small part of the community exists for the community at large. The compensation exists, in full plenitude, only in the case of a few persons who are, so to speak, consumers both of commodities and of services, and are not themselves producers or service-renderers. In the case of the persons who render the services, or who produce goods, without themselves hiring other servants, instead of compensation, there is cumulation, for they earn more money and their money purchases more goods. While in the case of other producers and service-renderers who are also employers, the cumulative influence generally exceeds the compensatory. Moreover, the wages of laborers hired for producing goods for sale, as before remarked, are already included in the prices of the products. Now if these wages rise while these prices fall, either this change is at the expense of the employers, causing their profits to fall, and so not affecting earnings in general, or both the rise and the fall are compensated to the employers by cheapened methods of production and improved machinery—that is, there is compensation of an entirely different sort.

§ 8. Of course in making the final measurement of the esteem-value of commodities in general, we must count earnings as equally important with the prices of commodities. But this is because earnings wholly belong in the measurement of the cost-value or esteem-value of money, and prices wholly belong in the measurement of the exchange-value of money. Then, when a single result is obtained from the two measurements, equal importance is to be attached to each.

But in the mixed method it is impossible to state what ought to be the weighting of commodities and of earnings as wholes relatively to each other. This problem has simply been ignored by most of the advocates of the mixed method. A couple of the early ones [6] weighted wages at about one third of commodity-prices, without telling us why, but apparently having in mind some estimate of the relative amount of money paid in wages and in other expenses by the then dominant class of landlords, or by their farmers.[7] Recent writers have been less definite. Some have implied that even wages, let alone earnings, should be treated as equally important with the prices of all commodities; while others have even asserted that greater importance should be assigned to wages than to prices. Of course these writers do not mean that people in general spend as much or more money in paying servants than in purchasing goods. What they have in mind is an indistinct notion that what they want to measure is rather the cost-value or esteem-value of money. Then, of course, they should give greater weight to wages than to prices, for the simple reason that prices should not be counted in that measurement at all. Prices no more belong in the measurement of the cost-value (or esteem-value) of money than do wages in the measurement of the exchange-value of money.[8] No wonder, then, that the persons who try to combine the two distinct measurements of the two distinct values are at a loss as to the amounts of importance they should attach to prices and to wages,

[6] Evelyn and Young.

[7] Some time before, Cantillon said it was a common opinion in England that of a farmer's income a third goes to his landlord, a third to himself, and a third to his laborers, *Essai sur le commerce*, about 1732, Harvard ed., pp. 159–160.

[8] Among the writers above cited as mixing up these things, Giffen at times inclines to recommend treating of wages and salaries in a separate table, B. 55, p. 131. But he does not recognize this as forming part in the measurement of something else than exchange-value.

and waver between them according as the idea of exchange-value or the idea of cost-value predominates in their minds.

§ 10. Slightly different, though at the bottom similar, is the position held by a few economists and many politicians to-day that wages are a better measure, or criterion, of the "value" of money than the prices of commodities. Here the opinion seems to be that we may make one measurement of the "value" of money by means of prices and another by means of wages, with different results, and that of the two the measurement by wages is the better or more trustworthy. This position is naïve. The measurement of the value of money by wages alone is wrong, in the first place, because wages are not the only earnings. And, in the second place, a correctly performed measurement by earnings would be the measurement of the esteem-value of money; wherefore, as such, it cannot be compared with the measurement by prices, which is not a measurement of the esteem-value of money at all. And the measurement by prices is the measurement of the exchange-value of money; wherefore, as such, it cannot be compared with the measurement by earnings, which is not a measurement of the exchange-value of money at all. Each of these methods of measuring the "value" of money, if carried out in the best manner possible, is the best in its kind. And neither has a better claim than the other to be the better measurement of the "value" of money singly, since the one kind of value is as much value as the other.[9]

[9] It is hardly necessary to notice that sometimes the same economists who make the above comparison, and sometimes others, also bring in still another measurement of the "value of money" as equally good as, or even as better than, either of the preceding. This is the measurement of something which is not value at all in any of the economic senses of this term, but to which this term is applied, in connection with a similarly uneconomic meaning of the term "money," in the slang of the "street." For certain classes of business men often mean by "money" loanable capital, and by the "value" of this "money" the rate of discount. Naturally the measurement of this "value of money" is by the rate of discount itself; and such a measurement is a perfectly good one of the "value of money" in these meanings of the terms. But it is not at all a measurement of the "value" of money in any of its economic senses; and so cannot properly be brought into comparison with any of the methods of measuring any of the economic values of money.—There is even another uneconomic meaning of "value," as used in the old monetary science, namely the meaning of "intrinsic value" in the case of metallic money, this being the fine metal in the coins. The measurement of this "value" is the simplest thing in the world, being made by finding the weight of

§ 11. Of course if either of these methods is thrust forward into the province of the other, it should be driven back. But what happens mostly is that one person in speaking of "value" is prone to think only of the one kind of value, and then he claims that the measurement properly applicable to that kind of value is the better measurement of "value." He ought really to say that it is the only measurement of "value," and dismiss the other altogether, confining himself to the one meaning of the term he uses; only this he cannot do, because the other meaning of "value" always does lurk in his mind. The best and only thing to do is to distinguish what kind of value we are treating of, and to assign to each kind its own proper kind of measurement. Otherwise when two persons meet each other in the combat of opinions, who happen each to be thinking of a different kind of value although they are both talking simply of "value," they will be apt to fall into behavior very much like that of the two knights on the opposite sides of the shield.

And yet, to repeat, unlike those two knights, each of these contending parties has an inkling of the other's meaning, and adopts it himself at times. Here is the fundamental explanation why any economists have advised such a mixing of incompatible elements in the measurement of the "value" of money. Using simply the one term "value," they have not distinguished which kind of value they were seeking to measure, and so have allowed elements proper in the measurement of distinct kinds to creep into a single measurement, which then is really a measurement of no one kind of value. Yet, as we have already had occasion to remark, many of the most prominent economists have affirmed that when they used the term "value" they intended to refer only to "exchangeable value." And in truth most of the efforts of economists have hitherto been directed to measure principally the exchange-value of money, with a little infiltration of cost-value elements. But cost-value itself, freed from any other kind of value, no one has ever yet attempted to measure, or so much as mentioned as a desirable object of mensu-

the coins and allowing for the alloy. It would be no more absurd to bring this into the comparison and to say that the best measurement of the "value" of money is by the weight and fineness of the coins.

ration by itself.[10] Thus the mensuration of exchange-value has been brought well along, while the measurement of cost-value has hardly so much as been begun. There have been, and especially nowadays are, a number of economists who look upon cost-value as a more important kind of value than exchange-value,—who, for instance, maintain that money, the recognized measure of "value," should be a fixed standard rather of cost-value than of exchange-value. Upon these, then, the duty is incumbent of instituting the attempt to measure the constancy or variation of the cost-value, or esteem-value, of money.

The present work, to repeat, is concerned only with the mensuration of exchange-value.

[10] To be sure, Ricardo and Malthus and others, we have seen, were mostly interested in the measurement of cost-value or of esteem-value. But they always expounded their procedure as a search after the "real" measure of "value in exchange," so that the measurement of exchange-value always interfered with their measurements of the other kinds of value.

CHAPTER V.

MATHEMATICAL FORMULATION OF EXCHANGE-VALUE RELATIONS.

I.

§ 1. For working out our problem of measuring the exchange-value of anything by combining its particular exchange-values into its one general exchange-value, or the many variations of these into one variation, the aid of mathematical formulation is indispensable.

The money-unit, being our unit for measuring particular contemporaneous exchange-values at the same place, has become also our practical small unit for measuring the comparative exchange-values of things in different places and their constancy or variation at different times. But it is a variable unit, comparable with metallic or wooden foot-sticks, which vary in length at different temperatures and which also wear down, except that the money-unit may vary in exchange-value indefinitely in either direction, and so is a much less trustworthy practical unit. To be precise, it is not the money-unit itself, not the coin-unit—the material thing,—that is the unit of exchange-value, as also not the material stick is the unit of length; but as the length of a particular stick is the unit of length, so the exchange-value of the money-unit is the unit of exchange-value.[1] Now just as the surveyor corrects the variations of his measuring instruments in their attribute of length, so we want to be able to correct the variations of our economic measuring instruments in their attribute of exchange-value, and wish to construct a true and unvarying unit underlying our practical units, the same in all places

[1] That, strictly speaking, we measure the value of things not by money, but by value of money, see Prince-Smith, *op. cit.*, p. 369; Knies, *Das Geld*, 2d ed., 1885, p. 150; cf. Rossi, *op. cit.*, pp. 145, 157.

and at all times. Some of the older writers on monetary matters asserted that because money is the measure of other things, we cannot measure it.[2] This is a false position in metrology, unhappily not yet wholly abandoned.[3] All our practical measures must themselves be measured—and not only our ordinary measures, but also the so-called standard measures.[4] In fact, metrology itself is the science of measuring our measures. Of course, the measurement of our measures is different from the measurement of other things. We measure other things by comparing them with our measures—a simple operation. We measure our measures by comparing them with other things, preferably with some selected other things—a complex operation. The object of measuring other things is to find their relative sizes, as well as the constancy or variation of their sizes, by finding their sizes compared with one given and constant size, the measure used. The object of measuring our measures is to find, except in the case of different units of the same measure, only their equality or inequalities and their constancy or variations. This is the labor we have before us in regard to exchange-value. We want especially to measure our measure of exchange-value, money.

As being the simpler form of our problem, we may devote most of the enquiry to the attempt to measure the constancy or variation of the money-unit in exchange-value at one place

[2] Javolinus in *Digest*, XLVI., I., 42; followed by Molinaeus and Budelius (in Thesaurus's collection of monetary tracts, Turin 1609, pp. 236, 461).—The doctrine was an inference from Aristotle's statement that money measures all things (*Eth. N.* V, v, 10). Hence Turgot also said that money can be measured only by other money, *op. cit.*, p. 76.

[3] L. A. Garnett, *The crux of the money question: has gold risen?* Forum, New York, Jan. 1895, p. 581.—And Mannequin has gone so far as to say not only that we do not measure our measures, but that it is not needed that measures be stable, this being a prejudice, and even, as measures should be like the measured, they ought to be variable! *op. cit.*, pp. 28, 70, *Uniformité monétaire*, 1867, pp. 10–11 *n.*, *La monnaie et le double étalon*, 1874, p. 39 (quoted by Walker, *Money*, p. 281 *n*).

[4] That money is measured by other things was said by Montanari, *Della moneta*, 1683, ed. Custodi, p. 91; Galiani, *op. cit.*, Vol. I., p. 52; Condillac, *Le commerce et le gouvernement*, 1776, ed., 1821, p. 93; (cf. Adam Smith, *op. cit.*, p. 190); Ricardo, p. 293; Levasseur, B. 18, pp. 137–138.—McCulloch concluded that money is therefore not a measure of value, as it no more measures other things than it is measured by them, *Treatises and essays*, 1859, p. 10.

the figures a, b, c, ······ will be various according to the mass-units which happen to be chosen, but, the mass-units once chosen being of course used at both periods, the variations represented by a', b', c', ······ are the same, no matter what be the mass-units used. Now at the second period we have

$$\text{M} \doteq a'a\,\text{A} \doteq b'b\,\text{B} \doteq c'c\,\text{C} \doteq \cdots\cdots .$$

These expressions, however, are not equations. They do not express equality between quantities of the same sort, but equivalence (*i. e.*, equality in one kind of quantity, exchange-value) between quantities of another or other kinds (weight, capacity, etc., of different things). Therefore they are not satisfactory.

§ 3. Let us proceed anew by representing the general exchange-value of M, of A, of B, of C, ······, by the italicized capital letters M, A, B, C, ······. These exchange-values we shall need to distinguish at the two periods, which we may do by appending a small number to them. When we wish to distinguish between general exchange-value in all *other* things and general exchange-value in *all* things (including the thing itself), we may do so by appending a small o, or a small a. Thus M_{o1} means the exchange-value of the money-unit in all other things at the first period. Therefore the expression M $\doteq a$A may (by Proposition VII.) be changed into $M_1 = aA_1$, which means either that the general exchange-value of M is equal to the general exchange-value of a A, or that the general exchange-value of M is a times the general exchange-value of A, at the first period. We may then have

$$M_1 = aA_1 = bB_1 = cC_1 = \cdots\cdots,$$

and

$$M_2 = a'aA_2 = b'bB_2 = c'cC = \cdots\cdots.$$

Here we have equations proper; for the terms all refer to equal homogeneous quantities, namely exchange-values, although these are attributes possessed by different quantities of various articles

[A]. If the figures were reversed, we should have $a' = \frac{2}{3}$, indicating a fall of $33\frac{1}{3}$ per cent. in M's exchange-value in [A]. If M purchases the same quantity at both periods, $a'a = a$, whence $a' = \frac{a}{a} = 1$, indicating constancy.

variously measured as regards their masses. To say the exchange-value of an ounce of gold is equal to the exchange-value of twenty bushels of wheat is to express equality as much as to say the weight of a bushel of wheat is equal to the weight of so or so many cubic inches of water. It is well known that a pound of feathers is as heavy as a pound of lead.[7]

These true mathematical equations admit of mathematical treatment. Thus from the equation $M_1 = aA_1$ we may derive $A_1 = \frac{1}{a} M_1$; and from $M_2 = a'aA_2$, $aA_2 = \frac{1}{a'} M_2$ or $A_2 = \frac{1}{a'a} M_2$ —which mean that a certain quantity of [A], and consequently [A] in general, has varied in exchange-value[8] compared with [M] by $\frac{1}{a'}$, the reciprocal of [M]'s variation in [A] (cf. Propositions IX. and XIII.). If we knew that $M_2 = M_1$, we should know that $a'aA_2 = aA_1$, whence $A_2 = \frac{1}{a'} A_1$ which means that A, or [A] in general, has varied in general exchange-value by $\frac{1}{a'}$ (in agreement with Proposition XXXIII.). Or if we knew that M_2 has varied from M_1 in a certain proportion, we should know that $a'aA_2$ has varied from aA_1 in the same proportion, the general exchange-value of A, and of [A], thus undergoing a double variation (in which the one may enhance, lessen, neutralize, or outdo the other). As yet, however, we do not know the relation of M_2 to M_1, this being what we want to find.

Still, for this purpose, the expressions are not serviceable in the meanings so far ascribed to them. For we cannot combine the general exchange-values of different things in order to find therefrom the general exchange-value of any one thing. What

[7] Bourguin says there are no equations between exchange-values, because exchange-value is not intrinsic, B. 132, p. 38. The reason is not adequate. We have seen that even weight is not intrinsic. Certainly wealth is not; yet two men may be equally rich.

[8] If we are dealing with M_a, the general exchange-value in which [A] so varies is likewise A_a. But if we are dealing with M_o, the general exchange-value in which [A] so varies is not A_o, but a general exchange-value of [A] in all things other to money, *i. e.*, in all commodities, including itself. Another distinction of a similar nature will be noticed in § 4.

we can so combine are the particular exchange-values of the thing in question in the other classes of things. Now by Proposition VII. a thing's particular exchange-value in anything else is equal to its general exchange-value; and by Proposition VI. the thing's general exchange-value is equal to the general exchange-value of the quantity of the other thing it exchanges for, by which quantity of that thing, according to Proposition I., its exchange-value in that kind of thing is measured. Therefore a thing's particular exchange-value in another thing is equal to the general exchange-value of that quantity of the other thing it is measured by. Hence the expression aA_1 may represent not only the general exchange-value of a A, but also the particular exchange-value M in [A], at the first period. And so the others. But it is necessary that the symbols should distinguish whether they refer to the general exchange-values of things, or to the particular exchange-value of M in those things. We may make them do so by always placing them in marks of parenthesis, thus (aA), (bB), (cC), ······, when referring to the particular exchange-value of M in [A], in [B], in [C], ······, at the first period. Then the expressions for the particular exchange-values of M in these things at the second period may be written either $(a'aA)$, $(b'bB)$, $(c'cC)$, ······, or $a'(aA)$, $b'(bB)$, $c'(cC)$, ······. The first of these forms would be merely the expressions for the particular exchange-values of M in [A], in [B], in [C], ······, as measured by the quantities of these things it exchanges for at the second period. But the second, while meaning this, mean, and express, something more, namely that the particular exchange-value of M in [A], for instance, is a' times what it was before, that is, at the second period a' times the particular exchange-value of M in [A] at the first period; wherefore in this form of the expression obviously—and also in the other—the contained expression (aA) still refers to the old exchange-value of M in [A]; and so with the rest. Hence it is no longer necessary here to number the expressions A, B, C, ······.[9]

[9] If we knew that $M_{02} = M_{01}$, then we should know that aA, if it refers to the general exchange-value of a A, refers to something different at the second period from what the general exchange-value of a A was at the first period, that is, it

We then have

$$M_1 = (aA) = (bB) = (cC) = \cdots\cdots,$$

which may be more specifically written

$$\mathrm{M}_{01} = (aA) = (bB) = (cC) = \cdots\cdots \text{ to } n \text{ terms},$$

n representing the number of all the (equally important) classes of things employed. This means that the general exchange-value of the money-unit in all other things is equal to every one of its particular exchange-values, which are stated as they are at the first period. And we have

$$M_{02} = (a'aA) = (b'bB) = (c'cC) = \cdots\cdots \text{ to } n \text{ terms},$$

which similarly expresses the equality between the general exchange-value of the money-unit in all other things and every one of its particular exchange-values, which are stated as they are at the second period. But if we write this as follows,

$$M_{02} = a'(aA) = b'(bB) = c'(cC) = \cdots\cdots \text{ to } n \text{ terms},$$

it expresses the same equality, but states the particular exchange-values at the second period as they relate to what they were at the first period.

§ 4. Using these running equations, we can now form expressions for the combination of the particular exchange-values of

refers to the general exchange-value of a A at the second period, wherefore it should be distinguished by a number, viz., aA_2. But in referring merely to the particular exchange-value of M in [A] when we do not know what the relation between M_{02} and M_{01} is, but are seeking it, we are using the particular exchange-value of M in [A] at the first period as our standard, and must look upon the particular exchange-value of M in [A] as changed at the second period by a', and therefore we must take (aA) as the constant and $a'(aA)$ as the variable. But (aA) alone is not our whole standard; it is only part of the whole standard composed of all other things. Yet we have to treat each of these separately as a standard, before we can unite them. Then after we have united them, and when we use the whole composed of them as the true standard, it is no contradiction with the preceding that we may find aA at the second period to be different from aA at the first. The particular exchange-value of M in [A] *has* changed by a' (while the particular exchange-value of A in [M] has changed by $\frac{1}{a'}$); yet, in the case supposed, the particular exchange-value of M in [A], now $a'(aA)$, is of the same magnitude as was the particular exchange-value of M in [A] when it was (aA), because at each period it is equal to the general exchange-value of M, which is supposed to be unchanged. The contradiction is only apparent, because of the change in the standard used. Cf. Chap. II., Sect. I., § 1.

M in every other thing into its one general exchange-value in all other things. We can, in fact, mathematically form several such combinations, but may content ourselves with three. To begin with the first period. From

$$M_{01} = (aA) = (bB) = (cC) = \cdots\cdots \text{ to } n \text{ terms}$$

we perceive that the sum total of the exchange-values of nM will be

$$nM_{01} = (aA) + (bB) + (cC) + \cdots\cdots \text{ to } n \text{ terms};$$

whence

$$M_{01} = \frac{1}{n}\left\{(aA) + (bA) + (cC) + \cdots\cdots \text{ to } n \text{ terms}\right\}.^{10} \quad (1, 1)$$

Again, the serial equation may be converted into this,

$$\frac{1}{M_{01}} = \frac{1}{(aA)} = \frac{1}{(bB)} = \frac{1}{(cC)} = \cdots\cdots \text{ to } n \text{ terms},$$

(which means that the exchange-value of a A in [M] is equal to the exchange-value of b B in [M], and so on, and to that of M in [M], in agreement with Proposition IV.). By the same treatment this yields

$$\frac{n}{M_{01}} = \frac{1}{(aA)} + \frac{1}{(bB)} + \frac{1}{(cC)} + \cdots\cdots \text{ to } n \text{ terms},$$

whence

$$M_{01} = \frac{1}{\frac{1}{n}\left\{\frac{1}{(aA)} + \frac{1}{(bB)} + \frac{1}{(cC)} + \cdots\cdots \text{ to } n \text{ terms}\right\}}. \quad (1, 2)$$

Lastly, from the first serial equation, and also from its inverted form, the total product of the exchange-values of M taken n times will be

$$M_{01}^n = (aA)\cdot(bB)\cdot(cC)\cdot\cdots\cdots \text{ to } n \text{ terms},$$

whence

$$M_{01} = \sqrt[n]{(aA)\cdot(bB)\cdot(cC)\cdot\cdots\cdots \text{ to } n \text{ terms}}. \quad (1, 3)$$

[10] With different notation, and also with reference to the quantities of things instead of their exchange-values, this form of the formula was, perhaps first, given by Prince-Smith, *op. cit.*, p. 372. Prince-Smith reached this formula in somewhat the same way as it is here reached; but he failed to see that there are other formulæ equally well deducible from the same original running formula. The idea expressed by this formula is also alone advanced by Nicholson, B. 94, pp. 307–308, and by Fonda, B. 127, p. 161.

Naturally the serial equation for the second period may be treated in the same way, and, likewise with even weighting, in its first form, it will yield, with omission of the superfluous "to n terms,"

$$M_{02} = \frac{1}{n}\{(a'aA) + (b'bB) + (c'cC) + \cdots\cdots\}, \qquad (2, 1)$$

$$M_{02} = \frac{1}{\frac{1}{n}\left\{\frac{1}{(a'aA)} + \frac{1}{(b'bB)} + \frac{1}{(c'cC)} + \cdots\cdots\right\}}, \qquad (2, 2)$$

$$M_{02} = \sqrt[n]{(a'aA)\cdot(b'bB)\cdot(c'cC)\cdot\cdots\cdots}\,. \qquad (2, 3)$$

What is here done for M, be it here parenthetically remarked, can be done for any of the other classes. For example at the first period, if we want to express the general exchange-value of A directly in all the others, we may begin by representing the general exchange-value of a A by $a\,A_{01}$, and then (M) would represent the exchange-value of a A in money, and (bB), (cC), ······ will continue to express the exchange value of a A in [B], in [C], ······. Then from

$$aA_{01} = (M) = (bB) = (cC) = \cdots\cdots \text{ to } n \text{ terms},$$

we can derive

$$aA_{01} = \frac{1}{n}\{(M) + (bB) + (cC) + \cdots\cdots \text{ to } n \text{ terms}\},$$

as in the first way; and similarly in the other two ways, which may be omitted. This formula we could have derived directly from formula 1, by simply substituting aA_{01} for M_{01} and (M) for (aA); and so on in the other omitted ways. And from this formula we derive

$$A_{01} = \frac{1}{an}\{(M) + (bB) + (cC) + \cdots\cdots\}.$$

Similarly for the second period we can get

$$a'aA_{02} = \frac{1}{n}\{(M) + (b'bB) + (c'cC) + \cdots\cdots\},$$

and

$$A_{02} = \frac{1}{a'an}\{(M) + (b'bB) + (c'cC) + \cdots\cdots\}.$$

It deserves, however, to be noticed that if we should use these formulæ to calculate the variation of the general exchange-value of A in the way to be described presently, we should obtain a result different from that obtained by calculating first the variation of the general exchange-value of M and then the variation of A by means of its variation in the varied money. For these two methods use different standards, the class A itself forming part of the standard in the latter method, but not in the former, and money forming part of the standard in the former, but not in the latter. (Cf. Propositions XII. and XIII.)

The interpretation of the above three kinds of formulæ is as follows. They each express primarily a way in which *the exchange-value in all other things* of the money-unit (or of the mass-unit of anything else) *is to be conceived*, the first, that it is to be conceived *as equal to the sum of its exchange-values in every one of the other things, or to the sum of all its particular exchange-values, divided by their number;* the second, *as equal to the reciprocal of the sum of the reciprocals of all its particular exchange-values divided by their number;* the third, *as equal to the n^{th} root of the product of all its particular exchange-values, n in number.*

Each of these formulæ is nothing but the expression of a mathematical mean or average between two or more equally important quantities—the first, of the *arithmetic*, the second, of the *harmonic*, the third, of the *geometric*. And each is perfectly true as an expression of its own kind of average drawn between the quantities given. But not more than one of these averages—and perhaps none of them—can be the proper or correct average to draw for the purpose we are putting it to. Having three ways of combining or averaging a thing's particular exchange-values into its one general exchange-value in all other things, we have an *embarras de richesses*, and much of our subsequent labor will be to decide between these different forms.

§ 5. Now our purpose being to compare the two combinations or averages at the two periods, we may make this comparison by dividing the formula for the second period by that for the first. Here we evidently want that form of the expression for the second period which brings its contained particular exchange-values into relation with the particular exchange-values of the first period. The comparison then will be as follows:

$$\frac{M_{02}}{M_{01}}=\frac{\frac{1}{n}\{a'(aA)+b'(bB)+c'(cC)+\cdots\cdots\}}{\frac{1}{n}\{(aA)+(bB)+(cC)+\cdots\cdots\}}, \tag{3, 1}$$

$$\frac{M_{02}}{M_{01}}=\frac{\dfrac{1}{\frac{1}{n}\left\{\frac{1}{a'(aA)}+\frac{1}{b'(bB)}+\frac{1}{c'(cC)}+\cdots\cdots\right\}}}{\dfrac{1}{\frac{1}{n}\left\{\frac{1}{(aA)}+\frac{1}{(bB)}+\frac{1}{(cC)}+\cdots\cdots\right\}}}, \tag{3, 2}$$

$$\frac{M_{02}}{M_{01}}=\frac{\sqrt[n]{a'(aA)\cdot b'(bB)\cdot c'(cC)\cdot\cdots\cdots}}{\sqrt[n]{(aA)\cdot(bB)\cdot(cC)\cdot\cdots\cdots}}. \tag{3, 3}$$

Here we have the formulæ desired for the variation of the general exchange-value of M, but in forms more complex than necessary. At the first period the exchange-values (aA), (bB), (cC), ... are each equal to the exchange-value of one money-unit. The exchange-value of the money-unit we take for the unit of exchange-value. Therefore $(aA)=(bB)=(cC)=\cdots\cdots=1$. Substituting this value of the terms in the denominators in each of the above equations, they all reduce to 1, thus:

$$M_{01}=\frac{1}{n}(1+1+1+\cdots\cdots)=\frac{1}{n}(n1)=1,$$

$$M_{01}=\frac{1}{\frac{1}{n}\left(\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\cdots\cdots\right)}=\frac{1}{\frac{1}{n}\left(\frac{n}{1}\right)}=\frac{1}{1}=1,$$

$$M_{01}=\sqrt[n]{1\cdot1\cdot1\cdot\cdots\cdots}=\sqrt[n]{1^n}=1.$$

Therefore, all these denominators may be dropped. In the numerators the terms, if written $(a'aA)$, $(b'bB)$, $(c'cC)$, ..., are also each equal to the exchange-value of the money-unit at the second period, and so the numerators also reduce to 1—only this is a different unit. But, of course, we want the same unit to be employed in all the terms throughout the equations, and the unit wanted is the unit employed in the denominators—the exchange-value of the money-unit at the first or basic period. Now in the numerators the terms (aA), (bB), (cC), ……, though no longer equal to the money-unit at the second period, still are expressions for the particular exchange-values of M at the first period, and so are still equal to the old unit. Therefore, as we employ this unit throughout, these terms, each = 1, may be dropped from the numerators also, and the equations, thus freed from useless symbols, reduce to the following:

$$\frac{M_{02}}{M_{01}} = \frac{1}{n}(a' + b' + c' + \cdots\cdots), \qquad (4, 1)$$

$$\frac{M_{02}}{M_{01}} = \frac{1}{\frac{1}{n}\left(\frac{1}{a'} + \frac{1}{b'} + \frac{1}{c'} + \cdots\cdots\right)}, \qquad (4, 2)$$

$$\frac{M_{02}}{M_{01}} = \sqrt[n]{a' \cdot b' \cdot c' \quad \cdots\cdots} \;. \qquad (4, 3)$$

Here we have the workable forms of the formulæ desired. In these forms the formulæ each express one of the mathematical averages of the *variations*, from the first to the second period, of the exchange-values of M in [A], in [B], in [C], ……, that is, of the variations of all the particular exchange-values of the money-unit in every other class of things. This appears to be as it should be; for a thing's variation in general exchange-value in all other things would seem to be made up of the variations of all its particular exchange-values, and therefore to be an average of some sort between all the particular variations. But we reached this position only through really comparing two distinct averages, in each of which the hypothesis was that the classes are equally important—*i. e.*, equally important at each of the periods. And in both these averages we used, in obedience to the principle stated in Chapter IV., Sect. V., § 9, the same number of classes—the same n. If we had used a different number of classes in each averaging, we could not have made this reduction to a common averaging. Whether we could make this reduction if the classes were unequally important at each period singly, but were equally important over both the periods together, is questionable. Perhaps it may be proper, perhaps not. By most writers on the subject, however, the question has hardly been raised. The fine distinctions as to the periods when the weighting is to be measured have not been noticed. Generally the assumption has been made that we can average the variations of the particular exchange-values using a single weighting, even though the weights of the two periods separately are different. We may for the present ignore this question, and assume that we are justified in using—or are dealing with cases which permit us to use—even weighting in an averaging of the variations of the particular exchange-values.

§ 6. The averages of the variations may be represented in still another way. Instead of representing the numbers of mass-units of the various classes purchasable by M at the second period by $a'a$, $b'b$, $c'c$, ..., let us represent them by a_2, b_2, c_2, ..., in distinction from which we shall represent the numbers purchasable at the first period by a_1, b_1, c_1, ... In this notation a_2, b_2, c_2, ..., taking the place of $a'a$, $b'b$, $c'c$, ..., are equal to them respectively; wherefore, as $a_2 = a'a_1$, $\frac{a_2}{a_1} = a'$, and similarly $\frac{b_2}{b_1} = b'$, $\frac{c_2}{c_1} = c'$, and so on. In other words, the variations are represented, in the new notation, by $\frac{a_2}{a_1}$, $\frac{b_2}{b_1}$, $\frac{c_2}{c_1}$, ... Therefore these new terms may take the places of a', b', c', ... in the formula 4, 1-3, and we have equally good and valid formulæ in the following:

$$\frac{M_{02}}{M_{01}} = \frac{1}{n}\left(\frac{a_2}{a_1} + \frac{b_2}{b_1} + \frac{c_2}{c_1} + \cdots\cdots\right), \qquad (5,\ 1)$$

$$\frac{M_{02}}{M_{01}} = \frac{1}{\frac{1}{n}\left(\frac{a_1}{a_2} + \frac{b_1}{b_2} + \frac{c_1}{c_2} + \cdots\cdots\right)}, \qquad (5,\ 2)$$

$$\frac{M_{02}}{M_{01}} = \sqrt[n]{\frac{a_2}{a_1} \cdot \frac{b_2}{b_1} \cdot \frac{c_2}{c_1} \cdot \cdots\cdots}. \qquad (5,\ 3)$$

§ 7. Any of these averages being adopted, if the result in any comparison of two periods is larger than unity in a certain proportion, this means that the exchange-value of money in all other things is indicated by this kind of averaging to have risen in that proportion (*e. g.*, if the result is 1.15, the rise is by 15 per cent.); if the result is less than unity in a certain proportion, the exchange-value of money is indicated to have fallen in the same proportion (*e. g.*, if the result is .85, the fall is by 15 per cent.); and if the result is unity, the exchange-value is indicated not to have varied, *i. e.*, to have remained constant.

If the variations of all the particular exchange-values of M are the same, so that $a' = b' = c' = \cdots\cdots = r$, or $\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1} = \cdots\cdots = r$, so that this one quantity can stand for them all, the results of all the averages are the same, and are r, as shown by the following:

$$\frac{M_{02}}{M_{01}} = \frac{1}{n}(r + r + r + \cdots\cdots) = \frac{1}{n}(nr) = r,$$

$$\frac{M_{02}}{M_{01}} = \frac{1}{\frac{1}{n}\left(\frac{1}{r} + \frac{1}{r} + \frac{1}{r} + \cdots\cdots\right)} = \frac{1}{\frac{1}{n}\left(\frac{n}{r}\right)} = \frac{1}{\frac{1}{r}} = r,$$

$$\frac{M_{02}}{M_{01}} = \sqrt[n]{r \cdot r \cdot r \cdot \cdots\cdots} = \sqrt[n]{r^n} = r.$$

Thus all these results agree with the knowledge we already possess that if a thing varies in all its particular exchange-values in a certain proportion, it varies in exchange-value in all other things in the same proportion (see Proposition XVII.). Unfortunately, therefore, that piece of knowledge does not help us to decide between these methods of averaging. Of course, if there is no variation in any of the particular exchange-values, so that $a' = b' = c' = \cdots\cdots = 1$, or $\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1} = \cdots\cdots = 1$, the result in all cases is 1, meaning that there is no variation in the general exchange-value (in accordance with Proposition XXVII.).

But if there is variation in at least one particular exchange-value, or a variation in at least one particular exchange-value divergent from all the other variations,—if there is the slightest irregularity,—the results in the three kinds of averaging always differ from each other. Between the three mathematical averages the mathematical relations are well known. If the variations happen to be such that the geometric average indicates constancy, the arithmetic average will always indicate a rise, the harmonic a fall. If the geometric average indicates a rise, the arithmetic will indicate a greater rise, the harmonic a lesser rise (perhaps constancy, perhaps a fall). If the geometric average indicates a fall, the arithmetic will indicate a lesser fall (perhaps constancy, perhaps a rise), the harmonic a greater fall. The geometric average is always between the arithmetic above and the harmonic below. In fact, when there are only two equally important terms used, the geometric mean is the geometric mean between the arithmetic and the harmonic means; and in other cases the geometric average is near that mean.[11]

It is plain that in none of the methods can constancy be indicated if there is only a variation of one particular exchange-value of money, or if all the variations are in the same direction; but only in the case of opposite and compensatory variations. This also agrees with what we already know (see Propositions XX. and XXVIII.). Given a single variation, however, each of the methods, in order to indicate constancy, requires a different compensatory variation—the arithmetic the greatest if the compensatory variation be a fall and the smallest if a rise, the harmonic the smallest if a fall and the greatest if a rise, the geometric always an intermediate.

II.

§ 1. It is easy merely to introduce uneven weighting into the formulæ. We only have to remember what uneven weighting means. Uneven weighting means that one class is larger than another, that is, contains more individuals; while classes evenly weighted are supposed to contain an equal number of individuals. What these individuals are we need not here again discuss. Now if a class contains twice as many such individuals as another, it virtually consists of twice as many classes as the

[11] See Appendix A VI. §§ 1–5.

other; and if we treat the other as one class we may treat this as two classes. The double number of individuals which are combined into the first class are so combined into one instead of into two classes only because they all have the same name and the same exchange-value. They are one class nominally; but logically they are two homonymous classes—compared with the other as one class. Thus suppose the class [B] is twice, and the class [C] thrice, as large as the class [A]; the state of things then is the same as if we had two classes both called [B], in both of which M has always the same exchange-value, and three classes all called [C], in all of which M has always the same exchange-value. If this be supposed to be the state of things at the first period, then at the first period the formulæ would be as follows:

$$M_{a1} = \frac{1}{n''}\left\{(aA) + (bB) + (bB) + (cC) + (cC) + (cC) + \cdots\cdots \text{to } n'' \text{ terms}\right\}$$

$$= \frac{1}{n''}\left\{(aA) + 2(bB) + 3(cC) + \cdots\cdots \text{to } n \text{ terms}\right\}, \qquad (6, 1)$$

$$M_{h1} = \cfrac{1}{\frac{1}{n''}\left\{\frac{1}{aA} + \frac{1}{bB} + \frac{1}{bB} + \frac{1}{cC} + \frac{1}{cC} + \frac{1}{cC} + \cdots\cdots \text{to } n'' \text{ terms}\right\}}$$

$$= \cfrac{1}{\frac{1}{n''}\left\{\frac{1}{aA} + \frac{2}{bB} + \frac{3}{cC} + \cdots\cdots \text{to } n \text{ terms}\right\}}, \qquad (6, 2)$$

$$M_{g1} = \sqrt[n'']{(aA)\cdot(bB)\cdot(bB)\cdot(cC)\cdot(cC)\cdot(cC)\cdot\cdots\cdots \text{to } n'' \text{ terms}}$$

$$= \sqrt[n'']{(aA)\cdot(bB)^2\cdot(cC)^3\cdot\cdots\cdots \text{to } n \text{ terms}}; \qquad (6, 3)$$

in which n'' represents the number of the virtual classes, or n enlarged to cover all the coefficients or exponents.

If, then, at the second period the same relative conditions persisted, in spite of variations in the particular exchange-values, we should have formulæ similar to these, with addition only of the symbols indicative of the variations. Then the comparison is easy. We may, however, represent the weighting more generally by employing algebraic symbols throughout. Let **a** represent the weight of [A], **b** the weight of [B], **c** the weight of [C], and so on, all these weights being supposed to be the same at both periods; and let $n'' = \mathbf{a} + \mathbf{b} + \mathbf{c} + + \cdots\cdots$ to n terms, the same classes of course being used in each of the averages compared. Then we have

$$\frac{M_{02}}{M_{01}}=\frac{\frac{1}{n''}\{\mathbf{a}a'(aA)+\mathbf{b}\,b'(bB)+\mathbf{c}\,c'(cC)+\cdots\cdots\text{ to } n \text{ terms}\}}{\frac{1}{n''}\{\mathbf{a}\,(aA)+\mathbf{b}\,(bB)+\mathbf{c}\,(cC)+\cdots\cdots\text{ to } n \text{ terms}\}}, \quad (7, 1)$$

$$\frac{M_{02}}{M_{01}}=\frac{\dfrac{1}{\frac{1}{n''}\left\{\frac{\mathbf{a}}{a'(aA)}+\frac{\mathbf{b}}{b'(bB)}+\frac{\mathbf{c}}{c'(cC)}+\cdots\cdots\text{to } n \text{ terms}\right\}}}{\dfrac{1}{\frac{1}{n''}\left\{\frac{\mathbf{a}}{(a'A)}+\frac{\mathbf{b}}{(bB)}+\frac{\mathbf{c}}{(cC)}+\cdots\cdots\text{to } n \text{ terms}\right\}}}, \quad (7, 2)$$

$$\frac{M_{02}}{M_{01}}=\frac{\sqrt[n'']{\{a'(aA)\}^{\mathbf{a}}\cdot\{b'(bB)\}^{\mathbf{b}}\cdot\{c'(cC)\}^{\mathbf{c}}\cdot\cdots\cdots\text{to}\,n\,\text{terms}}}{\sqrt[n'']{(aA)^{\mathbf{a}}\cdot(bB)^{\mathbf{b}}\cdot(cC)^{\mathbf{c}}\cdot\cdots\cdots\text{ to } n \text{ terms}}}. \quad (7, 3)$$

In all these, again, the denominators reduce to unity; and also the expressions (aA), (bB), (cC), ······ in the numerators are the same units. Therefore we have

$$\frac{M_{02}}{M_{01}}=\frac{1}{n''}(\mathbf{a}\,a'+\mathbf{b}\,b'+\mathbf{c}\,c'+\cdots\cdots\text{ to } n \text{ terms}), \quad (8, 1)$$

$$\frac{M_{02}}{M_{01}}=\frac{1}{\frac{1}{n''}\left(\frac{\mathbf{a}}{a'}+\frac{\mathbf{b}}{b'}+\frac{\mathbf{c}}{c'}+\cdots\cdots\text{ to } n \text{ terms}\right)}, \quad (8, 2)$$

$$\frac{M_{02}}{M_{01}}=\sqrt[n'']{a'^{\mathbf{a}}\cdot b'^{\mathbf{b}}\cdot c'^{\mathbf{c}}\cdot\cdots\cdots\text{to}\,n\,\text{terms}}. \quad (8, 3)$$

Or if we prefer the other system of notation to represent the variations, we have

$$\frac{M_{02}}{M_{01}}=\frac{1}{n''}\left(\mathbf{a}\frac{a_2}{a_1}+\mathbf{b}\frac{b_2}{b_1}+\mathbf{c}\frac{c_2}{c_1}+\cdots\cdots\text{ to } n \text{ terms}\right), \quad (9, 1)$$

$$\frac{M_{02}}{M_{01}}=\frac{1}{\frac{1}{n''}\left(\mathbf{a}\frac{a_1}{a_2}+\mathbf{b}\frac{b_1}{b_2}+\mathbf{c}\frac{c_1}{c_2}+\cdots\cdots\text{ to } n \text{ terms}\right)}, \quad (9, 2)$$

$$\frac{M_{02}}{M_{01}}=\sqrt[n'']{\left(\frac{a_2}{a_1}\right)^{\mathbf{a}}\cdot\left(\frac{b_2}{b_1}\right)^{\mathbf{b}}\cdot\left(\frac{c_2}{c_1}\right)^{\mathbf{c}}\cdot\cdots\cdots\text{ to } n \text{ terms}}. \quad (9, 3)$$

Here again are formulæ representing averages of the variations of the particular exchange-values. The reductions by which they are obtained suppose that the weights are the same at each period separately. But generally in practice the weights are different at each period separately. Still if we are to average the variations, we must employ only a single system of weighting—for we cannot, for instance, have a variation of two classes of [A] *and* a variation of three classes of [A]. We must have either a variation of two classes of [A], *or* a variation of three classes of [A], *or* a variation of a number of classes of [A] that is a mean between the numbers of classes of [A] at each of the periods separately, as examined in the preceding Chapter. For the present we may assume that this course is a proper one, and that one of these weightings is the proper one. We may, then, continue to represent the weight of [A] by **a**, and that of [B] by **b**, and so on, and their sum by n'', in whatever way these weights be calculated.

The averages agree with much that we have already discovered concerning variations of exchange-value, and disagree with nothing we yet know. Thus it is plain that in any of these formulæ a change of weighting without any change of exchange-values causes no change in the result; for then the terms $a', b', c', \cdots\cdots$, or $\frac{a_2}{a_1}, \frac{b_2}{b_1}, \frac{c_2}{c_1}, \cdots\cdots$, are all 1, and its does not matter how large or small **a**, **b**, **c**, ······ be, the results are always 1, indicating constancy in the general exchange-value of money. This is in agreement with Propositions XXVII. and XLIV. Also if the variations are all alike, so that $a' = b' = c' = \cdots\cdots = r$, or $\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1} = \cdots\cdots = r$, no matter what be the weights, the results are always r,[1] in agreement with Propositions XVII. and XLV.

In all these formulæ the results are the same whether we use the weighting **a**, **b**, **c**, ······, with $n'' = \mathbf{a} + \mathbf{b} + \mathbf{c} + \cdots\cdots$ to n terms, or whether we use $r\,\mathbf{a}$, $r\,\mathbf{b}$, $r\,\mathbf{c}$, ······, with $n'' = r\,\mathbf{a} + r\,\mathbf{b} + r\,\mathbf{c} \cdots\cdots$ to n terms.[2] This means that it is indifferent how large or small be the weights, provided they always maintain

[1] Cf. Appendix A I. §§ 5, 6, II. § 6, III. § 6, V. § 5.
[2] Cf. Appendix A I. § 11.

the same relative sizes. Therefore the weights may be any numbers, larger or smaller than unity, integral or fractional. It is, then, not necessary that any of the classes should be taken as a unit class like [A] in the above numerical example, although our subsequent calculations might be slightly simpler if this were done and the other weights reduced accordingly. In this case the coefficient of the unit class, and of all the classes equal to it, namely 1, must be counted in the composition of the term n''.

§ 2. It may, however, turn out that this course of averaging the variations on a single weighting is not right, no weighting being discoverable that will in all cases yield a demonstrably correct result. If this be so, we shall not be able to make the reduction of the comparison of the two averages, one at each period, to a single average of the variations, but shall have to retain the comparison of the two averages. This constitutes what we have called the method of double weighting. In our formulæ we shall have to distingush **a** into $\mathbf{a}_1$ and $\mathbf{a}_2$, **b** into $\mathbf{b}_1$ and $\mathbf{b}_2$, and so on, representing the weights at the first and second periods respectively; and consequently also n'' into n_1'' and n_2'', although n must remain undivided. Then the formulæ will become the following:

$$\frac{Mo_2}{Mo_1}=\frac{\frac{1}{n_2''}\{\mathbf{a}_2a'(aA)+\mathbf{b}_2b'(bB)+\mathbf{c}_2c'(cC)+\cdots\cdots \text{ to } n \text{ terms}\}}{\frac{1}{n_1''}\{\mathbf{a}_1(aA)+\mathbf{b}_1(bB)+\mathbf{c}_1(cC)+\cdots\cdots \text{ to } n \text{ terms}\}}, \qquad (10, 1)$$

$$\frac{Mo_2}{Mo_1}=\frac{\dfrac{1}{\frac{1}{n_2''}\left\{\frac{\mathbf{a}_2}{a'(aA)}+\frac{\mathbf{b}_2}{b'(bB)}+\frac{\mathbf{c}_2}{c'(cC)}+\cdots\cdots \text{ to } n \text{ terms}\right\}}}{\dfrac{1}{\frac{1}{n_1''}\left\{\frac{\mathbf{a}_1}{(aA)}+\frac{\mathbf{b}_1}{(bB)}+\frac{\mathbf{c}_1}{(cC)}+\cdots\cdots \text{ to } n \text{ terms}\right\}}}, \qquad (10, 2)$$

$$\frac{Mo_2}{Mo_1}=\frac{\sqrt[n_2'']{\{a'(aA)\}^{\mathbf{a}_2}\cdot\{b'(bB)\}^{\mathbf{b}_2}\cdot\{c'(cC)\}^{\mathbf{c}_2}\cdots\cdots \text{ to } n \text{ terms}}}{\sqrt[n_1'']{(aA)^{\mathbf{a}_1}\cdot(bB)^{\mathbf{b}_1}\cdot(cC)^{\mathbf{c}_1}\cdots\cdots \text{ to } n \text{ terms}}}. \qquad (10, 3)$$

But here the denominators again all reduce to unity, and may be omitted; and if in the numerators the terms (aA), (bB), (cC), ······ be conceived as referring to the exchange-values of the money-unit at the first period, and therefore, as mere units, be omitted, the formulæ reduce to expressions for the averages of the variations with the weighting of the second period—which is absurd. And if, then, we conceive of the terms $a'(aA)$, $b'(bB)$, $c'(cC)$, ... as referring to the exchange-values in the money of the second period, whereupon they each become equal to unity, the numerators also reduce to unity, and all these formulæ are useless.

What we need is this. The terms a, b, c, ..., and $a'a$, $b'b$, $c'c$, ..., or better, a_1, b_1, c_1, ... and a_2, b_2, c_2, ..., express the numbers of times the money-unit is more valuable than certain mass-units of [A], [B], [C], ... at the first and at the second period respectively. Therefore we want to average these ratios. But to average them properly we need to know how often they are repeated at each period. Let x_1 and x_2 represent the numbers of times the mass-unit of [A] is repeated at each period, that is, the numbers of these mass-units that have been produced or consumed within these periods; and let y_1 and y_2, z_1 and z_2, ... represent the numbers of the mass-units of [B], [C], ... And let $n_1' = x_1 + y_1 + z_1 +$ ...... to n terms, and $n_2' = x_2 + y_2 + z_2 +$ ...... to n terms. Then the desired formulæ are

$$\frac{M_{02}}{M_{01}} = \frac{\frac{1}{n_2'}(x_2a_2 + y_2b_2 + z_2c_2 + \cdots\cdots \text{ to } n \text{ terms})}{\frac{1}{n_1'}(x_1a_1 + y_1b_1 + z_1c_1 + \cdots\cdots \text{ to } n \text{ terms})}$$

$$= \frac{n_1'(x_2a_2 + y_2b_2 + z_2c_2 + \cdots\cdots \text{ to } n \text{ terms})}{n_2'(x_1a_1 + y_1b_1 + z_1c_1 + \cdots\cdots \text{ to } n \text{ terms})}, \qquad (11, 1)$$

$$\frac{M_{02}}{M_{01}} = \frac{\dfrac{1}{\frac{1}{n_2'}\left(\frac{x_2}{a_2} + \frac{y_2}{b_2} + \frac{z_2}{c_2} + \cdots\cdots \text{ to } n \text{ terms}\right)}}{\dfrac{1}{\frac{1}{n_1'}\left(\frac{x_1}{a_1} + \frac{y_1}{b_1} + \frac{z_1}{c_1} + \cdots\cdots \text{ to } n \text{ terms}\right)}}$$

$$= \frac{n_2\left(\frac{x_1}{a_1} + \frac{y_1}{b_1} + \frac{z_1}{c_1} + \cdots\cdots \text{ to } n \text{ terms}\right)}{n_1\left(\frac{x_2}{a_2} + \frac{y_2}{b_2} + \frac{z_2}{c_2} + \cdots\cdots \text{ to } n \text{ terms}\right)}, \qquad (11, 2)$$

$$\frac{M_{02}}{M_{01}} = \frac{\sqrt[n_2']{a_2^{x_2} \cdot b_2^{y_2} \cdot c_2^{z_2} \cdot \cdots\cdots \text{ to } n \text{ terms}}}{\sqrt[n_1']{a_1^{x_1} \cdot b_1^{y_1} \cdot c_1^{z_1} \cdot \cdots\cdots \text{ to } n \text{ terms}}}. \qquad (11, 3)$$

These may also be written more fully by expanding the n's. The first so treated becomes

$$\frac{M_{02}}{M_{01}} = \frac{x_2a_2 + y_2b_2 + z_2c_2 + \cdots\cdots}{x_1a_1 + y_1b_1 + z_1c_1 + \cdots\cdots} \cdot \frac{x_1 + y_1 + z_1 + \cdots\cdots}{x_2 + y_2 + z_2 + \cdots\cdots},$$

which is the general formula for the method of double weighting with the arithmetic average.

But here an important remark must be made. It is evident that in all these last formulæ the results will be variously affected by the sizes of the mass-units we happen to choose. For if we choose large mass-units, the numbers of times they occur at each period (the expressions $x, y, z, \cdots\cdots$) and the numbers of times the money-unit is more valuable than they (the expressions $a, b, c, \cdots\cdots$) are smaller than the numbers of times smaller mass-units would occur and than the numbers of times the money-unit would be more valuable than smaller mass-units. In some of the terms, as in the parts within brackets in the second formula, these two changes neutralize each other; but not in others, and never in the terms n_1' and n_2', except only if all the mass-units be altered in the same proportion. But if the mass-units be variously and indiscriminately chosen, the terms n_1' and n_2', not to mention others in some of the formulæ, will variously relate to each other, and the results will be various, in a haphazard manner. Hence, it is necessary that some method of selecting the mass-units be established, as without such a method also these formulæ are worthless. Such a method need not be stated in the formula, and then the above formulæ become applicable only on the supposition that the method of selecting the mass-units has already been found and applied. Or again it may sometimes be possible to introduce into the formula itself the method of selecting the mass-units. This has actually been done in some methods dealing with the inverted cases of price measurements. These we shall examine later.

III.

§ 1. Certain warnings are needed against errors into which the use of mathematical formulæ in this subject may lead us, unless we be careful.

Two general principles are obtained from the preceding investigations. The one is that in cases when the weighting is the same at both the periods compared, it is immaterial whether we average all the particular exchange-values at the two periods separately and then find the variation of these averages, or whether we average all the variations of the particular exchange-values. The other is that when the weighting is different, and different weighting is used in the averaging, at each period, we can only find the variation of these averages, there being no one averaging of the particular variations that can use double weighting. We have seen, however, that we can draw an average between the two weightings; and now we might use this single weighting in averaging the particular variations. But we shall find that this last operation will yield the same result as the comparison of the two separate averages only in peculiar circumstances. We therefore have before us two distinct methods. The one is to compare the averages separately drawn at each period. The other is to average the particular variations. These two coincide only when the weighting actually is the same at both periods, or under other peculiar conditions. Even in the first case, when the weighting (reckoned according to the relative total exchange-values of the classes) is the same at both periods, there might be made averages at each period on other systems of weighting (*e. g.*, according to mass-quantities), which might be different at each period, or if the same at both periods, might be so only in case the weighting, properly reckoned, is different at each period.

Thus the subject is very complex. Running through it all are the similarity and difference between comparisons of averages that have varied and averages of the particular variations—a purely mathematical question, which has never been fully studied even by mathematicians, let alone economists. Hence it has been necessary in this work to make such a mathematical study, to serve as a foundation for the various applications of the mathematical principles which we shall have to make in dealing with the averages of exchange-values and of prices, and with their variations. This study is added as Appendix A.

There are many difficulties engendered by the many complexities in our subject, which we shall meet from time to time. But there are some simple errors into which we might easily slip at the commencement, and into which many persons have slipped. Some of them are even suggested by the second method of notation above given—a method which, in its general feature of distinguishing the exchange-values (or again the prices) of the same classes at the different periods only by different subscript numbers appended to the same letters, is the simplest and the most commonly used. Against the errors so suggested it is well here to give warning. These occur even when the weighting is the same at both periods—or when we are attempting to use a single weighting in averaging variations. Confinement may be made for the present to such cases.

§ 2. A general error is this. Because in these cases the method of averaging the variations properly agrees with the method of comparing the averages, and because the former is the simpler method, we might be tempted to say of exchange-values what Jevons said of prices,[1] that there is no such thing as an average of them, but only an average of their variations. Indeed, as all the particular exchange-values of anything at any time are always equal, not only the average of the particular exchange-values of money properly weighted at the first period is always unity, or $1M_{01}$, but at the second period also the average of the particular exchange-values of money properly weighted will also be unity in the new unit, M_{02}. Still there may be a different average of the exchange-values at the second period in the unit of the first period; which is just what we want, both because we ought to conduct the whole operation on the same unit, and because it is convenient that the average at the first period, with which that at the second is compared, should always

[1] B. 22, p. 23.

be 1, as it forms the denominator in the expression for the comparison. The process then is the following: At the first period we construct, as it were, a level of the particular exchange-values of money. This level breaks up at the second period into an undulating line, of the various heights of which an average may be drawn and compared with the uniform height of the first level. This operation evidently gives both an averaging of the variations and a variation of the average at the second period from the level or average at the first. It is difficult, in fact, to conceive of an average of variations without admitting an average of the starting and of the finishing points. But the averages do not exist in nature, nor are the positions given upon which averages merely need to be drawn. The whole operation has to be constructed, and error will ensue unless it be constructed in the right way. Our guiding consideration must be that the variation of the averages must, in these cases, give the same result as the proper average (of the same mathematical species) of the variations. This agreement is always obtained if we construct the average at the first period upon equal terms, so that the average itself is equal to them, and then use the same weighting in both methods.

But the agreement is also obtainable by comparing separate averages at each period without the average at the first period being so constructed upon equal terms. This occurs always in the geometric average if we use the same weighting in both methods; but in the arithmetic and harmonic averages never if we use the same weighting,[2] and only if we use different weighting in each method, a definite relation existing between those which have to be employed to get the same result. For in the arithmetic and harmonic averaging the inequality of the terms themselves at one of the periods forms part of the weighting in the comparison of the separately made averages, but has no influence on the weighting in the averaging of the variations; wherefore, unless it is allowed for in the former, an apparent similarity in the weighting will really be a difference, and real

[2] Except when all the variations are exactly alike, or when there are no variations. See Appendix A I. § 6. As these cases rarely happen, and as we have no control in regard to them, they are not of much importance.

similarity can be obtained only by employing different apparent weighting.

Now if we had at the start employed the form of notation later introduced, namely that of using a_2, b_2, c_2, ······ for the numbers of the mass-units of the classes purchasable with the money-unit at the second period, we might have fallen insensibly into the treatment of comparisons of averages employing even, or what we thought to be the proper uneven, weighting, and to have supposed that we were then employing even, or the same proper uneven, weighting in averaging the variation; or if we happened to notice the divergence in the results, we might have been nonplussed, and perhaps have given up the subject in despair, as some economists have done.

The errors we could thus fall into are different in the different kinds of averaging; wherefore these need to be treated separately.

§ 3. *Arithmetic averaging.*—Intending to employ even weighting at both periods and in the averaging of the variations—or to disregard weighting, using "unweighted" averages—we might represent the average at the first period thus,

$$M_{01} = \frac{1}{n}\{(a_1A) + (b_1B) + (c_1C) + \cdots\cdots\},$$

and that at the second period thus,

$$M_{02} = \frac{1}{n}\{(a_2A) + (b_2B) + (c_2C) + \cdots\cdots\};$$

wherefore the comparison of these would be

$$\frac{M_{02}}{M_{01}} = \frac{\frac{1}{n}\{(a_2A) + (b_2B) + (c_2C) + \cdots\cdots\}}{\frac{1}{n}\{(a_1A) + (b_1B) + (c_1C) + \cdots\cdots\}},$$

which reduces to

$$\frac{M_{02}}{M_{01}} = \frac{(a_2A) + (b_2B) + (c_2C) + \cdots\cdots}{(a_1A) + (b_1B) + (c_1C) + \cdots\cdots},$$

the variation of the arithmetic averages being the same as the variation of the sums. Both of these comparisons of the averages

...t, though meaningless without further ...we might have argued that, as A, B, C, ...unity, or merely to make reference to the ...]......, therefore they can be omitted, and

$$\frac{M_{02}}{M_{01}} = \frac{a_2 + b_2 + c_2 + \cdots\cdots}{a_1 + b_1 + c_1 + \cdots\cdots}.$$

...on, however, would be wrong, because it is ...which are equal to unity (the exchange-...ey-unit), but (aA), (bB), (cC), ... ; and ...ot refer to the classes, but, taken thus ...al exchange-values of the mass-units of the ...ain, if we dropped only the denominator as the remaining form

$$= \frac{1}{n}\{(a_2A) + (b_2B) + (c_2C) + \cdots\cdots\}$$

...ngless, because in the unit at the second period ...unity, and it contains no indication of the value ...$A)$, (b_2B), (c_2C), ... in the original unit M_{01}. ...er, know that $a_1A = b_1B = c_1C = \cdots\cdots = 1$,

$\frac{1}{a_1}$, $B = \frac{1}{b_1}$, $C = \frac{1}{c_1}$, and so on. Substituting

$$\frac{M_{02}}{M_{01}} = \frac{1}{n}\left(\frac{a_2}{a_1} + \frac{b_2}{b_1} + \frac{c_2}{c_1} + \cdots\cdots\right),$$

...ect form already obtained, (5, 1). ...argumentation by which the formula

$$\frac{M_{02}}{M_{01}} = \frac{a_2 + b_2 + c_2 + \cdots\cdots}{a_1 + b_1 + c_1 + \cdots\cdots}$$

...t also this formula itself, is wrong. By saying, ...ng we mean it is not universally right, although ...n some occasions. This formula is wrong be-... universally carry out—and in practice is rarely likely to carry out—the idea we intended to carry out, namely to employ even weighting. For (as is more fully shown in Appendix A, II. § 7) this formula really contains a special system of weighting in the case of averaging the variations, as it agrees with the formula

$$\frac{M_{02}}{M_{01}} = \frac{1}{n''}\left(a_1\frac{a_2}{a_1} + b_1\frac{b_2}{b_1} + c_1\frac{c_2}{c_1} + \cdots\cdots \text{ to } n \text{ terms}\right),$$

(in which $n'' = a_1 + b_1 + c_1 + \cdots\cdots$ to n terms), which is the formula for averaging the variations with the weights a_1 for [A], b_1 for [B], c_1 for [C], ... (so that there would be even weighting here only in case it happened that $a_1 = b_1 = c_1 = \cdots\cdots$). But we have probably chosen these terms at haphazard, that is, without paying regard to the importance of the classes: we have sought merely, in the different classes, of the usually employed mass-units the numbers that happen at the first period to be equivalent to the money-unit. The weighting for the variations which we are really using is, then, haphazard weighting, and the whole process is absurd.

No reliable result can be obtained in this way, and—which proves it—an indefinite number of different results may be so obtained. This may be shown by an example. Suppose the money-unit purchases two quarters of wheat at the first period and at the second three, at the first four bales of cotton and at the second two, at the first five tons of coal and at the second still five; and suppose we are trying to measure the exchange-value of money in these three other things only. We might then be tempted to represent the average exchange-value of money at each period, and consequently the variation of the averages, in the following way,

$$\frac{M_{02}}{M_{01}} = \frac{\frac{1}{3}(3 + 2 + 5)}{\frac{1}{3}(2 + 4 + 5)} = \frac{3 + 2 + 5}{2 + 4 + 5} = \frac{10}{11} = 0.9090,$$

which indicates a fall of 9.09 per cent. Now if it had happened that wheat had been measured in bushels instead of in quarters, exchange-value of the money-unit at the first period would 'e been equal to that of sixteen bushels of wheat and at the second to that of twenty-four. Then if these figures had been used, we might have represented the averages at each period and the comparison as follows,

$$\frac{M_{02}}{M_{01}} = \frac{\frac{1}{3}(24 + 2 + 5)}{\frac{1}{3}(16 + 4 + 5)} = \frac{24 + 2 + 5}{16 + 4 + 5} = \frac{31}{25} = 1.24,$$

which indicates a rise of 24 per cent.

The reason for the difference in the results is plain. It is that in the latter case a great deal more importance or weight is attached to the rise of the wheat than to the fall of the cotton and to the constancy of the coal, whereas in the former case there was more importance attached both to the fall of the cotton and to the constancy of the coal. Now if even weighting were employed, or intended, the proper measurement would be in this form,

$$\frac{M_{02}}{M_{01}} = \frac{1}{3}\left(\frac{3}{2} + \frac{2}{4} + \frac{5}{5}\right) = \frac{1}{3}\left(\frac{24}{16} + \frac{2}{4} + \frac{5}{5}\right) = 1,$$

which indicates constancy (on the supposition that the arithmetic average is the right one to employ, the rise of wheat by fifty per cent. being counterbalanced by the fall of cotton by fifty per cent.). The same result would be obtained if we put the comparison in this form

$$\frac{M_{02}}{M_{01}} = \frac{\frac{1}{3}(1\frac{1}{2}[2] + \frac{1}{2}[4] + 1[5])}{\frac{1}{3}(1[2] + 1[4] + 1[5])},$$

and used only the outer coefficients. This is what we did when we employed the algebraic symbols $a'a$, $b'b$, $c'c$, ... for the second period and a, b, c, ... for the first, and then counted only a', b', c', ... This method gives only one result—the only result which can be given with the employment of even weighting for the variations. The other gives as many results as there are mass-units that might be employed. That these various results are given by employing variously weighted averages of the variations, in which the weighting is according to the number of mass-units purchasable at the first period, may be shown in the above instances as follows,

$$\frac{M_{02}}{M_{01}} = \frac{3+2+5}{2+4+5} = \frac{1}{11}\left(2 \times \frac{3}{2} + 4 \times \frac{2}{4} + 5 \times \frac{5}{5}\right) = \frac{10}{11},$$

and

$$\frac{M_{02}}{M_{01}} = \frac{24+2+5}{16+4+5} = \frac{1}{25}\left(16 \times \frac{24}{16} + 4 \times \frac{2}{4} + 5 \times \frac{5}{5}\right) = \frac{31}{25}.$$

Thus, in using the arithmetic average, when we want to employ even weighting, we must divide the exchange-values of the money-unit in each class at the second period by its exchange-value in the same class at the first period, then add the quotients, and divide by the number of the classes. This method may, for short, be called the method of *dividing before adding.* The other would be the method of *adding before dividing.* The latter always gives the same result as the former, and correctly carries out the intention, if we first reduce the numbers of the mass-units purchasable at the first period to the same figure, preferably to unity, or to 100 (by creating, so to speak, a special mass-unit for each class). It is wrong simply to add up the purchasable numbers of any mass-units that happen to be employed. In doing so, we are really using uneven weighting for the classes according to the numbers of these units which happen to be purchasable at the first period. The larger the mass-unit which happens to be employed, the smaller is the number of these units purchasable, and the smaller the weighting; and the smaller the unit, the larger the weighting. Yet in spite of itself, it is conceivable that this method might happen to be right. For, though intending to employ even weighting, if things so happened that the uneven weighting really involved should be the correct weighting, then our operation would be much better than what we intended. But this would happen only on the very improbable supposition that the numbers of the mass-units of the different classes purchasable at the first period were directly proportional to the sizes of the classes properly estimated. It is especially unlikely to happen because the custom is to employ larger units for the more important classes, so that the weighting is more apt to be the opposite of what it ought to be. If we employed the same mass-unit throughout, for instance, a pound, the ng would be inversely according to the preciousness[3] of ses,—which would mostly be in the right direction, but usly exaggerated in some instances, and generally most r.

employment of a single mass-unit seems to have the endation that it does away with the multiplicity of obtainable, and so with our perplexity about deciding them. But it is not so; for if we employed a weight-oughout, or a capacity-unit throughout, the results in the s of each period separately and in comparison would be t. Evidently neither measure is more authoritative than r, and neither provides the weighting which any rational nt of the subject would recommend.

however, questionable whether in practice any of these s is worse than the method carrying out even weighting is, whether the weighting in any of these methods de-ore from the true weighting than does even weighting. If, then, we wish to employ proper weighting, and think have found what it ought to be, representing the weight | by **a**, that for [B] by **b**, that for [C] by **c**, and so on; roceeded as before and introduced *this* weighting for the in the averages at the periods separately, and then com-nem, as represented in this formula

$$\frac{M_{02}}{M_{01}} = \frac{\mathbf{a}(a_2A) + \mathbf{b}(b_2B) + \mathbf{c}(c_2C) + \cdots\cdots}{\mathbf{a}(a_1A) + \mathbf{b}(b_1B) + \mathbf{c}(c_1C) + \cdots\cdots},$$

n dropped the A, B, C, ..., wrongly taking them for r as merely referring to the classes, and so used the

$$\frac{M_{02}}{M_{01}} = \frac{\mathbf{a}\,a_2 + \mathbf{b}\,b_2 + \mathbf{c}\,c_2 + \cdots\cdots}{\mathbf{a}\,a_1 + \mathbf{b}\,b_1 + \mathbf{c}\,c_1 + \cdots\cdots},$$

nula would still be wrong, and would still not universally t we intended; for now the weights of the variations e (as shown in Appendix A, I. § 8), not **a**, **b**, **c**, ..., as

[3] iousness is an idea in economics very much like density in physics. s the ratio of the weight of a body to its volume. Preciousness is the he exchange-value of a body either to its volume or to its weight.

intended, but **a** a_1, **b** b_1, **c** c_1, ..., that is, something else, irregular and haphazard, containing the same accidental factors as before, namely, the quantities at the first period purchasable with one money-unit and measured in various mass-units. And there would also be an indefinite number of results obtainable, according to the possible combinations of possible mass-units that may be used. But, of course, if the reduction had been made so that a_1, b_1, c_1, ... are units, or any uniform quantity, then the weighting employed would be the one intended for the variations. Naturally, with the proper weighting rightly applied there can be only one result.

Thus in our previous numerical example, suppose at each period (or on the whole) we found the class wheat to be twice as important as the class cotton and three times as important as the class coal, and wanted to use this weighting. This weighting for the variations would be carried out in the following manner,

$$\frac{M_{02}}{M_{01}} = \frac{1}{6}\left(3 \times \frac{3}{2} + 2 \times \frac{2}{4} + 1 \times \frac{5}{5}\right) = 1.08\tfrac{1}{3},$$

indicating a rise of $8\frac{1}{3}$ per cent. But if we employed the same weighting in the method of averaging separately on the unequal figures above instanced, we should have in the one case

$$\frac{M_{02}}{M_{01}} = \frac{\frac{1}{6}(3 \times 3 + 2 \times 2 + 1 \times 5)}{\frac{1}{6}(3 \times 2 + 2 \times 4 + 1 \times 5)} = \frac{9 + 4 + 5}{6 + 8 + 5} = \frac{18}{19} = 0.947,$$

indicating a fall of 5.3 per cent.; and in the other,

$$\frac{M_{02}}{M_{01}} = \frac{\frac{1}{6}(3 \times 24 + 2 \times 2 + 1 \times 5)}{\frac{1}{6}(3 \times 16 + 2 \times 4 + 1 \times 5)} = \frac{72 + 4 + 5}{48 + 8 + 5} = \frac{81}{61} = 1.327,$$

indicating a rise of 32.7 per cent. These discrepancies are due to the fact that in the first operation, which carries out the weighting intended, we are attaching slightly more importance to the rise of wheat than to the fall of cotton and to the constancy of coal; in the second, we are really attaching slightly more importance to the fall of cotton (in the proportion of 8 to 6 and 5); in the third we are really attaching much more importance than we intended to the rise of wheat (in the proportion of 48 to 8 and 5). When we come to treat of the measurement of the exchange-value of money by means of prices, we shall find a convenient way by which the right result may in some cases be very conveniently reached by the use of this method of adding before dividing. But here no such method is forthcoming. For the operation represented in the formula

$$\frac{M_{02}}{M_{01}} = \frac{\frac{1}{6}(3 \times 1\frac{1}{3}[2] + 2 \times \frac{1}{2}[4] + 1 \times 1[5])}{\frac{1}{6}(3 \times 1\ [2] + 2 \times 1[4] + 1 \times 1[5])}$$

has nothing to recommend it, as its numerator performs the whole operation of the method of dividing before adding, and its denominator is pure waste, reducing to unity and not affecting the result.

§ 5. *Harmonic averaging.*—Intending to use even weighting, we might be led to average each period and to compare the averages in this way,

$$\frac{M_{02}}{M_{01}} = \frac{\dfrac{1}{\dfrac{1}{n}\left\{\dfrac{1}{(a_2A)} + \dfrac{1}{(b_2B)} + \dfrac{1}{(c_2C)} + \cdots\cdots\right\}}}{\dfrac{1}{\dfrac{1}{n}\left\{\dfrac{1}{(a_1A)} + \dfrac{1}{(b_1B)} + \dfrac{1}{(c_1C)} + \cdots\cdots\right\}}};$$

and if, as before, we incorrectly eliminated A, B, C, ..., we should have, after reducing and transposing,

$$\frac{M_{02}}{M_{01}} = \frac{\dfrac{1}{a_1} + \dfrac{1}{b_1} + \dfrac{1}{c_1} + \cdots\cdots}{\dfrac{1}{a_2} + \dfrac{1}{b_2} + \dfrac{1}{c_2} + \cdots\cdots}.$$

And here, too, not only the reasoning by which this result is reached, but this result itself, is wrong. For (as is shown in Appendix A, III. § 7) this is the formula for the harmonic averaging of the variations with weighting of $\frac{1}{a_1}$ for [A], of $\frac{1}{b_1}$ for [B], of $\frac{1}{c_1}$ for [C], and so on, that is, with weighting *inversely* according to the sizes of the figures at the first period. Hence a multiplicity of results would be obtained according to the mass-units we happen to employ. Here the larger the mass-unit, the smaller the number of it purchasable with the money-unit at the first period, and the greater the influence of the variation; and the smaller the mass-unit, the smaller the weight. And here, too, it is conceivable that such even weighting in the separate averages, giving uneven weighting in the average of the variations, might give the right uneven weighting, although this is exceeding unlikely to occur. It is not so unlikely, however, as in the preceding case; for we have seen that the weighting there was likely to be the opposite of what it ought to be. Here the weighting, being the inverse of that, is therefore likely to run more in the right direction, although, of course, only in an irregular way. On the other hand, the employment here of the same mass-unit throughout would tend to make the error run in the wrong direction.

If, instead, we wished to employ proper weighting, and, finding it the same at both periods, introduced it in the separate averages, and compared them in this form,

$$\frac{M_{02}}{M_{01}} = \frac{\dfrac{1}{\dfrac{1}{n'}\left\{\dfrac{\mathbf{a}}{(a_2A)} + \dfrac{\mathbf{b}}{(b_2B)} + \dfrac{\mathbf{c}}{(c_2C)} + \cdots\cdots\right\}}}{\dfrac{1}{\dfrac{1}{n'}\left\{\dfrac{\mathbf{a}}{(a_1A)} + \dfrac{\mathbf{b}}{(b_1B)} + \dfrac{\mathbf{c}}{(c_1C)} + \cdots\cdots\right\}}},$$

and in the same manner reduced this to

$$\frac{M_{02}}{M_{01}} = \frac{\dfrac{\mathbf{a}}{a_1} + \dfrac{\mathbf{b}}{b_1} + \dfrac{\mathbf{c}}{c_1} + \cdots\cdots}{\dfrac{\mathbf{a}}{a_2} + \dfrac{\mathbf{b}}{b_2} + \dfrac{\mathbf{c}}{c_2} + \cdots\cdots};$$

we should again be performing a wrong operation, as the weighting in this formula, for the variations (as shown in **Appendix A, III.** § 8) is *not* **a, b, c,** ······ as intended, but $\frac{\mathbf{a}}{a_1}, \frac{\mathbf{b}}{b_1}, \frac{\mathbf{c}}{c_1}$, ······, and the correct weighting is again perverted by a haphazard factor.

It is not worth while to give numerical examples here, or to expatiate further on the errors incurred through negligence in the treatment of harmonic averages, although this is the average which has been mostly adopted in measuring the exchange-value of money. But it has generally been adopted from the opposite view-point of the measurement of price-variations by means, as we shall see presently, of the arithmetic average, in treating of which we shall have to revert to this subject.

§ 6. *Geometric averaging.*—Here everything is simpler. The weighting in the comparison of the averages at the two periods and the weighting in the averaging of the variations, if apparently the same, are so really; for it does not now matter whether we multiply before dividing or divide before multiplying. If we employ even weighting in the one case, we have even weighting in the other, as is hereby shown,

$$\frac{M_{02}}{M_{01}} = \frac{\sqrt[n]{a_2 \cdot b_2 \cdot c_2 \cdot \ldots\ldots}}{\sqrt[n]{a_1 \cdot b_1 \cdot c_1 \cdot \ldots\ldots}} = \sqrt[n]{\frac{a_2}{a_1} \cdot \frac{b_2}{b_1} \cdot \frac{c_2}{c_1} \cdot \ldots\ldots}.$$

And if we employ the same uneven weighting in each separate average and in the average of the variations, we really have the same weighting in both cases.

The wrong reasoning by which the erroneous results were reached in the first two cases has no room here. In reducing

$$\frac{M_{02}}{M_{01}} = \frac{\sqrt[n]{(a_2 A) \cdot (b_2 B) \cdot (c_2 C) \cdot \ldots\ldots}}{\sqrt[n]{(a_1 A) \cdot (b_1 B) \cdot (c_1 C) \cdot \ldots\ldots}}$$

to the above expressions by dropping A, B, C, ... from the two sides of the fraction, we have a perfect right to do so; for although these expressions are not units, they are the same on both sides. The whole expression cannot be further simplified by dropping the radical signs, because the products are not in the same ratio as their roots.

Now the same apparent weighting being the same real weighting in this kind of averaging, a warning is still needed. Averaging the periods separately, we might be inclined to weight every figure according to the number of times it occurs, whatever the size of its class may be; and if these numbers happen to be the same at both periods, the comparison of the averages would be

$$\frac{M_{02}}{M_{01}} = \frac{\sqrt[n']{a_2^x \cdot b_2^y \cdot c_2^z \cdots\cdots \text{to } n \text{ terms}}}{\sqrt[n']{a_1^x \cdot b_1^y \cdot c_1^z \cdots\cdots \text{to } n \text{ terms}}},$$

in which the exponents x, y, z, ... refer to the numbers of the mass-units that appear in trade, and $n' = x + y + z + \cdots\cdots$ to n terms. We might then be inclined to adopt this weighting into the method of averaging the variations,

$$\frac{M_{02}}{M_{01}} = \sqrt[n']{\left(\frac{a_2}{a_1}\right)^x \cdot \left(\frac{b_2}{b_1}\right)^y \cdot \left(\frac{c_2}{c_1}\right)^z \cdot \cdots\cdots \text{to } n \text{ terms}}.$$

This would be wrong. The results would be various according to the sizes of the mass-units accidentally chosen. For, the larger the mass-unit chosen, say, for [A], the smaller would be, not only a_1 and a_2, (the numbers of times the mass-unit is purchasable with one money-unit, which are here immaterial, since their proportion is always the same), but also x (the number of times the mass-unit appears in trade), that is, the weight of this variation. Therefore this weighting would be haphazard. If, in order to avoid the multiplicity of results which would ensue, we adopted only one mass-unit throughout, the weighting would be influenced by the relative preciousness of the articles; and, as before, as preciousness may be measured in two ways (by comparison of the exchange-value either with weight or with bulk), there would still be two distinct acts of weighting. The proper weighting is that which we have already discovered,—the number of times, not the accidentally actual, but the real (or ideal) economic units occur in each class. This weighting should be employed in the averaging of the variations; and if we happen to prefer, for any reason, the other method of averaging each period separately and comparing their results, we ought to employ the same weighting there.

§ 7. One more point deserves to be considered here. We have seen that the errors we have been warning against we should have been apt to slip into had we employed the method of notating the quantities at the second period independently, without expressing their relation to the quantities at the first period. We should have been especially apt to fall into them had we additionally employed the imperfect method first noticed of formulating exchange-value relations by means of the equivalences of mass-quantities. For then—we may here confine our attention to the arithmetic method—from the formula for the first period,

$$M_1 \asymp a_1 A \asymp b_1 B \asymp c_1 C \asymp \cdots\cdots \text{ to } n \text{ terms},$$

we might have deduced

$$M_1 \asymp \frac{1}{n}(a_1 A + b_1 B + c_1 C + \cdots\cdots \text{ to } n \text{ terms});$$

and from the similar formula for the second period we might have deduced a similar formula of the combination, and then the comparison would be in this form,

$$\frac{M_2}{M_1} \asymp \frac{\frac{1}{n}(a_2 A + b_2 B + c_2 C + \cdots\cdots \text{ to } n \text{ terms})}{\frac{1}{n}(a_1 A + b_1 B + c_1 C + \cdots\cdots \text{ to } n \text{ terms})};$$

in which as the terms A, B, C, ... merely refer to mass-units of the different classes, and so represent units, we might—especially if only one mass-unit had been employed for all the classes—have thought ourselves justified in omitting them and reducing the formula to this,

$$\frac{M_2}{M_1} \asymp \frac{a_2 + b_2 + c_2 + \cdots\cdots}{a_1 + b_1 + c_1 + \cdots\cdots},$$

—the faulty formula already treated of. Here, too, however, our reasoning would have been wrong, as is obvious if A, B, C, ... referred to different mass-units. But even if they referred all to the same mass-unit, our reasoning would have been wrong, as we are not dealing with quantities of weight or of bulk, but with quantities of exchange-value (namely the exchange-values of certain weights or volumes of things). Here, for our purpose, a pound of lead is very different from a pound of feathers. It is only a dollar's worth of lead that is worth as much as a dollar's worth of feathers.

§ 8. This kind of formulation calls for some remarks. The arithmetic formula

$$M = \frac{1}{n}(aA + bB + cC + \cdots\cdots \text{ to } n \text{ terms})$$

may be read: "The money-unit (or anything else) is equivalent to the sum of the quantities of the things it can exchange for or purchase, divided by the number of their kinds"; or again: "The purchasing power of the money-unit (or of anything else) is the power of purchasing the sum of the quantities of the things it can purchase, divided by the number of their kinds," the supposition being that the kinds or classes are equally important. And similarly if we used this formulation for the other two kinds of averages. The peculiarity of this formulation, and of such interpretations of it, is that there is reference only to the mass-quantities of the things purchasable, and no reference to the exchange-values of these masses, or of money in their classes. Evidently such an average would have manifold results according to the sizes of the mass-units employed. Hence some method has to be added of selecting the mass-units. One is to use always the same mass-unit—either a weight-unit, or a capacity-unit. With the addition of such a method (the difference between the two not generally being noticed), such is the employment often made of the term "purchasing power." Yet not only this formulation does not yield a mathematically serviceable formula, but also these interpretations of it, with the added methods of selecting the mass-unit just noticed, and the term "general purchasing power" so used with reference merely to the mass-quantities of the things purchasable, do not yield clear ideas. The things purchasable whose mass-quantities alone are thus united and divided, or averaged, form only a promiscuous conglomeration of variously valuable things with many and great qualitative distinctions, which are entirely ignored. The partial similarity in this method of measuring general purchasing power with the true method of measuring particular purchasing powers, and the complete erroneousness of it, will be more fully pointed out in a later Chapter.

It is possible, however, to put a true interpretation upon the formulation by equivalence. The above-given arithmetical average may be made to read: "The general purchasing power of the money-unit (or of anything else) over all other things is equal to the sum of its particular purchasing powers over everything separately, divided by their number"—under the supposition of even importance. But here we merely have exchange-value under another name, based upon a formula mathematically imperfect. We must prefer, therefore, not only the other kind of formulation, but also the term "exchange-value." It is a canon in logic that we should avoid the use of two terms with identical meaning, and should choose the term which more clearly expresses the meaning.[4]

IV.

§ 1. Thus far we have made no use of prices. Yet in all monetary matters the use of prices is a great help. Almost all the attempted measurements of the exchange-value of money have proceeded by drawing averages of prices. We must therefore restate the previous procedures in terms of prices.

We have been dealing directly with the exchange-value of money in other things. Prices express the exchange-value of the other things in money. Therefore the variations of prices are the inverse of the variations of the exchange-values of money in the things priced. And a variation of an average of the variations of prices will indicate the inverse variation of an average of the variations of the exchange-values of money in the things priced. Hence the possibility of substituting measure-

[4] We can think of the "lifting power" of a derrick without thinking of a lifting power in the things it lifts. And so we can think of the "purchasing power" of money without thinking of the purchasing power of the things it purchases. In fact, this term "to purchase" always means that we give money for something. Hence money actually has purchasing power, in this proper sense, without anything else having purchasing power; and so "purchasing power" is not a correlative term. But we cannot employ the term "exchange-value" without the idea of exchange, which involves correlative exchanges, and consequently correlative exchange-values.

ments of prices for direct measurements of the exchange-value of money in other things.

If we start with mass-quantities equivalent to the money-unit, and with prices therefore at unity, the subsequent variations of the exchange-values of the money-unit and of prices will be, as we have seen (in Proposition X.), to reciprocals of each other. Now we commenced our previous enquiries always by supposing $M \Leftrightarrow aA \Leftrightarrow bB \Leftrightarrow cC \Leftrightarrow \cdots\cdots$, which means that the prices of aA, of bB, of cC, ... are one money-unit each (whence the price of one mass-unit of [A], namely A, is $\frac{1}{a}$, of B $\frac{1}{b}$, of C $\frac{1}{c}$, and so on). Therefore, at the first period, if we draw the averages of the prices of these equivalent mass-quantities, however large or small these be, we shall always get unity, as before. And at the second period, when $M \Leftrightarrow a'aA \Leftrightarrow b'bB \Leftrightarrow c'cC \Leftrightarrow \cdots\cdots$, the prices of aA, of bB, of cC ...... are $\frac{1}{a'}, \frac{1}{b'}, \frac{1}{c'}, \cdots\cdots$. Now letting P_1 and P_2 represent the averages of the prices at the first and second periods respectively, as P_1 in every case is 1, in every case $\frac{P_2}{P_1} = P_2$, that is, the comparison of the two averages is the same as the average at the second period. Therefore, again supposing that we are dealing with classes equally important at each period, or over both the periods together, by inverting the order and placing the harmonic average first and the arithmetic second, for a reason which will appear immediately, we have

$$\frac{P_2}{P_1} = \frac{1}{\frac{1}{n}\left(\frac{1}{\frac{1}{a'}} + \frac{1}{\frac{1}{b'}} + \frac{1}{\frac{1}{c'}} + \cdots\cdots\right)} = \frac{1}{\frac{1}{n}(a' + b' + c' + \cdots\cdots)}, \qquad (12, 1)$$

$$\frac{P_2}{P_1} = \frac{1}{n}\left(\frac{1}{a'} + \frac{1}{b'} + \frac{1}{c'} + \cdots\cdots\right), \qquad (12, 2)$$

$$\frac{P_2}{P_1} = \sqrt[n]{\frac{1}{a'} \cdot \frac{1}{b'} \cdot \frac{1}{c'} \cdot \cdots\cdots} = \frac{1}{\sqrt[n]{a' \cdot b' \cdot c' \cdot \cdots\cdots}}. \qquad (12, 3)$$

But from the exchange-value comparisons we had (4, 1-3)

$$\frac{M_{02}}{M_{01}} = \frac{1}{n}(a' + b' + c' + \cdots\cdots),$$

$$\frac{M_{02}}{M_{01}} = \frac{1}{\frac{1}{n}\left(\frac{1}{a'} + \frac{1}{b'} + \frac{1}{c'} + \cdots\cdots\right)},$$

$$\frac{M_{02}}{M_{01}} = \sqrt[n]{a' \cdot b' \cdot c' \cdot \cdots\cdots}.$$

Thus it is evident that in the order given the first formula for $\frac{P_2}{P_1}$ yields a result the reciprocal of the result of the first formula for $\frac{M_{02}}{M_{01}}$, the second formula for $\frac{P_2}{P_1}$ a result the reciprocal of the result of the second formula for $\frac{M_{02}}{M_{01}}$, and the third formula for $\frac{P_2}{P_1}$ a result the reciprocal likewise of the third formula for $\frac{M_{02}}{M_{01}}$. Now the first formula here expresses the harmonic average of the prices at the second peoiod (or of the variations of the prices), while the first formula there expressed the arithmetic average of the variations of the money-unit's exchange-values; the second formula here the arithmetic average, the second there the harmonic; but the third here and the third there both express the geometric average. Thus when the *harmonic* average of prices shows a rise of the exchange-values of other things in money by p per cent. to $1 + p$ times their former level, which means a fall of the exchange-value of money in the things priced to $\frac{1}{1+p}$ by $1 - \frac{1}{1+p}$ or $\frac{p}{1+p}$ per cent., this fall is indicated by the *arithmetic* average of the exchange-values of the money-unit in those other things; and reversely if the former shows a fall. When the *arithmetic* average of prices shows such a rise of the exchange-values of other things in money, which means such a fall of the exchange-value of money in the other things, this fall is indicated by the *harmonic* average of the exchange-values of the money-unit; and reversely if the former shows a fall. But when the *geometric* average of prices shows such a rise of the exchange-values of other things in money, which means such a fall of the exchange-value of money in other things, this fall is indicated also by the *geometric* average of the exchange-values of the money unit; and reversely if the former shows a rise. Again, if on one set of variations the *harmonic* average of prices shows constancy of the average of the exchange-values of other things in money, which means also constancy in the average of the exchange-values of money in other things, this result is, on the same variations, indicated by the *arithmetic* average of the exchange-values of the money-unit. If on another set of variations, the *arithmetic* average of prices shows such constancy, it is indicated, on this set of variations, by the *harmonic* average of the exchange-values of the money-unit. But if, on still another set of variations, the *geometric* average of prices shows such constancy, it is indicated, on the same set, likewise by the *geometric* average of the exchange-values of the money-unit.

This relationship between these averages is in accordance with a well-known mathematical theorem concerning these averages. The harmonic average of any given quantities is the reciprocal of the arithmetic average of the reciprocals of those quantities. Reversely the arithmetic average of any given quantities is the reciprocal of the harmonic average of the reciprocals of those quantities. But the geometric average of any given quantities is the reciprocal of the geometric average of their reciprocals. It is precisely with such quantities and their reciprocals that we are dealing.

We can therefore find the average variation of the exchange-values of money in all other things by drawing the average variation of the prices of all things, but noticing that the harmonic averages of prices is the one which gives the inverse of the arithmetic average of the exchange-values of money, the arithmetic average of prices the one which gives the inverse of the harmonic average of the exchange-values, while the geometric average of prices corresponds also with the geometric average of the exchange-values.

The problem still remains as to which—if any—of these averages is the right one. The proper order of the enquiry would seem to be as to which is the right average to express the variation of the exchange-values of the money-unit in all other things. If we find this to be the arithmetic average, we shall know that the proper average to draw of prices is the harmonic. If the right average for the exchange-values turns out to be the harmonic, then, and only then, will the arithmetic average of prices be the right one. But if the right average in the one case is found to be the geometric, the geometric will also be the right average in the other. It would seem that to turn the problem around and to start with, or to confine ourselves to, an enquiry concerning the proper averages of prices would give us a wrong point of view and put us at a disadvantage.

§ 2. The formulæ for the averages of prices, of course, are not to be confined to the form in which the prices at the second period are stated as reciprocals of the variations of the exchange-values of the money-unit in the things priced, that is, in the fractional forms $\frac{1}{a'}, \frac{1}{b'}, \frac{1}{c'}, \ldots\ldots$. We may give them upright integral forms. Let us employ the accented Greek letters, α', β', γ', ..., to represent the prices of a A, b B, c C, ... at the second period—these being the quantities which at the first period were all priced at 1.00, so that their prices at the second period directly express the variations of the prices. Then, as α' replaces $\frac{1}{a'}$, β' $\frac{1}{b'}$, γ' $\frac{1}{c'}$, ... in the preceding formulæ, we have the following:

$$\frac{P_2}{P_1} = \frac{1}{\frac{1}{n}\left(\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} + \ldots\ldots\right)}, \qquad (13, 1)$$

$$\frac{P_2}{P_1} = \frac{1}{n}(\alpha' + \beta' + \gamma' + \ldots\ldots), \qquad (13, 2)$$

$$\frac{P_2}{P_1} = \sqrt[n]{\alpha' \cdot \beta' \cdot \gamma' \cdot \ldots\ldots}. \qquad (13, 3)$$

We can also employ the other method of notation. We may represent the prices of *any* mass-units of the various classes at the first period by α_1, β_1, γ_1, ..., and the prices of the same quantities of the same classes at the second period by α_2, β_2, γ_2 .... Then the variation in the preceding formulæ represented by α' is here represented by $\frac{\alpha_2}{\alpha_1}$, and that represented there by β' is here represented by $\frac{\beta_2}{\beta_1}$, and so on. And the preceding formulæ become the following:

$$\frac{P_2}{P_1} = \frac{1}{\frac{1}{n}\left(\frac{\alpha_1}{\alpha_2} + \frac{\beta_1}{\beta_2} + \frac{\gamma_1}{\gamma_2} + \cdots\cdots\right)}, \qquad (14, 1)$$

$$\frac{P_2}{P_1} = \frac{1}{n}\left(\frac{\alpha_2}{\alpha_1} + \frac{\beta_2}{\beta_1} + \frac{\gamma_2}{\gamma_1} + \cdots\cdots\right), \qquad (14, 2)$$

$$\frac{P_2}{P_1} = \sqrt[n]{\frac{\alpha_2}{\alpha_1} \cdot \frac{\beta_2}{\beta_1} \cdot \frac{\gamma_2}{\gamma_1} \cdot \cdots\cdots}. \qquad (14, 3)$$ [1]

In these formulæ the terms are the reciprocals of the terms in formulæ 5, 1-3, that is, $\frac{\alpha_2}{\alpha_1} = \frac{a_1}{a_2}$, $\frac{\beta_2}{\beta_1} = \frac{b_1}{b_2}$, $\frac{\gamma_2}{\gamma_1} = \frac{c_1}{c_2}$, and so on, and inversely. Hence each formula here expresses for $\frac{P_2}{P_1}$ the reciprocal of the formula there given, in the same order, for $\frac{M_{02}}{M_{01}}$.

It is, of course, more natural to use the prices of the customarily employed mass-units of the different kinds of goods; and it is more convenient. In practice all reductions of prices at the first period are labor wasted.

V.

§ 1. The mere introduction of uneven weighting into these formulæ is likewise easy. The weighting is, of course, the same

[1] These three formulæ, confined to two terms, were first given by Jevons, B. 23, p. 121, and then, in the more general form, by Walras, B. 69, pp. 12-13, B. 70, p. 432.

as in the measurement of the exchange-values of money in the other classes. The weighting of the arithmetic average of the variations there will come into the harmonic average of the variations here as it there entered the harmonic average; and reversely. The formulæ, then, when we are justified in using, or when we do use, single weighting, will be

$$\frac{P_2}{P_1} = \frac{1}{\frac{1}{n''}\left(\frac{\mathbf{a}}{\alpha'} + \frac{\mathbf{b}}{\beta'} + \frac{\mathbf{c}}{\gamma'} + \cdots\cdots \text{to } n \text{ terms}\right)}, \qquad (15, 1)$$

$$\frac{P_2}{P_1} = \frac{1}{n''}(\mathbf{a}\alpha' + \mathbf{b}\beta' + \mathbf{c}\gamma' + \cdots\cdots \text{to } n \text{ terms}), \qquad (15, 2)$$

$$\frac{P_2}{P_1} = \sqrt[n'']{\alpha'^{\mathbf{a}} \cdot \beta'^{\mathbf{b}} \cdot \gamma'^{\mathbf{c}} \cdot \cdots\cdots \text{to } n \text{ terms}}, \qquad (15, 3)$$

in which $n'' = \mathbf{a} + \mathbf{b} + \mathbf{c} + \cdots\cdots$ to n terms; or again

$$\frac{P_2}{P_1} = \frac{1}{\frac{1}{n''}\left(\mathbf{a}\frac{\alpha_1}{\alpha_2} + \mathbf{b}\frac{\beta_1}{\beta_2} + \mathbf{c}\frac{\gamma_1}{\gamma_2} + \cdots\cdots \text{to } n \text{ terms}\right)}, \qquad (16, 1)$$

$$\frac{P_2}{P_1} = \frac{1}{n''}\left(\mathbf{a}\frac{\alpha_2}{\alpha_1} + \mathbf{b}\frac{\beta_2}{\beta_1} + \mathbf{c}\frac{\gamma_2}{\gamma_1} + \cdots\cdots \text{to } n \text{ terms}\right), \qquad (16, 2)$$

$$\frac{P_2}{P_1} = \sqrt[n'']{\left(\frac{\alpha_2}{\alpha_1}\right)^{\mathbf{a}} \cdot \left(\frac{\beta_2}{\beta_1}\right)^{\mathbf{b}} \cdot \left(\frac{\gamma_2}{\gamma_1}\right)^{\mathbf{c}} \cdot \cdots\cdots \text{to } n \text{ terms}}.^{1} \qquad (16, 3)$$

The same comments are suggested by these formulæ as by those for the exchange-values of money. It is indifferent how large or small, integral or fractional, the numbers **a**, **b**, **c**, ... be, so long as they are in the proper proportions to one another. And a change in the weighting without a variation in the prices (or with a uniform variation of all prices) produces no change in the result (in agreement with Propositions XLIV. and XLV.).

[1] These formulæ for the unevenly weighted averages have rarely been stated. Westergaard gives the arithmetic formula in a cumbrous form (see Appendix C, III. § 1); and adds that a "similar alteration" may be given to the other simple averages, but he does not give it, and the expression "similar" is misleading. They are unnoticed by all the other writers cited in the Bibliography except Edgeworth, who gives the third in its logarithmic form, as noticed below in Note 3.

We may here add that these and all the preceding proper formulæ also carry out Proposition XXXVI.[2]

With high numbers for the weights the third of these formulæ might seem unworkable. Indeed it would be unworkable even with small numbers for the weights, or with all the weights alike,—since their sum, when many classes are dealt with, would involve the extraction of a large root,—but for the help which may be rendered by logarithms. With the aid of logarithms the third formula is no longer difficult to execute, even with large, or with minutely fractional, figures for the weights. This use of logarithms was made by Jevons, although he employed only even weighting. With uneven weighting, the logarithm of the result may be obtained in several ways, among which the following may be noted:

$$(1)\ \log \frac{P_2}{P_1} = \frac{1}{n''}\left(\mathbf{a}\cdot\log\frac{\alpha_2}{\alpha_1} + \mathbf{b}\cdot\log\frac{\beta_2}{\beta_1} + \cdots\cdots \text{ to } n \text{ terms}\right),$$

$$(2)\ \log \frac{P_2}{P_1} = \frac{1}{n''}\left\{(\mathbf{a}\cdot\log\ \alpha_2 + \mathbf{b}\cdot\log\ \beta_2 + \cdots\cdots \text{ to } n \text{ terms}) - (\mathbf{a}\cdot\log\alpha_1 + \mathbf{b}\cdot\log\beta_1 + \cdots\cdots \text{ to } n \text{ terms})\right\},$$

$$(3)\ \log \frac{P_2}{P_1} = \frac{1}{n''}\left\{\mathbf{a}(\log\alpha_2 - \log\alpha_1) + \mathbf{b}(\log\beta_2 - \log\beta_1) + \cdots\cdots \text{ to } n \text{ terms}\right\}.$$

The first is the operation we should naturally adopt if we had already reduced the prices at the first period to 1.00 (or to 100), $\frac{\alpha_2}{\alpha_1}$ then being replaced by α' already ascertained, and so on. The second would be the simplest, if the weighting were even, the expressions **a**, **b**, **c**, ······ then dropping out.[3] The third is

[2] This and a remark in Appendix A, I. § 10 end, deserve attention. They show that when we know approximately the variation of the average, it is more important that we should seek to be accurate with the weighting of the classes whose variations are greatly divergent from that of the average than with the weighting of the classes whose variations are nearly the same as that of the average. See also Note 21 in Chapter IV., Sec. V., § 10.

[3] Very nearly in this form, with even weighting, the formula is given by Walras, B. 61, p. 7. A formula very nearly in the first form, with uneven weighting, has since been given by Edgeworth, B. 59, p. 287.

generally the most useful operation, saving the divisions needed in the first and the doubling of the multiplications in the second.[4]

§ 2. We may, of course, here as well as in the direct measurement of the exchange-value of money, need to use double weighting. The formulæ embodying double weighting in the case of prices may be directly derived from the formulæ previously given for double weighting in the other case (formulæ 11, 1-3). We must still employ some expressions—x_1, x_2, y_1, y_2, etc.—to represent the numbers that are produced or consumed at each period of the mass-units whose prices are given. And we must remember that we must make two inversions, first of the numerators and denominators, for $\frac{P_2}{P_1} = \frac{M_{01}}{M_{02}}$, and second of the exchange-value symbols and the price symbols, for $\frac{a_2}{a_1} = \frac{\alpha_1}{\alpha_2}$ and reversely, and so on. Then those formulæ become these, in the same order:

$$\frac{P_2}{P_1} = \frac{\frac{1}{n_1'}\left(\frac{x_1}{\alpha_1} + \frac{y_1}{\beta_1} + \frac{z_1}{\gamma_1} + \cdots\cdots \text{ to } n \text{ terms}\right)}{\frac{1}{n_2'}\left(\frac{x_2}{\alpha_2} + \frac{y_2}{\beta_2} + \frac{z_2}{\gamma_2} + \cdots\cdots \text{ to } n \text{ terms}\right)}, \qquad (17, 1)$$

$$\frac{P_2}{P_1} = \frac{\frac{1}{n_2'}(x_2\alpha_2 + y_2\beta_2 + z_2\gamma_2 + \cdots\cdots \text{ to } n \text{ terms})}{\frac{1}{n_1'}(x_1\alpha_1 + y_1\beta_1 + z_1\gamma_1 + \cdots\cdots \text{ to } n \text{ terms})}, \qquad (17, 2)$$

$$\frac{P_2}{P_1} = \frac{\sqrt[n_2']{\alpha_2^{x_2} \cdot \beta_2^{y_2} \cdot \gamma_2^{z_2} \cdot \cdots\cdots \text{ to } n \text{ terms}}}{\sqrt[n_1']{\alpha_1^{x_1} \cdot \beta_1^{y_1} \cdot \gamma_1^{z_1} \cdot \cdots\cdots \text{ to } n \text{ terms}}}. \qquad (17, 3)$$

The anticipatory remark may here be made that in the future course of these pages we shall find no use for any of these formulæ except the second, which is the general formula for double weighting in the arithmetic average of prices. This second formula may be expanded in full as follows,

[4] Jevons used three-place tables. It would be advisable, however, to use tables with not less than six places.

$$\frac{P_2}{P_1} = \frac{\dfrac{x_2a_2 + y_2\beta_2 + z_2\gamma_2 + \cdots\cdots}{x_2 + y_2 + z_2 + \cdots\cdots}}{\dfrac{x_1a_1 + y_1\beta_1 + z_1\gamma_1 + \cdots\cdots}{x_1 + y_1 + z_1 + \cdots\cdots}}, \qquad (17, 2, 2)$$

which reduces to

$$\frac{P_2}{P_1} = \frac{x_2a_2 + y_2\beta_2 + z_2\gamma_2 + \cdots\cdots}{x_1a_1 + y_1\beta_1 + z_1\gamma_1 + \cdots\cdots} \cdot \frac{x_1 + y_1 + z_1 + \cdots\cdots}{x_2 + y_2 + z_2 + \cdots\cdots}, \qquad (17, 2, 3)$$

this last being the best form for it.

Here, as before, it is evident that in all these formulæ the results will be variously affected by the sizes of the mass-units we happen to choose. Here also, therefore, a method of selecting the mass-units must be established. And this method need not, or it may be, indicated in the formula itself. It is not indicated in the formula used by Drobisch, who therefore employed the simple formula here given, after first telling how the mass-units are to be chosen. It is indicated in the formula used by Professor Lehr, which therefore becomes more complex, though the real difference between his method and Drobisch's is not in the formula, but in the method of choosing the mass-units, and hi method is not so difficult of application as Drobisch's. Bot these methods use the second formula, which admits of oth variations through the employment of several different methc of selecting the mass-units.

In this second formula there is the peculiarity that the s of the mass-units have no effect upon the terms x_1a_1, x_2a_2, that is, upon any of the terms in the first half of the forn in the form last given. For the larger the mass-units, the l will be a in the same proportion, and x will be smaller i inverse proportion, so that their product, xa, will be unaff and so in all the other classes, and at both periods.[5] H

[5] E. g., if [B] be wheat and y_1 represent a thousand quarters, then β_1 r the pr of a quarter—let us say eight dollars, wherefore $y_1\beta_1 = 1000$ If i we used bushels, y_1 would represent eight thousand bushe at one dollar, so that we should still have $y_1\beta_1 = 8000$ ty exists in the corresponding second formula for the r (11, 2), expressing the harmonic average; for there no the terms $\frac{y_1}{b_1}$, etc., since the larger is y_1, the larger also i , and the ratio remains the same. Thus in the abo

this form of this formula the first half remains always the same, and the modifications, if made at all, are to be made only in the second half.

This formula may be simplified. In the first half the whole denominator is merely the total valuation of all the goods produced and consumed in the first period at the prices of the first period; and the whole numerator is a similar total valuation for the second period. Naturally these parts of the formula remain unchanged whatever be the mass-units employed. In the second half the whole numerator is the simple inventory of all the mass-units of the goods produced or consumed in the first period; and the whole denominator is the similar inventory for the second period. Naturally these parts are affected by the sizes of the mass-units in which these numbers are reckoned. Now the parts in the first half represent the total money-values of the goods at the first and second periods, and hence, as wholes, may be represented by V_1 and V_2; and the parts in the second half represent the total mass-quantities of the goods at the two periods respectively, and hence, as wholes, may be represented by Q_1 and Q_2. Then

$$\frac{P_2}{P_1} = \frac{V_2}{V_1} \cdot \frac{Q_1}{Q_2} \qquad (17, 2, 4)$$

is a simplified form of the second formula.

The real nature of this second formula is indicated in the second form in which it has above been put, which may be simplified into this,

$$\frac{P_2}{P_1} = \frac{\frac{V_2}{Q_2}}{\frac{V_1}{Q_1}} . \qquad (17, 2, 5)$$

For here it is seen that the formula is merely that of a comparison between an average price of all goods at each period. For at each period the total money-value of all goods collectively is divided by the total *quantity* of all the goods; which is the plain method of (arithmetically) averaging the prices of all goods at each period. But the total "quantity" of all the goods is

$$\frac{P_2}{P_1}=\frac{\dfrac{x_2\alpha_2+y_2\beta_2+z_2\gamma_2+\cdots\cdots}{x_2+y_2+z_2+\cdots\cdots}}{\dfrac{x_1\alpha_1+y_1\beta_1+z_1\gamma_1+\cdots\cdots}{x_1+y_1+z_1+\cdots\cdots}}, \qquad (17, 2, 2)$$

which reduces to

$$\frac{P_2}{P_1}=\frac{x_2\alpha_2+y_2\beta_2+z_2\gamma_2+\cdots\cdots}{x_1\alpha_1+y_1\beta_1+z_1\gamma_1+\cdots\cdots}\cdot\frac{x_1+y_1+z_1+\cdots\cdots}{x_2+y_2+z_2+\cdots\cdots}, \qquad (17, 2, 3)$$

this last being the best form for it.

Here, as before, it is evident that in all these formulæ the results will be variously affected by the sizes of the mass-units we happen to choose. Here also, therefore, a method of selecting the mass-units must be established. And this method need not, or it may be, indicated in the formula itself. It is not indicated in the formula used by Drobisch, who therefore employed the simple formula here given, after first telling how the mass-units are to be chosen. It is indicated in the formula used by Professor Lehr, which therefore becomes more complex, though the real difference between his method and Drobisch's is not in the formula, but in the method of choosing the mass-units, and his method is not so difficult of application as Drobisch's. Both these methods use the second formula, which admits of other variations through the employment of several different methods of selecting the mass-units.

In this second formula there is the peculiarity that the sizes of the mass-units have no effect upon the terms $x_1\alpha_1$, $x_2\alpha_2$, etc., that is, upon any of the terms in the first half of the formula in the form last given. For the larger the mass-units, the larger will be α in the same proportion, and x will be smaller in the inverse proportion, so that their product, $x\alpha$, will be unaffected, and so in all the other classes, and at both periods.[5] Hence in

[5] *E. g.*, if [B] be wheat and y_1 represent a thousand quarters, then β_1 represents the price of a quarter—let us say eight dollars, wherefore $y_1\beta_1 = 1000 \times 8 = 8000$. If instead we used bushels, y_1 would represent eight thousand bushels, but β_1 would represent one dollar, so that we should still have $y_1\beta_1 = 8000 \times 1 = 8000$.—This peculiarity exists in the corresponding second formula for the exchange-value of money (11, 2), expressing the harmonic average; for there no influence is exerted upon the terms $\frac{y_1}{b_1}$, etc., since the larger is y_1, the larger also is b_1 in the same proportion, and the ratio remains the same. Thus in the above example $\frac{1000}{1/8} = \frac{8000}{1}$.

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$$\frac{P_2}{P_1} = \frac{\dfrac{x_2 a_2 + y_2 \beta_2 + z_2 \gamma_2 + \cdots\cdots}{x_2 + y_2 + z_2 + \cdots\cdots}}{\dfrac{x_1 a_1 + y_1 \beta_1 + z_1 \gamma_1 + \cdots\cdots}{x_1 + y_1 + z_1 + \cdots\cdots}}, \qquad (17, 2, 2)$$

which reduces to

$$\frac{P_2}{P_1} = \frac{x_2 a_2 + y_2 \beta_2 + z_2 \gamma_2 + \cdots\cdots}{x_1 a_1 + y_1 \beta_1 + z_1 \gamma_1 + \cdots\cdots} \cdot \frac{x_1 + y_1 + z_1 + \cdots\cdots}{x_2 + y_2 + z_2 + \cdots\cdots}, \qquad (17, 2, 3)$$

this last being the best form for it.

Here, as before, it is evident that in all these formulæ the results will be variously affected by the sizes of the mass-units we happen to choose. Here also, therefore, a method of selecting the mass-units must be established. And this method need not, or it may be, indicated in the formula itself. It is not indicated in the formula used by Drobisch, who therefore employed the simple formula here given, after first telling how the mass-units are to be chosen. It is indicated in the formula used by Professor Lehr, which therefore becomes more complex, though the real difference between his method and Drobisch's is not in the formula, but in the method of choosing the mass-units, and his method is not so difficult of application as Drobisch's. Both these methods use the second formula, which admits of other variations through the employment of several different methods of selecting the mass-units.

In this second formula there is the peculiarity that the sizes of the mass-units have no effect upon the terms $x_1 a_1$, $x_2 a_2$, etc., that is, upon any of the terms in the first half of the formula in the form last given. For the larger the mass-units, the larger will be a in the same proportion, and x will be smaller in the inverse proportion, so that their product, xa, will be unaffected, and so in all the other classes, and at both periods.[5] Hence in

[5] *E. g.*, if [B] be wheat and y_1 represent a thousand quarters, then β_1 represents the price of a quarter—let us say eight dollars, wherefore $y_1 \beta_1 = 1000 \times 8 = 8000$. If instead we used bushels, y_1 would represent eight thousand bushels, but β_1 would represent one dollar, so that we should still have $y_1 \beta_1 = 8000 \times 1 = 8000$.—This peculiarity exists in the corresponding second formula for the exchange-value of money (11, 2), expressing the harmonic average; for there no influence is exerted upon the terms $\frac{y_1}{b_1}$, etc., since the larger is y_1, the larger also is b_1 in the same proportion, and the ratio remains the same. Thus in the above example $\frac{1000}{1/8} = \frac{8000}{1}$.

quantities existed at both the periods compared. Now whenever this second method is employed, it is the same as some method of averaging the price variations with single weighting; and reversely, whenever single weighting is employed in averaging price variations,—or whenever the method of averaging the price variations is employed, since in this there can only be single weighting,—it is the same as some method of averaging the prices at each period separately and comparing the averages. Therefore the single weighting for the price variations which is involved in the method of separately averaging the prices and of comparing the averages ought to be the same as would be the weighting did we avowedly pursue the method of averaging the price variations; for it would be absurd to employ a method of comparing averages of prices that is really the same as a method of averaging the price variations, and to employ in it a system of weighting for the price variations which we would reject if we consciously employed the method of averaging the price variations. Thus, as before (Sect. III. § 2), we have a criterion that *the method of comparing separate price averages with the same weighting in each should always agree with the method of averaging the price variations with the proper weighting for these*, whatever this weighting may be; for at all events it is often easy to reject flagrantly improper weighting, or weighting that has no reason in its favor.

This criterion is of importance because the single weighting for averaging the price variations involved in comparing the separately drawn averages is not always the same as the weighting used in each of these averages. It is always the same if we construct at the first period rather a level of prices, which breaks up into an uneven range of prices at the second period; for in this case to draw an average of these later prices is really to draw an average of the variations of the prices, with the same weighting. But if we start at the first period with various prices, there is generally, though not always, a divergence between the weighting in the separate averages and the weighting for the price variations involved, and to the ignorant hidden, in the comparison of those. It is here that the opportunity for the total sum of the numbers of the mass-units of all the goods, which sum is affected by the sizes of the mass-units chosen to measure the goods by, so that the averages are arbitrary until the proper method of selecting the mass-units is found. This fact of its representing a comparison between two ordinary (arithmetic) averages is what gives this second formula its superior recommendation over the other two.

It might be thought that there would be a similar recommendation for the first formula here, because it corresponds to the first formula for the exchange-value of money, in which the separate averages are the arithmetic. But in (arithmetically) averaging at each period the exchange-value of money in all goods by dividing the sum of *its* exchange-values in them by *their* quantity there does not seem to be so much sense as in (arithmetically) averaging at each period the exchange-value of all goods in money by dividing the sum of *their* exchange-values in money by *their* quantity. There is congruity between the money-values of commodities and the quantity of commodities: there is none between the commodity-values of money and the quantity of commodities.

VI.

§ 1. The averaging of prices gives occasion for similar miscarriages of intention as the averaging of the exchange-values of money. Jevons's assertion that there is no average of prices, and consequently no variation of the average of prices, must be regarded as erroneous, except in the sense that there is no real or absolute average already given to us in nature. We must construct the average, and can do so wrongly or rightly; wherefore the problem really is to find how to construct it rightly. In the attempt to solve this problem two methods have presented themselves. The one is to construct separate averages, one at each period, on the mass-quantities that exist at each period, variously reckoned according to the mass-units chosen; whereupon the comparison lies between these two averages. The other is likewise to construct separate averages, and to compare them, but to use in each the same weighting—as if the same mass-quantities existed at both the periods compared. Now whenever this second method is employed, it is the same as some method of averaging the price variations with single weighting; and reversely, whenever single weighting is employed in averaging price variations,—or whenever the method of averaging the price variations is employed, since in this there can only be single weighting,—it is the same as some method of averaging the prices at each period separately and comparing the averages. Therefore the single weighting for the price variations which is involved in the method of separately averaging the prices and of comparing the averages ought to be the same as would be the weighting did we avowedly pursue the method of averaging the price variations; for it would be absurd to employ a method of comparing averages of prices that is really the same as a method of averaging the price variations, and to employ in it a system of weighting for the price variations which we would reject if we consciously employed the method of averaging the price variations. Thus, as before (Sect. III. § 2), we have a criterion that *the method of comparing separate price averages with the same weighting in each should always agree with the method of averaging the price variations with the proper weighting for these*, whatever this weighting may be; for at all events it is often easy to reject flagrantly improper weighting, or weighting that has no reason in its favor.

This criterion is of importance because the single weighting for averaging the price variations involved in comparing the separately drawn averages is not always the same as the weighting used in each of these averages. It is always the same if we construct at the first period rather a level of prices, which breaks up into an uneven range of prices at the second period; for in this case to draw an average of these later prices is really to draw an average of the variations of the prices, with the same weighting. But if we start at the first period with various prices, there is generally, though not always, a divergence between the weighting in the separate averages and the weighting for the price variations involved, and to the ignorant hidden, in the comparison of those. It is here that the opportunity for error enters in this method of comparing averages, systems of weighting for the price variations being admitted which would never have been allowed had their existence been perceived. To the errors so occasioned we may for the present mostly confine our attention, but also noticing an allied error in a method using double weighting. Again the averages need to be treated separately.

§ 2. *Harmonic averaging of prices.*—This corresponds to the arithmetic averaging of the exchange-values of money, but its form is the same as the harmonic averaging of those exchange-values. The possible error here is that of supposing we are using even weighting in using the following formula,

$$\frac{P_2}{P_1} = \frac{\frac{1}{\alpha_1} + \frac{1}{\beta_1} + \frac{1}{\gamma_1} + \cdots\cdots}{\frac{1}{\alpha_2} + \frac{1}{\beta_2} + \frac{1}{\gamma_2} + \cdots\cdots};$$

or that we are using weighting according to **a**, **b**, **c**, ······ in using the following,

$$\frac{P_2}{P_1} = \frac{\frac{\mathbf{a}}{\alpha_1} + \frac{\mathbf{b}}{\beta_1} + \frac{\mathbf{c}}{\gamma_1} + \cdots\cdots}{\frac{\mathbf{a}}{\alpha_2} + \frac{\mathbf{b}}{\beta_2} + \frac{\mathbf{c}}{\gamma_2} + \cdots\cdots};$$

—that is, in comparing the harmonic averages of *any* prices (the prices of *any* mass-units) at each period separately, in each of these averages the same even or uneven weighting being used. For in the first of these formulæ the weighting of the price variations is really according to $\frac{1}{\alpha_1}, \frac{1}{\beta_1}, \frac{1}{\gamma_1}$, ······, and in the second it is according to $\frac{\mathbf{a}}{\alpha_1}, \frac{\mathbf{b}}{\beta_1}, \frac{\mathbf{c}}{\gamma_1}$, ······, and in neither is the weighting what is intended unless $\alpha_1 = \beta_1 = \gamma_1 = \cdots\cdots$, that is, unless all prices have been reduced to the same figure at the first period.[1] We know that $\alpha_1 = \frac{1}{a_1}, \beta_1 = \frac{1}{b_1}, \gamma_1 = \frac{1}{c_1}$, and so on. Thus the weighting here is, in the first formula, according

[1] See Appendix A, III. §§ 7 and 8, already referred to.

to $a_1, b_1, c_1, \ldots\ldots$, and in the second according to $\mathbf{a}a_1, \mathbf{b}b_1, \mathbf{c}c_1, \ldots\ldots$; which is the same weighting as we discovered in the corresponding forms of the arithmetic averaging of the exchange-values of money (above in Section III. §§ 3 and 4).

Various complications could be added which would occur in comparisons of the results obtained for subsequent periods by comparing every one of these with the same basic period. But as this method has rarely been used we need not tarry over it.

§ 3. *Arithmetic averaging of prices.*—This corresponds to the harmonic averaging of the exchange-values of money, and its errors, while the same in form as those which may occur in arithmetically averaging the exchange-values, are really the same as those which may occur in harmonically averaging them. As these latter have only been lightly treated, and as this is the method which has mostly been used, more attention is to be paid to it here.

In the comparison of the price variations we can start with the prices of *any* mass-units. Hence the simplest form of error, and the one earliest to appear, was to think we may simply average the prices of any mass-units of commodities at the two periods and compare them. The operation is according to this formula,

$$\frac{P_2}{P_1} = \frac{\frac{1}{n}(\alpha_2 + \beta_2 + \gamma_2 + \ldots\ldots)}{\frac{1}{n}(\alpha_1 + \beta_1 + \gamma_1 + \ldots\ldots)},$$

which reduces to

$$\frac{P_2}{P_1} = \frac{\alpha_2 + \beta_2 + \gamma_2 + \ldots\ldots}{\alpha_1 + \beta_1 + \gamma_1 + \ldots\ldots};$$

so that this method consists merely in comparing the sums in lists of prices at two or more periods.

This form is wrong because, as a method of averaging the variations of prices it contains weighting according to the sizes of the figures in the denominator—the prices at the first period.[2] But as these prices have been hit upon without reference to the

[2] See Appendix A, II. § 7, already referred to.

sizes of the classes, the weighting is haphazard—as already noticed in the preceding Chapter. As the prices are larger the larger the mass-units chosen, the weighting will be larger the larger the mass-units. This agrees with what we have found in the case of the improper method of comparing harmonic averages of the exchange-values of money. Now if it be claimed that all we want is to compare averages of prices, and not to average variations, the error of this position is shown by the fact that this method would give as many different results as there are combinations of mass-units that might be employed. Among these the right result would be hit upon, or approached, only if the prices at the first period happened to be directly according to the sizes of the classes rightly determined.

This haphazard method, which may be named, after the writer who first employed it, Dutot's method (see Appendix C, I.), has rarely been employed completely. Yet it has frequently been involved in another method which was specially invented to avoid it, and which may be called Carli's method (see Appendix C, II.). This other method is to average variations with even weighting by reducing all the prices at the first period to a uniform level; and then,—in a variation introduced by Evelyn,—in order to avoid the trouble of repeating this operation for every comparison, the first period is used as a common basis with which every subsequent period is directly compared. This method carries out its intention when comparing each subsequent period with the basic period; but whenever the result obtained for one subsequent period is compared with the result obtained for another, the comparison acquires an entirely different character. The comparison of the second period, for instance, with the first, whose prices have been reduced to unity, is in accordance with this formula,

$$\frac{P_2}{P_1} = P_2 = \frac{1}{n}(\alpha_2' + \beta_2' + \gamma_2' + \cdots\cdots),$$

and the comparison of the third with the same first is

$$\frac{P_3}{P_1} = P_3 = \frac{1}{n}(\alpha_3' + \beta_3' + \gamma_3' + \cdots\cdots);$$

wherefore the comparison of these two is as follows,

$$\frac{P_3}{P_2} = \frac{\frac{1}{n}(a_3' + \beta_3' + \gamma_3' + \ldots\ldots)}{\frac{1}{n}(a_2' + \beta_2' + \gamma_2' + \ldots\ldots)} = \frac{a_3' + \beta_3' + \gamma_3' + \ldots\ldots}{a_2' + \beta_2' + \gamma_2' + \ldots\ldots}.$$

;hich may also be written thus,

$$\frac{P_3}{P_2} = \frac{\frac{a_3}{a_1} + \frac{\beta_3}{\beta_1} + \frac{\gamma_3}{\gamma_1} + \ldots\ldots}{\frac{a_2}{a_1} + \frac{\beta_2}{\beta_1} + \frac{\gamma_2}{\gamma_1} + \ldots\ldots}.$$

Iere the averaging of the price variations is no longer with ven weighting: it is with uneven weighting according to the ariations of the prices at the earlier of the two periods ompared. Consequently the result obtained in this way is not he same as would be the result if we compared the third period irectly with the second, employing even weighting; and the esult we do get depends upon the accident as to what period we appen to start with as the base, and what price variations have aken place since. Of course the price variations have no re-ard for the sizes of the classes, so that the uneven weighting ere is purely fortuitous. To be sure, by the time of the second eriod, the variations may not be large, so that the weighting rill not much depart from the even weighting intended. But he further we go from the original period, the more and more reakish becomes the weighting. This method has now been mployed by the *Economist* in a series covering fifty years. It s amusing to think what queer weighting may be involved in omparing the fiftieth with the forty-ninth year. Yet, after all, his weighting may not be more incorrect than the even weight-ng itself.

That the comparison of the averages of prices using *any* prices, rith apparent even weighting, may give an indefinite number f different answers is, of course, simply due to the fact that an ndefinite number of different weightings may be used, according o the prices that happen to exist and to be hit upon at the irst period. Yet this fact of the variability of the results ob-ainable has been urged as a reason for rejecting the system of index-numbers altogether. Writers have toyed with various prices that might be used at the first period, and on the same price variations have naturally obtained different results. Then, not perceiving the reason for the difference in the results, they have concluded that a method giving such a variety of answers can have no validity.[3] Of course this criticism, based upon ignorance, and merely playing off one incorrect form against another, has no validity against the correct form, whatever this may turn out to be when it is discovered.

Now the variability of the results, it might be thought, could be obviated by confining our comparisons always to the prices of the same mass-unit. If we did so with the intention of employing even weighting,—in fact, employing even weighting in the averages of prices at each period separately,—there would still be uneven weighting when we view the result as an average of the price variations. For the prices would be large or small directly according to the preciousness of the classes, and so the result would be the same as if we averaged the variations using weighting directly according to the preciousness of the classes at the first period—an absurd system of weighting, as the importance of the classes is very different from their preciousness, and, in fact, apt to be the opposite. Moreover this method would not obviate the difficulty, as there would be two results obtainable according as we used throughout the same weight-unit or the same capacity-unit. Another objection, but of minor importance, is that this method could not possibly be applied to land, and with difficulty to gases. That it could not be applied to labor, however, would be in its favor. We need not dwell longer on this method, since it has never been employed. But the notice of it is justified because its principal feature, the use of the same mass-unit, has been incorporated in another method, to be examined presently.

§ 4. Another way in which variability of the answers may be avoided is by taking account of the mass-quantities of the

[3] So Pierson in the paper referred to in a note to B. 122, and C. W. Oker, *The fallacy of index-numbers*, Journal of Political Economy, Chicago, Vol. IV., 1896, pp. 517-519.—The proper answer has recently been given by Padan, B. 141, pp. 108-109.

classes produced and consumed at one of the periods, or of some average mass-quantities, treating them as if they were the same at each period. This method has not infrequently been employed, and may be called Scrope's method, after the person who first suggested it (see Appendix C, IV.). Let x represent the number of the mass-units of the class [A] found to be produced or consumed by the community at large during a given period or periods, and y the number of the mass-units of the class [B] similarly found, and so on. Then at the first period the community can buy this total mass of goods by expending the following sum of money : $x\alpha_1 + y\beta_1 + z\gamma_1 + \cdots\cdots$ to n terms,—the prices α_1, β_1, γ_1, $\cdots\cdots$ of course being the prices of the same mass-units of [A], [B], [C], $\cdots\cdots$ whose numbers are represented by x, y, z, $\cdots\cdots$. And at the second period the same total mass of goods can be purchased for this sum of money : $x\alpha_2 + y\beta_2 + z\gamma_2 + \cdots\cdots$ to n terms. If we divide each of these sums of money by the number of mass-units purchased—namely by $x + y + z + \cdots\cdots$ to n terms, which we may represent by n',—the quotient may be taken to be the average price of a mass-unit (an average mass-unit, so to speak) of the goods ; and therefore a variation of these averages may be represented thus,

$$\frac{P_2}{P_1} = \frac{\frac{1}{n'}\left(x\alpha_2 + y\beta_2 + z\gamma_2 + \cdots\cdots \right)}{\frac{1}{n'}\left(x\alpha_1 + y\beta_1 + z\gamma_1 + \cdots\cdots \right)},$$

which reduces to

$$\frac{P_2}{P_1} = \frac{x\alpha_2 + y\beta_2 + z\gamma_2 + \cdots\cdots}{x\alpha_1 + y\beta_1 + z\gamma_1 + \cdots\cdots}.$$

Or we might have started simply with this last formula as the expression of the variation of the total prices of the same total quantity of goods. Therefore a method doing this, although it does not necessarily involve an arithmetic averaging of the prices at each period, is the same as a method which does make such arithmetic averaging.

There is little to recommend this method regarded as a comparison of averages ; for the averages are average prices of a rather nondescript mass-unit of goods. But these averages have the merit that, vary the mass-units employed as we may, the two averages vary in the same proportion, so that the result of the comparison of them is unaffected; wherefore also it is the same as that of a mere comparison of the total sums without any averaging at all.[4] Now viewed in this second form there is much to recommend this method, provided the mass-quantities produced or consumed of every class are constant over both the periods compared. For then we are really comparing, so to speak, the total price at each period of the same mass of goods actually produced or consumed at each period.

In this case we know, by our mathematical analysis,[5] that while in each of the separate averages of the prices the weights of the classes are x, y, z, ..., that is, the numbers of times the mass-units chosen are produced or consumed, yet in this method viewed as a method of arithmetically averaging the price variations the weighting is according to $x\alpha_1$, $y\beta_1$, $z\gamma_1$, ..., that is, according to the total exchange-values of the classes at the first period, or, more simply, it is what we have called the weighting of the first period. This we have already denounced in the preceding Chapter as an absurd system of weighting. We shall, however, also find that this method may be viewed as a method using other kinds of averaging of the price variations, with other weighting, and so avoids the absurdity of using the weighting only of the first period. But we must postpone further examination of this method, applied to this case, till a later Chapter. Here what needs to be noticed, is that in Scrope's method, in all its forms, regarded as a method of arithmetically averaging price variations, the weighting is according to $x\alpha_1$, $y\beta_1$, $z\gamma_1$, ...

[4] It is plain that all the terms used—$y\beta_1$, $y\beta_2$, $z\gamma_1$, etc.—are the same whole quantities whatever be the sizes of the mass-units chosen. Cf. above Sect. V., Note 5. Hence it is indifferent what mass-units be employed. This is shown at length by Padan, B. 141, pp. 195–198. Yet some writers employing this method have thought it necessary to state that we ought always to employ the same mass-unit. So Laspeyres, B. 26, p. 306, and Paasche, B. 33, p. 171. (Also Lehr has made a similar mistake with regard to his method, as will be pointed out on a later occasion.) In this method the variability of answers has already been avoided by use of the total money-values and elimination of the bare mass-quantities, and so does not need addition of the other method of avoiding variability by the use of a single mass-unit.

[5] See Appendix A, II. § 7 already referred to.

If the mass-quantities of any of the classes vary from the one period to the other, provided they do not all vary alike, the above recommendation for this method vanishes. For this case several variations upon this method have been suggested. Some writers have recommended using only the mass-quantities of the first period, others the mass-quantities only of the second period, and others a mean between these two, as we have seen in the preceding Chapter. Furthermore, almost all the economists and statisticians who have advised the adoption of this method, or have actually adopted it, in any of its varieties, have had the habit, similar to the treatment of Carli's method, already noticed, of stringing out the comparisons over many periods in a series, comparing each of the later periods directly with the first as a basic period. Some have used always the same mass-quantities, either those of the first period, or of some other one period, or a general average, in all these comparisons. Now if the mass-quantities of the first or of some other one period be used, the comparisons between the later periods really are averages of the price-variations with weighting according to the money-values of the mass-quantities of the given period at the prices of the less removed of the two periods compared (or of the further removed, if the two periods be prior to the basic period),—which is a wholly anomalous and senseless system of weighting, the weighting also being different in every comparison, though this was not intended. If a single general average mass-quantity of every class be used, the later comparisons are equally anomalous in form, but have the merit of not altogether disregarding the mass-quantities of the periods compared. There is little to recommend this method in theory, though it may perhaps turn out to yield answers near to the truth in practice. If, lastly, the mass-quantities of the second, or later period, be used in all the comparisons with the first or basic period, a peculiar complication arises. For now, when we compare two later periods with each other, we are employing the mass-quantities of both these periods—that is, we are employing double weighting, and double weighting without any authenticated method of selecting the mass-units, hence an altogether absurd method, struck upon by chance merely. This method, without knowledge of its consequences, was actually recommended by Paasche, and may be called Paasche's variety of Scrope's method.[6] The consequence which it brings about was evidently never intended; for if it were, the same method of using the weighting of both periods ought to be employed in comparing the subsequent periods with the first, and if this double weighting is not wanted in those comparisons, it is not wanted in the others. In fact, Paasche himself objected to the use of double weighting, and criticised Drobisch for using it.[7]

A similar inconsistency exists in another method. This is in the similar serial form of the method of arithmetically averaging the price variations with uneven weighting—the method first employed by Arthur Young. For if this method be employed in such a series always with the weighting of the later period in every comparison of a later period with the common first or basic period,—as it has been employed by Mr. Palgrave, wherefore this may be called Palgrave's variety of Young's method,—then in the inverse of every comparison of prices between any of the later periods (or in the direct comparison of the exchange-values of money) there is use of the weighting of each of the periods compared.[8] Here also the use of double weighting was never intended, and the same inconsistency exists; for if the comparisons between the later periods are correct, the comparisons of the later periods with the first period cannot be correct, and then correct comparisons would be dependent upon incorrect comparisons,—which is absurd.

§ 5. This unperceived merging of some of the common methods of employing single weighting into methods employing double weighting, suggests that we should here examine the methods avowedly employing double weighting (with the arithmetic averaging of prices), doing so in the comparison of the second period with the first as well as in every other comparison. But we may confine our attention for the present to the first of these methods ever invented, namely to Drobisch's, examining

[6] For its formula see Appendix C, IV. § 2 (2), and for the nature of its mass-ts, Appendix C, V. § 4.

B. 33, pp. 172–173.

mula see Appendix C, III. § 4.

this because it also involves another of the features above noticed, namely the use of a single mass-unit in all the classes for the sake of avoiding variability in the results. The mass-unit preferred by Drobisch was a weight-unit—a hundred-weight, though the size of the common weight is indifferent. He wanted the finding at each period of the average price of the weight-unit of all goods, the average being the arithmetic with weighting according to the numbers of the weight-units in every class at each period. He claimed that by so doing we should get a sort of absolute average price, and so disprove Jevons's denial of an average price.[9]

The formula for this method is the general formula for the arithmetic average of prices with double weighting above given (17, 2.3), the method of selecting the mass-unit being presupposed. The method itself may be illustrated by the following example.[10] If in a country during a certain first period say ten millions of the same weight-units of goods (this figure being obtained by adding up all the numbers of weight-units of all the classes—$x_1 + y_1 + z_1 + \cdots\cdots$ to n_1 terms $= 10$) are produced and consumed at a total valuation of fifteen million money-units (this sum being obtained by adding up the numbers of weight-units of every class multiplied by their prices—$x_1\alpha_1 + y_1\beta_1 + z_1\gamma_1 + \cdots\cdots$ to n_1 terms $= 15$), there is an average price, represented by $\frac{15}{10}$ or $1\frac{1}{2}$ money-units, for the weight-unit. Then if at the second period twelve million weight-units ($x_2 + y_2 + z_2 + \cdots\cdots$ to n_2 terms $= 12$) are produced and consumed at a valuation of twenty-one million money-units ($x_2\alpha_2 + y_2\beta_2 + z_2\gamma_2 + \cdots\cdots$ to n_2 terms $= 21$), the average of prices, or the average price of the mass-unit, would be $\frac{21}{12}$ or $1\frac{3}{4}$ money-units. Therefore, he would conclude, not merely an average of prices, but simply the price of the mass-unit of goods, has risen from $1\frac{1}{2}$ to $1\frac{3}{4}$, or as from 6 to 7, that is, by $16\frac{2}{3}$ per cent., and inversely money has depreciated as from 7 to 6, by 14.29 per cent.

In this method the use, not of the arithmetic average, nor of double weighting, but of a common mass-unit may be shown to be wrong.

[9] B. 29, pp. 44–45, B. 30, p. 153.
[10] See also Appendix C, V. § 1.

The use of a common mass-unit means that the average drawn at each period is an average price of an average mass, so to speak, of all goods, or rather, it is simply the price of a small mass made up of small fractions of all goods, these fractions being proportioned to one another, by weight or by bulk, according as are the weights or the bulks of the classes relatively to one another. This small compound mass, or Anaxagorian homœomeria, is a sort of representative mass-unit, being composed in the same way as the whole mass of goods. It may be differently composed at each period, and its price may be different at each period. Drobisch claimed that the variation of its price represents the variation in the money-value of all goods, and inversely the variation of the exchange-value of money in all goods (other than money). This claim is unfounded, and violates several principles.

In the first place no reason is offered why the use of a common weight-unit was chosen rather than a common capacity-unit. Yet two differently varying prices would be obtained according as the one or the other is used of these kinds of mass-units. Thus this method does not succeed, except arbitrarily, even in the attempt to avoid the fault of admitting variability of equally good and hence mutually destructive results, and succumbs to a criticism already advanced against several other suggested or possible methods.

In the second place, because of this feature in it, the following absurdities can be proved of this method: (1) between two periods between which no variation of any price whatsoever takes place, if any irregular change takes place in the mass-quantities, the result will indicate a variation of prices [11] (contrary to Propositions XXVII. and XLIV.); (2) between two periods between which all prices have varied in exactly the same proportion, if any irregular change takes place in the mass-quantities, the result will indicate a variation different from that common variation (contrary to Propositions XVII. and XLV.); (3) between two periods between which all prices have

[11] This is the criticism made by Laspeyres, B. 26, p. 308. Drobisch replied by reaffirming his position, asseverating that this is as it should be, B. 31, pp. 425–426.

risen somewhat, if certain changes take place in the mass-quantities, it is possible that the result may even indicate a fall of the average price; and conversely.[12] These absurdities are not due to the use of double weighting, but to the use of double weighting along with this method of selecting a common mass-unit. They are to be found in some other methods using double weighting, where they are due to its conjunction with some other (consequently) improper method of selecting the mass-units. Thus all three are found, upon inspection of the formulæ, in Paasche's variety of Scrope's method and in Palgrave's variety of Young's method, so far as these use double weighting, that is, in comparisons between later periods, in a series founded on one period as a base. In Lehr's method, rather curiously, the second and third absurdities are to be found, but not the first. Again all three are found in the last two forms into which a rather vague method invented by Professor Nicholson may be analyzed.[13] But a method using double weighting with the arithmetic average may be discovered that is free from all these absurdities.

In Drobisch's method it is not difficult to see why double weighting with use of a common mass-unit leads to these absurdities. All prices remaining constant, an increase in the quantity of expensive goods, the prices of which are above the average at the first period, tends to raise the average at the second period (and reversely a decrease); but an increase in the quantity of cheap articles, the prices of which are below the average at the first period, tends to lower the average at the second (and reversely a decrease); and the average will be changed according as the one or the other of these movements

[12] *E. g.*, in the above numerical example, the following variations would be possible. Suppose at the first period things were thus:

cheap goods	8 mill. cwts.,	value	8 mill.	dollars;	price	$\frac{8}{8} = 1.00$,
costly "	2 "	"	" 7	"	"	" $\frac{7}{2} = 3.50$,
total	10 "	"	" 15	"	"	" $\frac{15}{10} = 1.50$;

and at the second thus:

cheap goods	7 "	"	" 6	"	"	" $\frac{6}{7} = .86$,
costly "	5 "	"	" 15	"	"	" $\frac{15}{5} = 3.00$,
total	12 "	"	" 21	"	"	" $\frac{21}{12} = 1.75$.

Here the whole collection of cheap goods has fallen in price, and also the whole collection of costly goods, wherefore it is possible that every article has fallen in price. Yet the average of the whole has risen in price.

[13] See Appendix C, V. § 3.

predominates. And all prices rising, or their average variation being a rise, the former change of quantities will raise the average still higher, and the latter will prevent it from rising so high —perhaps will leave it at constancy, or even occasion a fall; and reversely if all prices fall, or their average variation is a fall.[14] It is evident that the disappearance of a cheap article and the appearance of a dear one have the same effect as the variation of a cheap article into a dear one; and reversely the disappearance of a dear article and the appearance of a cheap one. A single disappearance, or a single appearance, has half this influence. Thus, prices remaining constant, or varying in a given degree, whether this method shall indicate a rise or a fall of prices, or a greater or a smaller rise or fall, will depend upon whether the change in the weighting happens to fall more on expensive goods or more on cheap goods.[15]

Thirdly, even at each of the periods separately the weighting employed by Drobisch is wrong; for his weighting is according to the mass-quantities, which weighting we have seen to be improper (in Chapter IV., Sect. III. § 3). A reason why it is incorrect is that the mass-quantities used have different preciousness, which also deserves to be counted, or otherwise the height of the average will depend upon the accident whether dear or cheap goods happen to abound. Still, if we tried to correct this error by weighting the mass-units at each period according to their preciousness, as well as according to their masses, we should get no determinate result, because at each period the average would return to the lowest price we happened to take as our basis (probably a mass-unit of the poorest quality). The two averages would have no connection with each other, and the comparison of them would have no meaning. To get this connection we need to employ a mass-unit in every class that

[14] Thus in the example in Note 12 the supposed increase of the falling goods was in the expensive goods and the decrease in the cheap goods, and the result was a rise.—The general mathematical principles are given in Appendix A. VII. § 4, with reference back to I. §§ 8-10.

[15] Hence Drobisch's method can be true only if the mass-quantities do not vary, or all vary in the some proportion. This was perceived by Lehr, who likewise condemns Drobisch's method for using such economically dissimilar mixtures of things, B. 68, pp. 41-42. The last criticism is also made by Zuckerkandl, B. 115, p. 247, B. 116, p. 245.

has the same exchange-value over both the periods together as in every other class. This is the improvement introduced by Professor Lehr, whose method, however, as we shall find in a later Chapter, does not correctly carry out the improvement intended, wherefore the true method is still something else.

Lastly, a general reason why Drobisch's method is wrong is that it has been invented without any regard for the principles of simple mensuration. We have seen in Chapter III. that we must compare variations only in two similar worlds. Neither of the methods there pointed out for obtaining such similarity are here observed. The same trouble exists in the varieties of Scrope's and of Young's methods introduced by Paasche and by Palgrave. In comparing a later period with a first period on a given weighting, we are comparing variations in two similar worlds, which so far is correct; and in comparing another later period with the first on another weighting, we are comparing variations in another set of similar worlds; but in comparing the results of these two measurements we are comparing things in two different worlds.

It should be noticed that in many—in fact, in most—subjects of statistics we do wish to use double weighting with the quantities as they happen actually to be reported, that is, to compare averages at different periods built on varying quantities of things whose individuality is reckoned without regard to the attribute we are measuring. Here the above principles of simple mensuration are not violated, for the good reason that these are not subjects of simple mensuration. And if we are comparing things in dissimilar worlds, the very point is that we are dealing with dissimilar worlds and want to compare them. Thus to measure the measure of length at different periods, we must deal with similar worlds at both periods; but in measuring the average tallness of people at different times or places, we are expressly dealing with different peoples, and so must average their heights at each period or place separately, weighting each operation according to the numbers of persons measured, and then merely compare the results of the separate averagings—in all cases using a supposedly constant measure of length already examined and approved. Again, to use an example nearly allied to our own subject, in the measurement of the average wealth of a country at different periods we must take into account the different total wealths and the different numbers of the populations possessing them. Here, for instance, it would be perfectly proper for the result to indicate that, though there be no variation whatsoever in the wealth of the individuals in the different classes of society (as found by our using a measure of wealth already examined and approved), yet if the wealthier classes increased more in numbers, the average wealth of the country would increase, and if the poorer classes increased more in numbers, the average wealth of a country would decrease.

Not so, however, in regard to the exchange-value of money (which is the measure itself used in the preceding measurement). Suppose a creditor should argue thus: "Prices have not varied a particle, but I, and the whole community as well, are now purchasing more expensive articles and fewer cheap articles than we used to do; therefore, as we get a smaller quantity of goods for the same money, our money has fallen in purchasing power or exchange-value, and the average of prices has risen, and my debtor ought to pay me back more money than he borrowed, to make up the deficiency." It is plain that he would be talking nonsense, because the debtor could readily reply: "If you and I and the rest of us spend our money more on expensive articles and less on cheap articles than we used to, the fact that prices have not changed shows that, though we are getting smaller quantities, we are getting better qualities, and the gain in the latter makes up for the loss in the former;[16] wherefore there is no reason why I should give you back more money to enable you to get the same quantity of better articles."

Now if we already had a correct measure of exchange-value, if we wanted to measure variations of the average *preciousness*, not of money, but of all commodities in general, we should pursue exactly the course advised by Drobisch for measuring variations of the average exchange-value of money. It is perfectly proper

[16] Cf. Chapt. IV., Sect. V., Note 19.

that, the preciousness of individual things remaining unchanged (all prices being constant), an increase in the quantity of cheap things should lower the average of preciousness, and an increase in the quantity of costly things should raise it. The three absurdities in Drobisch's method as a measure of the exchange-value of money cease to exist in it as a measure of the preciousness of commodities. We have seen, also, that Drobisch's method gives two results, according as it is applied to a common weight-unit or to a common capacity-unit. This is perfectly proper in the method as a method of measuring the preciousness of goods, because there are two ways of conceiving of preciousness.[17] Thus Drobisch has hit one thing while aiming at another. But the proviso stated at the commencement is of importance. For the perfect working of this method of measuring variations in the preciousness of goods the exchange-value of money, by which we measure the preciousness of individual things, ought to be constant, and we should know this.[18] Otherwise its variations must be allowed for; for which purpose also its variations need to be known. Thus the proper method of measuring variations in the exchange-value of money is a prerequisite even for the employment of Drobisch's method as a method for measuring variations in the preciousness of all commodities.

§ 6. *Geometric averaging of prices.*—As in the case of averaging the exchange-values of money, the geometric average of prices does not fall into two different methods according as we compare averages of prices or average the variations of prices. What is an apparent weighting in the one is actually the weighting both in it and in the other. If we average prices at two periods separately by using weighting, the same in both cases, according to the relative mass-quantities, the same result is obtained as if we employed the same weighting in averaging the variations of prices. But we know that we do not want to weight either the prices or the variations according to the mass-quantities—or the mere numbers of times some mass-units happen to recur,—but according to the total exchange-values, which

[17] See above Sect. III., Note 3.

[18] Money serves here as water does for measuring the specific gravity of bodies.

correspond to the numbers of times equivalent mass-units are repeated. Hence the average should be in this form,

$$\frac{P_2}{P_1} = \sqrt[n'']{\left(\frac{\alpha_2}{\alpha_1}\right)^{\mathbf{a}} \cdot \left(\frac{\beta_2}{\beta_1}\right)^{\mathbf{b}} \cdot \left(\frac{\gamma_2}{\gamma_1}\right)^{\mathbf{c}} \cdot \text{...... to } n \text{ terms}},$$

in which $n'' = \mathbf{a} + \mathbf{b} + \mathbf{c} +$ ...... to n terms, and $\mathbf{a} = x\alpha$, $\mathbf{b} = y\beta$, and so on. At which periods the terms $x\alpha$, $y\beta$, ... should be chosen, we have examined in the preceding Chapter, and have concluded that the best weighting is for $\mathbf{a}$ to be equal to $\sqrt{x_1\alpha_1x_2\alpha_2}$, $\mathbf{b}$ to $\sqrt{y_1\beta_1y_2\beta_2}$, and so on. One reason why weighting according to the mere mass-quantities is not to be employed is because this weighting would be accidental according to the sizes of the mass-units used. Or if, to avoid this variability in the results, we used the same mass-unit throughout, there would still be a choice between two results, on a unit of weight or a unit of capacity. And in either of these cases, as, given the total exchange-values of two classes to be the same, the same mass-unit would be repeated less frequently in a more precious article than in a less precious one, the weighting would depart from the proper weighting inversely according to the preciousness of the articles.

§ 7. In a series of successive periods, if we employ even weighting or the same uneven weighting throughout, a comparison of subsequent periods all with a first basic period, on the geometric average, will yield results which, compared with one another, give the same results as when these periods are directly compared with one another with the same weighting. Thus the comparison between the second and the first will be as above indicated, that between the third and the first like unto it with change of numbering, and that between the results of these comparisons as follows,

$$\frac{P_2}{P_1} = \frac{\sqrt[n'']{\left(\frac{\alpha_3}{\alpha_1}\right)^{\mathbf{a}} \cdot \left(\frac{\beta_3}{\beta_1}\right)^{\mathbf{b}} \cdot \left(\frac{\gamma_3}{\gamma_1}\right)^{\mathbf{c}} \cdot \text{...... to } n \text{ terms}}}{\sqrt[n'']{\left(\frac{\alpha_2}{\alpha_1}\right)^{\mathbf{a}} \cdot \left(\frac{\beta_2}{\beta_1}\right)^{\mathbf{b}} \cdot \left(\frac{\gamma_2}{\gamma_1}\right)^{\mathbf{c}} \cdot \text{...... to } n \text{ terms}}},$$

which reduces to

$$\frac{P_2}{P_1} = \sqrt[n'']{\left(\frac{\alpha_3}{\alpha_2}\right)^{\mathbf{a}} \cdot \left(\frac{\beta_3}{\beta_2}\right)^{\mathbf{b}} \cdot \left(\frac{\gamma_3}{\gamma_2}\right)^{\mathbf{c}} \cdot \text{...... to } n \text{ terms}},$$

the formula for comparing the third period with the eriod directly.

agreement does not exist in the comparisons made in ıstructed on Carli's or on Young's method, or in a few ys, as in one variety of Scrope's method. The employ- these methods on the usual plan of comparing all later lirectly with an original basic period involves an incon- since the indirect comparisons between these later yield a different result from what would have been by the employment of the same method of averaging in measurement between these periods, or from what the neasurement itself would have been had a different basic een chosen.[19] This inconsistency appears never to n perceived[20] until it was pointed out a few years Professor Westergaard. Professor Westergaard also as- a general way, that this inconsistency does not occur ometric method, and advanced this difference as an ar- in favor of the geometric method.[21] Credit is due to Westergaard for calling attention to this difference. ıse of it as an argument for the geometric averaging of iations suggests the following remarks.

there are other methods beside the geometric that share reedom from this inconsistency. Thus in a series in e later periods are separately compared with the first, nsistency is not incurred by the use of Dutot's method, se of Scrope's method with the same mass-quantities in (either those of the first or of some other period, or a verage of all the periods or of some of them or of any [22] or by the use of Drobisch's method (provided it be to the same classes). Hence the argument for the average as exclusively possessing a certain excellence, invalid. Professor Westergaard had in mind only

ove in § 3 in regard to Carli's method. It is plain that the same holds ung's. Cf. Appendix C, II. § 2 and III. § 2.

d example of how it has been ignored is furnished by De Foville. See C, IV. § 2 (2).

), pp. 219–220, followed by Edgeworth, B. 65, p. 386, B. 66, p. 137.

e formulæ for a series of index-numbers using this method, in Appen- § 1.

Carli's and Young's methods; and his argument is due to this restriction. As for the methods which do incur this inconsistency, these are not only Carli's and Young's, including Palgrave's variety of the latter, but also Paasche's variety of Scrope's, Lehr's, Nicholson's (in its one original form), and another which is still to be described—in the last three provided they were to be employed in the serial form with dependence upon a common base.[23]

Secondly, it should be distinctly noticed that even the geometric average possesses this excellence only if used with the same weighting through the whole series. But this system of weighting evidently is slipshod, and cannot pretend to give perfectly true results. The true system of weighting is to employ special weighting in every single comparison, according to the conditions existing at the two periods compared. Now if we employed such proper weighting in comparing the second with the first, and again in comparing the third with the first, the weighting would most likely be different in these two comparisons. Then a comparison of the results so obtained would, as in some of the arithmetic averagings already noticed, be a comparison between these two periods with double weighting—and of such a kind as to lead to the three absurdities into which Drobisch's method fell. And now the direct comparison between the two periods, with single weighting, would be with weighting still different from that used in either of the preceding comparisons, and its result would not agree, except acciden-

[23] In all these methods the direct comparisons between the later periods, and the indirect comparisons between them through mediation of their comparisons with the basic period, may agree under certain conditions. It is easy to discover these conditions by comparing the formulæ, and it may be interesting to state them. Thus the indirect agrees with the direct comparison in *Carli's* method if between the basic and the nearest subsequent period (or furthest prior period) there be no price variations, or if all these be in the same proportion; in *Young's* method, the same; in *Palgrave's* variety of Young's method, if the mass-quantities be the same at the two subsequent periods, or if they have varied between these two periods in the same proportion; in *Paasche's* variety of Scrope's method, if between the two subsequent periods there be no variations either in the prices or in the mass-quantities, or if between these two periods both the prices and the mass-quantities have all varied in one and the same proportion. The other methods have never been suggested for use in this serial form. What would be the conditions of such agreement in them if so used will be examined for another purpose in a later Chapter.

tally, with the result obtained by comparing their results. Yet, supposing the geometric average the right one to use, among these comparisons it is only this direct one, if the periods be contiguous, that can give the correct result.

Thirdly, it would seem to be unquestionable that the agreement between the indirect and the direct comparisons ought to exist. Professor Westergaard based his recommendation of the geometric average upon the principle that prices measured from 1860 to 1870 and from 1870 to 1880 ought to show the same variation from 1860 to 1880 as would be shown by comparing the prices of 1880 directly with those of 1860.[24] By adding another supposition this can be made self-evident. Suppose that everything at the last period is in exactly the same state as at the first (or that everything diverges in the same proportion). Then it is evident that the exchange-value of money is the same at the last period as at the first period (or deviates inversely to a uniform price variation), whatever intervening variations may have taken place.[25] Constancy (or that variation) is shown by the direct comparison between these two periods; and constancy (or that variation) should be shown by a series of comparisons through intervening periods, no matter how many or how varied the intervening periods may be. But it does not follow that this agreement should be sought at the expense of other requirements, or that such agreement, if existing in any method, can prove it to be true, if we already know that this method fails in other respects. For instance, Drobisch's method provides this agreement, but Drobisch's method is not thereby proved to be correct. The need of this agreement can be taken only as a negative test, or touchstone. If a method has other things to recommend it, it may be disproved by being found not to stand this test. But unless it is recommended by other reasons, it is not proved by its satisfying this test. Perhaps, however, the

[24] B. 110, p. 219.

[25] This would seem to be so also if only all the prices were the same (or equally divergent) at the separate periods, let the mass-quantities be what they may—in accordance with Propositions XXVII. and XVII., and XLIV. and XLV. But if the mass-quantities as well as the prices are all exactly the same (or equally divergent), there can be no question about it.

perfectly correct method, for a long sequence of periods, will not be found to be workable, and the workable form that comes the nearest to it may not satisfy this requirement perfectly. Then this requirement may be used as a means of correcting the latter.

Lastly, this argument of Professor Westergaard's leads to the consideration whether we should continue in the usual course of comparing every subsequent period with the same original period as common basis for a whole series, or whether the series should be formed by comparing every period with the immediately preceding period, and again the next period with it, and welding them all together. The latter is as practicable a course as the former, and may yield the same sort of serial index-numbers. For example, if we find the result of the first comparison to show that between the first and the second periods prices have fallen on the average by 10 per cent., and the result of the second comparison to show that between the second and the third periods they have risen by 30 per cent., we may express these relations in the series 100, 90, 120, which indicates at once that, by passing through the second period, the level of prices at the third is 20 per cent. above what it was at the first. As to the first of these procedures, there is little in its favor but its convenience. The most natural and rational procedure, the moment we attach importance to correct weighting, would seem to be the second. At first sight the argument between the two appears to lie merely between the opinions that in the former we gain exactness in the comparisons of the later periods with the first and lose it in the comparisons between the later periods, and in the second we gain exactness in the comparisons between contiguous periods and lose it in all other comparisons. But of course both these positions are false, being self-contradictory. The question is really, In which of the two courses are we likely to gain greater exactness in the comparisons actually made? Here the probability seems to incline in favor of the second course; for the conditions are likely to be less diverse between two contiguous periods than between two periods say fifty years apart.

§ 8. There is still another reason for adopting the second course. This is that the first course is tied to the employment of a definite number of classes, decided once for all at the start; but the second course permits the dropping at any time of an old class or the introduction of a new one. In fact this second course is the only one permitting an accurate measurement of the exchange-value of money over a very long series—say of one or several centuries. For in periods separated by a very long interval there is much that is not common to both the periods, and to compare them by what is common to them both is to make a restricted and incomplete comparison. A complete comparison can be made only by comparing all that is in each of them with the intervening periods near to them, gradually dropping out the old things as they disappear and introducing the new as they appear. It is also only in the comparison of such temporally distant periods that the divergence between the two ways of conducting the measurement is likely to be great. Therefore for long series it is important to decide between the two courses. Now if the two periods be two years separated by a century, it is probable that if a rightly conducted measurement had been made of the variations from year to year filling up the whole century, the result at the end would be very nearly the true one, while a direct comparison between the two extreme years, which comparison might have to leave out altogether some classes that have been counted in the other comparisons, because they did not exist at the first period or no longer exist at the last, might be appreciably divergent from the truth, even though the comparison were made in proper manner and the data were correct.[26] Hence the procedure of comparing every successive period with the immediately preceding is the proper one to adopt if it be intended to continue the measurement over many years, or indefinitely.[27]

Incidentally we hereby see also that among the reasons why we cannot successfully compare the exchange-value of money

[26] Cf. above Chapt. IV., Sect. V. § 7, Note 18.

[27] Into this course De Foville was forced by circumstances in his calculations after 1862. It has been recommended by Lehr for his method, B. 68, pp. 45, 46, and in general by the British Association Committee, *First Report*, B. 99, p. 250, and Edgeworth, B. 59, pp. 268–269.

to-day with what it was five hundred years ago is not merely that we have not accurate data about the year 1400, but that we have not complete data about all the intervening years. Yet there is no reason why, if from now on the measurement be correctly made from year to year, the people living in 2400 should not know with almost perfect exactness the relation between the exchange-value of their money and ours.[28]

The brief hypothetical series of index-numbers above used as examples, namely 100, 90, 120, can be reduced to this: 111.11, 100, 133.33, or again to this: 83.33, 75, 100, or to any other series in which the figures observe the same proportions.[29] This fact shows that it is of no importance what period is selected as the so-called basic period of the series, whichever of the above two courses be adopted. Properly every period is basic in the comparison with the next period, and none is basic for a series of periods. A series with the index-number 100 at one period does not better represent the sequence of events than a series with this index-number at any other period. Nor is there rhyme or reason for making the basic period longer than the other periods. Much discussion as to the superiority of this or that base in the various attempts at measuring the course of the exchange-value of money during the latter half of the century just elapsed has been wasted.[30]

VII.

§ 1. There remains only to point out the mathematical formulation for the exchange-value of money in *all* things (or inversely for the level of all prices, including the price of money). This

[28] The not infrequent complaint about the incommensurability of the exchange-value of money at two long separated periods directly compared (*e. g.*, recently by W. Cunningham, *On the value of money*, Quarterly Journal of Economics, July 1899, pp. 379–385) should be allowed no influence in the question as to whether the variation of this exchange-value through successive periods is measurable. The difficulty is greater in the attempt to measure the contemporaneous levels of the exchange-value of money in two distinct countries, say an arctic and a tropical. But luckily we have not so much interest in measuring this.

[29] The table of Evelyn on the basis of 1550 was thus turned about and put on the basis of 1700 by J. P. Smith, B. 7, pp. 472–476.

[30] Concern for the base was first shown by Porter, B. 11, p. 440. That the base should be carefully selected is the first of the canons laid down by Martin, *op. cit.*, p. 626. It is said to be of great importance by Mayo-Smith, B. 137, p. 486.

is easy on the assumption that any of the mathematical averages is proper for expressing the exchange-value of money in all *other* things. We simply have to add to the right side of the equation the expression for the exchange-value of the money-unit in money, which is always unity, and invariable.

Thus, taking the arithmetic average and using M_a to express the exchange-value of the money-unit in *all* things, we have for the first period, with even weighting,

$$M_{a1} = \frac{1}{n+1}\{1 + (aA) + (bB) + (cC) + \cdots\cdots \text{ to } n+1 \text{ terms}\},$$

which reduces to unity. And for the second period, again with the same even weighting, we have

$$M_{a2} = \frac{1}{n+1}\{1 + a'(aA) + b'(bB) + c'(cC) + \cdots\cdots \text{ to } n+1 \text{ terms}\}.$$

And the comparison of these two reduces to

$$\frac{M_{a2}}{M_{a1}} = \frac{1}{n+1}(1 + a' + b' + c' + \cdots\cdots \text{ to } n+1 \text{ terms}). \quad (18, 1)$$

For the other kinds of averaging, still with even weighting, we should have, after similar reductions,

$$\frac{M_{a2}}{M_{a1}} = \frac{1}{\frac{1}{n+1}\left(1 + \frac{1}{a'} + \frac{1}{b'} + \frac{1}{c'} + \cdots\cdots \text{ to } n+1 \text{ terms}\right)}, \quad (18, 2)$$

and

$$\frac{M_{a2}}{M_{a1}} = \sqrt[n+1]{a' \cdot b' \cdot c' \cdot \cdots\cdots \text{ to } n \text{ terms}}. \quad (18, 3)$$

It is easy to restate these formulæ with the terms $\frac{a_2}{a_1}, \frac{b_2}{b_1}, \frac{c_2}{c_1}$, ... to represent the variations, in place of a', b', c', ...

Inversely, by means of averages of prices, the same results may be obtained by inserting in the formulæ for the averages of price variations the price also of the money-unit, which is invariably a unit.

With uneven single weighting, we have to add the weights of

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the commodity classes as before, and in addition we must give some weight to the class money. What the nature of this weight is, will be left over for a later discussion (in Chapter XIII.). Here we may represent it by **m**. Then, using the form for the variations of prices, with P_a to represent the level of prices with the price of money included, we have

$$\frac{P_{a1}}{P_{a2}} = \frac{1}{\frac{1}{n'' + \mathbf{m}}\left(\mathbf{m} + \mathbf{a}\frac{\alpha_1}{\alpha_2} + \mathbf{b}\frac{\beta_1}{\beta_2} + \mathbf{c}\frac{\gamma_1}{\gamma_2} + \cdots\cdots \text{ to } n+1 \text{ terms}\right)}, \qquad (19, 2)$$

$$\frac{P_{a2}}{P_{a1}} = \frac{1}{n'' + \mathbf{m}}\left(\mathbf{m} + \mathbf{a}\frac{\alpha_2}{\alpha_1} + \mathbf{b}\frac{\beta_2}{\beta_1} + \mathbf{c}\frac{\gamma_2}{\gamma_1} + \cdots\cdots \text{ to } n+1 \text{ terms}\right), \qquad (19, 2)$$

$$\frac{P_{a2}}{P_{a1}} = \sqrt[n''+\mathbf{m}]{\left(\frac{\alpha_2}{\alpha_1}\right)^{\mathbf{a}} \cdot \left(\frac{\beta_2}{\beta_1}\right)^{\mathbf{b}} \cdot \left(\frac{\gamma_2}{\gamma_1}\right)^{\mathbf{c}} \cdot \cdots\cdots \text{ to } n \text{ terms.}} \qquad (19, 2)$$

It is evident on inspection that, compared with the results of the formulæ for the corresponding averages of the variations of money in exchange-value in all other things, the results of these formulæ always show a smaller variation than those, in agreement with Proposition XXI.; that they always indicate constancy when those do, in agreement with Propositions XXII. and XXVI.; that they always vary in the same direction, in agreement with Proposition XXIII.; that their divergence, or falling short, from those is always smaller the greater the number of terms (or rather, the larger is n'' compared with **m**), in agreement with Proposition XXIV.; and that the proportion of this divergence is always the same, in agreement with Proposition XXV.; and lastly that their indications are unaffected by the numbers of other things that are constant in general exchange-value, if the indications be of constancy, or that vary in the same proportion in exchange-value in all things (their prices varying as $\frac{P_{a2}}{P_{a1}}$), in agreement with Propositions XXXII., XXXV., and XXXVI.

§ 2. Other formulæ may be altered in the same manner. In Scrope's method the want of body to money—the indifference we all feel to its mass—causes no trouble, because all the terms are the total money-values of the classes, and so long as we can get this in the case of money, without reference to its weight or bulk,[1] we are satisfied. Thus, with v representing the number of money-units employed, the formulæ for Scrope's method are the following,

$$\frac{P_{a2}}{P_{a1}} = \frac{v + x\alpha_2 + y_i\beta_2 + z\gamma_2 + \cdots\cdots}{v + x\alpha_1 + y_i\beta_1 + z\gamma_1 + \cdots\cdots},$$

and

$$\frac{M_{a2}}{M_{a1}} = \frac{v + x\alpha_1 + y_i\beta_1 + z\gamma_1 + \cdots\cdots}{v + x\alpha_2 + y_i\beta_2 + z\gamma_2 + \cdots\cdots}.$$

The same remarks apply to these as have just been made on the formulæ for averaging the price variations (except the reference to Proposition XXV.).

In Drobisch's method, however, where specific weights, or capacities, appear alone in one half of the formula, it would be impossible to make the alteration permitting this method to be applied to the measurement of the exchange-value of money in all things. Here is one more reason showing the falsity of that method.[2]

But there are methods employing double weighting that are applicable to the measurement of the exchange-value of money in all things, including itself.

[1] The mass-unit of money which we habitually use is nothing else than the money-unit itself, whose money-value is always one money-unit.

[2] As a measure of the preciousness of commodities it is no fault in this method that it cannot be applied to all things, including money. For money, in which mass is inessential, has no preciousness.

CHAPTER VI.

THE QUESTION OF THE MEANS AND AVERAGES.

I.

§ 1. We are now prepared for entering upon our subject proper, the search after the right method of measuring general exchange-value and its variations. We must deal first exhaustively with exchange-value in all other things, as the simpler.

When one thing rises or falls to a certain extent in exchange-value in every other thing alike, we know (by Proposition XVII.) that it rises or falls to the same extent in exchange-value in all other things, and (by Proposition XVIII.) that every one of the other things have reversely fallen or risen to a smaller extent in exchange-value in all other things. We also know (by Proposition XIX.) that the larger the number of the things, the smaller is the opposite variation of every one of the other things. But we do not as yet know how great the opposite variation of every one of the other things is, given the variation of one and their number. Here is a Proposition lacking, which if we shall be able to supply, we shall be able to measure all exchange-value variations. The problem, then, is to supply this Proposition.

In attempting to solve a problem it is well to put it in its simplest form. The simplest form in which we can put our problem is the following. Let us suppose that we are dealing, with a world in which are only three classes of exchangeable things (or with a part of our world which consists of only three classes). One of them let us suppose to be money, so that we have a world with money, represented by [M], and two classes of commodities, represented by [A] and [B]. Here all the *other* things beside the one class, money, whose exchange-value we are examining, are comprised in only two classes. These two classes of commodities we may also, for simplicity, suppose to be equally important, or equally large—not at one period only, but either at each of the periods compared or somehow over both the periods together, without reopening here the question how their importance or economic size is to be computed. Now if we succeed in solving the problem in this confined and simple form, we shall not be able immediately to extend the same solution to all cases; for what is true of two equally important classes may perhaps not admit of being applied to more complex cases. But if we succeed in the simplest case, we shall be well on the way to solving the complex cases. These, moreover, may be admitted to consideration as we advance further, even before we finally solve the simple cases.

Simplicity in the form of the problem will also be carried out by supposing that at the first period the money-unit is equivalent to a mass-unit of each of the commodities, so that M ⇔ A ⇔ B, and the prices of A and B are one money-unit each.[1] We may then suppose that at the second period A rises in exchange-value in the other two, individually and together, by one half, or by 50 per cent.; which means that its price rises to 1.50, while the price of B remains 1.00. At the same time M and B have each fallen by one third in exchange-value in [A], or by 33⅓ per cent., but as M retains its exchange-value in [B] and B its in [M], M has not fallen as much in [A] and [B] together, that is, in exchange-value in all other things, nor has B fallen so much in [A] and [M] together, that is, also, in exchange-value in all other things.

Digressively it may here be remarked that, as a corollary to Proposition XIV. [M] and [B], that is, *two or more classes that retain the same exchange-value in each other, vary alike in exchange-value in all other things beside themselves and in exchange-value in all things including themselves, but not in all other things (i. e., other to each class), unless these classes be equally large (over both the periods together)*, (Proposition XLVI.). The

[1] But we must be on our guard not to regard A and B, equivalent at the first period only, as the economic individuals in the classes [A] and [B] for the two periods compared; for the economic individuals must be equivalent over both the periods together.

reason why they do not necessarily vary (or remain constant) alike in exchange-value in all other things, is because for each the standard of other things, composed of the classes other to it, is different from what it is for every other.[2] Thus, in the example before us, if [M] be a class twice as large as [B], and consequently of [A], the rise of A will have less influence in depressing the exchange-value in all other things of [B] than in depressing the exchange-value in all other things of [A], since it is a rise of only one third of the things other to the class [B], while it is a rise of one half of the things other to the class [M]. But if the classes [M] and [B] are equally large or important, the class [A] will be of the same relative size among the things other to each of them, and hence will have the same influence in making them vary in exchange-value in all other things. We do not here concern ourselves with the size of the class [M], and confine our attention to enquiring what influence upon its exchange-value in all other things has a rise of the exchange-value of [A] both in it and in [B]. For this purpose we must presuppose knowledge of the relative sizes of [A] and [B]; and, to repeat, the simplest case is to suppose that they are equally large.

Our special problem then is: Given the above supposed conditions and variations of A, how much has M fallen in exchange-value in all other things, that is, in [A] and [B] together? Now the fall of M in [A] and [B] together is the same as its fall in [A] and [B] under the supposition that A and B both rose together to some extent. Therefore this problem is the same as to ask: *To what rise of A and B together is the rise of A alone equal in its influence upon the exchange-value of M in all (both these) other classes of things?* What we want is to reduce irregular variations in the particular exchange-values of the money-unit to a uniform variation of all of them, because when we have the latter, we know the variation in the exchange-value of the money-unit in all other things (by Proposition XVII.).

§ 2. To this problem an answer might be made which has the nature of an objection. It might be argued that, though M

[2] Cf. also Proposition XXXIII.

has fallen from the power of purchasing A to the power of purchasing only $\frac{2}{3}$A, yet as this $\frac{2}{3}$A has risen by one half in exchange-value in all other things, therefore M still commands as much exchange-value in all other things as before, and still possesses as much; and with M also B. But this argument assumes that A has risen by one half in exchange-value in all other things themselves remaining constant in exchange-value in all other things. This implied assumption is false. With the same evidence with which we perceive that A has risen by one half in exchange-value in all other things, we perceive that the other things have fallen somewhat in exchange-value in all other things. We know that M has fallen as well as we know that A has risen. And the two statements are perfectly consistent. For when M purchases two thirds of A, which has risen by one half in exchange-value in all other things, themselves fallen in all other things, M does not command so much exchange-value in all other things as it did before.

Then another solution might at once be offered, which, if correct, would settle our problem without further trouble. It might be argued that if at the second period M had become equivalent to $\frac{1}{2}(\frac{2}{3}A + \frac{2}{3}B)$, it would have fallen by one third in exchange-value in all other things; therefore, since it has fallen to equivalence with $\frac{1}{2}(\frac{2}{3}A + B)$, it has fallen by one sixth, a fall of one third in one thing being equal to a fall of one sixth in two things. But this argument makes a mistake similar to that in the preceding. That argument neglected to take account of the fall of M and of B in exchange-value in all other things. This argument neglects to take account of the fall of B in exchange-value in all other things. M purchases only one whole of an article which has fallen with it. It purchases two thirds of an article which has risen by one half in exchange-value in articles which have fallen. Therefore M would seem to have fallen by more than one sixth. But from these data alone no definite conclusion can be drawn concerning the extent of the fall of M (and of B) in exchange-value in all other things.

§ 3. If we could know how much A has risen in exchange-value in all things, this would give us sufficient information from which to calculate the exchange-values of M and of B. But we cannot know that until we know how much the exchange-values of M and B have fallen. Thus the beginning seems to be lacking.

The trouble is, we are as yet in possession of only one equation with two unknown quantities. We know the equality of the exchange-value of M at the second period to the exchange-value of $\frac{2}{3}$ A and of B. But we do not know the exchange-value of B any more than we know that of M itself; and although we know the exchange-value of A in all other things, that is, other to it, this is in another standard, and we do not know the exchange-value of A in the standard of other things for M, namely [A] itself and [B]. We cannot know this until we know the variation of B, or of M itself, in this standard. A further piece of information is wanting. In order to solve the problem we must supply ourselves somehow with this other information. Further consideration of the question, with variations in the manner of putting it, may perhaps yield us further insight into the nature of our subject, and so disclose that we are in possession of the information desired.

§ 4. Now we do possess a piece of information, already noticed, which is of some service here. This is the knowledge that the variation of M (and of B) in exchange-value in all other things, when A rises to $1\frac{1}{2}$, is *less* than the inverse variation from 1 to $\frac{2}{3}$, that is, it is to some quantity between 1 and $\frac{2}{3}$.[3] This provides us with a hint. Between two numbers we are acquainted with several mathematical means, and the idea suggests itself that the variation in question may perhaps be to one of the mathematical means between 1 and $\frac{2}{3}$. This suggestion becomes more plausible when we recall that the formulæ we have discovered for exchange-value variations are expressions of the three "classic" means or averages. Thus if it be true that the variation of M_{02} compared with M_{01} is to one of these means

[3] According to Proposition XVIII. Evidently to produce the same influence upon the exchange-value of M a common variation of A and B together cannot be so large as the variation of A alone; for if we had the variation of A alone in the first place, influencing M to a certain extent, an added variation of B in the same direction would increase the influence upon M. Compare with this the reasoning leading up to Proposition XXX.

ween 1 and ⅔, it would be either to $\frac{1}{2}\left(\frac{2}{3} + 1\right) = \frac{5}{6}(= .8333)$, arithmetic, or to $\frac{1}{\frac{1}{2}\left(\frac{3}{2} + 1\right)} = \frac{4}{5}(= .80)$, the harmonic, or to $\sqrt{\frac{2}{3} \cdot 1} = .8165$, the geometric; whence the percentage of the ll may be calculated, in hundredths, by subtracting the result rom unity. To be sure, we have here only a hint, but it is ufficient to incline us to try these suggested answers, and so at uce to narrow our enquiries; for, although there are other more complex mathematical means, we need not notice them unless the true answer be not found among the simpler ones.[4]

And this hint extends to the average of prices also. The relationship between these and the averages of the exchange-values of M we have already seen. Thus in our suppositional case the reciprocal of the arithmetic mean between 1 and ⅔, namely, .8333, is 1.20, which is the harmonic mean between 1.00 and 1.50; the reciprocal of the harmonic mean between those exchange-values, namely .80, is 1.25, which is the arithmetic mean between these prices; and the reciprocal of the geometric mean between those exchange-values, namely .8165, is 1.2247, which is likewise the geometric mean between these prices. Therefore if we find to which average between 1 and ⅔ the exchange-value of M in all other things has sunk when A alone rises to 1½, we shall know to which average between 1.00 and 1.50 the rise of both A and B together is, in its influence upon M, equal to the rise of A alone to 1.50.

Thus our simple problem becomes a question of mathematical means. And as it has been stated in the simplest way possible, it has been made to employ single weighting. The question has been put as a question of means between variations of exchange-

[4] The reader cannot be too often warned that we are not averging ⅔A and 1B, but ⅔*A* and 1*B*—we are not averaging, for instance, ⅔ pound of sugar and 1 pound of copper, which would be meaningless, but we are averaging the exchange-value of M in [A], say sugar, and the exchange-value of M in [B], say copper, in each case measured at the second period by comparison with what it was at the first. These exchange-values, or these variations of exchange-values, are similar and co-ordinate things between which it is perfectly proper to draw averages.

values, or of prices. In this form the problem may be extended to complex cases, embracing more than two classes, and classes unevenly important. It then becomes a question of mathematical averages, still with single weighting, between price variations—for in these cases the variations of the exchange-values of money have generally been relegated to the rear behind the variations of prices.

Now we may find that, while the question of the mean may be definitely answerable in the simple case posited, yet the question of the averages may not be in the complex cases, except on special occasions. It is plain that the above indication of the answer to our general problem is not necessarily complete. The universal solution may perhaps be yielded, not by a mathematical average of the price variations, but by a comparison of mathematical averages of prices at the two periods compared, possibly requiring the use of double weighting. We at least see that we have these more complex solutions in reserve, in case the more desirable, because simpler, solutions, which first suggest themselves, shall fail.

II.

§ 1. The suggestion that the general exchange-value of money has changed to some average of the variations of its many exchange-values, or to the reciprocal of some average of the price variations, is so plain that it has mostly been acted upon without question. Indeed it was followed for more than a century before the problem itself was even definitely raised. And it was followed very specifically. Among the three averages was selected the arithmetic, this being the simplest, the easiest to manage, the most familiar, and therefore the first to recommend itself. The adoption of this average has been due to no consideration of its special propriety for the subject it is applied to, but merely to a general preference for it in every case. This is shown by the way it has been applied. Statisticians have mostly conducted their investigations about variations in the exchange-value of money simply by noting variations in the prices of commodities; in doing which they have operated upon the variations of the exchange-values of commodities in money, and not directly upon the variations of the exchange-values of money in commodities, but inversely upon the reciprocals of these. It happened then that, dealing merely with prices, the early workers in this field applied the arithmetic average to the variations of prices; and most of the later ones have followed suit. Thus they have unwittingly employed the most difficult and least familiar average, the harmonic, for the direct averaging of the variations of the particular exchange-values of money. Had they happened to conduct their enquiries directly upon these, as measured by the quantities of the things purchasable with the money-unit, it is likely they would still have employed the arithmetic average, which then would be the harmonic average of the price variations,—and with equally plausible reason, if a reason be desired, since this average is the one which above presented itself to us as the most specious at the first glance, when we looked at the subject from this side. In fact this course has actually been pursued. In India it is customary to report, not the prices of corn, but the numbers of seers purchasable per rupee;[1] and in averaging these the arithmetic average has been used.[2] And not long ago at least one statistical historian, M. l'abbè Hannauer, presented his facts in this form and applied to them the arithmetic average, thus really using the harmonic average of the price variations.[3]

While many writers were so engaged in arithmetically averaging variations, generally of prices, with single weighting, another set of writers arithmetically averaged separately the prices at each of the periods compared, but, likewise using single weighting, on the assumption that the same mass-quantities were produced or consumed at all the periods compared. In doing so they employed a different method; for the others, whether they used even or uneven weighting, made no such assumption

[1] See *Prices and wages in India, compiled in the Statistical Branch of the Department of Finance and Commerce*, Calcutta 1885, (under the direction of J. E. O'Conor). Cf. D. Barbour, *Theory of bimetallism*, London 1886, p. 121.

[2] By Palgrave, copying from Charles C. Prinsep, B. 77, pp. 378–383.

[3] See B. 35. Also D'Avenel, in B. 117, seems to have pursued the same course.

about the mass-quantities. Yet none of the writers in these two sets seemed to recognize that they were pursuing two different (though convertible) lines of inquiry (the ones hailing from Carli, the others from Scrope,—for the followers of Dutot do not deserve further notice). And we have seen that both these sets have admitted many inconsistencies in their methods, consisting of kinds of weighting never intended (among them even double weighting) doing so because they carelessly did not examine what they were doing, and were led astray by mere convenience.

§ 2. The merit of raising the question between the three averages belongs to Jevons, who took up the subject in 1863. Jevons belonged to the first of the above two sets—and avowedly so, for we have seen that he denied the propriety of averaging prices at any one period alone. He posed the question in somewhat the same way as has above been done, but very barely.[4] At first he does not appear to have been acquainted with the work of his predecessors, and, coming to the subject without prejudice, he stumbled upon the problem at the outset, and at once decided it in favor of the geometric mean and average (with even weighting), from which he was not repelled by its difficulty, because, being a mathematician, he knew the aid to be derived from the use of logarithms. He was very brief in explaining why he chose this answer, and the reason which determined him is only very slightly indicated. It seems to have been that variations of prices are variations of ratios, and the proper method of averaging ratios is always the geometric.[5] Within a year his position was assailed by Professor Laspeyres, who advocated the arithmetic average (still with even weighting), and now for the first time advanced an argument in its behalf. This argument, which we shall examine in detail later, was that the arithmetic average of the price variations marks the variation in the sum of money needed to purchase the same goods at the two periods compared. Jevons was not converted, and yet he showed no

[4] He asked merely: If one thing rises in price from 100 to 150 and another from 100 to 120, what is the average rise of prices? B. 22, p. 23.

[5] B. 22, pp. 23–24. This reason was later more plainly stated in his *Principles of science*, 1874 (2d ed., p. 361), in a passage which will later be quoted (in Chapter VIII.); and again in a note added to B. 23, p. 128.

sign of detecting the error in Laspeyres's argument. He reaffirmed his position the next year, and now introduced notice of the harmonic mean of price variations, again for the first time, which mean (with even weighting), he pointed out, marks the variation in the quantity of goods the same sum of money evenly distributed will purchase at the two periods. This average he set over against the arithmetic average, and not being able to see why either should be preferred to the other, he found satisfaction in his geometric average as lying between those equally good extreme averages, and therefore combining the excellences of both.[6] He added: "It is probable that each of these is right for its own purposes when these are more clearly understood in theory." But instead of taking pains to analyze the subject further in order to reach the needed clearer understanding of it, he contented himself with the above reasons, to which he added the following: "because it [the geometric mean] presents facilities for the calculation and correction of results by the continued use of logarithms,"—a worthless reason in science, and a foolish one to oppose to the much more facile arithmetic average.[7] Consequently he never felt sure of his position, and often spoke doubtfully about it,[8] and took refuge in the thought that at all events he would be erring on the safe side; for he was writing to prove that there had been a general rise of prices, and therefore he preferred to underestimate rather than to run any risk of overestimating it.[9]

Following Jevons, the geometric average has been adopted by the eminent mathematical economist, Professor Walras, who, however, has found no better reason to offer for it than Jevons's second reason (about its being midway between the other two averages), even asserting this to be the only reason for employ-

[6] B. 23, pp. 120–121.

[7] This is given as the second in a summing up containing three arguments, *ibid.*, pp. 121–122. The third is the first one above noticed, the original one. The first is the later one just explained.

[8] "It may be a matter of opinion which result is the truer" (the geometric or the arithmetic), B. 24, p. 154. And he maintained similar reserve in regard to these averages, still looking upon the question as unsettled, in his *Money and the mechanism of exchange*, 1873, p. 332.

[9] Cf. B. 23, p. 122, B. 24, p. 154. This has sometimes been taken for his principal reason. It was, of course, an afterthought.

ing it.[10] Consequently he, too, has been weak in the faith; and he has inclined more and more to prefer the arithmetic average in one of its forms, without recognizing it.[11] But his choice has really lain only between the geometric average with even weighting and the arithmetic average in a form embracing uneven weighting, so that it is only natural he should have preferred the latter. With this restricted view he has not given the geometric average a fair trial, failing to perceive that it also can be used with allowance for the relative sizes of the classes.[12] The cause of the geometric average being thus feebly pleaded, it has met with little favor,[13] in spite of the high authority of its originator, who has even been treated with scant courtesy.[14] More recently Professor Westergaard has advanced for it the argument which we have already examined and found wanting. And occasionally Professor Edgeworth has put in a good word for it.[15]

§ 3. Modern workers, then, have continued to make use of the arithmetic average of prices, or of price variations. They have done so commonly without regard for Laspeyres's or for any other argument, properly so called, in its defense,[16] but, like the older writers, merely because it is the easiest to execute and

[10] B. 69, p. 15.

[11] See Appendix C, IV. § 1.

[12] The geometric average as used by Jevons, with even weighting, has similarly been rejected, without notice of the possibility of its being used with uneven weighting, by Wicksell, B. 139, p. 8.

[13] The only other investigator who has actually used it is Forbes. See B. 78.

[14] Jevons's use of the geometric average was treated with levity by Paasche, B. 33, p. 54, and as a curiosity by Lehr, B. 68, p. 41 n. Marshall, without argument, considers it "a mathematical error, the one flaw in his unrivalled contributions to the theory of money and prices," B. 93, p. 372 n.; Pierson pronounces it "clearly a mistake," p. 130 in the article referred to in note to B. 122; and Padan attacks it savagely, but with little comprehension, B. 141, pp. 171–180.—Walras's use of the geometric average seems to have been passed by unnoticed.

[15] The latter recommends it, with even weighting, in case we are seeking the "Determination of an Index irrespective of the quantities of commodities; upon the hypothesis that there is a numerous group of articles whose prices vary after the manner of a perfect market, with changes affecting the supply of money," B. 50, pp. 280–289. This seems to refer to cases when all prices vary alike, in which cases the weighting is indifferent. But in these cases the kind of average itself is also indifferent. In no other case do we want to seek any determination "irrespective of the quantities of commodities."

[16] Laspeyres's argument is reviewed, favorably, by Lindsay, B. 114, p. 12, and by Mayo-Smith, B. 137, p. 492.

because they have seen no proof of the superiority of any other. Moreover some experiments, by no means conclusive, show what has been deemed small divergence in the results yielded by the different methods. Hence it has seemed like a dictate of wisdom to adopt the easiest, so long as we do not know it to be wrong.[17] But it is not an exhibition of the proper scientific spirit to be content to remain in this ignorance.[18] Beside Professor Walras, a few writers, after avowedly rejecting all three averages, have lighted upon the arithmetic average again, in one of its forms, without recognizing it.[19]

§ 4. Consequently also the harmonic average of prices, which is the most difficult of all to manage, and the one to which people are the least accustomed, has met with the least favor, and, in fact, has generally been altogether ignored, even on the rare occasions it has virtually been applied. Yet, as we have seen, it is this average which is first likely to appeal to us when we approach the subject from the point of view of the particular exchange-values of money in other things, measured by the quantities of the other things the money-unit will purchase; where it corresponds to the arithmetic average. This reason, which still manifests a predilection for the arithmetic average, is the one which led Jevons to suggest the harmonic average of prices. And it has since led one prominent economist and statistician, Professor Messedaglia, to recommend its use in investigations about variations in the exchange-value of money. This writer thought he showed the suitability of the harmonic

[17] Thus its retention is advised by Edgeworth on the principle that "beggars cannot be choosers," B. 65, p. 386.

[18] It should be noticed that in the usual method of comparing every subsequent period with a single original base, the divergences, which are no greater in the last than in the first comparison, are not of much consequence—and this whole method being wrong, it hardly matters which average is used. But in the proper method of comparing each period with the immediately preceding and of forming a series from the results so obtained, a very slight error in every comparison, which might perhaps have a tendency to work in the same direction, caused by the use of a wrong average, could rapidly accumulate to an absurd extent. Hence the selection of the right average is of the utmost consequence. Having the right average, or the right method we should probably make small mistakes in every calculation, but by the law of probability these would fall about equally on each side and would compensate one another in the long run.

[19] See Appendix C, V. § 1 n. 2.

average of price variations for this purpose by pointing out—here like Jevons—that it indicates the common variation in the quantities of goods purchasable at the different periods with the same sum of money; for it thus indicates the variation in the "capacity of acquisition," or purchasing power, of this sum of money. On the other hand, here like Laspeyres, he pointed out another purpose served by the arithmetic average of price variations, namely to indicate the common variation in the sums of money needed to purchase the same goods at the two periods; only, unlike Laspeyres, he did not think this shows the suitability of this average for the previous purpose, for which it has ordinarily been used, as it indicates rather the variation in the capacity of goods to acquire money. The geometric average of price-variations he thought to be unsuitable for either of these purposes, and not finding any other purpose for which it is suitable, he considered it, in our subject, good for nothing.[20] Thus he has left the subject divided, with two distinct solutions, where Jevons tried to mediate and to give one solution. In doing so he has made no use of his knowledge that, in his own words, "the geometric mean . . . corresponds to a *dynamic* concept, of movement."[21] As exchange-value is a power, or *dynamis*, it is strange that his attention was not called to the peculiar appearance of fitness in the geometric average to serve our very purpose.

It may be added that a few writers have tentatively put forward the so-called "median mean" and the "mean of greatest thickness."[22] These are, properly speaking, not "means" at all, but only "averages," having no existence between two quantities. In an hypothesis concerning the mean between variations of only two prices, as they have no place, they do not call for attention.

§ 5. While Jevons and Laspeyres were debating the merits of

[20] B. 52, pp. 38–40.

[21] *Ibid.*, p. 30.

[22] The median mean is especially favored by Edgeworth. He recommends it when we are seeking the "Determination of an Index based upon quantities of commodities; upon the hypothesis that a common cause has produced a general variation of prices," B. 59, pp. 289–293. See also B. 61, pp. 360–363, and B. 65, p. 74.

the geometric, arithmetic and harmonic means applied to price variations with even weighting, the economist Roscher submitted the question for decision to the mathematician Drobisch. Drobisch decided it by summarily rejecting all three of the means —although what he really did was to reject them with even weighting for conditions demanding uneven weighting. He also did more. He rejected the averaging of price variations with single weighting, and rejected the use of the same mass-quantities at both periods in averaging the prices at each period. In their place he introduced the method of double weighting—with the arithmetic average, and with what we have seen to be a mistaken method of selecting the mass-units. Except for a vigorous reply by Laspeyres, this new method met with little notice, and was almost unknown outside of Germany. In Germany, however, an improvement upon it was later made by Professor Lehr. And in England another economist, Professor Nicholson, apparently without knowledge either of Drobisch or of Lehr, invented a method very similar—in fact, in some aspects quite similar—to Drobisch's. These three, Drobisch, Lehr and Nicholson, form another distinct line of theorists,—differentiated from the rest, however, rather on the subject of weighting than on the subject of the averages.

The position of these, the most difficult, and intended to meet the most difficult cases, we may leave to the last in our attempt to solve anew the problem above posed—a problem raised more than a third of a century ago and not yet settled. We may proceed circumspectly, beginning with the simplest and most easily suggested answers to the simplest form of the problem. The simplest and first suggested answer is that of a mean or average of the price variations, with single weighting. This we must examine first. If it fails us,—or when it fails us, for it may perhaps suffice for the very simplest form of the problem (with two classes and even weighting),—we must then pass on to seek other solutions on other lines—either with supposedly permanent mass-quantities or with double weighting.

CHAPTER VII.

BRIEF COMPARISON OF THE MEANS.

I.

§ 1. Before examining the arguments proper for the different mathematical means of the variations, it will be well to compare these means with one another. The comparison will not only make us better acquainted with them, but may perhaps even give some indication of the superiority of one of them by disclosing shortcomings in the other two, and so aid our understanding of the arguments. For this purpose it will be advisable to put our simple problem in another form. We may make use of another piece of information already possessed. This is that if one article rises or falls, its variation may be compensated by an opposite fall or rise of another article. As yet we do not know how great must be the opposite variation. Here also is a Proposition lacking, which we wish to find. The seeking after it is the same operation as the seeking after the missing Proposition referred to in the last Chapter, as will be seen in a moment. Keeping to our simplest suppositional case, we may ask this question :—

Among three classes of things, two being equally important, whose units, M, A and B, are at first equivalent, if one of them, A, rises in exchange-value in [M] by one-half, so that its price, from being 1.00, becomes 1.50, and so that M, from purchasing A, comes to purchase only $\frac{2}{3}$ A, how much of B must M purchase in order that its exchange-value in [A] and [B] together shall remain as it was before, and to what price, from 1.00, will the price of B then fall?

Here the new exchange-value of M in [A] and the new exchange-value of M in [B] will have the old exchange-values of

M in [A] and in [B], which were units, between them, and therefore possibly as a mean of some sort; and consequently the new price of A and the new price of B will have their old prices, 1.00, between them as a mean of a corresponding sort. Only instead of seeking the mean between two given extremes, we are seeking the absent extreme when one extreme and the mean are given. Hence this problem is the same as the problem set in the last Chapter.

In this problem we shall for the present assume that the classes are equally large or important, not at either of the periods alone, but at each of the periods, or somehow over both the periods. But we shall attempt to leave the question of weighting out of sight so far as possible. We have already treated the question of weighting by itself, as far as we could; and now we shall try to treat the question of the averages by itself, as far as we can. We shall, however, find, as we proceed, that we can not settle this question any more than that question separately. We shall then have to combine the two into one question about the variation of the general exchange-value of money under given conditions of variations both of prices and of mass-quantities. This course is adopted because it repeats the usual way the subject has been approached, although only three writers have as yet carried it to the end. In pursuing it all opinions hitherto advanced may be reviewed.

§ 2. The problem in its new form, may, for the exchange-values of M, be formulated in our three formulæ, in the order with which we are familiar, as follows:

$$M_{01} = \frac{1}{2}(1 + 1) \quad = M_{02} = \frac{1}{2}\left(\frac{2}{3} + b'\right),$$

$$M_{01} = \frac{1}{\frac{1}{2}\left(\frac{1}{1} + \frac{1}{1}\right)} = M_{02} = \frac{1}{\frac{1}{2}\left(\frac{1}{2/3} + \frac{1}{b'}\right)},$$

$$M_{01} = \sqrt{1 \cdot 1} \qquad = M_{02} = \sqrt{\frac{2}{3} \cdot b'}.$$

For these formulæ to be carried out, that is, for M_{02} to be equal to M_{01}, we want the resultants of the figures in the second halves to equal those in the first halves, which are all units. We see that this will be the case in the first formula if $b' = 1\frac{1}{3}$, in the second if $b' = 2$, in the third if $b' = 1\frac{1}{2}$. Thus if we later find any one of these figures to be the right answer, we shall know that its formula is the proper one, in similar cases, for averaging particular exchange-values of M in all other things at two periods for the purpose of comparing its exchange-value in all other things at those periods, as it alone gives the right answer in this simple case.

The corresponding formula for the prices are, in the same order:

$$P_1 = \frac{1}{\frac{1}{2}\left(\frac{1}{1.00} + \frac{1}{1.00}\right)} = P_2 = \frac{1}{\frac{1}{2}\left(\frac{1}{1.50} + \frac{1}{\beta'}\right)},$$

$$P_1 = \tfrac{1}{2}(1.00 + 1.00) = P_2 = \tfrac{1}{2}(1.50 + \beta'),$$

$$P_1 = \sqrt{1.00 \times 1.00} = P_2 = \sqrt{1.50\,\beta'}.$$

And similarly we want the results in the second halves to equal unity, which will be the case in the first formula if $\beta' = .75$, in the second if $\beta' = .50$, in the third if $\beta' = .66\frac{2}{3}$; which are the reciprocals of the preceding answers for b', in the same order. That one of these formulæ for averaging prices, if any, will be the proper one, in which the answer for β' is the reciprocal of the proper answer for b'.

An advantage in re-stating the question in this form is that the diversity of the proffered extremes is greater than the diversity of the proffered means. Also it brings in the idea of balancing, in which the idea of equality of influence is plainer than in the case of conjoint action. The review and comparison of the suggested answers will now disclose interesting relations. And two of the answers will manifest unmistakable signs of falsity.

II.

§ 1. Only the purchasable quantities of the articles being considered (which are directly according to the exchange-values of M in them), it is easy to answer that if M will purchase only ⅔ as much of [A] as before, it should purchase ⅓ more of [B] in order to make up for the ⅓ lost on [A]. Here the new quantities purchasable with M (and the new exchange-values of M in [A] and [B]) surround the old quantities (and exchange-values), which were units, so as to hold them in the arithmetic mean, the progression being ⅔, 1, 1⅓. The new prices, then, are 1.50 for A and .75 for B, which hold the old prices, 1.00, as their harmonic mean, in the progression 1.50, 1.00, .75. The proper formulæ, if this answer be correct, are

$$M_{02} = \frac{1}{2}\left(\frac{2}{3} + 1\frac{1}{3}\right) \quad \text{and} \quad P_2 = \frac{1}{\frac{1}{2}\left(\frac{1}{1.50} + \frac{1}{.75}\right)},$$

the result in each of these being unity.

Only the prices being considered, it is easy to answer that if A has risen to 1.50, B should fall to .50, so that the increase of money needed to purchase A shall be made up for by the decrease of money needed to purchase B. Here the new prices have the old prices as their arithmetic mean, the progression being 1.50, 1.00, .50. On examining the new quantities purchasable with M, we find that they are ⅔ A and 2 B, which have the old quantities, which were units, between them in the harmonic mean, according to the progression ⅔, 1, 2. The proper formulæ for this answer are

$$M_{02} = \frac{1}{\frac{1}{2}\left(\frac{1}{2/3} + \frac{1}{2}\right)} \quad \text{and} \quad P_2 = \frac{1}{2}(1.50 + .50).$$

Both the quantities and the prices being considered, it is possible to treat them uniformly by answering that if the price of A rises to $\frac{3}{2}$ times its former position, the price of B should fall to ⅔ of its former position, and that if the quantity of [A] purchasable with M falls to ⅔ of its former amount, the quantity of [B] purchasable with M should rise $\frac{3}{2}$ times its former amount, —whereby are produced the same variations in the exchange-values of A in [M] and of B in [M] as in the exchange-values [illegible] Mean [A]. Here both the new prices and [illegible] have their old prices and their old quantities, [illegible] between them as geometric means, in the pro[illegible] [illegible] ⅘, 1, 1¼. The proper formulae are

$$[illegible] \text{ and } P_2 = 1 \; 1.50 \times .66\tfrac{2}{3}.$$

[illegible] speak, the harmonic and arithmetic aver[illegible] [illegible] inseparable embrace, but the geometric [illegible] self-sufficient. And both its answers are mid[illegible] [illegible] the others. The peculiarities of these [illegible] answering may best be shown by drawing up [illegible] supposed variations in price of A and of the [illegible] variations in price of B according to the three [illegible] them with the variations in the quan[illegible] M.

[illegible]	[illegible]	[illegible]	[illegible]	Counterbalancing new quantities of B according to progressions		
				Harmonic	Geometric	Arithmetic
[illegible]	[illegible]	[illegible]		[illegible]	1.25	1.2
[illegible]	[illegible]	[illegible]		[illegible]	1.33	1.25
[illegible]	[illegible]	[illegible]		[illegible]	1.5	1.33
[illegible]	[illegible]	[illegible]		[illegible]	1.75	1.42
[illegible]	[illegible]	[illegible]		[illegible]	1.9	1.47
[illegible]	[illegible]	[illegible]		∞	2	1.5
[illegible]	[illegible]	[illegible]			2.5	1.6
[illegible]	[illegible]	[illegible]			3	1.66
	[illegible]	[illegible]			4	1.75
		[illegible]			5	1.8
	[illegible]				10	1.9
					100	1.99
					∞	2

[illegible] the price of A is [illegible] purchasable with M [illegible] of B according to [illegible] the correspond[illegible] [illegible] with M at these [illegible] harmonic progres[illegible] [illegible] and thereafter there [illegible] would have to be zero or less, and no counterbalancing quantity of B purchasable with M, as it would have to be infinite or become a negative quantity, which here is meaningless. And correspondingly, if we consider the fall in the price of B and ask for the counterbalancing rise in the price of A according to the harmonic progression, with the quantities in arithmetic progession, we find this rise so rapid as to reach infinity when the price of B falls to .50, and thereafter it would be a negative quantity, which again is meaningless here. In practice, however, these difficulties would not always be insuperable, as we should then be dealing with many articles, and the compensation might be obtained by distributing the counterbalancing change over two or more of them. Thus in the case of the arithmetic solution of prices, a fall of prices of 100 per cent. or greater could be obtained by dividing it into several lots. Yet a rise of 100 per cent. on an average over half the number of articles could not be counterbalanced by any fall of the others. In the case of the harmonic solution of prices, the rise to infinity would not be required if two or more articles rose in price, because a whole or greater quantity purchasable with M may be lost by being distributed over two or more articles without any considerable rise of their prices. For instance, the fall of B to .50 can be compensated by the rise of two articles to 2.00. Yet again the fall of half the number on an average to less than .50 could not be counterbalanced by any rise of the others. These great changes of prices on the part of a majority of commodities are not to be expected unless money be varying in its general exchange-value, and so the compensation may not be practically needed. Yet theory would seem to require its possibility always. The geometrical method satisfies this theoretical requirement. For every rise or fall of one article it provides a counterbalancing fall or rise of another.

§ 3. This tendency of the arithmetic and harmonic solutions to run into the ground or to fly into the air by their excessive demands is clear indication of their falsity. Error is often made patent by examining extreme cases. It is so here. Take the case when A rises in price to 1.90. You can then buy with one money-unit ten nineteenths as much of [A] as you could before, or a trifle over half. The arithmetic answer says you ought then to buy of [B] ten times as much as before. Thus, losing nine nineteenths or not quite half on [A], you gain nine wholes on [B]. The excessiveness is evident. Your money has appreciated. On the other hand consider the fall of the quantity of [B] to .52$\frac{12}{19}$. With your money-unit you can now get nine tenths more of [B] than before, or not quite double. Yet the harmonic method of calculating the terms of prices would require the price of A to rise to 10.00, which will enable you with your money-unit to get only one tenth so much of [A] as formerly. Again the excessiveness is evident. Your money has depreciated. The geometric method commits neither of these excesses. In the former case it does not require the price of B to fall so low as did the arithmetic—only to .55$\frac{12}{19}$ instead of to .10; and it enables you then to get nineteen tenths instead of ten wholes, so that, losing nine nineteenths on [A], you gain nine tenths on [B] instead of nine wholes. There is no appearance here of appreciation. In the latter case, this method does not require the price of A to rise so high as did the harmonic—only to 1.90 instead of to 10.00; and it enables you then to get with your money-unit ten nineteenths instead of only one tenth of [A], so that, gaining nine tenths on [B], or not quite as much again, you lose nine nineteenths on [A] instead of nine tenths, or nearly the whole. Neither is there here any appearance of depreciation. There is thus an appearance that the geometric mean may have rendered your money stable in general exchange-value, since it has removed the appearance both of appreciation and of depreciation.

As yet, however, we are but slightly advanced beyond Jevons in one of his moods. Jevons acknowledged himself at a loss between the opposite merits, as he admitted them, of the arithmetic and harmonic methods of averaging prices, and rested content with a compromise on the intermediate geometric method. We see the opposite failings of those methods, and welcome the geometric method for avoiding each extreme, and so suggesting at least the possibility of its being the correct one—under the assumed conditions at least.

CHAPTER VIII.

THE GENERAL ARGUMENT FOR THE GEOMETRIC MEAN.

I.

§ 1. The problem before us has two aspects. The one is the point of view of prices, the other the point of view of the exchange-values of money in the goods priced. Of two equally important classes of goods the rise of the price of the one is to be compensated by a fall of the price of the other; and the rise of the exchange-value of money in the latter is to be compensated by a fall of the exchange-value of money in the former. The second is really our main problem; but its place may be taken by the first, for the sake of convenience. No argument, however, is good showing the superiority of one kind of compensation in the problem of prices, unless it can show also the superiority of the kind of compensation thereby involved in the problem of the exchange-value of money. In order, then, to be able to apply the question of compensation and of averages to our double-faced problem, we must understand something of the nature of compensation in general, and of the nature of averages in reference to the kinds of subjects for which they are generally suitable.

In the question of compensatory variations, the following proposition at once strikes us as evident:—*A compensatory variation in the one direction must equal the variation in the other direction which it is to compensate.*

We generally treat variations by percentage, and we have an habitually established manner of reckoning percentages. We treat an increase or rise on the one hand, and on the other a decrease or fall, as variations from one common point or position, or whole, arbitrarily chosen as that which exists at the commencement. The percentage of the rise is the ratio of the later excess beyond the original whole to that whole, multiplied by one hundred; and the percentage of the fall is the ratio of the later deficiency below the original whole to that whole, multiplied by one hundred. Hereupon we are inclined to maintain—and this is, in almost all cases, the fundamental idea in the argument for the arithmetic average of variations—that *a compensatory fall should always have the same percentage as the rise it is to compensate, and reversely.* But the above is not the only way of reckoning percentages.

Of reckoning percentages there are three different ways—or better, two distinct ways, and a third constructed by uniting them; wherefore there are three different ways of obtaining sameness of percentage in compensatory variations. Consequently the argument for the arithmetic average has no validity unless it can be proved that the ordinary way of reckoning percentages is the proper way of reckoning them in the special case in question. The ordinary way of reckoning percentages, or of measuring variations, is the way we shall always continue to pursue, because of its greater convenience. But unless it be the proper way for indicating the equality of variations in the problem before us, we shall have to adapt it to our subject by adapting whatever turns out to be the proper method of measuring the variations to this convenient method of measuring them.

We must therefore examine the three ways in which we can conceive of equality in compensatory variations.

§ 2. A being supposed to rise in some attribute to a certain extent, equality to that rise in the fall of B in the same attribute may be represented in the following three ways: (1) B may fall so that the point to which it falls is as much below the point from which it falls as the point to which A has risen is above the point from which A has risen; (2) B may fall so that the point from which it falls is as much above the point to which it falls as the point from which A has risen is below the point to which A has risen; (3) B may fall so that either the point to which it falls is as much below the point from which it falls as he point from which A has risen is below the point to which A has risen, or the point from which it falls is as much above the point to which it falls as the point to which A has risen is above the point from which A has risen. In the first we measure the variations, or their percentages, in the usual way, in what the things vary *from*. In the other two we depart from the usual method. In the second we measure the percentages in what the things vary *to*. In the third we measure the variations either in what the one thing rises *to* and the other falls *from*, or in what the former rises *from* and the latter falls *to*. In the first two we make each measurement in an opposite direction—in the first from the start, at the mean, to the extremes; in the second from the extremes at the starting points to the mean,—the former being centrifugal, so to speak, and the latter centripetal. In the third we make both measurements in the same direction, either both upwards or both downwards, the starting point of the one being at the mean and that of the other at an extreme, or that of the latter at the mean and that of the former at the other extreme.

When the variations in which equality is obtained in these three ways are all measured in the usual way, the first are variations from the mean to the simple[1] arithmetic extremes around it. Thus, for example, A rising from 1.00 to 1.50 by 50 per cent. (reckoned in 1.00), B falls from 1.00 to .50 by 50 per cent. (also reckoned in 1.00). They may therefore be called simple *arithmetic variations*. The second are variations from the mean to the simple harmonic extremes around it. Thus, A rising from 1.00 to 1.50 so that 1.00 is 33⅓ per cent. below 1.50 (reckoned in 1.50), B falls from 1.00 to .75 so that 1.00 is 33⅓ per cent. above .75 (reckoned in .75). They may therefore be called simple *harmonic variations*. The third are variations from the mean to the simple geometric extremes around it. Thus, A rising from 1.00 to 1.50 by 50 per cent. above 1.00 (reckoned in 1.00), B falls from 1.00 to .66⅔ so that 1.00 is 50 per cent. above .66⅔ (reckoned in .66⅔); or, A rising so that 1.00 is 33⅓ per cent. below 1.50 (reckoned in 1.50), B falls from 1.00 to

[1] *I. e.*, with even weighting, only two figures being used.

.66⅔ so that .66⅔ is 33⅓ per cent. below 1.00 (reckoned in 1.00). They may therefore be called simple *geometric variations*.[2]

Thus when we use in all the cases the ordinary method of reckoning the percentages, the equality exists only in the first one. Yet this equality is no more real than the equality which appears in the other ways of reckoning the percentages.

§ 3. The equality manifested by the arithmetic variations seems to be of a peculiar nature. This is equality of difference, or of distance traversed, 1.50 and .50 being equidistant, so to speak, from 1.00, wherefore A in rising from 1.00 to 1.50 and B in falling from 1.00 to .50 have traversed equal distances. This property may be thought, at first sight, to belong only to the arithmetic variations. It does so belong only to them if the variations must be conceived as starting from 1.00, or from some common figure, that is, from the mean. But there is no reason, except mere convenience, why in all cases the variations must be so conceived. If they are conceived as starting from other figures, this property of traversing equal distances may belong to the other variations.

Thus if A rises from 1.00 to 1.50 and B falls from 2.00 to 1.50, A has risen by 50 per cent. and B has fallen by 25 per cent. Evidently the variation, merely as a variation, of ½ B from 1.00 to .75 is the same variation as that of B from 2.00 to 1.50. Here we have harmonic variations. Yet the distances traversed are equal.

And if A rises from 1.00 to 1.50 and B falls from 1.50 to 1.00, or, B falling from 1.00 to .66⅔, if A rises from .66⅔ to 1.00, A has risen by 50 per cent. and B has fallen by 33⅓ per cent., which are geometric variations; for A has risen to be 50 per cent. higher than it was, and B was 50 per cent. higher than it has come to be. But A and B have each traversed an equal distance—in fact, so to speak, the same road, only in opposite directions, so that in this case sameness is added to equality.

These properties are universal. All variations from the same starting point to equal distances on opposite sides, above and below, are arithmetic variations (variations to arithmetic ex-

[2] The universality of these relations is demonstrated in Appendix B, I.

tremes); and all arithmetic variations may be reduced to such. All variations from equal distances on opposite sides, above and below, to the same ending point are harmonic variations (variations to harmonic extremes, when measured in the usual way); and all harmonic variations may be reduced to such. All variations over the same distances, between the same extremes, in opposite directions, upwards and downwards, are geometric variations (variations to geometric extremes, when measured in the usual way); and all *simple* geometric variations may be reduced to such.[3]

§ 4. Thus in all these variations we find two kinds of equality: equality of proportion, and equality of distance traversed, or of difference. These may be represented algebraically as follows. Let a_1 and a_2 represent the figures at which A stands at the beginning and at the end of its variation, and b_1 and b_2 the similar figures for the variation of B. Then in the arithmetic variations we have $\frac{a_2 - a_1}{a_1} = \frac{b_1 - b_2}{b_1}$, and if $a_1 = b_1$, $a_2 - a_1 = b_1 - b_2$; in the harmonic variations, $\frac{a_2 - a_1}{a_2} = \frac{b_1 - b_2}{b_2}$, and if $a_2 = b_2$, $a_2 - a_1 = b_1 - b_2$; and in the geometric variations, both $\frac{a_2 - a_1}{a_1} = \frac{b_1 - b_2}{b_2}$, and if $a_1 = b_2$, $a_2 - a_1 = b_1 - b_2$ (whence also $a_2 = b_1$), and $\frac{a_2 - a_1}{a_2} = \frac{b_1 - b_2}{b_1}$, and if $a_2 = b_1$, $a_2 - a_1 = b_1 - b_2$ (whence also $a_1 = b_2$).[4]

When one or both of these kinds of equality are present, as they may appear in three different ways, it may be well to give to these distinctive appellations. Thus when we have the equality which appears in arithmetic variations, we may name it *arithmetic equality* of variations, or of percentages. When we have the equality which appears in harmonic variations, we may name it *harmonic equality* of variations, or of percentages. And when we have the equality which appears in geometric varia-

[3] For the universal demonstration see Appendix B, II.

[4] These formulæ make it clear that the above-used combinations exhaust all the possible ways of reckoning percentages and of getting equality between variations.

tions, we may name it *geometric equality* of variations, or of percentages.

Of the two general kinds of equality which may run through all these varieties of variations, the equality of proportion must always be present if the variations are to be as we have been describing them (*i. e.*, if their percentages are to be equal in the three ways described). But the equality of difference, or of distance traversed, need be present only under certain conditions, different in each of the three varieties. Of course when we have two variations presented merely as variations, we may twist about the starting and ending points as we please, and so whenever we have an equality of proportion, we may also conceive of a corresponding equality of difference. If, however, the variations are given us as the definite variations of an attribute in certain things, which have started from certain figures and have ended at certain figures, then an equality of proportion may be present, in either of the three forms, without the corresponding equality of difference, although the latter cannot be present without the former. The conditions for the presence of the equalities of difference are plain. In the case of arithmetic variations, the two things, or individuals, must be equally large (or important) *at the first period*, and at this period only; for they both start from unity, or from any other common figure, that is, from the same level of equality in size or amount. In the case of harmonic variations, the two things, or individuals, must be equally large (or important) *at the second period*, and then only; for they both end by being equal to unity, or to some other common figure, that is, they come to the same level of size. In the case of geometric variations, the two things, or individuals, must be alternately equal in size, the one at the first period with the other at the second, and again the former at the second with the latter at the first, so that they are equal in size *over both the periods together*. The connection between these conditions and the subject of weighting is obvious.

When we have these equalities not merely of proportion but of distances actually traversed, we virtually have three different kinds of compensation—*arithmetic*, *harmonic* and *geometric com-*

pensation. There may be agreement between these at times, as we shall see, and also disagreement. In the latter case it is the problem before us to decide which of these compensations is the proper one for our subject.

§ 5. So far we have been dealing with only two things (or classes) that vary, and all these relations have been found to be universal when we are dealing with two things (or classes), with even weighting in the averaging of their variations. All of them are, or may be, universal, when unity is the arithmetic, harmonic or geometric *mean* between two opposite variations. Furthermore, now, they are completely and unconditionally universal in the cases of the arithmetic and harmonic *averages*, no matter how many variations on the one side be opposed to no matter how many on the other. In these cases if unity be the arithmetic or the harmonic average (in each case with the weighting of the period proper for it) between all the variations, there is equality of proportion between all the variations on the one side together and all the variations on the other side together, and, with observance of the proper starting and finishing points, there is equality between the distance traversed by all the rising things and the distance traversed by all the falling things, the whole distance in each case being obtained by summing up the distances traversed by all the things individually.

Thus in the case of compound arithmetic variations, if instead of one B we have two B's, and both fall together from 1.00 in the compensation for the rise of A from 1.00 to 1.50, the figure to which they fall must be such that the arithmetic average between it twice repeated and 1.50 is unity. This figure is .75; for $\frac{1}{3}(1.50 + 2 \times .75) = 1.00$.[5] Or if two things fall unequally, suppose the one has fallen to .80, then the other must fall to a figure between which and .80 and 1.50 the arithmetic average is unity. This is .70; for $\frac{1}{3}(1.50 + .80 + .70)$

[5] It should be noticed that the *two* figures, 1.50 and .75, are the simple harmonic extremes around 1.00, but the *three* figures, 1.50 and .75 twice repeated, are arithmetic extremes around 1.00. The figure .75 repeated a fractional number of times may even be a geometric extreme opposed to 1.50; for we may have $\sqrt[1+x]{1.50 \times .75^x} = 1.00$, and it is not difficult to find x $\left(\text{it is } \frac{\log \frac{3}{2}}{\log \frac{4}{3}} = 1.4086\right)$.

= 1.00. And similarly in other cases.[6] Here the sums of the percentage of the two falls, namely 25 + 25, and 20 + 30, are equal to the percentage of the one rise, namely 50 ; and the sums of the distances traversed by the two falling things, namely .25 + .25, and .20 + .30, are equal to the distance traversed by the one rising thing, namely .50. These equalities exist—the latter if all the variations start from the same figure—whatever be the number of the arithmetically compensating variations on either side. The general formula with n equally rising and n' equally falling things is $n\left(\frac{a_2 - a_1}{a_1}\right) = n'\left(\frac{b_1 - b_2}{b_1}\right)$; and if $a_1 = b_1$, $n(a_2 - a_1) = n'(b_1 - b_2)$.

For compound harmonic variations, in a similar supposition, the formula is $n\left(\frac{a_2 - a_1}{a_2}\right) = n'\left(\frac{b_1 - b_2}{b_2}\right)$; and if $a_2 = b_2$, $n(a_2 - a_1) = n'(b_1 - b_2)$. For example, the rise of A alone from 1.00 to 1.50 is harmonically compensated by the equal falls of two B's from 1.75 to 1.50 ; for $\frac{1}{\frac{1}{3}\left(\frac{1.00}{1.50} + 2 \times \frac{1.75}{1.50}\right)} = 1.00$.

Or if one thing has already fallen from say 1.80 to 1.50, another must fall from 1.70 to 1.50 ; for $\frac{1}{\frac{1}{3}\left(\frac{1.00}{1.50} + \frac{1.80}{1.50} + \frac{1.70}{1.50}\right)}$ = 1.00. And so in other cases.[7] Here, too, the sums of the percentages of the two falls, measured from the ending point, which are $16\frac{2}{3} + 16\frac{2}{3}$, and $20 + 13\frac{1}{3}$, are equal to the percentage of the one rise similarly measured, which is $33\frac{1}{3}$; and the sums of the distances traversed by the two falling things, namely .25 + .25, and .30 + .20, are equal to the distance traversed by the one rising thing, namely .50. And these equalities exist—the latter if all the variations end at the same figure—whatever be the number of the compensating variations on either side.

[6] If the one has already fallen to .50, the other must remain at 1.00. If the one has already fallen beyond .50, the other must rise.

[7] If the one has already fallen from 2.00 to 1.50, the other must remain at 1.50. If the one has fallen from above 2.00, the other must rise to 1.50 from below.

§ 6. But in the compound geometric variations the equality of proportion ceases to exist, and the equality of distances traversed exists only at the sacrifice of some of the relations previously included, so that we no longer have the same universal relations as before. This becomes evident the moment we deal with two equal variations in compensation for one variation. Thus the rise of A from 1.00 to 1.50 is geometrically compensated by the falls of two B's from 1.00 to a figure between which twice repeated and 1.50 the geometric average is unity. This is .8165 ($= \sqrt{\frac{2}{3}}$; for $\sqrt[3]{1.50 \times .8165^2} = 1.00$. Here the percentage of the fall from 1.00 to .8165 reckoned in 1.00 is 18.35, and twice this is more than the percentage of the rise of A reckoned in 1.50, which is $33\frac{1}{3}$; and the percentage of the fall reckoned in .8165 is 22.47, and twice this is less than the percentage of the rise reckoned in 1.00, which is 50. As regards the distances traversed, the variations being kept the same merely as variations, the compensation may be by the two B's falling from 1.2247 to 1.00, or from 1.50 to 1.2247. Then in the former case they together traverse a shorter, and in the latter a greater, distance than A. Now if the one thing falls from 1.50 to 1.2247 and the other from 1.2247 to 1.00, they both together traverse the same distance as A, covering in two stages downwards the same road covered by A in one leap upwards. But then the two B's are not equal things belonging to the same class (they are rather B and C, and unequal in size over both the periods together so that we should not be justified in geometrically averaging the three variations with even weighting). It is, however, possible for the two B's to be alike, and falling together in the same proportion as before, to traverse each half the distance traversed by A. This is when they both fall from 1.3623 to 1.1123. But then the two B's are not each equally large or important with the one A over both the periods together; for the geometric mean between their starting and ending points is 1.23096, while the geometric mean between the starting and ending points of A is 1.2247. Thus the relations that hold true of *simple* geometric variations—variations of two equally important things (or classes), with even weighting, so that unity is the geometric *mean* between them, do not hold true of *compound* geometric variations—variations between several opposed things, with various weighting, such that unity is a geometric *average* between them. What has above been called geometric equality of variations or of percentages exists only between opposite variations of two things equally large or important over both the periods, and does not exist in more than two variations that geometrically compensate for one another (unless they do so in pairs). That this is so in general may be seen from the formulæ. With several things, n in number, falling together in compensation for the rise of A alone, the formulæ are either $\frac{a_2 - a_1}{a_1} = \frac{b_1^n - b_2}{b_2}$, which is true only if $a_1 = b_2 = 1$, whereupon $a_2 - a_1 = b_1^n - b_2$, and $a_2 = b_1^n$; or $\frac{a_2 - a_1}{a_2} = \frac{b_1 - b_2^n}{b_1}$, which is true only if $a_2 = b_1 = 1$, whereupon $a_2 - a_1 = b_1 - b_2^n$, and $a_1 = b_2^n$. Now while A traverses the distance $a_2 - a_1$, none of the opposing things traverse the distance $b_1^n - b_2$, or $b_1 - b_2^n$, but only the distance $b_1 - b_2$, which is not $\frac{1}{n}$ the distance $a_2 - a_1$. Still, trial shows that in ordinary cases, that is, with not more than two or three things opposing any one thing, the relations between compound geometric variations are very *nearly* the same as those above described.

What is here discovered, in differentiating the geometric *average* from the geometric *mean*, and segregating it from the arithmetic and harmonic averages, which behave universally in the same manner as do the arithmetic and harmonic means, will later call for our repeated attention. It shows that we cannot treat geometric averages as we do geometric means, and that the use of only two figures, with even weighting, will lead us into error if we treat them as examples good for all cases. After examining what is true of the geometric mean, with two things evenly weighted, we must always turn to more complex examples; and in them we shall find that convenient relations discovered in the former case no longer hold—except, in ordinary cases, only approximately. We thus have obtained a warning for all our future labors.

It is pleasing to notice that in the employment of the geometric mean already made in Chapter IV. we have nothing to revise; for we there dealt with the geometric mean proper, that is, with two figures—two periods—regarded as equally important.

And we may continue for the present to treat principally of means.

§ 7. Now in our own special subject people have generally confined their attention to price variations, and these price variations they have, of course, measured in the usual way, conceiving of them all as starting from the same figure. They have also generally used the arithmetic average, assuming that the proper equality in compensatory changes is the arithmetic. The adoption of the arithmetic average we have seen to be due principally to its convenience. But when people have looked for a reason to justify its use, they have lighted upon the fact that by it alone—measured in the usual way—does equality, especially the equality of distance traversed, exist in the opposite variations that are supposed to compensate for each other. We now see that it is only because of the method usually adopted for reckoning percentages that equality is apparent only in the arithmetic compensations; and yet this is only one of three possible methods. The usual method is the most convenient and the most suitable for measuring the proportions of the variations; but it is no better than either of the others for measuring the actual amounts, or distances traversed, of the variations. We therefore find ourselves in need of special reasons in our subject for showing why the one of these sets of starting points is the proper one rather than another. But for this purpose we must appeal to facts; for only facts will show what the starting points really are. Facts will give us the weighting, and the period, or periods, of the weighting. And the mean, or average, will then have to be chosen with dependence upon the period, or periods, whose weighting is chosen. Otherwise our procedure will be arbitrary.

In general these principles, derived from what has above been shown at the end of § 4, may be stated :—(1) If it happens that mencement. The percentage of the rise is the ratio of the later excess beyond the original whole to that whole, multiplied by one hundred; and the percentage of the fall is the ratio of the later deficiency below the original whole to that whole, multiplied by one hundred. Hereupon we are inclined to maintain—and this is, in almost all cases, the fundamental idea in the argument for the arithmetic average of variations—that *a compensatory fall should always have the same percentage as the rise it is to compensate, and reversely.* But the above is not the only way of reckoning percentages.

Of reckoning percentages there are three different ways—or better, two distinct ways, and a third constructed by uniting them; wherefore there are three different ways of obtaining sameness of percentage in compensatory variations. Consequently the argument for the arithmetic average has no validity unless it can be proved that the ordinary way of reckoning percentages is the proper way of reckoning them in the special case in question. The ordinary way of reckoning percentages, or of measuring variations, is the way we shall always continue to pursue, because of its greater convenience. But unless it be the proper way for indicating the equality of variations in the problem before us, we shall have to adapt it to our subject by adapting whatever turns out to be the proper method of measuring the variations to this convenient method of measuring them.

We must therefore examine the three ways in which we can conceive of equality in compensatory variations.

§ 2. A being supposed to rise in some attribute to a certain extent, equality to that rise in the fall of B in the same attribute may be represented in the following three ways: (1) B may fall so that the point to which it falls is as much below the point from which it falls as the point to which A has risen is above the point from which A has risen; (2) B may fall so that the point from which it falls is as much above the point to which it falls as the point from which A has risen is below the point to which A has risen; (3) B may fall so that either the point to which it falls is as much below the point from which it falls as the point from which A has risen is below the point to which A has risen, or the point from which it falls is as much above the point to which it falls as the point to which A has risen is above the point from which A has risen. In the first we measure the variations, or their percentages, in the usual way, in what the things vary *from*. In the other two we depart from the usual method. In the second we measure the percentages in what the things vary *to*. In the third we measure the variations either in what the one thing rises *to* and the other falls *from*, or in what the former rises *from* and the latter falls *to*. In the first two we make each measurement in an opposite direction—in the first from the start, at the mean, to the extremes; in the second from the extremes at the starting points to the mean,—the former being centrifugal, so to speak, and the latter centripetal. In the third we make both measurements in the same direction, either both upwards or both downwards, the starting point of the one being at the mean and that of the other at an extreme, or that of the latter at the mean and that of the former at the other extreme.

When the variations in which equality is obtained in these three ways are all measured in the usual way, the first are variations from the mean to the simple[1] arithmetic extremes around it. Thus, for example, A rising from 1.00 to 1.50 by 50 per cent. (reckoned in 1.00), B falls from 1.00 to .50 by 50 per cent. (also reckoned in 1.00). They may therefore be called simple *arithmetic variations*. The second are variations from the mean to the simple harmonic extremes around it. Thus, A rising from 1.00 to 1.50 so that 1.00 is $33\frac{1}{3}$ per cent. below 1.50 (reckoned in 1.50), B falls from 1.00 to .75 so that 1.00 is $33\frac{1}{3}$ per cent. above .75 (reckoned in .75). They may therefore be called simple *harmonic variations*. The third are variations from the mean to the simple geometric extremes around it. Thus, A rising from 1.00 to 1.50 by 50 per cent. above 1.00 (reckoned in 1.00), B falls from 1.00 to $.66\frac{2}{3}$ so that 1.00 is 50 per cent. above $.66\frac{2}{3}$ (reckoned in $.66\frac{2}{3}$); or, A rising so that 1.00 is $33\frac{1}{3}$ per cent. below 1.50 (reckoned in 1.50), B falls from 1.00 to

[1] *I. e.*, with even weighting, only two figures being used.

.66⅔ so that .66⅔ is 33⅓ per cent. below 1.00 (reckoned in 1.00). They may therefore be called simple *geometric variations*.[2]

Thus when we use in all the cases the ordinary method of reckoning the percentages, the equality exists only in the first one. Yet this equality is no more real than the equality which appears in the other ways of reckoning the percentages.

§ 3. The equality manifested by the arithmetic variations seems to be of a peculiar nature. This is equality of difference, or of distance traversed, 1.50 and .50 being equidistant, so to speak, from 1.00, wherefore A in rising from 1.00 to 1.50 and B in falling from 1.00 to .50 have traversed equal distances. This property may be thought, at first sight, to belong only to the arithmetic variations. It does so belong only to them if the variations must be conceived as starting from 1.00, or from some common figure, that is, from the mean. But there is no reason, except mere convenience, why in all cases the variations must be so conceived. If they are conceived as starting from other figures, this property of traversing equal distances may belong to the other variations.

Thus if A rises from 1.00 to 1.50 and B falls from 2.00 to 1.50, A has risen by 50 per cent. and B has fallen by 25 per cent. Evidently the variation, merely as a variation, of ½ B from 1.00 to .75 is the same variation as that of B from 2.00 to 1.50. Here we have harmonic variations. Yet the distances traversed are equal.

And if A rises from 1.00 to 1.50 and B falls from 1.50 to 1.00, or, B falling from 1.00 to .66⅔, if A rises from .66⅔ to 1.00, A has risen by 50 per cent. and B has fallen by 33⅓ per cent., which are geometric variations; for A has risen to be 50 per cent. higher than it was, and B was 50 per cent. higher than it has come to be. But A and B have each traversed an equal distance—in fact, so to speak, the same road, only in opposite directions, so that in this case sameness is added to equality.

These properties are universal. All variations from the same starting point to equal distances on opposite sides, above and below, are arithmetic variations (variations to arithmetic ex-

[2] The universality of these relations is demonstrated in Appendix B, I.

tremes); and all arithmetic variations may be reduced to such. All variations from equal distances on opposite sides, above and below, to the same ending point are harmonic variations (variations to harmonic extremes, when measured in the usual way); and all harmonic variations may be reduced to such. All variations over the same distances, between the same extremes, in opposite directions, upwards and downwards, are geometric variations (variations to geometric extremes, when measured in the usual way); and all *simple* geometric variations may be reduced to such.[3]

§ 4. Thus in all these variations we find two kinds of equality: equality of proportion, and equality of distance traversed, or of difference. These may be represented algebraically as follows. Let a_1 and a_2 represent the figures at which A stands at the beginning and at the end of its variation, and b_1 and b_2 the similar figures for the variation of B. Then in the arithmetic variations we have $\frac{a_2 - a_1}{a_1} = \frac{b_1 - b_2}{b_1}$, and if $a_1 = b_1$, $a_2 - a_1 = b_1 - b_2$; in the harmonic variations, $\frac{a_2 - a_1}{a_2} = \frac{b_1 - b_2}{b_2}$, and if $a_2 = b_2$, $a_2 - a_1 = b_1 - b_2$; and in the geometric variations, both $\frac{a_2 - a_1}{a_1} = \frac{b_1 - b_2}{b_2}$, and if $a_1 = b_2$, $a_2 - a_1 = b_1 - b_2$ (whence also $a_2 = b_1$), and $\frac{a_2 - a_1}{a_2} = \frac{b_1 - b_2}{b_1}$, and if $a_2 = b_1$, $a_2 - a_1 = b_1 - b_2$ (whence also $a_1 = b_2$).[4]

When one or both of these kinds of equality are present, as they may appear in three different ways, it may be well to give to these distinctive appellations. Thus when we have the equality which appears in arithmetic variations, we may name it *arithmetic equality* of variations, or of percentages. When we have the equality which appears in harmonic variations, we may name it *harmonic equality* of variations, or of percentages. And when we have the equality which appears in geometric varia-

[3] For the universal demonstration see Appendix B, II.

[4] These formulæ make it clear that the above-used combinations exhaust all the possible ways of reckoning percentages and of getting equality between variations.

tions, we may name it *geometric equality* of variations, or of percentages.

Of the two general kinds of equality which may run through all these varieties of variations, the equality of proportion must always be present if the variations are to be as we have been describing them (*i. e.*, if their percentages are to be equal in the three ways described). But the equality of difference, or of distance traversed, need be present only under certain conditions, different in each of the three varieties. Of course when we have two variations presented merely as variations, we may twist about the starting and ending points as we please, and so whenever we have an equality of proportion, we may also conceive of a corresponding equality of difference. If, however, the variations are given us as the definite variations of an attribute in certain things, which have started from certain figures and have ended at certain figures, then an equality of proportion may be present, in either of the three forms, without the corresponding equality of difference, although the latter cannot be present without the former. The conditions for the presence of the equalities of difference are plain. In the case of arithmetic variations, the two things, or individuals, must be equally large (or important) *at the first period*, and at this period only; for they both start from unity, or from any other common figure, that is, from the same level of equality in size or amount. In the case of harmonic variations, the two things, or individuals, must be equally large (or important) *at the second period*, and then only; for they both end by being equal to unity, or to some other common figure, that is, they come to the same level of size. In the case of geometric variations, the two things, or lividuals, must be alternately equal in size, the one at the first iod with the other at the second, and again the former at the s l with t latter at the first, so that they are equal in size b the ds together. The connection between these consubject of weighting is obvious.

ve equalities not merely of proportion but ly t ersed, we virtually have three different i —*arithmetic*, *harmonic* and *geometric* com-

pensation. There may be agreement between these at times, as we shall see, and also disagreement. In the latter case it is the problem before us to decide which of these compensations is the proper one for our subject.

§ 5. So far we have been dealing with only two things (or classes) that vary, and all these relations have been found to be universal when we are dealing with two things (or classes), with even weighting in the averaging of their variations. All of them are, or may be, universal, when unity is the arithmetic, harmonic or geometric *mean* between two opposite variations. Furthermore, now, they are completely and unconditionally universal in the cases of the arithmetic and harmonic *averages*, no matter how many variations on the one side be opposed to no matter how many on the other. In these cases if unity be the arithmetic or the harmonic average (in each case with the weighting of the period proper for it) between all the variations, there is equality of proportion between all the variations on the one side together and all the variations on the other side together, and, with observance of the proper starting and finishing points, there is equality between the distance traversed by all the rising things and the distance traversed by all the falling things, the whole distance in each case being obtained by summing up the distances traversed by all the things individually.

Thus in the case of compound arithmetic variations, if instead of one B we have two B's, and both fall together from 1.00 in the compensation for the rise of A from 1.00 to 1.50, the figure to which they fall must be such that the arithmetic average between it twice repeated and 1.50 is unity. This figure is .75; for $\frac{1}{3}(1.50 + 2 \times .75) = 1.00$.[5] Or if two things fall unequally, suppose the one has fallen to .80, then the other must fall to a figure between which and .80 and 1.50 the arithmetic average is unity. This is .70; for $\frac{1}{3}(1.50 + .80 + .70)$

[5] It should be noticed that the *two* figures, 1.50 and .75, are the simple harmonic extremes around 1.00, but the *three* figures, 1.50 and .75 twice repeated, are arithmetic extremes around 1.00. The figure .75 repeated a fractional number of times may even be a geometric extreme opposed to 1.50; for we may have $\sqrt[1+x]{1.50 \times .75^x} = 1.00$, and it is not difficult to find x $\left(\text{it is } \frac{\log \frac{3}{2}}{\log \frac{4}{3}} = 1.4086\right)$.

toward it, they posit merely that the classes in question should be equally important. They have not entered into the question at what period the classes should be equally important. We may, then, at present, take our own position on this subject, and require that the classes be equally important, or large, over both the periods together—either alternately, or constantly so, or on the average.

We may suppose that as A rises in price by one per cent. at a time (always reckoned on the original starting point at 1.00), B falls by one per cent. at a time (likewise always reckoned on the same original starting point at 1.00). According to the arithmetic averagist these opposite movements always leave the exchange-value of money and the level of prices unchanged—until 2.00 is reached by A, after which he does not concern himself further (or requires that another equally important class shall begin to fall). But, now, every successive rise of A is a smaller rise reckoned from *its* starting point, while every successive fall of B, similarly reckoned, is a larger fall. Thus, while the first rise and fall are each by 1 per cent., the second rise of A from 1.01 to 1.02 is a rise by $100\left(\frac{1.02-1.01}{1.01}\right)$ $= 0.990099$ per cent., and the corresponding fall of B from .99 to .98 is a fall by $100\left(\frac{.99-.98}{.99}\right) = 1.0101$ per cent. The difference will be shown more plainly by an extreme case. When A has risen to 1.98 and B fallen to .02, the next rise of A to 1.99 is a rise by $100\left(\frac{1.99-1.98}{1.98}\right) = 0.505$ per cent.; but the next fall of B to .01 is a fall by $100\left(\frac{.02-.01}{.02}\right) = 50$ per cent. Now suppose we begin with things in this condition. The price of A being 1.98, the price of 0.505 A is 1.00, and the price of B being .02, the price of 50 B is 1.00. We may, then, form a new unit for each of these quantities, say A′ priced at 1.00 and B′ priced at 1.00—and we may still suppose that the classes [A] and [B] are equally important. Then the arithmetic averagist, if he says the rise of the price of A to 1.99 is compensated by a fall of the price of B to .01, must say that the rise of the price of A′ to 1.00505 is compensated by the fall of the price of B′ to .50; which is absurd on his own principles. Thus the position of the arithmetic averagist leads him into inconsistency. And on going back to the start, we now see that even the first fall from 1.00 to .99 is a larger fall than the rise from 1.00 to 1.01, if we analyze each of these variations into component stages. Therefore all such price variations mean an appreciation of money.

On the other hand, in harmonic variations the fall demanded in compensation for a rise is always smaller than the rise. This is so obvious that it does not need explication. But the position of the harmonic averagist of prices may be shown to be wrong also in this way. His position is that we should have arithmetic equality in the compensatory changes in the quantities of things purchasable, whereby the exchange-value of money in their classes is measured. Here of course the same absurdity comes to light, with the consequent inconsistency, as in the case where the arithmetic average is applied to prices. And the falls in the exchange-values of money being too large, this kind of variations means a depreciation of money.

§ 2. The geometric method of averaging escapes such absurdity and such inconsistency. It provides, not equality of rate of variation (measured in the usual way), but correspondence. As the rate of the rise of A, supposed to be always at the same speed from the original starting point, grows smaller with every advance, so the rate of the fall of B is made to grow proportionally smaller. Thus when A rises from 1.00 to 1.01 by 1 per cent., the compensatory fall of B is from 1.00 to $\frac{1}{1.01} = .990099$, which is a fall by $100(1 - .990099) = 0.990099$ per cent. Then when A next rises to 1.02, the compensatory fall of B is to $\frac{1}{1.02} = .980392$. Here the percentage of the rise of A, reckoned in its starting point, has sunk to $100\left(\frac{1.02-1.01}{1.01}\right) =$ 0.990099, and the percentage of the fall of B, reckoned in its starting point, has sunk to $100\left(\frac{.990099 - .980392}{.990099}\right) =$ 0.980407. Again when A next rises to 1.03, the compensatory fall of B is to $\frac{1}{1.03} = .970873$. Here the percentage of the rise of A, reckoned in its starting point, has sunk to $100\left(\frac{1.03 - 1.02}{1.02}\right) = 0.980392$ (the same as the point to which B previously fell), and the percentage of the fall of B, reckoned in the same way, has sunk to 0.970938 per cent. And so the process will continue indefinitely, the percentage of the fall of A always being the same with the point to which B previously fell, and the percentage of the fall of B always being still smaller (though above the point to which B falls). For instance, in the ninety-ninth stage, the rise of A from 1.98 to 1.99 is compensated by the fall of B from $\frac{1}{1.98} = .505050$ to $\frac{1}{1.99} = .502512$, so that the rise by $100\left(\frac{1.99 - 1.98}{1.98}\right) = 0.505050$ per cent. is compensated by a fall by $100\left(\frac{.505050 - .502512}{.505050}\right) = 0.502524$ per cent.

Now the first compensatory fall of B from 1.00 to .990099 is the same variation as a fall from 1.01 to 1.00,—merely the reverse of the rise of A for which it is offered in compensation. And the next fall of B from .990099 to .980392 is the same variation as a fall from 102 to 101,—again merely the reverse of the rise of A for which it is offered in compensation. And so with all the compensatory variations required by the geometric averaging (with even weighting). They are merely the reverse of each other. Whenever A rises from any figure, a_1, to any figure, a_2, by a percentage (reckoned in a_1) obtained by dividing a hundred times the difference which measures the amount of the rise by the earlier figure, namely $100\left(\frac{a_2 - a_1}{a_1}\right)$, the compensatory fall of B is a fall from a_2 to a_1 by a percentage (reckoned in a_2) obtained by dividing a hundred times the difference which measures the amount of the fall by the earlier figure, namely $100\left(\frac{a_2-a_1}{a_2}\right)$. These percentages are not the same, but it is evident that this fall is equal to that rise. If A rises from a_1 to a_2 and then falls back to a_1, it is evident that the fall is equal to the rise, since it brings A back to its original position. Then the fall of B from a_2 to a_1, equalling the fall of A from a_2 to a_1, equals the rise of A from a_1 to a_2.[1] The usual method of reckoning percentages does not manifest this equality, which is shown mathematically only by reckoning the percentages from the opposite extremes, but in the same direction. That the mathematical equality should reside only in this kind of percentage is plain, since the compensatory variations must plainly be merely the reverse of each other. Therefore the correctness of the geometric *mean* (although not of the geometric average) is demonstrative.

§ 3. Jevons wrote in his *Principles of Science:*[2]—" In almost all the calculations of statistics and commerce the geometric mean ought, strictly speaking, to be used. If a commodity rises in price 100 per cent. and another remains unaltered, the mean rise of a price is not 50 per cent. because the ratio 150 : 200 is not the same as 100 : 150. The mean ratio is as unity to $\sqrt{1.00 \times 2.00}$ or 1 to 1.41." There is exaggeration in the first part of this statement, since the geometric mean is almost exclusively to be confined to the measurement of variations, and many calculations of statistics are not measurements of variations —though often providing the data for such measurements. In this statement we have our problem solved in the first form in which we approached it. Therefore we may find interest in noticing this also. As Jevons's brief statement has not met with acceptance on the part of statisticians of prices, it needs to be explicated.

The arithmetic averagist would say that the mean rise in the above suppositional case is to 1.50. Now suppose, the price of B remaining at 1.00, the price of A rises first of all to 1.50.

[1] Cf. the reasoning for Proposition XI.
[2] Second ed., p. 361.

Then the arithmetic averagist would say the mean price has risen to 1.25. Suppose that later the price of A rises from 1.50 to 2.00. This rise, measured from its starting point, is a rise by $33\frac{1}{3}$ per cent. Then, according to the principles of the arithmetic averagist, the mean, already risen to 1.25, ought to rise further by half of $33\frac{1}{3}$, or $16\frac{2}{3}$, per cent. above 1.25. This is a rise to $1.25 \times 1.16\frac{2}{3} = 1.45433 \cdots\cdots$. Therefore on his own principles the arithmetic averagist is mistaken in saying that when A rises to 2.00 the mean rise is to 1.50. And consequently, too, he was mistaken when he said it rose first to 1.25, and his whole position from beginning to end is inconsistent with itself, and wrong. The mean price at every rise is below the arithmetic mean.

On the other hand, the mean price is always above the harmonic mean, because if it rose only to the harmonic mean, the mean exchange-value of the two things would fall to the arithmetic mean, and a similar inconsistency would be found.

The error in the position of the arithmetic averagist is evident. The higher A rises from its original position, the smaller is its rise in each stage of its advance. Yet the arithmetic averagist accords to it the same influence upon the mean when it rises from 1.99 to 2.00 as when it rose from 1.00 to 1.01, although the mean has lagged behind somewhere below 1.50, where its rise by half of one per cent. at a time (reckoned in 1.00) is considerably more than half as large as the rise of A from 1.99 to 2.00. And reversely, when A alone falls, while it is falling from .02 to .01, the arithmetic averagist accords to it not so much influence upon the mean as when it fell from 1.00 to .99, although its fall is now fifty times greater than it was then.

But the geometric averagist, in placing the mean at 1.41 when A alone rises from 1.00 to 2.00, places it where the influence of the later rises is the same as that of the earlier, when in the same percentage. Thus if A alone rises first to 1.41 by 41 per cent., the mean is placed at 1.19, indicating a rise by 19 per cent. Then when A rises further to 2.00, this also being a rise by 41 per cent., the same influence as before is attributed to it; for the rise of the mean from 1.19 to 1.41 is by 19 per cent.

§ 4. Here we might pause, our labor done, but for the fact, above shown, that what is true of the geometric mean is not true of the geometric average. All that has just been demonstrated has been demonstrated on the supposition that we are dealing with only two classes of things, and these equally large or important over both the periods compared. If the two classes are not equally large over both the periods, or if there are many varying classes to be considered, nothing that has just been proved applies,—nor does it apply with accuracy if we use the geometric average with the proper weighting over both the periods. For instance, suppose the classes are equally important only at the first period, and as [A] rises in price it rises also in importance, and as [B] falls in price it falls also in importance. Then it is evident that a rise of A from 1.00 to 1.50 is not fully compensated by the fall of B to the other geometric term, 0.66⅔; but, for compensation, B must fall further. How much further it must fall will depend upon the extent of the alteration in the relative importance of the classes. It is possible, therefore, that the price of B may have to fall to the arithmetic term, 0.50. We shall in fact find this to be the case when the relative sizes of the classes vary exactly as their prices. Yet this will not be exactly indicated by the geometric average with the proper (uneven) weighting for both the periods.

Therefore we need to turn to the examination of the averages more closely in connection with the subject of weighting. Also it is always well to examine the arguments of persons who have advocated other opinions, and although we have already disproved the positions of the arithmetic and harmonic averagists under certain provisos, yet their general position still remains. It happens that both these objects may be pursued together.

CHAPTER IX.

REVIEW OF THE ARGUMENTS FOR THE HARMONIC AND ARITHMETIC AVERAGES OF PRICE VARIATIONS.

I.

§ 1. The arguments for the harmonic and arithmetic averages have been very imperfectly stated. We shall therefore have to try to understand them not merely as they have been presented, but as they turn out to be on fuller analysis. They have generally been adduced in the form which considers what constitutes variation. But this depends upon what constitutes constancy. Also little or no reference has been made to weighting. This we shall have to supply. We may, however, supply it later, first reviewing the arguments in their most general forms.

In the argument for the use of the harmonic average of price variations the idea, when there is supposed to be constancy, is as follows. The same total sum of money purchasing the same total quantity of all kinds of goods at both the periods compared, the purchasing power of this sum (and consequently the exchange-value of money) is considered not to have varied, whatever be the changes in the make-up of the total quantity of goods, an increase in the quantity of one class of things purchasable with the same particular sum of money devoted to purchasing it at both the periods being offset by an equal decrease in the quantity of another class of things purchasable with the particular sum devoted to purchasing it at the two periods. For the total of the quantities remains the same when these quantities vary oppositely to the arithmetic terms, and consequently when their prices vary oppositely to the harmonic terms; so that if the harmonic average of the price variations indicates constancy, it indicates this condition. As the quantities of the goods are in arithmetic progression, the compensation may be described as *arithmetic compensation by equal mass-quantities*—a loss of one third on A, for instance, being compensated by a gain of one third on B, or by a gain of one sixth both on B and on C. Now if a variation occurs in the total quantity of the goods purchasable at the second period from that at the first, this variation of the total is indicated by the arithmetic average of the variations of the particular quantities; for if the quantities purchasable all varied at the same common average rate, the same result in the total would be obtained. Thus if there is a loss of one third in the quantity of A alone, this is the same, among two classes of commodities, as a loss of one sixth on each; among three classes, of one ninth on each; and so on, just the same as if each of the two classes had lost one sixth, or each of the three classes one ninth, and so on.[1] Hence this same result is obtained, for the prices, by the harmonic average of the price variations. All this is a mathematical fact. Upon this fact is based the argument, if argument it may be called, or rather the claim, suggested by Jevons and insisted upon by Messedaglia, that the harmonic average of the price variations, because it inversely indicates this variation, or the preceding constancy, of the total quantity of goods purchasable with the same total sum of money, also inversely indicates the variation, or constancy, of the purchasing power of money (and consequently of its exchange-value), and so is to be taken as the proper method of measuring it.

In this argument it is always understood that the particular sums of money spent on every class of goods remains constant at both the periods, whatever be the variations in the particular mass-quantities therewith purchased.

§ 2. In the argument for the use of the arithmetic average of price variations the idea, when there is supposed to be constancy, is as follows. The same total quantity of goods being purchasable with the same total sum of money at both the periods, a total price is conceived of the total quantity, which total price

[1] This will be recognized as the argument we lighted upon at first glance above in Chapter VI. Sect. I. § 2.

has not varied whatever be the changes in its make-up, that is, in the particular prices, an increase in the price of one class of things being offset by an equal decrease in the price of another. For the total of the prices remains the same when they vary oppositely to the arithmetic terms (and consequently when the mass-quantities purchasable vary oppositely to the harmonic terms); so that if the arithmetic average of the price variations indicates constancy, it indicates this condition. And so there is *arithmetic compensation by equal sums of money*—a need, for instance, for one half more money to purchase A being offset by a need for one half less money to purchase B. Now if a variation occurs in the total price, or total sum of money needed to purchase the same quantities of the same classes of goods at the second period from what was needed at the first, this variation of the total is indicated by the arithmetic average of the particular prices; for if the prices all varied at the same common average rate, the same result in the total price would be obtained. Thus a rise in the price of one class of commodities by fifty per cent. has the same influence upon the total as a rise of two classes by twenty five per cent., of three classes by sixteen and two thirds per cent., and so on. Hence the result in the variation of the total price is obtained by the arithmetic average of the variations of the particular prices. Here again is a mathematical fact, upon which is based the argument, or claim, that the arithmetic average of the price variations, because it directly indicates this variation, or the preceding constancy, of the total price of the same goods, also inversely indicates the variation, or constancy, in the purchasing power of money (and consequently in its exchange-value) and so is to be taken as the proper method of measuring it.

In this argument it is always understood that the particular quantities purchased of every class remain constant at both the periods, whatever be the variations in the particular sums of money needed to purchase them.

§ 3. Thus the harmonic and arithmetic averages of prices, in the minds of their advocates, represent reversed positions. The former is directly applied to the measurement of the purchasing power of money by the total quantity of all commodities a given total sum of money, spent in the same way at both periods, will purchase. The latter is applied rather to the measurement of the power of all commodities over money by the total sum of money a total quantity of given particular commodities, composed in the same way at both periods, will command in exchange. The former makes use of arithmetic compensation by equal mass-quantities, the latter of arithmetic compensation by equal sums of money. They have in common the use of arithmetic compensation. They differ in applying this to opposite sides of the question, in each case excluding notice of the other side.

When we look at either of these positions by itself, it seems very strong. The one inversely measures variations in the total mass-quantity purchasable with a given sum of money. Does it not then inversely measure variations in the purchasing power or exchange-value of money, and directly measure variations in the general level of prices? The other directly measures variations in the total price of given quantities of all things together. Does it not then directly measure variations in the general level of prices, and inversely variations in the purchasing power or exchange-value of money?

Each of these methods is founded on a procedure which we employ with regard to single classes of commodities. We measure the constancy or variation of the particular exchange-value of money in any one class of commodities by the constancy or variation of the quantity of this commodity purchasable with a given sum of money at each of the periods (according to Proposition I.),—and by inverting the result so obtained we can measure the constancy or variation of the price of this thing (although we never adopt this roundabout course). Hereupon the harmonic averagist of prices concludes that we can measure the constancy or variation of the general exchange-value of money in all things by the constancy or variation of the total quantity of things purchasable at each of the periods with a given total sum of money (which he further specifies must be spent in the same way at both the periods),—and the inverse of the result so obtained he regards as the proper measure of the constancy or variation of prices in general. Again we measure the constancy or variation of the particular price of any one class of commodities by the constancy or variation of the price of a given quantity of this commodity at each of the periods,—and the inverse of this gives the constancy or variation of the particular exchange-value of money in that class (according to Proposition X.). Hereupon the arithmetic averagist of prices concludes that we can measure the constancy or variation of a total price of all things by the constancy or variation of the total price at each period of a given total quantity of all things (further specifying that these quantities must be individually the same at both periods),—and the inverse of the result so obtained (the inverse of the constancy or variation of this total price of all the same things) he regards as the proper measure of the constancy or variation of the general exchange-value of money in all things.

In the two particular measurements that serve as models for these two general measurements there is no disagreement possible. The results obtained by the one are universally the same as the results obtained by the other. Those two particular measurements are also always applicable to every one and the same case. The two copies, however, are really applicable each to a different state of things, so that they are not properly contradictory or antagonistic. Yet this fact has been mostly overlooked, and the advocates of these different averages have simply urged the use of the one or of the other method for all cases. When applied to the same cases, differently garbling them to fit them to the different requirements, these methods give different results. This disagreement shows that at least one of them is false, and probably both, when applied in such a loose way. But confined each to the cases that may happen to exist to which it is applicable, they may both be true, or they may both be false, or the one may be true and the other false. At all events this divergence of the copies from the models shows that something is wrong in the copying at least in one case.

§ 4. The faultiness of the copy in the position of the harmonic averagist of prices is not far to seek. In measuring the constancy or variation of the particular exchange-value or purchasing power of money in any class of commodities by the constancy or variation in the quantity of it purchasable at each period with a given sum of money, we are careful to note that the quality of this mass-quantity must not change, and guarding this, there is no possibility of divergence in our result, no matter what mass-unit we use. Now when we try to imitate this operation in measuring the constancy or variation in the general exchange-value or purchasing power of money in many or all classes of commodities, we at once strike upon a difficulty if there is the least change in the relative exchange-values or prices or preciousness of these classes. For if there is any such change, this is the same as a change in the quality of the total mass-quantity, which renders the comparison of its change of size nugatory, unless it can be allowed for. Furthermore, the comparison of the total mass-quantities purchasable, or actually purchased, at each period with the same sums of money, will now be different according to the mass-units that are used in each class. The harmonic averagist of prices adopts the usual practice in these matters and takes as his mass-unit in every class the mass that is equivalent to the money-unit at the first period. But this is only one of many possible ways of selecting the mass-units, and he has offered no reason for adopting it—and apparently has none, except the blind following of a convenient habit. Doing so, when he finds the mass-quantities purchasable with the same sums at each period to foot up to the same total quantity, so that he concludes that the exchange-value of money has not altered on the whole, it may be that the total weight, or the total bulk, of the goods so purchasable with the same sums of money may be very different at the two periods. Hence he is here not following his model. And if he should now try to return to his model, he would be confronted with the question. Is he to require the same total weight, or the same total bulk? Between these two there is nothing to decide. But if he does (arbitrarily) hit upon the one or the other, he will only be measuring the constancy or variation in the one or in the other kind of preciousness of the goods relatively to money.[2] Thus the

[2] He would be using Drobisch's method applied to cases in which $x_1 a_1 = x_2 a_2$,

measurement of the general exchange-value of money cannot be so simply made on the model of this method of measuring a particular exchange-value of money. The imitation will be exact, in case the total mass-quantities are the same at both periods, only if there are no price variations, and, in case the total mass-quantities are different, only if all the prices have varied in the same proportion—that is, only when any other average would be as good. There is, however, a more complex way in which this model may be imitated. But the harmonic averagist has not sought it, and nobody has hitherto pointed it out.

In the argument for the arithmetic average of prices there is a somewhat similar defect. But this seems to bear with it its own correction, in the way alluded to at the end of the preceding Chapter. Here, in its totality and in its details, identically the same (or similar) mass of goods is used, of which the constancy or the variation in the total price is taken as the measure of the constancy or inverse variation in the exchange-value of money. And consequently there is no difficulty here about the mass-units to be employed. But this total mass is not economically the same in all its parts at both periods, unless all prices have remained constant or varied alike; for otherwise some of its parts have become more or less precious, and also more or less important, than others, and so its economic make-up has altered. Still it is precisely the prices of the things that have grown more precious and more important that have risen, and the prices of the things that have become less precious and less important that have fallen, and all these variations are in exactly the same proportions; which is about as we should desire. Hence it is possible that this measurement is a good copy of its model. But it would probably be more difficult to prove this than to prove the correctness of the arithmetic average (in the special cases to which it is claimed to be applicable). Hence the above argument made by the arithmetic averagists is still unsatisfactory, and we must continue to probe it.

$y_1\beta_1$ $y_2\beta_2$, ... (because the same sums are supposed to be spent on every class at both periods), so that the first part of Drobisch's formula falls away, being reduced to unity.

II.

§ 1. Both the above arguments have been advanced only on very cursory inspection and incomplete analysis of the mathematical relations involved. The writers acquainted with them seem to have thought that the argument for the harmonic average is peculiar to that average, applicable neither to the arithmetic nor to the geometric, and that the argument for the arithmetic average is peculiar to that average, applicable neither to the harmonic nor to the geometric. And as these two arguments, attacking our problem from its two opposite sides, appear to occupy all the possible positions, it has seemed as if no room were left for the geometric average or mean. This has seemed to stand out in the cold, with no function to fulfil, and with no argument applicable to it. Hence the neglect with which it has been treated. Yet not much analysis is needed to show that these views are false.

The argument for the harmonic average of prices assumes a certain distribution of our spendings, and then finds compensation arithmetically by equal mass-quantities, so that, when constancy is indicated, an equal total mass-quantity, though differently made up, is purchasable at both periods with the same sum spent in the same way at both periods. Now suppose prices have changed to the arithmetic extremes, namely in our example from 1.00 to 1.50 and from 1.00 to .50. Then if we spend 1.50 on [A] at both periods and .50 on [B] at both periods, we get for 2.00 at the first period 1½ A and ½ B, and at the second period 1 A and 1 B,—losing ½ A and gaining ½ B. Thus there is compensation by equal mass-quantities, and ability with the same sum to purchase an equal total mass-quantity.

The argument for the arithmetic average of prices assumes a certain distribution of our purchases, and then finds compensation arithmetically by equal sums, so that, when constancy is indicated, the same total sum, though differently made up, is able to purchase at both periods the same total mass-quantity made up of the same particular mass-quantities at both periods. Now suppose prices have changed to the harmonic terms, namely in our example from 1.00 to 1.50 and from 1.00 to .75. Then if we purchase $\frac{2}{3}$ A and $1\frac{1}{3}$ A at both periods, we can do so at the first period by spending $.66\frac{2}{3}$ on [A] and $1.33\frac{1}{3}$ on [B], and at the second period by spending 1.00 on [A] and 1.00 on [B]—or $\frac{1}{3}$ M more on [A] and $\frac{1}{3}$ M less on [B]. Thus there is compensation by equal sums, and ability to purchase exactly the same particular mass-quantities with the same total sum.

Suppose, again, that prices have changed to the geometric terms, namely from 1.00 to 1.50 and from 1.00 to $.66\frac{2}{3}$. Then if we spend 1.20 on [A] at both periods and .80 on [B] at both periods, we get for 2.00 at the first period $1\frac{1}{5}$ A and $\frac{4}{5}$ B and at the second period $\frac{4}{5}$ A and $1\frac{1}{5}$ B,—losing $\frac{2}{5}$ A and gaining $\frac{2}{5}$ B. Thus there is compensation by equal mass-quantities, and ability with the same sum to get an equal total mass-quantity. Also if we purchase $\frac{4}{5}$ A and $1\frac{1}{5}$ B at both periods, we can do so at the first period by spending .80 on [A] and 1.20 on [B] and at the second period by spending 1.20 on [A] and .80 on [B]—or $\frac{2}{5}$ M more on [A] and $\frac{2}{5}$ M less on [B]. Thus there is compensation by equal sums, and ability to purchase the same particular mass-quantities with the same total sum.

Hence the argument which seems to be peculiar to the harmonic average applies also to the arithmetic and to the geometric; and the argument which seems to be peculiar to the arithmetic, applies also to the harmonic and to the geometric. And the geometric average, or at least the geometric mean, instead of being without any argument, and without any function, equally is subject to either of the arguments, and equally performs the same functions. Instead of standing out in the cold, it belongs in the fold; and we cannot examine either of the arguments far without taking also it into consideration.

Thus are there two arguments, in the forms hitherto used, for three means or averages, each argument being found to be applicable to each of the means. Therefore, so far as we yet see, any one mean can apparently be argued for as well as any other by either of these arguments.[1]

[1] That there is ability to purchase with the same sum at both periods an equal total quantity of commodities differently made up [hence with compensation by

§ 2. Furthermore, when prices change to the harmonic extremes and the harmonic average indicates constancy, or when prices change to the arithmetic extremes and the harmonic average indicates constancy, or when prices change to the geometric extremes and the geometric mean indicates constancy, although it is possible in each case with the same sum of money spent in the same way at both periods to purchase an equal total quantity of goods differently made up, the spendings and the purchases being different in each of the supposed changes, it is also possible in each case with the same sum of money spent in the same way at both periods to get both a larger and a smaller total quantity of goods at the second than at the first period, our spendings being variously distributed. Thus in our simple example of two classes of commodities both priced at 1.00 at the first period we can at that period itself with 2 M purchase 1 A and 1 B, or two given quantities of [A] and [B] together, and in any combination of spendings of the same sum we always can purchase two such quantities—*e. g.*, $\frac{2}{3}$ A and $1\frac{1}{3}$ B, $\frac{1}{2}$ A and $1\frac{1}{2}$ B, $1\frac{1}{4}$ A and $\frac{3}{4}$ B, 2 A and 0 B, 0 A and 2 B, etc. But at the second period with prices changed to the harmonic terms, 1.50 and .75, if we employ 2 M to purchase 1 A and some B, we must use $1\frac{1}{2}$ of it in purchasing 1 A and have left only $\frac{1}{2}$ with which we can purchase only $\frac{2}{3}$ B, or all told only $1\frac{2}{3}$ of [A] and [B] together; or if we purchase 1 B with $\frac{3}{4}$ M, with the remaining $1\frac{1}{4}$ M we can purchase only $\frac{5}{6}$ A, or $1\frac{5}{6}$ of [A] and [B] together —in both cases a smaller total than before. Or again, employing 2 M to purchase $\frac{1}{2}$ A and some B, we can purchase $1\frac{2}{3}$ B, or together $2\frac{1}{6}$, this time more than before. With the prices changed to the arithmetic terms, 1.50 and .50, by employing 1 M to purchase some A and 1 M to purchase some B, we get $\frac{2}{3}$ A and 2 B, or on the whole more than before; or by spending $1\frac{3}{4}$ M on [A] and $\frac{1}{4}$ M on [B] we get $1\frac{1}{6}$ A and $\frac{1}{2}$ B, or on the whole less than before. And lastly if the prices change to the geometric terms, 1.50 and .66$\frac{2}{3}$, we can also get sometimes more and sometimes less of [A] and [B] together—more,

equal mass-quantities] even when the simple harmonic average of the price variations indicates constancy, was perceived by Walras, B. 69, pp. 14–15. But Walras has not investigated further.

for instance, if we spend 1 M on [A] and 1 M on [B], getting $\frac{2}{3}$ A and $1\frac{1}{2}$ B, or all told $2\frac{1}{6}$, or less if we spend $1\frac{2}{3}$ M on [A] and $\frac{1}{3}$ M on [B], getting $1\frac{1}{9}$ A and $\frac{1}{2}$ B, or all told $1\frac{11}{18}$.

Hence, if the mere possibility of getting, with our money spent in the same way at both periods, an equal total quantity of commodities is a reason for thinking our money constant in purchasing power, the simultaneous possibility of getting both more and less is an equally good reason, so far as is yet shown to the contrary, in each case, to think that our money has both appreciated and depreciated; which is absurd.

Again, although it is possible in each of these cases with the same total sum of money differently spent to purchase at both periods exactly the same quantities of commodities, these being different in each of the supposed cases, it is also possible in each case that to purchase other exactly the same quantities of commodities, a larger or smaller sum of money is needed at the second period. This could be easily shown in our simple example. But enough has been shown already.

Hence, if the mere possibility of getting at both periods exactly the same quantities of commodities with the same total sum of money is a reason for thinking the level of prices constant, the simultaneous possibility that for getting exactly the same other quantities of commodities both a larger and a smaller total sum may be needed, is equally good reason, so far as is yet shown to the contrary, in each case to think that the level of prices has both risen and fallen; which again is absurd.

Therefore, so far as we yet see, these arguments apparently are equally defective whether applied to the harmonic, arithmetic or geometric averages or means.

§ 3. This defectiveness of the arguments seems to have been ignored. The conflicting possibilities have been overlooked. Hardly any advance has been made in this matter since the famous dispute between Jevons and Laspeyres. That dispute, therefore, deserves review; for it may provide a warning, still needed in our subject, against reasoning which stops at a few half-truths first lighted upon.

Jevons had noticed in his first work that the price of cocoa had recently risen 100 per cent., while that of cloves had fallen 50 per cent., and had said it would be "totally erroneous" to say the average change was a rise of 25 per cent., since the geometric mean in this case indicates no variation at all.[2] Hereupon Laspeyres commented and argued as follows:—"The geometric mean expresses neither the depreciation of commodities or appreciation of money, nor the appreciation of commodities or depreciation of money—that is, according to Jevons, increase or decrease in its 'potency in purchasing other articles.' Let us retain the example used by Jevons. Here, after the change in price of cocoa and cloves, the same sum of money has not the same purchasing power as before, but a smaller one, and exactly so much smaller as is indicated by the arithmetic mean. If a certain weight of cocoa (say 1 cwt.) previously cost 100 thalers, and a certain weight of cloves (say 1 cwt.) also cost 100 thalers, and the price of this amount of cocoa rises from 100 to 200 thalers, and that of the cloves falls from 100 to 50, then 200 thalers no longer have the same potency in purchasing cocoa and cloves. For this sum the purchaser procures only $\frac{3}{4}$ cwt. cocoa (= 150 th.) and 1 cwt. cloves (= 50 th.), or he procures 1 cwt. cocoa (= 200 th.) and no cloves at all. The purchasing power is now $\frac{1}{5}$ less, that is, the purchaser must add $\frac{1}{4}$ in order to get the same quantity; or the 250 thalers are now by $\frac{1}{5}$ (50 th.) less worth than formerly. Exactly this is expressed by the arithmetic mean $\frac{200 + 50}{2} = 125$; 125 thalers have only the same purchasing power as 100 before, or 250 only the same as 200 before. Money has depreciated 20 per cent.; commodities have risen 25 per cent. What is true of the average of two commodities, is true also for any number of commodities."[3]

Laspeyres thus found fault with the geometric average for indicating constancy in the "potency in purchasing" under the given conditions, because these conditions permit us with one whole sum, 200 thalers, to get at the second period only 1 cwt. cocoa and no cloves, that is, a smaller quantity than before,

[2] B. 22, pp. 23–24.
[3] B. 25, p. 97.

which fact he took for an indication that the "potency in purchasing" was smaller; to which came the added evidence that more money is required at the second period to buy the 1 cwt. of each article. He omitted to state that these conditions permit us at the later period to buy 4 cwts. cloves and no cocoa, that is, this time a larger quantity than at first, and one just doubles, as the other was half. Had he done so, the indication of depreciation would have been no stronger than that of appreciation. And he probably failed to see that under these conditions we could at the first period purchase with 66.66⅔ thalers ⅔ cwt. cocoa and with 133.33⅓ thalers 1⅓ cwts. cloves, or 2 cwts. with 200 thalers; and that at the second period we could purchase with 133.33⅓ thalers ⅔ cwt. cocoa and with 66.66⅔ thalers 1⅓ cwts. cloves, that is, exactly the same quantities of cocoa and cloves, amounting to 2 cwts., with the same total sum of money, 200 thalers. Had he noticed this, he would have seen that, so far as either he or Jevons had yet carried their investigations, the indication of constancy is as strong as the indication of depreciation, his argument offering nothing distinctive in proof of the indication of depreciation made by the arithmetic average over against the indication of constancy made by the geometric.

And when Jevons in reply suggested the harmonic average (in this case .80, indicating a fall of prices by 20 per cent.) on the ground that it marks the change in the total quantity of commodities the same sums of money will purchase at the two periods (here, for 100 thalers spent on each article, 1 cwt. cocoa and 1 cwt. cloves at the first period, and at the second ½ cwt. cocoa and 2 cwts. cloves, the arithmetic average being $\frac{1}{2}(\frac{1}{2}+2) = 1.25$, indicating appreciation of money corresponding to the above fall of prices), he probably failed to see that we could at the first period purchase with 133.33⅓ thalers 1⅓ cwts. cocoa and with 66.66⅔ thalers ⅔ cwt. cloves, or 2 cwts. with 200 thalers; and that at the second period we could purchase with 133.33⅓ thalers ⅔ cwt. cocoa and with 66.66⅔ thalers 1⅓ cwts. cloves, that is, with exactly the same expenditure of 200 thalers the same total quantity of 2 cwts.,—wherefore even his argument for the harmonic average, so far as he worked it out, would indicate constancy as readily as a fall of prices.

§ 4. We have, even, as yet by no means exhausted the possibilities which render such arguments ridiculous. The following propositions hold :—*The price of at least one class rising and the price of at least one class falling, no matter how large or small these variations be, it is possible by spending constant sums of money on the different classes, with the same total sum to purchase at both periods the same total quantity of goods;* and under the same conditions, *it is possible at both periods to purchase constant quantities of the different classes, consequently the same total quantity of goods, with the same total sum of money, differently spent at the two periods.* In the first of these cases the equality in the total quantities of goods at both the periods depends, given the price variations, upon two other factors. It depends both on the sizes of the mass-units between the numbers of which the compensation by arithmetically equal quantities is desired, and on the proportions between the special sums devoted at both periods to purchasing the different classes. In the second case the equality in the total sums of money at both periods depends upon only one factor beside the price variations. This one other factor is the proportion between the special quantities of the different classes purchased at the two periods; but as these quantities are affected, in their numerical expressions, by the sizes of the mass-units used, these also play a part, though a subordinate one, as we shall see. Now this factor, so far as it is a factor, of the sizes of the mass-units used, has generally been decided at the outset in the same way by the harmonic and by the arithmetic averagists. They both employ mass-units that are equivalent *at the first period* (being then priced at 1.00 or at 100 money-units)—as we have just seen done by Jevons and Laspeyres. These mass-units being settled upon, the first factor in the first case, and so far as it affects the second, is disposed of, and now the equality in question depends upon the other factor in both cases. In the first case, if an article rises much in price, so that the deficiency in the quantity of it purchasable at the second period, compared with the quantity purchasable at the first, is large, and if another falls slightly in price, so that the gain here, quantity for quantity, is small, we only have to extend the quantity of this class purchased at the first period until its gain at the second period equals the deficiency of the other. And similarly in the second case. The number of classes does not affect the matter. All but one may rise much and that one fall but slightly: still it is possible for the compensation by arithmetic equality to take place.

When we are dealing with only two oppositely varying classes it is easy to get formulæ, which may be of service. To begin with the first case:—Let the sum of money devoted at both periods to purchasing [A] be represented by **a**, and that devoted at both periods to purchasing [B] be represented by **b**; let us for the present adopt the usual course and take for our mass-unit for each class that mass of it which can be purchased at the first period for one money-unit, and let α_2' and β_2' be the prices at the second period of these mass-units of [A] and [B] respectively. We therefore purchase with **a** money-units **a** mass-units of [A] at the first period and at the second $\frac{\mathbf{a}}{\alpha_2'}$ mass-units, and with **b** money-units we purchase **b** mass-units of [B] at the first period and at the second $\frac{\mathbf{b}}{\beta_2'}$ mass-units. Assuming that the sums of these mass-quantities are the same at both periods, we have

$$\mathbf{a} + \mathbf{b} = \frac{\mathbf{a}}{\alpha_2'} + \frac{\mathbf{b}}{\beta_2'},$$

which gives us

$$\mathbf{b} = \frac{\mathbf{a}\,\beta_2'(\alpha_2' - 1)}{\alpha_2'(1 - \beta_2')}.$$

Now let us take **a** as a unit sum; then

$$\mathbf{b}' = \frac{\beta_2'(\alpha_2' - 1)}{\alpha_2'(1 - \beta_2')},$$

which means that for every sum of money spent on [A], the rising article, we must spend a so many times larger or smaller sum on [B], the falling article. Or if we represent the total sum of money to be spent by S, and the sum to be spent on [A] by s_a, we have

$$s_a \left\{ 1 + \frac{\beta_2'(a_2' - 1)}{a_2'(1 + \beta_2')} \right\} = S,$$

whence

$$s_a = \frac{Sa_2'(1 - \beta_2')}{a_2' - \beta_2'};$$

and representing the sum to be spent on [B] by s_b, we easily obtain

$$s_b = \frac{S\beta_2'(a_2' - 1)}{a_2' - \beta_2'}.$$

For example, suppose A rises in price from 1.00 to 1.99 and B falls from 1.00 to .99. Here all the employers of the above arguments would at once conclude that the level of prices has risen and money depreciated—the harmonic averagist amongst them. Yet it is possible with the same sums of money to get the same total quantity of the goods at both periods. For here

$$b' = \frac{.99(1.99 - 1)}{1.99(1 - .99)} = \frac{.9801}{.0199} = 49.25125.$$

Thus if we spend 100 money-units on [A] at each period we get at the first period 100 A and at the second 50.2512 A, and if we spend 4925.12½ money-units on [B] we get at the first period this quantity of B and at the second 4974.8737; and the total quantity of [A] and [B] bought at the first period is 5025.12½, and the total quantity of [A] and [B] bought at the second is the same.

It is plain that if at both periods we devote all our money to purchasing the article which has risen in price, we get the extreme diminution in the total quantity purchasable at the second period compared with the total quantity purchasable at the first; and if at both periods we devote all our money to purchasing the article which has fallen in price, we get the extreme augmentation in the total quantity purchasable at the second period compared with the total quantity purchasable at the first. Between these extremes it is evident, by the law of continuity, that there must be some distribution of our spendings which will give neither diminution nor augmentation in the total quantities. This is the distribution indicated by our formula. It is evident, further, that if we spend more on the article rising in price than the proportion indicated by the formula, we get a smaller total quantity at the second period, and more and more smaller as we depart from this proportion, up to the limit when we spend all on this article. And reversely if we spend more on the article falling in price than the proportion indicated, we get a larger total quantity at the second period, increasing up to the limit when we spend all on this article. Thus between the two limits there is an infinity of total quantities that may be purchased with exactly the same sums of money at both periods.

§ 5. In the second case, let the mass-unit used for each class be the same as in the preceding case, that is, the mass whose price at the first period is one money-unit, and again let the prices of these mass-units of [A] and [B] at the second period be α_2' and β_2' respectively; but let the number of the mass-units of [A] to be purchased at both periods be represented by x', and the number of the mass-units of [B] to be purchased at both periods be represented by y'. Then the sums of money needed to purchase x' A is at the first period x' money-units and at the second $x'\alpha_2'$ money-units; and the sum of money needed to purchase y' B is at the first period y' money-units and at the second $y'\beta_2'$ money-units. Assuming that the total sums of these sums are the same at both periods, we have

$$x' + y' = x'\alpha_2' + y'\beta_2',$$

which gives us

$$y' = \frac{x'(\alpha_2' - 1)}{1 - \beta_2'}.$$

And now again, if we take x' as a quantity-unit, we have

$$y'' = \frac{\alpha_2' - 1}{1 - \beta_2'},$$

which means that for every quantity of [A] we purchase we must purchase a so many times larger or smaller quantity of

[B] (or must spend so much more or less money on [B] than on [A] at the first period). Or if we represent the total quantity of both classes by Q, and the quantity to be purchased of [A] by q_x, we have

$$q_x\left(1 + \frac{\alpha_2' - 1}{1 - \beta_2'}\right) = Q,$$

whence

$$q_x = \frac{Q(1 - \beta'_2)}{\alpha_2' - \beta_2'};$$

and representing the quantity to be purchased of [B] by q_y, we easily obtain

$$q_y = \frac{Q(\alpha_2' - 1)}{\alpha_2' - \beta_2'}.$$

As these expressions give the quantities of equivalents of the first period, they also represent the sums that must be spent at the first period.

Thus in the above numerical example, in which A is supposed to rise in price from 1.00 to 1.99 and B to fall from 1.00 to .99, and in which the arithmetic averagist would probably see only a rise in the general level of prices, it is possible to get the same quantities of [A] and [B] at both periods. For here

$$y'' = \frac{1.99 - 1}{1 - .99} = \frac{.99}{.01} = 99;$$

that is, if we purchase 100 A at each of the periods, we spend for it 100 money units at the first period and 199 at the second; and if we purchase 9900 B at each of the periods, we spend for it 9900 money-units at the first period and 9801 at the second; and so at the first period we spend for the quantities of [A] and [B] together the total sum of 100 + 9900 = 10,000, and at the second the total sum of 199 + 9801 = 10,000, or in other words, for 10,000 money-units we can at each period purchase 100 A and 9900 B.

Here, too, it is plain that if at both periods we purchase only the article which rises in price, we have the extreme augmentation in the total sum of money needed to purchase the same quantity at the second period compared with the total sum needed to purchase it at the first; and if at both periods we purchase only the article which falls in price, we have the extreme diminution in the total sum of money needed to purchase the same quantity at the second period compared with the total sum needed to purchase it at the first. Between these extremes, again, it is evident, by the law of continuity, that there must be some distribution of our purchases which will require neither augmentation nor diminution in the total sum of money needed at the two periods to purchase the same quantities. This is the distribution indicated by our formula. Again it is evident that the further we depart from this distribution of our purchases by purchasing at both periods more of the article rising in price, the greater will be the total sum of money required at the second period compared with the first, up to the limit when we purchase only this article; and reversely, by purchasing at both periods more of the article falling in price the smaller will be the total sum needed at the second period compared with the first, up to the limit when we purchase only this article. Thus between the two limits there is an infinity of total sums that may be needed to purchase exactly the same quantities of articles at both periods.

Evidently no arguments have validity that rest merely on some possibilities—either on the potentiality of certain sums of money to purchase quantities of goods, or on the potentiality of certain quantities of goods to command sums of money.

III.

§ 1. Of the arguments as hitherto employed the defectiveness may be cured. It consists primarily in the neglect of weighting. Therefore we must first of all introduce into them consideration of weighting.

Disregard of weighting we have seen to be the fault with the objections that have been urged against the employment of any average as an indicator of constancy or variation of exchange-value. Some persons have played with the same price variations, differently representing them, and really representing them so as to express different weightings; and getting different results, have denounced the whole subject as intractable. Similar disregard of weighting has been the besetting sin in all the arguments for each of the averages, and the cause why none has been convincing.

Thus in the controversy above reviewed neither Jevons nor Laspeyres sought for any more data than the mere variations of prices. Their dispute was very much as if they posited that some men are six feet tall and others are five feet tall, and quarreled over the average tallness. Factors necessary for the solution of the problem were absent; yet they blissfully went on with the attempt to solve it. They each relied on various possibilities in the purchases, instead of requiring to have given, as real or as suppositional data, what were the actual purchases. Or if they did think they were agreeing upon the use of even weighting, they each conceived of this differently. Laspeyres supposed equal mass-quantities to be purchased of both articles, and constantly so at both periods (and also equal thaler's worths, but only at the first period). Jevons, when he suggested the harmonic average, supposed equal sums of money to be spent on both classes, and constantly so at both periods; but when he advocated the geometric average, he gave no hint what conception he had of even weighting. Such argumentation disparages the whole subject of exchange-value mensuration, and strengthens the hands of its opponents.

In general, writers on the subject have made arguments for the different averages without regard to weighting, and they have made arguments—if arguments they deserve to be called—for several kinds of weighting without regard to the averages. They have never combined them. Up to this point separate argumentation has been employed in this work also. We now need to make the combination.

§ 2. It is plain that in the argument made by the harmonic averagist the weighting is according to the constant sums of money that are supposed to be devoted at both periods to purchasing variable quantities in the different classes.

This being so, certain peculiarities arise, due to the other factor in this case, namely the sizes of the mass-units used, whose numbers are the mass-quantities between which compensation is desired by their equality. These peculiarities may here be briefly indicated.

Let x_1 and x_2, y_1 and y_2, ... represent the numbers of mass-units, whatever these be, of the classes [A], [B], ..., purchased at the first and at the second periods respectively with the constant sums **a**, **b**, ... The only conception we as yet have of the mass-quantities are represented by these symbols x_1, x_2, y_1, y_2, ... And so the argument from compensation by equal mass-quantities for constancy in the exchange-value of money, or for variation in this exchange-value by the variation in the total mass-quantities, calls for formulation as follows,

$$\frac{M_{02}}{M_{01}} = \frac{x_2 + y_2 + \cdots\cdots}{x_1 + y_1 + \cdots\cdots}, \qquad (1)$$

whence, by inversion,

$$\frac{P_2}{P_1} = \frac{x_1 + y_1 + \cdots\cdots}{x_2 + y_2 + \cdots\cdots}, \qquad (2)$$

But unless we specified that we had already selected the proper mass-units for this purpose, that is, if we only used the ordinary mass-units such as may be variously used by merchants, it is evident that in such formulæ the result would be variable according to the sizes of the mass-units we happened to use,[1] and we should be committing the same sort of absurdity as committed, from the other side, by Dutot.[2] We must add a restriction to these formulæ by including in them the method of selecting the mass-units. Let α_1 and α_2, β_1 and β_2, ... be the prices at the first and at the second periods respectively of the mass-units of [A], of [B], ... which we do happen to use. Then $x_1 = \frac{\mathbf{a}}{\alpha_1}$ and $x_2 = \frac{\mathbf{a}}{\alpha_2}$, $y_1 = \frac{\mathbf{b}}{\beta_1}$ and $y_2 = \frac{\mathbf{b}}{\beta_2}$, and so on. If now we converted the last formula into this,

[1] See Chapt. V. Sect. III. § 5.
[2] See Chapt. V. Sect. VI. § 3.

$$\frac{P_2}{P_1} = \frac{\dfrac{\mathbf{a}}{\alpha_1} + \dfrac{\mathbf{b}}{\beta_1} + \cdots\cdots}{\dfrac{\mathbf{a}}{\alpha_2} + \dfrac{\mathbf{b}}{\beta_2} + \cdots\cdots}, \qquad (3)$$

we should still have no single determinate result.[3] The principle of selecting the mass-units is still missing. A simple suggestion is that we should select mass-units that are equivalent. But then the questions arise, Equivalent at one of the periods only? and at which? or over both the periods together? Now without consideration, without offering a reason, without argument, people have agreed upon the convenient practice of employing mass-units that are equivalent at the first period. This being done, and in the still more convenient form of employing mass-units that are equivalent to the money-unit at that period, so that $\alpha_1' = \beta_1' = \cdots\cdots = 1$, the last formula reduces to

$$\frac{P_2}{P_1} = \frac{\mathbf{a} + \mathbf{b} + \cdots\cdots}{\dfrac{\mathbf{a}}{\alpha_2'} + \dfrac{\mathbf{b}}{\beta_2'} + \cdots\cdots},$$

or by using n'' to represent $\mathbf{a} + \mathbf{b} + \cdots\cdots$,

$$\frac{P_2}{P_1} = \frac{1}{\dfrac{1}{n''}\left(\mathbf{a}\dfrac{1}{\alpha_2'} + \mathbf{b}\dfrac{1}{\beta_2'} + \cdots\cdots\right)}, \qquad (4)$$

and, by restoring the price variations to their original forms,

$$\frac{P_2}{P_1} = \frac{1}{\dfrac{1}{n''}\left(\mathbf{a}\dfrac{\alpha_1}{\alpha_2} + \mathbf{b}\dfrac{\beta_1}{\beta_2} + \cdots\cdots\right)}. \qquad (5)$$

These two are formulæ (15, I) and (16, I) given in Chapter V. Section V. § 1, as the formulæ for the *harmonic* average of price variations with weighting according to **a**, **b**, $\cdots\cdots$.

This is why this argument, applied in this way, is an argument for the harmonic average of price variations. But there is no necessity for it to be applied in this way, for which no

[3] Cf. Chapt. V. Sect. VI. § 2.

reason has been offered, so as to become an argument specially for the harmonic average. Applied to other mass-units, it may become an argument for other averages.

Thus suppose we select mass-units that are equivalent to the money-unit at the second period. Then $\alpha_2' = \beta_2' = \cdots\cdots = 1$, and the above formula (3) reduces instead to

$$\frac{P_2}{P_1} = \frac{\frac{\mathbf{a}}{\alpha_1'} + \frac{\mathbf{b}}{\beta_1'} + \cdots\cdots}{\mathbf{a} + \mathbf{b} + \cdots\cdots},$$

or, by using n'' to represent $\mathbf{a} + \mathbf{b} + \cdots\cdots$, and restoring the price variations to their simpler forms,

$$\frac{P_2}{P_1} = \frac{1}{n''}\left(\mathbf{a}\frac{\alpha_2}{\alpha_1} + \mathbf{b}\frac{\beta_2}{\beta_1} + \cdots\cdots\right), \tag{6}$$

which is formula (16, *i*) given in Chapter V. Section V. § 1, as the formula for the *arithmetic* average of price variations with weighting according to **a**, **b**, ...

Or again we might equally well, for all we have as yet heard to the contrary, and perhaps better, select mass-units that are equivalent over both the periods compared. The method of getting these has been examined in Chapter IV. They are such that the geometric means of their prices are equal. This suggests some relationship with the geometric mean or average of price variations. Now we do not find any exact connection here with the geometric *average*—that is, when we are dealing with several classes or with uneven weighting. But when we deal with only two classes evenly weighted (that is, in this case, equally large at each period), so as to be able to employ the geometric *mean* of the price variations, the connection is perfect. For now we have $\mathbf{a} = \mathbf{b}$, wherefore we may represent them each as 1; and from the nature of the mass-units (because of whose peculiarity we may distinguish their prices and mass-quantities by doubly priming them) we have $\alpha_1''\alpha_2'' = \beta_1''\beta_2''$, whence $\alpha_1'' = \frac{\beta_1''\beta_2''}{\alpha_2''}$ and $\alpha_2'' = \frac{\beta_1''\beta_2''}{\alpha_1''}$. By substituting the first of these values in formula (3) we get

$$\frac{P_2}{P_1} = \frac{\frac{\alpha_2''}{\beta_1''\beta_2''} + \frac{1}{\beta_1''}}{\frac{1}{\alpha_2''} + \frac{1}{\beta_2''}} = \frac{\frac{\alpha_2'' + \beta_2''}{\beta_1''\beta_2''}}{\frac{\beta_2'' + \alpha_2''}{\alpha_2''\beta_2''}} = \frac{\alpha_2''}{\beta_1''}, \tag{6}$$

and by substituting the second,

$$\frac{P_2}{P_1} = \frac{\frac{1}{\alpha_1''} + \frac{1}{\beta_1''}}{\frac{\alpha_1''}{\beta_1''\beta_2''} + \frac{1}{\beta_2''}} = \frac{\frac{\beta_1'' + \alpha_1''}{\alpha_1''\beta_1''}}{\frac{\alpha_1'' + \beta_1''}{\beta_1''\beta_2''}} = \frac{\beta_2''}{\alpha_1''}. \tag{7}$$

Hence we have, together,

$$\frac{P_2}{P_1} = \frac{\alpha_2''}{\beta_1''} = \frac{\beta_2''}{\alpha_1''},$$

whence

$$\frac{P_2}{P_1} = \frac{\alpha_2'' + \beta_2''}{\alpha_1'' + \beta_1''}, \tag{8}$$

which we should also have obtained directly, had we simultaneously substituted the two values.[4] (Here in parenthesis we may notice that, as $x_1''\alpha_1'' = x_2''\alpha_2'' = \mathbf{a} = 1$ and $y_1''\beta_1'' = y_2''\beta_2'' = \mathbf{b} = 1$, we have $\alpha_1'' = \frac{1}{x_1''}$, $\alpha_2'' = \frac{1}{x_2''}$, $\beta_1'' = \frac{1}{y_1''}$, and $\beta_2'' = \frac{1}{y_2''}$; wherefore by substituting these values in the formulæ (6), (7), and (8) we get

$$\frac{P_2}{P_1} = \frac{y_1''}{x_2''} = \frac{x_1''}{y_2''} = \frac{x_1'' + y_1''}{x_2'' + y_2''}, \tag{9}$$

the last part of which we knew already, according to the hypothesis.) Now from the above combination of formulæ (6) and (7) we also form

$$\left(\frac{P_2}{P_1}\right)^2 = \frac{\alpha_2''}{\beta_1''} \cdot \frac{\beta_2''}{\alpha_1''},$$

whence we draw

$$\frac{P_2}{P_1} = \sqrt{\frac{\alpha_2''}{\alpha_1''} \cdot \frac{\beta_2''}{\beta_1''}}, \tag{10}$$

[4] This is Dutot's method, which, therefore, is rational in this one special case.

which is the formula for the *geometric mean* of the price variations of the two classes, with even weighting, being like the formula (14, 3) in Chapter V. Section IV. § 2.

Thus, when attention is paid to the weighting, this argument from compensation by equal mass-quantities turns out to be still an argument equally well, as yet, applicable either to the harmonic or to the arithmetic means or averages or to the geometric mean. Therefore in order to decide what average or mean it really favors,—or if it favors the geometric mean in the case of two equally important classes, what is the method it favors in the case of many variously important classes,—we must pay attention not only to the weighting, but also to the mass-units it is proper to use. We must search for a reason why one set of mass-units is to be preferred, conducting an investigation which has hitherto been neglected.

§ 3. In the argument made by the arithmetic averagist the weights cannot be according to the constant mass-quantities that are purchased at each period by variable sums of money (unless we know some way of properly selecting the mass-units for this purpose), nor can they be merely according to the sums of money spent on them, because these are different at the two periods. Yet they must be somehow connected with these sums of money, as we have already examined in Chapter IV. The formula which represents the conditions to which this arguments is applied and expresses its treatment of them is easily seen to be either this,

$$\frac{P_2}{P_1} = \frac{\mathbf{a}_2 + \mathbf{b}_2 + \cdots\cdots}{\mathbf{a}_1 + \mathbf{b}_1 + \cdots\cdots}, \qquad (11)$$

or, more definitely,

$$\frac{P_2}{P_1} = \frac{x\alpha_2 + y\beta_2 + \cdots\cdots}{x\alpha_1 + y\beta_1 + \cdots\cdots}, \qquad (12)$$

in which x, y, ⋯⋯ represent the numbers of times the ordinary commercial mass-units of [A], [B], ⋯⋯, whose prices are α_1 and α_2, β_1 and β_2, ⋯⋯, at the first and second periods respectively. Now this last is the formula for Scrope's method, discussed in Chapter V. Section VI. § 4, and there shown to represent, or to be identical with, the *arithmetic* mean or average of the price variations with weighting according to the sums of money spent on the constant mass-quantities in the different classes at the *first* period. But that analysis was not complete. We shall later find that this formula equally well represents the *harmonic* average or mean of the price variations with weighting according to the sums spent at the *second* period, and again, in some cases (namely when we are dealing with only two equally important classes), the *geometric mean* with (even) weighting according to the geometric mean of the sums at *both* the periods.

Thus, again in this case, with attention paid to the weighting, the argument is still applicable as well either to the arithmetic or to the harmonic means or averages or to the geometric mean. Here, however, we are no longer bothered by the question of the mass-units to be used; and yet we shall find that the selection of the mass-units has had something to do with producing the appearance of this argument being more specially in favor of the arithmetic average. Here, too, we shall see that the question is not between different kinds of averages yielding different results, but between three different interpretations of, or three different ways of paralleling, one result yielded by one method.

§ 4. Thus in general we have, not an argument for the harmonic average and an argument for the arithmetic average of price variations, as their employers have hitherto conceived them to be, but two arguments for either of the three averages, or means, applicable the one to one state of things and the other to another,—between which arguments, therefore, when rightly confined each to its own field, there cannot even be contradiction, or antagonism.

The fields to which these arguments are by their own natures confined remind us of the first two divisions in the question of weighting which we discussed in Chapter IV. Section V. But those two divisions in the question of weighting we recognized to be incomplete. Similarly these two arguments, corresponding to those two divisions, are incomplete. The argument made by the harmonic averagist supposes that we spend the same sums of money on every class at both periods in spite of the variations in their prices, which we rarely, if ever, do. The argument made by the arithmetic averagist supposes that we buy the same quantities of every class at both periods in spite of the variations in their prices, which we rarely, if ever, do. As a rough proposition, we—a community—generally spend more on articles that have risen in price and get less of them, and spend less on articles that have fallen in price and get more of them. At all events, we almost always spend our money, and buy goods, in different proportions at any two periods. Thus the more usual state of things is neglected by the arguments as made by the harmonic and the arithmetic averagists, except in a possible supplement to each argument by which the more complex state is reduced by curtailment to the one or to the other of the states required by these arguments. Then, as already remarked, the arguments become antagonistic; but, so used, neither will have much claim for our respect.

Now the third—the more usual, the more complex—state of things is really made up of each of the other two, in this way:—In the first state of things above described the particular mass-quantities are different at the different periods, and in the second state of things the particular sums of money are different at the different periods; and this double difference is precisely what exists in the more complex state.

Hence, even though the two states of things required in the two arguments are not likely ever to happen in reality, it is well for us to examine these arguments thoroughly, just as if their states were likely to occur; because after reaching the right method for each of these states, we shall be in a position to combine them, and so form the right universal method, applicable to the complex states which generally exist.

Thus our future work is mapped out for us. We must examine the two arguments separately, each applied to its own state of things; and then we shall seek what can be united of the two for the complex state. The three divisions of weighting, which we previously disposed of, so far as then possible, in one Section of one Chapter, now become divisions in the question of the averages, will occupy the next three Chapters.

CHAPTER X.

THE METHOD FOR CONSTANT SUMS OF MONEY.

I.

§ 1. Having examined in a general way the argument from compensation by equal mass-quantities, we must now examine it in detail, applying it to many particular examples, for the purpose of discovering in them a principle that will give us a clear indication of the true average or mean or method for measuring the constancy or variation in the exchange-value of money in the cases when constant sums of money are spent on every class at both the periods compared.[1] Also we may look for some crucial instances, serviceable as tests, that shall render our conclusions demonstrative. We must survey first of all the different ways in which the argument is applicable to the different averages, or means, in order that we may later be in a position to judge in which application the argument is valid.

By compensation by equal mass-quantities it cannot be meant to judge things by actual masses, counted in any of the usual weights or measures; but the idea is to compare things by proportions of masses. If, for example, A, purchasable at the first period with one money-unit, be a quarter of barley, and B, likewise then purchasable with a money-unit, be a bushel of wheat, a loss of one third on [A] at the second period is a loss of $2\frac{2}{3}$ bushels of barley, while a gain on [B] by a third is a gain of

[1] All that is said in this Chapter may be extended also to cases in which at both periods there are expended on all the classes sums in the same proportion, whether at the second period they be all smaller or all larger than at the first, *provided* all the reasoning (and all the formulæ later to be described, except two) be applied only to the sums of the one or of the other period—or, theoretically better, only to what is common to both periods. For the sums in excess, all being in the same proportion, have no influence to alter the exchange-value of money already determined.

only $\frac{1}{3}$ bushel of wheat. With two money-units evenly distributed we could purchase at the first period 1 A + 1 B, or 9 bushels of grain, and at the second $\frac{2}{3}$ A + $\frac{4}{3}$ B, or $6\frac{2}{3}$ bushels of grain—a loss in bulk. Or if A be one pound of copper and B ten pounds of iron, a loss of $\frac{1}{3}$ on [A] is a loss of $\frac{1}{3}$ pounds of copper, and a gain of $\frac{1}{3}$ on [B] is a gain of $3\frac{1}{3}$ pounds of iron, so that from purchasing 11 pounds of metal our two money-units evenly distributed will come to purchase 14 pounds of metal—this time a gain in weight. Thus even the arithmetic equality in the compensation offered by the harmonic system disappears as a compensation by quantities of masses literally taken—or rather, taken at haphazard. The equality is systematically obtained only by treating the masses of [A] and [B], whatever they be, purchasable with the same sum of money at some period or periods, as equal because they are equivalent. That is, these masses are conceived as equal, not as weights or capacities, but as exchange-values. For then the equal mass-quantities are conceived as composed of equal numbers of such equivalent individuals. A loss of one third on one article is supposed to be compensated by a gain of one third on another, even though this one third be but a small fraction by weight or bulk of the other one third, or even though it far exceed the other in weight or bulk, provided it be as much more or less valuable as it is physically smaller or larger, so as to be equivalent to the other, its greater preciousness making up for its lack of weight or bulk, or reversely,—at some period or periods. The argument is really in its proper form when it passes from the quantities purchasable to the powers of purchasing them, and claims that a loss of a third in the purchasing power of money over [A] (or its exchange-value in [A]) is to be compensated by a gain of a third in the purchasing power of money over [B] (or its exchange-value in [B]). But as we measure particular exchange-values or purchasing powers of money by the quantities of the things a constant sum will purchase, we may continue to treat of the compensation by direct reference to the quantities.

Now as we are dealing only with the proportions of the quantities, we have three ways of conceiving of the proportions, when avowedly dealing with equally important classes :—either (1) as proportions of variation *from* an equal condition at the first period, or (2) as proportions of variation *to* an equal condition at the second period, or (3) as proportions of variation, in any two classes, in the one *from* a certain condition to another and in the other from the latter condition *to* the former. The same three positions are obtained by supposing the arithmetic compensation by equal quantities to be in equal numbers of mass-units (ideally constructed for the purpose) that are equivalent (1) at the *first* period, or (2) at the *second* period, or (3) over *both* the two periods together. It is more usual to adopt the first of these methods of measurement and this is the reason why this argument from equal mass-quantities has seemed to be an argument specially favoring the harmonic average of price variations. But we must examine all three of these methods in turn.

§ 2. Using mass-units that are *equivalent at the first period*, we may construct the following schemata illustrative of the conditions when there is compensation by equal numbers of such mass-units. We may still at first confine our attention to two classes supposed to be equally large in some respect. In the schemata are supposed to be expended at each period, marked I and II, certain constant sums, which are stated on the right-hand side. On the left are stated the mass-quantities, that is, the numbers of these mass-units purchasable with these sums at their prices, also stated, at each period ; and on their right, or in the middle, are added the sums or totals of these mass-quantities. Thus, on mass-units equivalent at the first period we have compensation by equal mass-quantities when the price variations are to the simple harmonic extremes, as follows :

I	100 A @ 1.00	100 B @ 1.00	— 200	100 for [A]	100 for [B],
II	66⅔ A @ 1.50	133⅓ B @ .75	— 200	100 for [A]	100 for [B];

when the price variations are to the simple arithmetic extremes, as follows :

I	100 A @ 1.00	33⅓ B @ 1.00	— 133⅓	100 for [A]	33⅓ for [B],
II	66⅔ A @ 1.50	66⅔ B @ .50	— 133⅓	100 for [A]	33⅓ for [B];

when the price variations are to the simple geometric extremes, as follows:

I	100 A @ 1.00	66⅔ B @ 1.00	—166⅔	100 for [A] 66⅔ for [B],
II	66⅔ A @ 1.50	100 B @ .66⅔	—166⅔	100 for [A] 66⅔ for [B].[1]

Here we have the peculiar features with which we are already familiar. In the first the numbers of mass-units are equal at the first period, and the compensation is by arithmetically equal variations in them—over equal distances away from this equal condition. In the second the numbers of mass-units are equal at the second period, and the compensation is by harmonically equal variations in them—over equal distances going toward this equal condition. In the third the numbers of the mass-units alternate and change places, traversing not only equal distances, but, so to speak, the same road, so that in them the compensation is by geometrically equal variations.

The universality of these relations existing in the first of these particular examples has been demonstrated near the end of the preceding Chapter. They all admit of demonstration by means of one of the formulæ discovered earlier in the same Chapter—in § 4 of Section II. In the formulation there made the mass-units were supposed to be equivalent at the first period, so that it is applicable here. Let the price of A always be supposed to rise from 1.00 to a_2'. Then if the price of B falls from 1.00 to the harmonic extreme, it will fall to $\frac{a_2'}{2a_2' - 1}$; if to the arithmetic extreme, it will fall to $2 - a_2'$; if to the geometric extreme, it will fall to $\frac{1}{a_2'}$. Supplying these values of β_2' in the formula

[1] Here in all the schemata the differences in the numbers of A and of B are the same, and the totals are different. Arrangement can be made so that the totals would be the same, as follows for the second and third schemata:

I	150 A @ 1.00	50 B @ 1.00	– 200	150 for [A] 50 for [B],
II	100 A @ 1.50	100 B @ .50	200	150 for [A] 50 for [B];

I	120 A @ 1.00	80 B @ 1.00	200	120 for [A] 80 for [B],
II	80 A @ 1.50	120 B @ .66⅔	– 200	120 for [A] 80 for [B].

But here the differences are different. The arrangement employed in the text is more perspicuous.

$$\mathbf{b}' = \frac{\beta_2'(\alpha_2' - 1)}{\alpha_2'(1 - \beta_2')},$$

we find, when the price variations are to the harmonic extremes,

$$\mathbf{b}' = 1 ;$$

when they are to the arithmetic extremes,

$$\mathbf{b}' = \frac{2 - \alpha_2'}{\alpha_2'} ;$$

when they are to the geometric extremes,

$$\mathbf{b}' = \frac{1}{\alpha_2'} .$$

The first of these expressions means that when the price variations of two classes are to the opposite harmonic extremes, in order to have compensation by equal numbers of mass-units equivalent at the first period we must spend our money evenly on the two classes at both periods; wherefore we must purchase at the first period equal numbers of their mass-units, and at the second period $\frac{1}{\alpha_2'}$ A and $\frac{2\alpha_2' - 1}{\alpha_2'}$ B, which are arithmetic extremes around 1, since half their sum is 1. The second means that when the price variations are to the opposite arithmetic extremes, in order to have compensation by equal numbers of such mass-units we must spend at both periods for every 1 M on [A] $\frac{2 - \alpha_2'}{\alpha_2'}$ M on [B]; wherefore the numbers of the mass-units purchased at the first period are in these proportions, and at the second period they are $\frac{1}{\alpha_2'}$ A and $\frac{\frac{2 - \alpha_2'}{\alpha_2'}}{2 - \alpha_2'} = \frac{1}{\alpha_2'}$ B, that is, they are equal numbers at this period; and now $1 - \frac{1}{\alpha_2'} = \frac{1}{\alpha_2'} - \frac{2 - \alpha_2'}{\alpha_2'}$, which shows that the numbers at the first period are arithmetic extremes around the common number at the second. The third means that when the price variations are to the opposite geometric extremes, in order to have compensation by equal numbers of such mass-units, we must spend at both periods, for every 1 M on [A] $\frac{1}{a_2'}$ M on [B]; wherefore the numbers of the mass-units purchased at the first period are in these proportions, and at the second they are $\frac{1}{a_2'}$ A and $\frac{\frac{1}{a_2'}}{\frac{1}{a_2'}} = 1$ B, so that these numbers alternate over the two periods.[3] We may therefore use the above particular examples, and the relations found in them, as universally illustrative.

Now if on the price variations in them supposed we employ the harmonic average on the first, the arithmetic on the second, and the geometric on the third, all with even weighting, we get in every instance an indication of constancy. But we have no right to use even weighting in every case. The only reason we can find for using even weighting in all these cases is that in all of them the same numbers of these mass-units are purchased at some period or periods,—in the first at the first period, in the second at the second, in the third at each of the periods alternately. But of course there is nothing to recommend such a combination of weighting and of averaging, it being remembered that these mass-units are equivalent only at the first period.[4]

[3] In this last case if the total sum to be spent on the two classes is 2.00, we find that at both the periods we must spend $\frac{2\,a_2'}{a_2'-1}$ M on [A] and $\frac{a_2'}{a_2'-1}$ M on [B], getting these numbers of A and B at the first period, and at the second the reverse. These figures, rather curiously, are the harmonic means, the first between 1 and a_2', the second between 1 and $\frac{1}{a_2'}$ $(=\beta_2')$. They are also arithmetic extremes around 1. (Thus the *harmonic* means between unity and *geometric* extremes around unity are *arithmetic* extremes around unity.)

[4] Another identity in the results deserves notice. In all three cases if we use the harmonic average with weighting in each case according to the numbers of these mass-units at the first period, or if we use the arithmetic average with weighting in each case according to the numbers of these mass-units at the second period, or if we use the geometric *mean* with weighting according to the geometric means of the numbers of these mass-units at both periods (whenever it happens that these are equal), all these means (and the first two averages in all cases) always give, applied to the same cases, identically the same results, (and in the more complex cases the geometric average, with its weighting, generally gives

The only system of weighting required by this argument from compensation by equal quantities is the system of weighting according to the constant sums devoted to purchasing each class at both periods. There is no use claiming that because the mass-quantities would then be different, we ought to correct this difference in the two worlds compared by reducing the mass-quantities either by taking only the smaller quantity in any class at either period (so as to get the largest quantity common to both the periods) or by taking some average of them. For then the sums paid for such reduced or averaged constant quantities would be different, and the two worlds would be no more alike than before. The truth is, we are now engaged in measuring the variation in the purchasing power (exchange-value) of given sums of money by the variations—not of prices (except as these indicate the others)—but of the mass-quantities purchased. Hence the weighting is not to be according to the mass-quantities, but according to the sums devoted to purchasing them. The economic worlds we are considering are really made up of these sums, which are supposed to be the same at both periods; and what we are measuring is the variation in their purchasing powers. In the next Chapter, when we have under examination the argument from compensation by equal sums, the economic worlds will be made up of the mass-quantities (but still to be conceived as exchange-values), the variables then being the sums they will command, indicative of their varying exchange-values in money and of money's inversely varying exchange-values in them.

Here, then, the weighting being according to the constant sums expended on each class at both periods, it is only the first of the above schemata in which even weighting can be used. In the second the weighting is 1 for [A] and $\frac{1}{3}\left(=\frac{2-a_2'}{a_2'}\right)$ for [B];

very nearly the same results as the other two with theirs). Here in the first two averagings the *periods* of the weightings are inverted from those given as general principles in Chapt. VIII. Sect. I. § 7. This is because the variations of the mass-quantities are the inverse of the price variations. A similar identity will occupy our attention in the next Chapter (where the periods of the weightings are the proper ones, and the weighting itself is proper). At present we have little interest in these relations.

and in the third it is 1 for [A] and ⅔ $\left(=\frac{1}{a_2'}\right)$ for [B]. Now if we use the harmonic average on each of these cases with these weightings, we again always get an indication of constancy. But if we use the other averages with these weightings, we get very different results. The arithmetic average, in each case with its proper weighting, indicates for the first a rise of 12½ per cent., for the second a rise of 25 per cent., and for the third a rise of 16⅔ per cent. And the geometric average, in each case with its proper weighting, indicates for the first a rise of 6.066 per cent., for the second a rise of 14 per cent., and for the third a rise of 8.44 per cent.

As neither of these other two systems pretends to claim constancy when the proper weighting is used with each average, it is not easy to compare the different averages on these schemata. To compare them we shall want rather the schemata in which equal constant sums are spent on every class, so that even weighting may be used in each case, wherefore each average, the price variations being as before, will indicate constancy.

§ 3. Let us now notice the same price variations with equal numbers of mass-units gained and lost when these are *equivalent at the second period.* It is easy to adapt the preceding schemata to these new mass-units. We need to change only the mass-unit of [B], which we shall represent by B′. It must be remarked that we are not rearranging the presentation of the same facts as in the previous schemata, but we are presenting different facts. The schema for the harmonic price variations is

I	100 A @ 1.00	100 B′ @ 2.00 — 200	100 for [A]	200 for [B],
II	66⅔ A @ 1.50	133⅓ B′ @ 1.50 — 200	100 for [A]	200 for [B];

that for the arithmetic price variations,

I	100 A @ 1.00	33⅓ B′ @ 3.00 — 133⅓	100 for [A]	100 for [B],
II	66⅔ A @ 1.50	66⅔ B′ @ 1.50 — 133⅓	100 for [A]	100 for [B];

that for the geometric price variations,

I	100 A @ 1.00	66⅔ B′ @ 2.25 — 166⅔	100 for [A]	150 for [B],
II	66⅔ A @ 1.50	100 B′ @ 1.50 — 166⅔	100 for [A]	150 for [B].

Here the varying numbers of the mass-units are the same as in the previous schemata,[5] but their prices being variously higher in each case, the constant sums expended on [B] are in each case variously different. That the above noticed peculiarities in regard to the numbers of the mass-units in this rearrangement are universal, could easily be proved in a manner similar to that before used, by finding first the general formula for the cases when, β_2 equalling a_2, β_1 is the figure that has varied harmonically, arithmetically or geometrically,[6] and applying this as before.

Here again the harmonic average applied to the first case, the arithmetic applied to the second, the geometric applied to the third, each with even weighting, all indicate constancy of the general exchange-value of money. But in this schematization it is only the second case, in which the price variations are arithmetic, that has a right to the use of even weighting. In the first case the weighting is 1 for [A] and 2 for [B]; and with this weighting the harmonic average indicates a fall of prices by 10 per cent. And in the third the weighting is 1 for [A] and 1½ for [B]; and with this weighting the geometric average indicates a fall of prices by 7.79 per cent. But, always used with the proper weighting, the arithmetic average indicates constancy in every case. Therefore also these schemata are not suitable for comparing the averages; and when we readapt them all to the same (even) weighting only the second will remain.

§ 4. Lastly we wish to schematize the compensation by equal mass-quantities when, in the same price variations, the mass-units are *equivalent over both the periods together*. As A is supposed in all the cases to rise from 1.00 to 1.50, its (geometric) mean price is always 1.2247. Therefore the (geometric) mean price of B″ is desired always to be 1.2247. This is obtained in the following schemata—for the harmonic price variations:

[5] Hence what is stated in the last note is still applicable also to the cases with *these* mass-units. And the numbers of these mass-units are no better criteria of weighting than the numbers of the mass-units there used.

[6] This formula, with a_1' at 1.00 and with a taken as the unit sum, is

$$b' = \frac{\beta_1 \ (a_2' - 1)}{\beta_1 - \beta_2},$$

in which $\beta_1 = a_1'$

I	100	A @ 1.00	100	B″ @ 1.4142 — 200	100 for [A]	141.42 for [B],
II	66⅔	A @ 1.50	133⅓	B″ @ 1.0606 — 200	100 for [A]	141.42 for [B],

(in which 1.4142 : 1.0606 :: 1.00 : 0.75, and 1.4142 × 1.0606 = 1.00 × 1.50); for the arithmetic price variations:

I	100	A @ 1.00	33⅓	B″ @ 1.732 — 133⅓	100 for [A]	57.73 for [B],
II	66⅔	A @ 1.50	66⅔	B″ @ .866 — 133⅓	100 for [A]	57.73 for [B],

(in which 1.732 : 0.866 :: 1.00 : 0.50, and 1.732 × 0.866 = 1.00 × 1.50); for the geometric price variations:

I	100	A @ 1.00	66⅔	B″ @ 1.50 — 166⅔	100 for [A]	100 for [B],
II	66⅔	A @ 1.50	100	B″ @ 1.00 — 166⅔	100 for [A]	100 for [B].

Here also the numbers of the mass-units are the same in these still other circumstances as in the preceding two sets of schemata;[7] but with still other constant sums expended on [A] and [B]. By the same method of proof the above-noticed peculiar relations between these numbers can be proved to be universal.[8] Of course, as before, with even weighting, the harmonic average of the price variations in the first case, the arithmetic average of them in the second, and the geometric *mean* of them in the third, all indicate constancy. But here it is only in the third that even weighting is proper. In the others, each average with its proper weighting, gives a various result—the harmonic in the first indicating a fall of prices by 5.41 per cent., and in the second the arithmetic indicating a rise of prices by 13.33 per cent. Therefore again we must re-adapt the schemata, only the third here given being serviceable. It may be added that in the other two cases, with their proper weighting, the geometric *average* yields results indicating slight divergences from constancy (in the first a fall of prices by 0.06 per cent., in the second a rise by 0.3 per cent.). The reason for these divergences is already known. Their meaning will be explained later.

[7] Hence again the statement in Note 4 is applicable also to the cases with *these* mass-units. But now the numbers of these mass-units are good criteria of weighting, as we shall see presently.

[8] The general formula for the relation between the sums, still with a_1' at 100 and with **a** as the unit sum, is

$$\mathbf{b}' = \frac{a_2' - 1}{\beta_1 - \beta_2},$$

in which $\beta_1 \beta_2 = a_2'$.

II.

§ 1. When the argument from compensation by equal mass-quantities is made an argument especially for the harmonic average of price variations, and the harmonic price variations, that are supposed to represent constancy are contrasted with the other variations in a manner detrimental to the argument for them, we have, for the simplest cases, which use even weighting throughout, the following schemata—for the harmonic price variations:

I	100 A @ 1.00	100 B @ 1.00—200	100 for [A]	100 for [B],
II	66⅔ A @ 1.50	133⅓ B @ .75—200	100 for [A]	100 for [B];

for the arithmetic price variations:

I	100 A @ 1.00	100 B @ 1.00—200	100 for [A]	100 for [B],
II	66⅔ A @ 1.50	200 B @ .50—266⅔	100 for [A]	100 for [B];

for the geometric price variations:

I	100 A @ 1.00	100 B @ 1.00 —200	100 for [A]	100 for [B],
II	66⅔ A @ 1.50	150 B @ .66⅔—216⅔	100 for [A]	100 for [B].

Here, the mass-units all being equivalent at the first period, there is compensation by equal mass-quantities (equal numbers of these mass-units) only in the case of the harmonic price variations, so that it is only in this case that the purchasing power or exchange-value of money seems to be constant, wherefore the harmonic average seems to be the right one, as it alone indicates constancy in this case; while in the other price variations the compensation by mass-quantities is, in the arithmetic price variations, for every one third lost on [A] a gain of one whole on [B], which seems very much too much, and in the geometric price variations for every loss of one third on [A] a gain of one half on [B], which still seems too much, wherefore the arithmetic and the geometric averages or means seem to be wrong because each in its own case, with the proper weighting, indicates constancy. It is wholly and solely on account of this special arrangement of the mass-quantities, due to the selection of the mass-units, which are equivalent at the first period, that some persons have been led to suggest the harmonic average of price variations as the right one.

But it is precisely because of this special arrangement that the harmonic average may be proved to be wrong.

§ 2. The argument for the harmonic average claims that, the classes [A] and [B] being constantly equally large or important over both the periods, if A and B, the equivalent mass-units at the first period, are equally precious at the first period (being equally heavy or bulky), a loss of one third on [A] is correctly compensated by a gain of the same mass (by weight or by bulk) on [B]; or if they are not equally precious at the first period, a loss of one third on [A] is correctly compensated by a gain in the same proportion, namely by one third, on [B], this addition of $\frac{1}{3}$ B being as much larger or smaller than $\frac{1}{3}$ A as B was less or more precious than A *at the first period*. Evidently there is here a tacit assumption, which is belied by the very supposition itself, of continuance at the second period of the same relative preciousness as at the first period. A and B are supposed to be equivalent at the first period. Then $\frac{1}{3}$ A and $\frac{1}{3}$ B are equivalent at the first period. Therefore at the first period if we distribute our purchases so as to get with two money-units $\frac{2}{3}$ A and $1\frac{1}{3}$ B instead of 1 A and 1 B, we have perfect compensation in the gain of $\frac{1}{3}$ B making up for the loss of $\frac{1}{3}$ A, because this gain is equivalent to this loss. But at the second period the supposition is that A, having risen in price to 1.50, while B has sunk to .75, has become more valuable than B—in fact, just twice as valuable in this example. Therefore $\frac{1}{3}$ B is no longer equivalent to $\frac{1}{3}$ A, and the gain of $\frac{1}{3}$ B is no longer sufficient to compensate for the loss of $\frac{1}{3}$ A.

Or let us put the question as one concerning purchasing power. If at the first period, possessing two money-units, we give up one third of our purchasing power over [A] by using only $\frac{2}{3}$ M to purchase [A], and if at the same time, without any intervening changes of prices we gain one third in our purchasing power over [B] by using the $\frac{1}{3}$ M saved from use on [A], there is perfect compensation; for the counter-balancing purchasing powers are equal. But the supposition we are dealing with is that at the second period we can get only $\frac{2}{3}$ A with 1 M, and it is claimed that the compensation is good if we can get $1\frac{1}{3}$ B with the other 1 M. Now under these circumstances it is true that the particular exchange-value of M in [A], or its particular purchasing power over [A], has fallen by one third, and that the particular exchange-value of M in [B], or its particular purchasing power over [B], has risen by one third, and so, if we were dealing with particular purchasing powers, this compensation might appear to be good. But we are really dealing with the general exchange-value of M, and so with its general purchasing power. This has not fallen by one third because of M's purchasing only $\frac{2}{3}$ A, unless M also purchases only $\frac{2}{3}$ of everything else; nor has it risen by one third because of M's purchasing $1\frac{1}{3}$ B, unless M also purchases $1\frac{1}{3}$ of everything else —which conditions are contradictory to each other and to the original supposition. We do not then as yet know what the compensation ought to be. But we do perceive this, that when A alone rises in price and B alone falls in price, M in gaining $\frac{1}{3}$ in quantity of [B], which has fallen in exchange-value, has gained less in exchange-value than it has lost in losing $\frac{1}{3}$ in quantity of [A], which has risen in exchange-value. The compensation offered is no longer an addition of one third of an equal exchange-value or purchasing power in place of a subtraction of one third of a given purchasing power; it is the addition of one third of a smaller-grown purchasing power in place of a subtraction of one third of a larger-grown purchasing power.

In such cases, therefore, our money has fallen in purchasing power or exchange-value—it has depreciated. And inversely the general level of prices has risen. The fall in the price of B to the harmonic term (a fall by $\frac{1}{4}$) is not great enough to compensate for the rise in the price of A (a rise by $\frac{1}{2}$). And the harmonic average of the price variations is wrong in indicating constancy.

The fallacy in the argument for the harmonic average of prices from compensation by equal mass-quantities is that it employs a compensation which is good only on the assumption that the relationship between the exchange-values or preciousness of the articles has remained the same at the second as at the first period, although the data argued upon preclude this continuance (except in case there are no irregular price variations).

§ 3. On the other hand when the argument from compensation by equal mass-quantities is applied to the arithmetic average of price variations, there is a similar fallacy reversed. The argument now tacitly assumes that the conditions existing at the second period existed also at the first; for it employs the compensation which is good only at the second period.

Thus in the above schema for the arithmetic price variations, where A and B are equivalent at the first period, the compensation offered is a gain of one whole A for every one third B lost, the gain being three times as large as the loss, measured in those mass-units. But we perceive that at the second period when the price of A has risen to 1.50 and the price of B fallen to .50, A has come to be, at the second period, three times as valuable as B, wherefore, at the second period itself, a compensation by a gain of three times as much of B as is lost of A is the proper compensation then. At this second period we are compensated by getting just as many times as much more of the fallen article than less of the risen as the former has become less valuable than the risen.[1] Here is a semblance of correctness in the position of the arithmetic averagist. The semblance is brought out more plainly by the following rearrangement of the schemata, in which merely the mass-units of [B] are altered, and consequently the numbers of them purchasable with the same sums of money, the facts represented being the same as before. The schema for the harmonic price variations is:

I	100 A @ 1.00	50 B′ @ 2.00 — 150	100 for [A]	100 for [B],
II	66⅔ A @ 1.50	66⅔ B′ @ 1.50 — 133⅓	100 for [A]	100 for [B];

that for the arithmetic price variations:

I	100 A @ 1.00	33⅓ B′ @ 3.00 — 133⅓	100 for [A]	100 for [B],
II	66⅔ A @ 1.50	66⅔ B′ @ 1.50 — 133⅓	100 for [A]	100 for [B];

[1] The universality of this relationship, given conditions permitting of even weighting, is evident when we remember that the mass-quantities are in harmonic progression; for in this progression around unity as the mean we know that $1 - a : b - 1 :: a : b$.

that for the geometric price variations:

I	100 A @ 1.00	$44\frac{4}{9}$ B′ @ 2.25 — $144\frac{4}{9}$	100 for [A]	100 for [B],
II	$66\frac{2}{3}$ A @ 1.50	$66\frac{2}{3}$ B′ @ 1.50 — $133\frac{1}{3}$	100 for [A]	100 for [B];

among which it is only the arithmetic price variations that gives compensation by arithmetically equal numbers of these mass-units; while in the harmonic price variations the compensation seems to be very much too small (although it is really the same as before, and proper at the first period itself), and in the geometric still too small; wherefore now, by a mere change in the size of the mass-units, the arithmetic average of the price variations, indicating constancy in this case, alone seems to be justified in indicating constancy, and so seems to be the proper average to use in all cases.

The fault with this argument for the arithmetic average of price variations is that the compensation by mass-quantities which it offers is what ought to take place *at the second period alone.* At this period when A = B′ (or when A = 3 B), in an even spending of three money-units on [A] and [B] we get 1 A and 1 B′ (or 3 B), and again in a spending of one third less on [A] and of one third more on [B] we gain just as much as we lose; for we gain $\frac{1}{3}$ B′ (or 1 B) in place of $\frac{1}{3}$ A lost. Here the compensation is perfect because the quantity lost is equivalent to the quantity gained in both the transactions compared (whether the masses gained and lost happen to be expressed in equal or unequal numbers of mass-units, according to the sizes of these). But in our suppositional case we are comparing a transaction at the first period when A (being equivalent to B) was equivalent only to $\frac{1}{3}$ B′, with a transaction at the second period, at which alone A is equivalent to B′ (or to 3 B). An offered compensation of $\frac{1}{3}$ B′ (or 1 B) for a loss of $\frac{1}{3}$ A at the first period would be three times too great. It is still too great, though not so much in excess, when it is offered at the second period in comparison with the first. We must reflect that $\frac{1}{3}$ B′ (or 1 B) has fallen in price, that is, it *was* more valuable, and that A has risen, that is, it *was* less valuable. The gain of a numerically equal amount (or three times as much) on an article which, although its mass-unit is equally (or three times less) valuable than the other at the second period, was three times more (or equally) valuable at the first period, is too great a gain. Similar would be the conclusion if we treated the subject from the point of view of purchasing power. Therefore, as this proffered compensation is too large for the quantity gained over the quantity lost, our money purchases too much to permit it to be stable: its purchasing power, and its exchange-value, has risen: it has appreciated. And inversely the arithmetic compensation by equal prices is too great for the loss over the gain, and the general level of prices has fallen, instead of being constant, as is wrongly indicated by this average.

The generalization may therefore be made that in all cases when constant sums are spent on the classes at both periods, these sums being taken for the weights, the indication concerning the general level of prices offered by the harmonic average of the price variations is lower than it ought to be; and the indication offered by the arithmetic average of the price variations is higher than it ought to be.

§ 4. The true position must take account of the conditions at both the periods, and so the compensation must lie between those offered by the harmonic and the arithmetic terms, being larger than the former and smaller than the latter for the loss over the gain by mass-quantities. It is here that lies the compensation offered by the geometric mean and average.

The schemata to illustrate the argument for the geometric *mean* are as follows—for the harmonic price variations:

I 100 A @ 1.00	70.71 B″ @ 1.4142	— 170.71	100 for [A]	100 for [B],
II $66\frac{2}{3}$ A @ 1.50	94.28 B″ @ 1.0606	— 160.94	100 for [A]	100 for [B];

for the arithmetic price variations:

I 100 A @ 1.00	57.73 B″ @ 1.732	— 157.73	100 for [A]	100 for [B],
II $66\frac{2}{3}$ A @ 1.50	115.47 B″ @ .866	— 182.13	100 for [A]	100 for [B];

for the geometric price variations:

I 100 A @ 1.00	$66\frac{2}{3}$ B″ @ 1.50	— $166\frac{2}{3}$	100 for [A]	100 for [B].
II $66\frac{2}{3}$ A @ 1.50	100 B″ @ 1.00	— $166\frac{2}{3}$	100 for [A]	100 for [B].

Here, the real facts represented being the same as in the two preceding sets of schemata, the only compensation by an equal number of *these* mass-units is in the simple geometric price variations, the compensation offered by the harmonic price variations being too small, and that by the arithmetic too large. Thus we have the appearance, equally good as such, that it is only the geometric mean of price variations that in indicating constancy in this case gives the right result. The question now arises, Are there any reasons why the appearance is better in this case than in the others?

The reasons are plain. They flow from the principles already examined in Chapter VIII., which principles are of direct application to our present subject. We are dealing with a subject in which no matter how far the price of A rises, the quantity of [A] purchasable with a given sum cannot fall to zero, and as B falls in price, the quantity of [B] can be limited by no figure short of infinity. Our subject then is suitable for the use of the geometric averaging of the price variations, in which constancy is shown when the quantities offered in compensation vary to the geometric extremes. Here the variations are all the reverse of each other. The mass-unit of [B] is no longer the equivalent of the mass-unit of [A] either at the first period or at the second period, but it at the second period is equivalent to A at the first and it at the first is equivalent to A at the second. And in the variations of the numbers of the mass-units there is compensation not only by equal quantities but by equality of distance traversed over the same road in reversed directions. These inverted relations are universal (for all simple geometric price variations of two classes in opposite directions) when we have the conditions required by even weighting. It follows from them that there is alternation of preciousness conjointly with, and oppositely to, the alternation of the mass-quantities. We lose one third on the whole of [A] while it gains in preciousness by half (reckoning from the first period), and (reckoning in the same way) we gain one half on the whole of [B] while it loses in preciousness by one third. This leads to the really fundamental reason, which is:—As the mass-unit here used of [B] is the equivalent of the mass-unit of [A] *over both the periods*, it is obviously correct that we should gain as many of these mass-units of [B] as we lose of these mass-units of [A]. The compensation by mass-quantities is in this case perfect.

The superiority of the geometric mean to the other means of price variations may be shown more clearly by bringing together the three forms of the schemata on which the argument for each of the averages has relied. These are—for the harmonic average of price variations:

I	100 A @ 1.00	100 B @ 1.00 — 200	100 for [A]	100 for [B],
II	66$\frac{2}{3}$ A @ 1.50	133$\frac{1}{3}$ B @ .75 — 200	100 for [A]	100 for [B];

for the arithmetic average of price variations:

I	100 A @ 1.00	33$\frac{1}{3}$ B′ @ 3.00 — 133$\frac{1}{3}$	100 for [A]	100 for [B],
II	66$\frac{2}{3}$ A @ 1.50	66$\frac{2}{3}$ B′ @ 1.50 — 133$\frac{1}{3}$	100 for [A]	100 for [B];

for the geometric mean of price variations:

I	100 A @ 1.00	66$\frac{2}{3}$ B″ @ 1.50 — 166$\frac{2}{3}$	100 for [A]	100 for [B],
II	66$\frac{2}{3}$ A @ 1.50	100 B″ @ 1.00 — 166$\frac{2}{3}$	100 for [A]	100 for [B].

In the first of these it is plain that the mass-unit used of [B] is of smaller exchange-value than the mass unit of [A] over both the periods together. Hence in gaining only an equal number of these mass-units of [B] for the mass-units of [A] lost, we gain less exchange-value than we lose. Therefore our money has diminished in exchange-value or purchasing power, or has depreciated, and the general level of prices has risen; wherefore the harmonic average of the price variations errs below the truth in indicating constancy.

In the second it is plain that the mass-unit used of [B] is of greater exchange-value than the mass-unit of [A] over both the periods together. Hence in gaining an equal number of these mass-units of [B] for the mass-units of [A] lost, we gain more exchange-value than we lose. Therefore our money has augmented in exchange-value or purchasing power, or has appreciated, and the general level of prices has fallen; wherefore the arithmetic average of the price variations errs above the truth in indicating constancy.

But in the third the mass-unit used of [B] is of the same exchange-value as the mass-unit of [A] over both the periods together. Hence in gaining an equal number of these mass-units f [B] for the mass-units of [A] lost, we gain exactly the same xchange-value as we lose.[2] Therefore our money has remained onstant in exchange-value or purchasing power, and the genral level of prices is also constant; wherefore the geometric nean is right in indicating constancy.

The geometric mean also rightly indicates the variations in he other cases. Let us go back to the first set of schemata in his paragraph, which, in the same order, represent the same tates of things as the last. For there we have reduced the nass-unit of [B] to equivalence over both the periods with the nass-unit of [A]. In the first schema, where the price variaions are the harmonic, the purchasing power of money, or its xchange-value, relatively to these two classes, is evidently acording to the totals of these equivalent mass-units which the iven total sum of money will purchase at each period, so that ve have $\frac{M_{02}}{M_{01}} = \frac{160.94}{170.71} = 0.9428$, indicating a fall of 5.72 per ent., while the general price variation is the inverse, thus $\frac{P_2}{P_1} = \frac{170.71}{160.94} = 1.0606$, indicating a rise of 6.06 per cent. Now he geometric mean of these price variations is $\frac{P_2}{P_1} = \sqrt{\frac{3}{2} \times \frac{3}{4}} = \frac{3}{\sqrt{8}} = 1.0606$, likewise indicating a general rise of prices by 6.06 per cent. In the second, where the price variations are he arithmetic, we have $\frac{M_{02}}{M_{01}} = \frac{182.13}{157.73} = 1.1547$, indicating a ise of 15.47 per cent., and $\frac{P_2}{P_1} = \frac{157.73}{182.13} = 0.8660$, indicating a

[2] These mass-units are not the economic individuals described, for the present upposition, in Chapter IV. Sect. V. § 3, namely constant exchange-values with ariable masses. But they are substitutes therefor; and they are like the econmic individuals described in Chapt. IV. Sect. V. § 6, which will be used in the upposition to be treated of in the next Chapter. They may very properly be sed for the purpose they are here put to. To be sure, the weighting of the classes t each period cannot be measured by the numbers of these mass-units they conain. Yet the weighting of the classes in the averaging of their price variations, ver two periods together, is according to the geometric means of the numbers of uch mass-units they contain at each period. Another example will be given ater (see below in Sect. IV. Note 1).

fall of 13.40 per cent.; and the geometric mean of the price variations is $\frac{P_2}{P_1} = \sqrt{\frac{3}{2} \times \frac{1}{2}} = \frac{\sqrt{3}}{2} = 0.8660$, likewise indicating a general fall of prices by 13.40 per cent. That the geometric mean of the price variations of two classes equally important over both the periods universally agrees with the inverse of the indication for $\frac{M_{02}}{M_{01}}$ rendered by the numbers of the mass-units equivalent over both the periods purchasable at each period, has already been demonstrated near the end of the preceding Chapter in formula (10) there given.[3] Therefore the geometric mean of price variations, whenever it is applicable to cases in which for two classes of goods the same sums of money are spent at both periods, universally gives the right indication.

§ 5. The fault with the argument from arithmetic compensation by equal mass-quantities, as it has generally been employed by the writers who have suggested or advocated or employed the harmonic average of price variations, has lain in the fact that utter neglect has been paid to the exchange-value of the mass-quantities whose loss and gain have been compared. To be sure, the mass-units have generally been chosen equivalent at the first period. But that has been due merely to convenience, and to the habit, itself due to convenience, of starting with units priced at 1.00 or at 100. The query why the compensation should be by equal numbers of such mass-units, has never been raised; wherefore the wrongness of such compensation has escaped notice. Possibly it has not been noticed that the so-called "quantities," or mass-quantities, as numerical figures, are not absolute figures, according to the masses given, but are de-

[3] Furthermore, in both the above examples, in which the mass-quantity of [A] is 100 and the price of A 1.00, at the first period, the figure for the variation of the general exchange-value of money is identical with the hundredth part of the mass-quantity of [B] at the second period, and also it is the quotient of the price of B″ at the first period divided by the price of A at the second; while the figure for the general price variation is identical with the price of B″ at the second period, and also it is the quotient of the price of A at the second period divided by the price of B″ at the first. The universality of these relations, and of some others which it would be too long to notice, is also demonstrated by formulæ (6), (7), and (9) (and by their inversions for $\frac{M_{02}}{M_{01}}$) given near the end of the preceding Chapter.

termined also by the sizes of the mass-units in which the masses are measured.

Many advocates not only of the harmonic, but even of the arithmetic average, have failed, as we have seen, at a still earlier stage. They would measure the general exchange-value of money, under the name of its "general purchasing power," by the mere mass-quantities that a given total sum of money *can* purchase, or has the power of purchasing, at each period, without regard either to the mass-units used or to the actual spendings of our money at the two periods compared. Our own completer analysis has shown that, when there is any change of prices, there is always, within fixed limits, a great variety of total mass-quantities that can be purchased with the same total sum of money, according as different spendings of its parts (the same at both periods) be hit upon. Therefore this argumentation proves only a great variableness in "general purchasing power" so conceived, whenever there is any change of prices. Thus if we look upon purchasing power as something to be measured only by the quantities of things that can be purchased, we see that for such a thing as the "general purchasing power" of money, so conceived, to remain stable, absolutely no change of prices must take place. So far Roscher and others are right in denying the possibility of stability of money in general purchasing power except—to use Martello's phrase—by "petrifaction" of the economic world. But they are wrong who extend this denial to the general exchange-value of money. For it is evident that if later in any distribution of spendings one's money gets the same exchange-value which it got before, it gets the same exchange-value in any other distribution of spendings.

It might, however, be admitted that we can have such a concept as this of "general purchasing power" measured only by quantities of things purchasable, if we desire to distinguish "purchasing power" from "exchange-value." For, as just remarked, we know that it remains constant so long as all prices remain unchanged; and we also perceive that if all prices rise or fall in exactly the same proportion, it rises or falls in that proportion—in the former case coinciding with exchange-value

between inessential attributes in things, this idea has only caused confusion, and it is responsible for many of the aberrations in our subject.[4]

III.

§ 1. The measurement of the constancy or variation in the exchange-value of money by the numbers of equivalent mass-units is not, of course, confined to the case of two equally important classes. It is as applicable to any number of classes, with any amount of unevenness in their sizes, provided they satisfy the condition required for its application. Therefore we reach this general method of measurement:—*When constant sums of money are expended at both periods, find in all the classes masses, to be used as units, that have the same exchange-value (or money-value) over both the periods together; and measure the constancy or variation in the exchange-value of money by the constancy or variation in the total number of such mass-units purchased at each period.*

If we have already executed the first part of this injunction, the true formula for the price variations, in these cases, is

$$\frac{P_2}{P_1}=\frac{x_1''+y_1''+z_1''+\cdots\cdots}{x_2''+y_2''+z_2''+\cdots\cdots}, \qquad (1)$$

or

$$\frac{P_2}{P_1}=\frac{\dfrac{\mathbf{a}}{\alpha_1''}+\dfrac{\mathbf{b}}{\beta_1''}+\dfrac{\mathbf{c}}{\gamma_1''}+\cdots\cdots}{\dfrac{\mathbf{a}}{\alpha_2''}+\dfrac{\mathbf{b}}{\beta_2''}+\dfrac{\mathbf{c}}{\gamma_2''}+\cdots\cdots}, \qquad (2)$$

in which, as before, the doubly accented letters represent the numbers and the prices of such mass-units at each of the periods.[1]

[4] And yet, as the term contains reference to *power*, reference to which is lacking in the term "exchange-value," the term "purchasing power" ought to have been of assistance. That it has not been is probably due to the limitation in the meaning of the term "to purchase" noticed in Note 4 in Chapt. V. Sect. III. § 8.

[1] The second of these formulæ is like a formula cited in Chapt. V. Sect. VI. § 2 as a miscarried harmonic average of price variations. But there the formula was applied to *any* mass-units, while here it is applied only to special mass-units. The weighting here in this formula, viewed as a harmonic averaging of price variations, is according to $\frac{\mathbf{a}}{\alpha_1''}, \frac{\mathbf{b}}{\beta_1''}, \cdots\cdots$, which agrees with what was there said. Another interpretation of this weighting will be noticed presently (for $\frac{\mathbf{a}}{\alpha_1''}=x_1''$, $\frac{\mathbf{b}}{\beta_1''}=y_1''$, and so on).

The mass-units may be of any sizes, provided they be equivalent over both the periods. Thus in the examples above used in which the price of [A] rose by 50 per cent., we supposed A to be such that its price rose from 1.00 to 1.50, so that the geometric mean price of this and of every equivalent mass-unit was 1.2247. We might equally well have supposed its price to rise from .60 to .90, so that its geometric mean price and that of every mass-unit equivalent to it would be .7348; or we might have supposed any other rise by 50 per cent. we pleased. For such changes affect all the mass-units and all the prices in the same proportion, so that their numbers are all altered in the same proportion (the inverse of the preceding), and consequently the sums of their numbers; wherefore the results are the same.

§ 2. As $x_1'' = \frac{\mathbf{a}}{a_1''}$ and $x_2'' = \frac{\mathbf{a}}{a_2''}$, therefore $\frac{x_1''}{x_2''} = \frac{a_2''}{a_1''}$, and $\frac{x_2''}{x_1''} = \frac{a_1''}{a_2''}$: and similarly in the other classes. Also, of course, the prices a_1'' and a_2'' are in the same ratio as the prices a_1 and a_2 (the prices ordinarily quoted), so that $\frac{a_2''}{a_1''} = \frac{a_2}{a_1}$ and $\frac{a_1''}{a_2''} = \frac{a_1}{a_2}$, and so on. Now then

$$\frac{x_1'' + y_1'' + \cdots\cdots}{x_2'' + y_2'' + \cdots\cdots} = \frac{1}{\frac{1}{x_1'' + y_1'' + \cdots\cdots}\left(x_1''\frac{x_2''}{x_1''} + y_1''\frac{y_2''}{y_1''} + \cdots\cdots\right)}$$

$$= \frac{1}{\frac{1}{x_1'' + y_1'' + \cdots\cdots}\left(x_1''\frac{a_1}{a_2} + y_1''\frac{\beta_1}{\beta_2} + \cdots\cdots\right)}; \quad (3)$$

and

$$\frac{x_1'' + y_1'' + \cdots\cdots}{x_2'' + y_2'' + \cdots\cdots} = \frac{1}{x_2'' + y_2'' + \cdots\cdots}\left(x_2''\frac{x_1''}{x_2''} + y_2''\frac{y_1''}{y_2''} + \cdots\cdots\right)$$

$$= \frac{1}{x_2'' + y_2'' + \cdots\cdots}\left(x_2''\frac{a_2}{a_1} + y_2''\frac{\beta_2}{\beta_1} + \cdots\cdots\right). \quad (4)$$

The interpretation of these two equations, along with what we have discovered and presently shall discover about the third kind of mean and average, gives us the following proposition:—

When constant sums of money are expended at both periods, the true result is always obtained (1) by the *harmonic* average of the price variations with weighting according to the numbers of the mass-units equivalent over both the periods which the classes contain at the *first* period, or (2) by the *arithmetic* average of the price variations with weighting according to the numbers of such equivalent mass-units which the classes contain at the *second* period, or (3) in cases with only two equally important classes, by the *geometric* mean, and in most cases (provided there be no great irregularity in the sizes of the classes) very nearly by the geometric average with weighting according to the geometric means of the numbers of such equivalent mass-units which the classes contain at each period[2]—which weighting is the same as the weighting of the classes according to the constant sizes of the classes at *both* periods.[3]

But the first two of these final solutions of our problem in regard to the proper averages of price variations (and their weightings) are of no practical utility, since they presuppose that we have performed the labor of finding the masses that are equivalent over both the periods together and the numbers of these contained in the classes at one of the periods. If we have performed this labor, it would be easier to continue the operation in the way prescribed in the first of the above formulæ. What we want is a method that will save us the great labor of such reductions. Here the geometric mean finds its recommendation in the cases where it is applicable; and here the geometric average would be exactly what we want, were it only accurate in all cases.

§ 3. It is, however, possible to make a formula that will in all cases yield the true result without requiring the finding of such equivalent mass-units—or rather a modification of the above formulæ such as to contain within itself, so to speak, the method of selecting the mass-units,—a formula therefore, that

[2] These general statements account for the particular statements made in Notes 4, 5, and 7 in Sect. I. of this Chapter.

[3] For $x_1'' = \frac{\mathbf{a}}{\alpha_1''}$, and so on, and if the condition is given that $\sqrt{\alpha_1''\alpha_2''} = \sqrt{\beta_1''\beta_2''} = \dots\dots$, it follows that $\sqrt{x_1''x_2''} = \frac{\mathbf{a}}{\sqrt{\alpha_1''\alpha_2''}}$, $\sqrt{y_1''y_2''} = \frac{\mathbf{b}}{\sqrt{\beta_1''\beta_2''}} = \frac{\mathbf{b}}{\sqrt{\alpha_1''\alpha_2''}}$, and so on; wherefore $\sqrt{x_1''x_2''} : \sqrt{y_1''y_2''} : \dots\dots :: \mathbf{a} : \mathbf{b} : \dots\dots$.

is applicable to *any* mass-units we happen to use in the various classes, requiring of us only that we know the prices of these mass-units, and the numbers of them purchased, at each period.

Let us continue to use the doubly accented letters, x_1'', x_2'', a_1'', a_2'', and the like, for the numbers and the prices of the mass-units which are equivalent over both the periods; and let us, as generally, use the simple letters, x_1, x_2, a_1, a_2, and the like, for the numbers and the prices of the ordinary mass-units which we find employed by merchants and referred to in statistical reports. Now it is possible to prove that

$$x_1\sqrt{a_1a_2} : y_1\sqrt{\beta_1\beta_2} : z_1\sqrt{\gamma_1\gamma_2} : \cdots\cdots :: \frac{\mathbf{a}}{a_1''} : \frac{\mathbf{b}}{\beta_1''} : \frac{\mathbf{c}}{\gamma_1''} : \cdots\cdots,$$

and

$$x_2\sqrt{a_1a_2} : y_2\sqrt{\beta_1\beta_2} : z_2\sqrt{\gamma_1\gamma_2} : \cdots\cdots :: \frac{\mathbf{a}}{a_2''} : \frac{\mathbf{b}}{\beta_2''} : \frac{\mathbf{c}}{\gamma_2''} : \cdots\cdots.$$

We may begin with the first two terms on each side of these proportions. As already explained,

we have $\qquad \dfrac{a_2}{a_1} = \dfrac{a_2''}{a_1''} \qquad$ and $\qquad \dfrac{\beta_2}{\beta_1} = \dfrac{\beta_2''}{\beta_1''}$,

whence $\qquad a_2 = \dfrac{a_1a_2''}{a_1''} \qquad$ and $\qquad \beta_2 = \dfrac{\beta_1\beta_2''}{\beta_1''}$.

Also by the hypothesis we have $a_1''a_2'' = \beta_1''\beta_2''$,

whence $\qquad a_2'' = \dfrac{\beta_1''\beta_2''}{a_1''} \qquad$ and $\qquad \beta_2'' = \dfrac{a_1''a_2''}{\beta_1''}$.

Substituting these values in the preceding expressions, we have

$$a_2 = \frac{a_1\beta_1''\beta_2''}{a_1''^2} \qquad \text{and} \qquad \beta_2 = \frac{\beta_1a_1''a_2''}{\beta_1''^2}.$$

And substituting these last values in the first expression to be proved, we have

$$x_1\frac{a_1}{a_1''}\sqrt{\beta_1''\beta_2''} : y_1\frac{\beta_1}{\beta_1''}\sqrt{a_1''a_2''} :: \frac{\mathbf{a}}{a_1''} : \frac{\mathbf{b}}{\beta_1''}.$$

But $x_1a_1 = \mathbf{a}$ and $y_1\beta_1 = \mathbf{b}$, and as $\sqrt{\beta_1''\beta_2''} = \sqrt{a_1''a_2''}$, they may be eliminated. Thus this proportion is evident. Again, by a similar derivation from the known conditions, we obtain

$$\alpha_1 = \frac{\alpha_2\beta_1''\beta_2''}{\alpha_2''^2} \quad \text{and} \quad \beta_1 = \frac{\beta_2\alpha_1''\alpha_2''}{\beta_2''^2},$$

and substituting these values in the second expression to be proved, we have

$$x_2\frac{\alpha_2}{\alpha_2''}\sqrt{\beta_1''\beta_2''} : y_2\frac{\beta_2}{\beta_2''}\sqrt{\alpha_1''\alpha_2''} :: \frac{\mathbf{a}}{\alpha_2''} : \frac{\mathbf{b}}{\beta_2''};$$

in which $x_2\alpha_2 = \mathbf{a}$ and $y_2\beta_2 = \mathbf{b}$, and the remainders of the expressions in the first half eliminate as before, so that this proportion also is evident.

The same operation may be conducted upon the classes [A] and [C] (or upon [B] and [C]), and so on through the whole list, with the same result. Q. E. D.

Consequently

$$\frac{x_1\sqrt{\alpha_1\alpha_2} + y_1\sqrt{\beta_1\beta_2} + z_1\sqrt{\gamma_1\gamma_2} + \cdots\cdots}{x_2\sqrt{\alpha_1\alpha_2} + y_2\sqrt{\beta_1\beta_2} + z_2\sqrt{\gamma_1\gamma_2} + \cdots\cdots} = \frac{\dfrac{\mathbf{a}}{\alpha_1''} + \dfrac{\mathbf{b}}{\beta_1''} + \dfrac{\mathbf{c}}{\gamma_1''} + \cdots\cdots}{\dfrac{\mathbf{a}}{\alpha_2''} + \dfrac{\mathbf{b}}{\beta_2''} + \dfrac{\mathbf{c}}{\gamma_2''} + \cdots\cdots}.$$

But we know by the second formula above given that the last half of this equation is the expression for the general price variation. Therefore

$$\frac{P_2}{P_1} = \frac{x_1\sqrt{\alpha_1\alpha_2} + y_1\sqrt{\beta_1\beta_2} + z_1\sqrt{\gamma_1\gamma_2} + \cdots\cdots}{x_2\sqrt{\alpha_1\alpha_2} + y_2\sqrt{\beta_1\beta_2} + z_2\sqrt{\gamma_1\gamma_2} + \cdots\cdots}. \qquad (5)$$

Here we have the formula desired, applicable to any mass-units and their prices and quantities, in all cases when constant sums happen to be spent on all the classes at both periods.

This formula may be further metamorphosed. It may be so stated,

$$\frac{P_2}{P_1} = \frac{\frac{\mathbf{a}}{\alpha_1}\sqrt{\alpha_1\alpha_2} + \frac{\mathbf{b}}{\beta_1}\sqrt{\beta_1\beta_2} + \frac{\mathbf{c}}{\gamma_1}\sqrt{\gamma_1\gamma_2} + \cdots\cdots}{\frac{\mathbf{a}}{\alpha_2}\sqrt{\alpha_1\alpha_2} + \frac{\mathbf{b}}{\beta_2}\sqrt{\beta_1\beta_2} + \frac{\mathbf{c}}{\gamma_2}\sqrt{\gamma_1\gamma_2} + \cdots\cdots},$$

which reduces to

$$\frac{P_2}{P_1} = \frac{\mathbf{a}\sqrt{\frac{\alpha_2}{\alpha_1}} + \mathbf{b}\sqrt{\frac{\beta_2}{\beta_1}} + \mathbf{c}\sqrt{\frac{\gamma_2}{\gamma_1}} + \cdots\cdots}{\mathbf{a}\sqrt{\frac{\alpha_1}{\alpha_2}} + \mathbf{b}\sqrt{\frac{\beta_1}{\beta_2}} + \mathbf{c}\sqrt{\frac{\gamma_1}{\gamma_2}} + \cdots\cdots}; \qquad (6)$$

and also to

$$\frac{P_2}{P_1} = \frac{\mathbf{a}\sqrt{\frac{x_1}{x_2}} + \mathbf{b}\sqrt{\frac{y_1}{y_2}} + \mathbf{c}\sqrt{\frac{z_1}{z_2}} + \cdots\cdots}{\mathbf{a}\sqrt{\frac{x_2}{x_1}} + \mathbf{b}\sqrt{\frac{y_2}{y_1}} + \mathbf{c}\sqrt{\frac{z_2}{z_1}} + \cdots\cdots}. \qquad (7)$$

These are more curious than useful, except for theoretical purposes. The last leads on through

$$\frac{P_2}{P_1} = \frac{x_2\alpha_2\sqrt{\frac{x_1}{x_2}} + y_2\beta_2\sqrt{\frac{y_1}{y_2}} + z_2\gamma_2\sqrt{\frac{z_1}{z_2}} + \cdots\cdots}{x_1\alpha_1\sqrt{\frac{x_2}{x_1}} + y_1\beta_1\sqrt{\frac{y_2}{y_1}} + z_1\gamma_1\sqrt{\frac{z_2}{z_1}} + \cdots\cdots}$$

to

$$\frac{P_2}{P_1} = \frac{\alpha_2\sqrt{x_1x_2} + \beta_2\sqrt{y_1y_2} + \gamma_2\sqrt{z_1z_2} + \cdots\cdots}{\alpha_1\sqrt{x_1x_2} + \beta_1\sqrt{y_1y_2} + \gamma_1\sqrt{z_1z_2} + \cdots\cdots}, \qquad (8)$$

which we recognize as a form of Scrope's method. It obviously is Scrope's method applied to the geometric means of the mass-quantities in every class at each period. Thus the method we have just elaborated turns out to be a new form of Scrope's method.

§ 4. That this method does not violate any of our Propositions concerning exchange-value in all other things goes almost without the need of notice. Taking any one of the formulæ representing it (but most conveniently 6 and 8), we see plainly that it indicates constancy if no prices vary (Propositions XXVII. and XLIV.), and if all prices vary alike, it indicates the same variation (Propositions XVII. and XLV.), no matter what be the weights (provided, of course, they be the same at both periods); nor can it indicate constancy if there is only one price variation, or if all price variations are in one direction—and in the latter event it naturally cannot indicate a variation in the opposite direction (Propositions XX. and XXVIII.). But there is one Proposition by which it especially needs to be tested. This is Proposition XXXVI. There is perfect evidence that if our method, applied to all but a few classes, indicates a variation (or constancy) in the level of prices so that $\frac{P_2}{P_1} = r$, and if we find that the remaining classes have all varied in price individually so that the price of each at the second period is r times its price at the first (or even if they have collectively done so), our method ought still, with these classes added, to indicate the same variation. Let [C] embrace all the individuals left over. Formula (6) is here the most convenient. If our method, applied to all classes except [C], indicates a variation in the general level of prices from 1 to r, we must have

$$\mathbf{a}\sqrt{\frac{\alpha_2}{\alpha_1}} + \mathbf{b}\sqrt{\frac{\beta_2}{\beta_1}} + \cdots\cdots = r\left(\mathbf{a}\sqrt{\frac{\alpha_1}{\alpha_2}} + \mathbf{b}\sqrt{\frac{\beta_1}{\beta_2}} + \cdots\cdots\right).$$

But the price of [C] has varied so that $\frac{\gamma_2}{\gamma_1} = r$, and $\frac{\gamma_1}{\gamma_2} = \frac{1}{r}$. Hence, extending our method to include [C], we should have

$$\mathbf{a}\sqrt{\frac{\alpha_2}{\alpha_1}} + \mathbf{b}\sqrt{\frac{\beta_2}{\beta_1}} + \cdots\cdots + \mathbf{c}\sqrt{r} = r\left(\mathbf{a}\sqrt{\frac{\alpha_1}{\alpha_2}} + \mathbf{b}\sqrt{\frac{\beta_1}{\beta_2}} + \cdots\cdots + \mathbf{c}\sqrt{\frac{1}{r}}\right),$$

which is evidently true, because it reduces to

$$\mathbf{a}\sqrt{\frac{\alpha_2}{\alpha_1}} + \mathbf{b}\sqrt{\frac{\beta_2}{\beta_1}} + \cdots\cdots + \mathbf{c}\sqrt{r} = r\left(\mathbf{a}\sqrt{\frac{\alpha_1}{\alpha_2}} + \mathbf{b}\sqrt{\frac{\beta_1}{\beta_2}} + \cdots\cdots\right) + \mathbf{c}\sqrt{r}.$$

Here we are supposing that $z_2\gamma_2 = z_1\gamma_1 = \mathbf{c}$, in order that the class [C] may be properly subject to this method. If things were not so—if we had to distinguish $\mathbf{c}$ into $\mathbf{c}_1$ and $\mathbf{c}_2$,—the desired equality would not exist. Now that Proposition[4] does not require the omitted class or classes to be of the same money-value at both the periods. Still this method is not shown to be false because it would not be true if we applied it to conditions to which it does not pretend to be applicable. Confined to the conditions to which it is applicable, it stands the test of that Proposition perfectly.[5]

§ 5. That our formula (5), upon which depend the ones following, is correctly derived, may be negatively shown in this way. If the mass-units have already been chosen that are equivalent over both the periods together, this formula reduces to formula (1), and so is correct.

That this formula and the ones following it will reduce to formulæ expressing the harmonic and the arithmetic averages, and in some cases the geometric mean, of the price variations, with the weightings above described (in § 2), is proved by the operation by which these formulæ themselves were proved, since formula (2) to which they were proved to be equal is equal to formula (1).

But with more than two classes, even if equally important, this formula will not universally reduce to the geometric average of the price variations.

Some special cases in which the geometric average of the price variations will regularly agree with this method and give the true result may easily be seen to be as follows :—(1) if all the classes are equally important and if there are 4, 8, 16, 32, 64, ... such classes; for then the common variation of two classes may be compared with the common variation of two other classes, and the common variation of all these four with

[4] Nor Proposition XXXII., which comes into play if $r = 1$.

[5] A qualification, however, must be added. Above we used formula (6). If we used formulæ (1), (2), (5) or (7), the same inability to satisfy that Proposition would remain. But if, instead, we used formulæ (3), (4) or (8), (which contain nothing that shows their confinement to cases with constant sums), and treated the additional mass-quantities as in any of them prescribed, this method would carry out that Proposition.

ommon variation of four others, and so on ;—(2) if some of lasses are doubly, quadruply, octuply, ... large and in the eding schema take the place of sets of 2, 4, 8, ... classes ; f beside such two or more classes all the rest are without variation, or all have the same price variation, and if the variations of those classes, measured apart by themselves, general constancy, or that same price variation ; for then tancy, or that variation, is the true result for all, and this still be indicated both by the geometric average and by this od, as just shown, no matter (in the case of the geometric aging) what be the weighting of these additional classes. shall also find a more general account of other more irreg- cases in which the geometric average may be exactly true. e must turn to the examination of the geometric average ice variations, this being generally so nearly true as to de- e our attention by itself.

IV.

1. A couple of illustrative examples may be examined in We need to treat only of the next simplest cases. These when we have three equally important classes to deal with, of which vary alike, or—what is the same thing—two classes of which is twice as important as the other. Here is room two examples according as it is the more or the less impor- class that falls in compensation for the rise of the other. us begin with the former. In the usual form of starting prices at 1.00, our familiar example with A rising by 50 cent. may be schematized as follows—for the harmonic e variations :

0 A @ 1.00 200 B @ 1.00 —300 | 100 for [A] 200 for [B],
6⅔ A @ 1.50 233⅓ B @ .8571 (= $\frac{6}{7}$)—300 | 100 for [A] 200 for [B];

he arithmetic price variations :

100 A @ 1.00 200 B @ 1.00—300 | 100 for [A] 200 for [B],
66⅔ A @ 1.50 266⅔ B @ .75—333⅓ | 100 for [A] 200 for [B];

for the geometric price variations :

I	100 A @ 1.00	200 B @ 1.00	—300		
II	$66\frac{2}{3}$ A @ 1.50	244.948 B @ .8165 $(=\sqrt{\frac{2}{3}})$	—311.614		
				100 for [A]	200 for [B],
				100 for [A]	200 for [B].

In the first of these, in which the harmonic average of the price variations with the weights 1 for [A] and 2 for [B] indicates constancy, the compensation is (arithmetically) by equal quantities of the mass-units of [A] and of [B] that are equivalent at the first period—a compensation which, as before, is good at the first period only, and is too small for both the periods together. In the second, in which the arithmetic average of the price variations with the same weighting indicates constancy, the compensation is by double gain on [B] over the loss on [A] in the same mass-units, this double gain being conformable to the fact that the price of this mass-unit of [B] has sunk at the second period to half that of the mass-unit of [A], so that this compensation is the compensation of the second period alone,—as may be perceived also by this arrangement :

I	100 A @ 1.00	100 B′ @ 2.00—200	100 for [A]	200 for [B].
II	$66\frac{2}{3}$ A @ 1.50	$133\frac{1}{3}$ B′ @ 1.50—200	100 for [A]	200 for [B].

in which, A and B′ now being mass-units that are equivalent at the second period, the compensation is by an equal number of such, and so is too great for both the periods together. Thus the argument from compensation by equal mass-quantities, applied to the harmonic and arithmetic averages in these more complex cases, has the same reversed errors we have seen in the simple cases with two equally important classes.

In the third schema, in which the geometric average of the price variations with the same weighting indicates constancy, there is no appearance of anything to recommend it, except the fact that its compensatory price and mass-quantity fall between the others. It has to be altered so that the mass-quantities compared be brought into better shape for the comparison. In it the price of B at the second period is the reciprocal of the metric mean between the two prices of A, namely 1.2247. formulation for [B] may then be changed into either

I	133⅓ B^{I} @ 1.50	200 for [B],
II	162.299 B^{I} @ 1.2247	200 for [B],

I	162.299 B^{II} @ 1.2247	200 for [B],
II	200 B^{II} @ 1.00	200 for [B].

l we have not what will help us. Suppose then, making of a hint given in Chapter VIII. Section I. § 6, we take of each of these alternative schematizations, as follows:

I	66⅔ B^{I} @ 1.50 + 81.65 B^{II} @ 1.2247	200 for [B],
II	81.65 B^{I} @ 1.2247 + 100 B^{II} @ 1.00	200 for [B].

re, however, there are two different mass-units of [B], the nd being .8165 of the first. Therefore this arrangement s not in itself help us. But we may combine the numbers en for the mass-units, and form a new single mass-unit, with following numbers and prices:

I	148.32 B^{III} @ 1.34846	200 for [B],
II	181.65 B^{III} @ 1.10102	200 for [B].

re the peculiarity exists that the difference in the numbers hese mass-units is 33⅓, so that we have compensation offered equal mass-quantities. Now if the geometric mean of the es of this mass-unit were the same as the geometric mean of prices of A, we should have exactly what we want; for this mass-unit of [B] would have the same exchange-value r both the periods as the mass-unit of [A], and the gain of of these mass-units of [B] would exactly equal and comsate the loss of 33⅓ of the mass-units of [A]. Unfortuely this is not the case. By the nature of the formation of mass-unit and its prices, its prices have 1.2247 for their hmetic mean only, and their geometric mean is a trifle er, being in effect 1.2185. Thus these prices of B^{III} incline the same direction as the prices in the harmonic extremes, offering compensation by equal mass-quantities; for the in between those prices of B is far below that between the es of A. We know that in those variations the price of

[B] has not fallen far enough. So, too, here we have an indication that the price of [B] has not fallen quite far enough. There is compensation by gain of an equal number of mass-units slightly less valuable than the mass-units lost. Therefore money has slightly depreciated, and the price-level has risen somewhat.

Another hint offered in the same place in Chapter VIII. leads to the same conclusion. There we saw that it is possible to arrange things so that the prices of two similar mass-units of [B], falling as from 1.00 to .8165, together traverse a distance equal to that traversed by one A, namely when they fall from 1.3623 to 1.1123. The schema for [B] would then be

I 146.81 B^{IV} @ 1.3623	200 for [B],
II 179.80 B^{IV} @ 1.1123	200 for [B].

Here the difference in the quantities is a gain of 32.99 such mass-units. The geometric mean of these prices is 1.23096. Thus while the compensatory quantities are slightly less, the exchange-value of their units over both the periods is slightly greater. Which of these differences preponderates? On [A] we lose $1.22474 \times 33\frac{1}{3} = 40.824$ money-unit's worths (over both the periods together); on [B] we gain $1.23096 \times 32.99 = 40.609$ similar money-unit's worths. So we gain less than we lose, and, as before, our money has depreciated, the fall in the price of [B] not being quite great enough.

Now it is easy to place the prices of a mass-unit of [B], falling as from 1.00 to .8165, around 1.2247 as their geometric mean, and to find the numbers of these mass-units purchasable with 200 money-units at each period. They are

I 147.5579 B″ @ 1.3554	200 for [B],
II 180.7174 B″ @ 1.1067	200 for [B].

But here the gain in such mass-units is 33.1595, or a trifle too little. This confirms the previous conclusion that the price of [B] has not fallen quite far enough; for it indicates a slight diminution in the purchasing power of our money, since we are not permitted to gain quite so many equally valuable mass-units of [B] as we lose of [A].

What we want is to combine certain features which have so far not come together, and which will not come together so long as we suppose the price of [B] to fall as from 1.00 to .8165. We want a mass-unit such that its prices, falling, have 1.2247 for their geometric mean, and such that the numbers of them purchasable with 200 money-units at the first and at the second period increase by 33⅓. If these two objects were obtainable with a fall of the price of [B] as from 1.00 to .8165, the geometric average of price variations would be true in this and all such cases. But they are not, and we must seek for another fall of the price of [B] that will unite these requirements. It is not difficult to find the terms of this desired fall. The following is the schema for [B]:

I 147.480987 B″ @ 1.356107	200 for [B],
II 180.814333 B″ @ 1.106107	200 for [B].

Here we have a class, [B], twice as large as the class [A], and the price variations are such that for what we lose on the mass-quantity of [A], the price of whose unit varies around 1.2247 as its geometric mean, we gain exactly as much on the mass-quantity of [B], the price of whose unit varies around the same figure, 1.2247, also as its geometric mean, and whose units gained are equal in number to the other units lost.[1] Therefore our gain is exactly equivalent to our loss *over both the periods compared.* Therefore our money has retained exactly the same purchasing power or exchange-value. But what is this price variation? It is as from 1.00 to $\frac{1.106107}{1.356107} = .815648$—a figure slightly lower, as we have already found reason to expect, than the figure required by the geometric method. Therefore if the price of B falls only from 1.00 to .8165, at which point the geometric average indicates constancy, it has not fallen quite far enough to counterbalance the rise of A from 1.00 to 1.50, so that the general level of prices has slightly risen, and the geometric average

[1] It may be noticed that the geometric mean between the numbers of these mass-units of [B] is 163.3, which is just double the geometric mean between the numbers of the mass-units of [A], namely 81.65. This is in argument with Note 2 above in Sect. II. It should also be noticed that the prices of every two such mass-units of [B] together traverse in the opposite direction exactly the same distance as the price of the one mass-unit of [A].

has, in this case, made an indication slightly below the truth.

We should have been pleased if one of the familar and easily working averages, applied to the price variations with weighting according to the constant sums expended, in this case 1 for [A] and 2 for [B], would indicate constancy only when the price of [B] has fallen from 1.00 to .815648. Unfortunately none of the three do so.[2] But of the three the geometric average, with this weighting, comes the nearest to it.

In default of a true average with the only weighting conveniently indicated, we can, of course, resort to one of the formulæ above discovered, preferably (5), and directly find the correct result without going through the long operation above pursued, which has been done to illustrate the principle and to show the defect in the geometric average. Thus applied for the purpose of finding the proper variation of the price of B from 1.00 when A rises from 1.00 to 1.50 and at both periods 100 money-units are spent on [A] and 200 on [B], in order to have constancy, the formula is

$$\frac{100 \times 1.224744 + 200\sqrt{{}_{1}\beta_2}}{66\frac{2}{3} \times 1.224744 + \frac{200}{{}_{1}\beta_2}\sqrt{{}_{1}\beta_2}} = 1.00.$$

This reduces to ${}_{1}\beta_2 + 0.204124\sqrt{{}_{1}\beta_2} = 1.00$, which being worked out in the ordinary algebraical manner (with only positive terms) gives $\sqrt{{}_{1}\beta_2} = .903132$, and ${}_{1}\beta_2 = .815648$.

§ 2. The second example, in which the class with rising price is the more important, may be disposed of more briefly. Let us suppose that twice as much money is spent on [A] at each

[2] Or we should be almost as well pleased if we had some easily indicated system of weighting which with one of the familiar averages would give the true result. Now in the above example, when B falls from 1.00 to .815648, the weighting wanted with the geometric average is 1 for [A] and 1.9897 for [B]; with the harmonic, 1 for [A] and 1.4748 for [B]; with the arithmetic, 1 for [A] and 2.7122 for [B]. Here the weighting for the geometric average is hopeless. That for the harmonic average is the ratio of the number of the (over both periods) equivalent mass-units of [A] to those of [B] purchased at the first period; and that for the arithmetic average the ratio between them at the second period. This is in agreement with what is above shown in Sect. II. § 2, which gives us a general principle for the weighting of these averages, but one, as there remarked, too laborious to be of utility.

period as on [B]. Now for constancy to be indicated by the three kinds of averages each with the weights 2 for [A] and 1 for [B], the schemata must be as follows—for the harmonic average:

I	200 A @ 1.00	100 B @ 1.00—300	200 for [A]	100 for [B],
II	133⅓ A @ 1.50	166⅔ B @ .60—300	200 for [A]	100 for [B];

for the arithmetic average:

I	200 A @ 1.00	100 B @ 1.00—300	200 for [A]	100 for [B],
II	133⅓ A @ 1.50	∞ B @ 0 —···	200 for [A]	············;

for the geometric average:

I	200 A @ 1.00	100 B @ 1.00 —300	200 for [A]	100 for [B],
II	133⅓ A @ 1.50	225 B @ .4444 ($=\frac{4}{9}$)—358⅓	200 for [A]	100 for [B].

In the first the compensation is still arithmetically by equal mass-quantities, measured in equivalents at the first period, and therefore only the proper compensation at the first period itself. In the second the required change of price of B is to zero, which makes A at the second period infinitely more valuable than B; wherefore compensation is impossible. The third again needs to be restated. The restatement may be made, this time, by changing the mass-units of [A]. The price of A at the second period, 1.50, is the reciprocal of the geometric mean between the two prices of B. We may then have

I	225 A^{I} @ .4444 + 150 A^{II} @ .6666	200 for [A],
II	150 A^{I} @ .6666 + 100 A^{II} @ 1.00	200 for [A];

whence, by combining the quantities, and forming a new mass unit:

I	375 A^{III} @ .5333 ($=\frac{8}{15}$)	200 for [A],
II	250 A^{III} @ .80	200 for [A];

Here the number of these mass-units lost is the same as of the mass-units of [B] gained. But the prices are arithmetical, not geometrical, terms around .6666, and their geometrical mean is lower, being .6532. The conditions no longer resemble those in the harmonic schema, in which the mean of the prices of the mass-unit which in equal numbers compensates for the gain of mass-units of [B] is above the mean of the prices of the latter. The suggestion from this difference is that the price of A has not risen quite high enough. We lose an equal number of mass-units that are not quite so valuable (over both the periods) as the mass-units gained. Therefore our money has augmented in purchasing power, and the price-level has fallen somewhat. The same suggestion is made when we find the mass-unit of [A] whose prices, rising in the same proportion as before, do vary around .6666 as their geometrical mean, as follows:

I 367.42349 A″ @ .544331	200 for [A],
II 244.95007 A″ @ .816497	200 for [A],

the number of these mass-units lost being smaller than that of the equally (over both the periods) valuable mass-units of [B] gained, so that money has slightly appreciated. To get the proper difference, 125, of mass-units of [A] whose prices are in the same variation around .6666 as their geometric mean, we find the following conditions required:

I 368.942 A″ @ .54209	200 for [A],
II 243.941 A″ @ .81987	200 for [A].

Here at last we have the proper compensation, and the proper price variation of the class [A].[3] The proper price variation is a rise as from 1.00 to $\frac{.81987}{.54209} = 1.5124$. The same figure is obtained more directly by use of the above-discovered formula (5). Applied to mass-units priced at 1.00 at the first period, this becomes

$$\frac{200\sqrt{a_2} + 100 \times \frac{2}{3}}{\frac{200}{a_2}\sqrt{a_2} + 225 \times \frac{2}{3}} = 1,$$

which reduces to $a_2 - 0.416666\sqrt{a_2} = 1$, and works out, giving $\sqrt{a_2} = 1.229804$ and $a_2 = 1.5124$. The figure is, this time, as we have found reason to expect, slightly higher than required by the geometric average. The geometric average is satisfied,

[3] The geometric mean between the numbers of these mass-units of [A] is double the geometric mean between the numbers of the mass-units of [B]; and the prices of every two such mass-units of [A] together traverse the same distance as the one mass-unit of [B]. Cf. above, Note 1.

and indicates constancy, when the price of A has risen only to 1.50, or before it has risen to its proper height, that is, when the compensation offered by its variation is not yet complete, and the general level of prices has fallen. Thus in this case the geometric average yields a result slightly above the truth.[4]

§ 3. These examples, in conjunction with many others of a similar nature to be given later, permit us to make an induction and to advance the following rule:—*When the prices that fall below the general average are of preponderating classes, the geometric average, with weighting according to the constant sums of money spent, yields a result below the truth; and when the prices that rise above the general average are of preponderating classes, it yields a result above the truth.*

We may form some idea of the probable amount of error incurred by the geometric average, by measuring the errors in the above examples. In the first, the sums of the (over both periods) equivalent mass-units of [A] and [B] purchasable with 100 and 200 money-units at the first and second periods were respectively 247.5579 and 247.3840; by dividing the former by the latter we get $\frac{P_2}{P_1} = 1.0007$. Thus the geometric average, weighted according to the constant sums, indicated constancy where there really was a rise of 0.07 per cent. In the revision we saw that there would really be constancy if the price of B fell to .81564. If it did so, the geometric average would be 0.99932, indicating a fall of 0.068 per cent. In the second example, the sums of the (over both periods) equivalent mass-units of [A] and [B] purchasable with 200 and with 100 money-units were successively 467.42349 and 469.95007, whence $\frac{P_2}{P_1} = 0.9946$, indicating a fall of 0.54 per cent., although the geometric average, weighted as above described, indicated constancy. In the revision we found that to obtain constancy the

[4] In the corrected example, with A rising from 1.00 to 1.5124, to indicate constancy, the weight wanted for [A], that for [B] being 1, with the geometric average is 1.9602, with the harmonic 3.689, with the arithmetic 1.084—the latter two being the ratios between the numbers of the equivalent mass-units at the first and at the second periods respectively. Cf. above, Note 2.

price of A should rise to 1.5124. If it did so, the geometric average would be 1.0055, indicating a rise of 0.55 per cent.

These trifling errors, we may notice, are on larger price variations than ordinarily occur in practice between two successive periods. It is only with very large price variations that the deviations become appreciable. But also inequality in the sizes of the classes has something to do with the matter. If the preponderating class varies much in price, while the smaller class varies little, the deviation may be appreciable, if the preponderance is moderate, but may be diminished almost to nothing if the preponderance is excessive (the double excess having the effect of dragging both the measurements after it till they almost coincide)—while, of course, a very slight preponderance also unites the results, approaching toward evenness of weighting. On the other hand, if the preponderating class varies little and the smaller class varies excessively,[5] the deviation may be great, and the geometric method becomes wholly untrustworthy.

Examples of these rules may be given at random as follows:

I	10 A @ 1.00	40 B @ 1.00	10 for [A]	40 for [B].
II	5 A @ 2.00	400 B @ .10	10 for [A]	40 for [B].

The proper result is $\frac{P_2}{P_1} = 0.2005$; the geometric method yields 0.1821.

I	10 A @ 1.00	40 B @ 1.00	10 for [A]	40 for [B].
II	5 A @ 2.00	4000 B @ .01	10 for [A]	40 for [B].

The proper result is $\frac{P_2}{P_1} = 0.0445$; the geometric method yields 0.0288.

I	10 A @ 1.00	400 B @ 1.00	10 for [A]	400 for [B],
II	5 A @ 2.00	40000 B @ .01	10 for [A]	400 for [B].

The proper result is $\frac{P_2}{P_1} = 0.0114$; the geometric method yields 0.0110.

[5] Remember that falling variations are larger than rising variations the percentages of which, reckoned in the usual way, are equal. And this superiority rapidly increases with the amount of the variation. A fall from 1.00 to 0.99 is only slightly greater than a rise from 1.00 to 1.01. A fall from 1.00 to 0.50 is twice as great as a rise from 1.00 to 1.50. A fall from 1.00 to 0.01 is one hundred times as great as a rise from 1.00 to 1.99.

I	100 A @ 1.00	10 B @ 1.00	100 for [A]	10 for [B],
II	50 A @ 2.00	1000 B @ .01	100 for [A]	10 for [B].

The proper result is $\frac{P_2}{P_1} = 0.8342$; the geometric method yields 1.2355.

Among many classes, if some large ones rise in price while other large ones fall, and several small ones similarly offset one another, and if the balances left over of large and small classes that vary in the same direction are small compared with the whole, the source of error again is small.

For the errors are not cumulative. That is to say, if the class [A] is twice as large as the class [B], and the class [C] twice as large as the class [D], and the prices of both [A] and [C] are the rising ones, so that in each comparison of [A] and [B] and of [C] and [D] the error may be, say, about half of one per cent., the error will be no greater in the comparison of all four together. On the contrary, the error will be an average somewhere between all the errors in the comparisons of any two unevenly matched pairs. And so, instead of accumulating, *the errors tend to neutralize one another*. This follows by the principle of continuity from the two halves of the statement above advanced. For if in one pair the preponderating class is the rising one, the error is above the truth; and if in another pair the preponderating class is the falling one, the error is below the truth. These two pairs of variations taking place together, and being averaged together, the error must lie between them, and so must be diminished. It may perhaps disappear altogether.

There is also another way in which there may be neutralization. This is through a series of successive periods. If during one comparison of two periods a large class rises in price through abnormal causes, its variation will tend to cause error in the geometric average above the truth. If then in the comparison of the next or some soon succeeding period its price falls back to its usual state, this reverse variation will tend to cause error in the geometric average below the truth. Hence in the series the later error will make up for the earlier. That this kind of compensation, if the reverse change is back to the same figure, the classes remaining of the same importance, is exact, will be shown presently.

In the harmonic and arithmetic averages there is no such tendency toward neutralization in successive periods. Their errors may accumulate indefinitely.

V.

§ 1. What has been proved in the preceding Sections may be confirmed on a new line of reasoning, which also yields instruction on points not yet proved.

Suppose that in our simple hypothesis of a world with money and two equally important commodities we have the following conditions, (the mass-units being chosen in the usual way, as equivalent at the first period), over three successive periods :

I	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B],
II	66⅔ A @ 1.50	133⅓ B @ .75	100 for [A]	100 for [B],
III	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B].

Here we have two sets of price variations, the first consisting of these, $\frac{3}{2}$ and $\frac{3}{4}$, the second of these, $\frac{2}{3}$ and $\frac{4}{3}$. In addition we have the comparison between the prices at the first period and the prices at the third period. This comparison shows constancy ; for evidently, the state of things being the same at the third as at the first period, we have the same conditions as if the state of things at the first period had remained constant. We *know*, therefore, that after drawing the average of the first set of price variations and that of the second set, if we then attempt from these averages to find the average of the whole variation (or constancy) from the first to the third period—which is done by multiplying together the two single averages of the price variations,—the result should be unity, indicative of constancy. Here we have control over the result, and may use these conditions as a test case.

It may be noticed at the outset that in all such cases (confined to three periods) the formulæ above obtained satisfy this test. For the quantities and prices being exactly the same at the third as at the first period, the first variation is indicated, by formula (5), to be

$$\frac{P_2}{P_1} = \frac{x_1\sqrt{a_1 a_2} + y_1\sqrt{\beta_1\beta_2} + \cdots\cdots}{x_2\sqrt{a_1 a_2} + y_2\sqrt{\beta_1\beta_2} + \cdots\cdots},$$

and the second

$$\frac{P_3}{P_2} = \frac{x_2\sqrt{a_1 a_2} + y_2\sqrt{\beta_1\beta_2} + \cdots\cdots}{x_1\sqrt{a_1 a_2} + y_1\sqrt{\beta_1\beta_2} + \cdots\cdots}.$$

These are reciprocals; wherefore $\frac{P_3}{P_1}\left(=\frac{P_2}{P_1}\cdot\frac{P_3}{P_2}\right) = 1.00$. It is plain that if the prices are *irregularly* different at the third period, the direct comparison of the third with the first will not yield the same result as the indirect comparison of them through the intervening period. We shall not pause to consider this inconsistency here, as it will again call for attention in the complete method involving this partial method. We shall at present devote our attention to the certain test yielded by a revision of everything at a later period to what it was at the first. Let us now see how the averages of the price variations stand this test.

Averaging the above sets of price variations harmonically, we get, for the first, $\frac{2}{\frac{2}{3} + \frac{4}{3}} = 1.00$, and for the second, $\frac{2}{\frac{3}{2} + \frac{3}{4}} = \frac{8}{9}$ $= 0.8888$; and the product of these results indicates that between the first and the third period the general level of prices has fallen by 11.11 per cent., which we know to be wrong—and to err by placing the level of prices too low.

But suppose, after drawing the harmonic average of the first set, we draw the arithmetic average of the second, likewise with even weighting. We then have $\frac{1}{2}\left(\frac{2}{3} + \frac{4}{3}\right) = 1.00$; and each of these results indicating constancy, the whole result indicates constancy, which is right.

In effect, the second price variations, from the harmonic terms on opposite sides of unity to unity, are (simple) arithmetic variations—that is, are the same as variations from unity to the opposite arithmetic terms. Thus at the second period the price of $\frac{2}{3}$ A is 1.00 and the price of $1\frac{1}{3}$ B is 1.00, and at the third the prices of these parts of A and B are $.66\frac{2}{3}$ and $1.33\frac{1}{3}$ respectively, which are the arithmetic terms around unity. Naturally, therefore, only the arithmetic average, with even weighting, will indicate constancy when prices vary in this way.

Turning our attention to the quantities purchased, we see that with every two money-units we purchase at the first and third periods 1 A and 1 B, and at the second $\frac{2}{3}$ A and $\frac{4}{3}$ B. Thus at the second period, compared with the first, on the quantities of the first, the loss on A is one third and the gain on B one third, the gain equalling the loss. But at the third period, compared with the second, and with the gain and loss measured in the same way, that is, on the quantities at the earlier of the two periods compared as the unit wholes, the gain on $\frac{2}{3}$ A is one half and the loss on $\frac{4}{3}$ B is one quarter, constituting the compensation by harmonic proportions. This agrees with the fact that the price variations in this case are arithmetic variations. But if the price variations really had been harmonic variations—namely, a fall of the price of $\frac{2}{3}$ A from 1.00 to $.66\frac{2}{3}$ (bringing the price of A back to 1.00) and a rise of the price of $\frac{4}{9}$ B from $.33\frac{1}{3}$ to $.66\frac{2}{3}$ (carrying the price of B up to 1.50), so that the second harmonic average would indicate constancy, then not only would the level of prices at the third period be openly higher than at the first, but it is plain that the compensation offered would be a gain on [A], now the article falling in price, smaller than the loss on [B], now the rising article, so that, losing equally on [A] in the first variation, but gaining less in the second, we should lose on the whole.

§ 2. Again if we had these conditions:

I	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B],
II	$66\frac{2}{3}$ A @ 1.50	200 B @ .50	100 for [A]	100 for [B],
III	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B],

and averaged the two sets of price variations arithmetically, with even weighting, the result for the first set would be 1.00, indicating constancy, and that for the second would be $1.33\frac{1}{3}$ indicating a rise of $33\frac{1}{3}$ per cent. But if we averaged the second set harmonically, with even weighting, we should have 1.00, and therefore 1.00 for the whole variation, correctly indicating constancy. In effect, the second set of price variations are harmonic variations. If they really were arithmetic, such that the arithmetic average would indicate constancy, the price-level would really have fallen at the third period; for though the price of A would be back at 1.00, the price of B would be .66⅔; and our money, still spent evenly, would purchase more than before.

§ 3. The order in which the two averages have been alternately applied might be inverted, and yet, although the particular results would be different in every instance, yet the whole results would be the same. Thus in the first example the arithmetic average of the first set of price variations would indicate a rise of 12½ per cent., and the harmonic average of the second set would indicate a fall of 11.11 per cent.; but a fall from 1.125 by 11.11 per cent. is a fall to 1.00. And in the last example the harmonic average of the first set of price variations would indicate a fall of 25 per cent., and the arithmetic average of the second set would indicate a rise of 33⅓ per cent.; but the net result of the two variations, $\frac{3}{4} \times \frac{4}{3} = 1$, indicates constancy.

From these particular examples we get the following general rules, the universality of which is proved in Appendix B IV. §§ 4, 5, 7, that *when the prices of two equally important classes after varying at the second period revert at the third to what they were before, the continuous use of the harmonic average of the price variations, with even weighting, gives a final result known to be too low, and the continuous use of the arithmetic average of the price variations, with even weighting, gives a final result known to be too high; but the alternate use of these two averages, similarly weighted, gives the final result known to be right.* We have sufficient acquaintance with these averages to infer that these statements will extend to include all cases with any number of classes of any sizes, the only provisos being that constant sums be expended on the classes at all the periods, and that the weighting be according to them.

It must not be inferred, however, that an alternate use of the arithmetic and harmonic averages would at the end of every other period give the right result. Unless all the prices revert to what they were at the start, and the weighting is the same throughout, the correction offered by the error of the one average for the error of the other has not full opportunity, or has too much opportunity, to work itself out. Yet this use of them would depart less from the truth than the continuous use of either of them alone.

The inverse nature of these two averages shows the mistake of a procedure which has not infrequently been pursued. Several statisticians, taking some period as a base at which all prices are reduced to 1.00 or 100 and with which all subsequent periods are compared by use of the arithmetic average of the price deviations, with even weighting, have also worked backward to still earlier periods, comparing these with the same base, now later or second to them, still using the arithmetic average; and have then strung out all the results in a single series. We now see that in order for the series to be of uniform nature, while the arithmetic average is used in comparing the later periods with the earlier base, the harmonic average should be used in comparing the earlier periods with the later base; or conversely. Yet as these statisticians have used even weighting where it does not belong, it does not much matter what average they use.

§ 4. Returning to our argument, we see that if the continuous use of the harmonic average over two periods gives a result too low, each single use of it gives a result too low; wherefore in the first example, when the first use of the harmonic average indicated constancy, there was really a rise of prices. And if the continuous use of the arithmetic average over two periods gives a result too high, each single use of it gives a result too high; wherefore in that example, when the second use of the arithmetic average indicated constancy, there was really a fall of prices. And reversely in the second example. That the two kinds of averages together give the right result at the end, infallibly shows that each one alone is wrong—and the one as much wrong on the one side as the other on the other. Now, to review, in our first example the indications are,

by the harmonic average:			by the arithmetic average:		
for the 1st variation		1.00,	for the 1st variation		1.125,
"	2d "	.8888,	"	2d "	1.00,
"	whole "	.8888;	"	whole "	1.125.

Here the right result for the whole comparison, 1.00, is the *geometric* mean between these two wrong results. Similarly in the second example the right result, 1.00, is the geometric mean between the other two, 1.33⅓ and .75.

Suppose now in the first example we average the price variations geometrically. The geometric mean of the first set of price variations is $\sqrt{\frac{3}{2} \times \frac{3}{4}} = \frac{3}{\sqrt{8}} = 1.0606$,[1] indicating a rise of 6.06 per cent. The geometric mean of the second set is $\sqrt{\frac{2}{3} \times \frac{4}{3}} = \frac{\sqrt{8}}{3} = 0.9428$,[2] indicating a fall of 5.72 per cent. The product of these two results is unity, meaning that the fall from 106.06 by 5.72 per cent. is a fall to 1.00. In the second example, with the prices at the second period altered to the arithmetic terms around 1.00, the geometric mean of the first set of price variations indicates a fall to $\frac{\sqrt{3}}{2} = 0.8660$ by 13.40 per cent., and the geometric mean of the second, a rise to $\frac{2}{\sqrt{3}} =$ 1.1547 by 15.47 per cent. The product of these again is unity, the rise from 0.8660 by 15.47 per cent. being a rise to 1.00.[3] Thus in these cases—and the universality of this rule is proved in Appendix B IV. § 6—*the continuous use of the geometric mean gives the final result known to be right.*

Let us, again, suppose these conditions:

I	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B],
II	66⅔ A @ 1.50	150 B @ .66⅔	100 for [A]	100 for [B],
III	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B].

[1] This is the geometric mean between 1.00, the harmonic mean of these variations, and 1.125, the arithmetic. Therefore, if the geometric mean of the price variations be the right one, the harmonic mean has erred as much below as the arithmetic has erred above, these proportions being measured geometrically.

[2] This is the geometric mean between 1.00, the arithmetic mean of these variations, and 0.8888, the harmonic. Therefore—as in the preceding note.

[3] Here, too, the figures, 0.8660 and 1.1547, are geometric means, the first between 1.00 and 0.75, the second between 1.00 and 1.3333.

The geometric mean of each of these sets of price variations, $\frac{3}{2}$, $\frac{2}{3}$, and $\frac{2}{3}$, $\frac{3}{2}$, is unity; and the result for the whole is also constancy. In effect, not only the first price variations, but the second likewise are geometric variations; for at the second period the price of $\frac{2}{3}$ A is 1.00 and the price of $1\frac{1}{2}$ B is 1.00, and at the third period the prices of these parts of A and B are .66$\frac{2}{3}$ and 1.50. Here $\frac{2}{3}$ A has fallen just as B fell in the first set of variations, and $1\frac{1}{2}$ B has risen just as A there rose. As regards the quantities purchased with every two money-units evenly distributed, at the second period compared with the first, and on the quantities of the first, they are $\frac{2}{3}$ A and $\frac{3}{2}$ B. Thus in the first quantity variations the loss is of $\frac{1}{3}$ A and the gain of $\frac{1}{2}$ B; and in the second, reckoned in the same way, the gain is of $\frac{1}{2}$ A and the loss of $\frac{1}{3}$ B,—so that the greater gain and the smaller loss (as appears from this way of estimating them) fall alternately on the one and on the other. Therefore, as the ultimate result of the two geometric averagings is correct, we have a clear advertisement either that the result of each of the geometric averagings is correct, or that the error in each is corrected by an equal error in the other.

But in the preceding examples, in which we dealt with only two classes equally important, the geometric mean gave a result geometrically midway between the two erroneous results given by the harmonic and arithmetic averages, which averages corrected each other at the end of the second variation, showing that they diverged equally from the truth on each occasion. Therefore, in these cases, under these conditions, the geometric mean of each set of price variations, coinciding with the mean between their equal errors, was exactly correct. And therefore, in general, *the geometric mean of price variations, whenever it is applicable, is correct.*

§ 5. So far we have dealt with only two equally important classes. Let us now briefly extend the investigation to cover uneven weighting. We may suppose the following conditions:

I	100 A @ 1.00	200 B @ 1.00		100 for [A]	200 for [B].
II	66$\frac{2}{3}$ A @ 1.50	233$\frac{1}{3}$ B @ .8571	(= $\frac{6}{7}$)	100 for [A]	200 for [B].
III	100 A @ 1.00	200 B @ 1.00		100 for [A]	200 for [B].

Iere the harmonic average indicates for the first price variations onstancy, for the second a fall to 0.9333 by 6⅔ per cent. The rithmetic average indicates for the first price variations a rise › 1.0714 by 7.14 per cent., for the second constancy. The al-ernating use of these two averages, in either order, gives the ght final result. The geometric average indicates for the first rice variations a rise to $\sqrt[3]{\frac{5}{4}\frac{4}{9}} = 1.0329$ by 3.29 per cent., for ıe second a fall as from 1.00 to $\sqrt[3]{\frac{4}{5}\frac{9}{4}} = 0.9681$ by 3.19 per ent. The continuous use of the geometric average gives the ight final result. But here the geometric results in the indi-idual instances no longer are geometric means between the ther two. In fact, we now have no reason to expect even the ight result to be the geometric mean between those other re-ults, since we are now dealing with higher powers. We do, owever, know from our previous reasoning that the geometric *verage* is not exactly right in its single uses. Hence the other lternative remains. We thus have demonstrative confirmation f our previous inductions; for, as the final result is right, we ee (1) that *the error in the geometric averages of the price varia-ions must be above the truth in the one instance and below it in he other*—and the determining factors can only be found in the acts that in the one instance the larger class has fallen in price while the smaller class has risen and in the other the larger has isen and the smaller fallen, although the connection is not ap-arent; and (2) that *these opposite errors must be (geometrically) qual to each other.* In the particular case before us these con-lusions are borne out by applying any one of the above-dis-overed formulæ. Any one of these gives for the first price ariations a rise to $\frac{307.634}{297.669} = 1.0335$ by 3.35 per cent., and for he second a fall as from 1.00 to $\frac{297.669}{307.634} = 0.9676$ by 3.24 er cent. Thus the geometric average was slightly below the ruth in the first instance (where the preponderating class fell in rice), and in the second (where the preponderating class rose) lightly above the truth (so that we may induct that such will e the connection between the causes and effects in all cases).

And the geometric indications are above and below the truth in equal proportions; for 1.0335 : 1.0329 :: 0.9681 : 0.9676 (with as close approximation as may be obtained without using more decimals). These relations will be found to hold in all cases.[4]

The general conclusion from this investigation is that in dealing with many classes in a long series of periods, where the many tendencies toward neutralization of the errors have room to play, it is extremely probable that the result given by the geometric average will never depart much from the true figure.

But of course this conclusion is purely theoretical, since the condition presupposed, of constant sums (or sums constantly in the same proportion) being always expended on all the classes, is never fulfilled in practice.

§ 6. The formula above discovered for finding the general price variation between *two* periods, in all cases when constant sums are expended at both periods, we have seen, satisfies this test whenever it is confined to a reversion of the state of things at a third period to what it was at a first period after a single deviation at a second period. Now suppose that the same state of things reverted to is at a still later period, after three or more variations through two or more intervening periods. Here we have a test for this formula applied in a series of measurements.

Let us consider the next simplest case possible, illustrative of these complex cases—consisting of two classes with three variations extending over four periods. We may posit them in the universal, or algebraic form, as follows:

$$\text{I}\quad \frac{\mathbf{a}}{\alpha_1}\text{ A @ }\alpha_1 \qquad \frac{\mathbf{b}}{\beta_1}\text{ B @ }\beta_1 \quad \Big|\quad \mathbf{a}\text{ for [A]} \quad \mathbf{b}\text{ for [B]},$$

$$\text{II}\quad \frac{\mathbf{a}}{\alpha_2}\text{ A @ }\alpha_2 \qquad \frac{\mathbf{b}}{\beta_2}\text{ B @ }\beta_2 \quad \Big|\quad \mathbf{a}\text{ for [A]} \quad \mathbf{b}\text{ for [B]},$$

$$\text{III}\quad \frac{\mathbf{a}}{\alpha_3}\text{ A @ }\alpha_3 \qquad \frac{\mathbf{b}}{\beta_3}\text{ B @ }\beta_3 \quad \Big|\quad \mathbf{a}\text{ for [A]} \quad \mathbf{b}\text{ for [B]},$$

$$\text{IV}\quad \frac{\mathbf{a}}{\alpha_1}\text{ A @ }\alpha_1 \qquad \frac{\mathbf{b}}{\beta_1}\text{ B @ }\beta_1 \quad \Big|\quad \mathbf{a}\text{ for [A]} \quad \mathbf{b}\text{ for [B]}.$$

[4] The argument here used concerning the geometric mean and average has been suggested by Westergaard's argument for the geometric average above examined near the end of Sect. VI. in Chap. V. But the details and the conclusion are different.—That the geometric average works backwards as well as forwards, has also been noticed by Wicksell, B. 139, p. 8, but without pointing out the conditions and without employing this fact as an argument in favor of the geometric average in comparison with the other two.

As the conditions are exactly the same at the fourth period as at the first, it is evident that the exchange-value of money and the level of prices are then the same as at the first period, and ought to be so indicated in the series of measurements in which the above formula is applied to the three successive price variations—as, indeed, it is indicated by a direct comparison of the fourth with the first period. For our present purpose formula (6) is the most serviceable. To find the result reached through the intermediate comparisons we have to multiply together the three applications of this formula,—one to each of the three sets of variations. To do this with the formula in its full form would be tedious. We may simplify the operation by substituting the following shortened symbols. Let $a = \sqrt{\frac{\alpha_3}{\alpha_1}}$, $b = \sqrt{\frac{\alpha_2}{\alpha_1}}$, $c = \sqrt{\frac{\alpha_3}{\alpha_2}}$, $d = \sqrt{\frac{\beta_3}{\beta_1}}$, $e = \sqrt{\frac{\beta_2}{\beta_1}}$, $f = \sqrt{\frac{\beta_3}{\beta_2}}$. The reciprocals will equal the reciprocals, and may stand for them. Also $bc = a$, and $ef = d$. The formula will now be

$$\frac{P_4}{P_1} = \frac{(\mathbf{a}\, b + \mathbf{b}\, e)\,(\mathbf{a}\, c + \mathbf{b} f)\left(\frac{\mathbf{a}}{a} + \frac{\mathbf{b}}{d}\right)}{\left(\frac{\mathbf{a}}{b} + \frac{\mathbf{b}}{e}\right)\left(\frac{\mathbf{a}}{c} + \frac{\mathbf{b}}{f}\right)(\mathbf{a}\, a + \mathbf{b}\, d)},$$

which, upon multiplying out, reducing and rearranging, becomes

$$\frac{P_4}{P_1} = \frac{\mathbf{a}^3 + \mathbf{a}^2\mathbf{b}\frac{a}{d} + \mathbf{a}\mathbf{b}^2\frac{d}{a} + \mathbf{a}^2\mathbf{b}\frac{e}{b} + \mathbf{a}\mathbf{b}^2\frac{b}{e} + \mathbf{a}^2\mathbf{b}\frac{f}{c} + \mathbf{a}\mathbf{b}^2\frac{c}{f} + \mathbf{b}^3}{\mathbf{a}^3 + \mathbf{a}\mathbf{b}^2\frac{a}{d} + \mathbf{a}^2\mathbf{b}\frac{d}{a} + \mathbf{a}\mathbf{b}^2\frac{e}{b} + \mathbf{a}^2\mathbf{b}\frac{b}{e} + \mathbf{a}\mathbf{b}^2\frac{f}{c} + \mathbf{a}^2\mathbf{b}\frac{c}{f} + \mathbf{b}^3}. \quad (9)$$

This can equal unity: (1) if $a = d$, $b = e$, and $c = f$, that is, if there are no price variations at all, or if all the price variations at any or every stage are in the same proportion; (2) if $a = b$, $d = e$, and $c = f$, that is, if there are no price variations between the second and third periods (which virtually reduces the series to three periods with two sets of variations), or if $a = rb$ and $d = re$, while $c = f$, that is, if all the price variations are alike between the second and third periods; (3) if **a** = **b**, that is, if the two classes are equally important. This last condition cannot be extended to cases with three or more classes indefinitely; but it might be extended to cases with 4, with 8, with 16, with 32, ... classes; or even with other numbers of variously important classes if these arranged themselves suitably, in pairs or sets.[5] In other cases the formula does not equal unity unless it happens to contain elements that counterbalance one another in tending to deviations in opposite directions.

Thus in this simplest example of the complex cases, used in a series of only four periods, the method fails to stand the test universally, even if everything is the same at the last period as at the first.

§ 7. What is the reason for the failure of this method in a series of more than three periods?—for the method has, apparently, been demonstrated to be true in every single comparison of two contiguous periods. The trouble is that the demonstration has been directed at proving the formula to be true in every single comparison of two contiguous periods only in reference to each other and out of connection with any other periods.

Over a series of years it would seem as if the perfectly true method should be modelled on what we have done in the case of a single measurement. That is, we should seek in all the classes, for service as units, masses that are equivalent to one another over *all* the periods compared, and should measure at every period the total numbers of these mass-units contained in all the classes at that period. Representing these numbers by x''', y''', z''', and so on, it is plain that if over four periods we had these results:

$$x_1''' + y_1''' + z_1''' + \cdots\cdots \text{ to } n \text{ terms} = Q_1,$$
$$x_2''' + y_2''' + z_2''' + \cdots\cdots \text{ to } n \text{ terms} = Q_2,$$
$$x_3''' + y_3''' + z_3''' + \cdots\cdots \text{ to } n \text{ terms} = Q_3,$$
$$x_4''' + y_4''' + z_4''' + \cdots\cdots \text{ to } n \text{ terms} = Q_4,$$

then the comparisons (according to formula 1 in Sect. III.) going through this series, would yield

[5] Cf. the conditions for the correctness of the geometric average above pointed out in Sect. III. § 5.

$$\frac{P_4}{P_1} = \frac{Q_2}{Q_1} \cdot \frac{Q_3}{Q_2} \cdot \frac{Q_4}{Q_3} = \frac{Q_4}{Q_1},$$

just as if we compared the last period directly with the first. And not only this, but also the comparison through the serial forms between any distant periods would yield the same result as would a direct comparison between them, no matter what be the states of things at these periods or what the intervening variations, provided only the same classes be used throughout. Thus, so emended, the method universally satisfies Professor Westergaard's general test. It is not difficult to show that the same results would be obtained by using, between any periods in an epoch of n' periods, this formula, applied to any mass-units and their prices,

$$\frac{P_2}{P_1} = \frac{x_1 \sqrt[n']{a_1 a_2 a_3 \cdots\cdots} + y_1 \sqrt[n']{\beta_1 \beta_2 \beta_3 \cdots\cdots} + \cdots\cdots}{x_2 \sqrt[n']{a_1 a_2 a_3 \cdots\cdots} + y_2 \sqrt[n']{\beta_1 \beta_2 \beta_3 \cdots\cdots} + \cdots\cdots}.[6] \qquad (10)$$

But in comparing every period first with the period preceding and then with the period following, if the state of things at the latter period be irregularly different from that of the former, the numbers which express at the second period the mass-units then contained that are equivalent over the first and second periods alone are not the same, or in the same proportions, as the numbers which express at that period the mass-units then contained that are equivalent over the second and third periods alone. Therefore a direct comparison between the first and third periods in this same manner, namely by comparing the numbers contained in the classes at each period of the masses that are equivalent over the first and third periods alone, these numbers being again different, would not yield the same result as that yielded by the two former operations together. But if all prices are the same (or in the same proportion) at the third as at the first period, the mass-units used in both the comparisons are the same; and so, in such cases, there is necessary

[6] The same satisfaction would be given by

$$\frac{P_2}{P_1} = \frac{a_2 \sqrt[n']{x_1 x_2 x_3 \cdots\cdots} + \beta_2 \sqrt[n']{y_1 y_2 y_3 \cdots} + \cdots\cdots}{a_1 \sqrt[n']{x_1 x_2 x_3 \cdots\cdots} + \beta_1 \sqrt[n']{y_1 y_2 y_3 \cdots\cdots} + \cdots\cdots};$$

but this would not agree with the above in the individual results.

agreement between their final result and the direct comparison between the first and third periods. But, again, even if all prices are the same at a fourth or later period as at the first, there has been a break in the continuity of the mass-units used; and therefore the direct comparison, using one set of mass-units, and the indirect comparison, using two or more other sets of mass-units, will not necessarily coincide.

The direct comparison, however, we must observe, will be no more authoritative than the indirect comparison through the successive comparisons of the intervening periods. The operation which uses the mass-units that are equivalent over all the periods compared, would emend both those operations. Yet again, that even this operation would be authoritative, does not appear. For this procedure is similar to that of using the geometric average of the price variations with the same weighting over all the periods, this weighting to be according to the geometric averages of the money-values at all the periods—a procedure against which objection has already been raised.[7]

Moreover, in practice it is impossible to employ this consistent method in a long series, especially as it would have to be revised in full every year upon the adding of a new period. We are practically confined to the course of comparing every period with the periods immediately preceding and following, and of comparing distant periods through the mediation of such intervening measurements. It is desirable, then, to form an idea of the possible extent of the error in this procedure, and, if possible, of the principles that govern it.

§ 8. Suppose the class [B] is twice as large as the class [A], or, in other terms, **a** = 1 and **b** = 2. Then formula (9) becomes

$$\frac{P_4}{P_1} = \frac{\left[1 + 2\frac{a}{d} + 2\frac{d}{a} + 2\frac{e}{b} + 2\frac{b}{e} + 2\frac{f}{c} + 2\frac{c}{f} + 8\right] + 2\frac{d}{a} + 2\frac{b}{e} + 2\frac{c}{f}}{\left[1 + 2\frac{a}{d} + 2\frac{d}{a} + 2\frac{e}{b} + 2\frac{b}{e} + 2\frac{f}{c} + 2\frac{c}{f} + 8\right] + 2\frac{a}{d} + 2\frac{e}{b} + 2\frac{f}{c}}.$$

Again, suppose [B] is three times as large as [A]. Then

$$\frac{P_4}{P_1} = \frac{\left[1 + 3\frac{a}{d} + 3\frac{d}{a} + 3\frac{e}{b} + 3\frac{b}{e} + 3\frac{f}{c} + 3\frac{c}{f} + 27\right] + 6\frac{d}{a} + 6\frac{b}{e} + 6\frac{c}{f}}{\left[1 + 3\frac{a}{d} + 3\frac{d}{a} + 3\frac{e}{b} + 3\frac{b}{e} + 3\frac{f}{c} + 3\frac{c}{f} + 27\right] + 6\frac{a}{d} + 6\frac{e}{b} + 6\frac{f}{c}}.$$

7 In Chapt. V. § VI. near end.

livergent parts of the expressions are in the last three terms e numerators and denominators. If [B] be four times than [A], the common terms in the brackets would be $\frac{a}{d} + 4\frac{d}{a} + \cdots\cdots + 64$, and the three terms in excess would ıave 12 for their coefficient. Let $\left(\frac{d}{a} + \frac{b}{e} + \frac{e}{f}\right)$ be rep- ;ed by E′, and $\left(\frac{a}{d} + \frac{e}{b} + \frac{e}{f}\right)$ by E″. Then, with **b** = 2 **a**, ·st of the above formulæ may be expressed in this simple er,

$$\frac{P_4}{P_1} = \frac{[4\frac{1}{2} + E' + E''] + E'}{[4\frac{1}{2} + E' + E''] + E''};$$

ith **b** = 3 **a**, the other may be expressed thus,

$$\frac{P_4}{P_1} = \frac{[9\frac{1}{3} + E' + E''] + 2\,E'}{[9\frac{1}{3} + E' + E''] + 2\,E''};$$

ith **b** = 4 **a**, the formula would be

$$\frac{P_4}{P_1} = \frac{[16\frac{1}{4} + E' + E''] + 3\,E'}{[16\frac{1}{4} + E' + E''] + 3\,E''};$$

o on. In general, therefore, letting **a** = 1 and **b** = *r*, the rehensive formula will be

$$\frac{P_4}{P_1} = \frac{\left[r^2 + \frac{1}{r} + E' + E''\right] + (r-1)\,E'}{\left[r^2 + \frac{1}{r} + E' + E''\right] + (r-1)\,E''}, \qquad (11)$$

$$\frac{P_4}{P_1} = \frac{r^2 + \frac{1}{r} + rE' + E''}{r^2 + \frac{1}{r} + E' + rE''}. \qquad (12)$$

ıe other hand, if [A] be the larger class, everything will out merely to the inverse of the preceding, so that, with ′ and **b** = 1, the universal formula is

$$\frac{P_4}{P_1} = \frac{r'^2 + \dfrac{1}{r'} + E' + r'E''}{r'^2 + \dfrac{1}{r'} + r'E' + E''}. \tag{13}$$

In these general formulæ, if r (or r') = 1, then $\frac{P_4}{P_1} = 1.00$; which is in agreement with the third condition above noticed for the reduction of formula (10) to unity. Again, as r or r', increases toward infinity, the result, $\frac{P_4}{P_1}$, approaches toward unity. This is in agreement with what was to be expected; for as the one class infinitely predominates, the other sinks to nothingness, and the measurement is virtually by one class, but indirect (serial) comparisons with one class will always agree with direct comparisons with that class. These two facts show that somewhere between equality in size of the two classes and the infinite predominance of the one, there is an inequality at which the error is at its maximum. Now from the way E' and E'' are constructed, it is plain that they must always be *about* 3. This being so, it is easy to find that the greatest possible error must always happen when the one class is about four times as large as the other. Roughly, also, it may be perceived, the greatest possible error will lie between one third and one fourth of the difference between $\frac{E'}{E''}$ $\left(\text{or } \frac{E''}{E'}, \text{ as the case may be}\right)$ and unity.

Consider the following example:

I	100	A @ 1.00	200	B @ 1.00	100 for [A]	200 for [B].
II	66⅔	A @ 1.50	266⅔	B @ .75	100 for [A]	200 for [B].
III	50	A @ 2.00	300	B @ .66⅔	100 for [A]	200 for [B].
IV	100	A @ 1.00	200	B @ 1.00	100 for [A]	200 for [B].

To find $\frac{P_4}{P_1}$ as reached through the intervening comparisons, we need only calculate out E' and E'' and apply them in formula (11) or (12). They are found to be 3.2163 and 3.2556 respectively; whence $\frac{P_4}{P_1} = \frac{14.1882}{14.2275} = 0.9972$, indicating a fall of

0.28 per cent., and committing an error to that extent. If, keeping the same price variations, we supposed the mass-quantities to be such as to make the class [B] constantly three times larger than [A], we should have $\frac{P_4}{P_1} = \frac{22.2378}{22.3164} =$ 0.99647; if [B] were four times larger than [A], then $\frac{P_4}{P_1} = \frac{32.3708}{32.4887} = 0.99637$; if [B] were five times larger than [A], then $\frac{P_4}{P_1} = \frac{44.5371}{44.6943} = 0.99648$; — and thereafter the result will rise toward unity. Thus in this case the greatest possible error is about 0.36 per cent. below the truth. Here $\frac{E'}{E''} = 0.9879$, which is 1.21 per cent. below unity, and about $3\frac{1}{3}$ times as large as the greatest possible error.

If these examples were turned about, the same price variations being kept, but the class [A] being supposed the constantly larger, the results would be the reciprocals of the preceding. Thus with **a** = 2 **b**, $\frac{P_4}{P_1} = 1.0028$; with **a** = 3 **b**, $\frac{P_4}{P_1} =$ 1.00353; with **a** = 4 **b**, $\frac{P_4}{P_1} = 1.00363$; with **a** = 5 **b**, $\frac{P_4}{P_1} =$ 1.00352. The maximum error is about 0.36 per cent. above the truth.

In the former of these examples the erroneous results were below the truth, following the gradual fall from the first period of the larger class; and in the latter they were above the truth, following the gradual rise from the first period also of the larger class. If we inverted the intermediate periods, so that the third came second and the second third, the values of E′ and E″ would change places. Then in the former examples (with [B] larger) the erroneous results would be above the truth, following the gradual rise from the second period of the larger class; and in the latter examples (with [A] larger) the erroneous results would be below the truth, following the gradual fall from the second period still of the larger class.

If we make each class both rise and fall in price, we get the most satisfactory results by making them rise and fall together, as in the following:

I	100 A @ 1.00	200 B @ 1.00	100 for [A]	200 for [B],
II	66⅔ A @ 1.50	100 B @ 2.00	100 for [A]	200 for [B],
III	150 A @ .66⅔	266⅔ B @ .75	100 for [A]	200 for [B],
IV	100 A @ 1.00	200 B @ 1.00	100 for [A]	200 for [B].

Here $E' = 3.0153$ and $E'' = 3.0161$, whence $\frac{P_4}{P_1} = 0.99994$, indicating a fall of 0.006 per cent., with a possibility of error through larger size of [B] shown by $\frac{E'}{E''} = 0.9997$ not to exceed 0.01 per cent. below the truth. The class [A] being the larger, the results would be inverted,—as also were the intervening periods alternated. And we get the least satisfactory results by making the classes rise and fall in price oppositely,—thus:

I	100 A @ 1.00	200 B @ 1.00	100 for [A]	200 for [B].
II	66⅔ A @ 1.50	266⅔ B @ .75	100 for [A]	200 for [B].
III	150 A @ .66⅔	100 B @ 2.00	100 for [A]	200 for [B],
IV	100 A @ 1.00	200 B @ 1.00	100 for [A]	200 for [B]:

for here $E' = 3.5545$ and $E'' = 3.7339$, whence $\frac{P_4}{P_1} = 0.9884$, wrongly indicating a fall of 1.16 per cent. With [B] three times as large as [A], $\frac{P_4}{P_1} = 0.9851$; with [B] four times as large as [A], $\frac{P_4}{P_1} = 0.9845$; with [B] five times as large as [A], $\frac{P_4}{P_1} = 0.9846$; and thereafter $\frac{P_4}{P_1}$ grows greater toward unity. Thus the error may, in this extravagant example, be as great as about 1½ per cent. Again, if [A] were the larger class, the results would merely be inverted,—as also would be the case, again, if the second and third periods were reversed.

In a series of five periods, the formulæ would be still longer; and longer still in a series of six periods, and so on. These become too complex to work out. We may rely on trial. And trial still shows but slight errors.

In the above examples it is difficult to see a definite principle determining which way the error shall go, or how far. A statement more than once made that *in a series of four periods, if the second and third change places, the final result is inverted*, may be seen to be universal, by inspecting the universal formulæ. In a longer series this rule can apply only to the alternation of periods equally distant from the first and the last. It is certain, then, that there is nothing determining a continual error in one direction. And the errors, so small in the above extraordinary cases, will probably be but trifling in ordinary cases. In a general way there is some resemblance to the erroneousness already found in the geometric averaging of the price variations, although the errors are probably smaller here. When many classes are dealt with, there is no reason to suppose the errors will accumulate, but rather is there probability they will tend to neutralize one another. Also if the result happens to be above the truth at the fourth period, it is as likely as not to be below the truth at the fifth period, there being balancing from period to period. There is, then, extreme probability that in a long series this method will correct itself, and the results indicated by it will always hover in the neighborhood of the truth.

But the fact that this method shows error at all in a series of four or more periods—not to mention inconsistency in a series of three periods—casts suspicion upon its perfect correctness even in a comparison of two periods. And yet the reasoning by which it has been discovered has seemed to be sound. The method will later be involved in a complete method for all possible cases. Then will be the occasion to continue investigating the fault in it.

CHAPTER XI.

THE METHOD FOR CONSTANT MASS-QUANTITIES.

I.

§ 1. As in the case of the preceding argument, so the argument from compensation by equal sums of money has been seen to be applicable to all three averages not only when the subject of weighting is left out of account, but also when this is taken into consideration. The appearance of application solely to the arithmetic average is due to the selection of the mass-units in which the prices are reckoned; for attention is often transferred from compensation by equal sums to compensation by equal prices.

In a certain way by compensation by equal sums spent on constant mass-quantities[1] is meant a compensation by equal price variations; but it is an equal variation in the price primarily of the *whole* quantity of every class that is purchased at each period. Thus if we represent the prices at the two periods of the constant whole quantity of [A] by $\mathbf{a}_1$ and $\mathbf{a}_2$, and those of the constant whole quantity of [B] by $\mathbf{b}_1$ and $\mathbf{b}_2$, and confine our attention to these two classes, the position underlying the present argument postulates that $\frac{P_2}{P_1} = \frac{\mathbf{a}_2 + \mathbf{b}_2}{\mathbf{a}_1 + \mathbf{b}_1}$; and for constancy of the price-level, requiring that $\mathbf{a}_2 + \mathbf{b}_2 = \mathbf{a}_1 + \mathbf{b}_1$, it requires (supposing A rises and B falls in price) that $\mathbf{a}_2 - \mathbf{a}_1 = \mathbf{b}_1 - \mathbf{b}_2$.

But the whole masses of [A] and [B] may be divided into

[1] As in the preceding Chapter, all that is said here is equally applicable to cases in which at both periods are purchased mass-quantities in the same proportion, whether at the second period they be all larger or all smaller than at the first, *provided* all the reasoning (and all the formulæ later to be described) be applied only to the mass-quantities of the one or the other period—or only to what is common to both periods. Cf. also Appendix A, VII. § 5.

any numbers of mass-units of various sizes, the prices of which are the ones cited. Hence for constancy of the price level it is only the price variations of the aggregates of the mass-units that are required to equal each other,—not the particular price variations that happen to be reported. This is evident when we reflect that the one class may be much larger and more important than the other, in which case the price variation of any individual in the former is required by every one of the averages to be smaller than that of an equal individual in the latter. For the present, however, we shall suppose ourselves to be dealing with equally large or important classes, so as to have the convenience at first of dealing with even weighting.

Now when the whole mass-quantities of every class purchased at each period are constant, but there are variations of prices, we have seen that the weights of the classes are different at the two periods, because the weights are according to the total exchange-values (or money-values) of the classes, and these must have varied under the conditions supposed. Hence, wanting to use even weighting, we have the option between three systems: (1) even weighting of the *first* period, (2) even weighting of the *second* period, (3) even weighting of *both* the periods together. If it happens that the total money-values of the classes are equal at the first period, then the equal compensation in their variations are *from* equality to equally distant opposite positions —which we know to be arithmetic variations. If it happens that the total money-values of the classes are equal at the second period, then the equal compensation in their variations are from equally distant opposite positions *to* equality—which we know to be harmonic variations. If it happens that the total money-values of *two* classes are alternately equal, that of the one at the first period being equal to that of the other at the second, and reversely, then the equal compensation in their variations are from opposite positions to reversely opposite positions, traversing the same road in opposite directions,—which we know to be geometric variations.

If, further, we divide the total masses of each class into ideal mass-units that are equivalent at the first period, then if the total money-values of the classes happen to be equal at the first period, and the variations of these are to the opposite arithmetic extremes, the variations of the *prices* of these mass-units will also be to the opposite arithmetic extremes. But if the total money-values happen to be equal at the second period and their variations are from the opposite arithmetic terms, although we have compensation by equal sums, we do not in this case have compensation by equal prices of these mass-units, which have varied to the opposite harmonic extremes. And similarly there is no equal compensation in the prices of these mass-units when there may be such compensation in the sums, if the total money-values happen to be equal over both the periods together. Hence in these cases, although the compensation by equal sums really exists as well as in the first case, yet the compensation by equal variations of the prices that happen to be cited has disappeared. This is why the argument under consideration has seemed to be an argument peculiarly in favor of the arithmetic averaging of price variations; for statisticians, as we know, have formed the habit of using the variations of prices equal at the first period. But if we used mass-units equivalent at the second period, or over both the periods together, we should get compensation by equal prices (of such mass-units) even when the price variations are the harmonic or the geometric. We need now to review these possible positions.

§ 2. Using mass-units that are equivalent *at the first period*, we may construct the following schemata illustrative of the conditions when there is compensation by equality in the variations of the total money-values of the total mass-quantities—in the cases, to begin with, of two classes supposed to be equally important at some period or periods. The only changes in the construction of these schemata from those in the last Chapter is the omission of the sums of the mass-units at each period, and instead the insertion, at the extreme right, of the total sums of money expended at each period on both the classes. Thus on mass-units equivalent at the first period we have compensation by equal sums when the price variations are to the simple arithmetic extremes, as follows :

I 100 A @ 1.00	100 B @ 1.00	100 for [A] 100 for [B] — 200,
II 100 A @ 1.50	100 B @ .50	150 for [A] 50 for [B] — 200;

when the price variations are to the simple harmonic extremes:

I 100 A @ 1.00	200 B @ 1.00	100 for [A] 200 for [B] — 300,
II 100 A @ 1.50	200 B @ .75	150 for [A] 150 for [B] — 300;

when the price variations are to the simple geometric extremes:

I 100 A @ 1.00	150 B @ 1.00	100 for [A] 150 for [B] — 250,
II 100 A @ 1.50	150 B @ .66⅔	150 for [A] 100 for [B] — 250.[2]

Here it is only in the first schema that we have compensation by equal prices; but in all of them we have compensation by equal sums. We find also here the peculiarities with which we are familiar. In the first the sums are equal at the first period, and the compensation by equal sums is away from this equal condition. In the second the sums are equal at the second period, and the compensation is by equal variations from the first condition to this equal second condition. In the third the equal sums alternate and change places, traversing not only equal distances, but the same road.

That these peculiarities are universal may likewise be proved by the formula discovered in § 5 of Section II. in Chapter IX. In that formula the mass-units were supposed to be equivalent at the first period, wherefore it is applicable here. Letting the price of A always be supposed to rise from 1.00 to α_2', we know that if the price of B falls from 1.00 to the opposite arithmetic extreme it falls to $2 - \alpha_2'$, if to the harmonic, to $\frac{\alpha_2'}{2\alpha_2' - 1}$, if to the geometric, to $\frac{1}{\alpha_2'}$. Supplying these values of β_2' in the formula

$$y'' = \frac{\alpha_2' - 1}{1 - \beta_2'},$$

[2] We might also here make the total sums always add up to the same figures, as follows for the second and third:

I 66⅔ A @ 1.00	133⅓ B @ 1.00	66⅔ for [A] 133⅓ for [B]—200,
II 66⅔ A @ 1.50	133⅓ B @ .75	100 for [A] 100 for [B]—200;
I 80 A @ 1.00	120 B @ 1.00	80 for [A] 120 for [B]—200,
II 80 A @ 1.50	120 B @ .66⅔	120 for [A] 80 for [B]—200.

But here again the differences in the sums are different. Cf. Note 2 in Sect. I. of the last Chapter.

we find, when the price variations are to the arithmetic extremes,

$$y'' = 1\ ;$$

when they are to the harmonic extremes,

$$y'' = 2\,a_2' - 1\ ;$$

when they are to the geometric extremes,

$$y'' = a_2'.$$

The first of these expressions means that when the price variations are to the opposite arithmetic extremes, in order to have compensation by equal sums, we must purchase an equal number of mass-units of [A] and [B] ; but these mass-units are, by hypothesis, equivalent at the first period; therefore we must spend equal sums on [A] and [B], or spend our total sum evenly, at the first period. This, moreover, is evident, because the prices of these mass-units being supposed to vary to the opposite arithmetic extremes, it is only by these mass-units being purchased in equal numbers that the total sums will vary at the second period to the opposite arithmetic extremes. The second of these expressions means that when the price variations are to the opposite harmonic extremes, in order to have compensation by equal sums, we must purchase for every mass-unit of [A] $2\,a_2' - 1$ mass-units of [B] equivalent at the first period, that is, for every money-unit spent on [A] at the first period we must spend this number of money-units on [B] at that period; but at the second period we must spend a_2' on this 1 A and on these $(2\,a_2' - 1)$ B's we must spend $(2\,a_2' - 1) \cdot \dfrac{a_2'}{2\,a_2' - 1} = a_2'$, that is, we must spend equal sums on [A] and [B], or spend our total sum evenly, at the second period. The third means that when the price variations are to the opposite geometric extremes, in order to have compensation by equal sums, we must purchase at the first period for every 1 A a_2' B's, and these mass-units being equivalent at the first period, we must spend our money in these proportions at the first period, that is, a_2' money-units on [B] for every one money-unit on [A] ; but at the second period to purchase the 1 A we have to spend a_2' money-units, and to purchase the a_2' B's we have to spend $a_2' \cdot \frac{1}{a_2'} = 1$ money-unit, that is, the sums must at the second period be the reverse of what they were at the first.[3]

§ 3. If we use different mass-units, the formulæ and the results for y'' (the ratio between the mass-units in the classes) will be different; but the results above obtained for the total sums to be spent on [A] and [B] at each period will not be affected. Therefore these results are universal.

We may, then, pass quickly over the schemata for the conditions when we use mass-units equivalent at the second and over both periods. These may be easily obtained from the preceding by rearrangement of the mass-units, and their prices, of [B]. Thus with mass-units equivalent *at the second period*, we have, for the arithmetic price variations:

I	100 A @ 1.00	33⅓ B′ @ 3.00	100 for [A]	100 for [B]—200,
II	100 A @ 1.50	33⅓ B′ @ 1.50	150 for [A]	50 for [B]—200;

for the harmonic price variations:

I	100 A @ 1.00	100 B′ @ 2.00	100 for [A]	200 for [B]—300,
II	100 A @ 1.50	100 B′ @ 1.50	150 for [A]	150 for [B]—300;

for the geometric price variations:

I	100 A @ 1.00	66⅔ B′ @ 2.25	100 for [A]	150 for [B]—250,
II	100 A @ 1.50	66⅔ B′ @ 1.50	150 for [A]	100 for [B]—250,

Here the particular and total sums are the same as in the preceding examples, only the numbers of the mass-units of [B] being changed. It is now only in the harmonic price variations that compensation takes place by equal prices.

With mass-units equivalent *over both periods* we have for the arithmetic price variations:

I	100 A @ 1.00	57.73 B″ @ 1.7320	100 for [A]	100 for [B]—200,
II	100 A @ 1.50	57.73 B″ @ .8660	150 for [A]	50 for [B]—200;

[3] In this case, if out of 2 money-units we spend a_2' times more on [B] than on [A] at the first period, we must then spend $\frac{2}{a_2'+1}$ on [A] and $\frac{2a_2'}{a_2'+1}$ on [B]; and at the second period the reverse. These figures, as before, at arithmetic extremes around 1, are the harmonic means between 1 and $\frac{1}{a_2'}$, (*i. e.*, β_2') and 1 and a_2' respectively. Cf. Note 3 in Section I. of the last chapter.

for the harmonic price variations:

I 100 A @ 1.00	141.42 B″ @ 1.4142	100 for [A]	200 for [B]—300,
II 100 A @ 1.50	141.42 B″ @ 1.0606	150 for [A]	150 for [B]—300;

for the geometric price variations:

I 100 A @ 1.00	100 B′′ @ 1.50	100 for [A]	150 for [B]—250,
II 100 A @ 1.50	100 B′′ @ 1.00	150 for [A]	100 for [B]—250.

And here, too, the particular and total sums are the same as before, only the numbers of the mass-units, and their prices, of [B] being changed. And it is now only in the geometric price variations that compensation takes place by equal prices.

II.

§ 1. Now the argument from compensation by equal sums must claim that in every one of the three price variations above thrice schematized the exchange-value of money, or the general level of prices, remains constant, because this compensation exists in them all. But, in each of the three sets, in the first examples, which always illustrate the arithmetic price variations, the arithmetic average indicates constancy only with even weighting—and this is the weighting only of the *first* period. In the second examples, always illustrative of the harmonic price variations, the harmonic average indicates constancy only with even weighting—and this is the weighting only of the *second* period. In the third examples, illustrative of the geometric price variations, the geometric average indicates constancy only with even weighting—and this is the weighting only of *both* the periods together. And these conditions we know, by means of the formula above employed, to be universal.

We find also that if we use the arithmetic average upon every single one of these schemata with the weighting of the *first* period, which is always uneven in the second and third examples in each set, we always get indication of constancy; that if we use the harmonic average upon every single one of these schemata with the weighting of the *second* period, which is always uneven in the first and third examples, we always get indication of constancy; that if we use the geometric average upon single one of these schemata with the weighting of *both* eriods (geometrically calculated, according to the square of the products of the total money-values at each period, lained in Chapter IV. Section V. § 6), we always get, at ery nearly, indication of constancy.

m these particular facts, the first batch of which have al- been universalized, are to be derived two important infer- the first of which can be directly universalized, and the will be universalized presently so far as it admits of uni- zation.

The first of these is this:—Constant mass-quantities be- rchased at each period, we can make use of the argument compensation by equal sums in favor of the *arithmetic* e only if we confine ourselves to using this average with ighting of the *first* period; we can make use of this same ent in favor of the *harmonic* average if we confine our- to using it with the weighting of the *second* period; we ake use of the same argument in favor of the *geometric* e (with a modification) if we use it with the weighting of e periods together. The argument is as good for one of erages as for another—unqualifiedly in the case of the first -provided each be used with a special weighting; and for with other weightings the argument has no force whatever. is we see that on the assumption that this argument is t, and on the supposition that it has application (that con- nass-quantities are purchased at each period), each of the verages has for its own proper weighting each of the three of weighting possible—on the supposition of constant uantities and varying total money-values of the classes. se cases *the proper weighting to use with the arithmetic aver- price variations is the weighting of the first period: the weighting to use with the harmonic average of price varia- s the weighting of the second period; the proper weighting with the geometric average of price variations is the weight- both the periods* (the weight for every class being the tric mean between its two weights, one at each period).

e we might be inclined to argue in favor of the geometric average on the ground that the weighting it requires is alone the right one. But we are deterred from so doing by the second inference.

§ 3. This second inference is that *all three averages, each with its own proper weighting, applied to the same cases with constant mass-quantities sometimes yield the same results, and always the first two yield the same results, and in most (ordinary) cases the third departs but slightly from them.*

That, when the mass-quantities are constant, the arithmetic average of the price variations with weighting according to the sizes of the classes at the first period and the harmonic average of them with weighting according to the sizes of the classes at the second period are identically the same universally is demonstrated in Appendix A, IV. § 3. For they both, being averages of price variations, reduce to a comparison of the same price averages (arithmetic, therefore in the same ratio as the price totals). Thus, condensing that demonstration, we have

$$\frac{P_2}{P_1} = \frac{1}{x\alpha_1 + y\beta_1 + \cdots\cdots}\left(x\alpha_1\frac{\alpha_2}{\alpha_1} + y\beta_1\frac{\beta_2}{\beta_1} + \cdots\cdots\right) \qquad (1)$$

$$= \frac{1}{\dfrac{1}{x\alpha_2 + y\beta_2 + \cdots\cdots}\left(x\alpha_2\dfrac{\alpha_1}{\alpha_2} + y\beta_2\dfrac{\beta_1}{\beta_2} + \cdots\cdots\right)} \qquad (2)$$

$$= \frac{x\alpha_2 + y\beta_2 + \cdots\cdots}{x\alpha_1 + y\beta_1 + \cdots\cdots}. \qquad (3)$$

This is nothing else than Scrope's method of measuring variations in the level of prices, rightly confined to cases when constant mass-quantities are supposed to be given. A shorter, but less perspicuous, formula for this method is

$$\frac{P_2}{P_1} = \frac{\mathbf{a}_2 + \mathbf{b}_2 + \cdots\cdots}{\mathbf{a}_1 + \mathbf{b}_1 + \cdots\cdots}. \qquad (4)$$

Thus in all the schemata employed in this Chapter, to find the constancy or variation of the general level or average of prices (or the average variation of prices), we have only to divide the total sum supposed to be spent on all the classes at the second period by the total sum spent on them at the first period. The more laborious operations of averaging the price variations arithmetically with the weights of the first period, or of averaging them harmonically with the weights of the second period, may be dispensed with.

That, on the same assumption of constancy in the mass-quantities purchased at both periods, with additional confinement to the supposition that we are dealing with the variations of only *two* classes, which, furthermore, are such that the geometric means of their two weights (at the two periods) are equal, so that we can use the geometric average of their price variations with *even* weighting, the geometric average with this weighting, that is, the geometric *mean*, universally yields exactly the same results as the other two averages, applied to these same cases, each with its proper weighting, is demonstrated in Appendix A, VI. § 7. There also, in § 9, it is shown that in all cases, with any number of classes, with variously uneven weighting, with all the moderate irregularities in the sizes of the classes such as are likely to exist in our statistical lists, the geometric *average* with its proper weighting always yields results very *nearly* the same as the single results yielded by the other two averages, each with its proper weighting. Of these more complex cases we shall treat in a later Section. Here we may continue to confine our attention mostly to the simple cases where all three of the averages, or rather means, each with its proper weighting, coincide.

§ 4. A pretty illustration of this coincidence may be offered in the following, which provides us with what may be called an argument by transposition of prices. Suppose we have at each period 1 A and 1 B, and it happens that their prices become transposed, that is, that the price of A at the second period is the same as B's at the first and that of B at the second the same as A's at the first, so that $\alpha_2 = \beta_1$ and $\beta_2 = \alpha_1$. Then also the importance of these articles is transposed—and if there be the same number of them in their classes, the sizes of the classes are to each other as these prices, and are transposed (as in the third schema in the last set above). Under these variations it is perfectly evident that the exchange-value of money suffers no variation. Now this constancy of the exchange-value of money, as shown in the level of prices, is indicated by the arithmetic average of these price variations with the weighting of the first period, thus $\dfrac{\alpha_1\frac{\alpha_2}{\alpha_1}+\beta_1\frac{\beta_2}{\beta_1}}{\alpha_1+\beta_1}$, and by the harmonic average of them with the weighting of the second period, thus $\dfrac{\alpha_2+\beta_2}{\alpha_2\frac{\alpha_1}{\alpha_2}+\beta_2\frac{\beta_1}{\beta_2}}$, because each of these reduces to $\dfrac{\alpha_2+\beta_2}{\alpha_1+\beta_1}$, which, on account of the equality between α_2 and β_1 and between β_2 and α_1, is equal to 1.00; and also by the geometric average with the weighting of both the periods, which is even, since $\sqrt{\alpha_1\alpha_2}=\sqrt{\beta_1\beta_2}$, because the expression $\sqrt{\frac{\alpha_2}{\alpha_1}\cdot\frac{\beta_2}{\beta_1}}$ is equal to 1.00 also on account of the equality between those terms. Hence the argument based upon this perfect evidence is equally good for either of these averages, each with its own proper weighting.[1]

§ 5. On account of this coincidence, when the physical foundations are the same at both periods, it is futile to argue for any one average alone, or for any one weighting alone, in preference to any other. As regards mere convenience, Scrope's method is the most serviceable of all, in these cases; and this represents no one of the three averages of the price variations more than

[1] This example may be widened, showing the same identity between the averages differently weighted. Let the point O_1 somewhere divide the line AB at the

A O_1 O_2 B

first period, and the point O_2 somewhere divide it at the second period. Then the segment AO_1 has become AO_2, and the segment O_1B has become O_2B, but these variations of the segments have not affected the whole line: wherefore the average of the variations evidently is unity, indicating constancy. Now whatever be the variations, provided neither segment be zero in length at either period, constancy is shown by the arithmetic average of them with the weighting of the first period, and by the harmonic average of them with the weighting of the second period; and provided they be such that $AO_1 = O_2B$ and $AO_2 = O_1B$, also the geometric mean (with even weighting) indicates constancy (and in other cases the geometric average with the weighting of both periods, provided the weight of the one be not more than two or three times larger than that of the other, gives a result very nearly equal to unity).

another, except in complex cases, when the geometric average deviates. There is little to say in favor of the arithmetic average, or in favor of the weighting of the first period, each by itself; but if together they give the right result, we cannot complain. Similarly with the harmonic average and the weighting of the second period. The average and the weighting that have most reason in their favor singly are the geometric and the weighting of both the periods. Yet we find that these give no better results than the others properly combined,—and in complex cases we shall find that their results are not so good as those of the others. The greatest errors arise from using an average with weighting not suitable to it—the arithmetic with the weighting of the second or of both periods, the harmonic with the weighting of the first or of both periods, the geometric with the weighting either of the first or of the second period alone.

In reality, however, not one, but all the averages and weightings are used. In Chapter IV. (Sec. V. § 4) we found it absurd to use the weighting of either period alone. But now, in using Scrope's method, we are not using the weighting of either period alone. For we are using the weighting of the first period alone only with the arithmetic average; but equally well are we using the weighting of the second period—alone with the harmonic average. The one true method which combines both these averages, combines both these weightings.[2] And in the special cases where the result given by it is necessarily given also by the geometric mean, this is with the use of the combined weightings of both the periods.[3]

[2] In Note 12 in that Chapter and Section it was stated that a variation is not properly a variation of the individuals existing at either period alone, but it is *from* the individuals at the first period *to* the individuals at the second. We have now found that in averaging *from* individuals at the first period the proper average to use is the arithmetic with the weighting of the *first* period; and that in averaging *to* individuals at the second period the proper average to use is the harmonic with the weighting of the *second* period. Thus we have perfect harmony throughout.

[3] Students of German philosophy will notice a peculiar resemblance between the three averages and the three categories in each of Kant's four divisions, and the three terms in the trichotomy of Hegel. For the arithmetic and harmonic averages are *opposed* to each other (although with likewise opposite weighting

§ 6. Consequently in his dispute with Laspeyres, in which the price of one article was supposed to rise from 1.00 to 2.00 and that of another to fall from 1.00 to .50, Jevons would have been right in considering the exchange-value of money constant, as indicated by the geometric average with even weighting, if he could have shown, or had added as part of the supposition, that the two classes were equally important over both the periods together. But in this case the other two averages, each with its proper weighting, would make the same indication. On the other hand Laspeyres would have been right in using the arithmetic average with even weighting, and in concluding that the price level had risen by 25 per cent., had he been careful to specify that he was dealing with classes equally important at the first period, and only with such. But in this case the harmonic average with its proper weighting would make the same indication, while the geometric average with its proper weighting, would diverge only by indicating a rise of 26 per cent. Again, Jevons, in suggesting the use of the harmonic average, still with even weighting, which would indicate a fall by 20 per cent., would have been right, had he rested on the condition of the two classes being equally important at the second period only. But in this case also the arithmetic average with its proper weighting would make the same indication, and the geometric average with its proper weighting would diverge only by indicating a fall of 20.64 per cent. All this, however, is said only on the supposition that both these writers agreed in assuming that constant mass-quantities of each class were purchased at each period.[4]

they agree), and the geometric average *synthesizes* these opposites (with weighting which likewise synthesizes their weightings, provided this composite weighting be even). Unfortunately this is exactly so only when the geometric average is restricted to being a mean proper.

[4] This assumption was used in Laspeyres's reasoning, as also that the classes were equally important at the first period, and so he happened to be right in his conclusion, confined to these conditions. But he never recognized these conditions, nor confined his argument to them, so that in general his position was no better than Jevons's. Had Jevons made the assumption that constant equal sums were spent on the two classes, his choice of the geometric mean would have been exactly right, as we have seen,—but not so his choice of the geometric average in wider cases. But he made this assumption only in connection with the harmonic average, which then is wrong.

On this assumption we find Jevons's prophecy fulfilled. He supposed another case, in which the price of one article remains unchanged at 1.00 and that of another rises from 1.00 to 2.00, and said that "the mean rise of price might be variously stated" as the arithmetic at 50 per cent., as the geometric at 41, or as the harmonic at 33, and added the sentence already quoted: "It is probable that each of these is right for its own purposes when these are more clearly understood in theory."[5] Strictly speaking, none of the averages has any purposes of its own, but each one is subject to certain conditions. Thus in the case supposed, with further assumption of the mass-quantities being constant, the arithmetic mean, indicating a rise of 50 per cent., is right if the two articles were equally important at the first period, the harmonic, with rise by 33 per cent., if they were equally important at the second period, and the geometric, with rise by 41 per cent., if they were equally important over both the periods. If their importance was wholly uneven, the mean rise might be any figure between 0 and 100 per cent.—and the right figure would be indicated *either* by the arithmetic average with the weighting of the first period, *or* by the harmonic average with the weighting of the second period, *or* (approximately in many cases) by the geometric average with the weighting of both the periods.[6]

But we have anticipated somewhat, and must now seek to prove that this common result of the two averages always, and sometimes of all three, is the right one.

III.

§ 1. The reason why the argument from compensation by equal sums has been mistaken for an argument specially favoring the arithmetic average of price variations is because in making use of it the advocates of this average have had in mind such conditions as are illustrated in these three schemata,—for the arithmetic price variations:

[5] B. 23, p. 121.

[6] We have seen something similar in the preceding Chapter. But there the weightings corresponding to these were of another kind, and hidden (except in the third instance), so as not to be serviceable (being there, as here, inexact in the third instance).

I	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B] — 200,
II	100 A @ 1.50	100 B @ .50	150 for [A]	50 for [B] — 200;

for the harmonic price variations:

I	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B] — 200,
II	100 A @ 1.50	100 B @ .75	150 for [A]	75 for [B] — 225;

for the geometric price variations:

I	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B] — 200,
II	100 A @ 1.50	100 B @ .66⅔	150 for [A]	66⅔ for [B] — 216⅔;

in which, the classes always being supposed equally large at the first period, compensation by equal sums (and by equal prices) takes place only in the first example, where also the arithmetic average with even weighting indicates constancy. If, however, any one had wanted to use this argument for the harmonic average of price variations, he might have made use of the following schemata,—for the arithmetic price variations:

I	100 A @ 1.00	100 B′ @ 3.00	100 for [A]	300 for [B] — 400,
II	100 A @ 1.50	100 B′ @ 1.50	150 for [A]	150 for [B] — 300;

for the harmonic price variations:

I	100 A @ 1.00	100 B′ @ 2.00	100 for [A]	200 for [B] — 300,
II	100 A @ 1.50	100 B′ @ 1.50	150 for [A]	150 for [B] — 300;

for the geometric price variations:

I	100 A @ 1.00	100 B′ @ 2.25	100 for [A]	225 for [B] — 325,
II	100 A @ 1.50	100 B′ @ 1.50	150 for [A]	150 for [B] — 300;

in which, the classes always being supposed equally large at the second period, compensation by equal sums (and by equal prices) takes place only in the second example, where constancy is indicated, even weighting being used, only by the harmonic average. Again, if any one had wanted to use this argument for the geometric mean of price variations, he might have made use of these schemata,—for the arithmetic price variations:

I	100 A @ 1.00	100 B″ @ 1.7320	100 for [A]	173.20 for [B] — 273.20,
II	100 A @ 1.50	100 B″ @ .8660	150 for [A]	86.60 for [B] — 236.60;

for the harmonic price variations:

I	100 A @ 1.00	100 B″ @ 1.4142	100 for [A]	141.42 for [B] — 241.42
II	100 A @ 1.50	100 B″ @ 1.0606	150 for [A]	106.06 for [B] — 256.06;

for the geometric price variations:

I	100 A @ 1.00	100 B″ @ 1.50	100 for [A]	150 for [B]	— 250,
II	100 A @ 1.50	100 B″ @ 1.00	150 for [A]	100 for [B]	— 250;

n which, the classes always being supposed equally large over oth the periods together, compensation by equal sums (and by qual prices) takes place only in the third example, where contancy is indicated, even weighting being used, only by the eometric average.

The error in all these applications of the argument is mere gnoring of the fact that each of the averages with its own roper weighting applied to every one of these examples gives xactly the same results (except in those where the weighting or the geometric average is not even, its results then deviating omewhat).

§ 2. But although we cannot use the argument to distinguish etween the three averages, we can use it to show the correctess of all three averages (the third, however, only in special ases), each with its own proper weighting—always on condition f constant mass-quantities. If we bring together the three implest schemata on which the advocates of each average may ely, as follows—for the arithmetic price variations:

I	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B]	— 200,
II	100 A @ 1.50	100 B @ .50	150 for [A]	50 for [B]	— 200;

or the harmonic price variations:

I	100 A @ 1.00	100 B′ @ 2.00	100 for [A]	200 for [B]	— 300,
II	100 A @ 1.50	100 B′ @ 1.50	150 for [A]	150 for [B]	— 300;

or the geometric price variations:

I	100 A @ 1.00	100 B″ @ 1.50	100 for [A]	150 for [B]	— 250,
II	100 A @ 1.50	100 B″ @ 1.00	150 for [A]	100 for [B]	— 250;

ve see the reason for the correctness of each average with its own proper weighting—always indicating constancy with perfect learness only in the third example; for in the others, although ve have compensation by equal sums, it is on classes, and individuals in them, that have different exchange-values over both he periods together. But, as already noticed, there is a corection accompanying each deviation. In the first example, the lass [B] is less important than the class [A] over both the periods together; but its price falls more than the price of [A] rises, since we know that a fall from 1.00 to .50 is greater than a rise from 1.00 to 1.50. And in the second case the class [B] is more important than the class [A] over both the periods together; but its price falls less than the price of [A] rises, since we know that a fall from 1.00 to .75 is smaller than a rise from 1.00 to 1.50. Still we do not yet perceive that in the former case the price of [B] falls exactly as much more as it ought to do, to make up for the smaller importance of its class; nor that in the latter case, it falls exactly as much less as it ought to do, to allow for the greater importance of its class. But we may learn it with perfect certainty by the following reasoning.

The first example may be converted into the following, already used, by merely employing a different mass-unit of [B], with prices inversely altered:

I	100 A @ 1.00	57.73 B″ @ 1.7320	100 for [A]	100 for [B] — 200,
II	100 A @ 1.50	57.73 B″ @ .8660	150 for [A]	50 for [B] — 200.

Here the mass-unit of [B] is equivalent to the mass-unit of [A] over both the periods, so that A and B″ may be taken as economic individuals. Now we purchase 157.73 such individuals at each period, and we pay exactly the same sum for them at each period. Therefore our money has retained exactly the same purchasing power or exchange-value over both the periods.[1] And the second example may be converted into the

[1] The fact that 100 of these individuals are in [A] and 57.73 in [B] merely shows that [A] is $\frac{100}{57.73} = 1.732$ times larger than [B]. This we already know from the fact that $\sqrt{\frac{100 \times 150}{100 \times 50}} = \sqrt{3} = 1.732053$. With this weighting the geometric average indicates a rise by 0.3 per cent., and is by so much wrong.—In Chapt. VIII. Sect. III. § 1 we saw inconsistency in the argument of the arithmetic averagist on the supposition of the classes always being equally important. But here the two classes, equally important at the commencement, are not thereafter equally important, and the inconsistency vanishes. Thus the arithmetic averagist argues that if [A] and [B] are equally important at the first period their compensatory variations should be to equal distances from their equal starting points. Suppose A rises from 1.00 to 1.50 and B falls from 1.00 to .50. Then [A] is three times more important than [B], and therefore, starting from this position as a new first period, the price of ⅔ A should rise from 1.00 only one third as far as the price of 2 B falls from 1.00. This is precisely what takes place when A continues to rise from 1.50 to 1.51, while B continues to fall from .50 to .49.

following, also already used :

I	100 A @ 1.00	141.42 B″ @ 1.4142	100 for [A]	200 for [B]	— 300,
II	100 A @ 1.50	141.42 B″ @ 1.0606	150 for [A]	150 for [B]	— 300.

Here also the mass-unit of [B] is equivalent to the mass-unit of [A] over both the periods, so that A and B″ may be taken as economic individuals. Now we purchase 241.42 such individuals at each period, and we pay exactly the same sum for them at each period. Therefore our money has retained exactly the same purchasing power or exchange-value over both the periods.[2]

§ 3. We thus obtain also here—that is, applicable only to cases with constant mass-quantities—a precise method of measuring the constancy or variation of the general exchange-value of money. It is : *Find in all the classes, for use as units, masses that have the same money-value over both the periods together, and measure the constancy or variation of the exchange-value of money inversely by the constancy or variation in the total sum of money needed at each period to purchase the constant numbers of these mass-units supposed to be actually purchased.* Naturally this method is not confined to cases with only two classes.

In this method, however, the first part is superfluous, since, whatever be the mass-quantities in the various classes, we know that they must contain certain numbers of mass-units that are equivalent over both the periods, which numbers will be constant if the mass-quantities are constant ; but as we make no use of these numbers when ascertained, it is unnecessary to ascertain them. All we need, then, is to *measure the constancy or variation of the exchange-value of money inversely by the constancy or variation in the total sum of money needed at each period to purchase the constant mass-quantities of all the classes.*

The formula carrying out this method, we may repeat, in its simplest form, is the following,

[2] Here, [B] containing 141.42 and [A] 100 of these individuals, the former is 1.4142 times larger than the latter. This we also know from the fact that $\sqrt{\frac{200\times 150}{100\times 150}} = \sqrt{2} = 1.414213$. With this weighting the geometric average indicates a fall by 0.06 per cent., and is by so much wrong.

$$\frac{P_2}{P_1} = \frac{\mathfrak{a}_2 + \mathfrak{b}_2 + \mathfrak{c}_2 + \cdots\cdots}{\mathfrak{a}_1 + \mathfrak{b}_1 + \mathfrak{c}_1 + \cdots\cdots}, \qquad (4)$$

or this,

$$\frac{P_2}{P_1} = \frac{x\alpha_2 + y\beta_2 + z\gamma_2 + \cdots\cdots}{x\alpha_1 + y\beta_1 + z\gamma_1 + \cdots\cdots}, \qquad (3)$$

(in which it is evident the sizes of the mass-units used have no influence). The last, to repeat also, we recognize to be the formula for Scrope's method, which, therefore, is the correct method for the cases in question.

Now in the preceding Chapter (Sect. III. § 3) we found the method there discovered for cases with constant sums to be Scrope's method applied to the geometric means of the mass-quantities. But if we take the formula for that form of Scrope's method (there given as No. 8), and apply it to cases in which the two mass-quantities in every class, the one at the one period and the other at the other, are the same, the formula reduces to the last formula above. Or, reversely, by distinguishing x into x_1 and x_2, to represent the mass-quantities at each period notwithstanding that they are equal, and distinguishing the other symbols for the mass quantities in the same way, we may derive from the last expression this,

$$\frac{P_2}{P_1} = \frac{\alpha_2\sqrt{x_1x_2} + \beta_2\sqrt{y_1y_2} + \gamma_2\sqrt{z_1z_2} + \cdots\cdots}{\alpha_1\sqrt{x_1x_2} + \beta_1\sqrt{y_1y_2} + \gamma_1\sqrt{z_1z_2} + \cdots\cdots}. \qquad (5)$$

Thus *Scrope's method applied to the geometric means of the mass-quantities is a comprehensive method, applying both to the cases with constant sums of money and to the cases with constant mass-quantities.*

§ 4. Therefore, like the method examined in the preceding Chapter for the cases with constant sums, this method for the cases with constant mass-quantities satisfies all the Propositions that more or less definitely prescribe what the variation of money in exchange-value in all other things, and consequently the inverse variation of the general level of prices, must be. Thus we see plainly that it indicates constancy if no prices vary, and if all prices vary alike it indicates the same variation (Propositions XXVII., XLIV., XVII., XLV.), no matter what be the weights (provided, of course, they be such that, with the price variations, they show the mass-quantities to be constant); nor can it indicate constancy if there is only one price variation, or if all price variations are in one direction—and in the latter event it of course cannot indicate a variation in the opposite direction (Propositions XX. and XXVIII.). It also satisfies Proposition XXXVI., if its own condition be observed in the omitted classes. This is so plain as not to need to be demonstrated. But again a similar remark has to be made here as was made in the corresponding passage in the preceding Chapter (Sect. III. § 4). That Proposition[3] does not require the omitted class or classes to be of the same mass-quantities at both the periods. Yet this method requires that they should be; for if they are not and its principal formula, (3), is extended to them (but altered, so as to distinguish between the new mass-quantities at each period), the two results will not agree. Yet it is not *this* method which is being used and disproved, but *another* method; and *this* method satisfies the test offered by that Proposition perfectly.[4]

§ 5. Moreover, as above seen, the arithmetic average of the price variations with the weighting of the first period and the harmonic average of them with the weighting of the second period are, under the given condition, universally the same as this method. Therefore these averages, with these weightings, are both correct. But they are superfluous, since Scrope's method is simpler and more convenient.

And the geometric average with the weighting of both periods also reduces to this method in all cases when we are dealing with the variations of two equally (over both periods) important classes, so as to be able to use even weighting. This is so not only when the result indicated is constancy, but also when the result indicated is a variation. We have already noticed several

[3] Nor Proposition XXXII., when the prices are constant.

[4] But again a similar qualification has to be added here also. What is said in the text may be said with reference to formulæ either (1), (2), (3), or (4). But formula (5) satisfies the Proposition. In other words, this method satisfies the Proposition if we treat the additional different mass-quantities as this formula prescribes.

instances of the former kind; we may now notice a couple of the latter. These may be taken from the last set of schemata given above in § 1, where the labor has already been performed of obtaining not only classes, but individuals, equally important over both periods. In the first of these examples we see, therefore, that we have $\frac{P_2}{P_1} = \frac{236.6}{273.2} = 0.8660$, indicating a fall of 13.40 per cent. The geometric mean of the price variations is $\sqrt{\frac{3}{2} \times \frac{1}{2}} = \frac{\sqrt{3}}{2} = 0.8660$, indicating the same fall. In the second example we have $\frac{P_2}{P_1} = \frac{256.06}{241.42} = 1.0606$, indicating a rise of 6.06 per cent. The geometric mean is $\sqrt{\frac{3}{2} \times \frac{3}{4}} = \frac{3}{\sqrt{8}}$ $= 1.0606$, indicating the same rise.

In each of these examples it may be noticed that the variation of general prices is from 1.00 to the same figure as in the price of B″ at the second period. This relation is universal under the conditions supposed. These are that $\alpha_1'' = 1.00$, $x'' = y''$ and $\mathbf{a}_1\mathbf{a}_2 = \mathbf{b}_1\mathbf{b}_2$. For from the latter, with the aid of other known relations, we derive $\frac{\mathbf{a}_2}{\mathbf{b}_1} = \frac{\mathbf{b}_2}{\mathbf{a}_1} = \frac{\mathbf{a}_2 + \mathbf{b}_2}{\mathbf{a}_1 + \mathbf{b}_1} = \frac{x''\alpha_2''}{y''\beta_1''} =$
$\frac{y''\beta_2''}{x''\alpha_1''} = \frac{x''\alpha_2'' + y''\beta_2''}{x''\alpha_1'' + y''\beta_1''} = \frac{P_2}{P_1} = \frac{M_{01}}{M_{02}} = \frac{\alpha_2''}{\beta_1''} = \frac{\beta_2''}{\alpha_1''} = \beta_2'' = \frac{\alpha_2'' + \beta_2''}{\alpha_1'' + \beta_1''}$
$= \sqrt{\frac{\alpha_2}{\alpha_1} \cdot \frac{\beta_2}{\beta_1}}$. Q. E. D.[5]

With uneven weighting the geometric average of the price variations does not necessarily agree with the common result given by the other two averages, each with its proper weighting. In showing that the other two averages always agree with the proper method we have shown that their common result is right. We need, therefore, not so much to show that the geometric average is wrong when it diverges, as to investigate its deviations. It is evident at once that the geometric average will

[5] Compare these with corresponding relations in the preceding Chapter, Sect. II. § 4, Note 3.

agree with the true method here in all cases corresponding to the cases in which we have found that it would agree with the other true method, in the preceding Chapter (Sect. III. § 5). Some inferences similar to inferences in that chapter made about he deviations will also follow.

IV.

§ 1. What we desire to prove and to illustrate may be shown ere on a single example. Let us suppose the class [B] to be wice as important, over both the periods together, as the class A]. The price of A rising from 1.00 to 1.50, to have contancy according to the geometric average with this weighting, he price of B must fall from 1.00 to $\sqrt{\frac{2}{3}} = .8165$; and in order o have this weighting with these price variations, the massuantities must be in the following proportions:

I	100 A @ 1.00	271.08 B @ 1.00	100 for [A]	271.08 for [B]	— 371.08,
I	100 A @ 1.50	271.08 B @ .8165	150 for [A]	221.34 for [B]	— 371.34,

n which $\sqrt{100 \times 150} = \frac{1}{2}\sqrt{271.08 \times 221.34}$. Here Scrope's nethod shows a rise of prices, viz., $\frac{371.34}{371.08} = 1.000698$—a ise by 0.0698 per cent. That this is right is more apparent ipon rearranging the mass-unit of [B] as follows:

I	100 A @ 1.00	200 B″ @ 1.3554	100 for [A]	271.08 for [B]	— 371.08,
I	100 A @ 1.50	200 B″ @ 1.1067	100 for [A]	221.34 for [B]	— 371.34.

For here the mass-unit of [B] is of the same exchange-value over both the periods as the mass-unit of [A]; wherefore, as we iave to pay more for these equivalent mass-units at the second period, it is evident that their prices have risen. Thus Scrope's nethod is right in its indication, and the geometric method errs y indicating constancy when it ought to indicate a slight rise, and so, in this case, it is slightly below the truth (by 0.06976 per cent.).

In this example the figures for [B] could also be arranged as follows:

I 180.72 B′ @ 1.50,
II 180.72 B′ @ 1.2247,

or

I 221.34 B^{II} @ 1.2247,
II 221.34 B^{II} @ 1.00,

these falls being the same as from 1.00 to .8165, and the sums spent on [B] being the same as before. Or we could take half of each of these, as follows:

I 90.36 B^{I} @ 1.50 + 110.67 B^{II} @ 1.2247,
II 90.36 B^{I} @ 1.2247 + 110.67 B^{II} @ 1.00;

and here the price of [B] has fallen from 1.50 to 1.00, inversely as the price of A has risen from 1.00 to 1.50. But it is not equal masses of [B] that hand on this fall. We virtually have three classes. And in the quantities indicated the three classes are equally important over both the periods; and the geometric average and the true method give the same divergent results as before.

Now suppose another distinct case, in which it happens that at both periods we purchase 100 A, 100 B, and 100 C, and suppose the price of A rises from 1.00 to 1.50, and the price of B falls from 1.50 to 1.2247 and the price of C from 1.2247 to 1.00. Here the classes [B] and [C] together make a fall exactly the reverse of the rise of the single class [A], and as the individuals in each of these classes are equal in number, every rise of 1 A from 1.00 to 1.50 seems to be met by a fall of 1 B and 1 C from 1.50 to 1.00. Therefore we should expect constancy. And constancy is indicated by Scrope's method, which shows that

$$\frac{P_2}{P_1} = \frac{150 + 122.47 + 100}{100 + 150 + 122.47} = 1.00.$$

But if we apply the geometric method to these conditions we must give these weights to the classes—to [A] $\sqrt{1.00 \times 1.50} = 1.2247$, to [B] $\sqrt{1.50 \times 1.2247} = 1.3554$, to [C] $\sqrt{1.2247 \times 1.00} = 1.1067$, and now

$$\frac{P_2}{P_1} = \sqrt[3.6868]{\left(\tfrac{3}{2}\right)^{1.2247} \times \left(\sqrt{\tfrac{2}{3}}\right)^{1.3554 + 1.1067}} = 0.999302 \ldots\ldots,$$

indicating a fall by a trifle less than 0.069 per cent. To get constancy here by the geometric method we should have to weight the common fall of [B] and [C] as 2 to the rise of [A] as 1. This we could do if it were proper to weight [B] and [C] by doubling the geometric mean of their weights, the class [B] being as much more important as the class [C] is less important than the class [A]. But we can form no general principle of weighting of this sort. For instance, in the preceding example this kind of weighting would still weight the class [B] as two to [A] as 1, and yet with this weighting the geometric average there showed about the same error.

Hence it appears that in these complex cases, even on the principle of the geometric method itself, Scrope's method is correct, and the geometric average of price variations, with the best weighting we know of, is wrong so far as it diverges.

§ 2. In the examples reviewed the preponderating class or classes have been the ones that fall in price, and the geometric average has been found to err below the truth. Making [A] the larger class, we should find the error to lie on the other side. These facts, added to what we already know about the deviation of the geometric average,[1] lead to the inference also here that *when the prices that rise above the general average are those of preponderating classes, the geometric average of price variations yields a result below the truth; and when the prices that fall below the general average are those of preponderating classes, it yields a result above the truth.*

Luckily, as we have another method which is not only exactly correct but far more convenient, we are not so much interested in the error of the geometric method as we were in the preceding Chapter. We may be sure, however, here as well as there, that with moderate price variations such as usually take place, the geometric average will not much deviate from the truth.

§ 3. Some extraordinary cases deserve a moment's attention. Suppose the classes [A] and [B] are equally important at the first period, and the price of A rises from 1.00 to 1.99, to what figure ought the price of B to fall from 1.00 in order to compensate for that rise?—always supposing that the mass quanti-

[1] Cf. other instances in Notes 1 and 2 in the preceding Section.

ties are constant. We can now answer without hesitation: it ought to fall to .01; for the arithmetic average with the even weighting of the first period indicates constancy under these price variations, and this average with this weighting is correct.[2] Now we might expect, from the reasoning in Chapters VII. and VIII., that according to the geometric method this fall would be made out to be too great. On the contrary, the conditions are not what were there supposed, and they require the geometric average, applied to these conditions, to indicate a rise—and, because of the enormous variation in the price of B, with consequent influence upon the relative sizes of the classes, a considerable rise—in the general level of prices, meaning that there has not been sufficient compensation. In fact we find that the geometric average here indicates a general rise by 40.16 per cent. And if we suppose that B falls from 1.00 to .01, and want the compensatory rise for A so as to make money constant in exchange-value, instead of requiring A to rise from 1.00 to 1.99, the geometric method requires it to rise from 1.00 only to 1.463. The truth is that, when dealing with such enormous price variations of the smaller class, as shown in similar instances in the preceding Chapter, the geometric method becomes wholly unworkable. To show this we may compare the results given by the geometric average with its proper weighting with the true results, indicated by the arithmetic average with even weighting, when, the classes [A] and [B] being equally large at the first period, and the mass-quantities remaining constant, the price of A is supposed in all cases to rise from 1.00 to 1.99, and the

[2] As is rendered plain by the following schema:

I	100 A @ 1.00	7.092 B″ @ 14.10	100 for [A]	100 for [B] — 200,
II	100 A @ 1.99	7.092 B″ @ .141	199 for [A]	1 for [B] — 200,

in which the mass-units, A and B″, are equivalent over both the periods together, and the same total sum is paid at each period for purchasing these equivalent mass-units. (To have even weighting over both the periods together, the conditions would have to be:

I	100 A @ 1.00	100 B @ 14.10	100 for [A]	1410 for [B] — 1510,
II	100 A @ 1.99	100 B @ .141	199 for [A]	14.10 for [B] — 213.10.

Here the geometric average with even weighting, $\sqrt{1.99 \times .01} = 0.1411$, indicates a fall of 85.89 per cent.; but also the arithmetic average with the weighting of the first period indicates the same fall, for $\frac{213.1}{1510} = 0.1411$.)

price of B is supposed to fall variously to the following figures, stated in the first column, the true average being stated in the next column, and the geometric average in the last:

1.00	1.495	1.4958
.81	1.400	1.4021
.64	1.315	1.3200
.49	1.240	1.2502
.36	1.175	1.1947
.25	1.120	1.1563
.16	1.075	1.1403
.12	1.055	1.1436
.09	1.040	1.1562
.04	1.015	1.2250
.01	1.000	1.4016
.005	0.9975	1.4954

Here the geometric method remains approximately correct till β_2' descends beyond .50 and its variation becomes the greater of the two, while [B] becomes less than half as large as [A] over both the periods, after which it departs appreciably from the truth. A strange thing is that after β_2' reaches about .15, the further it descends, the more the geometric average rises.

An objection previously urged against the arithmetic average was that with even weighting it permits of no compensatory fall of B after A has risen to 2.00. That was on the supposition of even weighting over both the periods together. With conditions permitting of even weighting over both the periods together there should be possible a compensatory fall of B for every rise of A. But if the weighting is even only at the first period, and if the sizes of the classes rise and fall with the rise and fall of their prices, there being no change in our purchases of them, there is no reason why beyond a certain point in the rise of A there should be no compensatory fall of B possible. That objection against the geometric average was accompanied by an argument for the geometric average. We now see that under the circumstances supposed the geometric average is in this respect no better than the arithmetic, each with its proper weighting. It is even worse, as it takes away the possibility of compensation at an earlier point in the rise of A. Moreover it is peculiar in that after A has passed the point where the possibility of complete compensation ends, the descent of B gains its maximum compensation before reaching zero.

V.

§ 1. In closing this Chapter some tests may be employed similar to those at the end of the preceding.

I	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B]	— 200,
II	100 A @ 1.50	100 B @ .60	150 for [A]	60 for [B]	— 210,
III	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B]	— 200.

Here it is evident that the exchange-value of money is the same at the third period as at the first, whatever it be at the intermediate stage; and this constancy at the third period compared with the first should be indicated by the results of the two variations. The weighting at the first and at the third period is even, but at the second period it is 5 for [A] and 2 for [B]. Now if we use the arithmetic average in both measurements, and each time with the weighting of the earlier or first period of the two compared, that is, in the first price variations with the even weighting of the first period, we get an indication of a rise of 5 per cent. ($\frac{1}{2}\{\frac{3}{2} + \frac{3}{5}\} = \frac{21}{20} = 1.05$); and in the second price variations with the uneven weighting of the second period we get an indication of the inverse fall of 4.77 per cent. ($\frac{1}{7}\{5 \times \frac{2}{3} + 2 \times \frac{5}{3}\} = \frac{20}{21} = 0.9523$); and these two together indicate the same level of prices at the third period as at the first. And exactly the same indications are given by the harmonic average of the first price variations with the weighting of the second period $\left(\frac{7}{\frac{5}{\frac{3}{2}} + \frac{2}{\frac{3}{5}}}\right.$ … as printed: $\left(\frac{7}{5 \times \frac{2}{3} + 2 \times \frac{5}{3}} = \frac{21}{20}\right)$, and by the harmonic average of the second price variations with the even weighting of the third period $\left(\frac{2}{\frac{3}{2} + \frac{3}{5}} = \frac{20}{21}\right)$—as also directly by Scrope's method of comparing the total sums.[1] And, once more, the correct

[1] The same results could also be obtained by combining these averages differently and using the same weighting throughout:—thus either by using even weighting (of the earlier and of the later period) with the arithmetic average of the first variations and the harmonic average of the second; or by using the weighting 5 for [A] and 2 for [B] (that of the later and of the earlier period) with the harmonic average of the first variations and the arithmetic average of the second. But these are methods above rejected.

final result is obtained also by using the geometric average, in both cases with the same weighting, which is 1.5811 for [A] and 1 for [B]. With this weighting the geometric average of the first price variations is $\sqrt[2.5811]{(\frac{3}{2})^{1.5811} \times \frac{3}{5}} = 1.0516$, indicating a rise of 5.16 per cent., and of the second, $\sqrt[2.5811]{(\frac{2}{3})^{1.5811} \times \frac{5}{3}}$ $= 0.9508$, indicating a fall of 4.92 per cent. It would seem as if the same weighting ought to be used in both the averagings, since in each set of variations the only difference in the weightings is the order of their occurrence. Still the geometric average does not give such true indication in each set of price variations as do the other two methods.[2]

We perceive, however, that the error of the geometric method above the truth in the first price variations, where the rising price is of the predominating class, is exactly counterbalanced by the error of the geometric method below the truth in the second price variations, where the variations are reversed and the falling price is of the predominating class; for 1.0516 : 1.05 :: 0.9523 : 0.9508.

If the weights were uneven at the first and third periods, or altogether so different as to make the classes very unequal in size over both the periods together, in each comparison, while the price of the smaller class varies considerably, the error of the geometric method above or below the truth in each measurement would be greater than in the case here cited, and might be considerable. Yet the same counterbalancing would always take place, and the indication for the third period through the intermediate one would still indicate constancy.

§ 2. With two or more intermediate periods the geometric average does not necessarily give the right indication for the last period. For example:

I	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B]—200,
II	100 A @ 1.50	100 B @ .75	150 for [A]	75 for [B]—225,
III	100 A @ .66⅔	100 B @ 2.00	66⅔ for [A]	200 for [B]—266⅔,
IV	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B]—200.

[2] The true final result is also obtained through intermediate errors in two other ways—namely, by averaging both sets of variations arithmetically with the weightings of the later periods, or harmonically with the weightings of the earlier periods. But these are methods also above rejected. Thus there are seven different ways of obtaining the known final result.

CHAPTER XII.

THE UNIVERSAL METHOD.

I.

§ 1. The conditions presupposed by both the preceding arguments are unlikely to be met with in practice. The first argument, in supposing the sums of money to be constant in spite of the price variations, can have application only if prices vary through changes in supply. The second, in supposing the mass-quantities to be constant in spite of the price variations, can have application only if prices vary through changes in demand. Neither of these changes is likely to take place alone. Prices generally vary through changes both in supply and in demand. Both these conditions must be admitted; and only that argument is complete which takes them both into consideration.

Or the user of the first argument can defend his position only by claiming that the weighting should be according to the smaller money-values (or, more properly, the smaller exchange-values) at either period. And the user of the second argument can defend his position only by claiming that the weighting should depend upon the smaller mass-quantities at either period. Each must rely upon something that is common to both the periods, eliminating all the rest of the same kind.

The practical and theoretical objections to these positions have already been set forth in Chapter IV. Section V. Their impropriety is especially apparent when we place them side by side. For which of them has more reason for it than the other? If we want the same material world at both periods, do we not equally much want the same economic world at both periods? We can not have both of these things together. And trial shows that on posited variations of prices and of mass-quantities the two methods give very divergent results—sometimes the former giving the higher result, and sometimes the latter, and more or less so, and sometimes both giving nearly the same results, without order or principle. And to use both, drawing some mean between their results, would convey no special meaning, and has no justification.

Or again, the user of the first argument might defend his position by drawing a mean between the total money-values of a class at each period, and treating this mean total money-value *as if* it were the total money-value at each period,—and doing so with every class, he would weight them accordingly. And the user of the second argument might defend his position likewise by drawing a mean between the total mass-quantities of a class at each period, and treating this mean total mass-quantity *as if* it were the mass-quantity at each period,—and doing so with every class, he would employ these mass-quantities as the basis of his weighting (or would simply apply to them Scrope's method).

Now if the user of the first argument employed the geometric mean between the total money-values of every class at each of two periods as the weight of every class, he would be doing merely what has already been recommended in Chapter IV. He would do well, however, to avoid recommending his procedure on the ground of an *as if*. The total money-values of the classes are different at each period, and what we want is not what might have been a similar state of things if a mean total money-value had existed in every class at both periods; but, as explained in the earlier Chapter, what we want is the number of individuals in every class that, given the facts as they are, have the same exchange-value over both the periods. And if the user of the second argument employed a mean between the total mass-quantities at each period, he would be employing a method also admitted in that Chapter as a tenable one. But he, too, ought to find some other reason for his position than an *as if*. As for the kind of mean to use here, everything we have learned points to the geometric mean as the proper one. But we shall later find that the arithmetic mean gives almost similar results, wherefore its greater convenience is a recommendation for its practical employment.

§ 2. Thus all the methods as yet in these pages examined that have any claim to consideration as theoretically reasonable and as likely to yield truthful, or nearly true, answers, reduce to these two: (1) the geometric averaging of the price variations with weighting according to the geometric mean of the full total money-values at both periods—which, as before, for brevity, we shall call simply the geometric method; and (2) Scrope's method applied to the geometric means of the full mass-quantities at both periods—which, again for brevity, we shall call Scrope's emended method.

For the first of these the recommendation is that it takes into consideration the conditions at both periods, correctly weights the classes and uses the best average we have for averaging the price variations of the economic individuals, whose relative numbers have been determined in the weighting. The objection to it is that the geometric average ceases to possess the virtue of the geometric mean when it is applied to more than two classes, or to two unequal classes. It is true that all commodity-classes fall into two general classes: those whose prices rise above the average variation, and those whose prices fall below it,—not to mention a third class, whose prices vary with the average, the presence of which class is indifferent. And in practice those two general classes may mostly be nearly equal in size. Hence in practice the geometric average is not likely to depart much from the truth. Still, we have seen that when the classes are very unequal and the price variations are very great, this average may deflect considerably. Therefore we should prefer not to have to rely on this method alone.

The second of these methods has in its favor that it likewise takes into consideration the conditions at both periods, and that it avoids the use of the geometric average, using only the geometric mean, so that it escapes perversion arising from that source in the extraordinary cases of great inequality in the sizes and of great variation in the prices. The principal argument for it is that it has been found to be the fundamental method underlying both the partial methods. For, being the right method both for constant money-sums, when the mass-quantities vary, and for constant mass-quantities, when the money-sums vary, why should it not be extendible to all cases, and be the true universal method? Unfortunately only one of those partial methods was found to be perfect. And the very one by which this method is suggested was found not to be perfect. Hence there is a probability of this method also not being perfect in all cases.

There is, however, still another way of deducing a universal method from the two partial methods,—adding a third method with claim upon our attention as likely to give approximation to the truth. This new method, by slightly altering each of those methods in the same manner, applies them to all possible cases, getting the same result always, whether it modifies the one or the other. And this alteration, or modification, in each case, respects the principle of simple mensuration, that we must at both periods be dealing with the same whole, or with similar wholes, allowing the details to be different. Hence this method seems to have much in its favor. Since it modifies the imperfect one, there is some hope that it escapes the imperfection. Whether it does so, will be seen in the sequel. We must now develop this method. Then we may test it, and compare it with the other methods.

II.

§ 1. Both the partial methods examined in the preceding Chapters start out with the same injunction, that we should *find in all the classes mass-units that have the same exchange-value over both the periods together*. This having been done, the first method, applied to conditions where the sums of money spent on the classes are not the same at both periods, becomes this: *measure the constancy or variation of the purchasing power of money by the constancy or variation in the total number of these mass-units purchased or purchasable at each period with a given total sum of money spent in the same proportions as the total sums actually were spent at each period.* Of course we may take one of the total sums as it actually was, and reduce the other to it. The second method, applied to conditions where the mass-quantities purchased of the classes are not the same at both periods, becomes this: *measure the constancy or variation of the exchange-value of money inversely by the constancy or variation in the total sum of money needed at each period to purchase a given total number of these mass-units in the same proportions as the total numbers of them actually were purchased at each period.* Of course, again, we may take the one of the total numbers as it actually was, and reduce the other to it. The measurement of the constancy or variation of prices in general is the inverse of these, that is, by inversion in the first case and without inversion in the second. Both these methods, to repeat, applied to the same conditions where both the sums of money and the mass-quantities vary from period to period, yield the same result.

These two methods, which really form one bipartite method, call for illustration first by numerical examples. Suppose we find this state of things:

I	90 A @ 1.00	70 B @ 1.00	90 for [A]	70 for [B],
II	80 A @ 1.50	150 B @ .50	120 for [A]	75 for [B],

This admits of conversion into the following:

I	90 A @ 1.00	40.415 B″ @ 1.732 — 130.415	90 for [A]	70 for [B] — 160,
II	80 A @ 1.50	86.605 B″ @ .866 — 166.605	120 for [A]	75 for [B] — 195,

in which the mass-units, A and B″, have the same money-value, and the same exchange-value, over both the periods together. Now at the first period we bought 130.415 such mass-units for 160 money-units, and at the second we bought 166.605 such mass-units for 195 money-units. Therefore at the second period we could buy, spending our money in the same proportions as we did then actually spend it, 136.701 such mass-units for 160 money-units (as we learn by the simple use of the rule-of-three). Thus the purchasing power of 160 money-units has risen from purchasing at the first period 130.415 to purchasing at the second period 136.701 mass-units that are equivalent over both the periods. Hence its variation has been $\frac{136.701}{130.415} = 1.0482$;

and the variation of prices has been $\frac{130.415}{136.701} = 0.9540$, indicating a fall of prices by 4.60 per cent. And, again, at the first period we gave 160 money-units for 130.415 such mass-units, and at the second we gave 195 money-units for 166.605 such mass-units. Therefore at the second period we gave 152.64 money-units for 130.415 such mass-units, in the same proportions as we actually did then purchase them. Thus this mass-quantity of 130.415 mass-units equivalent over both periods has fallen in price from 160 at the first period to 152.64 at the second, and the variation of the general price-level is $\frac{152.64}{160} = 0.9540$, indicating a fall by 4.60 per cent., the same as before. It may be added that on these data the geometric method indicates a fall of prices by 4.49 per cent.;[1] and Scrope's emended method indicates a fall by 4.71 per cent.[2]

Another example may be supposed as follows:

I	75 A @ 1.00	70.71 B @ 1.00	75 for [A]	70.71 for [B],
II	60 A @ 1.50	133⅓ B @ .75	90 for [A]	100 for [B].

This may be converted into

I	75 A @ 1.00	50 B″ @ 1.4142—125	75 for [A]	70.71 for [B]—145.71,
II	60 A @ 1.50	94.28 B″ @ 1.0606—154.28	90 for [A]	100 for [B]—190.00,

in which the mass-units, A and B″, are equivalent over both the periods. At the first period we got 125 such mass-units for 145.71 money-units, and at the second 154.28 of them for 190.00. Therefore, in the same proportions, at the second period we should get 118.31 such mass-units for 145.71 money-units; and so the purchasing power of this sum of money has fallen from purchasing 125 to purchasing 118.31; and inversely the general level of prices has varied thus: $\frac{125}{118.31} = 1.0565$, which indicates a rise by 5.65 per cent. Again, at the first period we paid 145.71 money-units for 125 such mass-units, and at the second 190.00 for 154.28. Therefore at the second

[1] This is slightly above the other result. The weights are 1.4342 for [A] and 1 for [B]. Notice that the rising price is of the preponderating class.

[2] On the arithmetic means Scrope's method indicates a fall by 6.41 per cent.

period we had to pay, in the same proportions in which we actually did spend our money, 153.94 money-units for 125 mass-units. Thus the price of 125 such mass-units, which are equivalent over both the periods, has risen from 145.71 to 153.87, and the price variation is $\frac{153.94}{145.71} = 1.0565$, likewise indicating a rise by 5.65 per cent. Here, we may again add, the geometric method indicates a rise by 5.64 per cent.;[3] and Scrope's method also indicates a rise by 5.64 per cent.[4]

§ 2. That these two ways of making the calculation universally agree, and coalesce into one method, may be demonstrated as follows. Employing our usual symbols, we construct this comprehensive schema, representative of any possible state of things :

$$\begin{array}{lllll} \text{I} & x_1 \text{ A @ } \alpha_1 & y_1 \text{ B @ } \beta_1 & z_1 \text{ C @ } \gamma_1 \cdots\cdots & - \; x_1 + y_1 + \gamma_1 + \cdots\cdots \\ \text{II} & x_2 \text{ A @ } \alpha_2 & y_2 \text{ B @ } \beta_2 & z_2 \text{ C @ } \gamma_2 \cdots\cdots & - \; x_2 + y_2 + \gamma_2 + \cdots\cdots \end{array}$$

$$\begin{array}{llll} x_1\alpha_1 \text{ for [A]} & y_1\beta_1 \text{ for [B]} & z_1\gamma_1 \text{ for [C]} \cdots\cdots & - \; x_1\alpha_1 + y_1\beta_1 + z_1\gamma_1 + \cdots\cdots, \\ x_2\alpha_2 \text{ for [A]} & y_2\beta_2 \text{ for [B]} & z_2\gamma_2 \text{ for [C]} \cdots\cdots & - \; x_2\alpha_2 + y_2\beta_2 + z_2\gamma_2 + \cdots\cdots, \end{array}$$

in which the mass-units, A, B, C, ..., of the classes, [A], [B], [C], ... are the customary commercial ones, and x, y, z, ..., represent the numbers of them that are bought and sold at the periods indicated by the numerals attached to them, and α, β, γ, ..., represent the prices of these mass-units at the periods similarly indicated. The number of classes may be extended indefinitely, but must be the same at both periods. As the mass-units are various, we must reduce them to equivalence over both the periods. We may take the mass-unit of any one of them, say of [A], as our unit, and reduce the rest to equivalence with it. Represent the mass-unit of [B] that is equivalent over both periods to A by B,″ and its prices at each of the periods by ${}_{\prime}\beta_1''$ and ${}_{\prime}\beta_2''$ respectively ; and treat the equivalent mass-unit of [C] in the same way. The first half of the above schema then becomes the following :

[3] Almost exactly like the other, but, this time, slightly below. Notice that the weighting is almost even, but slightly preponderating on the side of the falling price, the weight for [A] being 82.16 and that for [B] 84.09.

[4] These, however, differ in further decimals. For their closeness the near evenness of the weighting will also be found to be the reason.—On the arithmetic means Scrope's method indicates a rise by 4.86 per cent.

$$\text{I } x_1 \text{ A @ } a_1 \quad \frac{y_1\beta_1}{\beta_1''} \text{B}'' \text{ @ } \beta_1'' \quad \frac{z_1\gamma_1}{\gamma_1''} \text{C}'' \text{ @ } \gamma_1'' \quad \cdots\cdots,$$

$$\text{II } x_2 \text{ A @ } a_2 \quad \frac{y_2\beta_2}{\beta_2''} \text{B}'' \text{ @ } \beta_2'' \quad \frac{z_2\gamma_2}{\gamma_2''} \text{C}'' \text{ @ } \gamma_2'' \quad \cdots\cdots.$$

Here the conditions are that $\beta_1''\beta_2'' = \gamma_1''\gamma_2'' = \cdots\cdots = a_1a_2$, and also that $\frac{\beta_2''}{\beta_1''} = \frac{\beta_2}{\beta_1}$, $\frac{\gamma_2''}{\gamma_1''} = \frac{\gamma_2}{\gamma_1}$, and so on. From the first condition we derive $\frac{\beta_2''}{\beta_1''} = \frac{a_1a_2}{\beta_1''^2}$; wherefore, by means of the second condition, $\frac{\beta_2}{\beta_1} = \frac{a_1a_2}{\beta_1''^2}$, whence $\beta_1''^2 = \frac{a_1a_2\beta_1}{\beta_2}$. In a similar manner we obtain $\beta_2''^2 = \frac{a_1a_2\beta_2}{\beta_1}$. Thus we have $\beta_1'' = \sqrt{\frac{a_1a_2\beta_1}{\beta_2}}$, and $\beta_2'' = \sqrt{\frac{a_1a_2\beta_2}{\beta_1}}$; wherefore

$$\frac{y_1\beta_1}{\beta_1''} = \frac{y_1\beta_1\sqrt{\beta_2}}{\sqrt{a_1a_2\beta_1}} = y_1\sqrt{\frac{\beta_1\beta_2}{a_1a_2}},$$

and

$$\frac{y_2\beta_2}{\beta_2''} = \frac{y_2\beta_2\sqrt{\beta_1}}{\sqrt{a_1a_2\beta_2}} = y_2\sqrt{\frac{\beta_1\beta_2}{a_1a_2}}.$$

And treating the prices of [C] in the same way, we obtain

$$\frac{z_1\gamma_1}{\gamma_1''} = z_1\sqrt{\frac{\gamma_1\gamma_2}{a_1a_2}},$$

and

$$\frac{z_2\gamma_2}{\gamma_2''} = z_2\sqrt{\frac{\gamma_1\gamma_2}{a_1a_2}};$$

and so on with all the other classes. By substituting these values in the last schema, it becomes:

$$\text{I } x_1 \text{ A @ } a_1 \quad y_1\sqrt{\frac{\beta_1\beta_2}{a_1a_2}}\text{B}'' \text{ @ } \beta_1'' \quad z_1\sqrt{\frac{\gamma_1\gamma_2}{a_1a_2}}\text{C}'' \text{ @ } \gamma_1'' \quad \cdots\cdots$$
$$= x_1 + y_1\sqrt{\frac{\beta_1\beta_2}{a_1a_2}} + z_1\sqrt{\frac{\gamma_1\gamma_2}{a_1a_2}} + \cdots\cdots.$$

$$\text{II } x_2 \text{ A @ } a_2 \quad y_2\sqrt{\frac{\beta_1\beta_2}{a_1a_2}}\text{B}'' \text{ @ } \beta''_2 \quad z_2\sqrt{\frac{\gamma_1\gamma_2}{a_1a_2}}\text{C}'' \text{ @ } \gamma_2'' \quad \cdots\cdots$$
$$x_2 + y_1\sqrt{\frac{\beta_1\beta_2}{a_1a_2}} + z_2\sqrt{\frac{\gamma_1\gamma_2}{a_1a_2}} + \cdots\cdots.$$

he second half of the schema remaining the same as in the first. Here A, B″, C″, ... are mass-units equivalent over both the periods. Now at the first period we get $x_1 + y_1\sqrt{\frac{\beta_1\beta_2}{\alpha_1\alpha_2}} + \sqrt{\frac{\gamma_1\gamma_2}{\alpha_1\alpha_2}} + \cdots\cdots$ such mass-units for $x_1\alpha_1 + y_1\beta_1 + z_1\gamma_1 + \cdots\cdots$ money-units; and at the second period we get $x_2 + y_2\sqrt{\frac{\beta_1\beta_2}{\alpha_1\alpha_2}} + \sqrt{\frac{\gamma_1\gamma_2}{\alpha_1\alpha_2}} + \cdots\cdots$ such mass-units for $x_2\alpha_2 + y_2\beta_2 + z_2\gamma_2 + \cdots\cdots$ money-units. Therefore, in the same proportions, we should get at the second period (as we learn by the rule-of-three)

$$\frac{\left(x_2 + y_2\sqrt{\frac{\beta_1\beta_2}{\alpha_1\alpha_2}} + \gamma_2\sqrt{\frac{\gamma_1\gamma_2}{\alpha_1\alpha_2}} + \cdots\cdots\right)\left(x_1\alpha_1 + y_1\beta_1 + z_1\gamma_1 + \cdots\cdots\right)}{x_2\alpha_2 + y_2\beta_2 + z_2\gamma_2 + \cdots\cdots}$$

such mass-units for $x_1\alpha_1 + y_1\beta_1 + z_1\gamma_1 + \cdots\cdots$ money-units. Thus the purchasing power of this sum of money has varied from purchasing the former number to purchasing the latter number of mass-units equivalent over both the periods; and the purchasing power of all sums of money, or of money in general, that is, the exchange-value of money in all the other things, has varied in the same proportion. And the general level of prices, varying inversely, has varied as is thus represented,

$$\frac{P_2}{P_1} = \frac{x_1 + y_1\sqrt{\frac{\beta_1\beta_2}{\alpha_1\alpha_2}} + z_1\sqrt{\frac{\gamma_1\gamma_2}{\alpha_1\alpha_2}} + \cdots\cdots}{\frac{\left(x_2 + y_2\sqrt{\frac{\beta_1\beta_2}{\alpha_1\alpha_2}} + z_2\sqrt{\frac{\gamma_1\gamma_2}{\alpha_1\alpha_2}} + \cdots\cdots\right)\left(x_1\alpha_1 + y_1\beta_1 + z_1\gamma_1 + \cdots\cdots\right)}{x_2\alpha_2 + y_2\beta_2 + z_2\gamma_2 + \cdots\cdots}}$$

$$= \frac{\left(x_1 + y_1\sqrt{\frac{\beta_1\beta_2}{\alpha_1\alpha_2}} + z_1\sqrt{\frac{\gamma_1\gamma_2}{\alpha_1\alpha_2}} + \cdots\cdots\right)\left(x_2\alpha_2 + y_2\beta_2 + z_2\gamma_2 + \cdots\cdots\right)}{\left(x_2 + y_2\sqrt{\frac{\beta_1\beta_2}{\alpha_1\alpha_2}} + z_2\sqrt{\frac{\gamma_1\gamma_2}{\alpha_1\alpha_2}} + \cdots\cdots\right)\left(x_1\alpha_1 + y_1\beta_1 + z_1\gamma_1 + \cdots\cdots\right)}.$$

Again, at the first period we give $x_1\alpha_1 + y_1\beta_1 + z_1\gamma_1 + \cdots\cdots$ money-

[illegible]

units for $x_1 + y_1\sqrt{\frac{\beta_1\beta_2}{a_1a_2}} + z_1\sqrt{\frac{\gamma_1\gamma_2}{a_1a_2}} + \cdots\cdots$ such mass-units, and at the second period we give $x_2a_2 + y_2\beta_2 + z_2\gamma_2 + \cdots\cdots$ money-units for $x_2 + y_2\sqrt{\frac{\beta_1\beta_2}{a_1a_2}} + z_2\sqrt{\frac{\gamma_1\gamma_2}{a_1a_2}} + \cdots\cdots$ such mass-units. Therefore, in the same proportions, we should at the second period have to give

$$\frac{\left(x_1 + y_1\sqrt{\frac{\beta_1\beta_2}{a_1a_2}} + z_1\sqrt{\frac{\gamma_1\gamma_2}{a_1a_2}} + \cdots\cdots\right)\left(x_2a_2 + y_2\beta_2 + z_2\gamma_2 + \cdots\cdots\right)}{x_2 + y_2\sqrt{\frac{\beta_1\beta_2}{a_1a_2}} + z_2\sqrt{\frac{\gamma_1\gamma_2}{a_1a_2}} + \cdots\cdots}$$

money-units for $x_1a_1 + y_1\sqrt{\frac{\beta_1\beta_2}{a_1a_2}} + z_1\sqrt{\frac{\gamma_1\gamma_2}{a_1a_2}} + \cdots\cdots$ such mass-units. Thus the price of this number of mass-units equivalent over both periods has varied from the first to the second sum, and in the same proportion has varied the price of any given number of such mass-units purchased at each period in the same proportions as each of the totals at those periods actually were purchased. Therefore

$$\frac{P_2}{P_1} = \frac{\left(x_1 + y_1\sqrt{\frac{\beta_1\beta_2}{a_1a_2}} + z_1\sqrt{\frac{\gamma_1\gamma_2}{a_1a_2}} + \cdots\cdots\right)\left(x_2a_2 + y_2\beta_2 + z_2\gamma_2 + \cdots\cdots\right)}{\left(x_2 + y_2\sqrt{\frac{\beta_1\beta_2}{a_1a_2}} + z_2\sqrt{\frac{\gamma_1\gamma_2}{a_1a_2}} + \cdots\cdots\right)\left(x_1a_1 + y_1\beta_1 + z_1\gamma_1 + \cdots\cdots\right)},$$

which is the same as the preceding. Q. E. D.

Above, in Chapter IX. (Section I.) we criticized the argument from compensation by equal mass-quantities and the argument from compensation by equal sums of money because, although modelled on correct methods of measuring variations in a particular exchange-value of money, which give the same result from two opposite points of view, those arguments, as hitherto employed, gave different results, disclosing error somewhere. We then saw that the error lay in the method usually adopted of employing the argument from compensation by equal mass-quantities. Thereupon we determined the right method of employing the argument from compensation by equal mass-quantities. We now find that this right use of this argument and the right use of the other argument yield identical results. We have, therefore, apparently, got rid of the error which made those two arguments seem opposed to each other.

§ 3. We have done more still. We have finally reached a universal formula such as we have been seeking. For the above expression for $\frac{P_2}{P_1}$, twice obtained, is a universal formula for the constancy or variation of the general level of prices, and the inverse of it is a universal formula for the constancy or variation of the general exchange-value of money in all other things.

The above expression may be simplified, and, being rearranged so as to be brought into conformity with certain other formulæ, it may be written thus,

$$\frac{P_2}{P_1} = \frac{x_2\alpha_2 + y_2\beta_2 + \cdots\cdots}{x_1\alpha_1 + y_1\beta_1 + \cdots\cdots} \cdot \frac{x_1\sqrt{\alpha_1\alpha_2} + y_1\sqrt{\beta_1\beta_2} + \cdots\cdots}{x_2\sqrt{\alpha_1\alpha_2} + y_2\sqrt{\beta_1\beta_2} + \cdots\cdots}. \quad (1)$$

And this may be restated thus,

$$\frac{P_2}{P_1} = \frac{\mathbf{a}_2 + \mathbf{b}_2 + \cdots\cdots}{\mathbf{a}_1 + \mathbf{b}_1 + \cdots\cdots} \cdot \frac{\mathbf{a}_1\sqrt{\frac{\alpha_2}{\alpha_1}} + \mathbf{b}_1\sqrt{\frac{\beta_2}{\beta_1}} + \cdots\cdots}{\mathbf{a}_2\sqrt{\frac{\alpha_1}{\alpha_2}} + \mathbf{b}_2\sqrt{\frac{\beta_1}{\beta_2}} + \cdots\cdots}; \quad (2)$$

or thus,

$$\frac{P_2}{P_1} = \frac{\mathbf{a}_2 + \mathbf{b}_2 + \cdots\cdots}{\mathbf{a}_1 + \mathbf{b}_1 + \cdots\cdots} \cdot \frac{\sqrt{\mathbf{a}_1\mathbf{a}_2\frac{x_1}{x_2}} + \sqrt{\mathbf{b}_1\mathbf{b}_2\frac{y_1}{y_2}} + \cdots\cdots}{\sqrt{\mathbf{a}_1\mathbf{a}_2\frac{x_2}{x_1}} + \sqrt{\mathbf{b}_1\mathbf{b}_2\frac{y_2}{y_1}} + \cdots\cdots}. \quad (3)$$

Both these, like the corresponding formulæ in Chapter X., Section III. § 3, namely those numbered (6) and (7), are more curious than useful. Here we have no additional form corresponding to formula (8) in that place. The nearest we can get to that formula is by altering the last into this,

$$\frac{P_2}{P_1} = \frac{\mathbf{a}_2 + \mathbf{b}_2 + \cdots\cdots}{\mathbf{a}_1 + \mathbf{b}_1 + \cdots\cdots} \cdot \frac{\alpha_2\sqrt{x_1x_2\frac{\mathbf{a}_1}{\mathbf{a}_2}} + \beta_2\sqrt{y_1y_2\frac{\mathbf{b}_1}{\mathbf{b}_2}} + \cdots\cdots}{\alpha_1\sqrt{x_1x_2\frac{\mathbf{a}_2}{\mathbf{a}_1}} + \beta_1\sqrt{y_1y_2\frac{\mathbf{b}_2}{\mathbf{b}_1}} + \cdots\cdots}. \quad (4)$$

Thus this method is distinct from Scrope's method in any of its forms; nor is it a combination of any of the forms of that method.

If it happens that the sums of money spent on the classes are constant, that is, that $x_2a_2 = x_1a_1$, $y_2\beta_2 = y_1\beta_1$, and so on, the formulæ reduce to

$$\frac{P_2}{P_1} = \frac{x_1\sqrt{a_1a_2} + y_1\sqrt{\beta_1\beta_2} + \cdots\cdots}{x_2\sqrt{a_1a_2} + y_2\sqrt{\beta_1\beta_2} + \cdots\cdots},$$

and consequently also to

$$\frac{P_2}{P_1} = \frac{a_1\sqrt{x_1x_2} + \beta_1\sqrt{y_1y_2} + \cdots\cdots}{a_2\sqrt{x_1x_2} + \beta_2\sqrt{y_1y_2} + \cdots\cdots},$$

which we know to be correct in these cases, as proved in Chapter X., the latter being Scrope's method applied to the geometric means of the mass-quantities.[5] If it happens that the mass-quantities purchased of the classes are constant, that is, that $x_2 = y_1$, $y_2 = y_1$, and so on, so that each may be represented simply as x, y, ……, or that they have all varied alike and $x_2 = ry_1$, $y_2 = ry_1$, and so on, wherefore, the r eliminating itself, either may be represented simply as x, y, ……, the formulæ reduce to

$$\frac{P_2}{P_1} = \frac{xa_2 + y\beta_2 + \cdots\cdots}{xa_1 + y\beta_1 + \cdots\cdots},$$

which we know to be correct for these cases, as proved in the last Chapter, this being Scrope's method applied to the constant mass-quantities. Thus the above complete formulæ enclose both those sets of formulæ, just as the universal conditions to which the universal formulæ are applicable enclose the two special sets

[5] Also the same reduction will take place if the sums of money are all in the same proportion so that $x_2a_2 = rx_1a_1$, $y_2\beta_2 = ry_1\beta_1$, and so on, provided also the mass-quantities are in the same proportion, that is, that $x_2 = rx_1$, $y_2 = ry_1$, and so on; but then there are no price variations. Otherwise, it is only formulæ (6) and (8) in Chapt. X. Sect. III. § 3, that are directly applicable in such cases, formula (8) remaining unchanged (because $\sqrt{r}$ eliminates itself from both sides of the fraction), and formula (6) merely reducing to a form in which **a**, **b**, ……, representing either all the smaller or all the larger sums spent, take the places of $\mathbf{a}_1$ and $\mathbf{a}_2$, $\mathbf{b}_1$ and $\mathbf{b}_2$, ……. But all the other formulæ are applicable under the proviso that only the sums, and only the mass-quantities, of one of the periods be used. Cf. Note 1 in Chapt. X. Sect. I.

of conditions to which each of those formulæ was separately applicable.

§ 4. Desiring to get clearer insight into the meaning of this formula, we may do so by putting it in another form, to which it easily reduces, as follows,

$$\frac{P_2}{P_1} = \frac{\dfrac{x_2\alpha_2 + y_2\beta_2 + z_2\gamma_2 + \cdots\cdots}{x_2 + y_2\sqrt{\dfrac{\beta_1\beta_2}{\alpha_1\alpha_2}} + z_2\sqrt{\dfrac{\gamma_1\gamma_2}{\alpha_1\alpha_2}} + \cdots\cdots}}{\dfrac{x_1\alpha_1 + y_1\beta_1 + z_1\gamma_1 + \cdots\cdots}{x_1 + y_1\sqrt{\dfrac{\beta_1\beta_2}{\alpha_1\alpha_2}} + z_1\sqrt{\dfrac{\gamma_1\gamma_2}{\alpha_1\alpha_2}} + \cdots\cdots}}, \qquad (5)$$

The sub-numerators are the total sums of money spent on all the goods at the second and at the first periods. The sub-denominators are the total numbers of mass-units purchased at the second and at the first periods, that are equivalent over both the periods together. Thus the formula expresses not an average of price variations, but the variation (or constancy) of averages—to wit, the variation of the average of the prices at the second period of the mass-units then purchased that are equivalent over both the periods from the average of the prices at the first period of the same mass-units then purchased. These averages of the prices are arithmetic averages. But the averages used in obtaining the equivalence of the mass-units are geometric means.

Thus this method of measuring the constancy or variation of the exchange-value of money falls into line—not behind the methods adopted by Carli and Young, Jevons, Laspeyres, and Messedaglia, and mostly employed hitherto, of averaging merely the variations of prices, nor behind the method suggested by Scrope, in any of its forms, of comparing the averages of prices at each period on the same mass-quantities taken as constant in spite of facts to the contrary—but behind the method first discovered by Drobisch, of comparing the averages of prices at each period on the mass-quantities of each period, and so employing what we have called *double weighting*.

Its resemblance to Drobisch's method—and also its difference—is especially plain if we suppose that the labor of finding the mass-units equivalent over both the periods has already been performed, and consequently the numbers of them purchased at each period are known. For then, as $\sqrt{\alpha_1''\alpha_2''} = \sqrt{\beta_1''\beta_2''} = \cdots\cdots$, formula (1) reduces to

$$\frac{P_2}{P_1} = \frac{x_2''\alpha_2'' + y_2''\beta_2'' + \cdots\cdots}{x_1''\alpha_1'' + y_1''\beta_1'' + \cdots\cdots} \cdot \frac{x_1'' + y_1'' + \cdots\cdots}{x_2'' + y_2'' + \cdots\cdots}, \quad (6)$$

which, in form, is the same as the formula for Drobisch's method. Drobisch, however, had to use this general formula for double weighting on the presupposition that the labor of obtaining the numbers of the mass-units recommended by him had already been performed. He was unable to put into the general formula itself his method of selecting the mass-units, or of obtaining the ratio between their numbers. But we have been able to do this with our formulæ. Our formula, (1), for instance, is applicable to the prices and numbers of any mass-units that happen to be employed by merchants, as is evident from the fact that the larger is the mass-unit employed, the larger will be its prices and the smaller its numbers, and conversely, so that every full term ($x_1\alpha_1$, or $x_1\sqrt{\alpha_1\alpha_2}$, etc.) remains unchanged in size whatever be the sizes of the mass-units. What this formula does is to reduce, in its second half, the numbers of the mass-units commonly used to the same proportions as are the numbers of the mass-units equivalent over both the periods (as shown in Chapter X. Section III. § 3), so that we are freed from the need of knowing either these mass-units themselves or their numbers.

In the different mass-units used, resulting in different numbers and proportions in the second half of the common formula (6), lies the fundamental distinction between the present method and Drobisch's. Drobisch sought to draw the average price at each period of the mass-units, in all the classes, that are equal according to weight (or capacity). This method draws the average price at each period of the mass-units, in all the classes, that are equal according to exchange-value. Drobisch's method used equiponderant mass-units. This method uses equivalent mass-units—that is, of course, mass-units that are equivalent over both the periods compared. That this method is more arly right and that method altogether wrong, is plain from is distinction; for it is plain that not equiponderant, but equiv-nt, (or equally important) mass-units are the economic in-viduals the variations in whose average price we desire to easure.

From this fundamental distinction flow other differences. In obisch's method different classes can be used at each period, obstacle being offered by it against counting a new class ap-aring at the second period any more than against counting a w individual in an old class. In the present method only the ne classes can be used at each period in the comparison of any periods; for otherwise a price quotation would be wanting. us this method must obey the principles laid down in Chapter . Section V. § 9; while Drobisch's method is free from sub-tion to those principles. Again, in Drobisch's method we ve seen a grave defect to be that, even though it obey those nciples, yet in cases when between two periods there are egular variations in the mass-quantities but no variations in r prices whatsoever, it may indicate a variation in the general change-value of money, thus violating Propositions XXVII. l XLIV.; and also may give two other indications which we ow to be wrong.[6] None of these errors is committed by the sent method. If no price variations occur, the mass-quan-es may vary as they please, this method indicates only con-ncy. Or if all prices vary in the same proportion, the mass-ntities may vary as they please, this method indicates only a iation of the general level of prices in the same proportion. is last statement, which really embraces the first, may be ved as follows. Suppose $\alpha_1 = r\alpha_2$, $\beta_2 = r\beta_1$, and so on, r ng the common ratio of variation of every price. (If there no price variations, $r = 1$.) Then formula (1) becomes

$$\frac{P_2}{P_1} = \frac{x_2 r\alpha_1 + y_2 r\beta_1 + \cdots\cdots}{x_1\alpha_1 + y_1\beta_1 + \cdots\cdots} \cdot \frac{x_1\alpha_1\sqrt{r} + y_1\beta_1\sqrt{r} + \cdots\cdots}{x_2\alpha_1\sqrt{r} + y_2\beta_1\sqrt{r} + \cdots\cdots}$$

$$= \frac{r(x_2\alpha_1 + y_2\beta_1 + \cdots\cdots)}{x_1\alpha_1 + y_1\beta_1 + \cdots\cdots} \cdot \frac{\sqrt{r}(x_1\alpha_1 + y_1\beta_1 + \cdots\cdots)}{\sqrt{r}(x_2\alpha_1 + y_2\beta_1 + \cdots\cdots)} = r,$$

[6] See Chapt. V. Sect. VI. § 5.

whatever be the variations between x_1 and x_2, y_1 and y_2, etc. And never will this method indicate a rise of the general level of prices, when all prices fall, or conversely.

§ 5. There is another method which has followed the general lines of Drobisch's,—one which is described near the end of Appendix C., but which has not yet attracted our attention. This is the method invented by Professor Lehr. In this method its author has made an effort to do what appears to be accomplished in the method here presented. He has tried to measure the variation in the average price of mass-units, in all the classes, that have the same exchange-value over both the periods together,—to which equivalent mass-units he has given the not inappropriate name of "pleasure-units." The fault with this method lies in the way it measures the equivalence of the pleasure-units. Instead of finding mass-units between whose prices at each period the simple geometric means are the same, it employs mass-units between whose prices at each period the unevenly weighted arithmetic averages are the same. No reason is assigned for this choice, and it seems to have been made as a matter of course. The position is that in a given class, certain different sums of money being expended at each of the two periods compared to purchase certain different numbers of weight-units, the total of these sums is expended over both the periods together to purchase the total of these numbers; wherefore a single money-unit, on the average over both the periods, purchases a number of weight-units represented by the quotient of the total sum of money divided by the total of the numbers of weight-units, so that we have here a mass whose average price over both the periods is one money-unit. A similar operation is performed on every class, in each of which a mass is obtained whose average price, so measured, over both the periods, is one money-unit; wherefore it is maintained that these masses, being equivalent to the money-unit over both the periods, are equal pleasure-units. Then the rest of the method is to draw the averages of the prices of all these pleasure-units of all the classes at each period, and to compare them.

In this position a first error is the use of uneven weighting.

the ordinary commercial mass-unit of anything[7] be priced at ɜ money-unit at the first period and at two money-units at the ond, and if the ordinary commercial mass-unit of anything e be priced at two money-units at the first period and at one ney-unit at the second, these two mass-units are, over these) periods, equivalent mass-units or equal pleasure units, with- . regard to the numbers of them that may be purchased at ıer period. The numbers of them purchased at each period ermine the relative importance of the classes, but not the rela- ɜ importance of the individuals in the classes. A consequence this error is that undue influence may be given to the con- ions existing at one of the periods (or in one of the countries ose money is being compared with another's), while, in truth, ılready pointed out in Chapter IV. (Sect. V.) we ought to be ecially careful to allow no greater influence, or weight, to one the periods than to the other. Some deficiencies in the thod before us following upon the neglect of this principle l be pointed out further on in this Section and later.

A second error in this method of obtaining the pleasure-units n the use of the arithmetic average instead of the geometric. such a measurement all the principles examined in Chapter :II. apply. If we prize a mass-unit of [B] twice as highly a mass-unit of [A] at the first period and at the second prize mass-unit of [A] twice as highly as the mass-unit of [B], s obvious that over these two periods together we prize the ɔ mass-units equally. Therefore the geometric, and not the thmetic, mean is to be used to indicate such ratios of impor- ıce.[8] Also in the arithmetic averaging of the prices at the

[7] Lehr follows Drobisch to the extent of wanting us first to reduce all mass- ts to the same weight-unit. This is superfluous in his method. Simple in- :tion of its formula will disclose that the *terms* in it are unaffected whatever he size of the mass-units employed. It is really a merit in Lehr's method that oes not require the use of the same weight-units).—Another merit, where it arts from Drobisch's method and agrees with the one here presented, is that quires the use of the same classes at both periods. Wicksell's criticism of ır's method, noticed in the Appendix, thus strikes at a point in it which :rves credit instead of censure.

[8] The case above cited would exist if at the two periods respectively the prices : of A 1.00 and 1.00, of B 2.00 and .50; of A 1.00 and 1.50, of B 2.00 and .75; A 1.00 and 2.00, of B 2.00 and 1.00; of A 1.00 and .50, of B 2.00 and .25; or in ıy other combinations. In all these combinations the geometric means are al; the arithmetic, only in one.

two periods a variation in the exchange-value of money will derange the result of the calculation. But such a variation has no influence upon the geometric mean, as pointed out in Chapter IV.[9] It may be added that in using the geometric mean in obtaining our pleasure-units, we escape the imperfections in the geometric average with more than two figures, or with uneven weighting, which are pointed out in Chapter VIII. (Sect. I. § 6), and which have twice troubled us since, and will trouble us again presently. For here we are using the geometric *mean* proper, between only two figures, the prices of the same thing at two periods, rightly attaching equal importance to each of them. Thus what of the geometric method is retained in our final formula is flawless.

Professor Lehr's method has the peculiarity that in consequence of its merits and demerits just pointed out, it shares some of the defects of Drobisch's method, and escapes others. Thus if no prices vary between the two periods, Professor Lehr's method always indicates constancy, no matter what be the variations in the mass-quantities. But if all the prices vary uniformly, this method does not necessarily indicate the corresponding variation in the general level of prices unless all the mass-quantities remain constant or also vary uniformly.[10] Thus, although, unlike Drobisch's method, it respects Propositions XXVII. and XLIV., yet, like Drobisch's method, it violates Propositions XVII. and XLV. This fact alone would be sufficient to show that there is something wrong in it. Yet of all methods hitherto suggested Professor Lehr's approaches the nearest to the truth in theory, if not also in practice.

[9] If we retained this second, but avoided the first error, the formula would be

$$\frac{P_2}{P_1} = \frac{x_2\alpha_2 + y_2\beta_2 + \cdots\cdots}{x_1\alpha_1 + y_1\beta_1 + \cdots\cdots} \cdot \frac{x_1(\alpha_1 + \alpha_2) + y_1(\beta_1 + \beta_2) + \cdots\cdots}{x_2(\alpha_1 + \alpha_2) + y_2(\beta_1 + \beta_2) + \cdots\cdots}.$$

This would be better than the method under consideration, but still not true.

[10] If $\alpha_2 = r\alpha_1$, $\beta_2 = r\beta_1$, and so on, Lehr's formula, which is given in Appendix C. VI. § 2, reduces to this,

$$\frac{P_2}{P_1} = \frac{r(x_2\alpha_1 + y_2\beta_1 + \cdots\cdots)}{x_1\alpha_1 + y_1\beta_1 + \cdots\cdots} \cdot \frac{x_1\alpha_1\left(\frac{x_1 + rx_2}{x_1 + x_2}\right) + y_1\beta_1\left(\frac{y_1 + ry_2}{y_1 + y_2}\right) + \cdots\cdots}{x_2\alpha_1\left(\frac{x_1 + rx_2}{x_1 + x_2}\right) + y_2\beta_1\left(\frac{y_1 + ry_2}{y_1 + y_2}\right) + \cdots\cdots}.$$

This reduces to unity if $r = 1$; but otherwise it reduces to r only if $x_2 = sx_1$, $y_2 = sy_1$, and so on.

§ 6. There is still another way in which these three methods, which use double weighting, may be compared. This is by comparing the relations of their results in a series of periods. In any such series we know with certainty that the results obtained serially ought to agree with the results obtained directly: (1) if the prices have all remained constant through the whole series, or have all varied alike at any or every stage, no matter what be the changes in the mass-quantities; (2) if the mass-quantities have all remained constant throughout, or have all varied alike at any stage, no matter what be the changes in the prices; (3) if at all consecutive periods except two the states of things are exactly the same, or the variations are all in the same proportion, this having the effect of reducing the irregular variations in the series to two sets; and (4) if the state of things at the last period is exactly the same as at the first, or even different always in the same proportion, no matter what intervening changes have occurred. Or even, if we grant Professor Westergaard's position, we may dispense with all these restrictions, and say that the agreement ought to take place in any and every possible case, no matter what be the changes in the prices and in the mass-quantities.

To begin with a series of three periods: taking the formula (1) above reached for $\frac{P_2}{P_1}$, and similarly framing the formula for $\frac{P_3}{P_2}$, and multiplying these by each other, we get the formula for the method here presented as it serially indicates the variation from the first to the third period $\left(\text{for } \frac{P_3}{P_2} \cdot \frac{P_2}{P_1} = \frac{P_3}{P_1}\right)$. This formula we find to be

$$\frac{P_3}{P_1} = \frac{x_3\alpha_3 + y_3\beta_3 + \cdots\cdots}{x_1\alpha_1 + y_1\beta_1 + \cdots\cdots} \cdot \frac{(x_1\sqrt{\alpha_1\alpha_2} + y_1\sqrt{\beta_1\beta_2} + \cdots\cdots)(x_2\sqrt{\alpha_2\alpha_3} + y_2\sqrt{\beta_2\beta_3} + \cdots\cdots)}{(x_2\sqrt{\alpha_1\alpha_2} + y_2\sqrt{\beta_1\beta_2} + \cdots\cdots)(x_3\sqrt{\alpha_2\alpha_3} + y_3\sqrt{\beta_2\beta_3} + \cdots\cdots)}.$$

But the formula for the direct comparison of the third with the first period is

$$\frac{P_3}{P_1} = \frac{x_3a_3 + y_3\beta_3 + \cdots\cdots}{x_1a_1 + y_1\beta_1 + \cdots\cdots} \cdot \frac{x_1\sqrt{a_1a_3} + y_1\sqrt{\beta_1\beta_3} + \cdots\cdots}{x_3\sqrt{a_1a_3} + y_3\sqrt{\beta_1\beta_3} + \cdots\cdots}.$$

These formulæ may agree by chance if it happens that in the former the last two thirds yield products in the numerator and in the denominator equal to, or in the same ratio as, the numerator and the denominator in the last half of the latter. They will regularly agree, as is easily perceived: (1) if $a_3 = a_2 = a_1$, and so on with all the prices, or if $a_3 = ra_2$ and $a_2 = sa_1$, and all the other price variations be in the same ratios, that is, if there be no price variations, or if all the price variations at each stage be in the same ratio, no matter what be the variations in the mass-quantities $\left(\text{in the former case } \frac{P_3}{P_1} = 1.00, \text{ in the latter } \frac{P_3}{P_1} = rs\right)$; (2) if $x_3 = x_2 = x_1$ and so on, or if $x_3 = rx_2$ and $x_2 = sx_1$, and so on, that is, if there be no variations in the mass-quantities, or if all such variations at each stage be in the same ratio, no matter what be the price variations; (3) if $a_2 = a_1$, or $a_2 = ra_1$, and so on, provided either $x_2 = x_1$ or $x_2 = sx_1$ and so on, or if $a_3 = a_2$, or $a_3 = ra_2$, and so on, provided either $x_3 = x_2$ or $x_3 = sx_2$ and so on, that is, if there be only one stage with irregular variations; (4) if $a_3 = a_1$ and so on, provided either $x_3 = x_1$ or $x_3 = sx_1$ and so on, in which cases both the formulæ give unity for result, indicating sameness of the price-level, or if $a_3 = ra_1$ and so on, under either of the same provisos, in which case both the formulæ give r for result, indicating a general price variation the same as all the particular price variations, the conditions here being merely that there be no irregular differences between the first and the last periods, no matter what intervening changes may have taken place; but in no other cases, that is, not universally or unconditionally.

Examining Drobisch's formula in the same way, we find that the two measurements universally and unconditionally agree. Thus Drobisch's method completely satisfies all these tests. This fact we have already noticed; but we have seen that the correctness of Drobisch's method cannot thereby be proved. It shares this advantage with other methods clearly false. And it fails before other simpler tests.

Again, examining Professor Lehr's method in the same way,[11] re find that the two measurements agree in cases restricted to alf of each of the four divisions in which the method here adanced is consistent. They agree only (1) if $a_3 = a_2 = a_1$ and so n, that is, if there be no price variations at all, no matter what e the variations in the mass-quantities; (2) if $x_3 = x_2 = x_1$ and o on, that is, if the mass-quantities be constant, no matter what e the price variations; (3) if $a_2 = a_1$ and so on, provided $x_2 = x_1$ nd so on, or if $a_3 = a_2$ and so on, provided $x_3 = x_2$ and so on, hat is, if there be only one stage with any variations at all; (4) f $a_3 = a_1$ and so on, provided $x_3 = x_1$ and so on, that is, if the tates of things be exactly the same at the last as at the first eriod, no matter what be the intervening changes. It will be bserved that in every one of the four divisions the proportional ariations are excluded.

There is still another possible method using double weighting, which deserves to be noticed here for the sake of completeess. This is a form in which Professsor Nicholson's method dmits of being stated (the third form given in Appendix C, VI. §3). A general reason for its not being a successful method s that it uses even weighting for the inverted variations of the ass-quantities. In particular, it falls most abjectly before hese tests. The agreement between its two measurements deends entirely upon the behavior of the mass-quantities. It akes place only if $x_2 = x_1$, or $x_2 = sx_1$, and so on, or if $x_3 = x_2$, r $x_3 = rx_2$, and so on, that is, only if there be no irregular variaions in the mass-quantities at least in one of the stages.[12]

[11] Or it may be easier to examine it in conformity with this formula: $\frac{P_3}{P_2} = \frac{\ _3}{\ _1} \div \frac{P_2}{P_1}$.

[12] If $x_3 = x_1$ and so on, x_2 being different, the arithmetic average of the variions of the mass-quantities (in the last half of the formula) becomes in the direct comparison, a harmonic average of them, likewise with even weighting. The method there also (in Note 4) suggested as a variant (with the geometric rerage of the inverted variations of the mass-quantities) has the merit that it iversally satisfies Westergaard's test; but it likewise has the defect of using en weighting in averaging the inverted variations. If we cured it of the latter fect by using uneven weighting adapted for every comparison (as there formuted in Note 5) it would again lose the former merit.—The method above sugsted in Note 9, submitted to these tests, yields results that agree in exactly the me cases as Lehr's method.

§ 7. In a series with four or more periods the employment of these tests becomes too cumbersome to make it worth while to pursue this enquiry into much detail. We must, however, examine what happens in such series to the method here advanced. The indirect comparison of the fourth period with the first, very much abbreviated, is as follows:

$$\frac{P_4}{P_1} = \frac{x_4 a_4 + \cdots\cdots}{x_1 a_1 + \cdots\cdots} \cdot \frac{(x_1 \sqrt{a_1 a_2} + \cdots\cdots)\ (x_2 \sqrt{a_2 a_3} + \cdots\cdots)\ (x_3 \sqrt{a_3 a_4} + \cdots\cdots)}{(x_1 \sqrt{a_1 a_2} + \cdots\cdots)\ (x_3 \sqrt{a_2 a_3} + \cdots\cdots)\ (x_4 \sqrt{a_3 a_3} + \cdots\cdots)},$$

while the direct comparison, equally abbreviated, is

$$\frac{P_4}{P_1} = \frac{x_4 a_4 + \cdots\cdots}{x_1 a_1 + \cdots\cdots} \cdot \frac{x_1 \sqrt{a_1 a_4} + \cdots\cdots}{x_4 \sqrt{a_1 a_4} + \cdots\cdots}.$$

These will regularly agree (1) if $a_4 = a_3 = a_2 = a_1$, or, more comprehensively, if $a_4 = ra_3$, $a_3 = sa_2$ and $a_2 = ta_1$, and so on in every case, that is, if there be no price variations, or if all the price variations at every stage be in the same ratio, no matter what be the variations in the mass-quantities; (2) if $x_4 = x_3 = x_2 = x_1$, or if $x_4 = rx_3$, $x_3 = sx_2$ and $x_2 = tx_1$, and so on in every case, that is, if there be no variations in the mass-quantities, or all such variations at every stage be in the same ratio, no matter what be the variations in the prices; (3) if irregular variations take place only at one stage, all the periods then virtually reducing to two.[13] And it is evident that the series may be extended to any length, exactly these relations will hold—the two comparisons will regularly agree if any of these conditions be observed; but not necessarily in other cases.

Now among the cases in which these comparisons will not necessarily agree is the one in which everything at the last period is exactly what it was at the first period—both all the prices and all the mass-quantities. In this case we know with absolute certainty that the two comparisons ought to agree, no matter what be the intervening changes. Our new method, therefore, may lead to error. But among the methods we have been reviewing it is only Drobisch's method (along with a few other false methods not here noticed [14]) that still holds out against this test.

[13] For Lehr's method, and its modification suggested in Note 9, there are the same conditions with exclusion, as before, of the proportional variations.

[14] *E. g.* the method above alluded to, suggested in Appendix C, V. § 3, Note 4.

§ 8. Thus our method fails us twice. It fails even in a series of three periods to satisfy Professor Westergaard's full test, although in that series it satisfies the certain test yielded by supposing an exact reversion. But in a longer series it does not even satisfy this latter test. It behooves us then to advert to this defect in it.

This defect in the complete method is obviously a survival of the defect in the method for constant sums of money, above examined at the end of Chapter X. The complete method we started out to construct upon each of the partial methods, by extending and by modifying them. But it turns out that it modifies much more the method for constant mass-quantities than the method for constant sums. Now it was precisely the method for constant mass-quantities that was perfect; and this has been modified, while the method for constant sums has been incorporated whole in half of every formula for our complete method. Hence the imperfection in the method for constant sums has come over entire into this complete method.

In that method we discovered the cause of the defect, and a way of getting rid of it, though not without loss of other qualities. The defect can be dispelled by taking the numbers at each period of the mass-units that are equivalent over all the periods—the pleasure-units, as Professor Lehr calls them, not of two periods at a time, but of the whole comprehensive epoch. Having already worked these out, we should have the following formula for our complete method, expressed in our usual symbols,[15]

$$\frac{P_2}{P_1} = \frac{\mathbf{a}_2 + \mathbf{b}_2 + \cdots\cdots}{\mathbf{a}_1 + \mathbf{b}_1 + \cdots\cdots} \cdot \frac{x_1''' + y_1''' + \cdots\cdots}{x_2''' + y_2''' + \cdots\cdots}. \qquad (7)$$

And now, as all the terms remain the same for every period in all the comparisons, the measurements will all agree whether made directly between any two distant periods or serially through the intervening periods. Here, then, we have an at least consistent method.

But the alteration we saw to be impracticable in the case of the partial method. It is equally impracticable for the com-

[15] Cf. the formula in Chapt. X. Sect. V. § 7.

plete method. We are obliged, therefore, to get along with a slightly imperfect form of the complete method—which, however, we shall find reason to believe no worse than if it were revised in this way.

In the earlier Chapter we examined the probable amount of error incurred by using this defective feature in the partial method. We found it very small for ordinary cases; and also we detected in the method conflicting and neutralizing tendencies, so that when we deal with many classes, and in a long series, the probabilities are that the results will never deviate to any great extent on either side of the truth, but that they will pass from the one side to the other of it, always keeping it close company, and often coinciding with it. Now the defective feature in the whole of that partial method, which here forms only half of the complete method, enters into a composition in which the other half is without defect. Hence it might seem as if the error incurred through that defectiveness would be diluted, and lessened by half. Unfortunately this is not so. The defectiveness of the half leavens the whole. Still the error cannot be greater here than there. The error being the same, we need not examine it again.

§ 9. The defect which has been pointed out in this method belongs to it in a series of years, the reason for its existence there being obvious. This reason for the defect does not touch the method used in making a direct comparison between two periods—especially between two contiguous periods. The question still remains: In comparing two periods is this a perfect method?—is it the one true method? The reasoning by which it has been reached seems to be faultless. Yet there was a fault in one of the premises—in one of the partial methods. And the method, even in comparing two periods, is not perfect. We have seen it stand certain tests. Unfortunately it does not stand all tests.

One of the tests to which we subjected the partial methods previously reached for cases with constant money-sums or with constant mass-quantities, was examination as to whether they carried out Proposition XXXVI., or not. This, it may be recalled, is the perfectly evident principle that if, making a calculation upon all but one or more classes, we find the general price variation to be indicated at a certain ratio, and if, later noticing the other class or classes, and finding them all to vary in price exactly in this ratio, we insert them in the calculation, which we perform over again, the result yielded in the later recalculation ought to agree with the result first obtained. Now we have seen that the present complete method is composed of those two incomplete methods; and we previously saw that each of those methods stood this test, provided its own conditions were observed, and not otherwise.[16] Those conditions cannot exist together (except in absence of all variations); and we are now investigating cases in which both are supposed to be broken. We have seen, also, and see, that that Proposition applies to all possible cases, no matter what may be the money-values of the classes, or their mass-quantities. Does, then, our present method satisfy this test? It does not.

That it does not is seen most easily by taking formula (5) and treating it in the way supposed. If in all but one class the indication is of a variation in the general level of prices from 1 to r, it must be that

$$\frac{x_2\alpha_2 + y_2\beta_2 + \cdots\cdots}{x_2\sqrt{\alpha_1\alpha_2} + y_2\sqrt{\beta_1\beta_2} + \cdots\cdots} = r\left(\frac{x_1\alpha_1 + y_1\beta_1 + \cdots\cdots}{x_1\sqrt{\alpha_1\alpha_2} + y_1\sqrt{\beta_1\beta_2} + \cdots\cdots}\right);$$

and now if $\gamma_2 = r\gamma_1$, it ought still to be that

$$\frac{x_2\alpha_2 + y_2\beta_2 + \cdots\cdots + z_2\gamma_1 r}{x_2\sqrt{\alpha_1\alpha_2} + y_2\sqrt{\beta_1\beta_2} + \cdots\cdots + z_2\gamma_1\sqrt{r}} =$$

$$r\left(\frac{x_1\alpha_1 + y_1\beta_1 + \cdots\cdots + z_1\gamma_1}{x_1\sqrt{\alpha_1\alpha_2} + y_1\sqrt{\beta_1\beta_2} + \cdots\cdots + z_1\gamma_1\sqrt{r}}\right).$$

But this is *not* necessarily true,—nor is it necessarily true even if $z_2 = z_1$; nor even if $r = 1$; nor even if both these conditions occur together (unless the numerators themselves, the total sums

[16] Except in one form of their formulæ, common to both, namely Scrope's method applied to the geometric means of their mass-quantities, which we have called Scrope's emended method, and from which the present method has been distinguished.

spent at each period, are equal). And the same result is obtained by using any of the other formulæ for this method.[17]

Here, then, in this method is a grave theoretical defect. It will hardly lead to any inconsistency in practice, since we are not apt to make such recalculations, nor are such coincidences likely to be found. But the existence of this theoretical defect shows that the method, even in a single comparison between two contiguous periods, is not perfect.

This defect exists in all the other methods using double weighting—including Drobisch's.[18] It is inherent in all methods using double weighting, no matter which of the three kinds of averages of the prices be used. But from it are totally free all methods using single weighting (averaging the price variations),[19] and Scrope's method in all its forms (this being reducible to the preceding).

Hence in our search after perfection we are thrown back upon these older styles, though upon methods never before employed or suggested, and must examine whether they are better.

III.

§ 1. That both the geometric method and Scrope's emended method carry out Proposition XXXVI. is evident upon simple inspection of their formulæ. They thus avoid one of the defects in the method with double weighting above investigated. We now need to examine whether they avoid the other defects in that method.

The geometric method does not universally satisfy Professor Westergaard's test, as we already know.[1] It satisfies it, as is easily perceived, in a series of three periods only under three of the four full conditions that we have seen to be required for the method with double weighting, and in a longer series only under two of the three there allowed. It satisfies it under the first of

[17] Its failure in this respect leads to a defect in this method when extended to measuring constancy (or variation) in exchange-value in *all* things, which will be noticed in the next Chapter.

[18] And the method suggested in Appendix C, § 3, Note 4.

[19] Cf. Appendix A, I, § 8.

[1] See Chapt. V. Sect. VI. § 7.

those conditions, because then the weighting is indifferent;[2] under the third, because all methods do so under that condition; and, in the former case, under the fourth, because then the weighting is the same in both the comparisons. But it does not satisfy it under the second condition, in either case; for here there may be price variations with different weighting in the comparisons. Thus this method does not behave quite so well as that method. Like it, it fails in a long series also in the certain case where sameness of the price-level should be shown at the end of the serial calculations, when everything at the last period returns to what it was at the first. We have seen, however, that all such inconsistency would cease, if the same weighting be employed through the whole epoch, such weighting properly being the geometric average of the full money-values of the classes at every one of the periods in the epoch. But we have objected to such a procedure as being less trustworthy than the use of its own weighting in every comparison of two periods.

Scrope's emended method fails also before this test, but only as did the method with double weighting. In a series of three periods, its indirect comparison of the third with the first is as follows:

$$\frac{P_3}{P_1} = \frac{(\alpha_2\sqrt{x_1x_2} + \beta_2\sqrt{y_1y_2} + \cdots\cdots)\ (\alpha_3\sqrt{x_2x_3} + \beta_3\sqrt{y_2y_3} + \cdots\cdots)}{(\alpha_1\sqrt{x_1x_2} + \beta_1\sqrt{y_1y_2} + \cdots\cdots)\ (\alpha_2\sqrt{x_2x_3} + \beta_2\sqrt{y_2y_3} + \cdots\cdots)},$$

while the direct comparison is

$$\frac{P_3}{P_1} = \frac{\alpha_3\sqrt{x_1x_3} + \beta_3\sqrt{y_1y_3} + \cdots\cdots}{\alpha_1\sqrt{x_1x_3} + \beta_1\sqrt{y_1y_3} + \cdots\cdots}.$$

It is plain that these regularly agree only under all the four full conditions above noticed for the method with double weighting. These may be briefly recapitulated. They are: (1) if $\alpha_3 = r\alpha_2 = rs\alpha_1$ and so on; (2) if $x_3 = rx_2 = rsx_1$ and so on; (3) if $\alpha_2 = r\alpha_1$, provided $x_2 = sx_1$, and so on, or if $\alpha_3 = r\alpha_2$, provided $x_3 = sx_2$, and so on; (4) if $\alpha_3 = r\alpha_1$, provided $x_3 = sx_1$, and so on.

[2] See Appendix A, I. § 8.

And in a series of four or more periods, its indirect comparison of the fourth with the first is as follows, abbreviated as before:

$$\frac{P_4}{P_1} = \frac{(a_2\sqrt{x_1x_2} + \cdots\cdots)}{(a_1\sqrt{x_1x_2} + \cdots\cdots)} \frac{(a_3\sqrt{x_2x_3} + \cdots\cdots)}{(a_2\sqrt{x_2x_3} + \cdots\cdots)} \frac{(a_4\sqrt{x_3x_4} + \cdots\cdots)}{(a_3\sqrt{x_3x_4} + \cdots\cdots)},$$

while the direct comparison is

$$\frac{P_4}{P_1} = \frac{a_4\sqrt{x_1x_4} + \cdots\cdots}{a_1\sqrt{x_1x_4} + \cdots\cdots}.$$

And again it is evident that these regularly agree only under the three conditions noticed for that method, namely (1) if $a_4 = ra_3 = rsa_2 = rsta_1$ and so on; (2) if $x_4 = rx_3 = rsx_2 = rstx_1$ and so on; (3) if there be irregular variations only at one stage. The series may be extended indefinitely: the formulæ and the conditions will merely be extensions of these.

It is plain, again, that also this emended form of Scrope's method can be still further emended and cured of this defect, if, instead of applying it to the geometric means of the mass-quantities at each of the two periods in every comparison, we apply it in every comparison to the geometric average of the mass-quantities over the whole epoch.

This is a remedy similar to those we had to invoke for the other two methods. To review: The geometric method of averaging price variations (with single weighting) can be made universally to satisfy Professor Westergaard's test by using it with the same weighting over the whole epoch,—say of n' periods,—that is, with weights that are the geometric averages of the sums spent on every class at every period (*e. g.* $\sqrt[n']{x_1a_1 \cdot x_2a_2 \cdot \cdots\cdots}$ to n' terms). The method with double weighting above expounded can be made universally to satisfy that test by using it with mass-units equivalent over the whole epoch, as found by geometrically averaging the prices of the ordinary mass-units in every class at every period (*e. g.* $\sqrt[n']{a_1 \cdot a_2 \cdot \cdots\cdots}$ to n' terms). And now the emended form of Scrope's method can be made universally to satisfy that test by applying it to mass-quantities that are the geometric average of the mass-quantities of every class at every period (*e. g.* $\sqrt[n']{x_1 \cdot x_2 \cdots\cdots \text{to } n' \text{ terms}}$). It is therefore incumbent upon us to examine this system of further emending and revising the three methods for the sake of remedying one of their defects—in the last its only known defect.

§ 2. In no case is this remedy satisfactory, for two principal reasons:—(1) Because the present epoch is extending every year, requiring recalculations; and it does not appear that a later recalculation will be more correct than an earlier. Besides, how is a past variation between two years several years ago to be affected by present variations? (2) Because we really do not know how to calculate weights, or to determine equivalence of mass-units, or to average mass-quantities, over more than two periods, since the geometric average loses its virtue when applied to more than two figures. Hence it may be that in working over these methods into methods universally satisfying Professor Westergaard's test we gain consistency between cross-measurements at the expense of other qualities.

This, in fact, can be shown to be the case with the method using double weighting. Suppose states of things over three periods as follows:

I	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B],
II	75 A @ 1.50	130 B @ .60	112.50 for [A]	78 for [B],
III	100 A @ 1.00	120 B @ 1.00	100 for [A]	120 for [B].

Going from period to period the method with double weighting gives these results: $\frac{P_2}{P_1} = 0.9890$ and $\frac{P_3}{P_2} = 1.0322$, whence $\frac{P_3}{P_1} = 1.0208$, which is 2.08 per cent. above what we take to be the true position. In the direct comparison of the third period with the first this method rightly indicates sameness. Now when this method is worked over so as to avoid this inconsistency, it gives these results: $\frac{P_2}{P_1} = 0.9686$ and $\frac{P_3}{P_2} = 1.0468$, whence $\frac{P_3}{P_1} = 1.0139$; and in the direct comparison of the third with the first period this method still indicates $\frac{P_3}{P_1} = 1.0139$. Thus, though consistent, the revised method twice gives a wrong result, being 1.39 per cent. too high.

On this schema the two other methods, going from period to period, give results very close to those given by the method with double weighting similarly used. The geometric method yields $\frac{P_2}{P_1} = 0.9902$ and $\frac{P_3}{P_2} = 1.0310$, whence $\frac{P_3}{P_1} = 1.0209$. And Scrope's emended method yields $\frac{P_2}{P_1} = 0.9885$ and $\frac{P_3}{P_2} = 1.0325$, whence $\frac{P_3}{P_1} = 1.0206$. But when they are worked over to cover this short epoch of three periods, in avoiding inconsistency in their final results, they also avoid the error incurred by the preceding method. For, so used, the geometric method yields $\frac{P_2}{P_1} = 0.9698$ and $\frac{P_3}{P_2} = 1.0311$, whence $\frac{P_3}{P_1} = 1.00$, which is also indicated in the direct comparison; and Scrope's emended method yields $\frac{P_2}{P_1} = 0.9953$ and $\frac{P_3}{P_2} = 1.0046$, whence $\frac{P_3}{P_1} = 1.00$, which is also indicated in the direct comparison.

The question, however, arises: Is Professor Westergaard's test correct universally? The case before us is of such a nature as to throw doubt upon it. Here the prices of both the classes and the mass-quantity of [A] alone have reverted at the third period to what they were at the first. Had this third period, with the sameness of its prices, immediately followed upon the first period, there would be no question but that the exchange-value of money is constant (in accordance with Proposition XLIV.). But an intervening period has separated the two; and now, while the same number of economic individuals in [A] fall in price during the second stage as rose during the first, in [B] a greater number of such individuals rose in price during the second stage than fell during the first. Had the mass-quantity of [B] fallen back at the third period to what it was at the first, there could again be no question but that the level of prices at the third period has returned to what it was at the first,

—which would be indicated by all these, and by several other, methods. But ought not the fact that, while the changes in [A] counterbalance each other, a greater number of individuals in [B] have risen than fallen, be allowed to show that the level of prices has risen more than it has fallen, so that it is rightly placed by all our three methods, in their serial use, slightly above its first position? This would involve also that the exchange-value of money has fallen somewhat,—which, seeing that prices are exactly the same, is somewhat hard to entertain.

Yet another example casts doubt upon Professor Westergaard's universal test. Suppose that both prices and mass-quantities vary irregularly between a first and a second period. And suppose that between the second and a third period there is irregular variation of mass-quantities, but no variation of prices. There is, then, no variation of the general price level between these last periods. Therefore, the indirect comparison of the third with the first period will show the same general price variation as between the second and the first. But the direct comparison of the third with the first will show a different price variation from that between the second and the first, and consequently from that indirectly obtained between the third and the first. Now of these two measurements the latter has more reason in its favor.

Still, even if we should deny Professor Westergaard's test in such cases, we should gain little comfort in regard to our methods, since there is one case in which his test is perfectly certain, and which none of the methods (except in their doubtful revised forms) can satisfy. This is when at any later period the prices and the mass-quantities both revert exactly (or proportionally) to what they were at some earlier period. This is a test which no sound method yet devised or suggested, in going from period to period over all the intervening periods, will stand.

That is, none of the known methods that hold out against other tests will stand this test in the world such as we have it —with varying mass-quantities as well as with varying prices. But if we had an economic world,—or supposed one,—in which forever the same mass-quantities, or mass-quantities in the same proportions, are bought and sold, at prices varying according to demand only, then both Scrope's emended method and the method with double weighting above described—both of which are in these cases the same as Scrope's method applied to the constant mass-quantities,—would completely and absolutely satisfy Professor Westergaard's test, and all other tests. In such a world, the argument for this method being convincing, we can be certain that we have the absolutely true method of measuring variations in the exchange-value of money. But in the world as it is, we have not yet reached the absolutely true method.

§ 3. We can, however, be sure that we have come pretty near to it.

In the first place, we have three distinct methods, for each of which much can be said, which in some cases regularly give the same results, and which in all ordinary cases give results very close together. The cases when they exactly agree are when we are dealing with two classes equally important over both the periods.

In these cases we may first prove that the geometric method (which now must use even weighting) exactly agrees with the method with double weighting above expounded. The condition to be observed is that the two classes are such that $\sqrt{x_1 a_1 x_2 a_2} = \sqrt{y_1 \beta_1 y_2 \beta_2}$, or, which is the same thing, $x_1 a_1 x_2 a_2 = y_1 \beta_1 y_2 \beta_2$. From this condition is derived

$$\frac{x_2 a_2}{y_1 \beta_1} = \frac{y_2 \beta_2}{x_1 a_1} = \frac{x_2 a_2 + y_2 \beta_2}{x_1 a_1 + y_1 \beta_1}.$$

Therefore in the first half of formula (1) for the method with double weighting, applied to two such classes, we may substitute one of these values of that half, and proceed to reduce thus,

$$\frac{y_2 \beta_2}{x_1 a_1} \cdot \frac{x_1 \sqrt{a_1 a_2} + y_1 \sqrt{\beta_1 \beta_2}}{x_2 \sqrt{a_1 a_2} + y_2 \sqrt{\beta_1 \beta_2}} = \frac{\beta_2 \left(x_1 y_2 \sqrt{a_1 a_2} + y_1 y_2 \sqrt{\beta_1 \beta_2}\right)}{a_1 \left(x_1 x_2 \sqrt{a_1 a_2} + x_1 y_2 \sqrt{\beta_1 \beta_2}\right)}.$$

From the given condition is also obtained $x_1 x_2 = y_1 y_2 \dfrac{\beta_1 \beta_2}{a_1 a_2}$, and

$y_1 y_2 = x_1 x_2 \frac{\alpha_1 \alpha_2}{\beta_1 \beta_2}$. Therefore the last expression becomes

$$\frac{\beta_2\left(x_1 y_2 \sqrt{\alpha_1 \alpha_2} + x_1 x_2 \frac{\alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}}\right)}{\alpha_1\left(y_1 y_2 \frac{\beta_1 \beta_2}{\sqrt{\alpha_1 \alpha_2}} + x_1 y_2 \sqrt{\beta_1 \beta_2}\right)} = \frac{\beta_2 \sqrt{\alpha_1 \alpha_2}\left(x_1 y_2 + x_1 x_2 \sqrt{\frac{\alpha_1 \alpha_2}{\beta_1 \beta_2}}\right)}{\alpha_1 \sqrt{\beta_1 \beta_2}\left(y_1 y_2 \sqrt{\frac{\beta_1 \beta_2}{\alpha_1 \alpha_2}} + x_1 y_2\right)}.$$

Again from the given condition is obtained $\frac{\alpha_1 \alpha_2}{\beta_1 \beta_2} = \frac{y_1 y_2}{x_1 x_2}$, and $\frac{\beta_1 \beta_2}{\alpha_1 \alpha_2} = \frac{x_1 x_2}{y_1 y_2}$. Therefore the last expression becomes

$$\frac{\beta_2 \sqrt{\alpha_1 \alpha_2}(x_1 y_2 + \sqrt{x_1 x_2 y_1 y_2})}{\alpha_1 \sqrt{\beta_1 \beta_2}(\sqrt{x_1 x_2 y_1 y_2} + x_1 y_2)} = \frac{\beta_2}{\alpha_1}\sqrt{\frac{\alpha_1 \alpha_2}{\beta_1 \beta_2}} = \sqrt{\frac{\alpha_2}{\alpha_1} \cdot \frac{\beta_2}{\beta_1}},$$

which last is the formula for the geometric average of the price variations with even weighting. Q. E. D.

Next we may prove that in these cases Scrope's emended method likewise exactly agrees with the geometric method. From the given condition is derived $\sqrt{x_1 x_2} = \sqrt{y_1 y_2 \frac{\beta_1 \beta_2}{\alpha_1 \alpha_2}}$, and on inserting this value of $\sqrt{x_1 x_2}$ in the formula for Scrope's emended method confined to the two classes, and reducing, we get

$$\frac{P_2}{P_1} = \frac{\alpha_2 \sqrt{y_1 y_2 \beta_1 \beta_2} + \beta_2 \sqrt{y_1 y_2 \alpha_1 \alpha_2}}{\alpha_1 \sqrt{y_1 y_2 \beta_1 \beta_2} + \beta_1 \sqrt{y_1 y_2 \alpha_1 \alpha_2}}$$

$$= \frac{\sqrt{\alpha_2 \beta_2}\,(\sqrt{\alpha_2 \beta_1} + \sqrt{\alpha_1 \beta_2})}{\sqrt{\alpha_1 \beta_1}\,(\sqrt{\alpha_1 \beta_2} + \sqrt{\alpha_2 \beta_1})} = \sqrt{\frac{\alpha_2}{\alpha_1} \cdot \frac{\beta_2}{\beta_1}}. \quad \text{Q. E. D.}$$

Thus, *when we deal with two classes equally important over both the periods, the method with double weighting, and Scrope's emended method, and the geometric method, all yield the same result.*

In all such cases (which permit both the money-sums and the mass-quantities to be different at the two periods, but require them to yield equal products) the three methods satisfy all our tests. We may be sure, then, that in these restricted cases the common result is *the* true one.

§ 4. In other ordinary cases trial shows that in their results the three methods do not diverge considerably from one another. Therefore we have reason to believe that they do not deviate considerably from the truth.

As regards their divergence amongst themselves, trial seems to show that, only two unequal classes being employed, the method with double weighting generally gives a middling result. The highest is given by the geometric method and the lowest by Scrope's emended method, when the preponderating class is with price rising above the general average; and reversely the lowest by the geometric method and the highest by Scrope's emended method, when the preponderating class is with price falling below the general average. With more classes this rule does not seem to hold, unless most of the larger ones have prices varying in the one direction and most of the smaller ones have prices varying in the opposite direction (in relation to the general average price variation). The greatness of the divergence is determined both by the greatness of the inequality in the sizes of the classes and by the greatness of the price variations—principally by the latter. It is greatest when the preponderating class varies little and the smaller class varies much; but it may also be considerable when the preponderating class varies much and the smaller class varies little, provided the preponderance is not excessive; for when it is excessive all the results are drawn so far with the variation of the excessively preponderating class that their divergence may be lessened almost to nothing,—and such, of course, is the event also in cases where the classes, in pairs or sets, are nearly equal in size. In the cases of moderate inequality in the sizes of the classes and of excessive variation in one of the prices, there seems to be a tendency on the part of the geometric method to deviate by itself, becoming untrustworthy, while the other two methods keep fairly close together. All this about the geometric method in general agrees with what has already been found to be the deviation of this method, in the partial cases, in comparison with the component parts of the method with double weighting.

The following examples, purposely extravagant, will illustrate some of the salient positions.

I	10 A @ 1.00	50 B @ 1.00	10 for [A]	50 for [B],
II	6 A @ 1.50	70 B @ .40	9 for [A]	28 for [B].

[B] is 3.94 times larger than [A]. The geometric method indicates $\frac{P_2}{P_1} = 0.5226$; the method with double weighting makes it 0.5240, and Scrope's emended method, 0.5273.

I	100 A @ 1.00	30 B @ 1.00	100 for [A]	30 for [B],
II	80 A @ 1.50	40 B @ .20	120 for [A]	8 for [B].

[A] is 7.071 times larger than [B]. The indications, in the same order, are 1.1686, 1.1547, and 1.1370.

I	100 A @ 1.00	3 B @ 1.00	100 for [A]	3 for [B],
II	80 A @ 1.50	4 B @ .20	120 for [A]	0.80 for [B].

[A] is 70.71 times larger than [B]. The indications are 1.4585, 1.4555, and 1.4515.

I	100 A @ 1.00	300 B @ 1.00	100 for [A]	300 for [B],
II	80 A @ 1.50	400 B @ .20	120 for [A]	80 for [B].

[B] is 1.414 times larger than [A]. The indications are 0.4608, 0.4634, and 0.4667.

I	100 A @ 1.00	50 B @ 1.00	100 for [A]	50 for [B],
II	80 A @ 10.00	60 B @ .90	800 for [A]	54 for [B].

[A] is 17.21 times larger than [B]. The indications are 9.4630, 6.6824, and 6.5369.

I	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B],
II	98 A @ 2.00	102 B @ .10	196 for [A]	10.20 for [B].

[A] is 4.38 times larger than [B]. This example may be compared with one in Chapter XI. Section IV. § 3. The mass-quantities have varied so little that it almost comes under the cases with constant mass-quantities. If it did so, the mass-quantities of the first period remaining constant, we know with certainty that the price variation would be $\frac{P_2}{P_1} = 1.05$. But the variations of the mass-quantities, slight as they are, make [B] larger relatively to [A] than it otherwise would be, and so give greater effect to the fall of price. Therefore we know with certainty that the indication should be slightly below 1.05. Now the method with double weighting yields 1.0442, and Scrope's emended method 1.0405, while the geometric method yields 1.1464, thus being certainly wrong. Which of the former two results is nearer the truth, it is impossible to tell. That the geometric method should fail in such an extravagant example, ought not to be counted much to its discredit. All these examples being extraordinary in their variations, the general closeness of the results yielded in such cases by the three methods is a warranty of their greater closeness in all ordinary cases.

§ 5. In the second place, what has been shown of the geometric method with reference to the partial method for constant sums, and of that method with reference to the truth, evidently belongs to the geometric method in general, and to the method with double weighting, which contains that partial method, and also to Scrope's emended method, which likewise comprehends that method. That is, there is neutralization both between the many classes that are measured together and between the successive periods in a series. In consequence of this last quality, even though a considerable error should be made at one stage, there is probability of its being corrected at another, and there is little likelihood of the error being any greater at the end of a long series than near the beginning—except in case of continual tendency of the level of prices in one direction, although even in this case there may be neutralization through changes in the sizes of the classes. A complex example illustrative of some of these inductions will be given later.

Lastly, the amount of the errors at any later period may be subjected to a certain test, which generally shows but slight deviation—although this test is not as satisfactory as we might desire. This test is to suppose the period in question to be followed by a period with everything exactly the same as at some earlier period, and then to calculate on to it, to see how the result serially reached for it compares with unity, which is known to be the true result. This, however, is by no means a perfect test, for two reasons. The one is that the error proved for the supposed period is not necessarily the error for the other periods which preceded, nor a definite increase upon their errors, whence their errors may be calculated. For there is no gradual accumulation of error, but irregularity, and some of the preceding results may be above, and some below, the truth. The other reason is that this last calculation is nothing but the inverse of a direct calculation from the earlier period to the last actual period; but such a direct calculation we know to have no greater validity than the indirect calculation. Still, the fact that in practice the direct and the indirect comparisons do not diverge much—and especially the fact that they do not diverge more at the end of a long series than near the beginning, is good evidence that none of the methods deviate much from the truth.

§ 6. Between the three methods our choice may be guided by what we have so far learnt. The geometric averaging of the price variations, with single weighting, is probably the least trustworthy, because we have seen that the geometric average between more than two equally important things is not to be depended upon—and we have sometimes caught it *flagrante delicto*. The method with double weighting, using the geometrically measured equivalent mass-units, has been led up to by a chain of reasoning which seems to be sound. Yet there was something defective in the reasoning at an early stage, since even the partial method for the cases with constant sums was found to fail before some of the tests. For the form of Scrope's method, in which it is applied to the geometric means of the mass-quantities, the argument is that this method underlies both the partial methods, and so is the one fundamental method, which, being true in those two cases, or at all events in one of them, ought to be, if not absolutely true, yet near to the truth, in all other cases. But the decisive argument for it is that, while it stands all the tests that are satisfied by the method with double weighting, it stands still another test, before which that method fails. Hence this method (although the geometric method shares with it this last quality) is, in all probability, the best of the three.

§ 7. The fact that we have not reached a perfectly certain method—except for the cases when the mass-quantities (the physical parts of the economic world) are constant, or all vary alike, and the very special cases when there are only two equally important classes, or pairs or sets of such classes, not to mention the cases when the prices are constant, or all vary alike—must not be misinterpreted. It does not mean that in cases when the mass-quantities and the prices irregularly vary, there is no one true variation of money in general exchange-value; for if that were so, there could, in these most common cases, be no variation of money in general exchange-value at all, which is absurd. What it means is that our mathematics, so far as yet carried in the subject of averaging, fail us. We have not yet found the right average—or the right weighting for averages already known. It may be there is no average that is perfectly correct—or no weighting that will make it so. Perhaps no method exists to be found that will stand all our tests. But from the fact that the perfect average, and the perfect weighting—or the perfect method of combining them—have not yet been discovered, it does not necessarily follow that they are never to be discovered. Or if finally we must abandon the search as hopeless and believe that no perfect method exists to be discovered, this failure of mathematics would not disprove the existence of one true variation. The fact, however, that we have three methods—not to mention two or three more, as will be shown presently,—which in all ordinary cases give results very close together, and which we have every reason to believe to be close to the truth, and to hold the truth between them, ought to make us fairly content.[3]

[3] Here may be inserted a suggestion of a line along which it might be thought that mathematicians may perhaps be able finally to solve our problem with precision, and at the same time a warning against over-expectancy. When the mass-quantities are constant, we have seen that the solution is perfect. Such cases may, then, be used as a touchstone for the rest. Now let mathematicians find the weighting—according to means of some sort between the full money-values at each of the two periods compared—which will make the geometric average of the price variations always agree with Scrope's method applied to the constant mass-quantities. If this task be accomplished, it might seem as if the geometric average of the price variations with the same kind of weighting would universally be correct, including the cases when the mass-quantities vary. An approach toward this solution may be indicated. By trial it is found, at least with two uneven classes, that the geometric average of the price variations with weighting accord-

IV.

§ 1. It will be well also to examine other methods, but especially the convenient form of Scrope's method in which it is applied to the *arithmetic* means of the mass-quantities at each period, or, which is the same thing still more conveniently, to the aggregates of the mass-quantities at both periods.

Submitting this method to Professor Westergaard's test, we have for the indirect comparison

$$\frac{P_3}{P_1} = \frac{\alpha_2(x_2 + x_2) + \beta_2(y_1 + y_2) + \cdots\cdots}{\alpha_1(x_1 + x_2) + \beta_1(y_1 + y_2) + \cdots\cdots} \cdot \frac{\alpha_3(x_2 + x_3) + \beta_3(y_2 + y_3) + \cdots\cdots}{\alpha_2(x_2 + x_3) + \beta_2(y_2 + y_3) + \cdots\cdots};$$

ing to the arithmetic means of the full money-values at each period gives results with error opposite to the error given by the geometric average with geometric weighting. For example:

100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B] – 200,
100 A @ 1.96	100 B @ .09	196 for [A]	9 for [B] – 205.

The known result is $\frac{P_2}{P_1} = \frac{205}{200} = 1.025$. The geometric weighting is 4.666 for [A] to 1 for [B]; and with this weighting the geometric average of the price variations is 1.1426. The arithmetic weighting is 2.715 for [A] to 1 for [B]; and with this weighting the geometric average of the price variations is 0.8552. It is plain that the proper weighting must be given by some mean lying between the geometric and the arithmetic. One such mean has been discovered by Gauss, who named it the arithmetico-geometric mean. If the classes be weighted according to this mean, the weighting will be 3.657 for [A] to 1 for [B]; and with this weighting the geometric average of the price variations is 1.014, which is slightly too low. Other examples likewise show that the geometric average with geometrico-arithmetic weighting comes nearer to the truth than the geometric average with geometric weighting; but that it still errs on the side of the geometric average with arithmetic weighting. In our example the proper weighting for the geometric average—that which will make it give the true answer—is 3.7526. A still closer approach to this is made by a mean which is a combination of all the three common means, and which may be called the arithmetico-geometrico-harmonic mean. This is the harmonic mean between the arithmetic and geometric means between the two quantities in question, the formula for which is $\frac{2(a+b)\sqrt{ab}}{a+b+2\sqrt{ab}}$. Weighting the classes according to this mean between their two full money-values, we find the weight for [A] to be 3.718 times that for [B]. With this weighting the geometric average of the price variations is 1.0201. In practice, however, this method would hardly differ from the preceding, which would differ very slightly from the completely geometric method. There is, moreover, a consideration which invalidates the idea that a mean perfectly good for our purpose exists: for if the money-values of all the classes be constant, all kinds of means are the same, and all are then defective, as proved in the next preceding Chapter on the method for constant sums.

while the direct comparison is

$$\frac{P_3}{P_1}=\frac{\alpha_3(x_1+x_3)+\beta_3(y_1+y_3)+\cdots\cdots}{\alpha_1(x_1+x_3)+\beta_1(y_1+y_3)+\cdots\cdots}.$$

These regularly agree: (1) if $\alpha_3=\alpha_2=\alpha_1$, and so on, whatever be the variations in the mass-quantities; (2) if $x_3=x_2=x_1$, and so on, whatever be the variations in the prices; (3) if $\alpha_3=\alpha_2$, provided $x_3=x_2$, and so on, or if $\alpha_2=\alpha_1$, provided $x_2=x_1$, and so on; (4) if $\alpha_3=\alpha_1$, provided $x_3=x_1$, and so on. These are the four divisions in which Scrope's method applied to the geometric means and the method with double weighting regularly agree; but in each division only half of the conditions are retained, uniform variations in the prices or in the mass-quantities being cut off. In a series of four or more periods it is easily seen that there will be necessary agreement only (1) if there are no variations in the prices, or (2) if there are no variations in the mass-quantities, or (3) if the periods fall into two sets in each of which there are no variations at all. In short, this method behaves in this matter exactly like Professor Lehr's method.

§ 2. Still other limitations may be discovered in this method,—by which limitations also Professor Lehr's method will be found to be restricted. The investigation may be opened with examination as to what is the relation between the arithmetic means, or between the aggregates, of the mass-quantities of the different classes in cases when the result is known with certainty. These are when two classes are dealt with that are equally important over both periods. For simplicity we shall begin with cases where the result is unity, indicating constancy in the exchange-value of money; and always we shall follow the usual practice of employing mass-units that are equivalent at the first period. The following is the simplest schema:

I	100 A @ 1.00	200 B @ 1.00	100 for [A]	200 for [B] — 300,
II	100 A @ 2.00	200 B @ .50	200 for [A]	100 for [B] — 300.
	200	400		

Here the mean, or aggregate, mass-quantities purchased over both the periods together are twice as many of these mass-units of [B] as of [A]. This is in accordance with the formula discovered in Chapter IX. Section II. § 5, namely,

$$y' = \frac{x'(\alpha_2' - 1)}{1 - \beta_2};$$

for here

$$y' = \frac{x'(2 - 1)}{1 - \frac{1}{2}} = 2x'.$$

Now the same result, constancy, appears also if things happened as represented in the following schemata:

I	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B]	— 200,
II	50 A @ 2.00	200 B @ .50	100 for [A]	100 for [B]	— 200;
	150	300			

and

I	100 A @ 1.00	150 B @ 1.00	100 for [A]	150 for [B]	— 250,
II	75 A @ 2.00	200 B @ .50	150 for [A]	100 for [B]	— 250;
	175	350			

and

I	100 A @ 1.00	300 B @ 1.00	100 for [A]	300 for [B]	— 400,
II	150 A @ 2.00	200 B @ .50	300 for [A]	100 for [B]	— 400;
	250	500			

and an indefinite number more of such examples might be made. In all these it is observable that, while the states of things permit of even weighting, and the price variations are geometric, wherefore the result is known to be constancy, with the further condition that the total sums of money spent on both the classes together at each period are the same, the aggregates, and hence the arithmetic means, of the mass-quantities of each class purchased over both the periods together are twice as many of [B] as of [A]. This constant relation of *two* A's to one B (these being equivalent at the first period) is comparable with the rise of A in price from one to *two* money-units. Now under these conditions this relation between the arithmetic means, or aggregates, of the mass-quantities of each class over both the periods may be proved to be universal. The general schema is

I	x_1' A @ 1	y' B @ 1	x_1' for [A]	y_1' for [B]	— $x_1' + y_1'$,
II	x_2' A @ α_2'	y_2' B @ β_2'	$x_2'\alpha_2'$ for [A]	$y_2'\beta_2'$ for [B]	— $x_2'\alpha_2' + y_2'\beta_2'$;

in which the conditions, beside the equivalence of the mass-units at the first period, are: (1) $x_1'x_2'a_2' = y_1'y_2'\beta_2'$, (2) $\beta_2' = \frac{1}{a_2'}$, (3) $x_2'a_2' + y_2'\beta_2' = x_1' + y_1'$. We wish to prove that

$$y_1' + y_2' = a_2'(x_1' + x_2') = x_1'a_2' + x_2'a_2'.$$

From the first and second conditions is derived

$$a_2' = \frac{y_1'y_2'\beta_2'}{x_1x_2} = \frac{y_1'y_2'}{x_1'x_2'a_2'}, \qquad \text{whence} \qquad a_2'^2 = \frac{y_1'y_2'}{x_1'x_2'};$$

and the third condition reduces thus,

$$x_2'a_2' + \frac{y_2'}{a_2'} = x_1' + y_1',$$

$$x_2'a_2'^2 + y_2' = a_2'(x_1' + y_1'),$$

$$\frac{y_1'y_2'}{x_1'} + y_2' = a_2'(x_1' + y_1'),$$

$$y_2'(y_1' + x_1') = x_1'a_2'(x_1' + y_1'),$$

$$y_2' = x_1'a_2'.$$

From the first condition is derived $y_1' = \frac{x_1'x_2'a_2'}{y_2'\beta_2'}$; wherefore

$$y_1' = \frac{x_1'x_2'a_2'}{x_1'a_2'\beta_2'} = \frac{x_2'}{\beta_2'} = x_2'a_2'.$$

Therefore the sums of these equals are equal. Q. E. D. This shows, incidentally, that our previous formula, which with the relation between the prices above supposed in the second condition reduces as follows,

$$y' = \frac{x'(a_2' - 1)}{1 - \frac{1}{a_2'}} = x'a_2',$$

is true, not only for the purpose for which it was invented, but also, with $y' = y_1' + y_2'$ and $x' = x_1' + y_2'$, for the purpose in which we are at present interested. Thus in our particular examples, where $a_2' = 2.00$, the arithmetic mean number of B's purchased over both the periods will always be twice the mean number of A's, when these conditions are fulfilled. And on these relative mass-quantities Scrope's method always indicates constancy; for

$$\frac{P_2}{P_1} = \frac{1 \times 2 + 2 \times \frac{1}{2}}{1 \times 1 + 2 \times 1} = \frac{2 + 1}{1 + 2} = 1.00.$$

And so in the universal cases: applied to the universal relations between the mass-quantities, when the above conditions are fulfilled, Scrope's method always indicates constancy, and therefore always yields the right result; for

$$\frac{P_2}{P_1} = \frac{1 \times a_2' + a_2' \times \beta_2'}{1 \times 1 + a_2' \times 1} = \frac{a_2' + 1}{1 + a_2'} = 1.00.$$

Now, if we used other mass-units, we should get different relations between their aggregate numbers in the two classes, but the new aggregate numbers would be equally constant. In general we know that the results obtained by Scrope's method are the same whatever be the mass-units used. Therefore, what has above been proved of Scrope's method using certain mass-units may be universalized of it using any mass-units, and we have this general proposition: *When the conditions are such that, with two classes, their prices varying from unity to the opposite geometric extremes, the weighting is even over both the periods, wherefore we know that the geometric mean of the price variations is true, and it here indicates constancy, then, provided further that the total sums spent on the two classes together at each period are the same, also Scrope's method applied to the arithmetic means (or to the aggregates) of the mass-units of each class purchased at each period is always correct.* It will be noticed that the cases are extremely limited in which Scrope's method, so used, can be exactly right, except by chance.

§ 3. Let us now widen the restriction by leaving off the last condition. We may take another numerical example, with the same price variations as in the preceding, but with different mass-quantities. Things might happen as represented in this schema:

I	100 A @ 1.00	80 B @ 1.00	100 for [A]	80 for [B] — 180,
II	88 A @ 2.00	440 B @ .50	176 for [A]	220 for [B] — 396.
	188	520		

Here we likewise have even weighting over both the periods (as 100 × 176 = 88 × 220), so that we know that the geometric mean of the price variations is correct, and this still indicates constancy,—as is also indicated by Scrope's method itself, applied to the geometric means of the mass-quantities. But the arithmetic means (or the aggregates) of the mass-quantities of each class over both the periods are no longer in the relation of 2 B's to 1 A. And now Scrope's method, applied to the arithmetic means of these mass-quantities, thus

$$\frac{P_2}{P_1} = \frac{94 \times 2 + 260 \times \frac{1}{2}}{94 \times 1 + 260 \times 1} = \frac{318}{354} = 0.8983,$$

indicates a fall of 10.17 per cent., which is far astray.

But the above numerical schema may be analyzed into the following :

I	100 A @ 1.00	80 B @ 1.00	100 for [A]	80 for [B] — 180.
II	40 A @ 2.00	200 B @ .50	80 for [A]	100 for [B] — 180
	40 A @ 2.00	200 B @ .50	80 for [A]	100 for [B] — 180
	8 A @ 2.00	40 B @ .50	16 for [A]	20 for [B] — 36.

Here we see that at the second period after we have spent the same total sum of money as at the first—180 money-units (but in money, notice, that is shown to have the same exchange-value at both the periods),—the fact that we go on spending more money on the two classes in the same proportions in no wise affects the exchange-value of our money at this period. Hence all that we are concerned with in the comparison is expressed in the following :

I	100 A @ 1.00	80 B @ 1.00	100 for [A]	80 for [B]—180,
II	40 A @ 2.00	200 B @ .50	80 for [A]	100 for [B]—180,
	140	280		

And here we again have the same relations that we had in our earlier examples, the arithmetic mean (or aggregate) number of B's purchased over both the periods being twice that of the A's.

Therefore Scrope's method applied to the arithmetic means (or to the aggregates) of *these* numbers of mass-units gives the true results—and only when applied to these.

That Scrope's method, so applied, is always correct when these conditions are fulfilled, may also be demonstrated. The universal schema is the one above used; but now the conditions are only the former two, namely (1) $x_1'x_2'\alpha_2' = y_1'y_2'\beta_2'$, and (2) $\beta_2' = \frac{1}{\alpha_2'}$. Now if we reduce the total sum spent on both the classes at the second period, namely $x_2'\alpha_2' + y_2'\beta_2'$, to what is spent on them at the first period, namely $x_1' + y_2'$, we may do so by reducing the particular sums spent on [A] and on [B] at the second period in the same proportion, and consequently also the numbers of the mass-units purchased of [A] and of [B] at the second period likewise in the same proportion; and doing this, we get the following schema:

$$\begin{array}{lll} \text{I} & x_1' \text{ A @ } 1 & y_1' \text{ B @ } 1 \\ \text{II} & \dfrac{x_2'(x_1'+y_1')}{x_2'\alpha_2'+y_2'\beta_2'} \text{ A @ } \alpha_1' & \dfrac{y_2'(x_1'+y_1')}{x_2'\alpha_2'+y_2'\beta_2'} \text{ B @ } \beta_2' \end{array}$$

$$\left|\begin{array}{ll} x_1' \text{ for [A]} & y_1' \text{ for [B]}, \\ \dfrac{x_2'\alpha_2'(x_1'+y_1')}{x_2'\alpha_2'+y_2'\beta_2'} \text{ for [A]} & \dfrac{y_2'\beta_2'(x_1'+y_1')}{x_2'\alpha_2'+y_2'\beta_2'} \text{ for [B]}. \end{array}\right.$$

From the given conditions we derive $x_1'x_2'\alpha_2' = y_1'y_2'\frac{1}{\alpha_2'}$, and $x_1'x_2'\alpha_2'^2 = y_1'y_2'$, and again $y_1'y_2'\beta_2' = x_1'x_2'\frac{1}{\beta_2'}$, and $y_1'y_2'\beta_2'^2 = x_1'x_2'$; and by reducing the expressions for the sums spent on [A] and [B] respectively at the second period, and by substituting these values, the former reduces to y_1', and the latter to x_1'; wherefore simpler expressions for the numbers of A's and B's purchased at the second period are $\frac{y_1'}{\alpha_2'}$, or $y_1'\beta_2'$ for the A's and $\frac{x_1'}{\beta_2'}$ or $x_1'\alpha_2'$ for the B's, and the schema becomes

$$\begin{array}{lll|ll} \text{I} & x_1' \text{ A @ } 1 & y_1' \text{ B @ } 1 & x_1' \text{ for [A]} & y_1' \text{ for [B]}, \\ \text{II} & \dfrac{y_1'}{\alpha_2'} \text{ A @ } \alpha_2' & x_1'\alpha_2' \text{ B @ } \beta_2' & y_1' \text{ for [A]} & x_1' \text{ for [B]}; \end{array}$$

and the total of the numbers of A's purchased over both the periods is $\frac{x_1'a_2' + y_1'}{a_2'}$, and that of the numbers of B's is $x_1'a_2' + y_1'$, showing that the B's are a_2' times the A's. Now Scrope's method applied to these relative total mass-quantities, and therefore also if applied to the arithmetic means, always indicates the right answer, constancy, as above shown. Q. E. D.

Thus we find that in all cases where, with two classes equally important over both the periods, the exchange-value of money remains constant, a condition for Scrope's method applied to the arithmetic mean of the mass-quantities being correct is that we must make use only of those mass-quantities which are together purchased at each period with the same total sum of money.[1] Hence we may induct that in all cases, when the exchange-value of money remains constant, a condition of Scrope's method, so applied, being correct, is that we must apply it only to the mass-quantities which we have so reduced. Thus *a necessary preliminary work for the employment, with expectation of the best results, of Scrope's method applied to the arithmetic means of the mass-quantities, is that we must reduce the mass-quantities to those which are purchased at both periods with the same total sum of money (provided this be constant in exchange-value).*

Of this another brief proof may be supplied by the use of a test case. Suppose these states of things:

I	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B] — 200.
II	75 A @ 1.50	130 B @ .60	112.50 for [A]	78 for [B] — 190.50.
III	50 A @ 1.00	50 B @ 1.00	50 for [A]	50 for [B] — 100.

Here it is evident that money has the same exchange-value at the third as at the first period, and this ought to be indicated by

[1] The same reduction is required, in these cases, also if we should use Scrope's method applied to the smaller mass-quantities at either period, that is, to the mass-quantities common to both the periods. Then, under the conditions above supposed, this method likewise gives the right results; for, supposing $a_2' > 1$ and $\beta_2' < 1$, the smaller mass-quantities are $\frac{y_1'}{a_2}$ A and y_1' B, that is, a_2' times more B's than A's, as before.—Also, in the same cases, the geometric average of the price variations with the weighting of the smaller total exchange-value of each class at either period will always give the right result; for the smaller total money-values are y_1' for each class, that is, the same, so that even weighting will have to be used. Thus the same reduction is required here also.

the two results obtained for the two sets of price variations. Our three superior methods give the correct final answer; for the geometric method indicates $\frac{P_2}{P_1} = 0.9902$ and $\frac{P_3}{P_2} = 1.0098$, the method with double weighting $\frac{P_2}{P_1} = 0.9890$ and $\frac{P_3}{P_2} = 1.0111$, and Scrope's emended method $\frac{P_2}{P_1} = 0.9885$ and $\frac{P_3}{P_2} = 1.0116$, whence in every instance $\frac{P_3}{P_1} = 1.00$. Not so Scrope's method applied to the arithmetic means of the mass-quantities; for its results are $\frac{P_2}{P_1} = 0.9888$ and $\frac{P_3}{P_1} = 1.0321$, whence $\frac{P_3}{P_1} = 1.0206$, wrongly indicating a rise of 2.06 per cent.[2] But if all the mass-quantities are reduced to those which, in the same proportions as actually purchased in, were purchasable at every period with the same total sum of money, applied to their arithmetic means Scrope's method yields $\frac{P_2}{P_1} = 0.9922$ and $\frac{P_3}{P_1} = 1.0078$, whence $\frac{P_3}{P_1} = 1.00$.[3]

This example is one in which Professor Westergaard's test has been shown to be satisfied by our three methods (all containing a geometric element), but not by any of the purely arithmetic methods—Professor Lehr's included,—illustrating condition (4) above in § 5 of Section II. and at the commencement of this Section. It is plain, then, that also Professor Lehr's method requires a similar reduction of the mass-quantities before it can bring out its best results. In the example just given this method employed in the direct comparison of the third with the first period shows sameness of exchange-value of money at the third as at the first. But employed in measuring each variation separately,

[2] Applied to the smaller mass-quantities at either period in each comparison, Scrope's method is still more wrong in these cases; for then its indications are $\frac{P_2}{P_1} = 0.9857$, $\frac{P_3}{P_2} = 0.9523$, and $\frac{P_3}{P_1} = 0.9387$.

[3] Applied, as before, to the smaller of these reduced mass-quantities at either period, Scrope's method again gives the right final result; for its intermediate indications now are $\frac{P_2}{P_1} = 0.9964$ and $\frac{P_3}{P_2} = 1.0035$.

its results are $\frac{P_2}{P_1} = 0.9880$ and $\frac{P_3}{P_2} = 0.9916$, whence $\frac{P_3}{P_2} = 0.9797$, wrongly indicating a fall of 2.03 per cent. Yet applied to the mass-quantities reduced as before at every period, it is consistent in always yielding $\frac{P_3}{P_1} = 1.00$; for its two serial indications now are reciprocals, being $\frac{P_2}{P_1} = 0.9895$ and $\frac{P_3}{P_2} = 1.0106$. Also in the several examples above given (in § 2), which are variations upon one theme, in all those in which the total sums at each period are the same, Professor Lehr's method rightly indicates constancy; but in the one in which the total sum spent at the second period is larger than at the first, Professor Lehr's method yields $\frac{P_2}{P_1} = 1.1081$, which is 10.81 per cent. too high.[4] Thus in order to get the result known to be true, Professor Lehr's method requires a preceding reduction (except in rare cases when the proper conditions happen to exist) before it is to be applied—a preliminary labor the need of which its author never contemplated.

§ 4. When money has not the same exchange-value at the second period as at the first, one of the conditions necessary for proving the correctness of Scrope's method used in the way under discussion (and also of Professor Lehr's method) is absent—since constancy of money's exchange-value resulted from the two conditions posited in the above demonstration and was indispensable to it. Hence even if we make this preliminary reduction, —or if it is superfluous, the state of things already being as desired,—we cannot use the above demonstration to justify a belief that in all cases Scrope's method applied to the arithmetic means of the mass-quantities will give the correct answer. A single negative instance is sufficient to dispel any such belief. We may suppose the following state of things:

I	100 A @ 1.00	100 B @ 1.00	100 for [A]	100 for [B]	— 200.
II	50 A @ 2.00	133⅓ B @ .75	100 for [A]	100 for [B]	— 200.

[4] Cf. this result with that above given for Scrope's method applied to the arithmetic means. From these and some other examples it would seem as if *this form of Scrope's method and Lehr's method err nearly equally on opposite sides of the truth.*

As the weighting is even and only two classes are dealt with, this is a case in which all the three superior methods agree. Their common result is most easily obtained by the geometric method. This is $\frac{P_2}{P_1} = \sqrt{2 \times \frac{3}{4}} = \sqrt{\frac{3}{2}} = 1.2247$, indicating a rise by 22.47 per cent. But Scrope's method applied to the arithmetic means of these mass-quantities is

$$\frac{P_2}{P_1} = \frac{75 \times 2 + 116\frac{2}{3} \times \frac{3}{4}}{75 \times 1 + 116\frac{2}{3} \times 1} = \frac{237\frac{1}{2}}{191\frac{2}{3}} = 1.2391,$$

which, indicating a rise by 23.91 per cent., is slightly above the truth.[5]

For further comparison another more irregular example may be subjoined :

I 40 A @ 1.25	70 B @ 1.50	50 for [A]	105 for [B] — 155,
II 50 A @ 1.80	100 B @ 1.30	90 for [A]	130 for [B] — 220.

Here the geometric method (in which the weighting is 1 for [A] and 1.7416 for [B]) indicates a rise of the price-level by 4.30 per cent., the method with double weighting a rise by 4.31 per cent., and Scrope's emended method a rise by 4.33 per cent. —all three almost exactly alike. On the arithmetic means of the full mass-quantities, Scrope's method indicates a rise by 4.21 per cent.; and on the arithmetic means of the mass-quantities reduced so that their total money-values are the same at both periods (the numbers of A's at the second period being reduced to 35.225, and that of B's to 70.45) Scrope's method is still slightly wrong, as it indicates a rise by 4.36 per cent.[6]

What is needed more than a reduction of the total money-values to the same figure is the reduction to the same figure of the total *exchange-values*. Thus the first of these examples would need to be altered into the following :

[5] Applied here to the smaller mass-quantities at either period, Scrope's method indicates a rise by only 16⅔ per cent.—a considerable error.—Lehr's method here indicates a rise by 21.05 per cent.

[6] Applied here to the smaller of the original mass-quantities at either period Scrope's method indicates a rise by 5.16 per cent.; and applied to the smaller of the reduced mass-quantities, a rise by 3.60 per cent.—Lehr's method applied to the original mass-quantities indicates a rise by 4.46 per cent.; and applied to the reduced mass-quantities, a rise by 4.33 per cent.

I	100 A @ 1.00	100 B @ 1.00			
II	61.235 A @ 2.00	163.293 B @ .75			
			100 for [A]	100 for [B]	—200,
			122.47 for [A]	122.47 for [B]	—244.94.

And now Scrope's method on the arithmetic means of these mass-quantities is:

$$\frac{P_2}{P_1}=\frac{80.617 \times 2 + 131.646 \times \frac{3}{4}}{80.617 \times 1 + 131.646 \times 1} = 1.2247,$$

which is exactly right.[7] This exact agreement, however, is due to some peculiarity in this example; for we do not find it always. Thus in the second example, after reducing the total sum at the second period to 161.68 (the number of A's then being 36.745 and that of B's 73.49), on the arithmetic means of the mass-quantities Scrope's method indicates a rise by 4.27 per cent.[8] But we have reason to believe that this method, so used, would generally give a more nearly correct result than when applied to the arithmetic means either of the full mass-quantities as they happen to be at each period or to the mass-quantities reduced merely to the same total money-values. Still, to employ Scrope's method in this way requires that we should first know the constancy or variation of the exchange-value of money. Hence Scrope's method, so used, is unsuitable for *finding* the constancy or variation of the exchange-value of money, since this use of it presupposes the knowledge we are searching for.

Nor can the method of approach be employed with any great security. Thus in the first of the above examples, after the measurement by Scrope's method applied to the arithmetic mean of the mass-quantities as they actually are, which has been seen to indicate a rise by 23.91 per cent., if we provisionally assume that prices have so risen, we might work over the schema into the following:

[7] Here also the right result is given both by Scrope's method applied to the smaller of these mass-quantities at either period, and by Lehr's method.

[8] This calculation has been made on the assumption that the indication given by the method with double weighting, at 1.0431, was the right one.—On these reduced mass-quantities Lehr's method now indicates a rise by 4.34 per cent.

I	100 A @ 1.00	100 B @ 1.00			
II	61.95 A @ 2.00	165.213 B @ .75			
			100 for [A]	100 for [B]	— 200,
			123.91 for [A]	123.91 for [B]	— 247.82;

nd now, applied to the arithmetic means of these mass-quan-ities, Scrope's method indicates a rise by 20.04 per cent. gain assuming this to indicate the price variation, we might epeat the operation thus :

I	100 A @ 1.00	100 B @ 1.00			
II	60.02 A @ 2.00	160.053 B @ .75			
			100 for [A]	100 for [B]	— 200,
			120.04 for [A]	120.04 for [B]	— 240.08;

hence the indication is of a rise by 22.61 per cent. And gain :

I	100 A @ 1.00	100 B @ 1.00			
II	61.305 A @ 2.00	163.48 B @ .75			
			100 for [A]	100 for [B]	— 200,
			122.61 for [A]	122.61 for [B]	— 245.22;

hence the indication is that prices in general have risen by 2.46 per cent. And if we repeated the operation once more, e should practically get the same result again, or perhaps a loser approach to 22.47, according to the extent to which we arry the decimals. Therefore, having got practically the same esult from two successive calculations, we should have no eason to go further, and should adopt this result,—which we lready know to be the right one. But here again there is omething peculiar in this example, that permits the getting f the right result; for the right result is not always got in his way. Thus in the second of the above examples, for hich the first employment of Scrope's method applied to the rithmetic means of the full actual mass-quantities gave us a ise by 4.21 per cent., if we assume this to be right and reduce he total money-values accordingly (that of the second period to 61.5255), and work out as before, the indication is of a rise by .34 per cent.; and again, assuming this, and reducing the total oney-values (of the second period to 161.727), and working ut as before, the indication is of a rise by 4.53 per cent.; and next, after reducing the total money-values (of the second period to 162.0215), the indication is of a rise by 4.35 per cent. This fourth answer hardly differs from the second, wherefore the next would hardly differ from the third, and pursuing these operations indefinitely we might perhaps reach some unvarying result somewhere between 4.34 and 4.53, but never could we get the right result, which lies outside these limits. Yet even if the right result could be reached in this way, the method would be unserviceable, as no more laborious method can be imagined.

§ 5. The peculiarity in the first example which caused it twice to yield the happy result, while the results in the other example were always divergent, is, in all probability, the same peculiarity in it that caused the three superior methods exactly to agree. This is the fact that in it the two classes are equally important over both the periods. Thus, in general, when two classes equally important over both the periods are dealt with, the same result is given not only by the superior methods, but also by Scrope's method applied to the arithmetic means of the mass-quantities after these have been reduced so as to be purchased at each period with sums of money possessing the same exchange-value, and by Professor Lehr's method applied to the mass-quantities similarly reduced, and even by the former of these methods (and probably by the latter also) when applied to the mass-quantities reduced so as to be purchased simply by the same sums of money, if thus continued through the method of approach. Here is additional confirmation that in such cases the common result so reached is the true result. But in other cases, or employed without such reductions or corrections, these two purely arithmetic averages diverge and scatter their results more than do the three at least partly geometric methods.[6] Hence the latter (or at least two of them) are probably nearer the truth than either of the purely arithmetic methods.

Now it may be conceded that the classes with rising prices and the classes with falling prices may frequently, or in the long run generally, be about even. Therefore the purely arithmetic

[6] And as above noticed in Note 4 generally on opposite sides of the three.

methods, if properly corrected through the method of approach, would generally yield results almost as good as those given by the partly geometric methods. Yet, to repeat, the use of those methods with such correction is impracticable.

But even without attempting such laborious correction, when, in practice, many classes are dealt with, and the variations both of prices and of mass-quantities are such moderate ones as generally take place from year to year, containing many sources of compensation, and the total money-values of the classes not varying much, nor the general exchange-value of money, then it is probable that Scrope's method applied to the arithmetic means will deviate but slightly from Scrope's method applied to the geometric means, and that Professor Lehr's method will deviate but slightly from the method with double weighting above advocated.

§ 6. For the purpose of illustrating these inductions, and also of casting a parting glance at other methods, we may use a more complex suppositional example than any used as yet. In order to magnify the errors, we may suppose very great variations both of prices and of mass-quantities; and also, to give opportunity for the working of compensatory errors, we may extend the comparisons over a short series of periods. Let us, then, posit the following schema:

I	6 A @ 2	12 B @ 3	10 C @ 1	3 D @ 8	6 E @ 4	1 F @ 6,
II	4 A @ 3	10 B @ 7	6 C @ 2	4 D @ 5	6 E @ 3	2 F @ 2,
III	5 A @ 3	13 B @ 4	8 C @ 2½	3 D @ 4	7 E @ 3	3 F @ 2,
IV	5 A @ 4	12 B @ 6	9 C @ 2	2 D @ 6	5 E @ 5	2 F @ 4,
V	7 A @ 2	11 B @ 7	7 C @ 3	5 D @ 3	8 E @ 4	3 F @ 3,
VI	6 A @ 2	12 B @ 3	10 C @ 1	3 D @ 8	6 E @ 4	1 F @ 6.

Here we have five periods with different states of things, between which are four different sets of variations. In addition it is supposed that at a sixth period everything returns to exactly the same state as at the first. The object in making the last supposition will appear presently.

Measuring every set of price variations separately by the three methods above advocated, we obtain the following percentages (the positive indicating rises and the negative falls)—by the geometric method:

+ 31.79, − 23.02, + 40.05, − 6.18, − 24.01; (1)

by the method with double weighting:

+ 31.72, − 22.92, + 39.98, − 6.36, − 23.64; (2)

by Scrope's emended method:

+ 31.11, − 23.00, + 39.93, − 6.35, − 23.39. (3)

These yield the following index-numbers, the columns representing the lines similarly numbered:

	(1)	(2)	(3)
I	100	100	100
II	131.79	131.72	131.11
III	101.45	101.53	100.95
IV	142.10	142.02	141.27
V	133.29	132.98	132.30
VI	101.29	101.54	101.35

Now by the arrangement of the prices and of the mass-quantities at the sixth period we know that the index-number for the sixth period ought to be 100. Our methods, then, err at the sixth period by between 1¼ and 1½ per cent. above the truth.[10] If they contained a single sort of error which gradually accumulated at every advance, we might correct the preceding figure by reducing it by four fifths of 1¼, 1½ or 1⅓ per cent., according to the method that is being corrected, and the next preceding by reducing it by three fifths, and so on. But this we cannot do, because we know that the errors in these

[10] Rather curiously it is with weighting according to the arithmetic means of the full money-values that the geometric average of the price variations gives, at the sixth period, in this example, the best result. Its indications then are of the following percentages:

+ 33.17, − 22.94, + 40.12, − 6.33, − 25.16;

forming these index-numbers:

100, 133.17, 102.60, 143.80, 134.69, 100.80.

With weighting according to Gauss's means (see Note 3 at the end of Sect. III. above) the geometric average gives these indications:

+ 32.48, − 22.98, + 40.13, − 6.26, − 24.60;

100, 132.48, 102.04, 143.00, 134.04, 101.07.

With weighting according to the arithmetico-geometrico-harmonic means (there also suggested), the results are nearly the same, being

+ 32.47, − 22.99, + 40.09, − 6.24, − 24.57;

100, 132.47, 102.03, 142.93, 134.01, 101.08.

ods are not cumulative, but alternating. The fact that methods show errors above the truth by certain amounts at ixth period, gives us no hint as to the errors at any of the ding periods. For instance, we cannot know whether the ations given for the fifth period are above or below the . In obtaining the sixth index-number from the fifth, we measured a variation of the general price-level from the to the sixth period, which is just the inverse of the whole tion from the first to the fifth period. Hence to test the period by the sixth period would be the same as to test the l method of reaching the result for the fifth period by the t method of comparing the fifth period immediately with irst—which latter method, as already said, has no better ι upon our approbation than the former.

›r would such a test be better than a test by comparing the period directly with the second. In fact, if we use such t comparisons at all, we ought to use all possible ones. ›ught to compare not only every period with the first, but ʳ period with the second, every period with the third, and . In the following table the index-numbers resulting from comparisons are stated, the method with double weighting being employed :

Every period compared with the first.	Every period compared with the second.	Every period compared with the third.	Every period compared with the fourth.	Every period compared with the fifth.
100.	75.91	94.42	66.48	76.36
131.72	100.	129.73	91.83	98.88
105.90	77.08	100.	71.43	79.97
150.40	108.89	139.98	100.	106.79
130.95	101.13	125.04	93.64	100.
100.	75.91	94.42	66.48	76.36

›etter comparison these figures may be re-arranged as follows, e same order.

100.	100.	100.	100.	100.
131.72	131.72	137.39	138.13	129.49
105.90	101.53	105.90	107.44	104.72
150.40	143.44	148.25	150.40	139.85
130.95	133.22	132.42	140.84	130.95
100.	100.	100.	100.	100.

ιo one of these series is preferable to another, we may average them,—doing so on the first table, doubling the first column in order to represent also the comparisons with the sixth period, and using the arithmetic average for want of a better. The series thus obtained is as follows:

100, 133.26, 105.28, 145.46, 132.84, 100.

Yet such a series cannot be accepted as authoritative.

Nor are better results obtained by using these methods adapted to a whole epoch—say of the five different periods. The geometric method, with weighting according to the geometric averages of the full money-values of the classes over all the five periods, yield these index-numbers:

100, 140.0, 109.5, 155.7, 141.9, 100.[11]

The method with double weighting applied to the geometric averages of the prices of the mass-units over all the five periods, these:

100, 131.01, 106.02, 149.49, 135.12, 100.[12]

Scrope's method applied to the geometric averages of the mass-quantities over all the five periods, these:

100, 130.12, 104.38, 141.04, 133.93, 100.[13]

None of these is satisfactory. It is plain that if another period were included in the epoch at the beginning or at the end, all the indications would be shifted. Thus, for example, with the sixth period added to the epoch, the method with double weighting applied to the geometric averages of the prices over all the six periods yields these index-numbers:

100, 134.49, 105.07, 149.76, 132.15, 100.[14]

No correction of the figures first given, therefore, seems to be possible. All that we can do is to take them as approxi-

[11] With weighting according to the arithmetic averages:
100, 140.3, 109.6, 155.8, 142.3, 100.

[12] Applied to the arithmetic averages:
100, 136.77, 105.89, 146.57, 135.12, 100.

[13] Applied to the arithmetic averages:
100, 128.76, 100.17, 140.80, 132.43, 100.

[14] Applied to the arithmetic averages:
100, 140.18, 104.95, 149.13, 141.28, 100.

mately representing the true variation in the general level of prices.

§ 7. Let us now turn to the indications yielded by other methods applied to this same schema, beginning with those which make most pretension to accuracy.

Professor Lehr's method gives the following percentages of the separate price variations:

+ 31.69, — 22.86, + 39.84, — 5.70, — 28.09;

which yield these index-numbers:

100, 131.69, 101.58, 142.05, 133.96, 96.33.

These are very close to those given by the three methods similarly used in serial form, except only in the last indication.[15]

Scrope's method presents many varieties. We have seen that two ways of drawing a mean between the mass-quantities at the two periods have previously been recognized. One of these is to apply it to the arithmetic means of the mass-quantities. This gives the following percentages:

+ 30.80, — 23.00, + 40.04, — 6.96, — 22.43. (1)

Another is to make the calculations once on the mass-quantities

[15] The method above suggested in Note 9 in Sect. II. gives these percentages:

+ 31.71, 23.06, + 40.26, - 6.58, 24.09,

which yield index-numbers as follows:

100, 131.71, 101.34, 142.14, 132.78, 100.79,

which also are very close. On the other hand, the method suggested in Appendix C, VI. § 3 as a possible form of Nicholson's method gives the following percentages:

+ 33.91, - 12.95, + 55.70, - 9.10, 2.41,

which yield these index-numbers:

100, 133.91, 116.57, 181.50, 164.98, 160.99,

—the worst of all. The method (with geometric average of the inverted mass-quantity variations) there also (in Note 4) suggested as a variant upon the last, would give these percentages:

+ 23.83, 22.17, + 48.00, 19.57, 12.86,

forming these index-numbers:

100, 123.83, 96.38, 142.64, 114.72, 100,

which are better, but still not good. It may be noticed that it is impossible to apply Drobisch's method to this example without further specifying the weights (or capacities) of the mass-quantities. Various suppositions with regard to these would make Drobisch's method yield various results, but would have no influence upon the results given by any of the other methods.

of the earlier periods and again on the mass-quantities of the later periods, and to average the results. Thus Scrope's method on the mass-quantities of the earlier period in every comparison (the same as the arithmetic average of the price variations—Young's method—with the full weighting of the earlier period) gives these percentages:

+ 40.18, − 22.79, + 42.06, − 1.29, − 14.29. (2)

And Scrope's method on the mass-quantities of the later period in every comparison (the same as the harmonic average of the price variations with the full weighting of the later period), these:

+ 21.43, − 23.17, + 37.78, − 11.58, −30.87. (3)

The arithmetic means between these percentages are the following percentages:

+ 30.80, − 22.98, + 39.92, − 6.44, − 22.58. (4)

In stringing these results out in a series of index-numbers there is a split into two ways of forming the mean series. The one is to form the index-numbers on the (arithmetic) mean percentages, as given in line (4). The other is to draw the (arithmetic) means of the index-numbers in the two series formed on lines (2) and (3). The series so formed is added in column (5).

	(1)	(2)	(3)	(4)	(5)
I	100.	100.	100.	100.	100.
II	130.80	140.18	121.43	130.80	130.80
III	101.72	108.22	93.29	100.74	100.75
IV	142.45	153.74	128.53	140.96	141.13
V	132.53	151.76	113.65	131.88	132.70
VI	102.80	130.07	78.57	102.10	104.82

The results in the average methods (in columns 1, 4 and 5) are, like those of Professor Lehr's method, very close to those given by the three more theoretically correct methods, showing that in practice all these methods are likely to give results very nearly alike. The figures in columns (2) and (3) are, singly, extravagant and absurd. But there is order in their extravagance; for the nearness of their means to the more truthful results shows that they straddle the true course, the one varying on the one side about as the other does on the other.[16] This is as we might expect from reasoning *à priori;* for we have seen that the one has no more reason for it than the other, and therefore the one has no more reason against it than the other. Between the various ways of drawing the mean there is little choice, except that the first is the most convenient.[17]

As for Young's method, when it is employed with what we have found to be proper weighting, though not proper for it, this gives most absurd results. Thus, employed with weighting according to the geometric means of the full money-values at both periods, its indications for the price variations are

+ 55.66, − 19.76, + 43.62, + 1.44, − 0.58,

forming these index-numbers:

100, 155.66, 124.90, 179.38, 181.97, 180.91.

Or with weighting according to the arithmetic means, the percentages are

+ 57.12, − 19.35, + 43.67, − 0.57, − 1.88,

[16] Of course the proper mean to draw here is really the geometric. The percentages of the geometric means between the variations whose percentages are given in lines (2) and (3) are

+ 30.47, − 22.92, + 39.90, − 6.58, − 23.02

which yield these index-numbers:

100, 130.47, 100.48, 140.57, 131.23, 101.10.

[17] We might also use the geometric average of the price variations in the way the arithmetic has just been used—twice applying it to the same price variations, once with the weighting of the earlier, and once with the weighting of the later period, and then drawing the geometric mean between the two results. This yields the following percentages—of the price variations on the weighting of the earlier periods:

+ 16.94, − 26.01, + 38.11, − 7.25, − 33.03;

of the price variations on the weighting of the later periods:

+ 48.12, − 19.49, + 41.80, − 5.47, − 11.58;

of the geometric mean of these variations:

+ 31.61, − 22.82, + 39.94, − 6.36, − 23.05;

and the following index-numbers on the last percentages:

100, 131.61, 101.58, 142.15, 133.11, 102.42.

But these are no better than those directly obtained by the geometric average with the geometric means of the weights of both periods.

and the index-numbers

100, 157.12, 126.72, 182.05, 181.02, 177.61.

Even with the weighting of the first period alone in every comparison,—weighting which is proper for the arithmetic average of the price variations when the mass-quantities are constant,—this method still errs largely, although it is now much better than with mean weighting. The figures have already been given—under Scrope's method, with which, in one form, this method, so weighted, is identical.

Furthermore there is the method of using the geometric average of the price variations with the weights that are the smaller at either of the two periods compared, and the variety of Scrope's method which applies it to the smaller of the mass-quantities at either of the periods compared. The former gives these percentages and index-numbers:

+ 26.30, − 23.24, + 39.33, − 1.60, − 19.72,

100, 126.30, 96.95, 135.08, 132.92, 106.71;

and the latter these:

+ 31.63, − 22.90, + 39.09, − 3.45, − 28.09,

100, 131.63, 101.16, 141.16, 136.29, 97.92.

These figures, although sometimes nearly right, are erratic, and untrustworthy. Nor is anything trustworthy reached by taking an average between them.

§ 8. Lastly it remains to see what plan should be adopted when we are reviewing the course of the exchange-value of money in times past, for which complete and accurate data are not obtainable. To attempt to use the complex methods above recommended as the best would be pedantic; for the figures posited in our example, especially of the mass-quantities, are now to be taken as only approximately correct. We must still use uneven weighting: but it can only be rough uneven weighting. There are two separate and distinct systems ready for our adoption.

We can either (1) draw some average of the total money-values of the classes during an epoch of years, and with weighting so determined employ the geometric average of the price variations; or (2) draw some average of the mass-quantities of the classes during the epoch, and apply to them Scrope's method. In each case, in getting the average weights or mass-quantities, we might as well, for greater convenience, employ the arithmetic average, because in these complex instances there is little difference in the results whether we use the arithmetic or the geometric average, and neither is exactly true. But *if we use an average of the total money-values for our weighting, we must use the geometric average of the price variations; and if we use an average of the mass-quantities, we must use Scrope's method.* There must be no intermingling of these operations.

Both these methods so used, we have seen, universally satisfy Professor Westergaard's test. Hence it is indifferent whether we compare the periods successively, or compare each succeeding period with the same original base (whether at the beginning, end, or in the middle, of the series). The calculations whereby the rough weighting is obtained should be renewed from time to time—say in every decade. This will require the starting of new chains, from new bases. But the results may, if desired, all be strung out in one series.

On our schema these two methods have already been worked out with the exact weighting, and applied to the exact averages of the mass-quantities, over all the five periods.[18] The results so obtained by Scrope's method, in this particular instance, are better than those obtained by the geometric method; and in other cases we may be pretty sure they will be as good, if not better. Scrope's method is, then, recommended by its greater convenience. Over the five periods in question the exact arithmetic averages of the mass-quantities are: $5\frac{2}{5}$ A, $11\frac{3}{5}$ B, 8 C, $3\frac{2}{5}$ D, $6\frac{2}{5}$ E, $2\frac{1}{5}$ F. If in their places we used only rough averages, as *e. g.*, in the same order, 5, 11, 8, 3, 6, 2, applied to these Scrope's method would give the following series of index-numbers:

100, 127.77, 100.92, 140.74, 132.40, 100,

[18] Above in Notes 12 and 13.

which, it will be noticed, are still fairly accurate. But there are occasions when, having a general idea of the relative importance of the classes, we may more conveniently use the geometric average of the price variations weighted accordingly.

That the distinctive elements in these methods should not be mixed together, in violation of our above canon, may be shown by the results obtained from the application of Young's method to this schema. The following index-numbers are obtained by arithmetically averaging the price variations at the successive periods, all compared with the first, in every comparison with the same weighting, which is the arithmetic average of the total money-values over the five periods: [19]

100, 163.66, 123.93, 165.87, 172.16, 100,

—the extravagance of which runs to absurdity.

By the same schema may be tested a few more methods that have been put into practice. Jevons's method, also employed by Walras and Simon, using the geometric average of the price variations with even weighting, gives these index-numbers:

100, 101.50, 92.47, 130.76, 104.63, 100,

indicating these percentages of variation between the periods:

+ 1.50, − 8.90, + 41.40, − 19.98, − 4.41.

And Carli's method, as extended by Evelyn, and as employed, for instance, in the *Economist*, drawing the arithmetic average of the price variations with even weighting, gives the index-numbers:

100, 125.70, 115.28, 144.44, 136.80, 100,

indicating these percentages of variation between the periods:

+ 25.70, − 8.29, + 25.30, − 5.29, − 26.90.

Thus in this example, as Jevons himself claimed, his method gives lower results than the ordinary arithmetic method. But its results are more aberrant even than those given by the latter method.

[19] The same weighting as above used with the geometric average of the price variations in Note 11.

The correction of the latter method offered by Mr. Palgrave, thmetically averaging the price variations of the later periods mpared with the first, in every comparison with the weighting the later period, appears, in this example, worse than what it uld improve upon; for its index-numbers are

O, 171.07, 131.41, 170.05, 174.58, 100,

ich are comparable only with those above given by Young's ethod, of which this method itself is a variety. In fact, oung's method, in every form, has been found to be bad.[20]

It must be remembered, however, that in actual practice the leviations of these methods would be much smaller than they re in this example.

V.

§ 1. As a *résumé* of the little which, after all these investigations, we have learned with absolute exactness and positiveness, we can now supply, in part, two propositions that have been lacking hitherto.

The one of these, which was missed in Chapter VII., is in regard to compensatory price variations. Because of the perfect accuracy of the geometric average with even weighting, or the geometric mean, of the price variations in all cases dealing with two classes between whose full weights at each of the two periods compared the geometric means are equal, we have this: *Of two classes equally important over both the periods compared compensatory price variations are simple geometric variations,* that is, variations from unity to the opposite geometric extremes around unity, so that the geometric *mean* between them is unity, indicative of constancy (Proposition XLVII.). In other words, if the price of [A] rises by p per cent. to $1 + p$ times its former price, in order that the exchange-value of money remain constant, the price of [B] must fall by $\frac{p}{1+p}$ per cent. to $\frac{1}{1+p}$ of its former price; and reversely a fall of [A] by p' per cent. to

[20] With such weighting so used the harmonic average of the price variations yields better, but still unsatisfactory, results, its index-numbers being

100, 147.02, 99.21, 150.48, 116.65, 100.

$1 - p'$ of its former price would be compensated by a rise of [B] by $\frac{p'}{1 - p'}$, per cent. to $\frac{1}{1 - p'}$, times its former price,—provided the accompanying states, or variations, of the mass-quantities are such as to make these classes equally important over both the periods (which variations, if they be of the mass-quantities together purchased at each period with the same total sum of money, will likewise be found to be simple geometric variations). This proposition is true in a world with only two classes of commodities beside money, if the classes so vary in price (and in mass-quantities). It is true also in a world with any number of classes of commodities, referring to two of them, provided the others do not vary in price, or vary in such wise that they compensate for one another (in pairs, or otherwise) and do not cause an alteration in the exchange-value of money (according to Proposition XXXI.). These others being such, their numbers will, as we have seen (in Proposition XIX.), cause the general exchange-value of money to fall less through the single rise in price of the one class than it would fall without them; but they will cause the general exchange-value of money to rise equally less through the single fall in price of the other class. And the influence of these price variations singly to deflect the general exchange-value of money being reduced in the same proportion (according to Proposition XXV.), the compensation is the same, no matter how numerous the other classes may be, or how large. Or, if the other classes have varied so as to cause a variation in the general exchange-value of money, the influence of these two classes is such as it would be if their prices had remained constant. This last, however, belongs to the next Proposition.

But we cannot extend this Proposition so as to say that among three equally important classes (over both the periods compared) the rise of [A] by p per cent. to $1 + p$ times its former price is compensated by a fall of [B] and [C] each by a percentage carrying its price down to $\sqrt{\frac{1}{1 + p}}$ of its former price, that is, to the figure between which twice repeated and the other the geometric *average*, with even weighting for the three classes, is unity, indicative of constancy; for this is not true,—and yet it is very near the truth. Nor can we extend it to any more complex cases, or to cases with uneven weighting. Yet in all these, with the amount of unevenness in weighting and of variation in prices likely to be met with in practice, the statement that compensatory price variations must be such that the geometric average of them, with proper weighting, should be unity, is likely to be nearly correct. The peculiarity of the geometric average must always be borne in mind, that, while the geometric *mean* proper, between only two equally important classes, and therefore with even weighting, in all cases where it can rightly be used, yields the true result, the geometric *average*, between more than two classes, or between unequally important classes, wherever uneven weighting has to be employed, yields a result deviating above or below the truth according to what seem to be fixed laws, but only slightly erring in ordinary cases. Wherever in practice, among a large number of classes, some large and some small, some rising and some falling in price, the geometric average with its proper weighting indicates constancy, we have good reason to believe that the true answer would show at most but a trifling variation.

The other Proposition, which was missed in Chapter VI., is more comprehensive,—in fact, it embraces the preceding. It is this: *When of two classes equally important over both the periods compared, the price of the one remains stationary while that of the other varies, or the prices of both vary, the influence of these price variations upon the general exchange-value of money is the same as it would be if both the classes varied alike in the geometric mean between their actual price variations* (Proposition XLVIII.). This is the only definite answer that can universally be given to Jevons's problem, to which a threefold answer was given in the last Chapter, on the condition there posited that the mass-quantities are constant over both the periods. The present Proposition is applicable whenever the geometric mean between the full weights of one class at the two periods is the same as the geometric mean between the full weights of one other class, whatever be the variations of the mass-quantities, provided, of course, they be such as with the price variations to permit of equality between the means. It directly covers, therefore, all the cases in Chapter X. in which equal sums are supposed to be constantly spent on two classes at both periods. It covers also those cases in Chapter XI. in which the mass-quantities are supposed to be constant, provided the sums spent at each period have this property of equality between their geometric means.

But again this Proposition cannot be extended to permit us to say that the influence of the price variations of *any* two classes, or of any number of classes, is the same as it would be if they all changed alike in the geometric average between their actual price variations weighted in the manner always prescribed. And yet in most cases their influence would be very nearly such. Again what is true of the geometric *mean* is not exactly true of the geometric *average*.

In the absence of an absolutely true method of measuring variations of general exchange-value, it is not convenient to improve upon the preceding general statements by attempting to formulate propositions in accordance with either of the other two methods of measurement which we found reason to believe to be slightly better than the geometric method. Yet by means of the formulæ for these methods the influence of the price variations of any numbers of classes, or, given certain price variations, the compensatory price variations of any numbers of other classes, all the other conditions being stated, is easily found with still closer approximation to the truth.

§ 2. If, however, it ever happens, or if we suppose, that the mass-quantities are constant through both the periods, thus covering all the cases in Chapter XI., it is easy to summarize the principles there discovered and to state the lacking Propositions, confined to these special cases, in twofold forms with perfect definiteness and universality. The first is: *With constant mass-quantities, compensatory price variations are arithmetic variations with the weighting of the first period, or harmonic variations with the weighting of the second period* (Proposition XLIX.). The second: *With constant mass-quantities, the influence upon the* *eneral exchange-value of money of any two or more price varia-ons is the same as if the prices all varied in the arithmetic verage with the weighting of the first period, or in the harmonic verage with the weighting of the second period* (Proposition L.). 'he third forms in which these Propositions can also be stated ave already been included in the preceding Propositions. The ıst two Propositions have little chance of application. But 'hat we know about Scrope's emended method shows that ımilar, but over-complex, Propositions could be framed, applied › the geometric means (and in most cases even to the arith-ıetic means) of the mass-quantities of every class at the two ›eriods, that would very closely approximate to the truth.

CHAPTER XIII.

THE DOCTRINE OF THE CONSERVATION OF EXCHANGE-VALUE, AND THE MEASUREMENT OF EXCHANGE-VALUE IN ALL THINGS.

I.

§ 1. In Chapter II. our Proposition XXXIX. was worded: "All things collectively, provided they be the same (or similar and in equal quantities) at all the periods compared, are constant in exchange-value." Examination of this principle was deferred. We are now prepared to examine it.

This proposition is not new. In one form or another it has often been intimated. Originally, and most frequently, it has been stated in the form that it is impossible for all things to rise or fall in value. It was so stated by Senior as early as 1836, and by J. S. Mill in 1848, after the latter of whom it has constantly been repeated.[1] Sometimes little more has been meant than the obvious assertion that neither all commodities together can rise in exchange-value nor all commodities together can fall in exchange-value. But usually there has also been an impli-

[1] Senior, *Political economy*, p. 21 (originally in the Encyclopædia Metropolitana, 1836: also in Arrivabene's translation of Senior's writings, Paris 1836, p. 104); Mill, *op. cit.*, Vol. I., pp. 540, 588; Levasseur, B. 18, pp. 138, 158; Courcelle Seneuil, *op. cit.*, Vol. I., pp. 259, 272; J. Bascom, *Political economy*, Andover 1860, p. 224; Fawcett, *op. cit.*, p. 312; Bolles (quoting Fawcett), *op. cit.*, p. 55: Cairnes, *op. cit.*, p. 12; Macleod, *Theory of banking*, Vol. I., p. 70, *Theory of credit*, Vol. I., p. 176; A. L. Perry, *Introduction to political economy*, 1881, p. 65; Mannequin, *Question monétaire*, 1881, p. 7; Jourdain, *Cours d'économie politique*, 1882, p. 435; Hansard, B. 67, p. 1; R. T. Ely, *Introduction to political economy*, 1889, p. 179; S. N. Patten, *Theory of dynamic economics*, 1892, pp. 64–65, and *Cost and expense*, 1893, p. 57 (of "objective values"); Denis, B. 125, p. 171; Fonda, B. 127, p. 6; Laughlin, *op. cit.*, pp. 147-148; Bourguin, B. 132, p. 25; Parsons, B. 136, pp. 114, 115 (of "internal exchange-values," offering a demonstration which is mathematically incorrect).

cation of counterbalancing,[2] with the meaning that all commodities together can neither rise nor fall in exchange-value—in other words, they are constant in exchange-value.[3] In this sense it is true only if the commodities are the same (or similar, in equal mass-quantities) at all the periods compared. This proviso has been noticed by Cairnes, who added that "the aggregate amount of values" may increase or decrease with an increase or decrease of commodities.[4]

The need of this proviso may be made apparent by analogy. If we have a universe consisting of three atoms endowed with equal attractive force, the attractive force of each atom is one third of the whole attractive force in the universe. If then a fourth atom is created in this universe, endowed with attractive force equal to that of each of the others, although the force in each of the atoms now sinks to one fourth of the whole, it is one fourth of a whole larger by one third, and $\frac{1}{4} \times (3 + 1) = \frac{1}{3} \times 3$, so that each of the three atoms continues to have the same force as before, and as there is one more such atom, the attractive force in the universe has increased. Similarly if we have an economic world consisting of three articles, exchanging equally for one another, so that they are endowed with equal purchasing power or exchange-value, the purchasing power possessed by any one of these is one third of the whole purchasing power in this world. Then if another article is produced in this world,

[2] Sometimes erroneously. Thus Ely: "Let us suppose that to-day two bushels of wheat exchange for three of oats, and to-morrow for four bushels of oats. We may say that wheat has risen in value, but it is obvious that exactly in the same proportion oats have fallen in value," *loc. cit.* Wheat has risen 33⅓ per cent. in exchange-value in oats; oats have fallen 25 per cent. in exchange-value in wheat. These proportions are not the same. There is geometric equality of proportion; but this should be specified.—A mistake like this is responsible for Parson's attempted demonstration.

[3] This inference has been directly drawn by Fonda (without the proviso), B. 27, p. 18.—In this sense, but ambiguously about "value," the proposition has been combated by A. Clément, *De l'influence exercée par la hausse ou la baisse des valeurs sur la richesse générale*, Journal des Économistes, July 1854, pp. 8–10, because he perceived that use-values can rise or fall together; and by Pollard, *op. cit.*, Ch. IX., because it is not true of cost-value, in which sense alone he prefers to use the term "value." Cf. also Ricardo above quoted in Chapt. I. Sect. V. Note 8.

[4] *Op. cit.*, pp. 12–13. The proviso has also been incorporated in the statement by Mannequin, *loc. cit.*

which will exchange for any one of the others, so that it is endowed with purchasing power equal to theirs, although the purchasing power of each of the articles is now one fourth of the whole, yet as it is one fourth of a whole larger by one third as in the former instance, therefore the purchasing power of each article continues the same as before (cf. Propositions XLII. and XLIII.), and so the total amount of purchasing power, or exchange-value, in this world has increased.

§ 2. There is, however, a form in which this principle may be presented which seems to make it true even under any changes in the mass-quantities.[5] This form is, in fact, in accordance with the way in which the principle was originally hit upon. Let us conceive an economic world containing two articles, A and B, equally valuable at first. If A comes to rise in exchange-value compared with B, we know that B must fall in exchange-value in A. Therefore there cannot be a rise both of A and of B in exchange-value each in the other, or of both in both together; nor contrariwise a fall of both together. Now if A becomes twice as valuable as B, B becomes half as valuable as A, and from having A = B and B = A at the first period, we have at the second A = 2B and B = ½A. How are we to conceive of a constancy here of the exchange-values of A and B together? If we add together the equal exchange-values of A and B, each taken as a unit, at the first period, we get the sum $1 + 1 = 2$; and if we add together the unequal exchange-values at the second period, to wit, 2 for A and ½ for B, we get $2 + \frac{1}{2} = 2\frac{1}{2}$, which is not the same as before. Or again, A has risen by 100 per cent. in B, and B has fallen by 50 per cent. in A, that is, A has varied by + 100 per cent. and B by − 50 per cent., and if we add these we get + 50 per cent., which does not indicate constancy. The indication of constancy is to be got in another way. At the first period we may multiply

[5] This wider position cannot be argued for on the ground that the exchange-values of all things could change only relatively to something else, which is self-contradictory; for they also could not remain constant, in this way, except in relation to something else, which is equally self-contradictory (cf. Chapt. Sect. II. §§ 4 and 9). What we are here dealing with is not the exchange-value of all things as a whole compared with something else, but a total of some of all the exchange-values of all things within a whole.

hange-value of each article as expressed in the other, and oduct is unity, for $1 \times 1 = 1$; and at the second period ultiplication of the exchange-value of each article as ex- l in the other still yields unity as its product, for $= 1$. Yet if a third article had come into existence, say he exchange-value of B, on multiplying together the ex- ns for its exchange-value in A and A's in it, for its in B 's in it, and, as before, for A's in B and B's in A, as fol- $\frac{1}{2} \times 2 \times 1 \times 1 \times 2 \times \frac{1}{2}$, the product still is unity.

se considerations lead to two propositions, which are s, but of little utility. They are: *The product of the of all particular exchange-values is unity* (Proposition and, *The product of all variations of particular exchange- is unity* (Proposition LII.). To prove these propositions y try them on some simple case, so far as possible in uni- form. Let us suppose that we have an economic world ning three classes, such that we have, at the first period, ass-unit of [A] for every two mass-units of [B] and for three mass-units of [C], and that the mass-unit of [A] is b_1 times the mass-unit of [B] and c_1 times the mass-unit]. Then all the ratios of exchange-value between these are expressed in the following list:

$$A = b_1 B = b_1 B = c_1 C = c_1 C = c_1 C,$$

$$B = 1B = \frac{1}{b_1} A = \frac{c_1}{b_1} C = \frac{c_1}{b_1} C = \frac{c_1}{b_1} C,$$

$$B = 1B = \frac{1}{b_1} A = \frac{c_1}{b_1} C = \frac{c_1}{b_1} C = \frac{c_1}{b_1} C,$$

$$C = \frac{b_1}{c_1} B = \frac{b_1}{c_1} B = \frac{1}{c_1} A = 1C = 1C,$$

$$C = \frac{b_1}{c_1} B = \frac{b_1}{c_1} B = \frac{1}{c_1} A = 1C = 1C,$$

$$C = \frac{b_1}{c_1} B = \frac{b_1}{c_1} B = \frac{1}{c_1} A = 1C = 1C.$$

now in multiplying all these ratios together we find we have two b_1 to multiply by two $\frac{1}{b_1}$, three c_1 to multiply by three $\frac{1}{c_1}$, six $\frac{c_1}{b_1}$ to multiply by six $\frac{b_1}{c_1}$, and eight 1 to multiply together, the product of all which is unity. Then at the second period if we suppose A to be worth b_2 times B and c_2 times C, and if we suppose the same or any other number of mass-units of these classes, or even any other classes with any other ratios of exchange-value, treating them in the same way, we should get the same result. But when we multiply together the variations, it is plain that we have variations only of the same numbers of mass-units in each of the classes. Let us then suppose we have at both periods those already supposed. The variation of A relatively to each B is $\frac{b_2}{b_1}$ and of each B relatively to A $\frac{b_1}{b_2}$, of A relatively to each C $\frac{c_2}{c_1}$ and of each C relatively to A $\frac{c_1}{c_2}$, and of each B relatively to each C $\frac{b_1c_2}{b_2c_1}$ and of each C relatively to each B $\frac{b_2c_1}{b_1c_2}$, while the variation of each B to each B and of each C to each C is 1. Thus the full list is:

Variations of A — $\frac{b_2}{b_1}, \frac{b_2}{b_1}, \frac{c_2}{c_1}, \frac{c_2}{c_1}, \frac{c_2}{c_1}$,

" " B — 1, $\frac{b_1}{b_2}, \frac{b_1c_2}{b_2c_1}, \frac{b_1c_2}{b_2c_1}, \frac{b_1c_2}{b_2c_1}$.

" " B — 1, $\frac{b_1}{b_2}, \frac{b_1c_2}{b_2c_1}, \frac{b_1c_2}{b_2c_1}, \frac{b_1c_2}{b_2c_1}$.

" " C — $\frac{b_2c_1}{b_1c_2}, \frac{b_2c_1}{b_1c_2}, \frac{c_1}{c_2}$, 1, 1,

" " C — $\frac{b_2c_1}{b_1c_2}, \frac{b_2c_1}{b_1c_2}, \frac{c_1}{c_2}$, 1, 1.

" " C — $\frac{b_2c_1}{b_1c_2}, \frac{b_2c_1}{b_1c_2}, \frac{c_1}{c_2}$, 1, 1.

ind here, too, it is perceived that we have only even numbers f reciprocals, so that the multiplication of all these variations y one another gives the product, unity. And the same would e the result with any numbers of things, with any particular ariations.

These Propositions, then, are of little importance theoretical r practical; for as the products of all the ratios at every period, nd of all the variations between any periods, are always unity -always the same,—we gain nothing by comparing them.[6]

But there is more in the Proposition placed at the head of iis Chapter, something which renders it truly a doctrine of the onservation of exchange-value (of a certain kind), under the roviso that the classes and their mass-quantities are the same t both periods,—or that we confine our attention to those lasses, and to those portions of them, that are the same (or milar) at both the periods compared. In order to understand iis we must find the method of measuring, under the proviso ated, the variation of the exchange-value of every other thing ıder consideration, as well as that of money, and not only of eir exchange-values in all *other* things, but also of their exange-values in *all* things.

II.

§ 1. Under the condition that the mass-quantities are the me at both periods (or that we are dealing only with the massantities that are common to both periods) we know that rope's method, applied to the constant mass-quantities, is the oper method for measuring the variation of the price-level, id the inverse of this is the proper measurement of the variion of the exchange-value of money in all *other* things. Let s now examine, under its universal aspect, the simplest hypothis possible, namely the familiar one of a world with money and vo commodities. We have generally stated it in the following hema

$$\text{I} \quad x \text{A} @ \alpha_1 \quad y \text{B} @ \beta_1,$$
$$\text{II} \quad x \text{A} @ \alpha_2 \quad y \text{B} @ \beta_2.$$

[6] Cf. a similar useless position in Chapt. V. Sect. I. § 5.

This is sufficient when we are treating of the exchange-value of money in all *other* things; but it is incomplete when we are treating of the exchange-value of money in *all* things. Then we must include a statement of money itself, the quantity of which must, in accordance with the hypothesis, be the same at both periods, and the price of its money-unit is, of course, at both periods a unit. Hence, with the volume of money represented by v, the schema becomes

$$\begin{array}{llll} \text{I} & x\,\text{A} @ a_1 & y\,\text{B} @ \beta_1 & v\,\text{M} @ 1, \\ \text{II} & x\,\text{A} @ a_2 & y\,\text{B} @ \beta_2 & v\,\text{M} @ 1. \end{array}$$

Scrope's formula for the variation of the exchange-value of money in all *other* things is

$$\frac{M_{02}}{M_{01}} = \frac{xa_1 + y\beta_1}{xa_2 + y\beta_2},$$

this being the same as

$$\frac{M_{02}}{M_{01}} = \frac{1}{n_2''}\left(xa_2\frac{a_1}{a_2} + y\beta_2\frac{\beta_1}{\beta_2}\right),$$

in which $n_2'' = xa_2 + y\beta_2$, and this again the same as

$$\frac{M_{02}}{M_{01}} = \frac{1}{n_2''}\left(xa_2\frac{a_2}{a_1} + y\beta_2\frac{b_2}{b_1}\right),$$

that is, as the arithmetic average of the exchange-value variations with the weighting of the second period. We have already learned, in Chapter V. Section VII., to turn such a formula for exchange-value in all *other* things into the formula for exchange-value in *all* things. This is, returning to the second of the above formulæ,

$$\frac{M_{a2}}{M_{a1}} = \frac{1}{n_2''}\left(xa_2\frac{a_1}{a_2} + y\beta_2\frac{\beta_1}{\beta_2} + v1 \times \frac{1}{1}\right),$$

in which $n_2'' = xa_2 + y\beta_2 + v1$. This reduces to

$$\frac{M_{a2}}{M_{a1}} = \frac{xa_1 + y\beta_1 + v}{xa_2 + y\beta_2 + v},$$

which is Scrope's formula for the variation of the exchange-value of money in *all* things (including itself).

des this we want Scrope's formulæ for the variation exchange-value in *all* things both of [A] and of [B]. we may obtain by simply imitating what we have done ney. In obtaining the formula for money we first stated ditions in terms of money, the money-unit being taken unit of exchange-value at each period, and the exchange- of the other mass-units being stated as they relate to the ge-value of this money-unit at each point. Therefore, the formula for [A], we must first of all state the rela- f the exchange-value of the other mass-unit and of the unit to the exchange-value of the mass-unit of [A], s unit at each period. The schema then is

$$\text{I} \quad x \text{ A} @ 1 \quad y \text{ B} @ \frac{\beta_1}{\alpha_1} \quad v \text{ M} @ \frac{1}{\alpha_1},$$

$$\text{II} \quad x \text{ A} @ 1 \quad y \text{ B} @ \frac{\beta_2}{\alpha_2} \quad v \text{ M} @ \frac{1}{\alpha_2}.$$

have here modelled everything upon the previous schema ey, merely changing the units, we may merely copy the for the variation of the exchange-value of money in *all* and so we have

$$\frac{A_{a2}}{A_{a1}} = \frac{x + y\dfrac{\beta_1}{\alpha_1} + v\dfrac{1}{\alpha_1}}{x + y\dfrac{\beta_2}{\alpha_2} + v\dfrac{1}{\alpha_2}},$$

reduces to

$$\frac{A_{a2}}{A_{a1}} = \frac{\alpha_2(x\alpha_1 + y\beta_1 + v)}{\alpha_1(x\alpha_2 + y\beta_2 + v)} = \frac{\alpha_2}{\alpha_1} \cdot \frac{M_{a2}}{M_{a1}}.$$

or the case of [B] we have, first of all, the schema :

$$\text{I} \quad x \text{ A} @ \frac{\alpha_1}{\beta_1} \quad y \text{ B} @ 1 \quad v \text{ M} @ \frac{1}{\beta_1},$$

$$\text{II} \quad x \text{ A} @ \frac{\alpha_2}{\beta_2} \quad y \text{ B} @ 1 \quad v \text{ M} @ \frac{1}{\beta_2};$$

the formula,

$$\frac{B_{a2}}{B_{a1}} = \frac{x\frac{\alpha_1}{\beta_1} + y + v\frac{1}{\beta_1}}{x\frac{\alpha_2}{\beta_2} + y + v\frac{1}{\beta_2}}$$

$$= \frac{\beta_2(x\alpha_1 + y\beta_1 + v)}{\beta_1(x\alpha_2 + y\beta_2 + v)} = \frac{\beta_2}{\beta_1} \cdot \frac{M_{a2}}{M_{a1}}.$$

What is thus shown for money and two classes of commodities is plainly perceived to be extendible to money and any number of classes. With any number of classes, for any class, say [L], whose prices are λ_1 and λ_2, we shall have

$$\frac{L_{a2}}{L_{a1}} = \frac{\lambda_2}{\lambda_1} \cdot \frac{M_{a2}}{M_{a1}}.$$

We thus have a convenient method for converting the formula for the variation of the exchange-value of money in all things, under the given proviso, into a formula for the variation of any one class (that was included in the formula for money) in exchange-value in all things. We only have to multiply the expression for the variation of the exchange-value of money in all things by the price variation of the class in question.[1]

§ 2. Having obtained these formulæ, we may now discover the meaning of the Proposition at the opening of the Chapter. We have seen, by a particular example, which is sufficient to prove a negative, that the expressions for the exchange-values of all things (or classes) in all *other* things do not sum up to the same figure at each period, even when the same things are in existence at both periods. We shall now find that, given this condition, the universal expressions for the exchange-values of all things, or classes, in *all* things do sum up to the same figure,

[1] A similar result cannot be obtained with exchange-value in all *other* things; for, while we have

$$\frac{M_{O2}}{M_{O1}} = \frac{x\alpha_1 + y\beta_1}{x\alpha_2 + y\beta_2},$$

we have

$$\frac{A_{O2}}{A_{O1}} = \frac{\alpha_2(y\beta_1 + v)}{\alpha_1(y\beta_2 + v)} \quad \text{and} \quad \frac{B_{O2}}{B_{O1}} = \frac{\beta_2(x\alpha_1 + v)}{\beta_1(x\alpha_2 + v)},$$

and there is no such reduction.—All this is in agreement with Proposition XXXIII.: and it suggests a still broader Proposition.

when treated in a special manner. This is to multiply the expressions for the exchange-values at each period by the weights at the first period. These expressions at the first period are to be taken as units, so that the sum of these units multiplied by the weights of the first period is thus expressed, $x\alpha_1 + y_i\beta_1 + v$. And at the second period, compared with the first, the expressions for the exchange-value in *all* things of all the classes are the expressions above discovered for the variations of the exchange-values of all the classes in *all* things. Hence what we want to prove is

$$x\alpha_1 \cdot \frac{\alpha_2}{\alpha_1} \cdot \frac{M_{a2}}{M_{a1}} + y_i\beta_1 \cdot \frac{\beta_2}{\beta_1} \cdot \frac{M_{a2}}{M_{a1}} + v \cdot \frac{M_{a2}}{M_{a1}} = x\alpha_1 + y_i\beta_1 + v.$$

The first half reduces, and we get

$$(x\alpha_2 + y_i\beta_2 + v)\frac{M_{a2}}{M_{a1}} = x\alpha_1 + y_i\beta_1 + v,$$

which is evident when we restore the full expression for $\frac{M_{a2}}{M_{a1}}$. Q. E. D. Thus *at any two periods in which the mass-quantities are constant the sums of the exchange-values in all things of all the classes* (*or of all things*) *multiplied by the weights at the first period are the same, so that comparison of them gives unity, indicating constancy* (Proposition LIII.).

What is thus easily proved, may be varied in the following way. Of the three expressions

$$\frac{M_{a2}}{M_{a1}}, \quad \frac{\alpha_2}{\alpha_1} \cdot \frac{M_{a2}}{M_{a1}}, \quad \frac{\beta_2}{\beta_1} \cdot \frac{M_{a2}}{M_{a1}},$$

we know that at least one must be below, if one is above, unity, and reversely. We know this on general principles already reviewed.[1] It also follows as a necessary consequence from what has just been proved. But we may also gather it by simple inspection of the expressions, when stated in full. Let us suppose that the exchange-value of money has risen and that $\frac{M_{a2}}{M_{a1}} = r$ (r being > 1). The prices of [A] and [B] must now

[1] Propositions IX., XV., XVI., XXVIII., XXIX.

either have both fallen uniformly, or at least one of them must have fallen. If they have both fallen uniformly, then, in the full expression, r being a positive quantity that cannot be neglected, $\frac{\alpha_1}{\alpha_2}$ and $\frac{\beta_1}{\beta_2}$ must each be greater than r; and consequently $\frac{\alpha_2}{\alpha_1}$ and $\frac{\beta_2}{\beta_1}$ must each be smaller than $\frac{1}{r}$. But r multiplied by a figure smaller than $\frac{1}{r}$ gives a result smaller than unity. Or if one of the prices has fallen less (or remained constant, or even has risen), then the other must have fallen by still more than $\frac{1}{r}$, and at least the expression for the variation of the exchange-value in all things of the class (and everything in it) having this price variation must be below unity. The same result will follow if we suppose the exchange-value of money to have fallen, or if we started with either of the other classes; and also it will be obtained if we enlarge the supposition to include any number of classes. Always, if any one of the expressions is on one side of unity, at least one of them must be on the other side. (Wherefore if all but one are known to equal unity, that one also must equal unity, in obedience to Proposition XX.).

This being so, it admits of demonstration that the percentage of the variation, or the sum of the percentages of the variations, of exchange-value in all things, above unity, multiplied by the weights at the first period, is equal to the sum of the percentages of the variations, or to the percentage of the variation, of exchange-value in all things, below unity, likewise multiplied by the weights at the first period. We do not know, in the example before us, which of the above three expressions is larger, or which smaller, than unity. There are six possible permutations, and we may assume one of them as a specimen. Let us assume that money has risen in exchange-value in all things, and that the two commodity classes have each fallen in exchange-value in all things. Then the percentages of the rise of money (in hundredths) is $\frac{M_{a2}}{M_{a1}} - 1$, and the percentages of the falls of the commodities are $1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{M_{a2}}{M_{a1}}$ and $1 - \frac{\beta_2}{\beta_1} \cdot \frac{M_{a2}}{M_{a1}}$. We therefore wish to prove that

$$v\left(\frac{M_{a2}}{M_{a1}} - 1\right) = x\alpha_1\left(1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{M_{a2}}{M_{a1}}\right) + y_i\beta_1\left(1 - \frac{\beta_2}{\beta_1} \cdot \frac{M_{a2}}{M_{a1}}\right).$$

And this is easily done because the equation to be proved reduces, as before, to

$$(x\alpha_2 + y_i\beta_2 + v)\frac{M_{a2}}{M_{a1}} = x\alpha_1 + y_i\beta_1 + v,$$

and so again is evident when we restore the full expression for the variation of money. And if we suppose any of the five other permutations, we shall always find a similar demonstration. And what is thus proved of money and two classes, may be extended to money and any number of classes, or to cases without money; for, although the demonstration becomes more complex with a greater number of classes, it may be carried out in the same manner.

The meaning of what has just been proved may be expressed as follows: *The same mass-quantities of all classes existing at both periods, the sum of the percentages of the variation of every class in exchange-value in all things, each multiplied by the weight of the class at the first period (the percentages of the rises being treated as positive quantities, and the percentages of the falls as negative quantities), is zero.* (Proposition LIV.) This zero means that the common variation of all things together in exchange-value in all things is by zero per cent., which is constancy.

Here we have used percentages in the ordinary way, reckoning them in the starting points of the variations—at the *first* period. Hence it is only natural that importance should be assigned to the variations according to the importance of the variants at the first period. And conceiving the proportions of the variations in this the usual way, we find that, when the mass-quantities are constant, there is what we have called *arithmetic equality* between all the proportions of the rising variations of exchange-values in all things and all the proportions of the falling variations of exchange-values in all things, the weights of the classes at the *first* period being assigned to their variations. This is precisely what we should expect in such cases, because of Scrope's formula holding here. For we have learnt that Scrope's formula is the same as the formula for the *arithmetic* average of variations with the weighting of the *first* period.

We may also expect more, since we know more about Scrope's formula. We may suspect that there also is, in these cases, what we have called *harmonic equality* between all the proportions of the rising variations of exchange-values in all things and all the proportions of the falling variations of exchange-values in all things, the weights assigned to the variations being those of the *second* period. In other words, the sums of the percentages of all the rising and of all the falling variations harmonically reckoned, that is, reckoned in the finishing points, at the second period, multiplied by the weights at the second period, will be equal; or, if added together as positives and negatives, will yield zero as the total sum. This may be quickly proved by taking the same permutation in the above example, and stating its percentages (in hundredths) harmonically. We wish, then, to prove that

$$v\left\{\frac{\frac{M_{a2}}{M_{a1}}-1}{\frac{M_{a2}}{M_{a1}}}\right\}=x\alpha_2\left\{\frac{1-\frac{\alpha_2}{\alpha_1}\cdot\frac{M_{a2}}{M_{a1}}}{\frac{\alpha_2}{\alpha_1}\cdot\frac{M_{a2}}{M_{a1}}}\right\}+y\beta_2\left\{\frac{1-\frac{\beta_2}{\beta_1}\cdot\frac{M_{a2}}{M_{a1}}}{\frac{\beta_2}{\beta_1}\cdot\frac{M_{a2}}{M_{a1}}}\right\};$$

and can do so, because this reduces to

$$v\left(1-\frac{M_{a1}}{M_{a2}}\right)=x\alpha_2\left(\frac{\alpha_1}{\alpha_2}\cdot\frac{M_{a1}}{M_{a2}}-1\right)+y\beta_2\left(\frac{\beta_1}{\beta_2}\cdot\frac{M_{a1}}{M_{a2}}-1\right)$$

and to

$$(x\alpha_1+y\beta_1+v)\frac{M_{a1}}{M_{a2}}=x\alpha_2+y\beta_2+v,$$

which is evident when we restore the full expression for $\frac{M_{a1}}{M_{a2}}$ $\left(\text{the inverse of } \frac{M_{a2}}{M_{a1}}\right)$. Q. E. D. And again, as we may expect, we find that, with two things, or classes, constant in mass, and equally important over both the periods together, there is *geometric equality* between the proportions of their variations in exchange-value in all things (viz., the two things or classes themselves),—that is, the percentages of their variations in such exchange-value, geometrically reckoned (reckoned in opposite directions) are equal. For, supposing these articles to be priced in ideal money, [A] rising and [B] falling, what we have to prove is

$$\frac{\dfrac{y\frac{\beta_1}{\alpha_1}+x}{y\frac{\beta_2}{\alpha_2}+x}-1}{\dfrac{y\frac{\beta_1}{\alpha_1}+x}{y\frac{\beta_2}{\alpha_2}+x}}=1-\frac{x\frac{\alpha_1}{\beta_1}+y}{x\frac{\alpha_2}{\beta_2}+y};$$

but this reduces to

$$y^2\beta_1\beta_2=x^2\alpha_1\alpha_2,$$

which is the very condition supposed, so that the equation is true when this condition is satisfied. Q. E. D. We should find also, by trial, that in ordinary cases there is approximation to geometric equality, or equality between the sums of the geometric percentages of all the rising variations together and of all the falling variations, if these percentages are multiplied by the weights over *both* the periods. But these two positions, about harmonic and geometric equality, are of little utility, especially the last; and they do not merit being stated in formal propositions.

Thus we have found in what sense the Proposition at the opening of the Chapter is true. It is true of general exchange-value in *all* things rightly measured, and with proper allowance for the sizes of the classes. It is not true of general exchange-value in all *other* things.[3] Therefore we should add to it the words "*in all things.*"

§ 3. In evidence of what has here been universally proved, and as suggestive of other relations, a particular example may be offered. Suppose we have these states of things:

[3] Parsons tried to demonstrate it of this kind of general exchange-value. Hence the necessity of his failure. See Notes 1 and 2 in Section I. of this Chapter.

I 10 A@1 5 B@4 6 C@3 4 D@7 7 E@2,
II 10 A@1 5 B@4 6 C@6 4 D@7 7 E@2.

Here we have only one price variation, and this price variation shows that the class [C] has risen in exchange-value in all *other* things by 100 per cent. (reckoned in the ordinary way). Now if we measure the variation of money in exchange-value in all other things by means of Scrope's formula, we have

$$\frac{P_2}{P_1} = \frac{10 + 20 + 36 + 28 + 14}{10 + 20 + 18 + 28 + 14} = \frac{108}{90} = 1.20,$$

indicating a rise of 20 per cent., which means that money has fallen in exchange-value in all *other* things by $16\frac{2}{3}$ per cent. (for $\frac{90}{108} = 0.8333 \cdots\cdots$). This means also that all the commodities which have remained unchanged in exchange-value relatively to money have also fallen by $16\frac{2}{3}$ per cent. in exchange-value in all those things in which money has fallen by that percentage (that is, in all the things which are *other* to money, namely *in all commodities*). Now let us remove money from consideration,—or let us suppose we have been dealing only with ideal money (money of account). As it has not entered into the calculation, its removal causes no change. But now all commodities become all things. And the classes, [A], [B], [D], [E], which remained unchanged relatively to money and to one another, are still altered in exchange-value in all commodities, that is, now, in *all* things, just as they were before. Thus these classes have fallen by $16\frac{2}{3}$ per cent. in exchange-value in *all* things. And the class [C] has still risen by 100 per cent. in all *other* things; but as it has risen by 100 per cent. above things that have fallen by $16\frac{2}{3}$ per cent. in exchange-value in *all* things, it has risen from 1 to $2 \times 0.83\frac{1}{3} = 1.66\frac{2}{3}$, or by $66\frac{2}{3}$ per cent. in exchange-value in *all* things (not counting money, but including itself). This may be shown again by supposing that at the second period all the prices were $16\frac{2}{3}$ per cent. lower than they were supposed actually to be. Then we should have

$$\frac{P_2}{P_1} = \frac{8\frac{1}{3} + 16\frac{2}{3} + 30 + 23\frac{1}{3} + 11\frac{2}{3}}{10 + 20 + 18 + 28 + 14} = \frac{90}{90} = 1.00,$$

indicating constancy. For here, money remaining stable, the price, and consequently the exchange-value in all things other to money, that is, in all commodities,[4] of each of the classes has fallen by $16\frac{2}{3}$ per cent., except the price of C, which has risen by $66\frac{2}{3}$ per cent., showing that its exchange-value in all commodities has risen by $66\frac{2}{3}$ per cent. The same results could also have been obtained directly, by making use of the method above exploited. For here we could immediately take A as the unit, omitting money altogether, and should have

$$\frac{A_{a2}}{A_{a1}} = \frac{10 + 20 + 18 + 28 + 14}{10 + 20 + 36 + 28 + 14} = \frac{90}{108} = 0.83\frac{1}{3};$$

and for the variations in exchange-value in all things of B, D, E, we should have this multiplied by $\frac{4}{4}$, $\frac{7}{7}$, $\frac{2}{2}$, but for C we should have it multiplied by $\frac{2}{1}$, with answer $\frac{C_{a2}}{C_{a1}} = 1.66\frac{2}{3}$, showing the rise of [C] in exchange-value in all things to be by $66\frac{2}{3}$ per cent.

Now $66\frac{2}{3}$ is four times $16\frac{2}{3}$. But the weights of the classes [A], [B], [D], [E] are, four times larger than the weight of the class [C] at the *first* period. Thus four classes, four times larger than the class [C] at the first period, have together fallen $4 \times 16\frac{2}{3} = 66\frac{2}{3}$ per cent., arithmetically equalling the rise of [C] by $66\frac{2}{3}$ per cent., in exchange-value in all things (namely the five classes of commodities).[5]

§ 4. In this form the Proposition may be of some utility. Thus if we suppose only two classes equally important at the first period, and the mass-units of each, A and B, are equally valuable at the first period; if, the mass-quantities being constant, the exchange-value of A rises to double that of B, we see

[4] Also in all things (including money); for otherwise the method of conversion described in § 1 of this Section would be at fault. This is because money, here, is constant in general exchange-value of both kinds. It is plain that any number of classes could now be added with constant prices (or in the preceding case with prices rising 20 per cent.), without affecting any of the relations thus far used. See also Proposition XXXIV.—But this indication is *not* true of their exchange-values in all *other* things (other to them singly). See Proposition XXXIII.

[5] Again, at the *second* period the four classes together are twice as important as [C]. Reckoned harmonically, *i. e.*, in the finishing points, the percentage of their falls is 20 per cent., and the percentage of the rise of [C] is 40 per cent.—just twice as much.

that, whereas at the first period each class had half the total exchange-value in the world, at the second period [A] has two thirds of it and [B] one third; wherefore, although A has risen by 100 per cent. in exchange-value in the other thing, and B has fallen by 50 per cent. in exchange-value in the other thing —simple geometric variations,—the exchange-value of [A], and consequently of A, in all things has risen from $\frac{1}{2}$ to $\frac{2}{3}$ by $33\frac{1}{3}$ per cent., and the exchange-value of [B], and consequently of B, in all things has fallen from $\frac{1}{2}$ to $\frac{1}{3}$ by $33\frac{1}{3}$ per cent.—which are simple arithmetic variations.[6] And again, in an example like the one just used, if, [A] and four other classes being each equally important at the first period, A rises by 100 per cent. in all the others, the exchange-value of [A] (and of A) in all things rises from $\frac{1}{5}$ to $\frac{2}{6}$ by $66\frac{2}{3}$ per cent., while the exchange-value of each of the others in all things falls from $\frac{1}{5}$ to $\frac{1}{6}$, or together from $\frac{4}{5}$ to $\frac{4}{6}$, by $16\frac{2}{3}$ per cent.

We can sum up these relations in the following general statement:— *When in an economic world the mass-quantities are constant, the variation of the exchange-value in all things of any class (and of any individual in it) is the variation of its share of the constant total exchange-value or purchasing power in this world* (Proposition LV.).

And from this principle and the preceding examples we can form the following universal statements about the relationship

[6] Thus here, dealing with exchange-value in *all* things, we have the equality in the sums which we could not get in Sect. I. § 2 when dealing with exchange-value in all *other* things. Take another simple example. Suppose B is twice as valuable as A at the first period, and at the second A twice as valuable as B, and that the numbers of A's and B's are equal and constant. Then [A] from having $\frac{1}{3}$ of the total exchange-value in the world has come to have $\frac{2}{3}$ of it, and [B] from having $\frac{2}{3}$ has come to have $\frac{1}{3}$. Thus arithmetically reckoned, A has risen by 100 per cent., and B has fallen by 50 per cent., in exchange-value in all things; but the weight of [B] is twice that of [A] at the first period. But again, the weight of [A] is twice that of [B] at the second period; and now, reckoned harmonically (in the ending points), A has risen by 50 per cent. and B has fallen by 100 per cent., in exchange-value in all things. And again, the weights of [A] and [B] over both the periods together are equal; and now, reckoned geometrically (from the opposite points), the percentages of the rise and fall in exchange-value in all things are equal; for [A] has risen by 50 per cent. reckoned at the finish while [B] has fallen by 50 per cent. reckoned at the start, or reckoned in the opposite directions [A] and [B] have risen and fallen each by 100 per cent. This illustrates § 2 of this Section.

etween the two kinds of general exchange-value. The mass-uantities remaining constant, if the weight of a class at the *rst* period is $\frac{1}{r}$ the sum of the weights of all the other classes, id if this class rises by p per cent. (in hundredths, reckoned in ie ordinary way) in exchange-value in all *other* things, it rises om $\frac{1}{r+1}$ to $\frac{1+p}{r+1+p}$ in exchange-value in *all* things, or by $\frac{rp}{+1+p}$ per cent.; while the others, collectively, fall from $\frac{r}{+1}$ to $\frac{r}{r+1+p}$ in exchange-value in *all* things, or by $\frac{p}{+1+p}$ per cent., and if they do not vary amongst themselves, iey each fall by this percentage, or otherwise they fall by this ercentage on the average; so that the sum of the percentages f their falls equals the percentage of the rise of the one class. nd if this class falls by p' per cent. in exchange-value in all *her* things, it falls from $\frac{1}{1+r}$ to $\frac{1-p'}{r+1-p'}$, in exchange-ilue in *all* things, or by $\frac{rp'}{r+1-p'}$ per cent.; while the others, illectively, rise from $\frac{r}{1+r}$ to $\frac{r}{r+1-p'}$ in exchange-value in *l* things, or by $\frac{p'}{r+1-p'}$ per cent., and if they do not vary nongst themselves, they each rise by this percentage, or other-ise they rise by this percentage on the average; so that the im of the percentages of their rises equals the percentage of the ll of the one class.

Here we have an answer for one of our early problems, posed Chapter II. Section IV. § 2,—but an answer applicable ily to the cases when the mass-quantities are the same at both eriods. We see that the larger are the other classes (the larger e r), the smaller is their variation, in accordance with Proposi-on XIX., and the more nearly the variation of the one class exchange-value in *all* things comes to its variation in exchange-value in all *other* things, according to Proposition XXIV.

§ 5. We can, however, very much simplify the statement of the relationship between the two kinds of general exchange-value, by starting at the second period. In the first complex example above used (in § 3) another relationship stared us in the face, although we have not paid attention to it yet. This is that the exchange-value of [C] in *all* things (commodities) has varied two thirds as much as its exchange-value in all *other* things,—or its variation is one third less in the former than in the latter; but also the weight of this class at the *second* period is one third of the weights of all the classes,—or the weights of all the *other* classes are two thirds of the weights of *all* the classes. This relationship is universal, as is proved by the following equation expressing it in regard to money (on the supposition that money is rising in exchange-value),

$$\frac{\dfrac{v + x\alpha_1 + y_{\prime}\beta_1 + \cdots\cdots}{v + x\alpha_2 + y_{\prime}\beta_2 + \cdots\cdots} - 1}{\dfrac{x\alpha_1 + y_{\prime}\beta_1 + \cdots\cdots}{x\alpha_2 + y_{\prime}\beta_2 + \cdots\cdots} - 1} = \frac{x\alpha_2 + y_{\prime}\beta_2 + \cdots\cdots}{v + x\alpha_2 + y_{\prime}\beta_2 + \cdots\cdots},$$

which is easily seen to be true (or if we suppose money to be falling in exchange-value and reverse the terms in the first half, the same relationship is evident). Therefore

The mass-quantities all remaining constant, the variation of any class in exchange-value in all things relates to its variation in exchange-value in all other things as at the second period the weights of all the other classes relate to the weights of all the classes (Proposition LVI.).[7]

[7] This is the Proposition referred to under Proposition XXV. It shows the importance of the word "equal" in that Proposition. Here among constant mass-quantities, as one class alone varies in price, its size varies, and so therefore does the proportion between the variations of its two kinds of general exchange-value. Thus as the class rises in price, its size increases, and therefore the rise of its exchange-value in *all* things lags farther and farther behind the rise of its exchange-value in all *other* things. An ordinary class rising slightly, its exchange-value in *all* things rises almost as much as its exchange-value in all *other* things. But it is conceivable that the class could rise so high as to become, at the second period, equal to all the other classes together. Then its exchange-value in *all* things has risen only half as high as its exchange-value in all *other* things.

This Proposition applies to the variations in the two kinds of general exchange-value of every individual thing in the class in question, when by its "exchange-value in all *other* things" we mean its exchange-value in all things outside its own class.[8] But if by the exchange-value of an individual thing "in all *other* things" we mean absolutely its exchange-value in all other things (including all other individuals in its own class) beside itself, the Proposition must be modified. Its two variations will now relate to each other as the total exchange-values (or money-values) of all other things beside itself and the total exchange-values (or money-values) of all things relate to each other at the second period.

III.

§ 1. The mensuration of exchange-value in *all* things may be extended to all possible cases—with mass-quantities also varying. It is only necessary to insert the article itself, whose exchange-value in all things is being measured, among the things in which its exchange-value is being measured, and to invert the formulæ above found to be the best for measuring the variation of the general level of prices. Thus for money the universal schema is

$$\begin{array}{lllll} \text{I} & v_1\,\text{M}\,@\,1 & x_1\,\text{A}\,@\,\alpha_1 & y_1\,\text{B}\,@\,\beta_1 & \cdots\cdots, \\ \text{II} & v_2\,\text{M}\,@\,1 & x_2\,\text{A}\,@\,\alpha_2 & y_2\,\text{B}\,@\,\beta_2 & \cdots\cdots. \end{array}$$

The formula for the method using double weighting is

$$\frac{M_{a2}}{M_{a1}} = \frac{v_1 + x_1\alpha_1 + y_1\beta_1 + \cdots\cdots}{v_2 + x_2\alpha_2 + y_2\beta_2 + \cdots\cdots} \cdot \frac{v_2 + x_2\sqrt{\alpha_1\alpha_2} + y_2\sqrt{\beta_1\beta_2} + \cdots\cdots}{v_1 + x_1\sqrt{\alpha_1\alpha_2} + y_1\sqrt{\beta_1\beta_2} + \cdots\cdots}.$$

Now it is evident that this formula, in relation with the formula for the same method of measuring the variation or constancy of the exchange-value of money in all *other* things, may violate Propositions XXII., XXVI., XXXII. and XXXV. This is merely a continuation of the defect discovered in the preceding Chapter. But the other two superior

[8] This is the sense in which the term has generally been used in these pages.—Of course, the mass-quantities being constant, the variation in exchange-value in *all* things is the same of the individual as of its class.

methods are free from this defect; wherefore in dealing with exchange-value in *all* things the employment of one of them becomes imperative. And the one to be preferred is Scrope's emended method. The formula for this method is

$$\frac{M_{a2}}{M_{a1}} = \frac{\sqrt{r_1 r_2} + \alpha_1\sqrt{x_1 x_2} + \beta_1\sqrt{y_1 y_2} + \cdots\cdots}{\sqrt{r_1 r_2} + \alpha_2\sqrt{x_1 x_2} + \beta_2\sqrt{y_1 y_2} + \cdots\cdots}.$$

Again, reckoned in the mass-unit of [A], the schema is

$$\text{I}\quad r_1\ \text{M} @ \frac{1}{\alpha_1}\quad x_1\ \text{A} @ 1\quad y_1\ \text{B} @ \frac{\beta_1}{\alpha_1}\quad \cdots\cdots,$$

$$\text{II}\quad r_2\ \text{M} @ \frac{1}{\alpha_2}\quad x_2\ \text{A} @ 1\quad y_2\ \text{B} @ \frac{\beta_2}{\beta_1}\quad \cdots\cdots.$$

The formula is

$$\frac{A_{a2}}{A_{a1}} = \frac{\frac{1}{\alpha_1}\sqrt{r_1 r_2} + \sqrt{x_1 x_2} + \frac{\beta_1}{\alpha_1}\sqrt{y_1 y_2} + \cdots\cdots}{\frac{1}{\alpha_2}\sqrt{r_1 r_2} + \sqrt{x_1 x_2} + \frac{\beta_2}{\alpha_2}\sqrt{y_1 y_2} + \cdots\cdots}$$

$$= \frac{\frac{1}{\alpha_1}(\sqrt{r_1 r_2} + \alpha_1\sqrt{x_1 x_2} + \beta_1\sqrt{y_1 y_2} + \cdots\cdots)}{\frac{1}{\alpha_2}(\sqrt{r_1 r_2} + \alpha_2\sqrt{x_1 x_2} + \beta_2\sqrt{y_1 y_2} + \cdots\cdots)} = \frac{\alpha_2}{\alpha_1}\cdot\frac{M_{a2}}{M_{a1}}.$$

And, reckoned in the mass-unit of [B], the schema is

$$\text{I}\quad r_1\ \text{M} @ \frac{1}{\beta_1}\quad x_1\ \text{A} @ \frac{\alpha_1}{\beta_1}\quad y_1\ \text{B} @ 1\quad \cdots\cdots,$$

$$\text{II}\quad r_2\ \text{M} @ \frac{1}{\beta_2}\quad x_2\ \text{A} @ \frac{\alpha_2}{\beta_2}\quad y_2\ \text{B} @ 1\quad \cdots\cdots.$$

And the formula,

$$\frac{B_{a2}}{B_{a1}} = \frac{\frac{1}{\beta_1}\sqrt{r_1 r_2} + \frac{\alpha_1}{\beta_1}\sqrt{x_1 x_2} + \sqrt{y_1 y_2} + \cdots\cdots}{\frac{1}{\beta_2}\sqrt{r_1 r_2} + \frac{\alpha_2}{\beta_2}\sqrt{x_1 x_2} + \sqrt{y_1 y_2} + \cdots\cdots}$$

$$= \frac{\frac{1}{\beta_1}(\sqrt{r_1 r_2} + \alpha_1\sqrt{x_1 x_2} + \beta_1\sqrt{y_1 y_2} + \cdots\cdots)}{\frac{1}{\beta_2}(\sqrt{r_1 r_2} + \alpha_2\sqrt{x_1 x_2} + \beta_2\sqrt{y_1 y_2} + \cdots\cdots)} = \frac{\beta_2}{\beta_1}\cdot\frac{M_{a2}}{M_{a1}}.$$

And so on through all the classes. These results are the same as in the partial cases, with constant mass-quantities. They would be the same if we used the formula for the geometric method, or the formula for the method with double weighting just rejected, or any other formula whatsoever. Thus in general, *When the expression for the variation of the exchange-value of money in all things has been obtained, the expression for the variation of the exchange-value of any other class in all things may be obtained by multiplying the former expression by the expression for the price variation of this class* (Proposition LVII.).[1]

§ 2. Here it is impossible to find any equality in the sums of the percentages of the variations, multiplied by the weights of the first period, or in any other regular manner. For if what was found to be true in § 2 of the preceding Section were assumed to be true here, with regard to three classes, we should have to be able to show that

$$(v_2 + x_2\alpha_2 + y_2\beta_2)\frac{M_{a2}}{M_{a1}} = v_1 + x_1\alpha_1 + y_1\beta_1;$$

but on restoring the present value of $\frac{M_{a2}}{M_{a1}}$, we find that this is not necessarily true,—nor would it necessarily be so if we used either of the other methods for measuring variations in exchange-value. It may be true sometimes, but only accidentally. Therefore, as Cairnes said, the aggregate amount of purchasing power in an economic world may increase or decrease.

Such increase or decrease can be measured in the following manner. Take from all the classes the mass-quantities that are common to both the periods compared. As these are collectively constant in exchange-value in all things (amongst themselves), their presence may be ignored. To be examined are the excessive mass-quantities of some classes at the first period and of some classes at the second period. If the exchange-value of money has been found to be constant, all that is needed is to add up the money-values of the excessive mass-quantities at the first period, and the money-values of the excessive mass-quantities

[1] As before, this Proposition cannot be stated of exchange-value in all *other* things. It expands Proposition XXXIII.

at the second period; and the balance between them will show whether there is gain or loss, and to what extent. But if the exchange-value of money has been found to have varied, the operation must be corrected by reducing the total sum of the money-values at the one or the other period—say at the second by multiplying the sum first obtained by the expression for the variation of the exchange-value of money—and then striking the balance.

This procedure does not show the relative increase or decrease. After all, then, the completest method is simply to add up all the money-values of the classes at each period, and, after reducing the sum at the second period to a sum in money with constant exchange-value, to compare the two sums together.

Here we must be careful as to which general exchange-value of money it is whose variation is to be corrected. If we are measuring merely the increase or decrease in the aggregate exchange-value of commodities, having no interest in the increase or decrease in the total exchange-value of the class money, we should use merely the variation of money in exchange-value in all *other* things (that is, in all things other to it, therefore in *all* commodities), at the same time leaving out of account the quantity of money.

After such a measurement has been made, and the increase, decrease or constancy of the aggregate amount of exchange-value in all things in an economic world has been ascertained, it must not be supposed that any information has been acquired as to the increase, decrease or constancy of the aggregate amount of use-value or cost-value in that world; and whether the aggregate amount of esteem-value is thereby determined is a debatable subject. The truth of this statement may be seen by considering the conditions when the mass-quantities are all exactly the same at two different periods,—when we know the aggregate amount of exchange-value in all things is constant. For that the aggregate amount of the cost-values of these constant mass-quantities—determined by the aggregate amount of labor their production has cost—need not be the same at both periods, is evident. Equally evident is it that the aggregate amount of their use-values may not be the same. For instance, if the two periods are summer and winter, the aggregate of the total utilities of all the classes might be very different at the two periods. On the other hand, it would be a difficult question to decide whether in this latter case (on the supposition also that the numbers of the people are constant) the aggregate amount of the final utilities of all the classes, and hence the aggregate amount of the esteem-values of all the constant goods, are constant or not. For if the goods were, say, mostly adapted to summer, a few of them would have no (actual) esteem-value in summer and many no (actual) esteem-value in winter, while many would have moderate esteem-value in summer and a few excessive esteem-value in winter. Whether these would exactly counterbalance or not, is out of the province of this work to pass an opinion.

It is proper, however, here to state that the old and much debated question concerning the measurement of the *wealth* of a country is to be decided by saying that such measurement is the measurement of the increase, decrease or constancy of the aggregate amount of exchange-value (of all commodities in all commodities) in that country relatively to the numbers of its inhabitants.

§ 3. Both because the aggregate amount of the exchange-values of all things in all things is not constant, and because the measurement of general exchange-exchange is not known with absolute definiteness, when the mass-quantities as well as prices vary, it is impossible to construct such convenient statements for the comparison of the variations of the two kinds of general exchange-value as were above obtained for the cases when the mass-quantities are constant. But by means of the geometric average of the price variations, with its proper weighting, which method we know to be approximately accurate (except only in rare and extravagant cases), we may obtain almost equally convenient approximately correct formulæ also for these more usual and complex cases. Let the class [A] be $\frac{1}{r}$ the size of all the other classes of commodities over *both* the periods together, and let us suppose its price has risen by p per cent. (in hundredths), while the prices of all other commodities have remained unchanged. Then the exchange-value of money in all other things, *i. e.*, in all commodities, has fallen approximately from its first condition, represented as unity, to $\sqrt[r+1]{\frac{1}{1+p}}$ by $1 - \sqrt[r+1]{\frac{1}{1+p}}$ per cent. Therefore every one of the other commodities, remaining unchanged in relation to money, has fallen approximately by this percentage in exchange-value in all other things beside money, that is, in all commodities (including itself). But the class [A], while rising by p per cent. in exchange-value in all other things, has risen by p per cent. above $\sqrt[r+1]{\frac{1}{1+p}}$ of its former state, represented as unity, in exchange-value in all commodities (including itself, and excluding money), or to $\frac{p}{\sqrt[r+1]{1+p}}$ above its former state, by $\frac{p}{\sqrt[r+1]{1+p}} - 1$ per cent., approximately. Reversely if [A] has fallen by p' per cent. in price, and therefore in exchange-value in all other things, money has risen approximately in exchange-value in all things from 1 to $\sqrt[r+1]{\frac{1}{1-p'}}$, by $\sqrt[r+1]{\frac{1}{1-p'}} - 1$ per cent.; and the exchange-value of every one of the other commodities has fallen approximately by this percentage in exchange-value in all commodities (including itself). But the class [A] has fallen by p' per cent. below $\sqrt[r+1]{\frac{1}{1-p'}}$ of its former state in exchange-value in all commodities, or to $\frac{p}{\sqrt[r+1]{1-p'}}$ below its former state by $1 - \frac{p}{\sqrt[r+1]{1-p'}}$ per cent., approximately. Or if we desire to make similar measurements with money itself included in the standard, supposing its weight over both the periods to be s times that of [A], we merely have to add this to the indices of the roots in the preceding expressions. The larger r is, it is plain the smaller will be the opposite variation of money and the other commodities in exchange-value in all things, and the nearer will the rise of [A] in exchange-value in all things be to its rise in exchange-value in all other things, according to Propositions XIX. and XXIV., already referred to in similar cases above examined.

Again, if all prices vary equally by $+p$ or by $-p$ per cent. (rising in the former case and falling in the latter), or if they so vary on the average (geometrically measured), their variation may be comprehensively represented as a variation from 1 to $1 \pm p$. Then, while the exchange-value of money in all *other* things has inversely varied from 1 to $\frac{1}{1 \pm p}$, either by $\frac{-p}{1+p}$ or by $\frac{+p}{1-p}$ per cent., its exchange-value has varied approximately from 1 to $\frac{1}{(1 \pm p)^{\frac{r}{r+1}}}$ (in which r represents the number of times all the commodities are larger in total exchange-value over both the periods together than money). Therefore a commodity which has varied in price by $\pm p$ per cent. (the same as the average of all the price variations) has varied from 1 to $1 \pm p$ in money that has varied to the one or the other of the above expressions according as the one or the other of its general exchange-values is considered. Thus in all things other to money (*i. e.*, in *all* commodities, including itself) this commodity has varied from 1 to $\frac{1+p}{1+p} = 1$ or to $\frac{1-p}{1-p} = 1$, that is, it has remained constant,—and so, of course, also in exchange-value in all *other* commodities; but not so in exchange-value in all *other things* (other to itself, or to its own class, including money), or in *all things* (including money). In *all* things it has varied from 1 to $\frac{(1+p)}{(1+p)^{\frac{r}{r+1}}}$ or to $\frac{1-p}{(1-p)^{\frac{r}{r+1}}}$, or from 1 to $(1 \pm p)^{\frac{1}{r+1}}$, while in all *other* things it has varied from 1 to $(1 \pm p)^{\frac{1}{r}}$—approximately.

§ 4. In thus measuring the exchange-value of money in all things, or of anything else in all things, including money, we must know the quantity of money that has been employed at each period. Here is a problem of a peculiar nature, never yet discussed. The mere comparison of the quantity of money in a country with the quantities of commodities in the market at any one time is not satisfactory, as money is permanently in the market, while other goods merely flow through the market. Nor is there any propriety in the comparison of the quantity of money in the country with the quantity of goods (reckoned in money-values) bought and sold during each of the periods compared, because the lengths arbitrarily chosen for these periods will affect the quantities of the goods but not the quantity of money. Nor again between the quantities of exchanges effected with money and the quantity effected without money, because this comparison is affected by banking expedients, and banking expedients, although they make certain kinds of money less needed and more scarce, yet do not lessen the importance of money as a class—or if the quantity of money so decreased be counted for less, the quantity of the substitute, material or immaterial, provided by the banking expedients, ought to be counted, which is still more difficult to estimate. The following suggests itself as a possible solution of the question. Money, when used as a mere medium of exchange, is desired only for purchasing other things, hence not in itself, and so is incomparable with commodities. But money is desired in itself when it is wanted for paying debts or the interest on debts. Thus the quantity of debts falling due within a certain period (but only those contracted in an anterior period) along with the quantity of interest due on these and on unexpired debts, seems to be a quantity comparable with the quantities of commodities consumed within the period. Yet there is defect also here. One of the reasons for wishing money to be stable in exchange-value is because of this very debt-paying quality it possesses. But the larger the amount of debt in a country, and the larger the amount of weighting consequently assigned to the class money, the smaller will be the calculated variations of money's exchange-value in all things, whenever a variation takes place; so that the greater the quantity of debt and consequently the greater the nicety desired in the calculation, the duller becomes the calculation.

the other commodities in exchange-value in all things, and the nearer will the rise of [A] in exchange-value in all things be to its rise in exchange-value in all other things, according to Propositions XIX. and XXIV., already referred to in similar cases above examined.

Again, if all prices vary equally by $+p$ or by $-p$ per cent. (rising in the former case and falling in the latter), or if they so vary on the average (geometrically measured), their variation may be comprehensively represented as a variation from 1 to $1 \pm p$. Then, while the exchange-value of money in all *other* things has inversely varied from 1 to $\frac{1}{1 \pm p}$, either by $\frac{-p}{1+p}$ or by $\frac{+p}{1-p}$ per cent., its exchange-value has varied approximately from 1 to $\frac{1}{(1 \pm p)^{\frac{r}{r+1}}}$ (in which r represents the number of times all the commodities are larger in total exchange-value over both the periods together than money). Therefore a commodity which has varied in price by $\pm p$ per cent. (the same as the average of all the price variations) has varied from 1 to $1 \pm p$ in money that has varied to the one or the other of the above expressions according as the one or the other of its general exchange-values is considered. Thus in all things other to money (*i. e.*, in *all* commodities, including itself) this commodity has varied from 1 to $\frac{1+p}{1+p} = 1$ or to $\frac{1-p}{1-p} = 1$, that is, it has remained constant,—and so, of course, also in exchange-value in all *other* commodities; but not so in exchange-value in all *other things* (other to itself, or to its own class, including money), or in *all things* (including money). In *all* things it has varied from 1 to $\frac{(1+p)}{(1+p)^{\frac{r}{r+1}}}$ or to $\frac{1-p}{(1-p)^{\frac{r}{r+1}}}$, or from 1 to $(1 \pm p)^{\frac{1}{r+1}}$, while in all *other* things it has varied from 1 to $(1 \pm p)^{\frac{1}{r}}$—approximately.

§ 4. In thus measuring the exchange-value of money in all things, or of anything else in all things, including money, we must know the quantity of money that has been employed at commodities) has already been measured. Thus if the general level of prices is found to have fallen by 10 per cent., which fall means that .90 money-unit is now worth what 1.00 money-unit formerly was worth, it is evident that a commodity whose price has also fallen by 10 per cent., so that, instead of being held at 1.00, a certain quantity of it is sold for .90, has remained constant in exchange-value (in the commodity-standard). And a commodity that has fallen 5 per cent. in price, so that what of it before cost 1.00 is now got for .95, has really risen in such exchange-value from .90 to .95, which is a rise by 5.55 per cent.; while a commodity that has remained constant in price, has really risen in this exchange-value from .90 to 1.00, which is a rise by 11.11 per cent. On the other hand a commodity that has fallen 20 per cent. in price has really fallen in this exchange-value from .90 to .80, or by only 11.11 per cent.

Now as one of the uses we make of actual prices is to indicate the variations of commodities simply in exchange-value, and as this indication is false except in one given case, it may not be out of the way to call the corrected indication by the term *true-price.* Of course the actual prices are always true prices ; for they are always true in the only indication which they properly make, namely of the variation of the commodity in its particular exchange-value in money. By "true-prices" is meant something which is true, not necessarily in the first and proper indication made by actual prices, but in the secondary indication to which prices are habitually put.

This term, then, being permitted, we can most conveniently find true-prices by making use of the principle above enunciated in Proposition LVII. That Proposition, to be sure, refers only to the variation of the exchange-value of money in *all* things, and shows only the method of obtaining therefrom the variation of any commodity in exchange-value in all things. But when we know the variation of the exchange-value of money in all other things, which are all commodities, then by multiplying this by the price variation of any commodity, we obtain the commodity's variation in the same standard of all things other to money, namely all commodities. Thus the expression for

[...]haps, however, after all, we do not want a standard con[...] of all things, including money, but only a standard of [co]mmodities. This is the commodity-standard proper. [...] is something *sui generis*, which we commonly contrast [...] commodities. It is used as a measure of the exchange-[...] of commodities. Hence we want it to be stable in ex[...]ge-value in all commodities. To be sure, if it be stable in [...] exchange-value, it is also stable in exchange-value in all [...]gs, including itself; wherefore no room is left for dispute in [...]rd to the supreme desideratum. But when it varies in gen[...] exchange-value, and therefore differently in exchange-value [...]ll things and in exchange-value in all commodities, the lat[...]variation being slightly greater, the commodity-standard not [...] is the more convenient because of the greater and conse[...]ntly more apparent variation of money in it, but also it seems [...]e the standard in reference to which any correction in the [...]nent of debts ought to be made. Still, this is a fine ques[...], which will become of practical importance not till society [...]arge shall have grown much more sensitive to variations in [...] exchange-value of money than it is at present. For the [...]ent, then, we may assume the commodity-standard merely [...] its convenience.

5. Adopting this standard, we see that a variation in the [...]e of any commodity exactly represents the variation of this [...]modity in exchange-value in all commodities only if money [...] remained stable in exchange-value in all other (and in all) [...]gs, or, which is the same thing, only if the general level of [...]es has remained the same. Yet people almost always re[...]l a variation of the price of anything as an indication of its [...]ation simply in exchange-value in general, that is, they treat [...]ey as if it were stable in general exchange-value—as if it [...]e a good practical individual standard. But when it is per[...]ed that money has not remained stable in exchange-value, [...]e is need of correcting this false inference from the mere [...]e variation of any article. The correction can be easily [...]le if the variation of the general level of prices (inversely [...]wing the variation of the exchange-value of money in all

according as it is a rise or a fall (if there be constancy, $\pi = 0$); and let $\pm p$ similarly represent the percentage of the known variation of the price of the commodity in question. Then, the desired percentage of the variation of the true-price being represented by p_{tr}, the formula is

$$p_{tr} = \frac{1 \pm p - (1 \pm \pi)}{1 \pm \pi},$$

and this variation is a rise if the result be positive, a fall if the result be negative, and it is constancy if the result be 0. Or if the symbols be used to represent full percentages (in integers), the formula is

$$p_{tr} = 100 \left[\frac{100 \pm p - (100 \pm \pi)}{100 \pm \pi} \right].$$

It must always be borne in mind that when the variation of a commodity is so measured, the measurement is of its variation in exchange-value, not in all things, nor in all other things, nor even in all other commodities, but in all commodities (including itself). Of course, if the commodity be found to be constant in exchange-value in all commodities, and money also has been found to be constant in exchange-value in all other things, then the commodity is known to be constant in all the other kinds of general exchange-value. Otherwise, to find its variation in any of the other kinds of general exchange-value would require a separate and special measurement similar to that by which the variation of money was measured.

IV

§ [illegible]. When engaged in measuring the constancy or variation of the exchange-value of anything, it is essential that we should know which of the standards we are using, or else there is danger

[illegible] the supposition that the commodity in question is one of [illegible] taken into account in measuring the variation of the [illegible] the things necessarily left out of that [illegible] exchange-value measurement as above is its variation [illegible] commodities [illegible] "all" being meant all the other [illegible] in the standard, it being impossible to include [illegible] of our falling into inconsistencies. This risk, and this need, may be illustrated by the following example.

Suppose at the first period

$$A = B = C = D = E = F = \ldots\ldots,$$

and at the second

$$A = \tfrac{2}{3}B = 1\tfrac{1}{2}C = D = E = F = \ldots\ldots,$$

the only changes being in relation to B and C. And suppose the classes [B] and [C] are equally important over the two periods together. Then we know that the exchange-value of A has not varied. But at the second period

$$B = 2\tfrac{1}{4}C = 1\tfrac{1}{2}A = 1\tfrac{1}{2}D = 1\tfrac{1}{2}E = 1\tfrac{1}{2}F = \ldots\ldots.$$

If B had risen so as to become equivalent to 1½ of every other thing, we should say it had risen by 50 per cent. in general exchange-value (of some sort); and as it has risen by this amount in all the other things beside [A] and by more than this amount in [A], it has evidently risen in general exchange-value (of some sort) by slightly more than 50 per cent. Also at the second period

$$C = \tfrac{4}{9}B = \tfrac{2}{3}A = \tfrac{2}{3}D = \tfrac{2}{3}E = \tfrac{2}{3}F = \ldots\ldots;$$

and for a similar reason we perceive that C has fallen in general exchange-value (of same sort) by slightly more than 33⅓ per cent. Now then we might be led into the following argument. While, at the second period, A is exchanged for ⅔ B, it is exchanged for 33⅓ per cent. less of an article which has risen slightly more than 50 per cent. Therefore it has risen somewhat in general exchange-value; which is contrary to our first conclusion. And while, at the second period, A is exchanged for 1½ C, it is exchanged for 50 per cent. more of an article which has fallen slightly more than 33⅓ per cent. Therefore it has fallen somewhat in general exchange-value; which is contrary both to the first conclusion and to the immediately preceding. These two opposite conclusions cannot be reconciled by saying that they counterbalance each other; for they would do so only if it were necessary for the owners of [A] always to exchange it in equal portions for [B] and for [C]. But, according to the above reasoning, if any one exchanged all his [A] for [B], he would at the second period, get more general exchange-value than he got at the first; and if another exchanged all his [A] for [C], he would at the second period get less general exchange-value than he got at the first—which happenings are contradictory (because of Proposition VII.). Now the above reasoning would be perfect, and would lead us into a dilemma, or aporia, casting doubt over our whole subject, if the general exchange-values referred to in all the cases were the same. But they are not.

§ 2. In the first case A is constant in general exchange-value of both kinds—exchange-value in all other things and exchange-value in all things. In the second case B rises by more than 50 per cent. in a general exchange-value which is exchange-value in all the other things beside it, including C, but excluding itself. In the third case C falls by more than $33\frac{1}{3}$ per cent. in a general exchange-value which is exchange-value in all the other things beside it, including B, but excluding itself—and hence different from the exchange-value in *other* things in which B rose by more than 50 per cent.

On the other hand, B in exchanging for $2\frac{1}{4}$ C while exchanging for $1\frac{1}{2}$ of all the other things, exchanges for $2\frac{1}{4}$ of an article fallen in exchange-value, while it exchanges for $1\frac{1}{2}$ of other articles with constant exchange-value. If then C has fallen in general exchange-value of some sort by exactly $33\frac{1}{3}$ per cent., $2\frac{1}{4}$ C is equivalent to $\frac{9-3}{4} = 1\frac{1}{2}$ times the former C, and therefore to $1\frac{1}{2}$ A, to $1\frac{1}{2}$ D, etc. And C in exchanging for only $\frac{4}{9}$ B while exchanging for $\frac{2}{3}$ of the other articles, exchanges for $\frac{4}{9}$ of an article risen in exchange-value, while it exchanges for $\frac{2}{3}$ of other articles with constant exchange-value. If then B has risen in general exchange-value of some sort by exactly 50 per cent., $\frac{4}{9}$ B is equivalent to $\frac{4+2}{9} = \frac{2}{3}$ of the former B, and therefore to $\frac{2}{3}$ A, to $\frac{2}{3}$ D, etc. Thus in these cases everything comes out right, provided that B and C have so varied in some general exchange-value, and that this is the same general exchange-value.

But B and C have so varied, and in general exchange-value hich is the same; for they have both so varied in general ex-ange-value in *all* things (including themselves). For A has mained constant in exchange-value in all things, including B d C. Therefore, according to Proposition LVII., when B s risen by 50 per cent. in A, it has risen by 50 per cent. in change-value in *all* things, although it has risen by slightly ore than 50 per cent. in exchange-value in all things *other* to

And when C has fallen by $33\frac{1}{3}$ per cent. in A, it has fallen $33\frac{1}{3}$ per cent. in exchange-value in *all* things, although it has llen by slightly more than $33\frac{1}{3}$ per cent. in all things *other* to it. This is evident, as regards the standard of *all* things, also if we present the state of things at the second period in these ways,

$$B = 1\,B = 2\tfrac{1}{4}\,C = 1\tfrac{1}{2}\,A = 1\tfrac{1}{2}\,D = 1\tfrac{1}{2}\,E = \ldots\ldots,$$

d

$$C = 1\,C = \tfrac{4}{9}\,B = \tfrac{2}{3}\,A = \tfrac{2}{3}\,D = \tfrac{2}{3}\,E = \ldots\ldots;$$

here it is evident, in the first case, that for B to rise in ex-ange-value in all things by 50 per cent., it must rise still more exchange-value in at least one of the others in order to counter-lance the fact that it does not rise in exchange-value in itself, d in the second, that for C to fall in exchange-value in all ngs by $33\frac{1}{3}$ per cent., it must fall still more in exchange-lue in at least one of the other things in order to counter-ance the fact that it does not fall in exchange-value in inself.[1] This solution of the difficulty has been anticipated in Proposi-n XXXIII. It is also illustrative of Propositions XII. and II. For from the variations of B and C in relation to A we ht gather that B has become $2\frac{1}{4}$ times as valuable as C, and as valuable as B. This is true of their exchange-values in things, and in all other things beside them both. But it is true of their exchange-values each in all other things, since ll the things other to it B has become somewhat more than imes as valuable as C has become in all the things other to d inversely.

[1] will be noticed that in both these cases the extra variation of the one lass (equally important over both the periods with the class in question) is are of the common variation of all the other classes. This relation is al. Hereby is given a partial (and in complex cases an approximate) to a problem proposed in Chapter II. Sect. V. § 6.

CHAPTER XIV.

THE UTILITY OF MEASURING THE VARIATIONS IN THE EXCHANGE-VALUE OF MONEY.

I.

§ 1. Knowledge of the constancy or variation of the exchange-value of money is useful both for theoretical and for practical purposes.

For theoretical purposes it is useful in many scientific enquiries, which lead on to conclusions of great practical importance. Thus, for example, there is a prevalent opinion that a rise or fall in the exchange-value of money has considerable influence on industry and general prosperity, partly deleterious and partly beneficial, the one in some ways, the other in others. This opinion is to some extent based on experience in flagrant instances when there could be no doubt what course the exchange-value of money was taking; but as yet it has mostly been based on reasoning *à priori*. For it cannot be scientifically investigated until variations in the exchange-value of money are scientifically determined. Its scientific investigation, however, is of the greatest moment; for if there be truth in the doctrine that the deleterious influences are greater than the beneficial, and more so of a rise than of a fall, the detailed knowledge of such influences may lead to corrective and even to preventive measures. Attempt may perhaps be made to attack the source of the evil by regulating the exchange-value of money—both to prevent the insidious changes of metallic money over long stretches of years and the sudden changes of credit money during short periods. For this purpose also knowledge of the causes of the variations in the exchange-value of money will be necessary. But the prevalent opinions on such causes can likewise be scientifically confirmed or refuted only after more scientifically accurate measurements of the variations in question have been instituted than any yet made. On the whole, it is apparent that, as observed by Dana Horton, the theory of the multiple standard is "the key to the entire theory of money." [1] Of the corrective and preventive measures more will be said presently.

§ 2. Then again we have need of knowing how commodities themselves vary in general exchange-value; for we cannot well investigate the causes and consequences of their variations in such exchange-value imperfectly measured. To be sure, we can easily investigate the causes of the variations of one commodity in exchange-value in another, as such variations are plain. But, after all, to know the causes of these particular variations is not so important as to know the causes of the general variations, which latter have commonly been the subject really had in mind when economists have treated of the causes of variations in exchange-value—especially when specifying, as they so often do, that they are dealing with the causes of variations in prices under the supposition of money being constant in general exchange-value. Reasoning on this subject needs to be based upon experience, and therefore we should be able to convert the supposition of money being constant into reality by correcting its deviations in the instances taken from experience. Otherwise a commodity may appear to have risen or fallen in general exchange-value because its price has risen or fallen, although its general exchange-value may really have varied in the opposite direction. Money being habitually used as a measure of general exchange-value notwithstanding its own variableness in general exchange-value, we need to correct the results obtained from measurements with this imperfect instrument, after first measuring this instrument itself. For after finding its variations we can adapt the variations of the prices of commodities to variations in general exchange-value, in a manner already explained. We may thus obtain what have above been called the

[1] *Silver and Gold*, 2d ed., p. 40.

true-prices of commodities, namely the prices they would have had, had money remained stable in exchange-value, and had no other changes occurred.[2] This term, to repeat, is not inappropriate, because in spite of the variations of money we do continue to make use of prices for measuring variations in the exchange-values of commodities, not only in money, but in things in general; but it is only these adapted prices in an invariable imaginary money that are true for the latter purpose.

An example in point may be taken from a subject now agitating public opinion. In considering whether the present tendency of productive bodies in the same line of business to combine and thus avoid competition is beneficial to the country at large, or otherwise, one of the items discussed is whether the so-called "trusts" already formed have raised or lowered prices. Now to discuss whether they have raised or lowered merely actual prices is only to discuss whether they have raised or lowered the exchange-value of their products in relation to money. But actual experience of mere changes in price of any particular class of commodities shows only a change in the relation between the general exchange-value of the class in question and the general exchange-value of money; and does not show whether the change is on the side of the commodity or on the side of money, or how much on the one and how much on the other—that is, it does not show how much the change is due to the efforts of the producers of the commodity and how much to the efforts of the producers of money. This can be shown—for we are dealing with variations, not in cost-value or in esteem-value, but in exchange-value—only by investigating further the relation of both these classes to all commodities; which can be done very laboriously in regard to one of the things, preferably money, and then very easily in regard to the other. And after doing this we are not so much interested in the relative accomplishments

[2] This second proviso is necessary because the variation of money in general exchange-value may have had some influence, to be noticed later, to change relative exchange-values from what they would be, had money remained stable. But the variation having taken place, and exerted its influence, the *true-prices* still indicate the variations of the commodities in general exchange-value under such influences.

of the producers of the given commodity and of the producers of the metal used as money, as in the relative accomplishments of the producers of the commodity in question and the producers of all other commodities. Hence our interest is really in the variations of the commodity's *true-price;* for, although such variations do not show the commodity's variations in cost-value or in esteem-value, they do show the relation between its variations in these values and the variations in them of all commodities,—about which relation more will be said in the next Section. Yet this aspect of the question is mostly forgotten, and the question is often thought to be settled by an appeal merely to the actual variations of the prices of the given commodity. Thus, for instance, in a recently published work is to be found the following passage: "The price of cotton-seed oil has fallen, along with the economic improvement in its production introduced by the trust. In 1878 the average price of standard summer yellow oil was 47.94 cents per gallon. In 1883, the year before the organization of the trust, it had only fallen to 47.08 cents per gallon. In 1887, four years after the organization of the trust, it had fallen to 38.83 cents per gallon. In other words, during these four years the price of cotton-seed oil fell more than eight times as much as it did during the five years before the trust was formed."[3] Here no reference is made to the fact that after 1878 industrial conditions took an upward swing, which lasted till 1883, and was followed by depression. Now if the measurement of the course of the exchange-value of money during these years provided by the Aldrich Report were reliable, the true-prices of cotton-seed oil, calculated for the two later years, the price at the first year being taken as the base, would be 44.37 cents per gallon in 1883 and 41.89 cents per gallon in 1887. In other words, the fall before the organization of the trust was by 7.45 per cent., and the fall after the organization of the trust was by 5.52 per cent., so that the true fall, instead of being eight times greater in the later period, was only three fourths as great.[4] Unfortunately the index-numbers

[3] G. Gunton, *Trusts and the public*, 1899, p. 15.

[4] On p. 218 of the same work the prices of petroleum are given for a similar purpose (the Standard Oil Company being established in 1871 and the trust in

of the Aldrich Report not only do not refer only to prices in New York, where the above prices were reported, but also were calculated in an improper manner. Except that the figures of the Aldrich Report cannot in most cases be far from the truth, we are left in the dark as to the exact movement of the true-price of any commodity during the years preceding and succeeding the organization of its trust; and to the extent of this obscurity all argumentation on this subject is confused and confusing.

In view of the habitual inattention and neglect with which the subject of true-prices is treated not only by the people at large but by economists of repute, it is somewhat discouraging to find, as recalled by Dana Horton, that the need of observing them was pointed out more than two hundred years ago. In 1672 Pufendorf wrote: "When the price of any one and the same thing is said to vary, it must be carefully distinguished whether, properly speaking, the value of the thing or the value of money has varied."[5] And yet perhaps the first, and perhaps even the last, writer who has attempted to make a scientific investigation concerning true-prices is Professor Laspeyres, who wrote on

1880). Some of these prices, in cents per gallon, are here given, followed by the true-prices calculated from the figures in the Aldrich Report:

1863	30.7	30.7
1867	20.5	16.4
1871	21.7	18.1
1873	16.0	13.4
1877	15.0	14.7
1879	8.125	8.6
1880	9.125	8.7
1884	8.25	8.4
1889	7.125	7.7
1891	6.9	7.6

The inferences to be drawn from the latter figures are somewhat different from those from the former. In the eight years of open competition the true-price fell 41.05 per cent., at the average rate of 4.39 per cent. *per annum*; in the nine years of preponderance of the Company it fell 51.93 per cent., at the average rate of 4.75 per cent. *per annum*; and in the eleven years of quasi-monopoly under the trust it fell 12.64 per cent., at the average rate of 1.09 per cent. *per annum*.

[5] *De jure naturæ et gentium*, lib. V. cap. I. § 16. Pufendorf, however, would have judged the variation in the exchange-value of money by that of farm land. Yet the multiple standard was known still earlier, as we shall see presently.

this subject about thirty years ago.[6] He deserves credit for so doing, although his work was vitiated by a faulty method of calculating the variations of money. Other writers have but barely noticed the subject.[7]

§ 3. The practical purposes above mentioned are sought to be effected in the schemes for correcting the variations of money in its function as a measure of exchange-value, and, as far as is possible, for preventing such variations.

To begin with the former. It has been proposed that the mensuration of the exchange-value of money should serve as a guide in credit operations extending over at least six months, or a year, and longer. The design is that, by agreement between the parties at the time of contracting, debts of all sorts should be repaid in the same amount of exchange-value as was borrowed or bargained for, and therefore the sum of money pledged should be paid with addition or deduction according to the fall or rise of money in exchange-value. For example, if between the time of contracting and the time of solution money is found to have depreciated 10 per cent., a person owing 100 money-units, knowing that 100 of the new units are worth only 90 of the old units, and that 10 of the old are now worth $10 \times \frac{1.00}{.90} = 11.11$, must pay back 111.11 money units; and if money has appreciated, instead, by 10 per cent., the same debt is discharged by the repayment of only 90.91 money-units, the sum due for interest being increased or diminished in the same proportion.

This proposal, as is well known, was revived by Jevons, after having been suggested as a serious scheme, perhaps first, by Joseph Lowe, near the commencement of the century just elapsed, and soon afterwards, in dependence upon Lowe, by Poulett Scrope. It has been recommended by several other

[6] Especially in his essay *Welche Waaren werden im Verlaufe der Zeiten immer theurer?—Statistischen Studien zur Geschichte der Preisen*, in the Zeitschrift für die gesammte Staatswissenschaft, Tübingen 1872, 1ste Heft. He had previously touched upon the subject in B. 25.

[7] *E. g.*, Sauerbeck in B. 79, p. 599.—A casual use of a solitary true-price is to be found in the works of D'Avenel, B. 117, p. 6, B. 118, p. 4.

writers, and recently by Professor Laves as something new.[8] Yet hardly new, even when Lowe wrote, was the idea of it, which has been before the eyes of every jurist for nearly three hundred years. For in the great work of Grotius is the following passage: "Concerning money we must know that it naturally possesses the capacity to pay debts,[9] not in its material alone, nor in its denomination, but in a wider respect, namely, as it is compared either with all, or with the most necessary, things; which estimation, unless otherwise agreed upon, is to be made at the time and place of solution."[10]

§ 4. The other proposal is that the mensuration of the exchange-value of money should be employed as a guide for regulating the currency. A variation of money being detected before it has had time to go far, it has been suggested that the government can restore to money its former exchange-value by increasing or decreasing its quantity, and by performing this operation constantly, it can keep money always within very slight and inconsiderable deviations from a permanent exchange-value—as the helmsman steers his boat by arresting its

[8] Lowe, B. 8, pp. 278–279, 281–291; Scrope, B. 9, pp. 407–408 (followed by R. H. Walsh, B. 13, and reviewed by Maclaren, B. 17); Jevons, *Money and the mechanism of exchange*, pp. 328–333; Horton, *Silver and gold*, pp. 36–43; F. A. Walker (confining it to persons not in business), *Money*, pp. 161–163, *Money in its relation to trade and industry*, pp. 70–77; Marshall, in a paper read before the Industrial Remuneration Conference, pp. 185–186 of the Report, London 1885, and in B. 83, pp. 363–365; T. Laves, *Die "Warenwährung" als Ergänzung der Edelmetallwährung*, Schmollers Jahrbuch für Gesetzgebung, Verwaltung und Volkswirthschaft, Leipzig 1890, pp. 837–846. The scheme is entertained by Laughlin, *History of bimetallism*, 1885, pp. XI.–XII., and *Elements of political economy*, 1887, pp. 76–77, and by Giffen, *Recent changes in prices and incomes compared*, Journal of the Statistical Society, London 1888, pp. 54–55; and is recommended as a substitute, in case of failure to establish the next scheme, by H. Winn, *The multiple standard*, American Magazine of Civics, Dec. 1895, p. 584, and Parsons, B. 136, p. 333. The present scheme was also favored by Zuckerkandl in B. 115, pp. 249–250; but later he has found fault with it for not allowing for stability of money in cost-value (or esteem-value) when its exchange-value is rising because of improved production of commodities [and not of the money-material], B. 116, pp. 249–252.

[9] *Functio*—fungibleness, the ability of one portion of money to be paid back in return for another portion of money borrowed.

[10] *De jure belli et pacis*, 1625, lib. II. cap. XII. § 17. The emphasis in this passage lies upon the word "naturally." According to Grotius the multiple standard is the natural standard for paying debts, although it has never been employed within any State.

incipient deviations from the true course. The money whose quantity is to be regulated has generally been chosen to be paper money, issued by the government either directly or through the mediation of a bank or banks, and either inconvertible or convertible into a variable amount of metallic money or bullion. In the latter case this scheme is somewhat like the preceding, except that, the medium of exchange down to the smallest subsidiary coins being co-ordinate with the largest bills, this scheme will extend to all payments even of the smallest and shortest contracts, while that scheme was recommended only for large and long contracts; and, again, this scheme must from the beginning be compulsory, the paper money being legal tender, while in that scheme, the idea was that the contraction of obligations payable according to the multiple standard should, at least until the practice became customary, be voluntary. In the present scheme the suggestion has sometimes been made that the quantity of the outstanding money may be regulated by raising or lowering the rate of discount according as prices rise or fall; otherwise the regulation of the quantity, by some other method of extending and contracting the issues, would be anterior, itself influencing the prevalent rate of discount. Such a scheme, more or less definitely worked out, has been frequently recommended, more or less vigorously.[11] It has even been extended by Professor Walras to metallic money in one of its species, namely, to silver coins, in a system, as he calls it, of *billon régulateur*, for which

[11] It was hinted at by R. Walsh, *A Letter to Alexander Baring, Esq., on the present state of the currency of Great Britain*, in the American Review of History and Politics, Vol. II., 1811, pp. 275–277, and by Scrope, B. 9, pp. 418–419; and is said to have been recommended by W. Cross, *Standard pound versus pound sterling*, 1856. A very imperfect form of it was suggested by Jevons, *op. cit.*, pp. 327–328. It has been advocated more seriously by J. Barr Robertson before the Gold and Silver Commission, *Second Report*, 1888, qq. 6294–6304; A. Williams, *A 'fixed value of bullion' standard.—A proposal for preventing general fluctuations of trade*, Economic Journal, June 1892, pp. 280–289; J. Conrad, in *Wissenschaftliche Gutachten über die Währungsfrage*, Berlin 1893, pp. 33-34; O. J. Frost, *The question of a standard of value*, Denver 1894, p. 26; Osborne, *op. cit.*, p. 332; Fonda, B. 127, pp. 158–195; H. Winn, *op. cit.*, pp. 579–589; J. A. Smith, B. 129, pp. 33–42; Whitelaw, B. 130, pp. 20–22, 28-82; Parsons, B. 136, pp. 102 ff.; T. E. Will, *Stable money*, Journal of Political Economy, Chicago Dec. 1898, pp. 85–92.

the existence of "limping bimetallism" in many countries offers opportunity.[12]

§ 5. A warning should be given to the advocates of either of these schemes. This is that there is no such thing as a single variation in the exchange-value of money, or of gold or silver, the same throughout the whole world, or between adjoining countries, especially if separated by shifting tariff barriers, or even within the borders of any one fairly large country. For the changes in the charges of transportation and of intercourse cause the variation in the exchange-value of the same monetary system to be different in different regions. In our country, for instance, the variations in the exchange-value of money at New York, at New Orleans, at Chicago, and at San Francisco, would form four appreciably different series. Each of these series should be measured by itself, no prices in one part of the country being mixed up with prices in another part. Then, in the former voluntary scheme, the contractants could use the index-numbers of their own locality. But in the second compulsory scheme, as no one center alone should be favored, the standard for the whole country should be an average of the variations at the different centers, each being weighted according to, not so much the population, as the total wealth, of its region.

On the other hand, to the opponents of any such schemes may be given an admonition. There is a not uncommon opinion, to be found even in the works of very respectable economists, that the exchange-value of metallic money, and among English-speaking peoples, of gold money, is normally stable at certain levels, and that periods of variation are only transitions from one level to another. Accordingly, when nations have felt themselves suffering from monetary appreciation or depreciation, assertions have confidently been made that only a little patience is needed, as the condition is transitory and will

[12] In the Journal des Économistes, Dec. 1876, May 1881, Oct. 1882: B. 71, p. 11; B. 69, pp. 2-3, 12, 16-18; B. 70, pp. 411, 441-449, 478-484; B. 71, pp. 143-144, 148-151, 162-163, 484.—Walras is followed by Andrews, B. 107, pp. 36-46, B. 108, p. 141; and was partially anticipated by Mannequin, *La monnaie et le double étalon*, 1874, p. 59.

e followed by a settled condition when money shall have its new level; and therefore to do anything now will e to disturb that future happy state soon to be entered naturally. Nothing could be more false. There are no s of stability in the exchange-value of metallic money. easurement, among the measurements as yet made, of the of the exchange-value of money, has ever shown any- but perpetual movement. To be sure, it sometimes has ned that the upward and downward courses are short and counterbalance one another over an epoch of several But such occasions have been the exception, and not le. The so-called industrial cycles of eight or ten years rarely been at the same average level of prices. The rather that the fluctuations are uneven, and have a rising ling tendency for many years at a time, the durations of epochs themselves being various, so that anticipation is sible for more than a year or two in advance, and then great uncertainty. Natural money has no permanent level change-value; nor can we even rationally hope for such l. Perhaps, when the variations in exchange-value both ney and of commodities, the latter by means of their rices, have been better investigated, it will be found that ecious metals are among the products of human industry ost variable in exchange-value.

II.

. The principles upon which rests the scheme of regu- the exchange-value of money by exercising artificial con- ver its issuance, deserve our further attention. In this however, we are not concerned with the investigation of ; and so the quantity theory of the cause of the exchange- of money, which is the foundation for that scheme, is not discussed here. It may be assumed for the sake of argu- To be examined are mere relations between prices and nge-values.

the preceding pages the question has been investigated as at price variations ought to take place in order to com-

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Here, again, in our conception of the actual variations in the exchange-values of the commodities, we are no more concerned with the causes of these variations than we have been hitherto. [B] may fall in exchange-value in [A] because [B] is now produced more cheaply and abundantly than formerly, while [A] is still produced with the same difficulty and in no greater quantity, or because [B] is still produced with the same difficulty and in the same quantity, while [A] is produced with greater difficulty and in lesser quantity, or because of three more typical combinations of such changes; or again because the demand for [B] has diminished, while that for [A] has remained the same, or because the demand for [B] has remained the same while that for [A] has augmented, or because of three more typical combinations of such changes—making five in each kind, which may act singly or in unison, thus providing us with as many as thirty five typical combinations of causative changes. But these causes, so far as operating between [A] and [B], or between these and all other commodities, even if perfectly and completely known to us, in no wise inform us what are to be the changes in the prices of [A] and of [B], or explain to us why they are such as they are, except as regards their relation to each other; for these changes are in relation also to money, and depend upon a similar interaction of causes (again in thirty five typical combinations) operating between [A] and [B] and all other commodities on the one hand and money on the other. In other words, without the factors affecting money, the factors affecting commodities cannot alone determine prices; nor the former alone without the latter. The two sets of factors cannot operate independently of each other. Of course the changes in the causes may be in only one of the two sets, the other remaining untouched. But in whichever set it be, the effect upon the exchange-value or exchange-values on the one side is, *ipso facto*, an inverse effect upon the exchange-values or exchange-value on the other. For the exchange-value of money is affected as well by the causes affecting the production and quantity of commodities as, inversely, by the causes affecting the production and quantity of money; and the level of prices is only a manifestation of the effect, in whichever way it be brought about.[3] Therefore the residence of the changes in the causes on the one side or on the other should have no influence upon our opinion concerning the variation or constancy of the exchange-value of money. This is to be decided only by a measurement of the actual conditions,—most conveniently by the measurement of the inverse variation of the general level of prices. Changes in the causes, moreover, if taking place on both sides simultaneously, and influencing both sides in the same direction, may neutralize each other and so leave the exchange-value of money constant. The exchange-value of money will be altered only (1 and 2) by the existence of such changes on either side alone, either with an upward or with a downward influence, this influence being neither aided nor impeded by an opposite or by a similar influence coming from changes on the other side; or (3) by the influences of such changes on both sides in opposite directions, either upwards for commodities and downwards for money or downwards for commodities and upwards for money, these influences combining to increase the divergence; or (4 and 5) by an excess of the influence of such changes on either side alone operating in either direction over the influence of such changes operating in the same direction on the other side. And of course what is here said of the general exchange-value of money may be said of the general exchange-value of any class of things over against all other things.

§ 2. Now what everbody wants in regard to all commodities is that they should individually and collectively become cheaper in cost-value and in esteem-value. Many persons, then, have inclined to think that what is desirable is that all commodities should become individually and collectively cheaper in exchange-value also. Here is a mistaken inference due to confusion of thought, itself due to the confusing use of a single term "value" (and of the allied adjectivial terms "dear" and "cheap") to express what really are distinct ideas, the genus being used for the species because the species have not been

[3] See above. Chapt. II. Sect. III. § 1.

sufficiently distinguished, in the error which logicians call the fallacy of undistributed middle.[4]

The inference is mistaken, as is plain from the fact that it violates Proposition LII., and what is inferred is impossible. To renew the proof of this, let us suppose that the alleged desire for the cheapening of everything in exchange-value is satisfied by the successive equal cheapening of everything. If [A] becomes cheaper in exchange-value first, *ipso facto*, as we know, everything else becomes slightly dearer in exchange-value. If later [B] becomes equally cheaper in exchange-value, again everything else (including [A]) becomes slightly dearer in exchange-value. Also [B] is then as cheap as [A], wherefore [A] is as dear as [B]. Again, if [C] becomes equally cheaper in exchange-value, everything else (including [A] and [B]) becomes slightly dearer in exchange-value ; and now [A], [B] and [C] are equally cheap and equally dear. And the successive fall of everything else will tend to raise slightly the exchange-value of everything else (including the things already fallen). Every single thing, therefore, at some time falling in one jump the full extent of the common fall supposed, and then effecting a slight rise in all the others, will itself be raised in very small stages by the falls of all the others, whether before or after, or partly before and partly after, its own fall. Thus when all but one have fallen, that last thing will be dearer in exchange-value than it was at the commencement by the inverse of the fall of all the rest. When it too falls, the last rise in the others will be consummated, and as the last thing has fallen back to equivalence with all the rest, [A] and [B] and [C] and [D] and the rest through the whole list down to the last one now being equivalent, their exchange-values will be in exactly the same condition as they were at the commencement, that is, everything will have the same exchange-value as before, and the very supposition by which everything is made successively to fall equally in exchange-value shows that when the operation is completed nothing is fallen in ex-

[4]Notice that we do not want things to fall in use-value. Such a fall would mean an approach, on our part, to insensibility. Rather we would prefer that things should rise in use-value.

change-value. Of course if we should attempt to suppose that all things fall equally in exchange-value together and at once, the same result would be involved, which means that nothing is fallen, although all are supposed to have fallen, so that such a supposition is a contradiction in terms. And if the things are all supposed to fall in exchange-value unequally, or if only some of them are supposed to fall, the supposition remains true that these latter are fallen, or, in the former case, that those which were supposed to fall most, still are fallen. But at the same time it is true that the things supposed to fall least are risen in exchange-value. It is impossible to carry out the conception of all things falling together, equally or unequally, in exchange-value.

The reason why any difficulty should appear in this subject is that it is possible for people to make the supposition, or to entertain the thought, that some or all things have fallen in exchange-value, and at the same time to forget, or perhaps never to perceive, that every fall they suppose in exchange-value involves also a rise in exchange-value. Of course, then, while entertaining the one thought, and excluding the other, they may *think* of all things falling in exchange-value. But they can do so only because they are thinking imperfectly.

And, to repeat, the reason why people have thought so imperfectly is that they confound exchange-value with cost-value or with esteem-value; for in these latter kinds of exchange-value it is possible for everything to fall, and to remain fallen, the fall of one thing in such values not having any necessary influence to raise any other things in such values. A passage which may serve as a *locus classicus* of this confusion of thought is the following, taken from the *Political Economy* of Malthus. In it, to be sure, Malthus uses the term "exchangeable value"; but it must be remembered that he had no terms whereby to distinguish from exchange-value the other kinds of value except use-value, so that by "exchangeable value," as by "value" itself, he variously referred to all the kinds of value except the last. Now after supposing a case of equal improvement in the production of all goods, he proceeds to ask and to

-wer as follows:—"Can it be asserted with any semblance of rectness, that an object which under these changes would mmand the same quantity of agricultural and manufactured oducts of the same kind, and each in the same proportion as fore, would be practically considered by society as of the ame exchangeable value? On the supposition here made, no person would hesitate for a moment to say, that cottons had fallen in value, that linen had fallen in value, that silks had fallen in value, that cloth had fallen in value, etc., and it would be a direct contradiction in terms, to add that an object which would purchase only the same *quantity* of all these articles, which had confessedly fallen in value, had not itself fallen in value."[5] Here, of course, if by "value" be meant cost-value, or esteem-value, the conclusion would be perfectly correct. And here Malthus himself did have in mind a mixture of cost-value and esteem-value,[6] and his confusion consisted in still calling it "exchangeable value," which nine-tenths of his readers, and himself too at times,[7] take to mean what the term itself properly means. His conclusion, then, is both correct and incorrect. For the reasoning is perfect on condition that the falls of all the separate classes of things are independent of one another, and that there are no rises really involved, but suppressed. Such is the case with the falls in cost-value, as also in esteem-value (or at least to some extent,—that is, the condition is a possible one). But such is not the case with falls in exchange-value. With regard to this, the appearance of correctness in the reasoning is acquired only by suppressing the counterbalancing rises necessarily involved in every fall, wholly forgetting or ignoring the interdependence and correlation of all variations in this kind of value.

§ 3. The desire, then, that all things should become cheaper in exchange-value is impossible, absurd, and inane. But there is another confusion of thought possible in this connection, and, because possible, often fallen into, which is not so empty and harmless. Variations in prices are variations in exchange-

[5] Second ed., p. 58.

[6] Cf. *ibid.*, p. 60.

[7] Cf. *ibid.*, pp. 50 and 61.

values; and therefore variations in exchange-values have been confounded with variations in prices. And as people have entertained the contentless desire that all commodities should fall in exchange-value, they have identified this with the desire that all commodities should fall in price. Or perhaps this desire has been reached directly from the sound and respectable desire that all commodities should fall in cost-value and in esteem-value, since variations in the latter are frequently, and in isolated cases almost always, accompanied by variations in the former. Here is something possible. But it involves something else, which also is often ignored. This is that if every commodity becomes cheaper in price, that is, in exchange-value in money, money becomes dearer in exchange-value; and if the former is still desired when this is perceived, the latter must also be. Why are these two things to be desired? The only possible reply as regards the first is that the desire for the fall of prices is conjoined with the desire for the fall of all commodities in cost-value or in esteem-value. It is thought, with or without good reason, that the desired fall in these values, if occurring, should be marked and measured by a corresponding fall in their prices. And this thought necessarily involves the idea that money is to be considered the standard measure, not of exchange-value, but either of cost-value or of esteem-value.

It is not within the province of this work to argue as to whether money should be considered the standard of the one or of the other of the various kinds of value. But it is essential here to point out that money cannot possibly be the standard of more than one kind of value. With but the rarest exceptions, all the world has hitherto been agreed that money is a measure of "value," and consequently that in order to serve this function properly and to be worthy of being taken as a standard, it ought to be stable in "value." It becomes necessary now to distinguish in which sense the term "value" must be understood in this connection. To continue speaking of money merely as the measure of "value" will be to perpetuate an equivocation of thought that may lend aid to either of opposite sides on many important practical questions according to the desires selfishly uppermost in the minds of partisans. Here is a point which all economists should pause to decide and settle, before they attempt to take a further step in their science—or at least in that branch of it which is the science of money.

§ 4. As, however, this work specially deals with exchange-value, we may here assume, for the sake of theory, that money is properly the standard measure of exchange-value. Then money ought to be stable in exchange-value; and if it is not so naturally, it ought to be made so artificially, if this be possible. And, as another statement of the same thing, the general level of prices ought to be constant; and if it is not so naturally, it ought to be made so artificially, if this be possible. And this in spite of, and through, the fact that it is desirable for all commodities to fall in cost-value and in esteem-value.

The collective will of the people, as organized in government, cannot properly control the causes at work upon commodities, except for the purpose of unfettering and aiding production, and of making all commodities fall in cost-value and in esteem-value.[8] In fact, the fall in cost-value is the aim of every producer of every class of commodities, while the fall in esteem-value, aimed at by the consumers, is the result of the free play of all these agencies.[9] But it is believed to be within the power of government, by assuming the issuance of money, to control the exchange-value of money in all things,—that is, also, the exchange-value of all things together in money,—without in any wise seeking to control their relative exchange-values amongst themselves. If this be so, the aim should be, neither to make money cheaper in commodities, that is, to make prices rise, nor to make commodities cheaper in money, that is, to make prices fall, but to keep money stable in exchange-value in

[8] *Properly*,—again explanation is needed,—because as a matter of fact governments do exactly the opposite and raise the cost-value and esteem-value of goods—by tariffs, keeping out the goods produced more cheaply abroad that could be exchanged for the goods produced more cheaply at home.

[9] Without this free play the producers may succeed, through monopoly, not only in lowering the cost-value, but even in raising the esteem-value of their goods, by curtailing production.

commodities and commodities as a whole stable in money, that is, to make the general level of prices constant, so that, while the esteem-values of all commodities are happily falling with the fall in their cost-values, the esteem-value of money shall fall neither more rapidly nor more sluggishly than the esteem-values of all commodities on the average.

Take, for instance, the case above supposed of one commodity rising 50 per cent. in exchange value in another, the latter then falling 33⅓ per cent. in exchange-value in the former. Let us revive our earlier supposition of a simple economic world with money and two classes of commodities. Now if the rise of [A] by 50 per cent. in [B] were manifested in prices by [A] rising 50 per cent. in price while [B] remained constant in price, this would mean that the average of prices has risen and the exchange-value of money has fallen ; which is a sign that money is too abundant, whatever were the other causes at work. Some money, therefore, should be abstracted from the circulation. This would have the effect of lowering the prices of both the commodities in (about) the same ratio. If the contraction should proceed until the price of [A] stood at 1.2247 above what it was at the start and the price of [B] at .8165 below what it was at the start, we know by our investigations that then, on the supposition of the two classes being equally important over both the periods together, the exchange-value of money is the same as it was in the beginning. On the other hand if the relative change between [A] and [B] had manifested itself by [B] falling in price by 33⅓ per cent. while [A] remained constant in price, this would constitute a fall of the average of prices, indicating a rise in the exchange-value of money, signifying an insufficiency of money, whatever be the other causes of these changes, and therefore giving warning of the need of issuing more money for the purpose of running the price of [A] up to 1.2247 and the price of [B] up to .8165 compared with what they were at the start. Or the prices may originally have changed in any of three other typical ways. But whatever their original changes, due to natural causes, as soon as it is discovered that they are such as to constitute a fall rise in the exchange-value of money, the issuance should ıltered so as to make both the prices move into their proper nterbalancing positions.

f one of these classes were more important than the other, r counterbalancing positions would be different; but these, can be calculated, and, when calculated, their attainment ıld be aimed at in the regulation of the currency. Unfortely, however, in this case, unless the mass-quantities are stant, the calculation cannot be made with absolute preci- . Also this is so when we are dealing, as we must in our plex world, with many and variously large classes. Yet know that still, the more complex the world, the more is the of the properly weighted geometric average of the price ations likely to approximate to the truth. Therefore it is possible to used the geometric method with practically icient accuracy.

'hus in general, in any complex economic world, if one s of commodities rises p per cent. (in hundredths) in exıge-value in all other commodities, then, for money to ain stable in exchange-value, money should rise in exıge-value in the other commodities from unity approxiely to the geometric average between 1 repeated r times and p, which is $\sqrt[r+1]{1+p}$, or, to use another form of expression, $+ p)^{\frac{1}{r+1}}$, (r representing the number of times all the other modities are more important over both the periods comd than the one class in question); wherefore the prices hese articles should fall approximately by $1 - \frac{1}{(1+p)^{\frac{1}{r+1}}}$ cent., and, as the price of the one article must rise to p times the other prices, it should rise approximately to $\frac{+p}{+p)^{\frac{1}{r+1}}}$, or $(1+p)^{\frac{r}{r+1}}$, times its former price, approximately $1+p)^{\frac{r}{r+1}} - 1$ per cent. And reversely, if one class falls er cent. in exchange-value in all other commodities, its e should fall, not by p' per cent., but approximately by $(1 - p')^{\frac{r}{r+1}}$ per cent., and the prices of the others, instead of remaining unchanged, should rise approximately by $\frac{1}{(1-p')^{\frac{1}{r+1}}} - 1$ per cent. Or, further, if out of all classes of commodities n'' in number, certain classes l in number (in both cases including the repetitions needed to represent their relative sizes over both the periods) rise in exchange-value in all other commodities evenly by p per cent. each, or unevenly so that the properly weighted geometric average of their rises is p per cent., then money should rise in exchange-value in the others approximately to the geometric average between 1 repeated $n'' - l$ times and $1 + p$ repeated l times, that is, to $(1+p)^{\frac{l}{n''}}$ times its former exchange-value in them; wherefore they should fall in price evenly or on the average approximately by $1 - \frac{1}{(1+p)^{\frac{l}{n''}}}$ per cent., and the l articles should rise in price evenly or on the average approximately by $(1+p)^{1-\frac{l}{n''}} - 1$ per cent. And reversely if the l articles fall p' per cent. in exchange-value in the others, money should fall in the latter approximately to $(1-p')^{\frac{l}{n''}}$ of its former exchange-value in them; wherefore their prices should rise approximately by $\frac{1}{(1-p')^{\frac{l}{n''}}} - 1$ per cent., and the prices of the l articles should fall approximately by $1 - (1-p')^{1-\frac{l}{n''}}$ per cent. Therefore in any of these cases, whether addition or subtraction of money be needed, it should stop when these prices are obtained.[10]

The actual operation of the system, however, would be somewhat different, and simpler, though less precise. We never know directly how much one or more articles vary in exchange-value in another or others; but we learn this only from their

[10] In practice things would not work so smoothly as in the theory. Prices would not be raised or lowered all in exactly the same proportion, for the reasons stated in Note 1. For instance, small alterations in the quantity of money might perhaps raise or lower some prices which were just on the verge of making the jump from one round figure to the next, while others at the verge of falling might merely be advanced to the verge of rising, or reversely, without showing any actual variation. Still the principle is the same, and the addition or subtraction of money should stop when the prices, whatever they be, approximately compensate.

prices already formed under causes coming from money as well as from the commodities themselves. And we do not generally know the causes which make the changes either between the commodities or between these and money ; nor for our present purpose is it necessary that we should know them. Their effects are manifested in the subsequent prices, and to know these is sufficient. Here is where the formulæ for finding the new level of all prices compared with the former level come into practical use,—and now one of the other two is preferable to the geometric. But perfect precision in regard to the exact extent of a variation upward or downward is not of great importance, provided the result be always given in the right direction. If the level of prices is shown to have risen, it is a sign that the quantity of money is too great and should be diminished ; and if the level of prices is shown to have fallen, it is a sign that the quantity of money is too small and should be increased. Of course it is well that the measurement should show whether the rise or fall is great or small ; for then it will be known whether a great or a small alteration needs to be made in the quantity of money.[11] The alteration must continue until a subsequent application of the formula shows that the original level is again obtained. The possibility of any great variation in the level of prices should be cut off by making the measurements at short intervals.[12]

[11] Walras once thought that, the variation of the level of prices and the quantity of money in a country being known, he could calculate exactly the new quantity of money needed, his conclusion being that the former quantity [illegible] uld be multiplied by the reciprocal of the variation, B. 69, p. 17. But [illegible] o not work with such exactness because (1) the principle upon which his calcu[illegible] is founded, namely that, given the quantity of goods, the level of pri[illegible] exactly in inverse proportion to the quantity of money, is not true, and (2) e[illegible] f it were, there would be intervening changes going on in the goods before [illegible] addition or subtraction of money could be carried out or before its in[illegible] e [illegible] be felt. Walras has since modified his views on this subject, in B. [illegible]

[12] Walras wants the determining measurements to be made for e[illegible] decade, i. e., for an industrial cycle, B. 61, pp. 6, 18—this is too [illegible] A[illegible] ex-treme Williams, op. cit., p. 289, Fonda, B. 127, p. 165, [illegible] pp. [illegible] 51, want the me[illegible] made daily—[illegible] were [illegible] by Laves, [illegible] op. cit., p. [illegible] risional

§ 5. On the other hand, let us for a moment suppose that as a measure of "value" money is to be considered the measure of cost-value or of esteem-value. Then either the payment of contracts may be regulated by the ascertained variations of money in the one or the other of these kinds of value, or, the preventive system being adopted, the aim to be held in view would be, by a similar regulation of the quantity of the currency, to keep money stable in one of these values, forcing the prices of all commodities individually to rise or fall according as the commodities individually rise or fall in cost-value (although here there would be difficulty), or in esteem-value. Cost-value being selected as the norm, the cost of production of the money-metal (for here the cost-value of paper money is out of the question) ought on the average to be constant.[13] In the case of esteem-value being chosen, the gross-money earnings of an hour's labor of all the workers in a country (or their net money-earnings, along with the money-incomes of an hour, in a day of the number of hours the others work on the average, of all the non-workers) ought to be constant. The former of these standards has never been suggested either as guide for the payment of money-contracts or as guide for regulating the issuance of money—the last being possible only by interference with the working of the mines of the precious metal or metals, which has never been recommended in this connection. The latter standard has been suggested for both these uses. The suggestion has been made both that contracts should be payable in the quantity of money that is constant in esteem-value (imperfectly measured by wages only),[14] and that the issuance of money should be regulated so as to keep money stable in such value (likewise imperfectly measured by wages only).[15] In either case, as remarked in an earlier chapter, after having made such measurement in regard to money alone, if we have a desire to know whether, and how much, commodities in general have risen or fallen in cost-value or in esteem-value, we could calcu-

[13] *On the average*, and not at the least fertile mine, because we are now not dealing with relations of exchange or of esteem.

[14] By Shadwell, *Principles*, p. 260.

[15] Both indifferently, by Pollard, *op. cit.*, pp. 74-75.

late this (in cost-value only imperfectly) by inverting the calculation of the variation of such money in exchange-value in all commodities. Thus in these cases also the measurement of the general exchange-value of money would still be useful. Only now its use would be merely theoretical and to satisfy an idle curiosity, while the really important measurement would be the measurement either of cost-value or of esteem-value. But, on the contrary, if money is properly the measure of exchange-value, it is these other measurements that are practically useless and serve only to please our vanity by showing that we are making progress.

§ 6. Both the above discussed practical schemes for making use of the measurement of variations in the general exchange-value of money, or in the general level of prices, are still embryonic, and no attempt to apply them will probably be made for centuries to come.[16] But a stumbling block in their way is the fact that the measurement of exchange-value has never been perfected so as to win unanimous assent on the part of economists, as is necessary before scientific knowledge can be claimed concerning the very variations which those schemes propose to correct or to prevent. Also this lack of science, which is still greater in regard to cost-value and to esteem-value, is no doubt a reason why economists have not turned their attention to a more careful consideration of the question whether money is the measure of exchange-value or of cost-value or of esteem-value.

[16] Of course the mensuration of exchange-value should not be postponed until practical use can be made of it; for practical use can be made of it only after it has been performed. Hence it is advisable even now that in the mass of statistics which every civilized government makes it a duty to collect there should be an effort made to measure the variations in the exchange-value of money. The creation of an official bureau for this purpose has been recommended by Scrope, B. 9, p. 424; Newcomb, B. 76, p. 213; Marshall, B. 93, p. 365; British Association Committee, *Fourth Report*, B. 102, p. 487; Laves, *op. cit.*, p. 846; Zuckerkandl, B. 115, p. 249.

APPENDIX A.

ON VARIATIONS OF AVERAGES AND AVERAGES OF VARIATIONS.

The supposition is that we have two sets of figures, a_1, b_1, ……, at a first moment or period, and at a later moment or period certain other figures, a_2, b_2, ……, to which those have varied.[1] We may have any numbers of these figures at both periods, but always the same number in each set; for otherwise we should have a figure in the first which does not vary to anything in the second, or a figure in the second to which nothing in the first has varied. The number of the a's, whether one or more, may be represented by x, that of the b's by y, and so on. For convenience, although this phraseology is not strictly accurate, we may speak of the symbols a_1, a_2, b_1, b_2, ……, as *figures*, and of the symbols x, y, ……, as *numbers* (this referring to the numbers of times the figures occur in each set). The figures constitute, so to speak, classes, there being as many classes as there are distinct figures. We may have reason, provided by the problems we are dealing with, to divide even similar quantities into distinct classes, and then to represent them by distinct figures.[2]

The figures, of course, are quantities. But sometimes it will be convenient to speak of their *sizes*, when it is desired to call attention to their quantitative relations. The classes also have sizes, made up of their figures multiplied by their numbers,—*e. g.*, xa_1, xa_2, yb_1, etc.

The variations of the figures from the first to the second period are represented by the expressions $\frac{a_2}{a_1}$, $\frac{b_2}{b_1}$, ……. The variation of a_1 to a_2 is a variation of a_1 by $\frac{a_2}{a_1}$; for $a_1 \cdot \frac{a_2}{a_1} = a_2$. It is the same also as a variation of 1 $\left(= \frac{a_1}{a_1}\right)$ to $\frac{a_2}{a_1}$. Of course the variation of the whole class xa_1 to xa_2 is nothing else as a variation; for $\frac{xa_2}{xa_1} = \frac{a_2}{a_1}$. But the variations of all the figures in a class, thus

[1] If $a_2 = a_1$, this may be viewed as an infinitely small variation. Therefore we may include it in speaking of variations.

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$\frac{a_2}{a_1}, \frac{a_2}{a_1}, \cdots\cdots$ to x terms, when added together, are $x\frac{a_2}{a_1}$, or when multiplied by one another, are $\left(\frac{a_2}{a_1}\right)^x$.

We may average each set of figures separately. Representing the average of the figures at the first period by $\mathbf{A}_1$, and that of the figures at the second period by $\mathbf{A}_2$, we may represent the variation of these averages by $\frac{\mathbf{A}_2}{\mathbf{A}_1}$, the variation departing from constancy as this expression departs from unity. Or we may average the variations of the figures as they change from what they were at the first period to what they become at the second. Representing this by $\mathbf{V}$, we shall find sometimes that $\frac{\mathbf{A}_2}{\mathbf{A}_1} = \mathbf{V}$, and sometimes that these expressions are different. It is our purpose to study the relationship between these forms.

In averaging, to assign a certain importance to one term—a class of figures, or a kind of variations—relatively to others, is to *weight* it. The most natural way to weight classes of figures is by the numbers of times the figures occur in each class. Similarly it would seem most natural to weight kinds of variations by the numbers of variations in each kind. But as the figures may be reported capriciously, and the steady element be the sizes of the classes, we shall find the need of weighting the variations according to the latter, in several ways.

The number of all the classes, and consequently of all the kinds of variations, will be represented by n, so that $n = 1\left[\text{for } a_1,\ a_2, \text{ or } \frac{a_2}{a_1}\right] + 1\left[\text{for } b_1,\ b_2, \text{ or } \frac{b_2}{b_1}\right] + \cdots\cdots$ through the whole lists. The number of individual figures at each period, and the number of the individual variations, will be represented by n', so that $n' = x + y + \cdots\cdots$ to n terms. If there is only a single figure in every class, then $n' = n$; otherwise $n' > n$. Thus n' represents the sum of what we have just described as the most natural weights of the classes of figures we are dealing with. The sum of the other weights which we shall need for the variations (and sometimes for the figures) will always be represented by n'', although these weights are different on different occasions.

Before proceeding to our subject proper, it may be well to state certain principles which are common to all kinds of averages.

I. ON AVERAGES IN GENERAL.

§ 1. *An average of any set of terms is always smaller than the largest and larger than the smallest*, whatever be the number of terms. Consequently

§ 2. *If the terms are all equal, the average is equal to each term*, whatever be their number.

These two principles are explicative, that is, they flow from the definition of "average," and no formula pretending to be a formula of a kind of average is so unless it yields results in accordance with these principles. That the formulæ for the arithmetic, harmonic and geometric averages satisfy the second principle has incidentally been shown in Chapter V. Sect. I. §§ 5 and 6.

Consequently also, if we are dealing with sets of terms in classes,

§ 3. *The average is always between the highest and the lowest terms appearing in any of the classes*, whatever be the numbers of terms in any classes ; and

§ 4. *If the terms are all equal, but arranged in classes with different numbers in each, the average is equal to each of the figures*, whatever be the numbers of the figures in any of the classes.

A meaning in this last proposition is that the statement is universally true, whatever be the weighting of the classes. In other words, *when all the terms are equal, the weighting is indifferent.*

§ 5. What has been said of averaging any terms is true whether the terms be *figures* (as above described) or *variations* (of figures). For example, applied to variations, the fourth proposition becomes: *If the variations are all equal, the average is equal to each of them, no matter what be the weighting of the variations.*

That the formulæ for the three averages satisfy this principle will be indicated in the course of the separate treatment of them.

§ 6. Furthermore, *if the variations are all equal, the variation of the averages separately drawn of the figures which have varied and of the figures to which they have varied* $\left(\frac{\mathbf{A}_2}{\mathbf{A}_1}\right)$ *is the same as the average of the variations* ($\mathbf{V}$), *itself the same as any one of the variations* $\left(e.\ g.\ \frac{a_2}{a_1}\right)$.

If all the figures in each set are equal, this is self-evident. In other cases a demonstration may be needed for each kind of average. Indications of the demonstration will be added in the treatment of each kind of average. (See below, notes in II. § 6, III. § 6, and V. § 5).

Of course, if the variations are all $\frac{1}{1}$, or 1, that is, if there are no variations, this principle applies, and is now self-evident, whatever the figures. For in this case the corresponding figures in the two sets are identical, the sets themselves are so, and so must be their averages ; wherefore there is no variation of the averages.

We shall be interested, in what follows, principally with sets of figures in at least one of which at least one of the figures, and one of the variations, is different from the rest—or rather, with general conditions which admit the possibility of such divergence.

§ 7. Even now there is another case, common to all the kinds of averages, in which the variation of the averages and the average of the variations always agree. If all the figures *at the first period* are units, wherefore the average also is a unit, the figures in the second set themselves express their variations from the first period, since in all the expressions for the variations, $\frac{a_2}{a_1}$, $\frac{b_2}{b_1}$, ..., the denominators are 1, and may be dropped. Therefore the expression for the variation of the averages, $\frac{\mathbf{A}_2}{\mathbf{A}_1}$, in which the denominator is also 1, and may be dropped, and in which the numerator is the average of the figures at the second period, is also the expression for the average of the variations. Thus

If all the figures at the first period are units, the variation of the averages and the average of the variations are always the same.

This proposition, however, is very different from the preceding. In the conditions there posited it is indifferent what weighting is used. Here it is important that correct weighting be used, and it is necessary that the same weighting be used in averaging the figures (of the second period) and in averaging the variations.

Moreover this proposition is true also of all cases *when the figures at the first period are equal,* whether they all be units or any other quantity. For this other quantity may be taken as a unit, and the figures in the second set be reduced on the same scale, and then the proposition applies.

Consequently this principle will form part of all our separate treatment of the three kinds of averages (except in the third, where it will be swallowed up in a wider principle). The particular demonstrations will therefore be given later. (See II. § 6, III. § 6, V. § 5).

Applying to all cases, the following principles are also plain.

§ 8. Given a set of figures (or variations) not all alike, of which the average (of any kind) is known, if we add to it a figure equal to the average, or substract from it a figure which happens to be equal to the average, we do not alter the average. Hence it does not matter how often a figure equal to the average be added or subtracted, or whether it be omitted altogether. In other words, *the weighting of a figure equal to the average is indifferent.*

The similarity of this to § 4 is patent.

§ 9. Given a set of figures (or variations) as before, if we add *a figure larger than the average,* we raise the average somewhat; and if we subtract such a figure we take away one of the influences that have made the average as high as it is, and so we lower the average. Hence it does matter how often such a figure is repeated, and to increase the number of times it is repeated or *to enlarge its weight, is to raise the average nearer to it,* and nearer and nearer the more we repeat it (but the average can never reach it, short of an infinite number of repetitions, for in that case its weight would be indifferent); and to decrease the number of times it is repeated, or *to diminish its weight, is to lower the average* (until the figure is omitted altogether, when the average is what it would have been without this figure).

§ 10. Given a set of figures (or variations) as before, if we add *a figure smaller than the average,* we lower the average somewhat; and if we subtract such a figure we take away one of the influences that have made the average as low as it is, and so we raise the average. Hence it does matter here, too, how often such a figure is repeated, and to increase the number of times it is repeated, or *to enlarge its weight, is to lower the average nearer to it,* and nearer and nearer the more we repeat it (but without ever reaching it, as before); and to decrease the number of times it is repeated, or *to diminish its weight, is to raise the average* (until the figure is omitted, when the average is what it would have been without this figure).[1]

[1] Of course, if we add or subtract a figure equal to one average and unequal to another average, we alter that other average. Or if we add or subtract a figure unequal to this average, and so alter this average, we may perhaps not affect another average (to which this figure may happen to be equal). We are dealing with each kind of average carried throughout.

These three principles can easily be demonstrated in the case of each of the three kinds of averages. But they are too plain to need demonstration.

A *résumé* of them is that when a figure is the same as the average its weighting is indifferent; otherwise its weighting counts, and an alteration in the weighting of the classes without any variation in the sizes of the figures (but *like* a variation in the size of the figures) causes the average to change, the influence of an alteration in the weighting being different according as the figure operated on is larger or smaller than the average. To increase the repetitions of a larger figure has the same influence as to enlarge the figure; and to increase the repetitions of a smaller figure has the same influence as to diminish the figure; and conversely.

A practical application of these three principles is that an error in our weighting of a figure equal to the average is of no account (*i. e.*, if the average without a figure is the same as the average with it, we need not concern ourselves about its weight). And the nearer a figure is to the average, the less an error in its weighting will count; and the more it will count, the more the figure is removed from the average. Of course in the case of an average of variations, what is here said applies to a variation according as it is the same as, near to, or far from, the average of the variations.

Hence lastly,

§ 11. *If all the weights are altered in the same proportion, there is no effect upon the average*, or, in other words, if we have the proper weights, we may alter them as we please, so long as we keep them in the same proportion (multiplying or dividing them all by the same quantity).

For if we increase all the weights in the same proportion, the increase of the weights of the figures equal to the average has no influence, and the increase of the weights of the figures larger than the average tends to raise the average, while the increase of the weights of the figures smaller than the average tends to lower it; but as the influence of all the figures below the average to lower it is equal to the influence of all the figures above the average to raise it (for otherwise the average would not be where it is), so the influence of the changes in the numbers of the former is equal to the opposite influence of the proportionally equal changes in the numbers of the latter, and the average remains where it was. And reversely if we decrease all the weights in the same proportion.

If this general explanation be not sufficient, the proposition may be demonstrated in the case of each of the averages separately. This will be done incidentally for two of them in the following pages. (See I. § 7, II. § 7).

Or another general proof may be made as follows. If, for instance, we double all the weights, we may segregate all the new terms, and so form two distinct sets of figures (or variations) exactly duplicating each other, one of them being the original set. Then the average of each set, separately drawn, will be the same. Consequently the one average of the two together will be the same. And similarly, whatever be the multiplier.

Thus the general system of weighing depends upon the relative sizes of the weights, and not upon their absolute sizes. However large or small the weights, if in the same proportion, we practically have the same weighting.

This being so, we have *even weighting* whether we count every figure in the sets (or every variation) only once, or an equal number of times.

II. ARITHMETIC AVERAGING.

§ 1. With single figures in the two sets, the arithmetic average at the first period is $\frac{1}{n}(a_1 + b_1 + \cdots\cdots$ to n terms), and at the second it is $\frac{1}{n}(a_2 + b_2 + \cdots\cdots$ to n terms); wherefore the variation of the averages is

$$\frac{\mathbf{A}_2}{\mathbf{A}_1} = \frac{\frac{1}{n}(a_2 + b_2 + \cdots\cdots)}{\frac{1}{n}(a_1 + b_1 + \cdots\cdots)},$$

which reduces to

$$\frac{\mathbf{A}_2}{\mathbf{A}_1} = \frac{a_2 + b_2 + \cdots\cdots}{a_1 + b_1 + \cdots\cdots},$$

(thus showing, incidentally, that *it is indifferent whether we compare the arithmetic averages or the sums*).

Here the weighting of the figures in each average is even, each figure counting once.

§ 2. Given the same single figures, the arithmetic average of the variations, likewise with even weighting, each variation counting once, is

$$\mathbf{V} = \frac{1}{n}\left(\frac{a_2}{a_1} + \frac{b_2}{b_1} + \cdots\cdots\right).$$

This is a different expression, not universally reducing to the preceding.

§ 3. With classes of figures, the arithmetic average at the first period may be expressed in full thus,

$$\mathbf{A}_1 = \frac{1}{n'}\{(a_1 + a_1 + \cdots\cdots \text{ to } x \text{ terms}) + (b_1 + b_1 + \cdots\cdots \text{ to } y \text{ terms}) + \cdots\cdots \text{ to } n \text{ classes}\},$$

which may be abbreviated to $\frac{1}{n'}(xa_1 + yb_1 + \cdots\cdots$ to n terms), or may also be written $\frac{xa_1 + yb_1 + \cdots\cdots}{x + y + \cdots\cdots}$. And the average at the second period is like unto it, with change only in the numbering of a, b, $\cdots\cdots$. Therefore the variation of the averages is

$$\frac{\mathbf{A}_2}{\mathbf{A}_1} = \frac{\frac{1}{n'}(xa_2 + yb_2 + \cdots\cdots \text{ to } n \text{ terms})}{\frac{1}{n'}(xa_1 + yb_1 + \cdots\cdots \text{ to } n \text{ terms})},$$

which reduces to

$$\frac{\mathbf{A}_2}{\mathbf{A}_1} = \frac{xa_2 + yb_2 + \cdots\cdots}{xa_1 + yb_1 + \cdots\cdots},$$

(in which, again, the variation of the arithmetic averages is the same as the variation of the sums).

Here the weighting of the figures in each average is directly according to the numbers of times the figures are repeated in each class. Naturally an average of the figures taken only once each, as in § 1, is not an average of the figures here supposed, but only of the figures there supposed. (Yet the evenly weighted average might be said to be the average of the classes simply as classes, each class counting as an individual, without regard to the numbers of figures in them)

§ 4. Given the same classes of figures, the similarly weighted arithmetic verage of the variations in full is

$$V = \frac{1}{n'}\left\{\left(\frac{a_2}{a_1}+\frac{a_2}{a_1}+\cdots\cdots \text{ to } x \text{ terms}\right)+\left(\frac{b_2}{b_1}+\frac{b_2}{b_1}+\cdots\cdots \text{ to } y \text{ terms}\right)+\cdots\cdots \text{ to } n \text{ classes}\right\},$$

hich may be abbreviated to $\frac{1}{n'}\left(x\frac{a_2}{a_1}+y\frac{b_2}{b_1}+\cdots\cdots \text{ to } n \text{ terms}\right)$, or may be ritten

$$V = \frac{x\frac{a_2}{a_1}+y\frac{b_2}{b_1}+\cdots\cdots}{x+y+\cdots\cdots}.$$

Again this expression is different, not universally reducing to the preceding.

§ 5. When dealing with single figures, we may say that we are dealing ith classes in each of which there is only one individual. Therefore we may ıbsume sets of single figures (or variations) under sets of classes of figures or variations), and treat only of the latter. Thus in both the preceding cases e have been dealing with arithmetic averages of classes in which the weight ıg is according to the numbers of figures in the classes, and of averages of ariations with weighting likewise according to the numbers of varying figures ı the classes.

The conclusion from the preceding paragraphs then is that *the variation of e arithmetic averages and the arithmetic average of the variations, in all cases with eighting according to the numbers of figures in the classes, are not universally the me.*

§ 6. *The variation of the arithmetic averages and the arithmetic average of the iriations, all with weighting according to the numbers of figures in the classes, are e same when all the figures at the first period are equal.*

Let $a_1 = b_1 = \cdots\cdots = s$ (s being *any* figure above, below, or at unity, in- egral or fractional). Then the expression for the comparison of the averages that for the sums) in § 3 reduces to $\frac{xa_2+yb_2+\cdots\cdots}{s(x+y+\cdots\cdots)}$; and the expression or the average of the variations in § 4 reduces to the same. Q. E. D.

Hence if we reduce all the actual figures in an irregular set at the first eriod to the same ideal figure (by taking some common divisor of them) educing the figures at the second period in the same proportions, and if we hange the numbers of the figures in the classes in the inverse proportions (so s to keep unchanged the sizes of the classes), the weighting desired in order o make the two methods agree will be in accordance with the adjusted num- ers of these ideal individuals in the classes.

Thus a factor which affects the variation of the averages is the number of imes the figures occur in each class, or weighting of the variations directly ccording to the numbers of the figures.

In all other cases (except of course when all the variations are equal— ee I. § 6[1]) the average of the variations will require different weighting of he variations to make it agree with the variation of the averages.

[1] In the formula above given in § 4 let $\frac{a_2}{a_1} = \frac{b_2}{b_1} = \cdots\cdots = r$, and the expression educes to r. Now $a_2 = ra_1$, $y_2 = rb_1$, and so on. Introduce these into the for- aula in § 3, and this also reduces to r.

§ 7. *When the numbers of the figures in the classes are all equal, the variation of the arithmetic averages (each with weighting according to the numbers of figures in the classes—in this case with even weighting) is the same as the arithmetic average of the variations with weighting according to the sizes of the figures at the first period.*

Let $x = y = \cdots\cdots = t$. Then the expression for the variation of the averages reduces to $\frac{t(a_2 + b_2 + \cdots\cdots)}{t(a_1 + b_1 + \cdots\cdots)}$, and this, by dropping the t from both sides of the fraction, to the expression for the variation of the figures taken singly, *i. e.*, with even weighting. We want now to prove that this expression is the same as the expression for the average of the variations with weighting according to a_1, b_1, $\cdots\cdots$.

The expression for the average of the variations with weighting according to $a_1, b_1, \cdots\cdots$ is $\frac{1}{n''}\left(a_1\frac{a_2}{a_1} + b_1\frac{b_2}{b_1} + \cdots\cdots \text{ to } n \text{ terms}\right)$ in which $n'' = a_1 + b_1 + \cdots\cdots$ to n terms. This expression, by reducing, and restoring the value of n'', becomes $\frac{a_2 + b_2 + \cdots\cdots}{a_1 + b_1 + \cdots\cdots}$, which is the same as the expression for the variation of the averages. Q. E. D.

Hence a factor which affects the variation of the averages is the size of the figures in the classes, or weighting of the variations directly according to the sizes of the figures.

The reason for this it is important that we should perceive. Take the simple case of two figures in each set, or two reported variations. Suppose a_1 is larger than b_1 by an integral number of times, say q. Now the single variation of a_1 to a_2, compared with the single variation of b_1 to b_2, is a variation not merely of a q times greater quantity, but of q times more quantities. Hence it virtually contains q times more variations. Therefore, as, according to the preceding proposition, the average of classes of variations, when the figures at the first period are equal, must be drawn with the classes weighted according to the numbers of figures in them, if in the present case the weight 1 be given to the figure b_1, the weight q should be given to the figure a_1, as really being a class composed of q b_1's, each of which varies by $\frac{a_2}{a_1}$. In fact, the formula for the variation of the averages may be analysed into this,

$$\frac{\left(b_1\frac{a_2}{a_1} + b_1\frac{a_2}{a_1} + \cdots\cdots \text{ to } q \text{ terms}\right) + b_1\frac{b_2}{b_1}}{(b_1 + b_1 + \cdots\cdots \text{ to } q \text{ terms}) + b_1},$$

which reduces to $\frac{b_1\left(q\frac{a_2}{a_1} + \frac{b_2}{b_1}\right)}{b_1(q+1)}$ and to $\frac{1}{n''}\left(q\frac{a_2}{a_1} + \frac{b_2}{b_1}\right)$, which is the expression for the average of the variations with q and 1 for the weights. What is here shown of two figures so conveniently related may be generalized as follows. Having one reported variation of a_1 to a_2, one reported variation of b_1 to b_2, and so on, we may view the variation of a_1 to a_2 as consisting of a_1 variations of $1\left(=\frac{a_1}{a_1}\right)$ to $\frac{a_2}{a_1}$, and the variation of b_1 to b_2 as consisting of b_1 variations of $1\left(=\frac{b_1}{b_1}\right)$ to $\frac{b_2}{b_1}$, and so on with all the other reported single variations. Then the weight of a_1 variations of 1 to $\frac{a_2}{a_1}$ is a_1, and that of b_1 variations of 1 to $\frac{b_2}{b_1}$ is b_1, and similarly in all the other cases; and the rigorous expression for the arithmetic average of these variations, according to § 3, is as above given.

This way of viewing the variations is a perfectly correct way of viewing them, though not the only correct way. In a variation, for instance, of 15 to 20, the variation element is a variation of 1 to 1⅓, or a variation by ⅓. But the nominally single variation of 15 to 20 differs from the variation of 1 to 1⅓ in that it contains fifteen such variations; for $\frac{20}{15} = \frac{1\frac{1}{3} \times 15}{1 \times 15}$, and we have a variation, not of 1 by ⅓, but of 15 by ⅓. Now in a comparison of the arithmetic averages at the two periods (with weighting according to the numbers of figures reported in the classes, their reported sizes also being used) this difference between the variation of 15 to 20 and the variation of 1 to 1⅓—these figures being supposed to appear in two otherwise similar sets—shows itself by the fact that a different result is obtained according as we use the one or the other of these sets, although they contain the same variation elements. But in the arithmetic average of the variations (with weighting likewise according to the numbers of figures and variations reported in the classes) this difference is not allowed for, the variation of 15 to 20 and the variation of 1 to 1⅓ having exactly the same influence upon the result.

The former may, then, be the truer expression even for the average of the variations—and the average of the variations must be adapted to it by using a different weighting—in all those problems in which we wish the variation to count according to the sizes of the figures;[2] but not otherwise.

[2] In general we want the size factor in the weighting of variations in all those cases in which greater effort is needed to produce the same variation in a greater than in a smaller quantity. Also the average of the variations with weighting according to the numbers actually reported in the classes can obviously be correct only in those problems in which it is possible to state the figures only in one way; for otherwise the average of the variations would depend upon the accidental way in which the figures that vary happen to be reported. It may happen even that we wish the variations to count inversely according to the sizes of the figures. Then the weights of the variations would be $\frac{1}{a_1}, \frac{1}{b_1}, \cdots\cdots$, and the formula,

$$V = \frac{\frac{a_2}{a_1^2} + \frac{b_2}{b_1^2} + \cdots\cdots}{\frac{1}{a_1} + \frac{1}{b_1} + \cdots\cdots}.$$

Or if in any cases the weighting should be inversely according to the sizes of the classes, viz., $\frac{1}{xa_1}, \frac{1}{yb_1}, \cdots\cdots$, the formula would be

$$V = \frac{\frac{a_2}{xa_1^2} + \frac{b_2}{yb_1^2} + \cdots\cdots}{\frac{1}{xa_1} + \frac{1}{yb_1} + \cdots\cdots}.$$

$\frac{a_2}{a_1}$, $\frac{a_2}{a_1}$, ······ to x terms, when added together, are $x\frac{a_2}{a_1}$, or when multiplied by one another, are $\left(\frac{a_2}{a_1}\right)^x$.

We may average each set of figures separately. Representing the average of the figures at the first period by $\mathbf{A}_1$, and that of the figures at the second period by $\mathbf{A}_2$, we may represent the variation of these averages by $\frac{\mathbf{A}_2}{\mathbf{A}_1}$, the variation departing from constancy as this expression departs from unity. Or we may average the variations of the figures as they change from what they were at the first period to what they become at the second. Representing this by $\mathbf{V}$, we shall find sometimes that $\frac{\mathbf{A}_2}{\mathbf{A}_1} = \mathbf{V}$, and sometimes that these expressions are different. It is our purpose to study the relationship between these forms.

In averaging, to assign a certain importance to one term—a class of figures, or a kind of variations—relatively to others, is to *weight* it. The most natural way to weight classes of figures is by the numbers of times the figures occur in each class. Similarly it would seem most natural to weight kinds of variations by the numbers of variations in each kind. But as the figures may be reported capriciously, and the steady element be the sizes of the classes, we shall find the need of weighting the variations according to the latter, in several ways.

The number of all the classes, and consequently of all the kinds of variations, will be represented by n, so that $n = 1\left[\text{for } a_1, a_2, \text{ or } \frac{a_2}{a_1}\right] + 1\left[\text{for } b_1, b_2, \text{ or } \frac{b_2}{b_1}\right] +$ ····· through the whole lists. The number of individual figures at each period, and the number of the individual variations, will be represented by n', so that $n' = x + y +$ ····· to n terms. If there is only a single figure in every class, then $n' = n$; otherwise $n' > n$. Thus n' represents the sum of what we have just described as the most natural weights of the classes of figures we are dealing with. The sum of the other weights which we shall need for the variations (and sometimes for the figures) will always be represented by n'', although these weights are different on different occasions.

Before proceeding to our subject proper, it may be well to state certain principles which are common to all kinds of averages.

I. ON AVERAGES IN GENERAL.

§ 1. *An average of any set of terms is always smaller than the largest and larger than the smallest*, whatever be the number of terms. Consequently

§ 2. *If the terms are all equal, the average is equal to each term*, whatever be their number.

These two principles are explicative, that is, they flow from the definition of "average," and no formula pretending to be a formula of a kind of average is so unless it yields results in accordance with these principles. That the formulæ for the arithmetic, harmonic and geometric averages satisfy the second principle has incidentally been shown in Chapter V. Sect. I. §§ 5 and 6.

Consequently also, if we are dealing with sets of terms in classes,

§ 3. *The average is always between the highest and the lowest terms appearing in any of the classes*, whatever be the numbers of terms in any classes; and

§ 4. *If the terms are all equal, but arranged in classes with different numbers in each, the average is equal to each of the figures*, whatever be the numbers of the figures in any of the classes.

A meaning in this last proposition is that the statement is universally true, whatever be the weighting of the classes. In other words, *when all the terms are equal, the weighting is indifferent.*

§ 5. What has been said of averaging any terms is true whether the terms be *figures* (as above described) or *variations* (of figures). For example, applied to variations, the fourth proposition becomes: *If the variations are all equal, the average is equal to each of them, no matter what be the weighting of the variations.*

That the formulæ for the three averages satisfy this principle will be indicated in the course of the separate treatment of them.

§ 6. Furthermore, *if the variations are all equal, the variation of the averages separately drawn of the figures which have varied and of the figures to which they have varied* $\left(\frac{\mathbf{A}_2}{\mathbf{A}_1}\right)$ *is the same as the average of the variations* ($\mathbf{V}$), *itself the same as any one of the variations* $\left(e.\ g.\ \frac{a_2}{a_1}\right)$.

If all the figures in each set are equal, this is self-evident. In other cases a demonstration may be needed for each kind of average. Indications of the demonstration will be added in the treatment of each kind of average. (See below, notes in II. § 6, III. § 6, and V. § 5).

Of course, if the variations are all $\frac{1}{1}$, or 1, that is, if there are no variations, this principle applies, and is now self-evident, whatever the figures. For in this case the corresponding figures in the two sets are identical, the sets themselves are so, and so must be their averages; wherefore there is no variation of the averages.

We shall be interested, in what follows, principally with sets of figures in at least one of which at least one of the figures, and one of the variations, is different from the rest—or rather, with general conditions which admit the possibility of such divergence.

§ 7. Even now there is another case, common to all the kinds of averages, in which the variation of the averages and the average of the variations always agree. If all the figures *at the first period* are units, wherefore the average also is a unit, the figures in the second set themselves express their variations from the first period, since in all the expressions for the variations, $\frac{a_2}{a_1}, \frac{b_2}{b_1}, \ldots\ldots,$ the denominators are 1, and may be dropped. Therefore the expression for the variation of the averages, $\frac{\mathbf{A}_2}{\mathbf{A}_1}$, in which the denominator is also 1, and may be dropped, and in which the numerator is the average of the figures at the second period, is also the expression for the average of the variations. Thus

These three principles can easily be demonstrated in the case of each of the hree kinds of averages. But they are too plain to need demonstration.

A *résumé* of them is that when a figure is the same as the average its reighting is indifferent; otherwise its weighting counts, and an alteration in he weighting of the classes without any variation in the sizes of the figures but *like* a variation in the size of the figures) causes the average to change, he influence of an alteration in the weighting being different according as the gure operated on is larger or smaller than the average. To increase the epetitions of a larger figure has the same influence as to enlarge the figure; nd to increase the repetitions of a smaller figure has the same influence as o diminish the figure; and conversely.

A practical application of these three principles is that an error in our reighting of a figure equal to the average is of no account (*i. e.*, if the averge without a figure is the same as the average with it, we need not concern urselves about its weight). And the nearer a figure is to the average, the ess an error in its weighting will count; and the more it will count, the more he figure is removed from the average. Of course in the case of an average f variations, what is here said applies to a variation according as it is the ame as, near to, or far from, the average of the variations.

Hence lastly,

§ 11. *If all the weights are altered in the same proportion, there is no effect upon he average*, or, in other words, if we have the proper weights, we may alter hem as we please, so long as we keep them in the same proportion (multilying or dividing them all by the same quantity).

For if we increase all the weights in the same proportion, the increase of he weights of the figures equal to the average has no influence, and the inrease of the weights of the figures larger than the average tends to raise the verage, while the increase of the weights of the figures smaller than the verage tends to lower it; but as the influence of all the figures below the verage to lower it is equal to the influence of all the figures above the averge to raise it (for otherwise the average would not be where it is), so the influence of the changes in the numbers of the former is equal to the opposite nfluence of the proportionally equal changes in the numbers of the latter, nd the average remains where it was. And reversely if we decrease all the reights in the same proportion.

If this general explanation be not sufficient, the proposition may be demonstrated in the case of each of the averages separately. This will be done ncidentally for two of them in the following pages. (See I. § 7, II. § 7).

Or another general proof may be made as follows. If, for instance, we louble all the weights, we may segregate all the new terms, and so form two listinct sets of figures (or variations) exactly duplicating each other, one of hem being the original set. Then the average of each set, separately drawn, rill be the same. Consequently the one average of the two together will be he same. And similarly, whatever be the multiplier.

Thus the general system of weighing depends upon the relative sizes of he weights, and not upon their absolute sizes. However large or small the reights, if in the same proportion, we practically have the same weighting.

This being so, we have *even weighting* whether we count every figure in the ets (or every variation) only once, or an equal number of times.

§ 4. Given the same classes of figures, the similarly weighted arithmetic erage of the variations in full is

$$V \quad \frac{1}{n'}\left\{\left(\frac{a_2}{a_1}+\frac{a_2}{a_1}+ \cdots\cdots \text{ to } x \text{ terms}\right)+\left(\frac{b_2}{b_1}+\frac{b_2}{b_1}+\cdots\cdots \text{ to } y \text{ terms}\right)+ \cdots\cdots \text{ to } n \text{ classes}\right\},$$

iich may be abbreviated to $\frac{1}{n'}\left(x\frac{a_2}{a_1}+y\frac{b_2}{b_1}+\cdots\cdots \text{ to } n \text{ terms}\right)$, or may be itten

$$V \quad \frac{x\frac{a_2}{a_1}+y\frac{b_2}{b_1}+\cdots\cdots}{x+y+\cdots\cdots}.$$

Again this expression is different, not universally reducing to the preceding.

§ 5. When dealing with single figures, we may say that we are dealing th classes in each of which there is only one individual. Therefore we may sume sets of single figures (or variations) under sets of classes of figures r variations), and treat only of the latter. Thus in both the preceding cases have been dealing with arithmetic averages of classes in which the weight- g is according to the numbers of figures in the classes, and of averages of riations with weighting likewise according to the numbers of varying figures the classes.

The conclusion from the preceding paragraphs then is that *the variation of arithmetic averages and the arithmetic average of the variations, in all cases with ghting according to the numbers of figures in the classes, are not universally the ne.*

§ 6. *The variation of the arithmetic averages and the arithmetic average of the iations, all with weighting according to the numbers of figures in the classes, are same when all the figures at the first period are equal.*

Let $a_1 = b_1 = \cdots\cdots = s$ (s being *any* figure above, below, or at unity, in- gral or fractional). Then the expression for the comparison of the averages at for the sums) in § 3 reduces to $\frac{xa_2 + yb_2 + \cdots\cdots}{s(x+y+\cdots\cdots)}$; and the expression the average of the variations in § 4 reduces to the same. Q. E. D.

Hence if we reduce all the actual figures in an irregular set at the first riod to the same ideal figure (by taking some common divisor of them) lucing the figures at the second period in the same proportions, and if we ange the numbers of the figures in the classes in the inverse proportions (so to keep unchanged the sizes of the classes), the weighting desired in order make the two methods agree will be in accordance with the adjusted num- rs of these ideal individuals in the classes.

Thus a factor which affects the variation of the averages is the number of nes the figures occur in each class, or weighting of the variations directly cording to the numbers of the figures.

In all other cases (except of course when all the variations are equal— I. § 6[1]) the average of the variations will require different weighting of variations to make it agree with the variation of the averages.

[1] In the formula above given in § 4 let $\frac{a_2}{a_1} = \frac{b_2}{b_1} = \cdots\cdots = r$, and the expression luces to r. Now $a_2 = ra_1$, $y_2 = rb_1$, and so on. Introduce these into the for- la in § 3, and this also reduces to r.

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Then the weight of a_1 variations of 1 to $\frac{a_2}{a_1}$ is a_1, and that of b_1 varia-[tions] of 1 to $\frac{b_2}{b_1}$ is b_1, and similarly in all the other cases; and the rigorous [ex]pression for the arithmetic average of these variations, according to § 3, [is] as above given.

This way of viewing the variations is a perfectly correct way of viewing [the]m, though not the only correct way. In a variation, for instance, of 15 [to] 20, the variation element is a variation of 1 to $1\frac{1}{3}$, or a variation by $\frac{1}{3}$. [Bu]t the nominally single variation of 15 to 20 differs from the variation of 1 [to] $1\frac{1}{3}$ in that it contains fifteen such variations; for $\frac{20}{15} = \frac{1\frac{1}{3} \times 15}{1 \times 15}$, and we [ha]ve a variation, not of 1 by $\frac{1}{3}$, but of 15 by $\frac{1}{3}$. Now in a comparison of the [arit]hmetic averages at the two periods (with weighting according to the num[ber]s of figures reported in the classes, their reported sizes also being used) this [di]fference between the variation of 15 to 20 and the variation of 1 to $1\frac{1}{3}$—[the]se figures being supposed to appear in two otherwise similar sets—shows [it]self by the fact that a different result is obtained according as we use the one [or] the other of these sets, although they contain the same variation elements. [Bu]t in the arithmetic average of the variations (with weighting likewise ac[co]rding to the numbers of figures and variations reported in the classes) this [di]fference is not allowed for, the variation of 15 to 20 and the variation of 1 [to] $1\frac{1}{3}$ having exactly the same influence upon the result.

The former may, then, be the truer expression even for the average of the [v]ariations—and the average of the variations must be adapted to it by using [a] different weighting—in all those problems in which we wish the variation [to] count according to the sizes of the figures;[2] but not otherwise.

[2] In general we want the size factor in the weighting of variations in all those [ca]ses in which greater effort is needed to produce the same variation in a greater [th]an in a smaller quantity. Also the average of the variations with weighting according to the numbers actually reported in the classes can obviously be correct only in those problems in which it is possible to state the figures only in one way; for otherwise the average of the variations would depend upon the accidental way in which the figures that vary happen to be reported. It may happen even that we wish the variations to count inversely according to the sizes of the figures. Then the weights of the variations would be $\frac{1}{a_1}, \frac{1}{b_1}, \cdots\cdots$, and the formula,

$$V = \frac{\frac{a_2}{a_1^2} + \frac{b_2}{b_1^2} + \cdots\cdots}{\frac{1}{a_1} + \frac{1}{b_1} + \cdots\cdots}.$$

Or if in any cases the weighting should be inversely according to the sizes of the classes, viz., $\frac{1}{xa_1}, \frac{1}{yb_1}, \cdots\cdots$, the formula would be

$$V = \frac{\frac{a_2}{xa_1^2} + \frac{b_2}{yb_1^2} + \cdots\cdots}{\frac{1}{xa_1} + \frac{1}{yb_1} + \cdots\cdots}.$$

We have discovered two factors in the weighting of the variations—the numbers of the figures employed, and their sizes at the first period. We have discovered each of these upon eliminating the other. We must now unite the two. As they both act directly, their influences work together and strengthen each other. Therefore, in all cases,

§ 8. *The variation of the arithmetic averages of classes of figures, each of these averages with weighting according to the numbers of figures in the classes, is the same as the arithmetic average of the variations with weighting according to the sizes of the classes at the first period.*

The expression for the average of the variations with weighting according to xa_1, yb_1, ······, is $\dfrac{xa_1\frac{a_2}{a_1} + yb_1\frac{b_2}{b_1} + \cdots\cdots}{xa_1 + yb_1 + \cdots\cdots}$, which reduces to $\dfrac{xa_2 + yb_2 + \cdots\cdots}{xa_1 + yb_1 + \cdots\cdots}$, which is the expression for the variation of the averages. Q. E. D.

In any such expression as the last we should notice that the mathematical terms are not x and y, ······, a_1 and b_1, ······, but xa_1, yb_1, ······. Therefore in the denominator the one term xa_1 may be replaced by $\mathbf{a}_1$, the one term yb_1 by $\mathbf{b}_1$, and so on. Then in the numerator the terms become $\mathbf{a}_1\frac{a_2}{a_1}$, $\mathbf{b}_1\frac{b_2}{b_1}$, and so on. The expression then falls under § 7.

It may be remarked that the method of averaging the variations has this advantage, that it tells us what we are doing, while the method of comparing the averages hides this. We have now discovered that when we compare the arithmetic averages of the figures which have varied and of the figures to which these have varied, we are virtually averaging the variations with weighting according to the total sizes of the classes at the first period. Hence we may view this weighting of the variations as hidden or latent in the method of comparing the averages.

From this follows a simple corollary:

§ 9. *If the sizes of the classes at the first period are all equal* (i. e., $a_1 = b_1 = \cdots\cdots$), *the variation of the arithmetic averages, each with weighting according to the numbers of figures in the classes, is the same as the arithmetic average of the variations with even weighting.*

This condition is brought about when the numbers of the figures in the classes are in inverse *proportion* to the sizes of the figures at the first period.

III. HARMONIC AVERAGING.

§ 1. With single figures in the two sets, the harmonic average at the first period is $\dfrac{1}{\frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{b_1} + \cdots\cdots\right)}$: and that at the second is like unto it. Therefore the variation of the averages is

$$\frac{\mathbf{A}_2}{\mathbf{A}_1} = \frac{\dfrac{1}{\frac{1}{n}\left(\frac{1}{a_2} + \frac{1}{b_2} + \cdots\cdots\right)}}{\dfrac{1}{\frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{b_1} + \cdots\cdots\right)}},$$

which reduces to

$$\frac{\mathbf{A}_2}{\mathbf{A}_1}=\frac{\frac{1}{n}\left(\frac{1}{a_1}+\frac{1}{b_1}+\cdots\cdots\right)}{\frac{1}{n}\left(\frac{1}{a_2}+\frac{1}{b_2}+\cdots\cdots\right)},$$

and to

$$\frac{\mathbf{A}_2}{\mathbf{A}_1}=\frac{\frac{1}{a_1}+\frac{1}{b_1}+\cdots\cdots}{\frac{1}{a_2}+\frac{1}{b_2}+\cdots\cdots},$$

(which shows that it is indifferent, in comparing harmonic averages, whether we inversely compare the arithmetic averages of the reciprocals of the figures or the sums of the reciprocals of the figures).

Here the weighting of the figures in each average is even, each figure counting once.

§ 2. Given the same single figures, the harmonic average of the variations, likewise with even weighting, each variation counting once, is

$$\mathbf{V}=\frac{1}{\frac{1}{n}\left(\frac{\frac{1}{a_2}}{\frac{1}{a_1}}+\frac{\frac{1}{b_2}}{\frac{1}{b_1}}+\cdots\cdots\right)},$$

which reduces to

$$\mathbf{V}=\frac{1}{\frac{1}{n}\left(\frac{a_1}{a_2}+\frac{b_1}{b_2}+\cdots\cdots\right)}.$$

This is a different expression, not universally reducing to the preceding.

§ 3. With classes of figures, the harmonic average at the first period may be expressed in full thus,

$$\mathbf{A}_1=\frac{1}{\frac{1}{n'}\left\{\left(\frac{1}{a_1}+\frac{1}{a_1}+\cdots\cdots \text{ to } x \text{ terms}\right)+\left(\frac{1}{b_1}+\frac{1}{b_1}+\cdots\cdots \text{ to } y \text{ terms}\right)+\cdots\cdots \text{ to } n \text{ classes}\right\}},$$

which may be abbreviated to $\frac{1}{\frac{1}{n'}\left(\frac{x}{a_1}+\frac{y}{b_1}+\cdots\cdots \text{ to } n \text{ terms}\right)}$, or may also be written $\frac{x+y+\cdots\cdots}{\frac{x}{a_1}+\frac{y}{b_1}+\cdots\cdots}$. And the harmonic average at the second period is like unto it. Therefore the variation of the averages, after reductions similar to those used in § 1, is

$$\frac{\mathbf{A}_2}{\mathbf{A}_1}=\frac{\frac{x}{a_1}+\frac{y}{b_1}+\cdots\cdots}{\frac{x}{a_2}+\frac{y}{b_2}+\cdots\cdots},$$

Here, too, the weighting of the figures in each average is directly according to the numbers of times the figures are repeated in each class. No other way of averaging the sets would average the figures supposed.

[illegible] universally reducing to the p[illegible]

[illegible] figures [illegible] and [illegible]

[illegible] harmonic average [illegible] the [illegible] numbers of figures in the classes, [illegible]

[illegible] harmonic average [illegible] numbers of figures in the classes, [illegible] are equal.

[illegible] the expression for the comparison of [illegible]

[illegible] the expression for [illegible]

[illegible] the same. Q. E. D.

[illegible] the variation of the harmonic averages [illegible] the variation of the arithmetic averages, namely [illegible] figures are [illegible] in each class, or weighting of [illegible] the numbers of the figures.

[illegible] and the variations are equal [illegible] variations will require different weighti[illegible] agree with the variations of the averages.

[illegible] are all equal, the variation[illegible] ages in § 3 reduces to $\dfrac{t\left(\frac{1}{a_1}+\frac{1}{b_1}+\cdots\cdots\right)}{t\left(\frac{1}{a_2}+\frac{1}{b_2}+\cdots\cdots\right)}$, and this, by dropping the t from both sides of the fraction, to the expression for the variation of the averages of the figures taken singly, *i. e.*, with even weighting. We want now to prove that this expression is the same as the expression for the average of the variations with weighting according to $\frac{1}{a_1}, \frac{1}{b_1}, \cdots\cdots$.

The expression for the average of the variations so weighted is $\dfrac{1}{\frac{1}{n''}\left(\frac{1}{a_1}\cdot\frac{a_1}{a_2}+\frac{1}{b_1}\cdot\frac{b_1}{b_2}+\cdots\cdots \text{ to } n \text{ terms}\right)}$, in which $n''=\frac{1}{a_1}+\frac{1}{b_1}+\cdots\cdots$ to n terms; wherefore this expression, by reducing, restoring the value of n'', and converting, becomes $\dfrac{\frac{1}{a_1}+\frac{1}{b_1}+\cdots\cdots}{\frac{1}{a_2}+\frac{1}{b_2}+\cdots\cdots}$, which is the same as the preceding. Q. E. D.

Thus here, too, a factor which affects the variation of the averages is the size of the figures in the classes; only here the weighting of the variations is *inversely* according to the sizes of the figures.

This is the opposite of what was the case in the variation of the arithmetic averages. There the larger a figure, the more its variation counts. Here the larger a figure, the less its variation counts.

Of course in the harmonic average, as in all averages, the larger a figure, the larger is the average of the set of figures in which this figure occurs; and similarly, the larger a variation, the larger is the average of the set of variations in which this variation occurs. The proposition before us is that the larger the figure which varies, the smaller is the influence of its variation upon the variation of the averages. The reason of this is because the harmonic average is the reciprocal of the arithmetic average of the reciprocals; but the larger a figure, the smaller is its reciprocal.

We have, then, discovered two factors in the weighting of the variations —the numbers of figures, and their sizes at the first period. But as the one of these acts directly and the other inversely, their combined influence is the balance left over as the one neutralizes the other. Therefore, taking both into account, in all cases,

§ 8. *The variation of the harmonic averages of classes of figures, each of these averages with weighting directly according to the numbers of the figures in the classes, is the same as the harmonic average of the variations with weights which are the ratios of the numbers of figures in the classes to the sizes of these figures at the first period.*

The expression for the average of the variations with weighting according to $\frac{x}{a_1}, \frac{y}{b_1}, \cdots\cdots$, is $\dfrac{\frac{x}{a_1}+\frac{y}{b_1}+\cdots\cdots}{\frac{x}{a_1}\cdot\frac{a_1}{a_2}+\frac{y}{b_1}\cdot\frac{b_1}{b_2}+\cdots\cdots}$, which reduces to $\dfrac{\frac{x}{a_1}+\frac{y}{b_1}+\cdots\cdots}{\frac{x}{a_2}+\frac{y}{b_2}+\cdots\cdots}$, which is the expression for the variation of the averages. Q. E. D.

From this also follows the simple corollary:

§ 9. *If the ratios of the numbers of figures in the classes to the sizes of the figures at the first period are all equal* $\left(i.\ e.\ \frac{x}{a_1} = \frac{y}{b_1} = \cdots\cdots\right)$, *the variation of the harmonic averages, each with weighting according to the numbers of figures in the classes, is the same as the harmonic average of the variations with even weighting.*

This condition is brought about when the numbers of the figures in the classes are in *direct proportion* to the sizes of the figures at the first period.

§ 10. A remark deserves to be added.

We have seen that in the arithmetic averaging the expression for the variation of the averages was in some cases truer than the expression for the average of the variations (unless this has its weighting specially adapted), because it gives weight to the figures according to their numbers and sizes in the denominator. Now in the same case we might here still want the figures to count directly according to their sizes as well as directly according to their numbers—we might want the variation of a larger number to count more instead of less. Therefore the comparison of the harmonic averages would not be serviceable for this purpose, and in order to carry it out we must insert the desired weighting in the expression for the harmonic average of the variations, thus,

$$V = \frac{1}{\frac{1}{n''}\left(xa_1\frac{a_1}{a_2} + yb_1\frac{b_1}{b_2} + \cdots\cdots \text{ to } n \text{ terms}\right)},$$

which reduces to

$$V = \frac{xa_1 + yb_1 + \cdots\cdots}{x\frac{a_1^2}{a_2} + y\frac{b_1^2}{b_2} + \cdots\cdots},$$

or, if we represent xa_1 by **a**, yb_1 by **b**, and so on, (or supposing the weights are sometimes wanted to be something else, still letting **a** and **b** $\cdots\cdots$ represent them) we should have

$$V = \frac{1}{\frac{1}{n''}\left(\mathbf{a}\frac{a_1}{a_2} + \mathbf{b}\frac{b_1}{b_2} + \cdots\cdots \text{ to } n \text{ terms}\right)},$$

in which $n'' = \mathbf{a} + \mathbf{b} + \cdots\cdots$ to n terms. The expression for the variation of the averages, however, is serviceable on condition that the figures at the first period are reduced to equality; for then the numbers of times they are repeated in their classes are proportional to **a**, **b**, $\cdots\cdots$.

IV. CASES OF AGREEMENT BETWEEN THE ARITHMETIC AND THE HARMONIC AVERAGES OF VARIATIONS.

We have noticed in the comparison of the arithmetic averages what happens when $a_1 = b_1 = \cdots\cdots$, and when $xa_1 = yb_1 = \cdots\cdots$, and in the comparison of the harmonic averages what happens also when $a_1 = b_1 = \cdots\cdots$, and when $\frac{x}{a_1} = \frac{y}{b_1} = \cdots\cdots$. There remains to see what happens, in the former case, when $a_2 = b_2 = \cdots\cdots$, and when $xa_2 = yb_2 = \cdots\cdots$, and in the latter, when $a_2 = b_2 = \cdots\cdots$, and when $\frac{x}{a_2} = \frac{y}{b_2} = \cdots\cdots$. A few other coincidences will also call for attention.

The variation of the arithmetic averages being expressed thus,

$$\frac{A_2}{A_1} = \frac{xa_2 + yb_2 + \cdots\cdots}{xa_1 + yb_1 + \cdots\cdots},$$

ow this to be the same as the arithemetic average of the variations with ting according to xa_1, yb_1, $\cdots\cdots$.

$a_2 = b_2 = \cdots\cdots = s$. Then $xa_1 = xa_2\frac{a_1}{a_2} = xs\frac{a_1}{a_2}$, and similarly $yb_1 =$ and so on. Therefore this expression becomes $\frac{xs + ys + \cdots\cdots}{xs\frac{a_1}{a_2} + ys\frac{b_1}{b_2} + \cdots\cdots}$,

reduces to $\frac{x + y + \cdots\cdots}{x\frac{a_1}{a_2} + y\frac{b_1}{b_2} + \cdots\cdots}$, which we recognize as the expression e harmonic average of the variations with weighting according to $\cdots\cdots$. Therefore,

it happens that the figures at the second period are all equal, the arithmetic of the variations with weighting according to the sizes of the classes at the first is the same as the harmonic average of the variations with weighting according numbers of figures in the classes.

Let $xa_2 = yb_2 = \cdots\cdots = t$. Then $xa_1 = xa_2\frac{a_1}{a_2} = t\frac{a_1}{a_2}$, and similarly yb_1 nd so on. Therefore the above expression becomes $\frac{t + t + \cdots\cdots}{t\frac{a_1}{a_2} + t\frac{b_1}{b_2} + \cdots\cdots}$,

reduces to $\frac{1}{\frac{1}{n}\left(\frac{a_1}{a_2} + \frac{b_1}{b_2} + \cdots\cdots\right)}$, which we recognize as the expression harmonic average of the variations with even weighting. Therefore, *it happens that the sizes of the classes at the second period are all equal, the etic average of the variations with weighting according to the sizes of the classes first period is the same as the harmonic average of the variations with even ng.*

is theorem may be extended, as follows:

The harmonic average of the variations with weighting according to mbers of figures in the classes being this, $\frac{x + y + \cdots\cdots}{x\frac{a_1}{a_2} + y\frac{b_1}{b_2} + \cdots\cdots}$, suppose nstead of this weighting we use weighting according to the sizes of the at the second period. Then we must substitute xa_2 for x, yb_2 for y, and and we have $\frac{xa_2 + yb_2 + \cdots\cdots}{xa_2\frac{a_1}{a_2} + yb_2\frac{b_1}{b_2} + \cdots\cdots}$, which reduces to $\frac{xa_2 + yb_2 + \cdots\cdots}{xa_1 + yb_1 + \cdots\cdots}$, we recognize as the expression for the variation of the aritmetic average ith weighting according to x, y, $\cdots\cdots$, and which we know to be the same arithmetic average of the variations with weighting according to $_1 \cdots\cdots$. Therefore, in all cases,

arithmetic average of any variations with weighting according to the sizes of the at the first period is the same as the harmonic average of the variations with ng according to the sizes of the classes at the second period.

to $\dfrac{\dfrac{x}{a_1}+\dfrac{y}{b_1}+\cdots\cdots}{\dfrac{x}{a_2}+\dfrac{y}{b_2}+\cdots\cdots}$, which we recog-
of the harmonic averages with weight-
we know to be the same as the har-
weighting according to $\dfrac{x}{a_1}, \dfrac{y}{b_1}, \cdots\cdots$.

with weighting according to the ratios of size of the figures at the first period is the ations with weighting according to the ratios at the second period.

the variations.

; and the harmonic average of the varia-

to $x, y, \cdots\cdots$ becomes $\dfrac{\dfrac{a_2}{a_1}+\dfrac{b_2}{b_1}+\cdots\cdots}{\dfrac{a_2}{a_1}\cdot\dfrac{a_1}{a_2}+\dfrac{b_2}{b_1}\cdot\dfrac{b_1}{b_2}+\cdots\cdots}$,

$\cdots\cdots)$, which we recognize as the arithmetic with even weighting. Therefore,

of figures in every class equals the variation, the with even weighting is the same as the harmonic weighting according to the numbers.

equal the reciprocals of the variations.

and so on; and the arithmetic average of the varia-

according to $x, y, \cdots\cdots$ becomes $\dfrac{\dfrac{a_1}{a_2}\cdot\dfrac{a_2}{a_1}+\dfrac{b_1}{b_2}\cdot\dfrac{b_2}{b_1}+\cdots\cdots}{\dfrac{a_1}{a_2}+\dfrac{b_1}{b_2}+\cdots\cdots}$,

$\dfrac{1}{\frac{1}{n}\left(\frac{a_1}{a_2}+\frac{b_1}{b_2}+\cdots\right)}$, which we recognize as the harmonic variations with even weighting. Therefore,

that the number of figures in every class equals the reciprocal of the harmonic average of the variations with even weighting is the same as average of the variations with weighting according to the numbers.[1]

aside from our subject, yet as throwing light upon it, the following theorems may be added:

The arithmetic average of any quantities is the harmonic average of weighting directly according to their sizes.

quantities $a, b, \cdots\cdots, k$, the harmonic average with weights according to their sizes is $\dfrac{a+b+\cdots\cdots+k}{a\frac{1}{a}+b\frac{1}{b}+\cdots\cdots+k\frac{1}{k}}$, which reduces to

$\cdots\cdots + k$), which is their simple arithmetic average. Q. E. D.

§ 4. The variation of the harmonic averages being expressed thus,

$$\frac{\mathbf{A}_2}{\mathbf{A}_1}=\frac{\frac{x}{a_1}+\frac{y}{b_1}+\cdots\cdots}{\frac{x}{a_2}+\frac{y}{b_2}+\cdots\cdots},$$

we know this to be the same as the harmonic average of the variations with weighting according to the ratios $\frac{x}{a_1}, \frac{y}{b_1}, \cdots\cdots$.

Let $a_2=b_2=\cdots\cdots=s$. Then $a_1=s\frac{a_1}{a_2}$ and $\frac{x}{a_1}=\frac{xa_2}{sa_1}$, and similarly $\frac{y}{b_1}=\frac{yb_2}{sb_1}$, and so on. Therefore this expression becomes $\frac{\frac{xa_2}{sa_1}+\frac{yb_2}{sb_1}+\cdots\cdots}{\frac{x}{s}+\frac{y}{s}+\cdots\cdots}$, which reduces to $\frac{x\frac{a_2}{a_1}+y\frac{b_2}{b_1}+\cdots\cdots}{x+y+\cdots\cdots}$, which we recognize as the expression for the arithmetic average of the variations with weighting according to $x, y, \cdots\cdots$. Therefore,

If it happens that the figures at the second period are all equal, the harmonic average of the variations with weighting according to the ratios of the numbers of figures in the classes to the sizes of the figures at the first period is the same as the arithmetic average of the variations with weighting according to the numbers of figures in the classes.

§ 5. Let $\frac{x}{a_2}=\frac{y}{b_2}=\cdots\cdots=r$. Then $x=ra_2$, and $\frac{x}{a_1}=\frac{ra_2}{a_1}$; and similarly $\frac{y}{b_1}=\frac{rb_2}{b_1}$, and so on. Therefore the above expression becomes $\frac{r\frac{a_2}{a_1}+r\frac{b_2}{b_1}+\cdots\cdots}{r+r+\cdots\cdots}$, which reduces to $\frac{1}{n}\left(\frac{a_2}{a_1}+\frac{b_2}{b_1}+\cdots\cdots\right)$, which we recognize as the expression for the arithmetic average of the variations with even weighting. Therefore,

If it happens that the ratios of the numbers of figures in the classes to the sizes of the figures at the second period are all equal, the harmonic average of the variations with weighting according to the ratios of these numbers to the sizes of the figures at the first period is the same as the arithmetic average of the variations with even weighting.

This theorem also may be extended, as follows:

§ 6. The arithmetic average of the variations with weighting according to the numbers of figures in the classes being this, $\frac{x\frac{a_2}{a_1}+y\frac{b_2}{b_1}+\cdots\cdots}{x+y+\cdots\cdots}$, suppose that instead of this weighting we use weighting according to the ratios of the numbers of figures in the classes to the sizes of the figures at the second period. Then we must substitute $\frac{x}{a_2}$ for x, $\frac{y}{b_2}$ for y, and so on, and we have

$$\frac{\frac{x}{a_2}\cdot\frac{a_2}{a_1}+\frac{y}{b_2}\cdot\frac{b_2}{b_1}+\cdots\cdots}{\frac{x}{a_2}+\frac{y}{b_2}+\cdots\cdots}$$, which reduces to $$\frac{\frac{x}{a_1}+\frac{y}{b_1}+\cdots\cdots}{\frac{x}{a_2}+\frac{y}{b_2}+\cdots\cdots}$$, which we recognize as the expression for the variation of the harmonic averages with weighting according to x, y, ……, and which we know to be the same as the harmonic average of the variations with weighting according to $\frac{x}{a_1}$, $\frac{y}{b_1}$, …… . Therefore, in all cases,

The harmonic average of the variations with weighting according to the ratios of the numbers of figures in the classes to the sizes of the figures at the first period is the same as the arithmetic average of the variations with weighting according to the ratios of these numbers to the sizes of the figures at the second period.

§ 7. Suppose the numbers equal the variations.

Then $x=\frac{a_2}{a_1}$, $y=\frac{b_2}{b_1}$, and so on; and the harmonic average of the variations with weighting according to x, y, …… becomes $$\frac{\frac{a_2}{a_1}+\frac{b_2}{b_1}+\cdots\cdots}{\frac{a_2}{a_1}\cdot\frac{a_1}{a_2}+\frac{b_2}{b_1}\cdot\frac{b_1}{b_2}+\cdots\cdots},$$ which reduces to $\frac{1}{n}\left(\frac{a_2}{a_1}+\frac{b_2}{b_1}+\cdots\cdots\right)$, which we recognize as the arithmetic average of the variations with even weighting. Therefore,

If it happens that the number of figures in every class equals the variation, the arithmetic average of the variations with even weighting is the same as the harmonic average of the variations with weighting according to the numbers.

§ 8. Suppose the numbers equal the reciprocals of the variations.

Then $x=\frac{a_1}{a_2}$, $y=\frac{b_1}{b_2}$, and so on; and the arithmetic average of the variations with weighting according to x, y, …… becomes $$\frac{\frac{a_1}{a_2}\cdot\frac{a_2}{a_1}+\frac{b_1}{b_2}\cdot\frac{b_2}{b_1}+\cdots\cdots}{\frac{a_1}{a_2}+\frac{b_1}{b_2}+\cdots\cdots},$$ which reduces to $\frac{1}{\frac{1}{n}\left(\frac{a_1}{a_2}+\frac{b_1}{b_2}+\quad\right)}$, which we recognize as the harmonic average of the variations with even weighting. Therefore,

If it happens that the number of figures in every class equals the reciprocal of the variation, the harmonic average of the variations with even weighting is the same as the arithmetic average of the variations with weighting according to the numbers.[1]

[1] Although aside from our subject, yet as throwing light upon it, the following two theorems may be added:

The simple arithmetic average of any quantities is the harmonic average of them with weighting directly according to their sizes.

Of the quantities a, b, ……, k, the harmonic average with weights directly according to their sizes is $\frac{a+b+\cdots\cdots+k}{a\frac{1}{a}+b\frac{1}{b}+\cdots\cdots+k\frac{1}{k}}$, which reduces to $\frac{1}{n}(a+b+\cdots\cdots+k)$, which is their simple arithmetic average. Q. E. D.

V. GEOMETRIC AVERAGING.

§ 1. With single figures in the two sets, the geometric average at the first period is $\sqrt[n]{a_1 \cdot b_1 \cdots\cdots \text{ to } n \text{ terms}}$, and that at the second is like unto it. Therefore the variation of the averages is

$$\frac{A_2}{A_1} = \frac{\sqrt[n]{a_2 \cdot b_2 \cdots\cdots}}{\sqrt[n]{a_1 \cdot b_1 \cdots\cdots}}.$$

(Here it may incidentally be remarked that the comparison of these averages cannot be replaced by comparison of the products except when the result is unity; for the products are in a ratio the nth power of the ratio between the averages.)

Here the weighting of the figures in each average is even, each figure counting once.

§ 2. Given the same single figures, the geometric average of the variations, likewise with even weighting, each variation counting once, is

$$V = \sqrt[n]{\frac{a_2}{a_1} \cdot \frac{b_2}{b_1} \cdots\cdots}.$$

This is an expression which universally has the same value as the preceding.

§ 3. With classes of figures, the geometric average at the first period may be expressed in full thus,

$$A_1 = \sqrt[n']{(a_1 \cdot a_1 \cdots\cdots \text{ to } x \text{ terms}) \cdot (b_1 \cdot b_1 \cdots\cdots \text{ to } y \text{ terms}) \cdots\cdots \text{ to } n \text{ classes}},$$

which may be abbreviated to $\sqrt[n']{a_1^x \cdot b_1^y \cdots\cdots \text{ to } n \text{ terms}}$. And, the average at the second period being similar, the variation of the averages is

$$\frac{A_2}{A_1} = \frac{\sqrt[n']{a_2^x \cdot b_2^y \cdots\cdots \text{ to } n \text{ terms}}}{\sqrt[n']{a_1^x \cdot b_1^y \cdots\cdots \text{ to } n \text{ terms}}}.$$

Here the weighting of the figures in each average is directly according to the numbers of times the figures occur in each class. No other way of averaging the sets would average the figures supposed.

§ 4. Given the same classes of figures, the similarly weighted geometric average of the variations in full is

The simple harmonic average of any quantities is the arithmetic average of them with weighting inversely according to their sizes.

Let k be the largest of these quantities. Then the weights of these quantities according to the inverse of their sizes are $\frac{k}{a}$ for a, $\frac{k}{b}$ for b, ······, and $\frac{k}{k}$ for k. The arithmetic average of these quantities so weighted is

$$\frac{1}{\frac{k}{a} + \frac{k}{b} + \cdots\cdots + \frac{k}{k}} \left(\frac{k}{a} \cdot a + \frac{k}{b} \cdot b + \cdots\cdots + \frac{k}{k} \cdot k \right)$$

which reduces to $\frac{1}{\frac{1}{n}\left(\frac{1}{a} + \frac{1}{b} + \cdots\cdots + \frac{1}{r}\right)}$ which ...

age. Q. E. D.

$$\mathbf{V}=\sqrt[n']{\left(\frac{a_2}{a_1}\cdot\frac{a_2}{a_1}\cdot\cdots\cdots \text{ to } x \text{ terms}\right)\cdot\left(\frac{b_2}{b_1}\cdot\frac{b_2}{b_1}\cdot\cdots\cdots \text{ to } y \text{ terms}\right)\cdot\cdots\cdots \text{ to } n \text{ classes}},$$

which may be abbreviated to

$$\mathbf{V}=\sqrt[n']{\left(\frac{a_2}{a_1}\right)^x\cdot\left(\frac{b_2}{b_1}\right)^y\cdot\cdots\cdots \text{ to } n \text{ terms}}.$$

Again this is an expression which universally has the same value as the preceding.

§ 5. Here also subsuming single figures under classes, we see that

The variation of the geometric averages and the geometric average of the variations, in all cases with weighting according to the numbers of figures in the classes, are the same.

We see also that this agreement will universally take place with any weighting whatsoever, provided it be the same in all the three averagings.

Thus, unlike the other two kinds of averaging, in the geometric averaging for this agreement to take place it is not necessary that $a_1 = b_1 = \cdots\cdots$, or that $\frac{a_2}{a_1} = \frac{b_2}{b_1} = \cdots\cdots$;[1] since the agreement takes place not only in these but in all cases.

§ 6. If we employed here the weighting which we found necessary, except in the above two cases, for the arithmetic average of the variations in order that it should agree with the variation of the arithmetic averages, namely, according to xa_1, yb_1, $\cdots\cdots$, which would be in this form,

$$\sqrt[n'']{\left(\frac{a_2}{a_1}\right)^{xa_1}\cdot\left(\frac{b_2}{b_1}\right)^{yb_1}\cdot\cdots\cdots \text{ to } n \text{ terms}},$$

in which $n'' = xa_1 + yb_1 + \cdots\cdots$ to n terms; or if we employed here the similarly necessary weighting in the harmonic averaging, namely $\frac{x}{a_1}$, $\frac{y}{b_1}$, $\cdots\cdots$, which would be in this form,

$$\sqrt[n'']{\left(\frac{a_2}{a_1}\right)^{\frac{x}{a_1}}\cdot\left(\frac{b_2}{b_1}\right)^{\frac{y}{b_1}}\cdot\cdots\cdots \text{ to } n \text{ terms}},$$

in which $n'' = \frac{x}{a_1} + \frac{y}{b_1} + \cdots\cdots$ to n terms; either of these expressions for the average of the variations would agree with the expression for the variation of the averages (apart from the case when all the variations are alike) only in a particular case, namely if the figures are all the same at the first period (*i. e.*, $a_1 = b_1 = \cdots\cdots$)—the very condition previously found necessary to make averages of the variations with weighting like the weighting in the separate averages of the figures agree with the variations of these averages.

[1] That in this particular case the results are always the same as any one of the variations, whatever be the weighting employed (if only it be the same in the two averages of the figures that are compared), may easily be seen by letting $\frac{a_2}{a_1} = \frac{b_2}{b_1} = \cdots\cdots = r$, and by introducing r into the above expressions with *any* weights. These expressions then always reduce to r.

Therefore if here, with uneven variations, the figures being unequal at the first period, we reduce these to equality and adapt their numbers accordingly, and employ weighting according to the adapted numbers of figures in the classes, we shall get a result for the average of the variations thus weighted (really according to xa_1, yb_1, $\cdots\cdots$) different from that of the variation of the averages (in which the weighting is according to x, y, $\cdots\cdots$).

§ 7. The reason for the universal agreement in this case is that here the sizes of the figures at the first period, in the comparison of the averages, is offset by their sizes at the second period, and do not affect the result. Hence there is here no weighting in the comparison of the averages except the numbers of times the figures occur in the classes. There is no hidden weighting.

Therefore if we wish to employ weighting which also allows for the sizes of the figures at the first period (or at any other period), we have to introduce this into the separate averages which we compare as well as into the averages of the variations.

VI. COMPARISON OF THE GEOMETRIC AVERAGE WITH THE OTHER TWO.

§ 1. Of the same given numbers of the same given figures, at least one of which is unequal to the others, the arithmetic average is always greater than the harmonic. Thus

$$\frac{xa + yb + \cdots\cdots}{x + y + \cdots\cdots} > \frac{x + y + \cdots\cdots}{\frac{x}{a} + \frac{y}{b} + \cdots\cdots}.$$

The demonstration of this, which involves the demonstration that

$$(xa + yb + \cdots\cdots)\left(\frac{x}{a} + \frac{y}{b} + \cdots\cdots\right) > (x + y + \cdots\cdots)^2,$$

is somewhat elaborate, and need not be given here.

This being true when the terms are figures (or integers), it is also true when the terms express the variations of figures (or are fractions). Hence

With the same weighting, the arithmetic average of the same variations is always greater (or higher) than the harmonic. That is, when the averages are above unity, the arithmetic average is a greater variation than the harmonic; and when the averages are below unity, the arithmetic is a smaller variation than the harmonic; or the arithmetic may be above while the harmonic is at unity, or may be at unity while the harmonic is below, or may be above while the harmonic is below.

§ 2. Given only two figures (or two figures each repeated the same number of times, so that we have even weighting), the arithmetic mean being $\frac{a+b}{2}$, and the harmonic $\frac{2}{\frac{1}{a}+\frac{1}{b}} = \frac{2ab}{a+b}$, the geometric mean between these means is $\sqrt{\frac{a+b}{2}\cdot\frac{2ab}{a+b}} = \sqrt{ab}$, which is the geometric mean between the given figures. The same holds good if the figures are fractions (representing variations of figures). Therefore

Between two figures (or variations), each taken singly (or in equal numbers), the geometric mean is the geometric mean between the arithmetic and the harmonic means. Hence also, in these cases,

The geometric mean is smaller (lower) than the arithmetic, and greater (higher) than the harmonic.

§ 3. With three figures each taken singly (or in equal numbers), or with two figures taken uneven times (so that we must employ uneven weighting), the first of these propositions does not hold good of the *averages* between them. For between three given figures, each taken singly, the arithmetic average is $\frac{a+b+c}{3}$, and the harmonic, $\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}=\frac{3abc}{ab+bc+ca}$; and the geometric mean between these two averages is $\sqrt{\frac{a+b+c}{3}\cdot\frac{3abc}{ab+bc+ca}}=\sqrt{abc\left(\frac{a+b+c}{ab+bc+ca}\right)}=\sqrt{\frac{a+b+c}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}}$, which does not reduce to the geometric average between the three figures, which is $\sqrt[3]{abc}$. Or with two figures of which the one occurs x times, and the other y times, the arithmetic average is $\frac{xa+yb}{x+y}$, and the harmonic, $\frac{x+y}{\frac{x}{a}+\frac{y}{b}}=\frac{(x+y)ab}{ya+xb}$; and the geometric mean between these two averages is $\sqrt{\frac{xa+yb}{x+y}\cdot\frac{(x+y)ab}{ya+xb}}=\sqrt{ab\left(\frac{xa+yb}{ya+xb}\right)}=\sqrt{\frac{xa+yb}{\frac{x}{a}+\frac{y}{b}}}$, which does not reduce to the geometric average between the two uneven classes, which is $\sqrt[x+y]{a^xb^y}$. In general, a set of uneven classes of figures or variations may be analyzed into a larger number of even classes, some being homonymous. We may, therefore, confine our attention to sets of even classes, or of single figures or variations. Of a number, n, of such figures, the geometric mean between the arithmetic and the harmonic means is $\sqrt{\frac{a+b+c+\cdots\cdots}{n}\cdot\frac{n}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\cdots\cdots}}=\sqrt{\frac{a+b+c+\cdots\cdots}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\cdots\cdots}}$, which does not reduce to the geometric average of the given figures, which is $\sqrt[n]{abc\cdots\cdots}$, (provided the figures, or variations, do not constitute, or reduce to, two classes with even weighting).

But in these cases the second of the above propositions still holds, and we may also specify it more definitely, though without exact demonstration. Thus we have the two following propositions, the first of which is demonstrable.

§ 4. *Of any figures, or variations, with any weighting, the same in all the averagings, the geometric average is smaller (lower) than the arithmetic, and greater (higher) than the harmonic.*

The demonstration of the first part of this proposition, at least for cases with even weighting (whence it can easily be extended) is generally given in works on algebra,[1] although notice of the second is generally neglected. Of both a brief indication is given by Walras (B. 61, p. 15). In full the demonstration is too long to give here.

§ 5. *Of any figures, or variations, with any weighting, the same in all the averagings (provided it be not even weighting for only two classes of distinct figures or variations), the geometric average is sometimes above, and sometimes below, the geometric mean between their arithmetic and harmonic averages.*

This is shown by trial. But it is evident that, if different at all, the geometric average must vary on both sides of the geometric mean in question; for if in any one set of figures or variations it be above, it must be below in a set of their reciprocals (*e. g.*, it is above in 2, 3, 4, 5, and below in $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$).

The easiest way to make the comparison is as follows. Place the expressions for this average and for this mean side by side, the former on the left and the latter on the right, thus,

$$\sqrt[n]{abc\cdots\cdots}, \qquad \sqrt{\frac{a+b+c+\cdots\cdots}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\cdots\cdots}};$$

square each, and raise each the n^{th} power, thus,

$$(abc\cdots\cdots)^2, \qquad \left(\frac{a+b+c+\cdots\cdots}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\cdots\cdots}\right)^n;$$

multiply each by the denominator of the one on the right, thus,

$$(abc\cdots\cdots)^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\cdots\cdots\right)^n, \qquad (a+b+c+\cdots\cdots)^n.$$

The side on which superiority appears in the last line will also be the side on which it exists in the first line. But this method does not exhibit the proportionate amount of the difference.

Trial seems to show that when all the terms are above unity, or when the terms in the numerators (including those virtually divided by unity) sum up to a greater amount than, or outweigh, the terms in the denominators (including the omitted units), the geometric average is above the geometric mean in question (as in $\frac{1}{2}$, 3, $\frac{1}{4}$, 5); and if reversely, below (as in 2, $\frac{1}{3}$, 4, $\frac{1}{5}$).

In intermediate cases the geometric average may coincide with the geometric mean between the other averages.

Trial also shows that the difference is generally very slight, although it may be appreciable if only a few of the figures or variations are very different from the rest, or very excessive compared with the average.

§ 6. Now of given numbers of given figures that vary between two periods, we have seen that the arithmetic average of their variations with weighting according to the sizes at the first period is

$$\frac{xa_1\frac{a_2}{a_1}+yb_1\frac{b_2}{b_1}+\cdots\cdots}{xa_1+yb_1+\cdots\cdots}=\frac{xa_2+yb_2+\cdots\cdots}{xa_1+yb_1+\cdots\cdots};$$

and the harmonic average of their variations with weighting according to the sizes of the classes at the second period is

[1] *E. g.* Todhunter's *Algebra for the use of schools and colleges*, § 680.

$$\frac{xa_2 + yb_2 + \cdots\cdots}{xa_2\frac{a_1}{a_2} + yb_2\frac{b_1}{b_2}} = \frac{xa_2 + yb_2 + \cdots\cdots}{xa_1 + yb_1 + \cdots\cdots}.$$

We now see (from § 4) that the geometric average of these variations with weighting according to the sizes of the classes at the first period is smaller than the arithmetic average with this weighting, thus,

$$\sqrt[xa_1 + yb_1 + \cdots\cdots]{\left(\frac{a_2}{a_1}\right)^{xa_1} \cdot \left(\frac{b_2}{b_1}\right)^{yb_1} \cdots\cdots} < \frac{xa_2 + yb_2 + \cdots\cdots}{xa_1 + yb_1 + \cdots\cdots};$$

and that the geometric average of them with weighting according to the sizes of the classes at the second period is larger than the harmonic average with this weighting, thus,

$$\sqrt[xa_2 + yb_2 + \cdots\cdots]{\left(\frac{a_2}{a_1}\right)^{xa_2} \cdot \left(\frac{b_2}{b_1}\right)^{yb_2} \cdots\cdots} > \frac{xa_2 + yb_2 + \cdots\cdots}{xa_1 + yb_1 + \cdots\cdots}.$$

And we see (from § 5) that, generally, the geometric average with the latter weighting is larger *nearly* in the same proportion as with the former weighting it is smaller, than this one and the same average. Therefore if we take the geometric mean between the weights in these two systems of weighting, and employ the geometric average of the variations with weighting according to these means, thus,

$$\sqrt[x\sqrt{a_1a_2} + y\sqrt{b_1b_2} + \cdots\cdots]{\left(\frac{a_2}{a_1}\right)^{x\sqrt{a_1a_2}} \cdot \left(\frac{b_2}{b_1}\right)^{y\sqrt{b_1b_2}} \cdots\cdots},$$

the result will be *nearly* the same as that of the other average. But because, in § 5, the geometric average is not exactly at the geometric mean between the other two, but inclines to the one side or to the other according as the classes preponderate that rise or fall, so the geometric average with this intermediate weighting will incline to the one side or to the other in similar cases.

With only two classes with even weighting the geometric average is exactly at the geometric mean between the other two, as seen in § 2. Hence in these cases we should expect the following proposition to be demonstrable, and, in fact, find it so.

§ 7. *Of the variations of two figures, or classes of figures, such that the products (and hence the square roots of these products) of their sizes at both periods are equal, the geometric average with even weighting is the same as the arithmetic average with weighting according to the sizes at the first period, or the harmonic average with weighting according to the sizes at the second period.*

Treating of classes, as the more complex case including the other, we wish to prove that

$$\sqrt{\frac{a_2}{a_1} \cdot \frac{b_2}{b_1}} = \frac{xa_2 + yb_2}{xa_1 + yb_1},$$

given that $xa_1 \cdot xa_2 = yb_1 \cdot yb_2$ (this condition being necessary in order that the weighting of the geometric average may be even).

From the condition we obtain $a_2 = \frac{y^2b_1b_2}{x^2a_1}$; and by substituting this value of a_2 in the equation to be proved, we have

$$\sqrt{\frac{y^2b_1b_2}{x^2a_1^2}\cdot\frac{b_2}{b_1}}=\frac{\frac{y^2b_1b_2}{xa_1}+yb_2}{xa_1+yb_1},$$

which reduces to

$$\frac{yb_2}{xa_1}=\frac{yb_2(yb_1+xa_1)}{xa_1(xa_1+yb_1)},$$

which is evident; wherefore the first equation is correct. Q. E. D.

With only two figures, that is, with one figure in each class, x and y in the above are units (so that the condition is $a_1a_2=b_1b_2$), and the result works out the same.

§ 8. With three or more figures, or classes of figures, even though the products of their sizes at both periods be all equal, the statement in the preceding proposition does not universally hold.

It holds when, there being no variations of the figures in the other classes, the figures in two such classes, or in any pairs of such classes (up to all the classes, provided there is no odd one that varies), vary to opposite geometric extremes so that the geometric average is unity; for then the arithmetic average with the weighting of the first period (or the harmonic with that of the second) is also unity, being so for each of the pairs and for the unchanged figures.

It holds also in another case, which may be shown as follows for three classes. We wish to prove that under certain conditions

$$\sqrt[3]{\frac{a_2}{a_1}\cdot\frac{b_2}{b_1}\cdot\frac{c_2}{c_1}}=\frac{xa_2+yb_2+zc_2}{xa_1+yb_1+zc_1},$$

a given condition being that $xa_1\cdot xa_2=yb_1\cdot yb_2=zc_1\cdot zc_2$. From this condition we obtain $a_2=\frac{z^2c_1c_2}{x^2a_1}$ and $b_2=\frac{z^2c_1c_2}{y^2b_2}$. Upon substitution of these values in the equation to be proved, it reduces to

$$\sqrt[3]{\frac{1}{(xa_1)^2\cdot(yb_1)^2\cdot(zc_1)^2}}=\frac{yb_1\cdot zc_1+xa_1\cdot zc_1+xa_1\cdot yb_1}{xa_1\cdot yb_1\cdot zc_1(xa_1+yb_1+zc_1)}.$$

This we see to be true when $xa_1=yb_1=zc_1$; for then

$$\sqrt[3]{\frac{1}{(xa_1)^6}}=\frac{3(xa_1)^2}{(xa_1)^3\,3xa_1},$$

which is evident. But then, according to the condition, we must also have $xa_2=yb_2=zc_2$, and both these conditions together mean that $\frac{a_2}{a_1}=\frac{b_2}{b_1}=\frac{c_2}{c_1}$, that is, that *the variations must all be alike.*

The same result is obtained if we analyze the case with four figures or classes, again with five, and so on.

We know, moreover, the correctness of this result, because we know in general that when variations are all alike any average of them with any weighting is the same as the common variation.

§ 9. In other cases trial shows that the geometric average of the variations with weighting according to the geometric means between the sizes of the classes at each period does not, in ordinary cases, much differ from the arithmetic average of them with weighting according to the sizes of the classes at the first period or (which is the same thing) the harmonic average with weighting according to the sizes at the second period (there being a common form to which both of these averages so weighted reduce).

More specifically, trial seems to show that the geometric average is above the other common form when the preponderating classes have variations rising above the average, and below the other common form when the preponderating classes have variations falling below the average.[2] Here the preponderance is to be determined by comparing the sums of the weights (measured as above described) of the former with those of the latter classes.

The divergence of the geometric average from the common form seems to be greatest either when, amidst ordinary variations, the variations of the classes whose combined weights are about one fifth of the whole weighting are nearly the same, or at least in the same direction, while the rest are in the opposite direction, or when the variations of the smaller classes in the same direction are excessive. In all other cases the results are very close together. Especially so if none of the variations be great or unusual, or if great and extraordinary variations of some large classes are met by great opposite variations of other nearly equally large classes, or of many small ones in combination nearly equalling them, or if the great variations of small classes are met by great opposite variations of nearly equally small classes. In general, if the combined weights of the classes rising above the average and the combined weights of the classes falling below the average are nearly even, these conditions tend to bring the results together; and it is possible that they may coincide.

VII. COMPARISON OF AVERAGES OF UNEQUAL SETS.

For completeness we may add consideration of another case. So far we have all along supposed that the numbers of figures in all the classes separately and together are the same at both periods; for only in this case can all the figures at the second period be regarded as variations of figures at the first. But it may happen that we can have different numbers of figures in the whole sets, or even a different number of classes (since, so long as we are having a different number of figures, it may not matter whether the new ones be in old classes or whether they be altogether new ones forming new classes); and we may still have reason for wishing to compare the averages of these sets of figures.

This case can be represented by numbering all the symbols previously left undistinguished. The numbers of classes of figures at the two periods may be represented by n_1 and n_2 respectively; the numbers of times the figures a_1, b_1, ······ occur at the first period, by x_1, y_1, ······, and the numbers of times the figures a_2, b_2, ······ occur at the second, by x_2, y_2, ······; and the total numbers of the individuals at the first period by n_1', so that $n_1' = x_1 + y_1 +$ ······ to n_1 terms, and at the second by n_2', so that $n_2' = x_2 + y_2 +$ ······ to n_2 terms.

[2] If the geometric average shows a rise from 1 to 2, the rise of a figure from 1 to 1½ is virtually a fall compared with the variations of all the figures; while if the geometric average shows a fall from 1 to ½, the fall of a figure from 1 to ¾ is virtually a rise compared with the variations of all the figures.

§ 1. *Arithmetic averaging.* The average at the first period is

$$\frac{x_1a_1 + y_1b_1 + \cdots\cdots \text{ to } n_1 \text{ terms}}{x_1 + y_1 + \cdots\cdots \text{ to } n_1 \text{ terms}}, \quad \text{or} \quad \frac{1}{n_1'}(x_1a_1 + y_1b_1 + \cdots\cdots \text{ to } n_1 \text{ terms});$$

and at the second it is similar, with all the numbers changed. Therefore the variation of the averages is

$$\frac{\mathbf{A}_2}{\mathbf{A}_1} = \frac{\frac{1}{n_2'}(x_2a_2 + y_2b_2 + \cdots\cdots \text{ to } n_2 \text{ terms})}{\frac{1}{n_1'}(x_1a_1 + y_1b_1 + \cdots\cdots \text{ to } n_1 \text{ terms})},$$

which reduces to

$$\frac{\mathbf{A}_2}{\mathbf{A}_1} = \frac{n_1'(x_2a_2 + y_2b_2 + \cdots\cdots \text{ to } n_2 \text{ terms})}{n_2'(x_1a_1 + y_1b_1 + \cdots\cdots \text{ to } n_1 \text{ terms})}.$$

(Here, incidentally, we see that the variation of the averages is not the variation of the sums, and can not be replaced by that, since the variation of the sums does not allow here for the change in the numbers of the figures.)

By restoring the values of n_1' and n_2', and rearranging, the expression may be stated in full, as follows:

$$\frac{\mathbf{A}_2}{\mathbf{A}_1} = \frac{x_2a_2 + y_2b_2 + \cdots\cdots \text{ to } n_2 \text{ terms}}{x_1a_1 + y_1b_1 + \cdots\cdots \text{ to } n_1 \text{ terms}} \cdot \frac{x_1 + y_1 + \cdots\cdots \text{ to } n_1 \text{ terms}}{x_2 + y_2 + \cdots\cdots \text{ to } n_2 \text{ terms}}.$$

§ 2. *Harmonic averaging.* This being worked out in the same way, the variation of the averages reduces to

$$\frac{\mathbf{A}_2}{\mathbf{A}_1} = \frac{\frac{x_1}{a_1} + \frac{y_1}{b_1} + \cdots\cdots \text{ to } n_1 \text{ terms}}{\frac{x_2}{a_2} + \frac{y_2}{b_2} + \cdots\cdots \text{ to } n_2 \text{ terms}} \cdot \frac{x_2 + y_2 + \cdots\cdots \text{ to } n_2 \text{ terms}}{x_1 + y_1 + \cdots\cdots \text{ to } n_1 \text{ terms}}.$$

§ 3. *Geometric averaging.* The variation of the averages obviously is

$$\frac{\mathbf{A}_2}{\mathbf{A}_1} = \frac{\sqrt[n_2']{a_2^{x_2} \cdot b_2^{y_2} \cdots\cdots \text{ to } n_2 \text{ terms}}}{\sqrt[n_1']{a_1^{x_1} \cdot b_1^{y_1} \cdot \cdots\cdots \text{ to } n_1 \text{ terms}}}.$$

§ 4. All three of these variations of averages have the following common properties.

Without any variation in the sizes of the figures between the first and the second period, a variation in the numbers of the figures in the classes—a change of weighting—may produce a variation in the average. These cases come under the principles in I. §§ 8, 9, 10. Thus, without any variation in the sizes of the figures, an increase or decrease in the number of any figure above the average at the first period will raise or lower the average at the second period; and reversely an increase or decrease in the number of any figure below the average at the first period will lower or raise the average at the second period; while any change whatever in the number of a figure equal to the average at the first period has no influence upon the average at the second period.

Also, there being some variation in the average at the second period from the first, due to variations, all alike or otherwise, in the sizes of the figures (supposing no change in their numbers), then a superadded change in the numbers of the figures may accentuate, lessen, nullify, or outbalance the variation of the average due to the other influences alone.

§ 5. Naturally, if the numbers have not changed at all from the first to the second period, these formulæ all reduce to the corresponding formulæ with the same numbers at both periods.

They do so also *if the numbers of figures in the classes all vary in the same proportion*, and the number of the classes remains the same ($i.\ e.\ n_2 = n_1$). We have then the state of things described in I. § 11—retention of the same weighting; and so, for instance, impossibility of a variation in the average without any variation in the sizes of the figures. In effect, suppose from the first to the second period every number increases or decreases in the proportion r; then $x_2 = rx_1$, $y_2 = ry_1$, and so on, and the expression for the arithmetic average becomes

$$\frac{\mathbf{A}_2}{\mathbf{A}_1} = \frac{rx_1a_2 + ry_1b_2 + \cdots\cdots \text{ to } n \text{ terms}}{x_1a_1 + y_1b_1 + \cdots\cdots \text{ to } n \text{ terms}} \cdot \frac{x_1 + y_1 + \cdots\cdots \text{ to } n \text{ terms}}{rx_1 + ry_1 + \cdots\cdots \text{ to } n \text{ terms}},$$

which reduces to

$$\frac{\mathbf{A}_2}{\mathbf{A}_1} = \frac{x_1a_2 + y_1b_2 + \cdots\cdots \text{ to } n \text{ terms}}{x_1a_1 + y_1b_1 + \cdots\cdots \text{ to } n \text{ terms}},$$

that is, to what is sometimes the proper form of the average of the variations on the numbers which are common to both the periods—in this case the numbers at the first period. Or if we let r' be the reciprocal of r, then $x_1 = r'x_2$, $y_1 = r'y_2$, and so on, and the expression will still reduce to what is in the same cases the proper form of the average of the variations on the numbers which are common to both the periods—this time, the numbers at the second period, wherefore it is indifferent whether we use the numbers of the first or of the second period. And similar would be the reductions in the expressions for the variations of the harmonic and the geometric averages, except that these would yield forms for the averages of the variations not proper when the above forms of the arithmetic averages are proper, the weighting of the variations being different.

§ 6. In all other cases there is no reduction of these formulæ to any formulæ for averages of variations. It is impossible to find any formulæ for any averages of variations which agree with any of the above formulæ for the variations of averages. The reason is simple. There is no average of variations in these cases, because, although there may be some variations contained in the two sets of figures, yet there are other figures which, appearing only in the one or in the other set, do not represent a variation of anything.

The forms here reviewed are averages of the figures *as reported*—in the numbers that happen to be reported. In some matters it is necessary to alter these numbers into other numbers, in various ways, always according to some established principle. After such reduction, the new numbers being used in place of the old, the formulæ will remain the same in form, although, being applied to different numbers, their results will be different. But it is sometimes possible to introduce into the formulæ themselves the principle by which the reduction of the numbers is made. One condition for this is that the new formulæ may be made to contain something which necessarily confines them to the same number of classes at each period. Then the formulæ themselves become different in form. And now the new formulæ may even have some points of contact with the formulæ for averages of variation.

APPENDIX B.

ON COMPENSATORY VARIATIONS.

The supposition is that we have two series of three terms, a, m, b, which are such that $a = m = b$ at one period and at another a and b are opposite terms, compensating for each other, in one of the three progressions, arithmetic, harmonic or geometric, so that m is always either the arithmetic, harmonic or geometric mean between a and b. As m is always the mean, its presence or absence in the calculations is indifferent; and we are virtually dealing with two figures, a and b, and, treating them as equally important, are using even weighting. 'We may suppose that one of these terms, always a, when not equal to m, is *given*, and that this term is always larger than m (which is also a *given* term). Then the other, always smaller than m at the same period when a is larger, may always be expressed in these given terms. Thus when the terms are in arithmetic progression, $b = 2m - a$; when in harmonic progression, $b = \frac{ma}{2a - m}$; when in geometric progression, $b = \frac{m^2}{a}$.

We may suppose either (I) that the terms are equal at the first period and a and b become divergent at the second, or (II) that a and b are divergent at the first period and by converging become equal at the second. There will then be occasion for some explanatory and amplificative remarks.

I.

§ 1. In this supposition we have $a_1 = m = b_1$ and $a_2 > m > b_2$. Here a_2 and b_2 have varied *from* the mean to opposite extremes, a_2 having risen and b_2 fallen.

The variation of a_2 from a_1 is always $\frac{a_2}{a_1} = \frac{a_2}{m}$. The variation of b_2 from b_1 to the arithmetic extreme is $\frac{b_2}{b_1} = \frac{2m - a_2}{m}$; to the harmonic ... $\frac{b_2}{b_1} = \frac{\frac{ma_2}{2a_2 - m}}{m} = \frac{a_2}{2a_2 - m}$; to the geometric, $\frac{b_2}{b_1} = \frac{\frac{m^2}{a_2}}{m} = \frac{m}{a_2}$. N... mean between the variations to the arithmetic terms is ... wherefore these may be regarded as (simple) ... arithmetic mean indicates constancy). T...

variations to the harmonic terms is $\dfrac{1}{\frac{1}{2}\left(\frac{m}{a_2}+\frac{2a_2-m}{a_2}\right)}=1$; wherefore these may be regarded as (simple) *harmonic variations* (whose harmonic mean indicates constancy). The geometric mean between the variations to the geometric terms is $\sqrt{\frac{a_2}{m}\cdot\frac{m}{a_2}}=1$; wherefore these may be regarded as (simple) *geometric variations* (whose geometric mean indicates constancy).

§ 2. The percentage (in hundredths) of the rise of a_2 above m, reckoned in m, is always $\frac{a_2-m}{m}$. The percentage of the fall of b_2 below m, likewise always reckoned in m, in the arithmetic variation is $\frac{m-(2m-a_2)}{m}=\frac{a_2-m}{m}$; in the harmonic variation, $\dfrac{m-\frac{ma_2}{2a_2-m}}{m}=\frac{a_2-m}{2a_2-m}$; in the geometric, $\dfrac{m-\frac{m^2}{a_2}}{m}=\frac{a_2-m}{a_2}$. Thus, *the percentage being reckoned in the mean or the uniform starting point, it is only the rise and fall from the mean to the arithmetic extremes, or the arithmetic variations, that are in equal percentage.*

§ 3. But if we reckon the percentage in the terms reached, the percentage of the rise of a_2, in a_2, is always $\frac{a_2-m}{a_2}$. The percentage of the fall of b_2, always in b_2, in the arithmetic variation is $\frac{m-(2m-a_2)}{2m-a_2}=\frac{a_2-m}{2m-a_2}$; in the harmonic variation, $\dfrac{m-\frac{ma_2}{2a_2-m}}{\frac{ma_2}{2a_2-m}}=\frac{a_2-m}{a_2}$; in the geometric, $\dfrac{m-\frac{m^2}{a_2}}{\frac{m^2}{a_2}}=\frac{a_2-m}{m}$. Thus, *the percentage being reckoned in the opposite extremes reached, it is only the rise and fall from the mean to the harmonic extremes, or the harmonic variations, that are in equal percentage.*

§ 4. If again we reckon the percentage always in the same direction, that is, (1) from the lowest terms, or (2) from the highest terms, or in other words, (1) from the starting point at the mean for the rising term and from the extreme point reached for the falling term, or (2) from the extreme point reached for the rising term and from the starting point at the mean for the falling term, we have the following percentages. (1) The percentage of the rise of a_2, reckon[illegible] $\frac{a_2-m}{m}$; and the percentages of the falls of b[illegible] We see that, *the percentage* being reck[illegible] *and fall from the mean to the* [illegible] *equal percentage.* (2) The [illegible] § 3, is $\frac{a_2-m}{a_2}$; and [illegible] in § 2. We see [illegible] *the rise and fall* *from the mean to the geometric extremes, or the geometric variations, that are in equal percentage.*

II.

§ 1. In this supposition we have $a_1 > m > b_1$ and $a_2 = m = b_2$; and a_2 and b_2 have varied *to* the mean from opposite extremes, a_2 now having fallen and b_2 risen.

The variation of a_2 from a_1 is always $\frac{a_2}{a_1} = \frac{m}{a_1}$. The variation of b_2 from b_1, from the arithmetic extreme, is $\frac{b_2}{b_1} = \frac{m}{2m - a_1}$; from the harmonic extreme, $\frac{b_2}{b_1} = \frac{m}{\frac{ma_1}{2a_1 - m}} = \frac{2a_1 - m}{a_1}$; from the geometric extreme, $\frac{b_2}{b_1} = \frac{m}{\frac{m^2}{a_1}} = \frac{a_1}{m}$. Thus these variations are the reciprocals of the preceding; for we may suppose a_1 here to be equal to a_2 there, and b_1 here equal to b_2 there.

§ 2. Now unity is the harmonic mean between the variations from the arithmetic extremes; for $\frac{1}{\frac{1}{2}\left(\frac{a_1}{m} + \frac{2m - a_1}{m}\right)} = 1$; wherefore *the variations from the arithmetic extremes to their mean are harmonic variations* (whose harmonic mean indicates constancy). Unity is the arithmetic mean between the variations from the harmonic extremes; for $\frac{1}{2}\left(\frac{m}{a_1} + \frac{2a_1 - m}{a_1}\right) = 1$; wherefore *the variations from the harmonic extremes to their mean are arithmetic variations* (whose arithmetic mean indicates constancy). But unity is the geometric mean between the variations from the geometric extremes; for $\sqrt{\frac{m}{a_1} \cdot \frac{a_1}{m}} = 1$; wherefore *the variations from the geometric extremes to their mean are geometric variations* (whose geometric mean indicates constancy).

§ 3. In effect, if we reduce the terms at the first period, a_1 and b_1, to equality to m, and observe the same variations, we may reduce the terms at the second period, a_2 and b_2, to figures no longer equal to m, as follows. We reduce a_1 to m by multiplying a_1 by $\frac{m}{a_1}$; therefore we must reduce a_2 in the same manner; but $a_2 \cdot \frac{m}{a_1} = \frac{m^2}{a_1}$. Thus the variation of a_1 to m is the same variation as that of m to $\frac{m^2}{a_1}$. We reduce b_1 to m by multiplying b_1 by $\frac{m}{b_1}$; and we must reduce b_2 in the same manner. Now when the terms were in arithmetic progression, we had $b_1 = 2m - a_1$; therefore we have $b_1 \cdot \frac{m}{b_1} = m$, $\frac{m}{2m - a} = \frac{m^2}{2m - a_1}$. Thus the variation from b_2, when it is at the arithmetic extreme, or from $2m - a$, to m is the same variation as that of m to $\frac{m^2}{2m - a_1}$. Similarly when the terms were in harmonic progression, the variation from the harmonic extreme, b_1, to the mean, or from $\frac{ma_1}{2a - m}$ to m, is the same as that from m to $\frac{m(2a - m)}{a}$. And when the terms were in geometric progression, the variation from the geometric extreme, b_1, to the mean, or from $\frac{m^2}{a}$ to m, is the same variation as that of m to a_1. Now $\frac{m^2}{a_1}$ and $\frac{m^2}{2m-a_1}$ are the harmonic terms around m; for $\frac{1}{\frac{1}{2}\left(\frac{a_1}{m^2}+\frac{2m-a_1}{m^2}\right)}=m$. Therefore *variations from the arithmetic extremes to their mean are the same as variatians from the mean to harmonic extremes.* Consequently it is the percentages of these variations, reckoned in the extremes, the positions originally at the first period, but at the second period in the reductions, that are equal. And $\frac{m^2}{a_1}$ and $\frac{m(2a_1-m)}{a_1}$ are arithmetic terms around m; for half the sum of these is m. Therefore *variations from the harmonic extremes to their mean are the same as variations from the mean to arithmetic extremes.* Consequently it is the percentage of these varitions reckoned in the mean, the position originally at the second period, but at the first period in the reductions, that are equal. But $\frac{m^2}{a_1}$ and a_1 are geometric terms around m as their mean; for $\sqrt{\frac{m^2}{a_1}\cdot a_1}=m$. Therefore *variations from the geometric extremes to their mean are the same as variations from the mean to geometric extremes.* Consequently it is the percentages of variations reckoned either from the top or from the bottom that are equal.

§ 4. Thus if two quantities, equal at first, vary to the opposite arithmetic extremes, and then vary back to the same figure as at the start, their first variations, diverging, are arithmetic variations, and their later variations, converging, are harmonic variations. If two quantities, equal at first, vary to the opposite harmonic extremes, and then vary back to the same figure as at the start, their first variations, diverging, are harmonic variations, and their later variations, converging, are arithmetic variations. But if two quantities, equal at first, vary to the opposite geometric extremes, and then vary back to the same figure as at the start, both these diverging and converging variations are geometric variations.

III.

The following explanations may be offered.

§ 1. In I. we have been considering variations of states which may be represented thus,

$$\frac{a_2 > m > b_2}{a_1 = m = b_1} = \frac{m}{m} = 1.$$

But we have been averaging the variations, with even weighting. Hence, in accordance with Appendix A, I. § 7 (II. § 6, III. § 6, and V. § 2), we have been performing the same operation as if we averaged the figures at each period separately, with even weighting in each case, and then compared the results.

§ 2. In II. we have been considering variations of states which may be represented thus,

$$\frac{a_2 = m = b_2}{a_1 > m > b_1} = \frac{m}{m} = 1,$$

in each of which averages even weighting is employed. But we have been averaging the variations with even weighting, and now the operations are not the same in two out of the three cases.

§ 3. Thus if the terms change from the arithmetic extremes, applying the arithmetic average with even weighting to the figures at each of the periods separately, we have

$$\frac{\mathbf{A}_2}{\mathbf{A}_1} = \frac{\frac{1}{2}(m + m)}{\frac{1}{2}\{a_1 + (2m - a_1)\}} = \frac{m}{m} = 1.$$

But we do not obtain this result by arithmetically averaging the variations with even weighting; for

$$\mathbf{V} = \frac{1}{2}\left(\frac{m}{a_1} + \frac{m}{2m - a_1}\right) = \frac{m^2}{a_1(2m - a_1)};$$

but only by harmonically averaging them, if we are to use even weighting, as shown in II. § 3. This is because the comparison of the arithmetic averages with even weighting, when the figures are unequal at the first period, is the same as the arithmetic average of the variations, not with even weighting, but with weighting according to the sizes of the terms at the first period, according to Appendix A, II. § 7;[1] and, according to Appendix A, IV. § 2, this is the same as the harmonic average of the variations with even weighting.

§ 4. If the terms change from the harmonic extremes, applying the harmonic average, we have

$$\frac{\mathbf{A}_2}{\mathbf{A}_1} = \frac{\dfrac{1}{\frac{1}{2}\left(\frac{1}{m} + \frac{1}{m}\right)}}{\dfrac{1}{\frac{1}{2}\left(\frac{1}{a_2} + \frac{2a_1 - m}{ma_1}\right)}} = \frac{m}{m} = 1.$$

And again we do not obtain this result by harmonically averaging the variations; for

$$\mathbf{V} = \frac{1}{\frac{1}{2}\left(\frac{a_1}{m} + \frac{a_1}{2a_1 - m}\right)} = \frac{m(2a_1 - m)}{a_1^2};$$

but only by arithmetically averaging them, if we are to use even weighting, as shown in II. § 3. This is because the comparison of the harmonic averages with even weighting, when the figures are unequal at the first period, is the same as the harmonic average of the variations, not with even weighting, but with weighting inversely according to the sizes of the terms at the first

[1] Here, with even weighting, with a limited between m and $2m$, the result is *larger* than unity, indicating a rise. This is because the average of the variations can be brought down to unity only by giving greater weight to the fall of a_1.

period, according to Appendix A, III. § 7;[2] and, according to Appendix A, IV. § 5, this is the same as the arithmetic average of the variations with even weighting.

§ 5. But if the terms change to the geometric extremes, we have

$$\frac{A_2}{A_1} = \frac{\sqrt{m \cdot m}}{\sqrt{a_1 \cdot \frac{m^2}{a_1}}} = \frac{m}{m} = 1;$$

and we obtain the same result by geometrically averaging the variations with even weighting, as shown in II. § 3. This is because the comparison of the geometric averages is the same as the geometric average of the variations with the same weighting, according to Appendix A, V. § 5.

IV.

The consequences indicated in II. § 4 may be amplified as follows.

§ 1. Suppose we have these three series at three different periods:

1st period—	$a_1 = m = b_1$;
2d "	$a_2 > m > b_2$;
3d "	$a_3 = m = b_3$;

in which $a_3 = a_1$ and $b_3 = b_1$. Here we have two variations of the series, which, with substitution of a_1 for a_3 and of b_1 for b_3, are

$$\text{1st variation—} \quad \frac{a_2 > m > b_2}{a_1 = m = b_1};$$

$$\text{2d} \quad " \quad \frac{a_1 = m = b_1}{a_2 > m > b_2}.$$

But the two variations together produce a return to constancy; for

$$\frac{a_3 = m = b_3}{a_1 = m = b_1} = \frac{m}{m} = 1.$$

For what follows, however, it is not necessary that a_2 and b_2 be on opposite sides of m. They may be *any* quantities, with any variation of their mean (now to be represented by m_2) from m_1. All that is needed is that they return at the third period to what they were at the first; whereupon also their mean, m_3, will return to m_1.

§ 2. If we average these series separately and then compare the averages, we get inverted results, which are reciprocals of each other. Then the comparison of the third period with the first by means of the two intermediate comparisons—which comparison consists in multiplying together the results of those comparisons—turns out correctly; for these reciprocal results multiplied by each other give unity.

§ 3. We have two sets of variations, which are

[2] Here, with even weighting, with a larger than m, the result is *smaller* than unity, indicating a fall. This is because the average of the variations can be brought up to unity only by giving a greater weight to the rise of b_1.

1st set of variations— $\frac{a_2}{a_1}, \frac{b_2}{b_1}$;

2d " " $\frac{a_1}{a_2}, \frac{b_1}{b_2}$;

and the whole variations from the first to the third period are $\frac{a_3}{a_1} = \frac{a_1}{a_1} = 1$ and $\frac{b_3}{b_1} = \frac{b_1}{b_1} = 1$; wherefore the average of these whole variations is always unity, indicating constancy. But if we average each set of variations, and seek thence the average of the whole variations, we obtain the following results.

§ 4. The arithmetic averages of the variations, with even weighting, are, of the first set, $\frac{1}{2}\left(\frac{a_2}{a_1} + \frac{b_2}{b_1}\right)$, and of the second, $\frac{1}{2}\left(\frac{a_1}{a_2} + \frac{b_1}{b_2}\right)$. The multiplication of these gives $\frac{1}{2} + \frac{1}{4}\left(\frac{a_1b_2}{a_2b_1} + \frac{a_2b_1}{a_1b_2}\right)$, which by elimination of a_1 and b_1, which are equal, reduces to $\frac{1}{2} + \frac{1}{4}\left(\frac{b_2}{a_2} + \frac{a_2}{b_2}\right)$. This does not reduce to unity unless $a_2 = b_2$. In all other cases its result is above unity,[3] indicating a rise, instead of constancy.

§ 5. The harmonic averages of the variations, with even weighting, are, of the first set, $\frac{1}{\frac{1}{2}\left(\frac{a_1}{a_2} + \frac{b_1}{b_2}\right)}$, and of the second $\frac{1}{\frac{1}{2}\left(\frac{a_2}{a_1} + \frac{b_2}{b_1}\right)}$. The product of these is $\frac{1}{\frac{1}{2} + \frac{1}{4}\left(\frac{a_2}{b_2} + \frac{b_2}{a_2}\right)}$, which does not reduce to unity except, as before, in the simple case when $a_2 = b_2$. In all other cases its result is below unity, indicating a fall, instead of constancy.

§ 6. The geometric averages of the variations, with even weighting, are, of the first, $\sqrt{\frac{a_2}{a_1} \cdot \frac{b_2}{b_1}}$, and of the second, $\sqrt{\frac{a_1}{a_2} \cdot \frac{b_1}{b_2}}$. The product of these is unity, indicating constancy. Therefore *the continuous use of the geometric mean through both the variations gives the right result for the two variations together.*

§ 7. But again, if we used the arithmetic average of the variations for the first set and the harmonic for the second, or the harmonic for the first and the arithmetic for the second, all with even weighting, the product of the results thereby obtained give unity, indicating constancy. Therefore *the alternate use of the arithmetic and of the harmonic means of the variation gives the right result for the two variations together.*

This proposition is plain if we remember that the harmonic average of given figures is the reciprocal of the arithmetic average of their reciprocals, and reversely; and that when variations take place from the mean to the

[3] To demonstrate this it is necessary to show that $\frac{a}{b} + \frac{b}{a} > 2$. Now $\frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab}$. Let this equal x. Then $\frac{a^2 + b^2}{ab} - \frac{2ab}{ab} = x - 2$; whence $\frac{(a-b)^2}{ab} = x - 2$. But $\frac{(a-b)^2}{ab}$ is always positive, a and b being positive. Therefore $x - 2$ is positive, and x is greater than 2.

arithmetic or to the harmonic extremes, the return variations from these extremes to the mean are the reciprocals of the first variations.

V.

§ 1. Lastly we may analyze the variations in the following series:

$$\text{1st period— } a_1 > m > b_1\,;$$
$$\text{2d \quad `` \quad } a_2'(= b_1) < m < b_2(= a_1)\,;$$

in which m represents any one of the three means. Here the figures are supposed simply to change places, the one falling to the other and the other rising to it. The variations are $\frac{b_1}{a_1}$ and $\frac{a_1}{b_1}$, which are reciprocals of each other.

§ 2. It is plain that when a_1 is becoming equal to b_1, it must pass through the mean between them, that is, it must fall first *to* the mean and then *from* the mean; and when b_1 is becoming equal to a_1, it must pass through the mean between them, that is, it must rise first *to* the mean and then *from* the mean.

§ 3. Therefore if the mean be the arithmetic, the approaches of a_1 and b_1 *to* this mean are harmonic variations (above, II. § 2), and their departures *from* it are arithmetic variations.

If the mean be the harmonic, the approaches of a_1 and b_1 *to* this mean are arithmetic variations, and their departure *from* it are harmonic variations.

But if the mean be the geometric, both the approaches *to* it and the departures *from* it are geometric variations.

§ 4. Therefore also the whole variations of a_1 to b_1, and of b_1 to a_1, are geometric variations. This is plain also from the fact that the variations are reciprocals of each other. It may be shown, further, in the following way. The percentage of the rise of b_1 to a_1 may be represented as a rise by p per cent. (reckoned in b_1). Then the percentage of the fall of a_1 to b_1 (reckoned in a_1), being a fall by p from $1 + p$, is a fall by $\frac{p}{1+p}$ per cent. The rise of b_1 is the same variation as the rise of 1 to $1 + p$, and the fall of a_1 is the same variation as the fall of 1 to $1 - \frac{p}{1+p} = \frac{1}{1+p}$; that is, the rise and the fall are from 1 to geometric extremes, and are geometric variations.

§ 5. Moreover all geometric variations in opposite directions are expressible as rises and falls to reciprocals, that is, as rises from 1 to $1 + p$ and as falls from 1 to $\frac{1}{1+p}$. Therefore *all geometric variations in opposite directions are the same as variations from some one figure to another and from the latter to the former.*

§ 6. Still, in these geometric variations, a_1 in becoming b_1 and b_1 in becoming a_1 must pass through the arithmetic and the harmonic means. Therefore *all geometric variations may be viewed as composed of two variations, the one harmonic and the other arithmetic,* in either order as we please.

§ 7. If a and b vary at the same uniform speed so that a_1 becomes b_1 and b_1 a_1 in the same time, they pass through the arithmetic mean at the same moment, which is the halfway point in the time for the whole alternation.

If a and b vary at the corresponding uniform rates so that a_1 becomes b_1 and b_1 a_1 in the same time, they pass through the geometric average at the same moment, which also is the timal halfway point. There is a uniform variation which will carry them through the harmonic mean at the same moment at the timal halfway point.

§ 8. If the variations be the ones at the uniform rates, with a and b passing through the geometric mean at the timal halfway point, the rising figure, b, crosses the harmonic mean first and the arithmetic mean last, and the falling figure, a, crosses the arithmetic mean first and the harmonic mean last. At every moment a and b are at geometric extremes around their halfway point. Therefore (because of Appendix C, VI. § 2) at the moment when b crosses the harmonic average, a crosses the arithmetic, and at the moment when b crosses the arithmetic, a crosses the harmonic.

APPENDIX C.

REVIEW AND ANALYSIS OF THE METHODS EMPLOYING ARITHMETIC AVERAGING FOR MEASURING VARIATIONS IN THE EXCHANGE-VALUE OF MONEY.

Arithmetic averaging is the common feature in so many widely differing methods of calculating variations in the exchange-value of money from the variations of prices, that it may be well to review these methods by expounding them all in the same system of notation ; for many have been made without any formulation at all, and the others have been formulated with so many different symbols that comparison is difficult. Also some of the methods are so complex that analysis is necessary to disclose their real nature.

The notation here employed is that employed in the body of this work. Prices of the classes [A], [B], etc., are represented at the first and second periods respectively by numbered Greek letters—α_1, α_2, β_1, β_2, ······. These, unless otherwise specified, are prices of the ordinary mass-units used in commerce. When prices are taken at the first period all as 1.00 and the prices at the second period are reduced accordingly, the Greek letters are accented, thus, $\alpha_1' = \beta_1' = \cdots\cdots$ (= 1.00), and α_2', β_2', ······ (these figures by their excess above, or deficiency below, 1.00 directly indicating the variation of the prices ; for $\alpha_2' = \frac{\alpha_2}{\alpha_1}$, $\beta_2' = \frac{\beta_2}{\beta_1}$, and so on). The numbers of these mass-units (or masses whose prices are used) in the classes respectively, as they are supposed to occur in trade, or amounts all reduced in the same proportion, are represented by x, y, ······, or x', y', ..., (which are the same symbols as in Appendix A, while α, β, ······ here take the place of a, b, ······ as there used). These letters, x, y, ······, we must notice, do not represent the weights of the classes (except in the averagings, generally eliminated, at the different periods separately; they never represent the weights of the price variations). The weights of the price variations are represented by **a**, **b**, ····· for the variations of the prices of the classes [A], [B], ······ respectively. These letters, **a**, **b**, ······, represent, supposedly, the numbers of ideal individuals in the classes, or amounts reduced in the same proportion,—that is, the relative sizes of the classes measured by their exchange-value (or money-value) importance. In other words **a** $= x\alpha$, **b** $= y\beta$, and so on, or **a**:**b**:······::$x\alpha:y\beta$······

without specifying at which period, or how, the prices are used. The number of classes used in the calculation is represented by n. The number of actual individual prices (those of the mass-units employed) in all the classes together, by n'; so that $n' = x + y + \cdots\cdots$ to n terms. The number of ideal individual prices in all the classes together, by n''; so that $n'' = \mathbf{a} + \mathbf{b} + \cdots\cdots$ to n terms. The terms x, y, $\cdots\cdots$, $\mathbf{a}$, $\mathbf{b}$, $\cdots\cdots$, n, n' and n'', when the same for both the periods compared, need not be numbered. But when they are employed distinctively for the conditions at one period alone, they are also to be distinguished by subscript numbers.

I. DUTOT'S METHOD.

Dutot, 1738; see B. 2, especially pp. 370–373.

The prices of articles are taken at their market quotations on the mass-units that happen to be employed by merchants, however large or small, and are added together, the sums for the two periods being compared. It may be expressed thus,

$$\frac{P_2}{P_1} = \frac{a_2 + \beta_2 + \cdots\cdots}{a_1 + \beta_1 + \cdots\cdots}.$$

Here only sums are used. But we know that this gives the same result as if the arithmetic average were drawn for each period (see Appendix A, II. § 1). We also know that it contains hidden weighting for the price variations, viz. according to a_1, β_1, $\cdots\cdots$, with arithmetic averaging, (see Appendix A, II. § 7). Therefore this method contains what we have called haphazard weighting (in Chapt. IV. Sect. II. § 2).

This method has since been employed by Levasseur, B. 18, pp. 179, 180, and by A. Walker, B. 27, even in series of many years; by a writer in the London *Bankers' Magazine*, B. 37; by Kral, B. 98; and by Fraser and Sergel, B. 120.

(As the mass-quantities—the mass-units hit upon—are constant, this method is really a variety of Scrope's method, to be treated of later. But as it was invented and has been used without any reference to the mass-quantities, it deserves to be classed by itself.)

II. CARLI'S METHOD AND ITS VARIETIES.

§ 1. Carli, 1766; see B. 4, especially pp. 350–351.

All the prices being taken at the first period as equal to 1.00, the prices at the second period are taken on the same masses, and the percentages of their variations noted; then the average is drawn between these. The procedure in full is described in the following formula, upon the prices of any mass-units. The average percentage is represented by %.

$$\% = \frac{100}{n}\left(\frac{a_2 - a_1}{a_1} + \frac{\beta_2 - \beta_1}{\beta_1} + \cdots\cdots\right),$$

in which the answer indicates a rise if positive, a fall if negative, and constancy if zero.

The same result is yielded by the following formula,

$$\frac{P_2}{P_1} = \frac{1}{n}\left(\frac{a_2}{a_1} + \frac{\beta_2}{\beta_1} + \cdots\cdots\right),$$

in which the answer indicates constancy if unity, a rise if above unity, a fall if below unity; and indicates the percentage of the rise in the second decimal, and the percentage of the fall in the second decimal of the remainder when the answer is substracted from unity. If the prices at the first period are all taken at 1.00, and those at the second period are reduced accordingly, the same result is given, in the same way, by the following,

$$\frac{P_2}{P_1} = \frac{1}{n} (\alpha_2' + \beta_2' + \cdots\cdots),$$

which is practically the form employed by Carli. Or if the prices at the first period are all taken at 100, by this,

$$\frac{100P_2}{P_1} = \frac{1}{n} (100\alpha_2' + 100\beta_2' + \cdots\cdots),$$

in which the answer indicates constancy if 100, and the percentage of a rise or fall according to the units it is above or below 100.

This last form of the method—the one which has usually been employed —was introduced by Evelyn, 1798; see B. 5. This variation of Carli's method is only formal, and, when it is employed, the method still deserves to be called Carli's method. In it, it is plain that when we are comparing only a second with a first period, there is even weighting of the price variations.

§ 2. Evelyn, however, compared several periods all with the same first period (some of the second periods being later and some earlier than the common first or basic period); and this also has become the usual practice. Now in this case—confining our attention to the simplest form of the formulæ, that in which prices are reduced to 1.00 at the basic period,—after comparing a second period with the first and getting this result,

$$\frac{P_2}{P_1} = \frac{1}{n} (\alpha_2' + \beta_2' + \cdots\cdots),$$

and comparing a third period with the same first, with this result,

$$\frac{P_3}{P_1} = \frac{1}{n} (\alpha_3' + \beta_3' + \cdots\cdots),$$

if we compare these with each other, our comparison of these is now as follows,

$$\frac{P_3}{P_2} = \frac{\alpha_3' + \beta_3' + \cdots\cdots}{\alpha_2' + \beta_2' + \cdots\cdots};$$

or

$$\frac{P_3}{P_2} = \frac{\frac{\alpha_3}{\alpha_1} + \frac{\beta_3}{\beta_1} + \cdots\cdots}{\frac{\alpha_2}{\alpha_1} + \frac{\beta_2}{\beta_1} + \cdots\cdots}.$$

But here we know that the weighting of the price variations is no longer even, —it is accidental, being according to α_2', β_2' ⋯⋯, or $\frac{\alpha_2}{\alpha_1}$, $\frac{\beta_2}{\beta_1}$, ⋯⋯, that is according to the price variations between the base and the earlier period.

Thus this method—mostly without the knowledge of those who employ it—switches off into something quite different.

This method of Carli was so employed by Porter, B. 11; and, with Evelyn's formal variation, it has been employed by J. P. Smith, B. 7; the *Economist*, on the lists of prices begun by Newmarch, B. 19 and 20; Laspeyres, B. 25 and 26; Bourne, at first, B. 46; Burchard, B. 53; Hansard, B. 67; Sauerbeck, B. 79–90; Rogers, B. 92, pp. 787, 789, 790, 791; Falkner, partly, B. 111–113; Wiebe, partly, B. 124, pp. 369–382; Whitelaw, B. 130, pp. 32–33; Wetmore in B. 119; Barker, B. 128; Powers, partly, B. 131, pp. 27, 28, 29, etc. Geyer employed it with another formal variation, reducing the first prices to *about* 2.00, B. 23, pp. 321–322.

§ 3. Evelyn introduced furthermore a real variation. He counted always wheat by itself, butcher's meat by itself, and day labor by itself, all as equally important, and also as equally important with each of these a figure made up of the prices of twelve other agricultural products, each of which had been counted as of equal importance in making up this figure. This is irregular weighting by classification (in which classes count more the fewer there are in the divisions). It is not worth while to give a formula for this. It would be exceedingly complicated if we attempted to find the weighting employed in the comparison of two later periods with each others. Whenever, with the rest of Carli's method, instead of absolutely even weighting in the comparison of the other periods with the basic one, there is employed such irregular weighting by classification, this may be called *Evelyn's method*, as it employs a feature really distinctive in Evelyn's procedure.

This method was sometimes employed by Young (who in doing so abandoned his own method), B. 6, pp. 84, 118; also by Soetbeer, B. 19; by Kral, B. 98; and by Wiebe, partly, B. 124, pp. 383–386.

III. YOUNG'S METHOD AND ITS VARIETIES.

§ 1. Arthur Young, 1812; see B. 6, p. 72.

Prices at the first or basic period are taken at 100 and at the other periods reduced accordingly. Percentages of the variations are used, and these are multiplied by the weights (according to the relative total exchange-values of the classes, in general, at no particular period), and the sum of the products divided by the sum of the weights. Thus

$$\% = \frac{100}{n''}\left(\mathbf{a}\,\frac{a_2 - a_1}{a_1} + \mathbf{b}\,\frac{\beta_2 - \beta_1}{\beta_1} + \cdots\cdots \text{ to } n \text{ terms}\right),$$

in which the answer has the same meaning as in Carli's.

The same result, with our usual interpretation of the answer, is yielded by the following,

$$\frac{P_2}{P_1} = \frac{1}{n''}\left(\mathbf{a}\,\frac{a_2}{a_1} + \mathbf{b}\,\frac{\beta_2}{\beta_1} + \cdots\cdots \text{ to } n \text{ terms}\right);$$

or, if we start with prices at 1.00, by

$$\frac{P_2}{P_1} = \frac{1}{n''}\left(\mathbf{a}\,a_2' + \mathbf{b}\,\beta_2' + \cdots\cdots \text{ to } n \text{ terms}\right).$$

Either of these may be regarded as the formula for Young's method.

This method was approved and recommended by Lowe, B. 8; and has since been reinvented, with slight variations in the procedure, by Ellis, B. 36, and Wasserab, B. 105; by Sauerbeck at times (using the weights of 1889–1891), B. 82, p. 218, and later; by Falkner at times (using the weights of periods at the end of the series), B. 112, p. 63; by Fonda, B. 127, pp. 160–161; and a correct formula has been made for it by Westergaard, B. 110, p. 220, in this cumbrous form,

$$\frac{P_2}{P_1} = \frac{\mathbf{a}}{\mathbf{a} + \mathbf{b} + \cdots\cdots} \cdot \frac{\alpha_2}{\alpha_1} + \frac{\mathbf{b}}{\mathbf{a} + \mathbf{b} + \cdots\cdots} \cdot \frac{\beta_2}{\beta_1} + \cdots\cdots,$$

which easily reduces to the simpler form above given.

§ 2. Young himself happened not to do so, but the rest of these writers have employed, or recommended, this method in a series of periods, all the later being compared directly with the first. Then a comparison between any of the later periods would be as follows,

$$\frac{P_3}{P_2} = \frac{\mathbf{a}\alpha_3' + \mathbf{b}\beta_3' + \cdots\cdots}{\mathbf{a}\alpha_2' + \mathbf{b}\beta_2' + \cdots\cdots},$$

or, with the prices of any mass-units,

$$\frac{P_3}{P_2} = \frac{\mathbf{a}\frac{\alpha_3}{\alpha_1} + \mathbf{b}\frac{\beta_3}{\beta_1} + \cdots\cdots}{\mathbf{a}\frac{\alpha_2}{\alpha_1} + \mathbf{b}\frac{\beta_2}{\beta_1} + \cdots\cdots},$$

in which, as we know, the weights of the price variations are no longer **a**, **b**, ······, but $\mathbf{a}\alpha_2'$, $\mathbf{b}\beta_2'$, ······, or $\mathbf{a}\frac{\alpha_2}{\alpha_1}$, $\mathbf{b}\frac{\beta_2}{\beta_1}$, ······ (see Appendix A, II. § 8). For this difference between the comparisons of the later periods with one another and the comparisons of each of the later periods with the first, Westergaard has found fault with this method (see here Ch. V. Sect. VI. § 7).

§ 3. Variations upon this method have been made by Giffen and by Palgrave. Giffen's variation is mostly formal, though it is real as regards the choice of weighting. The formula for it, for the second period, is

$$\% = \frac{100}{n_7''}\{\mathbf{a}_7(\alpha_2' - 1) + \mathbf{b}_7(\beta_2' - 1) + \cdots\cdots \text{ to } n \text{ terms}\},$$

or, with ordinary prices,

$$\% = \frac{100}{n_7''}\left(\mathbf{a}_7\frac{\alpha_2 - \alpha_1}{\alpha_1} + \mathbf{b}_7\frac{\beta_2 - \beta_1}{\beta_1} + \cdots\cdots \text{ to } n \text{ terms}\right).$$

(A formula for it somewhat like the last, but not specifying the period at which the weights are chosen, is given by Edgeworth, B. 59, p. 265.) It is confined to custom-house returns, and so, like James's method already (see next section), has the fault of representing the relative weights only of goods exported or imported. He makes one calculation each for exports and for imports. His weights, which are according to the total money-values of the classes, are reduced to percentages of the total money-value of all the exports, or imports, not at the basic period, 1861, nor on an average, but at another period, 1875, the seventh in his series (which skips every other year). These percentages should add up to 100; but as he could not manage all the classes, his weights for exports in that year added up only to 73.1 and those for imports only to 84.38; wherefore these figures are taken instead of 100 for his starting points (in 1861, the prices then being reduced to unity). [He said it is unimportant of what year he chose the weighting, because the totals of the percentages varied very slightly—between 72.7 and 80.4 for exports, and between 82.42 and 88.88 for imports. B. 40, p. 8, cf. B. 39, p. 5. This is no reason at all, as it overlooks that the particular weights of the individual classes varied considerably.] In extending his investigations backwards, in B. 36, he had to confine himself to fewer classes still, so that the figures (for the weights of 1875, and the price averages of 1861) reduce to 65.8 for exports, and to 81.16 for imports. (He describes his method best in B. 45, qq. 709–716.)

§ 4. Palgrave, B. 77, has introduced the real variation of employing the weighting of every subsequent year in the comparison of it with the first or basic period; and he appears to have been followed by Sauerbeck in one of the latter's "tests" (see B. 82, p. 218). Prices are taken for the basic period at 100; but here, for simplicity, we may take them at 1.00. The formula for comparing the second period with the first is

$$\frac{P_2}{P_1} = \frac{1}{n_2''}(\mathbf{a}_2\alpha_2' + \mathbf{b}_2\beta_2' + \cdots\cdots \text{ to } n \text{ terms});$$

or, adapted to the prices of any mass-units,

$$\frac{P_2}{P_1} = \frac{1}{n_2''}\left(\mathbf{a}_2\frac{\alpha_2}{\alpha_1} + \mathbf{b}_2\frac{\beta_2}{\beta_1} + \cdots\cdots \text{ to } n \text{ terms}\right),$$

which in full is

$$\frac{P_2}{P_1} = \frac{1}{n_2''}\left(x_2\alpha_2\frac{\alpha_2}{\alpha_1} + y_2\beta_2\frac{\beta_2}{\beta_1} + \cdots\cdots \text{ to } n \text{ terms}\right),$$

(like this it is given by Edgeworth, B. 59, p. 265), which reduces to

$$\frac{P_2}{P_1} = \frac{1}{n_2''}\left(x_2\frac{\alpha_2^2}{\alpha_1} + y_2\frac{\beta_2^2}{\beta_1} + \cdots\cdots \text{ to } n \text{ terms}\right).$$

When dealing with a third, or any subsequent, period, Palgrave compares it with the first, in the same manner, thus,

$$\frac{P_3}{P_1} = \frac{1}{n_3''}(\mathbf{a}_3\alpha_3' + \mathbf{b}_3\beta_3' + \cdots\cdots \text{ to } n \text{ terms}).$$

Therefore a comparison between the results so obtained, as representing a comparison between the subsequent periods themselves, means a comparison in this form,

$$\frac{P_3}{P_2} = \frac{\frac{1}{n_3''}(\mathbf{a}_3\alpha_3' + \mathbf{b}_3\beta_3' + \cdots\cdots \text{ to } n \text{ terms})}{\frac{1}{n_2''}(\mathbf{a}_2\alpha_2' + \mathbf{b}_2\beta_2' + \cdots\cdots \text{ to } n \text{ terms})},$$

which, by restoring the values of n_2'' and n_3'', and converting, reduces to

$$\frac{P_3}{P_2} = \frac{\mathbf{a}_3\alpha_3' + \mathbf{b}_3\beta_3' + \cdots\cdots}{\mathbf{a}_2\alpha_2' + \mathbf{b}_2\beta_2' + \cdots\cdots} \cdot \frac{\mathbf{a}_2 + \mathbf{b}_2 + \cdots\cdots}{\mathbf{a}_3 + \mathbf{b}_3 + \cdots\cdots};$$

and the same may be more generally stated thus,

$$\frac{P_3}{P_2}=\frac{a_3\frac{\alpha_3}{\alpha_1}+b_3\frac{\beta_3}{\beta_1}+\cdots\cdots}{a_2\frac{\alpha_2}{\alpha_1}+b_2\frac{\beta_2}{\beta_1}+\cdots\cdots}\cdot\frac{a_2+b_2+\cdots\cdots}{a_3+b_3+\cdots\cdots};$$

or, lastly, in this form,

$$\frac{P_3}{P_2}=\frac{x_3\frac{\alpha_3^2}{\alpha_1}+y_3\frac{\beta_3^2}{\beta_1}+\cdots\cdots}{x_2\frac{\alpha_2^2}{\alpha_1}+y_2\frac{\beta_2^2}{\beta_1}+\cdots\cdots}\cdot\frac{x_2\alpha_2+y_2\beta_2+\cdots\cdots}{x_3\alpha_3+y_3\beta_3+\cdots\cdots}.$$

The inverse of this formula gives

$$\frac{Mo3}{Mo2}=\frac{x_3\alpha_3+y_3\beta_3+\cdots\cdots}{x_2\alpha_2+y_2\beta_2+\cdots\cdots}\cdot\frac{x_2\frac{\alpha_2^2}{\alpha_1}+y_2\frac{\beta_2^2}{\beta_1}+\cdots\cdots}{x_3\frac{\alpha_3^2}{\alpha_1}+y_3\frac{\beta_3^2}{\beta_1}+\cdots\cdots},$$

which we recognize as a formula with double weighting (with very curious weighting in the separate averages at each period).

IV. SCROPE'S METHOD AND ITS VARIETIES.

§ 1. Scrope, 1833 ; see B. 9.

The method there described may be thus formulated,

$$\frac{P_2}{P_1}=\frac{\frac{1}{n'}(x\alpha_2+y\beta_2+\cdots\cdots\text{ to } n \text{ terms})}{\frac{1}{n'}(x\alpha_1+y\beta_1+\cdots\cdots\text{ to } n \text{ terms})},$$

this representing the variation in the "mean"; or "in the sum" as follows,

$$\frac{P_2}{P_1}=\frac{x\alpha_2+y\beta_2+\cdots\cdots}{x\alpha_1+y\beta_1+\cdots\cdots}.$$

Scrope did not specify at which period, or how, the mass-quantities are to be chosen. All he had in mind was a general average of some sort.

We know that this method is the same as the arithmetic average of the price variations with weighting according to $x\alpha_1$, $y\beta_1$, ······ (Appendix A, II, § 8), and as the harmonic average of the price variations with weighting according to $x\alpha_2$, $y\beta_2$, ······ (Appendix A, IV. § 3), and, in some cases, as the geometric average of the price variations with weighting according to the geometric means between these weights (Appendix A, VI. §§ 7–8). As it virtually employs arithmetic averages of the prices at each period, it is simpler to regard it as the arithmetic average of the price variations with the weighting of the first period, and it could be expressed thus, in the comparison of the second period with the first,

$$\frac{P_2}{P_1}=\frac{1}{n_1''}\left(a_1\frac{\alpha_2}{\alpha_1}+b_1\frac{\beta_2}{\beta_1}+\cdots\cdots\text{ to } n \text{ terms}\right).$$

In this form it might be regarded as a variety of Young's method, and it was originally no doubt suggested without an idea of its being other than Young's method. But it is essentially different.

Its differentia from Young's method is that it uses mass-quantities, with hidden weighting of the price variations, instead of direct weighting according to total money-values. This rests on the difference that it compares price averages (or price sums) at two or more periods, instead of directly averaging price variations between two periods.

But just as Young's method uses single weighting in every average of price variations, so Scrope's method uses the same mass-quantities at both periods in every comparison.

Used, as recommended by Scrope himself, in a series of years, always on the same mass-quantities, if the total sum of the money-values at the first period is reduced, as usually is the case, to 100, then the others are to be reduced accordingly merely by dividing them by the same figure. Suppose this figure is r. Then the index-numbers (I) for a series of years are

$$I_1 = \frac{1}{r}(x\alpha_1 + y\beta_1 + \cdots\cdots) = 100,$$

$$I_2 = \frac{1}{r}(x\alpha_2 + y\beta_2 + \cdots\cdots),$$

$$I_3 = \frac{1}{r}(x\alpha_3 + y\beta_3 + \cdots\cdots),$$

and so on. A comparison between any of these index-numbers obviously yields the same result as a direct comparison between the corresponding periods (using the same mass-quantities).

This method, with more or less undefined mass-quantities, has been recommended and formulated by Fauveau, B. 54, p. 356, and by Walras (who calls it the method of "the multiple standard"), B. 69, pp. 15–16, B. 70, pp. 432, 468, B. 71, p. 131; and has been recommended by Newcomb, B. 76, p. 211, Andrews, B. 107 and 108, J. A. Smith, B. 129, pp. 27–29, Pomeroy, B. 135, p. 332, and Parsons, B. 136, pp. 134–136.

§ 2. Definiteness in regard to the mass-quantities may be given to this method in several ways, among which three have been noticed; (1) by employing the mass-quantities of the first (or basic) period; (2) by employing the mass-quantities of the other periods which are compared with the first (or basic) period; or (3) by employing some mean between these methods, two such having been suggested. All these have been formulated, most of them recommended, and some of them applied.

The first two were first distinguished and formulated by Drobisch, B. 29, pp. 36–39, who added notice also of one form of the third, B. 31, pp. 423–425, but who rejected them all, except in an accidental case when it happens that the mass-quantities are the same at both the periods compared. They have all been similarly condemned by Wicksell, unless it accidentally happens that this method tried in the first way and in the second way yields the same result, B. 139, pp. 11–12. On the other hand all three were considered equally good by Sidgwick, who therefore regarded none as authoritative, B. 56, pp. 67–68, (cf. Edgeworth, B. 59, p. 264). Similarly Padan, B. 141

But Lindsay, likewise considering all three equally good, and wanting only one to be employed, is indifferent which be adopted, B. 114, pp. 25-26.

(1) The method with the mass-quantities of the first or basic period was formulated and recommended by Laspeyres, B. 26, p. 306 (who seems to have been unaware that Drobisch had already formulated and condemned it). Adapted, and modified (by omitting use of 100), his formula is

$$\frac{P_2}{P_1} = \frac{x_1 a_2 + y_1 \beta_2 + \cdots\cdots}{x_1 a_1 + y_1 \beta_1 + \cdots\cdots}.$$

In a series of years later comparisons would be in this form,

$$\frac{P_3}{P_2} = \frac{x_1 a_3 + y_1 \beta_3 + \cdots\cdots}{x_1 a_2 + y_1 \beta_2 + \cdots\cdots},$$

in which the weighting of the price variations (arithmetically averaged) is rather curious, being according to the money-values of the mass-quantities of the first or basic period at the prices of the earlier of the two periods compared.

This variety of the method has been used by Bourne, in one of his operations upon custom-house returns, B. 47-49. The next variety, as used by Sauerbeck for periods prior to his basic period, is like this in using the mass-quantities of the earlier periods (but not those of the basic period), (cf. Edgeworth, B. 59, p. 264).

(2) The method with the mass-quantities of the later periods (or, more generally, of the periods other than the basic) was first consciously employed by Paasche, B. 33, pp. 171-173, who has been followed by v. d. Borght, B. 55, and Conrad, B. 96 and 97. The same method has also been employed, in a very general way, and dogmatically, by Mulhall, B. 74, p. 1157, B. 75, p. 1; and in his second "test" by Sauerbeck, B. 82-90, who is, partly, followed by Powers, B. 131, see p. 29.

The formula for the method in this form is

$$\frac{P_2}{P_1} = \frac{x_2 a_2 + y_2 \beta_2 + \cdots\cdots}{x_2 a_1 + y_2 \beta_1 + \cdots\cdots}$$

(cf. Edgeworth, B. 59, p. 264). Thus the weighting in (arithmetically) averaging the price variations is according to the money-values of the mass-quantities of the other period at the prices of the first or basic period.

In a series of years, a comparison between any two of the later years would be as follows,

$$\frac{P_3}{P_2} = \frac{\dfrac{x_3 a_3 + y_3 \beta_3 + \cdots\cdots}{x_3 a_1 + y_3 \beta_1 + \cdots\cdots}}{\dfrac{x_2 a_2 + y_2 \beta_2 + \cdots\cdots}{x_2 a_1 + y_2 \beta_1 + \cdots\cdots}}$$

$$= \frac{x_3 a_3 + y_3 \beta_3 + \cdots\cdots}{x_2 a_2 + y_2 \beta_2 + \cdots\cdots} \cdot \frac{x_2 a_1 + y_2 \beta_1 + \cdots\cdots}{x_3 a_1 + y_3 \beta_1 + \cdots\cdots},$$

which uses double weighting in a curious form (the weighting of the average at each period compared being according to the money-values of its mass-quantities at the prices of the basic period). If the prices have all been reduced so that they are at unity (or 100) at the first period, the mass quantities being correspondingly (inversely) altered, the formula becomes

$$\frac{P_3}{P_2}=\frac{x_3'a_3'+y_3'\beta_3'+\cdots\cdots}{x_2'a_2'+y_2'\beta_2'+\cdots\cdots}\cdot\frac{x_2'+y_2'+\cdots\cdots}{x_3'+y_3'+\cdots\cdots}.$$

This method had already been employed in a particular application to custom-house returns, in England by James, B. 10, (his method being based upon the peculiarity that British exports were officially rated always at the same prices at which they had been originally rated in 1694,[1] and that since 1798 there were added to the records also the "declared values" according to the current prices set upon their goods by the merchants themselves, so that his weighting for each year was according to the money-values of its mass-quantities at the prices of 1694); and in France (on the "official values" based on the prices of 1826 and the "actual values" recorded after 1847), both for imports and exports, separately and combined, in one jump from 1826 to 1847 and thereafter annually to 1856, by Levasseur, B. 18, pp. 181–184, 188–192. This work was resumed by De Foville, B. 50, who continued it down to 1862, when the "official values" on the old prices of 1826 were discontinued. (Thereafter the French Custom-house every year first published "provisional values" consisting of this year's mass-quantities at the last year's prices, and later the "actual values" consisting of this year's mass-quantities at this year's prices. To compare the latter with the former for every year is always to do the same operation as here indicated in the formula for $\frac{P_2}{P_1}$, a new basis being employed every time. De Foville continued his series in this way down to 1880, stringing them out in a series following upon the previous series.)

(3) A mean between the first two varieties may be by merely using each separately and drawing the (evenly weighted arithmetic) mean between their results. The operation is expressed by this formula,

$$\frac{P_2}{P_1}=\frac{1}{2}\left(\frac{x_1a_2+y_1\beta_2+\cdots\cdots}{x_1a_1+y_1\beta_1+\cdots\cdots}+\frac{x_2a_2+y_2\beta_2+\cdots\cdots}{x_2a_1+y_2\beta_1+\cdots\cdots}\right).$$

This appears to be the form Drobisch had in mind. (It is so understood by Edgeworth, B. 59, p. 265.)

Another is to form one calculation on the (evenly weighted arithmetic) mean of the two mass-quantities of every class, according to this formula,

$$\frac{P_2}{P_1}=\frac{\frac{1}{2}(x_1+x_2)a_2+\frac{1}{2}(y_1+y_2)\beta_2+\cdots\cdots}{\frac{1}{2}(x_1+x_2)a_1+\frac{1}{2}(y_1+y_2)\beta_1+\cdots\cdots}.$$

This formula, which reduces to

$$\frac{P_2}{P_1}=\frac{(x_1+x_2)a_2+(y_1+y_2)\beta_2+\cdots\cdots}{(x_1+x_2)a_1+(y_1+y_2)\beta_1+\cdots\cdots},$$

[1] Or 1696 or 1697; cf. Flux, B. 140, p. 67, who has recently employed the same method.

was independently suggested by Marshall and Edgeworth (according to the latter, B. 51, p. 265), and is recommended by the latter, *ibid.*, p. 266 (but not decisively, cf. p. 255), and by the British Association Committee, *First Report*, B. 75, pp. 249–250. Lindsay recommends it when comparing periods long separated, B. 114, p. 25. It is formulated by Zuckerkandl, B. 115, p. 248, B. 116, p. 247. And it is experimented with by Powers, B. 131, pp. 28–29.

As regards the relationship between these two sub-varieties, it is obvious that if in the formula for the first the denominators happen to be equal as wholes, this formula reduces to the formula for the second; but not otherwise.

The arithmetic average of the mass-quantities over the whole epoch investigated (in his case of thirty five years) has been recommended as the best, and employed, by Powers, B. 131, p. 24.

Another mean of the two mass-quantities of every class is the evenly weighted geometric. Scrope's method applied to such mean mass-quantities is recommended in this work (Chapt. XII.) under the name of Scrope's emended method. Its formula is

$$\frac{P_2}{P_1} = \frac{a_2\sqrt{x_1x_2} + \beta_2\sqrt{y_1y_2} + \cdots\cdots}{a_1\sqrt{x_1x_2} + \beta_1\sqrt{y_1y_2} + \cdots\cdots}.$$

§ 3. It may here be added that it would be obviously absurd to attempt to use the mass-quantities at both periods in the following way,

$$\frac{P_2}{P_1} = \frac{x_2a_2 + y_2\beta_2 + \cdots\cdots}{x_1a_1 + y_1\beta_1 + \cdots\cdots};$$

for this would be merely a comparison between the total valuations or inventories at the two periods, which might be altered by a change in the mass-quantities without any price variations. But if we suppose that the mass-quantities have not changed, *i. e.*, that $x_2 = x_1$, $y_2 = y_1$, and so on, this formula would be the same as any of the preceding. On the supposition that the mass-quantities have remained *nearly* the same, it is employed by Edgeworth, B. 59, p. 272, cf. pp. 264, 293; also, inverted for the exchange-value of money, by Nicholson, see here below, V. § 3. Furthermore, if the mass-quantities all change alike, so that $\frac{x_2}{x_1} = \frac{y_2}{y_1} = \cdots\cdots = r$, this formula would also be serviceable if we multiply the denominator by r, in order to counteract the change of the mass-quantities; for then

$$\frac{P_2}{P_1} = \frac{x_2a_2 + y_2\beta_2 + \cdots\cdots}{r(x_1a_1 + y_1\beta_1 + \cdots\cdots)}$$

reduces to

$$\frac{P_2}{P_1} = \frac{x_2a_2 + y_2\beta_2 + \cdots\cdots}{x_2a_1 + y_2\beta_1 + \cdots\cdots};$$

or if we multiply the numerator by $\frac{1}{r}$; for then it reduces to

$$\frac{P_2}{P_1} = \frac{x_1a_2 + y_2\beta_2 + \cdots\cdots}{x_1a_1 + y_1\beta_1 + \cdots\cdots},$$

and these two reduced expressions are equal, having really been reached in the same way, namely by multiplying the original expression by $\frac{1}{r}$. Cf. Appendix A, VII. § 5. These properties have been made use of by Nicholson, see below V. § 3.

On the same two suppositions the following formula (of Drobisch's method) would be serviceable,

$$\frac{P_2}{P_1}=\frac{\frac{1}{n_2'}(x_2a_2+y_2\beta_2+\cdots\cdots \text{ to } n \text{ terms})}{\frac{1}{n_1'}(x_1a_1+y_1\beta_1+\cdots\cdots \text{ to } n \text{ terms})};$$

for, in the second, more comprehensive, case, this would reduce to

$$\frac{P_2}{P_1}=\frac{\frac{1}{n_2'}(x_2a_2+y_2\beta_2+\cdots\cdots \text{ to } n \text{ terms})}{\frac{r}{rn_2'}(x_2a_1+y_2\beta_1+\cdots\cdots \text{ to } n \text{ terms})}=\frac{x_2a_2+y_2\beta_2+\cdots\cdots}{x_2a_1+y_2\beta_1+\cdots\cdots},$$

or to

$$\frac{P_2}{P_1}=\frac{\frac{r}{rn_1'}(x_1a_2+y_1\beta_2+\cdots\cdots \text{ to } n \text{ terms})}{\frac{1}{n_1'}(x_1a_1+y_1\beta_1+\cdots\cdots \text{ to } n \text{ terms})}=\frac{x_1a_2+y_1\beta_2+\cdots\cdots}{x_1a_1+y_1\beta_1+\cdots\cdots};$$

but not otherwise.

Thus in cases when the mass-quantities are the same, or proportional, at both or all the periods compared, all the varieties of the method reviewed in this Section (and some of the methods to be reviewed in the next) are the same.[2]

Now we know (see Chapt. XI. Sect. III. §§ 2-3) that on the suppositions here made, namely of the mass-quantities being the same, or proportional, at both or all the periods compared, Scrope's method, applied to the same mass-quantities, or to the proportional ones at either period, or to any mean or average between them, is perfectly correct. This has been recognized, not only by Edgeworth and Nicholson, as above, but also by Drobisch, B. 29, pp. 30, 37, B. 30, pp. 147-148; Sidgwich, B. 56, pp. 66-67; Lehr, B. 68, p. 40; Zuckerkandl, B. 115, p. 247, B. 116, pp. 244-245; Wicksell, B. 139, pp. 8-9; Padan, B. 141.

V. METHODS EMPLOYING DOUBLE WEIGHTING.

§ 1. **Drobisch's Method.** Drobisch, 1871; see B. 29 and 30 (31 being apologetic).

He always considered the variation of the arithmetic averages to be this, $\frac{xa_2+y\beta_2+\cdots\cdots}{xa_1+y\beta_1+\cdots\cdots}$, *only* when $xa_1=y\beta_1=\cdots\cdots$, (B. 29, p. 36, cf. pp. 31, 39) *i. e.*, only when this expression is equivalent to $\frac{1}{n}\left(\frac{a_2}{a_1}+\frac{\beta_2}{\beta_1}+\cdots\cdots\right)$, (see

[2] This is not recognized by Powers, B. 131, pp. 27-28, where, by faulty arithmetic, he divides into two distinct methods what is really, in the case supposed, only one method.

Appendix A, II. § 9). In other words, he always took "arithmetic average" (applied to price variations) to mean only arithmetic average *with even weighting*. And then he rightly objected to the general use of the arithmetic average so understood, B. 29, p. 32, and rightly condemned Laspeyres for using it, B. 30, p. 145, this average being correct only in the particular case when the mass-quantities (numbers of the mass-units) happen to be in inverse proportion to the prices (of the same mass-units) at the first period. His objection, evidently, was only against the use of even weighting in averaging price variations; but he thought it was against the use of the arithmetic average in general.[1] He also objected to the method of using uneven weighting when only the mass-quantities at one of the periods are considered, the change in the mass-quantities at the other period being disregarded, and also to the method of using the mean between the two methods (or the mean of the mass-quantities?), all these methods giving different results, none better than another, B. 29, pp. 36–39, B. 31, pp. 423–425. He thought that "the following consideration does away with all difficulties," B. 29, p. 39.

Taking his prices always to be the prices of the same mass-unit (preferably a hundredweight) of every class of goods, and consequently his mass-quantities to be the numbers of these mass-units of all the classes of goods marketed at the place in question during the period in question, B. 29, p. 34, B. 30, p. 148, he represented the total mass-quantities of the classes [A], [B], ······ marketed at the first period by $x_1 + y_1 + \cdots\cdots$, and the sums spent on them by $x_1\alpha_1 + y_1\beta_1 + \cdots\cdots$; and hereupon he asserted that the average price of the mass-unit is *not* $\frac{1}{n}\left(\frac{x_1\alpha_1}{x_1} + \frac{y_1\beta_1}{y_1} + \cdots\cdots\right)$, $\left[\text{not } \frac{1}{n}(\alpha_1 + \beta_1 + \cdots\cdots)\right]$, but $\frac{x_1\alpha_1 + y_1\beta_1 + \cdots\cdots}{x_1 + y_1 + \cdots\cdots}$, (and remarked that even here he was not using the arithmetic average, and concluded that his method does not use the arithmetic average at all, any more than it uses the harmonic or the geometric, but only the rule-of-three, B. 29, p. 40). Proceeding in the same way for the second period, he represented the then average price of a mass-unit of goods by $\frac{x_2\alpha_2 + y_2\beta_2 + \cdots\cdots}{x_2 + y_2 + \cdots\cdots}$. Therefore, neglecting any average of the price variations, he represented the variation of the price-average by the variation of the average price of the common mass-unit of goods, thus,

$$\frac{P_2}{P_1} = \frac{\dfrac{x_2\alpha_2 + y_2\beta_2 + \cdots\cdots}{x_2 + y_2 + \cdots\cdots}}{\dfrac{x_1\alpha_1 + y_1\beta_1 + \cdots\cdots}{x_1 + y_1 + \cdots\cdots}}$$

$$= \frac{x_2\alpha_2 + y_2\beta_2 + \cdots\cdots}{x_1\alpha_1 + y_1\beta_1 + \cdots\cdots} \cdot \frac{x_1 + y_1 + \cdots\cdots}{x_2 + y_2 + \cdots\cdots};$$

[1] He also noticed that if the mass-quantities happen to be in inverse proportions to the prices at the second period, the above expression reduces to the simple harmonic average of the price variations (cf. Appendix A, IV. § 2); and concluded that even on the assumption of the mass-quantities being the same at both periods, the "arithmetic average" is no better than the "harmonic average," and *vice versâ*, each being proper in certain circumstances and improper in all

and the variation of the exchange-value of money (*Geldwert*), he added, is the inverse of this, B. 29, p. 39, B. 30, pp. 148-149.

We see that this is nothing but a method using the arithmetic average with different numbers of figures at each period—*i. e.*, with double weighting (cf. Appendix A, VII. § 1). The arithmetic average used by Drobisch, when he is drawing the average price of the mass-unit of all goods at each period separately, is the least imperfect distinct average (for each period separately) that he could draw, since it takes into account the numbers of times the mass-unit is repeated in every class, while the simple average, with even weighting of the classes, in the form rejected by him, would obviously be imperfect. But the fact that the average he rejected is an arithmetic average, does not prevent the average he adopted from being an arithmetic average.[2] He thought he disproved the arithmetic and harmonic, and also the geometric, averages, and proved that a method using none of them is the right one. As a matter of fact, his arguments never touch upon the question as to which kind of average is the correct one, but only show that all of them are wrong with even weighting—which everybody admits. And he re-adopted the arithmetic average without any argument, but assumed it. His attention was really concentrated upon the question of weighting, and it was here that he made advance.

In this method it is material that the mass-unit should be the same in all the classes of things. For, although in the numerator for the average at each period every term (*e. g.*, $x_1 a_1$) would be the same whatever mass-units are employed, yet in the denominator the corresponding term (*e. g.*, x_1) would be larger or smaller according as the mass-unit is smaller or larger.[3]

The peculiarity of Drobisch's method, then, is that it uses the arithmetic average (1) with double weighting (2) on the same mass-unit for all the classes.

This method is praised by Roscher, B. 32, p. 275; criticized by Laspeyres (see here Chapt. V. Sect. VI. § 5, Note 11), by Paasche (see here Chapt. IV. Sect. V. § 1), and by Lehr and Zuckerkandl (see here Chapt. V. Sect. VI. § 5, Note 15); noticed by Edgeworth, B. 59, p. 265; and reviewed by Lindsay (who said it rested on the harmonic mean!), B. 114, pp. 18-21. It has even been used, in application to goods commonly measured in the

others, B. 29, pp. 36-37—a conclusion perfectly correct concerning these averages of the price variations, each with even-weighting, but without any application whatever against either with its proper weighting.

[2] Like Drobisch in condemning the "arithmetic average" while using it themselves (their objection being really against only the arithmetic average of the price variations with even weighting) are Geyer, B. 28, p. 321, and Paasche, B. 33, p. 171, cf. p. 172, B. 34, p. 60, cf. p. 64. Even Walras does not appear to perceive that the formula for Scrope's method (using the same mass quantities at each period) is one employing arithmetic averages (and is connected with the arithmetic average of price variations), sometimes contrasting it with the geometric, the harmonic and the arithmetic averages of the price variations (always with even weighting), as if these alone were averages. For the references, see above in IV. § 1 end.

[3] Drobisch's method is the only one yet invented in which this feature is essential.

same mass-unit, without knowledge of Drobish, by Powers, B. 131, see p. 29.

§ 2. **Lehr's method.** Lehr, 1885; see B. 68.

Starting out, like Drobisch, by supposing the prices of all articles to be expressed on the same mass-unit (by weight), although the rest of his method makes this unnecessary, Lehr notices that over the two periods together there is spent the sum of $(x_1a_1 + x_2a_2)$ money-units for $(x_1 + x_2)$ mass-units of the class [A], wherefore the average price of one mass-unit of [A] for both the periods is $\frac{x_1a_1 + x_2a_2}{x_1 + x_2}$ money-unit, and the money-unit purchases on the average during both the periods $\frac{x_1 + x_2}{x_1a_1 + x_2a_2}$ mass-unit of [A]. And similarly for the article [B] over the two periods together, there is spent the sum of $(y_1\beta_1 + y_2\beta_2)$ money-units for $(y_1 + y_2)$ mass-units, wherefore the average price of one mass-unit of [B] for both the periods is $\frac{y_1\beta_1 + y_2\beta_2}{y_1 + y_2}$ money-unit, and the money-unit purchases on the average during both the periods $\frac{y_1 + y_2}{y_1\beta_1 + y_2\beta_2}$ mass-unit. And so on with the other classes. Now in every class the mass-quantity which the money-unit purchases on the average over the two periods together—or the mass-quantity of which the average price during the two periods together is one money-unit—he calls a "pleasure-unit," p. 38, having first defined a pleasure-unit to be an equivalent quantity of any goods, pp. 37–38. The total quantity of [A] which comes into trade during the first period consists of $\frac{x_1}{\frac{x_1 + x_2}{x_1a_1 + x_2a_2}}$ or $x_1\left(\frac{x_1a_1 + x_2a_2}{x_1 + x_2}\right)$ such pleasure-units; and the total quantity of [A] which comes into trade during the second period consists of $x_2\left(\frac{x_1a_1 + x_2a_2}{x_1 + x_2}\right)$ such pleasure-units. Similarly the total quantities of [B] coming into trade during each of the periods consist of $y_1\left(\frac{y_1\beta_1 + y_2\beta_2}{y_1 + y_2}\right)$ and $y_2\left(\frac{y_1\beta_1 + y_2\beta_2}{y_1 + y_2}\right)$ such pleasure-units respectively. And so on. During the first period, then, there come into trade

$$x_1\left(\frac{x_1a_1 + x_2a_2}{x_1 + x_2}\right) + y_1\left(\frac{y_1\beta_1 + y_2\beta_2}{y_1 + y_1}\right) + \cdots\cdots \text{ to } n \text{ terms}$$

pleasure-units, for which are paid $(x_1a_1 + y_1\beta_1 + \cdots\cdots$ to n terms) money-units; wherefore the pleasure-units cost on the average, or their average price, expressed in money-units, is

$$P_1 = \frac{x_1a_1 + y_1\beta_1 + \cdots\cdots}{x_1\left(\frac{x_1a_1 + x_2a_2}{x_1 + x_2}\right) + y_1\left(\frac{y_1\beta_1 + y_2\beta_2}{y_1 + y_2}\right) + \cdots\cdots}.$$

And during the second period there come into trade

$$x_2\left(\frac{x_1a_1 + x_2a_2}{x_1 + x_2}\right) + y_2\left(\frac{y_1\beta_1 + y_2\beta_2}{y_1 + y_2}\right) + \cdots\cdots \text{ to } n \text{ terms}$$

pleasure-units, costing $(x_2\alpha_2 + y_2\beta_2 + \cdots\cdots$ to n terms) money-units; wherefore

$$P_2 = \frac{x_2\alpha_2 + y_2\beta_2 + \cdots\cdots}{x_2\left(\dfrac{x_1\alpha_1 + x_2\alpha_2}{x_1 + x_2}\right) + y_2\left(\dfrac{y_1\beta_1 + y_2\beta_2}{y_1 + y_2}\right) + \cdots\cdots}.$$

Therefore the variation of these average prices is the quotient of the expression for P_2 divided by the expression for P_1, which reduces to this,

$$\frac{P_2}{P_1} = \frac{x_2\alpha_2 + y_2\beta_2 + \cdots\cdots}{x_1\alpha_1 + y_1\beta_1 + \cdots\cdots} \cdot \frac{x_1\left(\dfrac{x_1\alpha_1 + x_2\alpha_2}{x_1 + x_2}\right) + y_1\left(\dfrac{y_1\beta_1 + y_2\beta_2}{y_1 + y_2}\right) + \cdots\cdots}{x_2\left(\dfrac{x_1\alpha_1 + x_2\alpha_2}{x_1 + x_2}\right) + y_2\left(\dfrac{y_1\beta_1 + y_2\beta_2}{y_1 + y_2}\right) + \cdots\cdots}$$

and the reciprocal of this, Lehr adds, gives the variation of the exchange-value of money, pp. 38-40.

Comparing this formula with Drobisch's, we see that Lehr's formula differs from Drobisch's only in the second half, and there by his multiplying both x_1 and x_2 by a certain expression, which is the average price of the mass-unit of [A] during both the periods, and by multiplying both y_1 and y_2 by another expression, which is the average price of a mass-unit of [B] during both the periods, and so on, in every instance the average being the arithmetic average with weighting according to the mass-quantities of each period.

Hence the peculiarity of Lehr's method is that it uses the arithmetic average (1) with double weighting (2) on mass-units that have the same (unevenly weighted arithmetic) average price over both the periods compared.

That the actual mass-unit used in every class is indifferent is plain from the fact that not only the terms, *e. g.*, $x_2\alpha_2$, in the first half, but also in the second half the terms, *e. g.*, $x_1\left(\dfrac{x_1\alpha_1 + x_2\alpha_2}{x_1 + x_2}\right)$, have the same value whatever be the mass-unit whose price is α and whose number is x.

This method is reviewed by Edgeworth, B. 59, pp. 265-266; declared the best by Zuckerkandl, B. 115, p. 248, B. 116, p. 248, and Wiebe, B. 124, p. 165; and by Lindsay, B. 114, pp. 22-24, criticised for seeking to measure variations in the mass-quantities! Pointing out the error in this criticism, Wicksell tries to refute the method by a test case, which supposes the prices of certain articles to be zero at the one or the other period [and which therefore constitutes a case in which these articles ought to be excluded from the measurement], B. 139, pp. 10-11.

§ 3. **Nicholson's method.** Nicholson, 1887; see B. 94.

Independently, and seemingly without knowledge of Drobisch's method, or of Lehr's, Nicholson has invented a method which, being somewhat vaguely conceived, turns out, on analysis, to be either Drobisch's method mutilated, or an imperfect form of Scrope's, or something else.

Representing the aggregate wealth of a country expressed in the money-unit (the pound sterling) by £w, or by w, he sets out with a formula, for the first period, which in our notation is this, $£_1 w_1 = x_1\alpha_1 + y_1\beta_1 + \cdots\cdots$; wherefore the purchasing power [or exchange-value in all other things] of the money-unit is $\frac{1}{x_1a_1 + y_1\beta_1 + \cdots\cdots}$, pp. 307–308. Similarly at the second period, on the supposition that the mass-quantities are about the same as before, and no new kinds of articles have been added or subtracted, we should have $£_2w_2 = x_2a_2 + y_2\beta_2 + \cdots\cdots$, and the purchasing power of the money-unit is $\frac{1}{x_2a_2 + y_2\beta_2 + \cdots\cdots}$. Therefore we should have (representing the exchange-value of money in all other things by M_O),

$$\frac{M_{O2}}{M_{O1}} = \frac{\frac{1}{x_2a_2 + y_2\beta_2 + \cdots\cdots}}{\frac{1}{x_1a_1 + y_1\beta_1 + \cdots\cdots}} = \frac{x_1a_1 + y_1\beta_1 + \cdots\cdots}{x_2a_2 + y_2\beta_2 + \cdots\cdots}.$$

This he represents as $\frac{w_1}{w_2}$, and says that, on these suppositions, "the change in the purchasing power of the £ is equal to the fraction $\frac{w_1}{w_2}$," and calls this fraction "the coefficient of appreciation," pp. 309–310. The same is true, he further says, if the mass-quantities have increased uniformly, which uniform increase he represents by m, $\left[\text{wherefore } m = \frac{x_2}{x_1} = \frac{y_2}{y_1} = \cdots\cdots = \frac{x_2 + y_2 + \cdots\cdots}{x_1 + y_1 + \cdots\cdots}\right]$, on the ground that "the purchasing power of the standard will obviously be the same for the old inventory and for every uniform addition to it," so that, although the new coefficient of appreciation for these cases is $\frac{w_1}{w_2} \cdot m$, yet either "the change in the standard may be measured by the old inventory at the old prices divided by the old in-inventory at the new prices" $\left[i.\ e., \frac{M_{O2}}{M_{O1}} = \frac{x_1a_1 + y_1\beta_1 + \cdots\cdots}{x_1a_2 + y_1\beta_2 + \cdots\cdots}\right]$, pp. 310–311, or "the change in the purchasing power of the standard is found by dividing the value of the new inventory at the old prices by its value at the new" $\left[i.\ e., \frac{M_{O2}}{M_{O1}} = \frac{x_2a_1 + y_2\beta_1 + \cdots\cdots}{x_2a_2 + y_2\beta_2 + \cdots\cdots}\right]$, p. 313. [Cf. above IV. § 3.—Thus far we have merely Scrope's method inverted, because Nicholson is dealing with variations in the exchange-value of money in all other things.] All this he thinks accurate enough when we are dealing with periods near together. His new method he begins when dealing with periods far apart, in which the classes of goods increase and decrease, and old ones become extinct and new ones come in—and all deserve to be allowed for. Here he again lets m represent the increase in the mass quantities. He is now very vague as to how m is to be estimated. He is willing to estimate it "generally," p. 312, or "from various independent considerations," p. 316. When the classes are the same at both the periods, he describes the estimation more minutely, in a way expressible in our notation thus, $m = \frac{x_2a_1 + y_2\beta_1 + \cdots\cdots}{x_1a_1 + y_1\beta_1 + \cdots\cdots}$, pp. 312–313 (cf. Bourne in one of his methods, B. 47–49). When the classes are not all the same, he says "m should be proportioned to the increase in the population and to the increase in the efficiency of the labor and capital," p. 316. Such an increase would seem to be represented either by $\frac{x_2 + y_2 + \cdots\cdots}{x_1 + y_1 + \cdots\cdots}$ (provided the numbers be expressed as those of some common mass-unit, about which Nicholson is silent), or, again supposing the classes to be the same at both the periods, the increase would seem to be represented by $\frac{1}{n}\left(\frac{x_2}{x_1} + \frac{y_2}{y_1} + \cdots\cdots\right)$. However this be, the new coefficient of appreciation we find still to be $\frac{w_1}{w_2} \cdot m$, which cannot be reduced now as it could be before, p. 315. Hence, the variation of "general prices" being the inverse of the purchasing power of money, pp. 317, 318, his formula is

$$\frac{P_2}{P_1} = \frac{w_2}{w_1 m},$$

which, in full, according to the methods of estimating m, is either

$$\frac{P_2}{P_1} = \frac{x_2 a_2 + y_2 \beta_2 + \cdots\cdots}{x_1 a_1 + y_1 \beta_1 + \cdots\cdots} \cdot \frac{x_1 a_1 + y_1 \beta_1 + \cdots\cdots}{x_2 a_1 + y_2 \beta_1 + \cdots\cdots}, \qquad (1)$$

or

$$\frac{P_2}{P_1} = \frac{x_2 a_2 + y_2 \beta_2 + \cdots\cdots}{x_1 a_1 + y_1 \beta_1 + \cdots\cdots} \cdot \frac{x_1 + y_1 + \cdots\cdots}{x_2 + y_2 + \cdots\cdots}, \qquad (2)$$

or

$$\frac{P_2}{P_1} = \frac{x_2 a_2 + y_2 \beta_2 + \cdots\cdots}{x_1 a_1 + y_1 \beta_1 + \cdots\cdots} \cdot \frac{1}{n}\left(\frac{x_1}{x_2} + \frac{y_1}{y_2} + \cdots\cdots\right). \qquad (3)$$

The first reduces to

$$\frac{P_2}{P_1} = \frac{x_2 a_2 + y_2 \beta_2 + \cdots\cdots}{x_2 a_1 + y_2 \beta_1 + \cdots\cdots},$$

which is Paasche's variety of Scrope's method. It is plainly no better than

$$\frac{P_2}{P_1} = \frac{x_2 a_2 + y_2 \beta_2 + \cdots\cdots}{x_1 a_1 + y_1 \beta_1 + \cdots\cdots} \cdot \frac{x_1 a_2 + y_1 \beta_2 + \cdots\cdots}{x_2 a_2 + y_2 \beta_2 + \cdots\cdots},$$

which reduces to

$$\frac{P_2}{P_1} = \frac{x_1 a_2 + y_1 \beta_2 + \cdots\cdots}{x_1 a_1 + y_1 \beta_1 + \cdots\cdots},$$

which is Laspeyres's variety of Scrope's method. Hence it is evident that in the second half some mean must be employed of the prices at the two periods.

The second is the formula for Drobisch's method. But in this form Nicholson's method is not the same as Drobisch's, since Nicholson does not specify how the mass-units are to be chosen. In this form this method is wholly haphazard.

The third is the most original of the three forms.[4] But it contains a de-

[4] A variation upon it may be suggested in the following,

$$\frac{P_2}{P_1} = \frac{x_2 a_2 + y_2 \beta_2 + \cdots\cdots}{x_1 a_1 + y_1 \beta_1 + \cdots\cdots} \cdot \sqrt[n]{\frac{x_1}{x_2} \cdot \frac{y_1}{y_2} \cdots\cdots}.$$

fect in averaging the inverted variations of the mass-quantities with even weighting.[5]

Which of these three the method really is, depends upon the way the calculations have been made by the statisticians upon whom Nicholson is willing to rely. For a peculiarity in his method, as used by himself, is that he makes separate use of the two halves in the formula. He accepts some statistician's rough appraisal of the comparative wealth, expressed in money, of a country at two periods, calculated, *e. g.*, from the income tax, p. 318, (this giving $\frac{x_2\alpha_2 + y_2\beta_2 + \cdots\cdots}{x_1\alpha_1 + y_1\beta_1 + \cdots\cdots}$, although no particular things are noticed, and permanent things, such as real estate and other capital, are included, if indeed the appraisal is not principally of them); and again he accepts some other statistician's rough estimate of the comparative wealth, in commodities, of the country at the two periods, calculated, *e. g.*, from the increase of population and improvements in machinery and methods of production, *ibid.*, (this giving, by inversion, either $\frac{x_1 + y_1 + \cdots\cdots}{x_2 + y_2 + \cdots\cdots}$, or $\frac{1}{n}\left(\frac{x_1}{x_2} + \frac{y_1}{y_2} + \cdots\cdots\right)$, indefinitely, without notice of particular things, but this time with reference chiefly to products). In this way he even thought he could show that between 1848 and 1868 money appreciated, or "general prices" fell! pp. 318–320.

This method has been described by the Gold and Silver Commission, *Final Report*, London 1888, p. 23, as "totally distinct" from the usual methods. Its partial similarity with Drobisch's method seems to have escaped notice.

§ 4. **Other methods** involving double weighting we have found to be accidentally produced, in the comparison of later periods with one another, by methods using single weighting in the comparisons of the later periods with a common basic period. For these see above, III. § 4, and IV. § 2 (2). The latter of these may be described as a method the peculiarity of which is that it uses the arithmetic average (1) with double weighting (2) on mass-units that have the same price at some other period chosen as base.

A method suggested in this work in Chapt. XII. Sect. II. § 4, Note 9, as an improvement on Lehr's is

$$\frac{P_2}{P_1} = \frac{x_2\alpha_2 + y_2\beta_2 + \cdots\cdots}{x_1\alpha_1 + y_1\beta_1 + \cdots\cdots} \cdot \frac{x_1(\alpha_1 + \alpha_2) + y_1(\beta_1 + \beta_2) + \cdots\cdots}{x_2(\alpha_1 + \alpha_2) + y_1(\beta_1 + \beta_2) + \cdots\cdots},$$

the peculiarity of which is that it uses the arithmetic average (1) with double weighting (2) on mass-units that have the same (evenly weighted arithmetic) average price over both the periods compared.

[5] An improvement may therefore be suggested as follows,

$$\frac{P_2}{P_1} = \frac{x_2\alpha_2 + y_2\beta_2 + \cdots\cdots}{x_1\alpha_1 + y_1\beta_1 + \cdots\cdots} \cdot \frac{1}{n''}\left(\mathbf{a}\frac{x_1}{x_2} + \mathbf{b}\frac{y_1}{y_2} + \cdots\cdots \text{ to } n \text{ terms}\right),$$

or better still,

$$\frac{P_2}{P_1} = \frac{x_2\alpha_2 + y_2\beta_2 + \cdots\cdots}{x_1\alpha_1 + y_1\beta_1 + \cdots\cdots} \cdot \sqrt[n'']{\left(\frac{x_1}{x_2}\right)^{\mathbf{a}} \cdot \left(\frac{y_1}{y_2}\right)^{\mathbf{b}} \cdot \cdots\cdots \text{ to } n \text{ terms}},$$

in which **a**, **b**, ····· represent means between the weights of the two periods.

A method recommended in this work in Chapt. XII. Sect. II. is

$$\frac{P_2}{P_1} = \frac{x_2\alpha_2 + y_2\beta_2 + \cdots\cdots}{x_1\alpha_1 + y_1\beta_1 + \cdots\cdots} \cdot \frac{x_1\sqrt{x_1x_2} + y_1\sqrt{y_1y_2} + \cdots\cdots}{x_2\sqrt{x_1x_2} + y_2\sqrt{y_1y_2} + \cdots\cdots},$$

the peculiarity of which is that it uses the arithmetic average (1) with double weighting (2) on mass-units that have the same (evenly weighted geometric) average price over both the periods compared.

BIBLIOGRAPHY

OF WORKS DEALING WITH THE MEASUREMENT OF THE EXCHANGE-VALUE OF MONEY BY COMPARING MANY PRICES.

W. Fleetwood

1. Chronicon preciosum: or, an account . . . of the price of corn and other commodities . . . shewing from the decrease of the value of money . . . that a Fellow, who has an estate in land of inheritance, or a perpetual pension of five pounds *per annum*, may conscientiously keep his Fellowship, and ought not to be compelled to leave the same, tho' the Statutes of his College (founded between the years 1440 and 1460) did then vacate his Fellowship on such condition.—London 1707. (2d ed. 1745. See especially pp. 48–49, 135–137.)

   To find the number of pounds which have the same exchange-value as the five pounds formerly had, he inquires how much corn, meat, drink and cloth that sum would then purchase, and what sum is now needed to purchase them. [His proportions are so nearly the same, mostly six times, that he escapes the question of averaging the present prices of the mass-quantities formerly priced at five pounds, and also the question of weighting.]

Dutot

2. Réflexions politiques sur les finances et le commerce.—The Hague 1738, 18mo., Vol. I., pp. 365–377.

   Compares the total sums made up, at two periods (times of Louis XII and Louis XIV), of the prices of the customary mass-units of various articles (including wages). [Uses the arithmetic average of prices with haphazard weighting. See Appendix C, I.]

N. F. Dupré de Saint-Maur

3. Essai sur les monnoies, ou réflexions sur le rapport entre l'argent et les denrées.—Paris 1746, 4to. (The *Réflexions* occupy pp. 19–104. See also pp. 5–6, and in the second part pp. 161–182.)

   Uses many desultory price notices, but principally of grain, to compare the general prices of his day with those in the period before the discovery of America. His general conclusion is an average

of some sort, unspecified. But he uses the arithmetic average, with even weighting, in drawing the average price of single articles over many years. Criticizes Dutot for his data, not for his method.

G. R. Carli

4. Del valore e della proporzione de' metalli monetati con i generi in Italia prima delle scoperte dell' Indie col confronto del valore e della proporzione de' tempi nostri.—1764. (Ed. Custodi, Opere scelte di Carli, Vol. I., pp. 299–366 ; in particular § IV., pp. 335–354.)

Averaging the prices of grain, wine and oil, in the periods about 1500 and 1750, he compares them by taking the earlier as units and reducing the later to the proper figures in proportion, thereby representing the variations, and draws the arithmetic average, [thus using even weighting. See Appendix C, II.].

G. Shuckburgh Evelyn

5. An account of some endeavors to ascertain a standard of weight and measure. (Philosophical Transactions of the Royal Society of London, 1798, Part I., Art. VIII., 4to., pp. 133–182. Only pp. 175–176, less than one page of printed matter, besides a table, devoted to the subject of measuring exchange-value[1] the rest treating of weights and lengths.)

Calculates the "depreciation of money" from 1050 to 1800 by taking the prices of 1550 at 100 and reducing the prices at the other dates in the same proportions. Each price-figure—for wheat, butcher's meat, day labor, and twelve other agricultural products lumped together—is counted as equally important, and the arithmetic average is drawn. [Thus even weighting is used, but with subordination of each of the twelve articles, *i. e.* irregular weighting by classification. See Appendix C, II., § 3.]

Arthur Young

6. An enquiry into the progressive value of money in England as marked by the price of agricultural products : with observations upon Sir G. Shuckburgh's Table of Appreciation : the whole deduced from a great variety of authorities, not before collected.—London 1812, viii pp. and pp. 66–135.

Objects mostly to Evelyn's prices and authorities, but also to his method, for counting every article as equally important. Counting wheat five times "on account of its importance" by value, barley and oats twice, provisions four times, day labor five times, wool, coal and iron each once, he multiplies the percentages of the variations betwen two periods by these weights, adds up the products, and divides by nineteen, the sum of the weights, [thus introducing uneven

[1] All this has been reprinted by R. Giffen in the Bulletin de l'Institut international de Statistique, Rome 1887, pp. 132–134.

weighting, with the arithmetic average of the price variations. See Appendix C, III.]. But in some calculations he also uses Evelyn's irregular weighting by classification.

J. P. Smith

7. The elements of the science of money.—London 1813. (Appendix, Art. I., "Estimate of the effective debasement of money in the eighteenth century," pp. 471–476.)

   Applies Evelyn's [form of Carli's] method (with strictly even weighting), to the reduced prices furnished by Young. [Very careless.]

Joseph Lowe

8. The present state of England in regard to agriculture, trade, and finance.—London 1822, pp. 261-291, and Appendix, pp. 85–101.

   Seeking to measure the "power of money in purchase" in order that debts may be paid in the same value, or at least that it may serve as a "table of reference," forming a "standard from materials," he adopts Young's method, (but omits labor).

G. Poulett Scrope

9. Principles of political economy . . . applied to the present state of Britain.—London 1833, 18mo., pp. 405–408.

   Following Lowe in the object sought (the establishment of a "tabular standard"), suggests this method: "Take a price-current, containing the prices of one hundred articles in general request, in quantities determined by the proportionate consumption of each article—and estimated . . . in gold. Any variation from time to time in the sum or the mean of these prices will measure . . . the variations which have occurred in the general exchangeable value of gold" (p. 406). [For the formulation of this see Appendix C, IV.]

[Henry James]

10. The state of the nation. Causes and effects of the rise in value of property and commodities from the year 1790 to the present time.—London 1835, pp. 12–15 and a table.

   Measures the rise and fall in value [= money-value, price] of British produce, from 1798 to 1823, by the difference between the "official values" (always on the prices of 1694) and the "declared values" (according to current prices) of British exports. [Applies Scrope's method to custom-house returns. See Appendix C, IV., § 2 (2).]

G. R. Porter

11. The progress of the nation, in its various social and economical relations, from the beginning of the nineteenth century.—London 1838. (2d ed. 1847, pp. 439–440, 444–445.)

Employs [Carli's] method on monthly prices of fifty articles in London from January 1833 to December 1837, the reduced price-figures being carried out to four decimal places. Calls the unit-price started with, the "index price." [2]

M. C. Leber

12. Essai sur l'appréciation de la fortune privée au moyen age.—Paris, 2d ed. 1847, 335 pp.

Roughly measures the purchasing power of silver in relation to many commodities, etc., from the 8th and 11th centuries to the present, but differently for the poor and the rich. The method is not explained, [but is probably a rough arithmetic average of prices, reduced to francs].

R. H. Walsh

13. Elementary treatise on metallic currency.—Dublin 1853, pp. 94–96.

Quotes and reviews, with apparent approval, Scrope's scheme and method.

A. Soetbeer

14. Das Gold. (Die Gegenwart, Leipzig 1856. Tables, pp. 587–588.)
15. Ueber die Ermittelung zutreffender Durchschnittspreise. (Vierteljahrschrift für Volkswissenschaft und Kulturgeschichte, Berlin 1864, Band III., pp. 8–32.)
16. Materialien zur Erlaüterung und Beurteilung der wirtschaftlichen Edelmetallverhältnisse und der Währungsfrage.—Berlin 1886, 4to., pp. 81–117.

In the first gives tables of prices in Hamburg for 1831–40, 1841–50, 1854, 1855, and of the same reduced to 100 at the first period; but does not add or average them. In the second considers only the arithmetic average. In the third employs it on the prices, from 1851 to 1885, on the basis 1847–50, of one hundred Hamburg and fourteen British articles, arranged in different divisions in which the articles are evenly weighted, and which, in the final averaging, are evenly weighted. [Evelyn's irregular weighting by classification.]

J. Maclaren

17. Sketch of the history of currency.—London 1858, pp. 311–312.

Reviews Scrope's scheme and method.

[2] Porter's table was put in evidence by J. B. Smith before the Select Committee on Banks of Issue, 1840, p. 31, following q. 362. It was severely condemned, for the use of even weighting, before the same Committee, q. 3615, by Th. Tooke, who said that Porter had submitted to him the frame-work of his table before publication.

E. Levasseur

18. La question de l'or.—Paris 1858. (Measurements, pp. 179–195.)

Uses partly [Dutot's] method, and partly [James's], the latter applied to similar French "official values" (at the prices of 1826) and "actual values," from 1847 to 1856. [See Appendix C, IV. § 2 (2).]

W. Newmarch

19. Mercantile reports of the character and results of the trade of the United Kingdom during the year 1858; with reference to the progress of prices 1851–9. (Journal of the Statistical Society of London, Vol. XXII., 1859, pp. 76–100.)
20. Results of the trade of the United Kingdom during the year 1859; with statements and observations relative to the course of prices since the year 1844. (*Ibid.* Vol. XXIII., 1860, pp. 76–110.)
21. Commercial history and review of 1863. (Supplement to the Economist, Feb. 20th 1864, folio, pp. 4, 40–46.)

In the first reduces the prices of twenty articles in the year 1851 to 100, and gives their proportionate prices for the years following (omitting '53, '54, '55, '56). In the second reduces the prices of twenty two articles in the period 1845–50 to 100, and gives their proportionate prices for the years following (with the same omissions). In the third continues the latter operations, and begins the series published annually in the Economist. In none are the reduced prices added or averaged. (The addition was first made in the Supplement of March 13th 1869, where, on p. 44, appeared for the first time the "Total Index No.")

W. S. Jevons

22. A serious fall in the value of gold ascertained, and its social effects set forth.—London 1863. (Republished in *Investigations in currency and finance*, London 1884, pp. 15–118.)
23. The variation of prices and the value of the currency since 1782. (Journal of the Statistical Society of London, June 1865. Republished in *Investigations*, pp. 119–150.)
24. The depreciation of gold. (Letter in the Economist, May 8th 1869. Republished in *Investigations*, pp. 151–159.)

In the first raises the question whether the average of price variations should be the arithmetic or the geometric, and adopts the latter, which he always uses with even weighting. The table contains thirty nine chief articles, from 1845 to 1862. In the second defends the geometric average against Laspeyres (see below No. 25), and introduces consideration also of the harmonic average. The table is extended to cover the years 1782-1865, on many articles in ten divisions [apparently with irregular weighting by classification, like Evelyn's].

E. Laspeyres

25. Hamburger Waarenpreise 1850–1863 und die californisch-australischen Goldentdeckungen seit 1848. Ein Beitrag zur Lehre von der Geldentwerthung. (Jahrbücher für National-oekonomie und Statistik, Jena 1864, Band III., pp. 81–118.)
26. Die Berechnung einer mittleren Waarenpreissteigerung. (*Ibid.* 1871, Band XVI., pp. 296–314.)

Employs [Carli's] method on the prices of forty eight articles, the basis being 1831–40, and coming down to 1863. Combats Jevons's geometric average and gives an argument for the arithmetic, in the first work. In the second defends his own position against Drobisch (see below No. 30) and Geyer (No. 28), against whose methods he delivers a counter-attack. Admits that uneven weighting ought to be employed, and gives an emended formula [for a method like Scrope's, but with the mass-quantities specified to be those of the first period—one of the formulæ already rejected by Drobisch in No. 29].

A. Walker

27. The science of wealth.—Boston 1866. (5th ed. revised 1869. pp. 177–178, 481, 488.)

Employs [Dutot's] method on the prices of ten articles in New York from 1834 to 1859, of seven in Calcutta from 1850 to 1854 and from 1863 to 1867, and of sixteen in Boston from 1862 to 1865.

Ph. Geyer

28. Theorie und Praxis des Zettel-Bankwesens.—Munich 1867. (2d ed. 1874, Appendix, pp. 321–326.)

Condemns Laspeyres's arithmetic and Jevons's harmonic (*sic*) averages, and then reconstructs a method very much like Laspeyres's, differing only in reducing the first prices to *about* 2.00, [*i. e.* only nominally, and with slipshodness; virtually Carli's method].

M. W. Drobisch

29. Ueber Mittelgrössen und bie Anwendbarkeit derselben auf die Berechnung des Steigens und Sinkens des Geldwerths. (Berichte über die Verhandlungen der Königlich sächsischen Gesellschaft der Wissenschaften zu Leipzig; Mathematisch-physische Classe. Band XXIII., 1871, pp. 25–48.)
30. Ueber die Berechnung der Veränderungen der Waarenpreise und des Geldwerthes. (Jahrbücher für Nat.-oekon. und Statistik, 1871, Band XVI., pp. 143–156.)
31. Ueber einige Einwürfe gegen die in diesen Jahrbüchern veröffentlichte neue Methode, die Veränderungen der Waaren-

preise und des Geldwerthes zu berechnen. (*Ibid.* 1871, Band XVI., pp. 416–427.)

Would settle the dispute between Laspeyres and Jevons by rejecting both the arithmetic and the geometric averaging of price variations, on the ground that the mass-quantities ought to be taken into account at each period. Introduces a method employing double weighting, a feature in which is that the mass units are all the same. Thinks he rejects all averages, making use only of the rule-of-three. [But really compares the arithmetic average of the preciousness of commodities at each period. See Appendix C, V. § 1, and Chapt. V. Sect. VI, § 5.] Himself formulates for the first time [Scrope's] method applied to the mass-quantities of the first, and of the second, period, and mentions a mean between the two.

7. Roscher

32. Die Grundlagen der Nationaloekonomie.—Stuttgart, 10th ed. 1873, § 129, pp. 273–276.

Approves of Drobisch's solution of the problem, (himself having appealed to Drobisch for the solution of it).

. Paasche

33. Ueber die Preisentwickelung der letzten Jahre, nach den Hamburger Börsennotirungen. (Jahrbücher für Nat.-oekon. und Statistik, 1874, Band XXIII., pp. 168–178.)

34. Studien über die Natur der Geldentwerthung und ihre praktische Bedeutung in den letzten Jahrzehnten.—Jena 1878, 173 pp. (Conrad's Sammlung nationaloekonomischer und statistischer Abhandlungen, Band I.)

Follows Drobisch in condemning both the geometric and the arithmetic averages, but, unlike Drobisch, wants the mass-quantities to be considered only of one period, preferably the later, [thus falling back on Scrope's method, like Laspeyres himself, except for the difference in regard to the period at which the mass-quantities are chosen. See Appendix C, IV., § 2 (2)]. Employs this method, in the first work, on twenty two articles on the basis 1847–67 through the years 1868 to 1872, also with the average of these later years. [The table is continued by v. d. Borght, below No. 55, and Conrad, Nos. 96 and 97.]

. Hanauer

35. Études économiques sur l'Alsace ancienne et moderne. Vol. II., Denrées et salaires.—Paris 1878. (Chapt. XV., "Conclusion," giving a "résumé général" of the "pouvoir de l'argent en général," pp. 601–609.)

Employs unevenly weighted arithmetic average of the mass-quantities of ten articles purchasable with one franc (5 grammes silver at 9/10) at quarter-century periods from 1351–75 down, compared with the mass-quantities purchasable 1851–75 as units. [Thus virtually uses the unevenly weighted harmonic average of price variations.]

A. Ellis

36. The money value of food and raw materials. (The Statist, London June 8th 1878.)

Applies [Young's] method to twenty five articles for the years 1859, 1869, 1873, 1876 and the first quarter of 1878, the weighting being according to the importance of the articles in 1869, which is used as the base.

Anonymous

37. The rise in the value of gold. (Bankers' Magazine, London Oct. 1878, pp. 842–848.)

Uses [Dutot's] method, comparing the prices of thirty one articles in 1878 with their prices in 1868.

R. Giffen

38. Report to the Secretary of the Board of Trade on the prices of exports of British and Irish produce in the years 1861–1877. (Parliamentary documents, Session 1879, c. 2247, folio, pp. 2–15.)
39. On the fall of prices of commodities in recent years. (Journal of the Statistical Society of London, March 1879; republished in *Essays in Finance*, London 1880, § I, the extent of the fall, pp. 312–322.)
40. Report to the Secretary of the Board of Trade on the prices of exports of British and Irish produce, and the prices of imports, in the years 1861–78. (Parliamentary documents, Session 1880, c. 2484, pp. 3–27.)
41. Report to the Secretary of the Board of Trade on recent changes in the amount of the foreign trade of the United Kingdom and the prices of imports and exports. (Parliamentary documents, Session 1881, c. 3079, pp. 4–30.)
42. Report to the Secretary of the Board of Trade on recent changes in the amount of the foreign trade of the United Kingdom and the prices of exports and imports. (Parliamentary documents, Session 1885, c. 4456, pp. iii–viii, 1–53.)
43. Trade depression and low prices. (Contemporary Review, London June 1885; republished in *Essays in Finance, Second Series*, London 1886, § III., "The history of prices," pp. 16–22.)
44. Index numbers. (Bulletin de l'Institut international de Statisque, Rome 1887, 4to. pp. 126–131.)
45. Evidence before the Royal Commission on Gold and Silver, First Report—London 1887, qq. 663–793, folio, pp. 34–42.

Applies [Young's] method to custom-house returns, with the peculiarity that while his basis for prices in 1861, he takes the relative importance (according to total money-values) of the various classes in the year 1875 for the weighting through all the years, which are each compared directly with 1861. [See Appendix C, III., § 3.] Tables at first, in No. 38, confined to exports for the alternate years from 1861 to 1877; later, in No. 40, applied also to imports, and extended to the succeeding years, and finally, in No. 42, carried backwards to 1840 for exports and to 1854 for imports. The primary object is to measure the volume of foreign trade.

S. Bourne

46. On some phases of the silver-question. (Journal of the Statistical Society of London, June 1879; on the fall of prices, pp. 413–417.)
47. On the use of index numbers in the investigation of trade statistics. (British Association for the Advancement of Science, 55th Meeting, 1885; published in the Report, London 1886, pp. 859–873.)
48. Index numbers as illustrating the progressive exports of British produce and manufactures. (*Ibid.* 58th Meeting, 1888; in Report, 1889, pp. 536–540.)
49. Index numbers as applied to the statistics of imports and exports. (*Ibid.* 59th Meeting, 1889; in the Report, 1890, pp. 696–701.)

In the first employs [Carli's] method, with strictly even weighting, but with attempt to improve upon the table of the Economist by omitting some of the variations of cotton and adding coal, in a table from 1847 to 1879. In the others the object is rather to measure the volume of foreign trade, and the index numbers are of two kinds, referring to prices and referring to volumes. The former employ [Scrope's] method on the mass-quantities of 1883; the latter compare the mass-quantities which could have been bought for the sums actually spent at the prices of 1883—both on sixty five articles in the custom-house returns for several years from 1865 down, [the latter sometimes involving the harmonic average of price variations].

A. de Foville

50. La mouvement des prix dans le commerce exterieur de la France. (L' Économiste français, 1st article, July 5th 1879, folio, pp. 3–5; 2d, July 19th 1879, pp. 64–65; 3d, Nov. 1st 1879, pp. 533–534. Second series, 1st article, April 29th 1882, pp. 503–505, 2d, June 17th 1882, p. 504.)
51. Article "Prix" in the Nouveau Dictionaire d'Économie politique. Paris 1892, Vol. II. (On price-measurement, pp. 607-612.)

In the first applies [James's] method [as already done by Levasseur] to the French "official" and "actual values" so long as possible, from 1847 to 1862, and thereafter, down to 1880, a method by comparing the "provisional values" of each year (on the prices of the preceding year) with its "actual values." [3] [See in Appendix C, IV. (2). The object is rather to measure the volume of foreign trade]. The second is merely descriptive.

A. Messedaglia

52. Il calcolo dei valori medii e le sue applicazioni statistiche. (Archivio di Statistica, Anno V., 1880. Republished by itself, Rome 1883, 86 pp.; on prices, pp. 36–40.)

Examines the mathematics of the three means, and, pointing out the inverse relationship between the arithmetic and the harmonic, thinks that for measuring the purchasing power of money over goods we want the arithmetic average of the variations of the mass-quantities, and therefore the harmonic average of the price variations, or for measuring the power of goods over money, directly the arithmetic average of the price variations; but for the geometric average there is no place. Pays little attention to weighting.

H. C. Burchard

53. In the Finance Reports of the Secretary of the Treasury, Washington, for the years 1881 pp. 312–321, 1882 pp. 252–254, 1883 pp. 316–318; and in the Report of the Director of the Mint on the production of the precious metals in the United States, for the year 1884, pp. 497–502.

Uses [Carli's] method. The tables give average prices for 1883 and 1884 compared with 1870, and with 1882, and also with the general average for fifty six years ending 1880.

G. Fauveau

54. Comparaison du pouvoir de la monnaie à deux époques différents. (Journal des Économistes, Paris June 1881, pp. 354–359.)

Formulates [Scrope's] method.

R. v. d. Borght

55. Die Preisentwickelung während der letzten Decennien nach der Hamburger Börsennotirungen. (Jahrbücher für Nat.-oekon. und Statistik, 1882, N. F. Band V., pp. 177–185.)

Continues Paasche's tables [see above No. 33] down to 1880, using the same method.

H. Sidgwick

56. Principles of political economy.—London 1883. (Book I.)

[3] This method has been continued in several of the Bulletins du Ministère des Finances.

Chapt. II., "On the definition and measure of value," pp. 52-69.)

Recommends [Scrope's] method, with the mass-quantities of the first or of the second period, or of a mean [the arithmetic] between the two. None of these being better than another, there is no single authoritative measurement.

F. Y. Edgeworth

57. On the method of ascertaining a change in the value of gold. (Journal of the Statistical Society of London, Dec. 1883, Vol. XLVI., pp. 714–718.)
58. The choice of means. (The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science, Sept. 1887, pp. 268–271.)
59. Memorandum of the Secretary, attached to the First Report of the Committee of the British Association [see below No. 99], at the 57th Meeting, 1887. (Published in the Report of that Meeting, London 1888, pp. 254–301.)
60. Memorandum on the accuracy of the proposed calculations of index numbers, attached to the Second Report of the same Committee [see below No. 100], at the 58th Meeting, 1888. (Published in the Report, 1889, pp. 188–219.)
61. Some new methods of measuring variations in general prices. (Journal of the Royal Statistical Society, London, June 1888, pp. 346–368.)
62. Appreciation of gold. (Quarterly Journal of Economics, Boston Jan. 1889, pp. 151–169.)
63. Memorandum, attached to the Third Report, of the same Committee as above [see below No. 101], at the 59th Meeting, 1889. (Published in the Report, 1890, pp. 133–164.)
64. Recent writings on index numbers. (Economic Journal, London 1894, pp. 158–165.)
65. Articles "Average" and "Index Numbers" in Palgrave's Dictionary of Political Economy, Vol. I., London 1894, p. 74, Vol. II., 1896, pp. 384–387.
66. A defence of index numbers. (Economic Journal, March 1896, pp., 132–142.)

Treats of the measurement of the variations of the value of money under many [often artificial] conceptions of the *quæsitum*. Recommends generally the arithmetic average with even or uneven weighting, but also the median, and again the geometric (this principally with even weighting), according to the object sought.

L. Hansard

67. On the prices of some commodities during the decade 1874–83.

[illegible] Paper read before the Bankers' Institute, London Dec. 17th [illegible] and published in their Journal, Jan. 1885, pp. 1–42.)

[illegible] addition of the prices reduced on a common scale [illegible] at first, virtually Carli's method].

[illegible]

[illegible] zur Statistik der Preise, insbesondere des Geldes und [illegible]—Frankfurt a. M. 1885. (On the method, pp. 11, [illegible]

[illegible] in employing double weighting, but seeks a [illegible] both the periods compared, which he calls a [illegible] and which he finds in the mass-quantity of every [illegible] weighted arithmetic average price over both [illegible] money-unit. [See Appendix C, V, § 2.]

[illegible] Walras

[illegible] régularisation de la valeur de la monnaie. [illegible] before the Société vaudoise des Sciences natur[illegible] published, Lausanne 1885, 22 pp.; also [illegible] in No. 71.

[illegible] politique pure.—2d ed., Lausanne 1889. [illegible] partly incorporating the preceding.)

[illegible] appliquée.—Lausanne and Paris [illegible] with some new matter.)

[illegible] geometric average of price varia[illegible] compares it with the arithmetic and [illegible] even weighting, all which he for[illegible] what he calls "the formula of the [illegible] because it takes into account [illegible] he does not consider].

[illegible] Walras

[illegible] les variations des prix depuis la sus[illegible] les écus d'argent. (Mémoire read before [illegible] naturelles, June 3d 1885, [illegible] No. [illegible] Lausanne 1885, 11 pp.; also re[illegible]

[illegible] with even weighting to the prices [illegible] from 1871 to 1884 on the basis of the [illegible] during the period 1871–78.

[illegible] Wright

[illegible] of the Bureau of Statistics of Labor.— [illegible]

[illegible] Great Britain and in Massachusetts, for work[illegible] method twice applied, on the mass-quan[illegible]

M. G. Mulhall

74. On the variations of price-levels since 1850. (British Association, 55th Meeting, 1885, epitomized in the Report, 1886, pp. 1157–1158.)
75. History of prices since the year 1850.—London 1885, 190 pp.

Employs [Scrope's] method, with the mass-quantities of the later periods [in Paasche's form]. Claims to apply it to the whole world. Calls it "the volume of trade method."

S. Newcomb

76. Principles of political economy.—New York 1886. (Book III. Chapt. II., "The measure of value by an absolute standard," pp. 205–214.)

Recommends [Scrope's] method.

R. H. Inglis Palgrave

77. Currency and standard of value in England, France, and India, and the rates of exchange between these countries. (Memorandum laid before the Royal Commission on Depression of Trade and Industry, 1886, Third Report, Appendix B, folio, pp. 312–390.)

Gives various tables of index-numbers, correcting the Economist's figures for England by weighting the price variations according to the money-values at the later periods [Young's method, more specified as to the weighting, see Appendix C, III. § 4], also for India and France, mostly from 1865 to 1886.

F. B. Forbes

78. The causes of depression in the cotton industry of the United Kingdom. (Occasional Paper of the Bimetallic League, No. 3.) London August 1886, pp. 12, 18, 20.

Applies Jevons's simple geometric mean to Barbour's figures of quantities per rupee, comparing 1884–85 with 1875–76 on twelve classes of exported, and seven of imported, goods.

A. Sauerbeck

79. Prices of commodities and the precious metals. (Journal of the Statistical Society of London, Sept. 1886, pp. 581–631, Appendix, pp. 632–648.)
80. Prices of commodities in 1888 and 1889. (*Ibid.* March 1890, pp. 141–135.)
81. Prices of commodities in 1890. (*Ibid.* March 1891, pp. 128–137.)
82. Prices of commodities during the last seven years. (*Ibid.* June 1893, pp. 215–238, Appendix, pp. 239–247 and 254.)
83. Prices of commodities in 1893. (*Ibid.* March 1894, pp. 172–183.)

84. Prices of commodities in 1894. (*Ibid.* March 1895, pp. 140–154.)

85. Index numbers of prices. (Economic Journal, June 1895, pp. 161–174.)

86. Prices of commodities in 1895. (Journal of the Statistical Society, March 1896, pp. 186–201.)

87. Prices of commodities in 1896. (*Ibid.* March 1897, pp. 180–194.)

88. Prices of commodities in 1897. (*Ibid.* March 1898, pp. 149–162.)

89. Prices of commodities in 1898. (*Ibid.* March 1899, pp. 179–193.)

90. Prices of commodities in 1899. (*Ibid.* March 1900, pp. 92–106.)

Applies [Carli's] method to the prices of forty five articles on the bases of prices in 1867-77, going back to 1848 and continuing to the present. Also adds (beginning in No. 82) two "tests," which seem to be [Young's] method, on the relative money-values in 1889–91, and [Scrope's] method on the mass-quantities of the other years [like Paasche's]. In No. 86 (pp. 193-194) he experiments with the geometric mean.

F. Coggeshall

91. The arithmetic, geometric, and harmonic means. (Quarterly Journal of Economics, Oct. 1886, pp. 83–86.)

Discusses the three means, mostly from the point of view of avoiding error in the result arising from errors in the data, without attaching superiority to any one of them, regarding the mean of prices as a "fictitious mean."

J. E. Thorold Rogers

92. A history of agriculture and prices in England, Vol. V.—Oxford 1887. (Chapt. XXVI., "On prices generally between 1583 and 1702," pp. 778–800.)

Employs [Carli's] method.

A. Marshall

93. Remedies for fluctuations of prices. (Contemporary Review, London March 1887, Sect. V., "How to estimate a unit of purchasing power," pp. 371–375.)

Assumes the arithmetic average.

J. S. Nicholson

94. The measurement of variations in the value of the monetary standard. (Paper read before the Royal Society of Edinburgh, March 21st 1887; published in the Journal of the Statistical

Society of London, March 1887, and republished in *Treatise on money*, Edinburgh and London 1888, pp. 298–331.)

Invents a new method [which is partly a variation upon Drobisch's. See Appendix C, V. § 3].

A. Beaujon

95. Sur la question des "index numbers."—Propositions soumises à l'Institut international de Statistique en vue d'obtenir des tableaux de prix moyens comme base du calcul des index numbers.—Index numbers ou chiffres de prix de marchandises dans divers états, depuis 1870. (Bulletin de l'Institut international de Statistique, 1887, pp. 106–114, 115–116, 117–126.)

Discusses manner of collecting data. Leaves the question of averages to a future deliberation of the Institute.[4] In the third reports several tables of the writers above.

J. Conrad

96. Beiträge zur Beurteilung der Preisreduktion in den 80er Jahren. (Jahrbücher für Nat.-oekon. und Statistik, 1887, N. F. Band XV., pp. 322–331.)

97. Die Entwickelung des Preisniveaus in den letzten Decennien und der deutsche Getreidebedarf in den letzten Jahren. (*Ibid.* 1899, Dritte F. Band XVII., pp. 642–660.)

Continues Paasche's and v. d. Borght's tables down to 1885, and later to 1897, using the same method.

F. Kral

98. Geldwert und Preisbewegung im Deutschen Reiche 1871–1884. —Jena 1887. (On prices, pp. 63–111.)

Uses the arithmetic average with both [Dutot's] haphazard weighting and [Evelyn's] weighting by classification.

British Association for the Advancement of Science:

Committee consisting of S. Bourne, F. Y. Edgeworth, H. S. Foxwell, R. Giffen, A. Marshall, J. B. Martin, J. S. Nicholson, R. H. I. Palgrave and H. Sidgwick, appointed for the purpose of investigating the best methods of ascertaining and measuring variations in the value of the monetary standard.

99. First report, to the 57th Meeting, 1887. (Published in the Report, London 1888, pp. 247–254.)

[4] A *Comité de la Statistique des Prix* was appointed by the Institute, consisting of Beaujon, de Foville, de Inama-Sternegg, Giffen, de Neumann-Spallart, de Mayr, and Pantaleoni. But beside brief reports on recent works, made by Martin and Palgrave, October 1891 and September 1893 (in the Bulletin, 1892, pp. 245–246, and 1895, pp. 57–60), this Committee does not appear to have made an original report.

100. Second report, to the 58th Meeting, 1888. (*Ibid.* 1889, pp. 181–188.)

101. Third report, to the 59th Meeting, 1889. (*Ibid.* 1890, p. 133.)

102. Fourth report, to the 60th Meeting, 1890. (*Ibid.* 1891, pp. 485–488.)

Review several forms of [Scrope's] method, and discuss various questions connected with price-measurements.

E. Nasse

103. Das Sinken der Warenpreise während der letzten fünfzehn Jahre. (Jahrbücher für Nat.-oekon. und Statistik, 1888, N. F. Band XVII.; on methods of measurement, pp. 51–53.)

104. Das Geld und Münzwesen. (Schönbergs Handbuch der Politischen Oekonomie, Tübingen, Vol. II., 1890; on methods of measurement, pp. 331–332, folio.)

Reviews some of the methods using the arithmetic average.

K. Wasserab

105. Preise und Krisen. Gekrönte Preisschrift über die Veränderungen der Preise auf dem allgemeinen Markt seit 1875 und deren Ursachen.—Stuttgart 1889. (On price-measurements, pp. 75–128.)

Applies a method of his own [which is really Young's] to prices on the basis 1861–70 down to 1885.

F. Schmid

106. Bericht über die Thätigkeit des statistischen Seminars an der k. k. Universität Wien im Wintersemester 1888–89. (Statistische Monatschrift, XV Jahrgang, Vienna 1889. On variations in the purchasing power of money, aided by G. H. Thierl, pp. 643-650)

Reviews several methods.

E. B. Andrews

107. An honest dollar. (Publications of the American Economic Association, New York, Vol. IV., No. 6, Nov. 1889; see pp. 38–39.)

108. Institutes of economics.—Boston 1891, pp. 141–142.

Recommends [Scrope's] method, but allows the use of the geometric, arithmetic, or harmonic means.

K. T. von Inama-Sternegg

109. Der Rückgang der Waarenpreise und die oesterreichisch-ungarische Handelsbilanz 1875–1888. (Statistische Monatschrift, XVI Jahrgang, 1890. Tables, pp. 6–7.)

Gives tables applying [Carli's] method (in a form like the Economist's) to thirty articles of import and to twenty-five of export in Austria-Hungary from 1880 to 1888 on the basis of 1875–1879.

H. Westergaard

110. Die Grundzüge der Theorie der Statistik.—Jena 1890. (On price-measurements, pp. 218–220.)

Points out that the geometric average [he supposes the same weighting throughout, having even weighting mostly in mind] gives the same index-numbers in a series of periods whether applied to comparing each subsequent period with the original base or to comparing any of the subsequent periods with each other; and that this is not done by the usual methods employing the arithmetic average [Carli's and Young's]. Offers this as an argument for the geometric average.

R. P. Falkner

111. Report of the Statistician. (Pp. xi–c in Vol. I. of the Report on retail prices of Mr. Aldrich from the Committee on Finance, 52d Congress, 1st Session, No. 986. Washington 1892.)

112. Report of the Statistician. (Pp. 27–337 in Vol. I. of the Report on wholesale prices of Mr. Aldrich from the Committee on Finance, 52d Congress, 2d Session, No. 1394. Washington 1893.)

113. Wholesale prices: 1890 to 1899. (Bulletin of the Department of Labor, No. 27, Washington March 1900; pp. 237–313.)

Applies both [Carli's] and [Young's] methods to prices in the United States, in the first comparing 1891 with 1889, in the second extending the investigation to the years 1840–1891 on the basis of 1860, and in the third bringing it down to 1899.

S. McC. Lindsay

114. Die Berechnung der Edelmetalle seit 1850.—Jena 1893. (Conrad's Sammlung. On the method, pp. 9–28.)

Reviews several methods, and adopts [Scrope's] with the mass-quantities of the first or of the second period, or a mean between them, according as any of these best represents the importance of the classes.

R. Zuckerkandl

115. Die statistische Bestimmung des Preisniveaus. (Handwörterbuch der Staatswissenschaften Jena, Vol. V., 1893, 4to., pp. 242–251.)

116. La mesure des transformations de la valeur de la monnaie. (Revue d'économie politique, Paris 1894, pp. 237–253.)

Avenel.

... la propriété, des salaires, des den... depuis l'an 1200 jusqu'en ... 1894–1898. (On the method, ... pp. 27, 32, 137.)

... à travers sept siècles.—Paris 1895, 1 ... results, p. 37.)

... roughly, an average (the arithmet... goods purchasable with given amounts of ... from 1200 to 1890, [thus using the har... with uneven weighting (in budgets ... different classes of society). (The seco...

...

... British trade with Oriental coun... Royal Colonial Institute, Feb. 12th ... Proceedings of the Institute. Tab...

... compiled by W. S. Wetmore, on the pl... method], for twenty articles in China.

... Sargel

... 1895. (A measurement, p. 171 ... method to four articles, comparing

...

... on prices in the United St...

Is content with the arithmetic average. Prefers Soetbeer's results because of the great number of articles used.[5]

A. L. Bowley

123. Comparison of the rates of increase of wages in the United States and in Great Britain 1860–1891. (Paper read before the British Association, Sept. 12th 1895; published in the Economic Journal, 1895; price-measurements, p. 381.)

Uses the arithmetic average.

G. Wiebe

124. Zur Geschichte der Preisrevolution des XVI und XVII Jahrhunderts.—Leipzig 1895. (On the method, pp. 163–174; tables, pp. 369–386.)

Applies both [Carli's] and [Evelyn's] methods to European prices from 1451 to 1700. Recommends Lehr's method for present researches.

H. Denis

125. La depression économique et sociale et l'histoire des prix.—Ixelles-Bruxelles 1895. (On price-measurements, pp. 9–34.)

Uses the arithmetic average with even weighting [Carli's method]. This weighting he considers sufficiently approximative in practice, though wrong in theory.

G. M. Boissevain

126. La question monétaire. (Mémoire traduit du Hollandais.) —Paris 1895. (On the method, pp. 52–53; and two tables.)

Modifies Sauerbeck's index-numbers by calculations upon British exports and imports.

A. L. Fonda

127. Honest money.—New York 1895, 12 mo. (On "the standard of value," pp. 158–161, 165.)

For his standard would use [Young's] method, to be applied to a hundred staple commodities.

Wharton Barker

128. The course of prices. (The American, Jan. 25th 1896, folio, pp. 54–56, and quarterly since.)

Gives index-numbers for American prices on a hundred and one articles in thirteen groups from Jan. 1st 1891, quarterly, continued to the present, using the arithmetic average of the price variations with even weighting, all compared with the first period (following the example of the Economist in England).

[5] In *Further considerations on index-numbers*, in the same Journal, March 1896, pp. 127–131, he rejects the whole system of index-numbers because different results can be obtained on the same price variations [by using different weights]. (Reply by Edgeworth, No. 66.)

J. Allen Smith

129. The multiple money standard. (Publications of the American Academy of Political and Social Science. Philadelphia 1896. On the measurement of the standard, pp. 27–30.)

For his standard would use [Scrope's] method, the mass-quantities chosen to be revised from time to time (at long intervals), all commodities being included that can be accurately defined as to quantity and quality.

T. N. Whitelaw

130. A contribution to the study of a constant standard and just measure of value.—Glasgow 1896. (On the standard, pp. 18–19, 32–35.)

For his standard would use [Carli's] method, to be applied to about twenty of the chief agricultural products.

L. G. Powers

131. Fifth annual report of the Bureau of Labor of the State of Minnesota, 1895–1896.—St. Paul 1896, 524 pp (On the methods used, pp. 26–30.)

Uses (1) [Drobisch's] method in groups of articles whose prices are reported in the same mass-unit, (2) [Scrope's] method applied to the arithmetic average of the mass-quantities over thirty five years. (3) Sauerbeck's "corrected method" [Paasche's, or Scrope's applied to the mass-quantities of the later years singly], and (4) the "simple" arithmetic average of the price variations [Carli's method]. Calculations confined to agricultural products, principally in the West, extending from 1862 to 1895, on the basis of 1872. [Are vitiated by combining the prices of widely separated localities, and by attempting to eliminate the effects of reduced costs of transportation.]

M. Bourguin

132. La mesure de la valeur et la monnaie.—Paris 1896, 273 pp. (On index numbers, pp. 134–139.)

Although denying the existence of general exchange-value, wants to measure the average of the variations of all the particular exchange-values of money. Approves of the arithmetic average, with even weighting.

L. L. Price

133. Money and its relations to prices—being an enquiry into the causes, measurement, and effects of changes in general prices. London 1896, 12 mo. (On the measurement, pp. 9–36.)

Briefly surveys the subject.

F. J. Atkinson

134. Silver prices in India. (Journal of the Royal Statistical Society, March 1897, pp. 84–147.)

Applies [Young's] method to one hundred articles in forty groups in various parts of India from 1861 to 1895, on the price-basis of 1871, with weighting according to the total money-values of the groups in 1893-94.

Eltweed Pomeroy

135. The multiple standard for money. (Arena, Boston Sept. 1897. On the method, pp. 331-333.)

For his standard would use [Scrope's] method applied to the mass-quantities consumed by working men's families, employing two hundred staple articles, their prices being collected from one hundred centers of commerce.

F. Parsons

136. Rational money. A national currency intelligently regulated in reference to the multiple standard.—Philadelphia 1898. (On the standard, pp. 113-138.)

For his standard would use [Scrope's] method applied to a couple of hundred articles in the mass-quantities that are consumed by the average family, the list to be revised from time to time (frequently).

R. Mayo-Smith

137. Movements of prices. (Political Science Quarterly, New York Sept. 1898, pp. 477-494.)

138. Statistics and economics.—New York 1899. (Chapt. VI., "Prices"; on the measurement, pp. 199-228.)

Briefly reviews the problems connected with index-numbers. (The second slightly expanded from the first.)

K. Wicksell

139. Geldzins und Güterpreise.—Jena 1898. (On price-measurements, pp. 6-16.)

Recommends [Scrope's] method, provided the results are the same on the mass-quantities of each period. Otherwise the problem is unsolvable, as the measurement with the mass-quantities of the one period is as good as with those of the other, and the mean between the two can have only "conventional meaning."

A. W. Flux

140. Some old trade records re-examined: a study in price-movements during the present century. (Paper read Feb. 8th 1899 before the Manchester Statistical Society, and published in their Transactions, Session 1898-99, pp. 65-91.)

Applies [James's] method to British prices 1798-1869, to French prices 1873-97 [cf. De Foville, No. 50], and to German prices 1891-97.

R. S. Padan

141. Prices and index numbers. (Journal of Political Economy, Chicago March 1900, pp. 171–202.)

Considers the arithmetic average the only rigorous one. Attacks Jevons for "beclouding" the subject by introducing the geometric average and suggesting the harmonic. Advocates recognition of mass-quantity. This being done, the method [Scrope's] he thinks to be accurate provided the mass-quantities are proportional at all the periods compared, because then the same results are obtained of whatever period the mass-units be used. But not so, if the mass-quantities are irregular, so that in such cases no one result is authoritative.

INDEX

[*Simple numerals refer to pages. Following the letter B they refer to works in the Bibliography.*]