THE MODEL ALGEBRA FOR ELEMENTARY SCHOOLS

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THE

MODEL ALGEBRA

ARRANGED FOR

ELEMENTARY SCHOOLS

BY

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PREFACE.

In presenting this work to the teachership for approval the author assumes that pupils who have been studying arithmetic for several years have acquired a somewhat thorough knowledge of numbers, terms, principles, processes, and their applications; that they will, therefore, experience little difficulty in making the transition from arithmetic to algebra; and that they can readily pass from operations and forms of reasoning involving particular numbers to similar operations and processes of reasoning involving general numbers. For these reasons, only such definitions are given as seem to be really needed to make clear the use of terms repeated or introduced. Formal rules are not given because they are of little value unless deduced by the pupil from the solution of illustrative examples and problems.

It has not been deemed wise to require pupils to perform operations with abstract symbols in accordance with arbitrary rules, except so far as such drill is necessary to prepare them for the solution of the equation, which is the main purpose of all instruction in elementary algebra. At every step suitable problems are introduced to show the practical applications of operations leading through correct processes of reasoning to accurate and definite results. The numerous examples and problems presented have been selected or made expressly for pupils of higher grammar-school grades. They are sufficiently difficult to stimulate thought, and are so graded as to insure quickness and accuracy in operations, and to develop skill in discovering and applying the general conditions which govern the solution of problems. An unusually large number of exercises in algebraic notation and expression has been given, that the pupil may become familiar with the symbols used in this branch of mathematics and with the methods of generalizing processes and results.

As already stated, the main purpose of exercises in the elementary rules is to prepare the learner for the solution of the algebraic equation. That this purpose may be accomplished with the greatest economy of time without sacrificing the scientific sequence of subjects, operations in clearing the equation of fractions are introduced after those of changing fractions to higher and to lower terms and to the least common denominator, as applications of common factors and common multiples. Exercises in changing fractional forms to entire or mixed expressions, entire or mixed expressions to fractional forms, and exercises in addition, subtraction, multiplication, and division of fractions may be subsequently taken up for more extended drill. In order to emphasize the nature of the equation and to illustrate further its applications, a sufficient number of examples and problems containing two unknowns has been inserted to insure a satisfactory knowledge of this subject.

The author hopes that his manner of presenting the subject may encourage and assist his fellow teachers in their efforts to popularize this important branch of mathematics in elementary schools.

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ALGEBRA

FOR

ELEMENTARY SCHOOLS.

INTRODUCTION.

1. CHARACTERS used to represent numbers are called **number-symbols.**

2. In arithmetic, numbers are represented by *figures.*

Thus, seven is represented by *7;* twenty is represented by *20;* one hundred seventy-five, by *175;* etc.

3. Figures are number-symbols that have always definite or particular values.

Thus, the value of 2 is always *two;* the value of 4 is always *four;* the value of 6 standing alone, or in 26, or in 126, is always *six ones;* etc.

A number has always the same value when it is expressed by the same figures in the same places.

Thus, the value of 5 is always *five ones;* the value of 25 is always *two tens five ones,* or *twenty-five;* the value of 175 is *one hundred seventy-five;* etc.

4. Numbers represented by figures are called **numerical expressions, or numerical quantities.**

Thus, 6, 8, and 10 are numerical expressions; so, also, are $7 + 9 = 16$; $12 - 7$; 8×5; $20 \div 4$; $\frac{1}{4}$; $2\frac{1}{3}$; etc.

5. In algebra, numbers are represented by *letters,* or by *figures and letters.*

Thus, 5 may be represented by a, and 8 by b, and then $5 + 8$ is represented by $a + b$; also, the letters and figures in $3x - 2x$ and $m = 4$ represent numbers; etc.

6. Letters are used as general number-symbols that may have any particular values given to them.

Thus, a letter may represent a number having a particular value in one problem, and a different value in another problem.

If b represents 9 and c represents 6, then $b + c$ represents $9 + 6$; if $b = 12$ and $c = 5$, then $b - c$, or $12 - 5$, equals 7; etc.

If $x = 8$ and $y = 6$, then $x \times y$ stands for 8×6, or 48; if $x = 20$ and $y = 5$, then $x \div y$ stands for $20 \div 5$, or 4; etc.

7. Figures and letters used to represent numbers or quantities are called **symbols of quantity.**

Thus, 5, 12, 20, a when it stands for a number, and x when it stands for a number, are symbols of quantity; etc.

8. Numbers represented by letters are called **literal quantities.**

Thus, a, b, $b + c$, $x - y$ are literal quantities; so, also, are x in $3x$, and a and y in $2a + 3y$; etc.

9. Figures, and letters that have values given to them, are called **known quantities.**

The first letters of the alphabet, a, b, c, etc., are generally used to represent known values or quantities.

10. Letters used to represent numbers whose values are to be found, are called **unknown quantities.**

Thus, if three times John's money is 24 cents, John's money may be represented by x; then 3 times x, or $3x$, equals 24 cents, and $x = 8$ cents.

The last letters of the alphabet, x, y, z, are generally used to represent unknown values or quantities.

11. Numbers or quantities represented by letters, or by figures and letters, are called **algebraic quantities**, or **algebraic expressions.**

Thus, a, x, 3 times x, or $3x$, $\frac{1}{2}$ of x, or $\frac{1}{2}x$, $a + 2x$, $b - \frac{c}{2}$, etc., are algebraic quantities.

12. Algebra is a general method of representing numbers and of solving problems by means of letters, figures, and signs.

13. The fundamental signs used in arithmetic are generally used for the same purposes in algebra.

14. The sign of addition is $+$. It is read *plus*.

Thus, $5 + 7$ indicates that 5 and 7 are to be added.

$a + b$ indicates that the number represented by a and the number represented by b are to be added.

15. The sign of subtraction is $-$. It is read *minus*.

Thus, $12 - 5$ indicates that 5 is to be taken or subtracted from 12.

$x - y$ indicates that the number represented by y is to be subtracted from the number represented by x.

16. The sign of multiplication is $\times$. It is read *times*, or *multiplied by*.

Thus, $a \times b$ indicates that the number represented by a is to be taken b times. It is, however, generally read a times b.

17. In algebra, multiplication is sometimes indicated by a dot instead of the sign $\times$.

Thus, $a \cdot b$ means $a \times b$; $2 \cdot x$ means $2 \times x$; etc.

Generally, letters, or a figure and one or more letters that are to be multiplied together, are written side by side without any sign between them.

Thus, $a \times x$, or $a \cdot x$, is written ax; $4 \times c$, or $4 \cdot c$, is written $4c$; also, abx means $a \times b \times x$, or $a \cdot b \cdot x$; etc.

18. The number prefixed to a literal quantity is called a **coefficient.**

Thus, in $2a$, read two a, 2 is the coefficient of a; in $5bc$, read five bc, 5 is the coefficient of bc; in $4a + 10xy$, 4 and 10 are coefficients; etc.

If a literal quantity is written without a number prefixed, 1 is understood to be its coefficient.

Thus, a means $1a$; $b + b$ means $1b + 1b$, which equals $2b$; $6x - x = 5x$; etc.

19. The coefficient of a quantity shows how many times the quantity is taken additively.

Thus, a means that a is taken once, or 1 time; $3b$ means $b + b + b$, or b taken 3 times; in $4ax$, 4 shows that ax is taken 4 times; etc.

20. The sign of division is $\div$. It is read *divided by*.

Thus, $x \div a$ indicates that the number represented by x is to be divided by the number represented by a.

Division is sometimes indicated by placing the dividend over a horizontal line with the divisor under it, in the form of a fraction.

Thus, $a \div b$ may be written $\frac{a}{b}$, read a divided by b; $2 \div x$ may be written $\frac{2}{x}$; $(2 + x) \div a$, $\frac{2 + x}{a}$; etc.

21. The fundamental signs used in arithmetic and in algebra are called **symbols of operation.**

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22. The sign of equality is $=$. It is read *equals*, or *is equal to.*

Thus, $a + b = 2x$ indicates that the sum of the quantities represented by a and b equals twice the quantity represented by x.

23. The expression which shows that two quantities are equal, is called an **equation.**

Thus, $5 + 7 = 8 + 4$ is an equation; so, also, is $a + b = c$; also, $3x = 30$; etc.

24. The two quantities connected by the sign of equality are called the **members** of the equation.

Thus, in the equation $a + b = 5 + 8$, the quantities $a + b$ and $5 + 8$ are the members of the equation.

$a + b$, on the left of the sign of equality, is called the *first member;* and $5 + 8$, on the right of the sign, is called the *second member.*

PROBLEMS ILLUSTRATING THE NATURE OF ALGEBRA.

25. *Illustrative Problem.*—I wish to divide 21 marbles between Charles and Henry, so that Henry shall have twice as many as Charles.

EXPLANATION.

I wish to give to Charles *a certain number*, and to Henry *twice that number.* Hence, I must give to both of them *three times that number.* Then, since both together are to have 21, I know that *three times that number* is 21. Therefore, *once that number* is $\frac{1}{3}$ of 21, or 7. Hence, the number which Charles is to receive is 7 marbles; and twice that number is 14 marbles, which Henry is to receive.

ARITHMETICAL SOLUTION.

A certain number		=	the number given to Charles,
Then	2 times that number	=	the number given to Henry,
And	3 times that number	=	the number given to both,
Or	3 times that number	=	21 marbles.
Hence,	the number	=	7 marbles,
And	2 times the number	=	14 marbles.

Therefore, Charles received 7 marbles, and Henry received 14 marbles.

It will be observed that at the beginning of the solution the number of marbles to be given to Charles *is not known.* It was called *a certain number,* and *that number.*

Now, for more than two hundred years mathematicians have been in the habit of using the last letters of the alphabet, x, y, z, etc., to stand for numbers that are *not known.* If we adopt their practice, the solution just given may be changed to the following:

ALGEBRAIC SOLUTION.

Let $x =$ the number given to Charles,

Then $2x =$ the number given to Henry.

Hence, $x + 2x =$ the number given to both.

$$3x = 21$$

$$x = 7$$

$$2x = 14$$

Therefore, Charles received 7 marbles, and Henry received 14 marbles.

It will be noticed that the algebraic solution is very much like the arithmetical solution, except that letters are used instead of words and figures to represent numbers.

The letter y, z, a, b, or any other letter in the alphabet, may be used in the same manner as x is used to represent an unknown number or quantity, and the same result will be obtained.

26. *Illustrative Problem.* — Divide 60 cents among Fred, Harry, and Frank, so that Harry shall have twice as many as Fred, and Frank shall have as many as Fred and Harry together.

FIRST SOLUTION.

Let $x =$ Fred's share,

Then $2x =$ Harry's share,

And $x + 2x =$ Frank's share.

Hence, $x + 2x + x + 2x = 60$ cents.

$$6x = 60$$
$$x = 10$$
$$2x = 20$$
$$x + 2x = 30$$

Therefore, Fred received 10 cents, Harry 20 cents, and Frank 30 cents.

In this problem, as there are three numbers to be found, x may represent any one of them. In the first solution, x was used to represent Fred's money.

SECOND SOLUTION.

Let $x =$ Harry's share,

Then $\frac{1}{2}x =$ Fred's share,

And $x + \frac{1}{2}x =$ Frank's share.

Hence, $x + \frac{1}{2}x + x + \frac{1}{2}x = 60$ cents.

$$3x = 60$$
$$x = 20$$
$$\frac{1}{2}x = 10$$
$$x + \frac{1}{2}x = 30$$

Therefore, Harry received 20 cents, Fred 10 cents, and Frank 30 cents.

THIRD SOLUTION.

Let $x =$ Frank's share,

Then $\frac{1}{3}x =$ Fred's share,

And $\frac{2}{3}x =$ Harry's share.

Hence, $x + \frac{1}{3}x + \frac{2}{3}x = 60$ cents.

$$2x = 60$$
$$x = 30$$
$$\frac{1}{3}x = 10$$
$$\frac{2}{3}x = 20$$

Therefore, Frank received 30 cents, Fred 10 cents, and Harry 20 cents.

The second solution and the third are not quite so easy as the first, because they require the use of fractions.

In problems of this kind, it is best, whenever possible, to let the unknown represent a number which will not involve the use of fractions in the solution.

PROBLEMS.

1. George and Thomas have the same number of apples, and both together have 36 apples. How many apples has each?

2. If $5x = 40$, to what is x equal?

3. Mary has a certain number of books, and Anna has four times as many. If both together have 25 books, how many has each?

4. If five times John's money is 60 dollars, how much money has he?

5. A and B have 96 dollars, and A has 7 times as many as B. How many dollars has each?

6. What number added to itself equals 24?

7. William and Henry caught three dozen fish. If William caught 8 times as many as Henry, how many fish did each catch?

8. What number added to five times itself equals 144?

9. A man bought a horse and a carriage for 160 dollars, and the horse cost 3 times as much as the carriage. What did each cost?

10. If $x + 3x + 5x = 81$, what is the value of x?

11. Thomas rode a certain distance one day, and twice as far the next day. If he rode 51 miles altogether, how far did he ride each day?

12. Emma is 14 years older than Ida, and her age is 8 times Ida's age. How old is each?

13. Three times Walter's marbles increased by two times his marbles equal 60 marbles. How many marbles has he?

27. *Illustrative Problem.*—If $\frac{2}{3}$ of John's age is 10 years, how old is he?

SOLUTION.

Let x = John's age,

Then $\frac{2}{3}x = 10$ years.

$$\frac{1}{3}x = 5$$

$$\frac{3}{3}x = 15$$

Therefore, John is 15 years old.

14. The number of Harry's pigeons added to one-half the number is 30. How many pigeons has he?

15. George and Frank together have 24 rabbits. If George has one-fifth as many as Frank, how many rabbits has each?

16. Anna has $\frac{1}{4}$ as many oranges as Martha, and both together have 15 oranges. How many oranges has each girl?

17. John's age and Mary's age together equal 30 years. If Mary is $\frac{2}{3}$ as old as John, how old is each?

18. Fanny spent $\frac{2}{5}$ of her money, and had 30 cents left. How much money had she at first?

19. Thomas caught $\frac{2}{5}$ as many fish as James. If both together caught 35 fish, how many did each catch?

20. A is $\frac{3}{4}$ as old as B, and the difference of their ages is 9 years. How old is each?

21. I bought a horse and a carriage for $140, and the carriage cost $\frac{3}{4}$ as much as the horse. How much did I pay for each?

22. The sum of two numbers is 44. The less number is $\frac{2}{9}$ of the larger. Find each number.

23. Anna picked $\frac{3}{8}$ as many berries as George. If George picked 15 quarts more than Anna, how many quarts did each pick?

24. A farmer bought a cow and a sheep for $27. If the cow cost $3\frac{1}{2}$ times as much as the sheep, how much did he pay for each?

25. Divide 56 cents among three girls so that the second shall have twice as many as the first, and the third twice as many as the second.

26. Divide 60 inches into three parts so that the second shall be four times the first, and the third shall equal the sum of the two others.

27. Emma's age is $\frac{4}{7}$ of Ida's age, and the difference of their ages is 6 years. How old is each?

28. Mr. Jones has $\frac{5}{8}$ as many sheep as Mr. Brown has. If Mr. Brown has 30 sheep more than Mr. Jones, how many sheep has each?

29. Frank has $\frac{2}{3}$ as many oranges as Harry, and Harry has three times as many as Minnie. If all have two dozen oranges, how many has each?

30. John gave $\frac{1}{4}$ of his marbles to one boy, and $\frac{1}{5}$ of them to another boy. If he gave 18 marbles to both boys, how many had he at first?

31. A mother gave Fanny $\frac{1}{3}$ of her oranges, and $\frac{1}{4}$ of them to Ella, and she kept 5 oranges for Walter. How many oranges had she at first?

32. If $\frac{3}{4}$ of my money decreased by $\frac{3}{8}$ of my money equals 30 cents, how much money have I?

33. What number increased by the difference between its $\frac{1}{4}$ and its $\frac{1}{5}$ equals 42?

34. A boy on being asked his age, said that his age increased by $\frac{1}{3}$ of his age and $\frac{2}{3}$ of his age equaled 28 years. How old was he?

35. Find the number of cents whose $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$ are together equal to 47 cents.

28. *Illustrative Problem.*—Three rings cost \$36. The first cost $\frac{2}{3}$ as much as the second, and the third cost $\frac{1}{3}$ more than the second. How much did each cost?

SOLUTION.

Let x = cost of second ring,

Then $\frac{2}{3}x$ = cost of first ring,

And $\frac{4}{3}x$ = cost of third ring.

Whence, $x + \frac{2}{3}x + \frac{4}{3}x = \$36.$

$$\frac{9}{3}x = 36$$

$$\frac{1}{3}x = 4$$

$$x = 12$$

$$\frac{2}{3}x = 8$$

$$\frac{4}{3}x = 16$$

Therefore, the first ring cost \$8, the second \$12, and the third \$16.

36. A pole 27 feet long was broken into two parts; the larger part decreased by $\frac{1}{4}$ of itself equaled 12 feet. How long was each part?

37. What number increased by $\frac{2}{3}$ of itself and then decreased by $\frac{4}{5}$ of itself equals 22?

38. Frank's pony and cart cost \$120. If the cost of the cart equals $\frac{1}{2}$ of the cost of the pony, how much did each cost?

39. A certain number is $2\frac{1}{2}$ times another number, and the sum of the two numbers is 14. What are the two numbers?

40. If Harry's pigeons were decreased by $\frac{2}{5}$ of the number and by $\frac{1}{3}$ of the number, he would have 12 pigeons. How many pigeons has he?

41. What number increased by its $\frac{5}{6}$ and its $\frac{2}{9}$ equals 37?

42. After paying $\frac{3}{8}$ of her money for a doll and $\frac{1}{4}$ of her money for the doll's dress, Lillie had 30 cents remaining. How much money had she at first?

43. I sold my watch at a gain of 25% of the cost, and received $50 for it. How much did it cost?

SUGGESTION.—$25\% = \frac{25}{100} = \frac{1}{4}$.

44. John sold his sled to me at a loss of 20% of the cost. If I paid $8 for it, how much did he pay for it?

29. *Illustrative Problem.*—My money increased by $10 equals $30. How much money have I?

EXPLANATION.

If my money increased by $10 equals $30, then my money must be $30 decreased by $10, or $30 – $10, which equals $20.

SOLUTION.

$$\begin{aligned} \text{Let} \quad & x = \text{my money,} \\ \text{Then} \quad x + \$10 &= \$30. \\ x &= 30 - 10 \\ x &= 20 \end{aligned}$$

Therefore, I have $20.

45. Kate's age increased by 12 years equals 20 years. How old is Kate?

46. If two times Robert's marbles increased by 8 marbles equal 32 marbles, how many marbles has he?

47. Frank and Harry together have 25 rabbits. If Frank has 5 rabbits more than Harry, how many rabbits has each boy?

48. A horse and a carriage are worth \$120, and the horse is worth \$20 more than the carriage. How much is each worth?

49. Alice and Lucy together have 50 cents. If Alice had 10 cents more, she would have as many as Lucy. How many cents has each girl?

50. If the distance that I rode in the cars and the 8 miles that I walked equal 25 miles, how far did I ride in the cars?

51. Henry is $\frac{1}{4}$ as old as his father, and the sum of their ages is 10 years less than 60 years. How old is each?

52. To three times a number, I add 7 and obtain 40. What is the number?

53. John and William gathered 18 quarts of chestnuts. John gathered 3 quarts more than twice as many as William. How many quarts did each gather?

54. 7 feet of a pole are in the ground, $\frac{1}{6}$ of it is in water, and $\frac{3}{5}$ of it in the air. What is the length of the pole?

30. *Illustrative Problem.*—If $\frac{2}{3}$ of my money decreased by 12 cents equals 20 cents, how much money have I?

EXPLANATION.

If $\frac{2}{3}$ of my money decreased by 12 cents equals 20 cents, then $\frac{2}{3}$ of my money must be 20 cents increased by 12 cents, or 20 cents + 12 cents, which equals 32 cents. Since $\frac{2}{3}$ of my money equals 32 cents, $\frac{1}{3}$ of my money is, etc.

SOLUTION.

Let $x =$ my money,

Then $\frac{2}{3}x - 12 \text{ cents} = 20 \text{ cents}.$

$$\frac{2}{3}x = 20 + 12$$

$$\frac{2}{3}x = 32$$

$$\frac{1}{3}x = 16$$

$$x = 48$$

Therefore, I have 48 cents.

55. Edgar sold 6 quarts of berries less than Walter. If both together sold 30 quarts, how many quarts did each sell?

56. If from four times a number 15 is subtracted, the remainder is 25. What is the number?

57. Ida and Emma have two dozen oranges. If Emma has 4 oranges less than Ida, how many has each girl?

58. If 30 rabbits are 10 rabbits more than Frank and Emma have, and Emma has $\frac{2}{3}$ as many as Frank, how many has each?

59. The cost of my watch decreased by $\frac{2}{5}$ of its cost and by \$10 is \$50. How much did my watch cost?

60. Harry's money decreased by $\frac{5}{9}$ of his money and increased by 8 cents is 32 cents. How much money has he?

61. Twice Mary's age decreased by $\frac{1}{2}$ of her age equal 18 years increased by 3 years. How old is she?

62. A watch and a chain cost \$75. If $\frac{4}{5}$ of the cost of the watch decreased by \$20 is \$28, what did each cost?

31. *Illustrative Problem.*—A's money increased by \$30 equals four times his money. How much money has A?

EXPLANATION.

If A's money must be increased by \$30 to equal four times his money, then his money is \$30 less than four times his money, and the difference between his money and four times his money, or three times his money, is \$30, etc.

FIRST SOLUTION.

Let $x =$ A's money,

Then $x + \$30 = 4x.$

$3x = \$30$

$x = 10$

Therefore, A had \$10.

SECOND SOLUTION.

Let $x =$ A's money,

Then $x + \$30 = 4x.$

$30 = 3x$

$10 = x$

$x = 10$

63. Four times the number of sheep in a field decreased by 30 sheep equals twice the number. How many sheep are in the field?

64. A number increased by twice itself and by 24 equals five times the number. What is the number?

65. If 30 years be added to John's age multiplied by 3, the sum will be five times his age. Find John's age.

66. Twice the number of Walter's marbles increased by 16 and then doubled will equal six times the number. How many marbles has he?

67. Four dozen oranges cost as much as 8 dozen lemons, and a dozen oranges cost 12 cents more than a dozen lemons. What is the price of each per dozen?

68. Martha had $\frac{2}{3}$ as much money as Anna; Nellie had $\frac{1}{2}$ as much as Anna less 10 cents. If all together had 42 cents, how many cents had each?

69. A certain number can be separated into two factors, one of which is 6 less than $\frac{2}{3}$ of the other. What are the two factors if their sum is 14? What is the number?

70. John bought some apples at 2 cents each, and Thomas bought $\frac{1}{2}$ as many at 3 cents each. If they all cost 77 cents, how many apples did each buy?

71. Minnie bought some oranges for 50 cents. For one-third of them she paid 4 cents each, and for the remainder she paid 3 cents each. How many did she buy?

32. *Illustrative Problem.*—Twice Fred's money increased by $12 equals three times his money increased by $4. How much money has Fred?

SOLUTION.

Let $x =$ Fred's money,

Then $2x + \$12 = 3x + \$4.$

$$12 = x + 4$$

$$8 = x$$

$$x = 8$$

Therefore, Fred had 8 dollars.

Give the explanation of this problem (*Illustrative Problem* 31).

72. Three times a certain number increased by 8 equal four times the number decreased by 7. What is the number?

73. One-half the distance from Ella's home to the schoolhouse decreased by 40 yards, equals one-third of the distance increased by 20 yards. Find the distance.

74. One-third the length of a pole plus 4 feet equals $\frac{3}{4}$ of the length minus 6 feet. How long is the pole?

75. Three times the length of a string increased by 6 inches, equals four times the length decreased by 2 inches. How long is the string?

76. The width of a slate is 2 inches more than one-half the length, and the whole distance around the slate is 12 inches less than 40 inches. How long and how wide is the slate?

77. Harry earned as much money as he had in his bank. He then spent 25 cents more than 20% of all his money, and had 55 cents remaining. How much was in his bank?

78. Four times a certain number, decreased by 8, equals $2\frac{1}{2}$ times the number, increased by 10. What is the number?

79. George and William have 20 pigeons. If $\frac{1}{2}$ of George's number equals $\frac{1}{3}$ of William's, how many pigeons has each?

80. Four years less than twice Anna's age equaled sixteen years less than three times her age. How old was she?

81. William paid $\frac{1}{4}$ of the price of his skates, and he then owed 60 cents more than he had paid. How much did his skates cost?

82. Mr. Smith gave 40 cents to his two children so that $\frac{1}{2}$ of Frank's amount equaled $\frac{3}{4}$ of Fanny's. How much did he give to each?

83. Mary spent $\frac{1}{3}$ of her money for a doll, and had remaining 25 cents more than the doll cost. How much did she pay for the doll?

84. One week Frank earned $\frac{1}{2}$ as much as his father and $1 more than Harry. If all together earned $19, how much did each earn?

85. The sum of the ages of father, mother, and child is 60

years. The mother is five times as old as the child and is five years younger than the father. How old is the father?

86. In a class of 56 pupils there are 4 girls for every 3 boys. How many girls and how many boys are in the class?

SUGGESTION.—Let $4x$ = number of girls and $3x$ = number of boys; or, let x = number of girls and $\frac{3}{4}x$ = number of boys.

87. A's money is to B's money as 2 is to 3. If both together have 75 cents, how many cents has each?

88. Two boys bought a bicycle for $25. George paid $2 every time Henry paid $3. How much did each pay altogether?

89. Two men hired a boat for $30. A used it 9 days and B used it 6 days. How much should each of them pay?

90. William sold a watch for $50 at a gain of $10 less than 25% of the cost. Find the cost and the gain.

91. Walter gained 20% by selling his bicycle for $30. What would have been his selling price at a loss of 20%?

92. The amount of a sum of money on interest for a year at 5% is $210. What is the principal?

93. I paid my agent $52 for some wheat that he had bought and for his commission at 4%. What was the cost of the wheat?

94. Horace and Herbert bought five dozen marbles. Horace paid 9 cents and Herbert paid 6 cents. How many marbles should each boy have?

95. Two boys bought 50 newspapers. One boy paid 60 cents and the other paid 40 cents. How many papers belonged to each boy?

96. A's money equals $\frac{1}{2}$ of B's, and B's equals $\frac{2}{3}$ of C's. What part of C's money equals A's money?

97. If you were to make change for a dollar, and should give twice as many 5-cent pieces as dimes, how many nickels and how many dimes would you give?

98. John's age is $\frac{3}{4}$ of Harry's age, and Harry's age is $\frac{2}{3}$ of Frank's age. What part of Frank's age equals John's age?

99. Three boys in partnership printed some cards and cleared $1.40. The first boy had put in to buy the press twice as much as the second boy, and one-half as much as the third. How much did each make?

100. Three times the length of a fishing pole increased by 12 feet, equals twice the length of the line decreased by 6 feet. How long is each, if the pole is one-third as long as the line?

ALGEBRAIC EXPRESSION.

33. The method of expressing number or quantity by letters, or by figures and letters, is called **algebraic notation.**

Thus, the expression of a number by the letter a, whatever its value may be, is algebraic notation; so, also, is the use of b, $2c$, x, $3x$, $\frac{y}{2}$, $\frac{1}{3}y$, etc., to represent numbers.

34. A number or quantity represented by algebraic symbols is called an **algebraic expression.**

Thus, a, $2b$, $\frac{x}{2}$, $\frac{1}{3}y$, $2x + 3y$, $a + b = 15$, etc., are algebraic expressions.

35. The parts of an algebraic expression connected by the sign $+$ or $-$ are called **terms.**

Thus, in the algebraic expression $a + 2b$, a and $2b$ are terms; so, also, in $2ax - \frac{y}{2}$, the terms are $2ax$ and $\frac{y}{2}$, etc.

36. An algebraic expression consisting of but one term is called a **monomial.**

Thus, $3b$ is a monomial; so, also, are $4abx$, $\frac{x}{4}$, $-\frac{2}{3}y$, etc.

37. An algebraic expression consisting of two terms is called a **binomial.**

Thus, $a + b$ is a binomial; so, also, are $b + 3c$, $3x + 4y$, $a + \frac{x}{2}$, $4b + \frac{1}{3}y$, etc.

38. An algebraic expression consisting of three terms is called a **trinomial.**

Thus, $a + 2b + c$, is a trinomial; so, also, are $b - 3c + 5x$, $2x + \frac{y}{2} - \frac{3}{4}z$, etc.

Algebraic expressions of two or more terms are called, in general, **polynomials.**

39. A coefficient of a quantity may be numerical, or literal, or both.

Thus, in $4a$, the coefficient is 4, a numerical coefficient; and $4a$ means that a is taken 4 times.

In ab, the coefficient of b is a, a literal coefficient; and ab means that b is taken a times.

In $2ac$, the coefficient of ac is 2; and the coefficient of c is $2a$. In $3(x+y)$, the coefficient is 3, etc.

If a quantity is written without any numerical coefficient, 1 is understood to be the coefficient of the whole literal quantity.

Thus, in a, the coefficient is understood to be 1; and a means $1a$, or a taken but 1 time; so, also, 1 is the coefficient of abx, etc.

40. When a number or quantity is multiplied by itself one or more times, the product is called a **power.**

Thus, the product of 5×5, or 25, is the second power of 5; 64, the product of $4 \times 4 \times 4$, is the third power of 4; xxx is the third power of x; etc.

The second power of a number or quantity is called the *square* of that number or quantity; the third power is called the *cube.*

The product of $a \times a$, or aa, is generally written a^2. Instead of xxx, x^3 is written, etc.

41. The number written to the right of the upper part of a number or quantity is called an **exponent.**

Thus, in a^2, ^{2} is the exponent of a; in b^3, ^{3} is the exponent of b. a^2 is read a square; b^3 is read b cube; c^4 is read c fourth power; etc.

The exponent shows how many times the number or quantity is used as a factor.

If a letter or quantity is written without an exponent, 1 is understood to be the exponent.

Thus, a is the same as a^1. The exponent of b or c or x is understood to be ^{1}; also of a and of b in ab; etc.

42. An equation which contains only the first power of the unknown quantity, is called an **equation of the first degree, or a simple equation.**

Thus, $x + 2x + 3x = 24$ is a simple equation; so, also, is $y + \frac{2}{3}y + 4y = 34$; $2a + b = 12$; etc.

EXERCISES.

Read each of the following algebraic expressions:—

1. a.
2. a^2b.
3. $4ab^2$.
4. $5b^3c^2d$.
5. $a + 2b$.
6. $b^2c - d$.
7. $2a + 3x^3$.
8. $\frac{a}{3} - b^2c^2$.
9. $x^2 + \frac{2x}{3}$.
10. $\frac{1}{4}c - 5ab^3$.
11. $3b^3c - \frac{ax}{2}$.
12. $5y + ax^3$.
13. $r^2s - xy^3$.
14. $b^2 + 3a^2x^3$.
15. $5a^3 - c^2y$.
16. $4c^2 + 3b^3d$.
17. $4a^3 - b + d$.
18. $b + c^2 + 3d$.
19. $\frac{1}{4}xy + c^2$.
20. $3c - \frac{1}{3}a^2b$.
21. $cd^2 + \frac{2a}{b}$.
22. $2a - b^3 + \frac{c}{4}$.
23. $3 - a^2x + x^3$.
24. $b^3 - 2b^2y - \frac{1}{2}$.
25. $b^2 - 2c + 1$.
26. $x + y^2 - \frac{2}{3}$.
27. $\frac{1}{5} - a^2b^3 + 2y$.
28. $ax^3 + \frac{ac}{3}$.
29. $\frac{2}{b} + 4ab^2c^3$.
30. $2x^2 - \frac{1}{3}a^3bx^2$.
31. $b^3 + \frac{2a^2b}{3}$.
32. $4a - \frac{a}{b} - c^3$.

In exercises 1–32, name each numerical coefficient; each exponent; each monomial; each binomial; and each trinomial.

Express algebraically each of the following:—

1. Two times a times b square.
2. The cube of x divided by 4.
3. Two times a, increased by b.
4. The cube of c, decreased by four times d square.

5. The sum of *a* and three times *b* times *c* cube.
6. Three times *b* times *c*, decreased by the cube of *d*.
7. The sum of four times *c* square times *d*, and *b* divided by 3.
8. Four times *c* cube, increased by *x* square times *y*.

9. From one-half of *d* square, take twice *a* cube times *b*.
10. Find the sum of *a* and three times *b*, and the cube of *c*.
11. From one-third of *b* times *c*, take *x* cube.
12. To four times *b* square times *c*, add one-fourth of *a* square.
13. From *b* cube divided by 4, take 5 times *c* square times *x* cube.
14. Find the product of four times *x* cube, and to it add one-half of *x* square.
15. Take three from the sum of two times *a*, plus *b*.
16. From three times *b* take two-thirds of *c* times *d*, and then add *x* cube.
17. Take one-fourth of *y* square from five times *s*.
18. To one-fifth of *a* square times *x*, add three times *y* cube.
19. Find the product of four times *b* cube times *c*, and then subtract *y* square divided by two.
20. Find the sum of twice *a* plus three times *b* plus four.
21. To three times *b* square add five, and then subtract two divided by *a*.
22. From *c* cube take four times *d* square, and then add *a* times *x* divided by five.
23. Multiply four and *b* square and *c* cube, and from the product take twice *a* divided by *b*.
24. To five times *a* cube times *x* square add 4, and from the sum take two-thirds of *y*.

Read each of the algebraic expressions 1–24 after writing them; name each numerical coefficient and exponent, and each monomial, binomial, and trinomial.

EXAMPLES AND PROBLEMS.

1. If 2 is represented by a and 3 is represented by b, how is the sum of 2 and 3 expressed algebraically?

2. If b stands for 3 and c stands for 5, how is the sum of 3 and 5 expressed algebraically?

3. If a stands for 4 and c stands for 6, what algebraic expression stands for the sum of 4 cents and 6 cents?

SUGGESTION: 4 cents is represented by a cents, etc.

4. Let 3 be represented by b and 8 be represented by d. If I pay 3 cents for an orange and 8 cents for a melon, what algebraic expression shows what I pay for both?

5. If 5 is represented by a and 3 is represented by b, how is $5 - 3$ expressed?

6. If b stands for 8 and c stands for 5, how is $8 - 5$ expressed? The difference between 8 cents and 5 cents?

7. Let a stand for 10 and c for 5. What algebraic expression stands for the difference between 10 dollars and 5 dollars?

8. John has 12 marbles and Harry has 8 marbles. What algebraic expression stands for the difference between John's number and Harry's number, if x stands for 12 and y stands for 8?

9. If 1 is represented by a, how is $1 + 1$, or 1 taken two times, expressed? 1 taken three times, or three times 1?

SUGGESTION: $a + a = 2a$.

10. If a number is represented by b, how is three times the number expressed? Five times the number?

11. If 5 is represented by c, what algebraic expression stands for three times 5? For four times 5?

12. How is the sum of a number represented by d and another number represented by $3d$ expressed?

13. If x stands for 4, how is the difference between 12 and 8, or $12 - 8$, expressed?

14. Let y stand for 2. What algebraic expression stands for the cost of 3 apples at 2 cents each?

15 If three times a number is a, what is the number?

SUGGESTION: $a \div 3$, or $\frac{a}{3}$.

16. If I pay c cents for 5 oranges, what algebraic expression stands for the cost of one orange?

17. Frank gave x cents for a dozen oranges. How much did each orange cost?

18. The difference between two numbers is a, and the larger one of them is 4. What algebraic expression stands for the other number?

19. If the product of two numbers is b, and one of them is 3, what algebraic expression stands for the other number?

20. The sum of two numbers is c, and one of them is two times the other number. Find each of the two numbers.

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21. Find the sum of a and $4a$.

What is the literal part of each number or quantity added? The coefficient of each part? The coefficient of the sum?

22. What is the sum of $2a$ and $3a$? Of $a + 2a + a$?

23. Find the sum of $4b$ and $3b$; of $b + b + b$.

24. $5bc + 3bc + bc =$?

Name the literal part of each term in this trinomial. Name each coefficient. Name the literal part and the coefficient of the sum.

43. The value of an algebraic expression obtained by using for each letter in the expression the number for which the letter stands, and then performing the operations required, is called the **numerical value** of the expression, or the **numerical result.**

Thus, if a stands for 3, b for 4, and c for 10, the numerical value of $a^2 + 2ab - \frac{c}{2}$ is $9 + 24 - 5$, or 28.

In examples 21–24, let $a = 4$, $b = 5$, and $c = 10$; find the numerical result of each example.

25. From $5a$ take $3a$. $6a - 2a =$? $8a - 5a =$?
26. Find the difference between $7b$ and $3b$. $9b - 4b =$?
27. Take $3b$ from $10b$.

What is the coefficient of the minuend? Of the subtrahend? Of the remainder?

28. What is the result of $9c - 4c$? Of $12c - 5c$?
29. $4ac - 2ac =$?

Name the literal part and the coefficient of the minuend; of the subtrahend; and of the remainder.

In examples 25–29, let $a = 5$, $b = 7$, and $c = 12$; find the numerical result of each example.

30. What is the product of $a \times 2$? Of $3 \times a^2$?
31. What is the product of b taken 5 times?

What is the literal part of the product? What is the coefficient of the product?

32. What is the result of $b \times 3$? Of $a \times b$? Of $b^2 \times c^3$?
33. Find the product of $2c \times 4$. 5 times $3b =$?
34. $4a \times 3c =$?

Name the literal part and the coefficient of each factor and of the product.

In examples 30–34, let $a = 3$, $b = 5$, and $c = 7$; find the numerical result of each example.

35. Express the quotient of $a \div 2$ in the form of a fraction.
36. Find the quotient of $12c \div 6$; of $15a \div 5a$.
37. What other algebraic expression shows the quotient of $2a \div 3$? Of $16 \div b$?
38. $12ac \div 3a =$?

Name the literal part and the coefficient of the dividend; of the divisor; and of the quotient.

$a = 6$
$b = 8$
$c = 10$

In examples 35–38, let $a = 6$, $b = 8$, and $c = 10$; find the numerical result of each example.

In problems 39–43, let some letter stand for each number used; indicate, by the proper algebraic expression, the operation and the result of each problem.

39. Mary gave 5 cents for some paper and 3 cents for a lead-pencil. How many cents did she give altogether?

40. A tinsmith made 8 cans for one milkman and 6 cans for another. How many cans did he make for both?

41. How far does a man go who drives 6 miles one hour, 5 miles the next hour, and 7 miles the third hour?

42. If John pays 12 cents for a knife, and sells it at a gain of 6 cents, what does he get for it?

43. A lady bought 10 yards of dress goods, 4 yards of linen, and 6 yards of muslin. How many yards did she buy?

In problems 44–48, let some letter stand for each number used; indicate, by the correct algebraic expression, the operation and the result of each problem.

44. James is 12 years old and his brother is 5 years younger. How old is James's brother?

45. A newsboy bought some papers for 10 cents, and sold them for 15 cents. How many cents did he gain?

46. A dealer had 20 tons of coal, and sold 8 tons to one man and 6 tons to another. How many tons remained?

47. From a cask containing 25 gallons of oil, 6 gallons leaked out and 9 gallons were sold. How many gallons remained?

48. If the sum of three numbers is 18, and two of the numbers are 9 and 6, what is the third number?

49. If the fare on a street-car is 5 cents, how many cents does a conductor collect from 9 passengers?

In problems 49–53, use some letter for each number; indicate, by the correct algebraic expression, the operation and the result of each problem.

50. If in a piece of muslin there are 15 yards, how many yards are there in 4 such pieces?

51. How much will a dozen oranges at 3 cents each, and a dozen lemons at 2 cents each, cost?

52. Mary bought 10 pounds of sugar at 5 cents a pound, and had 10 cents left. How many cents had she at first?

53. Two boys started from the same place and walked in the same direction. One boy walked 3 miles an hour, and the other walked 2 miles an hour. How far apart were they in 5 hours?

In problems 54–58, use some letter for each number; indicate, by the correct algebraic expression, the operation and the result of each problem.

54. A dozen oranges were divided among some boys so that each boy had 3 oranges. How many boys were there?

55. If the dividend is 20 pounds and the divisor is 4, what is the quotient? If the divisor is 10?

56. If a man earns 10 dollars in 5 days, how many dollars will he earn in 7 days at the same rate?

57. The product of three numbers is 24, and two of the numbers are 3 and 4. What is the third number?

58. A boat ran up stream 6 hours at the rate of 8 miles an hour, but returned in 4 hours. How fast did it run down stream?

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59. Anna gave a cents for a pencil and b cents for paper. How many cents did she spend?

60. Fred had c oranges, and bought 5 oranges more. How many oranges had he then?

61. A slate is c inches long and b inches wide. How long a string will reach entirely around it?

62. Mabel had c cents, and she spent a cents for stamps. How many cents had she left?

63. From a ribbon $2b$ inches long, 3 inches were cut off. How long was the ribbon then?

64. A stick c feet long was driven a feet into the ground. How many feet were above the ground?

65. One factor of a number is $3a$, and the other factor is c. What is the number?

66. Harry sold 10 quarts of berries at b cents a quart. How many cents did he get for them?

67. What is the entire cost of 4 pencils at a cents each and 3 note books at c cents each?

68. If I divide 12 apples among a girls, how many apples does each girl get?

69. Mary paid $5b$ cents for sugar worth 4 cents a pound. How many pounds did she buy?

70. How many hats at a dollars each can be bought for $4c$ dollars?

71. If b yards of ribbon cost 32 cents, what is the price per yard?

72. Robert bought a sled for c dollars, and sold it at a gain of 25%. How many dollars did he get for it?

73. What is the interest on $5c$ dollars for 1 year at 5%? Find the interest for 2 years; for 3 yr. 6 mo.

In problems 59–73 let $a=4$, $b=8$, and $c=12$; find the numerical result of each of these problems.

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74. If 5 times a number equals a, what is the number?

75. If $\frac{1}{2}$ a number equals b, what is the number?

76. A certain number increased by 3 times the number equals c. Find the number.

77. What number increased by $\frac{2}{3}$ of itself equals c?

78. $\frac{1}{2}$ a certain number added to $\frac{1}{4}$ the number equals b. What is the number?

79. If 6 times A's money equals c dollars plus 10 dollars, how much money has he?

80. William paid $15b$ cents for some berries, and sold them for $15a$ cents. How much did he lose?

81. If the two factors of a number are a and b, what is one-half of the number?

82. The product of two numbers is x, and one of the numbers is a. What is the other number?

83. Frank has c dimes in his bank. How many cents are they worth? How many dollars?

84. The sum of two numbers is c, and one of the numbers is three times the other. What are the two numbers?

85. If John is c years old to-day, how old was he 4 years ago? How old will he be in a years?

86. Mary had c pet rabbits. She bought a rabbits more, and then she sold b rabbits. How many rabbits had she left?

87. I had x dollars. After paying $\frac{a}{2}$ dollars for shoes and b dollars for a coat, how much had I left?

88. Martha paid $3c$ cents for ribbon worth b cents a yard. How many yards did she buy?

89. William is four times as old as Elsie, and the sum of their ages is c years. How old is each?

90. The divisor is b, the quotient is c and the remainder is a. What is the dividend?

91. The dividend is x, the divisor is b, and the remainder is 2. What is the quotient?

92. If I work a hours a day for b days at c cents an hour, and then spend 5 dollars, how much money have I left?

93. George's money increased by c dollars equals five times his money. How much money has he?

94. A boy bought x quarts of berries at a cents a quart, and sold them at b cents a quart. How many cents did he gain?

95. A tailor paid c dollars for a yards of cloth, and sold it at a gain of 10 dollars. What was the selling price per yard?

96. Three times Anna's age increased by twice her age equal c years. How old is she?

97. Thomas has a number of marbles, and Henry has four times as many. If both together have x marbles, how many has each boy?

98. Martha had x cents; she spent one-half of her money for a knife, and a cents for a pencil. How many cents had she left?

99. A farmer sold $\frac{1}{2}b$ pounds of butter at c cents a pound, and with the money he bought some tea at $6a$ cents a pound. How many pounds did he buy?

100. If a tailor buys c yards of cloth at two dollars a yard, and sells it at a gain of 10%, what is the total selling price?

In problems 74–100, let $a = 10$, $b = 12$, $c = 20$, and $x = 50$; find the numerical result of each problem.

POSITIVE AND NEGATIVE QUANTITIES.

44. In arithmetic, the sign + always indicates the operation of addition.

Thus, $4 + 6 + 8$ indicates that 4, 6, and 8 are to be added.

So, also, the algebraic expression $3a + 4b + 5c$ indicates that the sum of the numbers expressed by $3a$, $4b$, and $5c$ is to be found.

45. In arithmetic, the sign — always indicates the operation of subtraction.

Thus, $12 - 7$ indicates that 7 is to be subtracted from 12.

So, also, the algebraic expression $5a - 3b$ indicates that the difference between the numbers expressed by $5a$ and $3b$ is to be found.

46. In algebra the signs + and − are *signs of operation*, and are also **signs of opposition** in kind or nature.

47. A quantity or term which has the sign + prefixed, is called a **positive quantity** or **term**.

When a quantity or term has no sign prefixed, the sign + is understood, and the quantity or term is a positive quantity or term.

Thus, a, $2b$, $\frac{1}{4}a^2x^3$ are positive quantities.

Also, in the algebraic expression $a - 3b^2 + \frac{1}{2}x^3$, a and $\frac{1}{2}x^3$ are positive terms.

48. A quantity or term which has the sign − prefixed, is called a **negative quantity** or **term.**

A negative quantity or term is always written with the sign − prefixed.

Thus, $-2b$, $-3x^2$, $-\frac{1}{3}ac^3$ are negative quantities.

Also, in the algebraic expression $-a + 5c^2 - \frac{1}{4}x^3$, $-a$ and $-\frac{1}{4}x^3$ are negative terms.

49. Positive and negative quantities are opposite in meaning or nature; they indicate opposite operations; they are reckoned in opposite directions or ways; etc.

Thus, if a positive quantity is used to indicate gain, then a negative quantity indicates loss; if a positive quantity is used to indicate north, a negative quantity indicates south; if a positive quantity indicates a number above zero, a negative quantity indicates a number below zero; etc.

50. A positive quantity is additive, and indicates that something is to be increased by this quantity.

A negative quantity is subtractive, and indicates that something is to be decreased by this quantity.

Thus, a gain of a dollars is expressed by $+a$ dollars, and a loss of a dollars is expressed by $-a$ dollars.

A number of degrees above zero is indicated by +, and a number of degrees below is indicated by −; etc.

51. In algebraic expressions having more than one term, the kind of quantity forming each term may be indicated by enclosing it, with its proper sign, within a parenthesis.

Thus, $(+a) + (-2b)$ or, enclosing only the negative term, $a + (-2b)$ indicates that the first term is positive and the second term negative, and that they are to be added. Or,

The parenthesis may be omitted from such expressions without changing the meaning or the value of the expression (see p. 44, par. 3, 4, 5).

Thus, $a - 2b$ means that $2b$ may be regarded as a negative quantity to be added to a, or as a positive quantity to be subtracted from a. Hence,

The signs + and − connecting the terms of an algebraic expression are regarded either as indicating the kinds of quantities forming the several terms, or as indicating the operations to be performed.

Thus, $x^2 + (-2ax) + y^2$, is usually written $x^2 - 2ax + y^2$, in which x^2 and y^2 are positive and are to be added, and $2ax$ is negative and is to be added, or is positive and is to be subtracted.

52. Terms which have the same literal parts are called **similar terms.**

Thus, $4ab$ and $-7ab$ are similar terms or quantities; also $-a^2x$ and $\frac{1}{2}a^2x$; etc.

53. Terms which have different literal parts are called **dissimilar terms.**

Thus, $4ab$ and $5ac$ are dissimilar; also, $3a^2b$ and $-4ab^2$; also, in the expression $2ax - 7a^2b + \frac{1}{2}ax$, the terms $2ax$ and $-7a^2b$, and $-7a^2b$ and $\frac{1}{2}ax$, are dissimilar; but $2ax$ and $\frac{1}{2}ax$ are similar.

EXERCISES.

1. How many units does $+6$ express? How many does -6 express?

2. What kind of units does $+6$ express? What kind does -6 express?

3. How is the kind of units expressed by $+6$, or 6, distinguished from the kind expressed by -6?

4. How many and what kind of units does $5a$ express? How many and what kind does $-5a$ express?

If the positive units in $5a$ and the negative units in $-5a$ be combined, what is the result?

5. How many and what kind of units does $7x$ express? How many and what kind does $-10x$ express?

If the units in $7x$ and in $-10x$ be combined, which quantity shows the greater number of units? How many more units are in the greater than in the less, and what kind are they?

54. The number of units which a number expresses is called the **real** or **absolute value** of the number, no matter what the kind of units may be.

Thus, the absolute value of $+4$, is 4, or four units; also, the absolute value of -4 is 4, or four units.

If a stands for 2, then the absolute value of a is two positive units; but the absolute value of $-a$ is two negative units.

So, in arithmetic, the absolute value of 5 is five units, whatever kind or order they may be; as, 5 expresses five ones; .5 expresses five tenths; 5 in 50 expresses five tens, etc.

55. The sum of similar positive quantities or of similar negative quantities is found by adding the quantities, or their coefficients, and prefixing the sign that all of them have.

Thus, the sum of $+3$, $+4$, and $+6$, or 3, 4, and 6, is $+13$, or 13; the sum of $-2a$, $-5a$, and $-7a$ is $-14a$; etc.

The sum of two or more numbers or quantities either positive or negative is always composed of the same kind of units as the numbers or quantities added.

56. The sum of similar quantities or terms, some of which are positive and some negative, is obtained by finding the sum of the positive and the sum of the negative quantities, then combining these two sums and prefixing the sign of the quantity which expresses the greater number of units, or has the greater absolute value.

Thus, in the expression $5 + (-3)$, the three units expressed by -3 cancel three of the five positive units expressed by 5, and there remain two positive units, or 2.

In $-5 + 3$, the three units expressed by 3 cancel three of the five negative units expressed by -5, and there remain two negative units, or -2.

In $2a - 3a + 4a - 5a$, the sum of the positive terms is $6a$, and the

sum of the negative terms is $-8a$; the difference between these two sums is two a, which is a part of the negative sum, and, therefore, is $-2a$.

The sum of similar quantities, some of which are positive and some negative, is composed of the same kind of units as those which form the quantity or sum expressing the greater number of units, or having the greater absolute value.

57. The result of collecting both positive and negative quantities into one quantity or term, is called the **algebraic sum** of the quantities.

Tell which of the following numbers or quantities are positive and which are negative:—

1. 4.	6. 12.	11. $2ab^3$.	16. $-5a^2x^3$.
2. -5.	7. $-2a$.	12. $-x^2$.	17. $\frac{abc}{3}$.
3. a.	8. $4a$.	13. $-4a^2b$.	
4. $-x$.	9. -15.	14. $3ac$.	18. $-\frac{b^2y}{5}$.
5. -6.	10. $6ax$.	15. $-2x^2y$.	

In expressions 1–18, let $a=6$, $b=5$, $c=4$ $x=3$, and $y=2$; find the numerical value of each.

Tell which of the following numbers or terms are positive, and which are negative:—

19. $6+4$.	23. $(-7)+(-6)$.	27. $10+3+5$.
20. $7+5$.	24. $(-8)+(-7)$.	28. $-12-6-7$.
21. $8+6$.	25. $(-10)+(-8)$.	29. $14+7+8$.
22. $10+7$.	26. $(-12)+(-10)$.	30. $-16-8-10$.

Express, with its proper sign, the sum of each of the expressions 19–30.

In each of the following expressions, tell which of the terms are positive and which are negative; which are similar and which are dissimilar:—

31. $a + b$.
32. $b + c$.
33. $2a + a$.
34. $b + 3y$.
35. $-2b - 3b$.
36. $-3x - 4y$.
37. $-5y - 6y$.
38. $-18 - a$.
39. $+b + 6$.
40. $-8x - 4$.
41. $2x + 9$.
42. $-10y - y$.

In expressions 31–42, let $a = 10$, $b = 8$, $c = 5$, $x = 3$, and $y = 2$; find the numerical value of each expression.

Read each of the following expressions; tell which terms are positive and which are negative:—

43. $-6 - 2$.
44. $-2 + 7$.
45. $8 - 3$.
46. $-4 + 6$.
47. $9 - 4$.
48. $5 + a$.
49. $-a + 6$.
50. $7 - b$.
51. $-c + b$.
52. $2x - 6$.
53. $a - 2b + c$.
54. $-2c + 4x - y$.
55. $c - 6x + 5y$.
56. $-3x + 5c - 3c$.
57. $4c + 2y - 3x$.

In expressions 43–57, let $a = 2$, $b = 3$, $c = 4$, $x = 5$, and $y = 6$; find the algebraic sum and the numerical value of each expression.

Write the following algebraic expressions, using a, b, c, x, or y, or combinations of these letters, in forming the literal parts; let $a = 5$, $b = 4$, $c = 3$, $x = 2$, and $y = 1$; find the numerical value of each expression:—

58. A monomial positive literal quantity.
59. A monomial negative literal quantity.
60. A monomial positive literal quantity, with a coefficient.
61. A monomial negative literal quantity, with a coefficient.
62. A binomial expressing the sum of positive literal terms.
63. A binomial expressing the sum of negative literal terms.
64. A binomial composed of similar positive terms, with coefficients.
65. A binomial composed of dissimilar negative terms, with coefficients.

66. A binomial of a positive and a negative term, similar, with coefficients.

67. A binomial of dissimilar terms, the first of which shall be negative and the second positive, with coefficients.

68. A trinomial of similar negative terms, with coefficients.

69. A trinomial of dissimilar terms, two of which shall be negative, and one positive, with coefficients.

70. An algebraic expression of three terms, the first and the third of which shall be positive and similar, and the second negative and unlike the two others.

ILLUSTRATIONS OF POSITIVE AND NEGATIVE QUANTITIES.

58. *Illustrative Problem.*—A ship sailed from the equator 3 degrees north, then 2 degrees south, then 5 degrees north, and then 9 degrees south. In what latitude was it then?

OPERATION.

3 degrees
– 2 degrees
5 degrees
– 9 degrees
– 3 degrees

EXPLANATION.—If a number of degrees toward the north be regarded as a positive number, then a number of degrees toward the south must be regarded as a negative number; they are opposite in nature.

The ship sailed north 8 degrees, and south 11 degrees; therefore, its course ended 3 degrees south of the equator.

59. *Illustrative Problem.*—From a certain place, one boy rode north 10 miles, and another boy rode south 8 miles. Express their relation to each other.

OPERATION 1.

+ 10 miles
– 8 miles
+ 18 miles

OPERATION 2.

– 8 miles
+ 10 miles
– 18 miles

EXPLANATION.—Since one boy rode north 10 miles, and the other rode south 8 miles, the difference in position between them is 18 miles.

Considering the first boy as 18 miles north of the second, his relation to the second is indicated by + 18 miles; but considering the second as 18 miles south of the first, his relation to the first is indicated by – 18 miles.

Explain each of the following; place the proper sign before each number, and find the result:—

1. A man gained 10 dollars on one sale of goods, and lost 10 dollars on another sale.

2. A man gained 10 dollars on one sale, and lost 5 dollars on another sale.

3. A man gained 10 dollars on one sale, and lost 15 dollars on another sale.

4. A thermometer showed a rise of 10 degrees in temperature, and then a fall of 10 degrees.

5. A thermometer showed a rise of 10 degrees in temperature, followed by a fall of 8 degrees.

6. A thermometer showed a rise of 10 degrees in temperature, followed by a fall of 12 degrees.

7. Anna has no money, and Emma has none, but owes 10 cents.

8. A boat moved at the rate of 5 miles an hour up a stream whose current was 3 miles an hour.

9. A boat moved at the rate of 3 miles an hour up a stream whose current was 5 miles an hour.

10. A boat moved at the rate of 5 miles an hour down a stream whose current was 3 miles an hour.

11. On one bundle of papers a newsboy gained 50 cents, but on the next bundle he lost 10 cents.

12. Sarah had 25 cents; she spent 10 cents for candy, and 12 cents for paper.

13. William has 12 cents in his pocket, 15 cents at home, and a playmate owes him 13 cents. He owes 12 cents to Fred, 10 cents for some paper, and 3 cents to an apple-woman.

14. A caterpillar crawls 6 feet up a tree, then falls back 2 feet, and then crawls up 8 feet.

15. A man earns a dollars; he then spends 10 dollars; and afterwards he earns b dollars.

16. Thomas walked 50 feet to the right from a street, then walked back x feet, and then walked again y feet to the right.

60. Note to Teachers.—1. Positive and negative quantities may be regarded as two series of numbers increasing in opposite directions from 0.

Positive numbers begin at 0, and *increase above* zero.

Thus, $0 + 1$ is ${}^+1$, or 1; $0 + 1 + 1$ is ${}^+2$, or 2; $0 + 1 + 1 + 1$, or 1 added three times to 0, is ${}^+3$, or 3; etc.

Negative numbers begin at 0, and *increase below* zero.

Thus, $0 - 1$ is ${}^-1$; $0 - 1 - 1$ is ${}^-2$; $0 - 1 - 1 - 1$, or 1 taken or subtracted three times from 0, is ${}^-3$; etc.

2. The scale of positive and negative numbers may be written,—

$$\cdots {}^-5 \quad {}^-4 \quad {}^-3 \quad {}^-2 \quad {}^-1 \quad 0 \quad {}^+1 \quad {}^+2 \quad {}^+3 \quad {}^+4 \quad {}^+5 \cdots$$

${}^-5$, or -5, is read *negative five;* ${}^+3$, or $+3$, is read *positive three,* etc.

The sign ${}^+$ or $-$ prefixed to a number shows to which series it belongs, while its absolute value shows its place in the scale.

Thus, in ${}^-4$, ${}^-$ shows that 4 belongs in the negative series, and the absolute value of 4 places it fourth in the scale.

3. The sum of two algebraic numbers is found by beginning with the number first named and counting in the direction indicated by the sign of the second number, as many units as there are in the absolute value of the second number.

Thus, the sum of $+5$ and -3 is found by beginning at $+5$ and counting three units in the negative, or subtractive, direction, giving the algebraic sum $+2$.

Also, $+5) + (+3)$, or $5 + 3 = 8$; $(-5) + (-3)$, or $-5 - 3 = -8$; $(+5) + (-3)$, or $5 - 3 = 2$; $(-5) + (+3)$, or $-5 + 3 = -2$; etc.

The *algebraic sum* of a positive and a negative number or quantity is the *numerical difference* of their absolute values, and is of the same kind as that quantity which has the greater absolute value.

Thus, the algebraic sum of -6 and $+2$ is -4; $6a + {}^-8a = {}^-2a$; etc.

4. In algebra, as in arithmetic, subtraction is the opposite of addition, and the operation is performed in a manner opposite to that of addition.

5. The difference between two algebraic numbers is found by beginning at the minuend and counting in the direction opposite to that indicated by the sign of the subtrahend, as many units as there are in the absolute value of the subtrahend.

Thus, the difference between $+5$ and -3 is found by beginning at $+5$ and counting three units in the positive, or additive, direction, giving the algebraic difference $+8$.

Also, $(+5) - (+3)$, or $5 - 3 = 2$; $(-5) - (-3)$, or $-5 + 3 = 2$; $(+5) - (-3)$, or $5 + 3 = 8$; $(-5) - (+3)$, or $-5 - 3 = -8$; etc.

If a negative term follows the sign $-$, the sign of quantity and the sign of operation must be changed to find the algebraic difference.

Thus, $6-(-2)$, or $6 - {}^-2 = 6 + 2 = 8$; ${}^-4a - {}^-2a = {}^-2a$; etc.

The *algebraic difference* between a positive and a negative number or quantity is the *numerical sum* of their absolute values, and is of the same kind as that to which both belong, after the sign of the subtrahend is changed.

The algebraic difference is the number or quantity that must be added to the subtrahend to equal the minuend.

Thus, $-6-(+4)$, or $-6 - {}^+4 = -10$; $8a - {}^-4a = 12a$; etc.

From the preceding, it is evident that the signs $+$ and $-$ may be used to indicate either operations or kinds of quantity.

Thus, $8 - {}^+3$, or $8 + {}^-3$, is equivalent to $8 - 3 = 5$; and $5 - {}^-3$, or $5 + {}^+3$, is equivalent to $5 + 3 = 8$; etc.

61. Additional principles of positive and negative quantities, and their applications, are given under Addition (pp. 48–53), Subtraction (pp. 56–63), Multiplication (pp. 81–89), and Division (pp. 98–107).

EQUATIONS.

62. *Illustrative Example.*—If $9x-7x+5x-x=40-15+5$, what is the value of x.

SOLUTION.

$$9x - 7x + 5x - x = 40 - 15 + 5$$
$$14x - 8x = 45 - 15$$
$$6x = 30$$
$$x = 5$$

Find the value of x in each of the following:—

1. $x+3x+5x=36$.
2. $8x-6x+4x=24$.
3. $x-10x+12x=30$.
4. $2x+9x-7x=48$.
5. $3x+8x-6x=5+20$.
6. $4x-20x+8x=5-29$.
7. $15x-8x+5x=60-12$.
8. $x-12x+7x=10-70$.

In equations 1–8, substitute the values of x and prove each result.

9. Divide a yard of ribbon into two such parts that one part shall be 8 inches longer than the other.

10. How can you divide 30 oranges among three boys so that Harry shall have 3 oranges more than Frank, and Walter 9 oranges more than Harry?

11. The sum of one-half, one-third, and one-fourth of the number of which I am thinking is 65. What is the number?

12. A boy walked 40 miles in three days. He walked 3 miles more the second day than he did the first, and 7 miles more the third day than he did the second. How far did he walk each day?

13. If Anna's age were increased by twice her age and 9 years, she would be 30 years old. How old is she?

14. One-sixth of the price of my watch is $4 more than one-tenth of the price. What did my watch cost?

15. If $4x-3$ stands for 45, for what number does $x+8$ stand?

16. James bought 16 apples at one time and 5 apples at another time, all at the same rate. The second time he paid 33 cents less than he did the first time. What was the price of each apple?

17. Harry has $1 less than Mary, and both together have $15. How much has each?

18. James has five times as many marbles as George, and William has three times as many as George. If James has 22 marbles more than William, how many marbles has each boy?

19. A farmer has five times as many cows as horses, and three times as many sheep as cows. How many of each has he, if there are 105 altogether?

20. A man bought 3 horses and 4 cows for $600. Each horse cost twice as much as each cow. What was the price of each?

21. Two men are 42 miles apart, and travel toward each other. One goes 3 miles an hour, and the other goes 4 miles an hour. In how many hours will they meet?

all

22. William bought 4 oranges and 3 lemons for 22 cents. He gave as much for one orange as he gave for two lemons. How many cents did he pay for each?

23. A boy put in his bank an equal number of nickels and dimes, which were together worth $3. How many of each had he?

24. Two men put an equal sum of money in business. B lost $400, and A gained $700. If B then had one-half as much as A, how much did each invest?

25. I bought a set of harness, a horse, and a carriage for $325. I paid four times as much for the carriage as for the harness, and twice as much for the horse as for the carriage. What did each cost?

26. What number increased by its one-half, and then diminished by its two-thirds, equals 30?

27. If Harry's pigeons were increased by as many more and half as many more and 5 pigeons, he would have a hundred. How many pigeons has he?

28. Mary and Martha bought 5 dozen oranges. Every time Mary paid 2 cents for them, Martha paid 3 cents. How many oranges should each girl take?

29. By selling a pair of skates at a loss of 10%, John received $1.80 for them. What did he pay for his skates?

30. This year the interest on a sum of money is 5% more than it was last year. If the interest for both years is $164, what was the interest for each year?

31. John bought some oranges at 3 cents each, and Thomas bought a dozen more than one-third as many at 4 cents each. If all cost $1.26, how many did each buy?

32. Two-thirds of the length of a piece of ribbon increased by 8 inches, equals eight-ninths of its length decreased by 8 inches. How long is the piece of ribbon?

ADDITION.

1. What is the sum of 5 apples and 7 apples? The sum of 5 cents and 7 cents?

2. The sum of 5 times a quantity and 7 times the quantity is how many times that quantity?

What is the sum of 5 times a and 7 times a? The sum of $5a$ and $7a$? $5a + 7a =$?

What kind of quantity is $5a$? What kind is $7a$? What kind is the sum of these two quantities?

3. What is the sum of 6 times $-b$ and 8 times $-b$? Of $-6b$ and $-8b$? $(-6b) + (-8b) =$? $-6b - 8b =$?

What kind of quantity is $-6b$? What kind is $-8b$? What kind is the sum of these two quantities?

4. Find the sum of $3x$, $5x$, and $7x$, or $3x + 5x + 7x$.

OPERATION.

$$\begin{array}{r} 3x \\ 5x \\ 7x \\ \hline 15x \end{array}$$

What kind of quantity is each?
What is the sum of the units in all three quantities?
What kind of quantity is the sum?

5. Find the sum of $-4y$, $-6y$, and $-8y$, or $-4y - 6y - 8y$.

OPERATION.

$$\begin{array}{r} -4y \\ -6y \\ -8y \\ \hline -18y \end{array}$$

What kind of quantity is each?
What is the sum of the units in all three quantities?
What kind of quantity is the sum? How is the sum found?

6. What kind of quantity is $6a$? What kind is $-6a$?

If $6a$ and $-6a$ be combined or added, what is the result? If $7b$ and $-7b$ be added? $8c$ and $-8c$?

What is the algebraic sum of $10x$ and $-10x$? Of $12y$ and $-12y$? Of $15z$ and $-15z$? How is the algebraic sum of each found?

7. What kind of quantity is $+6c$, or $6c$? What kind is $-8c$? Which of these two quantities expresses the greater number of units?

If the number of units in $6c$ and in $-8c$ be combined, how many of the units in $-8c$ are canceled by the number in $6c$? How many of the units in $-8c$ remain uncanceled?

8. What kind of quantity is $8a$? What kind is $-10a$?

If $8a$ and $-10a$ be combined, or added, what is the result? If $9b$ and $-12b$ be added? $10c$ and $-15c$?

What is the algebraic sum of $12x$ and $-14x$? Of $13y$ and $-17y$? Of $15z$ and $-20z$? How is each of these algebraic sums found?

9. Find the algebraic sum of $-3z$, $5z$, $-7z$, and $9z$, or $-3z + 5z - 7z + 9z$.

OPERATION.

$$\begin{array}{r} -3z \\ 5z \\ -7z \\ 9z \\ \hline 4z \end{array}$$

What is the sum of the positive quantities?

What is the sum of the negative quantities?

How many of the units in the greater sum are canceled by the number in the less sum? How many units in the greater sum remain uncanceled by the number in the less sum?

What is the algebraic sum of the four quantities?

63. In arithmetic, the sum of two or more numbers is found by the process of addition only.

Thus, the sum of 4, 6, and 8, is 18; $6 + 8 + 10 = 24$; 5 feet + 7 feet + 9 feet = 21 feet.

64. In algebra, the sum of two or more positive or two or more negative quantities is found by the process of addition only.

Thus, the sum of $2a$, $3a$, and $4a$, is $9a$; $4b + 6b + 8b = 18b$; $5ax + 7ax + 9ax = 21ax$; etc.

Also, the sum of $-3a$, $-4a$, and $-5a$, is $-12a$; $(-6b) + (-8b) + (-10b)$ or $-6b - 8b - 10b = -24b$.

65. The algebraic sum of quantities, some of which are positive and some negative, is found by the processes of both addition and subtraction.

Thus, the algebraic sum of $4x - 6x + 8x - 10x$, or $12x - 16x$, is $-4x$; $5ac - 7ac + 10ac = 8ac$; etc.

66. Only similar numbers or quantities can be collected into one sum or term.

67. The operation of adding dissimilar numbers or quantities may be indicated, but not actually performed.

Thus, the addition of 3 pints and 4 feet is indicated by the expression 3 pints + 4 feet, but the sum can not be expressed by a single number.

Also, the sum of $3ab$ and $4ax$ is indicated by the expression $3ab + 4ax$; the sum of $4ac$ and $-5xy$ is $4ac + (-5xy)$, or $4ac - 5xy$; etc.

Although the sum of dissimilar terms can not be expressed as a single term, the operation of addition may be regarded as really performed, as indicated by the expression showing the sum of the dissimilar terms.

Thus, $2ax + 3ab^2$ may be regarded as the sum of $2ax$ and $3ab^2$, whatever value these terms may have; $4ac - 5bx$ is regarded as the sum of $4ac$ and $-5bx$; etc.

Tell the algebraic sum of each of the following:—

$5a$	$3a$	$-5a$	$-3a$	$-5a$	$-3a$	$5a$	$3a$
$3a$	$5a$	$-3a$	$-5a$	$3a$	$5a$	$-3a$	$-5a$

68. *Illustrative Example.*—Find the algebraic sum of $-a$, $3a$, $6a$, $-5a$, and $-7a$.

OPERATION.

$$\begin{array}{r} -a \\ 3a \\ 6a \\ -5a \\ \underline{-7a} \\ -4a \end{array}$$

EXPLANATION.—The terms, being similar, are written in a single column.

The sum of the positive terms is $9a$, and the sum of the negative terms is $-13a$.

The sum of the negative terms is greater than the sum of the positive terms by $-4a$, which is the algebraic sum required.

69. *Illustrative Example.*—Find the algebraic sum of $2a + 5x$, $-4a + 2x + y$, and $9a - 7x - 5y$.

OPERATION.

$$\begin{array}{rrr} 2a & +5x & \\ -4a & +2x & +y \\ \underline{9a} & \underline{-7x} & \underline{-5y} \\ 7a & & -4y \end{array}$$

EXPLANATION.—The quantities are written so that similar terms shall stand in the same column. Beginning at the right or the left, the algebraic sum of each column is found. The several sums connected by their proper signs form the entire algebraic sum required, which is $7a - 4y$.

70. *Illustrative Example.*—Find the algebraic sum of ax, $-bx$, and cx.

OPERATION.	
ax	**EXPLANATION:** a, $-b$, and c may be regarded as literal coefficients, and x as the similar literal part of the several quantities.
$-bx$	The sum of the coefficients is $a-b+c$, and the entire algebraic sum is $(a-b+c)x$.
cx	
$(a-b+c)x$	

EXAMPLES.

1	2	3	4	5	6
$3ab$	$-5ax$	$8bc$	$-7bx$	$8ad$	$-7ay$
$4ab$	$-8ax$	$-4bc$	$3bx$	$-ad$	$8ay$

7	8	9	10	11	12
$2ab$	$-3ax$	$4bc$	$3ay$	$-4xy$	$8am$
$-3ab$	$4ax$	$-8bc$	$-8ay$	$-8xy$	$-3am$
$4ab$	$-5ax$	$2bc$	$15ay$	$14xy$	$-7am$

13	14	15	16	17	18
$-4bd$	$7ay$	$4b^2x$	$-cm^2$	$4by^3$	$2a^3c$
$-5bd$	$-8ay$	$6b^2x$	$-6cm^2$	$-7by^3$	$\frac{1}{2}a^3c$
$8bd$	$6ay$	$-7b^2x$	$-9cm^2$	$8by^3$	$3a^3c$
$-2bd$	$-9ay$	$-8b^2x$	$10cm^2$	$-6by^3$	$\frac{1}{4}a^3c$

Collect the following:—

19. a, a, a, a, and a.
20. $-x$, $-x$, $-x$, $-x$, $-x$.
21. $3b+2b+5b+10b$.
22. $-4c-5c-2c-12c$.
23. $-x+2x$.
24. $x-2x$.
25. $-b-2b+3b-4b$.
26. $ax-4ax+3ax-5ax$.
27. $b^2-3b^2-2b^2+4b^2$.
28. $2a^2b+3a^2b-5a^2b$.

29	30	31
$3a-5cx-7ay$	$5b+7ac$	$4c-9b+xy$
$-4a-6cx$	$-9b-10ac+x$	$-4c+8b+4xy$
$-9a+8cx+8ay$	$-3b-2ac-4x$	$-9c+2b-8xy$

Find the algebraic sum of the following:—

32. $10ax$ and $-12ax$.
33. $2x+3y$ and $4x-7y$.
34. $a-c$, $-3a+4c$, $-7c$.
35. ax, $-4ax$, $3ax-y$.
36. $x+y$, $x-y$, and y.
37. $2abx$, $-3abx$, $4abx$.
38. $9az$, $14(a+x)$, $-9az$.
39. $-5a^2x$, $9a^2x$, $-a^2x$.

Add the following quantities:—

40. $10x+3y$, $5x-2y$, $3x-4y$, and $4x+6y$.
41. $a-3x$, $6a-5x$, $7a-3x$, and $-8a+10x$.
42. $5b+a$, $-3b+x$, $6x-a$, and $4x$.
43. $13ab+2cd$, $25a^2b+3cd$, and $3ab-8a^2b$.
44. $12by+6ac$, $-11by+4ac$, and $-by-11ac$.
45. $9(a+y)$, $-14(a+z)$, $-9(a+y)$, and $(a+z)$.
46. $10ay+y^2-abx$, $-3ay+3y^2$, $7ay+5abx$, $-8ay-2y^2$, and $6y^2-4abx$.
47. $21a^2x^2y$, $3a^4xy^2-2a^2x^2y$, $13a^4xy^2-11a^3x^5y+3a^2x^2y$, $12a^4xy^2-10a^3x^5y$, and $5a^2x^2y$.
48. $17ax+13abc+2ay$, $13abc+14ax-5ay$, and $3ay-27abc+3ax$.
49. $9a^2b^3c^4+8a^3b^2c^4-7a^4b^2c^3$, $21a^3b^2c^4-5a^2b^3c^4$, $-11a^2b^3c^4+2a^3b^2c^4$, $21a^4b^2c^3$, and $5a^3b^2c^4-2a^2b^3c^4$.
50. $8(a+b)$, $-9(x+y)$, $-10(a+b)$, and $12(x+y)$.
51. $5(a-b^2+c^3)$, $-7(a-b^2+c^3)$, $9(a-b^2+c^3)$, and $-10(a-b^2+c^3)$.

52	53	54	55	56	57
$2x$	$-3a$	ax	$3bc$	ac	ay
$3y$	$4b$	$-2y$	$-ax$	bc	$-by$

58	59	60	61	62
y	$5(a+b)$	$-(x+1)$	$c(x+y)$	$3a+2a^2b$
$-y$	$-3(a+b)$	$6(x+1)$	$-2(x+y)$	$-2b-4a^2b$

63	64	65	66	67
$+4ab$	$-4bc$	$+3cd$	$+2d(a-x)$	$-b(x+y)$
$-5bc$	$7bc$	$-4ax$	$-(a-x)$	$+c(x+y)$
$-3ab$	$-3ax$	$+7ax$	$-a(a-x)$	$+3b(x+y)$

EXERCISES IN ALGEBRAIC EXPRESSION.

NOTE.—The following exercises in algebraic notation, and algebraic expressions, generally, may be stated or read less formally than has been suggested in previous exercises.

Thus, three times *a* times *b* square may be stated or read three *ab* square; $4x^3y$ may be read four *x* cube *y*; $\frac{x}{2}$ may be read *x* over 2; etc.

Express the following in algebraic notation:—

1. The sum of *a* and *b*.
2. *b* added to four times *c*.
3. *x* square added to *b*.
4. *a* plus *b* times *c*.
5. *b* square times *c*, plus *y*.
6. *a* times *y*, plus *b* cube.
7. Four times *a*, plus *x* decreased by five times *a*.
8. The sum of *b*, *c*, and *x*, divided by *y*.
9. The cube of *a* diminished by the square of *c*.
10. Four times *b* times *c*, decreased by the square of *a*.

11. Harry paid *x* cents for a bat and *y* cents for a ball. How much did both cost?

12. Mary is *a* years old to-day. How old will she be in 3 years? How old in *b* years?

13. A field is *x* rods long and *y* rods wide. How many rods of fence are needed to enclose it?

14. If I pay *x* dollars for a horse and sell him at a gain of *a* dollars, how much do I get for him?

15. Henry paid six times *y* dollars for a watch, and sold it at a gain of 10%. What did he get for it?

16. What is the amount of *x* dollars on interest for a year at 5%?

17. If a is an even number, what is the next larger odd number? The next smaller even number?

18. Fred was a years old b years ago. How old will he be in c years?

19. Mary bought a quarts of berries at b cents a quart. What is one-half of the price that she paid for them?

20. Thomas had a cents, William had y cents, and Susan had 12 cents. How many cents had they all?

21. Charles bought a bicycle for four times y dollars, paid c dollars for repairing it, and then sold it for five dollars more than it cost altogether. What was his selling price?

22. A grocer sold ten pounds of sugar at b cents a pound, and y pounds of rice at c cents a pound. How much did he get for all?

23. One week Robert earned a dollars, Edward one-half as much, and their father as much as both boys together. What did all earn that week?

24. I paid x cents for two yards of ribbon, and sold it at a gain of y cents. For how much per yard did I sell it?

In exercises 1–24, let $a = 8$, $b = 6$, $c = 4$, $x = 50$, and $y = 10$; find the numerical value of each of these exercises.

EQUATIONS.

1. $3x + \frac{1}{4}x = 45 - 6.$
2. $y + \frac{2}{3}y = 36 + 9.$
3. $5z + \frac{1}{2}z + z = 78 + 13.$
4. $3x + \frac{1}{3}x - 10 = 35 + 15.$
5. $2y - \frac{3}{4}y + 13 = 30 + 8.$
6. $\frac{1}{4}z + 5z - 10 = 40 - 8.$

In equations 1–6, substitute the value of each unknown, and prove each result.

7. A hat and a coat cost $18. If the coat cost five times as much as the hat, what did each cost?

8. My house and lot cost $2500, and the lot cost one-fourth as much as the house. What did I pay for each?

9. How can you make change for $3 using an equal number of dimes and five-cent pieces?

10. Two boys bought a sled for $12. John paid $3 as often as William paid $1. How much did each pay?

11. Mary spent one-fourth of her money for music and one-third of it for a pocket-book. If she paid $1.40 for both, how much money had she at first?

12. A grocer bought four boxes of figs and six boxes of raisins for $8. The figs cost one-half as much per box as the raisins. What did he pay for each box?

13. Ten yards of muslin and fifteen yards of linen cost $7, and the linen cost four times as much per yard as the muslin. What was the price of each per yard?

14. Three men hired a boat for $45. The first paid twice as much as the second, and the second paid two-thirds as much as the third. What did each pay?

15. The distance around an oblong field is 80 rods. If the length is three times the width, how long and how wide is the field?

16. Two boys start at the same time from places 75 miles apart, and ride towards each other. If one rides six miles an hour while the other rides nine miles, how soon will they meet?

17. Two friends start at the same time from Philadelphia. One goes northward on a train moving 30 miles an hour, and the other goes southward 20 miles an hour. In how many hours will they be 300 miles apart?

18. Mary has in her bank $2 in dimes and nickels. If she has three times as many nickels as dimes, how many of each has she?

19. A boy bought an equal number of marbles, tops, and balls for 70 cents. The marbles cost one cent, the tops three cents, and the balls ten cents. How many of each did he buy?

20. Three boys owned 36 rabbits. Frank had one-third as

many as George, and Thomas had twice as many as Frank. How many had each boy?

21. Three girls earned \$50. Mary earned twice as much as Anna, and Emma earned \$10 less than Mary. How much did each earn?

22. One-third the cost of my watch, and \$10 more than one-fifth of the cost equal \$50. How much did it cost?

23. Edward spent three-fourths of his money for clothes, and then earned \$5 more than he had spent. If he then had \$25, how much had he at first?

24. By selling a watch for \$50, a jeweler gained \$5 less than 25% of the cost. What did he pay for it?

25. I saved \$100 a year by borrowing a sum of money at 5% instead of 6%. How much did I borrow?

SUBTRACTION.

1. What is the difference between 7 cents and 5 cents? 6 feet − 4 feet = ? 5 − 2 = ?

2. What is the remainder of 9 books less 6 books? Of 8 pints − 4 pints? 9 − 3 = ?

3. The difference between 10 times a quantity and 7 times that quantity is how many times the quantity?

What is the difference between $10a$ and $7a$? What kind of quantity is $10a$? What kind is $7a$? What kind is the difference between them?

4. What quantity remains after taking 6 times $-b$ from 11 times $-b$? $-6b$ from $-11b$ leaves what remainder? $(-11b) - (-6b) =$?

What kind of quantity is $-11b$? What kind is $-6b$? What kind is the remainder?

5. From $13x$ take $7x$.

OPERATION.	
$13x$	What kind of quantity is the minuend?
$7x$	What kind of quantity is the subtrahend?
$6x$	What is the remainder? What kind of quantity is it?

6. From $-15y$ take $-7y$.

OPERATION. What kind of quantity is the minuend? What kind is the subtrahend?

$$\begin{array}{r} -15y \\ \underline{-7y} \\ -8y \end{array}$$

What is the remainder? What kind of quantity is it?
Of what is the minuend the sum? How is it found?

7. What quantity must be added to $4a$ to equal $9a$? To $6b$ to equal $11b$? To $8c$ to equal $14c$?

What quantity must be added to $-5x$ to equal $-10x$? To $-8y$ to equal $-15y$? To $-9z$ to equal $-18z$?

8. What quantity must be added to $+a$, or a, to equal 0? To equal $-a$? To equal $-2a$? To equal $-4a$?

What quantity must be added to $-3b$ to equal 0? To equal b? To equal $2b$? To equal $3b$? To equal $6b$?

9. From $2x$ take $5x$.

OPERATION. What kind of quantity is the minuend? The subtrahend?

$$\begin{array}{r} 2x \\ \underline{5x} \\ -3x \end{array}$$

Which expresses the greater number of units?
What quantity must be added to the subtrahend to equal the minuend?
What is the remainder? What kind of quantity is the remainder?

10. Take $-6y$ from $3y$.

OPERATION. What kind of quantity is the minuend? The subtrahend?

$$\begin{array}{r} 3y \\ \underline{-6y} \\ 9y \end{array}$$

Which has the greater numerical or absolute value?
What quantity must be added to the subtrahend to equal the minuend?
What is the remainder? What kind of quantity is the remainder, and how is this remainder found?

11. From $-5ax$ take $-9ax$.

OPERATION. Which of the two quantities has the greater absolute value?

$$\begin{array}{r} -5ax \\ \underline{-9ax} \\ 4ax \end{array}$$

What quantity must be added to the subtrahend to give the minuend? What kind of quantity must be added?
What is the remainder, what kind of quantity is it, and how is it found?

71. In arithmetic, the difference between two numbers is found by taking the subtrahend from the minuend.

The subtrahend is never greater than the minuend; and the process of finding the difference, or remainder, involves the idea of subtraction only.

Thus, the difference between 12 yards and 5 yards is 7 yards; the remainder of 15 cents − 6 cents is 9 cents; $6 - 6 = 0$; etc.

In algebra, when the terms are similar and of the same kind, and the value of the subtrahend is not greater than the value of the minuend, the difference, or remainder, is found by taking the subtrahend from the minuend; and the process involves the idea of subtraction only.

Thus, the difference between $-15ax$ and $-7ax$ is $-8ax$; the remainder of $20by - 15by = 5by$; $c - c = 0$.

In algebra, the numerical or absolute value of the subtrahend is sometimes greater than that of the minuend; and then the process of finding the difference, or remainder, involves the idea of both subtraction and addition.

Thus, taking $12ax$ from $7ax$ gives $-5ax$, the quantity which must be added to the subtrahend $12ax$ to equal $7ax$, the minuend.

72. The difference between similar quantities or terms of the same kind only, can be expressed as one quantity or term.

Thus, 7 dollars − 4 dollars = 3 dollars; $5 - 5 = 0$; $8ax - 3ax = 5ax$; $y - y = 0$; etc.

The operation of finding the difference between two dissimilar terms may be indicated, but it can not be actually performed.

Thus, the difference between $3ab$ and $2x$ is indicated by $3ab - 2x$; also, $2b^2 - 3xy$ indicates the difference between $2b^2$ and $3xy$, whatever value these two quantities may have.

73. In algebra, taking from the minuend a positive quantity gives the same result as adding to the minuend a negative quantity of the same numerical value as the subtrahend.

Thus, taking $2a$ from $5a$ leaves $3a$, and adding $-2a$ to $5a$ gives $3a$; taking $7b$ from $-4b$ gives $-11b$, the same as adding $-7b$ to $-4b$; etc.

Subtracting a negative quantity gives the same result as adding to the minuend a positive quantity of the same absolute value as the subtrahend.

Thus, taking $-3a$ from $-5a$ leaves $-2a$, and adding $+3a$, or $3a$, to $-5a$ gives $-2a$; taking $-7b$ from $4b$ gives $11b$, the same as adding $7b$ to $4b$; etc.

74. In examples 5, 6, 9, and 10, above, the same result or remainder in each can be found by changing the sign of the subtrahend and adding the subtrahend to the minuend.

Since the correct difference, or remainder, can *always* be found in this manner, it is best to adopt it as a uniform method. Hence,

In finding the remainder in algebra, imagine or conceive the sign of the subtrahend to be changed, and then find the algebraic sum of the minuend and the subtrahend.

75. The difference between positive and negative quantities or terms is called the **algebraic difference** or **remainder.**

Find the algebraic remainder of each of the following:—

$5x$	$3x$	$-5x$	$-3x$	$-5x$	$-3x$	$5x$	$3x$
$3x$	$5x$	$-3x$	$-5x$	$3x$	$5x$	$-3x$	$-5x$

76. *Illustrative Example.*—Take $8ab$ from $5ab$.

OPERATION.

$$\begin{array}{r} 5ab \\ 8ab \\ \hline -3ab \end{array}$$

EXPLANATION.—After taking as much of $8ab$ from $5ab$ as can be taken, there still remains $3ab$ to be subtracted; this remainder is expressed as a subtractive or negative quantity, and the algebraic remainder is $-3ab$. Or,

Since subtracting a positive quantity gives the same result as adding a negative quantity of equal absolute value, $8ab$ may be subtracted from $5ab$ by changing $8ab$ to $-8ab$, and then finding the algebraic sum of $5ab$ and $-8ab$, which is $-3ab$.

77. *Illustrative Example.*—From $-5ax+8y$ take $3ax+15y$.

OPERATION.

$$\begin{array}{rr} -5ax & +\ 8y \\ 3ax & +15y \\ \hline -8ax & -\ 7y \end{array}$$

EXPLANATION.—Since the minuend and the subtrahend are composed of more than one term, the similar terms in the two expressions are placed in the same column.

Changing $15y$ to $-15y$, the algebraic sum of $8y$ and $-15y$ is $-7y$; changing $3ax$ to $-3ax$, the sum of $-5ax$ and $-3ax$ is $-8ax$.

Hence, the algebraic remainder is $-8ax - 7y$.

78. *Illustrative Example.*—From ax take $-bx$.

OPERATION.

$$\begin{array}{r} ax \\ -bx \\ \hline (a+b)x \end{array}$$

EXPLANATION: a and b are the literal coefficients of x, the similar literal part of the minuend and the subtrahend.

Changing the sign of the subtrahend, and adding the minuend and the subtrahend gives $(a + b)x$, the algebraic remainder.

EXAMPLES.

Find the algebraic remainder of the following:—

1	2	3	4	5	6
7	-4	-8 feet	10	$4a$	-7 cents
3	-5	4 feet	-6	$-5a$	4 cents
7	**8**	**9**	**10**	**11**	**12**
$-5b$	$-6c$	$-4a$	$9d$	$-5b$	$9ac$
$-6b$	$3c$	$-8a$	$-4d$	$8b$	$-4ac$
13	**14**	**15**	**16**	**17**	**18**
$-5a^2$	$6x^2$	$-4y^2$	$5ab^2$	$-7a^2x$	$4a^2c^2$
$9a^2$	$-8x^2$	$-8y^2$	$-8ab^2$	$9a^2x$	$-8a^2c^2$
19	**20**	**21**	**22**	**23**	**24**
$4ax$	a^2b	$-3xy^2$	b^2x	$-4a^2y^2$	b^2c^3
$-ax$	$3a^2b$	xy^2	$5b^2x$	a^2y^2	$-5b^2c^3$

From

25. a take $-3a$.
26. $-4b$ take $5b$.
27. c take c.
28. $-2c$ take $5b$.
29. xy take $-xy$.
30. $-7c^3$ take $-c^3$.
31. $4b^2$ take $5b^2$.
32. $2a$ take $-2a$.
33. $-\frac{1}{2}x^2$ take $\frac{1}{4}x^2$.

Find the algebraic remainder of—

34. $4a-(-2a)$.
35. $-6b-(+7b)$.
36. $8c-(-2c)$.
37. $-9x-(-4x)$.

38. $-8y^2-(+8y^2)$.
40. $-b^2x^2-(+2a^2b^2)$.

39. $a^2b-(-a^2b)$.
41. $a^3-(-4a^3)$.

42. $-12a^2xz^3-(-10a^2xz^3)$.

43	44	45	46
$x+5$	$17y+3$	$-10+8xy$	$-9ax-3bc$
$-x+5$	$16y-6$	$-12-7xy$	$-6ax+5bc$

47	48	49	50
$4a+b$	15	$4a^2x+y^3$	$2ab-\frac{1}{4}c^3$
$-4a$	$3ax-15$	$-a^2x+4y^3$	$-\frac{1}{2}ab+c^3$

From

51. $a+b$ take $a-b$.
54. $-5x^2-3ay$ take $6x^2+4y^3$.
52. $-3b+c$ take $4b-2c$.
55. a^2x+5 take $-5a^2x-10$.
53. $4ax-y$ take $-6ax+y$.
56. $4b^2c-2a^3$ take $-7b^2c+4a^3$.

57. From $5x-3y-2z$ take $7x-5y+z$.
58. Subtract $-8a^2x+7a^2y$ from $9a^2x-4a^2y-a$.
59. Subtract $ax^2+5ax-3a$ from $-2ax^2-4ax+a$.
60. Take $6a^2bc-5ay+8$ from $2a^2bc-8$.
61. Take $-5de+3f^2-6$ from $3de-f^2$.
62. From $9a^2xy$ take $-3a^2xy-4a+9$.
63. From $7ab^2-5x^2y+3a^2b$ take $5ab^2+8a^2b$.
64. Subtract $-6ay-3x+6$ from $4x+5ay$.
65. Subtract $8ay^2+6x^2y^3+3x^3$ from $-6ay^2-$ [illegible].
66. From $-4a^2x+5c-1$ take $-5a^2x-3c$.
67. Take $7bxy-2c-5$ from $-4bxy+c-5$.
68. From $-9x^3+6x^2y+4y^2$ subtract $x^3-8x^2y+y^2$.
69. Subtract $5cd^2+6cy-a^2x$ from $-cy+2a^2x-cd^2$.
70. Take $-6x^2-a$ from $5+11x^2+a$.

71. From $5h - 6c^2d + 8a^2$ take $-6a^2 + 3h + c^2d$.

72. Subtract $-\frac{1}{2}a^2b + 2\frac{1}{4}b^2c^3$ from $-\frac{1}{4}b^2c^3 - a^2b$.

73. From $-8a - 3x^2y^3$ subtract $4a - \frac{1}{2}x^2y^3 + 7$.

74. Take $\frac{1}{4}a^2x - y^3 + \frac{1}{2}$ from $-4y^3 + \frac{1}{2}a^2x$.

75	76	77	78	79	80
ac	bx	$-2b$	ac	d	$-ay$
bc	x	$-ab$	$-4c$	$-cd$	$-by$

81	82	83	84	85	86
y	$-3cd$	$4bc$	$-a^3$	$3(x+1)$	$-(a+x)$
$-2y$	bd	bc	$2a^3$	$a(x+1)$	$a(a+x)$

87. From the sum of $2a + 3b - 4c$ and $4b - 2a + 7c$, take $-4b + 6a - 2c$.

88. Take $4a - 7x + 3y$ from the sum of $-7a - 6y + 4x$ and $-5y + 6a - x$.

89. From $8b - 4c + 8d$ take the sum of $-4d + 5b - 3c$ and $4c - 8d - 7b$.

90. Subtract $5ab - 4bc + 5cd$ from the sum of $4cd + 5bc - 6ab$ and $7ab - 6cd - 8bc$.

91. Take the sum of $a^2b - 2ax^2 + 5by$ and $-6by - a^2b + 3ax^2$ from $-5a^2b + 2by - 8ax^2$.

92. From the sum of $5bc^2 - 8cy^2$ and $cy^2 + 4bx - 4bc^2$, take $-cy^2 + 2bc^2 + 10bx$.

93. Add $-4c^2d + xy^2$ and $8xy^2 - 4c^2d + 4a^2x$, and from the sum take $-3a^2x + 5xy^2$.

94. Take $2dx - \frac{1}{2}cy^2 + \frac{1}{4}xy^2$ from the sum of $\frac{1}{4}dx + \frac{1}{2}cy^2$ and $2cy^2 + 3dx - \frac{1}{4}xy^2$.

95. Find the sum of $\frac{1}{8}a^2b + 5c^2x^3$ and $\frac{2}{3}c^2x^2 + \frac{1}{4}bc^2 + 2a^2b$, and subtract $\frac{1}{3}a^2b - \frac{1}{2}bc^2$.

96. Subtract the sum of $\frac{1}{2}b - \frac{2}{3}c$ and $3c - \frac{1}{4}b + \frac{1}{2}a$ from the sum of $b + \frac{3}{4}a$ and $\frac{1}{3}c + 3b$.

97. From the sum of $2a^2b - a^3$ and $3a^3 - a^2b - 4bc$ take the sum of $\frac{1}{2}bc + 4a^3$ and $a^2b + a^3$.

In examples 90–97, find the results again after changing the signs of the terms in every quantity forming a subtrahend.

98. From the same seaport, one vessel sailed a miles northward, and another sailed b miles southward. How far apart were they then?

99. On Monday the temperature was x degrees, and on Tuesday it fell y degrees. What was the temperature then?

100. If you take or subtract 10 dollars from a man's debt, what change do you make in his financial condition?

ALGEBRAIC EXPRESSION.

Express in algebraic notation the following:—

1. The difference between a and b.
2. Three times b minus $\frac{1}{2}c$.
3. a square decreased by y.
4. Two times x less $\frac{1}{4}a$ square.
5. a times c diminished by $\frac{3}{5}y$. Three b less $\frac{3}{4}c$.
6. $\frac{4}{5}b$ times y minus c cube.
7. Three times b plus a, minus one-half of four times c.
8. The sum of a, b, and y, divided by c.
9. Four times b square minus one-fourth of the cube of c.
10. Six times a times c, diminished by two times b square.

State or read exercises 1–10 without using the word *times*.

11. John bought x marbles and lost y marbles. How many marbles had he left?

12. Express the equation, four times a diminished by y equals ten.

13. One-half of my money decreased by one-third of my money equals a dollars. How much money have I?

14. Frank was y years old four years ago. How many years old will he be in c years?

15. The top of a stand is y inches long, and its width is b inches less. How long a string will reach around it?

16. If a stands for an odd number, express the next two smaller even numbers.

17. Mary has x cents, and Martha has y cents less. How many cents has Martha? How many cents have they both?

18. If a boy pays a times c cents for a knife, and he sells it at a loss of b cents, how much does he get for it?

19. I paid x dollars for my watch and I sold it at a loss of 20%. What was the selling price of the watch?

20. An agent sold for me some shoes worth five times y dollars, and charged me 5 per cent. commission. How much did he pay over to me?

21. John bought b times c pencils at a cents each, and he paid y cents on account. How much does he yet owe?

22. A man earned x dollars a month. He paid four times a dollars for his board, and y dollars for a coat. How many dollars had he left?

23. William was x miles from home. On Monday he walked y miles towards his home, and on Tuesday four times b miles. How far from home was he then?

24. If Harry walks a miles north from his home, and his brother b miles south, how far apart are they?

25. A grocer paid a cents a pound for b pounds of cherries, and sold them at a loss of c cents a pound. What did he get for them?

In problems 1–25, let $a=5$, $b=3$, $c=2$, $x=50$, and $y=10$; find the numerical result of each of these problems.

EQUATIONS.

1. $x - \frac{x}{3} = 25 + 5.$
2. $2y - y + \frac{1}{2}y = 24.$
3. $3z - \frac{1}{4}z = 42 - 9.$
4. $4x + \frac{1}{2}x - 3x = 16 + 5.$
5. $2\frac{1}{2}y + y = 42 - 10 + 3.$
6. $4z + \frac{1}{2}z - 10 = 60 - 7.$

In equations 1–6, substitute the value of the unknown, and prove each result.

7. If the difference between four times x and two times x equals the sum of 35 and 9, what is the value of x?

8. Four times y plus the difference between six times y and three times y is 42 cents. Find the value of y.

9. If the difference between 5 times x and 2 times x be taken from the sum of 6 times x and 2 times x, the result is 30 miles. What is the value of x?

10. A carriage cost two-thirds as much as a horse, and the difference in the prices paid for them is $80. What was the price of each?

11. A and B started business with the same amount of capital. A lost one-third of his, and B gained one-fourth as much as his, and then A had $700 less than B. What capital had each at first?

12. If John's money were increased by its two-thirds and then diminished by its one-half, he would have $14. How much money has he?

13. If George were to buy as many pigeons as he now has, and should then sell three-fourths of them all, he would have 8 pigeons left. How many pigeons has he?

14. Mary bought some oranges at 4 cents apiece. If she had paid 6 cents apiece, they would have cost her 24 cents more. How many did she buy?

15. The difference in the values of an equal number of five-dollar bills and two-dollar bills is $27. How large a debt could be paid with all the bills?

16. By paying a third and a sixth of my money for a watch and chain, I spent $16 more than a sixth and a ninth of my money which I had paid for some books and music. How much money had I?

17. The distance around an oblong piece of paper is 60 inches. The length decreased by its one-third equals the width. How long and how wide is the paper?

18. I invested some money in stocks expecting them to pay a 5% dividend; but they paid only a 4% dividend, and I received $50 less than I expected. How much did I invest?

19. Henry bought a wheel and paid 20% of its cost for repairs. By selling it for $10.80, he did not gain or lose. What did he pay for it?

20. The number of papers sold by a newsboy on Monday, decreased by one-fourth of this number, equals the number sold on Tuesday. If the whole number sold was 70 papers, how many did he sell on each day?

21. Frank sold some oranges at 3 cents each, and Fred sold two-thirds as many at 4 cents each. If Frank received 20 cents more than Fred, how many did each sell?

22. Mary and Anna have 50 cents, of which Anna has 10 cents less than Mary. How many cents had each?

23. Harry and William have 32 marbles. If Harry's were doubled, he would still have 8 marbles less than William. How many marbles has each?

24. A horse, a carriage, and a set of harness cost $225. The carriage cost $50 less than the horse, and the harness $50 less than the carriage. What did each cost?

25. A man and a boy received $25 for 10 days' work. If the boy received 50 cents a day less than the man, how much did each receive altogether?

PARENTHESES.

79. It is often more convenient to indicate the addition or the subtraction of a quantity by enclosing it within a parenthesis, than to place the quantities with similar terms in the same column to be added or subtracted.

Thus, if the sum of 2 and 3, or $2 + 3$, is to be taken from 8, the operations may be briefly indicated by the expression $8 - (2 + 3)$.

Also, the sum of $2a + 3b$ and $3a - 5b$ may be indicated by the expression $(2a + 3b) + (3a - 5b)$.

The difference between $2a + 3b$ and $3a - 5b$ may be indicated by the expression $(2a + 3b) - (3a - 5b)$.

80. The terms within the parenthesis are to be regarded as forming a single quantity, which is to be treated as indicated by the sign placed before the parenthesis.

Thus, $x + (a + b)$ indicates that the sum of a and b is to be added to x; $x - (a - b)$ indicates that the difference between a and b is to be taken from x.

Also, $a(x + y)$ indicates that the sum of x and y is to be multiplied by a; $(x - y) \div b$ indicates that the difference between x and y is to be divided by b; etc.

81. Brackets [ ], braces { }, and the vinculum ——— are used for the same purpose as the parenthesis.

The parenthesis, brackets, braces, and the vinculum, used to enclose a quantity which is to be treated as a whole, are called **signs of grouping, or signs of aggregation.**

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82. *Illustrative Example.*—Find the result of $[8 - (\overline{2 + 4} \div 3)] \times 5$.

OPERATION.

$$\begin{aligned}&[8 - (\overline{2 + 4} \div 3)] \times 5\\ &= [8 - (6 \div 3)] \times 5\\ &= [8 - 2] \times 5\\ &= 6 \times 5 = 30.\end{aligned}$$

EXPLANATION.—Beginning with the inner pair of terms, the sum of 2 and 4 is 6; the quotient of 6 divided by 3 is 2; the difference between 8 and 2 is 6; and the product of 6 multiplied by 5 is 30, the result required.

Operations of multiplication and division must be performed before those of addition and subtraction, unless signs of grouping indicate otherwise.

The difference ...
expression ...

80. The ...
forming a ...
by the sign ...

Thus, x ...
$x-(a-b)$ ...

Also, $a(x-$ ...
$(x-y)\div$ ...
divided by b, etc.

81. Brackets [ ] ...
used for the same ...

The ...
to enclose ...
called ...

... c

... c.

... m

... n a

..., how

...2; find

... be re-
..., if the
...anged.
... by the

EXAMPLES.

Find the result of each of the following indicated operations:—

1. $8 + (3 - 2)$.
2. $(8 - 3) + 2$.
3. $(8 - 3) + \overline{5 - 2}$.
4. $8 - (\overline{3 + 5} - 2)$.
5. $20 - 5 \times 3$.
6. $(20 - 5) \times 3$.
7. $(5 + 4 - 3) \times 2$.
8. $\overline{5 + 4} - 3 \times 2$.
9. $5 + (\overline{4 - 3} \times 2)$.
10. $\overline{5 + (4 - 3}) \times 2$.
11. $(20 - 5) \div (3 + 2)$.
12. $(\overline{20 - 5} \div 3) + 2$.
13. $20 - [(6 \div 3) + 2]$.
14. $4 \times 10 - 6 + 2$.
15. $(4 \div \overline{10 - 6}) + 2$.
16. $4 \times \overline{10 - 6} + 2$.
17. $4 \times [(10 - 6) + 2]$.
18. $18 - (3 \times 4 \div 2)$.
19. $[\overline{18 - 3} \times 4] \div 2$.
20. $(18 - 3) \times \overline{4 \div 2}$.
21. $[(6 + 3 - 4) \div 5] + 2$.
22. $(6 + 3 - \overline{4 \div 2}) + 5$.
23. $6 + 3 - [(4 \div 2) + 5]$.
24. $6 + [5 - (4 \div 2 + 3)]$.

In examples 1–24, the signs + and − are signs of operation merely, and do not indicate the kind or nature of the numbers used, all of which are regarded as positive numbers.

In examples 25–37, let $a = 8$, $b = 4$, $c = 3$, and $d = 2$; perform the indicated operations to find the numerical value of each example:

25. $(a - b) + c$.
26. $a - (b + c)$.
27. $a + b - (c - d)$.
28. $[(a + b) - c] - d$.
29. $\frac{a}{b} \times d + (2c - d)$.
30. $\frac{a}{b} \times (d + 2c) - d$.
31. $[(a + b) - c] + d$.
32. $2a \times (b - c) \div d$.
33. $\overline{2a \times b - c} \div d$.
34. $2a \times [c - (b \div d)]$.
35. $\frac{a}{b} \times [(d + 2c) - d]$.
36. $\left(\frac{a + b}{c} - d\right) + \overline{a \div b}$.
37. $\left[\left(\frac{a + b}{c} - d\right) + a\right] \div b$.

Indicate the following operations by the proper use of the parenthesis and the fundamental signs:—

38. From a take the sum of b and d.

39. To the difference between a and b, add c.

40. From the sum of a and b, take the product of c and d.

41. Add the product of b and c to the difference between b and c.

42. Multiply the difference between a and b by the sum of b and c.

43. From twenty, take two times the sum of c and d.

44. Take c from the sum of a and two times b, and multiply the remainder by 4.

45. Divide a by c (in fractional form), and to the quotient add a times b times c.

46. To the product of four times b times c, add the quotient of c divided by d.

47. Take the sum of b and c from the quotient of six times a divided by d.

48. From two times b times c, take the quotient of a divided by c.

49. Express in fractional form the quotient of six times c divided by the sum of c and d; to the quotient add b times c.

50. From two times a take b; add the quotient of the sum of a and c, divided by b.

51. To four times a times b, add the difference between a and c; from the sum take two times a.

52. If one-fourth of John's marbles is $a + 10$ marbles, how many marbles has he?

In problems 38–52, let $a = 8$, $b = 6$, $c = 4$, and $d = 2$; find the numerical value of each of these problems.

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83. A parenthesis with the plus sign before it may be removed without changing the value of the expression, if the signs of the terms within the parenthesis remain unchanged.

Thus, the expression $7 + (5 + 3)$ means that 7 is to be increased by the sum 5 + 3, or 8; but this gives the same result as increasing 7 by 5 and by 3 separately; that is,

$$7 + (5 + 3) = 7 + 5 + 3 = 15.$$

In like manner, 7 + (5 − 3) means that 7 is to be increased by the remainder of 5 − 3, or 2; but this gives the same result as increasing 7 by 5, and then decreasing this sum by 3; that is,

$$7 + (5 - 3) = 7 + 2 = 9$$
$$7 + (5 - 3) = 7 + 5 - 3 = 9$$

Also, $a + (b + c) = a + b + c$; $a + (b - c) = a + b - c$; etc.

1. What is the sum of $(2a + 3b) + (3a - 5b)$?

OPERATION.

$$\begin{array}{r} 2a + 3b \\ \underline{3a - 5b} \\ 5a - 2b \end{array}$$

It is more convenient to find the sum by omitting the parentheses, and collecting similar terms as in addition; thus,

$$2a + 3b + 3a - 5b = 5a - 2b.$$

84. A parenthesis with the minus sign before it may be removed without changing the result of the expression, if every term within the parenthesis be changed.

Thus, the expression 7 − (5 − 3) means that 7 is to be decreased by the remainder of 5 − 3, or 2; but this gives the same result as decreasing 7 by 5, and then adding 3; that is,

$$7 - (5 - 3) = 7 - 2 = 5$$
$$7 - 5 + 3 = 2 + 3 = 5$$

Also, $a - (b + c) = a - b - c$; $a - (b - c) = a - b + c$; etc.

But, if the parenthesis with the minus sign before it be removed without changing the signs of the terms in the subtrahend, the value of the expression is changed.

Thus, $7 - (5 - 3) = 7 - 2 = 5$

But, $7 - 5 - 3 = 2 - 3 = -1$

2. Find the result of $(2a + 3b) - (3a - 5b)$.

OPERATION.

$$\begin{array}{r} 2a + 3b \\ \underline{3a - 5b} \\ -a + 8b \end{array}$$

It is more convenient to find the result by omitting the parentheses, changing the sign of each term of the subtrahend, and then collecting the similar terms; thus,

$$2a + 3b - 3a + 5b = -a + 8b.$$

85. A quantity may be placed within a parenthesis with the plus sign before it, without changing the value of the expression, if the signs of the terms placed within the parenthesis be not changed.

Thus, $8 - 7 + 6 - 5 = 8 + (-7 + 6 - 5) = 8 + (-6) = 8 - 6 = 2$
$= 8 - 7 + (+6 - 5) = 1 + (+1) = 1 + 1 = 2$
$= 8 - 7 + 6 + (-5) = 7 + (-5) = 7 - 5 = 2$

86. A quantity may be placed within a parenthesis with the minus sign before it, without changing the value of the expression, if the sign of every term placed within the parenthesis be changed.

Thus, $8 - 7 + 6 - 5 = 8 - (+7 - 6 + 5) = 8 - (+6) = 8 - 6 = 2$
$= 8 - 7 - (-6 + 5) = 1 - (-1) = 1 + 1 = 2$
$= 8 - 7 + 6 - (+5) = 7 - (+5) = 7 - 5 = 2$

87. Coefficients of literal parts, that can not be collected into a single term, may be placed with their proper signs within the parenthesis as the coefficient of the remaining parts of the terms.

Thus, $ax + bx = (a + b)x$; $ab - b = (a - 1)b$; etc. Or,

The part common to the several terms may be made the coefficient of the remaining parts of the terms placed within the parenthesis.

Thus, $ax + bx = x(a + b)$; $ab - b = b(a - 1)$; etc.

---

88. *Illustrative Example.*—Find the value of $20 - [8 + 7 - (6 - 5)]$.

OPERATION.

$20 - [8 + 7 - (6 - 5)]$
$= 20 - [8 + 7 - 6 + 5]$
$= 20 - 8 - 7 + 6 - 5$
$= 26 - 20 = 6.$

EXPLANATION.—First the inner parenthesis which has the minus sign before it is removed, and the sign of every term within this parenthesis is changed.

Next the brackets which have the minus sign prefixed, are removed, and the sign of every term within these brackets is changed.

Then, collecting, the result is 26 − 20, or 6.

The sign prefixed to the parenthesis, the brackets, etc., is not the sign of the first term within the parenthesis; it is the sign of operation for the quantity regarded as a whole, and it disappears when the parenthesis is removed.

EXAMPLES.

Find the value of each of the following expressions:—

1. $6 \times (5-4)$.
2. $6 \times 5 - 4$.
3. $\overline{6-5} \times 4$.
4. $(6+4) \div 2$.
5. $6 + (4 \div 2)$.
6. $6 \div (4+2)$.
7. $6 \div 4 + 2$.
8. $10 + (6-4)$.
9. $10 - \overline{6+4}$.
10. $(-10-6)+4$.
11. $(-10+6)-4$.
12. $12-(4+2)$.
13. $-12-(4+2)$.
14. $(-12-4)+2$.
15. $-12+(4-2)$.
16. $-12-(-4-2)$.
17. $-15+5-3$.
18. $15-(-5-3)$.
19. $(-15-5)+3$.
20. $15-\overline{5+3}$.
21. $\overline{-15-(-5)}-3$.
22. $8-6+5-3$.
23. $8+(-6+5-3)$.
24. $\overline{8-6}+(-5-3)$.
25. $(8-\overline{6+5})-3$.
26. $(\overline{8-6}+5)-(-3)$.
27. $(12+\overline{6-4}-3)-5$.
28. $12+(\overline{6-4}+3)-5$.
29. $12-(6-\overline{4-3})+5$.
30. $(12-\overline{6-4}+3)-5$.
31. $12+6-4-[-3+(-5)]$.

Simplify and collect the following:—

32. $a+(b-\overline{c+d})$.
33. $a-(\overline{b+c}-d)$.
34. $a-(b-\overline{c-d})$.
35. $[a-(-b-c)]-d$.
36. $5a+[4a-\overline{3a-2a}]$.
37. $(5a-\overline{4a-3a})-2a$.
38. $(\overline{5x-4x}-3x)+2x$.
39. $5x-(4x-\overline{3x-2x})$.
40. $5x-[4y+(-3x+2y)]$.
41. $(5x-\overline{4y-3x})-2y$.
42. $6y-5z-[4y+(-3z)]$.
43. $[6y-(5b-4y)]-(-3b)$.

In expressions 32–43, let $a=4$, $b=3$, $c=2$, $d=1$, $x=15$, $y=10$, and $z=5$; find the numerical value of each of these expressions.

In expressions 44–49, consider the letters that precede x, y, and z as literal coefficients, and place them within parentheses as coefficients of the remaining literal parts (87):—

44. $ax + bx + cx$.
45. $ay + cy + az - 4z$.
46. $10x - ax + by - cy$.
47. $\overline{ax + bx} \times 4$.
48. $\overline{ay - cy} \div 3$.
49. $6 \times (bx - cx)$.

In expressions 44–49, let $a = 8$, $b = 6$, $c = 5$, $x = 4$, $y = 3$, $z = 2$; find the numerical value of each expression.

50. $2x - 3a + 4b - 5c$.
51. $x + y - 2a + 3c$.
52. $3x - y - 3a - 4c$.
53. $x - y + 2a - 3b$.
54. $4x + 2y - 3b - 2c$.
55. $2x - 3y - 2a + c$.

In 50–55, let $a = 4$, $b = 3$, $c = 2$, $x = 10$, and $y = 5$.

Using the terms as given,—

(1) Express each as two binomials within parentheses.

(2) In each, enclose within the parenthesis a trinomial, using the first terms.

(3) In each, enclose within the parenthesis a trinomial, using the last terms.

Find the numerical value of each expression formed.

56. Express the following quantity in several different ways by placing any two or three terms within the parenthesis; let a, b, c, x, and y each stand for some number; find the numerical value of each expression formed:—

$$5a - 4b + 3c - 2x - y.$$

TRANSPOSITION IN EQUATIONS.

1. If a number increased by 4 equals 10, what is the number? How is it found?

2. If a number decreased by 4 equals 10, what is the number? How is it found?

3. If $x+4=10$, what is the value of x? How is this value found?

4. If $x-4=10$, what is the value of x? How is this value found?

5. In the equation $x+5=15$, how much more than the value of x is 15? How is the value of x found? What is the value of x?

6. In the equation $x-5=15$, how much less than the value of x is 15? How is the value of x found? What is the value of x?

7. If in the equation $x+8=12$, 8 be taken from the first member, what does this member become?

What must be done to the second member that the members of the new equation shall be equal? What will the new equation be?

8. If in the equation $x-8=12$, 8 be added to the first member, what does this member become?

What must be done to the second member that the members of the new equation shall be equal? What will the new equation be?

89. If in any equation the first member contains all the unknown quantities, and the second member contains all the known quantities, the value of the unknown can easily be found. Thus,

What is the value of x in the equation $4x-5x+6x=12-15+18$?

OPERATION.

$$4x - 5x + 6x = 12 - 15 + 18.$$
$$10x - 5x = 30 - 15$$
$$5x = 15$$
$$x = 3$$

In which member are all the unknown quantities? In which are all the known quantities?

What is the result of collecting the unknown quantities in the first member? Of collecting the known quantities in the second member?

What new equation is formed? What is the value of x?

90. Sometimes an equation does not have all the unknown quantities in the first member, with all the known quantities in the second member.

In such equations, all the terms containing the unknown quantities must be gotten into the first member, and all the known quantities into the second member. Thus,

If $5x - 4 = 6$, what is the value of x?

OPERATION.

(1) $5x - 4 = 6$

(2) $5x = 6 + 4$

$5x = 10$

$x = 2$

6 is how much less than the value of $5x$? The value of $5x$ is how much more than 6?

What equation shows the value of $5x$?

In which member was 4 in the first equation? What was its sign? In which member is 4 in the second equation? What is its sign?

Tell whether or not the equality of the members has been destroyed by removing or transposing $+ 4$ from the first member and placing it as $- 4$ in the second member.

What is the value of x? Prove that the value of x is 2 in the first equation, and, also, in the second equation.

91. If any number be added to one member of an equation, the same number may be added to the other member without destroying the equality.

Thus, $5 + 3 = 8$ Also, $5 + 3 + 4 = 8 + 4$

Or, $8 = 8$ Or, $12 = 12$

92. If any number be taken from one member of an equation, the same number may be taken from the other member without destroying the equality.

Thus, $15 = 8 + 7$ Also, $15 - 5 = 8 + 7 - 5$

Or, $15 = 15$ Or, $10 = 10$

9. What is the value of x in the equation $7x - 14 + 3x + 5 = 31$?

OPERATION.

(1)	$7x - 14 + 3x + 5 = 31,$
Adding,	$+ 14 \quad - 5 = \quad + 14 - 5.$
(2)	$7x \quad + 3x \quad = 31 + 14 - 5$
Collecting,	$10x = 45 - 5$
Or,	$10x = 40$
And,	$x = 4$

What is the result of adding to any term the same number or quantity with the sign changed?

What must be added to -14 to remove this term from the first member? What must be added to $+ 5$ to remove it from the first member? What must be added to the second member to preserve the equality?

What is the value of x? Prove by substituting the value of x, that it is the same in equations (1) and (2).

10. If the same number or quantity be both added to and taken from a number or quantity or member, what is the effect on the value of that number or member?

What is the value of $x+3-3$? Of $10-x+x$? Of $5x-2x+2x=20+4-4$?

93. The operation of adding equal terms, with their signs changed, to both members of an equation need not be actually performed. Since this addition removes or transposes terms from one member to the other without destroying equality in the new equation, the transposition of unknown terms from the second member to the first, and known terms from the first member to the second, may be made at once. Thus,

Given $12x-14+3x=72+5x+4$ to find the value of x.

	Operation.
(1)	$12x-14+3x=72+5x+4.$
(2) Transposing,	$12x+3x-5x=72+4+14$
Collecting,	$15x-5x=90$
Or,	$10x=90$
And,	$x=9$

What unknown quantity is in the second member? How may it be transposed to the first member?

What known term is in the first member? How may it be transposed to the second member?

What is the value of x? Prove by substitution that the value of x is the same in equations (1) and (2).

94. The terms which are to remain in either member should be placed first in the new equation, and after them should be placed the terms transposed from the other member.

95. The process of removing a term from one member of an equation to the other without destroying the equality, is called **transposition**.

96. The members of an equation may be multiplied or divided by the same number without destroying the equality of the members.

Thus, if $2x = 12$, the value of x is found by dividing both members by 2, and the equation becomes $x = 6$.

Also, if $y = 5$, the value of $4y$ is found by multiplying both members by 4, and the equation becomes $4y = 20$.

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97. *Illustrative Example.*—Given $5x + 15 - 4x = 25 - 6x + 32$ to find the value of x.

OPERATION.

Given	$5x + 15 - 4x = 25 - 6x + 32.$
Transposing,	$5x - 4x + 6x = 25 + 32 - 15$
Collecting,	$11x - 4x = 57 - 15$
Or,	$7x = 42$
And,	$x = 6$

PROOF, OR VERIFICATION.

Substituting the value of x,

	$30 + 15 - 24 = 25 - 36 + 32$
Collecting,	$45 - 24 = 57 - 36$
And,	$21 = 21$

EQUATIONS.

1. $x + 5 = 11.$
2. $x - 5 = 4.$
3. $2x + 4 = 14.$
4. $3x - 5 = 13.$
5. $2x + 5 = 17.$
6. $4x - 7 = 21.$
7. $3x + 5 = 10 + 4.$
8. $5x - 6 = 12 + 7.$
9. $6x + 7 = 25 - 6.$
10. $4x = 3x + 6 + 2.$
11. $5x = 6 + 6x - 7.$
12. $7x + 5 = 5x + 17.$
13. $6x - 7 = 2x + 9.$
14. $4x + 8 = 7x - 10.$

15. $x + 9 = 2x - 1.$
16. $14 + 3x = 5x - 10.$
17. $20 - 5x = 47 - 14x.$
18. $4x - 10 = 6x - 22.$
19. $x = 56 - 8 - 3x.$
20. $2x + 7 = x + 15.$
21. $3x + 5 - 2x = 6.$
22. $5x + 11 - 2x = 20.$
23. $14 + 3x = 7x - 10.$
24. $3x - 26 = 39 - 8.$
25. $39 + 2x = 4x - 47.$
26. $27 = 50 - 7x + 61.$
27. $82 - 13x = 92 - 15x.$
28. $4x - 6 + 5 = 33 - 14.$
29. $12 + 5x = 108 - 5x - 6.$
30. $14 = 60 + 4x - 8x + 2.$
31. $3x - 30 = 41 - 2 + 2x.$
32. $86 - 12 = 14 + 9x + 15.$
33. $8x = 14x + 14 - 55 - 85.$
34. $3x - 6 = x + 16 - 6.$
35. $6 + 2x + 36 = 5x + 21.$
36. $41 - 6x - 17 = 120 - 14x.$
37. $3\frac{1}{2}x - 15 = 2x + 3 - 6.$
38. $5x + 10 = 2\frac{1}{2}x + 15.$
39. $\frac{1}{2}x + 6 - x = 1\frac{1}{2}x - 10.$
40. $\frac{2}{3}x + 12 = 1\frac{1}{3}x + 4.$

In examples 1–20, substitute the value of x, and verify the equations.

In examples 21–30, use y instead of x, and find the value of y in each; use a instead of x.

In examples 31–40, use z instead of x, and find the value of z in each; use b instead of x.

In examples 21–40, substitute the value of each unknown used, and verify the equation.

41. What number increased by 10 equals 15?

42. What number decreased by 5 equals 20?

43. Twice a certain number, increased by 7, equals 25. What is the number?

44. What number increased by itself and by 15 equals 33?

45. From three times a number, 5 is subtracted, and the remainder is 13. What is the number?

46. Four times a number, plus 8, equals six times the number, minus 10. Find the number.

47. A and B together earn \$40 a week, of which B earns \$4 more than A. How much does each earn?

48. I paid $50 for a watch and a chain. The watch cost $25 more than the chain. What did each cost?

49. Twice Harry's age, increased by 12 years, equals three times his age, increased by 4 years. How old is he?

50. Five times Robert's marbles, increased by 6 marbles, equals six times his marbles, decreased by 4 marbles. How many marbles has he?

51. A number increased by $\frac{1}{2}$ of itself and by $\frac{1}{4}$ of itself and by 7, equals 28. What is the number?

52. Thomas and William bought a printing-press. Thomas put in $10 more than William, making his payment three times as much as William's. What did each pay?

53. Four times the length of a fishing-rod, increased by 4 feet, equals six times its length, decreased by 20 feet. How long is the rod?

54. George is five times as old as John. The difference of their ages, plus 24 years, is twice the sum of their ages. How old is each?

55. There are three numbers, the second of which is four times the first, and the third is twice as much as the first and the second together. If the difference between the second and the third is 36, what are the numbers?

56. Divide $47 among George, John, and Charles, so that Charles may have $1 more than George, and John $3 more than Charles.

57. Harry has two-thirds as many pigeons as Fred; Robert has 10 less than one-half as many as Fred. If all have 55 pigeons, how many has each?

58. Divide the number 43 into two such parts that twice the less part shall be 4 less than the greater.

59. Mary bought some oranges at 3 cents each, and Mabel bought 5 less than $\frac{1}{2}$ as many at 4 cents each. If they all cost 80 cents, how many did each buy?

60. A pole 23 feet long is stuck up in the centre of a pond, going 2 feet into the ground at the bottom. Twice the length

in the water, multiplied by 3, is the length in the air. How many feet are in the air?

61. Three boys had an equal number of marbles. Harry bought one-third of Fred's and three-fourths of William's, and then he had 75 marbles. How many had each at last?

62. Of $120, I spent a part. What I spent was $4 less than three times the remainder. How many dollars did I spend?

63. A man loaned a sum of money at 6 per cent. and an equal sum at 5 per cent. If he received altogether $110 interest, what was the sum loaned at each rate?

64. A person spent $24 for clothing and $10 for books, and then had remaining one-third of what he had at first. How much had he at first?

65. A farmer sold 10 acres more than three-eighths of his farm, and then he had 4 acres less than $\frac{1}{2}$ of it remaining. How many acres were in the farm at first?

66. At an election, 2750 votes were cast, and the successful candidate had a majority of 370. How many persons voted for each?

67. A man worked 6 days at a certain sum per day. One son earned three-fourths as much, and another son one-half as much. If all received $27, what wages did each receive daily.

68. A man earned $14 in three days. The second day he earned $1 more than on the first, and on the third day, as much as on the two other days. How much did he earn each day?

69. A merchant sold a bill of goods. On one-half he gained 20 per cent. and on the other half he gained 10 per cent. If the whole selling price was $115, what did the goods cost?

70. Two boys earned $10 for 8 days' work. One boy received 25 cents a day more than the other. How much did each receive?

MULTIPLICATION.

1. What kind of unit does $+a$ represent? How many such units does $2a$ express? What kind of quantity is $2a$?

2. What kind of unit does $-a$ represent? How many such units does $-2a$ express? What kind of quantity is $-2a$?

3. What is the result of adding $2a$ and $2a$ and $2a$? Of taking $2a$ three times additively? Of multiplying $2a$ by 3? Of $2a \times 3$? What kind of quantity is each result?

4. What is the result of adding $-2a$ and $-2a$ and $-2a$? Of taking $-2a$ three times additively? Of multiplying $-2a$ by 3? Of $-2a \times 3$? What kind of quantity is each result?

5. What is the result of subtracting $2a$ and $2a$ and $2a$? Of taking $2a$ three times subtractively? Of multiplying $2a$ by -3? Of $2a \times -3$? What kind of quantity is each result?

6. What is the result of subtracting $-2a$ and $-2a$ and $-2a$? Of taking $-2a$ three times subtractively? Of multiplying $-2a$ by -3? What kind of quantity is each result?

98. Taking any quantity a positive number of times, means that the sum of the units in the quantity taken the given number of times is to be found additively, and suggests the process of *addition* only.

Taking any quantity a negative number of times, means that the sum of the units in the quantity taken the given number of times is to be found subtractively, and suggests the processes of both *addition* and *subtraction*.

7. Multiply 5 by 3.

OPERATION.

$$\begin{array}{r} +5 \\ +3 \\ \hline +15 \end{array}$$

What kind of number or quantity is $+5$? What kind is $+3$? What kind is $+15$?

Since multiplication is a short method of addition, 5×3 means that 5, a positive number or quantity, is *added*, or taken, *three times*. That is,

$$\begin{aligned} +5 \times 3 &= (+5) + (+5) + (+5) \\ &= +(5 + 5 + 5) \\ &= +(5 \times 3) = +15, \text{ or } 15. \end{aligned}$$

8. Multiply -5 by 3.

OPERATION.

$$\begin{array}{r} -5 \\ +3 \\ \hline -15 \end{array}$$

What kind of number or quantity is -5? What kind is $+3$? What kind is -15?

Since multiplication is a short method of addition, -5×3 means that -5, a negative number or quantity, is *added*, or taken, *three times*. That is,

$$\begin{aligned} -5 \times +3 &= (-5) + (-5) + (-5) \\ &= -(5 + 5 + 5) \\ &= -(5 \times 3) = -15 \end{aligned}$$

9. Multiply 5 by -3.

OPERATION.

$$\begin{array}{r} +5 \\ -3 \\ \hline -15 \end{array}$$

What kind of number or quantity is the multiplicand? What kind is the multiplier? The product?

The operation means that $+5$, a positive number or quantity, is taken three times; but, since the multiplier is negative, $+5$ is taken in a manner *opposite* to that in which $+5$ is taken when the multiplier is positive; that is, $+5$ is taken *subtractively three times*. Thus,

$$\begin{aligned} +5 \times -3 &= -(+5) - (+5) - (+5) \\ &= +(-5 - 5 - 5) \\ &= -(5 \times 3) = -15 \end{aligned}$$

10. Multiply -5 by -3.

OPERATION.

$$\begin{array}{r} -5 \\ -3 \\ \hline +15 \end{array}$$

What kind of number or quantity is the multiplicand? The multiplier? The product?

The operation means that -5, a negative number, is taken three times; but, since the multiplier is negative, -5 is taken in a manner *opposite* to that in which -5 is taken when the multiplier is positive; that is, -5 is taken *subtractively three times*. Thus,

$$\begin{aligned} -5 \times -3 &= -(-5) - (-5) - (-5) \\ &= +(+5 + 5 + 5) \\ &= +(5 \times 3) = +15, \text{ or } 15. \end{aligned}$$

99. When a positive or a negative number or quantity is multiplied by a positive multiplier, the number or quantity is taken additively the given number of times, and the product is a number or quantity *like the multiplicand.*

When a positive or a negative number or quantity is multiplied by a negative multiplier, the number or quantity is taken subtractively the given number of times, and the product is a number or quantity *unlike the multiplicand.*

11. What kind of quantity is the multiplicand b? The multiplier a? What is the product, and what kind of quantity is it?

12. What kind of quantity is the multiplicand $-b$? The multiplier a? What is the product, and what kind of quantity is it?

13. What kind of quantity is the multiplicand b? The multiplier $-a$? What is the product, and what kind of quantity is it?

14. What kind of quantity is the multiplicand $-b$? The multiplier $-a$? What is the product, and what kind of quantity is it?

15. What is the product of $xy \times a$? Of $-ay \times x$? Of $ax \times -y$? Of $-y \times ax$? Of $x \times -ay$? Of $-a \times -xy$?

16. In the multiplicand a^3, how often is a used as a factor? How often is a used as a factor in the multiplier a^2? How often is a used as a factor in the product of $a^3 \times a^2$?

17. The exponent of a letter in the product is the sum of the exponent of the letter in the multiplicand and the exponent of the letter in the multiplier.

Thus, $x^4 \times x^2 = x^{4+2}$, or x^6; $a^2b^3c^4 \times ab^2c^3 = a^3b^5c^7$; etc.

100. In finding the product of factors having coefficients, it is easiest to find the product of the coefficients first, and then the products of the literal parts.

Thus, $2a^2x^3 \times 3ax^2 = 6a^3x^5$. The product of 2×3 is 6; of a^2 and a is a^3; of x^3 and x^2 is x^5; and the whole product is $6a^3x^5$.

101. From the preceding operations and results, it is clear that—

1. *When the signs of both factors are alike, the sign of the product is +; and when the signs of the factors are not alike, the sign of the product is —.*

2. *The product of the coefficients in the factors is the coefficient of the product.*

3. *The sum of the exponents of any letter in both factors is the exponent of this letter in the product.*

Tell the product of each of the following:—

$+5b$	$3b$	$-5b$	$-3b$	$5b$	$-3b$	$-5b$	$-3b$
$+3b$	$5b$	$3b$	$5b$	$-3b$	$5b$	$-3b$	$-5b$

Multiplication of Monomials.

102. *Illustrative Example.*—Multiply $4ab$ by $-2xy$.

OPERATION.

$4ab$
$-2xy$
$-8abxy$

EXPLANATION.—The positive quantity $4ab$ is to be taken $-2xy$ times, or 2 times and x times and y times, subtractively.

Since the product is the same in whatever order the factors may be arranged, the operation may be expressed $4 \times 2 \times a \times b \times x \times y$, and the product is $8abxy$; and, since the multiplicand is taken subtractively, the result is $-8abxy$.

103. *Illustrative Example.*—Multiply $-5a^2x$ by $2a^3x^2$.

OPERATION.

$-5a^2x$
$2a^3x^2$
$-10a^5x^3$

EXPLANATION.—The negative quantity $-5a^2x$ is to be taken $2a^3x^2$ times, additively.

The coefficient -5 taken 2 times equals -10.

The product of a^2 and a^3 is a used 5 times as a factor, or a^5.

The product of x and x^2 is x used 3 times as a factor, or x^3.

The entire product is $-10a^5x^3$.

EXAMPLES.

Multiply,—

1	2	3	4	5	6	7	8
6	-6	6	-6	6	x	$-x$	-6
7	7	-7	-7	x	-6	6	$-x$

9	10	11	12	13	14	15	16
$4ab$	$-3a^2$	$5x$	$-a^2$	$7a^3$	$-4b^2$	a^2b	$-a^4$
$6c$	$4y$	$-a$	$-b^2$	bc	$2x^2$	-5	-6

17. $3x^2$ by $-2y^2$.
18. $-3c^2d$ by $2ab$.
19. $-4a^2b$ by $-3cd$.
20. $5ab^3c$ by $4d^2$.
21. $7b^2c^3d$ by $-3x^2$.
22. $-9c^2d^2$ by $4a^3b$.
23. $-11b^3c^2d$ by $-3x^2y$.
24. $4a^2b^2c$ by $8x^2$.
25. $-6x^3y^2z$ by $4a^2b$.
26. $8u^2v^3$ by $-5b^2c^3$.
27. $-10y^3z^2$ by $-6c^2d^3$.
28. $3v^2wx^3$ by $5a^2b$.
29. $-5w^3x^3y$ by $7b^2c^3$.
30. $7xy^2z^3$ by $-9ac^2$.
31. $-9y^3z^2$ by $-2a^3b^2$.
32. $8a^2b$ by $3a^3b$.
33. $-6b^2c^3$ by $5bc^2$.
34. $4c^2d^3$ by $-7c^3d$.
35. $-2d^2e^4$ by $-9d^3c^2$.
36. $10x^2y$ by $-5x^2y^2$.
37. $-8a^2x^3$ by $4ab^2x^2$.
38. $-6b^3x^2$ by $-6bx^3y$.
39. $-4a^2y^2$ by $-3a^3xy$.
40. $6b^2x^3$ by $-4bxy^2$.
41. $-8c^3xy^2$ by $4x^2y^3$.
42. $-10x^3z^2$ by $-2ay^2z$.
43. $12xy^2z^3$ by $-xz^2$.
44. $-3a^3z^2$ by $12b^3z^2$.
45. $-5c^3d^2$ by $-10c^4$.
46. $-4a^2$, $5ab$, and $6a^2c$.
47. $8b^3$, $2bc^2$, and $-a^2c$.
48. $-6a^2x$, $-4x^3$, and $-a^3y$.
49. $4b^2c^3$, $-a^2c$, and $2ab^3$.
50. $-5a^3x^2$, $-2x^3$, and $5a^2y$.
51. $-6a^4y^2$, $4a$, and $-a^2x$.
52. $-8b^3$, $-2a^2b$, and $-b^2x^3$.
53. $9a^3y$, $-3a^2b$, and $4b^3$.
54. $-2b^3$, $4xy^2$, and $-4x^2$.
55. $3bc^2$, $-c^3x$, and $2b^2x^2$.
56. $-m$, $-m$, $-m$, and $-m$.
57. a, $-a$, a, and $-a$.
58. ax, $-ax$, $-ax$, and ax.
59. $3a$, $-2x$, $-a$, and $6x$.
60. -4, $-a$, $-ax$, and a^2x^2.

In examples 1–20, tell what kind of quantity each multiplicand, each multiplier, and each product is. Give to each letter a small numerical value, perform each operation, and find the numerical value of each example.

MULTIPLICATION OF A POLYNOMIAL BY A MONOMIAL.

104. When a polynomial is to be multiplied, every term in the multiplicand must be multiplied.

Thus, $a - b$ multiplied by $4a$, means that a is to be taken $4a$ times, giving the partial product $4a^2$; then, $-b$ is to be taken $4a$ times, giving the partial product $-4ab$; and these partial products united, give the entire product $4a^2 - 4ab$.

As in arithmetic, every figure in the multiplicand must be multiplied by the multiplier.

Thus, 25 × 6, means that 5 ones are to be taken 6 times, and 2 tens are to be taken 6 times, giving the entire product 150.

105. *Illustrative Example.*—What is the product of $3a^2x - 2a^3$ by $4ax^2$?

OPERATION.

$$\begin{array}{l} 3a^2x \ - 2a^3 \\ \underline{4ax^2} \\ 12a^3x^3 - 8a^4x^2 \end{array}$$

EXPLANATION.—The product of $3a^2x$ by $4ax^2$ gives the partial product $12a^3x^3$.

The product of $-2a^3$ by $4ax^2$ gives the partial product $-8a^4x^2$.

Hence, the partial products united give the entire product $12a^3x^3 - 8a^4x^2$.

EXAMPLES.

Multiply,—

61	62	63
$5a + 2b$	$6ax - 2y^2$	$4a^2b - 5bc^2 - y$
4	$-a^2$	$2a^2b^2$

64	65	66
$-6a^2b + 3c^2$	$5ax - 3xy^2$	$-3bc^2 - a^2c + 2c^4$
$-2ac$	$-2xy$	$-3b^2c$

67	68	69
$5a + 2x - 3y$	$8ax - 3a^2 + x^2$	$4a^3 - 5a^2b - 7ab^3$
-4	x^2y	$5a^2b^2$

70. $5ab^2 - 7a^3b - 12$ by $7abc$; by $-3a^2b$.
71. $7c - 8a + 9a^2c^3$ by $-7a^3c^2$; by $6ac^2d$.
72. $9abc - 8ab - 7$ by $-5a^2b^3$; by $10b^2c^3$.
73. $7a^3y - 5a^2y^2 + 3ay^3$ by $-2ay$; by $3a^2y$.
74. $5ax^3 - 8a^2x^2 - 2a^3y$ by $-4ay^2$; by $2a^2y$.

75. $-6x^4y-4x^2y^3+5y^5$ by $-3x^2y$; by $4xy^2$.
76. $5b^2x-7bx^2+9a^2b$ by $-6a^2b$; by $4ab^2$.
77. $-7a^3b^2x-5a^2bx^2+3ax^3$ by $-a^2x$; by $2ax^2$.
78. $-8ax^3-5a^3x-xy$ by $-3ax^2y^3$; by $4a^2x^2$.
79. $3a^5-5a^3b-7a$ by $-9ab^3c^5$; by $6b^2y^3$.
80. $-9c^3d+7cx-x^3$ by $-3c^2x^2$; by $4cx^3$.
81. $2x^3y^2z-4x^2yz^2-6xz^3$ by $-3xyz$; by $5x^2y^3$.
82. $-4a^5x^3y-2a^3xy^2+ay^3$ by $-6axy^2$; by $7a^2x^2$.
83. $6a^3m^2+12am^4-4m^6$ by $-7a^2b$; by $5a^3$.
84. $4\frac{1}{2}a^2c-2ac^2+5\frac{1}{4}c^3$ by $8ac$; by $-4a^3$.
85. $4(a-b)+5(a+b)-\frac{1}{2}c$ by $4a^2$; by $-8b$.
86. $3a^2y^3-5ax^3+3y^2$ by $-2a^3$; by $4x^2y^3$.
87. $-12-bc^2d^2+3ac^3d^5$ by $-a^2d$; by $4a^3c$.
88. $-5xy^2+2x^3y-5ay^3$ by $4axy$; by $-a^3x^2$.
89. $-7a^4m+5a^3m^2-3a^2m^3$ by $-5am^2$; by $4a^3$.
90. $2\frac{1}{2}x^3y-\frac{3}{4}y^3z+3x^2z^2$ by $-4x^2y^2$; by $8xy^2$.

Multiplication of a Polynomial by a Polynomial.

106. When a polynomial is to be multiplied by a polynomial, every term in the multiplicand must be multiplied by every term in the multiplier.

The sum of the partial products gives the entire product required.

107. *Illustrative Example.*—Multiply $a+b$ by $a+b$.

Operation.

$$\begin{array}{lll} a + b & & \\ \underline{a + b} & & \\ a^2 + ab & & = \text{the product of } a+b \text{ by } a, \\ \underline{\quad\quad ab + b^2} & & = \text{the product of } a+b \text{ by } b, \\ a^2 + 2ab + b^2 & & = \text{the product of } a+b \text{ by } a+b. \end{array}$$

108. *Illustrative Example.*—Multiply $2a + 3x$ by $2a - 3x$.

OPERATION.

$$\begin{array}{l} 2a + 3x \\ 2a - 3x \\ \hline 4a^2 + 6ax \\ \quad - 6ax - 9x^2 \\ \hline 4a^2 \quad\quad - 9x^2 \end{array}$$

EXPLANATION.—The product of $2a + 3x$ by $2a$ gives the partial product $4a^2 + 6ax$.

The product of $2a + 3x$ by $-3x$ gives the partial product $-6ax - 9x^2$.

Hence, the sum of the partial products is the entire product required, which is $4a^2 - 9x^2$.

EXAMPLES.

Multiply,—

91	92	93	94
$2a + 3b$	$a - b$	$3x - 4y$	$2x + 4y$
$a + b$	$3a - 2b$	$2x + 3y$	$4x - 8y$

95	96	97	98
$2b + 3c$	$3x + 4y$	$2x^2 + 3y^2$	$3a + 4x$
$4b - 6c$	$x - y$	$3x^2 + 2y^2$	$4a + 3x$

99. $a + b$ by $c - d$.
100. $4a + 3b$ by $2a + b$.
101. $4x + 3y$ by $2x + y$.
102. $3a - 2b$ by $a + 2b$.
103. $a^2 + b^2$ by $a + b$.
104. $x + 4$ by $5 - x$.
105. $1 + 2a$ by $3a - 5$.
106. $2m - 3n$ by $m + n$.
107. $4y - 3z$ by $2y - 2z$.
108. $2a + 5$ by $3a - 6$.
109. $x^2 + 2y$ by $x + 3y^2$.
110. $3ab - 4c$ by $2ab - c$.
111. $-4cd + 2d$ by $cd + 3d$.
112. $m + 10$ by $m - 6$.
113. $1 + a + c$ by $1 - a$.
114. $x - y - 1$ by $x + 1$.
115. $a + b + c$ by $a - c$.
116. $x - y - z$ by $y - z$.
117. $2a + b - c$ by $2a + c$.
118. $2ay - 6y$ by $4ay + 12y$.
119. $a + b - c$ by $a + c$.
120. $a^2 + 2ab + b^2$ by $a - b$.
121. $x^2 - 2xy + y^2$ by $x + y$.
122. $a^6 + a^4 + a^2$ by $a^2 - 1$.
123. $x^2 - ax + a^2$ by $a + x$.
124. $x^2 + xy + y^2$ by $x - y$.
125. $x - y + 1$ by $x + y - 1$.
126. Add $12x + 40$ to the product of $x + 4$ by $x - 10$.

127. Subtract the product of $2a - 5$ by $2a + 5$, from $4a^2 + 10a - 50$.

128. Multiply $6 - 2a$ by $6 - 5a$, and take the product from $10a^2 - 42a + 40$.

129. To $30 + 4x - 2x^2$ add the product of $x + 3$ by $x - 5$.

130. From $a^2 + ab$ take $ab + b^2$, and multiply the remainder by $a^2 + b^2$.

131. Multiply $x^2 - xy + y^2$ by $x + y$, and from the product take $x^3 - 2xy + y^3$.

132. Add $4x^2 + 10x - 50$ to the product of $2x - 10$ by $2x + 5$; from the sum take $8x^2 + 5y^2$.

133. From the product of $a - 3x$ by $a + 3x$, take $-a^2 + 6ax - 9x^2$; add $6ax - 2x^2$ to the remainder.

134. Multiply $a^2 - 2ab + b^2$ by the sum of $-4ab + 2a^2 - 2b^2$ and $2a^2 + 4ab + 2b^2$; subtract $-8a^3b + 8a^2b^2$.

135. From $2c^2 - 2c - 1$ take $1 - 2c - 2c^2$, and multiply the remainder by $2 + 4c^2$; add $16c^4 - 8c^2 + 4$.

136. Multiply $a^2 - ab + b^2$ by $a + b$, and subtract $a^3 + 2a^2b^2 + b^3$ from the product; subtract $2b^3 - 4a^2b^2$.

137. Take $4a^2 + 8ab + 4b^2$ from the product of $2a + 2b$ by $2a - 2b$; add $12a^2 + 8ab - 4b^2$.

138. Add $b^3 - 2b^2c^2 + c^3$ to the product of $b^2 + bc + c^2$ by $b - c$; subtract $4b^3 - 4b^2c^2$.

139. Multiply together $x + y$, $x - y$, and $x^2 + y^2$, and to the product add $x^4 + 2x^2y^2 - y^4$; add $4y^4 - 2x^4$.

140. From the product of $a - 2$ by $a - 2$ take the product of $a + 2$ by $a + 2$; subtract $-8a^2 + 16a - 8$.

EXPANDING ALGEBRAIC EXPRESSIONS.

109. The product of literal factors is indicated by writing them side by side.

Thus, the product of a and b is written ab; $x \times y = xy$; etc.

The product of a quantity within a parenthesis by another quantity within a parenthesis may be indicated by writing the parentheses side by side.

Thus, $(a + b)(a + b)$ indicates that the sum of a and b is to be multiplied by the sum of a and b; $(x - y)(x - y)$ indicates the product of $x - y$ by $x - y$; etc.

110. The process of finding the product of quantities enclosed within parentheses is called **expanding** algebraic expressions.

Thus, expanding the expressions $(a + b)(a + b)$ means that the indicated product of $a + b$ by $a + b$ is to be found; $(a + b)(a - b)$ is expanded by finding the product of the sum of a and b by their difference.

111. When a parenthesis is preceded by a quantity used as a coefficient, it means that every term of the quantity within the parenthesis is to be multiplied by the quantity prefixed to the parenthesis.

Thus, $2ax(3ab - 7bx)$ expanded equals $6a^2bx - 14abx^2$.

112. The product of a quantity used twice as a factor is indicated by the exponent ^{2}.

Thus, the product of x by x is indicated by x^2; 3×3 is indicated by 3^2; $(a + x)(a + x)$ is indicated by $(a + x)^2$, read the square of the sum of a and x; etc.

1. Expand $(x + y)(x + y)$; or, square $(x + y)$.

OPERATION.

$$\begin{array}{l} x + y \\ \underline{x + y} \\ x^2 + xy \\ \qquad \underline{xy + y^2} \\ x^2 + 2xy + y^2 \end{array}$$

Observe each term of the factors, and each term of the product.

How is each term of the product obtained from the terms of the factors?

What sign connects the terms of each factor? What sign connects the terms of the product?

113. From the operation of finding the square of the sum of two quantities, the following principle or law is derived:—

The square of the sum of two quantities is the square of the first term, plus twice the product of the first term by the second, plus the square of the second term.

2. Expand $(x-y)(x-y)$; or, find the result of $(x-y)^2$.

OPERATION.

$$\begin{array}{l} x - y \\ \underline{x - y} \\ x^2 - xy \\ \underline{\quad - xy + y^2} \\ x^2 - 2xy + y^2 \end{array}$$

Observe each term of the factors, and each term of the product.

How is each term of the product obtained from the terms of the factors?

What sign connects the terms of each factor? What signs connect the terms of the product?

114. From the operation of finding the square of the difference of two quantities, the following principle or law is derived:—

The square of the difference of two quantities is the square of the first term, minus twice the product of the first term by the second, plus the square of the second term.

3. Expand $(x+y)(x-y)$.

OPERATION.

$$\begin{array}{l} x + y \\ \underline{x - y} \\ x^2 + xy \\ \underline{\quad - xy - y^2} \\ x^2 \qquad - y^2 \end{array}$$

Observe each term of the factors, and each term of the product.

How is each term of the product obtained from the terms of the factors?

What sign connects the terms of each factor? What sign connects the terms of the product?

115. From the operation of finding the product of the sum of two quantities by their difference, the following principle or law is derived:—

The product of the sum of two quantities by their difference, is the square of the first term, minus the square of the second term.

65. $(x+2)(x+6)$.
66. $(x+3)(x+7)$.
67. $(2+3)(2+4)$.
68. $(a^2+4)(a^2+6)$.
69. $(ax+3)(ax+5)$.
70. $(2x^2+1)(2x^2+3)$.
71. $(a-2)(a-3)$.
72. $(b-4)(b-5)$.
73. $(c-3)(c-6)$.
74. $(d-4)(d-7)$.
75. $(x-5)(x-2)$.
76. $(y-4)(y-5)$.
77. $(4-2)(4-3)$.
78. $(x^2-5)(x^2-4)$.
79. $(ax-2)(ax-5)$.
80. $(2a^2-1)(2a^2-3)$.
81. $(b+3)(b-4)$.
82. $(c+2)(c-6)$.
83. $(a+3)(a-7)$.
84. $(x+4)(x-6)$.
85. $(y+5)(y-7)$.
86. $(z-3)(z+5)$.
87. $(ax-5)(ax+8)$.
88. $(5-2)(5+4)$.
89. $(a^2-4)(a^2+6)$.
90. $(3x^2-5)(3x^2+4)$.
91. $(a^2+2b)(a^2+3b)$.
92. $(3+5)(3+2)$.
93. $(x-2a)(x-4a)$.
94. $(5-3)(5-2)$.
95. $(3x+1)(3x-4)$.
96. $(2ax^2+4)(2ax^2-5)$.
97. $(7-2a^2b)(7+3a^2b)$.
98. $(2ab+5)(2ab+7)$.
99. $(5-2ax)(5-1)$.
100. $(6x^2y^2-7)(6x^2y^2+5)$.

EXERCISES IN ALGEBRAIC EXPRESSION.

Express in algebraic notation the following:—

1. The product of a by a.
2. a square b cube.
3. c cube multiplied by 5.
4. Six times a by y.
5. Ten b square c cube.
6. Five c minus x over 10.

7. What is the cost of two boxes each containing b dozen of lemons, at y cents a dozen?

8. If a man walks c miles an hour for b hours and then rides y miles, how far does he travel?

9. If 6 men do a piece of work in b days, how many days would it take one man to do it?

10. How many quarts of chestnuts are in b bushels c quarts?

11. The length of an oblong is represented by $a+3$, and the width by $a-2$. What is the area?

12. A farmer sold 12 quarts of berries at y cents a quart, and with the money he bought 2 pounds of coffee at x cents a pound. How much had he left?

13. Harry rides b miles an hour, and Frank rides $\overline{b+1}$ miles. In c hours, how much farther does one of them go than the other?

14. A man has a cows, and three times as many sheep, less 8. How many cows and sheep has he in all?

15. If a represents the number of tens in a number, and b the number of ones, how is the number itself represented?

16. How many more square inches are in the surface of a piece of paper a inches long and b inches wide, than in a square piece b inches long?

17. A clerk's salary is x dollars a month. He pays b dollars a week for his board, and c dollars a week for other expenses. How much does he save in 2 months?

18. At $b\%$ a year, what is the interest on x dollars for c years?

19. How much must a merchant charge for an article that cost him b dollars, that he may make 5% profit?

20. A commission merchant sold butter for x dollars, at 2% commission. How much did he pay over to his employer?

In problems 1–20, let $a=8$, $b=6$, $c=4$, $x=50$, and $y=10$; find the numerical result of each of these problems.

EQUATIONS.

1. $3(x+2)=7(x-2)$.
2. $(x+2)4=(x-1)8$.
3. $\overline{x-6}\times 5=\overline{x-4}\times 15$.
4. $\overline{x-3}\times 10=(x+5)\times 2$.
5. $2(3x-10)=5x-15$.
6. $3(12-x)=(x-3)6$.

7. $(x+6)\times 9-2=(x-1)\times 40-1.$

8. $(3-7)(5+x)+13=-9x+15-7.$

9. $x(x-7)+18=(x-3)x-36+14.$

10. $(5-x)(6-x)=(7-x)(8-x)-10.$

11. $(x-4+4x)(5-2)=11x+16.$

12. $(x-3)(x-20)=x^2-5x-84.$

In equations 1–12, substitute y for x; z for x. Substitute the value of the unknown used, and verify each equation.

13. Find two numbers whose sum shall be 60, and the first number shall be three times the second.

14. Divide 36 cents into two such parts that four times the less shall be 6 less than the greater.

15. From four times a number 8 is subtracted, the remainder is multiplied by 5, and the product is 20. What is the number?

16. If you add 6 to a certain number and multiply the sum by 3, the product will be 14 less than 7 times the number itself. What is the number?

17. If William's money were increased by twice as much money as he now has, and by 5 cents less than four-fifths as much, he would have 10 cents less than a dollar. How much money has he?

18. A merchant sold one-fourth of a roll of carpet to one man, and twice as much to another. If he then had 12 yards left, how many yards were in the roll at first?

19. A father is 28 years old and his son's age is 4 years. In how many years will the father be three times as old as his son?

20. A is now 40 years old and B is 10 years younger. How many years since A was three times as old as B?

21. Four men hired a boat, but one of them was unable to join the party, and the expenses of each of the other men were increased a dollar. What did they pay for the boat?

22. A father and his son earned $3.25 a day. If the son's wages had been doubled, he would still have earned 25 cents less than his father. How much did each earn?

23. A and B together had $40. A gave $4 to B, and then B had four times as much as A. How many dollars had each at first?

24. Three pears, four lemons, and five oranges cost 43 cents. A lemon cost one cent more than a pear, and an orange two cents more than a lemon. What did each cost?

25. If each side of a square piece of paper were 4 inches longer than it is, the surface of the paper would be 80 square inches greater than it is. What is the area of the paper now?

26. An oblong piece of paper is 10 inches longer and 5 inches narrower than a square piece containing the same area. Find the dimensions of the oblong piece.

27. The difference between the two factors of a number is 16, and one of them is 8 less than $\frac{2}{3}$ of the other. What is the number?

28. Frank had five times as many pigeons as George, and William had three times as many as George. If Frank had 10 pigeons more than William, how many had each?

29. A man invested a sum of money at 6 per cent., and twice as much at 5 per cent. If the income from all was $128, how much money did he invest?

30. Harry and Fred start from the same place and walk in the same direction, Harry having two hours' start. If Harry walks 2 miles an hour, and Fred 3 miles an hour, how soon will Fred overtake Harry?

31. The width of a table is one-half the length. If the length were 3 feet less, and the width 3 feet more, the table would be square. Find the dimensions of the table.

32. A newsboy bought some papers at 2 cents each, and had 12 cents left. If he had paid 3 cents each, he would have needed 13 cents more to pay for them. How many papers did he buy?

33. John has two-thirds as much money as James, and Robert has twice as much as John and James together. If Robert has 70 cents more than James, how much money have they all?

34. Divide 130 marbles among three boys so that Frank shall have three-fourths as many as Thomas, and George six-sevenths as many as both of the other boys.

35. A man has to wait 5 hours for a train. How far can he ride away from the station at the rate of 9 miles an hour, so that he can come back at the rate of 6 miles an hour and be in time for the train?

DIVISION.

117. Division is the reverse of multiplication.

In multiplication, two factors, called the *multiplicand* and the *multiplier*, are given to find their *product*.

Thus, 5 apples taken 2 times = 10 apples; $-b$ taken a times $= -ab$; $7 \times 3 = 21$; $-2b \times -c = 2bc$; etc.

In division, the product of two factors and one of the factors are given to find the other factor.

The product of the two factors is called the *dividend;* one factor is called the *divisor;* the other factor is called the *quotient.*

118. In algebra, the main purpose of division is to find by what quantity or factor the divisor must be multiplied to produce the dividend.

Thus, $12 \div 4$ equals 3, means that 4 must be taken 3 times to equal 12; $ab \div a = b$, means that a must be taken b times to equal ab; $-bc \div b = -c$; $-ax \div -a = x$; etc.

1. Divide $+12$ by $+4$.

OPERATION.	
$+4)\underline{+12}$	What is the dividend? The dividend is the product of what factors?
$\quad +3$	By what factor must the divisor, $+4$, be multiplied to produce the dividend, $+12$? What is the quotient?

What is the sign of the dividend? Of the divisor? Of the quotient?

2. Divide $+12$ by -4.

OPERATION.

$$\begin{array}{r} -4)\underline{+12} \\ -3 \end{array}$$

What is the dividend? By what factor must the divisor, -4, be multiplied to produce the dividend, $+12$? What is the quotient?

What kind of quantity is the dividend? The divisor? The quotient?

3. Divide -12 by $+4$.

OPERATION.

$$\begin{array}{r} +4)\underline{-12} \\ -3 \end{array}$$

Observe the dividend; the divisor; the quotient. What kind of quantity is each?

By what factor must the divisor, $+4$, be multiplied to produce the dividend, -12. What is the quotient?

4. Divide -12 by -4.

OPERATION.

$$\begin{array}{r} -4)\underline{-12} \\ +3 \end{array}$$

Observe the dividend; the divisor; the quotient. What kind of quantity is each?

By what factor must the divisor, -4, be multiplied to produce the dividend, -12? What is the quotient?

119. When a positive or a negative divisor is to be taken a number of times to produce a positive dividend, the quotient is a quantity *like the divisor*.

120. When a positive or a negative divisor is to be taken a number of times to produce a negative dividend, the quotient is a quantity *unlike the divisor*.

5. What kind of quantity is the dividend ab? The divisor a? What is the quotient, and what kind of quantity is it?

6. What kind of quantity is the dividend ab? The divisor $-a$? What is the quotient, and what kind of quantity is it?

7. What kind of quantity is the dividend $-ab$? The divisor a? What is the quotient, and what kind of quantity is it?

8. What kind of quantity is the dividend $-ab$? The divisor $-a$? What is the quotient, and what kind of quantity is it?

9. What is the quotient of $axy \div a$? Of $axy \div -x$? Of $-axy \div y$? Of $-axy \div ax$? Of $axy \div -ay$? Of $-axy \div -xy$?

10. In the dividend x^6, how often is x used as a factor? How often is x used as a factor in the divisor x^2? How often is x used as a factor in the quotient of $x^6 \div x^2$?

11. The exponent of a letter in the quotient is the difference between the exponent of the letter in the dividend and the exponent of the letter in the divisor.

Thus, $a^5 \div a^3 = a^{5-3}$, or a^2; $x^2y^4z^6 \div xy^2z^3 = xy^2z^3$; etc.

121. In finding the quotient of one quantity by another when both have coefficients, it is easiest to find the quotient of the coefficients first, and then of the literal parts.

Thus, $-6a^3b^4c^5 \div 2ab^3c^2 = -3a^2bc^3$; the quotient of $-6 \div 2 = -3$; of a^3 by a is a^2; of b^4 by b^3 is b; of c^5 by c^2 is c^3; and the whole quotient is $-3a^2bc^3$.

122. From the preceding operations and results, it is clear that—

1. *When the signs of the dividend and the divisor are alike, the sign of the quotient is* $+$; *when the signs of the dividend and the divisor are not alike, the sign of the quotient is* $-$.

2. *The quotient of the coefficient of the dividend by the coefficient of the divisor is the coefficient of the quotient.*

3. *The difference of the exponent of a letter in the dividend and the same letter in the divisor is the exponent of the letter in the quotient.*

Division of a Monomial by a Monomial.

Find the quotient of,—

1	2	3	4	5	6
$3)9$	$-9)27$	$2)-6a$	$-2)-6x$	$9)-27y$	$9y)63y$

7	8	9	10	11
$-3y)-9y$	$-3y)-6y$	$3)-6c$	$-xy)-2xy$	$-4)-4a$

12. abc by ab.
13. $9x^6$ by $-3x^2$.
14. $-6x^3$ by x^3.
15. $-4a^6$ by $-2a^5$.
16. $-a^5x^7y$ by $-a^2x^4y$.
17. $11a^4xy^2$ by $-a^4$.

18. $15x^2y^3z^2$ by $5x^2y^2z$.

19. $24a^3c^7d^5$ by $8ac^5d^5$.

20. $-10a^4x^2y^3$ by $-2a^3x^2y^2$.

21. $-12x^5y^7z^4$ by $-3xy^3z^4$.

22. $12a^5b^4c^3$ by $-4a^5b^4c$.

23. $-16b^5c^3d^4$ by $16b^4c^2d^4$.

24. $9a^4x^5y^7$ by $-9ax^5y^5$.

25. $-4x^2y^3z^4$ by $-xy^2$.

26. $14a^5b^3c^2$ by $-2a^4b^3c$.

27. $12b^5d^7x^5$ by $4b^2d^5$.

28. $-8c^5d^2f^5$ by $-2c^4d^2f^2$.

29. $26a^7c^4m^3$ by $13a^6c^4m^2$.

30. $-42a^7x^5y^4$ by $-6a^7x^4y^3$.

31. $-8m^5n^2x^7$ by $4mn^2$.

32. $10a^7m^5z^2$ by $-5am^5z$.

33. $81c^5x^7y^4$ by $9c^5x$.

34. $-8b^5c^4x^7$ by $2b^4c^4x^2$.

35. $-8a^8y^7z^3$ by $8a^8y^7z^3$.

36. $16m^2n^4x^5$ by $-2nx^5$.

37. $-20u^5vx^7$ by $5u^5vx^3$.

38. $9w^5y^4z^2$ by $-w^4y^4z$.

39. $-10x^5y^7z^5$ by $2x^5z^2$.

40. $-18a^5c^7d^4$ by $-6a^3c^6d^4$.

41. $8c^5d^6n^3$ by $-2c^5d^6$.

42. $-15a^7n^5x^4$ by $5a^7nx^3$.

43. $-8a^6b^5c^4$ by $-4a^6bc^3$.

44. $12b^4c^3d^5e^3$ by $-4b^2c^2de^3$.

45. $-8n^3x^5y^3$ by $2n^3xy^2$.

46. $a^5c^5d^6e^7$ by $a^4c^5de^7$.

47. $-9d^7e^4x^5y$ by $9e^4x$.

48. $-21a^7b^5c^8d^2$ by $3a^7b^5d$.

49. $-35x^5y^4z^2$ by $7xy^4z$.

50. $24b^3c^5d^2x$ by $-6b^2c^5dx$.

Division of a Polynomial by a Monomial.

123. When the dividend contains several terms and the divisor contains only one term, every term in the dividend must be divided by the divisor.

124. *Illustrative Example.*—Divide $6a^3x^5y - 12a^2x^3y^2 + 8a^4x^2y^5$ by $2a^2x^2y$.

Operation.

$$2a^2x^2y\overline{)6a^3x^5y - 12a^2x^3y^2 + 8a^4x^2y^5}$$
$$3ax^3 - 6xy + 4a^2y^4$$

Explanation.—The first term of the dividend divided by the divisor gives the factor or quotient $3ax^3$; the second term divided by the divisor gives $-6xy$; the third term divided by the divisor gives $+4a^2y^4$. Hence, the entire quotient is $3ax^3 - 6xy + 4a^2y^4$.

EXAMPLES.

51	52	53	54
$3)\underline{3a + 6b}$	$4)\underline{8x - 12}$	$-5)\underline{10x - 15}$	$6)\underline{-6a + 18x^2}$

55	56	57	58
$a)\underline{a^2 - a^3}$	$-b)\underline{b^2c - b^3x}$	$c)\underline{-c^3 - cd^3}$	$-x)\underline{-x^2 + 2x^3y}$

59. $-7b^3 + b^2c$ by $-b^2$.
60. $4bc^2 - 8b^2c$ by $2bc$.
61. $6x^2 - 3x^3z$ by $-3x^2$.
62. $-3y^2 + 6yz$ by $-3y$.
63. $ab^2x^3 - axy^2$ by $-ax$.
64. $-xyz - 5xy^2z^3$ by $-xyz$.
65. $5a^5c^3 - 10a^4c^2$ by $-5a^2c^2$.
66. $-9a^2b - 12ab^3$ by $3ab$.
67. $6bc^2 - 18b^3c$ by $-2bc$.
68. $10x^5y^3 - 15x^4y^2$ by $-5x^4y^2$.
69. $8ab^2x^3 - 12a^2b^3x^2$ by $-4ab^2$.
70. $-a^6 - 2a^5 - a^4$ by $-a^4$.
71. $2x^3 - 4x^2y - 2xy^2$ by $-2x$.
72. $-xy^3 - 3x^2y^2 + x^2y^5$ by $-xy^2$.
73. $ax - bx + cx$ by $-x$.
74. $xy + y - xy^2$ by $-y$.
75. $-6(a + x) + 12(a + x)$ by 6.
76. $5\overline{a - x} - 10\overline{a - x}$ by -5.
77. $x^4y^2 - 2x^3y^3 + x^2y^4$ by $-x^2y^2$.
78. $-9b^4c + 6b^3c^2 - 3b^2c^3$ by $-3b^2c$.
79. $4a^2x^3 - 6a^3x^2 + 8a^4x$ by $-2a^2x$.
80. $-15x^3y^4 - 12x^4y^3 + 9x^5y^2$ by $-3x^3y^2$.
81. $6x^5y^2 - 4x^3y^3 - 2xy^4$ by $2xy^2$.
82. $-25a^3c^2 + 15a^2c^3 - 10ac^4$ by $-5ac^2$.
83. $9a^2x^6 - 6a^4x^5 + 12a^6x^4$ by $-3a^2x^3$.
84. $-20b^2c^2 - 15ab^3c + 10b^4c^3y$ by $-5b^2c$.
85. $15a^2cy^2 - 9a^3dy^3 + 12a^2ey^5$ by $-3a^2y^2$.
86. $-7ab^5x - 14ab^5y - 7ab^5z$ by $7ab^5$.
87. $6a^3cm - 9a^4cm^2 + 12a^5cm^3$ by $-3a^3cm$.
88. $2a^2 - (a^3 - 3a)$ by a.
89. $6(a + x) - 9(a + x) + 3(a + x)$ by -3.
90. $-2a^2b(a - c) + 4ab^2(b + c)$ by $-2ab$.
91. $7(a + x) - 14a(a + x) - 21b(a + x)$ by $-7(a + x)$.

92. $-6xy-(-4x^2y+2xy)$ by $-2xy$.

93. $2abcx^2+4abc^2x^3-6abc^3x^4$ by $-2abcx^2$.

94. $-4p^2x^3y-6p^3x^2y-8p^4xy$ by $2p^2xy$.

95. $-21r^4s^2t+14r^3s^3t^2-7r^2s^4t^3u$ by $-7r^2s^2t$.

96. $15ab^4y^2z-10a^2b^3y^3z+5a^3b^2y^4z^2$ by $-5ab^2y^2$.

97. $-8a^2m^4o^3x^5+16a^3m^3o^4x^4-24a^4m^2o^5x^3$ by $8a^2m^2o^2x^2$.

98. $5a(x-y)-10b(x-y)+15(x-y)$ by $-5(x-y)$.

99. $\frac{1}{2}a^2b^4c+\frac{1}{4}a^3b^3cd-\frac{1}{3}a^4b^2c$ by $2a^2b^2c$.

100. $2a^2b(x+y)-4a^4b^2(x+y)+6a^2b^3(x+y)$ by $-2ab(x+y)$.

Division of a Polynomial by a Polynomial.

125. When both dividend and divisor have two or more terms, the process of dividing is much like the process of long division in arithmetic, and is the reverse of multiplication.

1. MULTIPLICATION.

Multiplicand,	$c+b$
Multiplier,	x
Product,	$cx+bx$

2. DIVISION.

Divisor. Dividend.

$c+b)cx+bx(x$, Quotient.

Product, $cx+bx$

EXPLANATION 1.—The first term of the product is the product of the first term of the multiplicand by the monomial multiplier; and the second term of the product is the product of the second term of the multiplicand by the multiplier. Hence,

EXPLANATION 2.—Dividing the first term of the dividend, or product, by the first term of the divisor gives the other factor, or quotient, x. Multiplying every term of the divisor by the monomial quotient x, gives the entire dividend, or product.

1. MULTIPLICATION.

Multiplicand,	$3a+2x$
Multiplier,	$a+x$
1st partial product,	$3a^2+2ax$
2d partial product,	$3ax+2x^2$
Entire product,	$3a^2+5ax+2x^2$

EXPLANATION 1.—The first term of the first partial product is the product of the first term of the multiplicand by the first term of the multiplier; and all of the first partial product is

2. DIVISION.

	Divisor. Dividend. Quotient.
	$3a + 2x)3a^2 + 5ax + 2x^2(a + x$
1st partial product,	$3a^2 + 2ax$
Remainder of dividend,	$3ax + 2x^2$
2d partial product,	$3ax + 2x^2$

the product of every term of the multiplicand by the first term of the multiplier.

The first term of the second partial product is the product of the first term of the multiplicand by the second term of the multiplier; and all of the second partial product is the product of every term of the multiplicand by the second term of the multiplier. Hence,

EXPLANATION 2.—Dividing the first term of the dividend, or product, by the first term of the divisor, gives the first term of the other factor, or quotient, a. Multiplying every term of the divisor by the first term of the quotient gives all of the first partial product, $3a^2 + 2ax$.

Dividing the first term of the remainder of the dividend, or product, by the first term of the divisor, gives the second term of the other factor, or quotient, x. Multiplying every term of the divisor by the second term of the quotient, gives the second partial dividend, or product, $3ax + 2x^2$. Therefore, the entire quotient is $a + x$.

126. From the preceding operations and results, it is clear that—

The terms of the quotient are found by dividing the first term of the dividend, and of each remainder in order, by the first term of the divisor.

Every term of the divisor is multiplied by each term of the quotient as it is found, for each partial dividend, or product.

127. *Illustrative Example.*—Divide $-2ab + a^2 + b^2$ by $-b + a$.

FIRST OPERATION.

$$a - b)a^2 - 2ab + b^2(a - b$$
$$a^2 - ab$$
$$- ab + b^2$$
$$- ab + b^2$$

SECOND OPERATION.

$$- b + a)b^2 - 2ab + a^2) - b + a$$
$$b^2 - ab$$
$$- ab + a^2$$
$$- ab + a^2$$

EXPLANATION.—For convenience, the terms of the dividend are arranged according to the descending exponents of the leading letter, a, and the terms of the divisor are arranged in the same order. Then,

Dividing the first term of the dividend by the first term of the divisor for the first term of the quotient, etc.

Dividing the first term of the first

remainder, or second partial dividend, by the first term of the divisor for the second term of the quotient, etc.

Hence, the entire quotient is $a - b$.

The same result or quotient is obtained by arranging the terms of the dividend according to the descending exponents of b as the leading letter, since the second quotient, $-b+a$, and the first quotient, $a-b$, are precisely the same in value.

128. *Illustrative Example.*—Divide x^3+y^3 by $x+y$.

OPERATION.

$$
\begin{array}{l}
x+y)x^3 \qquad\qquad + y^3(x^2 - xy + y^2 \\
\quad\;\; \underline{x^3 + x^2y} \\
\qquad\; - x^2y \qquad\; + y^3 \\
\qquad\; \underline{- x^2y - xy^2} \\
\qquad\qquad\quad xy^2 + y^3 \\
\qquad\qquad\quad \underline{xy^2 + y^3}
\end{array}
$$

EXPLANATION.—In the dividend, the second power and the first power of x, and the first power and the second power of y, are wanting.

Dividing the first term of the dividend by the first term of the divisor, etc.

Dividing the first term of the first remainder by the first term of the divisor for the second term of the quotient, etc.

Dividing the first term of the second remainder by the first term of the divisor for the third term of the quotient, etc.

Hence, the entire quotient is $x^2 - xy + y^2$.

EXAMPLES.

101. $ac+bc$ by $a+b$.
102. $xz-yz$ by $x-y$.
103. $a^2+2ab+b^2$ by $a+b$.
104. $b^2-2bc+c^2$ by $b-c$.
105. a^2-2a-8 by $a+2$.
106. x^2-2x-8 by $x-4$.
107. $a^2+4ac+4c^2$ by $a+2c$.
108. $a^2-7a+12$ by $a-4$.
109. $a^2+7a+12$ by $a+3$.
110. a^2-a-12 by $a-4$.
111. x^2+x-2 by $x-1$.
112. $7-6x-x^2$ by $1-x$.
113. $x^2-7x-30$ by $x+3$.
114. a^3-b^3 by $a-b$.
115. x^3+1 by $x+1$.
116. a^2-x^2 by $a-x$.
117. b^2-c^2 by $b+c$.
118. x^4-9 by x^2-3.
119. y^6-16 by y^3+4.
120. x^4-y^4 by x^2+y^2.
121. m^2-n^2 by $m-n$.
122. $-r^2+s^2$ by $-r+s$.
123. $4a^4x^4-1$ by $2a^2x^2+1$.
124. m^2-n^2 by $m+n$.
125. $8a^3+27b^3$ by $2a+3b$.
126. x^3+y^3 by $x+y$.

127. $a^3 - y^3$ by $a - y$.

128. $a^3 + 1$ by $a + 1$.

129. $x^3 - 27$ by $x^2 + 3x + 9$.

130. $x^3 + 1$ by $x^2 - x + 1$.

131. $9x^2 - 18xy + 9y^2$ by $3x - 3y$.

132. $12a^2 + 10a - 8$ by $4a - 2$.

133. $12a^2 + 22a + 8$ by $3a + 4$.

134. $6a^5 + a^4b - 2a^3b^2$ by $3a^2 + 2ab$.

135. $2x^2y^2 - 6xyz - 8z^2$ by $xy - 4z$.

136. $3a^4b^2 - 11a^2bc + 10c^2$ by $a^2b - 2c$.

137. $-8a^2x^2 + 26axy - 15y^2$ by $4ax - 3y$.

138. $-20r^4s^2 - 44r^2s^2 - 24s^2$ by $-4r^2s - 4s$.

139. $x^4 + x^3y + xy^3 + y^4$ by $x + y$; by $x^3 + y^3$.

140. $ac - bc - ad + bd$ by $a - b$; by $-d + c$.

141. $m^2 - mx - x - 1$ by $1 + m$; by $m - x - 1$.

142. $b^6 - 1$ by $b^2 - 1$.

143. $x^3 - 1 + x^2 - x$ by $1 + x$.

144. $x^8 - 256$ by $x^2 - 4$.

145. $2a + 4a^2 - 12$ by $2a - 3$.

In examples 101–145, find the dividend from the divisor and the quotient.

146. To $a^2 + 2ab + b^2$ add $a^2 - 2ab + b^2$ and divide $4a^4 - 4b^4$ by the sum.

147. Divide $a^3 + x^3$ by $a + x$ and take the quotient from $a^2 + ax + x^2$; multiply $2a^2 - 2x^2$ by the remainder.

148. Divide $4c^2 - 1$ by $2c - 1$ and multiply the quotient by $4c^2 + 1$; subtract $8c^3 + 2c$.

149. From $2a^2 - 2a + 2$ take $a^2 - 4a + 1$ and divide the remainder by $a + 1$; multiply by $8a^3 - 1$.

150. Divide $x^3 - y^3$ by $x - y$ and subtract the quotient from $x^2 + 4xy + y^2$; multiply $3x^2y - 2x + 1$ by the remainder.

151. Multiply $x^2 + xy + y^2$ by $x - y$ and from the product take $x^3 + y^3$; add $4y^4 + 2y^3 - y^2$.

152. From $2-4x^2+2x^3$ take $1-2x^2+3x^3$, and divide the remainder by $1+x$; multiply by $2+2x$.

153. Divide a^6-1 by a^2-1, and to the quotient add a^4-a^2-1; subtract $2a^4-2a^2+1$.

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129. *Illustrative Example.*—Divide $a^2+2ab+b^2$ by $a+b$.

OPERATION.

$$\frac{a^2+2ab+b^2}{a+b}=a+b, \text{ Quotient.}$$

EXPLANATION.—Arrange the dividend and the divisor in the manner most convenient for inspection.

It is seen that the first term of the dividend, or product, is the square of the first term of the divisor, the third term of the dividend is the square of the second term of the divisor, the second term is twice the product of the first term of the divisor by the second term, and the sign of the second term of the dividend, or product, is plus. Hence (**113**), the quotient is $a+b$, a factor or quantity similar to the divisor, showing the sum of the two terms of the divisor.

130. *Illustrative Example.*—Divide a^2-b^2 by $a+b$.

OPERATION.

$$\frac{a^2-b^2}{a+b}=a-b, \text{ Quotient.}$$

EXPLANATION.—Arrange the dividend, or product, etc.

The first term of the dividend, or product, is the square of the first term of the divisor, the second term of the dividend, or product, is the square of the second term of the divisor, and the sign connecting the terms of the dividend, or product, is minus. Hence (**115**), the quotient is $a-b$, a quantity showing the difference of the two terms of the divisor.

131. *Illustrative Example.*—Divide a^3+b^3 by $a+b$; and a^3-b^3 by $a-b$.

OPERATIONS.

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2$$

$$\frac{a^3-b^3}{a-b}=a^2+ab+b^2$$

EXPLANATION.—By actual division, the quotient of a^3+b^3 by $a+b$ is found to be a^2-ab+b^2. Also,

By actual division, the quotient of a^3-b^3 by $a-b$ is found to be a^2+ab+b^2.

132. Hence, the following principle or law is derived,—

The quotient of the sum of the cubes of two quantities by the sum of the two quantities, is the square of the first term of the divisor, *minus the product of the two terms of the divisor, plus the square of the second term of the divisor.* Also,

The quotient of the difference of the cubes of two quantities by the difference of the two quantities, is the square of the first term of the divisor, plus the product of the two terms of the divisor, plus the square of the second term of the divisor.

EXAMPLES.

Find the quotient in each of the following, first by actual division, and then by inspection merely:—

154. $\frac{b^2 + 2bc + c^2}{b + c}$.

155. $\frac{x^2 - 2xy + y^2}{x - y}$.

156. $\frac{a^2 - 2a + 1}{a - 1}$.

157. $\frac{4a^2 + 8ac + 4c^2}{2a + 2c}$.

158. $\frac{9x^2 - 18xb + 9b^2}{3x - 3b}$.

159. $\frac{16a^2c^2 + 16acy + 4y^2}{4ac + 2y}$.

160. $\frac{b^2 - c^2}{b - c}$.

161. $\frac{a^2 - y^2}{a + y}$.

162. $\frac{4a^2 - 1}{2a - 1}$.

163. $\frac{9x^4 - 4y^2}{3x^2 + 2y}$.

164. $\frac{4a^4b^2 - 4}{2a^2b - 2}$.

165. $\frac{9x^4y^2 - 4z^2}{3x^2y + 2z}$.

166. $\frac{b^3 + c^3}{b + c}$.

167. $\frac{a^3 - x^3}{a - x}$.

168. $\frac{8c^3 + 1}{2c + 1}$.

169. $\frac{27x^3 - 8}{3x - 2}$.

170. $\frac{8a^3x^3 + y^3}{2ax + y}$.

171. $\frac{64a^6 - 8x^6}{4a^2 - 2x^2}$.

In examples 154–171, find the dividend from the divisor and the quotient.

133. *Illustrative Example.*—Divide $x^3 - y^3$ by $x + y$.

OPERATION.

$$
\begin{array}{l}
x + y)x^3 - y^3(x^2 - xy + y^2 \\
\underline{x^3 + x^2y} \\
\quad - x^2y - y^3 \\
\quad \underline{- x^2y - xy^2} \\
\qquad\quad xy^2 - y^3 \\
\qquad\quad \underline{xy^2 + y^3} \\
\qquad\qquad - 2y^3
\end{array}
$$

EXPLANATION.—The quotient is found to be $x^2 - xy + y^2$, with a remainder of $-2y^3$. The divisor may be written under this remainder, indicating an unperformed division, and this indicated quotient added to the quotient previously found, giving the entire quotient, $x^2 - xy + y^2 - \frac{2y^3}{x+y}$.

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EXAMPLES.

Find the quotient of the following expressions:—

172. $\frac{x^2+y^2}{x+y}$.
173. $\frac{a^2+x^2}{a-x}$.
174. $\frac{1-2a^2}{1+a}$.
175. $\frac{2c^2+7}{c-4}$.
176. $\frac{x^2+4}{x-2}$.
177. $\frac{a^2+b^2}{a+b}$.
178. $\frac{x^3-y^3}{x+y}$.
179. $\frac{c^3+d^3}{c-d}$.
180. $\frac{b^2+2bc+c^2}{b-c}$.
181. $\frac{x^2+xy-y^2}{x-y}$.
182. $\frac{a^2-4ab+b^2}{a-b}$.
183. $\frac{ab^3-ab^4+c}{ab^2}$.

In examples 172–183, produce the dividend from the divisor, the quotient, and the remainder.

ALGEBRAIC EXPRESSION.

Express in algebraic notation the following:—

1. a square over 6.
2. One-half of b, plus c.
3. Two y over a, minus b.
4. c cube over $a+b$.
5. Two-thirds of c, plus y.
6. Six a square over d.

7. If c oranges cost x cents, what is the cost of b oranges?

8. Four times a number, increased by a, equals y. What is the number?

9. A farmer bought a cows at x dollars apiece, and sold them at a gain of 10 dollars. What did he get for each?

10. If b represents the tens' figure of a number, and a the ones' figure, express one-half the number.

11. The area of an oblong is y, and the length is $b+2$. What is the width?

12. It cost c dollars to paint a fence x feet long and y feet high. What was the cost per square foot?

13. A box a feet long and c feet deep contains $2x$ cubic feet of sand. How wide is the box?

14. If a boy rides x miles in c hours, how far does he ride in three-fourths of an hour?

15. The divisor of a number is d, the quotient is c, and the remainder is b. What is the number?

16. The dividend is $3x$, the divisor is d, and the remainder is b. What is the quotient?

17. How many yards are in three times x inches, minus b inches?

18. If John can do a piece of work in c days and Frank can do it in d days, what part of the work can both do in one day working together?

19. The sum of two numbers is y, and their difference is a. What are the numbers?

20. At 4 per cent. a year, the interest on $2x$ dollars is 4 dollars. What is the time?

In exercises 1–20, let $a = 2$, $b = 3$, $c = 5$, $d = 6$, $x = 25$, and $y = 10$; find the numerical value of each.

EXAMPLES AND PROBLEMS.

1. $\frac{1}{2}x + 2 = \frac{1}{4}x + 4.$
2. $\frac{1}{4}x + 2 = \frac{3}{8}x + \frac{1}{2}.$
3. $\frac{1}{3}x + 4 = \frac{5}{6}x - 5.$
4. $\frac{5}{6}x - 10 = \frac{1}{3}x + 20.$
5. $\frac{1}{2}x + 10 = (3x - 20) \div 2.$
6. $(4x - 8) \div 4 = (2x + 12) \div 6.$
7. $\frac{1}{2}(x + 6) - 3 = \frac{3}{4}(x + 8) - 9.$
8. $15 - \frac{3}{4}(x - 8) = \frac{2}{3}(x + 9) - 2.$

In examples 1–8, use y instead of x; use z instead of x; use a; use b; substitute the value of each unknown used, and verify each equation.

9. If the sum of x and $2x$ be divided by 2, the quotient equals $\frac{1}{2}x$ minus $\frac{1}{3}x$ plus 16. What is the value of x?

10. One-half the cost of my watch increased by 20 per cent. of the cost, is \$35. What did my watch cost?

11. If Mary's age were increased by two-thirds of her age and by 5 years, the sum would be twice her age. How old is Mary?

12. A boy started on his bicycle 4 hours after his brother who rode 6 miles an hour, and overtook him in 6 hours. How fast did he ride?

13. If you multiply a certain number by 8 and add 6, and then divide this sum by 2 and subtract 7, the remainder is 20. What is the number?

14. The sum of two numbers is 24, and their difference is 8. What are the numbers?

15. Mary's age added to her mother's age is 40 years. If Mary is 20 years younger than her mother, how old is each?

16. Divide 60 into two such parts that three-fourths of the first part added to one-half of the second shall be 40.

17. A man and his son earned \$80. The son earned \$8 more than one-half as much as his father. What did each earn?

18. Fred paid \$1.30 for three books. The second cost two-thirds as much as the first, and the third cost three-fourths as much as the second. How much did each cost?

19. If three times Harry's age be increased by five-sixths of his age and by 2 years, the sum will be four times his age. How old is he?

20. A woman paid \$88 with five-dollar bills and two-dollar bills, giving three times as many two-dollar bills as five-dollar bills. How many bills of each kind did she pay?

21. A man walked from his home at the rate of 3 miles an hour, and rode back at the rate of 6 miles an hour. If he was gone 6 hours, how far did he go?

22. Frank sold $\frac{2}{3}$ of his pigeons, and then he had 8 less than he had sold. How much were they all worth at 25 cents apiece?

23. Harry sold 12 kites for \$2. For some he received 20 cents each, and for the others 12 cents each. How many did he sell at each price?

24. James gave John 15 marbles, and then John had six times as many as James. If both had 35 marbles at first, how many had each?

25. A, B, and C bought a boat. A paid $\frac{1}{3}$ of the price, B $\frac{1}{4}$ of the price, and C \$50. What did the boat cost?

26. Two girls bought a bicycle. Emma paid \$10 less than Ethel. If Emma paid \$2 every time Ethel paid \$3, how much did each pay altogether?

27. By selling some stock for \$600, I gained 20 per cent. of the cost. What did I pay for the stock?

28. A father divided 60 cents among three children, so that the second child received two-thirds as much as the first, and the third one-half as much as the second. How much did each receive?

29. A man invested a sum of money at 5 per cent., and one-half as much at 6 per cent. His income from the first was \$80 more than his income from the second. How much did he invest at each rate?

30. I borrowed a sum of money at 5%. In ten years the interest amounted to \$200 less than the sum borrowed. How much did I borrow?

In problems 9–30, let x represent the unknown; let y; let z; substitute the value of the unknown used, and verify each solution.

FACTORING.

1. What is the product of 3 and 5? Of a and b? Of $a + b$ and a? How is the product of two numbers or quantities found?

2. In the expression $(x + y) \times 3a$, which is the multiplicand? Which is the multiplier? What is the product?

3. What numbers multiplied together produce 21? What numbers produce 26? What quantities produce $ax - ay$? What are the numbers or quantities multiplied together called?

134. The quantities multiplied together to produce a quantity are called the **factors** of that quantity.

Thus, 3 and 2 are the factors of 6; a and x are the factors of ax; b, b, and b are the factors of b^3; a and $a - y$ are the factors of $a^2 - ay$; etc.

The multiplicand and the multiplier are the factors of the product.

135. A quantity which can not be produced by any other factors than itself and 1, is called a **prime quantity.**

Thus, 5 is a prime number because it can be produced only by multiplying together 5 and 1; a is a prime quantity; so, also, are 7, x, $7 + x$; etc.

136. A quantity which can be produced by other factors than itself and 1, is called a **composite quantity.**

Thus, 6 is a composite number, because it is produced by the factors 3 and 2; ab is a composite quantity produced by the factors a and b; so, also, are a^2, $2ab$, b^2, $bx - x^2$; etc.

137. Prime quantities used to produce a composite quantity, are the **prime factors** of that quantity.

Thus, 2, 2, and 3 are the prime factors of 12; x and y are the prime factors of xy; 3, 2, a, and x, of $6ax$; x and $x + y$, of $x^2 + xy$; etc.

138. Since a composite quantity is the product of two or more factors, it can be separated into the prime factors which produce it.

Thus, 30 can be separated into its prime factors 2, 3, and 5; a^2, into a and a; $4x^2y$, into 2, 2, x, x, and y.

139. A factor of any number or quantity is an **exact divisor** of that number or quantity.

Thus, since 15 is the product of 3 and some other factor, that other factor is found by dividing 15 by 3; that is, 3 is an exact divisor, as well as a factor, of 15.

140. If the product of two factors is known, and one of the factors is known, the other factor is found by dividing the product of the two factors by the factor that is known.

Thus, since 42 is the product of at least two factors, one of which is 2, the other is found by dividing 42 by 2, which gives 21, the other factor; but 21 is itself the product of two factors, one of which is 3, and the other is 21 ÷ 3, or 7. Hence, all the prime factors of 42 are 2, 3, and 7.

$4a^2$ is the product of 4 and a^2, each of which is composite and can be separated into prime factors, giving 2, 2, a, and a as all the prime factors of $4a^2$; etc.

141. The process of finding the factors of a composite number or quantity is called **factoring**.

To Find Prime and Monomial Factors.

142. MULTIPLYING.

$2 \times 5 \times a \times x \times x = 10ax^2.$

FACTORING.

$10ax^2 = 2 \times 5 \times a \times x \times x.$

EXPLANATION 1.—The product of the prime factors 2, 5, a, x, and x is found by multiplying them together. Hence, the product $10ax^2$ is a composite quantity.

EXPLANATION 2.—In the composite quantity $10ax^2$, 10 is the product of the prime factors 2 and 5; a is a prime factor; and x^2 is the product of x and x. Hence, all the prime factors of $10ax^2$ are 2, 5, a, x, and x.

Find the prime factors of,—

1. 8.	4. ab.	7. ab^2c.	10. $12yz^3$.	13. $-18r^2s^3$.
2. 27.	5. b^2x.	8. $10mn^3$.	11. $-9a^2x$.	14. $-20y^3z^2$.
3. 45.	6. $5cd^2$.	9. $14x^3y^2$.	12. $-14az^3$.	15. $-27b^2y^3$.

In exercises 1–15, after finding the prime factors, multiply them together to produce the given composite quantities.

143. MULTIPLYING.

$4a^3 \times 4a^3 = 16a^6$

FACTORING.

$16a^6 = 4 \times 4 \times a^3 \times a^3$

EXPLANATION 1.—The product of 4 used twice as a factor, or the square of 4, is 16; the square of a^3 is a^6; and the entire square of $4a^3$ is $16a^6$.

EXPLANATION 2.—One of the two equal factors of 16 is 4; one of the two equal [illegible]s of a^6 is a^3; and the two equal factors of $16a^6$ are $4a^3$ and $4a^3$.

Find one of the equal factors of,—

16. a^2b^2.	19. $16b^2c^4$.	22. $\frac{1}{16}x^6z^8$.	25. 8.	28. $(x-y)^3$.
17. $4x^2y^2$.	20. $25a^4b^6$.	23. $36a^2x^6$.	26. 27.	29. $(a+x)^3$.
18. $9a^4b^2$.	21. $\frac{1}{9}a^2x^4$.	24. $9(a+b)^2$.	27. a^3b^3.	30. $27a^3b^9$.

In exercises 16–30, from the equal factors, produce the given squares and cubes.

144. Multiplying.

$$\begin{array}{l} 2a - 3x \\ \underline{4a^2b} \\ 8a^3b - 12a^2bx \end{array}$$

Factoring.

$$4a^2b)\underline{8a^3b - 12a^2bx}$$
$$\quad\quad 2a - 3x$$

Explanation 1.—Each term of the factor $2a - 3x$, multiplied by the monomial $4a^2b$, gives the entire product $8a^3b - 12a^2bx$.

Explanation 2.—The composite quantity $8a^3b - 12a^2bx$ is the product of two or more factors. 4 is the highest factor of both coefficients, and a^2 and b are the highest factors of the literal parts of both terms. Hence, the highest monomial factor is $4a^2b$; and, as a factor is also an exact divisor, the other factor is found by dividing the given quantity by $4a^2b$. Therefore, the two factors are $4a^2b$ and $2a - 3x$.

EXAMPLES.

Find the highest monomial factor, and the other factor of the following quantities:—

31. $ab + a$.	34. $2x^2 - 4x$.	37. $5ab - 25ax$.
32. $ab + b$.	35. $3a + 3ax$.	38. $2x^2y + xy^2$.
33. $4c + 4$.	36. $4a^3 - 8a$.	39. $6x^2 - 9ax$.

40. $8a^2b - 12a^2c$.	48. $7b^3 + 14b^4 - 21b^2$.
41. $7bc^2 - 14b^2cd$.	49. $6b - 12b^2c^2 + 18b^3c$.
42. $5m^2n^2 - 15mn^3$.	50. $4x^2 + 8xy - 12y^2$.
43. $6x^3y^2 - 12x^2y^3$.	51. $6ax^2 + 8a^2x^3 + 16a^3x^4$.
44. $18a^5x - 18a^3$.	52. $2xy + 3y^2 + y^3$.
45. $a(n+1) - b(n+1)$.	53. $6a^2 - 30ab + 12ad$.
46. $10 + 15x - 20x^2$.	54. $4b^2c + bcd - 2bce$.
47. $3x^4 - 9x^3 - 6x^2$.	55. $7a^2c - 14abc + 7b^2c$.

56. $xy^2z^3 + x^2y^3z^4 - x^3y^4z^5$.

57. $a^3bc^2 - a^2b^2c^2 - a^2bc^3$.

58. $x^2yz^3 + xz^2 - x^3y^2z$.

59. $\frac{1}{8}ab - \frac{1}{2}bc + \frac{1}{4}bcd$.

60. $\frac{1}{2}x^2y^5 - \frac{1}{3}x^5y^4z + \frac{1}{6}x^4y^2z^2$.

In examples 31–60, from the factors found, produce the quantities given.

To Find two Equal Binomial Factors of a Trinomial.

145. Multiplying.

$$\begin{array}{l} a - 2b \\ \underline{a - 2b} \\ a^2 - 2ab \\ \underline{\quad - 2ab + 4b^2} \\ a^2 - 4ab + 4b^2 \end{array}$$

Explanation 1.—Multiplying a binomial by itself, or squaring a binomial, produces the following results (**113-114**):—

1. The product is a trinomial.

2. The first term of the product is the square of the first term of the binomial.

3. The third term of the product is the square of the second term of the binomial.

4. The middle term of the product is twice the product of the two terms of the binomials.

If the second term of the binomial is negative, the second term of the product is negative, and the two other terms are positive.

If the second term of the binomial is positive, all the terms of the product are positive.

Factoring.

$a^2 - 4ab + 4b^2 =$
$(a - 2b)(a - 2b)$.

Explanation 2.—In the trinomial $a^2 - 4ab + 4b^2$, the first term is the square of a, the third term is the square of $2b$, and the second term is twice the product of a and $2b$. As the second term of the trinomial is negative, the second term of the binomial is negative. Hence, the two equal factors of $a^2 - 4ab + b^2$ are $a - 2b$ and $a - 2b$.

EXAMPLES.

Find the two equal factors of the following:—

61. $a^2 + 2ab + b^2$.

62. $b^2 - 2bc + c^2$.

63. $x^2 + 2x + 1$.

64. $1 - 2x + x^2$.

65. $4c^2 - 12c + 9$.

66. $4x^2 + 12xy + 9y^2$.

67. $9n^2 - 12nx + 4x^2$.

68. $a^2 + 4a + 4$.

69. $9 + 6c + c^2$.
70. $4y^2 + 8y + 4$.
71. $9a^2 - 24a + 16$.
72. $9c^2 - 18cx + 9x^2$.
73. $4s^4 + 4s^2r + r^2$.
74. $49 - 14ab + a^2b^2$.
75. $9x^4 - 18x^2 + 9$.
76. $a^2 - 6a + 9$.
77. $9a^4 + 6a^2b + b^2$.
78. $4m^4 + 4m^2n + n^2$.
79. $25x^2 - 10x + 1$.
80. $1 - 10x + 25x^2$.
81. $9b^2c^2 - 12bc + 4$.
82. $r^2 - 2rs + s^2$.
83. $9y^2 - 6yz + z^2$.
84. $1 + 6x + 9x^2$.
85. $4b^2 - 12b + 9$.
86. $a^4b^4 + 2a^2b^2c^2 + c^4$.
87. $4 - 12a^2b + 9a^4b^2$.
88. $1 - 4a^2b^3 + 4a^4b^6$.
89. $\frac{1}{4}a^2 - ab + b^2$.
90. $x^2 + \frac{2}{3}x + \frac{1}{9}$.

In examples 61–90, from the equal factors found, produce the given trinomials.

To Find two Binomial Factors of the Difference of two Squares.

146. Multiplying.

$$\begin{array}{l} a + 3x \\ a - 3x \\ \hline a^2 + 3ax \\ \quad - 3ax - 9x^2 \\ \hline a^2 \qquad\quad - 9x^2 \end{array}$$

Explanation 1.—Multiplying the sum of two quantities by the difference of the quantities, produces the following results (**115**):—

1. The product is a binomial.
2. The first term of the product is the square of the first term of the binomial.
3. The second term of the product is the square of the second term of the binomial.
4. The first term of the product is positive and the second term is negative.

Factoring.

$a^2 - 9x^2 =$
$(a + 3x)(a - 3x)$

Explanation 2.—The quantity $a^2 - 9x^2$ expresses the difference of the squares of two quantities; the first term is the square of a, and the second term is the square of $3x$. Hence, one binomial factor is the sum of a and $3x$, or $a + 3x$, and the other binomial factor is the difference of a and $3x$, or $a - 3x$.

EXAMPLES.

Find the two binomial factors of the following:—

91. $a^2 - x^2$.
92. $b^2 - c^2$.
93. $m^2 - n^2$.
94. $1 - z^2$.
95. $9m^2 - 16n^2$.
96. $x^2 - 1$.

97. $c^2 - 36t^2$.	105. $49x^2f^2 - 9$.	113. $m^2n^2 - \frac{1}{4}$.
98. $4x^2 - 1$.	106. $1 - 25x^2y^4$.	114. $\frac{1}{9}x^2 - \frac{1}{16}x^2y^2$.
99. $4c^2 - 9x^2$.	107. $81x^4z^2 - 1$.	115. $\frac{1}{16}m^2 - \frac{1}{25}y^2$.
100. $1 - 36a^2y^2$.	108. $3^2 - a^2b^2$.	116. $(a + x)^2 - a^2$.
101. $9x^2 - 64f^2$.	109. $4^2 - 3^2$.	117. $4 - (b + c)^2$.
102. $1 - 81x^2y^2$.	110. $\frac{1}{4}x^2 - \frac{1}{9}y^2$.	118. $1 - a^4$.
103. $4 - 9a^2x^2$.	111. $5^2 - 3^2$.	119. $a^4 - b^4$.
104. $9a^2 - c^2d^2$.	112. $\frac{1}{9} - a^2b^2c^4$.	120. $x^4 - y^4z^6$.

In examples 91–120, from the binomial factors found, produce the given quantities.

To Find two Unequal Binomial Factors of a Trinomial.

147. Multiplying.

$$\begin{array}{l} x + 2 \\ x - 3 \\ \hline x^2 + 2x \\ \quad - 3x - 6 \\ \hline x^2 - x - 6 \end{array}$$

Explanation 1.—Multiplying a binomial by another binomial when they have the first term common to both factors, produces the following results (116):—

1. The product is a trinomial.

2. The first term of the product is the square of the term common to the two factors.

3. The third term is the product of the unlike terms of the two factors.

4. The middle term is the product of the algebraic sum of the unlike terms by the common term.

Factoring.

$x^2 - x - 6 = (x + 2)(x - 3)$

Explanation 2.—In the trinomial $x^2 - x - 6$, x^2 is the square of x which must be the common term; -6 is the product of two factors, -1×6, or 1×-6, or -3×2, or 3×-2; and since the algebraic sum of these two terms multiplied by x gives $-1x$ or $-x$, the algebraic sum of the unlike terms must be -1, or the greater of these unlike terms must be -3 and the other must be 2. Hence, the two binomial factors are $x + 2$ and $x - 3$.

EXAMPLES.

Find the unequal binomial factors of the following trinomials:

121. $a^2 + 3a + 2$.	123. $y^2 + 9y + 20$.
122. $b^2 + 7b + 12$.	124. $a^2 - a - 2$.

125. $a^2 + a - 2$.
126. $x^2 + 5x + 6$.
127. $x^2 - 5x + 6$.
128. $x^2 + x - 6$.
129. $x^2 - x - 6$.
130. $x^2 - 9x + 18$.
131. $x^2 - 3xy + 2y^2$.
132. $4b^2 + 10b + 6$.
133. $4b^2 - 10b + 6$.
134. $4b^2 + 2b - 6$.
135. $4b^2 - 2b - 6$.
136. $9a^2 + 15a + 6$.
137. $9r^2 - 15r + 6$.
138. $a^2x^2 + 3ax - 4$.
139. $a^2x^2 - 3ax - 4$.
140. $a^4b^4 - a^2b^2 - 6$.
141. $16s^2 + 4s - 6$.
142. $16s^2 - 4s - 6$.
143. $9a^2 - 6a - 15$.
144. $9x^2 + 6x - 15$.
145. $9y^2 + 27y + 20$.
146. $9z^2 + 36z + 20$.
147. $4b^4 + 14b^2 + 12$.
148. $4b^4 - 14b^2 + 12$.
149. $c^2 - 6cd - 16d^2$.
150. $d^2 + 6de - 16e^2$.
151. $9x^2y^2 + 3xy - 20$.
152. $9a^2b^2 - 3ab - 20$.
153. $x^2 + ax - 12a^2$.
154. $y^2 - ay - 12a^2$.
155. $16a^4b^2 - 20a^2b - 6$.
156. $16b^4c^2 - 4b^2c - 6$.
157. $\frac{1}{4}a^2 + 5a + 24$.
158. $\frac{1}{9}m^2 - m - 18$.
159. $\frac{1}{16}y^2 - 3y + 35$.
160. $\frac{1}{16}z^2 + \frac{1}{2}z - 35$.

In examples 121–160, from the unequal binomial factors found, produce the given quantities.

To Find the two Factors of the Sum or the Difference of two Cubes.

148. The sum of the cubes of two quantities is divisible by the sum of the two quantities.

Thus, $a^3 + b^3$ can be divided by $a + b$, since $a^3 + b^3$ is the product of $a^2 - ab + b^2$ and $a + b$ (**131–132**); and, if $a + b$ is a factor of $a^3 + b^3$, it is also an exact divisor of $a^3 + b^3$.

149. The difference of the cubes of two quantities is divisible by the difference of the two quantities.

Thus, $a^3 - b^3$ can be divided by $a - b$, since $a^3 - b^3$ is the product of

$a^2 + ab + b^2$ and $a - b$(**125**); and, if $a - b$ is a factor of $a^3 - b^3$, it is also an exact divisor of $a^3 - b^3$.

150. *Illustrative Example.*—Find the two factors of $x^3 + y^3$.

OPERATION.

$$\begin{array}{l} x + y)x^3 + y^3(x^2 - xy + y^2 \\ \quad\ \ \underline{x^3 + x^2y} \\ \quad\quad - x^2y + y^3 \\ \quad\quad \underline{- x^2y - xy^2} \\ \quad\quad\quad\quad xy^2 + y^3 \\ \quad\quad\quad\quad \underline{xy^2 + y^3} \end{array}$$

EXPLANATION. — Since $x^3 + y^3$, the sum of the cubes of two quantities, is divisible by the sum of the two quantities (**148**), $x + y$, the exact divisor of $x^3 + y^3$ is also a factor of $x^3 + y^3$, and the other factor is found by dividing $x^3 + y^3$ by $x + y$. Hence, the two factors of $x^3 + y^3$ are $x + y$ and $x^2 - xy + y^2$.

151. From the preceding, it is clear that—

The sum of the cubes of two quantities can be separated into two factors, one of which is the sum of the two quantities, and the other is the square of the first quantity, minus the product of the first by the second, plus the square of the second quantity. Also,

The difference of the cubes of two quantities can be separated into two factors, one of which is the difference of the two quantities, and the other is the square of the first quantity, plus the product of the first by the second, plus the square of the second quantity.

EXAMPLES.

161. $a^3 + b^3$.
162. $x^3 + z^3$.
163. $b^3 - c^3$.
164. $c^3 - y^3$.
165. $1 + c^3$.
166. $x^3 - 1$.
167. $r^3 - s^3$.
168. $s^3 + r^3$.
169. $1 - a^3b^3$.
170. $x^3y^3 + z^3$.
171. $a^3x^3 + 1$.
172. $1 - b^3c^3$.
173. $b^3c^3 - x^3$.
174. $y^3z^3 + a^3b^3$.
175. $a^3b^3 - 8$.
176. $1 + 8a^3n^3$.
177. $r^3s^3 - 27$.
178. $y^3 + m^3n^3$.
179. $y^3z^3 - 8a^3$.
180. $a^3b^3 + x^3y^3$.
181. $8a^3 - b^3c^3$.
182. $a^3 + 27x^3$.
183. $27 - 8b^3c^3$.
184. $x^3y^3 + 64a^3$.
185. $125 - a^3z^3$.
186. $m^3n^3 + 64x^3$.
187. $216 - d^3e^3$.
188. $8x^3y^3 - 1$.
189. $\frac{1}{8}a^3b^3 - 64s^3$.
190. $27c^3d^3 + \frac{1}{64}z^3$.

In examples 161–190, from the factors found, produce the sum or the difference of the cubes given.

COMMON FACTORS.

1. What prime factors produce the quantity ab^2? The quantity $2b^2c$?

2. What are the prime factors of $3a^2b$? Of $6ab^2$? What prime factors are common to these quantities?

3. What prime factors are common to $4xy^2$ and $6x^3y$? What is the product of these common prime factors?

What is the highest factor common to the two coefficients? What is the highest degree or power of each letter in each quantity? What is the highest factor common to both quantities?

152. A **common factor** of two or more quantities is a quantity that exactly divides each of them.

Thus, 2 is a factor of 6 and also of 10, and, hence, is a common factor of 6 and 10; also, since a factor of a quantity is an exact divisor of that quantity (**139**), 2 exactly divides 6 and 10. 2 is also a common factor of 4 and 8; 10 and 12; 8, 14, and 20; etc.

$3a^2$ is a common factor of $6a^3$ and $9a^2$, and, therefore, exactly divides each of these quantities.

153. The **highest common factor** of two or more quantities is the quantity of highest degree that exactly divides each of them.

Thus, $4a^2xy^2$ is the highest common factor of $8a^3xy^3$ and $12a^2x^3y^2$; $2a^2b$ is the highest common factor of $4a^2b^3$ and $6a^3b + 8a^2b^4$; etc.

The highest common factor is the factor of the highest degree common to the given quantities, since it is the product of all the prime factors common to these quantities, and of no other factor.

Thus, the highest common factor of $8x^2y^3$ and $12x^3y^4z$ is $4x^2y^3$, since it can contain no higher factor of the numerical coefficients than 4, the product of their common prime factors; no higher power or degree of x than x^2; no higher degree of y than y^3; and it can not contain z which is found in only one of the given quantities.

Every letter common to the different quantities is used, each to the highest degree found in all of the quantities given.

154. Since a factor of a quantity is an *exact divisor* of that quantity, a common factor of two or more quantities is a *common divisor* of those quantities; and the highest common factor is the *highest common divisor.*

155. From the preceding illustrations and results, it is clear that,—

The highest common factor of two or more quantities contains all of the common prime factors of those quantities, but no other factors.

156. *Illustrative Example.*—Find the highest common factor of $12b^2x^3z$ and $20b^3x^2$.

OPERATION.

$$12b^2x^3z = 4 \times 3 \times b^2 \times x^2 \times x \times z$$
$$20b^3x^2 \;\;= 4 \times 5 \times b^2 \times b \;\times x^2$$
$$\text{H. c. f.} = 4 \times b^2 \times x^2 = 4b^2x^2$$

EXPLANATION.—The highest factors common to the given quantities are 4, b^2, and x^2, and the product of these factors, formed of all the prime common factors, is $4b^2x^2$, the highest common factor required. Or,

By inspection, 4 is the h. c. f. of the numerical coefficients; and the h. c. f. of the literal parts must contain b^2 and x^2, the highest degrees or powers of these letters common to the given quantities; and it can contain no other factor. Hence, the highest common factor is $4b^2x^2$.

The remaining factors 3, 5, b, x, and z are not common to the given quantities, and, therefore, do not affect the h. c. f.

It is not necessary always to separate the given quantities into prime factors; but if composite factors are used, they must be common to all the quantities.

EXAMPLES.

Find the highest common factors of,—

1. a, a^2b.
2. a^2x^3, a^3x^2.
3. b^2, $3b^3$, $5b^2$.
4. $4x^2y$, $8xy^3$, $6x$.
5. 5, $10bx^2$, $15b^2x$.
6. $8x^2$, $6x^3y^2$, $4x^2y$.
7. $15a^2c^3y$, $20a^3c^2y^2$.
8. $2a^4b$, $8a^2b^3$, $6ab^2$.
9. $21x^2y^3z^4$, $35x^3y^2z^3$.
10. $4x^2y^2$, $8x^4y^3$, $6x^2y^4$.
11. $5d^2e^3$, $10ad^3e^4$, $15d^2e^2$.
12. $7mx^3$, $14m^3x^2$, $21m^3xy$.
13. $18cr^2$, $6ac^4r^3$, $12c^2r^4$.
14. $a^2d^4x^3$, $2a^3d^2x^4$, $3a^2d^3x^2$.

15. $12c^2y^3$, $6c^3y^4$, $18c^4y^3$.

16. $4a^2b^3$, $6ab^4(x-1)$, $8ab^2$.

17. $8x^2(a-b)$, $12x^3(a-b)$.

18. $6ac^2$, $3a^2c+9c^3e-6c^2x$.

19. $2r^2s^3-6r^3s^4+4rs^2$, $8r^4s^2$.

20. $3a^2z^3$, $6az^2-9a^3z^4+a^2z^2$.

In examples 1–20, divide each of the given quantities by the highest common factor.

157. *Illustrative Example.*—Find the highest common factor of $a(x^2-1)$ and $2a(x^2-x-2)$.

OPERATION.

$$a(x^2-1) = a \times (x+1)(x-1)$$
$$2a(x^2-x-2) = 2 \times a \times (x+1)(x-2)$$
$$\text{H. c. f.} = a \times (x+1) = a(x+1).$$

EXPLANATION.—On separating the given quantities into their prime factors, a and $x+1$ are found to be the only prime common factors. Hence, their product $a(x+1)$ is the highest common factor required.

EXAMPLES.

Find the highest common factor of the following:—

21. a^2-b^2, $a^2+2ab+b^2$.

22. $a^2-2ax+x^2$, a^2-x^2.

23. $1-c^2$, $1-2c+c^2$.

24. x^2+4x+4, x^2-4.

25. $xy-y$, x^2-1.

26. x^2+xy, $4x(x+y)$.

27. a^2-2a^3, $2ab-4a^2b$.

28. $3a^2+3ax$, a^2-x^2.

29. x^2-2x+1, x^2-1.

30. a^2-1, a^2-2a+1.

31. x^2-1, x^3+1.

32. $1-x^3$, $1-x^2$.

33. x^3-1, x^2-1.

34. $3s^2(a+1)$, $6s^2(a^2-1)$.

35. m^3-n^3, m^2-n^2.

36. x^2-y^2, x^4-y^4.

37. b^4-c^4, b^2-c^2.

38. ab^3-a^4, b^2-a^2.

39. $a-b$, a^2-b^2, $(a-b)^2$.

40. a^3+b^3, a^2-ab+b^2.

41. a^2+ax+x^2, a^3-x^3.

42. a^3+b^3, $5ac^2+5bc^2$.

43. x^2-9, x^2-6x+9.

44. $(x+y)^2$, x^3+y^3.

45. a^2-4, a^2+6a+8.

46. a^2+2a-3, a^2-9.

47. x^2-2x-8, x^2-16.

48. r^2-4, r^2+r-6.

49. n^2-n-2, n^2-1.

50. $n+2$, n^2+n-2.

In examples 21–50, divide each quantity by the h. c. f.

COMMON MULTIPLES.

1. Name the product of $2a$ multiplied by some other factor. Name several products of which $2a$ is one of the factors.

2. What product is common to $3a$ and $5b$ used as factors? How is it obtained from these quantities or factors?

3. What are the prime factors of $4a^2b$? Of $6ab^3$? What prime factors are found in both of these quantities?

What is the highest degree or power of each letter in each quantity? What is the least or lowest product common to these two quantities obtained from their prime factors?

158. The product obtained by taking a quantity any number of times, is called a **multiple** of that quantity.

Thus, 12 is a multiple of 4, because 12 is the product of 4 taken 3 times; 20 is a multiple of 4; so, also, are 24, 40, 60, 80, 100, etc.

$3ab$ is a multiple of 3, a, and b, because it is the product of these factors, or of any of them taken as many times as the other factors indicate.

Since a multiple is the product of two or more quantities, it is also a *dividend* exactly divisible by each of them as a factor (**139**).

159. A **common multiple** of two or more quantities is a quantity exactly divisible by each of them.

Thus, 24 is a common multiple of 4 and 6, because it is a multiple of each of them; and, since it is the product of 4 by some factor, and of 6 by some factor, it is divisible by 4 and by 6. 24 is, also, a common multiple of 3 and 8; of 2 and 12; and of 2, 2, 2, 3, the prime factors of 4 and 6, 3 and 8, 2 and 12.

$6a^2b^3$ is a common multiple of 6, a^2, and b^3, or of their prime factors, and is, therefore, divisible by each of these quantities.

160. The **lowest common multiple** of two or more quantities is the quantity of lowest degree that is exactly divisible by each of them.

Thus, $12a^2b^3$ is the lowest common multiple of $4a^2b^3$ and $6ab^2$; $4a^4 - 4a^2x^2$ is the lowest common multiple of $4a^2$, $a - x$, and $a + x$; etc.

The lowest common multiple is the multiple of lowest degree common to the given quantities, since it is the product of the different prime factors found in any of the quantities.

Thus, the lowest common multiple of $4ab^2$, $3a^2b^3c$, and $6bc^4x$, is $12a^2b^3c^4$, since it can contain no lower common multiple of 4, 3, and 6 than 12; no lower degree or power of a than a^2; no lower degree of b than b^3; no lower degree of c than c^4; and it must contain x, found in one of the given quantities.

Every letter in the different quantities is used, each to the highest degree or power found in any of the quantities given.

161. Since a multiple of two or more quantities is a *dividend* exactly divisible by each of them (**158**), a common multiple of two or more quantities is a *common dividend* of those quantities, and the lowest common multiple is the *lowest common dividend.*

162. From the preceding illustrations and results, it is clear that,

The lowest common multiple of two or more quantities contains all the different prime factors of those quantities, each to the highest degree used in any of the quantities.

163. *Illustrative Example.*—Find the lowest common multiple of $12x^3yz^2$ and $18x^2z^3$.

OPERATION.

$$12x^3yz^2 = 6 \times 2 \times x^2 \times x \times y \times z^2$$
$$18x^2z^3 = 6 \times 3 \times x^2 \times z^2 \times z$$
$$\text{L. c. m.} = 6 \times 2 \times 3 \times x^2 \times x \times y \times z^2 \times z = 36x^3yz^3.$$

EXPLANATION.—The highest factors common to the two quantities are 6, x^2, and z^2; and the other factors not common, are 2, 3, x, y, and z, which must help to produce the l. c. m. Hence, the product of all the different factors of the given quantities is $36x^3yz^3$, the lowest common multiple required. Or,

By inspection, 36 is the l. c. m. of the numerical coefficients; and the l. c. m. of the literal parts must contain x^3 and z^3, the highest degrees or powers of these letters in either of the given quantities, and, also, y, found in one of the quantities. Hence, the lowest common multiple is $36x^3yz^3$.

It is not necessary to find all the prime factors of the given quantities; but if composite factors are used, they must be common to all of the quantities.

EXAMPLES.

Find the lowest common multiple of the following:—

1. $a, b.$
2. $2a, 6a^2b.$
3. $5x^2, 10ax^3.$
4. $4b^3y, 6by^3z.$
5. $12a^2bc^3, 16a^3b^2c.$
6. $15x^2yz^4, 15x^3y^2z.$
7. $18ac^4e^2, 12a^2c^3e.$
8. $10, 12xy^4, 15x^3y^2.$
9. $3cf, 6c^3f^2, 15c^2f^4s.$
10. $10x^3y, 15x^2z^3, 20xy^2z^4.$
11. $ab^4x^2, 12, 8a^2bx^3.$
12. $ab, bc, cd, ad.$
13. $15rs^4, 10s^2x, 20r^3s^2.$
14. $3ax^4, 4a^2b^3, 5a^3b^2x.$
15. $5c^5, 7b, 14b^3, 20c^2d.$
16. $12c^5d, 18c^3d^4x, 24cd^2.$
17. $8, a^2x^3, 16a^4x^2, 12ay^2.$
18. $10xy^2, 5x^3m^2, 15my^5.$
19. $6ax^3, 15a^2x(1+a).$
20. $2a, x+1, 3, x-1.$

In examples 1–20, prove that the lowest common multiple is exactly divisible by each of the given quantities.

164. *Illustrative Example.*—Find the lowest common multiple of $a^2 - x^2$ and $a^2 - 2ax + x^2$.

OPERATION.

$$\begin{aligned} a^2 - x^2 &= (a-x)(a+x) \\ a^2 - 2ax + x^2 &= (a-x)(a-x) \\ \hline \text{L. c. m.} &= (a-x)(a+x)(a-x) \\ &= a^3 - a^2x - ax^2 + x^3 \end{aligned}$$

EXPLANATION.—Each of the given quantities is separated into its prime factors.

The product of these different prime factors, $a - x$, $a - x$, and $a + x$, each taken as many times as it is found in any of the given quantities, is $a^3 - a^2x - ax^2 + x^3$, the lowest common multiple required.

EXAMPLES.

Find the lowest common multiple of the following:—

21. $x^2, x^2 + x.$
22. $2a^2, 3a^2 - 6ax.$
23. $4b^2, b^2 + b.$
24. $5y, y - 3, 3xy.$
25. $a + x, a^2 - x^2.$
26. $x^4 - y^4, x^2 - y^2.$
27. $a^2 + ab, ab + b^2.$
28. $a(x - y), bc(x - y).$
29. $a^2 + a, a^2 - 1.$
30. $b^2 - 1, b^2 - b.$

31. $ab(b + c)$, $ab^2 - abc$.
32. $2m(a + x)$, $4m^2(a - x)$.
33. $b^2 + bx$, $b^2 - x^2$.
34. $(1 + x)^2$, $a + ax$.
35. a, $a + 1$, $a - 1$.
36. $4a - 4b$, $3(a - b)^2$.
37. $4(x^4 - y^4)$, $6(x^2 + y^2)$.
38. $2x - 2b$, $5cx^2 - 5b^2c$.
39. $a + ax$, $(1 + x)^2$.
40. $2(x - b)$, $5c(x^2 - b^2)$.
41. $(6a^2x - 6a^2y)$, $4ab(x^2 - y^2)$.
42. $a(3x - 1)$, $ab(9x^2 - 1)$.
43. $4x^2(1 - 4a^2)$, $2x(1 - 2a)$.
44. $a + b$, $a - b$, $a^2 - b^2$.
45. $1 - x$, $1 - x^2$, $1 + x$.
46. $a + b$, $a^3 + b^3$.
47. $c^3 - 1$, $c - 1$.
48. $a + x$, $a^2 + 2ax + x^2$.
49. $x^2 - 2x + 1$, $x - 1$.
50. $1 + 2x + x^2$, $1 - x^2$.
51. x, $x^2 - 1$, $x^3 - 1$.
52. $a + 1$, $a^2 - 2a - 3$.
53. $x^2 - 3x - 10$, $x + 2$.
54. $a^2 - 4$, $a^2 + a - 6$.

In examples 21–54, divide the lowest common multiple by each of the given quantities.

In these examples, find, also, the highest common factor of the given quantities that are not prime to each other.

SOME APPLICATIONS OF COMMON FACTORS AND COMMON MULTIPLES.

I. To change fractions to higher terms.

165. OPERATIONS.

(a) $24 \div 6 = 4$.
(b) $(24 \times 2) \div 6 = 8$.
(c) $24 \div (6 \times 2) = 2$.
(d) $(24 \times 2) \div (6 \times 2) = 4$.

EXPLANATION 1.—The quotient of 24 divided by 6 is 4 (*a*).

2.—If the dividend 24 be multiplied by any factor, as 2, the quotient is multiplied by that factor (*a*) and (*b*).

3.—If the divisor 6 be multiplied by any factor, as 2, the quotient is divided by that factor (*a*) and (*c*).

4.—If both dividend and divisor be multiplied by the same factor, as 2, the quotient is not changed (*a*) and (*d*).

The numerator of a fraction may be regarded as a dividend and the denominator as a divisor.

Thus, $\frac{2}{3}$ may be regarded as $\frac{1}{3}$ of 1, in which case 1 is divided into 3 equal parts and 2 of these equal parts are used; or, it may be regarded as 2 divided into 3 equal parts, or as $2 \div 3$.

In $\frac{a^2 + a - 6}{a - 2}$, the numerator is a dividend, and the denominator is a divisor, to find the quotient, or the value of the fractional expression.

(e) $\frac{6}{24} = \frac{1}{4}$.

(f) $\frac{6 \times 2}{24 \times 2} = \frac{12}{48}$, or $\frac{1}{4}$.

EXPLANATION 5.—The value of $\frac{6}{24}$ is $\frac{1}{4}$ (e).

6.—If the denominator (divisor) and the numerator (dividend) be both multiplied by the same factor, as 2, the value of the fraction is not changed, since the value of $\frac{6}{24}$, and of $\frac{12}{48}$, is $\frac{1}{4}$ (d), (e), and (f). Hence,

A fraction may be changed to higher terms without changing the value of the fraction, by multiplying both denominator and numerator by the same factor.

166. *Illustrative Example.*—Change $\frac{6a^2b}{8ab^2}$ to a fraction whose denominator shall be $16a^2b^4c^3$.

OPERATION.

$$\frac{6a^2b \times 2ab^2c^3}{8ab^2 \times 2ab^2c^3} = \frac{12a^3b^3c^3}{16a^2b^4c^3}.$$

EXPLANATION.—The denominator $8ab^2$ of the given fraction must be multiplied by the factor $2ab^2c^3$ to produce the higher denominator required, $16a^2b^4c^3$. Hence, if the numerator $6a^2b$ of the given fraction be also multiplied by the same factor, the result is $\frac{12a^3b^3c^3}{16a^2b^4c^3}$, which has the same value as the given fraction, since each of them equals $\frac{3a}{4b}$.

167. *Illustrative Example.*—Change $\frac{a+1}{a-1}$ to a fraction whose denominator shall be $a^2 - 2a + 1$.

OPERATION.

$$\frac{(a + 1) \times (a - 1)}{(a - 1) \times (a - 1)} = \frac{a^2 - 1}{a^2 - 2a + 1}.$$

EXPLANATION.—The given denominator $a - 1$ must be multiplied by the factor $a - 1$ to produce the higher denominator required. Hence, if the numerator $a + 1$ of the given fraction be also multiplied by the same factor, the value of the fraction is not changed in the result, $\frac{a^2 - 1}{a^2 - 2a + 1}$.

EXAMPLES.

Change,—

1. $\frac{3}{4}$ to 8ths; to 12ths; to 20ths; to 100ths.
2. $\frac{a}{2}$ to a fraction having $2b^2c$ as its denominator.
3. $\frac{a+x}{xy}$ to a fraction having $6xy^3$ as its denominator.
4. $\frac{3ab}{x+a}$ to a fraction having x^2-a^2 as its denominator.
5. $\frac{1-x}{1+x}$ to a fraction having $1-x^2$ as its denominator.
6. $\frac{a+2}{a-3}$ to a fraction having a^2-a-6 as its denominator.
7. $\frac{6x}{a-b}$ to a fraction having a^3-b^3 as its denominator.
8. $\frac{b-1}{1+b}$ and $\frac{b+1}{1-b^2}$ to fractions having $3-3b^2$ as the denominator of each.

II. To change fractions to their lowest terms.

168. OPERATIONS.

(*a*) $24 \div 6 = 4$.

(*b*) $(24 \div 2) \div 6 = 2$.

(*c*) $24 \div (6 \div 2) = 8$.

(*d*) $(24 \div 2) \div (6 \div 2) = 4$.

EXPLANATION 1.—The quotient of 24 divided by 6 is 4 (*a*).

2.—If any factor, as 2, be taken from the dividend 24, the quotient is divided by that factor (*a*) and (*b*).

3.—If any factor, as 2, be taken from the divisor 6, the quotient is multiplied by that factor (*a*) and (*c*).

4.—If any factor, as 2, be taken from both dividend and divisor, the quotient is not changed (*a*) and (*d*).

(*e*) $\frac{6}{24} = \frac{1}{4}$.

(*f*) $\frac{6 \div 3}{24 \div 3} = \frac{2}{8}$, or $\frac{1}{4}$.

5.—The value of $\frac{6}{24}$ is $\frac{1}{4}$ (*e*).

6.—If a common factor, as 3, be taken from both denominator (divisor) and numerator (dividend), the value of the fraction is not changed (*d*), (*e*), and (*f*), since the value of $\frac{6}{24}$, and of $\frac{2}{8}$, is $\frac{1}{4}$. Hence,

A fraction may be changed to lower terms without changing the *value of the fraction, by removing or canceling a common factor from both denominator and numerator* Also,

If both denominator and numerator of a fraction be divided by their highest common factor, the fraction is changed to its lowest terms.

169. *Illustrative Example.*—Change $\frac{25a^2b^3xy^4}{35a^3b^2x^2y^3}$ to its lowest terms.

OPERATION.

$$\frac{25a^2b^3xy^4 \div 5a^2b^2xy^3}{35a^3b^2x^2y^3 \div 5a^2b^2xy^3} = \frac{5by}{7ax}.$$

EXPLANATION.—By observation, 5 is seen to be the h. c. f. of the numerical coefficients in both terms of the fraction; and a^2, b^2, x, and y^3 are seen to be the highest powers or degrees of the literal parts common to both terms. Hence, the product of these factors, or $5a^2b^2xy^3$, is the h. c. f. of both terms.

Dividing both terms of the fraction by their highest common factor, the result is $\frac{5by}{7ax}$, which is in its lowest terms.

170. *Illustrative Example.*—Change $\frac{x^2-1}{x^2-x-2}$ to its lowest terms.

OPERATION.

$$\frac{x^2-1}{x^2-x-2} = \frac{(x+1)(x-1)}{(x+1)(x-2)} = \frac{x-1}{x-2}.$$

EXPLANATION.—By separating each term of the fraction into its prime factors, $x + 1$ is found to be the only factor common to both of the terms.

Dividing both terms of the fraction by their highest common factor, the result is $\frac{x-1}{x-2}$, which is in its lowest terms.

EXAMPLES.

Change the following fractions to their lowest terms:—

1. $\frac{6b^2}{8b^3}$.
2. $\frac{10b^3d^2}{15b^2d^3}$.
3. $\frac{8cd^2x^4}{12c^2dx^3}$.
4. $\frac{12x^2yz^3}{15x^3yz^2}$.
5. $\frac{15r^3sx^2}{18r^2x^4y}$.
6. $\frac{2a}{4a^2-6ab}$.
7. $\frac{ab+ac}{ax}$.
8. $\frac{3(a+x)}{6(a+x)}$.
9. $\frac{abc-bc^2}{acd-c^2d}$.

10. $\dfrac{c^2-2cx}{2ac^3-4ac^2x}$.

11. $\dfrac{a+b}{a^2+2ab+b^2}$.

12. $\dfrac{a+b}{a^2-b^2}$.

13. $\dfrac{a-x}{a^2-x^2}$.

14. $\dfrac{ab+a}{b^2-1}$.

15. $\dfrac{r-s}{r^2-s^2}$.

16. $\dfrac{a^2-1}{4(a+1)}$.

17. $\dfrac{a(x^2-y^2)}{2a(x-y)}$.

18. $\dfrac{(x+y)^2}{x^2-y^2}$.

19. $\dfrac{a^2y-ay^2}{a^2-y^2}$.

20. $\dfrac{c^2-d^2}{2(c-d)^2}$.

21. $\dfrac{(a+b)^2}{(a+b)^3}$.

22. $\dfrac{x^2+xy}{x^2-y^2}$.

23. $\dfrac{ac-2c^2}{a^2-4ac+4c^2}$.

24. $\dfrac{x^2-y^2}{x^3-y^3}$.

25. $\dfrac{ab-a}{ab^2-a}$.

26. $\dfrac{4(x+y)}{12(x^2-y^2)}$.

27. $\dfrac{a^2-1}{2ab+2b}$.

28. $\dfrac{x^2+1}{x^4-1}$.

29. $\dfrac{a^2-x^2}{a^2-2ax+x^2}$.

30. $\dfrac{x^3+y^3}{x^2-y^2}$.

31. $\dfrac{3a-2y}{9a^2-4y^2}$.

32. $\dfrac{a^2-y^2}{a^3-y^3}$.

33. $\dfrac{s^2-s-6}{s^3+8}$.

34. $\dfrac{x^2-4}{x^2+x-6}$.

35. $\dfrac{r^2-9}{r^2-2r-15}$.

36. $\dfrac{x^2-1}{x^2-x-2}$.

III. To change fractions to others having the lowest common denominator.

171. A common denominator is a denominator common to two or more fractions.

Thus, 6 is the common denominator of $\frac{1}{6}$, $\frac{5}{6}$, and $\frac{5}{6}$; a^2b is the common denominator of $\frac{1}{a^2b}$, $\frac{xy}{a^2b}$, and $\frac{1+x}{a^2b}$; etc.

172. The lowest common denominator is the lowest denominator that two or more fractions can have.

Thus, 12 is the lowest common denominator to which $\frac{3}{4}$, $\frac{5}{6}$, and $\frac{2}{3}$ can be changed; ab^2c^2 is the lowest denominator to which $\frac{1}{b^2}$, $\frac{a+1}{ac}$, and $\frac{xy}{bc^2}$ can be changed.

The lowest common denominator of two or more fractions is the lowest common multiple of their denominators, since it is the least number or quantity that can be produced by multiplying each denominator by some factor.

173. Fractions having unlike denominators are called *dissimilar fractions;* fractions having a common denominator are called *similar fractions;* and fractions having the lowest common denominator are called *least similar fractions.*

174. *Illustrative Example.*—Change $\frac{2}{3ac}$, $\frac{3x^2}{b^2c}$, $\frac{5y}{c^2n}$ to fractions having the lowest common denominator.

OPERATION.

$$\frac{2 \times b^2cn}{3ac \times b^2cn} = \frac{2b^2cn}{3ab^2c^2n}$$

$$\frac{3x^2 \times 3acn}{b^2c \times 3acn} = \frac{9acnx^2}{3ab^2c^2n}$$

$$\frac{5y \times 3ab^2}{c^2n \times 3ab^2} = \frac{15ab^2y}{3ab^2c^2n}$$

EXPLANATION.—The l. c. m. of the denominators is $3ab^2c^2n$

To produce the lowest common denominator, the denominator of the first fraction must be multiplied by the factor b^2cn, by which the numerator also of this fraction must be multiplied, that the value of the fraction may not be changed.

To produce the lowest common denominator, the denominator of the second fraction must be multiplied by the factor $3acn$; etc.

To produce the lowest common denominator, the denominator of the third fraction must be multiplied by the factor $3ab^2$, etc. Hence,

The given fractions changed to the lowest common denominator are $\frac{2b^2cn}{3ab^2c^2n}$, $\frac{9acnx^2}{3ab^2c^2n}$, and $\frac{15ab^2y}{3ab^2c^2n}$.

NOTE.—Each fraction should be changed to its lowest terms, if necessary, before changing the fractions to their lowest common denominator.

EXAMPLES.

Change the following fractions to equivalent fractions having the lowest common denominator:—

1. $\frac{a}{b}$, $\frac{b}{c}$.

2. $\frac{a}{x}$, $\frac{b}{y}$, $\frac{c}{z}$.

3. $\frac{x}{c}$, $\frac{y}{d}$, $\frac{z}{r}$.

4. $\frac{ax}{y}$, $\frac{2b^2}{ay}$, $\frac{a^2b}{x^2y}$.

5. $\frac{2a}{a^2dx}$, $\frac{3b}{ad^2x}$, $\frac{4c}{a^2d^2x^2}$.

6. $\frac{c}{d^2}$, $\frac{d}{c^2}$, $\frac{a}{b^2}$.

7. $\frac{m}{n}$, $\frac{n^2}{m}$, $\frac{m^4}{r^2}$.

8. $\frac{2b}{xy^2}$, $\frac{1}{z}$, $\frac{3cd}{x^2y}$.

9. $\frac{c}{ax}$, $\frac{d}{cx}$, $\frac{r}{ac}$.

10. $\frac{ax}{dy}$, $\frac{dx}{ay}$, $\frac{xy}{ad}$.

11. $\frac{8}{ad}$, $\frac{9}{c^2}$, $\frac{5}{cd}$.

12. $\frac{c}{b}$, $\frac{ab}{a^2b^2}$, $\frac{a^2b}{ab^2}$.

13. $\frac{2a}{3xy}$, $\frac{3b}{4y^2}$, $\frac{4s}{8xz}$.

14. $\frac{m}{bcd}$, $\frac{r}{b^2c}$, $\frac{x}{ac^2d}$.

15. $\frac{2b}{3x}$, $\frac{3c}{4y}$, $\frac{d^2}{ax^2}$.

16. $\frac{4b}{a^2-b^2}$, $\frac{3x}{a+b}$.

17. $\frac{ab}{1+2a}$, $\frac{3}{1-4a^2}$.

18. $\frac{3}{9-x^2}$, $\frac{3-x}{3+x}$.

19. $\frac{1}{2}$, x, $\frac{y}{x-y}$.

20. $\frac{a+b}{2}$, $\frac{a-b}{6a}$, 3.

21. $\frac{x+y}{4x}$, $\frac{x-y}{3y}$, $\frac{ac}{6}$.

22. $\frac{2a}{1-a}$, $\frac{1}{1-a^2}$, $\frac{ab}{2a}$.

23. $\frac{2ac}{x+1}$, $\frac{4xy}{x^2-1}$, $\frac{y}{x-1}$.

24. $\frac{a}{a+x}$, $\frac{b}{a-x}$, $\frac{c}{a^2-x^2}$.

25. $\frac{2}{a}$, $\frac{3a}{a+3}$, $\frac{5}{a-3}$.

26. $\frac{a+b}{a-b}$, $\frac{4x}{a+b}$, $\frac{2a}{3x}$.

27. $\frac{3}{1-2a}$, $\frac{x}{1+2a}$, $\frac{1-2a}{1-4a^2}$.

28. $\frac{3a}{1-2x}$, 2, $\frac{2a}{1-4x^2}$.

29. $\frac{x}{b-a}$, $\frac{ax}{b^2-a^2}$, $2a$.

30. $\frac{3a}{x+y}$, $\frac{4b}{x-y}$, 3.

Note to Teachers.—A knowledge of the principles of the fundamental rules, of common factors and of common multiples and their applications in changing fractions to higher terms, to their lowest terms, and to the lowest common denominator, prepares the learner to clear the equation of fractions, which is the only remaining process necessary to insure expertness in the solution of the simple equation.

These several subjects have been, therefore, introduced in this sequence in order to economize the learner's time and energies. Teachers, however, who prefer to do so, may continue and complete the treatment of fractions generally (pp. 150–162), before presenting the process of clearing the equation of fractions.

IV. To clear the equation of fractions.

175. OPERATIONS.

(a) $2 = 2$

$2 \times 3 = 2 \times 3$

$6 = 6$

(b) $\frac{1}{5}x = 2$

$\frac{1}{5}x \times 5 = 2 \times 5$

$x = 10$

(c) $\frac{1}{2} + \frac{1}{4} = \frac{6}{8}.$

$\left(\frac{1}{2} \times 8\right) + \left(\frac{1}{4} \times 8\right) = \frac{6}{8} \times 8$

$4 + 2 = 6$

$6 = 6$

EXPLANATION 1.—If both members of the equation $2 = 2$ be multiplied by the same factor, as 3, the results will be equal (*a*).

2.—If both members of the equation (*b*) be multiplied by the same factor, as 5, the denominator of the given fraction, the results will be equal, and the new equation will contain no fraction.

3.—If each term of the members of the equation (*c*) be multiplied by 8, the lowest common multiple of the denominators, the results will be equal; and the new equation will contain no fraction. Hence,

176. *If equals be multiplied by the same factor, the results will be equal.* And,

If all the terms of an equation containing fractions be multiplied by the lowest common multiple of the denominators, the members will be equal, and the equation will be clear of fractions.

177. The process of changing an equation containing fractions to another which has no fractions and is of equal value, is called **clearing the equation of fractions.**

178. *Illustrative Example.*—Clear the equation $\frac{2x}{3} + \frac{x}{6} = \frac{5x}{12} + 5$ of fractions.

OPERATION.

(a) $\frac{2x}{3} + \frac{x}{6} = \frac{5x}{12} + 5.$

Clearing of fractions, (b) $8x + 2x = 5x + 60$

Transposing, $8x + 2x - 5x = 60$

Collecting, $5x = 60$

And, $x = 12$

EXPLANATION.—The l. c. m. of the denominators is 12.

Multiplying every term of each member by 12, and canceling factors common to the new numerators and denominators, the equation is cleared of fractions without destroying the equality of the members (**175–176**). Transposing and collecting, the value of x is found to be 12.

Substitute the value of x in equations (a) and (b) and verify each.

179. *Illustrative Example.* — Clear the equation $\frac{3x-6}{2} - \frac{7x-6}{8} = \frac{5x-2}{12}$.

OPERATION.

$$(a)\quad \frac{3x-6}{2} - \frac{7x-6}{8} = \frac{5x-2}{12}$$

Clearing of fractions,	(b) $36x - 72 - 21x + 18 = 10x - 4$
Transposing,	$36x - 21x - 10x = -4 - 18 + 72$
Collecting,	$5x = 50$
And,	$x = 10$

EXPLANATION.—The l. c. m. of the denominators is 24. The second term of the first member is to be subtracted, and, therefore, the sign of every term in its numerator must be changed.

Multiplying every term in each member of the equation by 24, etc.

Transposing and collecting, the value of x is found to be 10.

In equations (a) and (b), substitute the value of x, and verify each.

180. The sign prefixed to a fraction is not the sign of the first term of the numerator; it is the sign of operation for the fraction regarded as a whole, and it disappears when the equation is cleared of fractions. The line separating numerator and denominator has the effect of the parenthesis or the vinculum (**84**—p. 72).

EXAMPLES.

1. $\frac{x}{3} + \frac{x}{2} = 5.$
2. $\frac{x}{5} - \frac{x}{6} = 1.$
3. $x + \frac{x}{4} = 15.$
4. $\frac{x}{3} + x = 8.$
5. $3x - \frac{x}{2} = 15.$
6. $\frac{x}{3} + 2x = 16 - 2.$

7. $\frac{x}{4}-\frac{x}{6}+\frac{x}{8}=\frac{5}{12}$.

8. $25-\frac{x}{3}=-5$.

9. $\frac{x}{6}+\frac{x}{2}+\frac{x}{3}=12$.

10. $x-\frac{x}{2}+\frac{x}{3}=10$.

11. $\frac{x}{6}+x-\frac{x}{9}=6\frac{1}{3}$.

12. $\frac{6x}{8}-4+2x=\frac{9x}{4}$.

13. $x-\frac{7x}{8}+\frac{x}{6}=3\frac{1}{2}$.

14. $\frac{3x}{2}+\frac{x}{6}=2x-2\frac{2}{3}$.

15. $\frac{2x}{4}+2=\frac{9x}{2}-22$.

16. $\frac{3x}{5}+2=3\frac{4}{5}+\frac{3x}{10}$.

17. $\frac{x}{5}-\frac{x}{4}+x=\frac{x}{2}+4\frac{1}{2}$.

18. $\frac{x}{4}-\frac{x}{2}+4=\frac{5x}{8}-\frac{3}{8}$.

19. $5x-21\frac{1}{5}=x-17\frac{1}{5}$.

20. $\frac{x}{2}-1\frac{1}{4}+2x=6\frac{1}{4}$.

21. $\frac{5x}{12}+\frac{3x}{4}=\frac{7x}{2}-11\frac{2}{3}$.

22. $2x+\frac{3x}{4}-7\frac{1}{2}=\frac{3x}{7}-\frac{15}{28}$.

23. $\frac{3x-16}{x}=5-3\frac{1}{3}$.

24. $\frac{x-1}{6}+x=\frac{x+4}{12}+6$.

25. $\frac{11x}{4}+\frac{5x}{8}+5\frac{7}{8}=4x+\frac{3}{2}$.

26. $\frac{3x-2}{5}=\frac{3x+2}{7}$.

27. $\frac{x-1}{x-3}=\frac{x-4}{x-2}$.

28. $\frac{22x+2}{2}=\frac{47x-11}{3}$.

29. $\frac{16x-65}{5}=2x-7$.

30. $x+\frac{x-3}{4}=\frac{7x-3}{8}+12$.

31. $\frac{3x+2}{4}+\frac{4}{7}=\frac{3x}{14}+\frac{45}{28}$.

32. $\frac{3x-16}{x}=4-2\frac{1}{3}$.

33. $\frac{x^2}{4}+3=12-5$.

34. $\frac{5x+3}{2}-6=\frac{5x+6}{7}$.

35. $\frac{x-2}{x-3}=\frac{x-3}{x-2}$.

36. $\frac{x}{4}-\frac{x-1}{11}=x-10$.

37. $2x-\frac{x+5}{3}=\frac{4x+5}{3}$.

38. $\frac{x+3}{2}-\frac{x+4}{3}=\frac{2x-6}{4}$.

39. $2-\frac{x+8}{4}=2x-\frac{8x+1}{4}$.

40. $x-4-\frac{12-x}{7}=0$.

41. $\frac{x}{3}-\frac{x^2+5x}{3x-7}=\frac{2}{3}$.

42. $\frac{5x}{14}-\frac{10x-20}{21}=0$.

43. $\frac{2}{3}(x-1)=\frac{1}{2}(x+1)$.

44. $x-\frac{1-x}{3}=\frac{1-x}{8}+8\frac{7}{24}$.

45. $\frac{1}{x}+\frac{x-1}{x+1}=1$.

46. $\frac{1}{1+x}+\frac{3}{1-x}=\frac{8}{1-x^2}$.

47. $(a+c)x-(a-c)x=a^2$.

48. $\frac{a}{x}+\frac{b}{x}+\frac{c}{x}=1$.

49. $\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=1$.

50. $\frac{a-x}{a+x}=\frac{a^2}{a^2-x^2}$.

In examples 1–25, substitute the value of x, and verify each equation.

REVIEW EXAMPLES AND PROBLEMS.

1. Add $3ab^2-4d^3e^2+5a^2x^4$, $7d^3e^2-8a^2x^4$, $7ab^2-3d^3e^2$, and $5ab^2+2a^2x^4$.

2. Add $5c^2d-4e^3n^4$, $6m^4r^3+7c^2d$, $8e^3n^4+8m^4r^3$, and $-4e^3n^4+5c^2d-15m^4r^3$.

3. Find the sum of $6x^2y^3+3w^4z^3$, $8v^2x^3+10w^4z^3$, $5x^2y^3-7w^4z^3$, and $-9v^2x^3-6w^4z^3$.

4. Find the sum of $7a^4b^3-5d^3e^2$, $9c^2d^4-8n^2s$, $-7c^2d^4+4d^3e^2$, $7n^2s-9a^4b^3$, and $2c^2d^4+n^2s$.

5. Add $7s^2t^4-8y^2z^3$, $-8u^3v^2+9y^2z^3+7x^4y^3$, $-y^2z^3-8x^4y^3+5s^2t^4$, and $8s^2t^4+x^4y^3+9u^3v^2$.

6. Add $\frac{1}{3}a^2x-4b^2y^3-\frac{1}{2}c^4z^3$, $\frac{1}{4}b^2y^3+2c^4z^3$, $\frac{1}{8}c^4z^3+2a^2x$, and $-\frac{1}{8}b^2y^3+3a^2x$.

7. Find the sum of $\frac{3}{4}ab^2+\frac{1}{3}b^3c^2-5c^3d^4$, $-2ab^2+\frac{2}{3}b^3c^2+6a^4x^2$, $6c^3d^4+\frac{1}{4}a^4x^2$, and $-b^3c^2-3a^4x^2+1\frac{1}{4}ab^2$.

8. Find the sum of $b^3c^2 + \frac{1}{3}e^4x^3$, $\frac{1}{2}cd^3 - \frac{1}{6}e^4x^3$, $2a^2y^3 - \frac{3}{8}cd^3$, and $\frac{1}{4}a^2y^3 - \frac{1}{6}e^4x^3 - \frac{1}{8}b^3c^2$.

9. From $4bc^2 - 6d^2e^3 + 4c^3f^4$ take $3d^2e^3 - 7a^3x^3$.

10. From $6a^2x - 5b^3y^2$ take $6ax^3 + 7c^3z^4 - 5b^3y^2$.

11. Take $-6c^4y^3 + 8d^2z^4 - 5e^3n^5$ from $4b^3x^2 - 6c^4y^3$.

12. Take $\frac{1}{4}d^3x^4 - \frac{1}{2}e^3y^2 - 4a^4z^3$ from $\frac{1}{4}c^2w^3 - \frac{1}{2}e^3y^2 + 4d^3x^4$.

13. From $1\frac{1}{3}a^3x^2y - 4bx^2z^3 + \frac{1}{3}x^4y^3$ take $-\frac{3}{8}y^2z^4 - \frac{1}{6}x^4y^3 - 4bx^2z^3$.

14. Take $1\frac{2}{3}t^3u^2 + 3\frac{1}{2}st^3 - 7u^2v^4$ from $-3v^4w^3 + 3\frac{1}{2}st^3 + \frac{1}{3}t^3u^2$.

15. Add $5ab^2 - 6b^2c + 7c^2d^3$, $7b^2c - c^2d^3$, and $7ab^2 - 5c^2d^3$; subtract $12ab^2 - a^4x + 2c^2d^3 - 3b^2c$.

16. From $7b^2c^3 + 6a^4x^2 - 7c^2x^4$ take $-9d^3y^5 + 6a^4x^2 + 8b^2c^3$; add $7c^2x^4 - 3d^3y^5 - 2a^4x^2$, $6b^2c^3 - c^2x^4$, and $6d^3y^5 + 2a^4x^2$.

17. Add $\frac{1}{2}a^3x^2 - 3b^2y^4 + 1\frac{1}{4}c^4z^3$, $4b^2y^4 + \frac{1}{8}a^3y^4 - \frac{1}{2}c^4z^3$, and $\frac{1}{4}a^3x^2 - \frac{1}{2}a^3y^4 - b^2y^4$; subtract $a^3x^2 - \frac{3}{8}a^3y^4 - 1\frac{1}{4}b^2y^4$.

18. From $\frac{1}{3}a^3x^2 + 4b^3y^4 - 2\frac{1}{3}c^2z^3$ take $a^3x^2 + 4b^3y^4 + d^4s^3$; add $\frac{1}{3}a^3x^2 + 2c^2z^3 - 3b^3y^4$, and $-3d^4s^3 + \frac{1}{3}c^2z^3 - b^3y^4$.

19. Add $\frac{1}{2}bx^2 - c^3y^2 + 2d^3z^4$, $-6a^2y^3 + \frac{1}{2}d^3z^4 + 4c^3y^2$, and $\frac{1}{2}d^3z^4 + 6a^2y^3$; subtract $\frac{1}{2}bx^2 + 4d^3z^4 - 4c^3y^2$.

Simplify and collect the following :—

20. $20 - (\overline{10 - 4} + 8) - 6$.

21. $20 - (10 - \overline{4 - 8}) + 6$.

22. $(20 - \overline{10 - 4} + 8) - 6$.

23. $20 + 10 - \overline{4 - (-8 + 6)}$.

24. $30 - [-10 - (7 - \overline{10 - 2})]$.

25. $6x - [5y + (-4x + 3y)]$.

26. $(6x - \overline{5y - 4x}) - 3y$.

27. $8y - 7x - [6y + (-8x)]$.

28. $[8y - (7x + 6y)] - (-8x)$.

29. $2x - [2y - \overline{2x + 2y}] - 2x$.

30. $[10 - 8 - (6 - \overline{4 - 7})] - (-3)$.

31. $\{40 - \overline{8 + 2} - \overline{7 + 3}\} - [3 - (-2 + 1)]$.

32. $12 - [8 - \overline{7 + 6} - (5 + 4)] - [3 - (-2 + 1)]$.

33. $(3 - \overline{2a + 2}) - [1 - (3a + 2)] - 4a$.

34. $[a-(2b-3a)]-\{3b-(2a+b)\}-(-a)$.

35. $8x-[3y+(2x-y)]-[x-(2y-2x)]-3y$.

36. Multiply $2a+3bx$ by $4a-6bx$; add $2a^2-4abx+18b^2x^2$; subtract $3a^2x^2+4abx+10a^2$.

37. Multiply $3b^2+6ac$ by $4ac-2b^2$; subtract $6b^4-ab^2c$; add $12b^4-25a^2c^2-2ab^2c$.

38. Multiply $4cx+2y^2$ by $3cx-5y^2$; add $-9c^2x^2+10y^4$; subtract $-3c^2x^2-20y^4$.

39. Add $2ab^2-3bc+5c^3d^2$ and $-6c^3d^2+3bc$; subtract $-7bc+3ab^2-c^3d^2$; multiply by $2ab^2-3bc$.

40. From $4ab^2-3b^4c^2+2cd^2$ take $-3b^4c^2+4cd^2+2ab^2$; add $-2ab^2+2d^3e^2+cd^2$; multiply by $3cd^2-3d^3e^2$.

41. Add $3bx^2+4c^4y^3+5d^2z$ and $-5d^2z-4bx^2$; subtract $5bx^2+4c^4y^3+3d^2z$; multiply by $-4bx^2+2d^2z$.

Expand the following expressions according to general principles, by inspection:—

42. $(5a+7b)(5a+7b)$.
43. $(6x-8y)(6x-8y)$.
44. $(4ab+5c)(4ab+5c)$.
45. $(3b-4ax)(3b-4ax)$.
46. $(2x^2+5y^2)(2x^2+5y^2)$.
47. $(6a-5b)(6a+5b)$.
48. $(4x+6y)(4x-6y)$.
49. $(3ab+4y)(3ab-4y)$.
50. $(5x-7yz)(5x+7yz)$.
51. $(4a^2b-5c^2)(4a^2b+5c^2)$.
52. $(5a-7)(5a-5)$.
53. $(4b+3)(4b+6)$.
54. $(3ax-5)(3ax+4)$.
55. $(6+3a)(6-5a)$.
56. $(5a^2-6)(5a^2+8)$.
57. $(4ab^2+5)(4ab^2-7)$.
58. $(5+2xy)(5-4xy)$.
59. $(a+b)(a-b)(a^2+b^2)$.
60. $(x^2-1)(x-1)(x+1)$.
61. $(x^2-xy+y^2)(x+y)$.

In examples 42–61, prove each result by actual multiplication.

62. Divide $8a^2-18b^2x^2$ by $2a+3bx$; add $2a-6bx+3cy$; subtract $6a-4bx-cy$.

63. Divide $-6b^4 + 24a^2c^2$ by $-2b^2 + 4ac$; subtract $4b^2 + 5ax^2 + 6ac$; add $3ax^2 - b^2 - z^4$.

64. From $4a^2c^4 - 5b^4d^2 + 3c^2e^2$ take $-7c^2e^2 + 5a^2c^4$; add $5a^2c^4 + 5b^4d^2 - 19c^2e^2$; divide by $2ac^2 - 3ce$.

65. Multiply $2b + 3ax$ by $8b - 12ax$; divide by $4b + 6ax$; subtract $9b - 6ax - 7z$.

66. Add $4a^2b - 5s^2t^3 - 4y^4z^4$ and $y^4z^4 - 4a^2b$; subtract $-4a^2b^2 + 6y^4z^4 - 5s^2t^3$; divide by $2ab - 3y^2z^2$.

67. Divide $9a^4b^6 - 30a^2b^3x^3y^2 + 25x^6y^4$ by $3a^2b^3 - 5x^3y^2$; multiply by $3a^2b^3 + 5x^3y^2$; subtract $9a^4b^6 - 25\frac{1}{4}x^6y^4 + \frac{2}{3}a^2x^2$.

Find by inspection the quotient of each of the following expressions:—

68. $\dfrac{a^2 - 2ab + b^2}{a - b}$.

69. $\dfrac{4b^2 + 4by + y^2}{2b + y}$.

70. $\dfrac{4x^2 - 8xy + 4y^2}{2x - 2y}$.

71. $\dfrac{9a^2x^2 + 24abx + 16b^2}{3ax + 4b}$.

72. $\dfrac{16a^4 - 24a^3b + 9a^2b^2}{4a^2 - 3ab}$.

73. $\dfrac{25a^4b^2 + 40a^2bc^2 + 16c^4}{5a^2b + 4c^2}$.

74. $\dfrac{4x^2 + 4x - 15}{2x - 3}$.

75. $\dfrac{9a^2b^2 + 6ab - 63}{3ab - 7}$.

76. $\dfrac{25a^4x^2 - 10a^2x - 80}{5a^2x + 8}$.

77. $\dfrac{16b^2c^4 - 48bc^2 + 35}{4bc^2 - 7}$.

78. $\dfrac{36a^4y^4 - 24a^2y^2 - 32}{6a^2y^2 + 4}$.

79. $\dfrac{49b^2x^6 - 21bx^3 - 108}{7bx^3 - 12}$.

80. $\dfrac{9c^2 - 1}{3c + 1}$.

81. $\dfrac{16x^4 - 9y^2}{4x^2 + 3y}$.

82. $\dfrac{8a^3 + 1}{2a + 1}$.

83. $\dfrac{27x^3y^3 - 8z^3}{3xy - 2z}$.

84. $\dfrac{9a^4b^4 - 25x^6y^4}{3a^2b^2 - 5x^3y^2}$.

85. $\dfrac{8a^3 + 64a^6b^6}{2a + 4a^2b^2}$.

In examples 68–85, prove each result by actual division.

Find the quotient of each of the following expressions:—

86. $\frac{1+4a^2}{1+2a}$. 87. $\frac{4a^2+9b^2}{2a+3b}$. 88. $\frac{x^3+y^3}{x-y}$.

89. $\frac{25a^4b^2+16c^4}{5a^2b+4c^2}$. 92. $\frac{a^2-2ax+x^2}{a+x}$.

90. $\frac{8a^3b^3-27x^3y^3}{2ab+3xy}$. 93. $\frac{4b^4+8b^2c^2+4c^4}{2b^2-2c^2}$.

91. $\frac{27b^6c^3+64a^3x^6}{3b^2c-4ax^2}$. 94. $\frac{9a^4x^2-24a^2xy^2+16y^4}{3a^2x+4y^2}$.

In examples 86–94, from the quotient, the divisor, and the remainder, find the dividend.

Find the binomial, or the binomial and trinomial factors of the following quantities:—

95. $b^2-2ab+a^2$. 102. $x^2-2x-35$.

96. $a^2b^2+4abc+4c^2$. 103. $a^2y^2+2ay-24$.

97. $4a^2-12ax+9x^2$. 104. $4x^2+6x-40$.

98. $9b^4+24b^2c^2+16c^4$. 105. $9a^2b^2-6ab-48$.

99. $16a^2c^2-40acx+25x^2$. 106. $16b^2c^2-24bc-40$.

100. $25c^4+60ac^2y^2+36a^2y^4$. 107. $25a^4b^6+75a^2b^3+50$.

101. $36a^4x^4-60a^2x^4y^2+25x^4y^4$. 108. $36b^6y^4-78b^3y^2+42$.

109. $4a^4-9x^4$. 113. a^6+27b^3. 117. $2a^2+7a+5$.

110. $9a^2c^2-16b^2$. 114. $8a^3b^3+8$. 118. $4x^2-9x+2$.

111. $25b^2c^4-36a^2x^4$. 115. $27b^3x^6-64a^3$. 119. $6b^2+11b+4$.

112. $36c^6x^4-25a^2y^4$. 116. $64a^6y^6+125$. 120. $15y^2-2y-8$.

In examples 95–120, from the factors found, produce the given quantities.

Find the highest common factor in each of the following examples:—

121. $12a^2b^3c^4d$, $18b^2c^3d^2e$. 123. $2abc-2ab$, $6ac^2-6ac$.

122. $15ax^2$, $25a^2c^4x^5$, $10a^3b^2x^3$. 124. $3b^4c^3+6b^2c^4$, $9b^3c^2+18bc^3$.

125. $ab - b$, $2a^2b^2 - 2b^2$.
126. $1 - x^2$, $(1 - x)^2$.
127. $4ab - 4bx$, $6a^2c^2 - 6c^2x^2$.
128. $15x^2 - 12x$, $25a^2x^2 - 16a^2$.
129. $x^2 - y^2$, $x^2 - 2xy + y^2$.
130. $4x^2 - 9y^2$, $4x^2 + 6xy$.
131. $6a^4b - 18a^2b^2$, $8a^3b^3 - 24ab^4$.
132. $3a(2x - 3z)$, $6a^2(4x^2 - 12xz + 9z^2)$.
133. $2a^2 - 3a$, $6a(4a^2 + 4a - 15)$.
134. $2bx^2 - 4bx - 48b$, $4b^2(x^2 - 16)$.
135. $a + x$, $a^2 - x^2$, $a^3 + x^3$.
136. $x - 1$, $x^2 - 2x + 1$, $x^3 - 1$.
137. $3a^2 + 2a$, $9a^2 + 12a + 4$, $9a^2 - 9a - 10$.
138. $4 - 8x + 4x^2$, $6 - 6x^2$, $2(2 - 3x + x^2)$.
139. $2a^2(3a - 2b)$, $6ab(9a^2 - 4b^2)$, $3ab^2(27a^3 - 8b^3)$.

In examples 121–139, divide each quantity by the highest common factor.

Find the lowest common multiple in each of the following examples:—

140. $2ax^3y$, $3b^2x^4y^2$, $5a^2b^3y^4$.
141. $4b^2y^2z^5$, $6c^3y^4z^3$, $8b^3c^2z$.
142. $5cd^2x^3$, $15a^3x^4y^4$, $10c^2dy^5$.
143. $4ab^2c$, $6a^3c^2$, $ab^4(a + x)$.
144. $3xy^3(a + 1)$, $6x^2y(a^2 - 1)$.
145. $5a(1 - x)$, $10ab - 10abx$.
146. $3(a^2 - 1)$, $6ax^2 - 6x^2$.
147. $a - b$, $(a + b)(a - b)$.
148. $1 + x$, $1 - x$, $1 - x^2$.
149. $8y^2(a + x)^2$, $6x(a^2 - x^2)$.
150. $6ax^3 - 6ax^2y$, $4a^3x^2 - 4a^3y^2$.
151. $8b(a^2c - ac^2)$, $12(a^3 - ac^2)$.
152. $6b^2(1 - 2x + x^2)$, $4a(1 - x^2)$.
153. $4x^2 - 9y^2$, $4x^2 - 12xy + 9y^2$.
154. $2a(x + y)$, $4ac^2(x - y)$, $6b^3c(x^2 - y^2)$.
155. $3a^3 - 3a^3x$, $6a^2 + 6a^2x$, $9a - 9ax^2$.
156. $x - y$, $x^2 - y^2$, $(x + y)(x - y)$.
157. $8(a^2c - ac^2)$, $12(a^3 - ac^2)$, $16(a^3b + a^2bc)$.

158. $b+4,\ b^2-16,\ b^2-2b-24$.

159. $2a-5,\ (2a-5)(2a-5),\ 4a^2+2a-30$.

160. $1+x,\ 1-x^2,\ (1+x)(1-x+x^2)$.

In examples 140–160, divide the lowest common multiple by each of the given quantities; find the highest common factor in each example.

Find the value of the unknown in each of the following equations:—

161. $5x-15+6x-10-7x+12=3x-8$.

162. $2(x-8)+4x-10-(3x-12)=15-(3x+17)$.

163. $6x-(3x-6)-2(x-5)=[8x-(3x+8)]$.

164. $(x-4)(x-3)=(x+2)(x-4)$.

165. $8x(x-1)-4(x-2)=(4x-3)(2x+2)$.

166. $10x-2(3x-7)-[4-6x-(3x-3)]=33$.

167. $8(x+4)-6(x+5)+4x-5=4(x+1)-3$.

168. $\frac{1}{4}(x+8)=\frac{3}{4}(x+4)-5$.

169. $\frac{1}{3}(x+6)-\frac{1}{2}(16-x)=\frac{1}{4}(3x-8)-3$.

170. $\frac{1}{2}(x+12)-\frac{1}{4}(2x-4)=[\frac{1}{8}(3x-8)+5]$.

In examples 161–170, substitute the value of x, and verify each equation.

Clear the following equations of fractions; verify each equation:—

171. $\frac{x}{4}+\frac{x}{12}+\frac{x}{15}=9\frac{3}{5}$.

172. $\frac{25}{4x}+\frac{75}{3x}-\frac{25}{2x}=3\frac{3}{4}$.

173. $\frac{5x}{14}-\frac{5x-20}{21}=5$.

174. $3x-8-\frac{19-4x}{7}=0$.

175. $\frac{x-1}{x+1}+\frac{1}{x}=1$.

176. $\frac{2x-4}{5}-\frac{x-4}{6}=2\frac{2}{3}$.

177. $\frac{10-x}{6}-\frac{2(x-4)}{9}=\frac{x}{27}$.

178. $\frac{4}{x+1}-\frac{3}{x-1}=\frac{1}{x^2-1}$.

179. $\frac{x+3}{4} - \frac{x+4}{6} = \frac{2(x-3)}{8}$.

180. $\frac{3x+1}{4} - \frac{5x-1}{10} - 5\frac{3}{5} = 0$.

181. $\frac{16x}{3} - 5\frac{3}{4} = \frac{10x}{2} - 2x + 15\frac{1}{4}$.

182. $\frac{7x}{4} - 3\frac{1}{8} = -3x + 10\frac{7}{8} + 76\frac{1}{4}$.

183. $\frac{25x+7}{8} - \frac{9x+1}{5} = \frac{7x+5}{4} - 1$.

184. $\frac{1}{4}(3x-4) - \frac{1}{8}(2x-4) = \frac{1}{2}(x-2) - \frac{x+4}{6}$.

185. $\frac{x+3}{3} - \frac{3x-8}{5} = \frac{3x-8}{2} - \frac{2x+8}{5}$.

186. $\frac{x-6}{4} - \frac{2x+4}{5} = \frac{3x-14}{20} - \frac{2(3x-4)}{10}$.

187. $\frac{1}{8}[2x-(3x-12)] + 2x - 6 = \frac{2(3x+2)}{4}$.

188. If I multiply a number by 4, subtract 3 from the product, divide the remainder by 5, and then add 2, the sum will be 7. Find the number.

189. Divide 60 cents among three girls, so that Mary shall have $\frac{1}{2}$ as many as Ruth, and Ruth $\frac{2}{3}$ as many as Martha.

190. The sum of two numbers is 33; and $\frac{3}{5}$ of the less, increased by 3, is $\frac{2}{3}$ of the greater. What are the numbers?

191. Frank has 14 rabbits more than Harry; and $\frac{2}{5}$ of Frank's number equals $\frac{3}{4}$ of Harry's. How many has each?

192. Divide a dollar between John and James, so that $\frac{2}{5}$ of John's share, increased by 6 cents, shall equal $\frac{1}{4}$ of James's share.

193. Three girls together have a dollar. Ida has as many cents as both Fanny and Edith, and Fanny has 10 cents less than three times as many as Edith. How many has each?

194. If I put 15 bushels of wheat into a bin which is one-third full, the bin will then be one-half full. How many bushels can the bin hold?

195. In a mixture of coffee, $\frac{1}{8}$ of it plus 25 pounds is Java, and $\frac{1}{2}$ of it minus 5 pounds is Mocha. How many pounds of each are in the mixture?

196. One-third of a barrel of oil leaked out, and then 7 gallons were drawn out. If the barrel was then half full, how many gallons could it hold?

197. A son is one-fourth as old as his father. Four years ago, he was only one-sixth as old. How old is each now?

198. What sum of money must be invested at 5% interest to give a semi-annual income of $500?

199. A man sold 10 acres more than $\frac{3}{5}$ of his farm, and then had 12 acres less than $\frac{5}{8}$ of it remaining. How many acres had he at first?

200. B is 40 years of age and A is one-half as old as B. In how many years will A be two-thirds as old as B?

201. William can make a boat in 6 days, and his father can make it in 3 days. In what time can both working together make it?

202. If a cistern can be filled by a pipe in 4 hours, and by a smaller pipe in 12 hours, in how many hours can it be filled by both pipes running at the same time?

203. Two-thirds of Arthur's marbles, increased by 6 marbles, equals three-fourths of Walter's. How many has each if both together have 42?

204. A's salary added to one-half of B's salary is $2000. If A's salary is to B's as 3 to 4, what salary does each receive?

205. The width of a room is two-thirds of its length. If the length were 6 feet less, and the width 6 feet more, the room would be square. Find the dimensions of the room.

206. The width of a piece of paper is three-fourths of the length, and the area of the sheet is 108 square inches. Find the length and the width.

207. A boy sold two sleds at the same price, for $12. On one he gained 25%, and on the other he lost 25%. What did each cost?

208. William and George working together can make a bookcase in 2 days, and William alone can make it in 3 days. In what time can George by himself make it?

209. Mary and Anna together can make a dress in 4 days, and Mary alone can make it in 12 days. In what time can Anna make it?

210. A and B had the same salary. A saved 20% of his; B spent $100 a year more than A, and in five years he saved $500. What salary did each receive?

211. William bought three knives, paying the same price for each, and he sold them all for $1.95. On the first he gained 20%, on the second 25%, and on the third he lost 20%. Find the cost of each.

212. A man left $\frac{1}{3}$ of his property to his wife, $\frac{1}{2}$ to his children, and the remainder, $2000, to a hospital. How much was his property worth?

213. A and B together put $3200 in business, of which A put in $\frac{3}{5}$ as much as B. A afterwards bought 20% of B's share. How much capital had each then?

214. Three carpenters built a shed for $120. A received 1\frac{1}{4}$, B $2, and C 2\frac{1}{2}$ a day. How many days did they work? What did each receive in all?

215. Three boys bought four knives at a uniform price. John paid 25 cents, James 40 cents, and Joseph 55 cents. They lost one knife and sold the others for $1.20. Divide the money fairly among them.

216. Three men hired a boat for 20 days. The first used it 5 days, the second 6 days, and the third the remainder of the time. If the first man paid $10, how much did each of the others pay?

217. The interest on the sum of money loaned by A and B for 5 years at 6% was $180. If A loaned twice as much as B, how much did each loan?

218. A man rode into the country at the rate of 10 miles an hour. He there sold his bicycle and walked home at the rate of 4 miles an hour. If he was gone 7 hours altogether, how far from home did he ride?

219. Ned bought some plums at the rate of 5 for 2 cents. He sold $\frac{1}{3}$ of them at the rate of 2 for a cent, and the remainder for $\frac{3}{5}$ of a cent apiece. If he cleared 5 cents, how many plums did he buy?

220. Frank spent $\frac{1}{4}$ of his money for a sled, and then earned \$1. He spent $\frac{1}{4}$ of what he then had, and had \$3 remaining. How much money had he at first?

221. A can do a piece of work in 4 days, B in 8 days, and C in 12 days. In what time can all do it together?

222. A, B, and C can do a piece of work in 4 days. A alone can do it in 10 days, and B in 15 days. In what time can C alone do it?

223. What time is it when $\frac{1}{3}$ of the time past midnight equals the time to noon?

224. What time is it when $\frac{2}{3}$ of the time past noon equals the time to midnight?

225. Divide \$30 between George and Henry so that George shall receive a half-eagle as often as Henry receives a dollar.

226. A train runs 100 miles in a certain time. If it were to run 5 miles an hour faster, it would run 20 miles farther in the same time. At what rate does the train run?

227. The perimeter of a triangle is 36 inches. The hypothenuse is 6 inches longer than the base, and the base is 3 inches shorter than the perpendicular. Find the length of each side.

228. John borrowed as much money as he had, and spent 10 cents. He borrowed as much as he then had, and again spent 10 cents. If he had 10 cents remaining, how much money had he at first?

229. A baker sold $\frac{1}{2}$ a pie more than one-half of all his pies. He afterwards sold $\frac{1}{2}$ a pie more than one-half of the

remaining pies, and then he had 4 pies left. How many pies had he at first?

230. A man who walked 25 miles a day is followed four days later by another man who walks 30 miles a day. How soon will the second man overtake the first?

231. A clerk received $3 a day for his services, and was fined $1 for every day's absence from duty. At the end of 20 days he received $50. How many days was he absent?

232. The sum of three numbers is s, and two of the numbers are a and b. What is the third number?

233. The sum of two numbers is s, and their difference is d. What are the numbers?

234. The product of three numbers is p, and two of the numbers are a and b. What is the third number?

235. The dividend is d, the divisor is a, and the quotient is b. What is the remainder after dividing?

236. The divisor is a, the quotient is b, and the remainder is c. What is the dividend?

237. The area is a, and the length is b. What is the width?

238. The solidity is a, the length is b, and the width is c. What is the depth or thickness?

239. If an article is bought for c dollars, and is sold at a gain of $r\%$, what is the selling price?

240. An article is sold for s dollars, at a gain of $r\%$. Find the cost.

241. What is the interest on p dollars, at $r\%$ for t years?

242. What principal gains i dollars interest, at $r\%$ for t years?

243. In what time will p dollars give i dollars interest, at $r\%$?

244. At what rate will p dollars give i dollars interest, in t years?

245. Find the amount of p dollars, at $r\%$ for t years.

246. If the amount at $r\%$ for t years is a dollars, what is the principal?

247. Divide a into two such parts that $\frac{1}{m}$ of the first part shall equal the second part.

248. What number is that whose $\frac{1}{m}$ part exceeds its $\frac{1}{n}$ by a?

249. A can do a piece of work in a days, and B can do it in b days. In what time can both together do it?

250. A and B can do a piece of work in n days, and A alone can do it in a days. In what time can B do it?

251. A can do a piece of work in a days, B in b days, and C in c days. In what time can all together do it?

252. Two persons buy an article for c dollars. One pays a dollars and the other pays b dollars. What part of the cost does each pay?

253. What time of day is it when $\frac{1}{n}$ of the time past midnight equals the time to noon?

254. How far can a person ride at the rate of a miles an hour, so as to return at the rate of b miles an hour, and be gone n hours?

255. A man receives \$$a$ a day for his work, and forfeits \$$b$ a day for every day's idleness. At the end of r days, he receives \$$c$. How many days does he work?

256. A grocer mixed a pounds of tea worth x cents a pound, b pounds worth y cents a pound, and c pounds worth z cents a pound. Find the value of the mixture per pound.

In problems 232–256, find each algebraic formula; substitute an appropriate number for each letter, and again solve each problem; compare the numerical result with the formula previously obtained.

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