THE

MESSENGER OF MATHEMATICS,

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MESSENGER OF MATHEMATICS.

ON THE CYLINDROID,

By *T. C. Lewis, M.A.*, Fellow of Trinity College, Cambridge.

I. *Geometrical determination of distribution of pitch in a cylindroid, with some properties of the surface.*

THE use of the pitch-conic to determine the distribution of pitch in a cylindroid is not practically of much value, because a conic has to be constructed whose axes are inversely proportional to the square roots of the principal pitches—an operation by no means simple, and in fact when the cylindroid is known merely from two of its screws being given, the solution of the problem becomes altogether theoretical (Ball, *Theory of Screws*, § 16. Clifford, *Kinematic*, p. 131). Hence the following method, which at the same time determines, by an easy geometrical construction, the cylindroid of which two component screws are given, may be found interesting.

Let APB (fig. 1) be a circle, ABC being the diameter perpendicular to the line CD, which is in the same plane. Let a point P move uniformly round the circle whilst the plane rotates uniformly about CD with an angular velocity one-half of that of P about the centre of the circle. Then the ordinate PE generates the cylindroid. For the motion of E along CD is clearly harmonic, and the plane CEP rotates uniformly with a period twice that of the harmonic motion. Further, we see that the angle between two generating lines PE, QF is equal to the angle at the circumference of the circle subtended by the arc PQ over which P has travelled.

If the circle be such that CA, CB are the lengths of the two principal pitches, then *the length of every generating line of the cylindroid described as above will be equal to the pitch of the screw which has its axis along that line.*

For if θ be the angle between the generating lines EP, CA, and if a, b be the principal pitches, the pitch of the screw along EP regarded as a generating line is the ratio of the translation along EP to the rotation about it, due to the principal screws being compounded in the ratio requisite for this resultant to be a twist about EP. Taking a rotation ϖ about EP, the pitch is seen to be

$$\frac{a.\varpi \cos\theta.\cos\theta + b.\varpi \sin\theta.\sin\theta}{\varpi} = a \cos^2\theta + b \sin^2\theta.$$

And from the figure we have

$$PE = \tfrac{1}{2}(a+b) + \tfrac{1}{2}(a-b)\cos 2\theta$$
$$= a\cos^2\theta + b\sin^2\theta.$$

Therefore the length of EP is equal to the pitch along it.

Hence we see at once how to construct the cylindroid to which any two given screws belong.

Let EF be the shortest distance between them, it is therefore the axis of the cylindroid. Erect at E and F in the same plane EP, FQ perpendicular to EF and equal in length to the pitches of the screws at E and F respectively. Describe a circle passing through P and Q and having the angle in one segment equal to the angle between the given screw-axes. The cylindroid constructed as above from this circle is the required surface, and its generating lines are equal to the pitch along them.

This construction would give two circles through P and Q; the right circle to take is that in which the given angle at the circumference of the circle is subtended by that arc PQ over which the moving point P passes in the positive direction as the plane rotates in the positive direction round the axis CD from the generating line EP to the generating line FQ.

The circle introduced here may be called the *circle of pitches*, but it further gives the angle PAQ, which is the inclination of EP and FQ viewed as generating lines. The circle and its ordinates are the cylindroid and its pitches shut up fanwise into one plane. Hence a simple method of constructing a portable model of the surface is at once suggested.

Several properties of the surface are easily seen to be true.

1. The length of the axis is the difference of the two principal pitches.

2. If the pitch of one screw of the cylindroid vanishes so also does that of another; the corresponding twists reduce to spins.

3. If the pitch of one of the principal screws (that is a screw at C) vanishes, the product of the pitches at any point E is equal to the square on EC; or, in the general case, the product of the two pitches at E is greater than the product of the two principal pitches by the square on EC.

4. The ratio of the twists round the generating lines CA, CB, in order that their resultant may be a screw along EP, is equal to $\tan PAB$.

5. If S, S' are two intersecting screws of a cylindroid, and T a screw of the same cylindroid at right angles to S, the pitch of T is equal to that of S'.

6. Given the screw represented by EP, and a spin by a zero ordinate at a given point Q (fig. 2). Then the position, Q', of the second spin axis will depend upon the angle between the given screw and spin, but the angle it makes with the screw will be invariable; for it is equal to PAQ', which is equal to PQE, which is constant. Therefore if the motion of the one spin-axis is one of rotation about EQ, that of the other will be one of translation along it, and *vice versâ*.

7. The line through Q parallel to the spin-axis at Q' has no velocity in the direction of its length due to either of the two spins, and therefore none due to any combination of them, and therefore none due to a twist about any screw of the cylindroid; it is therefore reciprocal to every screw of the cylindroid, as is also a similarly constructed line at Q'.

Hence the reciprocal line at a distance EQ from any screw EP makes an angle $\tan^{-1}\dfrac{EP}{EQ}$ with EP.

8. We see that a given screw can be resolved into two spins in an infinite number of ways. If, however, we measure off lengths on the axes of the spins representing their magnitude, then the tetrahedron having the two lengths as opposite edges is of constant volume (Prof. Clifford, *Kinematic*, p. 135); for let α, β be the resolved parts about the spins at Q and Q' of the rotation ϖ about EP; and let θ, θ' be the angles the two spin-axes make with EP. Then

$$\frac{\alpha}{\sin\theta'} = \frac{\beta}{\sin\theta} = \frac{\varpi}{\sin(\theta'-\theta)},$$

and $$p = EP = EQ' \tan\theta = EQ\tan\theta';$$

therefore the volume of the tetrahedron

$$= \tfrac{1}{2}\alpha . QQ' . \tfrac{1}{3}\beta \sin(\theta' - \theta)$$

$$= \tfrac{1}{6}\varpi^2 \frac{\sin\theta \sin\theta'}{\sin(\theta' - \theta)} . p(\cot\theta' - \cot\theta)$$

$$= \tfrac{1}{6} p\varpi^2.$$

II. *Some geometrical properties.*

The following proof of known properties of the cylindroid is offered as depending upon a variation of Prof. Cayley's construction, which, so far as I can ascertain, has not hitherto been given, and which makes the demonstration very easy.

Prof. Cayley's construction, as given by Prof. Ball (*Theory of Screws*, p. 16), is: Cut the cylinder $x^2 + y^2 = r^2$ in an ellipse by the plane $z = x$, and consider the line $x = o$, $y = r$. If any plane $z = c$ cuts the ellipse in the points A, B and the line in C, then CA, CB are two generating lines of the surface. Here the nodal line passes through an extremity of the minor axis of the ellipse. Prof. Clifford (*Kinematic*, p. 127) varies the construction to make the nodal line pass through an extremity of the major axis. Now the same cylindroid would be described in different positions if the locus of C were *any* generating line of the cylinder. The construction in its most general form becomes: If any plane at right angles to the axis of a circular cylinder meet a fixed elliptic section in the points A, B and a fixed generating line in C, then CA, CB are two generating lines of the cylindroid.

For while the angular motion of CA about the nodal line is uniform, that of C along it is harmonic and of half the period.

Hence we see that if any cylinder touch the plane containing the nodal line and any generating line of the cylindroid along the nodal line, it will cut the surface in an ellipse, and that the generating line which intersects the given generating line will lie in the plane of the ellipse.

Hence every cylinder of which the nodal line is one generating line cuts the cylindroid in an ellipse; for it must also touch one generating line of the cylindroid.

Let LM be the generator in the plane of the ellipse (fig. 3), then this plane is clearly the tangent plane to the cylindroid at M. Therefore every tangent plane cuts the surface in an ellipse.

If LT be the diameter of the ellipse through L, the generator through T is as much above as that through M is below the middle point of the axis, and being drawn to points on the same side of the major axis of the ellipse the screws along them are of equal pitch. Otherwise, the generator through T is in a diametral plane of the cylinder; it is therefore perpendicular to the generator which intersects LM, and has therefore the same pitch as LM (see Part I.)

We see also that LM, MT are parallel to the minor and major axes of the ellipse respectively; and further, that the line joining the points where any two intersecting screws CA, CB meet the cylinder is parallel to the minor axis, whilst the line joining the points where any two screws CA, $C'A'$, of equal pitch, meet the cylinder is parallel to the major axis.

Most of the other properties of the surface due to Dr. Casey (see *Theory of Screws*, p. 26) are now self-evident. These are:—

The ellipse in which a tangent plane cuts the cylindroid has a circle for its projection on a plane perpendicular to the nodal line, and the radius of the circle is the minor axis of the ellipse.

The difference of the squares of the axes of the ellipse is constant wherever the tangent plane be situated. The constant value is clearly the square of the axis of the cylindroid.

The minor axes of all the ellipses lie in the same plane; this is the plane bisecting the axis at right angles.

Let QTt be a line through T parallel to the nodal line. Then from the cylinder we see that the line CA is perpendicular to the plane TAQ, and therefore QA is the perpendicular from Q upon any generator CA. Hence the locus generated by such perpendiculars from Q is the cone of the second order having its vertex Q and its base the ellipse LMT. This is called the *reciprocal cone* by Prof. Ball, being the locus of screws reciprocal to the cylindroid.

Hence we see that all the screws which lie in a plane and are reciprocal to a cylindroid envelope a conic; for from every point in the plane there can be drawn in that plane two and only two real or imaginary tangents of the required envelope, namely the intersection of the plane with the various reciprocal cones; the envelope is therefore a conic.

ON A CERTAIN SYSTEM OF SIMULTANEOUS DIFFERENTIAL EQUATIONS.

By *R. R. Webb, M.A.*

It will readily be conceded that the number of cases in which systems of simultaneous differential equations of the non-linear type can be completely solved is somewhat limited. In what follows, I do not think I can claim any merit of newness in the result, inasmuch as the system discussed is the analytical expression of the problem of the determination of the curve of constant curvature and tortuosity, but the method of integration is, as far as I am aware, unpublished at present.

The system in question is

$$x'^2 + y'^2 + z'^2 = \alpha \quad \text{......(i)},$$

$$x''^2 + y''^2 + z''^2 = \beta \quad \text{......(ii)},$$

$$\begin{vmatrix} x''', & y''', & z''' \\ x'', & y'', & z'' \\ x', & y', & z' \end{vmatrix} = \gamma \quad \text{......(iii)},$$

the differential coefficients with respect to the independent variable (s) being indicated by accents, while α, β, γ are constants.

On differentiating (i), we get

$$x'\,x'' + y'\,y'' + z'\,z'' = 0 \quad \text{......(iv)}.$$

Also (ii) is $\qquad x''\,x'' + y''\,y'' + z''\,z'' = \beta \quad \text{......(v)},$

and the result of differentiating (ii) is

$$x'''x'' + y'''y'' + z'''z'' = 0 \quad \text{......(vi)}.$$

On solving (iv), (v), (vi) as simultaneous equations giving $x''y''z''$, there results

$$\frac{1}{\beta}\begin{vmatrix} x''', & y''', & z''' \\ x'', & y'', & z'' \\ x', & y', & z' \end{vmatrix} = \frac{(y'z''' - z'y''')}{x''} = \&c.$$

Hence by (iii) we immediately have

$$\gamma x'' = \beta\,(y'z''' - z'y'''),$$

$$\gamma y'' = \beta\,(z'x''' - x'z'''),$$

$$\gamma z'' = \beta\,(x'y''' - y'x''').$$

The three equations just written down are exact differential equations and therefore give

$$\left.\begin{aligned}\gamma x' &= \beta\,(y'z'' - z'y'') + A_1\\ \gamma y' &= \beta\,(z'x'' - x'z'') + B_1\\ \gamma z' &= \beta\,(z'y'' - y'x'') + C_1\end{aligned}\right\}\ldots\ldots\ldots\ldots\text{(vii)}.$$

The constants of integration A_1, B_1, C_1 are, however, *not* independent, for on taking the β-term on the right-hand side over to the left squaring each equation and adding, we get

$$A_1^2 + B_1^2 + C_1^2 = \gamma^2\,(x'^2 + y'^2 + z'^2) + \beta^2\,\{(y'z'' - z'y'')^2 + \ldots + \ldots\}$$

$$= \gamma^2\alpha + \alpha\beta^3.$$

We are therefore at liberty to assume

$$\frac{A_1}{a_1} = \frac{B_1}{b_1} = \frac{C_1}{c_1} = \{\alpha\,(\beta^3 + \gamma^2)\}^{\frac{1}{2}} \equiv \delta,$$

where a_1, b_1, c_1 may be looked upon as the direction cosines of a right line referred to rectangular axes, so that equations (vii) are now

$$\left.\begin{aligned}\gamma x' &= \beta\,(y'z'' - z'y'') + a_1\delta\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\}\ldots\ldots\ldots\ldots\text{(viii)}.$$

On multiplying successively by x', y', z' and adding, these equations immediately give,

$$\alpha\gamma = \delta\,(a_1x' + b_1y' + c_1z');$$

therefore $$a_1x + b_1y + c_1z + p = \frac{\alpha\gamma}{\delta}s\ \ldots\ldots\ldots\ldots\text{(ix)},$$

where p is an arbitrary constant.

Next, let $a_2b_2c_2$, $a_3b_3c_3$ be so taken that in conjunction with $a_1b_1c_1$ they determine a system of three right lines mutually at right angles, and be in addition a *rotational* system, so that in

$$\begin{vmatrix} a_1, & b_1, & c_1\\ a_2, & b_2, & c_2\\ a_3, & b_3, & c_3\end{vmatrix}$$

each element equals its minor, we then have

$$(a_2x'' + b_2y'' + c_2z'')(a_3x' + b_3y' + c_3z') - (a_2x' + b_2y' + c_2z')(a_3x'' + b_3y'' + c_3z'')$$

$$= -[(y'z'' - z'y'')(b_2c_3 - b_3c_2) + (z'x'' - x'z'')(c_2a_3 - c_3a_2) + (x'y'' - y'x'')(a_2b_3 - a_3b_2)]$$

$$= -[a_1(y'z'' - z'y'') + b_1(z'x'' - x'z'') + c_1(x'y'' - y'x'')]$$

$$= \frac{\delta}{\beta} - \frac{\gamma}{\beta}(a_1x' + b_1y' + c_1z') \text{ by (viii)}$$

$$= \frac{\delta}{\beta} - \frac{\alpha\gamma^2}{\beta\delta} \text{ by (ix)} \qquad \text{(x)}.$$

Again,

$$(a_1x' + b_1y' + c_1z')^2 + (a_2x' + b_2y' + c_2z')^2 + (a_3x' + b_3y' + c_3z')^2 = 1;$$

therefore

$$(a_2x' + b_2y' + c_2z')^2 + (a_3x' + b_3y' + c_3z')^2 = 1 - \frac{\alpha^2\gamma^2}{\delta^2} \quad \text{(xi)}.$$

Dividing (x) by (xi), we immediately see that there results

$$\frac{d}{ds}\left[\tan^{-1}\left(\frac{a_2x' + b_2y' + c_2z'}{a_3x' + b_3y' + c_3z'}\right)\right] = \frac{\frac{\delta}{\beta} - \frac{\alpha\gamma^2}{\beta\delta}}{1 - \frac{\alpha^2\gamma^2}{\delta^2}} = \frac{\delta}{\beta}\left(\frac{\delta^2 - \alpha\gamma^2}{\delta^2 - \alpha^2\gamma^2}\right) \equiv \kappa;$$

therefore $$\frac{a_2x' + b_2y' + c_2z'}{a_3x' + b_3y' + c_3z'} = \tan(\kappa s + \epsilon),$$

where κ is an absolute and ϵ an arbitrary constant. This gives

$$\frac{a_2x' + b_2y' + c_2z'}{\sin(\kappa s + \epsilon)} = \frac{a_3x' + b_3y' + c_3z'}{\cos(\kappa s + \epsilon)}$$

$$= \pm\{(a_2x' + b_2y' + c_2z')^2 + (a_3x' + b_3y' + c_3z')^2\}^{\frac{1}{2}}$$

$$= \pm\left(1 - \frac{\alpha^2\gamma^2}{\delta^2}\right)^{\frac{1}{2}};$$

therefore $$a_2x' + b_2y' + c_2z' = \pm\left(1 - \frac{\alpha^2\gamma^2}{\delta^2}\right)^{\frac{1}{2}} \sin(\kappa s + \epsilon),$$

$$a_3x' + b_3y' + c_3z' = \pm\left(1 - \frac{\alpha^2\gamma^2}{\delta^2}\right)^{\frac{1}{2}} \cos(\kappa s + \epsilon);$$

therefore

$$a_2x+b_2y+c_2z+q=\mp\frac{1}{\kappa}\left(1-\frac{\alpha^2\gamma^2}{\delta^2}\right)^{\frac{1}{2}}\cos(\kappa s+\varepsilon),$$

$$a_3x+b_3y+c_3z+r=\pm\frac{1}{\kappa}\left(1-\frac{\alpha^2\gamma^2}{\delta^2}\right)^{\frac{1}{2}}\sin(\kappa s+\varepsilon).$$

Hence, finally, the complete integrals are

$$a_1x+b_1y+c_1z+p=\pm\left(\frac{\alpha\gamma^2}{\beta^3+\gamma^2}\right)^{\frac{1}{2}}s,$$

$$a_2x+b_2y+c_2z+q=\mp\frac{(\beta^3+\gamma^2-\alpha\gamma^2)^{\frac{3}{2}}}{\alpha^{\frac{1}{2}}\beta^2(\beta^3+\gamma^2)}\cos\left\{\pm\frac{\alpha^{\frac{1}{2}}\beta^2(\beta^3+\gamma^2)^{\frac{1}{2}}s}{\beta^3+\gamma^2-\alpha\gamma^2}+\varepsilon\right\},$$

$$a_3x+b_3y+c_3z+r=\pm\frac{(\beta^3+\gamma^2-\alpha\gamma^2)^{\frac{3}{2}}}{\alpha^{\frac{1}{2}}\beta^2(\beta^3+\gamma^2)}\sin\left\{\pm\frac{\alpha^{\frac{1}{2}}\beta^2(\beta^3+\gamma^2)^{\frac{1}{2}}s}{\beta^3+\gamma^2-\alpha\gamma^2}+\varepsilon\right\}.$$

In the particular case of the curve of constant curvature and tortuosity

$$\alpha=1,\quad \beta=\frac{1}{\rho^2},\quad \gamma=\frac{1}{\rho^2\sigma},$$

where $\frac{1}{\rho}$, $\frac{1}{\sigma}$ are respectively the curvature and tortuosity.

Putting $$\rho=\frac{a}{\cos^2 i},\quad \sigma=\frac{a}{\sin i\cos i},$$

then $$\left(\frac{\alpha\gamma^2}{\beta^3+\gamma^2}\right)^{\frac{1}{2}}=\sin i,$$

$$\frac{\alpha^{\frac{1}{2}}\beta^2(\beta^3+\gamma^2)^{\frac{1}{2}}}{\beta^3+\gamma^2-\alpha\gamma^2}=\frac{\cos i}{a},$$

$$\frac{(\beta^3+\gamma^2+\alpha\gamma^2)^{\frac{3}{2}}}{\alpha^{\frac{1}{2}}\beta^2(\beta^3+\gamma^2)}=a,$$

and the integrals would then be

$$a_1x+b_1y+c_1z+p=\pm s\sin i,$$

$$a_2x+b_2y+c_2z+q=\mp a\cos\left(\pm\frac{s\cos i}{a}+\varepsilon\right),$$

$$a_3x+b_3y+c_3z+r=\pm a\sin\left(\pm\frac{s\cos i}{a}+\varepsilon\right),$$

shewing, as they should, that the class of curves having constant curvature and tortuosity are helices.

January 25, 1879.

NOTES ON A METHOD IN AREAL COORDINATES, CONNECTED WITH THE GEOMETRICAL METHOD OF ORTHOGONAL PROJECTION.

By *J. W. Sharpe, M.A.*

If an equation in areal coordinates be interpreted with respect to different triangles, we shall obtain a series of related curves of the same degree. It can be proved that these curves are similar to those which are formed by taking any one of the series and projecting it orthogonally upon different planes.

Definition. When the same equation is interpreted with respect to different triangles, let those points which have the same coordinates be termed corresponding points; and loci of corresponding points, corresponding loci.

Prop. 1. Pairs of corresponding areas have to one another the same ratio as the two triangles of reference.

This follows at once from the fact that corresponding points have the same coordinates; thus, if P, P' be corresponding points, ABC, $A'B'C'$ the triangles of reference, area PBC : area $P'B'C'$: : area ABC : area $A'B'C'$.

Prop. 2. Straight lines corresponding to a system of parallel straight lines are themselves parallel.

For a pair of corresponding curves have the same equation.

Prop. 3. Parallel finite straight lines have the same ratio to their corresponding lines.

Let PQ, $P'Q'$ be two parallel straight lines; x, y, z the coordinates of P; x_1, y_1, z_1 those of Q; then

$$\frac{PQ^2}{P'Q'^2} = \frac{a^2(y-y_1)(z-z_1)+b^2(z-z_1)(x-x_1)+c^2(x-x_1)(y-y_1)}{a^2(y'-y'_1)(z'-z'_1)+b^2(z'-z'_1)(x'-x'_1)+c^2(x'-x'_1)(y'-y'_1)}.$$

Now PQ and $P'Q'$ are parallel; therefore

$$\frac{x-x_1}{x'-x'_1} = \frac{y-y_1}{y'-y'_1} = \frac{z-z_1}{z'-z'_1};$$

therefore

$$\frac{PQ^2}{P'Q'^2} = \left(\frac{x-x_1}{x'-x'_1}\right)^2 = \left(\frac{y-y_1}{y'-y'_1}\right)^2 = \left(\frac{z-z_1}{z'-z'_1}\right)^2,$$

but these ratios $x-x_1 : x'-x'_1$, &c., are unaffected by changing the triangle of reference.

From these three propositions it appears that corresponding loci are similar to those obtained by projecting any one of the system orthogonally upon proper planes.

Note. In the application of the method suggested by the above propositions we shall find ourselves using imaginary triangles of reference; in these cases the equations remain without geometrical interpretation, but, since our formulæ, equations, and arguments are essentially algebraical, and not geometrical, the results will be in no way vitiated. The nomenclature of the triangle is used for convenience of expression, and for succinct presentation of results, but the validity of the demonstrations depends solely upon considerations of algebra, and not at all upon the existence of corresponding geometrical relations.

We shall first give a few illustrations of the use of the method, and then add some general investigations as to the area, and the direction and magnitude of the axes, of a conic, represented by the general equation of the second degree.

We shall use a, b, c for the lengths of the sides of the triangle of reference, S for its area, R for the radius of the circumscribing circle, r for that of the inscribed, &c.; a', b', c', S', R', r', &c., referring to the new triangle of reference $A'B'C'$. Also let the general equation of the second degree be

$$ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy = 0;$$

then we shall put

$$\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} \equiv H,$$

$$\begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} \equiv K,$$

$$a^2u + b^2v + c^2w - (b^2 + c^2 - a^2)\,u' - (c^2 + a^2 - b^2)\,v' - (a^2 + b^2 - c^2)\,w' \equiv I,$$

i.e. H, K are the Hessian and the bordered Hessian respectively, and I is the third invariant of the conic.

Ex. 1. To find the area of the conic $x^2 - 2yz = 0$.

Here $K = -1$, and hence the conic is an ellipse, touching AB, AC at B and C respectively.

Now if we interpret the above with respect to a new triangle of reference $A'B'C'$, such that

$$\tfrac{1}{2}a'^2 = b'^2 = c'^2,$$

i.e. isosceles and right-angled at A', the locus is a circle, whose radius $= b'$ or c'.

Now, by Prop. 1,

$$\text{area of conic} : S :: \text{area of circle} : S';$$

therefore $$:: \pi b'^2 : \tfrac{1}{2}b'^2;$$

therefore $$\text{area of conic} = 2\pi S.$$

Ex. 2. To find the area of the inscribed ellipse

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = 0.$$

Here $$K = -12.$$

Interpreting with respect to a new triangle, such that the locus is a circle, we have

$$s' - a' = s' - b' = s' - c' = \tfrac{1}{3}s':$$

hence $$\text{area of conic} = \pi \frac{S'^2}{s'^2} \times \frac{S}{S'}$$

$$= \pi S \sqrt{\left\{\frac{(s-a)(s-b)(s-c)}{s^3}\right\}};$$

therefore $$= \frac{\pi S}{\sqrt{27}};$$

Ex. 3. To find a geometrical interpretation of the constants in the general equation of the inscribed conic

$$\sqrt{(lx)} + \sqrt{(my)} + \sqrt{(nz)} = 0.$$

Interpreting with respect to a new triangle $A'B'C'$, this equation denotes a circle, provided that

$$\frac{s'-a'}{\sqrt{l}} = \frac{s'-b'}{\sqrt{m}} = \frac{s'-c'}{\sqrt{n}};$$

hence $$\frac{a'}{\sqrt{m}+\sqrt{n}} = \frac{b'}{\sqrt{n}+\sqrt{l}} = \frac{c'}{\sqrt{l}+\sqrt{m}}.$$

Now let d_1, d_2, d_3 be the three diameters of the conic, which are respectively parallel to the sides BC, CA, AB of the triangle of reference ABC. Then, by prop. 3,

$$d_1 : a :: r' : a',$$
$$d_2 : b :: r' : b',$$
$$d_3 : c :: r' : c';$$

therefore

$$\frac{a}{d_1} : \frac{b}{d_2} : \frac{c}{d_3} :: \sqrt{m}+\sqrt{n} : \sqrt{n}+\sqrt{l} : \sqrt{l}+\sqrt{m};$$

therefore

$$\sqrt{l} : \sqrt{m} : \sqrt{n} :: \frac{b}{d_2}+\frac{c}{d_3}-\frac{d}{d_1} : \frac{c}{d_3}+\frac{a}{d_1}-\frac{b}{d_2} : \frac{a}{d_1}+\frac{b}{d_2}-\frac{c}{d_3};$$

and the equation to the conic may be written

$$\left(\frac{b}{d_2}+\frac{c}{d_3}-\frac{a}{d_1}\right)\sqrt{x}+\left(\frac{c}{d_3}+\frac{a}{d_1}-\frac{b}{d_2}\right)\sqrt{y}+\left(\frac{a}{d_1}+\frac{b}{d_2}-\frac{c}{d_3}\right)\sqrt{z}=0.$$

Ex. 4. To interpret the constants in the conic

$$lx^2+my^2+nz^2=0,$$

with respect to which the triangle of reference is self-conjugate. Interpeting it with respect to the triangle $A'B'C'$, it will be a circle if

$$\frac{a'^2}{m+n}=\frac{b'^2}{n+l}=\frac{c'^2}{l+m}.$$

As before, $\frac{a}{d_1}=\frac{a'}{r'},\ \frac{b}{d_2}=\frac{b'}{r'},\ \frac{c}{d_3}=\frac{c'}{r'};$

therefore $\dfrac{m+n}{\frac{a^2}{d_1^2}}=\dfrac{n+l}{\frac{b^2}{d_2^2}}=\dfrac{l+m}{\frac{c^2}{d_3^2}};$

therefore

$$\frac{l}{\frac{b^2}{d_2^2}+\frac{c^2}{d_3^2}-\frac{a^2}{d_1^2}}=\frac{m}{\frac{c^2}{d_3^2}+\frac{a^2}{d_1^2}-\frac{b^2}{d_2^2}}=\frac{n}{\frac{a^2}{d_1^2}+\frac{b^2}{d_2^2}-\frac{c^2}{d_3^2}}:$$

thus the equation to the conic may be written

$$\left(\frac{b^2}{d_2^2}+\frac{c^2}{d_3^2}-\frac{a^2}{d_1^2}\right)x^2+\left(\frac{c^2}{d_3^2}+\frac{a^2}{d_1^2}-\frac{b^2}{d_2^2}\right)y^2$$
$$+\left(\frac{a^2}{d_1^2}+\frac{b^2}{d_2^2}-\frac{c^2}{d_3^2}\right)z^2=0.$$

We shall now consider what relation must exist between the lengths of the sides of the two triangles ABC, $A'B'C'$, if one is to be the actual orthogonal projection of the other.

Let us imagine the triangle ABC to be orthogonally projected with the triangle $A'B'C'$, with the point A' coincident with A; then, clearly,

$$a^2 - a'^2 = (BB' - CC')^2.$$

Let $\qquad a^2 - a'^2 = \lambda, \quad b^2 - b'^2 = \mu, \quad c^2 - c'^2 = \nu;$

then $\qquad \lambda = \mu + \nu - 2\sqrt{(\mu\nu)};$

hence $\qquad \lambda^2 + \mu^2 + \nu^2 - 2\mu\nu - 2\nu\lambda - 2\lambda\mu = 0 \ldots\ldots\ldots\ldots(1),$

which is the condition that must be satisfied by a', b', c' if they are to refer to some possible orthogonal projection of the triangle ABC.

Now take the case of the circumscribed conic

$$lyz + mzx + nxy = 0.$$

Interpreting this with respect to the triangle $A'B'C'$, such that the equation represents a circle, we have

$$\frac{a'^2}{l} = \frac{b'^2}{m} = \frac{c'^2}{n} = k \text{ suppose.}$$

We require to determine k so that the circle may be the actual projection of the conic, and not merely similar to it.

Substituting for a', b', c' in the relation (1), we get

$$a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2$$
$$+ k^2(l^2 + m^2 + n^2 - 2mn - 2nl - 2lm)$$
$$+ 2k\{l(b^2 + c^2 - a^2) + m(c^2 + a^2 - b^2) + n(a^2 + b^2 - c^2)\} = 0,$$

But $\qquad a^4 + b^4 + \&c. = -16S^2,$

$$l^2 + m^2 + \&c. = K,$$

and $\qquad l(b^2 + c^2 - a^2) + \&c. = -I,$

I and K being calculated on the supposition that the equation to the conic is written in the form

$$2lyz + 2mzx + 2nxy = 0.$$

Hence we get

$$K.k^2 - 2I.k - 16S^2 = 0 \ldots\ldots\ldots\ldots\ldots(A)$$

as the required equation to determine k.

If we take the general conic

$$ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy = 0 \ldots\ldots (\Sigma),$$

we see, by adding the term

$$-(ux + vy + wz)(x + y + z),$$

that the similar and similarly situated conic circumscribing the triangle of reference is

$$(2u' - v - w)\,yz + (2v' - w - u)\,zx + (2w' - u - v)\,xy = 0 \ldots (\Sigma').$$

Call these two conics Σ and Σ' respectively.

Projecting them into circles, we get, for determining k,

$$K'.k^2 - 2I'.k - 16S^2 = 0 \ldots\ldots\ldots\ldots\ldots (2),$$

where
$$a'^2 = k(2u' - v - w),$$
$$b'^2 = k(2v' - w - u),$$
$$c'^2 = k(2w' - u - v),$$

and K', I' refer to the conic Σ'.

Now transform the equation of the cone Σ to a new triangle of reference $A_1B_1C_1$, similar and similarly situated to ABC, but such that it is inscribed in Σ.

Let the transformed equation be

$$2lyz + 2mzx + 2nxy = 0,$$

then the invariants of this equation are μ^2K, μ^2I, μ^2H, where K, I, H are those of the original equation to Σ; and $S_1 = \mu S$, μ being the determinant of transformation.

Hence, if $A_1'B_1'C_1'$ be the projection of $A_1B_1C_1$, we shall have

$$a_1'^2 = k'l,$$
$$b_1'^2 = k'm,$$
$$c_1'^2 = k'n,$$

where
$$K.k'^2 - 2I.k' - 16S^2 = 0 \ldots\ldots\ldots\ldots\ldots (3),$$

the μ^2 dividing out.

Now $$\frac{\text{area of } \Sigma'}{\text{area of } \Sigma} = \frac{A'B'C'}{A_1'B_1'C_1'} = \frac{k(2u' - v - w)}{k'l} = \&c.$$

$$= \frac{k}{k'}\left\{\frac{(2u' - v - w)(2v' - w - u)(2w' - u - v)}{lmn}\right\}^{\frac{1}{3}},$$

$$= \frac{k}{k'}\left\{\frac{H'}{H}\right\}^{\frac{1}{3}}.$$

Again, clearly,

$$\left(\frac{A'B'C'}{A_1'B_1'C_1'}\right)^2 = \frac{k^2 . K'}{k'^2 . \mu^2 K};$$

therefore $$\frac{1}{\mu^2} = \frac{k^2 K'}{k'^2 . \mu^2 K};$$

therefore $$k^2 . K' = k'^2 . K,$$

which eqnation gives the relation between the roots of the equations (2) and (3), viz.

$$\frac{k}{\sqrt{K}} = \frac{k'}{\sqrt{K'}}, \text{......................} (B).$$

Thus we get $$\left\{\frac{\text{area of } \Sigma'}{\text{area of } \Sigma}\right\}^2 = \frac{K}{K'} \times \frac{H'^{\frac{4}{3}}}{\mu^4 H^{\frac{4}{3}}},$$

or $$\frac{1}{\mu^2} = \text{..............};$$

therefore $$\mu = \frac{H(-K')^{\frac{3}{4}}}{H'(-K)^{\frac{3}{4}}}.$$

Here μ is the coefficient by which any linear magnitude connected with Σ' must be multiplied in order to obtain the similar magnitude connected with the given general conic Σ.

Note 1. Since the roots of equations (2) and (3) are connected by the above relation (B), we have simultaneously

$$K'.k^2 - 2I'.k - 16S^2 = 0,$$

and $$K'.k^2 - 2I.\sqrt{\left(\frac{K'}{K}\right)}.k - 16S^2 = 0;$$

whence we obtain the theorem

$$I'\sqrt{K} = I\sqrt{K'},$$

and therefore also $$\frac{k'}{I'} = \frac{k}{I}.$$

Here

$$K' = \begin{vmatrix} 0 & , 2w' - u - v, & 2v' - w - u, & 1 \\ 2w' - u - v, & 0 & , 2u' - v - w, & 1 \\ 2v' - w - u, & 2u' - v - w, & 0 & , 1 \\ 1 & , 1 & , 1 & , 0 \end{vmatrix},$$

$$I' = -(2u' - v - w)(b^2 + c^2 - a^2) - (2v' - w - u)(c^2 + a^2 - b^2)$$
$$- (2w' - u - v)(a^2 + b^2 - c^2)$$
$$= 2a^2(u + u' - v' - w') + 2b^2(v + v' - w' - u') + 2c^2(w + w' - u' - v').$$

Note 2. The equation to Σ' shews that if the general equation

$$ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy = 0$$

denote a circle, the necessary and sufficient conditions are expressed by the equations

$$\frac{v + w - 2u'}{a^2} = \frac{w + u - 2v'}{b^2} = \frac{u + v - 2w'}{c^2}.$$

We shall now investigate the area of the conic Σ', writing it in the form

$$2lyz + 2mzx + 2nxy = 0,$$

where

$$l = 2u' - v - w,$$
$$m = 2v' - w - u,$$
$$n = 2w' - u - v.$$

Projecting it into a circle, we have for the triangle $A'B'C'$ the equations

$$a'^2 = kl,$$
$$b'^2 = km,$$
$$c'^2 = kn.$$

Hence area of $\Sigma' = \pi R'^2 \times \frac{S}{S'}$

$$= \pi \frac{a'^2 b'^2 c'^2}{16 S'^2} \times \frac{S}{S'}$$

$$= \pi \frac{k^3 lmn}{k^3 (-K')^{\frac{3}{2}}} \times 4S$$

$$= \frac{2\pi SH'}{(-K')^{\frac{3}{2}}}.$$

Hence we at once derive that

$$\text{area of } \Sigma = (\text{area of } \Sigma') \times \mu^2$$

$$= \frac{2\pi SH}{(-K)^{\frac{3}{2}}}.$$

Again, we can find immediately the ratio of the axes of Σ', which ratio is, of course, the same for those of Σ.

Let ε denote the ratio of the axes, then

$$\varepsilon = \frac{S'}{S} = \frac{\sqrt{(-K')}}{4S}k;$$

therefore $$\varepsilon = \frac{I' \pm \sqrt{(I'^2 + 16S^2K')}}{4S\sqrt{(-K')}}.$$

The two values of ε given by the formula are, as they should be, reciprocals, the one of the other. When the conic is an ellipse, so that we have the case of real orthogonal projection, the two values of k, upon which of course the two values of ε depend, correspond, the one to the case in which the ellipse is projected into a circle equal to that on the major axis, and the other to the case in which it becomes that on the minor axis.

Hence the squares of the semi-axes are the two values of the expression

$$-\frac{I \pm \sqrt{(I^2 + 16S^2K)}}{4S\sqrt{(-K)}} \times \frac{2SH}{\{-K\}^{\frac{3}{2}}},$$

which reduces to $$-\frac{\{I \pm \sqrt{(I^2 + 16S^2K)}\}H}{2K^2}.$$

Note 1. The equation which determines k is

$$K'.k^2 - 2I\sqrt{\left(\frac{K'}{K}\right)}.k - 16S^2 = 0,$$

or $$K'.k^2 - 2I'.k \qquad - 16S^2 = 0;$$

therefore $$k = \frac{I' \pm \sqrt{(I'^2 + 16S^2K')}}{K'};$$

therefore $$k = \frac{I \pm \sqrt{(I^2 + 16S^2K)}}{\sqrt{(KK')}}.$$

Now K and K' must always have the same sign, since they belong to similar conics; hence the sign of k, and the real or imaginary form of its values, will depend upon those of the expression

$$\frac{I \pm \sqrt{(I^2 + 16S^2K)}}{K};$$

that is, will be the same as those of the roots of the equation

$$K.k^2 - 2I.k - 16S^2 = 0.$$

Now, if the general equation denote an ellipse, K is negative; thus giving a real value to the expression

$$\sqrt{(-K)} = \{(l'+m'+n')(m'+n'-l')(n'+l'-m')(l'+m'-n')\}^{\frac{1}{2}},$$

where

$$l'^2 = 2u' - v - w,$$

$$m'^2 = 2v' - w - u,$$

$$n'^2 = 2w' - u - v.$$

Also we see from the equation that the product of the roots is then positive, and therefore the values of k are of the same sign.

In the case of the hyperbola the roots are of opposite signs, K being positive; and in the case of the parabola we have simply

$$k = -\frac{8S^2}{I},$$

since here

$$K = 0.$$

Regarding this as the limiting case in which one root becomes infinite, we find for the semi-latus rectum of the parabola the expression

$$4S^2\sqrt{\left(\frac{H}{I^3}\right)}.$$

In the case of the rectangular hyperbola

$$I = 0,$$

and the roots are

$$\pm\frac{4S}{\sqrt{K}}.$$

Note 2. It may be shown that the expression $I^2 + 16S^2K$ is always positive.

This is the case if the conic be either an hyperbola or parabola. In the case of the ellipse K is negative. To examine this case reduce the general equation to the form

$$ux^2 - 2u'yz = 0 \ldots\ldots\ldots\ldots(\alpha),$$

which may always be done.

Further, we can always choose the triangle of reference in such a way that (α) reduces to

$$x^2 - 2yz = 0 \ldots\ldots\ldots\ldots(\beta).$$

To prove this, let us interpret geometrically the constants in the equation (α).

Projecting it into a circle, we find for the sides of the triangle $A'B'C'$

$$\frac{a'^2}{2u'} = \frac{b'^2}{u} = \frac{c'^2}{u} = k.$$

Let d_1, d_2, d_3 be the semi-diameters of (α) parallel to BC, CA, AB respectively; then

$$\frac{a^2}{d_1^2} = \frac{a'^2}{r'^2}, \quad \frac{b^2}{d_2^2} = \frac{b'^2}{r'^2}, \quad \frac{c^2}{d_3^2} = \frac{c'^2}{r'^2};$$

therefore

$$\frac{2u'}{\frac{a^2}{d_1^2}} = \frac{u}{\frac{b^2}{d_2^2}} = \frac{u}{\frac{c^2}{d_3^2}}.$$

Hence, if $u' = u$, it follows that

$$\frac{a^2}{d_1^2} = \frac{b^2}{d_2^2} + \frac{c^2}{d_3^2};$$

and we get in the projected figure

$$a'^2 = b'^2 + c'^2;$$

i.e. the triangle $A'B'C'$ is formed by two tangents at right angles and their chord of contact; hence we have only to choose, as triangle of reference in the original plane, any triangle which will correspond to any such triangle as $A'B'C'$ in the plane of projection, and the equation will be reduced to the required form

$$x^2 - 2yz = 0.$$

We may observe that, since we are dealing only with the case in which the general equation represents an ellipse, the projections are all real.

Our triangle of reference is now formed by taking the vertex at any point of the ellipse concentric with the given ellipse, and similar and similarly situated to it, having its linear dimension greater than those of the given ellipse in the ratio of $\sqrt{2} : 1$, the sides of the triangle being formed by the two tangents from the point and their chord of contact.

Now if K' refer to the new equation (β), and Δ denote the determinant of transformation from the form (α) to the form (β),

$$K' = \Delta^2 . K,$$

$$I' = \Delta^2 . I,$$

$$S'^2 = \Delta^2 . S;$$

hence the signs of the two expressions

$$I^2 + 16S^2K \text{ and } I'^2 + 16S^2K'$$

are the same.

Now $$I'^2 = b'^2 + c'^2,\quad K' = -1;$$

therefore $$I'^2 + 16S^2K' = (b'^2 + c'^2)^2 - 16S'^2$$
$$= (b'^2 + c'^2 - 4S')(b'^2 + c'^2 - 4S'),$$

which is positive; because

$$b'^2 + c'^2 - 4S' = b'^2 + c'^2 - 2b'c' \sin A',$$

which is positive.

Hence in all cases the values of k are real.

We shall now investigate the directions of the axes of the conic denoted by the general equation

$$ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy = 0.....(\Sigma).$$

The directions of these axes are of course the same as those of the similar and similarly situated conic

$$(2u' - v - w)\,yz + (2v' - w - u)\,zx + (zw' - u - v)\,xy = 0...(\Sigma'),$$

or, as we shall write it,

$$2lyz + 2mzx + 2nxy = 0,$$

where $$l = 2u' - v - w, \text{ \&c.}$$

Now if the straight line

$$\frac{y}{\mu} - \frac{z}{\nu} = 0,$$

which passes through the point A, cuts BC in P, then we have for the point P the equations

$$\frac{y}{\mu} = \frac{z}{\nu} = \frac{1-x}{\mu+\nu},$$

and hence $$AP^2 = \frac{b^2\nu(\mu+\nu) + c^2\mu(\mu+\nu) - a^2\mu\nu}{(\mu+\nu)^2}.$$

Now, projecting the conics into circles, and AP into $A'P'$, if AP and $A'P'$ be equal, they must be parallel to an axis of the conic Σ, and to the similar axis of the conic Σ'. Equating the expressions for AP^2 and $A'P'^2$, we get

$$(\mu+\nu)\{\nu(b^2 - b'^2) + \mu(c^2 - c'^2)\} = \mu\nu(a^2 - a'^2).$$

Now let
$$a'^2 = kl,$$
$$b'^2 = km,$$
$$c'^2 = kn,$$
$$\frac{\mu}{\nu} = \rho,$$

and determining k as usual, we get
$$\rho^2 (c^2 - kn) + \rho \{b^2 + c^2 - a^2 - k(m + n - l)\} + b^2 - km = 0.$$

But the equation for determining k is exactly the condition that the roots of this equation in ρ should be equal. Hence we may take as the value of ρ
$$\rho = \sqrt{\left(\frac{b^2 - km}{c^2 - kn}\right)};$$
where the radical must have the same sign as the expression $k(m + n - l) - b^2 - c^2 + a^2$; therefore the straight lines through the angular points A, B, C, parallel to that axis to which the root k_1 of the equation for k corresponds, are
$$\frac{y}{\sqrt{\{b^2 - k_1(2v' - w - u)\}}} = \frac{z}{\sqrt{\{c^2 - k_1(2w' - u - v)\}}},$$
$$\frac{z}{\sqrt{\{c^2 - k_1(2w' - u - v)\}}} = \frac{x}{\sqrt{\{a^2 - k_1(2u' - v - w)\}}},$$
$$\frac{x}{\sqrt{\{a^2 - k_1(2u' - v - w)\}}} = \frac{y}{\sqrt{\{b^2 - k_1(2v' - w - u)\}}},$$
the equation for k being
$$K'.k^2 - 2I\sqrt{\left(\frac{K'}{K}\right)}.k - 16S'^2 = 0.$$

Again, we may determine the condition that the straight line
$$\alpha x + \beta y + \gamma z = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\gamma)$$
be parallel to an axis of the conic Σ.

The line through the point A parallel to (γ) is
$$\frac{y}{\gamma - \alpha} = \frac{z}{\alpha - \beta};$$
hence we find that the required condition is
$$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 - (b^2 + c^2 - a^2)\beta\gamma - (c^2 + a^2 - b^2)\gamma\alpha - (a^2 + b^2 - c^2)\alpha\beta$$
$$= k\{(2u' - v - w)\alpha^2 + (2v' - w - u)\beta^2 + (2w' - u - v)\gamma^2$$
$$- 2(v' + w' - u' - u)\beta\gamma - 2(w' + u' - v' - x)\gamma\alpha$$
$$- 2(u' + v' - w' - w)\alpha\beta\}.$$

We can in a similar manner find the length d, of the semi-diameter of Σ, which is parallel to the straight line

$$\alpha x + \beta y + \gamma z = 0 \ldots\ldots\ldots\ldots\ldots\ldots (\gamma).$$

Let AP be parallel to (γ) and cut BC in P; as before, we get

$$AP^2 = \frac{-a^2(\gamma-\alpha)(\alpha-\beta) - b^2(\alpha-\beta)(\beta-\gamma) - c^2(\beta-\gamma)(\gamma-\alpha)}{(\beta-\gamma)^2}$$

$$= \frac{a^2\alpha^2+b^2\beta^2+c^2\gamma^2-(b^2+c^2-a^2)\beta\gamma-(c^2+a^2-b^2)\gamma\alpha-(a^2+b^2-c^2)\alpha\beta}{(\beta-\gamma)^2}.$$

Projecting the conic into a circle, we get, with the notation used above,

$$\frac{d'^2}{R'^2} = \frac{AP^2}{A'P'^2},$$

where d' is the semi-diameter of Σ' drawn parallel to (γ); therefore

$$d'^2 = \frac{a^2\alpha^2 + \&c.}{k(l\alpha^2 + \&c.)} \times \frac{k^3lmn}{-K'.k^2}$$

$$= \frac{a^2\alpha^2 + \&c.}{k(l\alpha^2 + \&c.)} \times \frac{H'}{-K'}.$$

Hence $$d^2 = \mu d'^2 \text{ or } d_2'^2 \times \frac{HK'^{\frac{3}{2}}}{H'K^{\frac{3}{2}}}$$

$$= -\frac{a^2\alpha^2 + \&c.}{(2u'-v-w)\alpha^2 + \&c.} \times \frac{HK'^{\frac{1}{2}}}{K^{\frac{3}{2}}}.$$

So far as the present writer knows, the above method is new.

Charterhouse School,
April, 1879.

ON THE CONNECTION OF CERTAIN FORMULÆ IN ELLIPTIC FUNCTIONS.

By Professor *Cayley*.

In reference to a like question in the theory of the double ϑ-functions, it is interesting to show that (if not completely, at least very nearly) the single formula

$$\Pi(u, a) = u\frac{\Theta' a}{\Theta a} + \tfrac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)},$$

that is

$$\int_0 \frac{k^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \operatorname{sn}^2 u\, du}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u} = u\frac{\Theta' a}{\Theta a} + \tfrac{1}{2}\log\frac{\Theta(u+a)}{\Theta(u+a)},$$

leads not only to the relation

$$\log\Theta u = \tfrac{1}{2}\log\frac{2k'K}{\pi} + \tfrac{1}{2}\left(1 - \frac{E}{K}\right)u^2 - k^2\int_0 du \int_0 du \operatorname{sn}^2 u,$$

between the functions Θ, sn, but also to the addition-equation for the function sn.

Writing in the equation a indefinitely small, and assuming only that $\operatorname{sn} a$, $\operatorname{cn} a$, $\operatorname{dn} a$ then become a, 1, 1, respectively, the equation is

$$k^2 a\int_0 \operatorname{sn}^2 u\, du = u\frac{a\Theta''0}{\Theta 0} + \tfrac{1}{2}\log\frac{\Theta u - a\Theta' u}{\Theta u + a\Theta' u},$$

$$= ua\frac{\Theta''0}{\Theta 0} - a\frac{\Theta' u}{\Theta u},$$

that is
$$\frac{\Theta' u}{\Theta u} = u\frac{\Theta''0}{\Theta 0} - k^2\int_0 du \operatorname{sn}^2 u,$$

or, integrating from $u = 0$, this is

$$\log\Theta u = C + \tfrac{1}{2}u^2\frac{\Theta''0}{\Theta 0} - k^2\int_0 du\int_0 du \operatorname{sn}^2 u,$$

which, except as regards the determination of the constants, is the required equation for $\log\Theta u$.

Next, differentiating twice the equation for $\Pi(u, a)$, and once the equation obtained for $\frac{\Theta' u}{\Theta u}$, we have

$$k^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \frac{d}{du}\left(\frac{\operatorname{sn}^2 u}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u}\right)$$

$$= \tfrac{1}{2}\frac{\Theta''\Theta - \Theta'^2}{\Theta^2}(u-a) - \tfrac{1}{2}\frac{\Theta''\Theta - \Theta'^2}{\Theta^2}(u+a),$$

and
$$\frac{\Theta''\Theta - \Theta'^2}{\Theta^2}u = \frac{\Theta''0}{\Theta 0} - k^2 \operatorname{sn}^2 u,$$

where, for shortness, $\frac{\Theta''\Theta - \Theta'^2}{\Theta^2}u$ is written to denote

$\dfrac{\Theta''u\Theta u-(\Theta'u)^2}{\Theta^2 u}$, and the like in the first equation; the right-hand side of the first equation therefore is

$$-\tfrac{1}{2}k^2\{\operatorname{sn}^2(u-a)-\operatorname{sn}^2(u+a)\},$$

or the equation becomes

$$2\operatorname{sn}a\operatorname{cn}a\operatorname{dn}a\frac{d}{du}\frac{\operatorname{sn}^2u}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2a}=\operatorname{sn}^2(u+a)-\operatorname{sn}^2(u-a),$$

that is $\dfrac{4\operatorname{sn}u\operatorname{sn}'u\operatorname{sn}a\operatorname{cn}a\operatorname{dn}a}{(1-k^2\operatorname{sn}^2u\operatorname{sn}^2a)^2}=\operatorname{sn}^2(u+a)-\operatorname{sn}^2(u-a).$

The numerator on the left-hand side must be a symmetrical function of u, a, and hence (even if the value of $\operatorname{sn}'u$ were unknown) it would appear that $\operatorname{sn}'u$ must be a mere constant multiple of $\operatorname{cn}u\operatorname{dn}u$; assuming, however, the actual value, $\operatorname{sn}'u=\operatorname{cn}u\operatorname{dn}u$, the formula is

$$\begin{aligned}\frac{4\operatorname{sn}u\operatorname{cn}u\operatorname{dn}u\operatorname{sn}a\operatorname{cn}a\operatorname{dn}a}{(1-k^2\operatorname{sn}^2u\operatorname{sn}^2a)^2}&=\operatorname{sn}^2(u+a)-\operatorname{sn}^2(u-a)\\&=\{\operatorname{sn}(u+a)+\operatorname{sn}(u-a)\}\{\operatorname{sn}(u+a)-\operatorname{sn}(u-a)\}.\end{aligned}$$

The factor $\{\operatorname{sn}(u+a)+\operatorname{sn}(u+a)\}$ becomes $=2\operatorname{sn}u$ for $a=0$, and this suggests that the factor $\operatorname{sn}u$ on the left-hand side is a factor of $\{\operatorname{sn}(u+a)+\operatorname{sn}(u-a)\}$. That $\operatorname{cn}u$ is *not* a factor hereof would follow from the properties of the period K; viz. for $u=K$, $\operatorname{cn}u=0$, but $\{\operatorname{sn}(u+a)+\operatorname{sn}(u-a)\}$, $=2\operatorname{sn}(K+a)$ is not $=0$; and, similarly, that $\operatorname{dn}u$ is *not* a factor from the properties of the period iK; hence, $\operatorname{cn}u$, $\operatorname{dn}u$ belong to the other factor $\{\operatorname{sn}(u+a)-\operatorname{sn}(u-a)\}$, and by symmetry $\operatorname{cn}a$, $\operatorname{dn}a$ belong to the first-mentioned factor. And we are thus led to assume

$$\operatorname{sn}(u+a)+\operatorname{sn}(u-a)=2M\operatorname{sn}u\operatorname{cn}a\operatorname{dn}a,$$

$$\operatorname{sn}(u+a)-\operatorname{sn}(u-a)=2M'\operatorname{sn}a\operatorname{cn}u\operatorname{dn}u,$$

where $\qquad \text{denom.}=1-k^2\operatorname{sn}^2a\operatorname{sn}^2u,$

and $MM'=1$. Some further investigation is wanting to show that M and M' are constants, but assuming that they are so and each $=1$, the formulæ give at once the ordinary expression for $\operatorname{sn}(u+a)$; that is, we have the addition-equation for the function sn.

NOTES ON OPTICS.

By *R. Pendlebury, M.A.*

1. *On the principal points of a system of lenses.*

It may be of service to notice (as a connexion between the method of Gauss and that of the ordinary English text-books, that in the case of a single lens the focal centres (Parkinson's *Optics*, p. 99) are the same as the principal points ("Hauptpunkte"). This may be shown in the following manner, which at the same time gives an expression for the position of the principal points of any system of lenses.

From the nature of the case the principal points may be defined as two points on the axis, such that the image of any object in a plane drawn through one perpendicular to the axis lies in a plane similarly drawn through the other, the magnification being $= +1$. It can be easily shown that for one refraction at a spherical surface the magnification is $\frac{v}{\mu u}$, where u, v, μ are the distances of the object and image from the surface, and μ the refractive index. Hence for a lens the magnification is $\frac{v_1 v_2}{u_1 u_2}$, the indices referring to the first and second refractions respectively. To find the principal points we have therefore $\frac{v_1 v_2}{u_1 u_2} = 1$. By the ordinary formula,

$$\frac{\mu}{v_1} - \frac{1}{u_1} = \frac{\mu - 1}{r_1},$$

$$\frac{\mu u_1}{v_1} = 1 + \frac{\mu - 1}{r_1} u_1.$$

Similarly, t being the thickness of the lens,

$$\frac{u_2}{\mu v_2} = 1 - \frac{\mu - 1}{\mu r_2} u_2$$

$$= 1 - \frac{\mu - 1}{\mu r_2} (v_1 + t)$$

$$= \left(1 - \frac{\mu - 1}{\mu r_2} t\right) - \frac{\mu - 1}{\mu r_2} v_1;$$

therefore

$$\frac{u_1 u_2}{v_1 v_2} = \left(1 - \frac{\mu - 1}{\mu r_2} t\right)\left(1 + \frac{\mu - 1}{r_1} u_1\right) - \frac{\mu - 1}{r_2} u_1 = 1,$$

if u_1 be the distance of the first principal point from the first surface. Solving the equation for u_1 we get

$$u_1 = \frac{tr_1}{\mu (r_2 - r_1 - t) + t}.$$

We have also

$$v_2 = \frac{u_1 u_2}{v_1} = u_1\left(1 + \frac{t}{v_1}\right) = u_1 + t \, . \left(\frac{1}{\mu} + \frac{\mu - 1}{\mu r_1} u_1\right)$$

$$= \frac{tr_2}{\mu (r_2 - r_1 - t) + t},$$

u_1 and v_2 are the distances of the two principal points from the first and second surfaces of the lens, and these values show (see Parkinson *l.c.*) that the principal points coincide with the focal centres.

The same method may be used to get the positions of the principal points in the most general case, *i.e.* for a series of spherical refracting surfaces on the same axis. Let μ_r be the index of refraction at the r^{th} surface, *i.e.* the index from the $(r-1)^{\text{th}}$ medium into the r^{th}; ρ_r the radius of the r^{th} surface; and t_r the distance from the vertex of the r^{th} surface to that of the $(r+1)^{\text{th}}$. Also let m_r be the *inverse* of the magnification produced by the r^{th} surface, *i.e.*

$$m_r = \frac{\mu_r u_r}{v_r}.$$

The total inverse magnification will be $m_1 m_2 \ldots m_n$, and to find the principal points this must be put $= +1$. As an abbreviation, put $\mu_r - 1 = \lambda_r$ and $m_1 m_2 \ldots m_k = f_k$. Now

$$m_n = \frac{\mu_n u_n}{v_n} = 1 + \lambda_n \frac{u_n}{\rho_n} = \left(1 + \frac{\lambda_n t_{n-1}}{\rho_n}\right) + \lambda_n \frac{v_{n-1}}{\rho_n}.$$

Putting $$v_{n-1} = \frac{\mu_{n-1} u_{n-1}}{m_{n-1}},$$

$$\rho_n m_n m_{n-1} = (\rho_n + \lambda_n t_{n-1})\, m_{n-1} + \lambda_n \mu_{n-1} u_{n-1}.$$

Putting for u_{n-1} the value $t_{n-2} + \frac{\mu_{n-2} u_{n-2}}{m_{n-2}}$, we get

$$\rho_n m_n m_{n-1} m_{n-2} = (\rho_n + \lambda_n t_{n-1}) m_{n-1} m_{n-2} + \lambda_n \mu_{n-1} m_{n-2} t_{n-2} + \lambda_n \mu_{n-1} \mu_{n-2} u_{n-2}.$$

[illegible]

Proceeding [illegible] reduction, we get finally

[illegible]

[illegible] by turning [illegible] we get

[illegible]

We thus have [illegible] between the quantities [illegible]

[illegible]

Hence [illegible]

[illegible]

and Δ′ = the same determinant as Δ omitting the last row and the last column. [illegible] is determined by a simple equation.

This, it may be noticed, gives at once an expression for the magnifying power [illegible] of such a system of surfaces.

In the case of an optical instrument the above expressions simplify, n must be even, and between the different quantities μ we have the relation

$$\mu_1\mu_2 = \mu_3\mu_4 = \ldots = \mu_{n-1}\mu_n = 1.$$

2. *On the condition of achromatism of a system of lenses.*

In considering the condition of achromatism of a system of lenses used to form the image of a plane object situated in a plane perpendicular to the axis of the system, it should be noticed that two things are necessary, viz. (i) that the focus conjugate to the point where the plane of the object cuts the axis should be the same for two (or more) colours, and (ii) that the magnifying powers for these colours should be the same. The first condition makes the image achromatic at its centre, and the second makes the image achromatic throughout (see Verdet, *Œuvres* IV. 935).

Take, for example, the image formed by two thin lenses. It can easily be seen that the magnifying power of a single lens is $\frac{v}{u}$. That of a combination of two lenses will therefore be $\frac{v}{u}\frac{v_1}{u_1}$ where v, u have their usual meanings, and v_1, u_1 refer to the second lens. If the lenses are close together $u_1 = v$, so that the magnification m is given by $m = \frac{v_1}{v}$. Taking the variation of this we get, since u is fixed, $\delta m = \frac{1}{u}\delta v_1$ so that if $\delta v_1 = 0$, $\delta m_1 = 0$ also. But $\delta v_1 = 0$ is the condition of achromatism at the centre of the image; so that, in this case, if the first of the above conditions is satisfied the second is also satisfied.

Now suppose the lenses separated by an interval a. We have then $u_1 = v + a$, and $m = \frac{v}{u}\frac{v_1}{v+a}$, or $\frac{1}{m} = \left(1 + \frac{a}{v}\right)\frac{v_1}{u}$. Taking the variation of this for different refractive indices,

$$\delta\left(\frac{1}{m}\right) = \left(1 + \frac{a}{v}\right)\frac{1}{u}\delta v_1 - \frac{v_1 a}{u}\frac{\delta v}{v^2},$$

and the expression on the right hand must $= 0$ in order to satisfy the second condition. The first condition requires that $\delta v_1 = 0$. Hence, if both conditions are satisfied, we must have

$$\delta v = 0, \quad \delta v_1 = 0.$$

The meaning of these equations is obviously that each lens must be achromatic independently of the other; that is to say, each lens must be a double one, made up of two separate lenses of different kinds of glass, and that the intermediate image must itself be completely achromatic.

The foregoing, of course, applies only to the cases where the inclination of any ray to the axis, and its distance at any point from the axes are small quantities whose cubes are neglected. It will therefore not apply to a camera lens where the field of view is very wide. Still in practice, the two pairs in a camera lens are separately achromatized.

It is obvious that the same method can be applied to any number of lenses.

ON RATIONAL FUNCTIONAL DETERMINANTS.

By Professor *Paul Mansion*.

1. *Notation.* Let

$$Fx = a + bx + cx^2 + dx^3,$$

$$\phi x = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + A_6x^6,$$

$$\psi x = B_0 + B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6,$$

$$\chi x = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + C_6x^6;$$

$$(m, n, p) = (mnp) = \begin{vmatrix} \alpha^m, & \alpha^n, & \alpha^p \\ \beta^m, & \beta^n, & \beta^p \\ \gamma^m, & \gamma^n, & \gamma^p \end{vmatrix},$$

$$(m, n, pF) = (mnpF) = \begin{vmatrix} \alpha^m, & \alpha^n, & \alpha^p F\alpha \\ \beta^m, & \beta^n, & \beta^p F\beta \\ \gamma^m, & \gamma^n, & \gamma^p F\gamma \end{vmatrix},$$

$$(\phi, \psi, \chi) = (\phi\psi\chi) = \begin{vmatrix} \phi\alpha, & \psi\alpha, & \chi\alpha \\ \phi\beta, & \psi\beta, & \chi\beta \\ \phi\gamma, & \psi\gamma, & \chi\gamma \end{vmatrix}.$$

If α, β, γ are the roots of $Fx = 0$, we have $(m, n, pF) = 0$, and it is easy, as we shall see, to find the value of $(\phi\psi\chi)$.

2. *Value of* (mnp). For the sake of simplicity we shall suppose m, n, p less than 7. Develope the first members of the equations

$$(010F) = 0, \quad (011F) = 0, \quad (012F) = 0, \quad (013F) = 0;$$

and we find

$$a\,(010)+b\,(011)+c\,(012)+d\,(013)=0,$$
$$a\,(011)+b\,(012)+c\,(013)+d\,(014)=0,$$
$$a\,(012)+b\,(013)+c\,(014)+d\,(015)=0,$$
$$a\,(013)+b\,(014)+c\,(015)+d\,(016)=0.$$

The functions (010), (011) being zero, these four homogeneous equations between

$$+(012),\quad -(013),\quad +(014),\quad -(015),\quad +(016),$$

give the ratios of these quantities. They are proportional to the determinants obtained by omitting successively the first, second, third, fourth, or fifth column of the rectangular table

$$\left|\begin{array}{ccccc} c, & d, & , & , & \\ b, & c, & d, & , & \\ a, & b, & c, & d, & \\ , & a, & b, & c, & d \end{array}\right|.$$

Write down the following table:

$$\left|\begin{array}{ccccccc} a, & b, & c, & d, & , & , & \\ , & a, & b, & c, & d, & , & \\ , & , & a, & b, & c, & d, & \\ , & , & , & a, & b, & c, & d \end{array}\right|,$$
$$[\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6\ ],$$

which is more extended, but of which the law of formation is more simple. The columns bear the indices 0, 1, 2, 3, 4, 5, 6. Denote by Δ_{mnp} the determinant obtained by omitting in this table the columns of indices m, n, p.

It is clear we shall have

$$(012):-(013):(014):-(015):(016)=\Delta_{012}:\Delta_{013}:\Delta_{014}:\Delta_{015}:\Delta_{01}\,.$$

The equations

$$(020F)=0,\quad (021F)=0,\quad (022F)=0,\quad (023F)=0,$$

give also

$$-(021):(023):-(024):(025):-(026)=\Delta_{012}:\Delta_{023}:\Delta_{024}:\Delta_{025}:\Delta_{02}\,.$$

In general

$$(mnp)\,:\,(012)=(-1)^{k+m+n-1+p-2}\Delta_{mnp}\,:\,\Delta_{012},$$

k indicating the number of inversions of the permutations (mnp).

3. *Value of the rational functional determinant* $(\phi\psi\chi)$ (Cf. Salmon, *Higher Algebra*, 3rd edition, p. 290). We have evidently

$$(\phi\psi\chi) = \begin{vmatrix} A_0, & A_1, & A_2, & A_3, & A_4, & A_5, & A_6 \\ B_0, & B_1, & B_2, & B_3, & B_4, & B_5, & B_6 \\ C_0, & C_1, & C_2, & C_3, & C_4, & C_5, & C_6 \\ a, & b, & c, & d, & , & , & \\ , & a, & b, & c, & d, & , & \\ , & , & a, & b, & c, & d, & \\ , & , & , & a, & b, & c, & d \end{vmatrix} \frac{(012)}{\Delta_{012}} \ldots (G),$$

for on comparing the coefficients of $A_m B_n C_p$ in $(\phi\psi\chi)$ and in the determinant of the second member, we find that they are equal to

$$(mnp), \quad (-1)^{k+m+n-1+p-2}\Delta_{mnp},$$

the ratio of which is $(012) : \Delta_{012}$.

The theorem is thus demonstrated. It was established in a different manner by Garbieri in 1878 (*Giornale di Battaglini*).

4. *Generalisation.* The method employed above enables us to find $(\phi\psi\chi)$, in the case in which α, β, γ are common roots of two equations; for example of

$$a+bx+cx^2+dx^3+cx^4=0, \quad a'+b'x+c'x^2+d'x^3+e'x^4+f'x^5+g'x^6=0.$$

In this case, the four last lines of the determinant of the second member of (G) are replaced by the following:

$$\begin{vmatrix} a', & b', & c', & d', & e', & f', & g' \\ a, & b, & c, & d, & e, & , & \\ , & a, & b, & c, & d, & e, & \\ , & , & a, & b, & c, & d, & e \end{vmatrix},$$

as is easily proved by the theory of elimination.

Antwerp, *May* 15, 1879.

THE SCALENE CONE.

By *C. Taylor, M.A.*

THE determination of the foci and directrices of the sections of the Oblique Cone has been very ably treated by Mr. John Walker in the *Cambridge and Dublin Mathematical Journal*, vol. VII. pp. 16–28 (1852), where it is shewn (as a special case of a more general theorem) that

The line of intersection of a pair of subcontrary planes of common section of a sphere with a cone is a directrix of the section made by a tangent plane to the sphere through that line, and its point of contact is the corresponding focus.

Since this special case has been more recently treated with overmuch elaboration, it may be worth while to give the proof over again in a simple form.

With V as vertex and any fixed circle as base describe a scalene cone (fig. 4); and let a plane through its "axis" (that is, the line joining the vertex to the centre of the base) at right angles to its base be taken as the plane of reference; and let AA' be the axis of the section made by a plane at right angles to the plane of reference.*

I.

Then if AD, $A'D'$ be diameters and O, O' the centres of the circular sections made by the planes drawn through A and A' parallel to the base, it may be shewn as in the case of the right cone that the conjugate axis is a mean proportional to AD and $A'D'$. Hence the focus S is determined by taking $AS.A'S$ equal $AO.A'O'$, or

$$AS : AO = A'O' : A'S.$$

II.

Let SO meet VA' in M, and let MNX be drawn parallel to the base to meet VA and $A'A$. Then since AD is bisected in O, the line AA' is divided harmonically in S and X;† and conversely NS passes through the middle point O' of $A'D'$.

Therefore by parallels, and from above (§ I.),

$$XM : XS = AO : AS = A'S : A'O'$$
$$= XS : XN;$$

and therefore the circle round MNS touches AA'.

* The *transverse* axis of the section being supposed to lie in the plane of reference, the two circular sections through A (or A') will lie on the same side of the plane of section.

† This of course shows at once that XZ is the S-directrix regarded as the polar of S.

In like manner it may be shown that the same circle meets VM and VN again upon the line Xmn drawn in the direction subcontrary to XNM.

III.

Draw VP to any point P of the section to meet the sphere in q and Q (fig. 5); and draw PRr parallel to the axis of the cone to meet the planes of the circles MQN, mqn; and let the axis of the cone meet mn in e and MN in E. Then by similar triangles, compounding,

$$PR.Pr : PQ.Pq = VE.Ve : VQ.Vq$$

where VE, Ve, and the product $VQ.Vq$ are constant.

Moreover, if PZ be drawn parallel to $A'A$ to meet the line of intersection of the planes of the two circles, it is evident that PR varies as PZ and likewise Pr varies as PZ for all positions of P.

Hence SP^2 or $PQ.Pq$ varies as $PR.Pr$, and therefore as PZ^2; that is to say, *SP varies as PZ and XZ is the directrix corresponding to the focus S.*

In like manner the second focus and directrix may be determined.

Conversely, the section made by the plane drawn through XZ to touch the sphere in S has S and XZ for Focus and Directrix.

IV.

If the tangent plane through V to the cone meet the directrices in T and T', and if VP meet the sphere corresponding to the second focus in Q' and q', it is evident that TQ, TQ' and Tq, Tq' (which touch parallel circular sections) are parallel each to each; whence it follows, antecedently to the proof of § III., that

$$PT^2 : PT'^2 = PQ.Pq : PQ'.Pq' = SP^2 : S'P^2,$$

and hence (since PST and $PS'T'$ may be shewn to be right angles) that the tangent at P makes equal angles with SP and $S'P$.

This subject is treated more at length by Mr. S. A. Renshaw in his treatise on *The Cone and its Sections* (London, 1875), to which the reader may be referred for some large and carefully drawn diagrams in illustration of the above constructions.

FLUID MOTION IN A ROTATING RECTANGLE, FORMED BY TWO CONCENTRIC CIRCULAR ARCS AND TWO RADII.

By *A. G. Greenhill, M.A.*

LET $r=a$, $r=b$ be the arcs, and $\theta=\pm\alpha$ the radii, and put $c^2=ab$, a being supposed greater than b.

We must put the current function

$$\psi=\tfrac{1}{2}\omega r^2\frac{\cos 2\theta}{\cos 2\alpha}$$

$$+\sum_0^\infty\left\{A_{2n+1}\left(\frac{r}{c}\right)^{(2n+1)\frac{\pi}{2\alpha}}+B_{2n+1}\left(\frac{c}{r}\right)^{(2n+1)\frac{\pi}{2\alpha}}\right\}\cos(2n+1)\frac{\pi\theta}{2\alpha},$$

and determine the A's and B's so that when $r=a$, $\psi=\tfrac{1}{2}\omega a^2$, and when $r=b$, $\psi=\tfrac{1}{2}\omega b^2$.

Put
$$q=\left(\frac{b}{a}\right)^{\frac{\pi}{2\alpha}}=e^{-\pi\frac{K'}{K}};$$

therefore
$$\frac{a}{c}=\sqrt{\left(\frac{a}{b}\right)}=q^{-\frac{\alpha}{\pi}}=e^{\alpha\frac{K'}{K}},$$

$$\frac{b}{c}=\sqrt{\left(\frac{b}{a}\right)}=q^{\frac{\alpha}{\pi}}=e^{-\alpha\frac{K'}{K}}.$$

Now put
$$\psi=\tfrac{1}{2}\omega r^2\frac{\cos 2\theta}{\cos 2\alpha}$$

$$+\tfrac{1}{2}\omega c^2\sum_0^\infty\left\{C_{2n+1}\cosh(2n+1)\frac{\pi}{2\alpha}\log\frac{r}{c}+D_{2n+1}\sinh(2n+1)\frac{\pi}{2\alpha}\log\frac{r}{c}\right\}$$
$$\times\cos(2n+1)\frac{\pi\theta}{2\alpha};$$

and putting (1) $r=a$ and $\psi=\tfrac{1}{2}\omega a^2$

$$=\tfrac{1}{2}\omega c^2e^{2\alpha\frac{K'}{K}}=\tfrac{1}{2}\omega c^2\left(\cosh 2\alpha\frac{K'}{K}+\sinh 2\alpha\frac{K'}{K}\right),$$

then

$$\Sigma\left\{C_{2n+1}\cosh(2n+1)\frac{\pi K'}{2K}+D_{2n+1}\sinh(2n+1)\frac{\pi K'}{2K}\right\}\cos(2n+1)\frac{\pi\theta}{2\alpha}$$
$$=\left(\cosh 2\alpha\frac{K'}{K}+\sinh 2\alpha\frac{K'}{K}\right)\left(1-\frac{\cos 2\theta}{\cos 2\alpha}\right)\ldots(1);$$

and putting $\Omega\gamma r = ...$, and therefore

$$2u = ...(\cosh 2x \frac{K'}{K} - \sinh 2x \frac{K'}{K});$$

then

$$\Sigma\{C_{2n+1}\cosh(2n+1)\frac{\pi K'}{2K} - D_{2n+1}\sinh(2n+1)\frac{\pi K'}{2K}\}\cos(2n+1)\frac{\pi\theta}{2a}$$

$$= (\cosh 2x\frac{K'}{K} - \sinh 2x\frac{K'}{K})\frac{\cos 2\theta}{\cos 2a} \ldots (2).$$

Multiplying equations (1) and (2) by $\cos(2n+1)\frac{\pi\theta}{2a}$, and integrating between the limits $-a$ and a, we shall find

$$C_{2n+1}\cosh(2n+1)\frac{\pi K'}{2K} - D_{2n+1}\sinh(2n+1)\frac{\pi K'}{2K}$$

$$= N\left(\cosh 2a\frac{K'}{K} - \sinh 2a\frac{K'}{K}\right),$$

$$C_{2n+1}\cosh(2n+1)\frac{\pi K'}{2K} + D_{2n+1}\sinh(2n+1)\frac{\pi K'}{2K}$$

$$= N\left(\cosh 2a\frac{K'}{K} + \sinh 2a\frac{K'}{K}\right),$$

where

$$N = 2(-1)^n\left\{\frac{1}{(2n+1)\pi - 4a} - \frac{2}{(2n+1)\pi} + \frac{1}{(2n+1)\pi + 4a}\right\}$$

$$= 64\frac{a^2}{\pi}(-1)^n\frac{1}{(2n+1)\{(2n+1)^2\pi^2 - 16a^2\}};$$

and therefore

$$C_{2n+1} = N\frac{\cosh 2a\frac{K'}{K}}{\cosh(2n+1)\frac{\pi K'}{2K}},$$

$$D_{2n+1} = N\frac{\sinh 2a\frac{K'}{K}}{\sinh(2n+1)\frac{\pi K'}{2K}}.$$

Therefore the current function

$$\psi = \tfrac{1}{2}\omega r^2 \frac{\cos 2\theta}{\cos 2\alpha}$$

$$+ \tfrac{1}{2}\omega c^2 \cosh 2\alpha \frac{K'}{K} \Sigma N \frac{\cosh\left\{(2n+1)\dfrac{\pi}{2\alpha}\log\dfrac{r}{c}\right\}\cos(2n+1)\dfrac{\pi\theta}{2\alpha}}{\cosh(2n+1)\dfrac{\pi K'}{2K}}$$

$$+ \tfrac{1}{2}\omega c^2 \sinh 2\alpha \frac{K'}{K} \Sigma N \frac{\sinh\left\{(2n+1)\dfrac{\pi}{2\alpha}\log\dfrac{r}{c}\right\}\cos(2n+1)\dfrac{\pi\theta}{2\alpha}}{\sinh(2n+1)\dfrac{\pi K'}{2K}};$$

and therefore the velocity function ϕ, being the conjugate function to ψ, can be immediately written down,

$$\phi = \tfrac{1}{2}\omega r^2 \frac{\sin 2\theta}{\cos 2\alpha}$$

$$+ \tfrac{1}{2}\omega c^2 \cosh 2\alpha \frac{K'}{K} \Sigma N \frac{\sinh\left\{(2n+1)\dfrac{\pi}{2\alpha}\log\dfrac{r}{c}\right\}\sin(2n+1)\dfrac{\pi\theta}{2\alpha}}{\cosh(2n+1)\dfrac{\pi K'}{2K}}$$

$$+ \tfrac{1}{2}\omega c^2 \sinh 2\alpha \frac{K'}{K} \Sigma N \frac{\cosh\left\{(2n+1)\dfrac{\pi}{2\alpha}\log\dfrac{r}{c}\right\}\sin(2n+1)\dfrac{\pi\theta}{2\alpha}}{\cosh(2n+1)\dfrac{\pi K'}{2K}}.$$

Therefore

$$\psi + i\phi = \tfrac{1}{2}\omega r^2 \frac{e^{2i\theta}}{\cos \alpha}$$

$$+ \tfrac{1}{2}\omega c^2 \cosh 2\alpha \frac{K'}{K} \Sigma N \frac{\cos(2n+1)\dfrac{\pi z}{2\alpha}}{\cosh(2n+1)\dfrac{\pi K'}{2K}}$$

$$+ \tfrac{1}{2}\omega c^2 \sinh 2\alpha \frac{K'}{K} \Sigma N \frac{\sin(2n+1)\dfrac{\pi z}{2\alpha}}{\sinh(2n+1)\dfrac{\pi K'}{2K}},$$

putting

$$z = \theta + i\log\frac{r}{c}.$$

We can easily verify that ϕ satisfies the required conditions, for when $\theta = \pm \alpha$, $\frac{\delta\phi}{r\delta\theta} = \omega r$; and, when $r = a$,

$$\frac{\delta\phi}{\delta r} = \omega a \frac{\sin 2\theta}{\cos 2\alpha} + 16\omega a\alpha \Sigma (-1)^{n+1} \frac{\sin (2n+1)\frac{\pi\theta}{2\alpha}}{(2n+1)^2\pi^2 - 16\alpha^2} = 0,$$

for all values of θ between $-\alpha$ and α; and, similarly, $\frac{\delta\phi}{\delta r} = 0$ when $r = b$.

In the infinite series for ψ and ϕ we can write down the values of the parts of the series which have $2n+1$ as a factor of the denominator of the n^{th} term; we have

$$\Sigma (-1)^n \frac{\cosh\left\{(2n+1)\frac{\pi}{2\alpha}\log\frac{r}{c}\right\} \cos (2n+1)\frac{\pi\theta}{2\alpha}}{(2n+1)\cosh (2n+1)\frac{\pi K'}{2K}}$$

$$= \frac{1}{k}\tan^{-1} \frac{k \operatorname{cn}\left(K\frac{\theta}{\alpha}, k\right)}{k' \operatorname{cn}\left(K'\dfrac{\log\frac{r}{c}}{\log\frac{a}{c}}, k'\right)};$$

$$\Sigma (-1)^n \frac{\sin\left\{(2n+1)\frac{\pi}{2\alpha}\log\frac{r}{c}\right\} \sin (2n+1)\frac{\pi\theta}{2\alpha}}{(2n+1)\cosh (2n+1)\frac{\pi K'}{2K}}$$

$$= \frac{1}{2k}\log \frac{\operatorname{dn} K\frac{\theta}{\alpha} \operatorname{dn} K' \frac{1}{\alpha}\log\frac{r}{c} + kk' \operatorname{sn} K\frac{\theta}{\alpha} \operatorname{sn} K\frac{1}{\alpha}\log\frac{r}{c}}{\operatorname{dn} K\frac{\theta}{\alpha} \operatorname{dn} K' \frac{1}{\alpha}\log\frac{r}{c} - kk' \operatorname{sn} K\frac{\theta}{\alpha} \operatorname{sn} K\frac{1}{\alpha}\log\frac{r}{c}};$$

$$\Sigma (-1)^n \frac{\cosh\left\{(2n+1)\frac{\pi}{2\alpha}\log\frac{r}{c}\right\} \sin (2n+1)\frac{\pi\theta}{2\alpha}}{(2n+1)\sinh (2n+1)\frac{\pi K'}{2K}}$$

$$= \tfrac{1}{2}\log \frac{\operatorname{dn} K\frac{1}{\alpha}\log\frac{r}{c} + k \operatorname{sn} K\frac{\theta}{\alpha}}{\operatorname{dn} K\frac{1}{\alpha}\log\frac{r}{c} - k \operatorname{sn} K\frac{\theta}{\alpha}};$$

$$\Sigma\,(-1)^n \frac{\sinh\left\{(2n+1)\dfrac{\pi}{2\alpha}\log\dfrac{r}{c}\right\}\cos\,(2n+1)\dfrac{\pi\theta}{2\alpha}}{(2n+1)\sinh\,(2n+1)\dfrac{\pi K'}{2K}}$$

$$=\tan^{-1}\frac{k\,\mathrm{cn}\,K\dfrac{\theta}{\alpha}\,\mathrm{sn}\,K\dfrac{1}{\alpha}\log\dfrac{r}{c}}{\mathrm{dn}\,K\dfrac{\theta}{\alpha}\,\mathrm{cn}\,K\dfrac{1}{\alpha}\log\dfrac{r}{c}};$$

the elliptic functions of $K\dfrac{\theta}{\alpha}$ being to the modulus k, and the elliptic functions of $K\dfrac{1}{\alpha}\log\dfrac{r}{c}$ or $K'\dfrac{\log\frac{r}{c}}{\log\frac{a}{c}}$ being to the complementary modulus k'.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, April 10th, 1879, C. W. Merrifield, Esq., F.R.S. *President*, in the Chair. Mr. Donald McAlister, B.A., B.Sc. was elected a Member, and Messrs. A. J. C. Allen, B.A., Scholar of St. Peter's College, Cambridge, and Edwyn Anthony, M.A., Christ Church, Oxford, were proposed for election.

The following communications were made: "Notes on Quantics of alternate Numbers, used as a means for determining the Invariants and Covariants of Quantics in general," by the late Prof. Clifford, F.R.S. (communicated by Dr. Spottiswoode, F.R.S.); "Note on Geometrical Maxima and Minima," by Mr. J. Hammond, M.A. (the paper related partly to the Note at the end of Williamson's *Differential Calculus;* Mr. J. W. L. Glaisher gave a reference to a somewhat similar investigation by Gauss in the *Theoria Motus*); "On certain Fractions," by Mr. R. Tucker, M.A.

Thursday, May 8th, 1879, C W. Merrifield, Esq., F.R.S., *President*, in the Chair. Messrs. A. J. C. Allen and Edwyn Anthony were elected Members.

The following communications were made: "On the Complex whose lines join Conjugate Points of two Correlative Planes," by Dr. Hirst, F.R.S.; "Note on a Geometrical Theorem connected with the function of an Imaginary Variable," by Prof. Cayley, F.R.S.; "Some Definite Integrals," by the late Prof. Clifford, F.R.S.; "A method of constructing by Pure Analysis Functions X, Y, &c. which possess the property that $\int XYd\sigma=0$, and such that any given function can be expanded in the form $\alpha X+\beta Y+\gamma Z+\ldots$," by Mr. E. J. Routh, F.R.S.; "The Numerical Calculation of a class of Determinants, and a Continued Fraction," by Mr. J. D. H. Dickson, M.A.; "On the inscription of the regular Heptagon," by Dr. Freeth.

R. Tucker, M.A., *Hon. Sec.*

NOTE ON THE INTEGRATION OF THE HIGHER TRANSCENDENTS WHICH OCCUR IN CERTAIN MECHANICAL PROBLEMS.

By *W. H. L. Russell, F.R.S.*

We must first express the functions to be integrated by means of infinite series.

As successive differentiation is excessively tedious in the cases before us, we must seek to express the irrational algebraic functions by means of linear differential equations with variable coefficients.

Thus if $u = \sqrt{s}$, we have $2s\dfrac{du}{dx} - s'u = 0$.

Again if $u = \sqrt[n]{(r + \sqrt{s})}$ where r and s are rational functions of x we proceed thus:

We easily find that

$$2n\{s + r\sqrt{s}\}\frac{du}{dx} = \{s' + 2r'\sqrt{s}\}\,u.$$

Multiplying this equation successively by $s - r\sqrt{s}$ and by $s' - 2r'\sqrt{s}$, we find the following:

$$2n(s^2 - rs)\frac{du}{dx} = (ss' - 2rr's)\,u + (2r's - rs')\,u\sqrt{s},$$

$$2n(2r's - rs')\sqrt{(s)}\frac{du}{dx} = 2n(ss' - 2rr's)\frac{du}{dx} - (s'^2 - 4r'^2 s)\,u.$$

Differentiating the first of these two equations and then eliminating $\sqrt{s}$ from the result by means of the equations themselves we arrive at the following equation:

$$\begin{aligned}
&4n^2 s(2r's - rs')(s^2 - rs)\frac{d^2u}{dx^2}\\
&\quad + [2ns(2r's - rs')\{(4n-1)ss' - 2(2n-1)rr's - 2nr^2 s'\}\\
&\quad - 2n^2(4r''s^2 + 4r'ss' - 2rss'' - rs'^2)(s^2 - rs)\\
&\quad + 2ns(2r's - rs')(2rr's - ss')]\frac{du}{dx}\\
&\quad + \{s(2r's - rs')(s'^2 - 4r'^2 s)\\
&\quad - 2ns(2r's - rs')(ss'' + s'^2 - 2r'^2 s - 2rr''s - 2rr's')\\
&\quad - n(2rr's - ss')(4r''s^2 + 4r'ss' - 2rss'' - rs'^2)\}\,u = 0.
\end{aligned}$$

But the most convenient method of expanding the function practically will be to expand $\sqrt{s}$ as far as we intend to expand u, and then calling the sum of the terms we retain v, to expand $\sqrt[3]{(r+v)}$.

If, however, we use the differential equations, we put $u=u_0+u_1x+u_2x^2+\&c.$, and determine the successive terms by the usual method.

The convergence must be determined by Cauchy's rule, or by the method for differential equations given by me in the *Proceedings of the Royal Society* for 1872, and afterwards discovered independently by Sir W. Thomson.

For very small values of x, this method will suffice, also for large values by expanding in terms of $\frac{1}{x}$. For other values of x consider this theorem:

$$\int \frac{dx}{\sqrt[3]{(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6)}}$$

$$=\mu_0\left(\frac{x-w}{x+w}\right)+\frac{\mu_1}{2}\left(\frac{x-w}{x+w}\right)^2+\frac{\mu_2}{3}\left(\frac{x-w}{x+w}\right)^3+\ldots,$$

where w is an arbitrary quantity, and $\mu_0\mu_1\mu_2$ are *à un facteur près*, the coefficients of expansion in

$$\frac{1}{\sqrt[3]{(A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+A_5x^5+A_6x^6)}},$$

where

$$A_0=\frac{1}{64}\left(a_6+\frac{a_5}{w}+\frac{a_4}{w^2}+\frac{a_3}{w^3}+\frac{a_2}{w^4}+\frac{a_1}{w^5}+\frac{a_0}{w^6}\right),$$

$$A_1=\frac{1}{16}\left(\frac{3a_6}{2}+\frac{a_5}{w}+\frac{a_4}{2w^2}-\frac{a_2}{2w^4}-\frac{a_1}{w^5}-\frac{3a_0}{2w^6}\right),$$

$$A_2=\frac{1}{64}\left(15a_6+\frac{5a_5}{w}-\frac{a_4}{w^2}-\frac{3a_3}{w^3}-\frac{a_2}{w^4}+\frac{5a_1}{w^5}+\frac{15a_0}{w^6}\right),$$

$$A_3=\frac{5a_6}{16}-\frac{a_4}{16w^2}+\frac{a_2}{16w^4}-\frac{5a_0}{16w^6},$$

$$A_4=\frac{1}{64}\left(15a_6-\frac{5a_5}{w}-\frac{a_4}{w^2}+\frac{3a_3}{w^3}-\frac{a_2}{w^4}-\frac{5a_1}{w^5}+\frac{15a_0}{w^6}\right),$$

$$A_5=\frac{1}{16}\left(\frac{3a_6}{2}-\frac{a_5}{w}+\frac{a_4}{2w^2}-\frac{a_2}{2w^4}+\frac{a_1}{w^5}-\frac{3a_0}{2w^6}\right),$$

$$A_6=\frac{1}{64}\left(a_6-\frac{a_5}{w}+\frac{a_4}{w^2}-\frac{a_3}{w^3}+\frac{a_2}{w^4}-\frac{a_1}{w^5}+\frac{a_0}{w^6}\right),$$

the most interesting case of this is

$$\int_{\rho}^{x} \frac{dx}{\sqrt{X}} = \mu_0 \left(\frac{x-\rho}{x+\rho}\right) + \frac{\mu_1}{2}\left(\frac{x-\rho}{x+\rho}\right)^2 + \frac{\mu_2}{3}\left(\frac{x-\rho}{x+\rho}\right)^3 + \&c.,$$

where $X = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6,$

and x, ρ are the limits of the integral.

MATHEMATICAL NOTES.

Note on geometrical conics.

The following very simple theorem, which occurred to the writer several years ago, seems to be useful in proving one or two of the elementary properties of the conic sections. It does not appear to be used in any of the books on the subject, but the writer recently noticed it among the examples in Mr. Taylor's "Geometrical Conics."

From S (fig. 6) the focus of a conic, draw SE perpendicular to the axis of the conic, making $SE : SX =$ the eccentricity, where X is the foot of the directrix. Join XE. Then, if any point P be taken on the conic, and the ordinate PM be produced to meet XE, or XE produced, in N, $SP = NM$.

For $SP : MX =$ eccentricity, and $NM : MX =$ eccentricity.

I. To find any number of points on the curve.

Take any point N (fig. 6) on XE. Draw NM perpendicular to SX. With centre S, radius $= NM$, describe a circle cutting NM in P, P'. Then P, P' manifestly lie on the conic. We have thus a convenient and accurate method of describing any conic. The general shape of the curve may be at once determined. For example, if the eccentricity is unity, $MN = MX$; and therefore SM is $< NM$ if M lies to the right, and $> NM$ if it lies to the left of A. Hence the curve consists of one infinite branch, no part of which lies to the left of the perpendicular through A.

In drawing tangents from any external point T (fig. 6), we take the radius of the circle $= NM$, and then proceed as usual with the help of Professor Adams's property.

II. To prove that $PK = SE$ (fig. 6), where PG is the normal at P, and GK is perpendicular to SP.

Draw SL parallel to XE. Then, by similar triangles

$SK : SM = SG : SP =$ eccentricity

$= ML : SM$, since SL is parallel to XN;

therefore $SK = ML$, and $PK = PS - KS = NM - ML = SE$.

III. To prove that $PQ.SE = 2SP.SQ$ (fig. 7).
Draw $L'SL$ parallel to XE. Then

$$SP : SQ = SL : SL' = ML : RL',$$

and $$ML = SP - SE;\ RL' = SE - SQ;$$

therefore $$SP(SE - SQ) = SQ(SP - SE);$$

therefore $$PQ.SE = 2SP.SQ.$$

IV. To shew that $XRSM$ (fig. 7) is an harmonic range.

$$XR : XM = DR : NM = SQ : SP = SR : SM.$$

Cor. If PQ be produced to meet the directrix in F, $PSQF$ is manifestly an harmonic range.

V. In any central conic, the conjugate axis divides the curves symmetrically. Draw PP' parallel to the axis (fig. 8), and take $DP = DP'$. Complete the figure which needs no explanation. Then $CO = CA$, since $CO : CX =$ eccentricity. And SO is perpendicular to OX', since $CX' : CO = CO : CS$;

therefore $$SN' = SR;$$

therefore $$SM'^2 + SP^2 = SM^2 + M'R'^2;$$

therefore $$SM'^2 + PM^2 + SM^2 = SM^2 + M'R'^2;$$

therefore $$SP'^2 = M'R'^2;$$

therefore P' is on the conic.

Cor. I. The conic has two foci and two directrices.

For $$S'P = SP' = R'M' = RM.$$

Cor. II. $$SP \pm PS' = AA'.$$

For $$SP \pm PS' = NM \pm R'M' = 2CO = AA'.$$

N.B. We have drawn the figure for the case of the ellipse, but the proof applies without change to the corresponding figure for the hyperbola.

VI. If $SP + SQ$ is constant (fig. 9), the locus of the middle point of PQ is a straight line perpendicular to the axis. For, bisect MR in O, then

$$SP + SQ = NM + DR = 2HO;$$

therefore HO is of constant length, and therefore HO is a fixed straight line.

If $SP \sim SQ$ is constant, the projection of PQ on the axis is constant. For projection of

$$PQ = MR = NL = e'DL = e'(DR \sim LR) = e'(DR \sim NM)$$
$$= e'(SQ \sim SP),$$

where e' is the reciprocal of the eccentricity.

EDWYN ANTHONY.

On the twist of a bar.

The twist of a narrow strip of surface, or of a rod, at any point may be found in the following simple way:

If a bar or band be unbent by rotation of the successive elements about normals to the osculating planes until the bar is straightened, the twist in the bar is unaltered by the process.

Hence, a rotation about the normal to the osculating plane does not alter the twist, nor does a rotation about the normal to the curve in the osculating plane alter the twist, but a rotation about the tangent line to the curve does alter the twist.

A straight untwisted narrow band, can be made to coincide with any given narrow band of the same length, and having any given twist, tortuosity, and curvature, by these three rotations from point to point. Therefore if the given band be a strip of a surface, its twist at any point is equal to the rotation of the tangent plane to the surface about the tangent line to the curve as we pass through that point along the curve; and the total twist is measured by the total rotation of the tangent plane about the tangent line, as we pass from one end of the curve to the other.

Hence, if we roll the surface along a plane, with the narrow band as its trace, the necessary angular rotation of the surface about the tangent line to the trace measures the twist of the band.

Or we may measure the twist by the rotation about the tangent of the normal plane to the surface which touches the curve at successive points, for the rotation of the normal plane is the same as that of the tangent plane.

For example, the twist along any geodetic is equal to the tortuosity of the geodetic, since the normal plane to the surface and the osculating plane of the curve are here identical.

In the case of a helix on a cylinder of radius r, and making an angle α with the generating lines, if the velocity of the point of contact be unity, the rotation of the tangent plane is about a generating line and is measured by $\frac{\sin\alpha}{r}$; therefore the resolved part about the tangent line is $\frac{\sin\alpha\cos\alpha}{r}$, and this therefore measures the twist.

To find generally the rotation of the normal plane about the tangent in passing from one point to another at distance δs from it along the curve, we rotate it (1) until it coincides with the osculating plane, through an angle ζ, (2) until it coincides with the osculating plane at the second point, through an angle $\delta\phi$, (3) back again, until it coincides with the normal plane at the second point, *i.e.* through an angle $\zeta+\delta\zeta$.

The twist is therefore measured by $\frac{\zeta+\delta\phi-(\zeta+\delta\zeta)}{\delta s}$, or, in the limit, by $\frac{d\phi}{ds}-\frac{d\zeta}{ds}$, where $\frac{d\phi}{ds}$ is the tortuosity and ζ is the angle between the normal and osculating plane at any point.

The whole twist, therefore, is equal to the whole tortuosity diminished by the increase of the inclination of the transverse to the osculating plane in the part of the band considered.

T. C. LEWIS.

Note on an expansion of Euler's.

At the end of his memoir *De partitione numerorum* (*Opera minora collecta*, t. I. p. 96), Euler gives the expansion of

$$(1-x)(1-x^2)(1-x^4)(1-x^8)(1-x^{16})\ldots,$$

which $=1-x-x^2+x^3-x^4+x^5+x^6-x^7-x^8+x^9+x^{10}-x^{11}$
$+x^{12}-x^{13}-x^{14}+x^{15}-x^{16}+x^{17}+x^{18}-x^{19}+x^{20}-\&c.,$

where all the coefficients are $+1$ or -1 "neque tamen legem obtinent solito modo assignabilem." Euler points out that the sign of any term may be readily found by means of the rules, (i) x^{2n+1} has the opposite sign to x^{2n}, and (ii) x^{2n} has the same sign as x^n. Thus, for example, considering only the signs of the terms,

$$x^{43}=-x^{42}=-x^{21}=+x^{20}=+x^{10}=+x^5=-x^4=-x^2=-x,$$

and the coefficient of x^{43} is therefore positive as the coefficient of x is negative.

If a denote a positive sign and b a negative sign, the signs of the terms are given by

abbabaabbaababbabaababbaabbabaab...,

and this sequence may be written down as follows. Calling the operation of writing b in place of a and a in place of b a change, then the first two letters are ab, the second two are the first two changed, viz. ba, the next four are the first four changed, viz. $baab$, the next eight are the first eight changed, viz. $baababba$, the next sixteen are the first sixteen changed, and so on. It may also be noticed, that if ab be denoted by A and ba by B, then the sequence is $ABBABAAB$..., which is similar to the original sequence; if A_1 denote the first four letters $abba$ and B_1, the second four $baab$, then the sequence is $A_1B_1B_1A_1B_1A_1A_1B_1$ ... as before, and the same is true generally if a denote the first 2^n letters, and β the next 2^n (so that β denotes a changed). It is scarcely necessary to remark, that the sequence $abbabaab$ ... is connected with the numbers of 1's in the series of numerals in the binary scale. For these numerals are 0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, &c., and a corresponds to a numeral containing an even number of 1's, and b to a numeral containing an uneven number.

J. W. L. GLAISHER.

Note on an example in Boole's 'Differential Equations' relating to orthogonal trajectories.

On pp. 246–247 of his *Differential Equations*, Boole investigates the orthogonal trajectory of a system of confocal ellipses as follows. Starting with the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ where } a^2-b^2=h^2 \text{ .... (1)},$$

the differential equation of the trajectory is found to be

$$xy\left(\frac{dy}{dx}\right)^2+(x^2-y^2-h^2)\frac{dy}{dx}-xy=0 \text{ .... (2)},$$

which, integrated, gives

$$y^2-c^2x^2=-\frac{h^2c^2}{c^2+1},$$

a result which, Boole remarks, may be reduced to the form

$$\frac{x^2}{a_1^{\,2}}-\frac{y^2}{b_1^{\,2}}=1,$$

a_1 and b_1 being connected by the condition $a_1^{\,2}+b_1^{\,2}=h^2$, so that the trajectory is a hyperbola confocal with the given system of ellipses.

In point of fact, the differential equation (2) is unaltered by the substitution of $-\frac{dx}{dy}$ for $\frac{dy}{dx}$, so that the curves and trajectories satisfy the same differential equation, and are therefore both represented by the same equation (1). In other words, (1) represents a system of curves cutting themselves orthogonally; this is at once seen to be true, for since we cannot introduce the condition that b^2 should be positive, (1) represents both the ellipses and hyperbolas.

It is clear that any equation of the form $U(p^2-1)+Vp=0$, U and V being any functions of x, y, or any differential equation which is not altered by the substitution of $-\frac{1}{p}$ for p represents a system of curves cutting themselves orthogonally.

A system of confocal and coaxial parabolas is self-orthogonal, for $y^2=4a(x+a)$ gives rise to the differential equation $y(p^2-1)+2xp=0$; this is a particular case of that of confocal conics.

It may be noticed, that by giving p equal roots in (2), we have

$(x^2-y^2-h^2)^2+4x^2y^2=0$, viz. $\{(x-h)^2+y^2\}\{(x+h)^2+y^2\}=0$,

which represents the foci; and these are in a certain sense an envelope, since a conic has double contact with its foci. The same result is obtained by giving b^2 equal roots in

$$\frac{x^2}{b^2+h^2}+\frac{y^2}{b^2}=1.$$

J. W. L. GLAISHER.

Theorem in partitions.

1. *Theorem.* If $P(u)$ denote the number of partitions of u into the n elements $1, 2, 3, \ldots n$, each partition containing exactly r parts, order attended to and repetitions not excluded, then

$$P(r) \quad +P(r+n) \quad +P(r+2n) \quad +\ldots=n^{r-1},$$
$$P(r+1) \quad +P(r+n+1) \quad +P(r+2n+1)+\ldots=n^{r-1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$P(r+n-1)+P(r+2n-1)+P(r+3n-1)+\ldots=n^{r-1}.$$

For example, let $n=3$, $r=3$, then, if order be *not* attended to, the partitions are:

of 3; 111, of 5; 113, 122, of 7; 133, 223, of 9; 333,
„ 4; 112, „ 6; 123, 222, „ 8; 233.

Therefore, order being attended to (so that *ex. gr.* 112, 121, 211 are to be regarded as different partitions),

$$P(3)=1, \quad P(5)=6, \quad P(7)=6, \quad P(9)=1,$$
$$P(4)=3, \quad P(6)=7, \quad P(8)=3,$$

and

$$P(3)+P(6)+P(9)=1+7+1=9,$$
$$P(4)+P(7) \qquad =3+6 \qquad =9,$$
$$P(5)+P(8) \qquad =6+3 \qquad =9.$$

Proof. If the expression $\omega^1+\omega^2+\omega^3...+\omega^n$ be multiplied by itself, and the product be reduced by means of the assumed equations $\omega^{n+1}=\omega^1$, $\omega^{n+2}=\omega^2$, ... $\omega^{n+m}=\omega^m$, ... the result is $n(\omega^1+\omega^2+\omega^3...+\omega^n)$; for $\omega^1+\omega^2...+\omega^n$ multiplied by ω^m, and so reduced, is equal to $\omega^1+\omega^2...+\omega^n$, whatever the value of m (supposed integral) may be. Thus, if the multiplication by $\omega^1+\omega^2...+\omega^n$ be performed $r-1$ times and the products continually reduced, we have

$$(\omega^1+\omega^2...+\omega^n)^r=n^{r-1}(\omega^1+\omega^2...+\omega^n) \text{ .... } (1),$$

and the theorem follows by comparing the coefficients of the powers of ω, the left-hand side being supposed to be completely expanded and then reduced.

[Of course ω is an n^{th} root of unity, and each side of (1) is equal to zero except when $\omega=1$; in the above process we merely consider the algebraical expressions arising from the multiplications, &c., reduced to the form $\omega^1+\omega^2...+\omega^n$ by means of the relation $\omega^n=1$].

2. The theorem may also be stated algebraically as follows:

If $(1+x+x^2...+x^{n-1})^r=1+a_1x+a_2x^2...+a_{nr-r}x^{nr-r}$, then

$$1+a_n \;+a_{2n} \;+...=n^{r-1},$$
$$a_1+a_{n+1}+a_{2n+1}+...=n^{r-1},$$
$$a_2+a_{n+2}+a_{2n+2}+...=n^{r-1},$$
....
$$a_{n-1}+a_{2n-1}+a_{3n-1}+...=n^{r-1},$$

and this can be proved by means of the usual rule for summing selected terms of a series; thus, m being $<n$,

$$a_m+a_{m+n}+a_{m+2n}+...=\frac{1}{n}\Sigma\omega^m(1+\omega+\omega^2...+\omega^{n-1})^r,$$

(where ω represents any n^{th} root of unity), $=\frac{1}{n}n^r$, since the expression following the Σ is zero except for the root $\omega=1$.

J. W. L. GLAISHER.

Note on some cases of the intersection of curves and surfaces by straight lines.

The ratios in which the line joining $P(x, y, z)$ or (x, y, z, w), and $Q(x', y', z')$ or (x', y', z', w') is cut by a plane curve or surface $U=0$ are determined by the equation

$$\lambda^n U + \lambda^{n-1}\mu\Delta U + \frac{1}{\underline{|2}}\lambda^{n-2}\mu^2\Delta^2 U + \frac{1}{\underline{|3}}\lambda^{n-3}\mu^3\Delta^3 U + \&c. = 0,$$

where $$\Delta^p U = \left(x'\frac{d}{dx} + y'\frac{d}{dy} + z'\frac{d}{dz}\right)^p U,$$

or $$\left(x'\frac{d}{dx} + y'\frac{d}{dy} + z'\frac{d}{dz} + w'\frac{d}{dw}\right)^p U,$$

according as $U=0$ represents a curve or a surface.

If any number P of the last terms vanish, so that Q is a p-ic point, the equation becomes identical in form with that which would have been obtained if U had been of the $(n-p)^{\text{th}}$ order.

I. Hence it follows (1) that as any line through any point is cut harmonically by the point, any conic or quadric and the polar of the point with respect to it, so an n-ic curve or surface and the first polar of an $(n-2)$-ic point upon it will cut any line through this point harmonically.

(2) The second polar of a point on a quartic, a double point on a quintic or generally an $(n-3)$-ic point on an n-ic cuts any line through the node so that the intercept is the harmonic mean between the three made by the locus, and the first polar of such a point cuts any line through it (Q) and meeting the locus again in P_1, P_2, P_3, so that

$$\left(\frac{1}{QR} - \frac{1}{QP_1}\right)\left(\frac{1}{QR} - \frac{1}{QP_2}\right) + \left(\frac{1}{QR} - \frac{1}{QP_2}\right)\left(\frac{1}{QR} - \frac{1}{QP_3}\right)$$
$$+ \left(\frac{1}{QR} - \frac{1}{QP_3}\right)\left(\frac{1}{QR} - \frac{1}{QP_1}\right) = 0.$$

II. If as before Q be a p-ic point and P be a q-ic point on U, the equation to determine $\lambda : \mu$ is an $(n-p-q)$-ic, and similar properties result.

Thus, if $p+q=n-2$, the line PQ is divided harmonically whenever $\Delta^{q+1}U$ also vanishes, *i.e.* at the points where $\Delta^{q+1}U=0$ intersects $U=0$.

And if $p+q=n-3$, PQ is divided by $\Delta^{q+1}U$, and $\Delta^{q+2}U$ in the same points as by the polars of a cubic intersecting PQ in the same three (additional) points.

Thus, in particular, all lines drawn from a point on a quartic, a double point on a quintic or an $(n-3)$-ic point on an n-ic, so as to pass through an intersection of the locus and the second polar of the point are divided harmonically. And, hence, five such lines can be drawn from an ordinary point on a quartic curve, whilst those from an ordinary point on a quartic surface form a quintic cone.

The full condition that the line joining P and Q should be cut harmonically by U (a quartic) is the invariant $T=0$ of

$$\lambda^4 U + \lambda^3 \mu \Delta U + \&c. = 0,$$

which reduces to

$$\Delta^2 U \{3\Delta U . \Delta^3 U - (\Delta^2 U)^2\} = 0,$$

when $U=0$ and $U'=0$.

This does not give any additional lines as

$$3\Delta U \Delta^3 U - (\Delta^2 U)^2$$

intersects U at the additional points where the lines PQ meet it.

W. J. CURRAN SHARP.

On directrices of conics represented by the homogeneous equation.

In a paper published in the *Quarterly Journal*, vol. XIII. p. 198, Dr. Eurenius discusses the foei and directrices of a conic whose equation is given in homogeneous coordinates. The directrices may be obtained perhaps more directly by the following method.

If the tangents from I, J be drawn to the conic, and if AB, $A'B'$ be the points of contact, the directrices are the two pairs of lines AA', BB'; AB', BA'. The equation to AB (supposing the conic to be $(u, v, w, u', v', w' \between x, y, z)^2$, and the coordinates of I, J to be $\xi, \eta, \zeta, \xi', \eta', \zeta'$), will be

$$\xi \frac{d\phi}{dx} + \eta \frac{d\phi}{dy} + \zeta \frac{d\phi}{dz} = 0.$$

Hence the equation

$$\left(\xi \frac{d\phi}{dx} + \eta \frac{d\phi}{dy} + \zeta \frac{d\phi}{dz}\right)\left(\xi' \frac{d\phi}{dx} + \eta' \frac{d\phi}{dy} + \zeta' \frac{d\phi}{dz}\right) = 0,$$

represents the two chords AB, $A'B'$, and the equation

$$k\phi + \left(\xi \frac{d\phi}{dx} + \eta \frac{d\phi}{dy} + \zeta \frac{d\phi}{dz}\right)\left(\xi' \frac{d\phi}{dx} + \eta' \frac{d\phi}{dy} + \zeta' \frac{d\phi}{dz}\right) = 0 \ldots \text{(i)},$$

represents a conic, such that two pairs of the chords common to it and $\phi=0$ are the directrices of the latter. Reducing (i), we have

$$0=k\phi+\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2+\left(\frac{d\phi}{dz}\right)^2$$
$$-2\cos A\frac{d\phi}{dy}\frac{d\phi}{dz}-2\cos B\frac{d\phi}{dz}\frac{d\phi}{dx}-2\cos C\frac{d\phi}{dx}\frac{d\phi}{dy}\ldots\text{(ii)}.$$

To find the directrices the discriminant of (ii) must be made $=0$. This will give a quadratic for k, whose two roots correspond one to the real, the other to the imaginary pair of directrices.

R. PENDLEBURY.

On the sign of any term of a determinant.

Consider a determinant

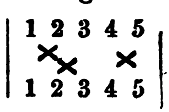
$$|\,a_{ij}\,|,$$

where
$$i=1,\,2,\ldots n,$$
$$j=1,\,2,\ldots n.$$

This is indicated in Sylvester's umbral notation by

$$\left|\begin{matrix}1,\,2,\ldots n\\1,\,2,\ldots n\end{matrix}\right|,$$

wherein the top line is composed of the first- or row-subscripts and the second line of the second- or column-subscripts. Any element a_{ij} may be denoted by a line joining i in the top line to j in the bottom line, and any term by a complete group of these lines; that is to say, by a group of lines joining all the top numbers to all the bottom numbers one to one. For example, the diagram



represents the term

$$a_{13}a_{21}a_{32}a_{45}a_{54}.$$

The theorem is that the term is positive or negative according as the number of intersections in the diagram is even or odd. For instance, the term just written is negative because there are 3 intersections in the diagram.

To prove this, observe that the principal term $a_{11}a_{22}\ldots$ is positive. Every other term is derived from this by transposing the second subscripts; and each implies a change of sign. In the diagram the right sign is given to the principal term, for there are no intersections. Also, every transposition of two adjacent numbers in the second line increases or diminishes the number of intersections by one, and so indicates a change of sign. For instance, if, in the term above written, we had $a_{12}\ldots a_{33}\ldots$ instead of $a_{13}\ldots a_{32}\ldots$ one intersection would be lost; if instead of $a_{13}\ldots a_{54}$ we had $a_{14}\ldots a_{53}$ one intersection would be gained. In each case the diagram indicates a change of sign, as it should.

H. W. LLOYD TANNER.

June, 1879.

Generalisation of a property of the pedal curve.

Let $M\mu m$, $N\nu n$ (fig. 10) be two positions of an angle of constant magnitude, sliding on two curves C, c. The straight lines $M\mu$, $N\nu$, tangents to C at M, N cut in T; the straight lines $m\mu$, $n\nu$, tangents to c at m, n cut in t. The four points T, t, μ, ν lie on the same circle, since the angle μ is equal to the angle ν. If the angle $N\nu n$ approaches indefinitely the angle $M\mu m$, the secant $\mu\nu$ tends to become the tangent to the locus of the point μ. Therefore *the tangent at μ to this locus is the tangent at this point to the circle circumscribing the triangle $M\mu m$*, the limit of $T\nu t$. The particular cases of the theorem in which the curve c reduces to a point or coincides with C are well known.

PAUL MANSION.

Note on the addition-equation in elliptic functions.

Euler's equation

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0,$$

where

$$X = a + bx + cx^2 + dx^3 + ex^4,$$
$$Y = a + by + cy^2 + dy^3 + ey^4,$$

reduces to

$$\frac{d\phi}{\sqrt{(1 - k^2 \sin^2\phi)}} + \frac{d\psi}{\sqrt{(1 - k^2 \sin^2\psi)}} = 0,$$

if we put

$$x = \sin\phi,\ y = \sin\psi,\ (a, b, c, d, e) = (1, 0, -1 - k^2, 0, k^2).$$

The solution of Euler's equation is

$$\left(\frac{\sqrt{X}-\sqrt{Y}}{x-y}\right)^2=c+d\,(x+y)+e\,(x+y)^2;$$

hence, the solution of

$$\frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}+\frac{d\psi}{\sqrt{(1-k^2\sin^2\psi)}}=0,$$

is, writing $\Delta\phi=\sqrt{(1-k^2\sin^2\phi)}$,

$$\left(\frac{\cos\phi\,\Delta\phi-\cos\psi\,\Delta\psi}{\sin\phi-\sin\psi}\right)^2=C+k^2\,(\sin\phi+\sin\psi)^2,$$

which transforms to

$$\left(\frac{\cos\phi\,\Delta\psi-\cos\psi\,\Delta\phi}{\sin\phi-\sin\psi}\right)^2=C;$$

therefore $$\frac{\cos\phi\,\Delta\psi-\cos\psi\,\Delta\phi}{\sin\phi-\sin\psi}=C';$$

putting $$\phi=\operatorname{am}u,$$ $$\psi=\operatorname{am}v,$$

we have, if $u+v=$ constant,

$$\frac{\operatorname{cn}u\operatorname{dn}v-\operatorname{cn}v\operatorname{dn}u}{\operatorname{sn}u-\operatorname{sn}v}=C',$$

this gives

$$\frac{\operatorname{cn}u\operatorname{dn}v-\operatorname{cn}v\operatorname{dn}u}{\operatorname{sn}u-\operatorname{sn}v}=\frac{\operatorname{cn}(u+v)-\operatorname{dn}(n+v)}{\operatorname{sn}(u+v)},$$

which is easily verified.

J. J. THOMSON.

Proof of Rodrigues's theorem.

Consider the expression

$$(x-1+h)^n\,(x+1+h)^n;$$

the coefficients of h^{n-m} and h^{n+m} are respectively

$$\frac{1}{(n-m)!}\left(\frac{d}{dx}\right)^{n-m}(x^2-1)^n \text{ and } \frac{1}{(n+m)!}\left(\frac{d}{dx}\right)^{n+m}(x^2-1)^n.$$

Now $$(x-1+h)^n=\Sigma_{r=0}^{r=n}\,a_r\,(x-1)^{n-r}h^r,$$

$$(x+1+h)^n=\Sigma_{r=0}^{r=n}\,a_r\,(x+1)^{n-r}h^r,$$

where $a_r=a_{n-r}$.

The coefficient of h^{n-m} in the product

$$= \Sigma_{t=0}^{t=n-m} a_{n-m-t} a_t (x-1)^{m+t} (x+1)^{n-t}$$

$$= (x-1)^m (x+1)^m \Sigma_{t=0}^{t=n-m} a_{m+t} a_{n-t} (x-1)^t (x+1)^{n-m-t}$$

$$= (x^2-1)^m \text{ coefficient of } h^{n+m},$$

therefore

$$\frac{1}{(n-m)!} \frac{d^{n-m}}{dx^{n-m}} (x^2-1)^n = (x^2-1)^m \frac{1}{(n+m)!} \frac{d^{n+m}}{dx^{n+m}} (x^2-1)^n.$$

E. W. Hobson.

On Euclid's numbers.

If 1, 2, 3, 5, ...n be the prime numbers in order up to the prime n, then numbers of the form $1.2.3.5...n+1, = \phi(n)$ say, may be conveniently termed *Euclid's numbers.** Of the first nine Euclid's numbers the first six are primes, and the next three are, I find, composite, viz. $\phi(1), = 2$; $\phi(2), = 3$; $\phi(3), = 7$; $\phi(5), = 31$; $\phi(7), = 211$; $\phi(11), = 2311$, are primes; and $\phi(13) = 30{,}031 = 59 \times 509$; $\phi(17) = 510{,}511 = 19 \times 97 \times 277$; $\phi(19) = 9{,}699{,}691 = 347 \times 27{,}953$.

R. Pendlebury.

On long successions of composite numbers. (*Correction of an error*).

In the list of sequences, each exceeding any preceding sequence, up to 100,000, given p. 175 of vol. VII. (March, 1878) a sequence of 21 between 1129 and 1151 was accidentally omitted. This sequence should have appeared between those of 19 and 33. I am indebted to Mr. Ferrers for calling my attention to this omission.

J. W. L. Glaisher.

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* *Elements* IX., 20. Euclid's proof that there exist an infinite number of primes is slightly different from the ordinary one (see *e.g.* Todhunter's *Algebra*). It is simply this: "If $a, b, c, ...n$ are primes, then either $abc...n+1$ is a prime or there exists a prime different from any of the series which divides the composed number."

ON RODRIGUES'S THEOREM.

By *J. W. L. Glaisher.*

§1. Rodrigues's theorem was first published in a paper by him, entitled *Mémoire sur l'attraction des sphéroides*, which occupies pp. 361–385 of tome III. of the *Correspondance sur l'École royale polytechnique...par M. Hachette*, Paris, 1816. A footnote to the title of the memoir runs "Ce Mémoire a été le sujet d'une thèse soutenue pour le doctoral, devant la Faculté des Sciences de Paris, le 28 juin 1815, sous la présidence de M. Lacroix, Doyen de la Faculté." The theorem itself occurs on p. 378, in the form

$$(-1)^n (m-n+1)(m-n+2)\dots(m+n) \frac{d^{m-n}(1-\mu^2)^m}{d\mu^{m-n}}$$

$$= (1-\mu^2)^n \frac{d^{m+n}(1-\mu^2)^m}{d\mu^{m-n}}.$$

The theorem was rediscovered by Jacobi, and published by him in his paper *Ueber eine besondere Gattung algebraischer Functionen die aus der Entwicklung der Function* $(1-2xz+z^2)^{\frac{1}{2}}$ *entstehen* (Crelle's Journal, t. II. pp. 223–226, 1827). Jacobi writes the theorem

$$\frac{d^{n-r}(x^2-1)^n}{1.2.3\dots(n-r)\,dx^{n-r}} = (x^2-1)^r \frac{d^{n+r}(x^2-1)^n}{1.2.3\dots(n+r)\,dx^{n+r}} \dots (1),$$

and also in the form

$$\frac{\int^r X^{(n)}dx^r}{1.2.3\dots(n-r)} = (x^2-1)^r \frac{d^r X^{(n)}}{1.2.3\dots(n+r)\,dx^r} \dots\dots (2),$$

where $X^{(n)}$ is defined by the equation

$$\frac{1}{\sqrt{(1-2xz+z^2)}} = 1 + X'z + X''z^2 + X'''z^3 \dots + X^{(n)}z^n + \&c.,$$

that is, $X^{(n)}$ is the n^{th} Legendrian coefficient, and

$$= \frac{1}{2^n}\frac{1}{n!}\frac{d^n}{dx^n}(x^2-1)^n \dots\dots\dots\dots\dots\dots (3).$$

This value of $X^{(n)}$ is obtained by Jacobi in his paper; it had, however, been previously published by Ivory in 1824, but was first given by Rodrigues in the memoir cited above. In his *History of the mathematical theories of attraction*...(1873), vol. II. pp. 248–250 (Nos. 1187–1189) Todhunter draws attention to the fact, that the theorems (1) and (3) are due to Rodrigues, and he there reproduces the analysis by means of which Rodrigues was led to the formula (1).*

Rodrigues's theorem forms Prop. VII. pp. 12–14 of Murphy's *Elementary principles of the theories of electricity, heat*,... *Part* 1, *on electricity*, Cambridge, 1833. The enunciation of the proposition is "To compare the *indefinite* integral of P_n of any order not higher than the n^{th}, and commencing with $t = 0$ with its differential coefficient of the same order." The theorem is proved by actually performing the differentiations by means of Leibnitz's theorem. A proof is also given by Mr. D. D. Heath on p. 25 of vol. VII. (1865) of the *Quarterly Journal of Mathematics*,† in which the author gives simplified forms of proof of some of Murphy's fundamental propositions.

Two proofs of the theorem are given by Todhunter on pp. 76–80 of his *Treatise on Laplace's functions* ... (1875). In the second, the slightly more extended theorem

$$\frac{(x+a)^m (x+b)^m}{(n+m)!} \frac{d^{n+m} (x+a)^n (x+b)^n}{dx^{n+m}}$$

$$= \frac{1}{(n-m)!} \frac{d^{n-m} (x+a)^n (x+b)^n}{dx^{n-m}} \quad \text{.........(4)},$$

is proved by direct differentiation by means of Leibnitz's theorem. The extension to the form (4) was noticed by Jacobi in his paper in t. II. of Crelle's Journal.

There is a proof of the theorem in Ferrers's *Treatise on Spherical Harmonics* (1877), pp. 13–16; and there are also two proofs by Mr. Walton in the *Quarterly Journal*, vol. XV. pp. 335–337 (June, 1878), and one by Mr. W. H. H. Hudson, *Messenger*, ante, vol. VII. p. 117 (December, 1877). This last proof, in which $\frac{1}{r!}\left(\frac{d}{dx}\right)^r (x^2-1)^n$ is treated as the coefficient of h^r in $\{(x+h)^2-1\}^n$, is substantially the same as Jacobi's; it is perhaps the most natural method of establishing the theorem, which is obviously one connecting the coefficients of h^{n-m} and h^{n+m} in this expansion.

* Heine in the new edition of his very valuable *Theorie der Kugelfunctionen* (1878), though he refers on p. 9 to Nos. 1187–1189 of Todhunter's *History* and assigns (2) to Rodrigues, still seems by inadvertence to attribute, on pp. 155 and 201, the theorem (1) to Jacobi.

† "On Laplace's coefficients and functions", pp. 23–36.

Mr. Hobson's proof contained in the note preceding this paper (pp. 53–54) is very elegant and simple; and, as far as I know, it is new.

§2. The theorem may be regarded as one connecting the m^{th} integral and the m^{th} derivative of the n^{th} Legendrian coefficient. [This is in fact the form in which it is presented by Jacobi in equation (2) and by Murphy].

Let

$$\frac{1}{(1-2hx+h^2)^{\frac{1}{2}}} = P_0 + P_1 h + P_2 h^2 + P_3 h^3 \ldots + P_n h^n + \&c. \ldots (5),$$

and let P_n^{-m} denote $\iiint \ldots P_n (dx)^m$, the lower limit of the integrations being unity, so that if $m <$ or $= n$, P_n^{-m} vanishes for $x = \pm 1$.

Then, integrating once,

$$\frac{(1-2hx+h^2)^{\frac{1}{2}} - (1-h)}{\frac{1}{2} \cdot -2h} = P_0^{-1} + P_1^{-1} h + P_2^{-1} h^2 \ldots + P_n^{-1} h^n + \&c.,$$

and, integrating m times,

$$\frac{(1-2hx+h^2)^{m-\frac{1}{2}} - (A + Bh \ldots + Kh^{2m-1})}{\frac{1}{2} \cdot \frac{3}{2} \ldots (m-\frac{1}{2})(-2h)^m}$$
$$= P_0^{-m} + P_1^{-m} h \ldots + P_m^{-m} h^m \ldots + P_n^{-m} h^n + \&c.,$$

viz. $(1-2hx+h^2)^{m-\frac{1}{2}} = A' + B'h \ldots + K'h^{2m-1}$

$$+ (-)^m 1.3 \ldots (2m-1) \{P_m^{-m} h^{2m} + P_{m+1}^{-m} h^{2m+1} \ldots + P_n^{-m} h^{n+m} + \&c.\}.$$

Differentiating $2m$ times with respect to h,

$$\left(\frac{d}{dh}\right)^{2m} (1-2hx+h^2)^{m-\frac{1}{2}} = (-)^m 1.3 \ldots (2m-1)$$
$$\times \Sigma_{n=m}^{n=\infty} (n+m)(n+m-1) \ldots (n-m+1)\, h^{n-m} P_n^{-m} \ldots (6).$$

Now differentiating the original equation (5) m times with regard to x,

$$\frac{1.3 \ldots (2m-1)\, h^m}{(1-2hx+h^2)^{m+\frac{1}{2}}} = \Sigma P_n^m h^n,$$

where P_n^m denotes $\dfrac{d^m P_n}{dx^m}$;

that is,
$$\frac{1.3 \ldots (2m-1)}{(1-2hx+h^2)^{m+\frac{1}{2}}} = \Sigma P_n^m h^{n-m} \ldots\ldots\ldots\ldots\ldots (7).$$

By Rodrigues's theorem,

$$(n+m)(n+m-1) \ldots (n-m+1)\, P_n^{-m} = (x^2-1)^m P_n^m,$$

so that, from (6) and (7),

$$\left(\frac{d}{dk}\right)^{2m}(1-2kx+k^2)^{m-\frac{1}{2}}=(-)^m 1^2 3^2\ldots(2m-1)^2\frac{(x^2-1)^m}{(1-2kx+k^2)^{m+\frac{1}{2}}};$$

or, since $\qquad 1-2kx+k^2=(k-x)^2+1-x^2,$

the theorem is

$$\left(\frac{d}{dk}\right)^{2m}(k^2+a^2)^{m-\frac{1}{2}}=1^2.3^2\ldots(2m-1)^2\frac{a^{2m}}{(k^2+a^2)^{m+\frac{1}{2}}};$$

or, replacing k by x and m by n,

$$\left(\frac{d}{dx}\right)^{2n}(x^2+a^2)^{n-\frac{1}{2}}=1^2.3^2\ldots(2n-1)^2\frac{a^{2n}}{(x^2+a^2)^{n+\frac{1}{2}}}\ldots(8),$$

a theorem which does not seem to be referred to in Heine's *Kugelfunctionen*, but which is not likely to be new.

§3. The result last written can be proved directly, for, putting $a=1$, which involves no loss of generality, the term involving x^{2p} in the expansion of $(1+x^2)^{n-\frac{1}{2}}$ is

$$\frac{n-\frac{1}{2}.n-\frac{3}{2}\ldots n-p+\frac{1}{2}}{p!}x^{2p}.$$

Differentiate this $2n$ times, and it becomes

$$\frac{\{2n-1.2n-3\ldots2n-2p+1\}\,2p.2p-1\ldots2p-2n+1}{2^p.p!}x^{2p-2n}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Since p is not less than n, the first factor of the numerator

$$=2n-1\ldots3.1.-1.-3\ldots-(2n-1).-(2n+1)\ldots-(2p-2n-1)$$
$$=1^2.3^2\ldots(2n-1)^2.(-)^{p-n}.2n+1\ldots2p-2n-1,$$

and the second factor, divided by the denominator,

$$=\frac{1}{2^{p-n}(p-n)!}2p-1.2p-3\ldots2p-2n+1,$$

so that the expression (9)

$$=A(-)^{p-n}\frac{2n+1.2n+3\ldots2p-1}{2^{p-n}(p-n)!}x^{2p-2n},$$

which = the term involving x^{2p-2n} in $A(1+x^2)^{-n-\frac{1}{2}}$, where $A=1^2.3^2\ldots(2n-1)^2$.

§4. The equation (8) may also be easily verified by the method employed by Mr. Hobson to prove Rodrigues's theorem, for $\frac{1}{(2n)!}\left(\frac{d}{dx}\right)^{2n}(x^2-1)^{n-\frac{1}{2}}$ = coefficient of h^{2n} in the expansion of $(x+1+h)^{n-\frac{1}{2}}(x-1+h)^{n-\frac{1}{2}}$

$$=a_{2n}(x+1)^{n-\frac{1}{2}}(x-1)^{-n-\frac{1}{2}}+a_1a_{2n-1}(x+1)^{n-\frac{3}{2}}(x-1)^{-n+\frac{1}{2}}+\ldots,$$

(if a_r denote the coefficient of h^r in the expansion of $(1+h)^{n-\frac{1}{2}}$)

$$=\frac{(x+1)^{n-\frac{1}{2}}}{(x-1)^{n+\frac{1}{2}}}\left\{a_{2n}+a_1a_{2n-1}\frac{x-1}{x+1}+a_2a_{2n-2}\left(\frac{x-1}{x+1}\right)^2+\&c.\right\}$$

$$=\frac{(x+1)^{n-\frac{1}{2}}}{(x-1)^{n+\frac{1}{2}}}(-)^n\frac{(n-\frac{1}{2}.n-\frac{3}{2}\ldots\frac{1}{2})^2}{(2n)!}\left\{1-2n\frac{x-1}{x+1}+\&c.\right\}$$

$$=(-)^n\frac{(n-\frac{1}{2}.n-\frac{3}{2}\ldots\frac{1}{2})^2}{(2n)!}\frac{(x+1)^{n-\frac{1}{2}}}{(x-1)^{n+\frac{1}{2}}}\left(1-\frac{x-1}{x+1}\right)^{2n}$$

$$=\frac{1}{(2n)!}(-)^n 1^2.3^2\ldots(2n-1)^2.\frac{1}{(x^2-1)^{n+\frac{1}{2}}},$$

which gives (8) on putting $xi:a$ in place of x.

The equation (8) was in §2 deduced from Rodrigues's theorem, and it is evident that the latter theorem can be deduced from (8).

§5. It is a theorem of Jacobi's that

$$\left(\frac{d}{dx}\right)^{n-1}(1-x^2)^{n-\frac{1}{2}}=(-)^{n-1}\frac{1.3\ldots2n-1}{n}\sin n\theta\ \ldots(10),$$

where $x=\cos\theta$,* so that

$$\left(\frac{d}{dx}\right)^{2n}(1-x^2)^{n-\frac{1}{2}}=(-)^{n-1}\frac{1.3\ldots2n-1}{n}\left(\frac{d}{dx}\right)^{n+1}\sin n\theta,$$

whence from (8)

$$\left(\frac{d}{dx}\right)^{n+1}\sin n\theta=-n\frac{1.3\ldots2n-1}{(1-x^2)^{n+\frac{1}{2}}}\ \ldots\ldots\ldots\ (11)$$

$$=n\left(\frac{1}{x}\frac{d}{dx}\right)^{n+1}(1-x^2)^{\frac{1}{2}},$$

* See Heine, *Kugelfunctionen* (1878), p. 157, Liouville's Journal, t. VI. (1841) pp. 69–73, or Jacobi, Crelle's Journal, t. XV. (1836) pp. 3–4.

viz. putting for x its value $\cos\theta$,

$$\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\right)^{n+1}\sin n\theta = n\left(\frac{1}{\sin\theta\cos\theta}\frac{d}{d\theta}\right)^{n+1}\sin\theta\ldots(12),$$

or, expressing (11) wholly in terms of x,

$$\left(\frac{d}{dx}\right)^{n+1}\sin(n\text{ arc}\cos x) = -n\frac{1.3\ldots2n-1}{(1-x^2)^{n+\frac{1}{2}}}.$$

The similarity of the formulæ (10) and (11) is noticeable; they differ, so to speak, only in the sign of n.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, June 12th, C. W. Merrifield, F.R.S, *President*, in the Chair. Mr. R. C. Rowe, B.A., Fellow of Trinity College, Cambridge, was proposed for election.

The following communications were made: "Notes on the reduction of a system of forces, and on plane curves," by J. J. Walker, M.A.; "Notes on determinants of n dimensions," by H. W. Lloyd Tanner, M.A.; "Curves for the inscription of a regular nonagon and undecagon in a circle," by Dr. Freeth; "On Clifford's graphs and on the twenty-one coordinates of a conic in space," by Dr. Spottiswoode, P.R.S.; "Two geometrical notes," by Prof. H. J. S. Smith, F.R.S. [A paper "Déduction de différents théorèmes géométriques d'un seul principe algébrique," by Dr. H. G. Zeuthen, Copenhagen, came to hand after the meeting.]

R. Tucker, M.A., *Hon. Sec.*

ON THE EQUALITY OF SYLVESTER'S AND CAUCHY'S ELIMINANTS.

By Professor *Paul Mansion.*

1. *Notation.* Let

$$A = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6,$$

$$\alpha_i = a_0 + \ldots + a_ix^i,$$

$$B = b_0 + b_1x + b_2x^2 + b_3x^3,$$

$$\beta_i = b_0 + \ldots + \beta_ix_i;$$

$$A = \alpha_0 + x\gamma_5 = \alpha_1 + x^2\gamma_4 = \alpha_2 + x^3\gamma_3,$$

$$B = \beta_0 + x\delta_2 = \beta_1 + x^2\delta_1 = \beta_2 + x^3\delta_0,$$

$$C_0 = A\delta_2 - B\gamma_5 = \alpha_0\delta_2 - \beta_0\gamma_5 = c_{00} + c_{01}x + \ldots + c_{05}x^5,$$

$$C_1 = A\delta_1 - B\gamma_4 = \alpha_1\delta_1 - \beta_1\gamma_4 = c_{10} + c_{11}x + \ldots + c_{15}x^5,$$

$$C_2 = A\delta_0 - B\gamma_3 = \alpha_2\delta_0 - \beta_2\gamma_3 = c_{20} + c_{21}x + \ldots + c_{25}x^5.$$

2. *First auxiliary eliminant.* If $A=0$, $B=0$ have at least one common root, we can consider the equations

$$C_0 = A(b_1 + b_2x + b_3x^2) - B(a_1 + a_2x + a_3x^2 + a_4x^3 + a_5x^4 + a_6x^5) = 0 \quad \ldots\ldots\ldots(1),$$

$$C_1 = A(b_2 + b_3x) - B(a_2 + a_3x + a_4x^2 + a_5x^3 + a_6x^4) = 0 \quad \ldots(2),$$

$$C_2 = Ab_3 - B(a_3 + a_4x + a_5x^2 + a_6x^3) = 0 \quad \ldots\ldots\ldots(3),$$

$$B = b_0 + b_1x + b_2x^2 + b_3x^3 = 0 \quad \ldots\ldots\ldots(4),$$

$$Bx = b_0x + b_1x^2 + b_2x^3 + b_3x^4 = 0 \quad \ldots\ldots\ldots(5),$$

$$Bx^2 = b_0x^2 + b_1x^3 + b_2x^4 + b_3x^5 = 0 \quad \ldots\ldots\ldots(6),$$

$$Bx^3 = b_0x^3 + b_1x^4 + b_2x^5 + b_3x^6 = 0 \quad \ldots\ldots\ldots(7),$$

$$Bx^4 = b_0x^4 + b_1x^5 + b_2x^6 + b_3x^7 = 0 \quad \ldots\ldots\ldots(8),$$

$$Bx^5 = b_0x^5 + b_1x^6 + b_2x^7 + b_3x^8 = 0 \quad \ldots\ldots\ldots(9),$$

as forming a system, of which the eliminant is

$$M = \begin{vmatrix} c_{00}, & c_{01}, & c_{02}, & c_{03}, & c_{04}, & c_{05} & & & \\ c_{10}, & c_{11}, & c_{12}, & c_{13}, & c_{14}, & c_{15} & & & \\ c_{20}, & c_{21}, & c_{22}, & c_{23}, & c_{24}, & c_{25} & & & \\ b_0, & b_1, & b_2, & b_3 & & & & & \\ & b_0, & b_1, & b_2, & b_3 & & & & \\ & & b_0, & b_1, & b_2, & b_3 & & & \\ & & & b_0, & b_1, & b_2, & b_3 & & \\ & & & & b_0, & b_1, & b_2, & b_3 & \\ & & & & & b_0, & b_1, & b_2, & b_3 \end{vmatrix}.$$

3. *Second auxiliary eliminant.* Multiplying the equations (4), ...(9) by convenient suitable constants and adding the results to (1), (2), (3), we shall obtain

$$A(b_1 + b_2x + b_3x^2) = d_0 + d_1x + \ldots + d_8x^8 = 0 \quad \ldots\ldots(1'),$$

$$A(b_2 + b_3x) = e_0 + e_1x + \ldots + e_7x^7 = 0 \quad \ldots\ldots\ldots(2'),$$

$$Ab_3 = f_0 + f_1x + \ldots + f_6x^6 = 0 \quad \ldots\ldots\ldots(3').$$

The eliminant

$$N = \begin{vmatrix} d_0, & d_1, & d_2, & d_3, & d_4, & d_5, & d_6, & d_7, & d_8 \\ e_0, & e_1, & e_2, & e_3, & e_4, & e_5, & e_6, & e_7 & \\ f_0, & f_1, & f_2, & f_3, & f_4, & f_5, & f_6 & & \\ b_0, & b_1, & b_2, & b_3 & & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & b_0, & b_1, & b_2, & b_3 \end{vmatrix}.$$

of (1′), (2′), (3′), (4), ...(9) can be obtained if we operate on the lines on M, as we have operated on the equations (1), (2), (3), (4), ...(9). From this remark, we conclude easily that

$$M = N.$$

4. *Sylvester's eliminant.* Again, divide (3′) by b_3, and subtract the result multiplied by b_2 from (2′), by b_1 from (1′), then divide the new equation (2′) [viz. $Ab_3x = 0$] by b_3, and subtract the result multiplied by b_2 from the new equation (1′) [viz. $A(b_2x + b_3x^2) = 0$]; finally, divide the equation (1′) so modified by b_3. We shall find, instead of (1′), (2′), (3′),

$$Ax^2 = a_0x^2 + a_1x^3 + a_2x^4 + a_3x^5 + a_4x^6 + a_5x^7 + a_6x^8 = 0 \quad \ldots(1''),$$
$$Ax = a_0x + a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 + a_5x^6 + a_6x^7 = 0 \quad \ldots(2''),$$
$$A = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 = 0 \quad \ldots(3'').$$

Eliminating $x^0, x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8$ from (1″), (2″), (3″), (4″), ...(9), the result is $S = 0$, S being the eliminant of Sylvester

$$S = \begin{vmatrix} & & a_0, & a_1, & a_2, & a_3, & a_4, & a_5, & a_6 \\ & a_0, & a_1, & a_2, & a_3, & a_4, & a_5, & a_6 & \\ a_0, & a_1, & a_2, & a_3, & a_4, & a_5, & a_6 & & \\ b_0, & b_1, & b_2, & b_3 & & & & & \\ & b_0, & b_1, & b_2, & b_3 & & & & \\ & & b_0, & b_1, & b_2, & b_3 & & & \\ & & & b_0, & b_1, & b_2, & b_3 & & \\ & & & & b_0, & b_1, & b_2, & b_3 & \\ & & & & & b_0, & b_1, & b_2, & b_3 \end{vmatrix}.$$

Again, we can obtain S, if we operate on the lines of N, as we have operated on the equations (1′), (2′), (3′). Thus we shall find $M = N = b_3^2 S$.

4. *Cauchy's eliminant.* If we put

$$C = \begin{vmatrix} c_{00}, & c_{01}, & c_{02}, & c_{03}, & c_{04}, & c_{05} \\ c_{10}, & c_{11}, & c_{12}, & c_{13}, & c_{14}, & c_{15} \\ c_{20}, & c_{21}, & c_{22}, & c_{23}, & c_{24}, & c_{25} \\ b_0, & b_1, & b_2, & b_3, & & \\ & b_0, & b_1, & b_2, & b_3 & \\ & & b_0, & b_1, & b_2, & b_3 \end{vmatrix},$$

we have also $M = b_3^2 C$. Therefore

$$S = C,$$

C being the eliminant of Cauchy.

5. *Extension.* Evidently, the above demonstration proves the equality of many minors of S and C, especially of those which are enclosed by lines in the symbolical relation (fig. 11).

6. *Conclusion.* The (principal) minors just written serve to express the conditions necessary and sufficient for the existence of 1, 2, or 3 common roots of $A = 0$, $B = 0$, to find the equations with common or non-common roots, &c. We can then conclude that the method of elimination of Bezout and Cauchy and the dialytical method of Sylvester are completely equivalent.

Antwerp, *July* 7, 1879.

INSIGNIORES ORBITAE COMETARUM PROPRIETATES.

By *C. Taylor, M.A.*

J. H. Lambert's brief work entitled *Insigniores Orbitae Cometarum Proprietates* was published at Augsburg (Augusta Vindelicorum) in the year 1761. Independently of the practical value of the results obtained in it, the treatise is of great interest to the pure geometer from the ingenuity and inventiveness which the writer displays in the course of his investigations, and it may therefore be worth while to give a general account of its contents with reference rather to its mathematical principles than to their technical applications.

The author observes in the preface that investigators when bent upon obtaining results for practical use are too apt to be content with the first method that presents itself, without considering whether it is the most direct and theoretically the best. Intricate problems require each its own method and special combination of artifices, failing which the desired end cannot be reached except by wearisome and circuitous ways. In problems of great complexity it is well to consider special cases, the solutions of which may be found to suggest and lead up to the complete solution of the general case. This principle is exemplified in the work under consideration, which is compared for its general method with the same writer's work (published three years earlier) on *Les* *propriétés remarquables de la route de la lumiere par les airs.* The discussion of the determination of the orbit of a comet from three observations was already "crambe bis, ter vel centies recocta," but there was still room enough for improvement in the methods employed. The task which the author sets before him is to treat the whole subject geometrically, and he meets with a success which surpasses his expectation:

"Mirum tamen videri poterat, an elegantissimae illae proprietates, quibus gaudent sectiones conicae, & in quibus eruendis summi geometrae veluti certatim ab antiquissimis retro temporibus operam collocarunt indefessam, prorsus inutiles atque superfluae forent, simul ac cometarum orbitis essent adplicandae? Res ipsa utique tentamine digna, spes haud inana, at superavit eventus."

Old things and new are intermingled in the work. As to what is claimed as original, it may be well to give the author's own words (which however themselves still leave us in doubt about some particulars) since the important theorem in parabolic motion so often ascribed to him had been previously discovered by Euler. See below, II. (A). Lambert however does not appear to claim more in this particular than to have demonstrated the theorem by new methods and to have extended it to the ellipse and the hyperbola. Speaking of what is new in the book he says:

"Huc referas meditata de projectione orthographica in planum eccliptices vel aliud quoduis utiliter ipsi substituendum. Similiter Lemmata quaedam in prima Sectione ipsi tractationi praefixa, & quae in Sectione IV. de differentia orbitarum parabolicarum, ellipticarum et hyperbolicarum brevibus disserui."

He goes on to remark that he has treated the parabolic orbit most at length, and has investigated the motion in elliptic orbits in the last section only so far as to shew how the properties of the former may be extended to the rest of the conics ("ceteris quoque sectionibus conicis"). The reader who wishes to pursue this subject further is referred to Euler's *Theoria motuum Planetarum & Cometarum.*

I.

The work is divided into four sections, the first of which contains twenty-two lemmas on the parabola, with numerous corollaries, and is in fact a brief treatise upon the geometry of that curve. Attention may be directed more particularly to the very ingenious geometrical transformation in lemma 20, which is eventually extended to the ellipse, and by implication to the hyperbola.

The following are the enunciations of the lemmas which constitute *Sectio* I.

1. If F be the focus of a parabola, T the point in which the tangent at N meets the axis, A the vertex and B a point in AF produced, the angle TNF is equal to $\frac{1}{2}NFB$.

2. If the tangent at A meets NT in S, the triangles AFS, SFN are similar.

It is added in a corollary that

$$AF = SF.\sin ASF = SF.\sin SNF = FN.\sin^2 FNT;$$

which implies the relation (see lemma 3),

$$FS : FN = \sqrt{(FA)} : \sqrt{(FN)}.$$

3. If RM and RN be any two tangents, the triangles NFR, RFM are similar, and the exterior angle between the tangents is equal to NFR or RFM.

It is added *inter alia* that

$$FN : FR = \sqrt{(FN)} : \sqrt{(FM)}.$$

4. *If the tangents at any three points L, M, N meet in P, Q, R, the points F, P, Q, R lie on a circle.*

This remarkable and now well known property is perhaps one of the novelties to which Lambert refers in general terms in the preface. The theorem was rediscovered many years later by Wallace, and again by Poncelet, the former of whom remarks upon it in his *Geometrical Treatise on the Conic Sections*, p. 167, note (Edinburgh, 1837):

"It may be proper to mention that this proposition was given by the author of this work in Leybourne's *Mathematical Repository*, about the year 1797, because it has since that time appeared as new in the *Annales des Mathématiques*."

The reference here is to Poncelet's article, *Théorèmes nouveaux sur les ligues du second ordre*, in Gergonne's *Annales*, VIII. 1—13 (1817), where the property is given as new; but in his *Traité des Propriétés Projectives*, § 466 (1822) Poncelet mentions Lambert as having anticipated him in the discovery of the theorem in question and some others, and acknowledges his obligations to M. Terquem (p. 269, note) for the reference to our author's researches.

5. If two of the three tangents as RL, RN remain fixed whilst the third PQ varies, the ratio of LP to RQ is constant.

It is added in a corollary that

$$\frac{PR}{LR} + \frac{RQ}{NR} = 1,$$

which is in fact a form of "tangential equation" to the parabola.

It is further remarked that the triangles *LRF*, *RNF*, *PFQ* are similar, and hence that (as *PQ* varies) the triangle *PFQ* is given in species.

6. Three fixed tangents to a parabola cut any fourth in a constant ratio.

7. Given three straight lines, it is required to draw a fourth which shall be cut by them in a given ratio.

The preceding lemma shews that this admits of an infinity of solutions.

8. Given the focus and two points of a parabola, it is required to construct the curve.

9. Given the triangle *NFM*, to find the area of the segment *NQM* cut off by *NM*.

If *MV* be the difference of the ordinates of *M* and *N*, the area is $\frac{MV^3}{24AF}$, which may be written in the form

$$\tfrac{1}{3}\sqrt{(ab)}.\sin c\,\{a+b-2\sqrt{(ab)}.\cos c\},$$

where $\quad a = FM;\ b = FN;\ 2c = \angle NFM.$

10. To find also *FA* and the area of the sector *FNQM*.

The area is

$$\tfrac{1}{8}\sqrt{\{k^2-(a-b)^2\}}.[a+b+\tfrac{1}{2}\sqrt{\{(a+b)^2-k^2\}}],$$

where *a* and *b* are used as before, and *k* denotes *NM*.

11. To find also the angle *RMF* or *SNF* (lemm. 2, 3).

12. If the diameter through *R* (lemma 3) meet the parabola in *Q* and *MN* in *G*, and if *FQ* meet *MN* in *E*, then $RQ = QG = QE$.

13. If the ordinate of *G* meet the curve in *g*, then

$$gF = \tfrac{1}{2}(FN + FM).$$

14. Also, $\quad Fg = FQ + QG.$

15. Given the triangle *FNM*, to find the length *FQ*.

16. If *RL* be a perpendicular to *FM* (lemma 12),

$$FL = FE \text{ and } RL = GK.$$

17. If Q be (as before) the vertex of the diameter which bisects MN, viz. in G, the triangles NFQ, QFM are as NE to EM; and the segments NMQ, NQ, QM are as NM^3, NG^3, GM^3.

18. To determine a straight line whose segments by four given straight lines shall be in given ratios.

Two solutions are given, with reference to the *Arithmetica Universalis* for a third.

19. To place a given quadrilateral with its vertices on four given straight lines.

20. *If MGN be a double ordinate of the diameter at Q and E its point of concourse with FQ, and if a line mEn equal to mn be placed at right angles to FQ so as to be bisected at E, the points m and n will lie on a parabola having F and Q for focus and vertex; and the areas of the sectors FNQM, FnQm will be in the subduplicate ratio of the latera recta of their parabolas.* [figure 12.

(i) By the property of ordinates and by lemma 12,

$$MG^2 = 4FQ\,.\,QG = 4FQ\,.\,QE,$$

and therefore $\quad mE^2 = nE^2 = 4FQ\,.\,QE,$

which proves the first part of the proposition.

(ii) Draw FS perpendicular to the tangent at Q in the first parabola. Then since the equal chords MN and mn in any two consecutive positions cut off from their parabolas elements of area which are as their own breadths and therefore in the *constant ratio* of FS to FQ (which is also the ratio of the triangles FMN and Fmn on equal bases MN and mn), therefore the whole segments MQN and mQn are in that ratio, and the sectors $FNQM$ and $FnQm$ are in that ratio.

That is to say, these sectors are as FS to FQ, or as $\sqrt{FA}$ to $\sqrt{FQ}$ (lemma 2), or in the subduplicate ratio of the latera recta of their parabolas.

21. Given the chord NM and the sagitta QG parallel to the axis, to find the areas of the segment NQM and the sector $FNQM$. [figure 12.

The area $FNQM$ may be evaluated as follows. If FZ be a perpendicular to NM, and MD a perpendicular to QG, then from the equality of the angles FEZ, MGD and by lemma 12,

$$\triangle\, NFM = MG\,.\,FZ = MD\,.\,FE = MD\,(FQ - QG).$$

Hence, the segment NQM being equal to $\frac{4}{3} MD . QG$,
sector $FNQM = \frac{4}{3} MD . QG + MD (FG - QG) = MD (FQ + \frac{1}{3} QG)$.

Therefore (putting MD^2 equal to $4AF . QG$),

$$\text{sector } FNQM = \tfrac{2}{3} (3FQ + QG) \sqrt{(AF . QG)}.$$

22. *Given NM equal to k, and $FM + FN$ equal to $a + b$, to find the area of the sector $FNQM$.*

Lambert obtains the result in the form

$$\frac{3}{\sqrt{AF}} . \left\{ \left(\frac{a+b+k}{2} \right)^{\frac{3}{2}} - \left(\frac{a+b-k}{2} \right)^{\frac{3}{2}} \right\},$$

by a series of reductions in place of which I propose the following direct process.

(i) The sum of the distances of M and N from the directrix being double of the distance of G (the middle point of MN) from the directrix, it follows that

$$SM + SN = 2 (FQ + QG).$$

Moreover, $$MN = 4 \sqrt{(FQ . QG)}.$$

Therefore

$$SM + SN \pm MN = 2 (\sqrt{FQ} \pm \sqrt{QG})^2,$$

and therefore, with the notation given in the enunciation,

$$\begin{aligned} \left(\frac{a+b+k}{2} \right)^{\frac{3}{2}} - \left(\frac{a+b-k}{2} \right)^{\frac{3}{2}} &= (\sqrt{FQ} + \sqrt{QG})^3 - (\sqrt{FQ} - \sqrt{QG})^3 \\ &= (6FQ + 2QG) \sqrt{QG} \\ &= \frac{3}{\sqrt{AF}} . \text{ sector } FNQM. \quad \text{[lemma 21.} \end{aligned}$$

(ii) Lambert himself first proves that

$$FQ = \tfrac{1}{4} [a + b + \sqrt{\{(a+b)^2 - k^2\}}],$$
$$QG = \tfrac{1}{4} [a + b - \sqrt{\{(a+b)^2 - k^2\}}];$$

then shews by substitution in lemma 21 that

$$\text{sector } FNQM = \frac{k [a + b + \frac{1}{2} \sqrt{\{(a+b)^2 - k^2\}}]}{3 \sqrt{[a + b + \sqrt{\{(a+b)^2 - k^2\}}]}} . \sqrt{AF};$$

and further reduces this last expression so as to obtain the result given above in the first proof. The result in this form is thus obtained only after a tedious process of reduction: it is however at once apparent from the expressions for FQ and QG together with lemma 21 that $\dfrac{\text{sector } FNQM}{\sqrt{(2AF)}}$ is a function of $a + b$ and k, and does not depend upon AF.

II.

The second and third sections contain *inter alia* numerous applications of the preceding lemmas to cases of motion in a parabolic orbit about the focus. We shall confine our attention to the determination of the general expression for the time in any arc of a parabola, which may be effected in the following ways with the help of lemmas 20 and 22 respectively.

(A) The time T of describing any arc NM varies as the area of the sector $FNMQ$ divided by the square root of the semi-latus rectum; or in other words, if m be a certain constant,

$$mT = \frac{\text{sector } FNMQ}{\sqrt{(2AF)}}.$$

Therefore by lemma 22,

$$3\sqrt{(2)}\, mT = \left(\frac{a+b+k}{2}\right)^{\frac{3}{2}} - \left(\frac{a+b-k}{2}\right)^{\frac{3}{2}}.$$

This result had been given eighteen years previously (with a different notation) in the course of L. Euler's "Determinatio Orbitæ Cometæ qui mense Martio hujus anni 1742 potissimum fuit observatus" (*Miscellanea Berolinensia*, tomus VII. pp. 19, 20. 1743).

(B). The same expression for T may also be deduced from lemma 20, by which Lambert very ingeniously reduces parabolic motion to rectilinear motion, as follows.

Since $$\frac{\text{sector } FNQM}{\sqrt{(2AF)}} = \frac{\text{sector } FnQm}{\sqrt{(2FQ)}}.$$

the arcs MQN and mQn are described in equal times.

Moreover, $$FN + FM = 2\,(FQ + QG) = 2\,(FQ + QE) = Fn + Fm;$$

and conversely *if $FN + FM$ be equal to $Fn + Fm$, and the chord NM to the chord nm, the curves NM, nm will be isochronous.*

Some consideration is required to see that this proposition is true generally, and not merely for the case in which one of the ordinates is at right angles to the axis of its parabola. But suppose any two varieties of the first parabola to be taken, and to be so placed as to have a common focal vector FQ.* In the one draw NM as before; and in the other

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* Since FQ may have any values between FA and ∞, if FQ be taken at random in the parabola which has the greater latus rectum an equal FQ can always be found in the other parabola.

draw $N'M'$ through the *same point* E parallel to the tangent to its parabola at Q. Then it may be shewn that the arc $N'M'$ is isochronous with nm, and therefore with NM: that the chord $N'M' = nm = NM$: and in like manner that $FN' + FM'$ is equal to $FN + FM$, and conversely.

It follows that the time in any arc NM of a parabolic orbit about the focus depends only upon the lengths NM and $FN + FM$, and not upon the latus rectum: the expression for the time will therefore be the same when the latus rectum vanishes and the motion becomes *rectilinear*.

α. Lambert applies this principle to the solution of the problem (p. 48):

Definire temporum interualla in lapsu parabolico, siue tempus quo cometa a data distantia ad datam quamuis aliam delabitur.

In this case $\dfrac{FN + FM \pm NM}{2}$ reduces to FM and FN respectively (the points F, N, M being in a straight line), and therefore

$$3\sqrt{(2)}\,.\,mT = FM^{\frac{3}{2}} - FN^{\frac{3}{2}}.$$

β. Conversely, having reduced the motion to motion in a straight line, we may find the expression for the time in the parabola without directly evaluating the sectorial area $FNQM$, viz. by solving the equation,*

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2};$$

whence it appears that (the velocity at infinity being zero) the time of traversing a distance MN varies as $FM^{\frac{3}{2}} - FN^{\frac{3}{2}}$, that is to say, as

$$\left(\frac{FN + FM + NM}{2}\right)^{\frac{3}{2}} - \left(\frac{FN + FM - NM}{2}\right)^{\frac{3}{2}}.$$

Therefore, with the same notation as before,

$$3\sqrt{(2)}\,mT = \left(\frac{a + b + k}{2}\right)^{\frac{3}{2}} - \left(\frac{a + b - k}{2}\right)^{\frac{3}{2}}.$$

Hence also we arrive at the curious result that the quadrature of the parabola may be made to depend upon an equation of motion in a straight line.

γ. Lambert also obtains the same formula by an ingenious double application of his geometrical principle (pp. 56, 57).

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* Lambert integrates for the ellipse, and gives the formula for the parabola as a limiting case in Sect. IV. §§ 210, 211.

Consider a variable parabolic sector FMN, and an isochronous portion MN of a straight line FM described under the rectilinear-parabolic law. Let the sector degenerate into the focal segment cut off by the chord MFN, and in the straight line let N coalesce with F: then the time in the arc of the segment is equal to the time in the straight line through the length MF equal to the chord of the segment. The time in the segment is easily found (by evaluating its area) to vary as $(MFN)^{\frac{3}{2}}$: therefore the time in the straight line from M to F varies as $MF^{\frac{3}{2}}$. Therefore in the general case the time from M to N in the straight line varies as $FM^{\frac{3}{2}} - FE^{\frac{3}{2}}$, that is to say as

$$\left(\frac{FM+FN+MN}{2}\right)^{\frac{3}{2}} - \left(\frac{FM+FN-MN}{2}\right)^{\frac{3}{2}},$$

and the same formula holds for motion in any parabolic sector FMN.

(C). The proofs in (B) may be somewhat shortened by substituting the following for lemma 20, whereby we avoid the slight difficulty arising from the apparent want of generality in the lemma itself.

Suppose any two separate parabolas to be drawn: in one of them (that which has the greater latus rectum) take FQ at random, and take $F'Q'$ equal to it in the other: draw the chords $NEGM$ and $N'E'G'M'$ parallel to the tangents at Q and Q', taking $Q'E'$ always equal to QE. It then follows by the same kind of reasoning that the chords NM, $N'M'$ are equal and subtend isochronous arcs, and that $FN+FM$ is equal to $F'N'+F'M'$, and conversely.

(To be continued.)

ON THE THEOREM CONNECTED WITH NEWTON'S RULE FOR THE DISCOVERY OF IMAGINARY ROOTS OF EQUATIONS.

By Professor *J. J. Sylvester.*

To save needless repetition in what follows I beg to refer the reader to Mr. Todhunter's section 26, p. 236, in the third edition of his *Treatise on the Theory of Equations*. It will there be seen that in order to provide against any loss of double permanences consequent upon any of the f's changing sign $\gamma_1, \gamma_2, \gamma_3, \ldots \gamma_{n-1}$ must all be positive; and in order to provide against the same thing happening consequent upon any of the G's changing sign we must have, from $i=2$ to

$i = n-1$ inclusive, $2-\gamma_i = \frac{1}{\gamma_{i+1}}$; and, moreover, $2-\gamma_{n-1}$ [denoted by $\frac{1}{\gamma_n}$, although strictly there is no γ_n, since G_n is simply a positive absolute], as well as $\gamma_1, \gamma_2, \ldots \gamma_{n-1}$, must be positive.

The solution of the equation $2-\gamma_i = \frac{1}{\gamma_{i+1}}$ is $\gamma_i = \frac{C+i-1}{C+i}$; and, in order that $\gamma_1, \gamma_2, \ldots \gamma_{n-1}, \gamma_n$ may all be *positive*, it is necessary that C shall be either positive or, if negative, of greater absolute value than n.

If we put $C=0$, $\gamma_1 = 0$; if we put $C=-n$, $\gamma_n = \infty$, so that, the condition of γ_i being *positive*, from 1 to n will not in either case be complied with, the signs of zero and of infinity being ambiguous. It is well known, however, that we may put $C=-n$; in fact, $-n$ is the value ordinarily attributed to C, for the corresponding value of γ_i, viz. $\frac{n-i+1}{n-i}$, it is which leads to that form of the theorem in which, when we put $\mu=\infty$ and $\lambda=0$, or $\mu=0$ and $\lambda=-\infty$ in the equation $pP(\mu)-pP(\lambda)=$ (the number of roots between λ and μ) $+2i$, gives Newton's rule as stated by Newton himself. Equally, we shall find it is lawful to put $C=0$, but each of these two suppositions requires to be subjected to a special examination before its validity can be admitted. Take the much more important case first, that where $C=-n$, we have then $\gamma_{n-1}=2$, and the only object of $2-\gamma_{n-1}$ being positive is to prevent mischief in the event of G_{n-1}, *i.e.* $(f_{n-1}x)^2 - 2f_{n-2}xf_nx$, changing its sign. But in this case $\frac{dG_{n-1}}{dx}=0$ by simple differentiation from $\frac{df_nx}{dx}=0$: in other words, G_{n-1} *is a constant* and never can change its sign. Thus, then, all necessity for $2-\gamma_{n-1}$ being positive is abolished by the very fact of its being zero.

It is worth noticing that this critical value of C, which makes $\gamma_i = \frac{n-i+1}{n-i}$, has the effect of lowering the degree of each G by two units; for if $\lambda = n-i+1$, we may write $f_{i-1} = px^\lambda + qx^{\lambda-1} + \ldots$, and then

$$G_i = f_i^2 - \frac{\lambda}{\lambda-1} f_{i-1} f_{i+1}$$

$$= \{p\lambda x^{\lambda-1} + q(\lambda-1)x^{\lambda-2} + \ldots\}^2$$

$$+ \frac{\lambda}{\lambda-1}(px^\lambda + qx^{\lambda-1} + \ldots)\{p\lambda(\lambda-1)x^{\lambda-2} + q(\lambda-1)(\lambda-2)x^{\lambda-3} + \ldots\};$$

so that the coefficient of $x^{2\lambda-2}$ becomes

$$p^2\left\{\lambda^2-\frac{\lambda}{\lambda-1}(\lambda^2-\lambda)\right\}=0,$$

and that of $x^{2\lambda-3}$ becomes

$$pq\left\{2\lambda(\lambda-1)+\frac{\lambda}{\lambda-1}\lambda(\lambda-1)+(\lambda-1)(\lambda-2)\right\}\doteq 0.$$

So again it will be found that C may be taken at the other extremity of the chasm or gap, which it is not permitted to enter; for if $C=0$ so that $g_1=0$, $G_1=(f'x)^2$.

Consider now the first three terms of the double series

$$fx,\ f'x,\ f''x,$$
$$I,\ I,\ G_2x,$$

where the two I's denote absolute positive quantities; at the moment of $f'x$ becoming zero, G_2x becomes positive, so that the succession of double permanences of sign for this double series is the same as of single permanences for fx, $f'x$, $f''x$, and consequently no double permanences can be lost by $f'x$ changing its sign. Since, then, we have shown that values of C giving rise to no negative but to an ambiguous sign, either of γ_1 or of γ_n are not prohibited, it might for a moment be imagined that any negative integer value of C, say $-\omega$, lying in the gap between 0 and $-n$ might also be admissible, seeing that such value would also *not introduce any negative value* of γ, but only two values of ambiguous signs, viz. for γ_ω and $\gamma_{\omega+1}$, ∞ and 0 respectively; all the other γ's will be positive. But it will be seen that this is inadmissible, for the course of the demonstration shows that every γ_i and $2-\gamma_i$ must both be positive, which conditions cannot be fulfilled for γ_ω, whether we consider it equal to plus or minus infinity.

As I have referred to Mr. Todhunter's treatise, I may notice the omission therein of the equation

$$\nu P\lambda-\nu P\mu=(\mu,\lambda)+2i',$$

where i' is any positive integer and (μ,λ) the number of real roots between λ and μ. This may be deduced *pari passu*, and in precisely the same way as the parallel equation

$$pP\mu-pP\lambda=(\mu,\lambda)+2i,$$

or either of these may be deduced from the other as follows. Let $fx=\phi(-x)$, and using the same parameter γ_i for the G's belonging to f and for those belonging to ϕ, let f_i, G_i for f become ϕ_i, T_i for ϕ. Then obviously

$$T_i(-c) = G_i c \text{ and } \phi_i(-c) = (-)^{n-1} f_i(c).$$

Hence, using π, Π in regard to ϕ in the same sense as p, P in regard to f, $\pi\Pi(-c) = \nu Pc$; also $(-\lambda, -\mu)$ in regard to ϕ is the same as (μ, λ) in regard to f. But remembering that if μ is greater than λ, then $-\lambda$ is greater than $-\mu$, the second equation above written applied to ϕ becomes

$$\pi\Pi(-\lambda) - \pi\Pi(-\mu) = (-\lambda, -\mu) \text{ in regard to } \phi + 2i.$$

Hence

$$\nu P(\lambda) - \nu P(\mu) = (\mu, \lambda) \text{ in regard to } f + 2i,$$

as was to be shown.*

One other point deserves mentioning. If any G, say G_i, becomes incapable of changing its sign (of which G_1 becoming f_1^2 when $C = 0$, offers a particular example), the necessity for the equation $2 - \gamma_i = \dfrac{1}{\gamma_{i+1}}$ is done away with for that value of i, so that γ_{i+1} becomes arbitrary (within limits), and we may start with a new definition of the values of the γ's lying beyond γ_i, viz. $\gamma_{i+i'} = \dfrac{C' - 1 + i'}{C' + i'}$ and so on, *toties quoties*, whenever in passing from G_1 to G_{n-1}, any of the G's becomes incapable of changing its sign.†

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* This equation is stated in the original memoir in the *Proceedings of the Mathematical Society of London*. Dr. Julius Petersen, of Copenhagen, in his treatise on Algebraical Equations, not having had the opportunity, as he has since informed me, of consulting this, and taking Mr. Todhunter's chapter on the subject as his anthority, was led to lay the fault of the omission at my door.

† Thus we see that iu the expression $\gamma_r = \dfrac{C - 1 + r}{C + r}$, C is not absolutely prohibited from entering the gap comprised between 0 and $-n$, but that C may be $-i$ where i is an integer, or any quantity between $-i$ and $-\infty$, provided that G_{i-1}, *i.e.* $f^2_{i-1} - \gamma_{i-1} f_{i-2} f_i$ is incapable of changing its sign. If $C = -i$, $\gamma_{i-1} = 2$.

As an application of the same principle we may make the γ series begin with G_2, *i.e.* make G a positive absolute so as to have two positive absolutes instead of one positive absolute at the beginning of the series of "the Quadratic elements," *i.e.* we may make $\gamma_1 = 0$ and $\gamma_{1+r} = \dfrac{C - 1 + r}{C + r}$, and continuing this process, $1 + k$ (any number) of the initial G's may be converted into positive absolutes; that is to say, we may make $\gamma_1 = 0$, $\gamma_2 = 0$, $\ldots \gamma_k = 0$, $\gamma_{k+r} = \dfrac{C - 1 + r}{C + r}$. If we make $k = n$, all the G's become positive absolutes, and *the theorem passes into Fourier's*. In connexion with this fact, it should be noticed that my theorem in its form as hitherto given does not logically contain Fourier's as a consequence; for it is possible that for certain values of λ and μ, $pP(\mu) - pP(\lambda)$ may be greater than $p(\mu) - p(\lambda)$, so that Fourier's theorem may indicate the passage of a smaller number of roots than the seemingly more stringent one; hence in

It will have been noticed in what precedes, that I have made no allusion to special forms of an equation, whether absolute or having reference to the assumed arbitrary parameter in G, but have confined myself to the general case where only one term in the double series can vanish for any given value of x. Nor is it necessary to do more than this in treating the theory; for 1°, if f contains no equal roots, we may, by infinitesimal or infinitely small variations attributed to the coefficients, cause those relations between them to subsist which are necessary in order that two or more of the terms may vanish simultaneously, and cannot thereby alter the character of the roots, which can only make the passage from real to imaginary, or *vice versâ*, after one or more pairs of them have passed through the state of equality; 2°, if f contains equal roots, we may vary the coefficients in such a manner as not to disturb the equalities which subsist between them, and shall have independent relations enough to spare to abolish as before the relations implied in the fact of the simultaneous evanescence above referred to.

Thus it seems to me that we need trouble ourselves with the discussion of the consequences of such simultaneous evanescence only if we wish to know what inferences to draw if we are unfortunate enough to find that event occurring at one or the other of the actual limits λ, μ we may be dealing with, and for no other purpose.

Baltimore, *Nov.* 2, 1878.

Postscript.

As I was on the point of despatching what precedes by post to England, it occurred to me, in consequence of the previously unnoticed depression of the degrees of the terms in the G series, to examine more closely their constitution

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applying my theorem, Fourier's should always be employed simultaneously with it, a practical direction which has hitherto been overlooked. Of course when the question concerns the total number of roots, Descartes' rule *is* logically contained in Newton's, or my generalisation of it as previously given.

It may be well to mention here, that a more general form of my theorem introducing a second arbitrary parameter will be found in some far back number of the *Educational Times* as the solution of a question proposed in a previous number. It is founded, if I recollect right, on the principle that if for the equation of the n^{th} degree in x, say $fx = 0$, we substitute $\epsilon x^{n+\nu} + fx = 0$, where ν is any positive integer (ϵ being an infinitesimal), no new real root is introduced if ν is even, provided ϵ be taken with the right sign, and only one (of infinite value) if ν is odd.

for the critical case, that namely where $\gamma_i = \frac{n-i+1}{n-i}$, and I have had the satisfaction of finding that every such G is proportional to the Hessian of the f antecedent to it, regarded as a homogeneous function of x and 1, being that Hessian multiplied by a negative number.

To prove this I have to show that if $F(x, y)$ is of the order λ, then

$$\lambda F \frac{d^2}{dx^2} F - (\lambda - 1)\left(\frac{dF}{dx}\right)^2$$

is a positive multiple of y^2 multiplied by the Hessian of F in regard to x, y.

Now $$\lambda F = x\frac{dF}{dx} + y\frac{dF}{dy},$$

and $$(\lambda - 1)\frac{dF}{dy} = x\frac{d}{dx}\frac{dF}{dy} + y\frac{d^2F}{dy^2}.$$

Hence $$y\frac{dF}{dy} = \lambda F - x\frac{dF}{dx},$$

$$y\frac{d^2F}{dx\,dy} = (\lambda - 1)\frac{dF}{dx} - x\frac{d^2F}{dx^2},$$

and $$y^2\frac{d^2F}{dy^2} = (\lambda - 1)\left(\lambda F - x\frac{dF}{dx}\right) - x\frac{d}{dx}\left(\lambda F - x\frac{dF}{dx}\right)$$

$$= (\lambda^2 - \lambda) F - (2\lambda - 2) x\frac{dF}{dx} + x^2\frac{d^2F}{dx^2}.$$

Hence $$y^2\left\{\frac{d^2F}{dx^2}\frac{d^2F}{dy^2} - \left(\frac{d^2F}{dx\,dy}\right)^2\right\}, \text{ i.e. } -y^2 H(F)$$

$$= (\lambda^2 - \lambda)\frac{d^2F}{dx^2}F - (2\lambda - 2) x\frac{d^2F}{dx^2}F + x^2\left(\frac{d^2F}{dx^2}\right)^2$$

$$- \left\{(\lambda - 1)\frac{dF}{dx} - x\frac{d^2F}{dx^2}\right\}^2$$

$$= (\lambda - 1)\left\{\lambda\frac{d^2F}{dx^2}F - (\lambda - 1)\left(\frac{dF}{dx}\right)^2\right\},$$

where the least value of λ is 2 so that $\lambda - 1$ is always positive.

Thus the f and G series may be put under the following form, where f_i of course means $\frac{d^i f}{dx^i}$ and $H\phi x$ signifies the Hessian of ϕ regarded as a quantic in x and 1,

$$f : f_1 : f_2 : f_3 : \ldots : f_{n-1} : f_n,$$
$$-1 : Hf : Hf_1 : Hf_2 : \ldots : Hf_{n-2} : -1.$$

I anticipate that it will be found possible to extend the theorem by the addition of a third series for the case of $n=4$ or 5, a third and fourth for that of $n=6$ or 7, and, in general, by the use of $\frac{1}{2}(n+2)$ or $\frac{1}{2}(n+1)$ series according as n is even or odd. And possibly it may turn out that the maximum number of series available for any given value of n will by the reckoning of the gain of complete permanences of sign (*i.e.* treble, quadruple...permanences for 3, 4...series) as x increases from λ to μ, afford not merely a superior limit to, but the actual number of, real roots passed over in the interval.

As I find that Mr. Todhunter uses a single symbol ϖ for the pP employed in my memoir in the second number of the *Proceedings of the London Mathematical Society*, it may be well to advise my readers that I use p, P to signify permanences of sign, and v, V variations of sign in the f and G series respectively; so that double permances, permanence variations, variation permanences and variation variations would be denoted by the compound symbols pP, pV, vP, vV respectively.

November 3, 1878.

The theorem above given is, I find, only a particular case of the one subjoined.

Let f_i denote $(a_0, a_1, a_2 \ldots a_i \between x, y)^i$ and $H_\varepsilon(f_{i+\varepsilon})$ that covariant of $f_{i+\varepsilon}$ whose highest powers of x bears the coefficient

$$\begin{vmatrix} a_0, & a_1, & a_2, & \ldots a_\varepsilon \\ a_1, & a_2, & a_3, & \ldots a_{\varepsilon+1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_\varepsilon, & a_{\varepsilon+1}, & a_{\varepsilon+2}, & \ldots a_{2\varepsilon} \end{vmatrix};$$

then is

$$\begin{vmatrix} f_{i-\varepsilon}, & f_{i-\varepsilon+1}, & f_{i-\varepsilon+2}, & \ldots f_{i+\varepsilon} \\ f_{i-\varepsilon+1}, & f_{i-\varepsilon+2}, & f_{i-\varepsilon+3}, & \ldots f_{i+\varepsilon+1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ f_{i+\varepsilon}, & f_{i+\varepsilon+1}, & f_{i+\varepsilon+2}, & \ldots f_{i+2\varepsilon} \end{vmatrix}$$

equal to $y^{\varepsilon^2+\varepsilon} H_\varepsilon(f_{i+\varepsilon})$.

The order in (x, y) of $H_\varepsilon f_{i+\varepsilon}$, since the weight of its leading coefficient is $\varepsilon^2+\varepsilon$ and its degree in the coefficients $\varepsilon+1$, will be $(\varepsilon+1)(i+\varepsilon)-2(\varepsilon^2+\varepsilon)$, *i.e.* $(\varepsilon+1)i-\varepsilon^2-\varepsilon$, so that multiplied by $y^{\varepsilon^2+\varepsilon}$ the order becomes $(\varepsilon+1)i$, as it ought to be.

The theorem may be proved as follows:

Let ϕ be any homogeneous function of λ dimensions in x, y, and denote $\frac{d}{dx}, \frac{d}{dy}$ by X, Y.

1°. I shall show that in respect of ϕ,

$$y^i.Y^i = {}^i\lambda - i^{i-1}(\lambda-1)\,xX + \frac{i.i-1}{2}\,{}^{i-2}(\lambda-2)\,x^2X^2 \ldots + (-)^i x^i X^i,$$

where ${}^im$ for any positive integer values of m and i denotes the factorial quantity $m(m-1)\ldots(m-i+1)$.

Suppose the equation to be true for any assigned value of i, it will be true for $i+1$. For $Y^i\phi$, it will be observed, is of $\lambda-1$ dimensions in x, y; hence

$$y^{i+1}Y^{i+1} = (\lambda - i - xX) * y^iY^i$$

for $(\lambda-i)\,Y^i\phi = (xX+yY)\,Y^i\phi$ by Euler's well-known theorem on homogeneous functions.

The $(j+1)^{\text{th}}$ and $(j+2)^{\text{th}}$ term in y^iY^i are respectively

$$\mp\frac{i(i-1)\ldots(i-j+1)}{1.2\ldots j}(\lambda-j)(\lambda-j-1)\ldots(\lambda-i+1)x^jX^j, \text{ say } -Ax^jX^j$$

and

$$\pm\frac{i(i-1)\ldots(i-j)}{1.2\ldots(j+1)}(\lambda-j-1)(\lambda-j-2)\ldots(\lambda-i+1)x^{j+1}X^{j+1},$$

say $Bx^{j+1}X^{j+1}$.

Now $xX * x^{j+1}X^{j+1} = x^{j+2}X^{j+2} + (j+1)\,x^{j+1}X^{j+1}$.

Hence the $(j+2)^{\text{th}}$ term in $y^{i+1}Y^{i+1}$ will be

$$A + (\lambda - i - j - 1)\,B,$$

i.e., putting $\frac{i(i-1)\ldots(i-j+1)}{1.2\ldots j\,(j+1)} = B'$ is $\mu B'$,

$$\begin{aligned}\text{where}\quad \mu &= (j+1)(\lambda-j) + (i-j)(\lambda-i-1-j)\\ &= \{-j^2+(\lambda-1)j+\lambda\} + \{j^2-(\lambda-1)j+\lambda i-i^2-i\}\\ &= (\lambda-i)(i+1).\end{aligned}$$

Thus the $(j+2)^{\text{th}}$ term in $y^{i+1}Y^{i+1}$ will be

$$\pm\frac{(i+1)\,i\ldots(i+1-j)}{1.2\ldots(i+1)}(\lambda-j-1)(\lambda-j-2)\ldots\{\lambda-(i+1)+1\};$$

and consequently the equation is true for $i+1$.

Hence, being true for $i=1$, it is true universally.

2°. Consider a persymmetrical determinant of the order $\varepsilon+1$ formed with the distinct constituents $\phi_0, \phi_1, \phi_2, \ldots\phi_{2\varepsilon}$, where ϕ_0 is a constant and in general $\frac{d}{dx}\phi_k = \pm k\phi_{k-1}$; as, for example, suppose $\varepsilon=2$, and let the determinant be

$$\begin{matrix} a\ , & ax+b, & P, \\ ax+b, & P\ , & Q, \\ P\ , & Q\ , & R, \end{matrix}$$

where P, Q, R stand for $ax^2+2bx+c$, $ax^3+3bx^2+3cx+d$, $ax^4+4bx^3+6cx^2+4dx+e$, and $\frac{d\phi_k}{dx}=k\phi_{k-1}$. If we made $\frac{d\phi_k}{dx}=-k\phi_{k-1}$ the effect would be to change the signs of all the odd-degreed functions, but the value of the determinant would not be altered by this change. Calling the columns P_0, P_1, P_2,

$$P_0,\ P_1-xP_0,\ P_2-2xP_1+x^2P_0$$

will represent a determinant equal to the given one, but of the form

$$\begin{matrix} a & , b & , c & , \\ ax+b & , bx+c & , cx+d & , \\ ax^2+2bx+c, & bx^2+2cx+d, & cx^2+2dx+e, \end{matrix}$$

and now, calling the lines L_0, L_1, L_2, the equivalent determinant

$$L_0,\ L_1-xL_0,\ L_2-2xL_1+x^2L_0$$

becomes

$$\begin{matrix} a, & b, & c, \\ b, & c, & d, \\ c, & d, & e, \end{matrix}$$

which is the same as if in the original form we made $x=0$.

So in general for the order $\varepsilon+1$, calling the ϵ columns

$P_0, P_1, P_2 \ldots P_\varepsilon$, we may pass to a new determinant by means of the combinations represented by

$$P_0,\ P_1 - xP_0,\ P_2 - 2xP_1 + x^2P_0, \ldots$$

$$P_\varepsilon - \varepsilon x P_{\varepsilon-1} + \frac{\varepsilon(\varepsilon-1)}{2} x^2 P_{\varepsilon-2} \ldots + (-)^\varepsilon x^\varepsilon P_0,$$

and calling the lines of these new determinants

$$L_0,\ L_1,\ L_2 \ldots L_\varepsilon,$$

$$L_0,\ L_1 - xL_0,\ L_2 - 2xL_1 + x^2L_0, \ldots$$

$$L_\varepsilon - \varepsilon x L_{\varepsilon-1} + \frac{\varepsilon(\varepsilon-1)}{2} x^2 L_{\varepsilon-2} \ldots + (-)^\varepsilon x^\varepsilon L_0,$$

will produce a determinant containing no power of x, and which is what the original one becomes on making $\varepsilon = 0$.

3°. If we take for our $2\varepsilon + 1$ distinct elements of the persymmetrical matrix, the quantities

$$X^{2\varepsilon}\phi,\ yYX^{2\varepsilon-1}\phi,\ y^2Y^2X^{2\varepsilon-2}\phi, \ldots Y^{2\varepsilon}\phi,$$

where ϕ is of λ dimensions in x, y, we shall find by virtue of 1° that they will be represented by

$$A_0,\ A_1 - A_0x,\ A_2 - 2A_1x + A_0x^2, \ldots$$

$$A_{2\varepsilon} - 2\varepsilon A_{2\varepsilon-1}x + \frac{2\varepsilon(2\varepsilon-1)}{2} A_{2\varepsilon-2}x^2 \ldots + A_0x^{2\varepsilon},$$

(where $\phi_k = -k\phi_{k-1}$) on making

$$A_0 = X^{2\varepsilon}\phi,\ A_1 = (\lambda - 2\varepsilon + 1)X^{2\varepsilon-1}\phi,$$

$$A_2 = (\lambda - 2\varepsilon + 2)(\lambda - 2\varepsilon + 1)X^{2\varepsilon-2}\phi,$$

....

$$A_{2\varepsilon} = \lambda(\lambda - 1)\ldots(\lambda - 2\varepsilon + 1).$$

Now obviously the persymmetrical determinant in question on striking out each power of y from its several constituents will be diminished in the proportion of 1 to $y^{2+4\ldots+2\varepsilon}$, *i.e.* $y^{\varepsilon^2+\varepsilon}$.

Hence
$$\left.\begin{matrix} A_0, & A_1, & A_2, & \ldots A_\varepsilon \\ A_1, & A_2, & A_3, & \ldots A_{\varepsilon+1} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ A_\varepsilon, & A_{\varepsilon+1}, & A_{\varepsilon+2}, & \ldots A_{2\varepsilon} \end{matrix}\right\}$$

$$= y^{\varepsilon^2+\varepsilon} \left| \begin{matrix} X^{2\varepsilon}\phi & , X^{2\varepsilon-1}Y\phi & , \dots X^{\varepsilon}Y^{\varepsilon}\phi & , \\ X^{2\varepsilon-1}Y\phi, & X^{2\varepsilon-2}Y^2\phi & , \dots X^{\varepsilon-1}Y^{\varepsilon+1}\phi, & \\ \dots\dots\dots & & & , \\ \dots\dots\dots & & & , \\ X^{\varepsilon}Y^{\varepsilon}\phi & , X^{\varepsilon-1}Y^{\varepsilon+1}\phi, & \dots Y^{2\varepsilon}\phi & , \end{matrix} \right.$$

This is true for any function ϕ homogeneous in x and y.*

If ϕ is a rational integral function of x, y, say

$$(a_0, a_1, a_2, \dots \between x, y)^{\lambda},$$

the last written determinant becomes a covariant whose leading coefficient is the persymmetrical determinant formed with the elements $a_0, a_1, a_2, \dots a_{2\varepsilon}$ multiplied by

$$\{\lambda(\lambda-1)\dots(\lambda-2\varepsilon+1)\}^{\varepsilon+1},$$

and if we write

$$\begin{aligned} \lambda(\lambda-1)\dots(\lambda-2\varepsilon+1)\, B_0 &= X^{2\varepsilon}\ \phi, \\ \lambda(\lambda-1)\dots(\lambda-2\varepsilon\quad)\, B_1 &= X^{2\varepsilon-1}\phi, \\ \lambda(\lambda-1)\dots(\lambda-2\varepsilon-1)\, B_2 &= X^{2\varepsilon-2}\phi, \\ \dots\dots\dots\dots\dots & \dots, \\ B_{2\varepsilon} &= \quad \phi, \end{aligned}$$

we shall have the persymmetrical determinant formed with the elements $B_0, B_1, \dots B_{2\varepsilon}$ equal to $y^{\varepsilon^2+\varepsilon}$ multiplied into the covariant of which the leading coefficient is the persymmetrical determinant formed with the elements $a_0, a_1, \dots a_2{}^{\varepsilon}$, as was to be proved.

Scholium. The theory of hyperdeterminants teaches us that every in- and co-variant has its source of being in a higher existence, viz. in a pure form typified by

$$F(X^{\mu}\phi,\ YX^{\mu-1}\phi,\ Y^2X^{\mu-2}\phi, \dots Y^{\mu}\phi),$$

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* It seems to me very likely or almost certain that every covariant of $f(x, y)$, or what becomes such when f is a quantic, may in like manner be converted into a function of f and of its derivatives in respect to one of the variables alone divided by an appropriate power of the other; and, if true, as it can hardly help being, the proof ought not to be far to seek.

It is indeed virtually contained in a formula obvious from inspection of the expression for y^iY^i in 1°, viz.

$$\left\{\frac{d}{dx}(y^iY^iX^j)\right\}_* = -i\,\{y^{i-1}Y^{i-1}X^{j+1}{}_*\},$$

whatever homogeneous function is supposed to follow the asterisk. In connexion with this it should be observed that the determinant in 2° is bound to vanish, from the mere fact that on putting $x=0$, it becomes an invariant of $(a_0, a_1, a_2, \dots a_{2\varepsilon})(\xi, \eta)^{2\varepsilon}$, and that its several terms are what the a elements become when X becomes when $\xi + x\eta$. We are thus led to view the whole subject of invariance under a somewhat broader aspect, as a theory not directly concerned with quantics, but with homogeneous functions in general.

ϕ being a perfectly general operand, or as we may phrase it, an operand absolute. This enables me to express the idea which was struggling into light when I wrote the antecedent footnote. It is this: Let ϕ now be made to do duty for any given homogeneous function of given order λ in x, y.

The value of F will remain unaltered when we write

$$\frac{\lambda-\mu+1}{y} \text{ in place of } Y,$$

$$\frac{(\lambda-\mu+2)(\lambda-\mu+1)}{y^2} \quad \text{,,} \quad Y^2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-\mu+1)}{y_\mu} \quad \text{,,} \quad Y^\mu.$$

This is an immediate consequence of the invariantive property of F combined with the fact that

$$\frac{d}{dx}y^iY^i = -iy^{i-1}Y^{i-1}X,$$

previously shown. The numerators in the above expressions are the first terms in the expression for y^iY^i as a function of xX modified by writing successively $\lambda-\mu+1, \lambda-\mu+2, \ldots\lambda$ in place of λ on account of the powers of X which precede $Y, Y^2, \ldots Y^\mu$ in F and lower the degrees of the operands in respect to these powers by $\mu-1, \mu-2, \ldots 0$ units respectively.

Thus *ex. gr.* the pure invariant

$$(X^4:)(Y^4:)-4(X^3Y:)(XY^3:)+3(X^2Y^2:)^2$$

where the colon (:) does duty for an operand absolute is equivalent to

$$\frac{1}{y^4}\begin{array}{ll}(\lambda-3)(\lambda-2)(\lambda-1)\lambda\,(X^4:)(X^0:) \\ -4(\lambda-3)^2(\lambda-2)(\lambda-1)\,(X^3:)(X:) \\ +3(\lambda-3)^2(\lambda-2)^2 \qquad (X^2:)(X^2:),\end{array}$$

the colon now representing a homogeneous function of order λ is x, y.

So in general we may say that a *pure invariant*, or it might be more correct to say the *Schema* of an invariant, *is a function of symbolic inverses* ($X, Y, \ldots$) *to any number of letters and of any number of unconditional absolutes possessing the property that when those absolutes become conditioned to* *stand for homogeneous functions of the letters, of* SPECIFIED *orders, it becomes a function of any one of the letters, of the symbolic inverses to the rest and of the absolutes so conditional.*

This property, which is certainly *necessary*, is in all probability sufficient to define a pure invariant, for I presume (nay I think it is obvious) that when it is satisfied, the only part the arbitrarily selected letter can play is that of contributing a power of itself as a factor to the function in which it figures. This definition of invariance, although it may appear abstruse, is in reality the most complete and simplest, in the sense of exemption from foreign ingredients and unnecessary specifications, that can be given, and may of course be extended without difficulty to systems of sets of letters $(x, y, \ldots)$. Nor should it be overlooked that in our great art, the *ars magna excogitandi*, a gain in expression is a gain in power.*

Returning from this rather wide excursus to our original theme of Newton's theorem, it may be useful to give the values of the G series as far as required for equations of the 5th order inclusive corresponding to the critical value of the arbitrary parameter, *i.e.* for the case of $C = -n$.

The given form being supposed to be $(a, b, c, \ldots \between x, y)^n$,

when $n = 2$, $\quad -G_1 = ac - b^2$,

when $n = 3$, $\quad -G_1 = (ac - b^2)\,x^2 + (ad - bc)\,x + (bd - c^2)$,

$\quad -G_2 = ac - b^2$,

when $n = 4$, $\quad -G_1 = (ac - b^2)\,x^4 + 2\,(ad - bc)\,x^3$

$\quad + (ae + 2bd - 3c^2)\,x^2 + 2\,(be - cd)\,x + (ce - d^2)$,

$\quad -G_2 = (ac - b^2)\,x^2 + (ad - bc)\,x + (bd - c^2)$,

$\quad -G_3 = ac - b^2$,

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* The object of pure Physic is the unfolding of the laws of the intelligible world. ["The unseen world" belongs to another province altogether.] The object of pure Mathematic (which is only another name for Algebra) that of unfolding the laws of the human intelligence. With Geometry it fares as it was thought to be probably about to fare with a certain distant land – it is "wiped out" between the two neighbouring powers. Algebra takes for its share Geometry in the abstract. Sensible or empirical Geometry (as, thanks to the Copernican genius of Lobatcheffsky and the sublimated practical sense of Helmholtz, is now beginning to be well understood) falls into the domain of Physic.

So already Logic is divided between Psychology and Algebra; and so eventually with Grammar, whilst linguistic is handed over to history, psychology and physiology; its theoretical part, the laws of syntax, declension or conjugation, regimen and collocation, must be eventually absorbed into Algebra.

when $n=5$, $-G_1=(ac-b^2)x^6+3(ad-bc)x^5+3(ae+bd-2c^2)x^4$
$+(af+7bc-8cd)x^3+(bf+ce-2d^2)x^2+(cf-de)x$
$+(df-e^2)$,

G_2, G_3, G_4 being the G_1, G_2, G_3 of the preceding case.*

In applying the series of these G's combined with the f series to ascertain the maximum possible number of real roots passed over in going *up* from λ to μ it is proper to use simultaneously the three *independent* superior limits 1° the gain of pP's, 2° the loss of vP's, 3° the gain of p's or loss of v's, which two latter numbers are of course identical.

December 9, 1878.

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* It is thus seen that the G series is formed of the second alliances or "überschiebungen" of the given form (made homogeneous in x, y), and of its successive derivatives each with itself; and I have great reason to believe (as already hinted) that we may append a 3rd, 4th, ... series by substituting the 4th, 6th, ... of such alliances in lieu of the second, filling up the vacant spaces with positive absolutes, and always reckoning the gain of the permanence-permanence-permanence...s in going up from λ to μ as one superior limit, and, as a consequence thereof, the loss of the variation-permanence-permanence...s as another. Thus, *ex. gr.*, for the case of $n=4$, the series would be three in number, viz.

$$\begin{array}{ccccc} f, & f_1, & f_2, & f_3, & f_4, \\ 1, & -Hf, & -Hf_1, & -Hf_2, & 1, \\ 1, & 1, & s, & 1, & 1, \end{array}$$

where $s=ae-4bd+3c^2$ (and it may be noticed that we know from the expression for s in terms of the roots that when they are real, s must be positive).

For $n=5$ the series would be

$$\begin{array}{cccccc} f, & f_1, & f_2, & f_3, & f_4, & f_5, \\ 1, & -Hf, & -Hf_1, & -Hf_2, & -Hf_3, & 1, \\ 1, & 1, & s, & s', & 1, & 1, \end{array}$$

where $s'=ae-4bd+3c^2$,

and $s=(ae-4bd+3c^2)x^2+(af-3be+2cd)x+(bf-4ce+3d^2)$.

When $n=6$ or $n=7$ a new series would dawn into existence, and so on continually. Thus we set a number of sieves, as it were, successively under each other; it is certain, however, that by this method we can never be assured that no more than the actual number of real roots have fallen through; but there is another method which might be studied, and is, I think, not unworthy of investigation, *i.e.* to take for our third series the covariants of f which have for their common leading coefficient the discriminant of the form $(a, b, c, d)(x, y)^3$, for the fourth series the covariants which have for their common leading coefficient the discriminant of $(a, b, c, d, e)(x, y)^4$, and so on indefinitely, always filling up the vacant spaces with positive absolutes.

In this way I think it not improbable that the gain of compound permanences may be found to give not merely a superior limit to, but the actual number of real roots passed over in any ascent from one value of x to another.

Such a theorem, however, would have no practical value as a method for separating the roots, as its application would entail much greater labour than the ordinary Sturmian process.

ERRATUM.—p. 79, line 4, *for* $i+1$ *read* $i=1$.

ON NORMALS TO ENVELOPES; AND ON THE ENVELOPES, IF ANY, TO WHICH A GIVEN DOUBLY INFINITE SET OF STRAIGHT LINES ARE NORMALS.

By *E. B. Elliott, M.A.* Queen's College, Oxford.

§ 1. It is an obvious property of any normal to a surface or normal plane to a curve that the perpendicular distance of any point on it from the tangent plane or line at its foot equals that on any consecutive tangent plane or line. Hence, ϖ being the analytical expression for the length of the perpendicular upon the tangent plane or line from any point, so that $\varpi + \delta\varpi$ is that upon a consecutive one, the locus given by $\delta\varpi = 0$ consists of the normal line or plane either alone or together with another locus, points of which possess the same property. Four cases deserve special consideration.

I. Firstly, in two dimensions. Given a series of straight lines enveloping a plane curve, the equation of the type of those lines referred to any system of plane coordinates will involve a single variable parameter, t say. Let the equation so prepared by a factor that its left-hand side is the analytical expression for the length of the perpendicular from any point upon the line it represents be $\varpi = 0$. Then $\delta\varpi = 0$ gives in this case $\frac{d\varpi}{dt} = 0$. And this is a straight line. Consequently, it is the normal at the point of the envelope where $\varpi = 0$ is the tangent. (This fact was given and treated at some length in a paper by Mr. Genese, *Quarterly Journal*, vol. XIII. p. 260, and it is this paper which first set me thinking on the subject).

II. Now in three dimensions with any kind of reference let us be given the equation of a plane involving a single variable parameter t. Its envelope is a developable. Its equation can also be prepared by a factor so as to take the form $\varpi = 0$, ϖ being the expression for the length of the perpendicular on it. In this case then $\delta\varpi = 0$, *i.e.* $\frac{d\varpi}{dt} = 0$

is the equation of a plane. It is therefore that of the plane which is normal to the developable along the line where $\varpi = 0$ touches it.

III. Given the equation of a plane involving two independent variable parameters t and t'. It envelopes a surface. Let the equation, prepared as before, be $\varpi = 0$. Then $\delta\varpi = 0$ gives in this case $\frac{d\varpi}{dt}\delta t + \frac{d\varpi}{dt'}dt' = 0$; that is to say, since δt and $\delta t'$ are independent, $\frac{d\varpi}{dt} = 0$ and $\frac{d\varpi}{dt'} = 0$ separately. These two equations taken simultaneously represent a straight line, being each of the first degree. This is then the normal line to the envelope of $\varpi = 0$ at its point of contact.

IV. Suppose the two equations of a straight line in space to involve a single variable parameter t. In general, two consecutive lines of the system of which it is the type do not intersect, so that the lines generate a skew surface and have no envelope. If, however, the variable parameter is so involved that a certain relation which can in each case be found is identically satisfied, every two consecutive lines will intersect, and the lines will generate a developable and envelope a curve, the edge of regression of that developable. Now in this case not only points in the normal plane at the point of contact to the curve enveloped, but also points in the normal plane through the line itself to the developable generated, clearly possess the property expressed by $\delta\varpi = 0$. And it will be found that, taking the analytical expression for the square of the perpendicular on the line and equating to zero its t-differential, we get a quadric equation in the coordinates, but that this, in virtue of the given identical relation, breaks up into factors. These equated to zero separately are two planes, the one of which that contains the line $\varpi^2 = 0$ is the normal plane through it to the developable, and the one that does not is the normal plane to the edge of regression at its point of contact with it.

Thus taking the most general equations of a straight line in rectangular coordinates,

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n},$$

let a, b, c and the direction-cosines l, m, n be given functions of a parameter t, and write a', b', c', l', m', n' for the t-differential coefficients of a, b, c, l, m, n.

If any plane

$$L(x-a)+M(y-b)+N(z-c)=0$$

contain it and the consecutive line, we have, expressing that it contains

$$a+a'\delta t,\quad b+b'\delta t,\quad c+c'\delta t,$$

and that it be parallel to the directions l, m, n and

$$l+l'\delta t,\quad m+m'\delta t,\quad n+n'\delta t,$$

$$La'+Mb'+Nc'=0,\quad Ll+Mm+Nn=0,\quad Ll'+Mm'+Nn'=0;$$

whence, eliminating, the condition that the lines have an envelope is

$$\begin{vmatrix} a', & b', & c' \\ l, & m, & n \\ l', & m', & n' \end{vmatrix}=0 \text{ ....(1).}$$

Now, writing momentarily for shortness ξ, η, ζ instead of $x-a$, $y-b$, $z-c$,

$$\varpi^2=\xi^2+\eta^2+\zeta^2-(l\xi+m\eta+n\zeta)^2;$$

therefore

$$\begin{aligned}-\varpi\frac{d\varpi}{dt}&=a'\xi+b'\eta+c'\zeta\\ &\quad+(l\xi+m\eta+n\zeta)(l'\xi+m'\eta+n'\zeta-la'-mb'-nc')\\ &=\xi\{a'-l(la'+mb'+nc')\}+\eta\{b'-m(la'+mb'+nc')\}\\ &\quad+\zeta\{c'-n(la'+mb'+nc')\}\\ &\quad+(l\xi+m\eta+n\zeta)(l'\xi+m'\eta+n'\zeta) \text{ .... (2).}\end{aligned}$$

But $$ll'+mm'+nn'=0,$$

by differentiating

$$l^2+m^2+n^2=1 \text{ ....(3),}$$

and by (1)

$$(b'n-c'm)\,l'+(c'l-a'n)\,m'+(a'm-b'l)\,n'=0,$$

whence, using (3),

$$\frac{l'}{a'-l(la'+mb'+nc')}=\frac{m'}{b'-m(la'+mb'+nc')}=\frac{n'}{c'-n(la'+mb'+nc')},$$

each of which equals

$$\frac{l'^2+m'^2+n'^2}{l'a'+m'b'+n'c'}.$$

Hence (2) becomes

$$-\varpi\frac{d\varpi}{dt}=\left(\frac{l'a'+m'b'+n'c'}{l'^2+m'^2+n'^2}+l\xi+m\eta+n\zeta\right)(l'\xi+m'\eta+n'\zeta).$$

Therefore equating to zero, and restoring to ξ, η, ζ their values at full,

$$l'(x-a)+m'(y-b)+n'(z-c)=0,$$

and $$l(x-a)+m(y-b)+n(z-c)+\frac{l'a'+m'b'+n'c'}{l'^2+m'^2+n'^2}=0,$$

of which the former is clearly the one that contains the line from which we started, are the planes that are normal, through it to the developable it generates, and through its point of contact with its envelope to that envelope respectively.

§ 2. Having in the above investigated a method for passing from the tangents of an envelope to its normals, it occurs to consider whether by reversing the process we can from the normals pass to the tangents. I. thus suggests a way of finding the envelope that is the orthogonal trajectory of a system of lines in a plane, and III. one of finding, when possible, an envelope which is orthogonal to a given doubly infinite set of straight lines in space. The latter follows:

Using rectangular coordinates, if the equation of an enveloping plane be

$$\varpi\equiv lx+my+nz-p=0\ \ldots\ldots\ldots\ldots\ldots(4),$$

where $$l^2+m^2+n^2=1,$$

the coordinates of any point on the corresponding normal will satisfy

$$\delta\varpi\equiv\left(x-\frac{dp}{dl}\right)\delta l+\left(y-\frac{dp}{dm}\right)\delta m+\left(z-\frac{dp}{dn}\right)\delta n=0,$$

δl, δm, δn being any variations consistent with

$$l\delta l+m\delta m+n\delta n=0.$$

The equations of the normal are consequently

$$\frac{x-\dfrac{dp}{dl}}{l}=\frac{y-\dfrac{dp}{dm}}{m}=\frac{z-\dfrac{dp}{dn}}{n}\ldots\ldots\ldots\ldots\ldots(5).$$

Now suppose we are given the straight line

$$\frac{x-A}{l}=\frac{y-B}{m}=\frac{z-C}{n}\ \ldots\ldots\ldots\ldots\ldots(6),$$

where A, B, C are known functions of the direction-cosines l, m, n. If this is identical with (5), we must have

$$A=\frac{dp}{dl}+\rho l,\ \ B=\frac{dp}{dm}+\rho m,\ \ C=\frac{dp}{dn}+\rho n,$$

ρ being the unknown distance between the points (A, B, C) and

$$\left(\frac{dp}{dl},\ \frac{dp}{dm},\ \frac{dp}{dn}\right).$$

Hence $Adl+Bdm+Cdn=dp+\rho\,(ldl+mdm+ndn)$

$$=dp.$$

Thus for the assumed equivalence of (5) with (6) to be possible, we must have $Adl+Bdm+Cdn$ (when expressed in terms of two of the cosines l, m, n by means of $l^2+m^2+n^2=1$) an exact differential, *i.e.*

$$\left(A-\frac{l}{n}\,C\right)dl+\left(B-\frac{m}{n}\,C\right)dm,$$

n meaning $\sqrt{(1-l^2-m^2)}$, must be an exact differential. Let it be the differential of $\phi\,(l, m, n)$. Then

$$lx+my+nz=\phi\,(l, m, n)+k\ \ldots\ldots\ldots\ldots(7),$$

is the equivalent of (4), *i.e.* is the type tangent plane of a surface to which the lines (6) are normals.

What is seen then is that the polar tangential equation of any such surface, *i.e.* the relation between the perpendicular on its tangent plane from the origin and the direction-cosines of this perpendicular, is

$$p=\phi\,(l, m, n)+k,$$

k being arbitrary.

Its Cartesian equation is found by elimination of l, m, n between (6), (7), and the the relation $l^2+m^2+n^2=1$, or, which is the same thing, by first preparing $\phi\,(l, m, n)$ by means of this relation so as to be of the first degree and homogeneous in l, m, n, in which prepared state call it $\psi\,(l, m, n)$, and then eliminating between

$$x=\frac{d\psi}{dl},\ y=\frac{d\psi}{dm},\ z=\frac{d\psi}{dn},$$

and

$$l^2+m^2+n^2=1.$$

This idea I borrow from a question by Mr. E. W. Symons in this month's *Educational Times.*

That the condition found above for the possibility of orthogonal surfaces is the same as the one more ordinarily obtained will be immediately clear. For if (ξ, η, ζ) be the point at which (6) is normal

$$\zeta = A + rl, \quad \eta = B + rm, \quad \zeta = C + rn,$$

r being some length. Therefore, since $ldl + mdm + ndn$ vanishes,

$$\xi dl + \eta dm + \zeta dn = Adl + Bdm + Cdn,$$

therefore

$$ld\xi + md\eta + nd\zeta = d\,(l\xi + m\eta + n\zeta) - (Adl + Bdm + Cdn.$$

Thus the condition $Adl + Bdm + CDn$, an exact differential is the same as the usual condition $ld\xi + md\eta + nd\zeta$ an exact differential (see Prof. Cayley, *Proc. Lond. Math. Soc.*, vol. VIII. p. 53).

Sept. 3, 1879.

A PHENOMENON OF THE KALENDAR.

By *W. Allen Whitworth, M.A.*

(*On the suggestion of Mr. W. M. Pendlebury*).

MR. PENDLEBURY has pointed out to me, *à propos* of a question in *Choice and Chance* (3rd edit., p. 118), that it is a mistake to assume that in a year taken at random, Christmas Day (or any other stated day of a stated month) is equally likely to fall on any day of the week.

For the period of 400 years in the Gregorian Kalendar consists of an exact number of weeks. For 400 years contain 97 leap years; therefore over and above the 52 weeks of each year, there are 400 days at the rate of a day a year, and 97 more days for the leap years.

But 497 days = 71 weeks exactly, therefore in 400 years there are 400 × 52 + 71 weeks = 20871 weeks exactly.

Consequently the 401st year commences with the same day of the week as the 1st year, and therefore every period of 400 years consists of the same cycle of years as regards their "Sunday letters," or, as regards the day of the week on which they begin, or on which Christmas day falls. And as 400 is not divisible by 7, the days of the week on which the years begin cannot be equally distributed in the cycle.

As $400 = 7 \times 57 + 1$, Christmas Day must fall 58 times on one day of the week and 57 times on each of the others, or else (as we shall shew is actually the case) 58 times on more than one day, and 57 or 56 on other days.

§ 2. Or we may regard the matter from a slightly different point of view. Since $365 = 7 \times 52 + 1$, any given date (such as Christmas Day) falls in consecutive years on consecutive days of the week, except in leap year, when a day is skipped.

Hence, in 400 years Christmas Day skips 97 days, and as $97 = 14 \times 7 - 1$, it cannot skip each day of the week an equal number of times, and whatever day it skips oftener than other days will be a day on which Christmas will fall less often than other days.

It is easy to see that Christmas skips

Sunday, if the S. Letter is			*bA*,
Monday	„	„	*aG*,
Tuesday	„	„	*gF*,
Wednesday	„	„	*fE*,
Thursday	„	„	*eD*,
Friday	„	„	*dC*,
Saturday	„	„	*cB*.

§ 3. *To determine the frequency of the several days of the week on which Christmas Day falls.*

Take the 400 years from 1501 to 1900 inclusive.

We proceed to consider what are the days which Christmas skips in the 97 leap years of the cycle.

As 1600 is a leap year, no irregularity occurs except in the years 1700 and 1800.

We may reject 7 periods of 28 years, 1501 to 1696, as containing no irregularity.

The next leap year is 1704, and we may reject 3 periods of 28 years commencing with 1704, as containing no irregularity. This brings us to the year 1787 inclusive. The remaining leap years of the 18th century with their Sunday Letters are as follows:

1788	*fE*,	skips	Wednesday,
1792	*aG*,	„	Monday,
1796	*cB*,	„	Saturday.

The next leap year after this is 1804, and we may again cast out 3 periods of 28 years, and there remain the following leap years to the end of the century:

1888	*aG*,	skips	Monday,
1892	*cB*,	„	Saturday,
1896	*cD*,	„	Thursday.

Hence, Monday and Saturday are skipped twice, Wednesday and Thursday are skipped once, and the other days not at all.

Therefore in 400 years Christmas Day falls

58	times on	Sunday,
56	„	Monday,
58	„	Tuesday,
57	„	Wednesday,
57	„	Thursday,
58	„	Friday,
56	„	Saturday.
400		

To determine the frequency of the several days of the week on which Feb. 29 falls.

Feb. 29 occurs 97 times in 400 years. We need only consider the six leap years of the last article.

	1788	*fE*,	Feb. 29	falls on	Friday,
	1792	*aG*,	„	„	Wednesday,
	1796	*cB*,	„	„	Monday.
Also	1888	*aG*,	„	„	Wednesday,
	1892	*cB*,	„	„	Monday,
	1896	*eD*,	„	„	Saturday.

Hence (combining these with the 13 complete cycles cast out)

Feb. 29	falls on	Sunday	13	times,
„	„	Monday	15	„
„	„	Tuesday	13	„
„	„	Wednesday	15	„
„	„	Thursday	13	„
„	„	Friday	14	„
„	„	Saturday	14	„
			97	

Belalp, *July* 18, 1879.

SOME CASES OF VORTEX MOTION.

By *T. C. Lewis, M.A.*, Fellow of Trinity College, Cambridge.

IF $2n$ straight vortex filaments, with alternately positive and negative rotations, be arranged within a cylinder, filled with fluid, along the generating lines of a coaxial cylinder, to find the ratio of the radii of the cylinders in order that the vortex filaments may be at rest, and that the motion of the fluid round them may be steady. This is the same problem as that of finding the stationary position of a single filament within a cylinder whose section is the sector of a circle having the angle at the vertex equal to $\frac{\pi}{n}$.

The motion of any one filament is the same as if it were due to two cylindrical arrangements of alternately positive and negative filaments at distances r and $\frac{a^2}{r}$ from the axis, if a be the radius of the cylindrical vessel. Each of these systems makes $\frac{dr}{dt}=0$.

And from the inner system $r\frac{d\theta}{dt}$ is equal to $\frac{m}{2\pi r}$ in magnitude for each filament besides that which is acted upon; there are $2n-1$ of these, and as m is alternately positive and negative, we have $r\frac{d\theta}{dt}=\frac{m}{2\pi r}$ from the inner system.

If B' (fig. 1) be a vortex of the outer system, A the filament whose motion is sought, A' its image, O the centre of the section, and ϕ the angle between OA and OB', we obtain from the outer system

$$\frac{\pi}{m}\cdot\frac{rd\theta}{dt}=\Sigma\frac{1}{AB'}\cos BAA',$$

where in Σ the terms are to be alternately positive and negative. Thus

$$\frac{\pi}{m}\frac{rd\theta}{dt}=\Sigma\frac{1}{AB'^2}(OB\cos\phi-OA)$$

$$=\Sigma\frac{(a^2\cos\phi-r^2)\,r}{a^4-2a^2r^2\cos\phi+r^4}.$$

Therefore for equilibrium we must have

$$\frac{1}{2r}+\Sigma\frac{(a^2\cos\phi-r^2)\,r}{a^4-2a^2r^2\cos\phi+r^4}=0;$$

therefore $$1+\Sigma\left\{-1+\frac{a^4-r^4}{a^4-2a^2r^2\cos\phi+r^4}\right\}=0;$$

therefore

$$1=2\Sigma(-1)^m\left(\frac{r^2}{a^2}\cos m\frac{\pi}{n}+\frac{r^4}{a^4}\cos 2m\frac{\pi}{n}+\frac{r^6}{a^6}\cos 3m\frac{\pi}{n}+\dots\right),$$

m having integral values from 1 to $2n$ inclusive; therefore

$$-1=2\frac{r^2}{a^2}\frac{\cos\left(\frac{\pi}{n}+\frac{2n-1}{2}\cdot\frac{n+1}{n}\pi\right)\sin(n+1)\pi}{\sin\frac{n+1}{2n}\pi}$$

$$+2\frac{r^4}{a^4}\frac{\cos\left(\frac{2\pi}{n}+\frac{2n-1}{2}\cdot\frac{n+2}{n}\pi\right)\sin(n+2)\pi}{\sin\frac{n+2}{2n}\pi}+\&c.$$

$$=\Sigma 2\left(\frac{r}{a}\right)^{2m}\frac{\cos\left(m\frac{\pi}{n}+\frac{2n-1}{2}\,\frac{m+n}{n}\pi\right)\sin(m+n)\pi}{\sin\frac{m+n}{2n}\pi},$$

m having here any integral value.

We see that every term of this last series vanishes for which m is not an odd multiple of n, and then each term of the series is equal to

$$-2\left(\frac{r}{a}\right)^{2m}2n;$$

therefore $$1=4n\left\{\left(\frac{r}{a}\right)^{2n}+\left(\frac{r}{a}\right)^{6n}+\left(\frac{r}{a}\right)^{10n}+\dots\right\};$$

therefore $$a^{4n}-r^{4n}=4na^{2n}r^{2n},$$

whence $$\frac{r}{a}=\{\sqrt{(4n^2+1)}-2n\}^{\frac{1}{2n}}.$$

If $n=1$, we obtain the case of two equal and opposite vortex filaments stationary within a cylinder (first given by Mr. Greenhill in *Quarterly Journal of Mathematics* for June, 1877) viz.

$$\frac{r}{a} = \cdot 485866.$$

Other results are $\quad \frac{r}{a} = \cdot 592342$ when $n=2$,

$$= \cdot 660160 \text{ when } n=3,$$

$$= \cdot 706752 \text{ when } n=4.$$

ON THE SUCCESSIVE EVOLUTES OF A CURVE.

By *W. J. Curran Sharp, M.A.*

1. If ρ, ρ_1, ρ_2, &c. be the radii of curvature of a curve and its successive evolutes, and ϕ be the angle which the tangent makes with any fixed line,

$$\rho = \frac{ds}{d\phi}, \quad \rho_1 = \frac{d\rho}{d\phi} = \frac{d^2s}{d\phi^2}, \quad \rho_2 = \frac{d\rho_1}{d\phi} = \frac{d^3s}{d\phi^3}, \text{ \&c.}$$

Hence, a geometrical meaning can be given to any differential equation, the variables of which are those of the intrinsic equation.

Thus, if $\rho_n = 0$, or $\frac{d^{n+1}s}{d\phi^{n+1}} = 0$, the curve is the $(n-1)^{\text{th}}$ involute of a circle, and its intrinsic equation is

$$s = a\phi^n + b\phi^{n-1} + \ldots + l.$$

It appears from this that every curve in which s can be expressed as a finite rational integral algebraic function of ϕ is an involute of a circle, and conversely.

Again, if $\frac{d^{n+1}s}{d\phi^{n+1}} = f(\phi)$, and σ be a particular solution,

$$\begin{aligned} s &= a\phi^n + b\phi^{n-1} + \ldots + l + \sigma \\ &= S + \sigma, \end{aligned}$$

where S is the corresponding arc of some involute of a circle; therefore, $s - \sigma = S$, or the difference of the arcs of any two particular involutes equals the arc of a corresponding involute of a circle.

2. From a figure it is evident that if P, P_1, P_2, &c. be corresponding points on a curve and its successive evolutes, the perpendicular from P_n on PT the tangent at p

$$=\rho+\rho_2+\&c.,$$

and that on PN the normal at p

$$=\rho_1+\rho_3+\&c.$$

Now if
$$\rho\;+\rho_2+\ldots+\rho_{2n-2}=0,$$
$$\rho_1+\rho_3+\ldots+\rho_{2n-1}$$

will also vanish, $P_{2n}P=0$ and the curve coincides with its $2n^{\text{th}}$ evolute.

The general differential equation to this curve is

$$\frac{ds}{d\phi}+\frac{d^3s}{d\phi^3}+\frac{d^5s}{d\phi^5}+\ldots+\frac{d^{2n-1}s}{d\phi^{2n-1}}=0,$$

the solutions of which for different values of n are the intrinsic equations to the curves, which are their own $2n^{\text{th}}$ evolutes.

3. A curve will be similar to its own n^{th} evolute if $\rho_n=p\rho$,

i.e.
$$\frac{d^{n+1}s}{d\phi^{n+1}}-p\frac{ds}{d\phi}=0,$$

an equation easily solved.

Thus if $n=1$, $s=a+be^{p\phi}$,

the logarithmic spiral,

if $n=2$, $s=a+be^{\sqrt{(p)}\phi}+ce^{-\sqrt{(p)}\phi}$,

or $=a+b\cos\sqrt{(p)}\,\phi+c\sin\sqrt{(p)}\,\phi$,

if p be negative of which the latter represents a cycloidal curve.

If n be even the n^{th} evolute is also similarly situated to the curve.

4. The same method of interpretation may be employed when the differential equation is given in terms of Cartesian coordinates.

With the usual notation for differential coefficients,

$$\rho=\frac{(1+p^2)^{\frac{3}{2}}}{q}\text{ and }\frac{dx}{d\phi}=\frac{1+p^2}{q};$$

therefore

$$\rho_1 = \frac{d\rho}{d\phi} = \frac{1+p^2}{q}\frac{d}{dx}\frac{(1+p^2)^{\frac{3}{2}}}{q} \,\&c.,$$

$$\rho_n = \left(\frac{1+p^2}{q}\frac{d}{dx}\right)^n \rho.$$

Hence $$\rho_1 = \frac{\rho}{q^2}\{3pq^2 - (1+p^2)r\},$$

(which gives $3pq^2 - (1+p^2)r = 0$, as the differential equation to the circle as it should, Boole, *Differential Equations*, p. 19, ed. 1), or

$$\rho_1 = \rho\left(3p - \frac{1+p^2}{q^2}r\right);$$

therefore

$$\rho_2 = \frac{1+p^2}{q}\frac{d}{dx}\left\{\rho\left(3p - \frac{1+p^2}{q^2}r\right)\right\}$$

$$= \rho_1\left(3p - \frac{1+p^2}{q^2}r\right)$$

$$+ \rho\left\{3(1+p^2) - \frac{(1+p^2)^2}{q^3}s - 2\frac{1+p^2}{q}\frac{pq^2-(1+p^2)r}{q^3}r\right\}$$

$$= \rho\left\{3 + 12p^2 - 8\frac{p(1+p^2)r}{q^2} + \frac{3(1+p^2)^2}{q^4}r^2 - \frac{(1+p^2)^2}{q^3}s\right\}.$$

So that the quantity in the brackets equated to 0 is the differential equation to the evolute to a circle.

Again

$$\rho_3 = \frac{1+p^2}{q}\frac{d}{dx}\rho_2$$

$$= \rho_1\left\{3 + 12p^2 - \frac{8p(1+p^2)}{q^2}r + 3\frac{(1+p^2)^2}{q^4}r^2 - \frac{(1+p^2)^2 s}{q^3}\right\}$$

$$+ \rho\frac{1+p^2}{q}\left\{24pq - \frac{8(1+3p^2)r}{q} - \frac{8p(1+p^2)}{q^2}s + \frac{28p(1+p^2}{q^3}r^2 + \frac{6(1+p^2)^2}{q^4}rs - \frac{12(1+p^2)^2}{q^5}r^3 - \frac{4(1+p^2)p}{q^2}s - \frac{(1+p^2)^2}{q^3}t + \frac{3(1+p^2)^2}{q^4}rs\right\}$$

$$= \rho \left\{ 33p + 60p^3 - \frac{(11+60p^2)(1+p^2)}{q^2} r \right.$$

$$+ \frac{45(1+p^2)^2 p}{q^4} r^2 - \frac{15(1+p^2)^3 r^3}{q^6}$$

$$\left. - \frac{15p(1+p^2)^2}{q^3} s + \frac{10(1+p^2)^3}{q^5} rs - \frac{(1+p^2)^3}{q^4} t \right\}.$$

When the quantity in the brackets equated to 0 gives the differential equation to the second involute of the circle.

5. The values of ρ_1, ρ_2 and ρ_3 above obtained may be employed to obtain a geometrical interpretation of the general differential equations to the parabola and to a conic.

The first is $$5r^2 = 3qs;$$

therefore

$$\rho_2 = \rho \left\{ 3 + 12p^2 - 8\frac{p(1+p^2)}{q^2} r + \frac{4}{3} \frac{(1+p^2)^2}{q^4} r^2 \right\};$$

therefore

$$3\rho_2 = \rho \left\{ 9 + 36p^2 - \frac{24p(1+p^2)}{q^2} r + 4\frac{(1+p^2)^2}{q^4} r^2 \right\}$$

$$= \rho \left\{ 9 + 4\left(3p - \frac{1+p^2}{q^2} r\right)^2 \right\}$$

$$= 9\rho + 4\frac{\rho_1^2}{\rho};$$

therefore

$$3\rho\rho_2 = 9\rho^2 + 4\rho_1^2 \quad \text{.......................................} \quad (1).$$

The general differential equation to a conic is

$$9q^2t - 45qrs + 40r^3 = 0;$$

therefore

$$\rho_3 = \rho \left\{ 33p + 60p^3 - \frac{(11+60p^2)(1+p^2)}{q^2} r \right.$$

$$+ 45\frac{(1+p^2)^2 p}{q^4} r^2 - 15\frac{(1+p^2)^3 r^3}{q^6} - 15\frac{p(1+p^2)^2}{q^3} s$$

$$\left. + 5\frac{(1+p^2)^3 rs}{q^5} + \frac{40}{9}\frac{(1+p^2)^3}{q^6} r^3 \right\},$$

and $5\left(3p-\frac{1+p^2}{q^2}r\right)\rho_2=\rho\left\{45p+180p^3\right.$

$$-\frac{(15+180p^2)(1+p^2)}{q^2}r+\frac{85p(1+p^2)^2}{q^4}r^2$$

$$\left.-\frac{15(1+p^2)^3r^3}{q^6}-\frac{15p(1+p^2)^2}{q^3}s+5\frac{(1+p^2)^3}{q^5}rs\right\};$$

therefore

$$5\frac{\rho_1\rho_2}{\rho}-\rho_3=\rho\left\{12p+120p^3-\frac{(4+120p^2)(1+p^2)}{q^2}r\right.$$

$$\left.+40\frac{p(1+p^2)^2}{q^4}r^2-\tfrac{40}{9}\frac{(1+p^2)^3}{q^6}r^3\right\}$$

$$=\rho\left\{4\left(3p-\frac{1+p^2}{q^2}r\right)+\tfrac{40}{9}\left(3p-\frac{1+p^2}{q^2}r\right)^3\right\}$$

$$=4\rho_1+\tfrac{40}{9}\frac{\rho_1^3}{\rho^2};$$

therefore $\qquad 45\rho\rho_1\rho_2=9\rho^2\rho_3+36\rho^2\rho_1+40\rho_1^3$ ....(2).

6. The relations (1) and (2) may be treated as differential equations in which the variables are s and ϕ, and may so be made to give the intrinsic equations to the parabola and to a conic.

14, Mount Street,
Grosvenor Square,
September 6, 1879.

SYMBOLIC POWERS AND ROOTS OF FUNCTIONS IN THE FORM $f(x)=\frac{ax+b}{cx+d}$.

By Professor *W. W. Johnson.*

THE symbolic products, and hence also the powers of these functions are functions of the same form; for if $\phi(x)=\frac{\alpha x+\beta}{\gamma x+\delta}$, we have

$$F(x)=\phi f(x)=\frac{\alpha ax+\alpha b+\beta cx+\beta d}{\gamma ax+\gamma b+\delta cx+\delta d}=\frac{Ax+B}{Cx+D},$$

a function of the same form in which

$$\left.\begin{array}{ll}A=\alpha a+\beta c, & C=\gamma a+\delta c\\ B=\alpha b+\beta d, & D=\gamma b+\delta d\end{array}\right\}\text{............ (1).}$$

Denoting the coefficients in $\phi^n(x)$ by α_n, β_n, γ_n and δ_n, we may, since $\phi^0(x)=x$, assume $\alpha_0=1$, $\beta_0=0$, $\delta_0=1$. Since $\phi^{n+m}=\phi^n\phi^m=\phi^m\phi^n$, we derive from (1)

$$\left.\begin{aligned}\alpha_{n+m}&=\alpha_n\alpha_m+\beta_n\gamma_m=\alpha_m\alpha_n+\beta_m\gamma_n\\ \beta_{n+m}&=\alpha_n\beta_m+\beta_n\delta_m=\alpha_m\beta_n+\beta_m\delta_n\end{aligned}\right\}\ \ldots\ldots\ldots(2),$$

from the first of which

$$\frac{\beta_n}{\beta_m}=\frac{\gamma_n}{\gamma_m},$$

and from the second

$$\frac{\alpha_n-\delta_n}{\alpha_m-\delta_m}=\frac{\beta_n}{\beta_m}.$$

Hence, putting $m=1$,

$$\frac{\beta_n}{\beta}=\frac{\gamma_n}{\gamma}=\frac{\alpha_n-\delta_n}{\alpha-\delta}=\rho\ \ (\text{say}),$$

and we may write

$$\left.\begin{aligned}\alpha_n&=\rho_n\alpha-v_n, & \gamma_n&=\rho_n\gamma\\ \beta_n&=\rho_n\beta, & \delta_n&=\rho_n\delta-v_n\end{aligned}\right\}\ \ldots\ldots\ldots\ldots(3),$$

in which ρ_n and v_n are functions of α, β, γ and δ.

From (2), putting $m=n=1$, we have

$$\alpha_2=\alpha^2+\beta\gamma,\quad \beta_2=\beta(\alpha+\delta).$$

From (3) we have $\rho_0=0$, $\rho_1=1$; and putting $n=2$ and comparing the values of α_2 and β_2 with those just written,

$$\rho_2=\alpha+\delta,\quad v_2=\alpha\delta-\beta\gamma.$$

Substituting the values of α_n and β_n from (3) in the second line of (2), we have

$$\beta_{n+m}=\rho_n(\alpha\beta_m+\beta\delta_m)-v_n\beta_m=\rho_n\beta_{m+1}-v_n\beta_m,$$

whence, since $\beta_n=\rho_n\beta$,

$$\rho_{n+m}=\rho_n\rho_{m+1}-v_n\rho_m.$$

Putting herein $n=2$,

$$\rho_{m+2}=\rho_2\rho_{m+1}-v_2\rho_m\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and putting $m=1$ and $m+1$ in place of n,

$$\rho_{m+2}=\rho_{m+1}\rho_2-v_{m+1}\rho_1.$$

Comparing these expressions we have, since $\rho_1=1$,

$$v_{m+1}=v_2\rho_m\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Putting $\sigma=\rho_1=\alpha+\delta$ and $\tau=v_2=\alpha\delta-\beta\gamma\ \ldots\ldots\ldots(6)$,

(4) becomes $$\rho_{m+2}=\sigma\rho_{m+1}-\tau\rho_m\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

and by means of (5), equations (3) become

$$\left.\begin{array}{ll}\alpha_n = \rho_n\alpha - \tau\rho_{n-1}, & \gamma_n = \rho_n\gamma \\ \beta_n = \rho_n\beta, & \delta_n = \rho_n\delta - \tau\rho_{n-1}\end{array}\right\} \text{.........(8)}.$$

Successive values of ρ may be computed by (7), and then (8) determines the coefficients for ϕ^n. For example, if $\phi(x) = \frac{x-2}{x-3}$, $\sigma = -2$, $\tau = -1$; and beginning with $\rho_0 = 0$, $\rho_1 = 1$, the successive values of ρ are $0, 1, -2, 5, -12$, &c.; hence, *e.g.* $\phi^4(x) = \frac{-7x+24}{-12x+41}$.

The form of equations (8) shows that $\rho_n = 0$ is a sufficient condition in order that $\phi^n = \phi^0$, for it makes $\alpha_n = \delta_n$, $\beta_n = 0$, $\gamma_n = 0$, whence $\phi^n(x) = x$. Moreover, this condition is a necessary one, except when $\alpha = \delta$, $\beta = 0$, $\gamma = 0$, which makes $\phi^1 = \phi^0$, while $\rho_1 = 0$ is impossible. Forming the values of ρ by (7), we have

$$\begin{array}{lll}\rho_0 = 0, & \rho_3 = \sigma^2 - \tau, & \rho_6 = \sigma^5 - 4\sigma^3\tau + 3\sigma\tau^2, \\ \rho_1 = 1, & \rho_4 = \sigma(\sigma^2 - 2\tau), & \quad = \sigma(\sigma^2 - \tau)(\sigma^2 - 3\tau), \\ \rho_2 = \sigma, & \rho_5 = \sigma^4 - 3\sigma^2\tau + \tau^2, & \quad \text{\&c.}\end{array}$$

When n is a composite number, of which n' is a divisor, $\phi^n = \phi^0$ is of course satisfied when $\phi^{n'} = \phi^0$; accordingly $\rho_{n'}$ is found to be a factor of ρ_n; thus ρ_6 contains ρ_3 and ρ_2 as factors, but it also contains a factor peculiar to itself, which put equal zero is the condition that ϕ^6, and no lower power of ϕ, shall equal ϕ^0.

The general expression for ρ_n is

$$\rho_n = \sigma^{n-1} - (n-2)\,\sigma^{n-3}\tau + (n-3)\,.\tfrac{1}{2}(n-4)\,\sigma^{n-5}\tau^2 \\ - (n-4)\,.\tfrac{1}{2}(n-5)\,.\tfrac{1}{3}(n-6)\,\sigma^{n-7}\tau^3 + \text{\&c.},$$

in which, however, all the terms containing negative binomials are to be rejected. For, denoting the coefficient of x in the expansion of $(1+x)^m$ by C_r^m, this equation becomes

$$\rho_n = \sigma^{n-1} - C_1^{n-2}\sigma^{n-3}\tau + C_2^{n-3}\sigma^{n-5}\tau^2 - \text{\&c.} \text{ .... (9)},$$

and assuming it to hold for n and $n-1$, the value of ρ_{n+1} formed by (7) is

$$\rho_{n+1} = \sigma^n - (C_1^{n-2} + 1)\,\sigma^{n-2}\tau + (C_2^{n-3} + C_1^{n-3})\,\sigma^{n-4}\tau^2 - \text{\&c.},$$

which, by the general relation,

$$C_r^m + C_{r-1}^m = C_r^{m+1} \text{ .... (10)},$$

becomes. $\rho_{n+1} = \sigma^n - C_1^{n-1}\sigma^{n-2}\tau^2 + C_2^{n-2}\sigma^{n-4}\tau^2 - \text{\&c.},$

which agrees with (9); but (9) is true for the two consecutive values $n = 1$ and $n = 2$, provided we regard C_r^m as vanishing when m is negative; hence, with this understanding, the equation is true generally.

The value of ρ_n may also be expressed as follows, put $\sigma = p + q$ and $\tau = pq$, then forming the values of ρ by (7), we find

$$\rho_n = \frac{p^n - q^n}{p - q},$$

in which p and q are the roots of the quadratic

$$x^2 - (\alpha + \delta)\,x + \alpha\delta - \beta\gamma = 0.$$

To determine the n^{th} *root* of a given function, $f(x) = \dfrac{ax + b}{cx + d}$, let $\phi^n = f$, then we must have α_n, β_n, γ_n, δ_n proportional to a, b, c, d, or employing (8)

$$\rho_n\alpha - \tau\rho_{n-1} = ka, \quad \rho_n\gamma = kc,$$
$$\rho_n\beta = kb, \qquad \rho_n\delta - \tau\rho_{n-1} = kd,$$

in which, as k is arbitrary, we may assume $k = \rho_n$. We may, therefore, put

$$\alpha = a + z, \quad \beta = b, \quad \gamma = c, \quad \delta = d + z \;\text{.......}\; (11),$$

where $$\rho_n z = \tau\rho_{n-1} \;\text{.......................}\; (12),$$

and if in this equation we express τ, ρ_{n-1} and ρ_n in terms of z, the result will determine z.

In expressing ρ_n, it is convenient to use s, t and r_n to denote the functions of a, b, c and d similar to σ, τ and ρ_n; we then have from (11)

$$\sigma = s + 2z, \quad \tau = t + sz + z^2 \;\text{.............}\; (13),$$

and shall prove that

$$\rho_n = nz^{n-1} + n\,.\tfrac{1}{2}(n-1)\,r_2 z^{n-2} + n\,.\tfrac{1}{2}(n-1)\,.\tfrac{1}{3}(n-2)\,r_3 z^{n-3} + \&c.,$$

or, as it may be written, since $r_0 = 0$, $r_1 = 1$,

$$\rho_n = r_0 z^n + C_1^{\,n} r_1 z^{n-1} + C_2^{\,n} r_2 z^{n-2} + \&c. \;\text{.......}\; (14).$$

Assuming this equation to hold for two consecutive numbers, $n - 1$ and n, we have also

$$\rho_{n-1} = r_0 z^{n-1} + C_1^{\,n-1} r_1 z^{n-2} + C_2^{\,n-1} r_2 z^{n-3} + C_3^{\,n-1} r_3 z^{n-4} + \&c.,$$

and may form the value of ρ_{n+1} by (7), which by means of (13) gives

$$\rho_{n+1} = (s + 2z)\,\rho_n - (t + sz + z^2)\,\rho_{n-1}.$$

The coefficient of z^{n-m+1} in the expansion of ρ_{n+1} is then

$$(2C_m^{\,n} - C_m^{\,n-1})\,r_m + s\,(C_{m-1}^{\,n} - C_{m-1}^{\,n-1})\,r_{m-1} - tC_{m-2}^{\,n-1} r_{m-2},$$

or by the general relation (10)

$$(2C_m^{\,n}-C_m^{\,n-1})r_m+(sr_{m-1}-tr_{m-2})(C_{m-1}^{\,n}-C_{m-1}^{\,n-1});$$

but since in conformity with (7)

$$sr_{m-1}-tr_{m-2}=r_m \;\ldots\ldots\ldots\ldots\ldots\ldots\; (15),$$

this reduces to $(2C_m^{\,n}-C_m^{\,n-1}+C_{m-1}^{\,n}-C_{m-1}^{\,n-1})r_m$,
and, since by (10) the negative terms $=-C_m^{\,n}$, we have finally

$$C_m^{\,n+1}r_mz^{n-m+1}$$

for a term of ρ_{n+1}. This agrees with (14); hence, since that equation is true for the consecutive values $n=0$ and $n=1$, it is true generally.

In forming the equation for z, it is convenient first to eliminate τ from (12) by means of $\rho_{n+1}=\sigma\rho_n-\tau\rho_{n-1}$; the result is

$$z\rho_n=\sigma\rho_n-\rho_{n+1},$$

or by (13) $\qquad \rho_{n+1}-(s+z)\rho_n=0.$

Employing (14), the coefficient of z^{n-m+1} in this equation is

$$C_m^{\,n+1}r_m-C_mr_m-sC_{m-1}^{\,n}r_{m-1};$$

but, since $\qquad C_m^{\,n+1}=C_m^{\,n}+C_{m-1}^{\,n},$

this reduces to

$$C_{m-1}^{\,n}(r_m-sr_{m-1})=-C_{m-1}^{\,n}tr_{m-1} \text{ [by (15)]}.$$

This gives the form of the coefficient of z^{n-m+1}, when $m>1$. The equation for z is therefore

$$z^n-C_1^{\,n}tr_0z^{n-1}-C_2^{\,n}tr_1z^{n-2}-\ldots-tr_{n-1};$$

or, since $r_0=0$, $r_1=1$,

$$z^n-n.\tfrac{1}{2}(n-1)tz^{n-2}-n.\tfrac{1}{2}(n-1).\tfrac{1}{3}(n-2)tr_2z^{n-3}-\ldots-tr_{n-1}=0.$$

Hence it appears that there are in general n forms of the function $f^{\frac{1}{n}}$. In particular, when $n=2$, the equation is $z^2-t=0$, whence $z=\sqrt{t}$, and we have

$$f^{\frac{1}{2}}(x)=\frac{[a\pm\sqrt{(ad-bc)}]\,x+b}{cx+d\pm\sqrt{(ad-bc)}}.$$

When $n=3$, the equation is

$$z^3-3tz-ts=0;$$

for example, if $f(x)=\dfrac{x-2}{2x-1}$, $s=0$, and $t=3$, the equation is $z^3-9z=0$, whence $z=0$ and $z=\pm3$; one of the values of $f^{\frac{1}{3}}$ is therefore identical with f, and the others are

$$f^{\frac{1}{3}}(x)=\frac{2x-1}{x+1} \text{ and } f^{\frac{1}{3}}(x)=\frac{x+1}{-x+2}.$$

ON THE MATRIX $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$, AND IN CONNEXION THEREWITH THE FUNCTION $\dfrac{ax+b}{cx+d}$.

By Professor *Cayley*.

In the preceding paper the theory of the symbolic powers and roots of the function $\dfrac{ax+b}{cd+x}$ is developed in a complete and satisfactory manner; the results in the main agreeing with those obtained in the original memoir, Babbage, "On Trigonometrical Series," *Memoirs of the Analytical Society* (1813), Note I. pp. 47-50, and which are to some extent reproduced in my "Memoir on the Theory of Matrices," Phil. Trans., t. CXLVIII. (1858), pp. 17-37. I had recently occasion to reconsider the question, and have obtained for the n^{th} function $\phi^n x$, where $\phi x = \dfrac{ax+b}{cx+d}$, a form which although substantially identical with Babbage's, is a more compact and convenient one; viz. taking λ to be determined by the quadric equation

$$\frac{(\lambda+1)^2}{\lambda} = \frac{(a+d)^2}{ad-bc},$$

the form is

$$\phi^n(x) = \frac{(\lambda^{n+1}-1)(ax+b)+(\lambda^n-\lambda)(-dx+b)}{(\lambda^{n+1}-1)(cx+d)+(\lambda^n-\lambda)(\ \ cx-a)}.$$

The question is in effect that of the determination of the n^{th} power of the matrix $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$; viz. in the notation of matrices

$$(x_1, y_1) = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix} (x, y),$$

means the two equations $x_1 = ax + by$, $y_1 = cx + dy$; and then if x_2, y_2 are derived in like manner from x_1, y_1, that is, if $x_2 = ax_1 + by_1$, $y_2 = cx_1 + dy_1$, and so on, x_n, y_n will be linear functions of x, y; say we have $x_n = a_n x + b_n y$, $y_n = c_n x + d_n y$: and the n^{th} power of $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$ is in fact the matrix $\begin{pmatrix} a_n, & b_n \\ c_n, & d_n \end{pmatrix}$.

In particular we have

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}^2, = \begin{pmatrix} a_2, & b_2 \\ c_2, & d_2 \end{pmatrix}, = \begin{pmatrix} a^2+bc, & b(a+d) \\ c(a+d), & d^2+bc \end{pmatrix},$$

and hence the identity

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}^2 - (a+d)\begin{pmatrix} a, & b \\ c, & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix} = 0;$$

viz. this means that the matrix

$$\begin{pmatrix} a_2-(a+d)\,a+ad-bc, & b_2-(a+d)\,b \\ c_2-(a+d)\,c \quad , & d_2-(a+d)\,d+ad-bc \end{pmatrix} = \begin{pmatrix} 0, & 0 \\ 0, & 0 \end{pmatrix},$$

or what is the same thing, that each term of the left-hand matrix is $=0$; which is at once verified by substituting for a_2, b_2, c_2, d_2 their foregoing values.

The explanation just given will make the notation intelligible and show in a general way how a matrix may be worked with in like manner with a single quantity (the theory is more fully developed in my Memoir above referred to); and I proceed with the solution in the algorithm of matrices. Writing for shortness $M = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$, the identity is

$$M^2-(a+d)\,M+(ad-bc)=0,$$

(the matrix $\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$ being in the theory regarded as $=1$); viz. M is determined by a quadric equation; and we have consequently $M^n =$ a linear function of M. Writing this in the form

$$M^n - AM + B = 0,$$

the unknown coefficients A, B can be at once obtained in terms of α, β, the roots of the equation

$$u^2-(a+d)\,u+ad-bc=0,$$

viz. we have

$$\alpha^n - A\alpha + B = 0,$$
$$\beta^n - A\beta + B = 0;$$

or more simply from these equations, and the equation for M^n, eliminating α, β, we have

$$\begin{vmatrix} M^n, & M, & 1 \\ \alpha^n, & \alpha, & 1 \\ \beta^n, & \beta, & 1 \end{vmatrix} = 0;$$

that is $M^n(\alpha-\beta) - M(\alpha^n-\beta^n) + \alpha\beta\,(\alpha^{n-1}-\beta^{n-1}) = 0.$

But instead of α, β it is convenient to introduce the ratio λ of the two roots, say we have $\alpha = \lambda\beta$; we thence find

$$(\lambda+1)\,\beta = a+d,$$
$$\lambda\beta^2 = ad-bc,$$

giving $\frac{(\lambda+1)^2}{\lambda} = \frac{(a+d)^2}{ad-bc}$ for the determination of λ, and then

$$\beta = \frac{a+d}{\lambda+1},$$

$$\alpha = \frac{(a+d)\,\lambda}{\lambda+1}.$$

The equation thus becomes

$$M^n(\lambda-1)\beta - M(\lambda^n-1)\beta^n + (\lambda^n-\lambda)\beta^{n+1} = 0,$$

or we have

$$M^n = \frac{\beta^{n-1}}{\lambda-1}\{(\lambda^n-1)M - (\lambda^n-\lambda)\beta\}.$$

It is convenient to multiply the numerator and denominator by $\lambda+1$, viz. we thus have

$$M^n = \frac{\beta^{n-1}}{\lambda^2-1}[(\lambda^{n+1}-1)M + (\lambda^n-\lambda)\{M-(\lambda+1)\beta\}],$$

the exterior factor is here $= \frac{1}{\lambda^2-1}\left(\frac{a+d}{\lambda+1}\right)^{n-1}$, moreover $(\lambda+1)\beta$ is $= a+d$: hence

$$M = \left(\begin{array}{cc} a, & b \\ c, & d \end{array}\right),$$

and $M-(\lambda+1)\beta = \left(\begin{array}{cc} a, & b \\ c, & d \end{array}\right) - \left(\begin{array}{cc} a+d, & 0 \\ 0, & a+d \end{array}\right), = \left(\begin{array}{cc} -d, & b \\ c, & -a \end{array}\right);$

the formula thus is

$$M^n = \frac{1}{\lambda^2-1}\left(\frac{a+d}{\lambda+1}\right)^{n-1}\left\{(\lambda^{n+1}-1)\left(\begin{array}{cc} a, & b \\ c, & d \end{array}\right) + (\lambda^n-\lambda)\left(\begin{array}{cc} -d, & b \\ c, & -a \end{array}\right)\right\},$$

viz., we have thus the values of the several terms of the nth matrix

$$M^n = \left(\begin{array}{cc} a_n, & b_n \\ c_n, & d_n \end{array}\right);$$

and, if instead of these we consider the combinations a_nx+b_n and c_nx+d_n, we then obtain

$$a_nx+b_n = \frac{1}{\lambda^2-1}\left(\frac{a+d}{\lambda+1}\right)^{n-1}\{(\lambda^{n+1}-1)(ax+b) + (\lambda^n-\lambda)(-dx+b)\},$$

$$c_nx+d_n = \quad ,, \quad ,, \quad \{(\lambda^{n+1}-1)(cx+d) + (\lambda^n-\lambda)(\,cx-a)\};$$

and in dividing the first of these by the second, the exterior factor disappears.

It is to be remarked that $n=0$, the formulæ become as they should do $a_0x+b_0=x$, $c_0x+d_0=1$; and if $n=1$, they become $a_1x+b_1=ax+b$, $c_1x+d_1=cx+d$.

If $\lambda^m-1=0$, where m, the least exponent for which this equation is satisfied, is for the moment taken to be greater than 2, the terms in { } are

$$(\lambda-1)(ax+b)+(1-\lambda)(-dx+b),$$

and
$$(\lambda-1)(cx+d)+(1-\lambda)(\ cx-a);$$

viz. these are $(\lambda-1)(a+d)x$, and $(\lambda-1)(a+d)$, or if for $(\lambda-1)(a+d)$ we write $(\lambda^2-1)\dfrac{a+d}{\lambda+1}$, the formulæ become for $n=m$

$$a_mx+b_m=\left(\frac{a+d}{\lambda+1}\right)x,$$

$$c_mx+d_m=\left(\frac{a+d}{\lambda+1}\right)^m;$$

viz. we have here

$$\frac{a_mx+b_m}{c_mx+d_m}=x,$$

or the function is periodic of the m^{th} order. Writing for shortness $\vartheta=\dfrac{s\pi}{n}$, s being any integer not $=0$, and prime to n, we have $\lambda=\cos 2\vartheta+i\sin 2\vartheta$, hence $1+\lambda=2\cos\vartheta(\cos\vartheta+i\sin\vartheta)$, or $\dfrac{(1+\lambda)^2}{\lambda}=4\cos^2\vartheta$; consequently, in order to the function being periodic of the n^{th} order, the relation between the coefficients is

$$4\cos^2\frac{s\pi}{n}=\frac{(a+d)^2}{ad-bc}.$$

The formula extends to the case $m=2$, viz. $\cos\frac{1}{2}(s\pi)=0$, or the condition is $a+d=0$. But here $\lambda+1=0$, and the case requires to be separately verified; recurring to the original expression for M^2, we see that for $a+d=0$, this becomes $\begin{vmatrix} a^2+bc, & 0 \\ 0\quad, & d^2+bc \end{vmatrix}$, $=(a^2+bc)\begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix}$; that is

$$\frac{a_2x+b_2}{c_2x+d_2}=x,$$

or the result is thus verified.

But the case $m=1$ is a very remarkable one; we have here $\lambda=1$, and the relation between the coefficients is thus $(a+d)^2=4(ad-bc)$, or what is the same thing $(a-d)^2+4bc=0$. And then determining the values for $\lambda=1$ of the vanishing fractions which enter into the formulæ, we find

$$a_n x+b_n=\frac{1}{2^n}(a+d)^{n-1}\{(n+1)(ax+b)+(n-1)(-dx+b)\},$$

$$c_n x+d_n=\frac{1}{2^n}(a+d)^{n-1}\{(n+1)(cx+d)+(n-1)(\quad cx-a)\},$$

or as these may also be written

$$a_n x+b_n=\frac{1}{2^n}(a+d)^{n-1}\{x[n(a-d)+(a+d)]+2nb\},$$

$$c_n x+d_n=\frac{1}{2^n}(a+d)^{n-1}\{x\,.\,2nc+[-n(a-d)+a+d]\},$$

which for $n=0$, become as they should do $a_0x+b_0=x$, $c_0x+d_0=1$, and for $n=1$ they become $a_1x+b_1=ax+b$, $c_1x+d_1=cx+d$. We thus do *not* have $\frac{a_1x+b_1}{c_1x+d_1}=x$, and the function is *not* periodic of any order. This remarkable case is noticed by Mr. Moulton in his edition (2nd edition, 1872) of Boole's *Finite Differences*.

If to satisfy the given relation $(a-d)^2+4bc=0$, we write $2b=k(a-d)$, $2c=-\frac{1}{k}(a-d)$, then the function of x is

$$\frac{ax+\frac{1}{2}k(a-d)}{-\frac{1}{2}k^{-1}(a-d)x+d}$$

and the formulæ for the n^{th} function are

$$a_n x+b_n=\frac{1}{2^n}(a+d)^{n-}\ \{(a+d)x+n(a-d)(x+k)\},$$

$$c_n x+d_n=\frac{1}{2^n}(a+d)^{n-1}\left\{(a+d)\quad -n(a-d)\left(\frac{x}{k}+1\right)\right\};$$

which may be verified successively for the different values of n.

Reverting to the general case, suppose $n=\infty$, and let u be the value of $\phi^\infty(x)$. Supposing that the modulus of λ is not $=1$, we have λ^n indefinitely large or indefinitely small. In the former case we obtain

$$u=\frac{\lambda(ax+b)+(-dx+b)}{\lambda(cx+d)+(\quad cx-a)},\quad =\frac{(\lambda a-d)x+b(\lambda+1)}{c(\lambda+1)x+\lambda d-a};$$

which, observing that the equation in λ may be written

$$\frac{\lambda a - d}{c(\lambda + 1)} = \frac{b(\lambda + 1)}{\lambda d - a},$$

is independent of x, and equal to either of these equal quantities; and if from these two values of u we eliminate λ, we obtain for u the quadric equation

$$cu^2 - (a - d)u - b = 0,$$

that is

$$u = \frac{au + b}{cu + d},$$

as is, in fact, obvious from the consideration that n being indefinitely large the n^{th} and $(n+1)^{th}$ functions must be equal to each other. In the latter case, λ^n indefinitely small, we have the like formulæ, and we obtain for u the same quadric equation: the two values of u are however not the same, but (as is easily shown) their product is $= -b \div c$; u is therefore the other root of the quadric equation. Hence, as n increases, the function $\phi^n x$ continually approximates to one or the other of the roots of this quadric equation. The equation has equal roots if $(a-d)^2 + 4bc = 0$, which is the relation existing in the above-mentioned special case; and here $u = \frac{1}{2c}(a - d), = \frac{-2b}{a-d}$, which result is also given by the formulæ of the special case on writing therein $n = \infty$.

NOTE ON THE FUNCTION $\phi(x) = \frac{ax + b}{cx + d}$.

By *H. W. Lloyd Tanner, M.A.*

If

$$\phi(t) = \frac{at + b}{ct + d},$$

$$\phi^2(t) = \frac{a\phi(t) + b}{c\phi(t) + d} = \frac{(a^2 + bc)t + (a + d)b}{(a + d)ct + bc + d^2}$$

$$= \frac{\dfrac{a^2 + bc}{a + d}t + b}{ct + \dfrac{bc + d^2}{a + d}}$$

$$= \frac{a_2 t + b}{ct + d},$$

where $$a_2=\frac{a^2+bc}{a+d},\quad d_2=\frac{d^2+bc}{a+d};$$

also $$a_2-a=d_2-d=\frac{bc-ad}{a+d}=(k_2 \text{ say}),$$

so that we may write

$$\phi^2(t)=\frac{at+b+k_2t}{ct+d+k_2}.$$

This holds generally; in other words

$$\phi^n(t)\equiv\frac{at+b+k_nt}{ct+d+k_n}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

To prove this, and at the same time find the form of k_n, we assume the theorem to be true for n, and thus obtain

$$\phi^{n+1}(t)=\frac{a\phi^n(t)+b}{c\phi^n(t)+d},$$

which, upon reduction, becomes

$$=\frac{at+b+\dfrac{bc-ad}{a+d+k_n}t}{ct+d+\dfrac{bc-ad}{a+d+k_n}},$$

this proves the law and gives

$$k_{n+1}=\frac{bc-ad}{a+d+k_n}.$$

If k_n be replaced by $h_n\div h_{n+1}$, this equation becomes

$$(bc-ad)\,h_{n+2}-(a+d)\,h_{n+1}-h_n=0,$$

so that $$h_n=C_1p^n+C_2q^n,$$

p, q being the roots of

$$(bc-ad)\,x^2-(a+d)\,x-1=0\ldots\ldots\ldots\ldots(a).$$

Thus $$k_n=\frac{p^n+Cq^n}{p^{n+1}+Cq^{n+1}};$$

but $C=-\dfrac{p}{q}$, because $k_1=0$; so that

$$k_n=\frac{p^{n-1}-q^{n-1}}{p^n-q^n}.$$

It is perhaps more convenient to write this

$$\left.\begin{array}{l} k_n = \dfrac{p^n q - p q^n}{p^n - q^n} \\ p, q \text{ now being the roots of} \\ x^2 + (a+d)\,x + ad - bc = 0 \end{array}\right\} \quad\ldots\ldots\ldots\ldots (2),$$

The above process fails when (a) has equal roots. In this case we have

$$\left.\begin{array}{l} (a+d)^2 = 4\,(ad-bc) \\ (a-d)^2 + 4bc = 0 \end{array}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

or

showing that b, c are of opposite signs. The solution of the difference equation is

$$h_n = (C_0 + C_1 n)\,p^n,\ p = \frac{a+d}{2\,(bc-ad)} = -\frac{2}{a+d},$$

and this easily leads to the result

$$k_n = -\frac{n-1}{2n}(a+d) \ldots\ldots\ldots\ldots\ldots (4).$$

The above discussion renders easy the determination of the conditions that ϕ should be periodic. Say the function is periodic of the m^{th} order, so that m is the smallest number for which the equations

$$\phi^m(t) = t,\ \phi^{m+1}(t) = \phi(t) \ldots\ldots\ldots\ldots\ldots (b)$$

are satisfied. These imply $k_m = \infty$ or $k_{m+1} = 0$, either of which gives

$$p^m - q^m = 0.$$

Hence, p, q is a primitive m^{th} root of unity; primitive, because otherwise m would not be the smallest number for which (b) are satisfied. Hence

$$\frac{p}{q} = \cos\frac{2r\pi}{m} + i\sin\frac{2r\pi}{m},$$

where r is prime to m, and different from zero; therefore

$$\frac{p}{q} + \frac{q}{p} = 2\cos\frac{2r\pi}{m}.$$

But, from (2),

$$\frac{p}{q}+\frac{q}{p}=\frac{(a+d)^2-2(ad-bc)}{ad-bc};$$

therefore
$$\cos\frac{2r\pi}{m}=\frac{a^2+d^2+2bc}{2(ad-bc)},$$

and
$$\frac{r}{m}=\frac{1}{2\pi}\cos^{-1}\frac{a^2+d^2+2bc}{2(ad-bc)}.$$

If the right-hand member is not commensurable the function is not periodic. When it is a commensurable quantity, form the fraction in lowest terms which is equal to it. The denominator of this fraction is the index of the periodicity of the function. Notice that the indeterminateness of value of $\cos^{-1}$... does not affect m, and that when

$$a^2+d^2+2bc=2ad-2bc,$$

the right side vanishes, so that $r=0$, and m is indeterminate. Indeed, in this case, the method is not legitimate, since r is assumed to be different from zero.

In the exceptional case just mentioned, we have, in fact, once more encountered the case in which (a) has equal roots, and the proper form for k_n is

$$-\frac{n-1}{2n}(a+d).$$

The equations $k_m=\infty$, $k_{m+1}=0$ give $m=0$, so that the function is not periodic when (3) is satisfied.

The supposition $a+d=0$, leads to $\phi(t)=\text{const.}$, and does not require further examination.

These results agree in substance with those given in Boole's *Finite Differences* (2nd edition), pp. 295-300, but the form is different.

August, 1879.

NOTES ON HYDRODYNAMICS.

By *A. G. Greenhill, M.A.*

I. *On Lord Rayleigh's paper on the irregular flight of a tennis-ball* (vol. VII. p. 14).

SUPPOSE a cylinder (fig. 2), centre C, and radius a, rotating with angular velocity ω, to have generated a vortex in the infinite surrounding liquid; then the velocity function $\phi = a^2\omega\theta$ at any point P, if there is no slipping of the liquid at the surface of the cylinder.

Suppose now the cylinder to have component velocities u and v parallel to the axes, then the velocity function

$$\phi = a^2\omega\theta - \frac{a^2}{r}(u\cos\theta + v\sin\theta),$$

or if x, y be the coordinates of P,

α, β .... C,

$$\phi = a^2\omega \tan^{-1}\frac{y-\beta}{x-\alpha} - a^2\frac{u(x-\alpha)+v(y-\beta)}{(x-\alpha)^2+(y-\beta)^2};$$

and therefore, since

$$\frac{d\alpha}{dt} = u, \quad \frac{d\beta}{dt} = v,$$

$$\begin{aligned}\frac{d\phi}{dt} = \quad & a^2\omega\frac{-v(x-\alpha)+u(y-\beta)}{(x-\alpha)^2+(y-\beta)^2} \\ & - a^2\frac{\frac{du}{dt}(x-\alpha)+\frac{dv}{dt}(y-\beta)}{(x-\alpha)^2+(y-\beta)^2} \\ & + a^2\frac{u^2+v^2}{(x-\alpha)^2+(y-\beta)^2} \\ & - a^2\left\{\frac{u(x-\alpha)+v(y-\beta)}{(x-\alpha)^2+(y-\beta)^2}\right\}^2\end{aligned}$$

or
$$\frac{d\phi}{dt} = \frac{a^2\omega}{r}(u\sin\theta - v\cos\theta)$$
$$-\frac{a^2}{r}\left(\frac{du}{dt}\cos\theta + \frac{dv}{dt}\sin\theta\right)$$
$$+\frac{a^2}{r^2}(u^2+v^2)$$
$$-\frac{a^2}{r^2}(u\cos\theta + v\sin\theta)^2;$$

and therefore, when $r=a$,
$$\frac{d\phi}{dt} = a\omega(u\sin\theta - v\cos\theta)$$
$$-a\left(\frac{du}{dt}\cos\theta + \frac{dv}{dt}\sin\theta\right)$$
$$+u^2+v^2-(u\cos\theta+v\sin\theta)^2.$$

Also, when $r=a$,
$$\frac{d\phi}{dr} = u\cos\theta + v\sin\theta,$$
$$\frac{d\phi}{rd\theta} = a\omega + u\sin\theta - v\cos\theta.$$

Therefore
$$\tfrac{1}{2}q^2 = \tfrac{1}{2}a^2\omega^2 + a\omega(u\sin\theta - v\cos\theta) + \tfrac{1}{2}(u^2+v^2).$$

The dynamical equation is
$$\frac{p}{\rho} + \frac{d\phi}{dt} + \tfrac{1}{2}q^2 + gy = H,$$

and therefore, when $r=a$,
$$\frac{p}{\rho} + 2a\omega(u\sin\theta - v\cos\theta) - a\left(\frac{du}{dt}\cos\theta + \frac{dv}{dt}\sin\theta\right)$$
$$-(u\cos\theta+v\sin\theta)^2 + \tfrac{3}{2}(u^2+v^2) + \tfrac{1}{2}a^2\omega^2$$
$$+ga\sin\theta + g\beta = H.$$

Let X, Y denote the components parallel to the axes of the resultant fluid pressure on the cylinder per unit of length of the cylinder:
$$X = -\int_0^{2\pi} p\cos\theta\, ad\theta, \quad Y = -\int^{2\pi} p\sin\theta\, ad\theta;$$

and, therefore, omitting terms which vanish when integrated from 0 to 2π,

$$\begin{aligned} X = \; & \rho \int_0^{2\pi} \{2a^2\omega (u \sin\theta - v \cos\theta) \cos\theta \\ & - a^2 \left(\frac{du}{dt} \cos\theta + \frac{dv}{dt} \sin\theta\right) \cos\theta \\ & - a (u \cos\theta + v \sin\theta)^2 \cos\theta \\ & + g a^2 \sin\theta \cos\theta\} \, d\theta \\ = & - 2\pi\rho a^2 \omega v - \pi\rho a^2 \frac{du}{dt}, \end{aligned}$$

and $$Y = 2\pi\rho a^2 \omega u - \pi\rho a^2 \frac{dv}{dt} + \pi\rho g a^2.$$

Therefore, if σ denote the density of the cylinder, the equations of motion of the cylinder are

$$\begin{aligned} \pi\sigma a^2 \frac{du}{dt} = \; & X \\ = & - 2\pi\rho a^2 \omega v - \pi\rho a^2 \frac{du}{dt}, \end{aligned}$$

or $$(\sigma + \rho) \frac{du}{dt} + 2\rho\omega v = 0 \text{ ....} (1);$$

and $$\begin{aligned} \pi\sigma a^2 \frac{dv}{dt} = \; & Y - \pi\sigma g a^2 \\ = \; & 2\pi\rho a^2 \omega u - \pi\rho a^2 \frac{dv}{dt} \\ & - \pi (\sigma - \rho) g a^2, \end{aligned}$$

or $$(\sigma + \rho) \frac{dv}{dt} - 2\rho\omega u + (\sigma - \rho) g = 0 \text{ ....} (2).$$

(1) Neglecting gravity, by putting $g = 0$; therefore

$$(\sigma + \rho) \frac{du}{dt} + 2\rho\omega v = 0,$$

and $$(\sigma + \rho) \frac{dv}{dt} - 2\rho\omega v = 0;$$

therefore $$u = - V \sin \frac{2\rho}{\sigma + \rho} \omega t,$$

$$v = V \cos \frac{2\rho}{\sigma + \rho} \omega t;$$

and
$$\alpha = c \cos \frac{2\rho}{\sigma+\rho}\omega t,$$
$$\beta = c \sin \frac{2\rho}{\sigma+\rho}\omega t;$$
and therefore the cylinder describes a circle with the *same* direction of revolution as the circulation of the liquid.

(2) Restoring g,
$$u = \frac{\sigma-\rho}{2\rho}\frac{g}{\omega} - V \sin \frac{2\rho}{\sigma+\rho}\omega t,$$
$$v = V \cos \frac{2\rho}{\sigma+\rho}\omega t;$$
therefore
$$\alpha = \frac{\sigma-\rho}{2\rho}\frac{gt}{\omega} + c \cos \frac{2\rho}{\sigma+\rho}\omega t,$$
$$\beta = c \sin \frac{2\rho}{\sigma+\rho}\omega t;$$
and therefore the cylinder describes a trochoid, moving from right to left with mean velocity $\frac{\sigma-\rho}{2\rho}\frac{g}{\omega}$.

(With an attraction to the centre proportional to the distance the differential equations of motion give the path of the centre of the cylinder a bicycloid.)

It is found that elongated projectiles with pointed heads, fired from guns rifled with a right-handed screw, deviate to the right of the vertical plane of fire, as seen from behind the gun.

This is in the opposite direction to that which is given by the above theory, namely, that the revolution of the projectile generates a vortex, and that differences of pressure arise from differences of velocity around the body.

Prof. Magnus, in his paper "On the Deviation of Projectiles," accordingly seeks for another explanation of the deviation of projectiles to the right of the vertical plane of fire.

He first proves experimentally that, with an elongated pointed projectile, the resultant pressure due to the air, when the projectile is moving very nearly in the direction of the axis, acts in a line which cuts the axis in a point (called the centre of effort) between the centre of gravity and the point of the projectile.

He also observed that, when the velocity was small enough for the eye to follow the motion, the axis of the projectile coincided very nearly with the tangent to the trajectory of the centre of gravity, but that the point of the projectile was a little higher and to the right of what it would have been had the axis been exactly tangential.

Consequently the resultant pressure of the air would be equivalent to a single force through the centre of gravity, called the "resistance of the air," and a couple, having one component, whose axis is horizontal and on the right-handed system, drawn to the right of the vertical plane of fire, arising in consequence of the point of the projectile being a little above the tangent to the trajectory; and another component whose axis is normal to the trajectory, arising in consequence of the point of the projectile being a little to the right of the tangent.

If we neglect the deviation of the axis of resultant angular momentum from the axis of figure, the first component couple will keep deflecting the point of the projectile to the right, and curve the path of the projectile to the right also; and the other component of the couple will be that required to keep depressing the point of the projectile so that it lies nearly in the tangent to the trajectory.

If the centre of effort had been behind the centre of gravity, the point would have to be deflected to the left in order that the point should also descend very nearly into the tangent to the trajectory, and the projectile would deflect to the left; it is asserted that this is the case with flat-headed projectiles.

II. *On the motion of a cylinder through frictionless liquid under no forces.*

In § 117 of Lamb's *Motion of Fluids* the equations of motion are

$$A\frac{du}{dt}=rBv,\quad B\frac{dv}{dt}=-rAu,$$

and
$$R\frac{dr}{dt}=(A-B)\,uv,$$

similar to Euler's equations of motion of a rigid body about a fixed point under no forces.

If I denote the resultant impulse of the motion, then the integrals of the first two equations are

$$Au = I\cos\theta \text{.......................}(1),$$

$$Bv = -I\sin\theta \text{.................. ....}(2),$$

where $r = \dot\theta$; and therefore,

$$R\ddot\theta + I^2\left(\frac{1}{B} - \frac{1}{A}\right)\sin\theta\cos\theta = 0 \text{ ....}(3).$$

Suppose $A > B$, and (1) suppose the cylinder to oscillate; then the solution of equation (3) is

$$\sin\theta = k\,\mathrm{sn}\,K\frac{t}{T} \text{.....................}(4),$$

when $k = \sin\alpha$, α being the extreme value of θ; and

$$\frac{K^2}{T^2} = \left(\frac{1}{B} - \frac{1}{A}\right)I^2.$$

Then
$$\begin{aligned}
\dot x &= u\cos\theta - v\sin\theta \\
&= I\frac{\cos^2\theta}{A} + I\frac{\sin^2\theta}{B} \\
&= \frac{I}{B} - I\left(\frac{1}{B} - \frac{1}{A}\right)\mathrm{dn}^2 K\frac{t}{T}, \\
\dot y &= u\sin\theta + v\cos\theta \\
&= -I\left(\frac{1}{B} - \frac{1}{A}\right)\sin\theta\cos\theta \\
&= -I\left(\frac{1}{B} - \frac{1}{A}\right)k\,\mathrm{sn}\,K\frac{t}{T}\,\mathrm{dn}\,K\frac{t}{T};
\end{aligned}$$

and therefore,

$$x = \frac{I}{B}t - I\left(\frac{1}{B} - \frac{1}{A}\right)\frac{T}{K}E\left(K\frac{t}{T}\right),$$

$$y = I\left(\frac{1}{B} - \frac{1}{A}\right)\frac{T}{K}k\,\mathrm{cn}\,K\frac{t}{T};$$

or putting
$$I\left(\frac{1}{B} - \frac{1}{A}\right)\frac{T}{K}k = c,$$

$$y = c\,\mathrm{cn}\,K\frac{t}{T},$$

and
$$x = \frac{IT}{BK}\mathrm{cn}^{-1}\frac{y}{c} - I\left(\frac{1}{B} - \frac{1}{A}\right)\frac{T}{K}E\left(\mathrm{cn}^{-1}\frac{y}{c}\right),$$

the equation of the curve described by the centre of the cylinder (fig. 3).

The time $2T$ of a small oscillation is obtained by putting $k=0$, $K=\frac{1}{2}\pi$; and supposing U the velocity of projection in the direction A of greatest effective inertia,

$$I=AU,$$

and
$$\frac{\pi^2}{4T^2}=\left(\frac{1}{B}-\frac{1}{A}\right)A^2U^2;$$

or
$$2T=\frac{\pi}{U}\sqrt{\left\{\frac{B}{A(A-B)}\right\}}.$$

(2). Suppose the cylinder to make complete revolutions; then the solution of equation (3) is

$$\theta=\text{am}\,K\frac{t}{T},$$

where $\frac{K}{T}=$ maximum angular velocity of the cylinder,

and
$$k^2\frac{K^2}{T^2}=\left(\frac{1}{B}-\frac{1}{A}\right)I^2.$$

Then

$$\dot{x}=\frac{I}{A}\text{cn}^2K\frac{t}{T}+\frac{I}{B}\text{sn}^2K\frac{t}{T}$$
$$=I\left(\frac{1}{B}-\frac{k'^2}{A}\right)\frac{1}{k^2}-I\left(\frac{1}{B}-\frac{1}{A}\right)\frac{1}{k^2}\text{dn}^2K\frac{t}{T};$$
$$\dot{y}=-I\left(\frac{1}{B}-\frac{1}{A}\right)\text{sn}\,K\frac{t}{T}\,\text{cn}\,K\frac{t}{T};$$

and therefore,

$$x=I\left(\frac{1}{B}-\frac{k'^2}{A}\right)\frac{t}{k^2}-I\left(\frac{1}{B}-\frac{1}{A}\right)\frac{T}{k^2K}E\left(K\frac{t}{T}\right),$$
$$y=I\left(\frac{1}{B}-\frac{1}{A}\right)\frac{T}{k^2K}\text{dn}\,K\frac{t}{T};$$

or putting
$$I\left(\frac{1}{B}-\frac{1}{A}\right)\frac{T}{k^2K}=c,$$

$$y=c\,\text{dn}\,K\frac{t}{T},$$

and
$$x=I\left(\frac{1}{B}-\frac{k'^2}{A}\right)\frac{T}{k^2K}\text{dn}^{-1}\frac{y}{c}$$
$$-I\left(\frac{1}{B}-\frac{1}{A}\right)\frac{T}{k^2K}E\left(\text{dn}^{-1}\frac{y}{c}\right),$$

the equation of the curve described (fig. 4).

In the separating case between cases (1) and (2), we have $k=1$, and the solution of (3) is

$$R\dot{\theta}^2 = I^2\left(\frac{1}{B}-\frac{1}{A}\right)\cos^2\theta$$

or
$$\dot{\theta} = \omega\cos\theta,$$

where
$$\omega^2 = \frac{I^2}{R}\left(\frac{1}{B}-\frac{1}{A}\right).$$

Therefore
$$y = c\cos\theta,$$

where
$$c = \frac{R}{I}\omega,$$

and
$$\dot{x} = I\frac{\cos^2\theta}{A} + I\frac{\sin^2\theta}{B},$$

therefore
$$\frac{dx}{d\theta} = \frac{I}{B\omega}\sec\theta - \frac{I}{\omega}\left(\frac{1}{B}-\frac{1}{A}\right)\cos\theta,$$

$$x = \frac{I}{B\omega}\log\sqrt{\left(\frac{1+\sin\theta}{1-\sin\theta}\right)} - \frac{I}{\omega}\left(\frac{1}{B}-\frac{1}{A}\right)\sin\theta$$

$$= \frac{I}{2B\omega}\log\frac{c+\sqrt{(c^2-y^2)}}{c-\sqrt{(c^2-y^2)}} - \frac{I}{\omega}\left(\frac{1}{B}-\frac{1}{A}\right)\frac{\sqrt{(c^2-y^2)}}{c},$$

the equation of the curve described in the separating case (fig. 5). This is the curve which would be described by the cylinder if originally projected in the direction of its length with velocity $\frac{I}{B}$, and indefinitely slightly displaced.

In an elliptic cylinder of mass M, density σ, and semi-axes a and b, moving in infinite liquid of density ρ,

$$A = M\left(1+\frac{\rho}{\sigma}\frac{b}{a}\right),$$

$$B = M\left(1+\frac{\rho}{\sigma}\frac{a}{b}\right).$$

$$R = M\tfrac{1}{4}(a^2+b^2)\left\{1+\frac{\rho}{\sigma}\frac{(a^2-b^2)^2}{2ab(a^2+b^2)}\right\}.$$

MATHEMATICAL NOTES.

Extension of Leibnitz's theorem in statics.

I have never seen the following extension of Leibnitz's theorem in statics noticed.

If a number of forces AO, BO, CO, &c. meeting at a point O, are in equilibrium, Leibnitz's theorem is that O is the centre of gravity of a system of equal particles at A, B, C, &c.

Now if any system of forces Aa, Bb, Cc, &c. are in equilibrium; a, b, c, &c. being their points of application, the centre of gravity of the points A, B, C, &c. must coincide with the centre of gravity of a, b, c, &c.

For let G be the centre of gravity of a, b, c..., then the system of forces aG, bG, cG... are in equilibrium; hence this system combined with the given one makes equilibrium; now the resultant of Aa and aG is AG, in magnitude and direction (though its line of action of course passes through a). Hence AG, BG, CG... must make equilibrium at G, and therefore G must be the centre of gravity of A, B, C..., by Leibnitz's theorem.

The *converse* is, that if the two centres of gravity coincide, the forces (if not in equilibrium) are equivalent to a couple.

M. W. CROFTON.

On duplication of results in maxima and minima.

When we have obtained a solution of the problem to make one of two given associated magnitudes a maximum or minimum, keeping the other constant, we are sometimes in doubt whether there is a corresponding solution of the problem, keeping the first constant to make the second a maximum or minimum, and if so, whether the singularity in the second problem is of the same kind as in the first, or of the opposite. The following simple theorem decides the question in a large class of cases.

Let the sizes of two connected varying magnitudes, expressed each in terms of any fixed unit magnitude of its own kind, be A, B; and suppose that while the size A is kept fixed at A', the greatest (or least) size which B can assume is B'. Then if the ratio $B' : A'$ is a constant number for all values of A', it shall be also true that if B retain the same fixed size B', the least (or greatest) size which A can assume is A'.

For, if possible (to be clear, expressing for the first case only), let a less A'' be assumed; then fixing A at A'' and letting B vary the greatest size which it can assume is $\frac{B'}{A'}A''$, that is to say, $B'\frac{A''}{A'}$, which is less than B'. Thus the supposition that $A=A'$ and $B=B'$ simultaneously is impossible, an absurdity. So, B being fixed at B', A cannot be less than A'; but it can equal A' by supposition. Thus A' is its least possible size.

In precisely the same way, if $B' : A'$ be not constant for all, but only for a limited range of values of A', then, B being kept fixed at B', A' is the least size among this limited range that A can assume.

As examples I write below two familiar duplications.

Of tetrahedra inscribed in a given ellipsoid one of maximum volume has its centre of gravity at the centre of that ellipsoid; and the ratio of the volume of such a maximum tetrahedron to that of the ellipsoid is constant, and equal to $2 : 3\pi\sqrt{3}$.

Of ellipsoids circumscribed to a given tetrahedron, one of minimum volume has that volume in a constant ratio $3\pi\sqrt{3} : 2$ to the volume of the tetrahedron. Consequently an ellipsoid whose centre is at the centre of gravity of the tetrahedron is such a minimum ellipsoid.

Of all closed curves having a given perimeter the circle is the one that has the greatest area.

Of all closed curves having a given area the circle has the least perimeter.

For in the last, taking square of perimeter for A and area for B, the ratio $B' : A' = 1 : 4\pi$, and is constant.

E. B. Elliott.

Note on Euclid II. 12, 13.

The following is a method of deducing the results of the forty-seventh Proposition of the first Book of Euclid's Elements, and those of the analogous theorems in the twelfth and thirteenth Propositions of the second Book, from the proportionality of the sides of similar triangles.

In any triangle ABC (fig. 6) let lines CD, CE be drawn from the angular point C to the side AB, making the angles CDA, CEB each equal to the angle at C.

Then the triangles ADC, CEB are similar to each other and to the triangle ACB.

Wherefore

$$\frac{AD}{AC}=\frac{AC}{AB}, \quad \frac{CD}{AC}=\frac{BC}{AB},$$

and

$$\frac{BE}{BC}=\frac{BC}{BA}, \quad \frac{CE}{CB}=\frac{AC}{AB},$$

or if we write a, b, c for the sides BC, CA, AB respectively,

$$AD=\frac{b^2}{c}, \quad BE=\frac{a^2}{c},$$

and

$$CD=CE=\frac{ab}{c}.$$

Now it is clear that D and E coincide, if C be a right angle; and that AD, BE overlap, if C be an acute angle, and fall short of each other, if C be an obtuse angle. We have, therefore, in the three cases,

$$AD+BE=AC,$$
$$AD+BE=AC+DE,$$
$$AD+BE+DE=AC,$$

or

$$b^2+a^2=c^2 \text{ in the first case}$$

$$DE=\frac{b^2+a^2-c^2}{c} \text{ in the second case,}$$

$$DE=\frac{c^2-a^2-b^2}{c} \text{ in the third case.}$$

Now if CF be drawn perpendicular to AB, it will bisect DE, and if BH be drawn perpendicular to CA, the two triangles CEF, BCH will be similar; therefore

$$\frac{CE}{EF}=\frac{CB}{CH},$$

or

$$DE=2EF=2\,\frac{CE.CH}{CB}=2\,\frac{AE.CH}{DB}$$
$$=\frac{2\,b\,CH}{c};$$

we have, therefore, in the case of C being acute or obtuse,

$$2\,b\,CH=a^2+b^2-c^2,$$
$$2\,b\,CH=c^2-a^2-b^2,$$

which are the results of Euclid, Book II. Proposition 12 and 13.

H. M. Taylor.

A trigonometrical identity.

The identity is

$$\sin\beta\sin\gamma\sin(\beta-\gamma)\{\sin^2\beta+\sin^2\gamma+\sin^2(\beta-\gamma)\}$$
$$+\sin\gamma\sin\alpha\sin(\gamma-\alpha)\{\sin^2\gamma+\sin^2\alpha+\sin^2(\gamma-\alpha)\}$$
$$+\sin\alpha\sin\beta\sin(\alpha-\beta)\{\sin^2\alpha+\sin^2\beta+\sin^2(\alpha-\beta)\}$$
$$+\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)$$
$$\times\{\sin^2(\beta-\gamma)+\sin^2(\gamma-\alpha)+\sin^2(\alpha-\beta)\}=0.$$

J. W. L. GLAISHER.

On an elementary integral.

If $$\frac{d\theta}{1+e\cos\theta}\equiv\frac{d\phi}{\sqrt{(1-e^2)}}\;\ldots\ldots\ldots\ldots\ldots\;(1).$$

Then it follows that

$$\tfrac{1}{2}\phi\equiv\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\tfrac{1}{2}\theta\right\};$$

and therefore $$\tan\tfrac{1}{2}\phi\equiv\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\tfrac{1}{2}\theta;$$

and therefore $$\tan\tfrac{1}{2}\theta\equiv\sqrt{\left(\frac{1+e}{1-e}\right)}\tan\tfrac{1}{2}\phi;$$

and therefore $$\tfrac{1}{2}\theta\equiv\tan^{-1}\left\{\sqrt{\left(\frac{1+e}{1-e}\right)}\tan\tfrac{1}{2}\phi\right\};$$

and therefore $$\frac{d\phi}{1-e\cos\phi}\equiv\frac{d\theta}{\sqrt{(1-e^2)}}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(2).$$

Hence $$(1+e\cos\theta)(1-e\cos\phi)\equiv(1-e^2),$$

and $$\frac{d\theta}{(1+e\cos\theta)^{n+1}}\equiv\frac{d\theta}{1+e\cos\theta}\cdot\left(\frac{1-e\cos\phi}{1-e^2}\right)^n$$
$$\equiv\frac{d\phi\,(1-e\cos\phi)^n}{(1-e^2)^{n+\frac{1}{2}}}.$$

R. R. WEBB.

On Legendre's coefficients.

If $\frac{1}{\sqrt{(1-2\mu h+h^2)}} \equiv 1 + P_1 h + P_2 h^2 + \ldots P_n h^n \ldots$ then P_n is defined to be Legendre's n^{th} coefficient.

Let $\qquad v \equiv 1 - \mu h, \quad V \equiv \frac{1}{\sqrt{(1-2\mu h+h^2)}}$.

The following identities are obvious:

$$\frac{\delta V}{\delta \mu} \equiv h V^3 \quad \text{(i)},$$

$$\frac{\delta V}{\delta h} \equiv (\mu - h) V^3 \quad \text{(ii)},$$

$$\frac{\delta}{\delta \mu}(Vv) \equiv h^2 \frac{\delta V}{\delta h} \quad \text{(iii)},$$

$$\frac{\delta}{\delta h}(Vv) \equiv -(1-\mu^2) \frac{\delta V}{\delta \mu} \quad \text{(iv)}.$$

Hence (iii) and (iv) give

$$-\frac{\delta^2}{\delta\mu\delta h}(Vv) \equiv +\frac{\delta}{\delta\mu}\left\{(1-\mu^2)\frac{\delta V}{\delta h}\right\} \equiv -\frac{\delta}{\delta h}\left(h^2\frac{\delta V}{\delta h}\right),$$

or $$\frac{\delta}{\delta\mu}\left\{(1-\mu^2)\frac{\delta V}{\delta\mu}\right\} + \frac{\delta}{\delta h}\left(h^2\frac{\delta V}{\delta h}\right) \equiv 0.$$

Hence, equating to zero the coefficient of h^n there results the well-known identity

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP_n}{d\mu}\right\} + n(n+1)P_n \equiv 0.$$

Next the following identities are obvious:

If $U \equiv V^m$ and $V \equiv \frac{1}{\sqrt{(1-2\mu h+h^2)}}$, $v = 1 - \mu h$ as before,

then $$\frac{\delta U}{\delta \mu} \equiv mhV^{m+2},$$

$$\frac{\delta U}{\delta h} \equiv m(\mu - h)V^{m+2};$$

therefore
$$h\frac{\delta U}{\delta h} \equiv \mu\frac{\delta U}{\delta\mu} - h\frac{\delta U}{\delta h} \quad\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)},$$

also
$$\frac{\delta}{\delta\mu}(Uv^m) \equiv v^{m-1}h^2\frac{\delta U}{\delta h} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)},$$

$$\frac{\delta}{\delta h}(Uv^m) \equiv -v^{m-1}(1-\mu^2)\frac{\delta U}{\delta\mu} \quad\ldots\ldots\ldots \text{(iii)}.$$

Hence, just as before,

$$-\frac{\delta^2}{\delta\mu\delta h}(Uv^m)$$

$$\equiv v^{m-1}\frac{\delta}{\delta\mu}\left\{(1-\mu^2)\frac{\delta U}{\delta\mu}\right\} - (m-1)\left\{v^{m-1}\mu\frac{\delta U}{\delta\mu} - v^{m-2}h\frac{\delta U}{\delta h}\right\}$$

$$\equiv -v^{m-1}\frac{\delta}{\delta h}\left\{h^2\frac{\delta U}{\delta h}\right\} + (m-1)\left\{v^{m-2}(1-v)h\frac{\delta U}{\delta h}\right\};$$

therefore
$$v^{m-1}\left[\frac{\delta}{\delta\mu}\left\{(1-\mu^2)\frac{\delta U}{\delta\mu}\right\} + \frac{\delta}{\delta h}\left(h^2\frac{\delta U}{\delta h}\right)\right]$$
$$-(m-1)v^{m-1}\left\{\mu\frac{\delta U}{\delta\mu} - h\frac{\delta U}{\delta h}\right\} \equiv 0,$$

or
$$\left[\frac{\delta}{\delta\mu}\left\{(1-\mu^2)\frac{\delta U}{\delta\mu}\right\} - (m-1)\mu\frac{\delta U}{\delta\mu}\right]$$
$$+\left[\frac{\delta}{\delta h}\left(h^2\frac{\delta U}{\delta h}\right) + (m-1)h\frac{\delta U}{\delta h}\right] \equiv 0,$$

or
$$\frac{\delta}{\delta\mu}\left\{h^{m-1}(1-\mu^2)^{\frac{m+1}{2}}\frac{\delta U}{\delta\mu}\right\} + \frac{\delta}{\delta h}\left\{h^{m+1}(1-\mu^2)^{\frac{m-1}{2}}\frac{\delta U}{\delta h}\right\} \equiv 0,$$

equating to zero the coefficient of h^n we have in the present case

$$\frac{d}{d\mu}\left\{(1-\mu^2)^{\frac{m+1}{2}}\frac{dQ_n}{d\mu}\right\} + (m+n+1)(m+n)(1-\mu^2)^{\frac{m-1}{2}}Q_n \equiv 0,$$

where
$$\frac{1}{(1-2\mu h+h^2)^{\frac{1}{2}m}} \equiv 1 + \ldots Q_n h^n \ldots$$

R. R. WEBB.

Theorem in elliptic functions.

The three equations considered by Prof. Cayley, pp. 17, 18 of vol. VII. (June, 1877), viz.

$$\frac{x^2}{x^2 - w^2} = \frac{y+z}{mw^2 - nyz},$$

$$\frac{y}{y^2 - w^2} = \frac{z+x}{mw^2 - nzx},$$

$$\frac{z}{z^2 - w^2} = \frac{x+y}{mw^2 - nxy},$$

which are such that any two of them imply the third, may be expressed in the following elliptic-function form:

$$\mathrm{dn}\,(\beta+\gamma)\,\mathrm{dn}\,(\beta-\gamma) - \mathrm{cn}\,(\beta+\gamma)\,\mathrm{cn}(\beta-\gamma) = \lambda\frac{\mathrm{sn}^2\alpha}{1-\mathrm{sn}^4\alpha},$$

$$\mathrm{dn}\,(\gamma+\alpha)\,\mathrm{dn}\,(\gamma-\alpha) - \mathrm{cn}\,(\gamma+\alpha)\,\mathrm{cn}\,(\gamma-\alpha) = \lambda\frac{\mathrm{sn}^2\beta}{1-\mathrm{sn}^4\beta},$$

$$\mathrm{dn}\,(\alpha+\beta)\,\mathrm{dn}\,(\alpha-\beta) - \mathrm{cn}\,(\alpha+\beta)\,\mathrm{cn}\,(\alpha-\beta) = \lambda\frac{\mathrm{sn}^2\gamma}{1-\mathrm{sn}^4\gamma},$$

α, β, γ, λ being any four quantities; these equations, therefore, are such that any two of them imply the third.

J. W. L. GLAISHER.

On the solution of some differential equations by Bessel's functions.

(1). Lommel in the last chapter of his *Studien über die Bessel'schen Functionen* has considered some equations which reduce to Bessel's form; a somewhat more general method leads to more general results.

We write the solution of the equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left(1 - \frac{n^2}{r^2}\right)u = 0 \ldots\ldots\ldots\ldots\ldots (a),$$

$$u = A\,J_n(r) + B\,Y_n(r),$$

where if n be an integer, $J_n(r)$ and $Y_n(r)$ are the Bessel's functions of the first and second order, but if n be a fraction $Y_n(r) = J_{-n}(r)$.

(2). Assume $r = \phi(x)$, we find

$$0 = \frac{d^2u}{dx^2} - \left\{\frac{\phi''(x)}{\phi'(x)} - \frac{\phi'(x)}{\phi(x)}\right\}\frac{du}{dx} + \left[\{\phi'(x)\}^2 - \frac{n^2\{\phi'(x)\}^2}{\{\phi(x)\}^2}\right] v \quad \text{..............}(b).$$

Take $\phi(x)\,\phi''(x) = \{\phi'(x)\}^2$,

or $\phi(x) = \alpha e^{\beta x}$;

therefore $\frac{d^2u}{dx^2} + (\beta^2\alpha^2 e^{2\beta x} - n^2\beta^2)\,u = 0.$

Thus the solution of the equation

$$\frac{d^2u}{dx^2} + (C_1 e^{\delta x} - C_2)\,u = 0 \text{..................}(c),$$

is
$$u = A\,J_{\frac{2\sqrt{C_2}}{\delta}}\left(\frac{2\sqrt{C_1}}{\delta}e^{\frac{1}{2}\delta x}\right) + B\,Y_{\frac{2\sqrt{C_2}}{\delta}}\left(\frac{2\sqrt{C_1}}{\delta}e^{\frac{1}{2}\delta x}\right).$$

If we put $C_2 = 0$, we have Lommel's equation, § 32.

(3). Let us further transform equation (b) by substituting

$$u = \chi(x)\,v,$$

we find

$$0 = \frac{d^2v}{dx^2} + \frac{dv}{dx}\left\{\frac{2\chi'(x)}{\chi(x)} - \frac{\phi''(x)}{\phi'(x)} + \frac{\phi'(x)}{\phi(x)}\right\}$$

$$+ v\left[\frac{\chi''(x)}{\chi(x)} - \frac{\chi'(x)}{\chi(x)}\left\{\frac{\phi''(x)}{\phi'(x)} - \frac{\phi'(x)}{\phi(x)}\right\} + \{\phi'(x)\}^2 - \frac{n^2\{\phi'(x)\}^2}{\{\phi(x)\}^2}\right] \quad \text{........................}(d);$$

or if we take

$$\frac{2\chi'(x)}{\chi(x)} = \frac{\phi''(x)}{\phi'(x)} - \frac{\phi'(x)}{\phi(x)},$$

and therefore $\chi^2(x) = \text{const} \times \frac{\phi'(x)}{\phi(x)}$.

We have the solution of the equation

$$\left.\begin{aligned} &\frac{d^2v}{dx^2} + v\,F(x) = 0, \\ &\text{where } F(x) = \tfrac{1}{2}\frac{\phi'''}{\phi'} + (\tfrac{1}{4} - n^2)\frac{\phi'^2}{\phi^2} - \tfrac{3}{4}\frac{\phi''^2}{\phi'^2} + \phi'^2, \\ &v = \sqrt{\left\{\frac{\phi(x)}{\phi'(x)}\right\}}\,[A\,J_n\{\phi(x)\} + B\,Y_n\{\phi(x)\}]. \end{aligned}\right\} \text{......}(e).$$

We proceed to apply this for different values of $\phi(x)$.

(4) Put $\phi(x) = Cx^p$,

$$F(x) = [\tfrac{1}{2}\{(p-1)(p-2)\} + (\tfrac{1}{4} - n^2)p^2 - \tfrac{3}{4}(p-1)^2]\frac{1}{x^2} + p^2x^{2p-2}C^2$$

$$= (\tfrac{1}{4} - n^2p^2)\frac{1}{x^2} + p^2 Cx^{2p-2}.$$

Take $\quad p = \tfrac{1}{2}(m+2),\ p^2C^2 = C_2,\ C_1 = \tfrac{1}{4} - n^2p^2.$

Thus the solution of

$$\frac{d^2v}{dx^2} + \left(\frac{C_1}{x^2} + C_2x^m\right)v = 0$$

is $\quad v = \sqrt{x}\left[A J_{\frac{2}{m+2}\sqrt{(\frac{1}{4}-C_1)}}\left\{\frac{2C_2}{m+2}x^{\frac{1}{2}(m+2)}\right\} + B Y_{\frac{2}{m+2}\sqrt{(\frac{1}{4}-C_1)}}\left\{\frac{2C_2}{m+2}x^{\frac{1}{2}(m+2)}\right\}\right].$

If we take $C_1 = 0$, $C_2 = 1$, this equation reduces to Riccati's equation (Lommel, § 31)

$$\frac{d^2v}{dx^2} + x^m v = 0,$$

with solution

$$v = \sqrt{x}\left[A J_{\frac{2}{m+2}}\left\{\frac{2}{m+2}x^{\frac{1}{2}(m+2)}\right\} + B Y_{\frac{2}{m+2}}\left\{\frac{2}{m+2}x^{\frac{1}{2}(m+2)}\right\}\right].$$

(5) $\phi(x) = \alpha e^{\frac{1}{x}}$. We find for the solution of

$$\frac{d^2v}{dx^2} + \frac{1}{x^4}(\alpha^2 e^{\frac{2}{x}} - n^2)v = 0,$$

$$v = x\{A J_n(\alpha e^{\frac{1}{x}}) + B Y_n(\alpha e^{\frac{1}{x}})\}.$$

Putting $n = 0$, $\alpha = 1$, we have for the solution of

$$x^4\frac{d^2v}{dx^2} + e^{\frac{2}{x}}v = 0,$$

$$v = x\{A J_0(e^{\frac{1}{x}}) + B Y_0(e^{\frac{1}{x}})\}.$$

(Lommel, § 32).

(6) $\phi(x) = A\log x$,

$$F(x) = \frac{1}{x^2}(A^2 + \tfrac{1}{4}) + (\tfrac{1}{4} - n^2)\frac{1}{x^2(\log x)^2}.$$

Take $$C_1 = A^2 + \tfrac{1}{4}, \quad C_2 = \tfrac{1}{4} - n^2,$$
and the solution of
$$\frac{d^2v}{dx^2} + \frac{1}{x^2}\left\{C_1 + \frac{C_2}{(\log x)^2}\right\} v = 0,$$
is $$v = \sqrt{(x \log x)}\,[A\, J_{\sqrt{(\frac{1}{4}-C_2)}}\{\sqrt{(C_1 - \tfrac{1}{4})} \log x\} + B\, Y_{\sqrt{(\frac{1}{4}-C_2)}}\{\sqrt{(C_1 - \tfrac{1}{4})} \log x\}].$$

As particular cases let $-C_1 = C_2 = 0$; here
$$\frac{d^2v}{dx^2} = 0 \quad \text{or} \quad v = Dx + E.$$

Thus $$Dx + E = \sqrt{(x \log x)}\,[A\, J_{\frac{1}{2}}\{\tfrac{1}{2}\sqrt{(-1)} \log x\} + BJ_{-\frac{1}{2}}\{\tfrac{1}{2}\sqrt{(-1)} \log x\}].$$

Putting $y = \frac{1}{2}\sqrt{(-1)} \log x$, we find
$$AJ_{\frac{1}{2}}(y) + BJ_{-\frac{1}{2}}(y) = \frac{1}{\sqrt{y}}(D' \sin y + E' \cos y),$$
a well-known result.

Next let $C_2 = 0$ only, then
$$\frac{d^2v}{dx^2} + \frac{C_1}{x^2} v = 0,$$
$$v = \sqrt{(x \log x)}\,[A\, J_{\frac{1}{2}}\{\sqrt{(C_1 - \tfrac{1}{4})} \log x\} + B\, J_{-\frac{1}{2}}\{\sqrt{(C - \tfrac{1}{4})} \log x\}];$$
or, using the expression for $A\, J_{\frac{1}{2}}(y) + B\, J_{-\frac{1}{2}}(y)$ just found, we have
$$v = \sqrt{x}\,[A \sin\{\sqrt{(C_1 - \tfrac{1}{4})} \log x\} + B \sin\{\sqrt{(C_1 - \tfrac{1}{4})} \log x\}],$$
which agrees with Lommel, § 31.

(7) $$\phi(x) = \frac{\beta}{\alpha + x},$$
$$F(x) = \frac{\beta^2}{(\alpha + x)^4} + (\tfrac{1}{4} - n^2)\frac{\beta^2}{(\alpha + x)^2};$$
or, if $$C_2 = \beta^2, \quad C_1 = (\tfrac{1}{4} - n^2)\beta^2,$$
we find for the solution of
$$\frac{d^2v}{dx^2} + \left\{\frac{C_1}{(\alpha + x)^2} + \frac{C_2}{(\alpha + x)^4}\right\} v = 0,$$
$$v = \sqrt{(\alpha + x)}\left\{A\, J_{\sqrt{(\frac{1}{4} - \frac{C_1}{C_2})}}\left(\frac{\sqrt{C_2}}{\alpha + x}\right) + B\, Y_{\sqrt{(\frac{1}{4} - \frac{C_1}{C_2})}}\left(\frac{\sqrt{C_2}}{\alpha + x}\right)\right\}.$$

(8) Lastly, if we take $n=\frac{1}{2}$, $X=\frac{1}{\phi'^{\frac{1}{2}}}$,

$$F(x)=\frac{1}{X^4}-\frac{1}{X}\frac{d^2X}{dx^2},$$

and the solution of

$$\frac{d^2v}{dx^2}+\left(\frac{1}{X^4}-\frac{1}{X}\frac{d^2X}{dx^2}\right)v=0$$

is
$$v=CX\sin\left(\int\frac{dx}{X^2}+\text{const.}\right).$$

By using the value obtained in (6) for $J_{\frac{1}{2}}(x)$, this result is however obtainable directly.

KARL PEARSON.

Integration of the rectangular equations of motion in the case of a central force varying inversely as the square of the distance.

The equations are

$$\frac{d^2x}{dt^2}=-\mu\frac{x}{r^3},\quad \frac{d^2y}{dt^2}=-\mu\frac{y}{r^3},$$

from which we obtain in the usual manner the first integral

$$x\frac{dy}{dt}-y\frac{dx}{dt}=h\text{....................}(1).$$

This may be written in either of the forms

$$\frac{d}{dt}\left(\frac{y}{r}\right)=h\frac{x}{r^3},\quad \frac{d}{dt}\left(\frac{x}{r}\right)=-h\frac{y}{r^3},$$

so that we have

$$\frac{d^2x}{dt^2}=-\frac{\mu}{h}\frac{d}{dt}\left(\frac{y}{r}\right),\quad \frac{d^2y}{dt^2}=\frac{\mu}{h}\frac{d}{dt}\left(\frac{x}{r}\right),$$

leading to the two first integrals

$$\left.\begin{aligned}\frac{dx}{dt}&=-\frac{\mu}{h}\left\{A+\frac{y}{r}\right\}\\ \frac{dy}{dt}&=\frac{\mu}{h}\left\{B+\frac{x}{r}\right\}\end{aligned}\right\}\text{...............}(2).$$

Substituting these values in (1) we have

$$Ay + Bx + r = \frac{h^2}{\mu} \quad\text{..................}\quad (3),$$

the equation of the orbit, which is a conic with focus at origin, the eccentricity being $\sqrt{(A^2+B^2)}$ and the semi-latus-rectum $\frac{h^2}{\mu}$. This mode of integrating the equations of motion seems to be worthy of notice for the following reasons: (i) the equation of the orbit appears in a form which shows that it is a conic with focus at origin, without assuming any of the properties of conics, *i.e.* the equation (3) shows that the orbit is a curve such that its distance from the origin is in a constant ratio to its distance from a straight line, the position of this line and the ratio being arbitrary; (ii) the velocities parallel to the axes are given in the convenient form (1) and show that the velocity at any point is compounded of two uniform velocities, $\frac{\mu}{h}$ perpendicular to the radius vector, and $\frac{\mu}{h}\sqrt{(A^2+B^2)}$, that is $\frac{\mu e}{h}$, parallel to the directrix $Ay + Bx - \frac{h^2}{\mu} = 0$ (Frost's *Newton*, 2nd edit. p. 235); (iii) the integration is effected without squaring, extraction of square roots, &c. HARRY HART.

Note on a point in the method of least squares.

When the number of equations exceeds the number of unknowns by one, then the probable error is a linear function of the observed quantities; for if $u_1 = a_1$, $u_2 = a_2$, $\ldots u_{n+1} = a_{n+1}$ be the $n+1$ equations, $a_1, a_2, \ldots a_{n+1}$ being the $n+1$ quantities observed, and the number of unknowns being n, then the probable error must vanish if $a_1, a_2, \ldots a_{n+1}$ are connected by a certain linear equation $\lambda_1 a_1 + \lambda_2 a_2 \ldots + \lambda_{n+1} a_{n+1} = 0$, this being, the condition that the $n+1$ equations should be consistent *i.e.* that they should be equivalent to only n independent equations. Now $[vv]$ the sum of the squares of the residuals being a quadric function of $a_1, a_2, \ldots,$ which vanishes if, and only if, $\lambda_1 a_1 + \ldots = 0$; it follows that $[vv] = \text{const.}\ (\lambda_1 a_1 + \ldots)^2$, and, therefore, probable error = const. $(\lambda_1 a_1 + \ldots)$.

J. W. L. GLAISHER.

Theorem connected with a certain figure inscribed in a circle.

Let a closed polygon of n sides be circumscribed about a circle of radius r, and let $\alpha, \beta, \gamma \ldots$ be the lengths of the lines joining the centre to its angular points; then if in a circle of diameter $\frac{\lambda}{r}$ (λ being arbitrary) chords equal to $\frac{\lambda}{\alpha}, \frac{\lambda}{\beta}, \frac{\lambda}{\gamma}, \ldots$ be placed, each starting from the end of the preceding one, the figure so formed will, if n be even, be a closed polygon, and, if n be uneven, the end of the last chord will be separated from the starting point of the first by half the circumference. In the former case the figure will extend $\frac{1}{2}(n-2)$ times round the circle; and in the latter case it will extend $\frac{1}{2}(n-3)$ times round the circle and, in addition, over a half circumference.

The proof is extremely simple; for if $A, B, C, \ldots K$ be the angles of the circumscribed polygon, and $\alpha, \beta, \gamma, \ldots \kappa$ the lines joining them to the centre, then

$$\sin\tfrac{1}{2}A = \frac{r}{\alpha}, \quad \sin\tfrac{1}{2}B = \frac{r}{\beta}, \ldots \sin\tfrac{1}{2}K = \frac{r}{\kappa},$$

so that

$$\text{arc}\sin\frac{r}{\alpha} + \text{arc}\sin\frac{r}{\beta} \ldots + \text{arc}\sin\frac{r}{\kappa} = \tfrac{1}{2}(A + B \ldots + K)$$
$$= \tfrac{1}{2}(n-2)\pi,$$

from which it follows that the chords $\frac{\lambda}{\alpha}, \frac{\lambda}{\beta}, \ldots \frac{\lambda}{\kappa}$, placed in a circle of radius $\frac{\lambda}{2r}$, subtend at the centre an angle $(n-2)\pi$; which is the above theorem.

The theorem was suggested by a proposition given by Mr. L. W. Meech, in the *Analyst*, vol. v. p. 8 (1878), which is in effect the particular case corresponding to $n = 3$.*

J. W. L. GLAISHER.

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* The proposition is "If a, b, c denote three lines drawn from the angles of a plane triangle to the centre of the inscribed circle whose radius is r, then will the reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{r}$ be the four sides of a trapezium inscribed in a semicircle, the latter side coinciding with the diameter."

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, November 13th. C. W. Merrifield, Esq, F.R.S, *President*, in the Chair.

The following changes were made in the Council: Messrs. C. Leudesdorf, M.A., and H. W. Lloyd Tanner, M.A., succeed Dr. Spottiswoode, P.R.S., and Prof. H. J. S. Smith, F.R.S., who retire.

The Chairman referred to the Society's Obituary of the past session, touching on the loss to Mathematics sustained by the deaths of Prof. Clifford, Prof. Clerk Maxwell, and Sir J. G. Shaw Lefevre.

Mr. R. C. Rowe, M.A. Fellow of Trinity College, Cambridge, was elected a member of the Society.

The following communications were made to the Society: "On the binomial equation $x^p - 1 = 0$: trisection and quartisection," Prof. Cayley, F.R.S. (founded on results in Reuschle's Tafeln complexer Primzahlen welche aus wurzeln der Einheit gebildet sind); "On cubic determinants and other determinants of higher class, and on determinants of alternate numbers;" Mr. R. F. Scott, M.A. "On a problem of Fibonacci's," Mr. S. Roberts, F.R.S.; "Notes on a class of definite integrals," Mr. T. R. Terry, M.A.

R. Tucker, M.A., *Hon. Sec.*

ON THE MOTION OF A VISCOUS INCOMPRESSIBLE FLUID.

By *A. R. Forsyth*, Trinity College.

Helmholz in his paper on Vortex Motion (translated by Prof. Tait in the *Philosophical Magazine* for 1867), shews that the kinetic energy of a mass of fluid in motion within a rigid envelope is constant. This, however, assumes that the fluid under consideration is a perfect fluid; and if account be taken of its viscosity it seems probable, *à priori*, that the theorem will no longer hold. That this is so, appears from the following investigation in which an expression for the rate of change of the kinetic energy is obtained, this rate depending directly on the coefficient of viscosity.

Let ρ be the density of the fluid, considered incompressible; u, v, w the component velocities at the point xyz; p the pressure at that point; μ the coefficient of viscosity; V the force function. Then the equations of motion are

$$\left.\begin{aligned}\rho\frac{\delta u}{\delta t} &= \rho\frac{dV}{dx} - \frac{dp}{dx} + \mu\nabla^2 u\\ \rho\frac{\delta v}{\delta t} &= \rho\frac{dV}{dy} - \frac{dp}{dy} + \mu\nabla^2 u\\ \rho\frac{\delta w}{\delta t} &= \rho\frac{dV}{dz} - \frac{dp}{dz} + \mu\nabla^2 w\end{aligned}\right\}\ \ldots\ldots\ldots\ldots (\text{I}),$$

where $\frac{\delta}{\delta t}$ indicates total differentiation with regard to t, Δ^2 has the usual meaning, and all the other differentiations are partial.

Let
$$\left.\begin{aligned} 2\xi &= \frac{dw}{dy} - \frac{dv}{dz} \\ 2\eta &= \frac{du}{dz} - \frac{dw}{dx} \\ 2\zeta &= \frac{dv}{dx} - \frac{du}{dy} \end{aligned}\right\} \text{..................(II)}.$$

Then
$$\frac{\delta u}{\delta t} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}$$
$$= \frac{du}{dt} + u\frac{du}{dx} + v\frac{dv}{dx} + w\frac{dw}{dx} + 2(w\eta - v\zeta).$$

Let
$$2T = u^2 + v^2 + w^2,$$

then
$$\frac{\delta u}{\delta t} = \frac{du}{dt} + \frac{dT}{dx} + 2(w\eta - v\zeta)$$

and similar expressions are obtained for $\frac{\delta v}{\delta t}\frac{\delta w}{\delta t}$. With the substitution of these in (I), the equations of motion are

$$\left.\begin{aligned} \rho\frac{du}{dt} + \rho\frac{d(T-V)}{dx} + \frac{dp}{dx} + 2\rho(w\eta - v\zeta) &= \mu\nabla^2 u \\ \rho\frac{dv}{dt} + \rho\frac{d(T-V)}{dy} + \frac{dp}{dy} + 2\rho(u\zeta - w\xi) &= \mu\nabla^2 v \\ \rho\frac{dw}{dt} + \rho\frac{d(T-V)}{dz} + \frac{dp}{dz} + 2\rho(u\xi - u\eta) &= \mu\nabla^2 w \end{aligned}\right\} \text{...(III)}.$$

The equation of continuity is

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0;$$

therefore
$$\nabla^2 u = \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} - \frac{d^2v}{dx\,dy} - \frac{d^2v}{dx\,dz}$$
$$= 2\left(\frac{d\eta}{dz} - \frac{d\zeta}{dy}\right).$$

Similarly $\nabla^2 v = 2\left(\frac{d\zeta}{dx} - \frac{d\xi}{dz}\right)$,

$$\nabla^2 w = 2\left(\frac{d\xi}{dy} - \frac{d\eta}{dx}\right).$$

Let $$U' = V - \frac{p}{\rho};$$

Substituting in (I), multiplying the equations by u, v, w respectively, and adding, we have

$$\frac{dT}{dt} = \left(u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}\right)U' + \frac{\mu}{\rho}(u\nabla^2 u + v\nabla^2 v + w\nabla^2 w).$$

If K be the kinetic energy

$$\frac{dK}{dt} = \iiint \frac{dT}{dt}\rho\, dx\, dy\, dz$$

$$= \rho\iiint\left(u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}\right)U'\, dx\, dy\, dz$$

$$+ \mu\iiint(u\nabla^2 u + v\nabla^2 v + w\nabla^2 w)\, dx\, dy\, dz \ldots\text{(IV)}.$$

The first of the integrals on the right-hand side of (IV) is

$$\rho\iint U'(lu + mv + nw)\, dS - \rho\iiint U'\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)dx\, dy\, dz$$

$$= \rho\iint U'(lu + mv + nw)\, dS,$$

by the equation of continuity; and since the boundary of the envelope is considered rigid

$$lu + mv + nw = 0,$$

at all points of the surface. Hence the first term in IV vanishes.

If we put

$$-X = w\eta - v\zeta, \quad -Y = u\zeta - w\xi, \quad -Z = v\xi - u\eta,$$

then

$$-2\left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right)$$

$$= 2w\left(\frac{d\eta}{dx} - \frac{d\xi}{dy}\right) + 2v\left(\frac{d\xi}{dz} - \frac{d\zeta}{dx}\right) + 2u\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right)$$

$$+ 2\eta\left(\frac{dw}{dx} - \frac{du}{dz}\right) + 2\zeta\left(\frac{du}{dy} - \frac{dv}{dx}\right) + 2\xi\left(\frac{dv}{dz} - \frac{dw}{dy}\right)$$

$$= -(w\nabla^2 w + v\nabla^2 v + u\nabla^2 u) - 4(\xi^2 + \eta^2 + \zeta^2).$$

If Ω be the resultant angular velocity of rotation, we have

$$u\nabla^2 u + v\nabla^2 v + w\nabla^2 w = 2\left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right) - 4\Omega^2.$$

Hence

$$\frac{dK}{dt} = \mu \iiint (u\nabla^2 u + v\nabla^2 v + w\Delta^2 w)\, dx\, dy\, dz$$

$$= -4\mu \iiint \Omega^2 dx\, dy\, dz + 2\mu \iiint \left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right) dx\, dy\, dz$$

$$= -4\mu \iiint \Omega^2 dx\, dy\, dz + 2\mu \iint (lX + mY + nZ)\, dS,$$

l, m, n being the direction cosines of a normal to the bounding surface.

Now if λ, μ, ν be the direction cosines of a vector whose components are X, Y, Z, the vector is $V\Omega \sin\chi$ where V is the resultant velocity at the point xyz, and χ is the angle between the direction of this velocity and the instantaneous axis of rotation; and if θ be the angle (less than two right angles) between the normal to the surface and the plane containing the axis of rotation and direction of resultant velocity,

$$\sin\theta = l\lambda + m\mu + n\nu,$$

so that

$$\frac{dK}{dt} = -4\mu \iiint \Omega^2 dx\, dy\, dz + 2\mu \iint V\Omega \sin\chi \sin\theta\, dS \ldots (\text{V}).$$

If now the boundary, in addition to being rigid, is so far distant that any motion at it would make the kinetic energy infinite, then at every point of the boundary V is zero, and

$$\frac{dK}{dt} = -4\mu \iiint \Omega^2 dx\, dy\, dz.$$

The equivalent of the latter form is given in an investigation in curvilinear coordinates by M. Bobilew in the *Mathematischen Annalen*, bd. VI.

Either of these expressions furnishes a proof of Helmholz's theorem that the kinetic energy of a perfect fluid under the action of a conservative system of forces and enclosed in a rigid envelope is constant; for in this case μ is zero; therefore

$$\frac{dK}{dt} = 0,$$

or

$$K = \text{constant}.$$

If the direction of motion at every point be along the instantaneous axis of rotation (a screw motion) then $\chi = 0$, and the second term of (V) vanishes. It also vanishes if θ be zero, *i.e.* if the plane containing the direction of motion and the instantaneous axis of rotation be everywhere normal to the surface.

Taking a cylindrical surface bounded by plane ends perpendicular to the axis, and if the axis of rotation be everywhere parallel to the cylindrical surface, then over the plane ends θ is zero, so that the surface integral over the plane ends vanishes. If the motion is symmetrical about the axis $\chi = \frac{1}{2}\pi$, and V, Ω are constant over the surface, thus

$$\frac{dK}{dt} = -4\mu\Omega^2 . \text{ volume} + 2\mu V\Omega . \text{ surface}.$$

But volume $= \frac{1}{2}$ surface $\times$ radius $= \frac{1}{2}a$ surface, a being the radius; therefore

$$\frac{dK}{dt} = -4\mu\Omega\left(\Omega - \frac{V}{a}\right) . \text{ volume}.$$

If it rotates as a solid body, V, being the velocity at the surface, is $a\Omega$; therefore

$$\frac{dK}{dt} = 0,$$

as we should have expected.

Putting $\frac{\mu}{\rho} = \mu'$, and inserting X, Y, Z in III, the equations of motion are

$$\frac{du}{dt} - \frac{dU}{dx} - \mu'\nabla^2 u = 2X,$$

$$\frac{dv}{dt} - \frac{dU}{dy} - \mu'\nabla^2 v = 2Y,$$

$$\frac{dw}{dt} - \frac{dU}{dz} - \mu'\nabla^2 w = 2Z.$$

Differentiating the first with respect to x, the second with respect to y, and the third with respect to z, adding and using the equation of continuity, we have

$$\nabla^2 U = -2\left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right);$$

therefore

$$\frac{p}{\rho} = C - T + V + \frac{1}{2\pi}\iiint \frac{\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'}}{r}\, dx'\, dy'\, dz',$$

where C is a function of the time, r the distance from xyz to $x'y'z'$, and the integration is to be performed throughout the volume.

This equation has the same form as that which would be obtained in the case of a perfect fluid; but X, Y, Z, T implicitly contain μ, and thus the expressions for the pressure will be different in the two cases. If there be any solids in the liquid, or boundaries of separation between two fluids, a coefficient of surface friction will be introduced by the boundary equations.

The above may also be put into the form

$$\frac{p}{\rho} = C - T \pm V - \frac{1}{2\pi}\iint \frac{V\Omega \sin\chi \sin\theta}{r}\, dS$$
$$- \frac{1}{2\pi}\iiint \frac{V\Omega \sin\chi \sin\theta'}{r^2}\, dx'\, dy'\, dz',$$

where θ' is the angle between the radius vector from xyz to $x'y'z'$, and the plane containing the direction of motion and the instantaneous axis of rotation at $x'y'z'$.

ON POWERS OF FUNCTIONS OF THE FORM $\dfrac{ax + b}{cx + d}$.

By *H. W. Lloyd Tanner, M.A.*

1. LET $\phi(x)$, $\psi(x)$ be two functions of the form considered, the coefficients being a, b, c, d and α, β, γ, δ respectively. Supposing ϕ to be given, let us find ψ such that

$$\phi\psi = \psi\phi.$$

The coefficients of $\phi\psi$ are

$$A = a\alpha + b\gamma,\quad B = a\beta + b\delta,\quad C = c\alpha + d\gamma,\quad D = c\beta + d\delta.$$

Comparing the A, B coefficients of $\phi\psi$, $\psi\phi$, we have

$$\frac{a\alpha + b\gamma}{\alpha a + \beta c} = \frac{c\beta + d\delta}{\gamma b + \delta d} = \frac{a\alpha + b\gamma + c\beta + d\delta}{\alpha a + \beta c + \gamma b + \delta d} = 1,$$

so that the coefficients of $\phi\psi$ must be severally equal to those of $\psi\phi$. Hence the B, C coefficients give the equations

$$a\beta + b\delta = \alpha b + \beta d,$$
$$c\alpha + d\gamma = \gamma a + \delta c;$$

i.e.
$$\frac{a-d}{\alpha-\delta} = \frac{b}{\beta} = \frac{c}{\gamma},$$

whence
$$\alpha : \beta : \gamma : \delta = a-k : b : c : d-k,$$

and
$$\psi(x) = \frac{(a-k)x+b}{cx+d-k}.$$

But we have for all values of n

$$\phi\phi^n = \phi^n\phi;$$

hence
$$\phi^n(x) = \frac{(a-k_n)x+b}{cx+d-k_n} \quad \text{..................(1)},$$

and this formula is true for all values of n commensurable or incommensurable, real or complex.

2. There is a certain quantity connected with every function $\frac{ax+b}{cx+d}$ to which Prof. Cayley has drawn attention. He has indicated it by the symbol λ, and has defined it by the equation

$$\frac{(1+\lambda)^2}{\lambda} = \frac{(a+d)^2}{ad-bc} \quad \text{.................. (2)};$$

or, what is the same thing,

$$\frac{\lambda}{(1-\lambda)^2} = \frac{ad-bc}{(a-d)^2+4bc},$$

whence
$$\left(\frac{1+\lambda}{1-\lambda}\right)^2 = \frac{(a+d)^2}{(a-d)^2+4bc} \quad \text{............... (3)}.$$

This quantity, λ, I propose to call the *modulus* of $\frac{ax+b}{cx+d}$.

If ϕ, ψ be any two functions of the class under discussion, $\phi\psi$ and $\psi\phi$ are equimodular. For, taking the coefficients of $\phi\psi$, we have

$$A + D = a\alpha + b\gamma + c\beta + d\delta,$$
$$AD - BC = (ad-bc)(\alpha\delta - \beta\gamma),$$

and these are evidently unchanged by a transposition of the Greek and Roman letters. Hence the equation (2) is the same for $\phi\psi$ and $\psi\phi$; or these compound fnnctions are equimodular. It thus appears that the modulus of $\phi\psi$ is a symmetrical function of the coefficients of ϕ, ψ; but it is not generally a function of the moduli of ϕ, ψ. If, however, ϕ, ψ are powers of one and the same function, then the modulus of $\phi\psi$ is the product of the moduli of ϕ, ψ. It is necessary, however, to define herein which of the two values given by (2) we take as the modulus: how this is done will appear in the course of the proof.

Solving (2), we find

$$\lambda = \frac{a^2 + d^2 + 2bc \pm (a+d)\sqrt{\{(a-d)^2 + 4bc\}}}{2(ad - bc)}.$$

The modulus of $\psi(x) = \dfrac{(a-k)x + b}{cx + d - k}$ is given by

$$\lambda' = \frac{(a-k)^2 + (d-k)^2 + 2bc \pm (a+d-2k)\sqrt{\{(a-d)^2 + 4bc\}}}{2(a'd' - bc)},$$

where, in the denominator, a', d' are written for $a-k$, $d-k$.

Similarly, the modulus of $\phi\psi$ is given by

$$\Lambda = \frac{2(A^2 + D^2 + 2BC) \pm 2(A+D)\sqrt{\{(A-D)^2 + 4BC\}}}{4(ad-bc)(a'd'-bc)},$$

and it only remains to show that the numerator of Λ is the product of the numerators of λ, λ', since the denominator of Λ is evidently the product of those of λ, λ'. It is easy to show that

$$(A-D)^2 + 4BC = (a+d-k)^2\{(a-d)^2 + 4bc\},$$

i.e. $\sqrt{\{(A-D)^2 + 4BC\}} = (a+d-k)\sqrt{\{(a-d)^2 + 4bc\}}$,

which rationally involves the same radical as before.

We shall leave the sign of this radical undetermined, but assume it to be the same in λ, λ', Λ, and then we may use only the + sign before it in each of the three formulæ. There is no difficulty, though the process is somewhat laborious, in verifying the identities

$$2(A^2 + D^2 + 2BC) = (a^2 + d^2 + 2bc)\{(a-k)^2 + (d-k)^2 + 2bc\}$$
$$+ (a+d)(a+d-2k)\{(a-d)^2 + 4bc\},$$

$$2(A+D)(a+d-k) = (a+d)\{(a-k)^2 + (d-k)^2 + 2bc\}$$
$$+ (a+d-2k)(a^2 + d^2 + 2bc),$$

the simplest method being to compare the coefficients of powers of k. This completes the proof that

$$\operatorname{mod}\phi^{n+1} = \operatorname{mod}\phi \times \operatorname{mod}\phi^{n}.$$

3. From the result of the last article we can at once infer that, λ being the modulus of $\phi(x)$, the modulus of $\phi^n(x)$ is λ^n, n being a real commensurable quantity. In fact, we have in succession

$$\operatorname{mod}\phi^2(x) = \{\operatorname{mod}\phi(x)\}^2 = \lambda^2,$$

$$\operatorname{mod}\phi^3(x) = \lambda^3,$$

and so on for all integral values of n. Then, taking $n = r : s$, we have

$$(\operatorname{mod}\phi^n)^s = \operatorname{mod}\phi^r = \lambda^r;$$

therefore $\operatorname{mod}\phi^n = \lambda^{\frac{r}{s}} = \lambda^n.$

If we assume $\operatorname{mod}\phi^i = \lambda^i,$

we have $\operatorname{mod}\phi^{n_2 i} = \lambda^{n_2 i},$

and therefore

$$\operatorname{mod}\phi^{n_1+n_2 i} = \operatorname{mod}\phi^{n_1} \times \operatorname{mod}\phi^{n_2 i}$$
$$= \lambda^{n_1}.\lambda^{n_2 i}$$
$$= \lambda^{n_1+n_2 i},$$

so that the equation $\operatorname{mod}\phi^n = \lambda^n$

may be extended to complex values of n.

4. Taking ϕ^n in the form given by (1) and remembering that its modulus is λ^n, we have by (3)

$$\left(\frac{1+\lambda^n}{1-\lambda^n}\right)^2 = \frac{(a+d-2k_n)^2}{(a-d)^2+4bc} \quad\ldots\ldots\ldots\ldots (3_n).$$

Eliminating $(a-d)^2+4bc$ by the help of (3), we have

$$\frac{a+d-2k_n}{a+d} = \pm\frac{1+\lambda^n}{1-\lambda^n}\,\frac{1-\lambda}{1+\lambda},$$

whence

$$k_n = \frac{a+d}{1+\lambda}\,\frac{\lambda^n-\lambda}{\lambda^n-1}, \text{ or } \frac{a+d}{1+\lambda}\,\frac{\lambda^{n+1}-1}{\lambda^n-1}.$$

The ambiguity in the value of k_n arises from the fact that we have found the value of k_{-n} as well as k_n, for the equation (3_n), with which we started, is the same for k_n as for k_{-n}.

Indeed, this ambiguity might have been anticipated from the circumstance that a function is equimodular with its inverse. That k_n is not ambiguous may be inferred from the fact that whichever value of λ we use, viz. whether we write λ or $\frac{1}{\lambda}$, each formula for k_n gives an unchanged value. There is no ambiguity in k_n by reason of the ambiguity of λ. On the other hand, by changing n into $-n$, the one form for k_n is changed into the other. It thus appears that the two values obtained for k_n really belong to k_n, k_{-n} respectively. To decide which corresponds to k_n, which to k_{-n}, it is only necessary to put $n=1$; and then, since $k_1 = 0$, $k_{-1} = a + d$, we find that

$$k_n = \frac{a+d}{1+\lambda}\,\frac{\lambda^n - \lambda}{\lambda^n - 1} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$k_{-n} = \frac{a+d}{1+\lambda}\,\frac{\lambda^{n+1} - 1}{\lambda^n - 1} = \frac{a+d}{1+\lambda}\,\frac{\lambda^{-n} - \lambda}{\lambda^{-n} - 1}.$$

Substituting this value of k_n in (1), we easily get Prof. Cayley's form for $\phi^n(x)$, viz.

$$\phi^n(x) = \frac{(\lambda^{n+1} - 1)(ax+b) + (\lambda^n - \lambda)(-dx+b)}{(\lambda^{n+1} - 1)(cx+d) + (\lambda^n - \lambda)(cx-a)}.$$

Writing this

$$\frac{k_{-n}(ax+b) + k_n(-dx+b)}{k_{-n}(cx+d) + k_n(cx-a)},$$

and noticing that

$$\phi(x) = \frac{ax+b}{cx+d}, \quad \phi^{-1}(x) = \frac{-dx+b}{cx-a},$$

the remarkable symmetry of the expression is manifest.

From (4) we learn that there are as many different values of ϕ^n as there are of λ^n, for two values of k_n cannot be equal unless the corresponding values of λ^n are equal.

5. The equation (4) may be obtained in another manner, which avoids the ambiguity noticed in the last article. This method we will briefly indicate. We call two functions ϕ, ψ *congruent*, or, say,

$$\phi(x) \equiv \psi(x),$$

when

$$\left.\begin{array}{r} a+d = \alpha + \delta \\ ad - bc = \alpha\delta - \beta\gamma \end{array}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

It will be noticed that a function and its inverse are not congruent, although equimodular.

If we write $\phi^n(x)$ as in (1), and $\psi^n(x)$ in the same form, but with κ_n in the place of k_n, we have

$$\phi^n(x) \equiv \psi^n(x) \quad\ldots\ldots\ldots\ldots (6),$$

if

$$k_n = \kappa_n \quad\ldots\ldots\ldots\ldots (7),$$

and *vice versâ*, for ϕ^n, ψ^n are congruent if the two expressions

$$a - k_n + d - k_n,$$

and $(a - k_n)(a - k_n) - bc, \quad = ad - bc - (a + d) k_n + k_n^2,$

are equal to the corresponding expressions in α, β, γ, δ, k_n; that is, by reason of (5), if and only if $k_n = \kappa_n$.

It hence follows that if two powers of a function be congruent they must be equal; and the converse is easily seen to be true, since two powers of a function have the same b, c.

We know that, when n is integral, (7) is satisfied; and therefore (6) follows. When n is commensurable, $= r : s$ say, we can show that every form of $\phi^n(x)$ is congruent with a form of $\psi^n(x)$. For taking a form of ϕ^n, select a power of ψ congruent with this selected form of ϕ^n; say it is

$$\chi \equiv \phi^n;$$

it is to be noted that χ is uniquely determined, its coefficient being $\alpha - k_n$, β, γ, $\delta - k_n$; and k_n being known when ϕ^n is given. To prove our theorem we have only to show that χ^s is identical with ψ^r, for then χ is a form of ψ^n $(n = r : s)$, and it has been assumed congruent with a form of ϕ^n. But

$$\chi^s \equiv \phi^{ns} \equiv \phi^r,$$

for s is integral. Also since r is integral,

$$\phi^r \equiv \psi^r,$$

therefore

$$\chi^s \equiv \psi^r,$$

and since χ^s, ψ^r are powers of the same function, ψ, they are equal. Thus (6) is proved for all real commensurable values of n.

We assume

$$\phi^i \equiv \psi^i,$$

and then we get

$$\phi^{n_2 i} \equiv \psi^{n_2 i}.$$

Moreover, it can be shown that

$$\phi^{n_1 + n_2 i} \equiv \psi^{n_1 + n_2 i},$$

this indeed being a particular case of the theorem

$$\phi^n\phi^{n'} \equiv \psi^n\psi^{n'},$$

if $$k_n = \kappa_n, \quad k_{n'} = \kappa_{n'}.$$

We may therefore take (6) to be generally true.

6. The simplest function ψ congruent with a given ϕ is that in which $\beta = \gamma = 0$. Then α, δ are, by (5), given by the relations

$$\alpha + \delta = a + d, \quad \alpha\delta = ad - bc,$$

viz. they are p, q, the roots of

$$p^2 - (a+d)p + ad - bc = 0.$$

We have then

$$\phi(x) \equiv \frac{px}{q};$$

therefore $$\phi^n(x) \equiv \frac{p^n x}{q^n}.$$

But by (1)

$$\frac{p^n x}{q^n} = \frac{(p - k_n)x}{q - k_n},$$

whence $$k_n = \frac{p^n q - p q^n}{p^n - q^n}.$$

This may be written in another form by putting

$$\lambda q = p,$$

so that $$(\lambda + 1) q = a + d;$$

viz. then we get

$$k_n = \frac{a+d}{1+\lambda} \frac{\lambda^n - \lambda}{\lambda^n - 1} \text{.....................} (4),$$

and since ϕ^n is congruent with $\frac{p^n x}{q^n}$ it has the same k_n.

By eliminating q between the equations

$$(\lambda + 1) q = a + d,$$
$$\lambda q^2 = ad - bc,$$

we see that λ satisfies (2); and thus our previous results are reproduced without any ambiguity.

7. It is known that ϕ^n considered as a function of n is in some cases periodic; viz. we have

$$\phi^{n+m} = \phi^n,$$

or, what is equivalent,

$$\phi^m(x) = x.$$

These equations are tantamount to

$$\lambda^{n+m} = \lambda^n \text{ or } \lambda^m = 1,$$

except in the case in which

$$\lambda = 1.$$

Writing $$\lambda = e^{\theta + i\eta},$$

and $$m = m_1 + m_2 i,$$

we have the condition that ϕ should be periodic of the m^{th} order

$$e^{(\theta+i\eta)(m_1+m_2 i)} = 1 = e^{2r\pi i},$$

r being an integer, therefore

$$\theta m_1 - \eta m_2 + (m_1\eta + m_2\theta)\, i = 2r\pi i,$$

whence $$\frac{m_1}{\eta} = \frac{m_2}{\theta} = \frac{2r\pi}{\eta^2 + \theta^2},$$

$$m = \frac{2r\pi}{\eta^2 + \theta^2}(\eta + i\theta) \quad \text{.................}(8).$$

We shall call the *period* that value of m which corresponds to $r = 1$; and then it appears that every function ϕ has a period, either real or complex, except when $\eta = \theta = 0$, (that is when $\lambda = 1$) when the period becomes indeterminate. The condition for a real period is $\theta = 0$, so that $\lambda = e^{i\eta}$. Substituting in (2) we get

$$4\cos^2 \tfrac{1}{2}\eta = \frac{(a+d)^2}{ad - bc}.$$

Hence, the conditions for a real period are

$$0 \leqq \frac{(a+d)^2}{ad - bc} < 4 \quad \text{.....................}(9),$$

this expression being real. It must not be equal to 4, for this would give $\eta = 0$.

The expression for m in terms of λ, when it is real, is

$$m = \frac{2\pi i}{\log \lambda},$$

which is real since $\log \lambda$ is a pure imaginary. It is usual to take for period the smallest multiple of m which is integral; but in some respects the above seems preferable.

If we take any power of ϕ the new function will have a new period, and it is always possible to take a power of ϕ which shall have any pre-determined period, save only when $\theta = \eta = 0$.

8. When $\lambda = 1$,

we have $\theta = 0, \quad \eta = 2s\pi,$

s being an integer. In this case then the period may be $1 : s$, by (8), or it may be infinite, the latter case corresponding to $s = 0$.

Reverting to the former notation we have, if λ is 1,

$$(a-d)^2 + 4bc = 0 \quad \text{..................(10)},$$

and (4) becomes

$$k_m = \tfrac{1}{2}(a+d)\frac{m-1}{m}.$$

giving $$\phi^m(x) = \frac{\{(m+1)a - (m-1)d\}x + 2mb}{2mcx + (m+1)d - (m-1)a}.$$

If this reduce to x, we must have either

$$b = c = a - d = 0,$$

or $$m = 0.$$

In the former case

$$\phi(x) = x,$$

and herein η is an indeterminate integral multiple of 2π. This function has an indeterminate period and we can always take a power of it such as to have any given period. It is in fact the m^{th} power of every periodic function $\psi(x)$ whose period is m.

In the latter case, when (10) is satisfied, but b, c do not both vanish, the function is non-periodic and no power of it is periodic. In this case then we have

$$\theta = 0, \quad \eta = 0.$$

9. In his paper upon these functions in the *Messenger of Mathematics*, vol. IX. p. 108, Prof. Cayley has given the values which $\phi^n(x)$ assumes when n becomes infinite. If we take that value of λ whose modulus is greater than 1, we have

$$\phi^{\infty}(x) = \frac{\lambda a - d}{c(\lambda + 1)} = \frac{b(\lambda + 1)}{\lambda d - a}.$$

If we take the other value of λ, we get

$$\phi^{\infty}(x) = \frac{a - \lambda_1 d}{c(1 + \lambda_1)} = \frac{b(1 + \lambda_1)}{d - \lambda_1 a}.$$

But, noticing that $\lambda_1 . \lambda = 1$, we see that these give the same value as before.

When $n = -\infty$ we get, taking λ to be that root of (2) whose modulus is greater than unity

$$\phi^{-\infty} = \frac{a - \lambda d}{c(1 + \lambda)} = \frac{b(1 + \lambda)}{d - \lambda a},$$

and we get the same result if we take the other value of λ though the form is different. The values of ϕ^{∞} and $\phi^{-\infty}$ satisfy the quadratic

$$cu^2 - (a - d)u - b = 0 \quad \text{................(11).}$$

Bearing in mind that c, $a - d$, b are the same for all powers of the same function, we see that to every infinite value of n, real or complex, corresponds a value of $\phi^n(x)$ independent of x and given by one, or it may be by both, of the roots of (11).

Taking any power $n_1 + n_2 i$ of ϕ, and supposing as before, λ to be written in the form $e^{\theta + \theta i}$, we infer

$$\phi^{\infty(n_1 + n_2 i)} = \frac{\lambda a - d}{c(\lambda + 1)} = \frac{b(\lambda + 1)}{\lambda d - a},$$

$$\phi^{-\infty(n_1 + n_2 i)} = \frac{a - \lambda d}{c(\lambda + 1)} = \frac{b(\lambda + 1)}{d - \lambda a},$$

λ being such that

$$n_1 \theta - n_2 \eta > 0 \quad \text{......................(12),}$$

for then the modulus of λ^n is greater than unity. Of course, it is assumed that a definite value is assigned to η, or we may put the matter by saying that the values of $\phi^{\infty(n_1 + n_2 i)}$ are definite for each value of $\phi^{n_1 + n_2 i}$.

If, however, $n_1\theta = n_2\eta$,

there seems to be ambiguity; it appears that both roots of (11) belong to $\phi^{\infty n}$, and both to $\phi^{-\infty n}$. We shall examine this in the case of $\theta = 0$, $n_2 = 0$, viz. we seek the value of $\phi^{\infty}(x)$ when the modulus of λ is unity. This may be considered as the limit of

$$\phi^{\infty(1+n_2 i)},$$

when n_2 becomes infinitesimal. But θ vanishing, it appears from (12) that when n_2 is positive, no matter how small it may be, we have one value of λ, when n_2 is negative we must take the other. That is

$$\phi^{\infty(1+n_2 i)}$$

is a discontinuous function of n_2, when $n_2 = 0$. It appears therefore that we may take this function to be really ambiguous; or, if we adopt the theorem

$$2\psi(a) = \psi(a+0) + \psi(a-0)$$

as applying when $\psi(a)$ is discontinuous, we should have

$$\phi^{\infty}(x) = \frac{a-d}{2c},$$

(this being the mean of the values of u given by (11)) when the modulus of λ is 1.

The same result may be obtained by seeking the real infinite power of the function whose modulus is $e^{\theta+i\eta}$; when $\theta = 0$ this is discontinuous.

When $\theta = \eta = 0$ there is no ambiguity, for then the roots of (11) are equal. But when $\theta = 0$, $\eta = 2r\pi$, that is, when

$$c = a - d = b = 0,$$

the roots of (11) become indeterminate. This is as it should be, for then

$$\phi(x) = x,$$

and is a power of every periodic function.

Another form may be given to ϕ^{∞}. We may propose the problem, what power of $\phi(x)$ is independent of x? In other words, for what value of k is

$$\frac{a-k}{c} = \frac{b}{d-k}?$$

But this is $k^2 - (a+d)k + ad - bc = 0,$

so that the values of the required powers of ϕ are

$$\frac{a-p}{c}=\frac{b}{d-p},\quad \frac{a-q}{c}=\frac{b}{d-q},$$

where p, q are the roots of

$$p^2-(a+d)p+ad-bc=0,$$

an equation we met with in Art. 6.

10. It has been shown that every power of $\phi(x)$ is of the form

$$\frac{(a-k)x+b}{cx+d-k}.$$

The converse is also true if we admit complex values of n. For supposing this function to be ϕ^n, we have, by (4),

$$k=\frac{a+d}{1+\lambda}\,\frac{\lambda^{\ }-\lambda}{\lambda^n-1},$$

whence $$n\log\lambda=\log\frac{k+\lambda(k-a-d)}{\lambda k+k-a-d},$$

which determines n when k is given.

A more convenient form is obtained from the equation of Art. 6

$$k=\frac{p^nq-pq^n}{p^n-q^n};$$

namely, $$n\log\frac{p}{q}=\log\frac{k-p}{k-q}.$$

Some particular cases may be noticed.

If p, q are real, so that $a+d$ is real, $(a-d)^2+4bc$ is real and positive, n is real provided k be real and do not lie between p, q in magnitude.

If p, q are conjugate complex imaginaries, we have $a+d$ real, and $(a-d)^2+4bc$ is real and negative. In this case n is real for every real value of k, for we may write the equation for n thus

$$n\tan^{-1}\frac{p-q}{a+d}=\tan^{-1}\frac{p-q}{a+d-2k}.$$

If p, q are equal, *i.e.* when $(a-d)^2+4bc$ vanishes, n becomes indeterminate.

When k is equal to p, q, n becomes infinite.

November, 1879.

THE BRACHISTOCHRONE PROBLEM OF A SYSTEM.

By *R. R. Webb, M.A.*, St. John's College.

The ordinary brachistochrone problem applies to a particle only, and may be stated thus. "To determine the smooth curve on which a particle must be constrained to move in order that it may pass from any one to any other position in the least possible time, the field of force being conservative." The problem here proposed is "to determine the nature of the constraints that must be imposed on any dynamical system in a conservative field that it may pass from any one to any other configuration in the least possible time." It is obvious that in all cases the constraints introduced will of necessity reduce the number of the degrees of freedom of the system to unity, so that our problem is really to find what functions of some independent variable (say s), each independent coordinate of the free system must be, that the time of transit should be a minimum.

Let θ, ϕ..., be the independent coordinates of the free system, then the equation of energy in the free system may be written $T=U$, where T is the kinetic energy and U the corrected force function, of course T is a homogeneous quadratic function of $\dot{\theta}$, $\dot{\phi}$, &c. Now when the system is constrained the equation of energy still holds good, but now must be written

$$T_1\dot{s}^2 = U,$$

where T_1 is the same quadratic function of $\frac{d\theta}{ds}$, $\frac{d\phi}{ds}$, ..., say $\underset{.}{\theta}$, $\underset{.}{\phi}$, ..., that T was of $\frac{d\theta}{dt}$, $\frac{d\phi}{dt}$, ..., that we called $\dot{\theta}$, $\dot{\phi}$, ...

From the above, then, we have

$$dt = ds\sqrt{\frac{T_1}{U}},$$

and therefore
$$t = \int_{s_0}^{s_1}\sqrt{\frac{T_1}{U}}\,ds.$$

Now when the time is a minimum the variation of t due to any small arbitrary variations of θ, ϕ, ..., as functions of s

must vanish to quantities of the first order of those variations. At this stage we have, therefore, to find the variation of an expression of the form $\int_{s_0}^{s_1} Vds$, where V is a function of the n *dependent* variables θ, ϕ, ..., and their differential coefficients $\dot{\theta}$, $\dot{\phi}$, ...; without entering needlessly into details fully laid down in treatises on the Calculus of Variations (Todhunter, *Integral Calculus*, §§ 364, 365). We may write down the required variation thus

$$\delta \int_{s_0}^{s_1} Vds = {}_0^1\left[\Sigma\delta\theta \frac{dV}{d\dot{\theta}}\right] + \Sigma \int_{s_0}^{s_1} \delta\theta \left\{\frac{dV}{d\theta} - \frac{d}{ds}\left(\frac{dV}{d\dot{\theta}}\right)\right\} ds.$$

Hence in the present case we have

$$\delta t = {}_0^1\left[\Sigma\delta\theta \frac{d}{d\dot{\theta}}\left(\sqrt{\frac{T_1}{U}}\right)\right]$$

$$+ \Sigma \int_{s_0}^{s_1} \delta\theta \left\{\frac{d}{d\theta}\left(\sqrt{\frac{T_1}{U}}\right) - \frac{d}{ds}\frac{d}{d\dot{\theta}}\left(\sqrt{\frac{T_1}{U}}\right)\right\} ds.$$

When, therefore, the time is a minimum it is essential that θ, ϕ, ..., should satisfy the equations

$$\frac{d}{ds}\frac{d}{d\dot{\theta}}\left(\sqrt{\frac{T_1}{U}}\right) - \frac{d}{d\theta}\left(\sqrt{\frac{T_1}{U}}\right) = 0.$$

....

Now we must remember that the differentiations $\frac{d}{d\theta}$, $\frac{d}{d\dot{\theta}}$, ..., and all the like are strictly partial and at the same time the differentiation $\frac{d}{ds}$ total, also,

$$\frac{d}{d\dot{\theta}}\sqrt{\frac{T_1}{U}} = \frac{d}{d\theta}\sqrt{\frac{T}{U}},$$

$$\dot{s}\frac{d}{d\theta}\sqrt{\frac{T_1}{U}} = \frac{d}{d\theta}\sqrt{\frac{T}{U}}.$$

The truth of these equations follows at once from the fact that $\dot{s}\sqrt{(T_1)} = \sqrt{(T)}$, and that T_1, T are like quadratic functions of $\dot{\theta}$, $\dot{\phi}$, ..., and θ, ϕ, ..., respectively. We may now advantageously change the independent variable from s to t, and as

$$\dot{s}\left\{\frac{d}{ds}\frac{d}{d\dot{\theta}}\left(\sqrt{\frac{T_1}{U}}\right)-\frac{d}{d\theta}\left(\sqrt{\frac{T_1}{U}}\right)\right\}$$

$$=\dot{s}\frac{d}{ds}\frac{d}{d\dot{\theta}}\left(\sqrt{\frac{T}{U}}\right)-\frac{d}{d\theta}\left(\sqrt{\frac{T}{U}}\right)$$

$$=\frac{d}{dt}\frac{d}{d\dot{\theta}}\left(\sqrt{\frac{T}{U}}\right)-\frac{d}{d\theta}\left(\sqrt{\frac{T}{U}}\right),$$

it follows that the general variation of t may be written in the form

$$\delta t=\Sigma_0^1\left(\frac{1}{2T}\frac{dT}{d\dot{\theta}}\delta\theta\right)$$

$$+\int_{t_0}^{t_1}\Sigma\left[\left\{\frac{d}{d\theta}\left(\sqrt{\frac{T}{U}}\right)-\frac{d}{dt}\frac{d}{d\dot{\theta}}\left(\sqrt{\frac{T}{U}}\right)\right\}d\theta\right]dt;$$

and hence that the general equations of the brachistochrone motion are

$$\frac{d}{dt}\frac{d}{d\dot{\theta}}\left(\sqrt{\frac{T}{U}}\right)-\frac{d}{d\theta}\left(\sqrt{\frac{T}{U}}\right)=0.$$

....

These equations are very much in appearance like Lagrange's equations for the motion of a free system.

Referring again to the brachistochronous particle we know that the effect of the constraints is a force opposite in direction to and of twice the magnitude of the component of the field of force normally to the path taken; we proceed now to investigate the analogue, or rather the extension of this property.

Let Θ, Φ, Ψ, ..., be the generalized components of the field of force Θ', Φ', Ψ', ..., the generalized constraints. Of course the constrained brachistochronous motion may be viewed as a free and unconstrained motion, provided we supplement the field of force by the constraints. Hence, applying Lagrange's equations, we see that the motion from this point of view is given by equations of the type

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\theta}}\right)-\frac{dT}{d\theta}=\Theta+\Theta'\equiv\frac{dU}{d\theta}+\Theta'.$$

But on expanding the equations of the motion before obtained, we get

$$\frac{1}{\sqrt{(TU)}}\left\{\frac{d}{dt}\left(\frac{dT}{d\dot{\theta}}\right)-\frac{dT}{d\theta}\right\}+\frac{\sqrt{T}}{\sqrt{U^3}}\frac{dU}{d\theta}+\frac{dT}{d\dot{\theta}}\frac{d}{dt}\left\{\frac{1}{(\sqrt{TU})}\right\}=0.$$

Now in such an expression as $\frac{d}{dt}\left\{\frac{1}{\sqrt{(TU)}}\right\}$, as the differentiation is total, and as throughout all time $T=U$, even in the brachistochrone, we may write

$$\frac{d}{dt}\frac{1}{\sqrt{(TU)}}=\frac{d}{dt}\left(\frac{1}{U}\right)=-\frac{1}{U^2}\left(\frac{dU}{d\theta}\dot{\theta}+\frac{dU}{d\phi}\dot{\phi}+\ldots\right)$$

$$=-\frac{1}{U^2}(\Theta\dot{\theta}+\Phi\dot{\phi}+\ldots\ldots.),$$

and the above equation may now be written

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\theta}}\right)-\frac{dT}{d\theta}+\Theta-\frac{1}{T}\frac{dT}{d\dot{\theta}}(\Theta\dot{\theta}+\Phi\dot{\phi}+\ldots)=0.$$

But $$\frac{d}{dt}\left(\frac{dT}{d\dot{\theta}}\right)-\frac{dT}{d\theta}-\Theta-\Theta' \qquad =0,$$

hence, on comparing, we get

$$\Theta'+2\Theta-\frac{1}{T}\frac{dT}{d\dot{\theta}}(\Theta\dot{\theta}+\Phi\dot{\phi}+\Psi\dot{\psi}+\ldots)=0,$$

this gives the generalized component of the constraint. It may be written in many ways; thus

$$\Theta'=\frac{1}{T}\left[\frac{dT}{d\dot{\theta}}(\Theta\dot{\theta}+\Phi\dot{\phi}+\ldots)-2T\Theta\right]$$

$$=\frac{1}{T}\left[\frac{dT}{d\dot{\theta}}(\Theta\dot{\theta}+\Phi\dot{\phi}+\ldots)-\Theta\left(\frac{dT}{d\dot{\theta}}\dot{\theta}+\ldots\right)\right]$$

$$=\frac{1}{T}\Sigma\dot{\phi}\left(\Phi\frac{dT}{d\dot{\theta}}-\Theta\frac{dT}{d\dot{\phi}}\right),$$

where the summation Σ extends to $(n-1)$ terms corresponding to $\dot{\phi}$, $\dot{\psi}$, ..., excluding $\dot{\theta}$, to which the component corresponds.

This latter form shows that $\Sigma\Theta'\dot{\theta}=0$, and thereby verifies, as it should, the fact that when the system moves as it does then the constraints do no work.

Again, if instead of calling the constraint Θ' we designate the whole component in the constrained motion Θ, then as $\Theta_1 = \Theta + \Theta'$ the above theorem takes the form

$$\Theta + \Theta_1 = \frac{1}{T}\frac{dT}{d\dot{\theta}}[\Theta\dot{\theta} + \Phi\dot{\phi} + \Psi\dot{\psi} + \ldots];$$

this gives an easy method of connecting the fields of stress in the two cases and contains the whole problem.

It is easy to see that the foregoing give the results so well known in the case of a single particle. In this case, on taking the particle's mass equal to unity,

$$T = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2),$$

$$U = \int Xdx + Ydy,$$

and hence, on substituting in the formula,

$$X' = \frac{1}{T}\{\dot{y}(Y\dot{x} - X\dot{y})\} = 2\frac{dy}{ds}\left(Y\frac{dx}{ds} - X\frac{dy}{ds}\right)$$

and $$Y' = \frac{1}{T}\{\dot{x}(X\dot{y} - Y\dot{x})\} = 2\frac{dx}{ds}\left(X\frac{dy}{ds} - Y\frac{dx}{ds}\right).$$

Now if N be the component of the field of force normally *outwards* at any point of the brachistochrone.

(i) If the brachistochrone be *concave* to the axis of x, then the direction-cosines of this outward drawn normal are

$$-\frac{dy}{ds}, \ +\frac{dx}{ds}$$

respectively, and here therefore

$$N = +Y\frac{dx}{ds} - X\frac{dy}{ds},$$

so that $$X' = +2N\frac{dy}{ds},$$

$$Y' = -2N\frac{dx}{ds},$$

showing that X', Y' are the components parallel to the axes of a force $2N$ normally *inwards*.

(ii) If, on the contrary, the curve be convex, then the outward normal is

$$+\frac{dy}{ds}, \ -\frac{dx}{ds},$$

and
$$N = +X\frac{dy}{ds} - Y\frac{dx}{ds},$$

and
$$X' = -2N\frac{dy}{ds},$$

$$Y' = +2N\frac{dx}{ds},$$

showing, as before, that X', Y' are the components of a force $2N$ normally inwards to the path. This verifies, as a particular case, the fundamental theorem in the case of a particle, viz. "that the constraint reverses the sign of the normal component of the field of force."

Or, again, as

$$X + X_1 = \frac{1}{T}\frac{dT}{d\dot{x}}(X\dot{x} + Y\dot{y})$$

$$= 2\frac{dx}{ds}\left(X\frac{dx}{ds} + Y\frac{dy}{ds}\right),$$

and a similar expression in Y, it follows that the resultant of the free and constrained fields of force is of magnitude

$$2\left(X\frac{dx}{ds} + Y\frac{dy}{ds}\right),$$

and is along the tangent to the path.

In three dimensions we have merely an extra term, and

$$X + X_1 = 2\frac{dx}{ds}\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right),$$

showing, as before, that the two fields of force have equal and like tangential components, and equal but unlike normal components.

Let us now take the simplest example we can, viz. the case of a rod moving in a vertical plane. Suppose, to fix ideas, that we wish to know how this rod must be constrained to move in order that starting from rest in a vertical position it may under gravity come into a given horizontal position in the least possible time.

Here on taking the mass unity, we get

$$\left.\begin{array}{l} T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \kappa^2\dot{\theta}^2) \\ U = gy \end{array}\right\},$$

and the formulæ give

$$X + X_1 = \dot{x}\,\frac{2g\dot{y}}{\dot{x}^2 + \dot{y}^2 + \kappa^2\dot{\theta}^2},$$

$$Y + Y_1 = \dot{y}\,\frac{2g\dot{y}}{\dot{x}^2 + \dot{y}^2 + \kappa^2\dot{\theta}^2},$$

$$\Theta + \Theta_1 = \kappa^2\dot{\theta}\,\frac{2g\dot{y}}{\dot{x}^2 + \dot{y}^2 + \kappa^2\dot{\theta}^2}.$$

But $X = 0$, $Y = g$, $\Theta = 0$; hence the equations determining the brachistochrone are

$$\ddot{x} = \dot{x}\,\frac{2g\dot{y}}{\dot{x}^2 + \dot{y}^2 + \kappa^2\dot{\theta}^2},$$

$$\ddot{y} = -g + \dot{y}\,\frac{2g\dot{y}}{\dot{x}^2 + \dot{y}^2 + \kappa^2\dot{\theta}^2},$$

$$\ddot{\theta} = \dot{\theta}\,\frac{2g\dot{y}}{\dot{x}^2 + \dot{y}^2 + \kappa^2\dot{\theta}^2}.$$

On multiplying by $\dot{x}$, $\dot{y}$, $\dot{\theta}$, we notice that one integral of the equations is the equation of energy, as it should be. On substituting for $(\dot{x}^2 + \dot{y}^2 + \kappa^2\dot{\theta}^2)$ its value $2gy$ and writing z for $\kappa\theta$, the equations are

$$\left.\begin{aligned} \ddot{x} &= \dot{x}\,\frac{\dot{y}}{y} \\ \ddot{y} &= -g + \frac{\dot{y}^2}{y} \\ \ddot{z} &= \dot{z}\,\frac{\dot{y}}{y} \end{aligned}\right\}.$$

These solve quite easily, thus:

$$\left.\begin{aligned} \dot{x} = ay, \quad \dot{z} = by \\ \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2gy \end{aligned}\right\};$$

and

therefore $$\dot{y}^2 + (a^2 + b^2)\,y^2 = 2gy;$$

and therefore $$\frac{dy}{\sqrt{\{2gy - (a^2 + b^2)\,y^2\}}} = +\,dt,$$

or $$\frac{\frac{1}{2}y^{-\frac{1}{2}}\sqrt{(a^2 + b^2)}\,dy}{\sqrt{\{2g - (a^2 + b^2)\,y\}}} = \tfrac{1}{2}dt\,\sqrt{(a^2 + b^2)},$$

therefore

$$y = \frac{2g}{a^2+b^2}\sin^2\{\tfrac{1}{2}t\sqrt{(a^2+b^2)}\} = \frac{g}{a^2+b^2}[1-\cos\{t\sqrt{(a^2+b^2)}\}].$$

The constant of integration is put zero, for initially y is zero.

On substituting in the values of $\dot{x}$, $\dot{z}$, we have

$$\dot{x} = \frac{ag}{a^2+b^2}\{1-\cos t\sqrt{(a^2+b^2)}\},$$

$$\dot{z} = \frac{bg}{a^2+b^2}\{1-\cos t\sqrt{(a^2+b^2)}\}.$$

These give

$$x = \frac{ag}{(a^2+b^2)^{\frac{3}{2}}}\{t\sqrt{(a^2+b^2)} - \sin t\sqrt{(a^2+b^2)}\},$$

$$\kappa\theta = z = \frac{bg}{(a^2+b^2)^{\frac{3}{2}}}\{t\sqrt{(a^2+b^2)} - \sin t\sqrt{(a^2+b^2)}\},$$

and the constants a, b, and the time of transit t will be given by making the equations in x, y, θ satisfied by $\theta = \frac{1}{2}\pi$, $x = x_1$, $y = y_1$, defining the final configuration. The path of the centre appears to be a portion of cycloid whose base is horizontal, cusp at the starting point, but whose vertical ordinates are increased in the ratio $\sqrt{(a^2+b^2)} : a$. This ratio here is $\sqrt{\{x_1^2 + \frac{1}{4}\pi^2\kappa^2\}} : x_1$.

INSIGNIORES ORBITAE COMETARUM PROPRIETATES.

By *C. Taylor, M.A.*

(Continued from page 71).

Having given some account of Lambert's work on comets so far as regards the parabola and motion therein, we shall next prove his theorem in elliptic motion by the projective method applicable to the ellipse only, reserving for a future occasion the less restricted geometrical proof by which the theorem is established in a way applicable to central conics in general.

III.

1. The geometrical equivalent of Lambert's theorem in elliptic motion is as follows:

If in any two ellipses described on equal major axes there be taken equal chords QQ' such that $SQ + SQ'$ (where S is a focus) is of the same magnitude in both ellipses, the areas of the two focal sectors SQQ' are in the subduplicate ratio of the latera recta of their ellipses.

In order to prove this property, and to deduce a formula for the time of describing any arc QQ' of an elliptic orbit under a force to either focus, we shall first establish the following lemmas, the former of which is proved by Lambert, in *Sectio* IV. § 180. p. 109.

LEMMA A.

2. *The chord of any arc of an ellipse intercepts on either of the focal vectors to the point of contact of the parallel tangent a length equal to the diametral sagitta of the projectively corresponding arc of the major auxiliary circle.*

For let SP (fig. 7) be any focal vector in an ellipse whose major axis is ACA'; QQ' any chord parallel to the tangent at P, and qq' the projectively corresponding chord of the circle on AA' as diameter; and let m and o be the middle points of QQ' and qq', and p the point on the circle corresponding to P, so that om and pP are both perpendicular to AA' and therefore parallel to one another.

Let SP meet QQ' in O, and let it meet the parallel diameter in K, so that (by a well-known property of the ellipse) PK is equal to CA or Cp.

Then by parallels,

$$PO : PK = Pm : PC = po : pC = po : KP,$$

or PO is equal to po, as was to be proved.

If now we suppose QQ' to slide parallel to itself, and in its new position to meet the same line KP in O'; and if o' be the new position of o; it follows that OO' is always equal to oo'.

For example, if O' coincide with S, and if s be the corresponding position of o', it follows that SO is equal to so; that is to say, *SO is equal to the altitude of the triangle qSq'.*

It is thus evident that every segment of the line $KSOP$ is equal to the corresponding segment of the radius $Csop$ of the auxiliary circle.

Hence also, if om meet AA' in M,

$$KS.KO = Cs.Co = CS.CM,$$

since the angles at s and M are right angles.

LEMMA B.

3. *The diameter parallel to any focal chord of an ellipse is equal to the projectively corresponding chord of its major auxiliary circle.*

If QQ' (in the same figure) be any chord of an ellipse parallel to the radius CD, and if qq' be the corresponding chord of the auxiliary circle and CA its radius, then

$$QQ' : qq' = CD : CA.$$

Now if QQ' be a focal chord,

$$QQ'.CA = 2CD^2.$$

Therefore in this case (if ff' be the corresponding position of qq', and s its middle point), we have

$$sf = \tfrac{1}{2}qq' = CD,$$

as was to be proved.

4. *Proof of Lambert's theorem.*

We may now prove Lambert's theorem in the geometrical form in which it has been stated above.

Having made the same construction as in the Lemmas, describe any second ellipse upon an *equal* major axis AA', and complete the second figure in the same manner as the first, with the proviso that SP shall be of the same magnitude in both ellipses,* and PO likewise of the same magnitude in both.

(*a*) Then by Lemma A, since PO is of the same length in the two ellipses, po is of the same length in the two circles; and therefore (the circles being equal) their chords qq' are likewise equal.

And since SP is of the same length in both ellipses, therefore CD, which is a mean proportional to SP and $AA' - SP$, is of the same length in both.

Hence, in virtue of the projective relation

$$QQ' : CD = qq' : CA,$$

it follows that QQ' is of the same length in both ellipses.

(*b*) It is now evident that the segments qpq' are equal in the two circles.

* Take SP at random in the less eccentric ellipse, then an equal SP can always be found in the other, since the range of values of SP, between AS and $A'S$, must increase with the eccentricity, when $AS + A'S$ is given.

Also the triangles Sqq' on equal bases qq', having also equal altitudes SO (Lemma A), are equal to one another.

Therefore the sectors Sqq' are equal, and therefore (the major axes of the two ellipses being equal) the focal sectors SQQ' are as the minor axes of their ellipses, or in the subduplicate ratio of their latera recta.

(c) Let Cn and Cn' be the abscissæ of Q and Q', and let e denote the eccentricity.

Then
$$\begin{aligned} SQ + SQ' &= (CA + e.Cn) + (CA - e.Cn') \\ &= AA' - \frac{2CS}{CA}.CM \\ &= AA' - \frac{2KS.KO}{CA}; \qquad \text{[Lemma A} \end{aligned}$$

and therefore $SQ + SQ'$ is of the same magnitude in both ellipses.

Conversely:

In any two ellipses described on equal major axes any pair of focal sectors SQQ' will be in the subduplicate ratio of their latera recta provided that the length of the chords QQ' and the sum of the terminal focal vectors $SQ + SQ'$ be the same in both ellipses.

Hence it appears that the area of any focal sector SQQ' of an ellipse divided by the square root of its latus rectum may be expressed in terms of $SQ + SQ'$, the chord QQ', and the major axis, and is *independent of the magnitude of the minor axis.*

And since the time in any arc QQ' of an elliptic orbit described under a given force to the focus S is proportional to the sectorial area SQQ' divided by the square root of the latus rectum (Newton's *Principia*, Lib. I. Sect. III. prop. 14, theorem 6), it follows that the time of describing the arc QQ' may be expressed in terms of $SQ + SQ'$, QQ', and the major axis, and is independent of the minor axis.

5. *To determine a formula for the time of describing any arc of an elliptic orbit.*

The formula may be written (with the notation of the proof given in the *Messenger*, vol. VII. 98),

$$nt = \{\beta + \alpha - \sin(\alpha + \beta)\} - \{\beta - \alpha - \sin(\beta - \alpha)\},$$

where t is the time in any arc QQ', n the mean motion, and α and β are certain angles which can be expressed in terms of AA', QQ', $SQ + SQ'$.

We shall give three methods of obtaining this result with the help of the geometrical properties proved above.

The expression for t in terms of AA', QQ', $SQ+SQ'$ has been shown to be *independent of the minor axis* (§ 4), and will therefore be the same when the minor axis vanishes and the motion takes place, under the same law, in a flat ellipse or *straight line.*

Now in any given ellipse, described under a force to the focus S, the time of describing any arc QQ' varies as the sectorial area SQQ', and therefore as the corresponding area Sqq' in the auxiliary circle.

Let the minor axis vanish—in which case the focus S coincides with an extremity A of the major axis: then the time of traversing the portion nn' (fig. 7) of the axis which is the projection of qq' thereupon varies as the area qAq', or as the difference of the circular segments cut off by Aq and Aq'.

The area of the segment ACq is equal to

$$\tfrac{1}{2}CA^2(\text{arc}Aq - \sin ACq);$$

and the area of the segment ACq' is equal to

$$\tfrac{1}{2}CA^2(\text{arc}ACq' - \sin ACq').$$

Hence if α denote half the difference and β half the sum of the angles ACq and ACq', we may at once deduce the above-mentioned formula,

$$nt = \{\beta + \alpha - \sin(\alpha+\beta)\} - \{\beta - \alpha - \sin(\beta - \alpha)\}.$$

And since (fig. 7)

$$\frac{An - \frac{1}{2}AA'}{\frac{1}{2}AA'} = \cos qCn = -\cos(\beta+\alpha);$$

and

$$\frac{\frac{1}{2}AA' - An'}{\frac{1}{2}AA'} = \cos q'Cn' = \quad \cos(\beta - \alpha);$$

and

$$An' - An = nn';$$

it follows that

$$1 - \cos(\beta \pm \alpha) = \frac{An + An' \pm nn'}{AA'}.$$

That is to say, $\beta+\alpha$ and $\beta-\alpha$ are determined as functions of $An+An'$, nn', and AA'; and therefore t is a function of the same three magnitudes.

Hence also in the general case of motion in an elliptic arc QQ', the time t may be expressed as the same function of $SQ+SQ'$, QQ', and AA'.

6. *Second method.*

Having reduced the motion to motion in a flat ellipse or straight line, we may determine the time t by integrating the equation

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}.$$

Hence, supposing the motion to commence at the distance $2a$ from the centre of force,

$$\sqrt{\left(\frac{\mu}{a^3}\right)}\,.\,t = \sqrt{\left(\frac{2x}{a} - \frac{x^2}{a^2}\right)} - \text{vers}^{-1}\frac{x}{a} + \pi,$$

where t is the time of passing from the distance $2a$ to the distance x from the centre of force.

The result may be written in the form

$$nt = \sin\theta - \theta + \pi,$$

where $$1 - \cos\theta = \frac{x}{a} = \frac{x + x' + k}{2a},$$

if $x - x'$ be put equal to k.

In like manner the time of reaching the distance x' from the centre of force is given by

$$nt' = \sin\phi - \phi + \pi,$$

where $$1 - \cos\phi = \frac{x'}{a} = \frac{x + x' - k}{2a}.$$

Hence $$n(t' - t) = \{\theta - \sin\theta\} - \{\phi - \sin\phi\},$$

and the time of traversing the length k may be obtained as a function of $x + x'$, k, $2a$, as in the former proof.

7. *Third method.*

Having made the same construction as in the Lemmas, let α denote the angle oCq and β the angle sCf (fig. 7). Then, referring to Lemma A and § 4 (c),

$$\cos\alpha.\cos\beta = \frac{Co.Cs}{CA^2} = \frac{CM.CS}{CA^2} = 1 - \frac{SQ + SQ'}{AA'};$$

and referring to Lemma B,

$$\sin\alpha.\sin\beta = \frac{\frac{1}{2}qq'.sf}{CA^2} = \frac{\frac{1}{2}qq'.CD}{CA^2} = \frac{QQ'}{AA'}.$$

Therefore $$1 - \cos(\beta - \alpha) = \frac{SQ + SQ' \pm QQ'}{AA'}.$$

Now in the circle,

$$\begin{aligned}\text{sector } Sqq' &= \text{sect. } Cqq' - \Delta\,(Cqq' - Sqq')\\ &= \text{sect. } Cqq' - qo.Cs\\ &= CA^2(\alpha - \sin\alpha.\cos\beta);\end{aligned}$$

whence we obtain as before,

$$nt = \{\beta + \alpha - \sin(\beta + \alpha)\} - \{\beta - \alpha - \sin(\beta - \alpha)\},$$

$\beta \pm \alpha$ having been already shown to be functions of $SQ + SQ'$, QQ', and AA'.

8. It appears from §4 (a) and (b) that the two elliptic arcs QQ' will be isochronous provided that SP and PO be of the same length in both.

This follows for the limiting case in which PO is *infinitesimal* from Newton's *Principia* Lib. I. Sect. II. prop. 6, theorem 5. Lambert shows that the arcs QQ' will still be isochronous when the PO's are equal and *finite*.

(To be continued.)

ON THE EXACT RELATION WHICH RESULTANTS AND DISCRIMINANTS BEAR TO THE PRODUCT OF DIFFERENCES OF ROOTS OF EQUATIONS.

By Professor *J. J. Sylvester*.

First, for Resultants.

Let there be two rational integral functions in x of the degrees r, s respectively; and, for greater simplicity, let the coefficients of x^r, x^s in these functions be each made equal to unity. Call ρ the roots of the one, σ of the other; and denote the product of the differences found by subtracting each σ from each ρ by $D_{\rho,\sigma}$.

Also, by the resultant $R_{r,s}$ understand that irreducible rational integral function of the coefficients, vanishing when the functions have a root in common, in which the highest power of the last coefficient of the "s" equation enters with the positive sign.

We must then have $R_{r,s} = \mu D_{\rho,\sigma}$; and it only remains to determine μ as a function of r, s.

To do this let the r function become x^r, and the s function $x^s + 1$.

For greater distinctness, suppose $r = 4$, $s = 2$.

Then, obviously, $R_{r,s}$ becomes the dialytic resultant of

$$\begin{array}{cccccc}
x^5 & & & & & \\
 & x^4 & & & & \\
x^5 & & +x^3 & & & \\
 & x^4 & & +x^2 & & \\
 & & x^3 & & +x & \\
 & & & x^2 & & +1
\end{array}$$

which is equal to 1.

And in like manner for all values of r, s,

$$R_{r,s} = 1.$$

Again, $$D_{\rho,\sigma} = \{0 - (-1)^{\frac{1}{s}}\}^{rs} = (-)^{rs+r}.$$

Hence μ, which is a function of r, s exclusively, $= (-)^{rs+r}$.

Next, for Discriminants.

By the discriminant of fx of the order n, and where, for greater simplicity, the coefficient of x^n is supposed to be unity, I mean the resultant of fx and $f'x$; or, which is the same thing, of $\frac{df(x,1)}{dx}$ and $\frac{df(x,1)}{d1}$, when the term in which the highest power of the last coefficient in fx appears is made positive. Let this be called R_n, and the product of the squared differences of the roots Z_n; we have then $R_n = \mu Z_n$, where μ is a function of n to be determined. To find it let us take $fx = x^n - 1$.

R_n is then the resultant of nx^{n-1}, $-ny^{n-1}$, *i.e.* is equal to

$$(-)^{n-1} n^{2n-2}.$$

Again,

$$\left.\begin{matrix} Z_n = (1-\rho) & (1-\rho^2) & \ldots(1-\rho^{n-1}) \\ (\rho-\rho^2) & (\rho-\rho^3) & \ldots(\rho-1) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ (\rho^{n-1}-1) & (\rho^{n-2}-\rho) & \ldots(\rho^{n-1}-\rho^{n-2}) \end{matrix}\right\} \div (-)^{\frac{1}{2}(n.n-1)},$$

ρ representing $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

Hence $$Z_n = n^n.(-)^{\frac{1}{2}n(n-1)}.\{\rho^{\frac{1}{2}n(n-1)}\}^{n-1}$$
$$= (-)^{\theta} n^n,$$

where $\theta = \frac{1}{2}\{n(n-1)\} + (n-1)^2 = \frac{1}{2}\{(n-1)(n-2)\}$,

and $$\theta + (n-1) = \frac{1}{2}\{(n-1)n\}.$$

Hence $R_n = (-)^{\frac{1}{2}(n-1)n} n^{n-2} Z_n$, or $\mu = (-)^{\frac{1}{2}(n-1)n}$, which was to be found.

For ordinary algebraical investigations the determination of μ has little importance, which may account for its value being omitted in the ordinary text books; but for certain investigations concerning the numerical divisors of cyclotomic functions, with which I am occupied, I found it necessary to pay attention to the numerical part at least of this factor, and I have thought that the publication of the result might save others some unnecessary trouble.

Johns Hopkins University, Baltimore,
27*th Jan.*, 1880.

ON CERTAIN SERIES WHOSE COEFFICIENTS ARE THE INVERSES OF BINOMIAL COEFFICIENTS.

By *Samuel Roberts, M.A., F.R.S.*

1. From the general equivalence

$$\frac{F(1)}{m+1}-\frac{F'(1)}{m}+\frac{F''(1)}{m-1.1.2}-\frac{F'''(1)}{m-2.1.2.3}+\&c.$$

$$=\frac{F(0)}{m+1}-\frac{F'(0)}{m+1.m}+\frac{F''(0)}{m+1.m.m-1}-\frac{F'''(0)}{m+1.m.m-1.m-2}+\&c.$$

we get, by putting $F(y)=\dfrac{1}{1-xy}$,

$$1-\frac{1}{m}x+\frac{1.2}{m.m-1}x^2-\frac{1.2.3}{m.m-1.m-2}x^3+\&c.$$

$$=(m+1)\left\{\frac{1}{m+1}\,\frac{1}{1-x}-\frac{1}{m}\,\frac{x}{(1-x)^2}+\frac{1}{m-1}\,\frac{x^2}{(1-x)^3}-\&c.\right\}.$$

The coefficients of the first member of the equation are the inverses of the coefficients of $(1-x)^m$, and I denote the expression analogously by $(1-x)_m$.

2. Let $m=\frac{1}{2}(2n-1)$, then

$$(1-x)_{\frac{1}{2}(2n-1)}=(2n+1)\left\{\frac{1}{2n+1}\,\frac{1}{1-x}-\frac{1}{(2n-1)}\,\frac{x}{(1-x)^2}+\&c.\right.$$

$$=(2n+1)\left\{\frac{1}{2n+1}\,\frac{1}{1-x}-\frac{1}{2n-1}\,\frac{x}{(1-x)^2}+\&c.(-)^n\,\frac{x^n}{(1-x)^{n+1}}\right\}$$

$$(-)^n(2n+1)\left\{\frac{x^{n+1}}{(1-x)^{n+2}}-\frac{1}{3}\,\frac{x^{n+2}}{(1-x)^{n+3}}+\frac{1}{5}\,\frac{x^{n+3}}{(1-x)^{n+4}}-\&c.\right\}$$

$$=(2n+1)\left\{\frac{1}{2n+1}\,\frac{1}{1-x}-\frac{1}{(2n-1)}\,\frac{x}{(1-x)^2}+\&c.(-)^n\,\frac{x^n}{(1-x)^{n+1}}\right\}$$

$$(-)^n(2n+1)\frac{x^{n+1}}{(1-x)^{n+2}}\left(\frac{1-x}{x}\right)^{\frac{1}{2}}\tan^{-1}\left(\frac{x}{1-x}\right)^{\frac{1}{2}},$$

or, writing $\sin^2\theta$ for x,

$$(2n+1)\,\theta = -\,(2n+1)\left\{\cot\theta - \tfrac{1}{3}\cot^3\theta + \&c.\ (-)^n \frac{1}{2n+1}\cot^{2n+1}\theta\right)$$

$$(-)^n \cot^{2n+2}\theta \cos\theta \sin\theta\,(1-\sin^2\theta)_{\frac{1}{2}(2n-1)}\ldots(A).$$

3. Next, let us suppose $m = -\tfrac{1}{2}(2n+3)$. Then

$$(1-x)_{-\frac{1}{2}(2n+3)} = (2n+1)\left\{\frac{1}{2n+1}\,\frac{1}{1-x} - \frac{1}{2n-1}\,\frac{x}{(1-x)^2} + \&c.\right\},$$

and $\displaystyle (2n+1)\frac{(1-x)^n}{x^n}\,\frac{1}{[x(1-x)]^{\frac{1}{2}}}\tan^{-1}\left(\frac{x}{1-x}\right)^{\frac{1}{2}}$

$$= (2n+1)\left\{\frac{(1-x)^{n-1}}{x^n} - \tfrac{1}{3}\frac{(1-x)\}^{n-2}}{x^{n-1}} + \&c.\ (-)^{n-1}\frac{1}{2n-1}\,\frac{1}{x}\right\}$$

$$(-)^n(2n+1)\left\{\frac{1}{2n+1}\,\frac{1}{1-x} - \frac{1}{2n-1}\,\frac{x}{(1-x)^2} + \&c.\right\}$$

$$= (2n+1)\left\{\frac{(1-x)^{n-1}}{x^n} - \tfrac{1}{3}\frac{(1-x)^{n-2}}{x^{n-1}} + \&c.\ (-)^{n-1}\frac{1}{2n-1}\,\frac{1}{x}\right\}$$

$$(-)^n(1-x)_{-\frac{1}{2}(2n+3)};$$

or, as before, writing $\sin^2\theta$ for x,

$$(2n+1)\,\theta = (2n+1)\left\{\tan\theta - \tfrac{1}{3}\tan^3\theta + \&c.\ (-)^{n-1}\frac{1}{2n-1}\tan^{2n-1}\theta\right\}$$

$$(-)^n \tan^{2n}\theta\cos\theta\sin\theta\,(1-\sin^2\theta)_{-\frac{1}{2}(2n+3)}\ldots(B).$$

Gregory's series corresponds to $n = \infty$.

4. Paying attention to the limits of convergency, we may apply these results to obtain various particular series. Thus, as to positive values of x and θ, we may take

$$x \leqq \tfrac{1}{2},\quad \theta \leqq \tfrac{1}{4}\pi.$$

The resulting series in simple cases, of course, come out in known forms, but the expressions exhibit in a striking manner the connection of different series.

For instance, if in (A) we put $n = 0$,

$$\theta = -\frac{\cos\theta}{\sin\theta}$$

$$+\frac{\cos^3\theta}{\sin\theta}\left(1 + \frac{2}{1}\sin^2\theta + \frac{1.2.2^2}{1.3}\sin^4\theta + \frac{1.2.3.2^3}{1.3.5}\sin^6\theta + \&c.\right);$$

or, substituting $1-\sin^2\theta$ for $\cos^2\theta$, and multiplying out,

$$\theta=\sin\theta\cos\theta\left(1+\frac{1}{1.3}2\sin^2\theta+\frac{1.2.2^2}{1.3.5}\sin^4\theta+\frac{1.2.3.2^3}{1.3.5.7}\sin^6\theta+\&c.\right\}.$$

If $n=1$,

$$\theta=-(\cot\theta-\tfrac{1}{3}\cot^3\theta)-\cot^5\theta\cos\theta\sin\theta$$

$$\times\left(1-\frac{2}{1}\sin^2\theta-\frac{1.2.2^2}{1.1}\sin^4\theta-\frac{1.2.3.2^3}{1.1.3}\sin^6\theta-\&c.\right).$$

In like manner, putting $n=0$ in (B), we get

$$\theta=\cos\theta\sin\theta(1+\tfrac{2}{3}\sin^2\theta+\&c.),$$

a result above given as derived from (A).

If $n=1$,

$$\theta=\tan\theta\left(1-\tfrac{1}{3}\sin^2\theta-\frac{1.2}{3.5}\sin^4\theta-\frac{1.2.2^2}{3.5.7}\sin^6\theta-\&c.\right).$$

which is Mr. Clarkson's series, *Proc. Roy. Soc.* XI. p. 489, and so on.*

The same series has been derived from both formulæ; in fact, we have obviously a linear relation between $(1-x)_{\frac{1}{2}(2n-1)}$, and $(1-x)_{-\frac{1}{2}(2n+3)}$. The relation is found by equating the two values of $(2n+1)\theta$, which is permissible since the limits are similar.

Let y stand for $1-x$, then we get

$$y^{2n+1}(1-x)_{\frac{1}{2}(2n-1)}-x^{2n+1}(1-x)_{-\frac{1}{2}(2n+3)}$$

$$=(-)^n(2n+1)x\left[x^{n-1}y^{n-1}(x+y)-\tfrac{1}{3}\{x^{n-2}y^{n-2}(x^3+y^3)\}+\&c.\right.$$

$$\left.(-)^{n-1}\frac{1}{2n-1}(x^{2n-1}+y^{2n-1})\right]+y^{2n}.$$

Although, therefore, the series given by one formula can be derived from the other, the process is complicated for higher values of n.

5. If $\theta=\frac{1}{4}\pi$ in (A), we have

$$(2n+1)\tfrac{1}{4}\pi=-2(2n+1)\left\{1-\tfrac{1}{3}+\&c.(-)^n\frac{1}{2n+1}\right\}$$

$$(-)^n(1-\tfrac{1}{2})_{\frac{1}{2}(2n-1)}.$$

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* Making $\theta=\frac{1}{4}\pi$ or $\frac{1}{6}\pi$, these series give results for the calculation of π. Compare Mr. Glaisher's list in *Messenger* vol. VII. p. 75 (1877), "On series and products for π."

But $(1-\frac{1}{2})_{\frac{1}{2}(2n-1)} = 1 - \frac{1}{2n-1} + \frac{1.2}{2n-1.2n-3} - \&c.$

$$(-)^n \left\{\frac{1.2\ldots n}{2n-1\ldots 3.1} - \frac{1.2\ldots n.1}{2n-1\ldots 3.1.(-1)} + \&c.\right\}$$

$$= 1 - \frac{1}{2n-1} + \frac{1.2}{2n-1.2n-3} - \&c. (-)^n \frac{1.2.3\ldots 2n+1}{2n-1\ldots 3.1.1.3\ldots 2n+1}$$

$$(-)^n \frac{1.2.3.2n+2}{(2n-1\ldots 3.1)^2 2n+1.2n+3}\left(1 + \frac{2n+3}{2n+5} + \frac{2n+3.2n+4}{2n+5.2n+7} + \&c.\right);$$

and by an independent process (by means of definite integrals or otherwise) it appears that

$$\tfrac{1}{2}\pi = \frac{2.4.6\ldots 2n+2}{1.3.5\ldots 2n+1}\,\frac{1}{2n+3}\left\{1 + \frac{2n+3}{2n+5} + \frac{2n+3.2n+4}{2n+5.2n+7} + \&c.\right\}.$$

Consequently, comparing this result with the previous one, we see that

$$1 - \frac{1}{2n-1} + \&c. (-)^n \frac{1.2.3\ldots 2n+1}{(2n-1.2n-3\ldots 3.1)^2 2n+1}$$

$$= 2(2n+1)\left\{\frac{1}{2n+1} - \frac{1}{2n-1} + \&c. (-)^n 1\right\}.$$

The left-hand expression is the sum of the first $2n+2$ terms of $(1-\frac{1}{2})_{\frac{1}{2}(2n-1)}$.

The foregoing results may be obtained by definite integrals in various ways. For example, in the case of $(1-x)_{-m}$, using the gamma function, we have

$$\frac{\Gamma 1\,\Gamma m}{\Gamma m} + \frac{\Gamma 2\,\Gamma m}{\Gamma m+1} + \&c.$$

$$= \int_0^1 [mv^{m-1}(1-v)^0 + (m+1)v^{m-1}(1-v)x + \&c.]\,dv$$

$$= \int_0^1 \frac{[m(1-x)+x]v^{m-1} + (m-1)xv^m}{(vx+1-x)^2}\,dv.$$

The process I have used is, however, more elementary.

If m is integer, the expression for $(1-x)_{-m}$ is

$$(-)^m (m-1)\frac{(1-x)^{m-2}}{x^{m-1}} \log\left(1 + \frac{x}{1-x}\right)$$

$$+ (m-1)\left\{\frac{1}{m-2}\,\frac{1}{x} - \frac{1}{m-3}\,\frac{1-x}{x^2} + \&c. (-)^{m-1}\frac{(1-x)^{m-3}}{x^{m-2}}\right\};$$

in other words, $(1-x)_{-m}$ is the series obtained by developing

$$(-)^m (m-1) \frac{(1-x)^{m-2}}{x^{m-1}} \log \left(1 + \frac{x}{1-x}\right),$$

and taking only terms whose arguments are positive powers of x.

No particular interest attaches apparently to this case or to $(1-x)_m$, where m is a positive integer.

SOME APPLICATIONS OF A THEOREM IN SOLID GEOMETRY.

By *R. R. Webb, M.A.*, St. John's College.

THE principal curvatures at any point of a surface are given in an exceedingly symmetrical form by the equation

$$\begin{vmatrix} \frac{dl}{dx} - \frac{1}{\rho}, & \frac{dl}{dy}, & \frac{dl}{dz} \\ \frac{dm}{dx}, & \frac{dm}{dy} - \frac{1}{\rho}, & \frac{dm}{dz} \\ \frac{dn}{dx}, & \frac{dn}{dy}, & \frac{dn}{dz} - \frac{1}{\rho} \end{vmatrix} = 0,$$

where l, m, n are the direction cosines of the normal at x, y, z, and the differential coefficients are all strictly partial; of course it need scarcely be remarked that the equation ought to be a quadratic, and that the above equation, though a cubic, has one value of $\frac{1}{\rho}$ zero, inasmuch as

$$\begin{vmatrix} \frac{dl}{dx}, & \frac{dl}{dy}, & \frac{dl}{dz} \\ \frac{dm}{dx}, & \frac{dm}{dy}, & \frac{dm}{dz} \\ \frac{dn}{dx}, & \frac{dn}{dy}, & \frac{dn}{dz} \end{vmatrix} \equiv 0,$$

in consequence of the identical relation $l^2 + m^2 + n^2 \equiv 1$.

This zero value does not correspond to anything in the present case and has to be rejected. If, then, ρ_1, ρ_2 be the two values of ρ, the above equation gives

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{dl}{dx} + \frac{dm}{dy} + \frac{dn}{dz}.$$

It is the object of this paper to give a few applications of this formula.

I. If ξ, η, ζ be any continuous functions of x, y, z, then

$$\iiint\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)dx\,dy\,dz=\iint(l\xi+m\eta+n\zeta)\,dS,$$

where the latter integration extends over the surface enclosing the volume in the former integration, and l, m, n are the direction cosines of the outward drawn normal at the element dS under consideration.

Now let us imagine a single closed surface, and imagine also the whole of the interior filled up with surfaces after any finite law, in such a way that the bounding surface is one of the set. Let ξ, η, ζ be the direction cosines of the normal at any point x, y, z of one of these; ξ, η, ζ must be expressed as functions of x, y, z alone, and the arbitrary parameter defining the surface through x, y, z must be eliminated by substituting the value it has as a function of x, y, z. Then, since in this case $\xi=l$, $\eta=m$, $\zeta=n$ at the boundary, the foregoing transformation gives

$$\text{surface of exterior}=\iiint\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)dx\,dy\,dz$$

$$=\iiint\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)dx\,dy\,dz.$$

Of course, if there be an interior surface, as there would be if the volume of integration were a shell, we should have

$$\text{exterior surface}-\text{interior surface}=\iiint\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)dx\,dy\,dz.$$

Hence we infer that the mean value of $\frac{1}{\rho_1}+\frac{1}{\rho_2}$ is

$$\frac{\text{exterior surface}-\text{interior surface}}{\text{volume}},$$

it being understood that the number of observations taken within an elementary volume is proportional to the volume of the element.

This theorem can be proved in an exceedingly elementary way. Thus, consider an element dS of one surface of the series, erect the normals at each point of the contour of the element, produce them to meet the consecutive surfac in the element dS'. Then, since ultimately these normals are normals to dS' as well, the two elements have in the end the same total curvature, and therefore

$$dS' = dS\left(1 + \frac{dn}{\rho}\right)\left(1 + \frac{dn}{\rho'}\right),$$

or
$$dS' - dS = dS.dn\left(\frac{1}{\rho} + \frac{1}{\rho'}\right),$$

or
$$dS' - dS = \left(\frac{1}{\rho} + \frac{1}{\rho'}\right) \text{ elementary volume,}$$

leading to the same as before.

Consider an example. Take a sphere and fill it with spheres, all touching at a given point; they will be

$$x^2 + y^2 + z^2 = 2cz.$$

Here, then,
$$\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{4z}{x^2 + y^2 + z^2},$$

and the volume integral

$$= 4\iiint \frac{z\,dx\,dy\,dz}{x^2 + y^2 + z^2};$$

changing to polar coordinates, it becomes

$$4\int_0^{2\pi}\int_0^{\frac{1}{2}\pi}\int_0^{2c\cos\theta} r\sin\theta\cos\theta\,d\phi\,d\theta\,dr$$

$$= 4\pi\int_0^{\frac{1}{2}\pi}(2c)^2\sin\theta\cos^3\theta\,d\theta$$

$$= 4\pi c^2 = \text{area of exterior sphere.}$$

Or, again, take the series of similar ellipsoids,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = k^2,$$

where k is a parameter defining the series, and $k = 1$ gives the exterior surface. Here

$$l = \frac{\frac{x}{a^2}}{\sqrt{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)}};$$

therefore

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}{\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{1}{2}}}-\frac{\frac{x^2}{a^6}+\frac{y^2}{b^6}+\frac{z^2}{c^6}}{\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{3}{2}}}.$$

Hence, applying our theorem, the surface of ellipsoid

$$=\iiint\frac{\frac{x^2}{a^4}\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+\frac{y^2}{b^4}\left(\frac{1}{c^2}+\frac{1}{a^2}\right)+\frac{z^2}{c^4}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}{\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{3}{2}}}\,dx\,dy\,dz,$$

where the volume integration is to be taken so as to include the whole of the ellipsoid. I think this expression was first obtained by Professor Tait, and communicated by him to Professor Cayley, who mentioned it at a meeting of the London Mathematical Society.

If we filled the ellipsoid with confocal ellipsoids, the formula would give the difference of the exterior surface and *twice* the area of the elliptic focal conic, inasmuch as the initial ellipsoid would be so flattened as to be of evanescent least principal axis, while its maximum principal section would be that conic. In this case, however, we are unable to lay down the triple integral in ordinary Cartesian coordinates, but it may be expressed in terms of elliptic coordinates.

II. In Routh's *Rigid Dynamics*, Arts. 523–532, many elegant theorems are proved with the ultimate object of getting the angular velocity in space of the plane containing the instantaneous axis and the invariable line. It seems to me that the whole may be made much more compact by first obtaining $\tan\rho$, where ρ is the great circle on a sphere of radius unity to the centre of curvature of the spherical ellipse, in which the cone described *in the body* by the invariable line meets the unit sphere at the fixed point.

This cone is

$$\frac{AT-G^2}{A}x^2+\frac{BT-G^2}{B}y^2+\frac{CT-G^2}{C}z^2=0;$$

or, calling the lengths of the principal semi-axes of the momental ellipsoid a, b, c, and p the perpendicular on the invariable plane,

$$(a^2-p^2)\,x^2+(b^2-p^2)\,y^2+(c^2-p^2)\,z^2=0.$$

Here, then,

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{(a^2-p^2)+(b^2-p^2)+(c^2-p^2)}{P^{\frac{1}{2}}}$$

$$-\frac{(a^2-p^2)^3x^2+(b^2-p^2)^3y^2+(c^2-p^2)^3z^2}{P^{\frac{3}{2}}}$$

$$=\frac{1}{P^{\frac{3}{2}}}\left[\{(b^2-p^2)(c^2-p^2)+(c^2-p^2)(a^2-p^2)+(a^2-p^2)(b^2-p^2)\}\right.$$

$$\times\{x^2(a^2-p^2)+y^2(b^2-p^2)+z^2(c^2-p^2)\}$$

$$\left.-(a^2-p^2)(b^2-p^2)(c^2-p^2)(x^2+y^2+z^2)\right],$$

where P denotes

$$(a^2-p^2)^2x^2+(b^2-p^2)^2y^2+(c^2-p^2)^2z^2.$$

Now, in the present case, one of the principal curvatures is zero and the other is the reciprocal of $r_1\tan\rho$, where r_1 is the radius vector. Hence, here

$$\frac{1}{r_1\tan\rho}=-\frac{r_1^2(a^2-p^2)(b^2-p^2)(c^2-p^2)}{\{x^2(a^2-p^2)^2+y^2(b^2-p^2)^2+z^2(c^2-p^2)^2\}^{\frac{3}{2}}},$$

or $$\cot\rho=-\frac{(a^2-p^2)(b^2-p^2)(c^2-p^2)}{\{(a^2-p^2)^2l^2+(b^2-p^2)^2m^2+(c^2-p^2)n^2\}^{\frac{3}{2}}},$$

where l, m, n are the direction cosines of p.

Now $$a^2l^2+b^2m^2+c^2n^2=p^2,$$

$$l^2+m^2+n^2=1,$$

and $$p^2r^2=a^4l^2+l^4m^2+c^4n^2,$$

where r is the radius vector to the point of contact; therefore

$$\cot\rho=-\frac{(a^2-p^2)(b^2-p^2)(c^2-p^2)}{p^3(r^2-p^2)^{\frac{3}{2}}}$$

$$=-\frac{(a^2-p^2)(b^2-p^2)(c^2-p^2)}{p^6}\cot^3\zeta$$

$$=-\left(\frac{a^2}{p^2}-1\right)\left(\frac{b^2}{p^2}-1\right)\left(\frac{c^2}{p^2}-1\right)\cot^3\zeta.$$

Now $$\frac{a^2}{p^2}=\frac{G^2}{AT},\ \frac{b^2}{p^2}=\frac{G^2}{BT},\ \frac{c^2}{p^2}=\frac{G^2}{CT};$$

therefore

$$\cot\rho=-\left(\frac{G^2}{AT}-1\right)\left(\frac{G^2}{BT}-1\right)\left(\frac{G^2}{CT}-1\right)\cot^3\zeta.$$

Hence, proceeding exactly as in Routh, Art. 532, where L, I are respectively the points where the invariable line and instantaneous axes meet the unit sphere, we get the velocity of the point I perpendicular to the arc LI equal to

$$\frac{\sin(\rho+\zeta)}{\sin\rho} \text{. velocity of } L \text{ along the cone in the body}$$

$$= \frac{T}{G}\tan\zeta(\cos\zeta + \cot\rho\sin\zeta).$$

Hence $$\sin\zeta\frac{d\phi}{dt} = \frac{T}{G}\sin\zeta(1+\cot\rho\tan\zeta);$$

therefore

$$\frac{d\phi}{dt} = \frac{T}{G}\left\{1 - \left(\frac{G^2}{AT} - 1\right)\left(\frac{G^2}{BT} - 1\right)\left(\frac{G^2}{CT} - 1\right)\cot^2\zeta\right\}.$$

III. The third application is the determination of the sum of the principal curvatures at any point of a surface differing very slightly from a sphere.

Let $r = a + bS_i$ be such a surface, where S_i is a spherical surface harmonic of order i and b so small that its square may be neglected. Then, of course, correctly to the first order of b, the surface may be written

$$r = a + \frac{b}{a^i}V_i,$$

where V_i is a spherical solid harmonic of order i.

Hence, putting $\frac{b}{a^i} = k$,

$$l : m : n :: \frac{x}{r} - k\frac{dV_i}{dx} : \frac{y}{r} - k\frac{dV_i}{dy} : \frac{z}{r} - k\frac{dV_i}{dz},$$

but $$\left(\frac{x}{r} - k\frac{dV_i}{dx}\right)^2 + \left(\frac{y}{r} - k\frac{dV_i}{dy}\right)^2 + \left(\frac{z}{r} - k\frac{dV_i}{dz}\right)^2$$

$$= 1 - 2\frac{k}{r}\left(x\frac{dV_i}{dx} + y\frac{dV_i}{dy} + z\frac{dV_i}{dz}\right)\ldots$$

$$= 1 - \frac{2ik}{r}V_i,$$

for V_i is homogeneous and of degree i in xyz.

Hence, correctly to the first power of k,

$$l = \frac{\frac{x}{r} - k\frac{dV_i}{dx}}{\sqrt{\left(1 - \frac{2ik}{r}V_i\right)}} = \frac{x}{r} - k\frac{dV_i}{dx} + \frac{ikx}{r^2}V_i,$$

so that

$$l = \frac{x}{r} - k\frac{dV_i}{dx} + i\frac{kx}{r^2}V_i,$$

$$m = \frac{y}{r} - k\frac{dV_i}{dy} + i\frac{ky}{r^2}V_i,$$

$$n = \frac{z}{r} - k\frac{dV_i}{dz} + i\frac{kz}{r^2}V_i,$$

correctly to the first power of k.

It is, perhaps, worth while remarking that these values of l, m, n satisfy $l^2+m^2+n^2=1$ correctly to the first powers of k, for

$$\begin{aligned} l^2+m^2+n^2 &= \left(\frac{x}{r} - k\frac{dV_i}{dx} + i\frac{kx}{r^2}V_i\right)^2 + \dots \\ &= \frac{x^2+y^2+z^2}{r^2} - \frac{2k}{r}\left(x\frac{dV_i}{dx} + y\frac{dV_i}{dy} + z\frac{dV_i}{dz}\right) \\ &\qquad + \frac{2ik}{r}V_i\left(\frac{x^2+y^2+z^2}{r^2}\right) \\ &= 1. \end{aligned}$$

Hence,

$$\frac{dl}{dx} = \frac{1}{r} - \frac{x^2}{r^3} - k\frac{d^2V_i}{dx^2} + i\frac{k}{r^2}V_i - 2i\frac{kx^2}{r^4}V_i + i\frac{kx}{r^2}\frac{dV_i}{dx},$$

and like expressions in y, z for $\frac{dm}{dy}$, $\frac{dn}{dz}$.

Hence, adding,

$$\begin{aligned} \frac{dl}{dx} + \frac{dm}{dy} + \frac{dn}{dz} &= \frac{2}{r} - k\nabla^2V_i + \frac{3ik}{r^2}V_i - \frac{2ik}{r^2}V_i + \frac{i^2k}{r^2}V_i \\ &= \frac{2}{r} + i(i+1)\frac{k}{r^2}V_i. \end{aligned}$$

But
$$r = a + kV_i;$$

therefore
$$\frac{1}{r} = \frac{1}{a} - \frac{k}{a^2}V_i.$$

Hence, substituting and putting $r=a$ in the small term, we get

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{2}{a}+(i^2+i-2)\frac{k}{a^2}V_i$$

$$=\frac{2}{a}+(i-1)(i+2)\frac{k}{a^2}V_i$$

$$=\frac{2}{a}+(i-1)(i+2)\frac{b}{a^2}S_i.$$

This result is very frequently required in such a case as the determination of the small oscillations of a sphere of liquid so far as depends on the capillarity at its surface. This problem has been given in a note appended to a paper on the "Stability of Jets" by Lord Rayleigh and printed in the *Proceedings of the Royal Society;* there the author uses exclusively the scientific method of energy and escapes the necessity of determining the curvatures directly. If on the other hand we merely use the old-fashioned equations of motion, we shall in satisfying the equations of pressure at the surface require the expression just obtained.

Thus, suppose the sphere $r=a$ of fluid of density σ held together in equilibrium under the tension at the surface, and let its form at any subsequent time be

$$r=a+S_i,$$

where S_i is expressible by means of surface harmonics and simple harmonic terms in the time. A suitable value of the velocity potential will be

$$\phi=\left(\frac{r}{a}\right)^i Q_i,$$

where Q_i is so far as the spherical coordinates are concerned a surface harmonic of order i.

Now as $$\frac{dr}{dt}=\frac{d\phi}{dr} \text{ when } r=a,$$

$$\frac{i}{a}Q_i=\frac{dS_i}{dt},$$

so that $$\phi=\frac{a}{i}\left(\frac{r}{a}\right)^i\frac{dS_i}{dt}$$

The equation of pressure within the fluid is

$$\frac{p}{\sigma} = C - \frac{d\phi}{dt},$$

provided we neglect the squares of the velocity, and at the surface

$$p = p_0 + T\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right).$$

Hence at the surface

$$\sigma\left(C - \frac{d\phi}{dt}\right) \equiv p_0 + T\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

$$\equiv p_0 + \frac{2T}{a} + \frac{(i+1)(i+2)}{a^2} TS_i.$$

Hence equating the surface harmonic term to zero we have

$$a\frac{\sigma}{i}\frac{d^2 S_i}{dt^2} + \frac{(i-1)(i+2)}{a^2} TS_i = 0.$$

Hence S_i goes through its period in a time

$$2\pi\sqrt{\left\{\frac{\sigma a^3 T^{-1}}{i(i-1)(i+2)}\right\}}.$$

This result is of course well known, and my only apology for reproducing it is to place it within the reach of the junior readers of the *Messenger*.

ON THE VECTOR POTENTIAL, AND ON SOME PROPERTIES OF THE SOLID HARMONICS.

By Professor *C. Niven*.

1. THE components F, G, H of the vector potential of a magnetic system are connected with the scalar potential by means of the relations

$$\left.\begin{aligned} \frac{dH}{dy} - \frac{dG}{dz} &= -\frac{d\Omega}{dx} \\ \frac{dF}{dz} - \frac{dH}{dx} &= -\frac{d\Omega}{dy} \\ \frac{dG}{dx} - \frac{dF}{dy} &= -\frac{d\Omega}{dz} \end{aligned}\right\} \quad \text{...............(1).}$$

If, when Ω is given, F_0, G_0, H_0 are a particular set of values of F, G, H satisfying these equations, the complete values are

$$F = F_0 + \frac{d\chi}{dx}, \quad G = G_0 + \frac{d\chi}{dy}, \quad H = H_0 + \frac{d\chi}{dz} \;\ldots.(2),$$

where χ is an arbitrary function of x, y, z. It is, therefore, the particular solution which is of primary importance.

If $$\Omega = \left(A\frac{d}{dx} + B\frac{d}{dy} + C\frac{d}{dz}\right)P,$$

where $\nabla^2 P = 0$, $\left(\nabla^2 \text{ standing for } \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)$, the values of F, G, H are

$$F = \left(B\frac{d}{dz} - C\frac{d}{dy}\right)P, \; G = \left(C\frac{d}{dx} - A\frac{d}{dz}\right)P, \; H = \left(A\frac{d}{dy} - B\frac{d}{dx}\right)P$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

If we substitute these values in the last of the three equations (1)

$$\frac{dG}{dx} - \frac{dF}{dy} = C\nabla^2 P - \frac{d}{dz}\left(A\frac{d}{dx} + B\frac{d}{dy} + C\frac{d}{dz}\right)P = -\frac{d\Omega}{dz},$$

and similarly for the first two.

As a particular case, if $\Omega = \frac{dP}{dz}$,

$$F = -\frac{dP}{dy}, \quad G = \frac{dP}{dx}, \quad H = 0 \;\ldots\ldots\ldots\ldots(4).$$

The above expressions may be considered as a generalisation of the formula given by Maxwell in his Electricity and Magnetism, vol. 2, Art. 405.

Maxwell has also thrown the general solid harmonic of negative degree into the form

$$\frac{d}{dh_1} \cdot \frac{d}{dh_2} \cdots \frac{d}{dh_s} \cdot \frac{1}{r},$$

where $$\frac{d}{dh} = A_1\frac{d}{dx} + B_1\frac{d}{dy} + C_1\frac{d}{dz}, \text{ \&c.} \;\ldots\ldots\ldots\ldots(5).$$

If we take for Ω an expression of this kind, we may write it in either of the forms

$$\Omega = \frac{d}{dh_1} P_1 \quad \text{or} \quad \Omega = \frac{d}{dh_2} P_2,$$

[illegible] containing all the remaining factors of the above [illegible].

[illegible] may be taken in forming the [illegible]. According to which of the two we [illegible], we shall get different expressions for the vector [illegible] which we may denote by the suffixes 1, 2 corresponding to the forms chosen. We may verify that their [illegible] have the forms prescribed in (2); for, if we put

$$Q = \frac{d}{dh_3} \cdot \frac{d}{dh_4} \cdots \frac{d}{dh_n} \cdot \frac{1}{r} \text{ ....(6)},$$

[illegible] $\left(\ldots - C_1 \frac{d}{dy} \right) \frac{dQ}{dh_2}$

$- \ldots - C_2 A_1) \frac{d}{dy}$

$\ldots - (B_1 C_2 - B_2 C_1) \nabla^2 Q.$

Since $\nabla^2 Q = 0$,

$$\ldots, \quad H_1 - H_2 = \frac{d\chi}{dz} \text{...(7)},$$

where $\chi = $ [illegible] ....(8).

9. Consider a very small magnet $P'N'$, the strength of whose positive pole P' is m, and whose length $P'N'$ is l, and let $\alpha\beta\gamma$ be the direction angles of $P'N'$

$$\Omega = -ml \left(\cos\alpha \frac{d}{dx} + \cos\beta \frac{d}{dy} + \cos\gamma \frac{d}{dz} \right) \frac{1}{R},$$

wherein $R = PP' = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$ ......(9).

The components of the vector potential are

$$F = -ml\left(\cos\beta\frac{d}{dz} - \cos\gamma\frac{d}{dy}\right)\frac{1}{R},$$

$$G = -ml\left(\cos\gamma\frac{d}{dx} - \cos\alpha\frac{d}{dz}\right)\frac{1}{R},\ H = \&\text{c.}$$

If the magnet be parallel to the axis of z,

$$\cos\alpha = \cos\beta = 0,\quad \cos\gamma = 1;$$

$$\Omega = -ml\frac{dR^{-1}}{dz},\quad F = ml\frac{dR^{-1}}{dy},\quad G = -ml\frac{dR^{-1}}{dx},\quad H = 0.$$

For a magnet $P'P''$ of finite length we may put $l = dz$, and integrate from P' up to P'',

$$F = -m(y-y')\int_{z'}^{z''}\frac{dz'}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{3}{2}}},$$

whence

$$\left.\begin{aligned} F &= \frac{m(y-y')}{(x-x')^2 + (y-y')^2}\left(\frac{z-z''}{PP''} - \frac{z-z'}{PP'}\right) \\ G &= -\frac{m(x-x')}{(x-x')^2 + (y-y')^2}\left(\frac{z-z''}{PP''} - \frac{z-z'}{PP'}\right) \\ H &= 0 \end{aligned}\right\}\ldots(10).$$

We may obtain the vector potential due to a single pole $(x'y'z')$ by making $z'' = \infty$,

$$\left.\begin{aligned} F &= -\frac{m(y-y')}{(x-x')^2 + (y-y')^2}\left(1 + \frac{z-z'}{PP'}\right) \\ G &= \frac{m(x-x')}{(x-x')^2 + (y-y')^2}\left(1 + \frac{z-z'}{PP'}\right) \\ H &= 0. \end{aligned}\right\}\ldots\ldots(11).$$

If we had made the second pole move off to infinity in the directions of the axes of x or y, or in any other direction, we should have obtained different values of F, G, H. These values, however, would have differed in any two cases by quantities of the form $\frac{d\chi}{dx}$, $\frac{d\chi}{dy}$, $\frac{d\chi}{dz}$. Though I have verified this result by actual differentiation and integration, I do not think it necessary to give the work here.

The most practically useful case of these formulæ is when Ω is expressed generally as a solid spherical harmonic, and we must in general suppose it expanded in a series of solid tesseral harmonics. If we employ the expressions (4), we have first to integrate and then differentiate P; and we have now to find how this may be done when P is a tesseral harmonic; let us commence by differentiating it.

Differentiation of a solid tesseral harmonic.

3. If H be a solid harmonic, it satisfies the equation $\nabla^2 H = 0$, and its differential coefficients likewise satisfy it, they are therefore solid harmonics. It might appear that the most obvious means of differentiating a tesseral harmonic would be found by employing Maxwell's expression for it

$$\frac{(-1)^n}{n!}\left(\frac{d}{dz}\right)^{n-m}\left(\frac{d^m}{d\xi^m}+\frac{d^m}{d\eta^m}\right)\frac{1}{r}\ \ldots\ldots\ldots\ldots(12),$$

in which $\xi = x + y\sqrt{(-1)}$, $\eta = x - y\sqrt{(-1)}$.

But this formula, though it serves to differentiate a negative harmonic, will not furnish with equal ease the corresponding results for a positive one; I shall therefore employ expressions derived from the forms of the 'associated functions' as definite integrals.

If $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\theta$, $\mu = \cos\theta$,

the general tesseral surface-harmonic may be written P_m^n multiplied by $\cos m\phi$ or $\sin m\phi$, where

$$P_m^n(\mu) = (\sin\theta)^m(\mu^{n-m} - \lambda\mu^{n-m-2} + \ldots)\ \ldots\ldots\ldots(13),$$

the coefficient of the highest power of μ being unity; and (see Heine, Kugelfunctionen, 2nd ed. p. 207)

$$\left.\begin{aligned} P_m^n &= \frac{2^n}{\pi}\frac{(n+m)!\,(n-m)!}{(2n)!}\frac{1}{i^m}\int_0^\pi(\cos\theta + i\sin\theta\cos\psi)^n\ \cos m\psi\,d\psi \\ &= \frac{2^n}{\pi}\ \ \frac{n!\,n!}{2n!}\ \ i^m\int_0^\pi(\cos\theta + i\sin\theta\cos\psi)^{-n-1}\cos m\psi\,d\psi \end{aligned}\right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

(in which $i = \sqrt{(-1)}$), the numerical coefficients of which will be denoted in future by A_m^n and B_m^{-n-1} respectively.

We shall also use the following notation for solid harmonics;

$$\left.\begin{aligned} C_m^n &= r^n\ \cos m\phi P_m^n,\quad S_m^n = r^n\ \sin m\phi P_m^n,\quad U_m^n = C_m^n + iS_m^n \\ C_m^{-n-1} &= r^{-n-1}\cos m\phi P_m^n,\quad S_m^{-n-1} = r^{-n-1}\sin m\phi P_m^n,\quad U_m^{-n-1} = C_m^{-n-1} + iS_m^{-n-1} \end{aligned}\right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15).$$

From (14)

$$U_m^n = A_m^n \; i^m \int_0^\pi e^{im\phi} \{z + i\sqrt{(x^2+y^2)} \cos\psi\}^n \cos m\psi \, d\psi \qquad \text{.........................(16)},$$

$$U_m^{-n-1} = B_m^{-n-1} i^m \int_0^\pi e^{im\phi} \{z + i\sqrt{(x^2+y^2)} \cos\psi\}^{-n-1} \cos m\psi \, d\psi \qquad \text{.........................(17)},$$

of which the expression $z + i\sqrt{(x^2+y^2)} \cos\psi$ will be denoted by M.

Differentiate equations (16) and (17) with respect to z,

$$\frac{dU_m^n}{dz} = n\, U_m^{n-1} \times A_m^n : A_m^{n-1} \text{..................(18)},$$

$$\frac{dU_m^{-n-1}}{dz} = -(m+1)\, U_m^{-n-1} \times B_m^{-n-1} : B_m^{-n-2} \text{....(19)}.$$

Differentiating (16) with regard to ϕ,

$$\frac{dU_m^n}{d\phi} = A_m^n i^m \int_0^\pi ime^{im\phi} M^n \cos m\psi \, d\psi,$$

integrating by parts, and putting $\rho = \sqrt{(x^2+y^2)}$,

$$\frac{dU_m^n}{d\phi} = nA_m^n i^{-m+2} \int_0^\pi e^{im\phi} \rho M^{n-1} \sin\psi \sin m\psi \, d\psi.$$

But $\qquad d\phi = \dfrac{xdy - ydx}{\rho^2}, \quad d\rho = \dfrac{xdx + ydy}{\rho};$

hence

$$\frac{dU_m^n}{dx} = nA_m^n i^{-m+1} \int_0^\pi e^{im\phi} M^{n-1}$$

$$\times (-i \sin\phi \sin\psi \sin m\psi + \cos\phi \cos\psi \cos m\psi) \, d\psi$$

$$= \tfrac{1}{2} nA_m^n i^{-m+1} \int_0^\pi$$

$$\times \{e^{i(m+1)\phi} \cos(m+1)\psi + e^{i(m-1)\phi} \cos(m-1)\psi\} M^{n-1} d\psi$$

$$= +\tfrac{1}{2} nA_m^n (U_{m-1}^{n-1} : A_{m-1}^{n-1} - U_{m+1}^{n-1} : A_{m+1}^{n-1}) \text{..................(20)}.$$

In like manner

$$\frac{dU_m^{-n-1}}{dx} = -\tfrac{1}{2}(n+1)\,B_m^{-n-1}\left(U_{m+1}^{-n-2} : B_{m+1}^{-n-2} - U_{m-1}^{-n-2} : B_{m-1}^{-n-2}\right)\ldots(21).$$

For the differential coefficients with regard to y,

$$\frac{dU_m^n}{dy} = nA_m^n i^{-m+1}\int_0^\pi e^{im\phi}$$

$$\times\,(i\cos\phi\,\sin\psi\,\sin m\psi + \sin\phi\,\cos\psi\,\cos m\psi)\,M^{n-1}\,d\psi$$

$$= \tfrac{1}{2}nA_m^n i^{-m+2}\int_0^\pi$$

$$\times\,\{e^{i(m-1)\phi}\cos(m-1)\,\psi - e^{i(m+1)\phi}\cos(m+1)\,\psi\}\,M^{n-1}\,d\psi$$

$$= \tfrac{1}{2}nA_m^n i\,(U_{m-1}^{n-1} : A_{m-1}^{n-1} + U_{m+1}^{n-1} : A_{m+1}^{n-1})\ldots\ldots\ldots\ldots(22),$$

$$\frac{dU_m^{-n-1}}{dy} = -\tfrac{1}{2}(n+1)B_m^{-n-1} i\,(U_{m-1}^{-n-2} : B_{m-1}^{-n-2} + U_{m+1}^{-n-2} : B_{m+1}^{-n-2})\ldots\ldots(23).$$

We have now to separate the real and imaginary parts of equations (18—23); we note also that

$$A_m^n : A_m^{n-1} = \frac{n^2-m^2}{n(2n-1)},\quad B_m^{-n-2} : B_m^{-n-1} = \frac{n+1}{2n+1},$$

$$A_m^n : A_{m-1}^{n-1} = \frac{(n+m-1)(n+m-2)}{n(2n-1)},$$

$$A_m^n : A_{m+1}^{n-1} = \frac{(n-m-1)(n-m-2)}{n(2n-1)},$$

$$B_{m+1}^{-n-2} : B_m^{-n-1} = \frac{n+1}{2n+1} = B_{m-1}^{-n-2} : B_m^{-n-1}.$$

The final results are

$$
\left.
\begin{aligned}
&\frac{dS_m^n}{dz} = \frac{n^2 - m^2}{2n-1} S_m^{n-1}, \quad \frac{dC_m^n}{dz} = \frac{n^2 - m^2}{2n-1} C_m^{n-1}, \\
&\frac{dS_m^n}{dx} = \frac{1}{2(2n-1)} \{(n+m-1)(n+m-2) S_{m-1}^{n-1} - (n-m-1)(n-m-2) S_{m+1}^{n-1}\} \\
&\frac{dC_m^n}{dx} = \frac{1}{2(2n-1)} \{(n+m-1)(n+m-2) C_{m-1}^{n-1} - (n-m-1)(n-m-2) C_{m-1}^{n-1}\} \\
&\frac{dS_m^n}{dy} = \frac{1}{2(2n-1)} \{(n+m-1)(n+m-2) C_{m-1}^{n-1} + (n-m-1)(n-m-2) C_{m+1}^{n-1}\} \\
&\frac{dC_m^n}{dy} = \frac{1}{2(2n-1)} \{(n+m-1)(n+m-2) S_{m-1}^{n-1} + (n-m-1)(n-m-2) C_{m+1}^{n-1}\}
\end{aligned}
\right\} \quad \ldots\ldots(24),
$$

and for the negative harmonics

$$
\left.
\begin{aligned}
&\frac{dS_m^{-n-1}}{dz} = -(2n+1) S_m^{-n-2}, \quad \frac{dC_m^{-n-1}}{dz} = -(2n+1) C_m^{-n-2} \\
&\frac{dS_m^{-n-1}}{dx} = \tfrac{1}{2}(2n+1)(S_{m-1}^{-n-2} - S_{m+1}^{-n-2}) \\
&\frac{dC_m^{-n-1}}{dx} = \tfrac{1}{2}(2n+1)(C_{m-1}^{-n-2} - C_{m+1}^{-n-2}) \\
&\frac{dS_m^{-n-1}}{dy} = -\tfrac{1}{2}(2n+1)(C_{m-1}^{-n-2} + C_{m+1}^{-n-2}) \\
&\frac{dC_m^{-n-1}}{dy} = +\tfrac{1}{2}(2n+1)(S_{m-1}^{-n-2} + S_{m+1}^{-n-2})
\end{aligned}
\right\} \quad \ldots\ldots(25).
$$

Integrals of the Solid Harmonics.

4. For the determination of the vector from the scalar potential, it is necessary in general, by equations (4), to determine only the integrals of the harmonics with regard to z.

Consider first the negative harmonics, and write $n+1$ for n in the first lines of (25) and integrate from $z = \infty$ to $z = z$,

$$
\int_\infty^z S_m^{-n-1}\, dz = -\frac{1}{2n-1} S_m^{-n}, \quad \int_\infty^z C_m^{-n-1}\, dz = -\frac{1}{2n-1} C_m^{-n} \quad \ldots\ldots(26).
$$

In integrating the first lines of (24) between $z = 0$ and $z = a$ we have to introduce the values of the solid harmonics when $z = 0$. Let P be the point whose coordinates are x, y, z, and let P_1, P_2, P_3 be the feet of the perpendiculars from P on the planes of yz, zx, xy, and let the solid harmonic which at P is denoted by H be denoted at these three points by H_1, H_2, H_3.

We have, in particular,

[illegible]

$= 0$ if [illegible] ....(27).

[illegible]

when [illegible]

It is unnecessary to write down [illegible] and [illegible]; integrating then the first [illegible] and writing [illegible] for n,

[illegible] ....(28).

[illegible]

5. To complete the [illegible] theory we shall now find the integrals [illegible] referring to the equations [illegible] that [illegible] and [illegible] are both zero when [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

the series terminating of itself when the suffix > n.

$$\left.\begin{aligned}\int_\infty^x S_{m-1}^{-n-2}\,dx &= \frac{2}{2n+1}S\\ \int_\infty^x C_{m-1}^{-n-2}\,dx &= \frac{2}{2n+1}C\\ \int_\infty^y S_{m-1}^{-n-2}\,dy &= \frac{2}{2n+1}C'\\ \int_\infty^y C_{m-1}^{-n-2}\,dy &= -\frac{2}{2n+1}S'\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(29),$$

no terms are to be added on the right-hand side because the expressions vanish when $r=\infty$.

If similarly, we put

$$s = S_m^n : A_m^n + S_{m+2}^n : A_{m+2}^n + \ldots,$$

$$c = C_m^n : A_m^n + C_{m+2}^n : A_{m+2}^n + \ldots,$$

$$s' = S_m^n : A_m^n - S_{m+2}^n : A_{m+2}^n + \ldots,$$

$$c' = C_m^n : A_m^n - C_{m+2}^n : A_{m+2}^n + \ldots,$$

and let the suffixes 1, 2 applied to $s, \ldots c'$ indicate that we put therein $x=0$ or $y=0$ respectively, we find

$$\left.\begin{aligned}1 \div A_{m-1}^{n-1}\,.\int_0^x S_{m-1}^{n-1}\,dx &= \frac{2}{n}(s-s_1)\\ 1 \div A_{m-1}^{n-1}\,.\int_0^x C_{m-1}^{n-1}\,dx &= \frac{2}{n}(c-c_1)\\ 1 \div A_{m-1}^{n-1}\,.\int_0^y S_{m-1}^{n-1}\,dy &= -\frac{2}{n}(c'-c_2')\\ 1 \div A_{m-1}^{n-1}\,.\int_0^y C_{m-1}^{n-1}\,dy &= \frac{2}{n}(s'-s_2')\end{aligned}\right\}\ldots\ldots\ldots(30).$$

Queen's College, Cork.
7th February, 1880.

[illegible] NOTES

[illegible]

[illegible]

$$[illegible]$$

[illegible] by [illegible]'s Theorem, [illegible]

$$[illegible]$$

results which [illegible] the known values of [illegible].

[illegible] be [illegible]

$$[illegible]$$

and consequently

$$[illegible]$$

But if [illegible] must be a function of the differences [illegible] as appears from the Lagrangian system corresponding to this partial differential equation. Hence if any function of the differences of [illegible] be operated on by

$$[illegible]$$

the result will vanish, and conversely, as is well-known.

The corresponding equation

$$\left(\alpha^2 \frac{d}{d\alpha} + \beta^2 \frac{d}{d\beta} + \dots\right) = p_1 \frac{d}{da} + 2p_2 \frac{d}{dp_1} + \dots + np_n \frac{d}{dp_{n-1}}$$

admits of a similar deduction from the expression

$$\left(\frac{1}{\alpha} - \frac{1}{x}\right)\left(\frac{1}{\beta} - \frac{1}{x}\right) \dots \left(\frac{1}{\lambda} - \frac{1}{x}\right).$$

If the operative function Δ be formed for any of the derived functions it will be found to be identical with that for the original functions, which proves that all functions of the differences of the roots of any of the derived equations are also functions of the differences of those of the original equation.

2. Evidently $\Delta S_r = rS_{r-1}$.

Now
$$S_{-1} = \frac{p_{n-1}}{p_n}$$

$$-S_{-2} = \Delta\left(\frac{p_{n-1}}{p_n}\right) = -\frac{p^2_{n-1} - 2p_{n-2}p_n}{p_n^2},$$

$$2S_{-3} = \Delta^2\left(\frac{p_{n-1}}{p_n}\right),$$

&c. &c.

$$S_{-r} = (-1)^{r-1} \frac{1}{r-1\,!} \Delta^{r-1}\left(\frac{p_{n-1}}{p_n}\right);$$

or
$$p_n S_{-1} - p_{n-1} = 0;$$

therefore

$$\Delta\ (p_n S_{-1} - p_{n-1}) = -p_n S_{-2} + p_{n-1} S_{-1} - 2p_{n-2} = 0,$$

$$\tfrac{1}{2}\Delta^2 (p_n S_{-1} - p_{n-1}) = \quad p_n S_{-3} - p_{n-1} S_{-2} + p_{n-2} S_{-1} - 3p_{n-3} = 0,$$

&c. &c.

Similarly if δ represent $\left(p_1 \frac{d}{da} + 2p_2 \frac{d}{dp_1}\ \&c.\right)$,

$$S_1 = \frac{p_1}{a},$$

$$S_2 = -\delta\left(\frac{p_1}{a}\right),$$

$$S_r = (-1)^{r-1} \frac{1}{r-1\,!} \delta^{r-1}\left(\frac{p_1}{a}\right),$$

and
$$aS_1 - p_1 = 0,$$
$$\delta\,(aS_1 - p_1) = -aS_2 + p_1S_1 - 2p_2 = 0,$$
$$\tfrac{1}{2}\delta^2\,(aS_1 - p_1) = \quad aS_3 - p_1S_2 + p_2S_1 - 3p_3 = 0,$$
$$\frac{1}{3\,!}\,\delta^3\,(aS_1 - p_1) = -aS_4 + p_1S_3 - p_2S + p_3S_1 - 4p_4 = 0,$$
$$\&c. \qquad \&c. \qquad \&c.$$
which are Newton's formulæ.

3. The results obtained in 1 may be employed to obtain the coefficients of the equation whose roots are the squares of those of $f(x)$. With the notation of Salmon's *Higher Algebra*, p. 52,
$$\frac{d}{d(\alpha^2)} = \frac{1}{2\alpha}\,\frac{d}{d\alpha} = \frac{1}{2\alpha}\,\frac{d}{dp_1} + \frac{q_1}{2\alpha}\,\frac{d}{dp_2} + \&c.$$
$$= \frac{1}{2\alpha}\,\frac{d}{dp_1} + \left(\frac{p_1}{2\alpha} - \tfrac{1}{2}\right)\frac{d}{dp_2} + \left(\frac{p_2}{2\alpha} - \tfrac{1}{2}p_1 + \tfrac{1}{2}\alpha\right)\frac{d}{dp_3} + \&c.;$$
therefore
$$\Sigma\left\{\frac{d}{d(\alpha^2)}\right\} = \tfrac{1}{2}\left\{S_{-1}\frac{d}{dp_1} + (p_1S_{-1} - n)\frac{d}{dp_2}\right.$$
$$+ (p_2S_{-1} - np_1 + S_1)\frac{d}{dp_3}$$
$$\left. + (p_3S_{-1} - np_2 + p_1S_1 - S_2)\frac{d}{dp_4} + \&c.\right\}$$
$$= \tfrac{1}{2}\left\{S_{-1}\frac{d}{dp_1} + (p_1S_{-1} - n)\frac{d}{dp_2}\right.$$
$$\left. + \{p_2S_{-1} - (n-1)p_1\}\frac{d}{dp_3} + \{p_3S_{-1} - (n-2)p_2\}\frac{d}{dp_4}\right\}$$
$$= \tfrac{1}{2}\left\{\frac{p_{n-1}}{p_n}\left(\frac{d}{dp_1} + p_1\frac{d}{dp_2} + p_2\frac{d}{dp_3} + \&c.\right)\right.$$
$$\left. - \left(n\frac{d}{dp_2} + (n-1)\,p_1\frac{d}{dp_3} + \&c.\right)\right\},$$
from which it follows that every function of the differences of the squares of α, β, &c. must be expressible in terms of the solutions of
$$\frac{dp_1}{p_{n-1}} = \frac{dp_2}{p_{n-1}p_1 - np_n} = \frac{dp}{p_{n-1}p_2 - (n-1)\,p_1p_n} = \frac{dp_4}{p_{n-1}p_3 - (n-2)p_2p_n}$$
$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{dp_n}{p^2_{n-1} - 2p_{n-2}p_n}\,.$$

If the equation whose roots are α^2, β^2, &c. be

$$x^n - P_1 x^{n-1} + P_2 x^{n-2} - P_3 x^{n-3} + \ldots + (-1)^n P_n = 0,$$

and if $\Sigma \left\{ \frac{d}{d(\alpha^2)} \right\}$ or its equivalent

$$\tfrac{1}{2} \left\{ \frac{P_{n-1}}{P_n} \left(\frac{d}{dp_1} + p_1 \frac{d}{dp_2} + p_2 \frac{d}{dp_3} + \&c. \right) - \left(n \frac{d}{dp_2} + (n-1) p_1 \frac{d}{dp_3} + \&c. \right) \right\}$$

be denoted by Δ'

$$P_r = \frac{1}{n-r\,!} \Delta'^{n-r} P_n$$

$$= \frac{1}{n-r\,!} \Delta'^{n-r} (p_n^2), \text{ since } P_n = p_n^2,$$

which determines the coefficients.

4. The condition that a function of the coefficients I. should be a homogeneous function of the roots of the m^{th} degree, which is usually secured by making the terms all of the same weight, may also be secured by observing that

$$\left(p_1 \frac{d}{dp_1} + 2p_2 \frac{d}{dp_2} + 3p_3 \frac{d}{dp_3} + \&c. \right) I = mI,$$

for
$$\left(p_1 \frac{d}{dp_1} + 2p_2 \frac{d}{dp_2} + \ldots \right) \equiv \alpha \frac{d}{d\alpha} + \beta \frac{d}{d\beta} + \&c.$$

W. J. C. Sharp.

14, Mount Street,
Grosvenor Square,
December 17th, 1879.

A trigonometrical identity.

If s, s_1 and c, c_1 denote respectively the sines and cosines of any two angles, then

$$(c_1^{\frac{4}{3}} s^{\frac{2}{3}} - c^{\frac{4}{3}} s_1^{\frac{2}{3}})^3 + (s_1^{\frac{4}{3}} c^{\frac{2}{3}} - s^{\frac{4}{3}} c_1^{\frac{2}{3}})^3 = (s^{\frac{2}{3}} c^{\frac{2}{3}} - s_1^{\frac{2}{3}} c_1^{\frac{2}{3}})^3 \ldots (1).$$

The above is a particular case of a more general theorem which may be enunciated.

If $$\lambda^2 + \mu^2 + \nu^2 = 1,$$

and $$\lambda_1^2 + \mu_1^2 + \nu_1^2 = 1,$$

then $$(\lambda^{\frac{4}{3}}\mu_1^{\frac{2}{3}}\nu_1^{\frac{2}{3}} - \lambda_1\mu^{\frac{2}{3}}\nu^{\frac{2}{3}})^3 + (\mu^{\frac{4}{3}}\nu_1^{\frac{2}{3}}\lambda_1^{\frac{2}{3}} - \mu_1^{\frac{4}{3}}\nu^{\frac{2}{3}}\lambda^{\frac{2}{3}})^3 + (\nu^{\frac{4}{3}}\lambda_1^{\frac{2}{3}}\mu_1^{\frac{2}{3}} - \nu_1^{\frac{4}{3}}\lambda^{\frac{2}{3}}\mu^{\frac{2}{3}})^3$$
$$= (\lambda_1^{\frac{2}{3}}\mu_1^{\frac{2}{3}}\nu_1^{\frac{2}{3}} - \lambda^{\frac{2}{3}}\mu^{\frac{2}{3}}\nu^{\frac{2}{3}})^3 + (\lambda^2 - \lambda_1^2)(\mu^2 - \mu_1^2)(\nu^2 - \nu_1^2).$$

If we put $\nu = \nu_1$ and divide throughout ν, this becomes

$$(\lambda^{\frac{4}{3}}\mu_1^{\frac{2}{3}} - \lambda_1^{\frac{4}{3}}\mu^{\frac{2}{3}})^3 + (\mu^{\frac{4}{3}}\lambda_1^{\frac{2}{3}} - \mu_1^{\frac{4}{3}}\lambda^{\frac{2}{3}})^3 + \nu^2(\lambda_1^{\frac{2}{3}}\mu_1^{\frac{2}{3}} - \lambda^{\frac{2}{3}}\mu^{\frac{2}{3}})^3$$
$$= (\lambda_1^{\frac{2}{3}}\mu_1^{\frac{2}{3}} - \lambda^{\frac{2}{3}}\mu^{\frac{2}{3}})^3,$$

which, on putting $\nu = 0$, gives the trigonometrical identity in question.

HARRY HART.

On the directrix of a parabola inscribed in a triangle.

Let three lines form a triangle ABC, and let a fourth line meet its sides in abc. Then if O be any point, the pencil $O\{AaBbCc\}$ is in involution; and to the same involution belong the tangents OP and OQ to any conic touching the four lines.

Remove the line abc to infinity, and let O be taken at the orthocentre of the triangle ABC. Then the conjugate rays of $O\{AaBbCc\}$ are at right angles, and therefore every other two conjugate rays OP and OQ in the same involution are at right angles; that is to say, the tangents from O to every parabola inscribed in the triangle ABC are at right angles, and therefore O is a point on the directrix of every such parabola.

C. TAYLOR.

THE END OF VOL. IX.

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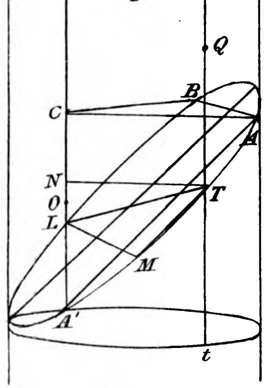

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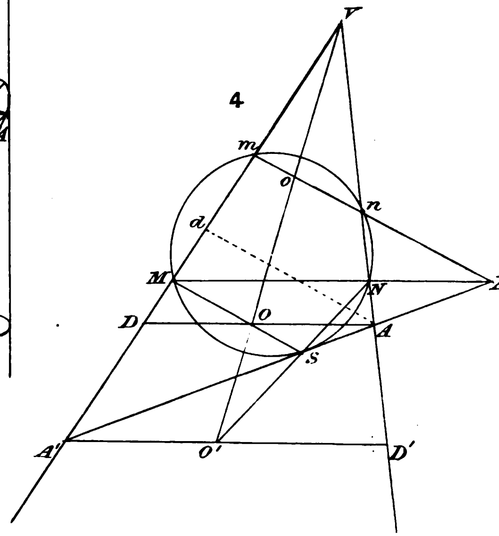

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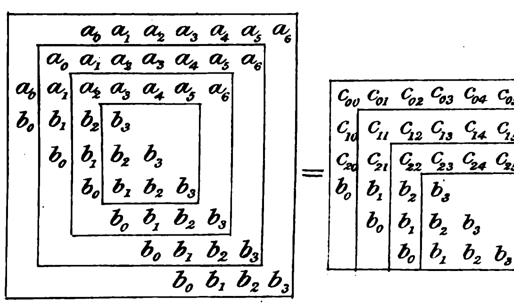

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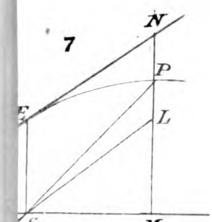

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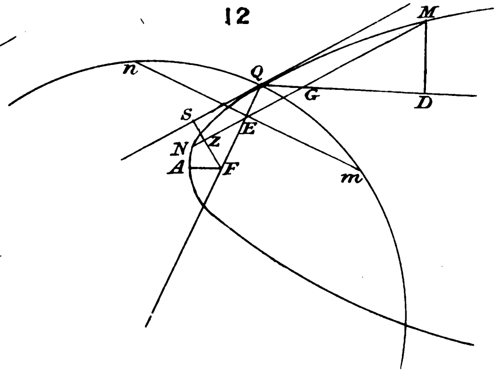

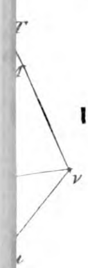

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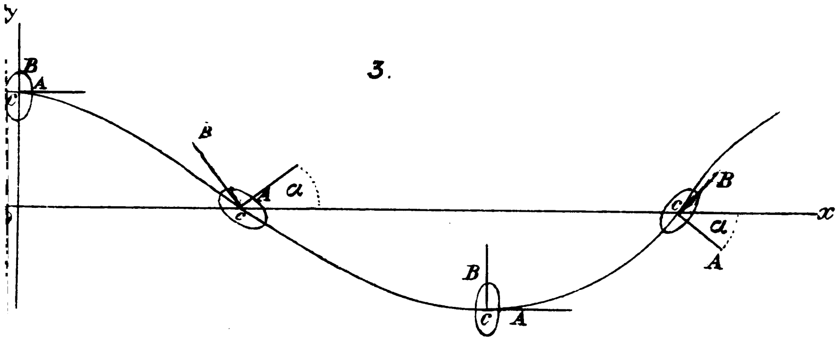

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