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KONINKLIJKE AKADEMIE
VAN WETENSCHAPPEN
:- TE AMSTERDAM :-

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KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS OF THE MEETING
of Friday December 24, 1909.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige
Afdeling van Vrijdag 24 December 1909, Dl. XVIII).

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Mathematics. — "*On the orbits of a function obtained by infinitesimal iteration in its complex plane.*" By M. J. VAN UVEN. (Communicated by Prof. W. KAPTEYN).

(Communicated in the meeting of November 27, 1909).

When a function $y = g(x)$ is iterated, each iteration $y_n = g_n(x)$ will give rise to a conform representation of the complex planes of x and y_n .

If we suppose $y = g(z)$ to be built up by means of infinitesimal iteration of the function $\lim_{m \rightarrow \infty} y_1 = \lim_{m \rightarrow \infty} g_1(x)$, so that y_n has also a meaning for broken and unmeasurable values of n , then the conform representation of $y = g(x)$ will gradually appear out of the identity belonging to $y_0 = g_0(x) = x$.

We now regard a plane V_0 as complex plane of the quantity x and we place the complex plane V_1 of the quantity $y = \varphi(x)$ parallel to V_0 at a distance h and in such a way, that the real axes and the imaginary axes are each other's orthogonal projection. Then to each point x of V_0 are conjugated by means of the function $y = \varphi(x)$ one or more points y of V_1 . By connecting corresponding points x and y by rays a congruence of rays is formed which can serve as the image of the function $y = \varphi(x)$.

For the case $y = \varphi(x) \equiv x$ we should obtain in this way the congruence of rays formed by all the normals on the planes V_0 and V_1 as representative of the identity.

If now we let the function $y = \varphi(x)$ gradually arise from the identity, then to each stage of the generating process a definite congruence of rays will belong. All these congruences form together a complex of rays. It is clear, that the formation of the function $y = \varphi(x)$ will now be represented by this complex of rays.

Let us first examine the complex cones of the points of V_0 . Each point $x = u + iv$ of this plane is the vertex of a cone counting in any case the normal in x on V_0 among its generatrices; this edge namely intersects the plane V_1 in $y = u + iv = x$.

The section of this complex cone with V_1 will pass through the point $z = x$ and all points representing the values taken by $y_n = \varphi_n(x)$ when n increases from 0 to 1. So this section also gives us a representation of the generating process of $\varphi(x)$. It goes without saying that we can continue the iteration also past $y = \varphi(x)$ and likewise that we can also regard negative values of n . The whole of the complex cone embraces in fact all functions $y_n = \varphi_n(x)$, where n varies from $-\infty$ to $+\infty$. Also the section regarded as a whole will contain all the values of the function $y_n = \varphi_n(x)$, where x is constant and n varies from $-\infty$ to $+\infty$. Each value of x possesses its own complex cone and therefore also its own section. We shall indicate this section by the orbit $x \rightarrow y_n$.

We might also have indicated the increase of $\varphi(x)$ by allowing the plane V_1 to grow gradually out of V_0 and that by allowing the distance of the planes to increase regularly from 0 to h , so that $\varphi_n(x)$ is represented in the plane V_n at a height nh above V_0 . Let us then suppose in each plane V_n the image $y_n = \varphi_n(x)$ belonging to some initial-point $x = u + iv$ to be constructed, then all these points will form in their regular succession a twisted curve. Each of the ∞^2 points x of V_0 gives rise to a suchlike *twisted curve* and the function $y = \varphi(x)$ with its different stages of development is thus represented by a *congruence of twisted curves*.

It is clear that the orthogonal projection of the twisted curve of x on the plane Γ_1 coincides with the orbit $x \rightarrow y_n$.

We shall for the present occupy ourselves only with the study of such an orbit $x \rightarrow y_n$.

To find the orbit $x \rightarrow y_n$ we have but to solve the functional equation of ABEL. We have namely to find that function $f'(x)$ of x increasing with n when for x is substituted $y_n = g_n(x)$; this function increases for the process of iteration with real contributions, i.e. the quantity $\xi = f'(x) = U + iV$ describes in its complex plane the right line $\Gamma = c$ parallel to the real axis. If once we know the form of the function $\xi = f'(x)$, then we also know the orbit of the quantity $x = f'_{-1}(\xi)$.

The value of V and the initial value ($n = 0$, of the real part U of ξ represent together two arbitrary constants, of which we do not dispose until we choose the initial value of x .

We shall indicate the current point (y_n) of the orbit $x \rightarrow y_n$ by z , whilst we shall point out x by z_0 ; we then have

$$f'(z) = f'(z_0) + n$$

or

$$U + iV = U_0 + iV_0 + n,$$

so that

$$U = U_0 + n, \quad V = V_0.$$

The choice of the initial point z_0 now determines the values U_0 and V_0 .

When working out some examples we shall not always follow the systematic way sketched above, as it is unnecessarily lengthy in simple cases.

In reference to the broken linear function $y = \frac{\alpha x - \beta^2}{\gamma x + \delta}$ we notice that this has been thoroughly investigated already by POINCARÉ¹⁾ and KLEIN²⁾, the latter having also included complex values of α, β, γ , and δ in the study. KLEIN too allows the function $y = \frac{\alpha x - \beta^2}{\gamma x + \delta}$ to arise gradually out of x and regards the orbit described thereby. For the non-parabolic cases he builds up the function by infinitesimal iteration in the sense indicated by us. For the parabolic case, on the other hand, he takes as parameter of the function in its orbit not the iteration-index n , but a complex multiple of it. In consequence of

¹⁾ POINCARÉ. Acta Mathematica I (1882), p. 1.

²⁾ KLEIN—FRICKE. Vorl. ü. d. Theorie der ell. Modulfunktionen (TEUBNER, 1890), p. 165.

this the orbit of z found by KLEIN differs a little from ours. Although after stating and annulling this difference we might suffice with a reference to the results of KLEIN, we will dwell a little longer on the function $y = \frac{ax + \beta}{\gamma x + \delta}$, the more so as, differing from KLEIN, who treats first simple cases and then applies the principle of transformation of the circle correspondence, we shall immediately investigate the most general case.

Examples:

I. $y = x + \beta$, $y_n = x + n\beta$ or $z = z_0 + n\beta$.

The point z describes the *right line* connecting the points $z = z_0$ and $z = z_0 + \beta$, in such a way that the distance from z to z_0 is proportional to n .

II. $y = ax$, $y_n = a^n x$ or $z = a^n z_0$.

Let us put $z = \rho e^{i\theta}$, $z_0 = \rho_0 e^{i\theta_0}$, $a = \sigma e^{i\tau}$, then

$$\rho e^{i\theta} = \sigma^n \rho_0 e^{i n \tau} \rho_0 e^{i \theta_0},$$

from which ensues

$$\rho = \sigma^n \rho_0, \quad \theta = \theta_0 + n\tau \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\rho = \rho_0 \sigma^{n - \theta_0/\tau} = e^{n\lambda}.$$

Point z describes a *logarithmic spiral* round the origin. The polar angle θ increases uniformly with n , i. e. w. the polar angle θ increases *arithmetically uniformly*; it is clear that the radius vector ρ increases *geometrically uniformly*.

If a is real, then $\tau = 0$. The second equation (1) tells us that the polar angle remains constant, so that point z moves along the *line connecting* O and z_0 and that with a geometrically uniform increase of ρ .

If $\text{mod } a = 1$, then $\sigma = 1$. The first equation (1) then indicates, that the radius vector remains constant, so that point z describes the circle round O as centre, passing through point z_0 . The polar angle θ increases arithmetically uniformly.

If τ is commensurable with π , i. e. if a is a root out of unity, then $y = ax$ leads back to x after a whole number of iterations.

III. $y = ax + \beta$, $f(x) = \frac{\log \left(x + \frac{\beta}{a-1} \right)}{\log a}$.

$$f(x) = \frac{-\alpha}{\log a} \text{ for } x = -\frac{\beta}{a-1} = g, \text{ therefore } f(z) = \frac{\log(z - g)}{\log a}.$$

If we displace the origin to g and if accordingly we call $z - g = \rho'e^{i\theta'}$, we find for the orbit of z the *logarithmic spiral* $\rho' = ce^{i\theta'}$ round the point g . If, however, α is real, then z describes the line from z_0 to $\alpha z_0 + \beta$, containing also point $g = -\frac{\beta}{\alpha-1}$. Is on the contrary $\text{mod } \alpha = 1$, then the orbit of z is a circle round g as centre.

$$IV. \quad g = \frac{\alpha z_0 + \beta}{\gamma z_0 + \delta}, \quad \text{where } (\alpha - \delta)^2 + 4\beta\gamma \gtrless 0.$$

$$f(x) = \frac{1}{\lambda} \log \frac{px+1}{qx+1}, \quad \text{where}$$

$$\lambda = \log \frac{(\alpha + \delta) + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}}{(\alpha + \delta) - \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}} = \log \frac{\{(\alpha + \delta) + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}\}^2}{4(\alpha\delta - \beta\gamma)},$$

$$p = \frac{(\alpha - \delta) + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}}{2\beta}, \quad q = \frac{(\alpha - \delta) - \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}}{2\beta}.$$

We shall take as general case, that β , γ , and δ are all complex; then λ , p , and q will also be complex.

$$f(z) = \frac{1}{\lambda} \log \frac{pz+1}{qz+1} = \frac{1}{\lambda} \log \frac{p}{q} + \frac{1}{\lambda} \log \frac{z+p^{-1}}{z+q^{-1}} = \frac{1}{\lambda} \log \frac{p}{q} + \frac{1}{\lambda} \log \frac{z_0+p^{-1}}{z_0+q^{-1}} + n.$$

From

$$\log \frac{z+p^{-1}}{z+q^{-1}} = \log \frac{z_0+p^{-1}}{z_0+q^{-1}} + \lambda n \dots \dots \dots (2)$$

ensues that for an infinite value of n the point z takes either the value $-p^{-1}$ or the value $-q^{-1}$. We shall call the points $z = -p^{-1}$ and $z = -q^{-1}$ the *limiting points* and we shall put $-p^{-1} = g'$, $-q^{-1} = g''$.

Thus our equation (2) becomes

$$\log \frac{z-g'}{z-g''} = \log \frac{z_0-g'}{z_0-g''} + (\mu + i\nu)n, \quad \dots \dots \dots (3)$$

where we have replaced λ by $\mu + i\nu$.

Let us choose g' and g'' as auxiliary origins and let us call

$$z-g' = z' = \rho''e^{i\theta''}, \quad z-g'' = z'' = \rho''e^{i\theta''},$$

we then find out of (3)

$$\log \frac{\rho''}{\rho''} + i(\theta' - \theta'') - \log \frac{\rho''_0}{\rho''_0} - i(\theta'_0 - \theta''_0) = \mu n + i\nu n,$$

where by separating the real part from the imaginary we find

$$\log \frac{\rho''}{\rho''} - \log \frac{\rho''_0}{\rho''_0} = \mu n, \quad (\theta' - \theta'') - (\theta'_0 - \theta''_0) = \nu n, \dots \dots (4)$$

or

$$\frac{g'}{g''} = \frac{g'_0}{g''_0} e^{2\mu n} \quad , \quad \theta' - \theta'' = \theta'_0 - \theta''_0 + 2\mu n \quad . \quad . \quad . \quad (5)$$

Elimination of n leads, when μ and r are neither of them equal to zero, to

$$g' e^{-\frac{2}{r} g'} : g'' e^{-\frac{2}{r} g''} = g'_0 e^{-\frac{2}{r} g'_0} : g''_0 e^{-\frac{2}{r} g''_0} = C \quad . \quad . \quad (6)$$

By putting

$$g' = e^t e^{\frac{2}{r} t} \quad , \quad g'' = e^u e^{\frac{2}{r} u} \quad , \quad . \quad . \quad . \quad (7)$$

we find

$$t = C r^u \quad . \quad . \quad . \quad (8)$$

The equations (7) and (8) determine together a so-called *logarithmic double spiral*¹⁾, with the points g' and g'' as poles.

From the second equation (5) ensues that the angle $\theta' - \theta'' = \varphi$ between the two auxiliary radii vectores $g'z$ and $g''z$ increases arithmetically uniformly, whilst the first equation (5) shows us that the quotient of the auxiliary radii vectores increases geometrically uniformly.

For the case α , β , γ , and δ real, some simplifications appear.

We shall distinguish three cases.

A. $(\alpha - \delta)^2 + 4\beta\gamma > 0$, $\alpha\delta - \beta\gamma < 0$.

The quantities ρ and q are real, so the points g' and g'' lie on the real axis. Furthermore we have $\rho < 0$, so that $r = \pi$.

Hence the orbit of z is a *logarithmic double spiral*, whose two poles lie on the real axis.

A special case is furnished by the condition $\alpha + \delta = 0$, or $\mu = 0$.

From the first equation (5) now ensues that the quotient of the auxiliary radii vectores is constant, so that the point z describes a circle of APOLLONIUS of the triangle $g'g''z_0$, whilst the angle $g'z g''$ increases uniformly with n . An example of the latter case is furnished by $y = \frac{1}{r}$; here $g' = +1$, $g'' = -1$.

B. $(\alpha - \delta)^2 + 4\beta\gamma > 0$, $\alpha\delta - \beta\gamma > 0$.

The points g' and g'' lie on the real axis, whilst $\rho > 0$, thus $r = 0$.

Now the second equation (5) shows us, that $\theta' - \theta'' = \varphi$ is constant, so that the point z describes the circle passing through g' , g'' and z_0 .

¹⁾ For the logarithmic double spirals the reader may consult: HOLZÄLLER, Ueber die logarithm. Abbildung etc. Zeitschr. f. Math. u. Physik., Vol. 16. (1871), p. 281.

All ∞^2 initial points z_0 furnish thus together all circles of the pencil of which g' and g'' are the base points.

Let us suppose point z determined on its orbit as point of intersection of this orbit with an element of the conjugated pencil of circles, intersecting the real axis a. o. in a point s , then evidently $g'z : g''z = g's : g''s$ holds, so that the equation (5) expresses that the quotient $g's : g''s$ increases geometrically uniformly. (This property enables us to construct easily the points z belonging to given values of n). Furthermore holds $g' = z_{-\pi}$ and $g'' = z_{+\pi}$.

$$C \quad (\alpha - \delta)^2 + 4\beta\gamma < 0.$$

The points g' and g'' lie symmetrically with respect to the real axis, p and q being conjugate complex. As mod. $\nu = 1$ we have $u = 0$. The ratio $q' : q''$ is now constant, so that the point describes a circle of APOLLONIUS of $\angle g'g''z_0$, i. e. a circle of the pencil with g' and g'' as point circles. We can again regard the point z as if originated by intersection of the orbit with a circle of the conjugated pencil of circles. As the angle $g'zg''$ increases uniformly with n we can easily construct with the aid of the conjugated pencil of circles the points z belonging to definite values of n . It is clear that the orbit of z when n increases indefinitely is described innumerable times, so that the function $g_n(x)$ has as a function of n a real period. If r is commensurable with π , then this period is a measurable number.

If particularly $\alpha + \delta = 0$ holds, then $r = \pi$. This case is a. o. realized in the function $y = \frac{-1}{x}$; here $g' = i$, $g'' = -i$.

$$V. \quad y = \frac{\alpha x + \beta}{\gamma x + \delta}, \text{ where } (\alpha - \delta)^2 + 4\beta\gamma = 0.$$

Here we are in the *parabolic* case.

$$f(x) = \frac{\alpha x + \beta}{\frac{\alpha - \delta}{2}x + \beta},$$

$$\text{or, if we put } -\frac{\beta}{\alpha} = a, \quad \frac{-2\beta}{\alpha - \delta} = \frac{\alpha - \delta}{2\gamma} = g,$$

$$f(x) = \frac{2\alpha}{\alpha - \delta} \cdot \frac{x - a}{x - g},$$

so

$$f(z) = \frac{2\alpha}{\alpha - \delta} \cdot \frac{z - a}{z - g} = \frac{2\alpha}{\alpha - \delta} \cdot \frac{z_0 - a}{z_0 - g} + n,$$

so that

$$z = \frac{a}{g} = \frac{z_0 - a}{z_0 - g} \cdot \frac{a - d}{2a} \cdot n = \frac{z_0 - a}{z_0 - g} \cdot (\mu + ir)n \dots (9)$$

The difference between our method and that of KLEIN arises from the fact that KLEIN allows the quantity $\frac{a-d}{2a} \cdot n$ to increase *really*.

If we take a and g as auxiliary origins and if we put

$$z - a = z' = \rho' e^{i\theta'}, \quad z - g = z'' = \rho'' e^{i\theta''},$$

then the equation (9) takes the form of

$$\frac{\rho' e^{i\theta'}}{\rho'' e^{i\theta''}} = \frac{\rho'_0 e^{i\theta'_0}}{\rho''_0 e^{i\theta''_0}} + (\mu + ir)n$$

or

$$\frac{\rho'}{\rho''} \{ \cos(\theta' - \theta'') + i \sin(\theta' - \theta'') \} = \frac{\rho'_0}{\rho''_0} \{ \cos(\theta'_0 - \theta''_0) + i \sin(\theta'_0 - \theta''_0) \} + (\mu + ir)n,$$

from which ensues

$$\left. \begin{aligned} \frac{\rho'}{\rho''} \cos(\theta' - \theta'') &= \frac{\rho'_0}{\rho''_0} \cos(\theta'_0 - \theta''_0) + \mu n, \\ \frac{\rho'}{\rho''} \sin(\theta' - \theta'') &= \frac{\rho'_0}{\rho''_0} \sin(\theta'_0 - \theta''_0) + r n. \end{aligned} \right\} \dots (10)$$

If we put $\mu = \sigma \cos \tau$, $r = \sigma \sin \tau$ (i. o. w. $\frac{a-d}{2a} = \sigma e^{i\tau}$) we find out of (10) when eliminating n :

$$\frac{\rho'}{\rho''} \sin(\theta' - \theta'' - \tau) = \frac{\rho'_0}{\rho''_0} \sin(\theta'_0 - \theta''_0 - \tau) = c. \dots (11)$$

It is clear, that the orbit as found by KLEIN follows from ours by putting $\tau = 0$. The orbit of KLEIN can thus serve as iteration-orbit for real values of the quantity $\frac{a-d}{2a}$, thus of $\frac{d}{a}$.

To investigate the curve determined by the equation (11) we imagine the circle passing through g and a and of which the arc ga amounts to 2τ , so that from each point of the supplementary arc the line ga is seen under the angle τ . (See fig. p. 511).

If we connect g with z_0 and z , the connecting lines will meet the circle in m_0 and m .

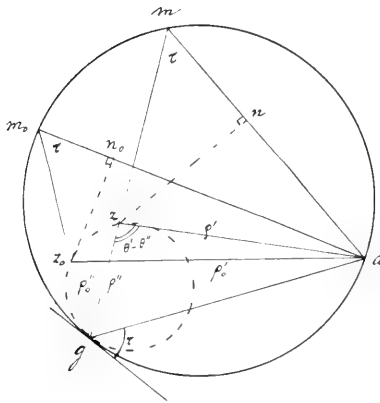
Now $\sphericalangle gma = \sphericalangle gma_0 = \tau$

Furthermore $\sphericalangle zam = \theta' - \theta'' - \tau$, $\sphericalangle za_0m_0 = \theta'_0 - \theta''_0 - \tau$.

If we let fall the normals z_0m_0 and zm on am_0 and am , then $z_0m_0 = \rho'_0 \sin(\theta'_0 - \theta''_0 - \tau)$ and $zm = \rho' \sin(\theta' - \theta'' - \tau)$.

The equation (11) now demands

$$\frac{zm}{z_0m_0} = \frac{z_0m_0}{z_0g} \quad \text{or} \quad \frac{zg}{z_0g} = \frac{zm}{z_0m_0}.$$



It is therefore evident that we arrive from points m to points z by diminishing or enlarging the chords gm in a definite ratio.

So the orbit of z is a circle touching the auxiliary circle (m) in g , whose tangent in g forms in that way the angle τ with the line ga .

If the quantities α , β , γ , and δ are real, then u and g are real, whilst $r=0$, therefore also $\tau=0$. The points u and g therefore lie on the real axis and the orbit of z touches the real axis in the point g . If on the other hand $u=0$, then the centre of the orbit lies on the line ga .

The way in which z changes with u we can read from the equations (10).

If we suppose the point z to be furnished by the circle, which passes through g and z and whose centre lies on ga , then the first equation (10) tells us that the *reciprocal* value of the radius of that circle increases arithmetically uniformly that i. o. w. the radii of the circles through g whose centres lie on ga and which pass through z_1, z_2 etc., form an *harmonic series*. If on the other hand we suppose that the point z is constructed as point of intersection of its orbit with the circle through z touching the line ga in g , it then follows easily out of the second equation that also the *reciprocal* value of the radius of this circle increases, arithmetically uniformly, that i.o.w. the radii of the circles touching ga in g and passing through the points z_1, z_2 etc. form an *harmonic series*.

It is clear that for the case α, β, γ , and δ real, thus $r=0$ and $\tau=0$, only the first determination of the course of z can serve,

whilst in the case $\mu = 0$ only the second determination retains its validity.

$$VI. \quad y = x^a, \quad y_n = x^{2n}, \quad f(x) = \frac{\log \log x}{\log a}.$$

$$f(z) = \frac{\log \log z}{\log a} = \frac{\log \log z_0}{\log a} + n.$$

Let us put $\log a = \mu + i\nu$, we then have

$$\log \log z = \log \log z_0 + (\mu + i\nu)n,$$

so

$$\log z = \log z_0 \cdot e^{\nu n} (\cos \nu n + i \sin \nu n),$$

or

$$\log \rho + i\theta = (\log \rho_0 + i\theta_0) e^{\nu n} (\cos \nu n + i \sin \nu n), \quad \dots \quad (12)$$

from which ensues

$$\left. \begin{aligned} (\log \rho)^2 + \theta^2 &= (\log \rho_0)^2 + \theta_0^2 e^{2\nu n}, \\ \log \rho &= \log \rho_0 \cos \nu n - \theta_0 \sin \nu n \\ \theta &= \log \rho_0 \sin \nu n + \theta_0 \cos \nu n \end{aligned} \right\} \dots \dots \quad (13)$$

Out of these equations follows by elimination of n the orbit of z . For the case a positive, so $\nu = 0$, the second equation passes into

$$\frac{\log \rho}{\theta} = \frac{\log \rho_0}{\theta_0} = \alpha.$$

or

$$\rho = \rho_0^\alpha.$$

The orbit of z is in this case a logarithmic spiral around the origin, which is *independent* of a .

If $\text{mod } a = 1$, then $\mu = 0$, so that the first equation (13) tells us that

$$(\log \rho)^2 + \theta^2 = (\log \rho_0)^2 + \theta_0^2 = \tau^2$$

or

$$\rho = e^{\pm \sqrt{\tau^2 - \theta^2}}.$$

This curve is likewise independent of the argument of a .

The function $y = x^{-1}$, which we have regarded on one hand under IV A, $\mu = 0$, and which then furnished for the orbit of z a circle, we can also range under the case treated last. If namely we take $y = x^{-1}$ as a special case of $y = x^a \pmod{a = 1, \text{arg. } a = \pi}$, we then find for the orbit of z quite a different curve.

To this remarkable property of $y = x^{-1}$ we hope to refer more explicitly later on.

Physics. — “*On the motion of the bridge of the violin*”. By J. W. GILTAY and Prof. M. DE HAAS. Communicated by Prof. H. KAMERLINGH ONNES).

(Communicated in the meeting of November 27, 1909).

I. In the following lines an account is given of an experimental research the object of which was to make a contribution to our knowledge of the manner in which the vibrations of the strings are transmitted to the roof of a violin by the bridge.

As far as we know the literature on the physics of bow instruments is very limited and leaves the true nature of the motion of the bridge undecided.

HELMHOLTZ¹⁾ says: “Der eine Fuss des Steges ruht auf einer relativ festen Unterlage, nämlich auf dem sogenannten Stimmstocke, einem festen Stäbchen, welches zwischen der oberen und unteren Platte des Körpers eingebaut ist. Der andere Fuss des Steges allein ist es, welcher die elastischen Holzplatten und mittels deren Hilfe die innere Luftmasse des Körpers erschüttert.”

From this description cannot be inferred whether the bridge vibrates principally in its own plane i. e. at right angles to the longitudinal direction of the strings, or at right angles to its own plane i. e. in the direction of the strings.

VAN SCHAIK²⁾ remarks: “By the vibrations of the bowed string a motion of the bridge is set up which consists in an oscillation about a line parallel to the length of the violin: in this manner the movable foot of the bridge communicates vibration to the roof of the violin and thus to the air.” His opinion therefore is that the bridge vibrates in its own plane perpendicularly to the direction of the strings.

APIAN-BENNEWITZ³⁾ observes: “dass nämlich der rechte Fuss eine viel geringere Bebung als der linke zu machen hat und dass die Thätigkeit des linken Fusses als eine hämmernde zu bezeichnen ist.” His view is thus the same as VAN SCHAIK's, as appears further from page 133 of his book.

BARTON⁴⁾ in conjunction with GARRET and afterwards with PENTZER has

¹⁾ Tonempfindungen, 3e Ausg. p. 146.

²⁾ Dr. J. BOSSCHA, Leerboek der Natuurkunde, III, bewerkt door Dr. W. C. L. VAN SCHAIK, 5th Ed., p. 170.

³⁾ Die Geige, der Geigenbau und die Bogenverfertigung, WEIMAR, BERNHARDT FRIEDRICH VOIGT, 1892, p. 125.

⁴⁾ Philosophical Magazine, 6th Series, Vol X, XII and XIII.

investigated the nature of the vibrations of the string, bridge, and roof of a sonometer as also of the air inside the sonometer. He examines both motions of the bridge and finds that for the same point of the bridge the displacement by the horizontal motion, i. e. in the direction of the string, is about 17 times the amplitude of the vertical motion ¹⁾. As the bridge of the sonometer is entirely different in shape from the bridge of the violin and the sonometer is moreover not fitted with a sound bar, the results of the investigation are not immediately applicable to the motion of the bridge of the violin.

SAVART ²⁾ in his very important memoir on string instruments does not refer to the motion of the bridge.

2. It seemed to us a priori somewhat improbable that as VAX SCHAIK and others suppose a comparatively massive object like the bridge by vibrating as a whole in its own plane about one of its corners should be able to follow completely the intricate motions of the strings and communicate them to the roof of the violin. It seemed to us more probable that, as BARTON found for the sonometer, both motions should be taken into account.

In order to investigate this experimentally we proceeded as follows.

Fig. 1 represents a violin-bridge manufactured by the well known makers CARESSA & FRANÇAIS of Paris. Fig. 2 shows a small

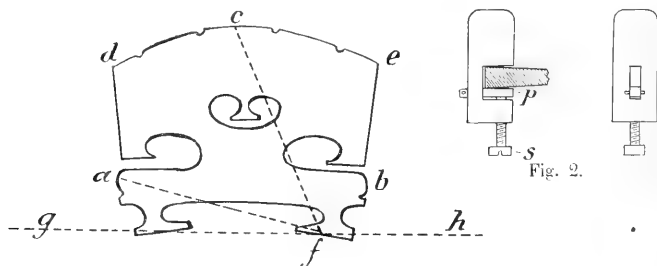


Fig. 1.

metal clamp which can be attached to the bridge at different points. In order not to damage the bridge the screw *s* does not press

¹⁾ Phil. Mag. Ser. 6, Vol. XIII, p. 451.

²⁾ "Mémoire sur la construction des instruments à cordes et à archet." A reprint of this paper is to be found in: "Nouveau Manuel complet du luthier", by MAUGIN and MARGNE. Paris, librairie encyclopédique de ROBERT, 1894, p. 333—398.

directly against the bridge but against a moveable piece of steel p . The weight of the clamp is rather more than 7 grammes.

If the bridge swings in its own plane about its right foot f' , then when we attach the clamp to the bridge at a , the moment of inertia of the bridge about the axis of rotation at f' perpendicular to the plane of the bridge will be much increased.

On the other hand when we fix the clamp at b , the effect on the moment of inertia will be much smaller.

We found however that there was very little difference in the sound of the violin in the two cases. By fixing the clamp at a some damping influence was noticeable in the g string; at b the e string was somewhat damped.

In view of the effect of the clamp being about the same in both cases it is difficult to conclude that the bridge swings principally in its own plane about one of its feet. Moreover the influence of the damper was in *both* cases very small.

The following experiment speaks even more clearly.

The distance between the middle of the right foot and the middle of the upper edge of the bridge $f'c$ is in our case 38 mms. The distance $f'a$ is 37 mms.

When the clamp is placed at c a strongly damped sound is obtained: this is the well known mute-effect, but even stronger in our case than with the ordinary mute which weighs only about 4 grammes as against ours which weighs over 7 grammes. At a the effect is as we saw, extremely small.

As $f'c$ and $f'a$ are approximately equal, the increase of the moment of inertia of the bridge is about equal in both cases. If the sound were transmitted by the bridge chiefly by its vibrations about an axis at f' , the damping effect of our clamp should be about equal in both positions.

As this appears not to be the case we cannot but infer from these experiments that the motion of the bridge in its own plane is not of primary importance for the transmission of the vibrations of the strings to the roof of the violin.

We subjoin as an instance some results obtained by two independent observers each playing his own violin.

Violin with strong sound, about 50 years old, maker unknown, model MAGGINI, very large.	Old violin by a pupil of STAINER'S, small strongly arched model, fine mellow sound, but not strong in tone, d string least fine, a string by far the best, e also very good.
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Metal damper at *a*. (Fig. 1).

Some damping effect, especially *g* string much less fine than without the *g* string. Rather strong out damper.
 nasal sound. *d* harder and inferior.
a inferior.
e improved.
 none of the strings damped, respond as promptly as without.

Metal damper at *b*.

Some damping effect, especially *g* string better than usual.
 on the *e* string. *e* better than *d* .. worse
 usual. *a* .. " " "
e .. " " "
g, *d* and *a* respond more promptly than otherwise. The *e* string is slightly damped.

Metal damper at *c*.

Damping much stronger than at *a*. Mute effect on all strings, but Effect the same as with a mute, much more strongly damped only less good than with an than with the ordinary mute. ordinary mute.

It will be seen that the two observers agree entirely as regards the main effect: the damper at *c* gives the ordinary mute effect. At *a* and *b* the effect is absent or at least only very small; again both observers find the effect of placing the clamp at *a* about the same as at *b*.

The small differences in the results of the two observers may be due to individual differences but also to the great difference between the two instruments.

The following observations prove also, that the parallel motion of the bridge has little influence in the transmission of the string motion to the roof of the violin.

The observers and violins were the same as in the previous experiments and the same damper of 7 grammes was used.

Metal damper at d . (Fig. 1.)

Mute effect, strongest on the g side. g string strongly damped.
 d string less, bad in tone.
 a string still less, bad.
 e " " " " "

Metal damper at e .

Damping, diminishing towards e damped, but much less than
the g side. The g string has the g in the d position of the
retained its original tone better the damper.
than the e string in the d posi- a less damped.
tion of the damper. d damped. gives the mute-sound
more than the a string, but is
still comparatively strong in
tone.
 g less damped than d , very ugly.

Both observers thus found, that in the position d the damping effect diminished towards e and vice versa.

Thus e. g. in the e position of the damper the g string was but little damped, although in this case assuming the bridge to vibrate chiefly in its own plane, the g string would act on a bridge with much increased moment of inertia which would involve strong damping.

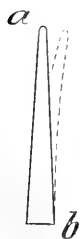


Fig. 3.

We think therefore that we may infer from these experiments that the motion of the bridge does not principally take place in its own plane about one of its feet, but that it vibrates chiefly transversely, as shown diagrammatically in Fig. 3 where ab represents the bridge in section. On this assumption the results of all the above experiments are completely explained:

I. A damper placed at a has much less damping influence than a damper at e , as the moment of inertia about gh is much less increased in the former case.

II. The effect is about the same whether the damper is attached at a or at b . It is clear that the moment of inertia of the bridge, with the clamp attached, about gh has about the same value in the two cases.

III. Again the results of the second set of experiments become

intelligible when a transverse vibration of the bridge is admitted: we found in that case that the damping effect diminished towards the right when the clamp is fixed at d and vice versa. By weighting the bridge at the top corners the vibration is no longer symmetrical; the part which is loaded at the top will vibrate less strongly than the unloaded part.

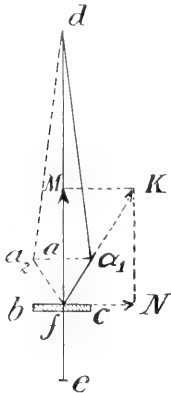


Fig. 4.

The sound of the violin is produced almost exclusively by the vibration of the roof; the string by itself imparts but a very small amount of energy to the air directly. If we suppose that the sound given by the string directly may be neglected in comparison to the much stronger sound which is due to the roof, and that the effect of the parallel motion of the bridge may also be neglected as against the much greater effect of the transverse motion, all the notes of the violin should be an octave higher than the pitch of the string, assuming that the strings deviate on both sides of the position of equilibrium.

The correctness of this conclusion however did not seem to us very probable: presumably if real, this striking fact would have been observed and communicated by previous observers.

We have therefore investigated the question experimentally by putting a steel string on a violin and making it vibrate electromagnetically.

We took a steel guitar string and put it in the position of the d string. Close to it a small electromagnet of the ROMERSHAUSEN type was fixed in a stand about vertically above the string, near the place where it is usually bowed. The coil of the electromagnet was in

circuit with three accumulators and a KÖNIG electromagnetic tuning fork ($Fa_3 = 682$ v. s.). The fork was placed in a distant room. The tension of the string was regulated until the violin when the string was bowed gave a note slightly lower than the fork. The fork was then started and the note of the string raised by pressing it with the finger until no beats were heard.

The note given out by the violin was now unmistakably Fa_3 .

Now if there really were a difference of an octave between the note of the violin (Fa_3) and the note of the string itself, the string ought under the influence of the electromagnet to have given the note Fa_2 . This is however impossible: an electromagnet magnetised by a fork Fa_3 can produce in a string the notes Fa_3 , Fa_4 , Fa_5 etc. but never the note Fa_2 . The experiment was thus by itself sufficient to show that the note given by the violin has the same pitch as the note of the string itself, even when the excursions of the string on the two sides of its position of equilibrium are about equal.

Thinking that the octave might perhaps appear, if the parallel motion of the bridge were damped down, we loaded the left foot of the bridge with our metal clamp, but even then the octave could not be heard.

As the question seemed to us of great importance we tried to solve it in a different more direct manner by an experiment in which the sound of the string was heard by itself.

On a heavy zinc-block of 80 by 40 cms and $3\frac{1}{2}$ cms thick (Fig. 5), two metal bridges are fitted (Fig. 6) at a distance from each other of $32\frac{1}{2}$ cms. An *a*-string 0,7 to 0,75 mm thick was tied

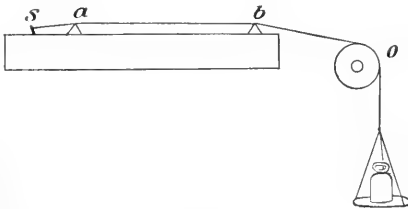


Fig. 5.

to a pin *s*, the other end being attached to a cord going over a pulley and a pan weighted with 6 kilogrammes. When bowed the string sounded a note near Ut_4 . The friction of the string on the bridges and of the cord on the pulley enabled us to slightly alter

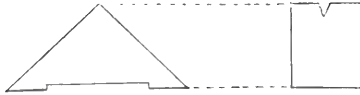


Fig. 6.

the pitch by turning the wheel of the pulley. In this manner the string was accurately tuned to Ut_4 (1023,9 v.s), so that it produced no beats in a resonator Ut_4 .

Next an a string of the same thickness was put on a violin (fig. 7). The distance of a to b was again 32,5 cms. The violin was clamped

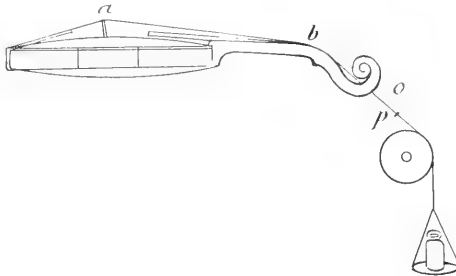


Fig. 7.

on the table with some wooden blocks; in the neck of the violin a hole O was bored, through which the string was made to pass. As the friction of the string on the usual ebony peg would have been too great, a metal peg was substituted which is represented in fig. 8.

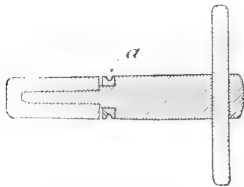


Fig. 8.

The string passes over the small metal wheel a . At p a cord was tied to the string which ran over a pulley and had a pan attached to it. The a string was now stretched by placing weights in the pan until the violin on bowing sounded Ut_4 accurately. It was found that a weight of 6 kilograms was required to do this, i. e. the same as with the zinc block.

That the violin and the string on the zinc block gave the same note, i. e. without a difference of an octave, was confirmed not merely by the ear but also by the aid of resonators: the resonator Ut_4 responded to both notes, the resonator Ut_3 did not. If the note given by the violin had been an octave higher than that of the string on the zinc plate (i. e. Ut_5) the resonator Ut_4 would not have responded to the violin note.

We have also determined the note which the string gave at the above tension by calculation.

For that purpose the string was cut at a and b (fig. 7) whereby its length shrunk to 30 cms. The weight of this piece was found to be 0,15 grams.

By substituting in the formula $t = \sqrt{\frac{pl}{gs}}$ where t is half the period, $p = 0,15$ gr, $l = 32,5$ cms, $g = 981,2$ cms sec⁻², $s = 6000$ gr, it follows $t = \frac{1}{1099}$ sec.

According to this calculation the string would have a frequency of $\frac{1099}{2} = 549,5$ complete vibrations whereas in reality the frequency was 511,9 (Ut).

These numbers agree sufficiently to show with certainty that in both cases the fundamental note of the string was heard. The comparatively small difference can be explained by assuming that the tension of the string was not exactly 6000 grams in consequence of the friction of the string on the bridges and of the cord on the pulleys.

From these experiments it appears that in the mixed sound which the violin produces the fundamental note produced by the parallel motion of the bridge and by the motion imparted to the air directly by the string is still present in sufficient intensity to give the sound the character of the fundamental as far as the pitch is concerned.¹⁾

It is indeed well known that the fundamental which determines the pitch of a composite note may be of smaller intensity than the overtones of the mixture, as HELMHOLTZ showed to be the case with the piano.²⁾

We thus know that the sound given by a violin must be ascribed to three distinct causes:

- a. a vibration imparted to the air by the string.
- b. a vibration which the roof of the violin acquires from the parallel swing of the bridge.
- c. a vibration communicated to the roof by the transverse vibration of the bridge.

The vibration mentioned under a will be left out of account as being of little importance.

¹⁾ Compare RAYLEIGH, "Theory of Sound". second ed. Vol. I p. 208 and BARRON and PENZER, Phil. Mag (6) XIII p. 452.

²⁾ Tonempfindungen, p. 134—135.

If a string is bowed the fundamental of which has a period T , the note will be accompanied by harmonics of periods $\frac{1}{2}T$, $\frac{1}{3}T$, $\frac{1}{4}T$ etc. respectively.

The parallel motion of the bridge will cause a periodical change of pressure of its left foot on the roof of the violin. When the bridge moves to the left the pressure increases and vice versa. The change of pressure may be represented by the following series:

$$a_1 \sin 2\pi \frac{t}{T} + a_2 \sin 2\pi \frac{t}{\frac{1}{2}T} + a_3 \sin 2\pi \frac{t}{\frac{1}{3}T} + a_4 \sin 2\pi \frac{t}{\frac{1}{4}T} \dots$$

The transverse motion of the bridge will also cause a change in the pressure between the left foot and the roof. When the bridge is pulled forward the front of the left foot will exert a greater pressure on the roof; when the bridge moves back the pressure diminishes. This change of pressure may be represented by a series of the form

$$b_2 \sin 2\pi \frac{t}{\frac{1}{2}T} + b_4 \sin 2\pi \frac{t}{\frac{1}{4}T} + b_6 \sin 2\pi \frac{t}{\frac{1}{6}T} + b_8 \sin 2\pi \frac{t}{\frac{1}{8}T} \dots$$

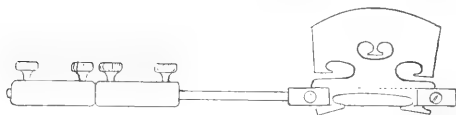
As the foot of the bridge has only a small area compared to the large surface of the violin which is set in motion, we may assume that the pressure changes which are due to the parallel and the transverse motions of the bridge respectively, occur *at the same point* of the roof. In order to find the total change of pressure produced by both motions together we must therefore add the two above series. If we assume that the excursion of the roof at the point where the left foot is attached to it is proportional to the change of pressure, the sum of the two series multiplied by a constant will give us the type of motion of the roof at that point.

It is well known that in general a sound becomes mellower according as the partial overtones become weaker and that the intensification of the even overtones especially renders the sound sharper. Many instances of this are to be found in HELMHOLTZ'S work already repeatedly quoted (p. 129—133 and p. 151—152). As an illustration of the influence of the overtones on a mixed sound we may also mention the sound of a piano when octaves are played. When an octave is struck on the piano the two notes cannot easily be heard separate, as they can be e. g. with thirds. But only very slight musical training is required to hear in a musical recital that running octaves are played: the sound is then sharper and rougher. The same holds for running octaves on the violin.

When in the above series we diminish the coefficients a_1 , a_2 , a_3 etc. while leaving the b_2 , b_4 , b_6 unchanged as far as possible, the

fundamental and odd harmonics are weakened more than the even harmonics. In accordance with the above results of HELMHOLTZ the sound will thereby be made sharper. We have proved this in the following manner by experiment :

To the bridge of a violin at the lowest possible point a metal clamp, represented half size in Figs. 9*a* and 9*b*, was attached. On the left side (i. e. on the side of the *g* string) a copper rod 3 mms

Fig. 9*a*.Fig. 9*b*.

thick and 10 cms long was screwed into this clamp. At the end of this rod two ordinary binding screws were fixed, weighing about 18 grammes each.

The moment of inertia of the bridge about the axis through the right foot, perpendicular to the bridge was naturally very much enlarged by these weights. The violin now gave a characteristic nasal sound, especially in the *g* and *d* strings; the timbre resembling most the note of a hautboy. Still notwithstanding the great weakening of the fundamental it continued to impart to the sound the character by which the pitch of a note is distinguished, in other words no change of an octave was perceptible.

When in addition to the clamp shown in Fig. 9*b* the bridge was loaded with two mutes fixed on top of each other and placed on the upper edge of the bridge, the original sound was approximately recovered, as now the transverse as well as the parallel motion of the bridge was damped. Of course the response of the violin at this load was difficult. The two mutes were an ordinary ebony mute with a metal mute, as often used, placed on top.

When a_1 , a_2 etc. and b_2 , b_4 etc. are all diminished in the same proportion the form of the curve of motion will not change, only the amplitude diminishes: the intensity is weakened, but the timbre remains the same.

If we could diminish the b 's and leave the a 's unchanged, the sound would become mellower, as in that case only the even upper partials would become weaker, including the first overtone which has the greatest intensity of all.

A mute placed on the bridge damps both motions. But from the fact that it renders the sound mellower we think we may infer that the *b*'s are reduced by it by a higher fraction than the *a*'s.

This would mean that the transverse motion of the bridge is damped to a higher degree by putting on the mute than the parallel motion.

5. We have also tried to show experimentally that the bridge in its parallel motion turns principally about its right foot.

For this purpose we screwed two metal rings into the clamp of fig. 9, which were placed in a horizontal position. The violin was fitted with a steel string, as before moved electromagnetically. While the string was moving a small leaden ball was placed alternately in the two rings; the two balls weighed 34 grms each. They were attached to a thin cord; as nearly as possible at the same moment that one ball was lifted out, the second ball was carefully placed in the other ring. We expected that the sound of the violin would be perceptibly weakened as the ball on the right was removed and the left ball simultaneously put in. But we did not succeed in arriving at a trustworthy result in this manner; in the first place a rattling noise was sometimes apparent while the balls were being exchanged and in the second place the tone of the steel string was not always of the same intensity.

6. The conclusion therefore to be derived from our experiments is that the bridge of a violin performs a parallel as well as a transverse motion and that the timbre of the tone, given by the violin, is modified greatly when the intensity of one of the motions is altered while leaving the other motion unchanged as nearly as possible.

Herewith we have at the same time given the physical explanation of the action of the mute and also of the influence which the use of too thick or too thin a bridge has on the sound of a violin.

The action of the mute is commonly described by calling it "damping" or "deadening"¹⁾. But if the mute caused nothing but a general damping or reducing of the bridge motion, the mute would only weaken the sound, and the same effect would be obtained by bowing softly on a violin without as by bowing hard on a violin with a mute. That however is by no means the case as every one knows.

Delft, November 1909.

¹⁾ BARTON. "Textbook on sound", p. 419: "The mute is a small apparatus of wood or metal which fits on the bridge, and thus deadens the sound considerably"

Anatomy. — “*On the slope of the Foramen magnum in Primates*”.

By Prof L. BOLK.

(2nd Paper on the Comparative Craniology of Primates).

In the first paper on the anatomy of the Primate-skull, the position and shifting of the occipital Foramen in Primates was treated. This paper will be devoted more especially to the consideration of the inclination of this plane.

All the writers who have dealt with this subject have pointed out that these two features: position and inclination, stand in a certain relationship to each other, in so far as the closer the Foramen lies to the occipital pole the more vertical a position does it assume, while as it gradually approaches the middle of the cranial base the tendency is towards a horizontal position. This variation in the slope, like the shifting, has been connected with the erect gait of the human body. In the typical quadruped, where the skull more or less hangs from the spinal column, the Foramen lies at the occipital pole of the skull, and the plane is vertical; in human beings, where the longitudinal axis of the body runs vertically, the occipital Foramen lies in the middle of the cranial base, while the plane is almost horizontal. Thus it is seen that this plane is disposed to take up a position perpendicular to the longitudinal axis of the body. Another point of view, first fully developed by HUXLEY, concerns the connection which is said to exist between the slope of the plane of the Foramen magnum and the degree of prognathism¹⁾. The more pronounced the prognathism — i. e. the longer the face-skull — the more perpendicular would the Foramen magnum stand. If now a rough comparison be made of an animal's skull with a human skull, the parallelism between these two features is at once noticeable. HUXLEY, however, believed he could show it even in the skulls of different races of men. From the superposition of the mediagrams of the highly prognathous skulls of an Australian and a Negro on the skull of a Tartar, it was seen that “the plane of the occipital Foramen forms a somewhat smaller angle with the basiscranial axis in those particular prognathous skulls than in the orthognathous”.

WELCKER²⁾ holds a somewhat similar opinion, though he does not express it as being a connection between prognathism and the slope of the Foramen magnum, but between prognathism and the position

¹⁾ T. HUXLEY, On some fossil remains of man. Collected Essays. VII, p. 198.

²⁾ H. WELCKER, Untersuchungen über Wachstum und Bau des menschlichen Schädels. Leipzig 1862.

of this opening, which, however, comes practically to the same thing, if a connection between position and slope be assumed. "Biegt am Vorderschädel", he says (l. c. p. 50), "der Oberkiefer des Menschen mehr nach vorn (Prognathismus) so rückt zugleich am Hinterschädel das Foramen medullare mehr nach rückwärts". AEBY¹⁾ does not agree with HUXLEY: "HUXLEY glaubte die Neigung mit dem Prognathismus in Verbindung bringen zu können. Die Steilheit der Stellung sollte in gleichem Masse wie die letztere wachsen. In unseren Tabellen findet sich keine Bestätigung dieser Ansicht" (l. c. p. 17). AEBY himself sees a connection between the degree of development of the occiput and the slope of the Foramen magnum: "Die Abflächung des Hinterhauptes führt eine Erhöhung des Foramen magnum im Gefolge." This opinion does not really differ in principle from WELCKER's, for if the occiput be markedly flattened the Foramen magnum will lie further back, and thus the opinions of WELCKER and AEBY coincide after all with the opinion already expressed by DAUBENTON, that the For. magn. is the more perpendicular in proportion as it is pushed further backwards. The connection which HUXLEY believed he had shown was, however, of another kind, and AEBY is therefore not correct in representing his opinion as being in contradiction to HUXLEY's. For it is not impossible that the slope is proportional on the one hand to the degree of prognathism, and on the other to the position. Is then the relation between position and slope of such a constancy as TOPINARD made it appear originally when he said²⁾: "qu'il suffit de mesurer l'un des deux termes par exemple l'inclinaison du trou occipital pour connaître l'autre, c'est à dire la quantité du déplacement du trou"? This seems à priori improbable, since TOPINARD's method of determining each of the two phenomena possesses merely a very relative degree of accuracy. Indeed TOPINARD himself saw this later³⁾, and then expressed himself more cautiously: "Toutefois il n'y a pas un parallélisme rigoureux entre les deux phénomènes."

In general the above writers determined the slope of the Foramen magnum by determining the angle which was formed between the base-line adopted by them, and the line which connects basion and opisthion. The base-line in these researches connected the basion with the nasion or typhlon, and therefore ran through the skull-base. This method has been contested by BROCA, and quite justly, for the size of the angle which is formed by these two lines is

¹⁾ G. AEBY, Die schädelformen der Menschen und Affen. Leipzig 1867.

²⁾ P. TOPINARD. L'Anthropologie. 4me Edition.

³⁾ P. TOPINARD. Éléments d'Anthropologie générale.

not dependent merely on the direction of the Foramen magnum because the direction of the base-line, i. e. one of the legs of the angle, depends on several factors, e.g. the angles of the basis cranii, the length of the skull, the length of the clivus, the position of the nasion, etc. To avoid this difficulty BROCA determined the slope of the For. magn. by an angle made by the plane of this opening with a plane which is entirely independent of the cranial base, viz. that which connects the axes of the two orbitae. He constructed his "angle orbito-occipital"¹⁾. BROCA here was proceeding from the postulate that the orbital-plane, in Primates at least, is the natural horizontal plane of the skull, as, in the case of normal sight, these animals look straight before them and the orbitae in this circumstance therefore will have the same direction. The correctness of this opinion will be discussed in a following paper.

RAUBER²⁾ in a recently published treatise, returns to the old, disused method, takes as base-line again the nasion-basion, and even says that: "eine Beziehung der Neigung des Foramen occipitale auf eine andere Linie als auf die Basallinie führt sehr leicht zu Unverständlichkeiten und entbehrt zugleich der morphologischen Bedeutung".

SCHWALBE, also, lately expressed as his opinion regarding the value of AEBY's base-line as follows: "So rationell auch die von AEBY gezogene Grundlinie ist, ist sie doch nicht geeignet über die Ausbildung der verschiedenen Teile des Schädelraumes Auskunft zu geben."³⁾

There is a certain contradictoriness in this criticism. A rational base-line of a craniometrical system must be able to serve as basis for at least a primary division of the skull. I have already briefly stated my objections to base-lines which are drawn through the skull-base, and will come back to this subject in a following paper. Such a line may have a certain value as boundary line between the cerebral- and facial-skull, but as basis of a craniometrical system it is absolutely useless.

HUBER⁴⁾ finally has determined the slope of the For. magn. in Hylobates with regard to the so-called German horizontal, a method which when the skulls to be examined cannot be halved medially is preferable to those of other investigators. The probable error will here be less than by the use of the basal line and certainly likewise less than by employing the horizontal auxiliary line made use of by

¹⁾ P. BROCA. Sur l'angle orbito-occipital. Revue d'Anthropologie 1897.

²⁾ A. RAUBER. Der Schädel von Kegel. Int. Monatsch. f. Anat. und Phys. 1906.

³⁾ G. SCHWALBE. Kritik zu KOHLBRÜGGE's: Morphologische Abstammung des Menschen. Globus 11 Juni 1908.

⁴⁾ L. HUBER. Vergleichung des Hylobates und Menschenschädels. München 1902.

LISSAUER¹⁾, running from the protuberantia occipitalis externa to the point where the ala of the vomer is joined to the rostrum sphenoidal.

By the method I have adopted in determining the slope of the For. magn. I have proceeded from the base-line which was described in the first paper, and in so-doing have answered the question as to what angle is made by the plane of the occipital foramen with this line. To express this angle in all Primates always as a positive value it is not possible to measure the angle directly. For in the Primates 3 conditions occur: *a.* the opisthion lies higher than the basion, the For. magn. looks backwards, and the angle is therefore an acute one closed at the back; *b.* basion and opisthion lie at equal distances from the base-line, the For. magn. looks downward, is parallel to the base-line, and the angle = 0; *c.* the basion lies higher than the opisthion, the For. magn. looks forward and the angle is an acute one closed in front. To prevent confusion between angles of equal size in cases *a* and *c*, a + or -- sign could be used. I think, however, the variations in the inclination might be represented more simply in determining the angle made by the plane of the For. magn. with a perpendicular drawn from the basion to the base-line. In case *a* this angle is always acute, in case *b* it is a right angle, and in case *c* it is an obtuse angle.

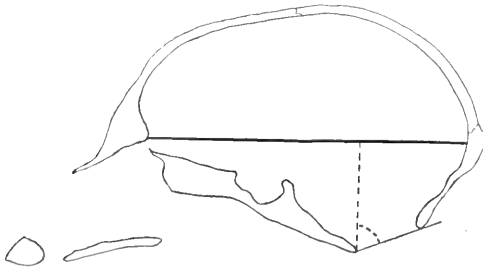


Fig. 1.

Mediagram of an *Ateles* skull, illustrating the method of determining the slope of the For. magn. ($\frac{2}{3}$ natural size).

In Fig. 1 this method is clearly seen on the mediagram of an *Ateles* skull. The following table gives the results of the researches on the skulls of full-grown monkeys.

¹⁾ LISSAUER. Untersuchungen über die sagittale Krümmung des Schädels. Arch. f. Anthrop. XV Bnd. Suppl.

- Lemur. 40. Propithecus 42.
 Mycetes. (18). 33. 45. 53. 59. Average **47.5**.
 Pithecia. 54. 56. 60. 64. Av. **58.5**.
 Hapale. 61. 61. 63. 64. 69. 72. Av. **65**.
 Chrysothrix. 60. 61. 63. 65. 66. 69. 70. 70. 71. 71. Av. **66.6**.
 Cebus. 63. 64. 64. 65. 67. 67. 68. 72. 73. 75. Av. **67.8**.
 Ateles. 66. 67. 68. 71. 77. 79. 82. Av. **72.7**.
 Cynocephalus. 63. 64. 66. Av. **64.2**.
 Inuus. 66. 68. 70. 76. 76. Av. **71.2**.
 Macacus ♂. 68. 70. 70. 74. 79. Av. **72.2**.
 Macacus ♀. 67. 73. 75. 78. 84. Av. **75.4**.
 Cercopithecus. 74. 80. 81. 82. Av. **79.2**.
 Colobus. 64. 72. Av. **68**.
 Semnopithecus. 60. 61. 61. 64. 68. Av. **62.8**.
 Siamanga. 55. 56. 56. 56. 58. 61. 63. 63. 67. 68. Av. **60.2**.
 Hylobates. 52. 60. 66. 73. 75. Av. **65.1**.
 Chimpanzee. 64. 79. 80. Av. **74.3**.
 Gorilla. 63. 63. 66. 70. 76. 77. 80. 80. Av. **71.8**.
 Orang. 58. 62. 68. 70. 72. 75. 79. 80. Av. **70.3**.

These figures show in the first place that the slope of the Foramen magnum varies greatly in individual cases, a fact which is apparent by merely looking at the skulls. This individual variability is especially noticeable in the large skulls such as those of the Anthropoids. And yet the general configuration of the skull is but little influenced by these great variations in the slope of the Foramen. As a proof

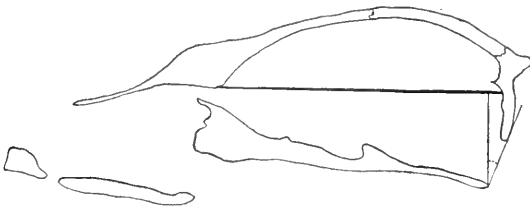


Fig. 2.

Mycetes. (♂) Angle of inclination of the For. magn. 18° .

of this, I have given in Figs. 2 and 3 the mediagrams of two Mycetes skulls, with slope-angles of 18° and 59° respectively.

From the figures it can also be seen that a slight shortening of the Clivus is of great influence on the angle of the slope. Now

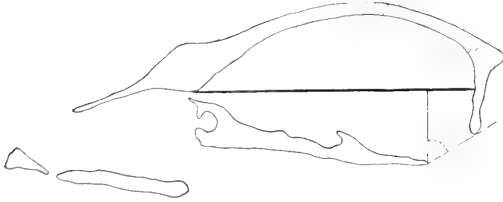


Fig. 3.

Mycetes. ($\frac{3}{4}$) Angle of inclination of the For. magn. 59° .

Mycetes occupies a foremost place in the variability of the inclination as in that of the position of the Foramen magnum for reasons fully given in the previous paper. For the other skulls, however, the same holds good. Another cause of the individual variations is the striking difference in sagittal measurement of the For. magnum especially in Anthropoids. In the Orang-outang skulls, for instance, which I used, this measurement varied from 25 to 41 mm.

Nevertheless, in spite of these individual variations some remarkable features are to be detected between the different primate-genera, especially if the series be compared as a whole with one another. It is noticeable that *Chrysothrix* does not seem to occupy the place attributed to this family in the literature on this subject. Among the Plathyrrhines, *Cebus*, and more especially *Ateles*, have greater angles, that is to say, in these genera the For. magn. lies more horizontally. In this respect the *Chrysothrix* is inferior even to most of the families of the Catarrhines. On an external observation, however, the For. magn. seems in this monkey's skull to lie horizontally in consequence of the enormous development of the occiput, and the large share that the squama occipitalis occupies in the formation of the cranial base. (See Fig. 4).

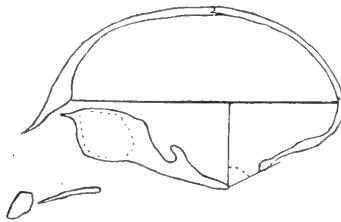


Fig. 4.

Mediagram of the skull of *Chrysothrix*. ($\frac{1}{3}$).

Among the Catarrhine monkeys, the greatest angles, 80° and more, occur among the Anthropoids and the genus *Cercopithecus*. This genus thus, also as regards the slope of the Foramen magnum, takes the high place which we have already awarded to it in the previous paper on account of the position. And similarly the genus *Siamanga* takes again the lowest place among this group of Primates. In this otherwise so highly developed monkey the Foramen magnum is inclined more vertically than in any other family of monkeys of the Old World, although it is closely followed by the genus *Semnopithecus*. A study of the skull base will afford us the opportunity of pointing out more particularly what a quite distinct place the *Siamanga* takes in the group of Primates, as regards the general form of the cranium. In the first paper I have already mentioned that it is difficult to believe that original conditions have been here persistent.

In the foregoing paper it was also pointed out that during the infantile and juvenile period the For. magn. shifts towards the occiput. It appears now that also the slope of the Foramen changes during growth. *For in the skull of a young ape the Foramen magnum lies more horizontally than in that of a full-grown one.* The following may serve as a proof of this. Whereas in a full-grown *Siamanga* the angle varied between 55° and 68°, I found in a juvenile skull (mixed dentition) an angle of 70°, and in an infantile skull (complete lactal dentition) an angle of 81°. In a Chimpanzee, with a complete set of milk teeth, the For. magn. lay almost horizontally with a angle of 88°. In three infantile Orang-outang skulls I found angles of 78°, 85°, and 86°, while finally a juvenile Gorilla skull had a angle of 87° and an infantile one even of 95°. In the case of this last skull, thus, the For. magn. looked forwards as in that of man. We shall soon see that as regards human beings also the plane of the For. magn. turns during infantile and juvenile periods in the same manner as with the Anthropoids, though I must here point out that this turning is much more pronounced in Anthropoids and *Siamanga* than in human beings.

Thus both in the position and the slope of the For. Magn. the young Anthropoid agrees more with the human conditions than the full-grown one.

In respect to the slope of the For. magn., man occupies a distinct place among all Primates, as in him the opening is not turned towards the back but towards the front. This fact, which has already been alluded to by DAUBENTON, and after him by all the writers on this subject, is illustrated by the figures below. I call to mind that

an angle of 90° obtained by my method agrees with a position of the For. magn. parallel to the base-line, i. e. a horizontal position.

Angle of the For. magn. in full-grown human skulls.

Papuans:	96°, 99, 99, 99, 100, 101, 103, 107, 107, 108, Av.	101,9°.
Negroes:	92°, 96, 07, 98, 99, 100, 100, 101, 103, 106, ,,	99,2°.
Frisians:	86°, 89, 90, 94, 95, 99, 100, 103, 103, 103, ,,	96,2°.
Zeelandians:	93°, 97, 99, 100, 101, 103, 104, 105, 109, 112, ,,	102,3°.
Javanese:	92°, 92, 97, 99, 100, 100, 103, 103, 103, 105, ,,	99,4°.

The averages of three of the groups lie comparatively near each other, and the existence of a difference between dolichocephalic skulls (the first three groups) and brachycephalic cannot be assumed on the ground of these figures, although the difference between the long dolichocephalic Frisian skulls and the short strongly brachycephalic Zeelandian skulls is very remarkable. It is also peculiar that among the Frisian skulls there were two in which the For. magn. looked slightly backwards (angles of 86° and 89°) and one where it lay exactly horizontal. This was caused by the particularly long clivus in these objects. That the degree of development of this part of the cranial base in human beings has a great influence on the slope of the For. magn. is proved by infant skulls. On an average the For. magn. in young human skulls has without exception a more considerable inclination towards the front than in full-grown ones, as will be seen from the following figures.

Angle of the For. magn. in children's skulls.

0—1 year.	110, 110, 109, 105, 104, 103, 102, 101, 100, 100, 92.
1—2 years.	100, 110, 110, 108, 106.
2 years.	107, 107, 106, 106, 103, 101, 95.
3 years.	110, 110, 108, 107.
4 years.	114, 109, 106, 105, 100.
5—6 years.	114, 113, 109, 107, 105, 103, 96, 96.
7 years.	108, 100, 100, 99, 98.
8—9 years.	104, 103, 101, 97.
10—11 years.	110, 104, 104, 101, 100, 92.

The average angle of the human full-grown skulls can from the preceding table be set at 100° . And now it is seen that of the 31

skulls of children under 5 years of age only 2 have a smaller angle while of the 23 skulls of children between 5 and 12 years of age this is so in 6 cases. From this it may be inferred that during infancy when, as has been shown in the 1st paper, a shifting of the position of the occipital foramen takes place in man, also the plane of the For. magn. turns, and in the same direction as with the Anthropoids. Yet, as has been said, this turning, like the accompanying shifting of position is more marked in Anthropoids than in human beings.

We have now seen twice over that a shifting of the For. magn. and a change in the angle go hand in hand during the individual development. For in human beings as well as in Anthropoids the shifting backwards diminishes the angle of inclination. To what degree this relation between these two features exists in comparative anatomy will be apparent from the following table. The 2nd column gives the average of the angle, while the first column shows the average basal-index as determined in the 1st paper. I may here call to mind that the greater this index is, the further backwards does the For. magn. lie.

	<i>Index basalis.</i>	<i>Angle of inclination of the For. magn.</i>
Lemur albifrons	87 (1)	40° (1)
Propithecus diadema	80 (2)	42° (2)
Mycetes	86 (3)	47.5° (3)
Pithecia	74 (6)	58.5° (4)
Hapale	71 (8)	65° (8)
Cebus	67 (10)	67.8° (11)
Ateles	64 (13)	72.7° (16)
Chrysothrix	59 (18)	66.6° (10)
Inuus	65 (12)	71.2° (14)
Cynocephalus	65 (12)	64.2° (7)
Macacus	64 (14)	73.8° (17)
Cercopithecus	57 (19)	79.2° (19)
Semnopithecus	74 (7)	62.8° (6)
Colobus	75 (5)	68° (12)
Siamanga	76 (4)	60.2° (5)
Hylobates	71 (9)	65.1° (9)
Chimpanzee	64 (15)	74.3° (18)
Gorilla	61 (16)	71.8° (15)
Orang	61 (17)	70.3° (12)

In brackets after the figures of both series is given the place number which each of the genera would take in a regular classification. A comparison of these place numbers shows at a glance in how far the position and the slope of the For. magn. go hand in hand. In general there appears to be a decided parallelism between these features in monkeys, and only in a few cases there is a fairly marked difference between position and slope. This is, for instance, the case in *Chrysothrix* where the angle is small in comparison to the position, and in *Colobus* where the reverse is the case.

At the beginning of this paper mention was made of the opinion held by HUXLEY, viz. that the slope of the For. magn. is in proportion to the degree of prognathism. In a following communication, which will deal with the prognathism of the primate skull, this view will be discussed at greater length.

Physics. — “*A short reply to Mr. VAN LAAR’S remarks.*” By Prof. PH. KOHNSTAMM. (Communicated by Prof. J. D. VAN DER WAALS).

In the proceedings of the preceding meeting of this Academy Mr. VAN LAAR made some remarks suggested by a paper by Mr. TIMMERMANS and me. Though these remarks do not call in question in any point the validity of our results, but exclusively deal with the question whether we have done sufficient justice to the share Mr. VAN LAAR has had in the construction of the theory, I think that both politeness to Mr. VAN LAAR and deference to the communicator of these remarks forbid me to leave them unanswered. So I shall try to state as shortly as possible the reasons why I still think I have done full justice to that share.

1. Mr. VAN LAAR writes in point *a* of his remarks:¹⁾ “Here I must remark that I have *never*²⁾ represented the special case $a_{12} = \sqrt{a_1 a_2}$ as the general case.”

In writing this Mr. VAN LAAR had certainly forgotten that he wrote in *These Proc.* Sept. 1906 p. 227: “In the *third* paper in *These Proceedings* (June 24, 1905) the equation:

$$\Delta = \frac{1}{T_1} \left(\frac{dT}{dx} \right)_0 = \theta \sqrt{\frac{1}{\pi}} \left[\theta \sqrt{\frac{1}{\pi}} \left(\sqrt[3]{\frac{1}{2}} - \sqrt[3]{\frac{1}{2}} \sqrt{\frac{1}{\pi}} \right)^2 - 1 \right]. \dots (3)$$

was derived . . . for the *quite general*³⁾ case $a_2 \leq a_1$, $b_2 \leq b_1$ ”, etc.

And on the same page: “Now the restricting supposition $\beta = 0$

¹⁾ *These Proc.* XII p. 455.

²⁾ Mr. VAN LAAR’S italics.

was relinquished for the determination of the *double point* of the plaitpoint line, and the *quite general case*¹⁾ $a_2 \gtrsim a_1$ $b_2 \lesssim b_1$ was considered.

And on p. 228: "We can, namely, characterize *all possible pairs*¹⁾ of substances by the values of θ and π , and finally it will *only*¹⁾ depend on *these values*,¹⁾ which of the three main types will appear."

And on p. 230: "The calculations get, however, so exceedingly intricate that they proved practically unfeasible for the *general case*¹⁾ $a_2 \gtrsim a_1$ $b_2 \lesssim b_1$."

And on p. 231: "This appears already from the fact that the substitution of the *quite general assumption*¹⁾ $b_1 \lesssim b_2$ for the simplified assumption $b_1 = b_2$ has made no change in the existence of a double point . . . , and that also the calculations for the limits of type III . . . may be carried out for the *quite general case*¹⁾ $b_1 \gtrsim b_2$."

And on p. 232: "The calculation proves that in the *quite general case*¹⁾ $b_1 \lesssim b_2$ " etc.

For, everywhere where the *general case* is spoken of here, it is the case $a^2_{12} = a_1 a_2$ that is meant, and also the quotation from p. 228 is possible only, by an identification of the general case and this special one.

2. In point *b* of his remarks Mr. VAN LAAR says in connection with our sentence that his investigations: "very onesidedly, lay the stress on the existence of open plaits, a circumstance which by no means can be considered as a *result*²⁾, as it immediately follows from the arbitrary, if not erroneous *supposition*²⁾ of the linear dependence of *b* and *x*": "Now I have never asserted that $\frac{d^2b}{dx^2} = 0$ would always agree with what actually happens; again I have simply *assumed*³⁾ this in order to make the *calculations*³⁾ possible."

Yet I read on p. 231 of the cited paper: "We shall once more emphatically point out that the *numeric*³⁾ results of our investigation will naturally be modified, when *b* is not assumed to be independent of *v* and *T* . . . but that *qualitatively*³⁾ everything will remain unchanged."

And on p. 233: "Then further increase of pressure makes the phases 1 and 2 again diverge . . . without the longitudinal plait ever closing again — *as was formerly considered possible*¹⁾ — [cf.

¹⁾ The italics are mine.

²⁾ T. and K's italics.

³⁾ Mr. VAN LAAR's italics.

inter alia VAN DER WAAALS, Cont. II p. 190 (1900)]. Only at temperatures higher than T_0 . . . there can be question of homogeneity to the highest pressures."

It seems to me that every unprejudiced reader of these lines must acknowledge that Mr. VAN LAAR thought that he gave a new *result* here, materially differing from the result of a closed plait as it was thought possible by VAN DER WAAALS, and that he cannot possibly have realized when writing these lines that this divergent result was only founded on his assumption $\frac{d^2b}{dx^2} = 0$.

3. As to point *c*, the sentence mentioned there really refers to a paper by Mr. VAN LAAR earlier than April 1905 (viz. of January 1905). I did not know, however, until the publication of the "Remarks", (and now I only know it from these "Remarks") that Mr. VAN LAAR has abandoned his views of this previous paper. Else we should, of course, not have mentioned it.

4. With regard to point *d* we must protect Mr. VAN LAAR against himself. We had said: "His results are of importance particularly because they showed that under certain circumstances non-miscibility can occur for perfectly normal substances, a fact which was generally doubted at the time." Mr. VAN LAAR remarks in this connection that it was by no means generally doubted up to now whether miscibility could occur for normal substances but only whether some special "abnormal" forms of non-miscibility could occur for perfectly normal substances. I must maintain in opposition to this that both LEHFELDT and VAN DER WAAALS, to whom we referred i.e., had by no means a special case of non-miscibility in view, but very decidedly all non-miscibility. So Mr. VAN LAAR's merit is decidedly greater than he will own here. On the other hand I must confess that in our endeavours to be perfectly objective to Mr. VAN LAAR, we have really got unjust in the above cited sentence to Mr. VAN LAAR's predecessors: VAN DER WAAALS and KORTEWEG. The above statement might lead one to think that Mr. VAN LAAR had been the first to demonstrate the possibility of non-miscibility for normal substances. As Mr. VAN LAAR justly remarks: this is incorrect, and it would have been better if our sentence had run like this: His results are of importance particularly because he adhered to the possibility of non-miscibility for normal substances in a time in which this was pretty generally doubted, and showed once more that for certain values of a 's and b 's, which could not a priori be considered as improbable, non-miscibility must really appear".

If I wanted to discuss also Mr. VAN LAAR's other remarks, I should

have to enter fully into the very heart of the matter, as I cannot assume the reader to be fully acquainted with the details of these investigations. But then I should think I abused the hospitality which this Academy so courteously extends in its publications also to non-members. So I think that the above will suffice. If Mr. VAN LAAR should, however, wish to pursue this discussion elsewhere, I am willing, though not desirous, to continue it.

Chemistry. — “*The equilibrium solid-liquid-gas in binary systems which present mixed crystals.*” By Dr. H. R. KRUYT. (Communicated by Prof. P. VAN ROMBERGH.) First communication.

In the Archives Néerlandaises [2] 5 (Jubilee number in honour of Prof. LORENTZ) p. 360 (1900) Prof. BAKHUIS ROOZEBOOM published an article “*Sur l'équilibre de cristaux mixtes avec la phase vapeur*” in which he described and illustrated the p - t - x surface of a binary system when exclusively homogeneous mixed crystals occur as a solid phase. He treats the case of unlimited miscibility in all phases and especially for a system in which the melting point line proceeds without a maximum or a minimum. He has, moreover, limited himself to the case that the three-phase line solid-liquid-gas (*SLG*) also occurs without a maximum or a minimum.

These matters have not been further investigated theoretically¹⁾; there was in fact no inducement to do so, as there has been an almost entire absence of experimental research. Only two investigators, SPERANSKI²⁾ and KÜSTER³⁾ furnished material as to the equilibrium of mixed crystals with a gas-phase, whereas the researches of HOLLMAN⁴⁾ belong to a category of more complicated phenomena.

I intend to carry out a series of investigations in order to extend our knowledge of the systems showing a miscibility in the solid condition. First of all, I will accept the facts already known and, therefore will discuss at present, *theoretically*, the various possibilities of the progressive change of the three-phase line indicated by ROOZEBOOM (i.e.) and communicate later the results of an *investigation*

¹⁾ The results obtained by A. SMITS (Proc. (1908) XI p. 165, and Zeitschr. f. physikal. Chem. (1909) 67, 464) do not differ from those of ROOZEBOOM. The only paper I know connected with this subject is a communication of MEYERHOPFER: “Ueber Reifkurven”, Zeitschr. f. physikal. Chem. 46, 379 (1903).

²⁾ Zeitschr. f. physikal. Chem. 46, 70 (1903) and 51, 45 (1905).

³⁾ Ibid. 51, 222 (1905).

⁴⁾ Ibid. 37, 193 (1901).

as to the three-phase equilibria in the system *p*-dichlorobenzene — *p*-dibromobenzene, the same system of which, thanks to KÜSTER and SPERANSKI (l.c.), we already know a series of solid-gas equilibria.

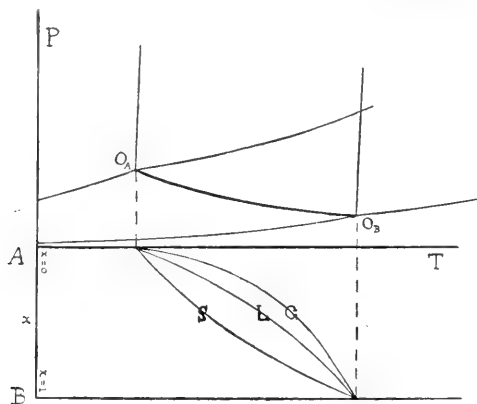


Fig. 1.

Fig. 1 is a combined PT and Tx -projection: O_A and O_B are the triple points of the components. They are connected by the three-phase line. In the Tx projection this line divides into three branches which indicate, respectively, the composition of the solid (S) liquid (L) and gas (G) phases.

Since the influence of the pressure on the equilibrium LS , is very trifling and as triple-point pressures are comparatively low, the branches S and L may, usually, be taken as being equal respectively to the melting-point curve and the freezing-point curve²⁾ at 1 atmosphere.

In fig. 1 is assumed $P_{O_A} > P_{O_B}$ ¹⁾ which case we will call chief type I. We will now ascertain under what conditions three conceivable cases might occur, namely:

- case *a* with a maximum pressure in the three-phase line
- .. *b* minimum
- .. *c* without a max. or min.

To get an insight as to the change of the pressure with the tem-

¹⁾ A (as is customary) is the name of the component with the lowest melting point and with a vapour pressure greater than that of B at the same temperature.

²⁾ In what follows we shall speak of these curves "as the branches of the melting diagram."

perature we must first of all proceed in the direction indicated by Prof. VAN DER WAALS¹⁾ where he treats of the three-phase equilibria of a binary compound with liquid and vapour.

To the ψv -surface of the liquid and vapour condition another one has to be added which shows the connection between those quantities in the homogeneous solid phase. If we consider the case occurring most frequently that the fusion takes place with an increase in volume this surface will be found between the liquid-vapour surface and the ψx -plane.

As to the form of this new ψv -surface it should be observed that it will practically be a plane with descriptive lines proceeding from the ψv -plane for $x = 0$, to that for $x = 1$. For the mixing of two solid substances to a homogeneous solid phase takes place either *without* a change in volume or *with a hardly appreciable one*²⁾.

If we now wish to know which are the coexisting phases we must allow tangent planes to move over these surfaces and thus cause the appearance of the derived surfaces and connodal lines³⁾.

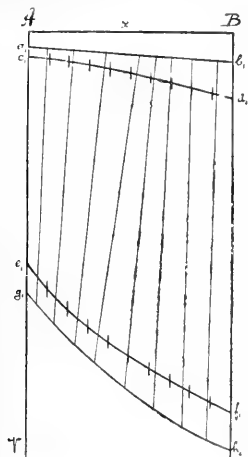


Fig. 2.

Let us commence by considering a surface for a temperature below the triple-point temperatures of the components. The surface for the solid condition will then be situated very low, the tangent plane will rest both on this surface and on the vapour part of vapour-liquid surface. The lines $a_1 b_1$ and $g_1 h_1$ in fig. 2 indicate the connodal lines so formed. The derived surface thus obtained will be situated lower than the derived surface which rests on the two parts of the vapour-liquid surface and which, therefore, does not represent stable conditions, but the vapour equilibria of "supercooled" liquids. The connodal lines ($c_1 d_1$ and $e_1 f_1$) proceeding therefrom are situated between the connodal lines of the solid-vapour equilibrium.

If we proceed to a higher temperature the correlated connodal lines

¹⁾ Verslagen Kon. Akad. V, p. 482, (1897).

²⁾ Cf. RETGERS, Zeitschr. f. physikal. Chem. **3**, 497 (1889) and GOSSNER, " " , Kristallographie **44**, 417 (1908).

³⁾ In what follows, the question whether a minimum or a maximum pressure is possible for the coexistence of two phases has not been considered. All nodal lines are therefore supposed to proceed in the same sense.

approach each other; and also the stable and metastable branches on the vapour part especially at the side of the component melting at the lowest temperature¹⁾. For if we approach the temperature of the triple point of this component the points

and g_1 of fig. 2 will have coincided to the point $e_2 g_2$ in fig. 3, which is intended for the temperature of O_1 (fig. 1). The two derived surfaces intersect each other in the ψv -plane of the component A ; that intersecting line is, of course, the tangent to the ψ -line for the gas-liquid condition of A and just the one which is also tangent to the ψ -line of solid A (triple point A).

By consulting fig. 4 it will be easily seen what happens at a temperature situated between that of the two triple

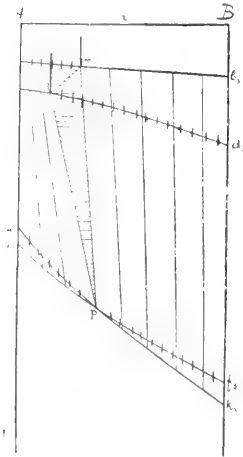


Fig. 4.

comment and a discussion of the configurations at higher temperatures will also be superfluous.

¹⁾ The non-related connodal lines ab (solid) and cd (liquid) diverge from each other because as a rule the coefficient of expansion of a substance is smaller in the solid than in the liquid state.

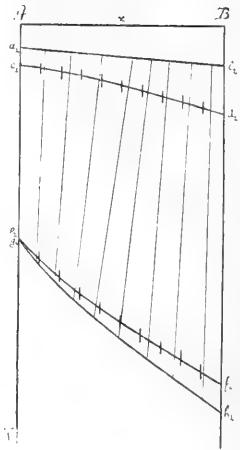


Fig. 3.

points. The rolling tangent plane coming from the A side will now rest first on the liquid- and vapour parts; but if a certain nodal line pq is thus reached the tangent plane will rest also on a point r of the surface of the solid phase. The angular points of the three-phase triangle pqr give us the composition of the three possible coexisting G , L , and S phases at that temperature. By further motion of the tangent plane a derived surface for GS equilibria is formed, whilst also a similar movement over the liquid part of the fluid surface and over the surface of the solid phase is possible in the direction of the small volumina. Hence a new system of connodal lines for LS equilibria is formed starting from r and q . Fig. 4, however, will be plainly understood without further

Prof. VAN DER WAALS (l. c. p. 490) has also taught us how to deduce an expression showing the relation between p, t and x .

From the three equations

$$V_S dp - \eta_S dt = dM_1 u_1 + x_S d(M_2 u_2 - M_1 u_1)$$

$$V_L dp - \eta_L dt = dM_1 u_1 + x_L d(M_2 u_2 - M_1 u_1)$$

$$V_G dp - \eta_G dt = dM_1 u_1 + x_G d(M_2 u_2 - M_1 u_1)$$

follows

$$\frac{dp}{dt} = \frac{\begin{vmatrix} x_S & \eta_S & 1 \\ x_L & \eta_L & 1 \\ x_G & \eta_G & 1 \end{vmatrix}}{\begin{vmatrix} x_S & V_S & 1 \\ x_L & V_L & 1 \\ x_G & V_G & 1 \end{vmatrix}} = \frac{x_S(\eta_L - \eta_G) + x_L(\eta_G - \eta_S) + x_G(\eta_S - \eta_L)}{x_S(V_L - V_G) + x_L(V_G - V_S) + x_G(V_S - V_L)}$$

This gives us a *quite general expression* for the three-phase line in the systems described. It will, however, not be easy to arrive through it to the desired elucidations. If, for instance, we wish to

know when $\frac{dp}{dt}$ will be equal to 0 the numerator thus becoming nought,

the question first arising is what do $\eta_L - \eta_G$ etc. really represent. KOHNSTAMM¹⁾ has rightly observed that such differences must not be simply called heat of condensation etc. because η_L and η_G do not relate to the same mixture. And the second question as to the numerical value of those quantities in a system to be investigated is still much more difficult to answer.

In order to get a first insight into these systems, I have taken another course though of less general applicability. We will see how the pressure changes in regard to the triple-point pressure of A , when the liquid phase has the composition x_L assuming that x_L has a very small value, in other words that but a very small quantity of B has been added to A .

The temperature T_2 at which that liquid is in equilibrium with a solid phase, the composition of which is x_S , is found from ROTHMUND'S formula²⁾ for very dilute mixtures:

$$T_2 = T_1 + \frac{RT_1^2}{q} (x_S - x_L) \dots \dots \dots (1)$$

¹⁾ Proc. Kon. Akad. IX p. 647 (1907).

²⁾ Zeitschr. f. physikal. Chem. 24, 710 (1897).

in which T_1 is the temperature of the triple point O_A .

The vapour pressure P_2 at the temperature T_2 is the sum of the partial pressures of the components p_A and p_B :

$$P_2 = p_A + p_B$$

for which we may write

$$P_2 = (1-x_L) P_{T_2} + p_B \dots \dots \dots (2)$$

if P_{T_2} represents the vapour pressure of liquid A at that temperature. If now we call P_{T_1} the vapour pressure of A at its triple point and use VAN DER WAALS' well known formula for the saturated vapour pressure we may write

$$l \frac{P_k}{P_{T_1}} = f \frac{T_k - T_1}{T_1}$$

$$l \frac{P_k}{P_{T_2}} = f \frac{T_k - T_2}{T_2}$$

By subtraction we get:

$$l \frac{P_{T_2}}{P_{T_1}} = f \frac{T_2 - T_1}{T_1}$$

$$l P_{T_2} = f \frac{T_2 - T_1}{T_1} + l P_{T_1}$$

If now we substitute the value found in (1) for T_2 we obtain

$$P_{T_2} = P_{T_1} e^{f \frac{RT_1}{q} (x_S - x_L)}$$

thus writing (2) in this form:

$$P_2 = (1-x_L) P_{T_1} e^{f \frac{RT_1}{q} (x_S - x_L)} + p_B \dots \dots (3)$$

If now case Ia (maximum pressure) is to occur, the three-phase line must rise from O_A to higher values of P and therefore $P_2 > P_{T_1}$. The chance of seeing this case realised in a certain system, therefore depends on P_2 having as great as possible a value in regard to P_{T_1} and relation (3) shows us when this will be the case. For the first term $\frac{T_1}{q}$ and $x_S - x_L$ will then be characteristic. The value of $x_S - x_L$ is indicated by the difference in initial direction of the branches of the melting point lines for solid and liquid and this difference is determined precisely by $\frac{T}{q}$. When therefore we pay special attention to $x_S - x_L$,

¹⁾ Compare VAN LAAR, Zeitschr. f. physikal. Chem. **64**, 257 (1908).

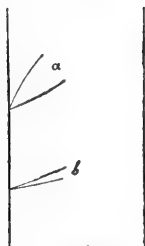


Fig. 5.

the first term of (3) will be large if $x_S - x_L$ is large, that is to say when the initial directions of the branches of the melting diagram line differ greatly (Fig. 5a).

The second term of (3) the partial \bar{x} pressure of the component B will as a rule be greater ¹⁾ when this component gets more volatile; as in the case of this chief type I we have assumed that its triple point pressure is smaller than that of A we shall have the most advantageous conditions when they differ as little as possible.

For the case Ia is, therefore, required 1. a type of melting diagram with greatly diverging branches near the A -axis and 2. about equal triple point pressures.

Case Ib (minimum pressure) makes two demands: from O_A an initial fall, but followed by a rise; if this second demand is not fulfilled we are dealing with Ic. This second demand means, of course, a small difference of the triple point pressures; the first demand, a small P_2 is, therefore, in regard to the value of p_B in (3), opposed to the second and is, in consequence, determined altogether by the first term of (3). In order that this may be as small as possible it is, of course, required that $x_S - x_L$ shall approach 0 as closely as possible, a demand which is complied with in a melting diagram with branches almost coinciding in the initial direction. (Fig. 2b).

We arrive at an identical result if we start from the triple point of B and examine the vapour pressure P'_2 of a liquid containing a little of A , when that liquid can also coexist with a solid phase. In this case the relations (1), (2), and (3) become:

$$T'_2 = T_1 - \frac{RT_1'^2}{q'}(x_S - x_L) \dots \dots \dots (1bis)$$

$$P'_2 = p_A + x_L P'_{T_2} \dots \dots \dots (2bis)$$

$$P'_2 = P'_{T_1} e^{-\frac{RT_1'}{q'}(x_S - x_L)} + p_A \dots \dots (3bis)$$

which will be readily understood on considering that the accentuated signs have the same significance for B as the non-accentuated ones had above for A .

In the case of Ib the three-phase line must descend from B , therefore $P'_2 < P'_{T_1}$. Now first of all p_A should be at a minimum

¹⁾ Apart, therefore, from special differences in the critical quantities and of special influences of the components on each other.

which, on the same supposition as above, again demands about equal triple-point pressures for A and B ; secondly, the exponent of e with a negative sign should be as large as possible, which requires widely diverging branches in the melting diagram at the side of the component B .

These demands put from two sides are brought into agreement by a conclusion of VAN LAAR (loc. cit. p. 265) that closely adjacent branches in the melting diagram at the side of the one component cannot meet a similar configuration at the side of the other.¹⁾ If this were possible, the occurrence of a maximum and a minimum in one three-phase line would be quite possible.

In the case of Ib we therefore, require:

1. Melting diagram with branches nearly coinciding at the side of the A -axis and 2. about equal triple-point pressures.

Case Ic finally occurs as an intermediate case between the two previous extreme cases. Of course, the line $O_A O_B$ may be concave or convex in regard to the temperature axis; this depends on whether the conditions for Ia or Ib have been partially fulfilled. Let us call these cases Ic_1 and Ic_2 respectively. For definite forms of the melting diagram points of inflection may probably occur, but our mode of treatment is inadequate for their investigation.

A single remark may be made as to the chance of observing a fall of the three-phase line starting from O_A . As stated, the following condition is required:

$$(1-x_L) P_{T_1} e^{-\frac{fRT_1}{q}(x_S-x_L)} + p_B < P_{T_1}.$$

If now we imagine the most favourable circumstance, in which p_B may be neglected (because the components differ, for instance, very much in their melting temperature) the factor $(1-x_L)$ will cause a

decrease and the factor $e^{-\frac{fRT_1}{q}(x_S-x_L)}$ an increase in the value of the first member in regard to that of the second one. For $1-x_L$ is always < 1 ; the other factor is > 1 and only in the case of $x_S = x_L$ it is equal to 1: in that case a fall may be expected, but as soon as x_S and x_L differ in value the enlarging factor appears and the said difference occurs therein *exponentially*. The enlarging influence will, therefore, very soon exceed the other, so that the chance for realising the case Ic will be diminished and that for Ib will be reduced to a minimum.

¹⁾ At least when we make the same suppositions as in the footnote on p. 543.

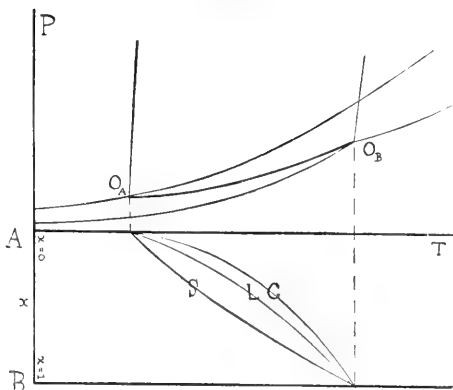


Fig. 6.

Let us now consider a second category of possibilities, namely $P_{O_A} < P_{O_B}$ which case we will call chief type II.

We again distinguish three possibilities, viz.

- | | |
|--------------------|----------------------------------|
| a. maximum | pressure in the three-phase line |
| b. minimum | " " " " " |
| c. no max. or min. | " " " " " |

It will be superfluous to repeat the previous arguments when we examine the initial directions in the equations (3) and (3bis). The conclusions arrived at are that we require for:

Case IIa: a melting diagram with closely joined branches at the side of the component *B*, and but slightly differing triple-point pressures.

Case IIb: a melting diagram with closely joined branches at the side of the component *A*, and but slightly differing triple-point pressures.

Case IIc will be again the intermediate case between the two previous ones; a concave (IIc₁) and a convex (IIc₂) course will again be possible.

In a future paper, I hope to communicate the results of an experimental investigation of the system *p*-dichlorobenzene — *p*-dibromobenzene which has been going on already for a considerable time.

November 1909.

Utrecht, VAN 'T HOFF-laboratory.

ERRATA.

p. 438 line 16 from the top: for 1000 read 10000.
(January 26, 1910).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS OF THE MEETINGS
of Saturday January 29 and February 26, 1910.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige
Afdeling van Zaterdag 29 Januari en 26 Februari 1910, Dl. XVIII.)

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Geology. — "*Pienaarite, a melanocratic foyaité from Transvaal.*"
By H. A. BROUWER. (Communicated by Prof. G. A. F. MOLENGRAAFF.)

(Communicated in the meeting of November 27, 1909).

Among the nepheline syenites on and to the west of the farm Leeuwfontein to the north-east of Pretoria, which show a complete series of varieties in chemical and mineralogical composition, the "collection MOLENGRAAFF" contains a variety very rich in titanite, which occurs $\frac{1}{4}$ mile to the west of the Pienaar's river near the boundary of the farm Zeekoegat.

Macroscopically the rock shows red feldspars to 1 cm. up in length, which have a tabular development after (010), and smaller crystals of red nepheline, with which contrast numerous slender prisms of aegirine and bright crystals of titanite, which make up over half of the rock.

Under the microscope the rock is seen to consist of felspar, nepheline, less sodalite, much aegirine, (aegirine augite) and titanite and small quantities of apatite, fluorine, calcite, analcime, and titanite iron ore.

The felspars are orthoclase and micropertthite in Carlsbad twins.

Nearly always nepheline and sodalite are transformed, respectively into pseudomorphoses of mica and zeolites. In the crystals of nepheline, which are not entirely transformed into mica, the transformation begins along the fissures, but nearly all the crystals are entirely altered. The sodalite pseudomorphoses consist of zeolites, in which we find distributed some small flakes of mica.

The aegirine is strongly pleochroic from olive-green to yellowish green, some crystals are homogeneous, other ones contain a centre of aegirine augite, which has for the greater part very low extinction angles; they are very rich in inclusions of small crystals of titanite and apatite, and they are strongly impregnated with fluorspar.

The titanite forms the well known twins after (001), in the rhombic sections the long diagonal is the twinning plane; both individuals are polysynthetically twinned. They are pleochroic from salmon coloured to colourless.

The apatite is the first product of crystallization, it is even formed as small idiomorphic inclusions in the titanite, for the greater part the crystallization of the other elements was simultaneous; the aegirine is idiomorphic in relation to felspars and feldspatoids but in general the contactlines are irregular and show simultaneous crystallization. The felspar includes some idiomorphic crystals of nepheline and sodalite, mainly it is the latest product of crystallization. Probably in pneumatolytical way, fluorine, calcite, and analcime crystallized in the remaining cavities.

It is evident how much the mineralogical composition of this rock differs from that of the normal types of nepheline syenite by its high content of aegirine and titanite. A. Lacroix¹⁾ gave the name of covite to the mesocratic form of this group and of teralite to the melanocratic form; as the type of covite he considers the rock of Magnet Cove in Arkansas, described by WASHINGTON and as type of teralite the alkali felspar-nepheline rocks from the Crazy Mountains in Montana.

The chemical composition of the rock here described is shown in I of the following table (analysed by F. Pisani); it is compared with the analyses of some covites and teralites.

¹⁾ Matériaux pour la Minéralogie de Madagascar. Extr. nouv. Arch. du Museum. 4e serie, Tome 1, pag. 184.



Explanation of Figure.

($\times 30$)

A part of a large individual of felspar shows poikilitic relation to aegirine, titanite, nepheline (at the top to the left) and sodalite (at the lower edge, in the middle and to the right).

The aegirine contains numerous idiomorphic inclusions of titanite, apatite and fluorspar

	I	II	III	IV	V	VI
SiO ₂	49.20	49.70	51.10	47.67	44.65	47.85
TiO ₂	7.13	1.33	4.38	—	0.95	—
Al ₂ O ₃	9.23	18.45	21.40	18.22	13.87	13.24
Fe ₂ O ₃	7.73	3.39	0.90	3.65	6.06	2.74
FeO	3.24	4.32	5.58	3.85	2.94	2.65
MnO	—	—	—	0.28	0.17	—
CaO	11.55	7.91	5.35	8.03	9.57	14.36
MgO	1.35	2.32	2.81	6.35	5.15	5.68
Na ₂ O	6.20	5.33	6.35	4.93	5.67	3.72
K ₂ O	1.96	4.95	4.21	3.82	4.49	5.25
P ₂ O ₅	0.06	0.40	—	2.97	2.10	2.74
H ₂ O	2.20	1.34	0.87	—	1.50	2.42
Som	99.85	99.44	99.65	100.15	99.93	100.65

- I. Pienaarite. Leeuwfontein (320) Pretoria. Transvaal.
- II. Covite. Magnet Cove. Arkansas cf. H. S. Washington. Journ. of Geol. IX. 614. 1901.
- III. Covite. Nosy Komba. cf. A. LACROIX Mat. Minéral. Madagascar Extr. Nouv. Arch. du Museum 4e Ser. I. 32.
- IV. Teralite. Crazy Mountains Bull. U. S. Geol. Surv. no. 150.
- V. Teralite. - " " " " " " " " " " " "
- VI. Teralite. (nepheline pyroxene malignite) cf. A. C. LAWSON Bull. Dep. of Geol. Univ. of California I. 337. 1896.

We see how the rock, here described, differs in mineralogical and chemical composition from the melanocratic nepheline syenites, which are hitherto known, its characteristic features are the large amount of Fe₂O₃, TiO₂, and CaO (abundance of aegirine, aegirine-augite and titanite) and its low content of lime (diminution of the feldspars and feldspatoids). Prof. MOLENGRAEFF proposed to me the name Pienaarite, after the Pienarsriver because the locality, where he collected this rock, is situated in a region between a tributary of the Pienarsriver called Mundtspruit, and the above river itself.

Physiology. — Prof. PEKELHARING offers a communication, also in the name of Dr. C. J. C. VAN HOOGENHUYZE: “*About the formation of creatine in the muscles at the tonus and at the development of rigidity.*”

(Communicated in the meeting of December 24, 1909).

On a preceding occasion¹⁾ in this Academy I have made a communication concerning an investigation by Mr. VAN HOOGENHUYZE and Mr. VERPLOEGH about the excretion of creatinine in man, from which it appeared that the excretion of this substance in a sufficiently nourished person was not increased by muscular labour. Since that time the investigation, which promised new results, now that a good method for the determination of the creatinine had been furnished by FOLIN, has been continued by VAN HOOGENHUYZE and VERPLOEGH and by several others. From these investigations has arisen the opinion that, through the consumption of protein in the tissues of vertebrate animals, creatine is formed, and that this matter is partly decomposed under oxidation, partly, particularly by the liver, changed into the anhydride, creatinine, that further the creatinine thus formed is for the greater part removed from the body through the kidneys²⁾.

If this opinion is correct, one cannot but assume that the creatinine excreted by the kidneys, originates for the greater part from the creatine of the muscles, not only because the muscles are richer in creatine than other organs, but also because it is especially the muscles which contain so considerable a part of the proteins which the body contains. This supposition did not seem unacceptable, notwithstanding it has been found that the excretion of creatinine is not increased by muscular labour. Observations were made which pointed to a connection between the excretion of creatinine and another phenomenon, which has to be distinguished from the muscular contraction in a narrower sense, the muscular tonus. VAN HOOGENHUYZE and VERPLOEGH found the excretion during the night to be smaller than in the daytime; likewise did they find remarkably little creatinine in the urine of old men and of patients who had a number of muscles paralyzed, whereas in case of fever the urine appeared to contain more creatinine than usual.

That really the tonic shortening of the muscle is brought about in another way than in case of rapid contraction, which has been examined so much more, and that, when a single stimulus is followed

¹⁾ Proceedings of the meeting of 30 Sept. 1905.

²⁾ See: *Zentralbl. f. d. ges. Physiol. und. Pathol. d. Stoffwechsels.* 1909. No. 8.

by a slow contraction, the two ways collaborate, is rendered very probable.

Already more than 20 years ago (GRÜTZNER¹⁾) made the supposition that the long continued contraction was brought about by another kind of muscular fibres than those which cause a more rapid contraction. In the first case fibres of the type of the red, in the second fibres of the type of the white muscles, were supposed to be brought into play. Afterwards others, more particularly BOTTAZZI²⁾, have defended the theory that the double contraction is caused by two different component parts of the same muscular fibre, the rapid by the double-refracting fibrils, the slow ones by the sarcoplasm. Mosso³⁾ had objections against this theory and drew attention to the double innervation of the muscular fibres, not long ago once more made clear in this Academy by BOEKE⁴⁾.

Meanwhile, whatever the opinion may be, at any rate there is some reason to assume that the contractions of two kinds must be accompanied by a chemical action of two kinds. Now that in the usual muscular labour, which is principally based on rapid and tetanic contractions, no increase in the consumption of protein and in the excretion of creatinine was found. it might be asked, whether perhaps in the tonic contraction formation of creatine in the muscles could be proved.

That under certain definite circumstances the muscles at their contraction yield more creatine to the blood than otherwise, has already been found by WEBER⁵⁾, with respect to the heart treated after LANGENDORFF'S method and beating in RINGER'S solution. He also found a considerable increase in the excretion of creatine with a dog, after violent cramps had been caused in the animal with cinchonine. Not only in the last case, in which the animal for an hour "was in violent tonic and clonic cramps", but also with respect to the heart taken from the body, it may be assumed that tonus has played a part.

However, in order to draw more certain inferences, we have examined the quantity of creatine in muscles, under circumstances which, as much as possible, allowed to judge about the influence, either of tonic, or of rapid contractions.

1) PFLÜGER'S Archiv. Bd. XLI, S. 280.

2) Journ. of Physiol. Vol. XXI, p. 1, Arch. f. Physiol. 1901, S. 377, Arch. Ital. de Biol. T. XLII, p. 169.

3) Arch. Ital. de Biol. T. XLI, p. 183.

4) Proceedings of the Meeting of 23 April 1909.

5) Arch. f. exp. Path. und Pharm. Bd. LVIII, S. 93.

The determinations always took place in the same way. The minced muscles were for some hours at a stretch boiled in 0.1% HCl, in consequence of which the tissue breaks altogether and all the creatine passes into the liquid. By evaporation the extract after being freed from protein, was concentrated and then with then double volume of normal HCl heated to 115° C. in the autoclave for half an hour, by which the creatine is completely changed into creatinine. Then the determination took place, according to the method of FOLIN, with the colorimeter formerly shown in the meeting of this Academy.

First of all we have some observations to mention about decrease of the quantity of creatine in muscles, of which the tonus, in consequence of section of the nerve, was eliminated. In the outset we came to very irregular results when examining the muscles of the hindlegs of rabbits and of a young dog, after a one-sided cutting of the nervus ischiadicus. Now less creatine was found in the muscles of the paralyzed leg, now more creatine in those of the paralyzed side. The cause of this irregularity appeared to be that we did not compare exactly corresponding muscles. Especially in the rabbit the difference in quantity of creatine in white and red muscles is rather considerable. With 10 rabbits we found in the gastrocnemius on an average 4.463, in red muscles (soleus, semitendinosus and semimembranosus examined together) 2.925 mgr. creatinine in 1 grm. of the muscle.

When this source of error was avoided, the influence of section of the ischiadicus became clear. With 3 rabbits we have, taking into consideration what has been said above, repeated the experiment and three days after the section of the nervus ischiadicus, we have examined the gastrocnemius of the paralyzed and of the not paralyzed leg. We found:

	Not paralyzed.	Paralyzed.	Loss after paralyzing.				
I	4.703	4.232	0.471	mgr.	cr.	p. gr.	of the muscle
II	4.448	4.289	0.159	"	"	"	"
III	4.983	4.013	0.970	"	"	"	"

Although the differences found lie without any doubt beyond the limits of the errors of observation, yet we have not continued these experiments, because too little can with certainty be concluded from them concerning the influence of the tonus. It is true, the muscles of the leg the ischiadicus of which has been cut through, are distinguished from those of the other side by the loss of tonus, but there are also other differences, which are perhaps of importance, as WEBER has already observed, who also found the quantity of creatine in

the muscles of the paralyzed leg smaller than in those of the normal leg in a dog after section of the ischiadicus. That the normal leg continues performing voluntary movements, is no objection, there being no ground to assume that then creatine is formed. But the muscles of the paralyzed leg degenerate. Although we killed the animals already three days after the section of the nerve, yet every time the paralyzed gastrocnemius appeared to be of a smaller weight than the normal one. Nothing is known about the formation and the destruction of creatine in degenerating muscles. Of no less importance seems to be the change in the circulation of the blood after section of the nerve, in consequence of which the removal of creatine from the muscles may be altered in a quite incalculable degree.

On account of these objections we thought of entirely giving up the attempt to inquire into the influence of the muscletonus in warm-blooded animals, and of being obliged to occupy ourselves only with cold-blooded vertebrates, in which without any trouble the blood-circulation can be shut out (in muscles of invertebrates no creatine has been found; they were not fit for our purpose accordingly) when my colleague Prof. R. MAGNUS drew our attention to a means of bringing muscles of one half of the body of a cat into strong tonus, whilst the corresponding muscles on the other side, without any disturbance in the action of the centrifugal nerves and in the circulation of the blood, remain slack.

SHERRINGTON¹⁾ has found that, when in deeply narcotised dogs, monkeys, cats, rabbits or caviars, the action of the cerebral hemispheres is excluded by a section in the region of the hindmost corpora quadrigemina, after a short time the so-called "decerebrate rigidity" develops itself, a long continuing tonic contraction of definite muscle-groups, among which especially the extensors of the extremities and the retractors of the head and the neck are the chief. This state of things is dependent on impulses which arise in the periphery, and by centripetal nerves are led to the spinal cord. That is why the stiffness does not arise in those parts of which the corresponding dorsal roots are severed.

Now Prof. MAGNUS had the kindness to operate upon five cats in such a way that one foreleg was brought in tonus two, three hours at a stretch, whilst the other leg remained slack. The perfectly narcotised animal, the narcosis being brought about first by means of ether, then by means of chloroform, after section on the left side of the hindmost roots of the four or five lowest cervical nerves and of the two highest thoracic nerves was decerebrated. Soon, the right

¹⁾ Journ. of Physiol. Vol. XXII, p. 319.

leg got into tonus, the left one remaining slack. Sometimes, in order to strengthen the tonus in the foreleg, also the spinal cord, at about the eleventh breast-vertebra, was cut through. When the tonus had lasted a few hours, the animal was killed by suffocation. Directly after the triceps brachii on both sides was prepared, minced and put into hydrochloric acid.

In all experiments we found the muscle that had been in tonus, richer in creatine than the one that had remained slack, and that, expressed in mgr. creatinine on 1 grm. muscle, as follows :

	Tonus	Slack	Difference
I	3.690	3.090	0.600
II	4.340	3.848	0.492
III	4.291	3.902	0.317
IV	3.806	3.185	0.621
V	3.198	2.963	0.235

It is remarkable that this difference pretty well keeps pace with the difference that in the experiment of the contraction of the muscles, right and left, was observed. In experiment I and still more in IV, the stiffness on the right was very beautifully developed, in II the tonus was strong, but of a shorter duration, in III the tonus on the right was good, but also the left foreleg occasionally showed some stiffness, which also occurred in V, though in a smaller degree, whilst the stiffness developed here slowly and to not so high a degree as otherwise.

We think we are entitled to derive from these experiments that by the muscles in tonus more creatine is formed than by those which are slackened. For the supposition that the difference may be attributed to an increased decomposition of creatine in the slackened muscles, it seems that there is not a single ground to be adduced.

Besides this we have made a number of experiments with frogs (*Rana esculenta*).

In the first place the influence of irritation with induction-currents on the quantity of creatine in muscles was examined. About this communications have been made by MELLANBY¹⁾ and by GRAHAM BROWN and CATHCART²⁾. By direct irritation MELLANBY brought the muscles in tetanus and then he found so slight an increase of the quantity of creatine that he came to the result: "that the perfor-

¹⁾ Journ. of Physiol. Vol. XXXVI, p. 447.

²⁾ Bio-Chemical Journ. Vol. IV, p. 420.

mance of muscular work leaves creatine unaffected". BROWN and CATHCART stimulated the muscles by means of the nerve and found a somewhat more considerable increase, of 7% à 13%, in four experiments, at which the circulation of the blood had been excluded. If the circulation was intact, they found a little diminution, not only with frogs, but also with rabbits, where it ought, however, to be taken into consideration that, in consequence of the stimulus, the muscle was more amply provided with blood, so that an exact comparison with resting muscles is scarcely possible.

We have made experiments with frogs, in three different ways, always excluding the current of blood. First, after destroying brain and spinal cord and after section of the heart, the nervus ischiadicus on one side, laid bare high in the thigh, was cut through and stimulated with a series of rapidly succeeding induction-strokes. Each stimulation lasted $\frac{1}{2}$ minute, after which $\frac{1}{2}$ minute of rest was afforded, during about one hour. In the second place the experiment was for the rest made in the same way, but the nerve was, for about half an hour, with the help of ENGELMANN'S rhythmic polyrheotome, stimulated 24 times per minute, alternately with a closing and an opening induction-stroke. At last two more experiments were made thus: the frog was cut through transversely in the lumbar region, after which the skin of the hind part was taken away. Now this was put astride on the partition of two basins of celluloid, standing against each other and filled with RINGER'S solution, so that each leg was immersed into the liquid to about half way up the thigh. Then the ischiadici were on one side, from the pelvis, stimulated for half an hour, 24 times a minute, with single closing and opening induction-strokes.

The quantity of creatine, expressed in mgr. creatinine per 1 gm. of muscle, was found as follows:

	Stimulated	Rest	Difference		Stimulated	Rest	Difference		
B	I	3.490	3.418	+ 0.072	A	I	3.366	3.386	- 0.020
	II	3.537	3.457	+ 0.080		II	3.616	3.683	- 0.067
	III	3.629	3.550	+ 0.079		III	3.796	3.856	- 0.060
	IV	3.567	3.560	+ 0.007					
C	I	3.203	3.230	- 0.027					
	II	3.585	3.593	- 0.008					

The differences are slight and do not fall, or fall scarcely, beyond the boundaries of the inevitable errors of observation. Moreover the difference is now in favour of the stimulated, now of the not-stimulated muscles. Even if one wishes to attach some importance to the greatest

differences found in these experiments, one need not yet derive from them that the muscle during the rapid contraction forms or loses creatine. For on the one hand the decomposition of creatine in the muscle is no doubt subject to quite unknown, but certainly varying influences, whilst on the other hand long continued irritating can also give rise to some lasting contraction, tonus.

That in the frog during the muscle-tonus in contradistinction to the rapid contractions, the quantity of creatine in the muscles increases, whilst, in default of tonus, it decreases, appears from the following experiments.

In the first place the influence of the elimination of the tonus, by excluding the innervation, was examined, first while the current of blood was stopped, then with undisturbed circulation of the blood. In the former case the ischiadicus was on one side cut through and then on both sides the root of the thigh was so well tied up with an elastic ligature that the blood in the vessels of the webs stood still, in such a way that the ligature ran below the ischiadicus not cut through. In the latter case the ischiadicus was simply cut through. Three days after the section of the nerve the animals were killed and the muscles of the hindlegs examined.

	intact	cut	difference		Intact	cut	difference		
Current of blood stopped	I	2.204	2.137	0.067	Current of blood undisturbed	I	3.784	3.342	0.442
	II	2.678	2.282	0.396		II	4.000	3.653	0.347
	III	2.990	2.790	0.200		III	4.146	3.688	0.458
	IV	2.987	2.887	0.100		IV	3.490	3.192	0.298
	V	2.726	2.551	0.175		V	3.434	3.131	0.303
	VI	2.833	2.688	0.145		VI	3.685	3.334	0.351
				VII		3.157	2.900	0.257	

Without any exception, therefore, there was found in the muscles that had lost the tonus for three days, less creatine than in the unhurt leg. If the current of blood was stopped, the difference was smaller than when it went on undisturbed. Yet in the tied up legs the quantity of creatine was, also on the side of the unhurt nerve, smaller than it is usually found in the frog. It is therefore probable that on both sides, after the current of blood had been stopped, creatine was decomposed. With respect to the experiments with undisturbed circulation of the blood the same objection may be raised which has been made mention of concerning similar experiments with the rabbit, viz. that it is unknown how far; perhaps

by a change of the current of blood, the removal of creatine from the muscles is altered. It is, however, not to be assumed that the differences observed should be attributed to this.

With much more certainty, however, the connection between tonic contraction and formation of creatine in the frog may, in our opinion, be derived from another series of experiments in which, with exclusion of the current of blood, muscle-tonus was caused. We have exposed the muscles to the action of substances of quite different nature, which, however, resemble each other in the fact that they cause tonus, viz.: veratrine, nicotine, calciumchloride rhodan-natrium and coffeine.

It is especially BOTTAZZI who has pointed out the tonicizing action of veratrine¹⁾. If the gastrocnemius of a frog is immersed in RINGER's solution containing 1 : 20000, or even less, veratrine, stimulation of the ischiadicus with a single induction-stroke causes a contraction which lasts much longer than with a muscle immersed in pure RINGER's solution. To the rapid a slow contraction is added.

In order to examine the influence on the formation of creatine the hindlegs of a frog were brought into the above mentioned celluloid basins, of which one was filled with RINGER's solution, the other with the same solution, in which a definite quantity of veratrine had been dissolved. Now the two ischiadici were, from the pelvis, during half an hour, stimulated 24 times per minute alternately with a closing and an opening induction-stroke. After that the muscles were prepared off, and with the liquid in which they had been immersed, treated in the usual way for the determination of creatine. The result was:

	RINGER's solution	Veratrine.	Difference.
		(1 : 40000)	
I	3.442	3.561	0.119
		(1 : 20000)	
II	3.189	3.389	0.200
		(1 : 5000)	
III	3.056	3.430	0.374
IV	3.250	3.670	0.420
		(1 : 1000)	
V	3.029	3.429	0.400

In III, IV and V the legs immersed in veratrine ceased to contract before the half hour had elapsed. Besides these legs showed in the end some stiffness in these experiments.

¹⁾ loc. cit.

The faculty of nicotine to cause tonic contraction of muscles, has been amply studied by LANGLEY in his experiments on receptive substances¹⁾. The forelegs of the frog, the flexors of which are so easy to bring in tonus, also by dripping with nicotine, would have been very fit for our purpose, if not the mass of the available muscles was so small that for a single determination of creatine a large number of frogs would be necessary. The experiments of LANGLEY, however, made us surmise that also the hindlegs would be fit for our purpose, which surmise was corroborated by the result.

First an experiment was made as follows:

After destroying brain and spinal cord 1 CC of a 1 % -solution of nicotine in RINGER's solution was injected into the abdomen, after which the tonic contraction of the forelegs soon made itself manifest. Half an hour after the injection the current of blood, by section of the heart, was brought to a standstill. Now the left ischiadicus was laid bare in the upper part of the thigh, cut through and for half an hour stimulated 24 times per minute with induction-strokes. Till the end the muscles reacted upon the stimulation of the nerve and at last a slight rigour was to be observed.

The stimulated muscles produced 3.491 mgr. of creatinine per grm. of muscle, the non-stimulated 3.090 mgr. Difference 0.401.

Then the experiments were made in the same way as with veratrine, with the following result:

	RINGER's sol.	Nicotine. (1 : 100)	Difference.
I	3.286	3.766	0.480
II	3.090	3.492	0.402
		(1 : 200)	
III	3.276	3.538	0.262
		(1 : 100)	
IV	3.037	3.401	0.364

At the end of the experiment the leg immersed in nicotine did not visibly contract any more and each time this leg was somewhat stiff.

Through an examination of the action of kalium- and calcium-salts also (GUENTHER²⁾) has come to the result that the muscular fibre possesses contractile substances of two kinds, one of which is made more susceptible to stimulation by K, the other by Ca.

¹⁾ Journ. of Physiol. Vol. XXXIII, p. 374, Vol. XXXVI, p. 347, Vol. XXXVII, p. 165, p. 285, Vol. XXXIX, p. 235. Proc. Royal Soc. B. Vol. LXXVIII, p. 170

²⁾ Amer. Journ. of Physiol. Vol. XIV, p. 73.

“The first contractile substance of the sartorius”, he says, “responds quickly with a contraction when subjected to a 1 percent solution of potassium chloride. Calcium chloride in a 1 percent solution produces no contraction of the first contractile element of the sartorius, gives rise to a slow contraction of the second contractile element, and produces quite a vigorous contraction of heart muscle.”

We had, therefore, to expect that excitation of muscles immersed in calciumchloride would make the quantity of creatine increase. Indeed this appeared to be the case. One basin was now filled with RINGER's solution, the other with a solution of CaCl_2 isotonic with it. For the rest the experiments were made in quite the same way as the preceding one. The results are the following :

	RINGER's sol.	CaCl_2 0.72 %	Difference
I	3.177	3.820	0.643
II	3.193	3.703	0.510
III	3.340	3.894	0.554
IV	3.040	3.647	0.607
V	3.156	3.501	0.345

In the first four experiments the contraction of the muscles immersed in CaCl_2 left off before the half hour was past and these muscles showed distinct stiffness. In V the contractions of the muscles immersed in CaCl_2 were at the end of the experiment clearly to be observed and stiffness was not to be perceived.

To the examination of the action of rhodane and coffeine we were led by a communication of vox FÜRTH and SCHWARZ¹⁾, from which it appeared that these substances, like e.g. veratrine, are able to considerably increase the labouring faculty of the muscles. The supposition that also here the tonus, the “innere Unterstützung”, of which GRÜTZNER spoke, was playing a part, we found corroborated. The two gastrocnemii of the same frog were hung, one in a vessel with RINGER's solution, the other in a vessel with the same liquid in which some citras coffeini was dissolved, or of which the sodium chloride had been replaced by rhodan-natrium. After both muscles, under the same tension, had been fastened to registering levers, they were now and then, by means of the nervi ischiadici laid upon a single couple of electrodes, excited with an induction-stroke. Now while the muscle immersed in RINGER's solution after each contraction returned to its former length, or was even somewhat lengthened,

¹⁾ PFLÜGER's Archiv, CXXIX, S. 525.

the muscle brought in contact with coffee or with rhodane, whilst it continued reacting well upon the excitation, became gradually not inconsiderably shorter.

The influence upon the quantity of creatine was examined in the usual way. The following figures were found :

	Ringers' sol.	NaCNS 0.614 %	Difference
I	2.822	3.098	0.276
II	3.106	3.354	0.248
III	3.051	3.537	0.486
IV	3.129	3.459	0.330
V	2.916	3.146	0.230

Towards the end of the experiment the muscles did not contract any more. Stiffening was not to be perceived.

	Ringer's liq.	Citr. Coff.		Difference
I	3.017	3.432	1 : 100	0.415
II	3.055	3.623	1 : 200	0.568
III	3.090	3.422	1 : 400	0.332
IV	3.194	3.551	1 : 400	0.357
V	3.316	3.519	1 : 800	0.203

The leg brought in contact with coffee was in I quite stiff after 10 minutes, in II after a quarter of an hour, the contractions leaving off. In III, IV and V the contractions remained visible till the end of the experiment. Also in those cases the rigour was clear, though in V not so strongly as in III and IV.

In all cases, without any exception, therefore, the quantity of creatine was found to be increased in the muscles that had been in tonus. The difference, except with coffee, may even be estimated somewhat higher than the figures given, because, leaving the mentioned exception out of consideration, the tonus appeared to be accompanied by a slight increase of the quantity of water. This difference is, however, so insignificant that it need not be taken into consideration.

Increase of the quantity of creatine was found only then when the muscles had been brought into tonus by excitation. Immersion of the legs, during half an hour, in the solutions, without excitation, had no influence upon the quantity of creatine. The following experiments were made in the usual way, only with this difference, that the nerves were not excited.

RINGER's sol.	Veratr. 1 : 5000	Difference
3.954	3.954	0
	Nicotine 1 : 100	
3.544	3.510	0.034
	CaCl, 0.72%	
3.399	3.394	0.005
	NaNCS	
3.340	3.336	0.004
	Coffeine 1 : 100	
3.295	3.327	0.032

In none of these cases was anything to be perceived of stiffness of the muscles.

Therefore our results are perfectly in keeping with the opinion that the muscular fibre, when reacting upon a stimulus with a rapid contraction, works in quite another way than when it is brought in tonic contraction. In the first case it consumes non-nitrogenous matter, in the second it forms creatine, consequently consumes protein. Against the supposition of GRÖTZNER that each of these actions should belong to a special kind of muscular fibres, tells among others our experience, that, with the rabbit, it is just the red muscles, which are distinguished for slowness in contraction, that contain less creatine than the white ones. Though the opinion of BOTTAZZI that muscular fibres show the phenomenon of tonus the more, as they are richer in sarcoplasma, as has already been pointed out by Mosso, is not quite in keeping with the observations, it may, however, especially after ENGELMANN's important researches, be assumed that the rapid contraction is performed by the anisotrope elements, accordingly by the muscular fibrils. The seat of the tonus must therefore be sought in the sarcoplasma or perhaps in the parts of the fibrils between which the anisotrope elements find a place. In a further investigation into the two different kinds of contraction of the muscular fibres it will certainly be of importance to keep the attention also directed to the double innervation again demonstrated by BOEKE.

As to the starting-point of our investigation we think we are entitled to give an affirmative answer to the question whether the formation of creatine, and consequently the consumption of protein in the body, is largely influenced by the tonus of the muscles. Already many years ago it was proved by PFLÜGER¹⁾ of how great an importance the muscular tonus is for the production of heat. If our opinion is correct, it also follows from this that limitation in the supply of protein with the food, which is at the present day aimed

¹⁾ PFLÜGER's Archiv., Bd. XVIII, S. 247.

at by many, has its dangerous side. Mechanical labour the muscles can perform at the cost of food free from nitrogen; however to be of service to the organism, also in other respects, by means of the tonus, they want protein.

The opinion has often been pronounced that the stiffening of the muscles after death should be considered as a last contraction of the muscles. Especially HERMANN has indicated the agreement between the changes the muscle undergoes at coagulation and those which are observed in the contraction. In the above mentioned paper of VON FÜRTH and SCHWARZ it is proved that it is such substances especially, which are capable of promoting the coagulation of the muscle-plasma, that raise the labouring-faculty of the muscles.

It seems that the agreement does not refer to the rapid contraction but to the tonus. We have found an increase of the quantity of creatine in frog-muscles which were stiffened by immersion in water of 42° or 45° C. In four experiments the increase amounted on an average to 0.305 mgr. creatinine on 1 gm. of the muscle (min. 0.204, max. 0.460 mgr.).

For the rabbit the investigation offered some difficulties, because here the decomposition of creatine, proved by GOTTLIEB and STANGASSINGER plays an important part and the so much thicker rabbit-muscle is not so rapidly coagulated as the thin muscles of the frog. When, however, the errors arising from this are avoided as much as possible, also in the majority of cases, both with the red and the white muscles of the rabbit, a distinct increase of the quantity of creatine was observed in the stiffened muscles.

Also in the investigation into the spontaneous stiffening of muscles after death, the postmortem disappearance of creatine has to be taken into consideration. When, however, the muscles of one side of the body were, directly after death, put in hydrochloric acid and the corresponding muscles of the other side after three or four hours when the stiffening had been well developed, each time there was found more creatine in the coagulated muscles than in those examined in a fresh condition. In the four cases dealt with in this way, we found an uncommonly great difference in one, and in the three others on an average 0.260 mgr. of creatinine more (min. 0.124, max. 0.336 mgr.). The description in details of these and the other observations mentioned we intend to give somewhere else.

From our investigation we think we are entitled to derive that in the muscles of vertebrate animals, at the heat-coagulation and the postmortem rigour as well as the tonus, a chemical process takes place which causes the origination of creatine.

Botany. — “*Sap-raising forces in living wood.*” By E. REINDERS.

(Communicated by Prof. J. W. MOLL.).

Of the many theories, which have been advanced in explanation of the transpiration-current of trees, most are at present only of historical importance in the literature. The imbibition theory of SACHS¹⁾; BÖHM's atmospheric pressure theory²⁾; the gas pressure theory of HARTIG³⁾; the views of WESTERMAIER⁴⁾, who regarded the xylem parenchyma as the water conduit and considered the vessels to be reservoirs; EWART's⁵⁾ hypothesis that the living elements help to overcome the resistance, the cohesion theory of ASKENASY⁶⁾, which neglected to adopt the continuity of water as a *conditio sine qua non* — all these have been given up. On the other hand opinion is still divided with regard to two hypotheses, the advocates of which combat the views of their respective opponents with remarkable asperity. GODLEWSKI⁷⁾ and his supporters defend the view that the transpiration-current cannot be explained without postulating the cooperation of the living elements of the wood; DIXON and JOLY⁸⁾ on the other hand advance the proposition that the living elements have not, and cannot have, anything to do with the process. They explain the phenomenon that water ascends up to the summits of the highest trees by assuming that in these trees the water, enclosed in the narrow water conduits, hangs like a thread from the surface of the leaf cells, where it is held by capillary or other physical forces. The thread does not break, because, as is supposed, it is nowhere in contact with air, and in these circumstances water can support a tension of 150 atmospheres. When the water evaporates in the leaves at the summit, this thread is drawn up through the tissues.

The keenness with which the two parties oppose each other is best illustrated by a couple of quotations.

SCHWENDENER⁹⁾, an advocate of the more physiological theory, says:

“An der Vorstellung, dass die Lebenstätigkeit der Zellen irgendwie in die Saftbewegung eingreift ist . . . unbedingt festzuhalten. Ohne dieses Eingreifen ist die Hebung des Wassers auf Höhen von 150-200 Fuss und darüber einfach unmöglich und alle Bemühungen, die vorhandenen Schranken mit unklaren physischen Annahmen zu durchbrechen, sind nicht viel mehr als ein Suchen nach dem Stein der Weisen”.

In the same year 1909 DIXON¹⁰⁾ writes:

“The adhesion of writers to the vital hypothesis . . . is so

remarkable that we must devote some space to examine fully the grounds for their contention”.

When we attempt to trace why opinions diverge so widely, the cause seems to lie principally in a different appreciation of certain experiments and in the somewhat adventurous aspect which the Dixonian explanation presents at first sight. It is necessary to become accustomed to the idea that the life of our trees hangs upon a water-thread, before we can become reconciled to it. GODLEWSKI¹¹⁾ indeed required a much more adventurous hypothesis in order to reconcile the anatomical structure of the wood with its power of pumping up water. This part of his theory has in consequence received adhesion from no one and so I will leave it out of discussion. In what follows below, “GODLEWSKI’s theory” will therefore mean the view that the living wood must be regarded as the cause of the transpiration current.

In order to facilitate a judgment of the state of affairs I will tabulate the most important arguments of the two parties side by side and will then discuss them in pairs. From this table I omit everything relating to the question whether the cohesion of water is sufficiently great to account for the work which DIXON and JOLY attribute to it. I will assume, if I may put it thus, that there is no technical objection to their theory and I think this assumption may be made with safety.

GODLEWSKI c. s.

DIXON and JOLY.

1*a*. There is not sufficient continuity in the water columns of the wood to admit cohesion as an explanation¹²⁾.

1*b*. There is no reason for doubting the continuity of the water columns¹³⁾.

2*a*. The remaining available physical forces are insufficient to raise the water more than 14 metres¹⁴⁾.

2*b*. STRASRURGER’s experiments in which the water ascended in poisoned trees, prove the contrary¹⁵⁾.

The cohesion theory has at its disposal forces which would be able to provide a tree of 200 metres and more with water¹⁶⁾.

3*a*. URSPRUNG’s experiments, with branches which had been killed for part of their length, after which the

3*b*. In URSPRUNG’s experiments the conduits become blocked and the leaves were poisoned because

leaves faded, prove that dead wood they got a decoction of wood for cannot transport enough water to their drink¹⁸). balance the transpiration¹⁷).

4*a*. The structure of the wood is in favour of GODLEWSKI'S theory²²). 4*b*. "The very structure of the wood offers the strongest evidence against GODLEWSKI'S theory"¹⁹).

Living wood offers the same resistance in either direction to the forcing through of water²⁰).

5*a*. Arguments from analogy²²). 5*b*. Arguments from analogy²¹).

6*a*. The distribution of pressure in living transpiring trunks is opposed to the cohesion theory²³). 6*b*. The measurements of pressure are considered unreliable or are left out of account.

Point 1. The question of the "continuity of the water-threads" in the wood amounts to the following. The cohesion theory requires the assumption that the water in the tree forms one connected mass from the root to the leaves. Every xylem vessel in which there is an air-bubble has according to this theory become useless for the conduction of water, for in such a vessel the water cannot be under negative pressure; it is at once sucked empty by the adjoining vessels. Every bubble of air therefore puts one vessel out of action.

Now if it could be shown that by far the largest proportion of vessels contain air bubbles, only a small percentage would remain available for the conduction of water, and perhaps here and there the required connection of the water would be entirely interrupted, so that there could be no question of the cooperation of cohesion.

It is of course difficult to prove the absence of air, for in the necessary manipulations preparatory to the examination there is always the chance that air bubbles in some way or other get into the vessels²⁴). If air is found in the majority of the vessels this does not prove that it was already present in the living plant, for it may have penetrated during manipulation.

For the further course of my argument it matters little, however, whether DIXON and JOLY or whether their opponents are right on this point. I will not therefore discuss it any further.

Point 2. The proposition, that physical forces alone are insufficient²⁵) to raise water higher than 13—14 metres is a very weak point in the defence of GODLEWSKI'S theory, for STRASBURGER'S intoxi-

cation experiments have proved in the most striking manner, that this proposition is untenable. He found that water still ascended to the highest tops of the poisoned trees, up to a height of 22 metres.

The attempts of GODLEWSKI's supporters to maintain their proposition in spite of this fact give a very unsatisfactory impression. STRASBURGER is attacked in vague terms²⁶); he is accused of a want of critical insight, he is reproached for not making any attempt at explanation: the fact itself remains.

The following argument appears to be somewhat more weighty. It is said²⁷): "with the help of a JAMIN chain atmospheric pressure may be imagined to force water up to 13—14 metres"; but fourteen is not twenty-two and moreover a JAMIN chain can by no way explain anything in this case. It might perhaps be applied to this purpose with some chance of success, if the vessels ran through continuously from the root to the leaf, but certainly not in a system of vesicles like the wood, where the bubbles cannot pass the partitions, dividing up the conducting tracts, to say nothing of the multitude of other clinching objections.

It is further adduced against STRASBURGER, that continuous liquid threads are formed when the trunk, having been sawn off, is placed in water²⁷), but in the first place it is not clear what objection is really meant by this and in the second place it is difficult to imagine how these threads are supposed to originate. The water which is sucked up cannot remove the air present, for the air is enclosed; it is moreover saturated with air, and is more likely to give off bubbles than to absorb them, as soon as it is exposed to a lower pressure at a certain height. Sawing off the tree will hardly affect its air-content except to increase it; the air which enters does not, however, endanger the cohesion, as it cannot ascend.

Point 3. URSPRUNG's experiments¹⁷) with branches, which had been killed by steam over part of their length, in consequence of which the leaves faded, do not prove much for GODLEWSKI either. The steam not only kills the living elements, but also induces other changes.

For some time the vessels must conduct a decoction of wood instead of water and a blocking of the membranes or even of the lumina of the vessels may be the consequence, so that the resistance increases. The cells of the leaves are further more or less poisoned by this liquid, so that it is very doubtful whether the death of the leaves may be attributed to a want of water¹⁸).

These experiments are therefore not of much importance in deciding the question under consideration.

Point 4. The anatomical structure of the wood is a better argument for DIXON¹⁹⁾ 20) than for GODLEWSKI, for as yet it is quite impossible to imagine in what way the living elements could really exert any successful pumping action. The unidirectional resistance without which such an action can hardly be conceived, has never been observed, in spite of a careful search for it.

This argument is therefore no longer always adduced in support of GODLEWSKI.

Point 5. In critical cases the arguments from analogy are hardly more valuable than illustrations. I will therefore not discuss them here.

We see therefore that the arguments which have been advanced so far give little support to GODLEWSKI's theory. On the other hand the striking and conclusive result of STRASBURGER's intoxication experiments is in favour of DIXON and JOLY. If to this be added the great convincing power which proofs from analogy exert, when well presented (and here DIXON and JOLY are much more fortunate than their opponents), we may readily understand, that the cohesion theory has many supporters.

There are, however, two facts which are adduced against this theory with more success.

In the first place a second series of experiments by URSPRUNG²¹⁾ in which he used ice instead of steam, in order to render part of a branch inactive. This series of experiments does not of course suffer from the objections which deprived the other series of its argumentative value. The fact, however, that fading only occurs after several days, makes the result less convincing.

Another objection is more important:

Point 6. The distribution of pressure in living trees is opposed to the theory²²⁾.

In a hanging water-thread the pressure decreases gradually as one ascends and the decrease is at least one atmosphere for an ascent of 10 metres. In living transpiring trees it has been impossible to demonstrate this; it was found on the contrary that manometers placed at different heights up the trunk, behave quite independently of one another. Sometimes one shows a lower pressure, sometimes the other.

It is true that objections can be raised against many of these measurements of pressure, but some of them in SCHWENDENER's opinion proved positively and undeniably that there can be no question of a regular decrease of pressure. For in this case it would be inconceivable, "dass ein Baumstamm der nach 2—3 Regentagen durch Nachschub von unten etwas wasserreicher geworden, in mittlerer

Höhe (wo vorher Saugen stattfand) Luft in das hier angebrachte Manometer hineinpreszt, während oben in der Krone und insbesondere unten am Stamm weder Saugung noch Pressung stattfindet" ²³).

It is remarkable that DIXON, in his review of the state of the problem in the "Progressus", does not at all refer to the ice experiments of URSPRUNG, nor to measurements of pressure, although he there considers at length and refutes much less important objections.

Thus we have traced the causes of the remarkable phenomenon mentioned in the introduction. The partisans of GODLEWSKI point to the measurements of pressure and maintain that STRASBURGER's experiments are invalid, whereas DIXON points to STRASBURGER and is not concerned with pressure measurements.

As will be seen the position is somewhat confused. In my opinion no advance can here be made along a theoretical road. Experiments alone can lead us out of the confusion.

I think I am able to supply conclusive, experimental proof that the normal living wood is able to pump water actively.

In order to give this proof I started from the following preliminary conception. If the irregularity of the results of pressure measurements is really caused by a pumping action of the living wood, this irregularity must at once disappear as soon as the experimental trees are killed or paralyzed. This was indeed found to be the case. Moreover, as soon as the trunk was dead the differences of pressure followed the same rule as would be expected to apply to a glass tube. When the conditions became unfavourable to evaporation, as in the evening and when rain supervened, the indications of the manometers approached each other more and more. At midday, in sunshine, on the other hand they differed more. This becomes intelligible, when we consider that a more rapid evaporation requires a stronger current; for a stronger current larger differences of pressure are however necessary, in order to overcome the greater resistances.

First I will describe the experiments somewhat more in detail. Later I hope to publish the curves of the positions of the manometers, together with the result of a more extensive investigation of this subject.

Of a $\pm 2\frac{1}{2}$ metres high specimen of *Sorbus latifolia*, which divided a little above the ground into two almost equal, strong branches, one branch was left intact as a control; to the other I fixed above one another some U-shaped open mercury manometers, in the following manner. Some lateral branches were cut off from the main branch in such a way that a stump of 5 cm. length remained. A tube was slid over this stump, and to it the manometer was afterwards fixed

this tube was blown out in the middle to a small bulb, and was hermetically fixed to the stump with a piece of india rubber tubing. It was then half filled with water, and momentarily pumped empty that we may inject the cut vessels. I then left it open for half an hour and finally closed it with the perforated rubber stopper, through which the manometer was stuck. Once a day the bulb-tubes had to be replenished, for the wood always leaks a little from the intercellular spaces. The bark leaks still more and for this reason I always removed it at the place where the rubber tube was to come.

As long as the tree was alive, no regularity could be perceived in the indications of the manometers: they all showed a pressure, smaller than that of the atmosphere, but sometimes one "sucked" more, sometimes another. After a few days I killed the portion of the branch bearing the manometers over its whole length by means of steam. At once the manometers followed the rule indicated above, and did not depart from it. The differences of pressure became very considerable towards midday, showing that the dead portion offered a great resistance to the strong current.

The crown and the base of the branch remained intact during this treatment. The leaves showed only after 3 weeks, that they had suffered from the operation; up to that time they remained perfectly fresh. When at last they began to change, they gave the impression that they were diseased, rather than that they suffered from want of water.

Two manometers were attached to the small trunk of a *Cornus* and fixed to almost equal stumps of branches, the one 66 cm. above the other. The whole tree was 2 metres high. Before I cut off the branches, which were to yield the stumps, I killed the trunk at these two places with steam over a length of 10—12 cm. The manometers were thus attached to dead branch stumps on dead pieces of the trunk, separated by a living portion.

I wished to investigate whether the living intermediate portion did pump or not. If so, it would always be occupied in diminishing the difference of pressure between the two dead pieces of the trunk. If it was then suddenly cooled with ice, the manometers would have to diverge suddenly and would once more approach each other if the tree was left to itself. Finally if it was killed, the well-known regularity would be bound to appear.

The result was different, however. The intermediate portion evidently did not pump, for the manometers behaved exactly as in a dead tree. At midday they sometimes differed by 24 cm. of mercury. However — on the fifth day their behaviour changed fairly

suddenly and on the sixth day it was as irregular as in living trees!

Evidently the intermediate portion had suffered too much by this treatment, to function immediately, but on the sixth day it had so far recovered, that it could work again. It lived on until the end of December, as could be seen by the perfectly fresh bark. Now, at the end of January, it is dead. The crown, however, still looks healthy, as also do the buds.

Although those facts, as far as I can see, do not permit of an explanation other than the one given here, a proof may still be adduced that such phenomena cannot be attributed to a change in the resistances. Such a change would moreover have to be of a very remarkable nature to be of any use as an explanation.

Four manometers were attached to the trunk of a lilac tree (*Syringa vulgaris*) 2 metres in height, and they were numbered in ascending order 1, 2, 3, and 4. After a short time they all showed an approximately equal "suction", which oscillated with diurnal periods between 48 and 28 cm. of mercury. Although the differences were small, some times one was the highest, some times another. After 15 days, when I knew the course of the pressure curves sufficiently, stump 2 was killed, together with the piece of the stem to which it was attached. This was done by passing through it for an hour the discharges of an induction coil capable of giving a spark of 10 cm. long, without sparking in the secondary circuit. The stump and the portion of the trunk became heated to nearly 60° C.: a few pieces of glass cement of that melting point, which I had fastened to it, just began to melt.

While the induction current was being passed, the suction of stump 2 first diminished greatly, as a result of the heating, the other manometers remained constant. Soon the fall of the mercury in no 2 stopped and the suction increased again. After the interruption of the current the mercury rose higher than usual; this abnormally high suction subsequently persisted; no 2 afterwards followed the periods of the other manometers, which went on without hindrance, but sucked always strikingly more. How we can deduce from this the proof that this phenomenon is not caused by changes in the resistances, will be explained presently.

Thus far the description of the experiments. I will now consider what may be deduced from the results.

The course of the manometers in *Sorbus* proves that the water current in a living tree is caused by quite different forces from those of a dead one. The result cannot be attributed to the imperfectness of measurements. Most of these are the same before and after death

and we cannot suppose that the circumstances which are changed in the operation, are altered exactly in such a way as to bring to light the observed regularity. Thus the distribution of pressure before death can only be explained on the assumption that there are pressor factors, i.e. pumping actions in the wood.

This view receives important support from quite a different side, through the experiments of ZIJLSTRA³⁰⁾. He allowed a solution of Säureviolett to ascend living and dead branches and then examined them microscopically. In the living ones only the tori of the bordered pits were stained, together with a thin layer of the walls of the vessels; in the dead ones, however, the whole of the wood was coloured uniformly. It follows from this that the water current takes quite a different course in dead wood from that taken in living wood.

That in the lilac only the one manometer was affected, which was attached to the portion killed by induction shocks, cannot in my opinion, be explained in any other way than by the aid of GODLEWSKI's theory. If one imagines, with DIXON and JOLY, that the whole trunk behaves like a dead tube, the phenomenon cannot be explained. An increase of resistance cannot be the cause, for then the other manometers would have undergone this influence. If on the other hand, we imagine a tree-trunk to be a system of tubes in which everywhere small pumps occur, the phenomenon becomes intelligible. The death of the piece of trunk puts the pumps out of action locally and the suction must there be somewhat greater to get the water through the piece of dead wood. This would not necessarily be observed at the following manometers, since the intermediate elements bring the pressure back to normal.

I regard all the above as proof positive that the living wood has a hydromotory power. The experiment with *Cornus* already proves this very clearly: one could almost see the recovering intermediate portion suddenly begin pumping, as it were before one's eyes.

After thus having given the positive proof that the living wood assists in the ascent of the water I will again take up the theoretical considerations with which I started, and see to what extent this proof can modify the condition of affairs.

We encounter the difficulty that STRASBURGER's intoxication experiments prove that help of living elements is not necessary, whereas the only theory which is not adversely affected by these experiments becomes untenable on account of the pressure measurements. The solution is clear from the preceding.

The adherents of GODLEWSKI are wrong in asserting that water

cannot ascend more than 14 metres without the help of life, for STRASBURGER's experiments show that this is indeed possible. But that is not the question. The position is simply this, that in a living tree the water is pumped up by living elements, whereas in a dead one it also ascends, but through other causes e.g., with the help of cohesion.

Let us test this view by the data in the table:

Point 1. The question of continuity is only of importance for the cohesion theory. As soon as this has been refuted in another way, the question of the continuity of water, may be left until it may perhaps arise again in connection with new questions.

Point 2. The intoxication experiments of STRASBURGER have been included in my thesis.

Point 3. Although the experiments of URSPRUNG do not prove anything certain in favour of GODLEWSKI's theory, they certainly prove nothing against it.

Point 4. The anatomical structure of the wood can never be adduced as an actual objection to the view here put forward. As soon as it has been proved that the living wood pumps, this fact cannot of course be weakened because we cannot at once imagine from its structure how this action may take place. The investigation of this point must simply be left for further research.

Point 6. The distribution of pressure is quite in agreement with GODLEWSKI's view. When pressor factors are everywhere present in the trunk, the distribution of pressure cannot be predicted as long as these factors themselves are not fully known.

We see therefore that the questions discussed here do not produce an objection to my view. In this preliminary communication I have of course limited myself to the most important matter; afterwards I hope to treat the same subject more completely.

I might have omitted the literature entirely, but it seemed desirable briefly to justify my quotations and references. The small figures in the text refer to the bibliography which is appended below.

I wish to conclude this preliminary communication by pointing out that the method which is here introduced, may also be of service in the solution of other questions. By its aid we might, for instance, ascertain whether the living elements cooperate, when a branch is made to transport water in the inverse direction; the influence of all sorts of stimuli (heat, cold, electricity, stimulant substances) on the activity of these elements can be examined. Should the intoxication experiments of STRASBURGER be repeated with manometers fixed to the experimental trees, they would at once constitute a

definite proof in favour of GODLEWSKI. A small tree would, however, be sufficient for this.

The method in which a living piece of wood is isolated between two dead portions is especially to be recommended. The portions to be killed should not, however, be heated above about 60°, in order to spare the intermediate position (compare the experiment with the induction current). If possible a leafy branch should be left attached to the intermediate portion, for otherwise it must soon die of hunger.

Groningen, January 28th 1910.

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Botany. — “*Contributions to the knowledge of the movement of water in plants.*” By Dr. K. ZIJLSTRA. (Communicated by Prof. J. W. MOLL).

(Communicated in the meeting of January 29, 1910).

For some time I have been occupied in the botanical laboratory at Groningen with the problem of the movement of water in plants and have carried out experiments of a somewhat diverse nature. Various circumstances have prevented me from continuing my experiments in this direction, so that the investigation has not been rounded off. I did not intend publishing it, but as I shall presumably have for some time no further opportunity of continuing my studies, I think I may be justified in publishing the data I have collected; possibly they may be of service to other investigators who have chosen for their researches the subject of the movement of water in plants.

The experiments referred to may be arranged under three heads, viz.:

- 1st. The trunk or stem of intact plants cooled to about 0° C.
- 2nd. The ascent of a dye solution in cut branches.
- 3rd. Interference with the movement of water in a tree-trunk by means of deep incisions.

First I propose to discuss the considerations which led to these experiments and the results obtained, and then I will give a more detailed account of the execution of the experiments.

1. *Trunk or stem of intact plants cooled to about 0° C.*

As is well known, GODLEWSKI (*Zur Theorie der Wasserbewegung in den Pflanzen. Jahrb. f. Wiss. Bot. Bd. 15*) attempts to find the cause of the movement of water in the activity of the living cells of the medullary rays and of the wood parenchyma; these cells would therefore have to act as it were as suction-pressure pumps. GODLEWSKI did not, however, adduce any direct experimental evidence in support of this theory. His theory is only made plausible with the aid of various data obtained by others, and it is urged that the theory does not conflict with the facts adduced by other investigators. Various botanists (JANSE, STRASBURGER, WEBER, URSPRUNG) have afterwards attempted to test the theory experimentally.

The most obvious method for such a test would be the following: to cut out the action of the living cells of medullary rays and wood parenchyma, and then to see whether the movement of water had become impossible.

This elimination of the action of living cells was most easily obtained by simply killing these cells by poisons or by a high temperature.

This method is, however, open to objection; such interference not only attains the elements which it is desired to put out of action; others also, especially the water-conducting vessels and tracheids will undoubtedly be affected, so that it is questionable whether the results of the experiments can only be attributed to the elimination of the activity of the living cells.

A method — already used by URSPRUNG but with a result opposite to mine — which meets this objection, is the cooling of the trunk or stem of the plant to about 0° . By this means it is possible to reduce the activity of the living cells to a minimum, while neither dead nor living elements undergo a permanent change. Moreover the advantage of being able to establish the original conditions after the conclusion of the experiment and therefore bring the plant back to normal conditions, should not be underrated. The experiment and its control can both be carried out on the same intact plant.

If by this means the plant could be made to fade, and to assume its original fresh appearance after the cooling had been stopped, GODLEWSKI's theory would receive considerable support.

According to this method I have myself carried out 3 experiments. The trunk of a small apple-tree, 2 stems of *Polygonum cuspidatum* and 2 stems of *Helianthus tuberosus* were cooled to about 0° C. over a length of 50 cm. The experiments lasted 6, 7, and 8 days, under conditions which were very favourable for a possible fading.

Nevertheless I have in no case been able to observe even incipient fading, although the transpiration from the leaves was strong, as shown by the cobalt test. Cut leafy branches, hung up near the plant, withered very rapidly.

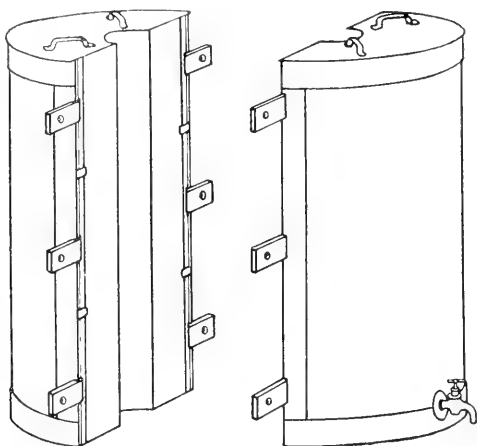
We may not, however, conclude from the negative result of these experiments that the living cells do not play a part in the movement of water. It is quite possible, even probable, that cooling a length of 50 cm. is not enough. This slight obstacle was perhaps easily overcome by differences of pressure present in the trunks. Had the results been positive it would have supported GODLEWSKI's theory. My negative results are, however, not able to oppose this theory.

The nature of the results notwithstanding, I think it may be useful to bring them to the notice of others.

Description of the experiments.

The cooling of the trunk or stem was brought about by melting ice, which was placed in an apparatus indicated by the figure.

The apparatus consisted of two equal parts, i.e. of two semicircular tin-plate reservoirs with fixed bottom and loose lid. The two half-



cylinders had on the middle of the flat side a portion which was bent like a half cylinder, so that the two reservoirs when joined to form one cylinder, left in the centre a space for the passage of the trunk, which was to be cooled. The height of the apparatus was 50 cm., its diameter 30 cm.; the space left free for the trunk had a diameter of 10 cm. Each reservoir was provided at the bottom with a tap, through which superfluous water could run off. The cylindrical surface of each reservoir and also the bottom and the lid, were covered on the outside with a layer of felt, 15 mm. thick and over this there was a covering of asbestos paper, 2 mm. thick. In an experiment the two reservoirs were placed round the stem and screwed together, after a piece of felt had been placed between the two flat surfaces in apposition.

The reservoirs were filled with ice. The space through which the trunk passed, was closed off above and below round the stem by a solid plug of cotton wool through which a thermometer passed. The temperature in the annular air space surrounding the trunk varied between 0° and $+3^{\circ}$ C.

The apparatus was sufficiently protected by the felt and the asbestos against the heat of the surrounding atmosphere. Even on hot days it was only necessary to renew the ice twice in 24 hours.

During the experiment the apparatus rested on some bricks, so that it was about 20 cm. from the ground.

EXPERIMENT I.

Apple tree.

The ice apparatus was fixed round the trunk having a diameter of $3\frac{1}{2}$ cm., of a small apple tree, about $2\frac{1}{4}$ metres high, on July 21st 1904, at noon, the weather being hot and sunny. The apparatus was filled with ice and in the course of the afternoon the temperature in the space round the trunk fell to about 1° C.; not the slightest fading of the leaves could be detected, although such fading would have at once been noticeable by comparison with two other apple trees which stood next to the tree experimented on.

Nor could any change be observed on the following days. The temperature of the air space round the stem remained continuously between 0° and 3° .

The maximum temperature of the atmosphere on the days of the experiment oscillated between 23° and 29° .

On the sixth day, when the temperature of the atmosphere was 20° and that round the trunk 9° , a strong transpiration of the leaves was demonstrated by means of the cobalt test. On the seventh day the trunk was sawn through immediately above the ice apparatus; a hole was drilled in the portion of the trunk still inside the apparatus, and a thermometer was placed in it.

In this way I was able to show that the temperature *inside the trunk* was the same as that of the annular air space *round the trunk*, i.e. in the course of three hours it oscillated between 2° and 3° , while the temperature of the atmosphere was 24° to 25° .

EXPERIMENT II.

Polygonum cuspidatum.

The ice apparatus was fixed round two immediately adjoining stems, 2 metres in height, on July 6th 1905, at noon, and it was filled with ice. In the course of the afternoon the temperature in the air space round the stems fell to 0° , without withering taking place. The numerous stems surrounding the apparatus served as controls. Nor was any change noticeable during the following days.

The temperature round the stems remained continuously between 0° and 3° . The maximum temperature of the surrounding atmosphere

on the various days during the experiment oscillated between 19° and 30°. The experiment was stopped on the seventh day.

EXPERIMENT III.

Helianthus tuberosus.

The ice apparatus was fixed round two immediately adjoining stems of plants, 1 $\frac{1}{4}$ M. in height, on July 14th 1905 at noon, and was filled with ice. In the course of the afternoon the temperature of the air space round the stems fell to 0°. No fading could be observed; several specimens of the same species, standing next to the plants experimented upon, served for comparison. Cut leaves, hung up on the plants were completely withered in a few hours. Nor was withering observable on the experimental plants on following days.

The temperature in the air space round the stems remained about 0°. The maximum temperature of the atmosphere in the days of the experiment oscillated between 17° and 26 $\frac{1}{2}$ °. The experiment was stopped on the eighth day.

2. *Ascent of a dye solution in living and dead cut branches.*

When cut branches, with a freshly cut surface, are placed with this surface in a dye solution, the liquid will in general ascend into the branches for some distance, and thus may be easily traced by cutting them across at different levels. Various elements of the wood are then found to have been stained. It matters little whether one takes for this experiment living or dead branches, with or without leaves; the fluid always ascends in the branches, even when these are upside down, i. e. are placed in the solution with their cut apex. I generally carried out such experiments with twigs 30—40 cm. long; sometimes with pieces of a branch, which had also been cut at its upper end. After some days the stain shewed itself on the surface of the upper section of these latter branches.

Although the dye ascends in all branches, the *way* in which the various elements are stained is not the same in living and dead branches. A sharp difference is observable.

In comparing living branches with dead ones, it was of course necessary to use a harmless stain; the experiments of STRACKE: investigation of the immunity of the higher plants towards their own poison (Dissertation), led me to choose *Säureviolett* of GRÜBLER. I used this stain in a $\frac{1}{10}$ ‰ aqueous solution. The twigs were placed

separately in a small bottle with the solution of the stain, the neck of the bottle being closed with a plug of cotton wool to prevent evaporation.

After the experiment the twigs were examined at different levels by microscopic sections. Transverse, radial and tangential sections were examined in *oil of cloves*, a medium in which Säureviolett is insoluble, so that the stain remained properly localized. The sections were cut without the use of any liquid and were at once placed in the oil of cloves. The slight water content of these preparations did not interfere. After a very short time the oil had thoroughly permeated. This method had moreover the advantage, that after most of the clove-oil had been wiped away, the preparations could be very well enclosed in Canada balsam, without further treatment.

A comparison of the behaviour of the xylem elements of living and dead branches brought out the following differences:

<i>living branch</i>	<i>dead branch</i>
a. torus of the closing membrane of the bordered pits deeply stained.	a. torus not stained, or only very slightly.
b. adjoining the lumen, a thin layer of the wall in the border of the pits is stained. The walls of vessels and fibres are only stained in a very thin layer, which is immediately adjacent to the lumen.	b. the walls of the vessels, fibres and parenchymatous cells are stained uniformly.
c. contents and wall of the cells of medullary rays and wood parenchyma are unstained.	c. contents of the cells are coloured.

The deep staining of the tori in living branches was especially noticeable, also in transverse sections, the more so because the staining of the layer next to the lumen in the walls of vessels and fibres was often difficult to see and because the living cells of the medullary rays and parenchyma were quite colourless.

In the wood of *Salix* and of *Fagus*, in which the tori cannot otherwise be seen at all, they were made very obvious by this staining of living branches.

The staining of the tori by eosine in a living branch of *Ginkgo* was already mentioned by JASSE in "Die Mitwirkung der Markstrahlen bei der Wasserbewegung im Holze" (Jahrb. f. Wiss. Bot. 1887 Bd. XVIII). In this case also the stain had ascended the branch:

the "primäre Wandlamelle" of the medullary ray cells was according to JANSE all that had been stained.

In my experiments with living branches the staining extended not only to the tori of the vessels and fibres, but also to those of the *half* bordered pits between the medullary ray cells on the one side and the vessels and fibres on the other side; the *contents* of the medullary ray cells however remained colourless, as stated above.

The results of other experiments carried out by me, agree well with these facts. Instead of taking dead branches, I caused to ascend in living branches a $\frac{1}{10}$ % solution of Säureviolett in strong alcohol, and also a $\frac{1}{10}$ % solution in water containing 4 % formaldehyde. As controls I employed living branches in a $\frac{1}{10}$ % solution of Säureviolett in water.

I now found that the living branches in the poisonous solutions were stained practically in the same way as the dead branches in innocuous ones, only not so completely. It was clear that the alcohol and the formaldehyde only gradually exercised their fatal action on the plant. The tori were always unstained; only a few were stained very faintly. The walls generally showed a uniform staining: the medullary ray and parenchyma cells with contents were coloured dark blue.

Finally I may add that microscopical transverse sections through living branches, which sections were afterwards placed for 20 hours in an aqueous Säureviolett solution of $\frac{1}{10}$ %, were stained quite uniformly dark blue, exactly in the same way as those sections made after the stain had ascended in *dead* branches; the colour was only somewhat more intense. The transverse sections through control branches, which had previously stood in the same solution for 4 days, on the other hand showed, as was to be expected, a staining quite similar to that which was described above for living branches.

Description of the experiments.

EXPERIMENT IV.

Fagus sylvatica.

A living leafy twig, about 4 mm. thick at its base, stood for 9 days in a solution of Säureviolett. The stain ascended to the top and into the leaves. The bark, the cambium and the pith remained quite unstained; the staining was limited to the wood and here the stain was only in the inner layer (adjoining the lumen) of the walls of vessels and fibres; the tori of the bordered pits were stained a very

deep blue-violet, and this was also the case with the half bordered pits between medullary ray cells and fibres. The medullary rays and the xylem parenchyma were quite unstained, both as regards wall and contents.

EXPERIMENT V.

Larix decidua.

A living leafy twig, 6 mm. thick at its base, stood for 5 days in the solution, after which the stain had penetrated to the apex. Staining completely limited to the wood, but no stain in the oldest of the 6 annual rings.

The stain only taken up by a very thin layer of the wall, adjoining the lumen of the tracheids and the cavities of the pits. Torus of the pits deep blue-violet, also in the half bordered pits between medullary rays and tracheids. For the rest everything unstained.

EXPERIMENT VI.

Salix spec.

Two living leafless branches, provided at either end with a cut surface, both 30 cm. long and more than $\frac{1}{2}$ cm. thick, stood for 2 days in the aqueous solution of Säureviolett; one of the branches had its lower end in the solution, the other its upper end.

The stain ascended readily, and *in the two branches simultaneously*. The stain only present in a thin layer of the wall adjoining the lumen of the vessels and fibres and the cavities of the pits. Tori deep blue-violet.

EXPERIMENT VII.

Fagus sylvatica. Taxus baccata.

Of each of these plants two similar 3-5 year old leafless branches were placed with the cut surface in the aqueous Säureviolett solution for 3 days. One of the branches of each species was alive, the other had been treated as follows. It had stood for $1\frac{1}{2}$ hours in boiling water. Then water was sucked through the boiled branch by means of a filter pump in order to remove possible obstructions, finally a fresh surface was cut.

After 3 days the stain had almost reached the top in all the four branches.

In the living branches staining was scarcely visible against the walls of the vessels and tracheids. The tori, including those of the half bordered pits were deeply stained. Medullary ray- und parenchyma cells quite colourless.

In the boiled branch of *Fagus* the walls of the libriform fibres and of the vessels were a uniform pale blue. Against the walls of the vessels in the spring wood a darker layer. Nowhere however coloured tori. The medullary rays also proved to be colourless.

In the boiled branch of *Taxus* the walls of many tracheids were stained a uniform pale blue; towards the inside against the walls a darker layer. The tori unstained. The medullary rays dark blue.

EXPERIMENT VIII.

Taxus baccata.

Two living branches were taken. One was placed with its cut surface in a solution of 0.1 gram of Säureviolett in 100 c.c. of *water*; the other in a solution of 0.1 gram of Säureviolett in 100 c.c. of *alcohol*.

Both branches remained standing in the solution for 43 hours, after which time sections were made through both at a height of 7 cm. The staining was as follows:

Branch in *aqueous solution*: staining only in the secondary xylem. A very thin blue layer against the walls of the tracheids, and of the cavities of the bordered pits. Tori dark blue, including those of the half bordered pits. Medullary rays unstained.

Branch in *alcoholic solution*: the stain had also penetrated into the cambium and the innermost layers of the cortex parenchyma, where both walls and contents were dark blue. In the secondary xylem the tracheid walls light blue; against the walls also clearly a blue layer, further in the cavities of the pits. Tori unstained. Medullary rays dark blue, both as regards walls and contents. The walls also coloured in the primary xylem.

EXPERIMENT IX.

Taxus baccata.

A living branch was placed with the cut surface in a solution of 0.1 gram of Säureviolett in 100 c.c. of a 4% *formaldehyde solution* (diluted formalin).

After 3 days the branch was examined: the stain had already reached the apex.

Staining only in the secondary xylem. Against the walls of the tracheids there was a thin blue layer, also in the cavities of the bordered pits. Tori colourless. Of the medullary rays both the walls and the protoplasm dark blue.

EXPERIMENT X.

Salix spec.

A living twig was placed with its cut surface in a solution of 0.1 gram of Säureviolett in 100 c.c. of a 4% formaldehyde solution. After 3 days the stain had penetrated to the apex and the twig was examined.

Staining only in the secondary xylem. The walls of the vessels coloured light blue with an indication of a somewhat darker layer adjoining the lumen. Tori practically colourless. The medullary ray cells, which adjoined the vessels, are coloured blue.

3. *Interference with the movement of water in a tree-trunk by means of deep incisions.*

EXPERIMENT XI.

An experiment with a small willow tree in the Botanic Gardens at Groningen showed, that in a trunk in which the transpiration current had been largely prevented or perhaps completely cut off as a result of transverse incisions on both sides at various heights, measures were taken in course of time which ultimately led to a complete recovery of this current.

The experiment was carried out as follows.

The trunk of the tree, 11½ cm. thick, was sawn into transversely to slightly beyond the centre at four places, alternately on either side of the trunk. The incisions were 22 cm. apart, and the lowest was 1.25 Metres from the ground. They were prevented from closing up again by the insertion of tin plates, which in future remained in position. At these four places the water current was therefore irreparably interrupted.

As the trunk had of course been greatly weakened by this operation the tree was supported by four iron wires, which were attached high up to the trunk and also to four pegs driven into the ground at some distance round the tree.

This experiment was started on July 14th 1908; the incisions were ready and the plates were pushed in at 9.30 a.m. At 10 a.m. the leaves were already drooping and they remained so throughout the day.

In the course of the five following days, in cool dry weather, the leaves gradually recovered. On the 7th day of the experiment the foliage began to wither from the top downwards; many yellow leaves also appeared in the crown. In all these days the temperature had not risen above 18° in the neighbourhood of the tree. On the 9th day the temperature rose in the afternoon to more than 26°, and probably as a result of this the number of yellow leaves now increased rapidly. Those leaves which had remained green also began to droop again. The tops of the branches in the upper part of the crown withered completely.

The 3 following days were warm and sunny with temperature maxima of 27° and 28°. Most of the leaves now fell off, while in the upper half of the crown the foliage withered completely.

After this time cooler weather supervened and the few remaining green leaves recovered and remained in good condition until the autumn.

That the tree had not suffered greatly however from the incisions, was shown in the following summer, for then the foliage developed as well as before the experiment, and remained fresh throughout the entire season.

Wageningen, Dec. 13th 1909.

Physics. — “*The magnetic separation of absorption lines in connexion with Sun-spot spectra.*” (I). By Prof. P. ZEEMAN and Dr. B. WINAVER.

1. As a consequence of the intimate connexion between emission and absorption, there exists closely corresponding to the magnetic separation of emission lines, a magnetic division of absorption lines. The dark lines which appear in a continuous spectrum, if a beam of white light traverses an absorbing flame, are divided and polarized under the influence of magnetic forces in exactly the same way as the emission lines. This correspondence between emission and absorption was shown to exist already in some of the first experiments on the subject by one of the present authors. Our knowledge of emission spectra under magnetic influence has since been extended considerably. The experimental study however of the inverse effect i. e. the magnetic division of absorption lines has less advanced.

After the first experiments of the first named of the authors of this paper, the change of absorption lines in a magnetic field was studied by KÖNIG ¹⁾ and COTTON ²⁾; RIGHI ³⁾ gave an elaborate study of the subject, to which we have to return later on. It contains the only investigation of the magnetic effect in a direction inclined to the lines of force. Closely connected with our subject are finally some observations by LODGE and DAVIES ⁴⁾ on the influence of a magnetic field on flames, emitting "reversed" lines.

The consideration of the inverse effect forms the basis of VOIGT's magneto-optical theories ⁵⁾; and it is considered also by LORENTZ ⁶⁾ in his investigation of the magnetic separation in a direction inclined to the line of force.

Theory indicates different points, which may be tested by experiment. The inverse effect has become of supreme interest in solar physics, since HALE's ⁷⁾ discovery that the dark lines of the sun-spot spectrum exhibit the characteristic phenomena of magnetic separation.

The experiments we intend to describe in the present communication, relate to the division of the sodium lines D_1 and D_2 . Some of our results may already be found in the work of the cited authors.

In order to present the subject in a connected form it seemed necessary not to exclude these.

The facts now ascertained in combination with former results appear to be of some value in explaining peculiarities observed in sun-spot spectra. Some instances will be given later on.

2. Type and relative amount of the magnetic division of the sodium emission lines, D_1 and D_2 , are given in Fig. 1.

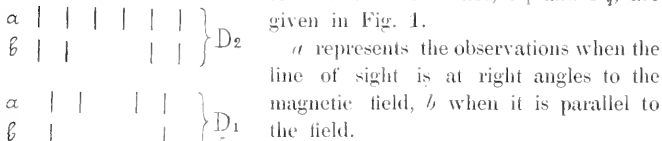


Fig. 1.

In a weak magnetic field D_2 exhibits the triplet type, at right angles to the

¹⁾ KÖNIG. Ann. d. Phys. Bd. 62. 240. 1897.

²⁾ COTTON. Éclairage Electrique. 5 et 26 mars. 1898.

³⁾ RIGHI. Sul fenomeno di ZEEMAN nel caso generale d'un raggio luminosa comunque inclinato sulla direzione della forza magnetica. Mem. di. Bologna. 17 Dicembre 1899.

⁴⁾ LODGE and DAVIES. Proc. R. Soc. 61 413. 1897.

⁵⁾ W. VOIGT. Magneto- und Elektrooptik. Chapter IV and the papers there cited.

⁶⁾ H. A. LORENTZ. These Proceedings, Vol. XII, p. 321, 1909.

⁷⁾ G. E. HALE. On the probable existence of a magnetic field in sun-spots. Contributions from the Mount Wilson Solar Observatory Nr. 30. 1908.

field: the doublet type if the light is examined parallel to the lines of force. D_1 seems to exhibit a doublet in both principal directions.

The FRAUNHOFER lines in the spectra of sun-spots investigated by HALE are either broadened, or changed to doublets (often incompletely resolved quartets), or to triplets. The resolutions exhibited by sodium vapour are therefore the very types of special importance to astrophysics; this and also the facility of producing sodium vapour in the magnetic field induced us to commence our experiments with this substance.

3. The explanation of the inverse effect is easily understood by means of the well known law of resonance. If there are in a flame under the influence of a magnetic field three periods of free vibrations, then we may expect that from incident white light vibrations of these very three periods will be taken away. The absorption is a selective one, with this peculiarity that the selection refers not only to the period but also to the direction of vibration. Consider for example the central component of a triplet which in the emission spectrum is due to vibrations parallel to the field. From incident white light only vibrations, corresponding as to period as well as to direction of vibration with the middle component, are absorbed. Vibrations, perpendicular to the field, though of the period of the unmodified line, pass unimpeded.

On the contrary white light of periods coinciding with those of the outer components is only deprived of its vertical constituents.

It will be clear from these very simple considerations what we may expect to observe with white light under the conditions of the experiment. The arrangement was the following: White light of the incandescent positive pole of an arc-lamp traverses a sodium flame, placed between the poles of a DU BOIS-electromagnet. This light is analysed by means of a stigmatic spectroscope with large ROWLAND grating. The observations are made in the first order.

If the observation is made at right angles to the lines of force, we see in the continuous spectrum 4 dark components in the case of D_1 , 6 dark components in the case of D_2 , as represented for both lines under *a* in the diagrammatical Figure 1.

In order to observe all these components the field must be strong and the vapour density adapted to the field.

The groups of lines indicated by *b* are seen, if the light is examined axially.

All these components, if narrow, are seen only diffuse and not black. From the considerations above given the reason will be clear at once; each of the components absorbs only *half* the incident natural light.

With very diluted vapour no absorption at all or only very weak traces of absorption are seen.

4. The introduction of a Nicol in the beam before or after the field entirely changes the phenomenon. The absorption lines can then be seen very narrow and black.

Let the observation be made at right angles to the horizontal field, then, if the Nicol is placed with its plane of vibration vertical D_1 exhibits its two, D_2 its four outer components.

After a rotation of the Nicol over 90° both D_1 and D_2 give only the two horizontally vibrating components.

Let a beam of natural white light traverse axially the magnetized vapour placed between the perforated poles of an electromagnet. Then by means of a quarter-wave plate and a Nicol we may quench either the right-handed or the left-handed circularly polarized component.

A combination of a quarter-wave plate and a Nicol, converting incident light into right-handed circularly polarized light may be called a right-handed circular analyser. The absorption line corresponding to a right-handed circularly polarized component is seen with both increased clearness and darkness by examining it with a right-handed circular analyser.

We introduce here this simple matter because there has been occasionally some confusion on this subject.

5. The behaviour of horizontal and vertical vibrations may be studied simultaneously by using according to the suggestion of CORX and KÖNIG a calcsparrhomb. By means of it we can obtain two oppositely polarized images of a horizontal slit of suitable width, placed near the magnetic field.

Right-handed and left-handed circular vibrations can be separated on the same plan by the introduction of a FRESNEL rhomb between the calcsparrhomb and the slit of the spectroscope.

It is, however, of considerable interest to examine also the behaviour of the lines in natural light. A separate examination after the removal of the polarizers might be made. The vapour density ought to be the same in both experiments. It seems difficult to realise this.

The desired end is secured more simply and surely, and with only half the labour, by adopting the width of the horizontal slit and the thickness of the calcsparrhomb in such a manner that the two images given by the calcsparrhomb partially overlap. We now obtain three stripes; the central one exhibits the phenomena as seen without polarizing apparatus. (See fig. 2).



Fig. 2.

The upper and lowest stripes show the influence of polarized light on the phenomenon. The observations given in this communication have been made by the described method. By its use all particulars of the phenomenon are simultaneously exhibited; we also succeeded in photographing the essential points. Examples of our photographs are given on the plates annexed to our paper.

6. If the absorption lines are not narrow or if the magnetic field is weak, the components of a magnetically divided line will partially overlap. This partial superposition is the cause of some particularities, especially manifest in the inverse effect and probably also apparent in sun-spot spectra.

The nature of these particularities may be illustrated by a few examples. We will consider the case of the magnetic triplet and the magnetic doublet.

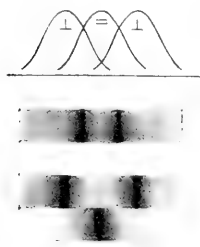


Fig. 3.

In Fig. 3 the curves show the distribution of intensity of the three components of a triplet, if the light is examined at right angles to the lines of force. If natural light traverses a source of light placed in a magnetic field, two black bands are seen, corresponding to the wavelength, for which vertical as well as horizontal vibrations are absorbed.

These black bands are surrounded by less dark parts, which absorb only one of the principal vibrations, the other proceeding unimpeded. (cf. §§ 3 and 4).

If now a Nicol with its plane of vibration vertical, is introduced two black bands are again seen. The darkest part of these components corresponds to the maximum of the curves relating to vertical vibrations.

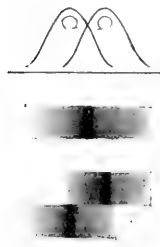


Fig. 4.

As a general rule the distance of the components exceeds that of the lines first considered.

7. Parallel to the lines of force a partial, not too small, overlapping of the components produces a black line limited by two less dark parts. This case is illustrated diagrammatically in Fig. 4.

The two components may be separated by a circular analyser.

These considerations may be applied to the magnetic division in

sun-spot spectra; as a general rule we may expect that the separation of lines in spot spectra becomes more distinct and of larger amount by the use of analysers.

The introduction of a Nicol in the beam may also reveal lines invisible without analyser.

Several peculiarities observed in the distribution of intensity in spot lines, remind one of the now specified superposition phenomena¹⁾; cf. § 19 below.

8. Superposition effects of nearly, though not exactly, the same nature occur if lines with the same direction of vibration are superposed and if the continuous source of light emits unpolarised light. In the more complicated divisions the now specified superposition occurs also. It is just possible that the superposition of the outer components of the sextet, type D_2 , produces only dark, that of the inner and the next outer components, black lines in the continuous spectrum.

It is easily seen that also in the case of the quartet, type D_1 , black lines may be produced. The darkest parts may be seen somewhat nearer to the middle of the complete figure, than the outer components of the quartet.

It seems unnecessary to illustrate this by figures. Examples of the specified actions will be given presently.

9. Our observations and spectrograms relate besides to the two principal directions (parallel and at right angles to the lines of force), also to directions inclined to the field.

In the present, first, communication, observations are discussed, relating to 5 different angles between the field and the direction of propagation of the beam (VOIGT's g , LORENTZ's ϑ).

These values are: 90° , 0° , 60° , 45° , $36^\circ-39^\circ$.

The results of the work relating to these angles have been recorded on nearly 100 spectrograms.

10. *Observations perpendicular to the field.*

In the upper of the three stripes which are present in the field of view (see § 5), the light vibrates vertically; in the lowest one horizontally, whereas the middle part relates to natural light.

Under the influence of the magnetic field we therefore see the vertically vibrating components as narrow black lines. The quartet of the D_1 line, the sextet of the D_2 line, may be seen very clearly

¹⁾ A figure equivalent to the one now given concerning the influence of superposition of magnetically divided components was already drawn for *emission* lines in ZEEMAN. Doublets and Triplets in the Spectrum produced by external magnetic forces. Phil. Mag. July 1897 § 7.

by this method. A small disturbance is produced by the narrow reversed lines due to the electric arc light. The intensity of these lines depends upon somewhat variable circumstances of the arc itself. In some cases these lines are almost invisible, in other ones more prominent. They are to be seen on some of our reproductions: with our present subject they have nothing to do.

As regards the central stripe we refer to the remark previously made, that the image of the separation must become, on account of the only partial absorption, rather indefinite and weak. (§ 3).

The partial superposition of components gives, at least in the case of diluted vapour, the most conspicuous lines. (§§ 6 and 7).

In the case of the quartet, for example, one sometimes sees instead of four, only two components, situated between the inner and outer ones.

We made experiments with different vapour densities. The observed phenomena may be classified under three phases:

1. The vapour is *very dilute*. The components are clearly visible in the upmost and lowest stripe. In the central stripe the absorption is either hardly perceptible (Plate I, Fig. 1) or the components of the quartet and the sextet are seen as separate, but weak lines. (Plate I, Fig. 2).

In this phase of the phenomenon the great difference of definiteness of the central and outer regions is very remarkable. This contrast is still more marked with eye observation.

In order to obtain good photographs, it was necessary to increase the density of the vapour above the one required for the observation of the very first trace of absorption.

2. Vapour of *intermediate* density.

The components in the upmost and lowest stripes are now no more separately visible or only in the case of the quartet. In the central stripe a superposition of the kind mentioned in § 6 takes place. In place of the quartet an apparent doublet is seen, the components of which are situated between the outer and inner components of the quartet. This case is very clearly represented in Plate I, Fig. 3.

The phenomena exhibited by the sextet (D_2 line) become rather complicated.

The superposition phenomenon is often very distinct. The D_2 line on Plate I, Fig. 3 shows sufficiently the appearance.

3. With still *denser* vapour, the components become very broad and the magnetic change hardly visible. The *polarisation of the edges* of the broad line may be recognized. This phase is represented in Plate I, Fig. 4. It corresponds to the emission effect as it was first discovered: a slight change of broad lines in a weak field.

With still greater absorption the influence of the field becomes imperceptible.

All these phases appear with great regularity. If the intensity of the field is known, it seems possible, the resolving power of the spectroscope being given, to deduce the density of the vapour from the nature of the observed phenomena.

The magnetic division phenomena hitherto observed in sun-spots appear to fall under the second and third phases above mentioned. HALE from measurements of spot lines, compared with laboratory experiments, deduces a maximum intensity of the spot field of 4500 Gauss. Hence, one would be inclined to think that the density in the layers, which bring about the absorption in the sun-spot spectrum can only be small. Moreover, the non-uniformity of the field of sun-spots produces by itself a widening of the components. Light from a limited portion of the spot would give perhaps very narrow spectral lines. In the light, however, of the critical remarks of KAYSER¹⁾ concerning our knowledge of the influence of pressure and of temperature on spectra all such considerations must be put forward with great diffidence.

11. *Observations parallel to the lines of force.*

In the present experiments the absorbing vapour subjected to magnetic forces is placed between perforated poles.

After putting on the current, one sees in the continuous spectrum, 2 dark bands in the case of D_1 , 4 in the case of D_2 , according to the diagrammatical figure 1. The absorption is incomplete also now, because of some wave-lengths only right-handed circularly polarized light, but not left-handed is absorbed and the reverse. In order to observe the separation and the polarization a FRESNEL rhomb is placed with its principal plane at an azimuth of 45° with the horizon, a horizontal slit being placed in one of the perforated poles. The FRESNEL rhomb converts circularly polarized into plane polarized light. By means of a calc spar rhomb also now three stripes are obtained. The first phase (very dilute vapour) is represented in Plate I, fig. 5.

Vapour of intermediate density (second phase) exhibits the superposition phenomena mentioned in §§ 7 and 8, and diagrammatically illustrated by Fig. 2. In the central strip *one* line, at the position of the unmodified one, surrounded by feebly absorbing regions, is

¹⁾ KAYSER, Handbuch. Kapitel V. Bd. II.

seen. Plate I, Fig. 6 shows these lines for the doublet and the quartet : especially with D_2 the effect is very marked.

12. Observations in directions inclined to the field.

According to LORENTZ's elementary theory of magnetic division one generally observes in a direction, which is oblique under an angle ϑ with the lines of force, a triplet with elliptically polarized outer components ¹⁾.

The ellipse, which characterizes the state of polarization of the components with period $T_0 + v$, is the projection on the wave-front of the circle perpendicular to the field, in which the electron with period $T_0 + v$ is moving. v is a small quantity. The direction of the motion of the moving electron also determines the motion in the ellipse. The ratio of the axes is as 1 to $\cos \vartheta$. For the other outer component with period $T_0 - v$ holds *mutatis mutandis* the same reasoning.

The central line with the unmodified period T_0 always remains linearly polarized. The vibrations of the middle component are in the plane determined by the ray and the line of force and the amplitude of the vibrations is proportional to $\sin \vartheta$.

If we put $\vartheta = 0$, i. e. in the case of the longitudinal effect, only circular motions remain.

All this applies to very narrow spectral lines in a strong field, the distance of the components being much greater than their width.

According to VOIGT and LORENTZ we must expect some interesting particularities if this restriction be discarded. We return to this point later on.

As a general rule the deductions from the elementary theory are verified. Also in the case of the quartet and the sextet the outer components become elliptically polarized, as has been observed already by RICH ²⁾.

In contradiction with the elementary theory, though not strictly applicable to the case, is the very slight diminution of intensity of the middle components of the quartet even for $\vartheta = 45^\circ$.

13. Observations at $\vartheta = 60^\circ$.

If the observation is made with a calc spar rhomb, the image

¹⁾ cf. RICH l. c.

²⁾ RICH's observations l. c. all refer to an angle of nearly 55° , the angle at which according to the elementary theory the three components of the triplet are of equal intensity.

remains as with the transversal effect. Yet the presence of elliptic polarization ought to manifest itself by the appearance in the lowest stripe of lines, corresponding to the outer components.

With very dilute vapour and with that of intermediate density as good as no trace of it is seen.

Fig. 7, Plate II shows the first phase with dilute vapour, Fig. 8 the second phase with denser vapour. Only traces of absorption, indicative of elliptic polarization can be seen near D_2 , Fig. 8.

The ellipticity is, however, undoubtedly proved by means of the FRESNEL rhomb, placed with its principal plane at an azimuth of 45° with the horizon. Fig. 9 shows the appearance.

The outer components of the quartet towards the red or towards the violet, dependent upon the stripe and the direction of the field, are now considerably weakened; in the case of the sextet they have vanished altogether. All this proves the elliptical polarization of the outer components. For, if the polarisation were linear, as might be inferred from observations with the calespar alone, then the observation with calespar and rhomb combined, ought to show no difference between the upmost and lowest stripe. The light of all plane polarized components would issue circularly polarized from the rhomb and, the calespar making no selection between right-handed and left-handed polarizations, the components towards red and towards violet would all be alike. Such a condition is disproved by photographs such as Fig. 9.

14. One point must be considered somewhat more in detail. What is the reason that the ellipticity is not shown by the calespar rhomb alone, whereas its existence is most clearly demonstrated by means of the FRESNEL rhomb?

Let an elliptic vibration with vertical axis b , horizontal axis a , be incident upon the rhomb, the principal plane of which is at an azimuth of 45° .

It is easily proved that the elliptic vibration issuing from the FRESNEL rhomb has its axes in the same direction as the original motion and a ratio of the axes $\frac{a_1}{b_1} = \frac{b-a}{b+a}$, the original ratio being $\frac{a}{b}$.

If a be small in relation to b (an elongated ellipse), then, the light issues from the FRESNEL as a more circular vibration, which is more easily analysed.

It depends upon the magnitude of a , whether $\frac{a}{b}$ is *greater* or *less* than $\frac{b-a}{b+a}$.

We distinguish the following cases :

1. a very small, then $\frac{b-a}{b+a} > \frac{a}{b}$.
2. $a = 0.414 b$, then $\frac{b-a}{b+a} = \frac{a}{b}$.
3. $a > 0.414 b$, then $\frac{b-a}{b+a} < \frac{a}{b}$.

We shall apply these results to the interpretation of our observations.

Two cases dependent upon the magnitude of a are of principal importance.

In the first case we can observe the effect of both the axes of the ellipse by means of the combination of the FRESNEL rhomb and the calespar (*this is the case of the quartet*) (D_1 , Fig. 9), whereas without FRESNEL rhomb no effect of the small axis is visible. In the second case the effect of the small axis becomes apparent by the use of the calespar, whereas its existence cannot be demonstrated with the FRESNEL, the value of $\frac{b-a}{b+a}$ being too small. *This case is represented by the sextet*, (D_2 , Fig. 9).

If the observation is made by means of the calespar rhomb, we indeed see with dense vapour new components in the lowest stripe (see Fig. 8, D_2). The theoretical import of this result will be discussed on another occasion.

After introduction of the FRESNEL rhomb the component to the left of the central line (small axis of the ellipse) remains invisible. (Fig 9, D_2 , inferior stripe).

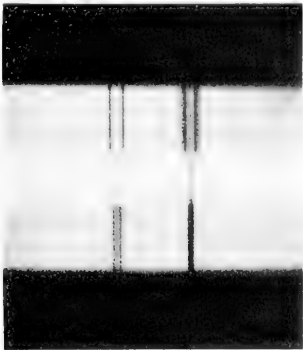
Hence we may conclude that at the angle now investigated the ellipticity of the outer components of the *sextet* (the ratio $\frac{a}{b}$) exceeds that of the *quartet* (and is also larger than 0.414).

15. Observations at $\vartheta = 45^\circ$.

The photographs taken with the calespar alone, show very clearly the ellipticity of the outer components.

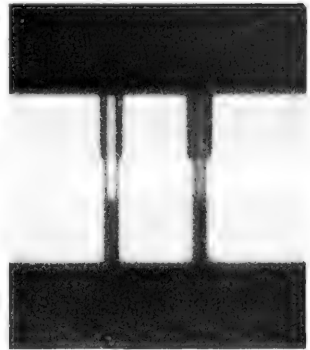
With vapour of intermediate density the phenomenon is already very marked, especially in the case of D_2 (Plate II, Fig. 10). Very remarkable is the slight diminution of intensity of the inner components of the quartet. According to the elementary theory the intensity of the central component of a *triplet* ought to have diminished already to less than *half* the original value.

1.



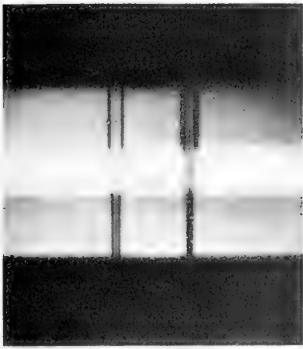
D_1 D_2

3.



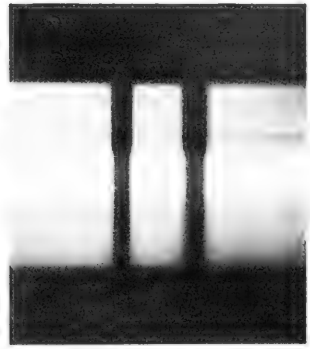
D_1 D_2

2.



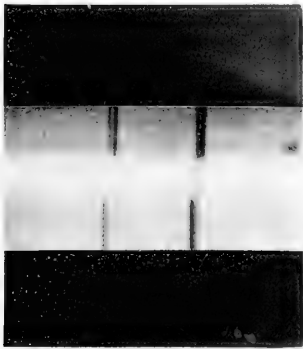
D_1 D_2

4.



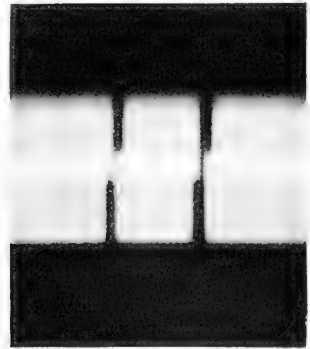
D_1 D_2

5.



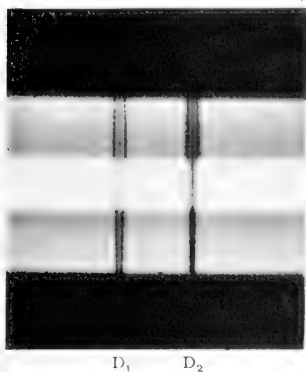
D_1 D_2

6.

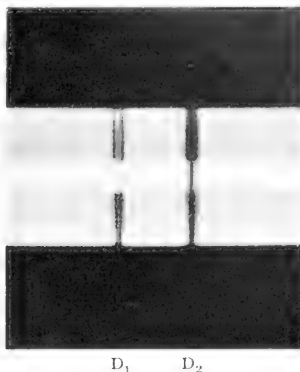


D_1 D_2

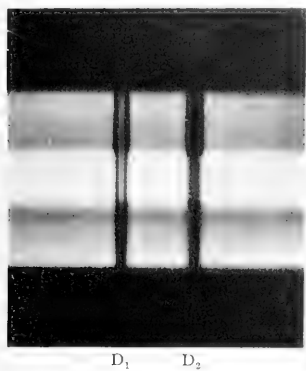
7. (60°)



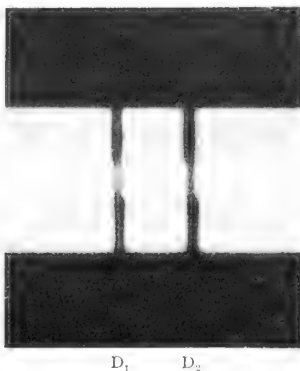
10. (45°)



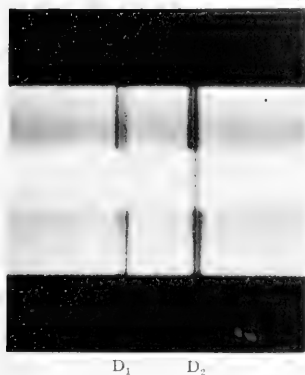
8. (60°)



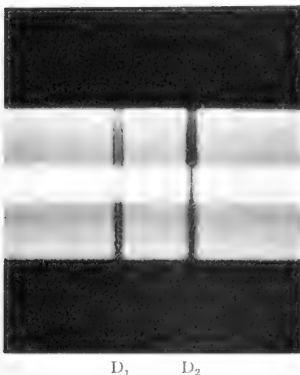
11. (45°)



9. (60°)



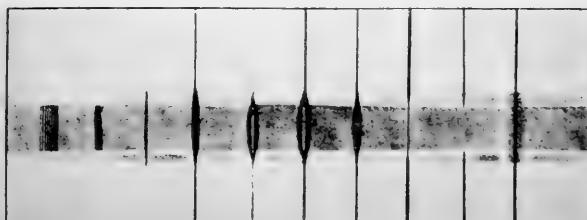
12. (60°)



13. (39°)



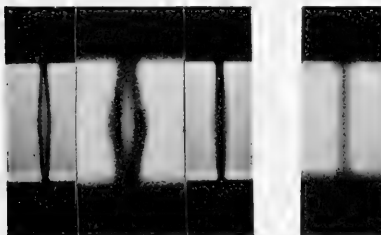
14. 1 2 3 4 5 6 7 8 9 10



Types of sun-spot lines. (Mitchell)

5, 6. Widened lines with centres reserved bright.
7. Widened and weakened line. 10. Winged line.

15. 5¹ 6¹ 7¹ 10¹



5¹, 6¹, 7¹. Types of magnetic resolutions in non-uniform fields.
10¹. Superposition of magnetic components.

16. If a FRESNEL rhomb combined with a calespar rhomb is introduced in the beam, one of the components of the quartet also entirely disappears. At an angle of 60° this was only the case with the sextet. (Plate II, Fig. 11)

17. *Observations at $\vartheta = 39^\circ$.*

The elliptic polarisation tested by means of the calespar rhomb is very marked, even with dilute vapour (Plate II, Fig. 12, Plate III, Fig. 13).

The inner components of the quartet are now decidedly less intense than the outer ones.

Plate III, Fig. 13 especially shows the smaller intensity of the components of D_1 in the lowest stripe. Indeed, they are unmistakably thinner than those in the upmost stripe.

18. According as the angle between the ray and the lines of force is diminished, the intensity of the field must diminish at the same time. In order to make it possible for the rays to traverse the field under smaller angles the vertex semiangle of the cones must deviate more and more from the theoretical optimum of nearly 55° .

The decrease of the magnetic separation is clearly shown in our photographs.

We intend to communicate on another occasion experiments under smaller angles ϑ and to enter upon some details concerning the case in which the components of the triplet are not neatly separated. Some measurements of the ellipticity of the components will also be given. On the present occasion we only intended to give a general survey of the inverse effect, illustrating it by some particular cases.

19. *Types of separation in spot and laboratory.*

In one direction we shall now enter upon some more details. The magnetic separation of lines in a *non-uniform* field has been treated on a former occasion.¹⁾ The results then obtained and our present observations may be of some interest in connection with certain phenomena observed by HALE. We intend to return to this subject. Presently it seems interesting to allude to MITCHELL'S descriptions of the various types of spot lines as indicated in the diagram published in the Transactions of the International Solar Union²⁾.

Our Fig. 14, Plate III has been copied from this source. The types 5, 6, 7, and 10 of the Figure are very characteristic. Type 9 perhaps falls under the type of lines invisible without Nicol mentioned

¹⁾ ZEEMAN. These Proceedings, April 1906, November 1907.

²⁾ Transactions Intern. Union Solar Research, p. 199 etc. 1908.

§ 7 above. In Fig. 15 are represented some separations observed in the laboratory *without* Nicol or other analyzer, 5', 6', 7' have been taken in non-uniform fields. 5' is the quartet of D_1 observed across the field; 6' the sextet of D_2 observed axially in a non-uniform, in the central part very strong, field; 7' also refers to D_2 in a weaker field, the observation being made across the lines of force. The type 10' refers to the D_2 line, when observed in a direction parallel to the field. The field is uniform. The separation gives an example of the superposition phenomenon mentioned in § 7.

The analogy of the type 10', Fig. 15 and the type of the "winged line" seems very remarkable. Of course observation of the state of polarization would be necessary in order to prove the analogy.

EXPLANATION OF PLATES I—III.

The figures 1—13 are about thirteenfold enlargements of the images given by the grating of the absorption lines D_1 and D_2 in a magnetic field.

The upmost and lowest of the three stripes of these figures relate to (oppositely) polarized light; in the central part the phenomenon is represented as it is seen in natural light.

PLATE I. 1, 2, 3, 4, observations \perp lines of force with different vapour density.

5, 6, observation // lines of force with different vapour density.

PLATE II. 7, 8, observation at $\mathcal{S} = 60^\circ$ calc spar rhomb alone.

9, $\mathcal{S} = 60^\circ$, calc spar combined with FRESNEL rhomb.

10, 11, $\mathcal{S} = 45^\circ$.

12, $\mathcal{S} = 39^\circ$.

PLATE III. 13, $\mathcal{S} = 39^\circ$.

14, Types of sun-spct lines (adopted from MITCHELL).

15, 5', 6' 7', separations in non-uniform laboratory fields. 10' superposition phenomenon § 7.

Physics. — "*The thermomagnetic properties of elements.*" By Prof. H. E. J. G. DE BOIS and Prof. KŌTARŌ HONDA. (Communication from the Bosscha-Laboratory).

(Communicated in the meeting of January 29, 1910.)

In 1895 CURIE¹⁾, though he had investigated relatively few substances, believed that he could formulate his results in the following rules:

1. For paramagnetic substances the specific susceptibility is inversely proportional to the absolute temperature.

2. For diamagnetic substances, on the contrary, the susceptibility is almost independent of temperature.

3. For the latter class of substances, changes of physical state generally have hardly any influence.

¹⁾ P. CURIE, Ann. de Chim. et de Phys. (7) 5 p. 289, 1895. — Oeuvres p. 232 Paris 1908.

4. The same holds for variations of chemical state (allotropy).

One of us in 1900 proposed to call the first of these thermomagnetic rules CURIE'S law and to introduce a *Curie's constant* such that:

$$\chi(\theta + 273) = C.$$

It was also expressly stated that very probably this was only a kind of "limit-law" in the sense of the analogous law for ideal gases. In addition it was very soon shown that the usual theory of directed magnecules leads to such a law, when generalised from a more magnetokinetic point of view; this was theoretically proved and experimentally confirmed in the LORENTZ- and BOSSCHA-volumes of the "Archives"¹⁾. With all due regard for CURIE'S important researches and for his first rule, the second can and could have no general signification, for it at once contradicted the results of other observers, e. g. as in the special case of water.

With regard to the third and fourth rules even their author pointed out several exceptions. As the values of the susceptibilities of the diamagnetic substances tested proved much less than those of the paramagnetic bodies, CURIE came to the conclusion that these two opposite forms of magnetic induction were due to completely different causes.

Starting from these experimental conclusions, LANGEVIN²⁾ in the year 1905 elaborated an electronic theory of magnetism; he also gave a kinetic representation of CURIE'S first law, completely analogous to the one mentioned above, without, however, mentioning it, and which is in addition perfectly independent of the introduction of electrons.

It appeared, therefore, desirable to investigate the thermomagnetic properties of more substances: in the first place those of elements, in order to judge whether CURIE'S conclusions admit of such an extensive generalisation. It may be at once remarked that such is not at all the case.

Experimental Arrangement. The method, previously used by CURIE and other investigators, of the torsion-balance combined with a non-uniform field was applied, employing the semicircular electromagnet recently described in these Proceedings. The axes of the two cores formed an angle of 10° to 20°; the maximum gradient of the field then lies at a certain distance to one side of their point of intersection. The field

¹⁾ H. DU BOIS, Rapp. Congr. d. Phys. 2 p. 486, Paris 1900. — Arch. Néerl. (2) 5 p. 246, 1900, also 6. p. 581, 1901. — Verh. nat. en gen. Congr. 8 p. 60, Rotterdam 1901. *Notations:*

a , Atomic weight.	θ , Temperature.
C , CURIE'S constant.	χ , Specific susceptibility.

²⁾ P. LANGEVIN, Ann. de Chim. et de Phys. (8) 5 p. 70, 1905. Journ. de Phys. (4) 4 p. 678, 1905.

itself at this particular point amounted to 25 kilogausses; it was measured from point to point by means of a small standardised spherical test-coil. The sensitiveness of the torsion-balance could be varied; it was determined in the usual way by means of applied additional moments of inertia.

The furnace consisted of a porcelain tube wound with platinum wire and insulated with kaolin powder and asbestos. With a consumption of 1.2 kilowatts a temperature of 1250° was attained, which was measured by means of a thermo-element, previously checked by observations on the melting-points of tin, tellurium, antimony, and gold.

Test-samples. The great difficulty with all experiments in this sphere of work is and always will be the prevalence of iron, with its overwhelming ferromagnetic properties, though it hardly ever seems to act quite freely. In the case of fifteen elements, their binary alloys with iron were examined in TAMMANN'S laboratory, not in the very diluted state, however, which generally corresponds to ferruginous impurities. Of 81 elements, 43 were tested; many of them were supplied as pure as possible by KAILBAUM; Prof. COHEN and Dr. HORTSEMA of Utrecht kindly placed several elements at our disposal; as yet the 10 gaseous elements have not been tested; Li, Rb, Cs, Ca, Sr, Ba could not be obtained sufficiently free of iron; while Be, Sc, Ga, Ge, Y, Rd, and the rare metals could not be procured. Fe, Co, Ni, of course, form a class by themselves. Dr. M. HANNA kindly determined the percentage of iron colorimetrically by the Berlin blue-reaction.

The experimental results, moreover, furnish certain physical criteria of their own reliability, for in so far as the susceptibility proves independent of the field there can hardly be question of a ferromagnetic ingredient. With about one third of the samples this was not the case, for the susceptibility diminished (in the algebraic sense) with an increasing field according to a hyperbolic law. From this Mr. MORRIS OWEN calculated the value χ_{∞} which would hold asymptotically for an infinite field; and, in addition, the influence of the ferromagnetic ingredient, which at most amounted to only one sixth — and generally much less even — of what could be imputed to the iron in the free state. The thermomagnetic properties also afford a test of purity up to a certain point; a few strongly ferruginous substances show a great diminution of susceptibility between 500° and 600°, whilst above 700° the influence of iron hardly need be feared. In no case is there reason to doubt that the value of the susceptibility of absolutely non-ferruginous elements would remain constant, at least within the usual

field-range. The full communication of the results obtained would require many tables and curves; we therefore draw attention to the principal points only.

Specific susceptibility ¹⁾ at 18°. The values found lie between -2 and $+5$ (amorphous carbon and palladium respectively). It cannot be maintained that the positive paramagnetic values are on the whole larger than the negative diamagnetic ones. Oxygen alone forms an exception with a value of about 100: the value for manganese was approximately 10: this contained, however, $\frac{1}{3}\frac{a}{a}$ of iron.

CURIE had already pointed out the influence of allotropy in the case of phosphorus and antimony, and also that there is no such influence with sulphur, though it is so well-known for its polymorphous properties. A difference was shown to exist between diamond (-0.49) and amorphous carbon (-2.02): silicium crystalline (0.12 and amorphous (-0.14); and especially between common tetragonal tin ($+0.03$) and grey tin (-0.29). In the case of tin, the first — the tetragonal — was KAHLBAUM'S very pure electrolytic material: it was afterwards inoculated with a small quantity of grey tinpest, kindly sent by Prof. COHEN from the stores of the VAN 'T HOFF Laboratory.

For weak fields indium seemed to be paramagnetic: in a field of 7 Kgs. the value of the susceptibility passed through zero and became negative, doubtless in consequence of 0.013% iron: this phenomenon is of no consequence because it is also discovered in complicated substances such as certain kinds of porcelain, glass, etc.

Notwithstanding many omissions, it was still possible to follow the general course of the curve $\chi = \text{funct. } (a)$: the curve appears to be rather intricate, but still shows a distinct relation to the periodic system. According to the arrangement of MENDELEJEFF-BRACNER'S table, the rows (1, 2, 3, 4), (5, 6, 7, 8), and (9, 10, 11, 12) each form a division I, II, III in which the shape of the curve repeats itself in a peculiar way. At the junction of I and II Cr, Mn, Fe, Co, Ni lie on a positive maximum: between II and III, in the same way, the "rare" metals: within I, II and III the diamagnetic negative peaks are occupied by the similar pentavalent elements P, Sb and Bi of the fifth group (3^d, 7th, 11th row). In more than one respect further magnetic analogies of secondary importance exist, which, however, must be left unmentioned in this communication.

Susceptibility at high temperatures. As a rule the path of the curve $\chi = \text{funct. } (b)$ for any substance proved to be the same when the temperature was increased or afterwards decreased; certain deviations probably depend on a change of condition of the iron present after

¹⁾ Everywhere below expressed in millionths

PARAMAGNETIC				DIAMAGNETIC			
C	I	D	C	I	D	C	D
Na (0°-97°)	Ti	[O]	B, cryst (400°-1200°)	B, cryst (0°-400°)	C, amorphous		
Al (657°-1100°)	V (500°-1200°)	Mg (0°-657°)	Si, crystalline	Diamond	Cu		
K (0°-150°)	Cr (500°-1100°)	[Fe] (800°-1350°)	P, white	Ag	Zn (300°-700°)		
V (0°-500°)	Mn	Co (1150°-?) ?	S, rhombic	I (0°-114°)	Zr (500°-1200°)		
Cr (0°-500°)	Mo	[Ni] (350°-800°)	Zn (0°-300°)		Cd (300°-700°)		
Nb (0°-400°)	Ru (550° 1200°)	Pd	As, Sublim.		In (0°-150°)		
W	Rh	Rare metals?	Se, Metallic		Sb (0°-631°)		
Os	Ir	Ta	Zr, cryst (0°-500°)		Tc		
	Th (0°-400°)	Pt	Cd (0°-300°)		I (114° 200°)		
		U	Au		Tl		
			Hg (0°-350°)		Pb (0°-327°)		
			Pb (327°-600°)		Bi (0°-268°)		

heating. Mg and Ru shewed the above mentioned diminution in a marked manner between 500 and 600°. The results are collected in the table p. 600. The elements in square brackets have previously been examined by others; the atomic weights in each column increase from top to bottom; the elements under column *C* show a constant susceptibility, under *I* a numeric increase on heating and under *D* a numeric decrease. The fewest number (4) of elements appears in the fifth column, in the case of which the susceptibility increases on heating, the increase being, however, very small in each instance.

From a thermomagnetic point of view a certain relation also exists in connection with the periodic curve $\chi = \text{funct. } (\theta)$: the paramagnetic elements under *D* all lie at the principal maxima or at the secondary peaks; on the contrary, those under *I* lie on the ascending branches of the curve. Therefore the sharpness of the bends would be flattened more and more at higher temperatures; probably at lower temperatures they would become more accentuated, and it may be that only then do they attain their most characteristic shape; of course the temperature of $+18^\circ$ is quite arbitrary. Concerning CURIE'S rules the following statements may be made:

1. Only palladium foil from KAHLBAUM, with 0,70% of iron and $\chi = +6,12$, on heating followed, more or less, CURIE'S law, but on cooling it shewed complications. With much purer palladium from Dr. HERAEUS, with 0,03% iron and $\chi = +5,79$, the susceptibility fell less rapidly than would follow from CURIE'S rule; temperature-hysteresis was not observed on cooling¹⁾.

2. There are only 6 diamagnetic elements which do not vary within the whole temperature-range. Of these P, S and Se had already been experimented upon by CURIE.

3. On melting or solidifying, sometimes — not always — a discontinuity appears, which can be classified under one or other of the two following divisions: I, a large or small leap in the curve of χ itself, as with P (44°), Ag (961°), Sn (233°), Sb (631°), Te (450°), Au (1064°), Tl (290°), Pb (327°), Bi (268°); II, a rather sudden change of $d\chi/d\theta$ as with Mg (633°), Cu (1065°), Cd (322°), I (114°); with regard to sulphur, the curve at the melting-point departs slightly from its otherwise absolutely rectilinear character, which variation was probably overseen by CURIE.

¹⁾ By chance palladium is the only paramagnetic element examined by CURIE; perhaps it was not pure enough. The important results for oxygen, for ferromagnetic metals at very high temperatures and also for their salts crystallised or in solution, of course continue to hold.

4. As regards the thermomagnetic examination of polymorphous transformations, a discontinuous diminution of $15\frac{1}{2}\%$ of the specific susceptibility was found at the transition-point of α -thallium and β -thallium at 234° . But the most remarkable properties are shown by tin: If diamagnetic grey tin is slowly heated, at 32° the specific susceptibility (-0.29) changes almost suddenly (like the density) and at 35° passes through zero. Possibly this change would wholly take place at the point of transformation (18°) but then at a much slower rate. Further heating continuously increased the susceptibility so that at about 50° the value ($+0.03$) for paramagnetic tetragonal tin was reached, which afterwards remained practically constant; according to DEGENS the point of transformation tetragonal \rightleftharpoons rhombic tin lies at 161° at which temperature nothing particular was noticed; at the melting point (233°) a discontinuity from $z = +0.03$ to $z = -0.04$ once more appeared; the diamagnetic liquid metal remained nearly unchanged.

An extension of these thermomagnetic investigations towards low temperatures is in preparation

From the above, especially from the conclusions arrived at under 1 to 4, it seems to follow that CURIE'S four corresponding rules are certainly devoid of the general meaning, which has rather rashly been ascribed to them. At the same time the experimental starting-points of LANGEVIN'S theory are evidently undermined; more solid and broad foundations for future theories can only be laid with the aid of more extensive research.

Chemistry. — "*Studies on Tellurium: 1. The mutual behaviour of the elements sulphur and tellurium*". By Prof. F. M. JAEGER. (Communicated by Prof. VAN ROMBURGH).

(Communicated in the meeting of January 29, 1910).

§ 1. Whilst we are in the main thoroughly informed as to the relation of selenium and sulphur, the views as to the mutual behaviour of the elements tellurium and sulphur still differ somewhat. KLAPROTH¹⁾ has already investigated this subject. He states that on melting together tellurium and sulphur leaden coloured masses are formed crystallising in rays, which, on heating, give off sulphur and yield a porous metallic looking mass, which he takes to be telluriumsulphide. BERZELIUS²⁾, thirty years later again broached the subject; he found that no compounds were formed on melting, but thought that the

¹⁾ KLAPROTH, Gelle's Ann. (1798). 191.

²⁾ BERZELIUS, Gilb.-Pogg. Ann. 8. (1826). 413.

compounds TeS_2 and TeS_3 are present in the brownish-black precipitates, formed when passing H_2S through solutions of tellurites and tellurates. He arrived at that conclusion on account of the solubility of these precipitates in aqueous potassium or sodium hydroxide, which is also the case with TeO_2 and TeO_3 .

BECKER¹⁾ was the first to analyze these precipitates and he finally arrived at the conclusion that their composition actually corresponds with TeS_2 and TeS_3 . He proved however, that nearly all the sulphur may be removed from these substances by treatment with carbon disulphide: TeS_3 yielded a residue containing 6.14% of sulphur instead of 42.85%, TeS_2 a residue containing 3.69% instead of 33.4%. He concludes that the black precipitates are only mixtures whose composition agrees nearly with those of the supposed compounds. According to him they are formed primarily as ephemeral compounds, which are at once decomposed by the solvent. BERZELIUS²⁾ and OPPENHEIM³⁾ obtained double sulphides to which they assigned the formulae $3\text{K}_2\text{S} + \text{TeS}_3$, etc. In more recent times, BRAUNER⁴⁾ and GUTBIER⁵⁾ again inclined to the opinion that we are dealing here with mixtures of the elements.

§ 2. Since DUMAS placed tellurium in the sulphur group, as the first homologue of selenium, and thus the well-known difficulty as to the position of tellurium, in regard to iodine, in the periodic system introduced later, was created, — the question as to the relation of tellurium on the one side and sulphur and selenium on the other has again become of actual importance. For now it is undoubtedly certain that the atomic weight of tellurium is 127.6 and therefore *greater* than that of iodine. On the other hand the differences between tellurium and the other two elements are so strongly pronounced that RETGERS on account of the isomorphism between tellurates and osmiates, thought it would be better to include tellurium in the group of the platinum metals. Tellurates to wit, are not isomorphous with sulphates, selenates, manganates, ferrates etc. On the contrary, PELLINI showed an isodimorphism in the case of $(\text{C}_6\text{H}_5)_2\text{SeBr}_2$ and $(\text{C}_6\text{H}_5)_2\text{TeBr}_2$, whilst NORRIS and MOMMERS noticed a direct isomorphism between the selenium- and tellurium double chlorides and bromides of dimethyl-

1) BECKER, Lieb. Ann. d. Chem. **180**. (1876). 257.

2) BERZELIUS, Traité de Chimie. (1830).

3) OPPENHEIM, Journ. f. prakt. Chem. **71**. (1857). 270

4) BRAUNER, Journ. Chem. Soc. **67**. (1895). 527.

5) GUTBIER, Berl. Ber. **34**, 2114. (1901).

amine. But on the other hand many objections have been raised to the position assigned to tellurium: for instance, the different constitution of telluric acid which, probably, must be looked upon as H_6TeO_6 and the totally different hydration of tellurates in comparison with sulphates and selenates. However this may be, it is highly desirable to obtain more *data* as to the position of tellurium among the other elements and for this reason, the relation to sulphur had to be ascertained in the first place.

§ 3. The tellurium was obtained from $1\frac{1}{2}$ kilo of crude tellurium probably derived from American ore. It appeared to contain the following elements: tellurium, selenium, sulphur, lead, copper, bismuth, iron, silicon and traces of antimony, zinc and a few other metals.

The first purification was carried out by oxidation with *aqua regia*, evaporation of the filtrate to dryness, and repeated extraction of the residue with strong hydrochloric acid. The boiling filtrate was then precipitated by sulphur-dioxide; the first portions of the precipitate being rich in selenium were each time rejected. This operation was repeated three times. The amorphous tellurium was divided into two parts: one portion was converted, by the process given by NORRIS, FAY and EDGERLEY¹⁾, into basic tellurium nitrate $Te_2O_3(OH)(NO_3)$ and by repeating the process five times, which operation lasted many weeks, it was finally obtained quite pure in the form of the said salt: from this, pure TeO_2 was then obtained by gentle ignition and this, dissolved in pure hydrochloric acid was precipitated by SO_2 . The other portion was converted into telluric acid by means of CrO_3 , according to STAUDENMAYER's process as modified by GUTBIER²⁾: this was purified by precipitating twelve times with nitric acid and then crystallising from water. It is necessary to reduce the adhering CrO_3 with alcohol, otherwise the telluric acid crystals retain a yellow colour which is caused by occluded solid CrO_3 ; this matter I hope to refer to shortly.

The pure telluric acid was then reduced completely by hydrazine hydrate.

The crystalline form of the basic nitrate has not been described up to the present. The following *data* have been obtained from the substance crystallised from nitric acid.

¹⁾ NORRIS, FAY and EDGERLEY, *Americ. Chem. Journ.* **23**, 105.

²⁾ GUTBIER, *Z. f. anorg. Chem.* **29**, 22. (1901); **32**, 96. (1902).

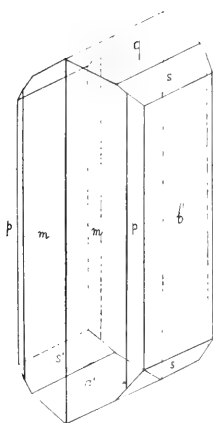


Fig. 1.
Crystalline form of basic
tellurium nitrate.

Colourless, very lustrous needles up to 5 m.m. in length and usually flattened along $\{010\}$. They exhibit many vicinal planes particularly in the vertical zone, and greater angular differences occur also in different individual crystals. The measurements must, therefore, be regarded only as approximations.

Rhombic-bipyramidal.

$$a : b : c = 0.590 : 1 : 0.607.$$

Forms: $m = \{110\}$, $b = \{010\}$ and $p = \{120\}$, very lustrous; particular b , which is also a cleavage plane and possesses a high lustre. On the other hand $q = \{011\}$ and $s = \{021\}$ are dull, the form $\{021\}$ is mostly absent. The crystals exhibit a pronounced inclination to tetragonal symmetry.

Angular values:	Measured:	Calculated:
$m : m = (110) : (\bar{1}\bar{1}0) =$	$61^{\circ} 5'$	—
$b : q = (010) : (011) =$	$58^{\circ} 44\frac{1}{2}'$	—
$m : p = (110) : (120) =$	$19^{\circ} 25'$	$19^{\circ} 18\frac{1}{2}'$
$p : b = (120) : (010) =$	$39^{\circ} 59'$	$40^{\circ} 2\frac{1}{2}'$
$q : q = (011) : (0\bar{1}\bar{1}) =$	$62^{\circ} 50'$	$62^{\circ} 31'$
$m : q = (110) : (011) =$	$74^{\circ} 54'$	$74^{\circ} 43'$
$m : b = (110) : (010) =$	$59^{\circ} 27\frac{1}{2}'$	$59^{\circ} 27\frac{1}{2}'$
$b : s = (010) : (021) =$	$39^{\circ} 46'$	$39^{\circ} 29'$
$s : q = (021) : (011) =$	$19^{\circ} 0'$	$19^{\circ} 16'$

Completely cleavable towards $\{010\}$.

The optical axial plane is $\{001\}$ with the a -axis as first diagonal. Strong rhombic dispersion with $q < c$; the apparent axial angle in cedar oil (1.51) was about 63° .

It may be observed here that the tellurium precipitated from telluric acid by hydrazinehydrate is distinguished from that precipitated from a hydrochloric acid solution by sulphurdioxide which is also amorphous, by a perceptible darker colour. It is, as yet, undecided whether this is merely due to another degree of division or to a real allotropism of the amorphous modification.

§ 4. Both masses of tellurium mixed with 5-6 times the amount of powdered, freshly prepared potassium cyanide were fused for some hours in large ROSE crucibles in an atmosphere of coal gas, with the aid of a large PERROT-furnace. In the course of a few months, about 5 kilos of these melts were obtained. When carefully powdered, the dark coloured masses dissolve in recently boiled, hot water to beautiful purple coloured solutions, which on cold oxidation by purified air deposit from the K_2Te all the tellurium in brilliant needles. On melting the masses, the poisonous influence of the hydrogen telluride, which is formed in small quantities, was experienced only too plainly, also the disagreeable consequences of breathing the very small quantities of $TeCl_2$ formed during the treatment with *aqua regia*. For weeks afterwards the breath has a powerful odour of $(CH_3)_2Te$, which resembles phosphine and is exceedingly sensitive to the olfactory nerve of bystanders.¹⁾

The crystalline and already very pure tellurium thus obtained is free from selenium as proved by the exceedingly delicate NORRIS' potassiumiodide-reaction and by the non-reduction of the TeO_2 by hydroxylamine in strong hydrochloric acid solution. All the selenium has been removed as $KCNSe$, whilst the tellurium has passed into K_2Te and then has again been liberated by the action of air free from H_2S .

The purified element was now distilled in vacuo at about 600---700° in long tubes made of hard glass and containing plugs of asbestos: a TÉLÉ furnace was used. This operation was repeated about seven times, each time about 10 grams were used. The pure tellurium thus obtained was silvery white and coarsely crystalline, much resembling crystallised antimony.

The determinations carried out have been made with the product obtained from telluric acid. The sulphur was recrystallised twice from boiling toluene and heated in a drying oven at 90° for some hours.

¹⁾ The opinions as to the physiological actions of tellurium are still very much divided. Although selenium is an element hardly less poisonous than arsenic, tellurium is considered by CZAPEK and WELL (Chem N (1893), 1098 2) to be comparatively harmless, owing to the much more rapid reduction of the tellurium compounds and the consequent localisation in the organism. The experience gained in my laboratory proves this view to be incorrect.

Tellurium is undoubtedly poisonous, but the individual sensitiveness to small traces varies widely with different persons. TeH_2 , in particular, is a poison causing severe headache and vomiting: other tellurium compounds such as $TeCl_2$, for instance are supposed to cause much inconvenience only, owing to their conversion into malodorous substances, but still there can be no doubt whatever as to their poisonous nature.

§ 5. The construction of the melting apparatus will be readily seen from fig. 2. The hard glass tubes always filled with 10 grams of the weighed and well mixed complex of the two elements were placed in iron cylinders filled with fine sand. Tube and cylinder were covered with asbestos; the requisite atmosphere of nitrogen was supplied by way of a hard glass gas-inlet-tube. The nitrogen was prepared from NH_4Cl and KNO_3 , freed from oxygen by means of alkaline pyrogallol and sodiumhydrosulphite and dried by sulphuric acid. The furnace was constructed of chamotte stone furnished

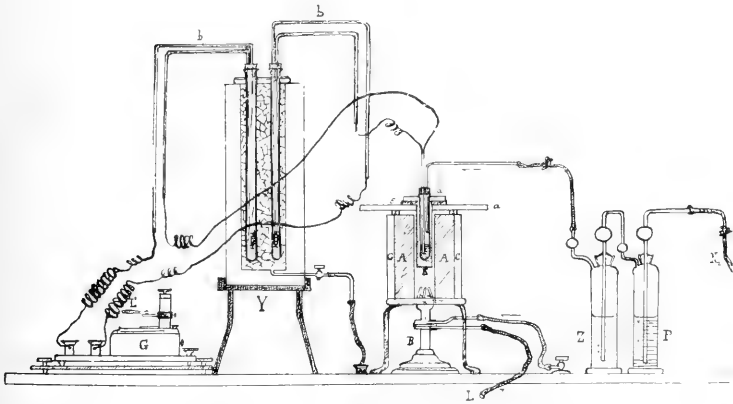


Fig. 2.

with an asbestos filling and a central cylinder of unglazed earthenware; it was covered with an asbestos board resting on three little chamotte blocks, which were either removable or not so, for the regulation of the velocity of cooling. The icekettle for the cold solderplace of the platinum-platinumrhodium thermoelement (3 mm. is double walled and allows of working for some six hours with the single supply of ice; all the conducting wires were isolated by glass tubes.

The galvanometer of SIEMENS and HALSKE was verified by determining the meltingpoints of tin, lead, bismuth, cadmium, zinc, antimony and silver and by making use of the values found by DAY and HOLBORN and by DAY and CLEMENT, which were compared with the gastermometer. The reading was taken with the aid of a lens, the counting of the time by means of a clockwork, which gave a signal every 10 seconds.

§ 6. Great difficulties were experienced in the determination; when we dealt with mixtures containing much tellurium every precaution had to be taken to prevent the boiling off of the sulphur, and in the case of complexes containing much sulphur trouble arose from the great viscosity of the fusions and the very slow crystallisation of the masses. If the percentage of sulphur exceeds 80, the deter-

MELTING POINT DIAGRAM OF SULPHUR-TELLURIUM COMPLEXES.				
Mol. % Sulphur	% by Weight Sulphur	Initial solidifying point C°	End solidifying point C°	Period of solidifying in seconds
0	0	452½	-	—
5	1.3	440	437	—
10	2.7	435	430	—
15	4.2	431	423	—
20	5.9	426	-	—
25	7.7	421	-	0
30	9.7	413	103	30
35	11.9	401	102	69
40	14.4	394	106	80
45	17.1	389	104	—
50	20	387	105	100
55	23.5	385	405	—
60	27.4	374	109	—
65	31.7	368	101	145
66.67	33.4	366	105	160
70	36.9	361	105	—
75	43	348	408	180
80	50	339	109	480
85	58.7	-	110	210
90	69.3	288	108	230
95	82.7	-	409	260
98	92.4	-	110	—
100	100	115	-	-

minations often become very uncertain; some of these mixtures only exhibited a sharp end-solidifying-point. Still it was generally possible to get concordant results on repeating the experiments.

The subjoined table shows the results of the experiments.

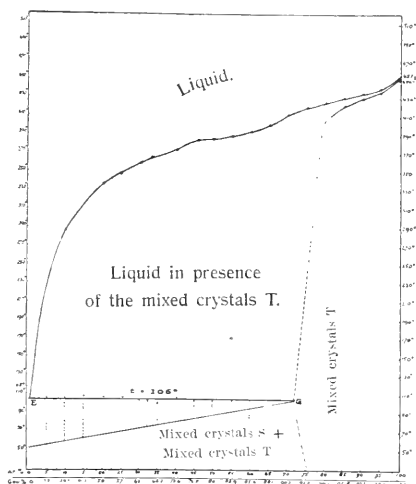


Fig. 3.

§ 7. These *data* represented graphically in Fig. 3 in the usual manner show, therefore, that the elements sulphur and tellurium when melted together yield, when solidifying, two series of mixed crystals of a different crystalline form. The condition diagram is that which has been noticed frequently with isodimorphous substances; there is a very extended hiatus starting from almost pure sulphur to perhaps 27% of sulphur at the side of the trigonal mixed crystals. The temperature of the eutectic point *E* is 106°: the time required for solidification, as far as could be ascertained, increases continuously with the percentage of sulphur until the pure sulphur is reached. The mixed crystals rich in sulphur have a slight ruddy colour; as very small amounts of tellurium impart to sulphur an intensely red colour, their tellurium content must be small indeed. They exhibit the thin needle shaped form of monoclinic sulphur. The transformation at 106° may be seen beautifully with the eye in the various melts on cooling as well as on warming. The monoclinic mixed crystals

rich in sulphur appear to change into the rhombic form at a lower temperature. In these circumstances nothing is noticed as to compounds between tellurium and sulphur: even at lower temperatures no heat effects are observed. The melts of the mixtures rich in tellurium are dark brownish black and in thin layers yellowish brown; unlike the melts rich in sulphur they are thin fluid up to their solidifying points.

§ 8. Considering all that is known up to the present as to the behaviour of the elements sulphur, selenium, and tellurium on being melted together, we may say that in this respect, tellurium certainly deserves the place assigned to it by DUMAS. Sulphur and selenium form, according to RINGER¹⁾, a trimorphous series of mixed crystals, selenium and tellurium, according to PELLINI and VIO²⁾ an uninterrupted series of trigonal mixed crystals: but no compounds are formed, as may be expected, looking at the experience gained, apart from the exceptions in such triads of homologous elements, — at any rate in the central groups of the periodic system. With sulphur and selenium the matter is even somewhat still more complicated, as three instead of two heteromorphous kinds of mixed crystals occur in this case. If we accept RETGERS' view according to whom a less stable form, mostly unknown in the free state, of each of the components should correspond to each of these forms, the isotrimorphism in the case of selenium and sulphur is certainly more difficult to explain than the dimorphism of sulphur and tellurium. For of the two monoclinic series in the system: sulphur-selenium one, according to MUTHMANN, is analogous to the form of γ -sulphur, whereas the trigonal series would already possess the form of metallic selenium. But neither of the two known monoclinic modifications of selenium is isomorphous with any monoclinic modification of sulphur³⁾, whilst the trigonal so-called ϵ -form of this element differs from the trigonal form of selenium. Looking from RETGERS' standpoint both these elements should be credited, in addition to their well known allotropic forms, with at least another two unknown, less stable modifications. In the trigonal series of the system: sulphur-tellurium we are dealing obviously with the same less stable trigonal form of sulphur as in RINGER's investigation, whilst the assumption of an unstable monoclinic form of tellurium cannot have anything artificial about it, in view of the fact that this symmetry occurs frequently

¹⁾ RINGER, Z. f. anorg. Chem. **32**, 183. (1902).

²⁾ PELLINI and VIO, Gazz. Chim. It. (1906). **II**, 476.

³⁾ GROTH, Chemische Krystallographie, Bd. I. (1906). p. 23—35.

both with selenium and sulphur. The research of PELLINI and VIO also does not introduce any further complications; the two elements are united there in all proportions to one trigonal series, so that only the sulphur-containing complexes of selenium and tellurium exhibit the hiatus on mixing in the solid condition.

All this admits of the conclusion that the elements sulphur, selenium and tellurium form indeed a natural triad of perfectly homologous elements which are more adjacent to each other than their group-fellow *oxygen* is to any one of them. There can be only question of true "compounds" when one of the elements combines with oxygen¹⁾.

§ 9. Now there is still the question: what must be thought of the tellurium-sulphur complexes which are formed, at the temperature of the room, by means of H₂S from solutions of tellurites and tellurates, and in what sense must the so-called double sulphides obtained by OPPENHEIM and BERZELIUS be regarded.

First of all, I soon succeeded in showing that the element tellurium and particularly its amorphous modification dissolves, without leaving any residue, when heated with a solution of alkali- or ammonium-sulphide, also that the solubility increases with the concentration of the sulphide; and further that the solubility also increases when potassium hydroxide is added to the sulphide solution, thus retarding the hydrolysis. Clear yellow solutions are so formed turning somewhat ruddy on boiling, and oxidising rapidly in contact with the air with formation of a black precipitate. They are strongly alkaline and give with hydrochloric acid a heavy, black precipitate with evolution of H₂S; the precipitate appears to contain tellurium as well as sulphur and is soluble in alkali-hydroxyde.

The analysis of these black precipitates did not give constant values; the tellurium content is much dependent on the *modus operandi* and oscillated between 46.9% and 80.9%.

Thereupon, the action of H₂S on different tellurium compounds was investigated: on the basic nitrate, on the finely divided dioxide suspended in absolute alcohol, on dioxide in hydrochloric acid solution, on telluric acid in water and on the tellurite- and tellurate-solutions obtained from TeO₂ or telluric acid. A beautiful, somewhat crystalline looking product was obtained from the alcoholic suspension of TeO₂; the analysis of the blue-black substance gave 80.1%—80.9% of tellurium whereas theory requires 79.9% for TeS, 66.6% for TeS₂,

¹⁾ It is, moreover, also known that O and S, for instance, never give isomorphous substitutions in organic compounds. S, Se and Te, however, behave differently as shown by the research of PELLINI, NORRIS, TUTTON, and others.

and 57% for TeS_3 , so that the composition came very near to that of TeS , but with an *excess* of tellurium.

The telluric acid was not reduced in the cold; the TeO_2 dissolved in hydrochloric acid yielded a black precipitate with 71.2% of tellurium and therefore situated between TeS and TeS_2 .

The basic nitrate is rapidly attacked by H_2S , but only at the surface; on the other hand it dissolved completely in boiling ammoniumsulphide, which solution, after being concentrated in vacuo at 40° , and allowed to crystallise in a vacuum desiccator over CaO , yields hard, pale yellow needles of a compound which may be recrystallised in vacuo without decomposition. The colour of the needles is greenish-yellow: they dissolve in water to a clear, yellow, strongly alkaline solution, which rapidly oxidises when exposed to the air; the crystals also soon turn black on exposure. The analysis gave a varying tellurium-content according to the method of preparation; in one instance were found 20.1% (NH_4) , 42% Te and 37.9% S , which leads to the formula $(\text{NH}_4)_6\text{Te}_2\text{S}_7$ ¹⁾.

In an analogous way the potassium compounds were prepared from the tellurite and tellurate with H_2S , by solution of the precipitate in the solution saturated with H_2S , or by solution in KOH , and by conduction of H_2S through it; the solutions were evaporated in vacuo, and were then left to crystallize over calcium oxide. Apparently the same yellow compounds are formed in all these cases, which crystallize in rosette-shaped aggregates of hard, fine needles, which in a humid condition smell strongly of H_2S , and yield clear, readily oxidisable solutions. Also the solid salts themselves oxidise rapidly, in which they become greenish yellow, and finally perfectly black. On close investigation the colour appears to assume different shades, even if to all appearances the same mode of preparation is used. Attempts to find a constant composition for these salts, have been unsuccessful; successively it was found to be: 35.4% Te (calculated

¹⁾ The analysis of these complexes is a very tedious operation. If tellurium only has to be estimated and no sulphur, the reduction process with SO_2 or NaHSO_3 is still to be preferred. In our case, the tellurium had to be precipitated from a boiling, strongly ammoniacal solution with hydrazine hydrate, which reduction proceeds very slowly and also incompletely, and had often to be repeated half a dozen times. The last traces of still dissolved tellurium betray themselves on heating by the fine steel blue colour of the colloidal tellurium present; this is generally completely precipitated on rendering the liquid acid, and by way of control the solution may then be heated once more with ammonia and hydrazine hydrate. The tellurium was collected on a weighed filter dried at 100° and weighed as such. The SO_3 was weighed as BaSO_4 , the K as KCl or KClO_4 , the NH_3 as NH_4Cl . For obvious reasons the analysis of the barium salt is a very tedious affair.

for $K_6Te_2S_7$ 35.7%); 33.5% Te, 33.4% S and 33.1% K, which answers to a formula $K_{10}Te_3S_{12}$; another time (for a product prepared from K_2TeO_4): 44.7% Te, 31.47% S, and 23.7% K, which would correspond to $K_{12}S_{20}Te_7$; again another time at somewhat higher temperature: 37.5% Te, 34.3% S and 28.1% K, which leads to a formula $K_{12}Te_3S_{19}$.

The behaviour is practically analogous to that found in the polysulphides of the alkalis towards sulphur, where, according to KÜSTER'S researches¹⁾, very complicated equilibria between different polysulphides and their dissociation products occur in the solution; and to that of selenium towards sulphides where, according to MESSINGER²⁾, a portion of the sulphur of the complex sulphohydrogen sulphides may be replaced by selenium, forming such compounds as Na_2SSe_3 which, therefore, belong to the type of a trisulphide. The behaviour of amorphous tellurium towards sulphide solutions as described previously also agrees with the notion that we are dealing here with salts of complex tellurohydrogen sulphides which in such solutions are in dissociation-equilibria with each other and are moreover split hydrolytically.

The type of the trisulphides becomes then of particular importance next to that of the disulphides: $K_{10}S_{12}Te_3$ may be derived from K_2S_3 by isomorphous substitution of $\frac{1}{5}$ of the S by Te; $K_6Te_2S_7$ and $(NH_4)_6Te_2S_7$ similarly from K_2S_3 , or $(NH_4)_2S_3$; on the other hand $K_{12}Te_3S_{19}$ has again the character of the type K_2S_4 etc.

§ 10. Although these compounds do not as a rule occur in measurable forms (the K-salt was obtained a few times as beautiful rhombic parallelipeds with faint double refraction and without perceptible dichroism) I finally succeeded in obtaining very large, yellow crystals, with many planes, of a barium salt prepared by dissolving the black precipitate formed by the action of H_2S on potassium tellurite in BaS-solution. The analysis indeed, did not always yield precisely the same results, but still the composition agreed closely with the formula $Ba_3S_7Te_2$; in one instance the normal composition; 45,8% Ba; 25% S and 29,1% Te was actually found. These crystals were accurately investigated and proved to be so well constructed that, in their habit, they did not remind us of mixed crystals but, on the contrary, made a vivid impression of belonging to a true chemical compound. The following data were found:

Large yellow transparent crystals fairly stable in the air but, after some time, assuming a greyish colour. They are well constructed,

¹⁾ KÜSTER, Z. f. anorg. Chem. 44. 431.

²⁾ MESSINGER, Berl. Ber. 30 805 (1897).

yield constant angular values and have, crystallonomically, quite the appearance of a well defined compound of constant composition.

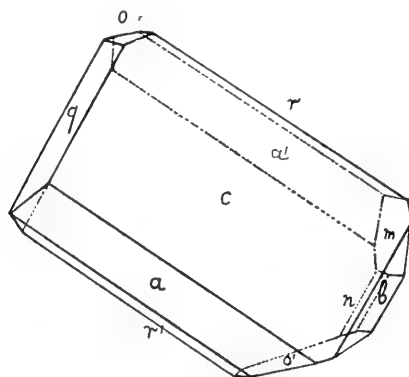


Fig. 4.

The compound has, in moist air, a strong odour of hydrogen-sulphide and is decomposed by water with separation of a black substance which contains tellurium and sulphur.

Triclinic-pinacoidal.

$$a : b : c = 1.6835 : 1 : 1.5515$$

$$\begin{array}{ll} A = 109^{\circ} 43' & \alpha = 113^{\circ} 7\frac{1}{2}' \\ B = 122^{\circ} 10\frac{1}{2}' & \beta = 124^{\circ} 13' \\ C = 90^{\circ} 32' & \gamma = 77^{\circ} 39' \end{array}$$

Forms observed: $c = \{001\}$, strongly predominating; $a = \{100\}$ and $b = \{010\}$ equally well developed and lustrous; $q = \{0\bar{1}1\}$ and $r = \{\bar{1}01\}$ quite as much developed as a and b , and yielding sharp reflexes; $o = \{11\bar{2}\}$, lustrous and fairly large; $n = \{01\bar{2}\}$ small but lustrous; $w = \{\bar{1}11\}$, small and subordinate, generally with but one plane; $m = \{\bar{1}10\}$, well developed and lustrous, also without the parallel opposite plane.

The habit is flattened towards $\{001\}$ with slight stretching in the direction of the b -axis. A complete or distinct plane of cleavage is not found.

The following angular values were measured:

	Measured :	Calculated :
$a : b = (100) : (010) =$	$89^{\circ} 28'$	—
$b : c = (010) : (001) =$	$70 \quad 17$	—
$c : a = (001) : (100) =$	$57 \quad 49\frac{1}{2}$	—
$c : r = (001) : (\bar{1}01) =$	$53 \quad 46\frac{1}{2}$	—
$c : q = (001) : (0\bar{1}1) =$	$65 \quad 45$	—
$o : b = (11\bar{2}) : (010) =$	$67 \quad 32$	$67^{\circ} 36'$
$r : b = (\bar{1}01) : (010) =$	$68 \quad 47$	$68 \quad 47\frac{1}{3}$
$a : q = (100) : (0\bar{1}1) =$	$67 \quad 29$	$67 \quad 26$
$c : o = (001) : (\bar{1}\bar{1}\bar{2}) =$	$50 \quad 5$	$49 \quad 49$
$a : r = (\bar{1}00) : (\bar{1}01) =$	$68 \quad 24$	$68 \quad 24$
$c : n = (00\bar{1}) : (01\bar{2}) =$	$38 \quad 17$	$38 \quad 27$
$n : q = (01\bar{2}) : (01\bar{1}) =$	$27 \quad 24$	$27 \quad 18$
$q : b = (01\bar{1}) : (010) =$	$43 \quad 56$	$43 \quad 58$
$r : o = (10\bar{1}) : (11\bar{2}) =$	$52 \quad 35$	$52 \quad 47$
$o : q = (11\bar{2}) : (01\bar{1}) =$	$32 \quad 21$	$32 \quad 19$
$m : b = (\bar{1}10) : (010) =$	$33 \quad 51$	$33 \quad 37$
$m : a = (\bar{1}10) : (\bar{1}00) =$	$56 \quad 41$	$56 \quad 55$
$m : r = (\bar{1}10) : (\bar{1}01) =$	$59 \quad 37$	$59 \quad 32\frac{1}{2}$
$m : q = (\bar{1}10) : (01\bar{1}) =$	$35 \quad 10$	$35 \quad 21\frac{1}{2}$
$\omega : b = (11\bar{1}) : (010) =$	$40 \quad 55$	$40 \quad 47$
$\omega : c = (11\bar{1}) : (00\bar{1}) =$	$85 \quad 46\frac{1}{2}$	$85 \quad 48\frac{1}{3}$
$\omega : a = (11\bar{1}) : (100) =$	$74 \quad 28$	$74 \quad 30\frac{1}{2}$
$\omega : q = (11\bar{1}) : (01\bar{1}) =$	$38 \quad 5$	$38 \quad 3\frac{1}{2}$

The agreement between the observation and the calculation is an excellent one.

Etching figures were not obtained. It may be, — looking at the acentric habit and plane-development of some of the forms, — that the symmetry is *triclinic-pedial*. The situation of the optical axial angle could not be determined. That of the optical main directions was such that the angle of extinction on $\{001\}$ with the side $(001) : (\bar{1}01)$, was about 15° , but on $\{\bar{1}01\}$ with the same side it amounted to about 12° , and that with an inclination which on $\{\bar{1}01\}$ proceeds from the left in front to the right at the back, and on $\{001\}$ from the right above to the left below.

Here we are, consequently, also dealing with a polysulphide of the type $Ba S_3$ in which $\frac{2}{3}$ of the sulphur has been replaced by tellurium.

Efforts to obtain this compound, prepared from BaS and S, in a measurable form, and thus to obtain an argument in favour of the said view, in the event of an isomorphism between the two substances, have been found so far unsuccessful.

§ 11. In the electrolysis of a dilute solution of the potassium salt, in which the platinum-dish acted as the cathode and a disc-shaped platinum-electrode as anode, it looked as if tellurium was precipitated at both electrodes. The liberation of the black substance at the anode is nothing else than an oxidationphenomenon.

The tension at the electrodes was 2.6 Volt, the current 0.05 Am-père; the oxygen formed at the anode oxidises the liquid, so sensitive to oxygen, with separation of tellurium-sulphur complexes which partly stick to the anode and partly collect above the same on the liquid; if the current passes for some time the precipitate redissolves and the anode again turns bright. I have ascertained that the adhering precipitate contains tellurium as well as sulphur.

On the other hand, the precipitation of a black substance at the cathode takes place continuously but very slowly; after twelve hours only a small portion of the salt, about one gram and a half in 50 cc of distilled water, had been decomposed by the current. This black precipitate has now proved to be pure tellurium and this observation would, therefore, go against the assumption that tellurium forms a constituent of the anion. This experiment, however, *cannot* be used as evidence against that view, since we know an analogous case in the electrolysis of sodium-sulphantimonate¹⁾, where the antimony also proceeds, apparently, to the cathode, although it acts, in the salt, as a constituent of the anion.

It has also transpired in these experiments that the metal does not wholly take part in the electric conductivity, but that in the electrolysis of the solution, the sodium sulphide is decomposed, and either the sodium liberated at the cathode, or the hydrogen which it causes to be evolved, precipitates the antimony by a secondary reaction. Only when a very little alkalisulphide is present, the anion $''SbS_4$, also moves towards the anode. Obviously, the explanation in our case is the same; the tellurium is formed secondarily at the cathode, whilst at the anode, as in the experiments cited, fairly complicated and somewhat obscure oxidationphenomena occur. In each case, this apparent contradiction does by no means prove the incorrectness of the view, that the said salts may be considered as derivatives of

¹⁾ OST und KLAPPROTH, Zeitschr. f. angew. Chemie (1900). p. 827.

complex tellurohydrogen sulphides. The investigation of this exceedingly complicated subject is being continued in the direction indicated.

§ 12. Summarizing the results of these investigations, I believe I may say :

1. That the elements tellurium and sulphur do not form compounds, but mixed crystals.

2. That the elements tellurium, selenium and sulphur behave in quite an analogous manner towards the sulphides of the alkali and alkali-earth metals, and cause the formation of complex sulpho-, seleno- or tellurohydrogen sulphides of a different type, and that it is quite unnecessary to presuppose the intermediate formation of selenium-sulphur or tellurium-sulphur compounds.

3. That the position, given by DUMAS, to tellurium in the sulphur group as the first homologue of selenium is quite justified so far as the mutual behaviour of these elements is concerned, and that sulphur, selenium, and tellurium form a natural triad of elements, which are more adjacent to each other than any one of them is to oxygen.

Groningen, Inorg. Chem. Lab. of the University.

Physics. — “*Some remarks on Prof. KOHNSTAMM’s reply.*” By Dr. J. J. VAN LAAR. Communicated by Prof. LORENTZ.

In these Proceedings of Jan. 6th 1910 Prof. KOHNSTAMM has inserted a reply to my remarks suggested by a paper by MESSRS. TIMMERMANS and KOHNSTAMM. Though I, too, very reluctantly continue the discussion, I feel obliged to briefly revert to this matter for the last time, in order to prevent further misunderstanding.

So I will just point out that Mr. KOHNSTAMM is quite silent about the *cardinal* point of my remarks, given in point *a* second part, point *e* and point *f*; viz. that Messrs. T. and K. in consequence of their disregard of the last five of my seven papers on the subject in question have wrongly asserted that the “abnormal” type III could not occur for *normal* substances, and that this would be due to my restricting supposition $a_{12} = \sqrt{a_1 a_2}$. Only to remove this misunderstanding — as I had asserted the very opposite of this — I wrote my preceding paper.

On the other hand a few minor questions are extensively discussed in the answer, viz. the question $a_{12} = \sqrt{a_1 a_2}$ and $\frac{d^2 b}{dx^2} = 0$. I must remark here that when I repeatedly spoke of the "quite general" case $a_2 \lesseqgtr a_1$, $b_2 \lesseqgtr b_1$, this expression "quite general" was obviously meant in *contrast* to the special case $a_2 \lesseqgtr a_1$, $b_2 = b_1$, treated by me before in the first two Papers, as would be clear to everybody, and that the "generality" meant by me according to the *whole tenour* of my papers, of course, only holds *within the area of the once assumed supposition* $a_{12} = \sqrt{a_1 a_2}$ (BERTHELOT'S). For this supposition I explicitly premised in all my papers, and I repeated it more than once (*loc. cit.*).

Now with regard to the question itself of the supposition $a_{12} = \sqrt{a_1 a_2}$ (which, however, is not the point at issue), I may be allowed to remind Mr. K. of a paper of his in the *Zeitschr. f. physik.* Ch. **36** p. 41 (1901), where it, *inter alia*, says at the end (p. 62): "So weit ich aus dem mir bekannten Material zu schlieszen vermag, scheinen mir jedoch die Tatsachen sehr zu Gunsten der (BERTHELOT'SCHE) Annahme zu sprechen..." I will add that I, too, consider the supposition $a_{12} = \sqrt{a_1 a_2}$ as very probable, and that seeming deviations from this supposition are attributed by me to the formation of compounds. But I hope to treat this more fully on a later occasion.

I now demonstrated that even on this simple supposition the abnormal type III can occur for *perfectly normal* substances. And this Mr. K. denied — as my later papers on this subject in which this was proved by me, had evidently escaped his notice.

With regard to the supposition $\frac{d^2 b}{dx^2} = 0$, Mr. K. refers to my statement that "qualitatively everything will remain the same if b is not assumed independent of v and T ". This, however, is quite beside the question whether the supposition $\frac{d^2 b}{dx^2} = 0$ is of influence on my results or not; for v and T are not the same thing as x . I fully maintain my contention, and Mr. K. will, no doubt, understand, that this dependence on v and T was only mentioned by me, because VAN DER WAALS' later investigations have shown that b still depends on this quantity. But this is not the point in question.

I, however, readily acknowledge that when writing the lines about the longitudinal plait closing again, quoted by Mr. K., I did

not sufficiently clearly state that the divergent result was only founded on the assumption $\frac{d^2b}{dx^2} = 0$. I knew, however, that VAN DER WAAALS in his *Continuität* II p. 24 has already treated this question. Yet on theoretical considerations I abide by my opinion that in the neighbourhood of the limiting volume, so at very high pressures, $\frac{d^2b}{dx^2}$ must be $= 0$.

And now I think that I for my part, have sufficiently elucidated Mr. KOHNSTAMM's Reply, so that further misunderstanding seems almost precluded.

Baarn, Febr. 21, 1910.

Mathematics. — “*The oscillations about a position of equilibrium where a simple linear relation exists between the frequencies of the principal vibrations.*” (1st part). By Mr. H. J. E. BETH. (Communicated by Prof. KORTEWEG).

Introduction.

§ 1. In his paper¹⁾: “On certain vibrations of higher order of abnormal intensity (vibrations of relation) for mechanisms with more degrees of freedom” (*Verhandelingen der Koninklijke Akademie van Wetenschappen*, Vol. V. N^o. 8, 1897; *Archives Néerlandaises* Vol. I, series II, pages 229—260) Prof. KORTEWEG has written down the expansions in series for the principal coordinates of an arbitrary mechanism with more degrees of freedom, performing small oscillations about a position of stable equilibrium. From these expansions in series could be deduced that in a certain case it was possible for some vibrations of higher order, having in general a small intensity with respect to the principal vibrations, to obtain an abnormally great intensity; this is the case when between the frequencies n , n_y etc. of the principal vibrations a relation exists of the form

$$pn_x + qn_y + \dots = \varrho;$$

where p, q etc. are positive or negative integers and ϱ is with respect to n_x, n_y etc. a small quantity, called residue of relation.

Furthermore however it became evident that, when $S \leq 4$ (S is the sum of the absolute values of p, q etc.) and at the same time $\varrho = 0$,

¹⁾ “Over zekere trillingen van hooger orde van abnormale intensiteit (relatie-trillingen) bij mechanismen met meerdere graden van vrijheid”.

the above-mentioned expansions in series lost their validity; we must therefore investigate in a different way what becomes of the movement in the case mentioned. In what follows we shall investigate this for a mechanism with two degrees of freedom. As a base for this investigation a very simple mechanism is selected, namely a material point which moves without friction yet under the influence of gravitation on a given surface in the vicinity of its lowest point. Every time one of the cases $S \leq 4$ is discussed we shall pass to an arbitrary mechanism with two degrees of freedom.

Movement on the bottom of a surface.

§ 2. We shall accordingly first pass on to the treatment of the simple mechanism we have chosen as a base for our investigation. When the surface has positive curvature in the vicinity of its lowest point O , when plane XY is the tangential plane in O , and the XZ - and YZ -planes are the principal sections of the surface in that point, whilst the Z -axis is supposed positive upwards, then the equation of the surface in the vicinity of O takes the form of:

$$z = \frac{1}{g}(c_1x^2 + c_2y^2 + d_1x^3 + d_2x^2y + d_3xy^2 + d_4y^3 + \dots); \quad (1)$$

where c_1 and c_2 are positive.

The equations of motion of the material point become:

$$\left. \begin{aligned} \ddot{x} + \frac{\partial z}{\partial x}(g + \ddot{z}) &= 0, \\ \ddot{y} + \frac{\partial z}{\partial y}(g + \ddot{z}) &= 0. \end{aligned} \right\}$$

Availing ourselves of (1) to eliminate \ddot{z} we find:

$$\left. \begin{aligned} \ddot{x} + \frac{\partial z}{\partial x}(g + \frac{\partial^2 z}{\partial x^2}x^2 + 2 \frac{\partial^2 z}{\partial x \partial y}xy + \frac{\partial^2 z}{\partial y^2}y^2 + \frac{\partial z}{\partial x}\ddot{x} + \frac{\partial z}{\partial y}\ddot{y}) &= 0, \\ \ddot{y} + \frac{\partial z}{\partial y}(g + \frac{\partial^2 z}{\partial x^2}x^2 + 2 \frac{\partial^2 z}{\partial x \partial y}xy + \frac{\partial^2 z}{\partial y^2}y^2 + \frac{\partial z}{\partial x}\ddot{x} + \frac{\partial z}{\partial y}\ddot{y}) &= 0. \end{aligned} \right\} \quad (2)$$

Let h be the small quantity (small e.g. with respect to the principal radii of curvature R_1 and R_2 of the surface in O) which determines the order of greatness of x and y , then the equations (2) become, omitting the terms of order h^2 and higher:

$$\left. \begin{aligned} \ddot{x} + 2c_1x &= 0, \\ \ddot{y} + 2c_2y &= 0. \end{aligned} \right\} \dots \dots \dots (3)$$

These equations are in general sufficient to arrive at the solution at first approximation. This then becomes:

$$\left. \begin{aligned} x &= Ah \cos(n_1 t + \lambda), \\ y &= Bh \cos(n_2 t + \mu); \end{aligned} \right\} \dots \dots \dots (4)$$

where $n_1 = \sqrt{2c_1}$, $n_2 = \sqrt{2c_3}$.

Here Ah, Bh, λ and μ are constants of integration; we suppose A and B to be of moderate greatness.

At first approximation therefore the horizontal projection of the moving point describes a Lissajous curve, which is closed when $m_1 = qn_2$, where p and q are integers. If $pn_1 = qn_2 + \varrho$, the curve described is not closed, but it consists of a succession of parts each of which differs but little from a closed curve. These last closed curves have however various shapes which answer to different values of the difference in phase. They are all described in the rectangle with $2Ah$ and $2Bh$ as sides.

§ 3. If we wish to take into consideration the terms of a higher order appearing in (2) we generally have but to apply small modifications to the first approximation.

These modifications are, however, not small in case a relation exists of the form:

$$pn_1 = qn_2 + \varrho;$$

where $S \equiv p + q \leq 4$ and $\frac{\varrho}{n_1}$ is very small (what is meant here by "very small" will be evident later on).

When by applying the method of consecutive approximations, starting from (4) as first approximation, we try to find expansions in series for x and y , we shall find, when substituting the expressions (4) into the terms of higher order of (2) and developing the products and powers of the cosines, in case $\frac{\varrho}{n_1}$ is very small, periodical terms which have about the same period as the principal vibration, to which the equation in which the indicated term appears relates more especially. Such terms in the equations of motion give rise in the expansions in series for x and y to terms with abnormally great amplitude. These amplitudes may reach the order h and even seem to be greater still.

This proves that in the case supposed our first approximation was not correct. It is evident that in the equations of motion there are terms of higher order, which are of influence even on the first

approximation. So we shall have to find in the equations (2) which terms give rise to the failure of the application of the method of consecutive approximations. These terms we shall have to include in the abridged equations, serving to determine the first approximation.

We shall consecutively discuss the cases:

$$S = 3 (2n_1 = n_2 + \varrho), \quad S = 4 (3n_1 = n_2 + \varrho), \quad S = 2 (n_1 = n_2 + \varrho).$$

$$S = 3.^1) \text{ Strict relation.}$$

§ 4. We suppose $\varrho = 0$; therefore

$$n_2 = 2n_1.$$

In the equations of motion appear for the first time among the terms of order h^2 terms which, according to what was said in § 3, must be included in the abridged equations. They are: in the first equation $2d_2xy$, in the second d_2x^2 . These are the most important among the terms referred to. Omitting the remaining terms of higher order we therefore have to consider:

$$\left. \begin{aligned} \ddot{x} + n_1^2 x + 2d_2xy &= 0, \\ \ddot{y} + 4n_1^2 y + d_2x^2 &= 0. \end{aligned} \right\} \dots \dots \dots (5)$$

We may also write this system as follows:

$$\left. \begin{aligned} \ddot{x} + n_1^2 x - \frac{\partial R}{\partial x} &= 0, \\ \ddot{y} + 4n_1^2 y - \frac{\partial R}{\partial y} &= 0; \end{aligned} \right\}$$

in which:

$$R \equiv -d_2x^2y.$$

To this we apply the method of the variation of the canonical constants. This means, as is known, that the equations, arising when the terms $\frac{\partial R}{\partial x}$ and $\frac{\partial R}{\partial y}$ are omitted, first are solved; in which solution 4 arbitrary constants appear; we then investigate what functions of the time must be the quantities just now regarded as constants, so that the expressions for x and y , taken in this way, represent the solution of the complete equations containing $\frac{\partial R}{\partial x}$ and $\frac{\partial R}{\partial y}$. The equations in which $\frac{\partial R}{\partial x}$ and $\frac{\partial R}{\partial y}$ are lacking, are solved according to

¹⁾ In a following paper we shall discuss the cases $S = 2$ and $S = 4$.

the method of HAMILTON-JACOBI in order that the constants we obtain may form a canonical system.

If $\alpha_1, \alpha_2, \beta_1$ and β_2 are the canonical constants then by substitution of the expressions found for x and y in R this R will become a function of $\alpha_1, \alpha_2, \beta_1, \beta_2$ and t . The variability of the α 's and β 's with the time is then given by:

$$\frac{d\alpha_1}{dt} = \frac{\partial R}{\partial \beta_1}, \quad \frac{d\alpha_2}{dt} = \frac{\partial R}{\partial \beta_2}, \quad \frac{d\beta_1}{dt} = -\frac{\partial R}{\partial \alpha_1}, \quad \frac{d\beta_2}{dt} = -\frac{\partial R}{\partial \alpha_2} \dots (6)$$

In case R is a function of the α 's and the β 's alone, and consequently does not contain t explicitly, the system has as an integral:

$$R = \text{constant.} \dots \dots \dots (7)$$

§ 5. If now we solve the equations

$$\left. \begin{aligned} \ddot{x} + n_1^2 x &= 0, \\ \ddot{y} + 4n_1^2 y &= 0, \end{aligned} \right\}$$

arising from (5) by omission of the terms $\frac{\partial R}{\partial x}$ and $\frac{\partial R}{\partial y}$, according to the method HAMILTON-JACOBI we may arrive at:

$$\left. \begin{aligned} x &= \frac{\sqrt{\alpha_1}}{n_1} \cos(n_1 t + 2n_1 \beta_1), \\ y &= \frac{\sqrt{\alpha_2}}{2n_1} \cos(2n_1 t + 4n_1 \beta_2); \end{aligned} \right\} \dots \dots \dots (8)$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ form a canonical system of constants. We must suppose α_1 and α_2 to be of order h^2 as the amplitudes of the x - and y -vibrations must be of order h .

Substitution of (8) in $\bar{R} \equiv -d_x x^2 y$ furnishes 3 terms:

$$\frac{\alpha_1 \sqrt{\alpha_2}}{4n_1^3} \cos(2n_1 t + 4n_1 \beta_2), \quad \frac{\alpha_1 \sqrt{\alpha_2}}{8n_1^3} \cos\{4n_1 t + 4n_1(\beta_1 + \beta_2)\} \text{ and}$$

$$\frac{\alpha_1 \sqrt{\alpha_2}}{8n_1^3} \cos 4n_1(\beta_1 - \beta_2),$$

each term multiplied by $-d_x$.

The first two terms contain t explicitly; setting aside the variability of the α 's and β 's we can say that those terms are periodical, whilst the period is comparable to that of the principal vibrations. The last term, however, does not contain t explicitly. Only this last term is of importance for the first approximation; the two others we omit (we shall revert to this in § 6).

We therefore take:

$$R = - \frac{d_2}{8n_1^3} \alpha_1 \sqrt{\alpha_2 \cos 4n_1 (\beta_1 - \beta_2)}.$$

Consequently system (6) takes this form :

$$\left. \begin{aligned} \frac{d\alpha_1}{dt} &= 2 N m_1 \alpha_1 \alpha_2^{\frac{1}{2}} \sin \varphi, \\ \frac{d\alpha_2}{dt} &= - 2 N m_1 \alpha_1 \alpha_2^{\frac{1}{2}} \sin \varphi, \\ \frac{d\beta_1}{dt} &= m_1 \alpha_2^{\frac{1}{2}} \cos \varphi, \\ \frac{d\beta_2}{dt} &= \frac{1}{2} m_1 \alpha_1 \alpha_2^{-\frac{1}{2}} \cos \varphi; \end{aligned} \right\} \dots \dots \dots (9)$$

where N is written for $n_2 = 2n_1$; further :

$$m_1 = \frac{d_2}{N^3},$$

$$\varphi = 2 N (\beta_1 - \beta_2).$$

As t does not appear explicitly in R we get according to what has been said at the close of § 4 as an integral :

$$\alpha_1 \sqrt{\alpha_2 \cos \varphi} = \text{constant} \dots \dots \dots (10)$$

Furthermore it appears at once from (9) that :

$$\frac{d\alpha_1}{dt} + \frac{d\alpha_2}{dt} = 0.$$

Therefore :

$$\alpha_1 + \alpha_2 = \text{constant} \dots \dots \dots (11)$$

is another integral.

The latter gives us reason to introduce a new variable ξ , in such a way that :

$$\alpha_1 = \frac{1}{4} R_0^2 N^2 h^2 \xi \quad , \quad \alpha_2 = \frac{1}{4} R_0^2 N^2 h^2 (1 - \xi);$$

ξ is then always situated between 0 and 1, R_0 is of moderate greatness.

By this (10) obtains the form :

$$\xi \sqrt{1 - \xi} \cos \varphi = K; \dots \dots \dots (12)$$

in which K represents a constant.

The first equation of (9) becomes :

$$\frac{d\xi}{dt} = \frac{d_2 R_0}{N} \xi \sqrt{1 - \xi} \sin \varphi \cdot h \dots \dots \dots (13)$$

By elimination of φ between (12) and (13) we arrive at :

$$\frac{d\xi}{\sqrt{\xi^2 (1 - \xi) - K^2}} = \pm \frac{d_2 R_0}{N} h \cdot dt.$$

Now put:

$$f(\xi) = \xi^2(1-\xi) - K^2,$$

then for the initial value of ξ we find $f(\xi) > 0$. For $\xi = 0$ and $\xi = 1$ we find $f(\xi) < 0$. Thus the equation $f(\xi) = 0$ has two roots between 0 and 1.

So K^2 cannot have all values; the possible values of K^2 lie between two limits; in § 9 we shall revert to this and to the special cases, corresponding to the limiting values of K^2 .

The roots between 0 and 1 which the equation

$$\xi^2(1-\xi) - K^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (14)$$

has in the general case will be called ξ_1 and ξ_2 , where we suppose $\xi_2 > \xi_1$. The third root is negative, we call it $-\lambda$.

The differential relation between ξ and t may now be written:

$$\frac{d\xi}{\sqrt{(\xi_2 - \xi)(\xi - \xi_1)(\xi + \lambda)}} = \pm \frac{d_1 R_0}{N} h \cdot dt \quad . \quad . \quad . \quad . \quad (15)$$

So with the aid of elliptic functions ξ may be expressed in t . It changes periodically between the limits ξ_1 and ξ_2 .

Now with the aid of (12) we can also calculate φ as function of t . And β_1 and β_2 likewise, it being possible to write the last two equations of (9):

$$\begin{aligned} \frac{d\beta_1}{dt} &= \frac{d_2 R_0 K}{2 N^2} \frac{h}{\xi}, \\ \frac{d\beta_2}{dt} &= \frac{d_2 R_0 K}{4 N^2} \frac{h}{1-\xi}. \end{aligned}$$

So now x and y are also known as functions of t ¹⁾.

In fig. 1 the relation (12) between ξ and φ is represented in polar coordinates, φ is taken as polar angle, $\sqrt{1-\xi}$ as radius vector. The circle drawn has unity as radius. The curves change with the value of K . For $K > 0$ the curves lie to the right of the straight line $\varphi = \frac{\pi}{2}$, for $K < 0$ to the left of it; $K = 0$ furnishes degeneration into the straight line $\varphi = \frac{\pi}{2}$ and the circle $\xi = 0$. By the maximal positive and negative value of K ($K = \pm \frac{2}{9} \sqrt{3}$) the curve has contracted into an isolated point. The special cases of the motion belonging to $K = 0$ and to $K = \pm \frac{2}{9} \sqrt{3}$ will be discussed in § 9.

¹⁾ These calculations will be found in my dissertation, which will appear before long.

§ 6. When astronomers try to obtain in the Theory of the disturbances of the movements of the planets by the application of the method of LAGRANGE expansions in series for the coordinates of the planets or the elements of their orbits, then terms may appear with abnormally large coefficients in consequence of small divisors, originating from the integration. This takes place when between the inverse values of the periods of revolution of some planets a linear relation with integer coefficients is almost fulfilled. Besides some other properties the terms are also distinguished according to their class, by which is meant:

$$\alpha - \frac{m}{2} - \frac{m'}{2};$$

where α represents the exponent of μ (a small quantity indicating the order of greatness of the disturbing function), m the exponent of t , m' the exponent of the small divisor, as they appear in the coefficient of the term indicated. Now it is the terms of the lowest class which we have to take into consideration if we wish to make the expansions in series to hold for a long space of time. By DELAUNAY a method is indicated to determine the terms of the lowest class. It consists principally in omitting all terms of short period (period comparable to the periods of the revolution of the planets) in the disturbing function and retaining the most important of the others. (Comp. e. g. H. POINCARÉ, *Leçons de mécanique céleste*, vol. I, page 341).

The problem under discussion has much resemblance with the one mentioned from the theory of disturbances. In the preceding § in omitting some terms in R we have imitated what is done in the theory of disturbances.

It is easy to see that the terms omitted have really no influence on the first approximation, when we consider the terms which appear e. g. in α_1 by introduction of such a term.

Osculating curves.

§ 7. In § 5 we have found that the movement of the horizontal projection of the material point might be represented by:

$$\begin{aligned} x &= \frac{\sqrt{a_1}}{n_1} \cos(n_1 t + 2n_1 \beta_1), \\ y &= \frac{\sqrt{a_2}}{2n_1} \cos(2n_1 t + 4n_1 \beta_2); \end{aligned}$$

where $\alpha_1, \alpha_2, \beta_1$ and β_2 are slowly variable; for $\frac{d\alpha_1}{dt}$ and $\frac{d\alpha_2}{dt}$ are of order h^3 , $\frac{d\beta_1}{dt}$ and $\frac{d\beta_2}{dt}$ of order h . (Comp. (9)).

For every arbitrary moment the α 's and the β 's have a definite value. These values determine a certain Lissajous curve. This curve we shall call the osculating curve for the moment indicated, which name is in use in the theory of disturbances. (See among others H. POINCARÉ, *Leçons de mécanique céleste*, vol. I, page 90). Thus in our problem the osculating curves are the wellknown Lissajous figures for 2 octaves.

By the change of the origin of time we may write the equations of an osculating curve:

$$\begin{aligned}x &= R_0 h \sqrt{\xi} \cos n_1 t, \\y &= \frac{1}{2} R_0 h \sqrt{1-\xi} \cos (2n_1 t - \varphi); \end{aligned}$$

where as in § 5 we have introduced ξ instead of α_1 and α_2 ; here too φ means $4n_1(\beta_1 - \beta_2)$.

We now see that φ is the value of the difference in phase, to which the osculating curve corresponds when the phase is calculated from the moment of the greatest deviation to the right.

The amplitudes of the x - and y -vibrations being respectively $R_0 h \sqrt{\xi}$ and $\frac{1}{2} R_0 h \sqrt{1-\xi}$, the vertices of the rectangles, in which the osculating curves are described lie on the circumference of an ellipse with its great axis along the x -axis and having a length of $2 R_0 h$, and its small axis along the y -axis and having a length of $R_0 h$.

Now ξ changes its value between ξ_1 and ξ_2 , so the rectangles in which the osculating curves are described also lie between two extremes.

Moreover as according to (12) to each value of ξ a value of $\cos \varphi$ belongs all osculating curves may now be constructed.

It follows from (13) that for the extreme values of ξ we find $\sin \varphi = 0$; so in the extreme rectangles parabolae are described.

The distance from OX of the node of an arbitrary osculating curve is $-\frac{R_0 h K}{2\xi}$, from which it is evident that the nodes and also the vertices of the parabolae lie all on the same side of O lying below O for positive values of K (see fig. 2).

Envelope of the osculating curves.

§ 8. If we perform the elimination of t and φ from:

$$x = R_0 h \sqrt{\xi} \cos n_1 t, \quad y = \frac{1}{2} R_0 h \sqrt{1 - \xi} \cos (2n_1 t - \varphi) \quad \text{and}$$

$$\xi \sqrt{1 - \xi} \cos \varphi = K,$$

we find for the equation of the osculating curves with ξ as parameter :

$$\xi^2 (X^2 + Y^2) + \xi (KY - X^2 - X^4) + \left(\frac{1}{4} K^2 - 2KX^2 Y + X^4 \right) = 0;$$

where for the sake of a simplified notation is put :

$$X \text{ for } \frac{x}{R_0 h}, \quad Y \text{ for } \frac{y}{R_0 h};$$

Thus the envelope has as equation

$$4(X^2 + Y^2) \left(\frac{1}{4} K^2 - 2KX^2 Y + X^4 \right) - (KY - X^2 - X^4)^2 = 0.$$

After reduction and division by X^2 (the Y -axis is the locus of the nodes) it may be written:

$$(K - 4Y^2 - 3X^2 Y + Y^2) = (X^2 + 4Y^2 - 1)^2 (X^2 + Y^2),$$

or if we solve K :

$$K = - (Y \pm \sqrt{X^2 + Y^2}) + (Y \pm \sqrt{X^2 + Y^2})^3.$$

Putting

$$Y \pm \sqrt{X^2 + Y^2} = \frac{K}{U},$$

it passes into

$$K = - \frac{K}{U} + \frac{K^3}{U^3},$$

$$U^2 (1 - U) - K^2 = 0.$$

Now this cubic equation has the same coefficients as (14), so it also has the same roots. So the envelope is degenerated into the 3 parabolae having as equations:

$$U = \xi_1, \quad U = \xi_2, \quad U = -\lambda;$$

which after reduction and reintroduction of x and y take the form of:

$$2 \frac{y}{R_0 h} - K \cdot \frac{x^2}{R_0^2 h^2} + \frac{K}{\xi_1} = 0 \quad \xi_1 \text{ parabola,}$$

$$2 \frac{y}{R_0 h} - K \cdot \frac{x^2}{R_0^2 h^2} + \frac{K}{\xi_2} = 0 \quad \xi_2 \text{ parabola,}$$

$$2 \frac{y}{R_0 h} + K \cdot \frac{x^2}{R_0^2 h^2} - \frac{K}{\lambda} = 0 \quad \lambda \text{ parabola.}$$

The parabolae are confocal and have O as focus. When K is positive the ξ_1 and the ξ_2 parabolae have their openings turned upwards,

the λ parabola has its opening turned downwards (this case is represented in fig. 2, where besides some osculating curves the enveloping parabolae are also given).

Special cases.

§ 9. At the close of § 5 we saw that two special cases may occur, viz. when $K=0$ and when $K=\pm\frac{2}{9}\sqrt{3}$.

A. $K=0$. We deduce from the relation

$$\xi\sqrt{1-\xi^2}\cos\varphi=K$$

three possibilities :

1. $\xi=0$. The movement remains confined to the YZ -plane.

2. $\xi=1$. The movement remains confined to the XZ -plane. This form of motion however proves to be impossible when $x\neq 0$ and $y=0$ is substituted in (5).

3. $\cos\varphi=0$, therefore $\varphi=\frac{\pi}{2}$ or $\varphi=\frac{3\pi}{2}$ invariably. The osculating curves have their nodes at O . The form of movement approaches asymptotically to a motion in the YZ -plane. What becomes of the enveloping parabolae has been represented in fig. 3, in which some osculating curves have been drawn too.

B. $K=\pm\frac{2}{9}\sqrt{3}$. Then $\xi_1=\xi_2=\frac{2}{3}$, $\lambda=\frac{1}{3}$. Now $\cos\varphi=\pm 1$ invariably, thus $\varphi=0$ or $\varphi=\pi$. The same parabola is continuously described, in which also the ξ_1 and ξ_2 parabolae have coincided. (Fig. 4). When K undergoes a slight change, ξ_1 and ξ_2 fall close together. So this form of movement is stable.

$$S=3, \quad \frac{g}{n_1} \text{ is of order } \frac{h}{R_1}.$$

§ 10. The expansions in series written down by Prof. KORTEWEG lose for $S=3$ their convergency as soon as $\frac{h}{R_1}$ passes into order $\frac{g}{n_1}$ (page 18 of his paper) or i. o. w. as soon as $\frac{g}{n_1}$ sinks into order $\frac{h}{R_1}$.

We shall now discuss this case.

We again take as first approximation :

$$x = \frac{\sqrt{a_1}}{n_1} \cos(n_1 t + 2n_1 \beta_1),$$

$$y = \frac{\sqrt{a_2}}{2n_1} \cos(2n_1 t + 4n_1 \beta_2);$$

and we must investigate what form the function R now assumes.

As we have supposed that

$$2n_1 = n_2 + \varrho,$$

the terms of the order h in the equations of motion would become

$$\ddot{x} + n_1^2 x$$

and

$$\ddot{y} + (2n_1 - \varrho)^2 y.$$

Because $\frac{\varrho}{n_1}$ is of order $\frac{h}{R_1}$ and we take no terms of higher order than h^2 in the equations, we may write for the latter:

$$\ddot{y} + 4n_1^2 y - 4n_1 \varrho y.$$

If we thus take the above expression for x and y as first approximation, then we must admit in the function R besides the term $-d_2 x^2 y$ also a term $2n_1 \varrho y^2$.

In the expression

$$-d_2 x^2 y + 2n_1 \varrho y^2$$

we substitute the above expressions for x and y and omit the terms containing t explicitly. In this way we arrive at:

$$R = -\frac{d_2}{N^3} \alpha_1 \sqrt{\alpha_2} \cos \varphi + \frac{\varrho}{2N} \alpha_2;$$

where again N is put for $n_2 + \varrho = 2n_1$.

The equations which serve to determine the α 's and β 's become:

$$\frac{d\alpha_1}{dt} = 2N m_1 \alpha_1 \alpha_2^{\frac{1}{2}} \sin \varphi,$$

$$\frac{d\alpha_2}{dt} = -2N m_1 \alpha_1 \alpha_2^{\frac{1}{2}} \sin \varphi,$$

$$\frac{d\beta_1}{dt} = m_1 \alpha_2^{\frac{1}{2}} \cos \varphi,$$

$$\frac{d\beta_2}{dt} = -\varrho' h + \frac{1}{2} m_1 \alpha_1 \alpha_2^{-\frac{1}{2}} \cos \varphi;$$

where

$$m_1 = \frac{d_2}{N^3}, \quad \varrho' = \frac{\varrho}{2Nh}.$$

We again see that

$$\frac{d\alpha_1}{dt} + \frac{d\alpha_2}{dt} = 0,$$

so

$$\alpha_1 + \alpha_2 = \text{constant};$$

for which reason we put:

$$\alpha_1 = \frac{1}{4} R_0^2 N^2 h^2 \zeta, \quad \alpha_2 = \frac{1}{4} R_0^2 N^2 h^2 (1 - \zeta).$$

Further we have according to § 4, as an integral of the system :

$$-\frac{d_2}{N^3} \alpha_1 \sqrt{\alpha_2} \cos \varphi + \frac{q}{2N} \alpha_2 = \text{constant}.$$

Introducing ζ , it becomes

$$\zeta \sqrt{1-\zeta} \cos \varphi - q'' (1-\zeta) = K;$$

where K is a constant and

$$q'' = \frac{qN}{d_2 R_0 h}.$$

In the same way as this was done for the case $q=0$ we may write down the differential relation between ζ and t and find x and y in the way indicated there as functions of the time; they get quite the same form as for $q=0$ ¹⁾.

In general ζ keeps changing periodically between two limits ζ_1 and ζ_2 ; ζ_1 and ζ_2 being the positive roots of

$$\zeta^2 (1-\zeta) - \{K + q'' (1-\zeta)\}^2 = 0.$$

Yet there is a considerable difference between the cases $q=0$ and q of order h .

§ 11. We notice this difference most distinctly when we represent the relation established between ζ and φ in polar coordinates.

If we put

$$q''' = -q'',$$

then we find :

$$\cos \varphi = \frac{K - q''' + q''' \zeta}{\zeta \sqrt{1-\zeta}}.$$

We take φ as polar angle, $\sqrt{1-\zeta}$ as radius vector and we investigate the site and shape of the curves for positive values of q''' and for all possible values of K .

For $K = q'''$ there is degeneration into the circle $\zeta = 0$ and a straight line normal to the origin of the angles at a distance q''' from pole O_1 . We have two cases now: $q''' < 1$ and $q''' > 1$.

$q''' < 1$. Let us now investigate the shape of the curves for different values of K . For $K > q'''$ they lie to the left of the straight line just mentioned, for increasing value of K they contract more and more until for the maximal value of K , belonging to a certain value of

¹⁾ Vide Chapter V of my dissertation.

q''' we get an isolated point. If $0 < K < q'''$ the curves surround point O_1 ; if $K = 0$ we have a curve through O_1 , for $K < 0$ they lie to the left of O_1 ; for the minimal value of K we again get an isolated point (fig. 5).

For increasing values of q''' the straight line separating the domains $K > q'''$ and $K < q'''$ moves to the right. The domain $K > q'''$ becomes smaller and vanishes for $q''' = 1$. For $q''' > 1$ we therefore have curves surrounding O_1 and curves to the left of O_1 only. When q''' increases still more the remaining isolated point approaches to O_1 and the curves farther from O_1 approach to circles.

For $q = 0$ we had (with the exception of the special case $K = 0$) only curves to the right of O_1 , and curves to the left of O_1 . For q of order h we have moreover curves around O_1 , which are even more frequent for great values of $\frac{q}{h}$.

The curves around O_1 point to a form of motion, where φ takes all values, the nodes of the osculating curves lie then above as well as below the point O of fig. 2; the osculating parabolæ have their openings turned to opposite sides.

That for increasing values of q''' the curves in general begin to resemble circles more and more, indicates that ξ is about constant; it changes between narrow limits.

This also appears in this way. From (16) we deduce:

$$K - q'''(1 - \xi_1) = \pm \xi_1 \sqrt{1 - \xi_1},$$

$$K - q'''(1 - \xi_2) = \pm \xi_2 \sqrt{1 - \xi_2}.$$

By subtraction we find:

$$\xi_2 - \xi_1 = \frac{\pm \xi_1 \sqrt{1 - \xi_2} \mp \xi_2 \sqrt{1 - \xi_1}}{q'''}$$

For greater values of q''' we find $\xi_2 - \xi_1$ becoming very small.

In this way we approach the general case where there is no question about relation.

§ 12. How the transition to this general case takes place is also clearly evident from the limitation of the domain of motion, which limitation we find by determining the envelope of the osculating curves. In the same way as this was done for the case $q = 0$, we find that the envelope degenerates into three parabolæ, of which the equations are:

$$\begin{aligned}
 2 \frac{y}{R_0 h} + \frac{K - \varrho''' }{\zeta_1} + \varrho''' &= \frac{\zeta_1}{K - \varrho'''} \cdot \frac{v^2}{R_0^2 h^2} & \zeta_1 \text{ parabola,} \\
 2 \frac{y}{R_0 h} + \frac{K - \varrho''' }{\zeta_2} + \varrho''' &= \frac{\zeta_2}{K - \varrho'''} \cdot \frac{v^2}{R_0^2 h^2} & \zeta_2 \text{ parabola,} \\
 2 \frac{y}{R_0 h} - \frac{K - \varrho''' }{\lambda} + \varrho''' &= - \frac{\lambda}{K - \varrho'''} \cdot \frac{v^2}{R_0^2 h^2} & \lambda \text{ parabola}
 \end{aligned}$$

The points of intersection of the λ parabola with the ζ_1 and ζ_2 parabolae lie again on the ellipse having $R_0 h$ and $2R_0 h$ as axes. The parabolae are confocal; the focus lies on the y -axis at the height of $-\frac{1}{2}R_0 h \cdot \varrho'''$. In fig. 6^a, 6^b, 6^c we find those parabolae (and also the osculating parabolae) corresponding to the cases $\varrho''' < 1$ and $K > \varrho'''$.

In fig. 7 we see how the limitation approaches more and more to a rectangle for increasing ϱ''' .

The ζ_1 and ζ_2 parabolae coincide for maximal and minimal K .

*Arbitrary mechanism with 2 degrees of freedom
for which $S = 3$.*

§ 13. Let q_1 and q_2 be the principal coordinates of the mechanism; they remain during the movement of order h and are zero in the position of equilibrium.

The kinetic energy T and the potential energy U may be written:

$$T = \frac{1}{2} \dot{q}_1^2 + \frac{1}{2} \dot{q}_2^2 + T_3; \quad U = \frac{1}{2} (n_1^2 q_1^2 + n_2^2 q_2^2) + U_3,$$

where T_3 and U_3 are expressions in whose terms h appears at least to the 3rd degree.

Let us write down the terms of order h^3 in T_3 :

$$T_3 = \frac{1}{2} (a q_1 \dot{q}_1^2 + b q_2 \dot{q}_1^2 + 2 c q_1 \dot{q}_1 \dot{q}_2 + 2 d q_2 \dot{q}_1 \dot{q}_2 + e q_1 \dot{q}_2^2 + f q_2 \dot{q}_2^2) + \dots$$

As far as and inclusive of the terms of order h^2 the equations of LAGRANGE now become:

$$\begin{aligned}
 \ddot{q}_1 + n_1^2 q_1 &= - \frac{1}{2} a \dot{q}_1^2 - a q_1 \ddot{q}_1 - b q_2 \ddot{q}_1 - b \dot{q}_1 \dot{q}_2 - c q_1 \ddot{q}_2 - d q_2 \ddot{q}_2 + \\
 &+ \left(\frac{1}{2} e - d \right) \dot{q}_2^2 - \frac{\partial U_3}{\partial q_1}, \\
 \ddot{q}_2 + n_2^2 q_2 &= \left(\frac{1}{2} b - c \right) \dot{q}_1^2 - c q_1 \ddot{q}_1 - d q_2 \ddot{q}_1 - e \dot{q}_1 \dot{q}_2 - e q_1 \ddot{q}_2 - f q_2 \ddot{q}_2 + \\
 &- \frac{1}{2} f \dot{q}_2^2 - \frac{\partial U_3}{\partial q_2}.
 \end{aligned}$$

In case the relation $n_2 = 2n_1$ is strictly satisfied or nearly so, the disturbing terms are:

$$\begin{aligned} & \text{in the first equation those with } \ddot{q}_2 \dot{q}_1, \dot{q}_1 \dot{q}_2, q_1 \ddot{q}_2, q_1 \dot{q}_2, \\ & \text{,, ,, second ,, ,, ,, } \dot{q}_1^2, q_1 \dot{q}_1, q_1^2. \end{aligned}$$

If at first approximation we try to satisfy the equations by:

$$q_1 = Ah \cos(n_1 t + \lambda) \quad , \quad q_2 = Bh \cos(n_2 t + \mu)$$

where A, B, λ and μ are functions of t , however in such a manner that $\dot{A}, \dot{B}, \dot{\lambda}, \dot{\mu}$ are of order h or smaller, we may replace in the second member of the equations:

$$\begin{aligned} \dot{q}_1^2 & \text{ by } n_1^2 (A^2 h^2 - q_1^2), & \dot{q}_2^2 & \text{ by } n_2^2 (B^2 h^2 - q_2^2), \\ \dot{q}_1 & \text{ by } -n_1^2 q_1, & \dot{q}_2 & \text{ by } -n_2^2 q_2. \end{aligned}$$

If we take this into account for the disturbing terms and if we omit the non-disturbing terms, the equations become:

$$\left. \begin{aligned} \ddot{q}_1 + n_1^2 q_1 &= (bn_1^2 + cn_2^2 + 2\rho) q_1 q_2 - b \dot{q}_1 \dot{q}_2 \\ \ddot{q}_2 + n_2^2 q_2 &= \left(2cn_1^2 - \frac{1}{2}bn_1^2 + \rho \right) q_1^2. \end{aligned} \right\}$$

The terms $2\rho q_1 q_2$ in the first equation and ρq_1^2 in the second originate from a term $-\rho \dot{q}_1^2 q_2$, appearing in U_3 .

To get rid of the term with $\dot{q}_1 \dot{q}_2$ we use the new variable q' so that:

$$q'_1 = q_1 + \frac{1}{2} b \dot{q}_1 q_2.$$

Then:

$$\begin{aligned} \ddot{q}'_1 &= \ddot{q}_1 + \frac{1}{2} b \ddot{q}_1 q_2 + \frac{1}{2} b \dot{q}_1 \ddot{q}_2 + b \dot{q}_1 \dot{q}_2 = \\ &= \ddot{q}_1 + b \dot{q}_1 \dot{q}_2 - \frac{1}{2} b (n_1^2 + n_2^2) q_1 q_2. \end{aligned}$$

Therefore:

$$\ddot{q}'_1 + b \dot{q}_1 \dot{q}_2 = q'_1 + \frac{1}{2} b (n_1^2 + n_2^2) q_1 q_2.$$

The equations now pass into:

$$\left\{ \begin{aligned} \ddot{q}'_1 + n_1^2 q'_1 &= (bn_1^2 + cn_2^2 - \frac{1}{2}bn_2^2 + 2\rho) q'_1 q_2, \\ \ddot{q}_2 + n_2^2 q_2 &= (2cn_1^2 - \frac{1}{2}bn_1^2 + \rho) q_1^2. \end{aligned} \right.$$

For we may replace in the second members q_1 by q'_1 , as their difference is of order h^2 .

Let us now suppose n_2 to be $= 2n_1$; then we get:

$$\begin{aligned} \ddot{q}_1 + n_1^2 q_1 &= (4c n_1^2 - b n_1^2 + 2\rho) q_1' q_2' \\ \dot{q}_2 + n_2^2 q_2 &= (2c n_1^2 - \frac{1}{2} b n_1^2 + \rho) q_1'^2. \end{aligned}$$

So we find:

$$\begin{cases} \ddot{q}_1' + n_1^2 q_1' + 2d_2 q_1' q_2 = 0, \\ \dot{q}_2 + n_2^2 q_2 + d_2 q_1'^2 = 0; \end{cases}$$

where

$$d_2 = -2c n_1^2 - \frac{1}{2} b n_1^2 - \rho$$

The equations determining the first approximation have exactly the same form as those found in § 4. What was formerly deduced for the simple mechanism holds consequently, if $n_2 = 2n_1$, for an arbitrary mechanism with two degrees of freedom in such a sense that the horizontal projection of the point moving over the surface may be regarded as the representative point for the arbitrary mechanism.

We finally observe that any mechanism for which

$$-2c n_1^2 + \frac{1}{2} b n_1^2 - \rho = 0$$

is not sensitive for the relation $n_2 = 2n_1$. So this is the condition requisite to make the mechanism for $n_2 = 2n_1$ a mechanism of exception in the sense indicated by Prof. KORTEWEG (§ 26 of his paper).

Mechanisms of exception therefore are among others the symmetrical mechanisms (§ 31 of that paper); for here c , b , and ρ are all equal to zero.

Microbiology. — “*Viscosaccharase, an enzyme which produces slime from cane-sugar*”. By Prof. Dr. M. W. BELJERINCK.

The emulsion reaction.

Many spore-producing and a few non spore-producing bacilli, cause, when growing in presence of cane-sugar or raffinose on neutral or feebly alkaline agarplates, a very peculiar “colloidreaction”, which is also valuable for the diagnosis of these bacteria. This reaction consists in the formation, in and also on the surface of the agar around the colonies or streaks, of a liquid “precipitate”, i. e. an emulsion, which can best be recognised in transmitted light, and at the same time in a swelling of the agar caused by the increase of volume produced by the emulsion.

The emulsion consists of drops (see plate) of different size, mostly very small, but sometimes growing to 0,2 mm. so that they may

be distinguished with a magnifying glass. At feeble magnification they might be taken for droplets of oil suspended in the agar, but as strong sulphuric acid dissolves these drops immediately, and a feebler acid more slowly, there can be no question of oil or fat.

Characteristic for the reaction is that it can only be distinctly observed in agar but imperfectly in gelatin. In the agar the process is impeded when acid is produced by the microbes. Thus bouillon-agar, yeastwater-agar, and wort-agar with cane-sugar can well be used, but the emulsion is more distinctly formed in agar with mixtures of substances that prevent the acidification, to which cane-sugar is so very apt. For that reason nitrates as nitrogen-food are especially favourable, as the withdrawing of nitrogen then necessarily must produce an alkali, while for example ammonium-salts, used as source of nitrogen, must promote the acid reaction.

A good experiment to produce the emulsion is the following: A plate is prepared of the composition: tapwater, 2 % of agar, 2 % of cane-sugar, 0,02 % KNO_3 and 0,02 % K_2HPO_4 . Nitrogen food may also be quite left out, so agar-plates with 10 % of cane-sugar and bikalium-phosphate only, are very well fit to demonstrate the emulsion with *Azotobacter* and the hereafter mentioned *Bacillus emulsionis*. The quantity of cane-sugar can vary between 0.1 % and 50 % without much difference in the result.

After the solidifying of the agar-plate and the removal of the adhering water, soil-bacilli are dispersed, obtained by shaking some garden-soil with water, and heating it a few minutes at 70° to 80° C. in order to kill the not sporulating microbes. Then the water is poured over the plate and allowed to flow off. The adhering germs, for so far they live, are nothing but spores of bacilli, which can germinate at 30° C.

After one or two days the colonies become visible and simultaneously the emulsion around some of them; the majority does not produce the emulsion.

Cane-sugar may be replaced by raffinose, which acts in the same way; but glucose, levulose, mannose, galactose, lactose, maltose, trehalose, melibiose, mannite, inulin, dextrin and xylose, do not give the emulsion.

The emulsion is distinct round the colonies of *Bacillus mesentericus vulgaris* (see plate Fig. 1), *B. megatherium* and a not yet described soil-bacillus, commonly also found in cane-sugar itself, recognisable by its small terminal spores, which may be called *Bacillus emulsionis* and whose transparent colony is likewise given on the plate (Fig. 2). The emulsion is wanting in *B. subtilis*, *B. mycoides*, *B.*

polymyxa, *B. nitrosum*, *B. splachosporus*, *B. luteus*, besides in the anaërobes *Granulobacter butylicum*, *Gr. saccharobutyricum* and *Gr. pectinovorum*.

The moulds, the various yeast species, even those which invert cane-sugar, besides all species of *Streptothrix*, and most of the non-spore producing bacteria, do not produce the emulsion either.

An exception to the last rule makes the non-spore producing *Azotobacter chroococcum*, which on plates of 2 % of agar, 2 to 10 % cane-sugar, and 0,02 % K_2HPO_4 , in water, gives a strong emulsion, which extends to a large distance round the colonies; later, in their vicinity, perhaps by the influence of a specific enzyme or an acid it vanishes, while near the colonies of the soil-bacilli the emulsion is permanent. With the exception of *B. chroococcum* the other forms of *Azotobacter* do not produce the emulsion. From cultures of *Azotobacter*, prepared with garden soil and destined for the absorption of free nitrogen, a species related to *B. radiobacter* can be obtained, which produces no spores, but does also give a strong emulsion.

An emulsion, from a physical view analogous but quite different by the manner in which it takes rise, was described by me on another occasion¹⁾. It appears when a 10% solution of gelatin in water is boiled with a 10% solution of soluble starch, or with a 2% agar-solution. Even by boiling the two watery solutions do not mix, which, of course, is also the case after solidifying. This reposes evidently on the fact that here two colloidal solutions are brought together, which cannot diffuse and whose emulsinated droplets constantly have a positive surface-tension with regard to each other. The same explanation must hold good for the emulsion formed by the viscosaccharase with regard to the agar, and as I may add, to culture-liquids wherein *Bacillus emulsionis* produces the emulsion also.

The emulsion is produced by an enzyme.

If from the emulsion field round a colony a small piece of agar is cut out, without touching the colony, and placed on an other cane-sugar-agar-plate, the emulsion itself does not diffuse out of it, but into the plate, a substance goes over, which produces the emulsion again and with regard to the quantities used rather strongly. This points with certainty to the presence of an enzyme as the cause of the emulsion, an enzyme which must have the property of moving through the agar by diffusion. This agrees perfectly well with the

¹⁾ Centralbl. f. Bacteriologie 2^{te} Abt. Bd. 2, p. 627, 1896.

origin of the emulsion round the colonies, for a substance which is evidently insoluble in the agar-plate, can only be found at the place where it is produced. This substance having in our case the nature of a plant slime, the enzyme may be called *viscosaccharase*.

The enzyme is prepared by filtering a culture of *B. mesentericus vulgatus* and precipitating the filtrate with alcohol, whereby, of course, other enzymes formed by this bacterium such as diastase, and also the slime substance itself, are precipitated. Whether to the enzymes, present in this mixture invertase must be reckoned, which is usually considered as a secretion-product of *B. mesentericus*, has become doubtful by the discovery of the viscosaccharase, at whose action, as will be seen below, together with the slime, the production of a reducing sugar is stated.

Even in presence of chloroform the emulsion reaction takes rise on cane-sugar agar-plates through the enzyme produced from the *mesentericus* cultures, without anything being perceived of the development of the germs of *B. mesentericus* itself, which may be still present after filtering and precipitating.

It is not difficult to prepare plates of any size containing the emulsion everywhere, and fit for experiments to demonstrate by what influences it may disappear.

To this end the required culture-agar is mixed before solidifying with a not too large number of germs, for example of *B. emulsionis*, and then placed one or two days in the thermostat; when the plate becomes quite turbid by the emulsion, the sugar is washed out and it is ready for the experiment. A drop of dilute acid thereon rapidly produces a clear space.

At the action of viscosaccharase, besides the slime a reducing sugar is found.

When small pieces of agar containing the emulsion are introduced into an experiment-tube and cautiously warmed with a little FÉHLING'S copper solution, a strong reduction is seen, which does not take rise with the same sugar-agar if the emulsion is wanting.

The question arose whether this reaction should be ascribed to the slime itself, or if at the same time, through the viscosaccharase, or in another way, some other reducing substance is formed. Therefore small pieces of the agar containing the emulsion were washed out with water, whereby the slime, which cannot diffuse from the agar into the water, remains behind, but the reducing power of the agar is lost, whilst the water used for the washing becomes itself strongly

M. W. BEIJERINCK. Viscosaccharase, an enzyme, which produces slime
from cane-sugar.

Fig. 1. *Bacillus mesentericus*

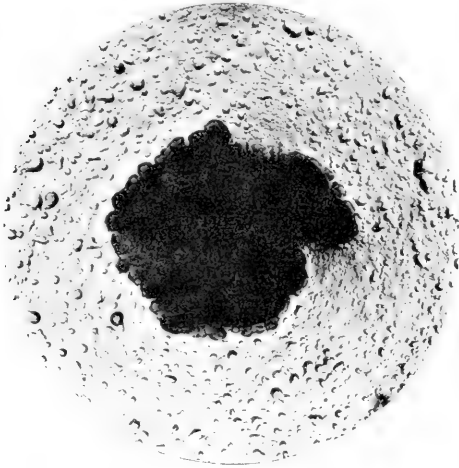
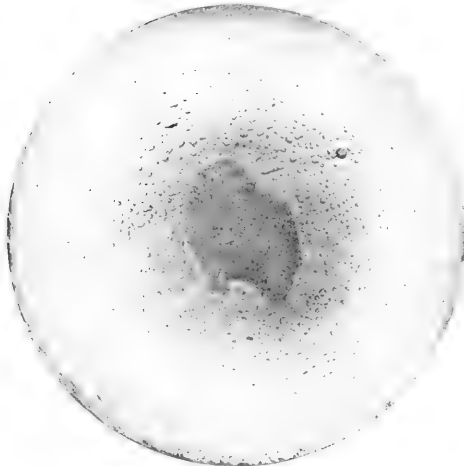


Fig. 2. *Bacillus emulsionis*.



The emulsion-reaction.

reducing. Hence it is sure that at the "emulsion reaction", together with the non-reducing slime, an easily diffusing and reducing substance (probably a sugar) is formed. The chemical composition of this substance is still unknown, just like that of the slime itself.

The possibility exists that the reducing substance is invert-sugar produced by invertase, which latter enzyme then should always accompany the viscosaccharase. Decisive experiments on this subject in progress.

Viscosaccharase is a synthetically acting enzyme.

As to the nature of the slime it must be accepted that its molecules are much larger than those of cane-sugar, else it would not be clear why the slime cannot diffuse through the agar, which cane-sugar does very easily. Viscosaccharase must therefore be a synthetically acting enzyme. This circumstance suggests a relation between the slime and "dextran"¹⁾. This is, however, a substance forming the cell-wall of the concerned microbes, which substance may spread in water, and even to some extent diffuse into agarplates, but is not the product of an exo-enzyme, i. e. of an enzyme able to leave the bacterial body and act outside of it like the viscosaccharase. In relation to this it is not astonishing that "dextran" can very well originate from glucose and some other sugars, which do not produce the emulsion.

Very remarkable is the fact that all the hitherto examined bacteria which show the emulsion-phenomenon, are able, at definite culture-conditions, for example on cane-sugar gelatin, when no emulsion is produced, to form non-diffusing "dextran", by which their colonies then become visible on the plates as large transparent drops. This also points to a narrow relation between the two phenomena and leads to the conclusion that the drops of the emulsion must be identic with, or related to dextran.

Perhaps by further research modifications of viscosaccharase will prove to exist, which also act on glucose and other sugars and from these may form "dextran", but which cannot leave the body, or rather the cell-wall of the microbes, and must be considered as endo-enzymes whose product, which itself does not diffuse, cannot be found beyond the limits of the colony.

If in accordance with my expectation, the emulsion is really brought about by "dextran", then light will be thrown on the formation of the wall-substances of plant cells in general; for there is no doubt

¹⁾ C. SCHEIBLER, Zeitschr. d. Vereins für Rübenzuckerindustrie, Bd. 24, p. 309, 1874. L. MAQUENNE, Les sucres et leurs principaux dérivés, p. 745, 1900.

that "dextran" is a modification of cellulose, and the till now not explained secondary changes, observed in so many cell-walls, may then freely be ascribed to the action of specific enzymes, related to the viscosaccharase.

Why the emulsion is distinctly observed in agar, and less easily in gelatin-plates, must probably be explained by the dimension of the molecules of viscosaccharase, which are small enough to enter without much trouble the relatively wide canals of the agar, but too large to pass through the much narrower ones of the gelatin.

Many of the experiments here related I owe to Mr. D. C. J. MINKMAX, assistant in my Laboratory.

EXPLANATION OF THE PLATE.

Fig. 1. Colony of *Bacillus mesentericus vulgaris* on: canal water, 2^o/₁₀ of agar, 1^o/₁₀ of cane-sugar, 0.02^o/₁₀ KNO₃ and 0.02^o/₁₀ K₂HPO₄, with emulsion around colony. Magnified 8 times.

Fig 2. Colony of *Bacillus emulsionis* n. sp., on canal water, 2^o/₁₀ of agar, 0.1^o/₁₀ of cane-sugar, 0.02^o/₁₀ ClNH₄, 0.02^o/₁₀ K₂HPO₄, with emulsion around colony, Magnified 9 times.

Microbiology. — "*Variability in Bacillus prodigiosus.*" By Prof. M. W. BELJERINCK.

In a former paper¹⁾ I showed how easily new constant variants of *Bacillus prodigiosus* and other microbes may be obtained. Here follow some further observations, made with the aid of Mr. H. C. JACOBSEN, assistant in my Laboratory.

The keeping constant of the cultures.

The principle on which the keeping constant of *B. prodigiosus* seems to repose is preventing the cultures from becoming alkaline by their own action. Thus, by re-inoculating in quick succession, for instance every 24 hours, into bouillon or on bouillon-agar at 30° C., each form of *Bacillus prodigiosus*, whether the natural or normal form, or a variant obtained from it, remains unchanged probably for an indefinite time.

For the transplantations only very little material must be used and an abundance of food.

If some lactic acid is added, for instance 0.5 to 1.5 cm³ normal per 100 cm³ of bouillon, the culture likewise remains unchanged

¹⁾ Royal Acad. of Sciences 21 Nov. 1900,

after a prolonged series of transports, if these are always carried out before the acid is neutralised by the alkali produced from the bouillon by the bacteria themselves¹⁾.

Addition of 1 to 2 pCt. of glucose acts in the same manner as free acid, *B. prodigiosus* therefrom producing acid which may rise, if sufficient glucose is added, to 3 to 4 cm³ normal per 100 cm³ of bouillon. As the titre of alkali, originating in the bouillon alone, can amount to 2.5 cm³ N per 100 cm³ of bouillon, and as from 1 pCt. of glucose there results no more than 1.5 to 2 cm³ N of acid, addition of 1 pCt. of glucose is sufficient to prevent variation, if the re-inoculations take place quickly; but not if effected with long intervals, for in the latter case more alkali may result from the bouillon than acid from the glucose.

If to the bouillon so much ammoniumcarbonate or natriumcarbonate is added that the titre of alkali amounts to about 3 cm³ N per 100 cm³ of the medium, *B. prodigiosus* likewise remains constant after repeated inoculations at 30° C., whilst the control culture, without carbonate but for the rest under the same conditions, strongly varies. The same result may be obtained with magnesiumhydrophosphate ($\text{Mg H PO}_4 \cdot 2 \text{ H}_2\text{O}$) to excess; this, however, quickly precipitates, and in order to be active should be used in a bouillon-agarplate or in a thin layer of liquid. In ordinary bouillon-agarplates 1 pCt. of this salt changes entirely into crystals of ammoniummagnesiumphosphate ($\text{Mg NH}_4 \text{ PO}_4 \cdot 6 \text{ H}_2\text{O}$) the plate becoming quite transparent; a plate with 3 to 4 pCt. on the other hand, remains white and turbid.

Although it may be admitted that by these various means the formation of secretion products by the bacteria is prevented, on whose stimulating action the variability probably reposes, yet it, is not clear how this preventing takes place. Evidently substances should be thought of here which, once produced, cannot or only with difficulty leave the bacterial body.

Of the said means quick transplantation is the simplest for always disposing of constant stocks for the experiments.

The origin of the variants in general.

When cultures, placed under favourable nutritive conditions, but for the rest prepared without special precautions, are growing older between 10° and 30° C., they exhibit a certain variability at which, as formerly described (l. c.), variants are thrown off, while beside

¹⁾ At 4 cm³ of acid per 100 cm³ of culture liquid the growth of *B. prodigiosus* is slackened, at 9 cm³ it is quite stopped.

these the original form is found unchanged. As by transplantations in rapid succession (and under constant and favourable conditions) no change occurs during thousands of cell-partitions, this variability cannot repose on some law governed by internal causes only, but a particular agency is wanted, which may have its seat within the cells, but which must yet be enacted on by external circumstances.

Although the variability can reveal itself already in an ordinary same well arranged culture, e.g. in bouillon or in maltwort, allowed to stand for a few weeks, yet this process may considerably be accelerated by repeated transplantations, *not* after a very short time, but with longer intervals, for example two days, with cultures kept at 30° C., a not too small quantity of the material for the inoculation being used, e.g. two loops of the platinum thread. After three or four repetitions, so after about a week, the variation can then be in full course, the first culture, left to itself, not yet showing any perceptible change.

This evidently reposes on the following circumstance. The influence which causes the variability in the culture when it gets older, acts in the chosen conditions already after two days. If now a re-inoculation is performed, the germs affected by that influence can increase as well as those that remained normal, whilst by not re-inoculating, thus in the first culture, the non-affected germs are by far more numerous and remain so as the cell-division slackens after the second day, because of want of food. At inoculation after two days there result at each time new modified germs, and those which are modified already, are enabled to augment without losing their modification.

In this explanation it must further be accepted, that a transplantation after two days gives no cause for atavism; for if this were the case, the reverse ought to take place of what is observed: after a week's growth the first culture should be more varied than that which has repeatedly been transplanted, but this is not so. This shows how carefully the variation experiments must be carried out in order not to become obscure.

Particularly the cultures on solid media must very accurately be observed. If these are allowed to stand for some days or weeks without further precautions, then in many cases, even with magnifying glass or microscope no variation at all can be detected, although it is actually going on, commonly to "rose" or "white".

Colony culture then shows that here and there varied germs or groups of such germs must be present, for from the seemingly homogeneous matter large numbers of white and rose variants are

obtained, which prove as constant as the normal form itself. However unchanged colonies, representing the pure stock and producing a material as fit for further experiments as the original culture, lie among the variants.

Experiences afforded by other bacteria seem to prove that the frequent repetition of the thus possible process of selection, produces a form which varies less than the original material. But it is not here the place to enter upon this important fact.

All colony cultures of *B. prodigiosus* are best made on bouillon-agar-plates, which after solidifying have been cautiously dried on a thermostat at circa 40° C. The water which then condenses on the glass cover can easily be removed; if this is neglected, *B. prodigiosus*, which is strongly motile, spreads over the surface of the agar and the colonies coalesce.

I shall now enter into a short discussion of the most important variants.

The obtained variants.

The variants derived from *B. prodigiosus* may be considered as plus- or gain-variants, minus- or loss-variants, and qualitative variants. This is exposed below in the table of descent, which shows the origin of the obtained forms; the qualitative variants (*auratus* and *hyalinus*) are placed on the same line with the normal form, the plus-variants above it, the minus-variants beneath. Hence, the arrows not only denote the descent but also whether the variability reposes on gain or loss of characters, or if it is qualitative. Dotted arrows indicate that atavism has with certainty been observed. The names indicate the chief qualities characterising the variants.

A survey of the variants without regard to their descent precedes: then follows their pedigree, which does not repose on hypothesis, but simply gives the result of the experiments.

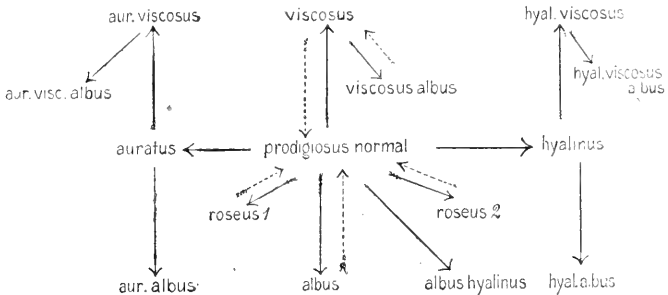
The obtained variants are :

- | | | |
|----|-------------------------------|---|
| 1. | <i>Bacillus prodigiosus</i> . | Normal form, isolated from nature ¹⁾ . |
| 2. | „ „ | <i>roseus</i> 1. |
| 3. | „ „ | „ 2. |
| 4. | „ „ | <i>albus</i> . |
| 5. | „ „ | „ <i>hyalinus</i> . |
| 6. | „ „ | <i>viscosus</i> . |
| 7. | „ „ | „ <i>albus</i> . |
| 8. | „ „ | <i>auratus</i> . |

¹⁾ About 1890 from mouldering bones of a gelatinfactory near Delft.

9. *Bacillus prodigiosus. auratus viscosus.*
 10. " " " " *albus* (= 7 ?)
 11. " " " " *albus* (= 4 ?).
 12. " " " *hyalinus.*
 13. " " " " *viscosus.*
 14. " " " " *albus.*
 15. " " " " *albus* (= 5 ?)

The relation and origin of these variants is given in the following table.



The upward arrows denote "gain-variation", the horizontal "qualitative variation", the downward arrows "loss-variation". Dotted arrows signify that atavism has been observed.

The two qualitative colour-variants, *auratus* which is orange-coloured and *hyalinus* of a deep vine-red, vary in a way quite corresponding to the normal form and like this throw off, under the same circumstances, slime-variants and white variants. Besides, the normal form may return by atavism as well from *auratus* and *hyalinus* themselves as from the variants derived from them. In the pedigree table atavism is indicated by dotted arrows for a few of the cases where it has been stated with certainty. But there is no doubt that also the other variants are disposed to atavism.

It should moreover be noted that the *auratus*-variant approaches, at least in colour, the natural variety *Bacillus Kiehlensis*, but that the latter possesses a stronger power of fermentation, and produces much gas ($\text{CO}_2 + \text{H}_2$) from maltwort with dextrose or cane-sugar, the former fermenting only dextrose.

For the rest, *B. Kiehlensis* itself, which varies in a way quite analogous to that of the normal form of *prodigiosus* here considered, has not yet been obtained as a variant from the latter.

A new character which may rise in addition to the already existing ones, is the production of a large quantity of slime substance by excessive growth of the cell-wall, which slime may spread through the liquids, and makes the individuals of the colonies on agarplates cohere into one tough mass. From *B. Kiehlensis* was even a variant obtained whose colonies appear on the agar plates as a very consistent, almost dry zoogloea, but the analogous variant did not till now arise from the common *prodigiosus*. The *viscosus* (6), derived from the latter, is an ordinary red slime bacterium.

This red-coloured, tough-slimy form, which may be called *B. prodigiosus viscosus*, is no doubt a plus-variant. Its production has been observed under the most different nutritive conditions, between the temperatures 10° (in a cellar) and 30° C., but always and exclusively in liquid media, never on a solid one. The latter circumstance is apparently the reason why the numerous experimenters, who have studied *B. prodigiosus*, have not seen this variant. It is true that SCHEUERLEN¹⁾ observed that old *prodigiosus*-cultures sometimes turn slimy, but he ascribed it to their becoming alkaline and overlooked that a new constant form was produced.

The only distinct condition which seems different in the liquid cultures compared with the solid, is the access of oxygen. In the depth of the liquid this access must, of course, be very deficient for a long time, or even be entirely lacking, as the upper layers of the culture, which are rich in bacteria, take up all the oxygen. Consequently anaërobiose becomes possible in the depth, which is not the case in cultures lying free on a solid medium, and this partial anaërobiose is apparently the stimulus which induces the formation of the slime variant. That here a rather complex influence and not a direct action must be ascribed to the partial withdrawing of the oxygen, follows from the fact that the culture of *B. prodigiosus* at complete exclusion of air, as in a closed bottle, does not, even with repeated transports, give rise to the slimy variant. At temperatures of about 35° C. this variant is no more formed, although the growth of *prodigiosus* is then still very strong; at 37° the growth slackens or ceases entirely, according to the food.

In the following liquid media the production of the slime variant has with certainty been observed, as well after repeated re-inoculations as after prolonged keeping of one and the same culture at 25° to 30° C.: in broth, in broth with 1 pCt of glucose, in malt-wort, in tap-water with 5 pCt of pure gelatin and 0,02 pCt K_2HPO_4 , and in

¹⁾ Archiv. für Hygiene. Bd. 26 p. 1.

tap-water with 2 pCt of glucose, 0.5 pCt of asparagine, 0.02 pCt K_2HPO_4 , always cultivated at 30° C. and with repeated transports after two days or longer. From this we also recognise that there is no question of a direct influence of the food on the production of the variant.

The *auratus*- and *hyalinus*-variants, also, have only taken rise in liquid cultures, namely in broth and in the glucose-asparagine solution. Moreover, *hyalinus*, which is of a deep vine red, is easily obtained from a solution of pure gelatin in tap-water with 0.02 pCt. K_2HPO_4 , after repeated re-inoculations, at 30° C., whereby also *hyalinus viscosus* results.

The colourless or white variants, which only differ from the original form in producing no pigment, should certainly be considered as minus-variants. They are obtained with more ease than the slime variants and, at least as to N° 4, have also been detected by other authors¹⁾.

Except under the said conditions, apt to keep them constant, all the cultures as well in liquid as on solid media, vary sooner or later towards white. The original form does remain preserved, but a colourless variant is thrown off, which is still more constant than the stock itself.

Not always does one and the same variant result in this case: two uncoloured constant forms, N° 4 and 5 can easily be distinguished if they originate at the same time, and their colonies are on the same agarplate so that they may be compared somewhat magnified. One, *albus hyalinus*, then looks more blueish transparent, the other, *albus*, is more of a cloudy and opaque white; under the microscope the former proves to consist of smaller cells than the latter.

The cause of the production of white variants cannot be a more or less abundant access of oxygen, but must probably be sought in a stimulus, exerted by secretion products which remain enclosed in the interior of the cells.

Although the presence of ammoniumcarbonate in the medium (broth-agar), as also cultivation at temperatures higher than 30° C. e. g. at 33° C., prevent pigment production, no hereditary variation at all is caused by these influences. If the thus treated colourless cultures are transported at 20° to 25°, no white variants are obtained from them, but the normal form is found back unchanged, if at least the above mentioned precautions to preserve the constancy of the stock are not neglected.

¹⁾ In LEHMANN and NEUMANN'S Atlas, 4th Ed. 1907, Table 30, Fig. 3, shows a coloured image of a "pure culture" of *prodigiosus*, consisting of red and white colonies.

When the white variants of the normal form are cultivated at 30° C. in bouillon or in malt-wort, the cultures will, after a few re-inoculations, turn slimy like those of the red normal form itself. Colony culture on bouillon-agar proves that white slime variants are thrown off, in the same way as the normal form throws off the red ones. The white slime variants (N°. 7 ½ and 14) correspond by the nature of their colonies to the two white forms, *albus* (4) and *albus hyalinus* (5), considered above.

There is still another method to obtain the colourless slime variant from the red one. If this latter is cultivated at 30° in malt-wort or in bouillon, we find after one or two transferrings, each time after two days, and when sown on bouillon-agar, many white slime colonies together with the unchanged red, moreover a considerable number of quite normal, not slimy red colonies, N°. 1, which is to be considered as atavism, but an atavism reposing on the loss of a character. The white slime variant, thus obtained by minus-variation, and found in the table as N°. 7, seems identic with the one produced by plus-variation from the not slimy white variant, which latter for that reason has not been specially mentioned.

Already in my earlier paper I spoke of rose variants, which so to say, keep the middle between the normal form and the white variant. They may be produced in various ways, for instance, by cultivating the normal form on plates of pure gelatin dissolved in distilled water (H₂O, 10% of gelatin) at room temperature, at which rapid growth and vigorous melting occur. By daily streaking off on a bouillon agarplate the same colony obtained on such pure gelatin, and provided the temperature be kept between 14° and 17° C., we find, on the fifth or sixth day, the first rose variants, either or not with the white, which under these conditions appear later. Two rose variants (table N°. 2 and 3) are easily distinguished, but it is possible that there are many more whose perception is beyond the reach of our observation. In any case, it is a fact that the character: "the faculty of producing pigment", is divisible in many ways. The hereditary constancy of at least one of these rose variants proved not to differ from that of the normal form.

Another method to obtain rose variants is cultivation of the normal form in bouillon, which by evaporation has been reduced to a threefold concentration. After a single transport already, a large number of rose variants (3) had appeared by the side of normal forms; by a much lighter colour they showed a disposition to lose their colour entirely. The variability of the different rose variants is not the same; the form, obtained by the concentration

experiment (3) produces, more readily than the rose variant (2), as well red normal forms (1) as white ones (4). For the rest, this more variable variant has also proved to remain constant when quickly transplanted.

Cases of atavism are frequently observed in these experiments. Thus, for example, the production of the normal form from *viscosus* (6) may easily be seen if the latter grows for a fortnight without transport on a bouillon-agarplate; along the margin of the streaks some few normal colonies (1) will then become perceptible.

The *albus*-variants, also have a disposition to throw off a few red normal forms, but they do so only after growing for weeks or months on bouillon-agar; at first they are very constant.

The to a certain extent completely regular production of the same variants of *Bacillus prodigiosus*, suggests the existence of variability in a special and determined direction, of orthogenesis, as ELMER expressed it.

As under different nutritive conditions the same variant may appear, the food itself cannot be the stimulus; there must be, as said above, another cause in the interior of the cells, which, for *B. prodigiosus*, seems only active in an alkaline environment.

On the other hand, the food, in a wider sense, has certainly a decisive influence on the variability, albeit indirectly. So we considered already the influence of the alkaline reaction of the medium if this alkali is produced by the microbes themselves. Another example is the following. As well in malt-wort as in bouillon the *viscosus* variant is regularly produced; but from malt-wort the *auratus* variant, which so readily takes rise in bouillon, is not obtained at all. Indeed, every culture condition gives a peculiar but constantly returning mixture of variants, differing both quantitatively and qualitatively from that found under any other conditions. But the real factors here active could not as yet be detected.

From the foregoing the following results may be derived.

1. *Bacillus prodigiosus* produces as well qualitative, as gain- and loss-variants, all obtained with certainty by determined experiments; the stock-form is always found unchanged in the same culture with the variants.

All the variants are from their origin as constant as their stock.

The true factors which govern the variability in these experiments are still unknown.

2. By rapidly repeated re-inoculations and by other methods, nor-

mal form and variants may be kept constant, as it seems for an unlimited length of time.

3. All the variants vary in a way analogous to that of the normal form, thus, the *auratus*-variant produces an *auratus*-slimevariant, which must be considered as a gain-variant, and an *albus*-variant, which must be taken for a loss-variant.

The natural variety *B. Kieliensis*, which approaches the *auratus*-variant, also varies in an analogous way. The variation thus seems to be directed or orthogenetic.

4. Gain-atavism in loss-variants and loss-atavism in gain-variants, can be obtained with certainty by determined experiments. Qualitative variants, too, may give rise to atavism.

5. The experimental variants of *B. prodigiosus* have not yet been found in nature. From another bacterium, *Bacillus herbicola*, a variant, took rise which I had before repeatedly isolated from nature and which I had taken for quite another species.

6. The variants of *prodigiosus*, and this holds good for many other microbes also, differ from each other and from their stock forms in the same way as closely related natural species or varieties do among each other. But their disposition to atavism is much more pronounced.

7. The sub-variants, e. g. the rose variants of different colour-intensity, arise in the same way as the chief variants and possess the same degree of constancy.

Physics. — "*Researches on magnetization at very low temperatures.*"

By PIERRE WEISS and H. KAMERLINGH ONNES. Communication N^o. 114 from the Physical Laboratory at Leiden.

§ 1. *Object of the research; results.*

a. Introduction. The extension of LANGEVIN'S¹⁾ kinetic theory of magnetism to all ferromagnetic phenomena by means of the hypothesis of the molecular field²⁾ rendered the testing of deductions from this hypothesis by experimental data of great importance. The first results of this comparison were very encouraging; in some respects a remarkable correspondence was found. For instance the curves

¹⁾ LANGEVIN. Ann. Chim. et Phys. 8 Sér. t. 5, p. 70; 1905.

²⁾ P. WEISS. Journ. de Physique 4e Sér. t. VI, p. 661; 1907.

calculated for the intensity of the magnetization at saturation as a function of the temperature corresponded very well with those which had been found experimentally for magnetite at temperatures above the ordinary. Moreover, the law determining the susceptibility above the CURIE-point¹⁾ developed from the hypothesis of the molecular field was found in CURIE's experiments, and in others which will soon be published, to be accurate over a temperature range of some hundreds of degrees. Finally the sudden changes in the specific heat at the CURIE-point were in correspondence with the values calculated from magnetic data. But other observations do not correspond so well with the theory. Fig 1, Pl. I in which the theoretical curve for the change of saturation-magnetization with temperature is shown by the full curve *a*, also shows the experimental results for magnetite, and the corresponding curve, *b*, for nickel²⁾. The last curve is drawn to such a scale that the best possible correspondence with the theoretical is obtained at the CURIE-point. In contrast with what was found for magnetite, nickel shows a deviation from the theoretical gradually increasing over the whole curve. Iron and cobalt behave practically the same as nickel. When all this is taken into consideration it is seen that the hypothesis of the molecular field is of the nature of a working hypothesis; the partial confirmation shows that the hypothesis contains a kernel of truth, and from the experimental deviations one will have to see how it should be modified or extended while still retaining its essential features.

It is not probable that these modifications will attack the property of reacting against the orientation by the magnetic field that has been ascribed to the kinetic energy, or that they will come into conflict with the manner in which the MAXWELL-BOLTZMANN partition law has been employed. Not only are these hypotheses of fundamental import, but they are still further forced upon our consideration by the ease with which they account for the fact that for paramagnetic substances the susceptibility varies inversely as the absolute temperature — an experimental law that is one of the most firmly established for a number of substances. In their important investigations upon the magnetization of the elements, of which an account was given at the last

¹⁾ In this Communication we shall give the name CURIE-point to the temperature at which spontaneous ferromagnetism ceases. This is by no means inconsistent with CURIE's idea that the transformation temperature is a function of the strength of the field, since the temperature at which *spontaneous* ferromagnetism ceases is the temperature obtained by reducing the field to zero.

²⁾ According to preliminary measurements. Accurate experiments upon the three metals and magnetite are in progress.

meeting¹⁾, H. DE BOIS and HONDA have, it is true, shown that this law of the dependence of susceptibility upon temperature is not generally valid, and that paramagnetism also occurs which is independent of the temperature or increases with increasing temperature. But it is by no means the case that the foregoing hypotheses should be discarded on that account; what we learn from experiment in this case is only that these suppositions are not sufficient to explain magnetism as a whole. In particular it will be necessary to revise LANGEVIN'S hypothesis that the magnetic moment of a molecule is constant, or at least quasi-constant, and also that concerning the nature of the mutual action of the molecules, which until now has been represented by the introduction of the molecular field. For an at low estimation of the value of both of these hypotheses, experiments temperatures are especially valuable.

For it is only at the absolute zero that the magnetization gives the sum of the molecular magnetic moments, as it is only then that heat-motion can no longer prevent the magnetization from attaining its full value; and at low temperatures, too, is the strongest demonstration of the mutual action of the molecules to be expected, since they are then at the smallest possible distance from each other.

b. Ferromagnetic substances. We have, therefore, aimed at the continuation of the curves connecting magnetization and temperature in the three ferromagnetic substances and in magnetite down to the neighbourhood of the absolute zero. By utilising the methods and appliances²⁾ suitable for long-continued accurate measurements at such constant temperatures as are obtainable with liquid hydrogen, we have been able in our measurements to reach a temperature of 20°,3 K. with hydrogen boiling under atmosphere pressure, and of 14°,0 K. with hydrogen near its melting point. The number of degrees on the absolute scale which separate these experimental temperatures from the absolute zero is but such a small fraction of the number between the absolute zero and the CURIE-point (even in the case of nickel this number is still so much as 648 Kelvin degrees) that, considering the nature of the curves, we may regard the saturation-magnetization at the absolute zero as being determined by our experiments. All this, of course, with the proviso that the phenomenon in the region to which extrapolation is carried should give no occasion for adopting another point of view. Since the object of the measurements was a determination of the saturation-magnetization, it seemed suitable to direct the exper-

¹⁾ These Proceedings Jan. 1910.

²⁾ H. KAMERLINGH ONNES, these Proceedings Sept. '06, Comm. Leyden N^o. 94f.

periments towards obtaining data for magnetization in strong fields, and from these the deduction of the law according to which the magnetization approaches its limiting value. But the method chosen for the magnetic measurements, viz: the determination of the maximum value of the couple exerted by a magnetic field of varying direction upon an ellipsoid of the experimental substance, was, as we shall presently show, less suitable for this determination of the law of approach than for comparisons of the magnetizations of the substance in the same field at different temperatures. The data to determine the law of approach were therefore made the subject of a separate investigation¹⁾ This gave the following values for the difference between the magnetization in a field of 10000 gauss and that in the limiting case:

Iron	0.08 %
Nickel	0.1 „
Cobalt (soft)	1.1 „
Magnetite	0.19 „

For these substances, the cobalt excepted, the approach of magnetization as a function of the strength of the field is hyperbolic, so that in a field of 20000 gauss, which we reached in our present experiments, the above differences were reduced to half their values. Observations by the ellipsoid method in different fields and at both low and ordinary temperatures have not, indeed, enabled us to test the law of approach, but they show sufficiently well that there is no essential difference between the behaviour in this respect at the two temperatures; and that at low temperatures, as could have been supposed the magnetic hardness does not assume an excessive value, the molecules hindering each other in assuming a new direction.

Further, by means of comparative measurements, magnetizations at ordinary and at low temperatures in fields of great strength were compared, and it was found that the ratio between the two is pretty well independent of the strength of the field. Thus, leaving the result uncorrected for the dilatation between the two temperatures (see note 2 on p. 11) we found for the ratio of the intensity of magnetization at 20°, 2 K. and at ordinary temperature the following:

Nickel	(17°.3 C.)	1.0548
Iron	(20°. C.)	1.0210
Magnetite	(15°.5 C.)	1.0569

The exact value of the ordinary temperature is given between brackets. In § 5 it will be explained why the experiments with

¹⁾ P. WEISS, Arch. des Sc. phys. et nat. février 1910 and Journ. de Phys. 4e Sér. t. IX. avril 1910.

cobalt have not been brought to a conclusion. It is difficult to know exactly the degree of accuracy of these results. Experimental work in every branch was carried out so that an accuracy of 1 in 1000 or even higher could be expected. But when one considers the disturbing influences which made themselves felt in the experiments upon cobalt, it seems rather incautious — and this is particularly the case with the magnetite measurements — to ascribe to the results an accuracy greater than 0.5 %, even though the occurrences which have thrown suspicion upon the cobalt measurements were nearly absent in the case of the other substances, and though in all its properties and particularly in its extraordinarily large magnetic hardness cobalt stands evidently alone. Since our experiments indicate these causes of uncertainty, they show how a higher degree of accuracy may be reached if so desired. The present accuracy is quite sufficient for the treatment of various problems.

The experiments with iron and magnetite were carried to 14° 0 K. The change of magnetization between 20° K. and 14° K. is too small to be expressed in figures. These experiments, therefore, only extend down to 14 K. the temperature region within which the diminution of the kinetic energy and the approach of the molecules to each other do not occasion the appearance of a single new phenomenon.

The portions of the curves for nickel and magnetite which have been newly obtained are given by broken lines in fig. I, Plate I.

Magnetite is of particular importance on account of the perfect correspondence between observation and theory over the greatest portion of the region between the CURIE-point and the absolute zero, and on account of the occurrence of a deviation of observation from theory only at low temperatures. Here, theory gives for the ratio between the magnetizations the value 1.139 instead of the value given above, 1.057. The result that theory and experiment clearly differ at these temperatures is corroborated by earlier experiments upon four samples of different kinds of magnetite, two obtained from natural crystals, the third from a fused natural crystal, and the fourth from artificial magnetite. These gave the following values for the ratio between the magnetizations at the temperature of solid carbon dioxide (−79° C.) and ordinary temperature:

1.033	ordinary temperature	16° C.
1.042		23° 2
1.043		24°
1.037		21° 5
<hr style="width: 50%; margin: 0 auto;"/>		
mean 1.039		21° 2 C.

while theory gives 1.053 for the same temperature.

An analogy thus seems to exist between this and compression and expansion by heat, for which VAN DER WAALS'S theory and law of corresponding states are supported better as a rule in the neighbourhood of the critical point than at low reduced temperatures where the ideal representations of the molecule and of molecular attraction no longer cover the phenomena sufficiently well and the differences between the specific properties of the real molecules appear.

The hypothesis that molecular magnets are essentially invariable would be established conclusively if there existed simple relations between the magnetic moments as calculated per atom, which one might be led to suspect from the increase by regular steps of the saturation-magnetization of the three metals.

The following table in which the numbers in the first column are taken from the paper¹⁾ referred to above and in which the relative increase for cobalt is estimated from comparison with iron and nickel shows that this is not the case. The data are not corrected for the dilation (see note 2 on p. 11).

	Specific saturation at temp. ().	Increase by reduction to saturation at low temp.	Specific to saturation at low temp.	Atomic weight or $\frac{1}{2}$ mol. wt.	Moment of gram-atom.
Ni	54.6 (17° C.)	1.0548	57.6	58.7	3381
Co	462 (17° C.)	1.01	163.6	59	9650
Fe	217 (20° C.)	1.0204	221.6	56	12410
FeO $\frac{1}{2}$	90.75 (15° 8 C.)	1.057	95.9	77.33	7417

In connection with this we must not lose sight of the fact that although the proof that the above magnitude is of great significance may have escaped us, still there is nothing whatever to justify an opposite conclusion.

When we look upon our measurements as a whole we remain inclined to retain the hypothesis that in ferromagnetic substances the magnetic atom does not in itself change much with temperature. There were indeed reasons for questioning if this approximate invariability, granting that it was proved in other circumstances, still existed at extremely low temperatures. Electrical resistance of metals, phosphorescence of sulphur compounds, absorption of light by the salts of the rare earths with or without magnetic field, all, at very low temperatures, exhibit characteristics that one may

¹⁾ P. WEISS, Arch. des Sc. phys. et nat. and Journ. de Phys. 1910.

try to explain by ascribing them to forces exerted by ponderable matter upon electrons; these forces in that explanation become of primary importance when the temperature sinks to that of liquid hydrogen, and it is ascribed to them in particular, that they make the current-carrying electrons in metals suffer an important diminution in number at very low temperatures by their being, as it were, frozen to the atom by the low temperature¹⁾.

It would also be possible that the motions of the electrons which cause magnetism while remaining constant or changing not much at other temperatures, begin to show considerable changes at very low temperatures.

The negative result that nothing happens even at the lowest temperatures, which should throw doubt upon the relative smallness of the variability of the magnetic atom itself, is not perhaps without importance when regarded as a means of weighing the value of the above assumptions regarding the phenomena mentioned, or as means of separating the group of electrons which occasion magnetism from groups which form the prime factors of other phenomena.

c. Vanadium, chromium, manganese. The question has often been asked if a gap which cannot be bridged over exists between the ferromagnetic metals of the iron group on the one hand and the paramagnetic metals of the same group on the other, or that the latter metals should also exhibit a very low CURIE-point if the temperature were sufficiently lowered.

CH. ED. GUILLAUME²⁾ says with reference to the HEUSLER alloys of Mn, Al, Cu and Mn, Sn, Cu which are ferromagnetic: "The reason for this can be found in the fact that aluminium or tin when compounded with manganese, a metal from the magnetic group, raises its transformation temperatures, which, following an hypothesis already suggested by FARADAY, ought to lie very low." It can indeed be seen that aluminium and tin raise the melting points of various alloys which they form with other metals (the series Al—Au, Al—Sb, Na—Sn) and seem to possess the general property of raising temperatures of transformation.

We might, therefore, expect that vanadium, chromium, and manganese should at very low temperatures exhibit either the characteristics of ferromagnetism (magnetization not proportional to strength of field,

¹⁾ Cf. H. KAMERLINGH ONNES, *Comm. fr. the Leyden labor. Suppl.* n^o. 9, p. 27 1904 and P. LENARD, H. KAMERLINGH ONNES and W. E. PAULI, *These Proceedings* June 1909, *Comm. fr. the Leyden Laborat.* n^o. 111, p. 3, note 2 1909.

²⁾ CH. ED. GUILLAUME, *Actes de la Soc. helv. der Sc. nat.* Vol. I p. 88. 1907.

saturation, hysteresis) or, in conformity with CURIE's law, a strongly increased paramagnetism. The susceptibility at the temperature of solid hydrogen should be about twenty times as great as at ordinary temperature¹⁾. At this time we were not yet aware of the results published last month by H. DU BOIS and HONDA²⁾, from which it appears that the inverse proportionality of paramagnetism to the absolute temperature is but one of the possible cases. To get an idea of the order of magnitude of the expected phenomena we may suppose that the paramagnetic γ iron still exists at 14° K. with the same CURIE constant (product of absolute temperature by susceptibility). In that case a value of about 400 is found for the magnetization of this substance in a field of 20000 Gauss.

Some time ago GEBHARDT³⁾ determined the susceptibility of manganese at ordinary temperature and found $K = 322.10^{-6}$ (density 6.4). The above calculation gives a value 134 for the magnetization of this substance in the same circumstances. And as the deflection in our apparatus is proportional to the square of the magnetization, one would obtain a deflection smaller in the proportion of 18 in the case of γ iron or 160 in the case of manganese than that which was found for iron at the ordinary temperature; as this was 100 cm. the manganese deflection should still be quite easily readable.

When we now introduced into our apparatus roughly formed ellipsoids of MOISSAN vanadium and GOLDSCHMIDT chromium and manganese in succession, the awaited change did not appear. In every case the deflection at the temperature of solid hydrogen as well as at that of hydrogen boiling under atmospheric pressure remained the same as it was at ordinary temperature, that is, to a few tenths of a millimetre, and these must be ascribed to the magnetism of the suspending apparatus. There was therefore no ferromagnetism and we were obliged to choose between the following two hypotheses for these substances. We were either dealing with paramagnetism of a new kind or with diamagnetism, which is also

¹⁾ A similar supposition formed the starting point of a research by H. KAMERLINGH ONNES and A. PERRIER, which will shortly be published, and is closely connected with the present research. This investigation has been taken to hand at the same time with the present subject. Using the method of the maximum couple and the hydrostatic rise the magnetizations of liquid oxygen at various temperatures and of solid oxygen at the temperatures of boiling and solidifying hydrogen were measured. The increase of the magnetization at low temperatures was found to be very great, though not so much as was expected, and a distinct deviation from CURIE's law and a characteristic curve were found.

²⁾ H. DU BOIS and HONDA l. cit.

³⁾ GEBHARDT. Inaug. Dissert. Marburg 1909.

found in copper while most of the salts of this metal are paramagnetic. DU BOIS and HONDA's paper in which these three metals are classified under those whose paramagnetism is invariable or increases with the temperature shows that the first assumption is the correct one. The behaviour of copper with the present research made us consider the other hypothesis a reasonable one.

One could always assume that the paramagnetism, which, as a general rule is ascribed to the metallic manganese, results from the presence of its oxides, which are strongly magnetic, or of a small quantity of iron. To put this assumption to the proof we prepared very pure manganese from MERCK's pure chloride, which had been proved to be free from iron. The preparation was accomplished by electrolysing the salt between a cathode of distilled mercury and an anode of iridium alloyed with 40% of rhodium which is not attacked by the chloridion. The amalgam obtained in this way was separated in a stream of pure, dry hydrogen. In this way a grey powder was obtained which when compressed in a glass tube as a mould took the shape of a solid rod. A rod prepared in this manner exhibited paramagnetism. A glass tube with the powdered manganese was also paramagnetic. The same manganese contained in a magnesia boat was thereupon fused in an electric resistance furnace and in an atmosphere of hydrogen. In this way an ingot was obtained which was covered with a light oxidised crust. It was found impossible to grind away this crust with quartzpowder, since the metal was of the same hardness as quartz. Emery could not be used as it is magnetic. The impure crust was therefore turned off with a diamond tool, and a small cylinder of pure substance was obtained.

This cylinder was found to be *ferromagnetic*. Fig. 2 Pl. I gives the hysteresis curve for this substance. The maximum value of the specific magnetization is 100 times weaker than that of iron, and the coercive field is 670 gauss, that is to say, 10 times as strong as the coercive field of steel which is used for the preparation of good permanent magnets. This peculiar substance seems moreover to have striking magneto-crystalline properties. The rod was strongly attracted between the poles of a magnet and placed itself perpendicular to the field.

Manganese of the same degree of purity can therefore occur in two states: paramagnetic and ferromagnetic. GEBHARDT's experiments give a susceptibility five times greater than that observed by DU BOIS. If GEBHARDT's powder was not impure or oxidised, it is thus possible that there are two paramagnetic states.

1) SECKELSON, Wied. Ann. LXVII, p. 37, 1899.

As regards the ferromagnetism of manganese, this had already been observed by SECKELSON¹⁾ with electrolytic manganese which was liberated at 100° C. from the chloride upon a platinum wire, and with a regulus prepared by BUNSEN from manganese fluoride. The very indefinite observations concerning the magnetization which he published do not contradict our measurements.

By a more direct method we have proved the absence of strong magnetism in vanadium, chromium and manganese at low temperatures. For this purpose we introduced ellipsoids of the three substances into a narrow unsilvered vacuum tube whose walls were separated by the smallest possible distance; this was placed in a second similar tube also as narrow as possible and filled with liquid air. We then determined the distance from the poles such that the ellipsoids were attracted from the bottom of the tube to the poles of the magnet. This experiment was made first with the inner tube empty, and then with the inner tube filled with liquid hydrogen. The following results were obtained:

	Ordinary temperature	In liquid hydrogen
Vanadium	Not attracted	} The same as at ordinary temperature
Manganese	Attracted from distance of 6 to 8 mm.	
Chromium <i>a</i>	" " " " 12 "	
Chromium <i>b</i>	" " " " 60 "	

The results for Chromium *b* which probably contained a small splinter of iron must be rejected. We also found further that a crystal of iron sulphate at ordinary temperature was attracted from a distance of 25 mm. while in liquid hydrogen it was attracted almost from the base of the magnet. Thus the weak magnetization of the three metals was found to be practically invariable, while the iron sulphate exhibited a very great increase in magnetic properties.

This experiment is well adapted for displaying the characteristic difference between the two groups of substances and is a typical example of the significance which even the simplest experiments acquire within the fallow region of very low temperatures.

§ 2. *Methods and apparatus.*

a. Discussion of the method of the maximum couple. We measured the intensity of magnetization by measuring the couple exerted on a prolate ellipsoid of revolution of the experimental substance arranged so that the angle of the field with the major axis of the ellipsoid might be varied. The expression for the couple is

$$M = (N_1 - N_2) I^2 v \sin \eta \cos \varphi$$

where N_1 and N_2 are the coefficients of demagnetization of the ellipsoid, I , the intensity of magnetization of the substance, v the volume, and φ the angle between I and the major axis of the ellipsoid. The maximum value of this couple is

$$M = \frac{N_1 - N_2}{2} \cdot I^2 v$$

for $\eta = 45^\circ$. Hence to measure I it is not necessary to know either the strength or azimuth of the field which yields the maximum couple. To make use of these methods the ellipsoid is suspended from a torsion-spring whose displacement is determined by a mirror-method, and an electromagnet turning round a vertical axis is used. The method has already been described¹⁾. Its advantages consist of the small range over which strong fields are necessary and the extreme simplicity of the relative measurements.²⁾ We shall now discuss two sources of error which affect it and which, although they may be made as small as one wishes in theory, render it less suitable for the search after the law of approach to saturation, although they do not take away from its value as a method of comparing in the same field two successive and slightly differing states of the same substance.

Influence of inhomogeneity of the field.

The ellipsoid is placed in the centre of a magnetic field possessing the symmetry of a body of revolution. The strength of the field at this centre is a maximum for a displacement in the plane of the equator, y , and a minimum for a displacement in the direction of the x axis. It is given by the series

$$H = H_0 + \frac{x^2}{1 \cdot 2} \left(\frac{\partial^2 H}{\partial x^2} \right)_0 + \frac{y^2}{1 \cdot 2} \left(\frac{\partial^2 H}{\partial y^2} \right) + \dots,$$

which, remembering the equation $\Delta V = 0$ for the magnetic potential V , and converting to polar coordinates r and θ , transforms into

¹⁾ P. WEISS, Journ. de Phys. 4 s^{er.} t. VI, p. 655, 1907.

²⁾ For comparing the intensities of magnetization I and I' at two temperatures we have to take into account that $v = m/d$, m being the mass of ellipsoid and d its density, so $\frac{I}{I'} = \frac{\sqrt{M}}{\sqrt{M'}} \cdot \frac{\sqrt{d}}{\sqrt{d'}}$ and $\frac{\sigma}{\sigma'} = \frac{\sqrt{M}}{\sqrt{M'}} \cdot \frac{\sqrt{d'}}{\sqrt{d}}$. The dilatation at the low temperatures and therefore the proportion of d and d' being unknown, we have omitted the correction for the difference of this proportion and unity, the value of which may be estimated at 0,004. [Added in Translation].

$$H = H_0 + \frac{r^2}{2} \left(\frac{\partial^2 H}{\partial x^2} \right)_0 \left(\cos^2 \theta - \frac{1}{2} \sin^2 \theta \right).$$

Now, the energy of a volume-element dv of the ellipsoid which we consider to be very long and magnetized with equal intensity I in the direction of the field, is

$$W = - I H dv$$

and therefore the moment of the couple exerted by the field on this element is

$$dM' = \frac{\partial W}{\partial \theta} = \frac{3}{2} dv \cdot I \left(\frac{\partial^2 H}{\partial x^2} \right)_0 r^2 \sin \theta \cos \theta.$$

The couple exerted by the field H_0 on the ellipsoid is

$$M = (N_1 - N_2) I^2 v \sin \varphi \cos \varphi.$$

In very strong fields the condition is fulfilled that the magnetization is parallel to the external field notwithstanding the demagnetizing forces of the ellipsoid, and therefore $\theta = \varphi$.

Hence the disturbing moment dM' varies with azimuth of the substance in exactly the same manner as the chief couple M . The maximum value of the couple dM' is

$$dM' = \frac{3}{2} dv I \cdot \left(\frac{\partial^2 H}{\partial x^2} \right)_0 r^2,$$

which for the whole ellipsoid gives

$$M' = \frac{3}{10} I \left(\frac{\partial^2 H}{\partial x^2} \right)_0 r \cdot a^3,$$

where a is the semi-major-axis of the ellipsoid.

Let us now make the assumption that the field changes conformally, and let us call the field 1 cm. from the axis in the direction of the y -axis $(1 - \epsilon) H_0$, then

$$\epsilon H_0 = \frac{1}{4} \left(\frac{\partial^2 H}{\partial x^2} \right)$$

and therefore

$$M' = \frac{6}{5} I \epsilon H_0 v \cdot a^2$$

and the ratio between the maximum values of the couples is

$$\frac{M'}{M} = \frac{6}{5} \epsilon a^2 \frac{H_0}{(N_1 - N_2) I}.$$

With constant magnetization, therefore, the second last equation shows that the disturbing couple increases proportionally to the strength of the field. In the III diagram, a sloping instead of a

horizontal asymptote will be found. This was shown clearly in some of the foregoing experiments. If the field is constant the disturbing couple increases with I . Therefore, if in the measurements with the greatest values of I for which the experiments are carried out, made with a certain apparatus the disturbing couple does not make its presence felt, then *a fortiori* it is negligible for the smaller values of I . The last equation shows that the *relative* value of the couple for non-uniformity of the field increases as the intensity diminishes. Hence it is to be feared particularly when one works with small magnetizations, and when, to increase the sensitivity of the apparatus, the torsion spring is replaced by a weaker one.

For the purposes of our measurements it is sufficient to get an idea of the order of magnitude of the error. For this purpose the non-uniformity of the field was measured for three different values of y ; it was found to be proportional to y^2 with $\epsilon = 0.0087$ as factor.

With $a = 0.15$ it follows that $\frac{M'}{M} = 0.00023 \frac{H_0}{(N_1 - N_2) I}$. For the ellipsoids used $N_1 = 1.90$ and $N_2 = 5.59$, and $(N_1 - N_2) I$ is almost 6600 gauss for iron and 1800 gauss for nickel. Hence, for iron the correction is scarcely 1 in 1000, while for nickel it increases to some thousandths.

Reaction of the ellipsoid on the pole-pieces.

When the ends of the ellipsoid come into the immediate neighbourhood of the end surfaces of the poles, they exert a noticeable influence upon the distribution of magnetism in the pole-pieces, and the couple becomes increased thereby. This fact was established by previous experiments with a larger electromagnet with flat pole-pieces of 15 cm. diameter. In these experiments was measured the couple exerted upon an ellipsoid with various distances between the poles by a field of the constant value of 9770 gauss regulated each time by passing the required current. In this way the following values were obtained for an iron ellipsoid 9 mm. long and 4 mm. thick.

Distance between poles	Maximum couple
9 mm.	335.6
15 "	320.45
23 "	319.32
35 "	319.18
47 "	319.08

The law according to which this magnitude changes shows that the change is not a consequence of the non-uniformity of the field; for just when the disturbance reaches its greatest value, the field is most regular owing to the closer approach of the flat pole-pieces. For distances of 23 mm. and greater the influence is insignificant, and the couple is constant.

b. Electromagnet. From what has been said about the influence of the ellipsoid and the pole-surfaces it follows that the distance between the poles should be about three times the length of the ellipsoid. The total thickness of the four walls of the DEWAR tubes and of the holder (§ 2c) could not be made smaller than 5 mm. Hence, keeping account of the difficulty on the one hand of obtaining strong fields of wide extension and on the other hand of reducing very small ellipsoids to the correct form, we decided upon an inter-pole distance of 9 mm. and a length of 3 mm. for the ellipsoids.

With this distance comparatively strong fields (up to 25000 gauss) may be excited with a magnet whose cores are 9 cm. in diameter. The electromagnet of this power which was used in these experiments has already served for magnetic experiments at high temperatures. It has already been described¹⁾ and is represented diagrammatically in fig. 1 Pl. II. Comparatively light (132 KG.) and, taking its power into account, easily transported, it was possible to study it in Zurich and to use it in Leyden. Hand-wheels, whose position is read from divided circles, communicate a horizontal micrometric movement to the pole-pieces.

The magnet turns upon a vertical axis and for that purpose is mounted upon a ball-bearing support. The azimuth is determined by means of a fixed mark on a cylindrical scale E_2 attached to the movable portion of the supporting base. Each of the coils has 1500 turns of 2.5 mm. wire and has a resistance of about 2 ohms. As the coils are arranged for a current of 10 amp. under ordinary circumstances, and as the current can for a short time be increased to 25 amp. the number of ampere-turns at one's disposal may reach as high as 75,000. The water circulation E_3 between the double walls of the coils has this immediate advantage that the duration of an experiment may be doubled, but it is chiefly of importance in protecting the pole-pieces from heat. Such a heating would lead to various difficulties, of which one of the worst would be that the strength of the field would noticeably change, for the expansion of the com-

¹⁾ G. ZINDEL. *Revue électrique* 20 Juin 1909 and *Elektrot. Zeitschr.* XXX, p. 446, 1909.

paratively long core by heat could distinctly alter the comparatively short distance between the poles.

c. Cryogenic apparatus. As it was necessary to shut off from the air the space in which the ellipsoid was freely suspended since it contained liquid hydrogen and its vapour, a fairly complicated cryogenic apparatus had to be employed. This is shown diagrammatically in Pl. II fig. 1 and in section in fig. 3. The apparatus consists chiefly of three tube-shaped portions which, naming from outside inwards, we call the *cover*, the *adjusting tube* f , and the *holder* b . The cover consists of a silvered vacuum tube A , a brass tube B , a glass tube C , and a cap D which shuts off the apparatus from the air.

Holder. The ellipsoid a (figs. 3 and 5) can turn round a vertical axis with the holder b in which it is fixed. For the greater part of its length the holder is made from a tube b_2 of german silver — a substance that is rigid, little magnetic, and a bad heat-conductor. The lower end is joined to a copper rod b_1 , which has only a very weak inherent magnetism. The holder is connected to the rod k by the spiral spring g_1 (g_1 was used for iron and cobalt; the weaker spring g_2 , which was used for nickel and magnetite is shown at the side). To make the equilibrium stable and to prevent the ellipsoid from being attracted to the poles of the magnet the holder is held fast underneath by a wire of platinum-iridium of 0.1 mm. diameter, for the torsion of which a correction need hardly be applied (§ 4).

The tube b_2 and the rod b_1 are carefully adjusted on the lathe, and the ellipsoid a (fig. 4) is fixed carefully in a cylindrical opening, the diameter of which is equal to the minor axis of the ellipsoid. If the ellipsoid is nickel or magnetite it can be fixed in position with a little wax. With iron and cobalt, however, the ellipsoid is subject to such strong forces that it is necessary to clamp it fast by covering it with a thin piece of sheet copper and then driving it forcibly into the opening. The turning of the ellipsoid is transmitted through the rod b_1 , and the thin-walled german-silver tube¹⁾ b_2 to the mirror h . From the mirror through the opening f_{10} and the window C_2 (figs. 1 and 3) the torsion of the spring g_1 is read. A

¹⁾ A slight twisting of this tube is of no account. Only that portion of the apparatus between the mirror and the cap acts as a spring. Twisting of the portion of the apparatus below the mirror only transmits the couple to that spring, its sole effect is to slightly, but not noticeably, alter the azimuth of the magnet.

glass scale 1.5 meters long, and subdivided into half millimeters is used; it is placed at a distance of 4.325 m. and is illuminated by spherical mirror strips¹⁾. The tension of the spring is regulated by the rod k (fig. 3), which passes through a stuffing box D_2 in the cap D . Vertical motion is communicated to k by turning the nut D_4 and at the same time preventing the motion of D_5 . The tension is read through the opening f_{21} from the pointer l on the scale b_{20} . Before mounting the apparatus, that division of the scale b_{20} is determined which corresponds with the tension that is to be used, by suspending known weights from the stretching wire.

The apparatus is, like a stretched string, very liable to start vibrating under the influence of small impulses. This tendency is counteracted by immersing the vanes of a vane-damper b_3 (fig. 6 and fig. 3) in oil contained in a circular vessel divided into different chambers by the partitions b_{34} . These partitions are attached to a cylinder which turns with slight friction in the adjusting tube and is therefore carried round by the vanes b_{30} whenever the holder must experience a somewhat greater torsion (§ 4²⁾). The vanes must be wholly immersed in the oil so as to ensure that capillary reactions do not bring forces into play (see § 4), whose torsional effect could not be neglected. In strong fields the torsion oscillations are damped extremely well by the FOUCAULT currents.

The whole holder and spring hang in the adjusting tube f' the upper end of which is screwed to the cap D ; this cap also carries the rod k , and is itself supported by the glass tube C . The adjusting tube, consisting of the portions f'_2, f'_3, f'_4, f'_5 , is three times diminished in cross-section. The lowest portion f'_5 is narrow and surrounds the rod b_1 of the holder as closely as possible. Against the bottom f'_6 (fig. 5) rests the cone c , which is soldered to the wire d and serves to keep it taut. A slit in the bottom allows the conical portion to be placed in position (fig. 5). When the apparatus is put together the adjusting tube sinks into the DEWAR vessel A so that the thin tube f'_5 is centred in the narrow portion of the vacuum tube. The adjusting tube as well as the tube b_2 of the holder is made of german-silver.

To mount the adjusting tube already containing the holder in the cover, the cap D is screwed to a bronze ring cemented to the glass tube C of the cover; the screws D_7 are tightened, and the junction

¹⁾ H. KAMERLINGH ONNES, *Comm. fr. the phys. Lab. Leiden*, n^o. 25. (1896).

²⁾ It is essential to free the oil beforehand from volatile substances, and also to prevent the accumulation of air bubbles under the oil, since the apparatus has to be completely evacuated after it is put together.

is made air-tight by means of the rubber sleeve D_4 which is smeared with rubber solution and bound with copper wire. The lower end of the glass tube C is cemented to a second bronze ring, which is soldered to the brass tube B of the cover. To the centre of this brass tube is attached a ring B_3 , carrying the bolts of the supporting rods B_4 which hold the vacuum glass in position.

The DEWAR tube itself consists of a narrow lower portion A_2 completely silvered and a wider upper portion that is silvered up to A_3 (the upper portion is left transparent so that we might be sure that we were not allowing too much liquid hydrogen to enter the glass). It fits into the brass tube B_1 and is protected by a wooden ring. The supporting rods B_4 keep the vacuum tube in position and at such a height that it is just clear of the wooden safety ring. Fig. 7 shows how, by means of the screw B_{50} , the vacuum glass protected by a layer of paper is clamped to the thin brass ring B_3 , to which are attached the ends of the supporting rods B_4 . The lower portion of the vacuum tube has an external diameter of 8 mm. and an internal diameter of 5 mm. The glass walls are 0,5 mm. thick, which leaves only 0,5 mm. as the distance between the two silvered walls.

The apparatus is centred by placing it on an auxiliary support by means of the ring B_2 . Before the vacuum tube is yet in position, the narrow portion f_3 of the adjusting tube is adjusted by a central ring in an adjustable centring-plate. The loose ring is then removed from the plate and a second is fitted such that it just fits the narrow portion of the lower end of the vacuum tube. The nuts B_{21} serve to bring the vacuum glass to its proper position, and, as before, it is made air-tight by a rubber sleeve B_6 , which is smeared with rubber solution and bound with copper wire. By adopting this method of attaching the vacuum tube one need not fear alteration of the cover when the apparatus is evacuated, and only small further adjustments are necessary for recentring the apparatus after evacuation.

In the tube B is soldered the steel capillary θ_3 (figs. 1 and 3) of a helium thermometer ¹⁾ with german silver reservoir θ_1 (figs. 3 and 7) and glass stem θ_4 , which is permanently attached to this portion of the cover. The quantity of helium is so chosen that at the boiling point of oxygen the mercury stands at a mark in the lower portion of the stem, and at the melting point of hydrogen at one in the upper portion. If, as is the case with hydrogen boiling under ordinary atmospheric pressure, the temperature is sufficiently well known

¹⁾ Compare the apparatus for the liquefaction of helium. H. KAMERLINGH ONNES These Proc. May/June 1908, Comm. Leid, N^o. 108.

without reading the thermometer, the thermometer is still necessary, however, to indicate the position of the upper surface of the liquid gas which is no longer visible beneath A_3 . As soon as the level sinks below the upper end of the reservoir θ_1 of the thermometer, the mercury in the stem θ_4 sinks.

d. First the electromagnet is adjusted which operation is independent of the centring of the adjusting tube, the holder and the vacuum tube. The axis round which it turns is made vertical, and then the pole distance is centred round this axis. Next the centre of the truncated spherical socket G'_{10} (fig. 1, 2 and 3) is made to coincide with the axis round which the magnet turns. It is supported by a plate which is attached by two beams to the freestone pillar \mathcal{G} . The cryogenic apparatus is then brought from its auxiliary support and arranged in its proper position by placing the ball-shaped portion of the surface of the ring B_2 in the concentric socket G'_{10} ; the centring of the narrow portion of the vacuum tube on the turning-axis of the magnet is completed by means of wing nuts on the ring B_2 . This centring must be done with great accuracy, for the magnet must turn freely and the distance between the vacuum tube and either pole is not more than half a millimetre. It can, however, easily be accomplished to 0.25 mm.

e. Liquid hydrogen is introduced into the apparatus by a german silver tube B_7 (cf. Comm. N^o. 94 f). The gas formed by evaporation escapes through B_8 (figs. 3 and 1) and through the valves K_1, K_2 (fig. 1) to a gasometer or to a vacuum pump. By means of the valves the vapour pressure is regulated, and its value is read on a manometer H which at the same time acts as a safety valve. In experiments made in the neighbourhood of the melting point of hydrogen the pressure was kept slightly above that of the triple point.

Before introducing liquid hydrogen through the tube B_7 , which is closed by a rubber tube with a glass stopper, the air is pumped out of the apparatus through the valve K_2 . It is absolutely essential that the apparatus should be air tight, for traces of air would solidify in the liquid hydrogen and, owing to magnetic attraction, would collect in the neighbourhood of the ellipsoid.

To prevent the cooling of the upper portion of the apparatus containing the torsion spring by the boiling hydrogen, a number of large openings are made in the tube f_4 (fig. 3) arranged in such a way that no injury is done to its resistance to torsion. In addition to this copper screens surrounding f_2 and soldered to B_1 , are arranged

so that the tube moves with slight torsion in them. A little cotton-wool placed on the bottom of the vacuumglass and attached to the holder lessens the sudden bubbling¹⁾ up of the hydrogen¹⁾).

Further additions of liquid hydrogen are made in the same way as the first. As a rule various series of measurements could be made with a single filling with hydrogen. The point of the vacuumglass which could not be silvered was protected by a small silvered vacuum beaker L containing liquid air. When the portion of the apparatus above the diaphragms B_{10} is again at ordinary temperature after a filling with liquid hydrogen, one can hardly notice that there is liquid hydrogen in the apparatus at all, if it is not above A_3 .

In the course of time a little mist is precipitated on the vacuum tube. By surrounding the tube at A_1 with blotting paper, the moisture is prevented from trickling down between the pole-pieces. Furthermore a stream of air is directed against the tube between the pole-pieces. Hence the pole-pieces are in no way affected by the cryogenic operations.

f. The springs are phosphorbronze. This substance is non-magnetic and acquires very little permanent set. Springs of the same constant can be made by winding a spiral either of a thin short wire or of a much longer thicker one. Of the two, the one which has the greater mass will experience the smaller specific changes, and consequently will be the more perfectly elastic in working. This circumstance has been duly taken into account. The springs are provided with straight extensions in the direction of their axis and are connected with the holder and the rod k (fig. 3) by screws. The turns of the spirals do not touch each other. The temperature of the spring is measured by a mercury thermometer that is clamped against the cap D and with it is insulated with wool. The constants of the two springs used are 261000 and 22300 dyne-centimetres per radian. The corrections for the influence of the stretching wire and for the temperature change of the spring will be discussed in § 4.

The ellipsoids of iron, nickel and cobalt are 3 mm. long and 1,333 mm. thick. They have been made with great accuracy by the Société Génèvoise pour la Construction d'Instruments de Physique. They were turned under a microscope giving a 30-fold magnification and provided with a camera lucida so that the image of the object and an enlarged drawing could be superposed. Measurements with

¹⁾ Should this occur one must ensure that the oil of the damper is not cooled by the drops that are thrown up.

the dividing-engine have shown that the ellipsoids are very accurately shaped.

The iron was obtained by melting pure electrolytic MERCK iron contained in a magnesia boat in an electrical resistance furnace and in an atmosphere of nitrogen. The nickel and cobalt were prepared in the same way, starting with the purest possible nickel and cobalt powder specially prepared by MERCK for these experiments. The magnetite was obtained by constructing an approximate ellipsoid from a drop of very pure magnetite obtained by melting very pure MERCK sesquioxide in an oxy-hydrogen flame. Since experiment showed that it was only at very high temperatures that the last trace of oxygen was driven out and real magnetite¹⁾ obtained an iridium cupola was used for this operation.

Ellipsoids of approximate shape were also constructed from GOLDSCHMIDT chromium and manganese and MOISSAN vanadium. As can easily be seen it is not necessary for comparative experiments that the ellipsoids should be constructed with particular accuracy. This was, moreover, experimentally demonstrated for magnetite, of which various samples roughly worked to various ellipsoidal shapes were used for obtaining curves for the thermal change at high temperatures, and these curves were in agreement with the theoretical curve, and consequently with each other.

§ 3. *Experimental method.* As mentioned in the introduction our aim was not to obtain absolute values for magnetization in strong fields at ordinary temperature and at the temperature of liquid hydrogen, but to compare the values at these temperatures; for we might expect that the change would be only a small fraction of the quantity to be measured. Hence it was an obvious procedure to make observations at these temperatures alternately in the same field. The change, however, from the one temperature to the other necessitated operations of such duration as to prohibit the use of this method. Hence we usually began with a series of measurements at ordinary temperature, in which the field was made the required series of strengths. Then an analogous series of measurements was made at a low temperature, and after the apparatus had returned to ordinary temperature, some individual measurements were repeated so as to make sure that the apparatus had not in the meantime undergone any change.

¹⁾ See also P. WEISS. Arch des Sc. phys. et nat. fevr. 1910 and Journ. de physique, 4e Sér. t. IX mars 1910.

Each series of measurements consists in turn of two branches. First by tentative approximation from both sides for all values of the field those values of the azimuth of the electromagnet are found for which the couple is a maximum. In this way two azimuths are found which are symmetrical with respect to the major axis of the ellipsoid and which exert couples of opposite sign. This determination can be made accurately to within $0,5^\circ$ to 1° , which is quite sufficient. Then follows the true measurement in which the magnet without current is placed in one of these positions, the circuit is closed and immediately afterwards the deflection is read. As soon as this is done, the circuit is broken, the magnet is placed in the symmetrical position; once more the current is allowed to flow and the new deflection is obtained. Since these operations occupy only a short time, the after effects in the spring are of no account. The difference between the scale readings gives twice the value of the couple to be measured, independent of the residual magnetism remaining after the current was broken, which however occasioned only an extremely small couple. The field was given as a function of the current indicated by the ammeter. For these observations the same ammeter (SIEMENS and HALSKE instrument, no temperature coefficient) was used which was employed in the study of the field. This method of evaluating the field was quite sufficient for our purpose. The distance between the pole-pieces was read off the divided cylinders of the magnet and was verified by passing between them callipers which had been previously adjusted to the desired distance. The fields given above are corrected for the demagnetizing fields of the ellipsoids.

§ 4. *Corrections and controls; auxiliary measurements.* The inherent magnetism of the holder is not so weak that the corrections necessary for it may be neglected. On that account a series of measurements was made with no ellipsoid in the holder at ordinary and liquid hydrogen temperatures. With the weaker spring we found:

TABLE I.

Correction for the magnetism of the holder.

	ordinary temperature	$t = 20^\circ.2 \text{ K.}$
4000 gauss	0.18 cm.	0.26 cm.
8000	0.29	0.48
12000	0.36	0.61
16000	0.43	0.73
20000	0.50	0.86
24000	0.57	0.98

For the stronger spring these corrections are multiplied by $\frac{22300}{261000}$; they are very small. Direct measurements have shown that the values calculated in this way are correct, which indicates that the inherent magnetism of the carrier is not changed by the various operations of mounting.

There is still a correction to be applied to the couple-ratio for the change in elasticity of the steadying wire under the carrier when its temperature changes from ordinary to that of liquid hydrogen. To obtain that correction the ratio of the torsion modulus of the platinum iridium wire and that of the weaker of the phosphorbronze springs was measured at the two temperatures. This was done in an apparatus similar to the one we have described with the exception that the cap D could turn relatively to the cover. By a mirror method the position of the cap was read on a scale at a distance of 175.9 cm. The cap was turned through an angle of about 360° , and the exact measurement of the angle was obtained from the same scale. This angle is the sum of the torsions of the spring and the wire caused by the same couple. The torsion of the wire was read from the mirror of the holder. In this way the ratio of the modulus of the wire to that of the spring was found to be

0.0125 at ordinary temperature

0.0144 in liquid hydrogen.

The fourth decimal is uncertain; hence the correction is two thousandths for the weak spring and two ten-thousandths for the stronger. The temperature coefficient of the phosphorbronze spring was obtained from determinations of the period of oscillation of the same oscillating system while the spring was first at the ordinary temperature and then surrounded with steam. By means of the temperature coefficient thus determined viz.:

$$k = -0.00053$$

the observations are reduced to the same temperature.

The temperature of the liquid bath in the vacuum tube was proved to be constant to 0.1 degree, by carrying out temperature measurements with a platinum resistance thermometer placed at different heights in a similar vessel. When placed alongside the thermometer θ it indicated temperatures corresponding with those deduced from the vapour pressures.

Capillary action in the oil damper.

Care was taken to fill the oil vessel to such a height that the cylindrical ring carrying the vanes of the damper was partly immersed

in the oil so that the vanes were completely immersed and should experience no capillary action. But still we wished to know the order of magnitude of the forces brought into play by capillary disturbances; for this purpose we greatly magnified them. A damper as like ours as possible was filled only to such a height that the vanes and partitions intersected the surface of the liquid. The movable portion was suspended by a platinum-iridium wire 20 cm. long and 0,1 mm. thick; deflections were read from a mirror on a scale 2 metres away. The oil vessel was placed successively in two different azimuths such that the approach of the vanes towards the partitions would bring into play couples of opposite moments. The scale deflection was 5 cm. The moment of the couple is therefore of the order of two thousandths of that of the couple exerted on the nickel ellipsoid.

§ 5. *Details of the observations.*

Nickel.

The first series of measurements was made at 17°.2 C.

T A B L E II.

H (gauss)	I^2 (cm. of the scale)
2230	89.42
6250	89.97
10270	90.12
13280	90.34
17760	90.50
20300	90.66
21540	90.79
22760	90.81

The scale reading was always corrected for the ratio of the tangent of the double angle to the double angle of the deflection. The zero as determined by the mean of readings to left and right remained constant to a few tenths of a millimeter.

After this series the apparatus was accidentally damaged; it had therefore to be taken to pieces and remounted. That occasioned a small change in the magnitude of the deflections. Since the change of I^2 with H is determined by the foregoing series, only two points were subsequently determined at ordinary temperature before and after determinations in liquid hydrogen.

T A B L E III.

$t = 19^{\circ}.5 \text{ C.}$		Hydrogen at atm. pressure ($20^{\circ}.2 \text{ K.}$)	
H (gauss)	l^2 (cm. of the scale)	H (gauss)	l^2 (cm. of the scale)
	before	1780	93.57
16100	91.74	5410	100.49
20540	92.09	5050	101.54
	after	11830	101.84
16100	91.79	16100	102.13
20540	92.20	19050	102.34
		20540	102.51
		22020	102.48
		22840	102.49

The zero determined from the mean of readings to right and left changed by about 2 mm.

$$\begin{array}{l} \text{For } H = 16100 \text{ gauss } \frac{I_{20^{\circ}.2K}}{I_{19^{\circ}.5C}} = 1.0549 \\ \text{,, } H = 20540 \text{ ,, ,, } \frac{1.0547}{\text{mean } 1.0548 \text{ not corrected for dilation.}} \end{array}$$

Cobalt.

The measurements with cobalt did not lead to the desired result. It was the extreme difficulty of bringing the magnetization of cobalt to saturation encountered in preliminary experiments that had led to the choice of an apparatus of such small dimensions. For the other substances a weaker field would have sufficed, and hence a greater distance between the poles would have served.

In the observations at ordinary temperature something unexpected already happened. Although the mean of the readings to right and left ought to have given the zero-point of the apparatus, the point was actually observed to vary with the field. This change was afterwards seen to be about twice as great at low temperatures. The following figures bring this out clearly. (In the cobalt measurements the external field is given uncorrected for the demagnetizing field of the ellipsoid. When saturation is reached this is 5000 gauss).

TABLE IV.

Cobalt I at ordinary temperature.

H_e (gauss)	I^2 (cm. of the scale)	calculated zero	
4025	17.16	76.73	
8050	38.14	77.47	the observed
12075	50.48	78.96	zero was
19560	53.24	78.37	not
23340	53.29	78.18	recorded
25650	53.30	78.63	

Cobalt I at temperature of solidifying hydrogen (14°.3 K.).

4025	13.5	77.62	
8050	32.59	78.84	
15820	53.23	81.93	observed zero
19560	54.33	81.40	78.26
21800	54.43	81.16	
23340	54.45	81.02	
24760	54.46	80.08	

From this it appears that asymmetric disturbing forces affect the main phenomenon. It is probable that we are here dealing with phenomena of crystal magnetism arising from the fact that in the small ellipsoid the crystalline elements of the cobalt are not sufficiently numerous to realize isotropy by compensation. The magnitude and sign of these subsidiary actions are independent of the main phenomenon, and they can even be of opposite effect for both azimuths of the electromagnet; they can become of very great importance if the substance possesses a more or less pronounced magnetic plane, and the example of pyrrhotine shows us that their influence becomes greater at lower temperatures. Further, the law of approach to saturation in cobalt which differs from that which holds for the other substances is consistent with the existence of strongly developed magneto-crystalline phenomena¹⁾.

These experiments were repeated with a second cobalt ellipsoid, and the same asymmetric action, but somewhat weaker, was observed. But in this case a disturbance of another nature was encountered, which shows how concomitant disturbing phenomena may affect the measurement of magnetization: the magnetization at low temperature was now found to be apparently smaller than at ordinary temperature. The following table contains an extract from the results obtained with this ellipsoid.

¹⁾ P. WEISS, Arch. des Sc. phys. et nat. février 1910, Journ. de phys. mars 1910.

T A B L E V.

Cobalt II at $t = 18^\circ \text{C}$.

H_e (gauss)	I^2 (cm. of scale)	Calc. zero	Obs. zero
4025	20.33	77.56	
12075	54.16	76.49	77.70
23340	59.76	76.86	
25560	59.94	76.90	

Cobalt II in H_2 at atm. pressure ($20^\circ.2 \text{K}$).

15080	53.53	76.61	
23340	58.09	77.07	78.90
25650	58.46	77.21	

The same ellipsoid was removed from the carrier and replaced with Khotinsky cement; one could easily understand that very strong strain-magnetic phenomena might be occasioned by forcibly driving it into its mount. At the same time it was for the new experiment displaced through a different angle of rotation with respect to its major axis; by this operation the sign of the change of zero point as a function of the field was reversed.

T A B L E VI.

Cobalt II $t = 16^\circ.5 \text{C}$.

H_e (gauss)	I^2 (cm. of scale)	Calc. zero	Obs. zero
8050	40.99	79.12	
19560	56.48	78.80	79.45
23340	57.07	78.89	
25650	57.34	78.90	

Cobalt II in H_2 at atm. pressure ($20^\circ.2 \text{K}$).

8050	34.07	78.88	79.20
19560	53.21	78.24	
23340	54.26	78.33	
25650	54.63	78.42	

The only conclusion one seems to be able to draw from these experiments with cobalt seems to be that the increase in magnetization of cobalt between ordinary and liquid hydrogen temperatures is very much smaller than that undergone by magnetite and nickel, for, if this were not the case, the increase could not have been obscured by the disturbing influences.

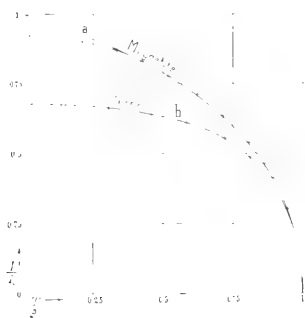
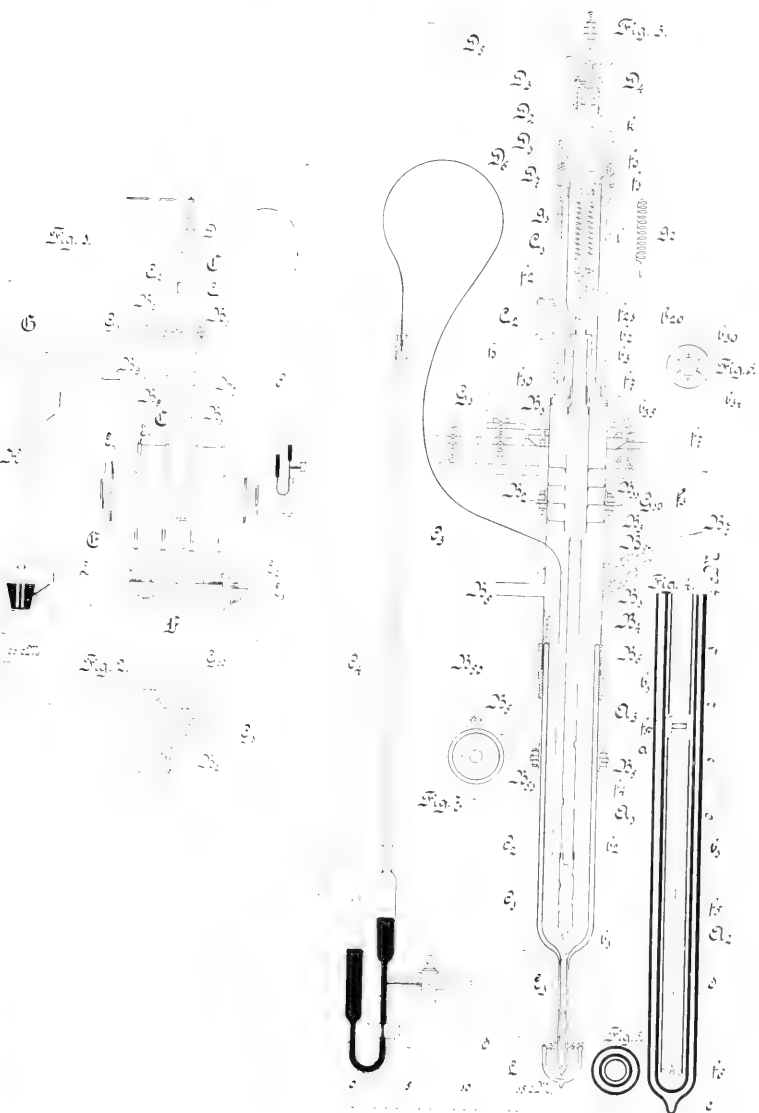


Fig. 1.



Fig. 2.



T A B L E VII.

<i>H</i> (gauss)	<i>Iron.</i>		
	<i>t</i> = 20° C.	<i>I</i> ² (cm of scale) <i>T</i> = 20°·3 K. (H ₂ atm. press.)	<i>T</i> = 14°·0 K. (H ₂ solidifying)
1700	95.23	101.98	101.95
5675	98.47	102.58	
8680	98.65	103.01	
13160	98.91	103.31	
15700	99.04	103.31	103.23
16940	99.08	103.27	
18360	99.06	103.27	
19250	99.07	103.25	103.25
	for <i>H</i> = 19250	$\frac{I_{20^{\circ} \cdot 3 \text{ K.}}}{I_{20^{\circ} \text{ C.}}} = 1.0209$	
	18360		1.0210
	16940		1.0209
	15700		1.0213
		mean not corrected for dilatation	1.0210

In all the iron experiments the zero as deduced by taking the mean of readings to right and left remained remarkably constant. As a rule its displacement was only a few tenths of a millimetre in any one series, and 6 mm. in proceeding from one series to another.

The few measurements at the temperature of solidifying hydrogen are sufficient to show that nothing particular happens between 20° K. and 14° K.

Magnetite.

We have already mentioned that the preparation of magnetite by heating the sesquioxide needs an extremely high temperature if one wishes to make sure that the last traces of oxygen are removed. A first ellipsoid obtained from iron oxide that had been insufficiently heated exhibited only little more than one half of the magnetization that was expected; it showed, too, a very distinct hysteresis, which was about three times as great at liquid hydrogen temperature as at ordinary temperature, while in all the experiments with the other substances hysteresis phenomena were insignificant. Moreover, the magnetization of this substance was the same at ordinary and liquid hydrogen temperatures, while between them it reached a maximum.

These peculiarities were not displayed by a second ellipsoid cut

out of well heated magnetite, but with this second ellipsoid further phenomena were observed which have not yet been explained but which seem to be of secondary importance. The zero point deduced from the mean of the two scale readings differed noticeably from the observed zero, while in any one series of measurements at the same temperature it remained practically constant. Further — and this is more worthy of notice — the deviations differ according to the direction of the field. It would clearly be very rash to attempt to ascribe to magnetite a hemimorphous symmetry like that of tourmaline from this sole observation. It seems more probable that some experimental error has here escaped our notice, and this can the more readily be accepted seeing that magnetite gives results much less regular than those of the metals. The following table contains an extract from the observations; the observations for the positive and negative directions of the field are given separately.

T A B L E VIII.

Magnetite. $t = 15^{\circ}.8 \text{ C.}$ Observed zero — calculated zero $\pm 0.9 \text{ cm.}$

H (gauss)	+ Field I^2 (cm. of scale)	-- Field I^2 (cm. of scale)
8600	71.40	71.72
18100	71.83	72.00
21800	71.75	72.57
23300	71.99	72.57
24200	71.77	72.45
H ₂ under atm. pressure (20 ^o .3 K.)		
8600	79.78	79.88
18100	80.69	80.79
21800	80.73	80.96
23300	80.34	81.10
24200	80.08	81.37
H ₂ solidifying (14 ^o .0 K.)		
18100	80.90	81.10
21800	81.12	81.64
24200	80.96	81.84

From these numbers follow these ratios of the intensities at 20^o.3 K. and 15^o.8 C.

TABLE IX.

<i>H</i> gauss	Field +	Field -
	$I_{20^{\circ}.3K.}$	$I_{20^{\circ}.3K.}$
	$I_{15^{\circ}.8C.}$	$I_{15^{\circ}.8C.}$
8600	1.0559	1.0553
18100	1.0591	1.0601
21800	1.0593	1.0567
23300	1.0564	1.0572
24200	1.0563	1.0628
	mean 1.0574	1.0564

Hence $\frac{I_{20^{\circ}.3K.}}{I_{15^{\circ}.8C.}} = 1.0569$ not corrected for dilatation.

Similarly for the ratio of the magnetization at $14^{\circ}.0 K.$ to that at $15^{\circ}.8 C.$ we find

$$1.0609 \qquad 1.0622$$

hence

$$\frac{I_{14^{\circ}.0K.}}{I_{15^{\circ}.8C.}} = 1.0616 \text{ not corrected for dilatation}$$

a ratio which deviates from the foregoing in the expected direction.

Collecting the foregoing results we find in this branch of the research for the ferromagnetic substances omitting the correction for dilatation (see note 2 pg. 11)

$$\text{Nickel} \quad \frac{I_{20^{\circ}.3K.}}{I_{17^{\circ}.3C.}} = 1.0548$$

$$\text{Iron} \quad \frac{I_{20^{\circ}.3K.}}{I_{20^{\circ}.C.}} = 1.0210$$

$$\text{Magnetite} \quad \frac{I_{20^{\circ}.3K.}}{I_{15^{\circ}.8C.}} = 1.0569.$$

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS OF THE MEETING
of Saturday March 26, 1910.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige
Afdeling van 26 Maart 1910, Dl. XVIII).

C O N T E N T S.

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- C VAN WISSELINGH: "On the tests for tanning in the living plant and on the physiological significance of tannin". (Communicated by Prof. J. W. MOLL), p. 685.
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- A. SMITS: "A new theory of the phenomenon allotropy". (Communicated by Prof. A. F. HOLLEMAN), p. 763. (With one plate).
- Erratum, p. 774.

Physiology. - "*Communications about the electrogram of the atrium cordis.*" By Dr. A. K. M. NOYONS. (Communicated by Prof H. ZWAARDEMAKER).

(Communicated in the meeting of November 27, 1909).

Involuntarily the ventricle-image, the tops *R* and *T* of which are prevailing, has, in the study of the electric phenomena of the heart, up till now been the principal subject. The top *P* by its smaller size drew less attention and at the outset was not even observed.

At present, however, there are in the literature already some data to be found here and there concerning the top *P*. Thus EINTHOVEN¹⁾ has pointed out how with increased action of the heart after great physical exertion *P* may gain in size, how under certain definite circumstances *P* may more or less be split up into a dim double-topped image, and besides how under pathological relations top *P* may be altered, which is demonstrated by cases of mitral stenosis. In this case of disease *P* would appear longer and enlarged, which EINTHOVEN thinks may be attributed to a stronger activity of the atrium for the sake of its compensative function. Division of the top may also appear. KRAUS and NICOLAI²⁾ have corroborated this find, just as SAMOJLOFF³⁾ and STESHINSKY⁴⁾, who have also been able to prove that the phenomenon is not pathognomonic, but depends on the relative welfare of the heart in case of mitral stenosis.

According to VAANDRAGER⁵⁾, top *P* in absolute measure would be higher with the dog than in man. With *N. vagi* cut through VAANDRAGER found in the dog that *P* grew three times its height and conversely could he make *P* smaller by stimulating the *N. vagi*. Besides this diminution he got at the same time an alteration in the form of top *P*.

With moderate bleeding of a sample-animal *P* increased in size, whilst after strong bleeding *P* grew smaller in the dog.

Top *P* was from the outset attributed by EINTHOVEN to the ventricles.

1) EINTHOVEN: See: *Onderzoekingen van het Physiol. Lab. te Leyden*. Second Series VII and the literature pointed out there.

2) KRAUS F. and NICOLAI G. F. Ueber das Electrocardiogramm unter normalen und pathologischen Verhältnissen. *Berl. klin. Wochenschr.* 1907 No. 25 and 26.

3) SAMOJLOFF A. *Electrocardiogramme*. Jena 1909. *Sammlung anat. und physiol. Vorträge*.

4) SAMOJLOFF A. und STESHINSKY. Ueber die Vorhoferhebung des Elektrokardiogramms bei Mitralstenose. *Münch. mediz. Wochenschr.* No. 38, 1909.

5) VAANDRAGER B. *Dissertatie* Leiden 1908.

The following grounds may be adduced for this, partially borrowed from my experiments:

1. *P* always appears with a definite interval of time before the mechanical change of the atria.
2. *P* continues existing at the registration of an isolated atrium (Rana, Emys).
3. *P* is absent when the electrogram is written of the isolated heart-ventricle of *Anguilla vulgaris*.
4. *P* continues existing when a certain detraction of the ventricle does not show itself. This may be observed both in the pathological heartblock and in the heartblock called into existence by experimental causes, among others:
 - a. by stimulating the N. vagus in the dog or the tortoise.
 - b. by administering toxic materials like chloroform.
 - c. by forming, resp. removing a ligature on the boundary of auricles and ventricles in Rana.
5. *P* may be made to disappear temporarily, when in appropriate sample-objects a heartblock is brought about by stimulating the N. vagus at the transition of the sinus to the atria, with which, it is true, the sinus-contractions are preserved, but the atrium-contractions with top *P* in the electrogram disappear.
6. *P* does not arise from the sinus, witness the fact that a small top may be registered before the appearance of top *P*, which may be attributed to the sinus.
7. Size and form of *P* depend upon the way in which the atrium is derived.

If we consider the electrocardiogram of man and animal superficially, we get the impression that *P* has a very simple form. Under quite peculiar circumstances this shape has been seen to alter. On closer investigation, however, it has appeared to me that the electric phenomenon of auricles is practically a whole complex. This becomes clear at the registration of an isolated pulsating atrium. Very fit for this purpose is one of the auricles of the heart of Emys.

Thus the adjoined figure 1 renders the electrogram obtained by derivation of apex and basis of an isolated right atrium of Emys with the appertaining myogram registered by simple suspension¹⁾.

This image, in many respects, makes us think of an electrogram

¹⁾ The registration of the electrograms was brought about by means of EINTHOVEN'S string-galvanometer (EDELMAAN'S small model) according to the method pointed out before; see: Proceedings of the Kon. Akad. v. Wetensch. 31 Oct. 1908.

of the ventriculus cordis, as it is to be registered with numbers of animals and with man.

The tops that are found in this atrioelectrogram are, as it were, analogous to the tops Q , R and P of the ventricle phenomenon and may respectively be called here P_z , P_{β} , P_c . The tops P_z and P_{β} , like Q and R , which are analogous to them, fall in the ventricle-image wholly before the commencement of the muscle-contraction. In the electrocardiogram of man and animals derived indirectly, we find, evidently on the ground of the P -elevation, only P_{β} expressed.

Again by derivation of a heart of *Rana* derobed from ventriculus, accordingly consisting only of sinus and atria, we get, a proportionately less large, but yet also a complicated electrogram of the atrium, as also by registration of the isolated atrium of the carp, where for example the first half of the atrioelectrogram shows a pronounced diphasic nature.

It is also possible with a sample-object to derive both the atria at the same time. The same thing I did also with the cut-out heart of *Emys* deprived of ventriculus, where one electrode invariably found a place on the backside of the sinus, whilst two other electrodes respectively caused the derivation of the atriatops. By means of a swing-apparatus the galvanometer was connected with the sinus and respectively with one of the atria or with both. The electrogram of the one atrium obtained in this way differs at such a derivation somewhat from the image got with the other atrium. The electrogram of the left atrium has a strong diphasic character; at derivation from the place of separation between the two atria with the back wall of the sinus, we get a less pronounced diphasic image, whilst the electrogram of the right atrium is only very feebly diphasic.

The action-current of the heart is considered as a summary utterance of the electric negativities, which show themselves in the tissue successively in different places and at different times. This negativity, as HERMANN formulated it, arises by the circumstance that every point of an irritable tissue at the moment of the stimulation stands in a negative relation to the parts that are in rest. At the derivation of such a tissue in different succeeding points we shall, therefore, every time get a deviating electrogram, but at the same time we shall be able to get an impression of the way which the proceeding stimulus has taken through the tissue. For this last purpose I have effected derivations of one atrium in different places, lying in regular order. In this experiment I made use of the heart of *Emys* deprived of its ventricle and then derived with an electrode constantly from the sinus, whilst the other electrode (store-electrodes with

movable pith) was placed: 1. at the apex; 2. $\frac{1}{2}$ cm. lower than the apex and 3. 1 cm. lower than the apex of the atrium. This took place both for the right and the left atrium, and also for the two atria combined.

Fig. 2 (*a*, *b*, and *c*) shows how the amplitude of top P_{β} from the atrio-electrogram diminishes in size as we descend with the electrode from the apex along the lateral side of the right atrium. The greatest potential difference, therefore, is manifest between basis and point, whilst each point of the atrium, lying lower than the apex, at derivation offers a smaller potential difference with the sinus. This is quite in accordance with the usual representation, at which it is supposed that, taking into consideration the fact that the contraction-stimulus arises from the sinus, the stimulus regularly goes on in the atrium tissue from the basis to the point of the atrium.

In the same figure 2*c* may also be demonstrated the appearance of a small elevation, with following slight fall in the electrogram, manifesting itself a full second before the commencement of the myogram of the atria.

This elevation may be attributed to the sinus, among others for this reason that this elevation in size increases according as we draw nearer to the sinus.

Already in a former communication I have alleged grounds to prove the independence of the electrical phenomena of the heart with respect to the changes of form. In the atria of *Emys* this quality can be demonstrated very clearly.

A part of the heart of *Emys* consisting of only atria and sinus is, isolated, brought into a gas-chamber, and derived in one place from the sinus, in another place from one of the two atrium-tops or from both tops at the same time. In this way different combinations of derivation may be brought about. The movements of the two atria are, by means of a simple suspension, registered by the silhouette of the little levers. If 2 cm³ of chloroform are administered which are evaporating in the gas-chamber, the mechanic movements are gradually growing smaller, so that at last they stop entirely, 11 minutes after the administration the chloroform. Also at an examination of the atria no trace of motion is to be observed, whilst however, the electric phenomena continue showing themselves periodically very clearly, through in shape they are a little more complicated than before the poisoning. In fig. 3*a* and fig. 3*b* the electrograms have been denoted as they are obtained by derivation of the sinus with the one electrode and derivation of the two atria-tops with a double other electrode. After evaporation of the chloroform by opening

the gas-chamber the atria begin to recover and after 39 minutes the electrical phenomena reach their original size again with their mechanical changes. Such poisoning-experiments can be repeated a few times without any great harm for the sample-object.

It is striking how the right atrium every time recovers first from the poisoning, and only later the left atrium begins to show mechanical changes. If the electrogram is examined during and after the poisoning, it is remarkable that the general form is not really altered here, but that the amplitude of the tops is greater than after the recovery of the poisoning, whilst the mechanical changes are altered in exactly the other way.

The form of the atrio-electrogram is evidently also dependent on nervous influences. EINTHOVEN and VAANDRAGER have already directed attention to this for the cardio-electrogram of a dog derived indirectly. For the atria of *Emys* this may be proved very distinctly at direct derivation under the influence of vagus-stimulation.

A heart of *Emys* derobed of ventriculus, so consisting only of the sinus and the two atria, is derived to the string-galvanometer. Derivation of one of the two atria separately or combined makes no particular difference. Therefore the derivation in the experiment takes place from sinus and right atrium-top. The right n. vagus in the neck is prepared free. The electrogram shows a fine double-topped image, accompanied by regular mechanical changes. At stimulation of the right n. vagus with induction-currents the mechanical utterance undergoes alterations. After a last contraction of the sinus, showing itself in a slight elevation in the myogram of the atria there begins for the object a vagus-standstill, which, as soon as the stimulus is put a stop to, is broken off and causes a new series of atrium-contractions with a strong sinus-contraction. During the vagus-standstill the object has produced no electrical phenomena that are to be registered. Directly after the vagus-standstill the atrio-electrograms show themselves again, but now altered in form. The double-topped image has been replaced by a phenomenon with a strongly pronounced diphasic character, which however, the stimulation being stopped, passes into the original double-topped image by a gradual alteration. In this experiment the n. vagus was stimulated by means of the sledge-inductorium of DU BOIS REYMOND, without a kernel, at a secondary coil-distance of 5.5 cm. and a LESSING-element in the primary chain.

When weaker currents are used for stimulation, a state of things may be obtained in which the vagus-standstill does not appear, but in which the peculiar alterations in the form of the electrogram of

the atrium show themselves, as they have been described above. These alterations in the form of the electrogram, apart from tonic changes, are not accompanied by changes in the motoric utterances. Fig. 4.

The tonic alteration cannot be considered as the cause of the changes in form of the electrogram, because, in using still weaker currents as stimuli for the n. vagus, the same tonic change may appear, without having any effect on the electric utterances of the atrium.

Botany. — “*On the tests for tannin in the living plant and on the physiological significance of tannin.*” By Mr. C. VAN WISSELENGH. (Communicated by Prof. J. W. MOLL.)

(Communicated in the meeting of February 26, 1910.)

In this paper a method will be described for demonstrating the presence of tannin in the living plant, a method which enables us moreover to obtain an idea of the amount of this substance in the living cells, and to ascertain whether after a given period of time the amount has increased or diminished: the method does not noticeably affect the living functions of the plant or damage the latter to an appreciable extent.

In addition a few results of experiments on the physiological significance of tannin will be communicated; these results are in my opinion a real contribution to our knowledge of this subject.

Before proceeding to a discussion of the method which I have worked out, I think it desirable to make some observations on the meaning of the word “tannin” and to give an account of the present state of the physiological tannin-problem.

As regards the meaning attached to the word tannin there is no uniformity. Botanists formerly meant by tannin every thing in the cell which was coloured blue or green by ferric salts¹⁾. This has led to confusion with other substances and to the view that tannin is a generally occurring constituent of plants. REINITZER²⁾ especially has drawn attention to this. As a result of his investigations he came to the conclusion that the word tannin is a misnomer, introduced into science from the leather industry. According to him it should

1) J. DEKKER, De looistoffen, Bot.-chem. monographie der tanniden, 1908, V. I, p. 197 and 210.

F. CZAPEK, Biochemie der Pflanzen, II. Bd. p. 576.

2) F. REINITZER, Bemerkungen zur Physiologie des Gerbstoffs, Ber. d. d. bot. Gesellsch. Bd. VII, 1889, p. 187.

again disappear from scientific terminology, but his suggestion did not receive any support. WAAGE¹⁾ especially has objected to it. With reference to this question DEKKER²⁾ rightly remarks in his botanico-chemical monograph of the tannins, that there certainly exist plant substances, which are sharply marked off from other carbon compounds by common characteristic properties, such as the property of transforming animal skins into leather, which depends on the property of forming with protein compounds insoluble in water, the adstringent taste, the presence of several phenolic hydroxyl-groups in the molecule, the power of precipitating alkaloids from aqueous solution and other properties; these substances must therefore be collected in a separate group. Until the chemical constitution of these substances is completely known, the group in which they are united, should not be split up.

Some authors, e. g. REINITZER³⁾ and BRAEMER⁴⁾ consider that a group of plant substances cannot be studied physiologically so long as our chemical knowledge of it is incomplete. WAAGE⁵⁾, in my opinion, is quite right in not agreeing with this. Of course it will be necessary in the physiological investigation of tannins to ascertain in each case with the means at our disposal, whether the plant under investigation actually contains a substance belonging to the tannin class, so that confusion with other bodies may be excluded.

The opinion of botanists concerning the physiological significance of tannins has always been much divided. TH. HARTIG⁶⁾ supposed that tannins contribute to the building up of the vegetable organism. SCHLEIDEN⁷⁾ on the other hand considered that tannin is only a decomposition product of the cell wall.

In agreement with HARTIG'S view tannin is, according to WIGAND⁸⁾, a real factor in the chemical process of plant life and belongs physiologically to the group of carbohydrates, on the formation and transformation of which the life process of the plant is especially based. In contradistinction to starch, which appears as reserve ma-

¹⁾ Th. WAAGE, Die Beziehungen des Gerbstoffs zur Pflanzenchemie, Pharm. Centralbl. f. Deutschl. N^o. 18, 1891, XII. Jahrg. N. F. p. 247.

²⁾ l. c. 1908, Vol. I. p. V; Vol. II. p. 66; Vol. I. pp. 211 and 212.

³⁾ l. c.

⁴⁾ L. BRAEMER, Les tannoïdes, 1890—91. Ref. Bot. Centralbl. Jahrg. XII, 1891, Bd. 47, p. 275.

⁵⁾ l. c.

⁶⁾ Th. HARTIG, Entwicklungsgeschichte des Pflanzenkeims, 1858, p. 103.

⁷⁾ M. J. SCHLEIDEN, Grundzüge der wissenschaftlichen Botanik, 1861, p. 141.

⁸⁾ A. WIGAND, Einige Sätze über die physiologische Bedeutung des Gerbstoffes und der Pflanzenfarbe, Bot. Zeitung, 20. Jahrg. 1862, N^o. 16, p. 121 and 129.

terial in the resting periods of vegetation, tannin generally belongs, according to WIGAND, to the fluid active substances necessary for growth. In some cases it appears, according to the same author, to act as reserve material. It thus follows that in WIGAND's opinion tannin is an extremely important product of vegetable metabolism. No other investigator has declared this so clearly and so emphatically.

WIGAND's view has been attacked, especially by SACHS, and has not received much support from botanists in general; this is evident, for instance, from the chapter "Die physiologische Bedeutung der Gerbsäuren" in CZAPEK's *Biochemie der Pflanzen* ¹⁾, where WIGAND's view and that of TH. HARTIG ²⁾ concerning "Gerbmehl" as carrier of tannin and organized reserve material is reckoned among the "irrigen Auffassungen über die physiologische Rolle der Gerbsäuren". WIGAND has not published the details of the observations on which his conclusions are based and this has probably contributed to the ready rejection of his results by other authors ³⁾.

The conception of the role of tannins arrived at by SACHS ⁴⁾ in his investigations on the germination of seeds, has received more support than that of WIGAND. SACHS considered the tannins formed in germination, to be merely excretory products, by-products or decomposition products. He thought it very improbable that tannins could serve in some way or other as material for the building up of cell-walls.

The results of some other observers agree with those of SACHS. Thus for instance KRAUS ⁵⁾, who was particularly interested in the physiological significance of tannins, arrived at the conclusion that tannin, once formed, in no case takes any further part in metabolism. According to GERBER ⁶⁾ the tannins disappear by oxidation, without the formation of carbohydrates from them. AF KLERCKER ⁷⁾ regards

¹⁾ l. c. p. 588.

²⁾ TH. HARTIG, *Das Gerbmehl*, Bot. Zeitung 23. Jahrg. N^o. 7, 1865, p. 53. Weitere Mitteilungen das Gerbmehl betreffend, Bot. Zeitung 23. Jahrg. N^o. 30, 1865, p. 235.

³⁾ Compare EMIL KUTSCHER, Ueber die Verwendung der Gerbsäure im Stoffwechsel der Pflanze, *Flora*, 66. Jahrg. N^o. 3, 4 and 5, 1883, p. 37.

⁴⁾ J. SACHS, Physiologische Untersuchungen über die Keimung der Schminkebohne (*Phaseolus multiflorus*), *Sitzungsber. d. kais. Akad. der Wiss. Wien*, 37. Bd., 1859, No. 17, p. 57. Zur Keimungsgeschichte der Dattel, Bot. Zeitung, 20. Jahrg., 1862, No. 31, p. 241 and 249. *Handbuch der Experimental-Physiologie der Pflanzen*, 1863, p. 360.

⁵⁾ G. KRAUS, *Grundlinien einer Physiologie des Gerbstoffs*, 1889, p. 38 and 44.

⁶⁾ C. GERBER, Rôle des tannins dans les plantes et plus particulièrement dans les fruits. *Compt. rend.* 124, p. 1106.

⁷⁾ J. E. F. AF KLERCKER, Studien über die Gerbstoffvacuolen. *Bihang till K. Svenska Vit.-Akad. Handlingar*, Bd. 13. Ald. III, No. 8, 1888. *Ref. Bot. Zeitung*, 47. Jahrg. 1889, p. 210.

tannins as excretion products. WAAGE¹⁾ calls them by-products of metabolism. BÜSGEN²⁾ insists, that the observations which have been made, afford no justification for the assumption that tannin acts as a plastic material. On the other hand SCHULZ³⁾ considers the tannin of evergreen leaves to play the part of reserve-material.

According to a few investigators tannins must, in some cases, be regarded as excretory products or as by-products of metabolism, whereas in other cases they take part in metabolism and serve as plastic material. Such was the conclusion of SCHELL⁴⁾, of KUTSCHER⁵⁾ and of WESTERMAIER⁶⁾.

According to SCHROEDER⁷⁾ the tannin of the birch and the maple is not a reserve material, but it is not an excretory product either. In the author's opinion it possibly in these cases constitutes a final product of metabolism. He does not, however, attempt to answer the question as to the physiological significance of tannin.

Many investigators have adopted the view, that tannins serve to protect plants against harmful external influences. These might be of very different kinds. STAHL⁸⁾ assumes that on account of its unpleasant taste tannin serves to protect the plants from the attacks of animals, especially slugs. KRAUS⁹⁾ also considers tannin to be a protective agent not exclusively against animals, but serving in addition to counteract the putrefaction of the plant.

In plants with evergreen leaves WARMING¹⁰⁾ regards the tannin content of the epidermis as protecting the plant from desiccation, while exposed to dangerous dry winds in winter, and as being at

¹⁾ l. c. p. 250.

²⁾ M. BÜSGEN, Beobachtungen über das Verhalten des Gerbstoffes in den Pflanzen, Jenaische Zeitschrift für Naturwissenschaft, 24. Bd., N. F. 17. Bd., 1890, p. 59. Erläuterung zu dem Referat über Beobachtungen etc., Bot. Zeitung, 1890, p. 381.

³⁾ E. SCHULZ, Ueber Reservestoffe in immergrünen Blättern unter besonderer Berücksichtigung des Gerbstoffes, Flora, 1888, p. 256.

⁴⁾ J. SCHELL, Physiologische Rolle der Gerbsäure, Kazan, 1874 (Russian), Botan. Jahresber. III. Jahrg., 1875, p. 876

⁵⁾ l. c. p. 73.

⁶⁾ M. WESTERMAIER, Zur physiol. Bedeutung des Gerbstoffes in den Pflanzen, Sitzungsber. d. königl. preuss. Akad. der Wissensch. zu Berlin, Jahrg. 1885, 2. Halbb. p. 1124 and 1125.

⁷⁾ J. SCHROEDER, Die Frühjahrsperiode der Birke (*Betula alba* L.) und der Ahorn (*Acer platanoides* L.), Die landwirthsch. Versuchs-Stationen, Bd. XIV, 1871, p. 146.

⁸⁾ ERNST STAHL, Pflanzen und Schnecken, Jenaische Zeitschrift für Naturwissenschaft, XXII. Bd., N. F. XV. Bd., p. 590 and 594.

⁹⁾ Grundlinien zu einer Physiologie des Gerbstoffes, 1889, p. 21.

¹⁰⁾ E. WARMING, Beobachtungen über Pflanzen mit überwinternden Laubblättern, Botan. Centralblatt, Jahrg. IV. Bd. 16, 1883, p. 350

the same time a means of rapidly restoring lost turgor. SCHELL¹⁾ considers that tannin in seeds is probably a protection against harmful influences from without. BÜSGEN²⁾ also supposes that tannin affords protection to the plant.

Other authors again have attributed different functions to tannins. According to GERBER³⁾ they prevent the transformation and fermentation of sugar in fruits and PFEFFER⁴⁾ thinks it very likely, that their role also consists in fixing sugars and other substances in the cell. KUTSCHER⁵⁾ considers it most plausible that tannin serves as a respiratory agent and is oxidized in respiration.

Various other functions have further been attributed to tannins in connexion with the metabolism of the plant. Thus WIGAND⁶⁾ supposed that the red colouring matters are formed, from tannins, a view shared by PICK⁷⁾, MIELKE⁸⁾, and TSCHIRCH⁹⁾ amongst others.

Some authors connect tannins with the formation of resin. WIESNER¹⁰⁾ thinks that starch and cell-wall may be transformed to tannin, and subsequently to resin. SCHELL¹¹⁾ and MIELKE¹²⁾ also regard tannin as an intermediate stage between starch and resin and between cellulose and resin. Both authors, however, also suppose, that tannin can be converted into starch. BASTIN and TRIMBLE¹³⁾ in their investigation of the resin-passages of conifers, have also received the impression, that tannin is connected with resin formation.

1) l. c. p. 877.

2) l. c. p. 58.

3) l. c.

4) W. PFEFFER, Über Aufnahme von Anilinfarben in lebende Zellen, Untersuchungen aus dem botan. Institut zu Tübingen, 2. Bd., 1886—1888, p. 310.

5) l. c. p. 73.

6) l. c.

7) H. PICK, Ueber die Bedeutung des rothen Farbstoffes bei den Phanerogamen und die Beziehungen desselben zur Stärkewanderung, Botan. Centralblatt, Jahrg. IV, 1883, p. 284.

8) G. MIELKE, Ueber die Stellung der Gerbsäuren im Stoffwechsel der Pflanzen, Programm der Realschule vor dem Holsthore in Hamburg, 1893. Ref. Botan. Centralblatt, Jahrg. XV, 1894, Bd. 59, p. 281.

9) A. TSCHIRCH, Schweiz. Wochenschr. f. Pharm. N. 7. Pharm. Centralbl. N. 10, 1891, p. 141.

10) J. WIESNER, Über die Entstehung des Harzes im Inneren der Pflanzenzellen, Sitzungsber. d. Wiener Akad., 1865, 52. Bd. II. Abt. p. 126 and 129. Ref. Jahresber. über die Fortschritte der Chemie etc., 1865, p. 627.

11) l. c.

12) l. c.

13) E. BASTIN and H. TRIMBLE, A contribution to the knowledge of some North Amerikan Coniferae, Amer. Journ. Pharm. 68, 1896.

According to BIGNET¹⁾ tannin in fruits contributes to the formation of sugar and according to STADLER²⁾ it supplies in the nectaries of *Oenothera* and *Saxifraga* the material for the formation of honey.

In connexion with the physiological significance of tannins in plant metabolism, I think it desirable to point out, what results botanists have arrived at with regard to the translocation and origin of tannins.

Some investigators suppose that tannin can be transported in the plant, namely KRAUS, MOELLER, and WESTERMAIER. According to KRAUS³⁾ tannin travels as such, MOELLER⁴⁾ thinks it probable, that carbohydrates are transported in the form of tannin compounds. WESTERMAIER⁵⁾ leaves it an open question whether tannin travels as such, and whether starch travels in the form of a soluble carbohydrate or in that of tannin.

The opinions of botanists are divided as to the origin of tannins in the plant. As was stated above, tannin is according to SCHLEIDEN⁶⁾ a decomposition product of the cell wall. According to TH. HARTOG⁷⁾ it arises from starch during the germination of *Quercus pedunculata*. Similarly according to SCHELL⁸⁾ tannin is formed from starch in the germination of the seeds of *Faba vulgaris* and *Pisum sativum*. MIELKE⁹⁾ supposes that tannin is formed from carbohydrates, from tannin glucosides, and also from starch and from cellulose. WESTERMAIER¹⁰⁾ regards it as an assimilation product but supposes that it is also formed by the decomposition of proteins. According to SCHROEDER¹¹⁾ it is formed by oxidation from organic material present in the plant. KRAUS¹²⁾ thinks that it is a decomposition product of amido-compounds formed during the synthesis of proteins. The observations of MOELLER¹³⁾

1) H. BIGNET, Recherches sur la matière sucrée contenue dans les fruits acides, son origine, sa nature et ses transformations, Compt. rend. 51, p. 894.

2) S. STADLER, Beiträge zur Kenntniss der Nectarien und der Biologie der Blüten, Berlin 1886.

3) G. KRAUS, Grundlinien zu einer Physiologie des Gerbstoffs, 1889, p. 20.

4) HERMAN MOELLER, Anatom. Untersuchungen über das Vorkommen der Gerbsäure, Ber. d. deutschen bot. Gesellsch. Bd. VI, p. LXXX.

5) M. WESTERMAIER, Neue Beiträge zur Kenntniss der physiologischen Bedeutung des Gerbstoffes in den Pflanzengeweben. Sitzungsber. d. königl. preuss. Akad. d. Wiss. zu Berlin. Jahrg. 1887. 1. Halbb. p. 134.

6) l.c. p. 141.

7) l.c. p. 102.

8) l.c.

9) l.c.

10) Zur physiol. Bedeutung des Gerbstoffes in den Pflanzen, l.c. p. 1124.

11) l.c. p. 146.

12) Grundlinien, p. 47.

13) MOELLER, Mitt. des naturw. Vereins f. Neu-Vorpommern und Rügen in Greifswald, 1887.

on the leaves of *Ampelopsis hederacea* and those of BÜSGEN¹⁾ on germinating seeds of *Vicia Faba* and on wounded leaves which were floated on a 10 percent grape sugar solution, have proved that tannin is formed from sugar.

KRAUS²⁾ and WESTERMAIER³⁾ have pointed out that in some cases the formation of tannin depends on the influence of light.

It is evident from the above, that botanical opinion is much divided on the subject of the physiological significance of tannins. It may be summarized as follows. According to some botanists tannins are of no value to the plant; they are merely excretory products. Others regard tannins as protective agents against various harmful external influences. A few believe that tannins contribute to the building up of the vegetable organism. A small number think that tannins can fulfil different functions.

Various authors e.g. CZAPEK⁴⁾ in his *Biochemie der Pflanzen* and DEKKER⁵⁾ in his *Botanisch-chemische Monographie der Tanniden*, have pointed out that the numerous investigations on the physiology of tannins have as yet produced but few results of any importance. DEKKER arrives at the conclusion, that if this group of substances is of significance to the plant, which he thinks probable, it is quite uncertain what function they fulfil. NOLL⁶⁾ in STRASBURGER'S *Lehrbuch der Botanik* expresses himself in the same way. The fact that the significance of the tannins is still so obscure, is attributed to various causes. Thus according to CZAPEK⁷⁾ a few observations of microscopical or chemical facts led to generalisations and to the construction of untenable theories. DEKKER⁸⁾ further points to the imperfection of the methods of investigation and the one-sided use of these methods, which sometimes causes tannins to be confused with other plant substances.

In my opinion the chief cause must be sought in the want of criticism, which often impairs the drawing of conclusions. The physiological tannin problem is most certainly a very difficult problem, the answer to which will have to take into account a large number of factors. These factors are known to us to a smaller or larger extent,

¹⁾ l. c. p. 34 and 35.

²⁾ Grundlinien, p. 20 and 44.

³⁾ Zur physiolog. Bedeutung des Gerbstoffes in den Pflanzen, l. c. p. 1117. Neue Beiträge etc. l. c. p. 128 and 133.

⁴⁾ l. c. p. 588.

⁵⁾ l. c. V. I, p. 220.

⁶⁾ l. c. 8. Aufl. 1906, p. 190.

⁷⁾ l. c. p. 588.

⁸⁾ l. c. V. I, p. 210 and 211.

but unknown factors may also come into play. Hence it is necessary to exercise the greatest caution in drawing conclusions. In advancing an explanation of an observed phenomenon, we must consider carefully whether it is the only one possible and we must attempt to prove it in various ways by means of comparative experiments. The extent to which these precautions have been observed by no means corresponds to the complexity of the problem. As a result of a few experiments many observers have put forward certain explanations, when other explanations were equally plausible, or they have combated the opinions of other investigators, who perhaps had a more correct insight, although they were unable to adduce sufficient proof for it. Even serious investigators have made this mistake. I will illustrate this very briefly, by showing the insufficiency of the reasoning which led to the rejection of the possibility that the tannins might serve as plastic material.

As was said above SACHS does not believe that tannins can act in any way as plastic material in the formation of the tissues. This opinion he has partly supported by observation and partly by drawing what is in my opinion an erroneous conclusion. SACHS¹⁾ found that in the germination of seeds which do not contain tannins in the endosperm or in the embryo, tannins are formed in metabolism and primarily there, where the formation of tissue has just started. He never saw the tannins diminish or disappear during germination. In other cases, namely in that of the acorn and of the chestnut, where the embryo contains tannin, he did not observe a diminution either, but rather an increase. He made similar observations on the development of buds. SACHS concludes from the above-mentioned facts, that tannins remain in plants in the places where they have been formed, and that therefore they do not take part in the formation of tissues, for if this were the case, a diminution would have been observed. I consider this conclusion to be incorrect. Quite a different conclusion might equally well be based on SACHS' observations, namely that the frequent appearance or presence of tannins in tissue-formation shows that these substances have probably a function to perform in this process. Nevertheless I do not at all consider that SACHS has proved, that tannins remain in the places where they are formed and that they do not serve as plastic material in tissue formation. For if in the germination of seeds more tannin is formed than is decomposed, a diminution of the tannin content need not occur, and an increase

¹⁾ Physiolog. Untersuchungen über die Keimung der Schminkebohne, l. c. p. 111. Zur Keimungsgeschichte der Dattel, l. c. p. 246. Handbuch der Experimental-Physiologie der Pflanzen, 1865, p. 361.

may even take place. Reserve-materials like starch and fatty oils may not be assumed to participate directly in the building of the cell wall. They must first be converted into soluble substances. Now suppose that tannins also belong to this category, i. e. to such a plastic material as is present in the plant in a dissolved state, then it is not at all surprising that for the maintenance of growth plenty of this material should always be present, and that occasionally, when more of it is being produced from the reserves than is used up in the growth of the cell-walls, the tannin content increases. Anyhow it has not been proved that, because the tannin does not diminish, it remains unused at the place, where it has been formed, and that it does not serve for the building up of cell walls.

Like SACHS, KRAUS¹⁾ also assumes that an increase in the tannin content in germination proves, that this substance is not used up and does not serve as building material. Thus with regard to the germination of the acorn KRAUS states, as a result of quantitative tannin determinations, that not only is tannin not used up, but that its quantity even increases, so that it cannot be of service in growth.

Whereas SACHS only observed an increase of the tannin content of germinating seeds, SCHELL²⁾ found in some plants an increase and in others a decrease or disappearance. In the first case SCHELL supposes, in agreement with SACHS, that the tannins are by-products of metabolism, but in the latter case he regards them as plastic material. With reference to what has already been said, it is a matter of course that I cannot either agree with SCHELL's conclusions. In my opinion it is not necessary to conclude, on the ground of an observed increase in the tannin content in some cases and a decrease in others, that tannins behave so differently in different plants. Supposing the tannin to be a plastic material in both cases, then the occurrence of an increase or decrease will depend on the quantities produced and used up. I also think it very plausible that in one and the same plant sometimes an increase, and sometimes a decrease takes place, according to circumstances.

Several botanists suppose that tannins can undergo translocation in the plant. How this might happen is still a moot point, but there can be no doubt that the possibility of translocation greatly complicates the question of the use of tannins as plastic material. The increase or decrease of the tannin content of a particular organ would then not depend wholly on production and consumption, but transport

¹⁾ Grundlinien, p. 38.

²⁾ l.c. p. 876.

to and from the organ would also have to be reckoned with. The mere increase or decrease of tannin in a seedling or a vegetable organ will not supply data of any value for the solution of the problem of the significance of tannin as a plastic material.

Hitherto botanists have chosen the higher plants for the study of the physiological significance of tannins. For the study of complicated vital processes and of the physiological significance of chemical constituents certain lower plants appear to me to offer advantages above those of the higher ones, the structure of which is so much more complicated. For such an investigation the thicker species of the genus *Spirogyra* seem particularly suitable. It is true that the tannin of *Spirogyra* has not yet been examined chemically, but numerous microchemical reactions allow us to conclude with a fair degree of certainty that *Spirogyra* contains in its cell sap a considerable quantity of tannin. DE VRIES¹⁾ has proved this, after abnormal plasmolysis, with various tannin reagents e. g. ferric salts, potassium bichromate, osmic acid. In addition to these reagents many others also give in the cell sap precipitates which agree completely with those caused in tannin solutions.

The advantages which the thicker species of *Spirogyra* have over the higher plants are the following. Pieces of the filaments may be examined microscopically without killing them or damaging them, and the changes in the cells can be studied in the living plant. They are particularly suited to all sorts of experiments. They are not too small to be handled and not too thick for microscopic examination. The various constituents of the cell can readily be observed under the microscope. As in the case of unicellular Algae a transport of foodmaterial from one cell to another is very probably excluded in *Spirogyra*. This important factor, which must be taken into account when dealing with the higher plants, need generally not be considered in the case of *Spirogyra*. By means of the centrifuge all sorts of abnormalities may be obtained, such as polynuclear cells, cells without nucleus, cells with a large and with a small chromatophores-mass, and even cells without chromatophores. In this way we can eliminate the assimilation process, i. e. the intake by the chromatophores of carbon from atmospheric carbon dioxide under the influence of light. With *Spirogyra* a number of comparative experiments may be made which are impossible in the case of the higher plants, and because

¹⁾ HUGO DE VRIES, Plasmolytische Studien über die Wand der Vakuolen, Pringsh. Jahrb. f. wissensch. Botanik, Bd. 16, 1885, Heft 4, p. 575. Over looistofreactiën van *Spirogyra nitida*, Maandblad voor Natuurwetenschappen, 1885, N^o. 7, Reprint p. 7.

certain factors are excluded or eliminated, others may be studied with a greater chance of success.

Because the investigation of the higher plants has yielded such unsatisfactory results for the knowledge of the physiological significance of tannins I have attempted to obtain more definite results for the solution of this problem in the case of the lower plants, particularly of *Spirogyra*; to this I was led by the above considerations.

The first question to present itself was, which method would be most satisfactory. In the case of the higher plants investigators have followed various methods. Of the many reagents which give precipitates or colour reactions with tannins, ferric salts and potassium bichromate have mostly been preferred. Potassium bichromate especially, which yields with tannins a reddish brown or orange precipitate, has often been used, e. g. by SCHROEDER ¹⁾, SCHELL ²⁾, KUTSCHER ³⁾, RULF ⁴⁾, SCHULZ ⁵⁾, MOELLER ⁶⁾ and BÜSGEN ⁷⁾. KUTSCHER made a dish with 8 sections, the colour of which agreed with that of the precipitate, but shaded in such a way that the intensity of the colour in two successive sections always differed by the same amount. This dish was used for the determination of the strength of the precipitates.

KRAUS ⁸⁾ determined the amount of tannin by means of titration with potassium permanganate or precipitated the tannin with cupric acetate and weighed the precipitated copper as copper oxide. The titration with potassium permanganate was also employed by RULF ⁹⁾.

These titrimetric and gravimetric methods cannot, of course, be applied to a small object like *Spirogyra*; moreover no method satisfied the demand which I had imposed upon myself. I desired a method which would enable me to determine the tannin content of one and the same cell at different periods, with sufficient accuracy to allow me to decide whether an increase or decrease had taken place, and this without killing the cell or harming it appreciably. The want of such a method had made itself felt in the investigation of various abnormal cells such as polynuclear and anuclear ones,

1) l. c. p. 140.

2) l. c. p. 873.

3) l. c. p. 38 and 39.

4) P. RULF, Ueber das Verhalten der Gerbsäure bei der Keimung der Pflanzen, Zeitschrift für Naturwiss. in Halle, LVII. Bd. Vierte Folge Bd. III, 1884, p. 42.

5) l. c. 227.

6) HERMANN MOELLER, Anatomische Untersuchungen über das Vorkommen der Gerbsäure, Ber. d. deutschen botan. Gesellsch., Bd. VI, 1888, p. LXVI.

7) l. c. p. 13.

8) Grundlinien, p. 61.

9) l. c. p. 42.

and cells containing many, few or no chromatophores. While I could determine the growth of such cells by measurement and could deduce from the size of the starch foci whether the starch content had increased or decreased, I was unable to obtain for one and the same cell an idea of the tannin content during the various periods of its existence. The usual reagents only permit of a single examination being made, because during it the cells are killed. I had therefore to look for another method.

I wondered whether methylene blue might perhaps satisfy the above requirements. According to PFEFFER¹⁾ this substance forms a compound with tannin in the living cell, and this compound separates as a fine blue precipitate. For various physiological investigations PFEFFER strongly recommends aniline dyes particularly methylene blue. Of this he says i. a. the following²⁾: "In allen Fällen werden also Methylenblau und andere Farbstoffe wertvolle Reagentien sein, mit deren Hülfe, ohne Schädigung Aufschlüsse über Vorkommen und Verteilung gewisser Körper in der Zelle zu erhalten sind. Mit solcher vielseitig ausnutzbaren Methode lässt sich unter richtiger Erwägung nach vielen Richtungen hin eine Kontrolle des jeweiligen Zustandes des Zellsaftes und der Veränderungen dieses im Laufe der Entwicklung erreichen." When dilute solutions are used, the penetration of methylene blue into the body of the plant and its accumulation in the cell sap continue, according to PFEFFER³⁾, without any harm to life and even when complete saturation has taken place, it is still innocuous. *Spirogyra* was one of the objects with which PFEFFER experimented.

PFEFFER'S experiments were repeated by me a few times with *Spirogyra maxima*, but with very unsatisfactory results. Even after several days only a slight granular precipitate was obtained in the cells, and at least the greater part of the tannin remained in solution; moreover, even very dilute solutions were found to be harmful. I cannot therefore agree with PFEFFER in praising his method of investigation, and after this disappointment a better method was sought.

Preliminary experiments were carried out on *Spirogyra maxima* with various tannin precipitants, such as alkaloids, antipyrine, ammonium vanadate and many others. Of all the substances examined, caffeine and antipyrine were found to be the least harmful, and therefore the action of these two substances was investigated more

¹⁾ l. c. p. 190.

²⁾ l. c. p. 191.

³⁾ l. c. p. 195, 196 and 197.

closely, in order to ascertain their value for the study of the physiological tannin problem. In doing this, special attention was directed to the following points: whether the substances penetrated rapidly into the cell and the cell sap, whether the tannin was completely precipitated, and what concentration of solutions was required for this; the nature of the precipitate and whether it redissolved on removal of the precipitant, whether the strength of the precipitate corresponded to the quantity of tannin in the cells and whether the method was sufficiently innocuous. After a number of experiments with antipyrine and caffeine solutions of various concentrations, which were allowed to act for a longer or shorter time, I came to the following conclusion:

The antipyrine and caffeine solutions penetrate rapidly into the cells and in sufficient concentration produce in the cell sap a precipitate, consisting of minute grains or globules, which are in constant motion to and fro. In order to precipitate the tannin as completely as possible, it is desirable to have the antipyrine solutions not more dilute than 1% and caffeine solutions not weaker than $\frac{1}{10}$ %. The greater the tannin content, the heavier the precipitate. Not infrequently the precipitate is so heavy, that the nucleus, which ordinarily can be readily discerned in *Spirogyra maxima*, cannot be distinguished at all and sometimes the precipitate is even heavier. If the *Spirogyra* filaments are placed in ditch water or in distilled water, the precipitate disappears in a short time, say in 10 minutes, and the *Spirogyra* threads are as before the experiment. No change whatsoever can be detected. If the *Spirogyra* filaments remain in the solution, the precipitate settles down and the small globules or spheres, of which it consists, gradually coalesce to larger globules, which appear perfectly colourless and may sometimes be very large closely resembling fat globules. This was generally the appearance of the precipitate after a few days. The settling down of the precipitate in the cells and the fusion of the globules to larger, purely spherical masses, proves that it is heavier than water and that it is liquid. From several data I deduce that it is not thinly liquid but viscid. The fusion to larger globules proceeds slowly and cannot, for instance, be brought about by a few minutes centrifuging. When the *Spirogyra*-cells with the globular precipitate are placed in water, the globules dissolve. Solution takes place more slowly, however, than in the case of a recently formed and still finely divided precipitate. If the preparations are placed in ferric chloride solution, instead of in water, the globules are coloured blue, while the cell sap is not coloured. It is rational to use caffeine as precipitant for this experiment, since antipyrine gives

a reddish-violet coloration with ferric chloride. Since the coloured compound is soluble and easily diffuses through the preparation, the ferric chloride-tannin reaction of the globules may also be detected when antipyrine is used, and the non-appearance of the reaction in the cell-sap may be observed, at least when the ferric chloride acts sufficiently rapidly. If the preparations are transferred from the antipyrine- or caffeine solution to a one percent solution of osmic acid, the globules are first coloured blue and soon afterwards black, whereas the cell sap remains colourless.

It is evident from the experiments with ferric chloride and with osmic acid, that the tannin is completely or almost completely precipitated by a one percent antipyrine solution and by a 0.1 percent caffeine solution, for otherwise the cell sap should have shown a blue or black coloration. If the antipyrine or caffeine precipitate, whether it be a finely divided recent precipitate or one which has fused to globules, is dissolved by placing the *Spirogyra*-filaments in water, and if ferric chloride- or osmic acid solution is then added, the cell-sap is coloured blue or black, just as is the case with cells which have not been treated with antipyrine- or caffeine solutions. When the cells finally die off in antipyrine- or caffeine solution, the globules are stained brown; their solubility in water has then decreased, but they still give with ferric chloride and osmic acid the reactions referred to.

By means of comparative experiments with antipyrine- and caffeine solutions, and various other tannin reagents, such as potassium bichromate, osmic acid and ferric salts, with *Spirogyra* cells containing a varying amount of tannin, I was able to show that the strength of the antipyrine- and caffeine precipitates agreed with the strength of the precipitates and colorations, given by the above-mentioned reagents. For these experiments I used *Spirogyra* filaments, which had been centrifuged a few weeks before, and in which there were also all sorts of abnormal cells, such as cells without a nucleus, without chromatophores, with several nuclei etc. The tannin content of the cells of these filaments varied very much. First the filaments were treated with antipyrine- or caffeine solution and the strength of the precipitate in the various cells was noted; afterwards the filaments were placed in water, and when the precipitates had dissolved, they were placed in a solution of potassium bichromate, osmic acid or ferric chloride, and the intensity of the reaction in the various cells was noted. On comparing the various notes it was found that the strength of the antipyrine- and caffeine precipitates agreed with the intensity of reaction obtained with the other reagents, and therefore corresponded to the quantities of tannin present in the various cells.

The strength of the precipitates with antipyrine and caffeine was judged in various ways. Thus it was noted, whether the nucleus, which in normal circumstances is very clearly visible in *Spirogyra maxima*, could still be distinguished after precipitation of the tannin. Furthermore it was noted whether the suspensory threads, the chromatophores and the starch foci above and below in the cell could still be discerned. In order to judge in which cells the precipitates were strongest, the various cells were not only compared after precipitation, but it was also noted in which cells the precipitate first appeared and remained visible for the longest time after the filament had been transferred to water. I had previously found that the precipitate first appeared in the cells with the largest tannin content and that after the filaments had been placed in water, it could be observed in these cells for the longest time.

In connexion with the use which I wished to make of antipyrine- and caffeine solutions, it was very important to know to what extent these solutions are harmful to life and whether a short stay in these solutions, sufficiently long to obtain an idea of the tannin content, might be regarded as harmless or practically harmless to the *Spirogyra* filaments. I found that, if a one percent solution of antipyrine, or a $\frac{1}{10}$ percent solution of caffeine were used, made up with ditch water or with distilled water (a solution of such concentration therefore that all or nearly all the tannin was precipitated in the cells) and that if the *Spirogyra* filaments remained in this solution, no further divisions took place and growth was soon arrested or was stopped at once. If, on the other hand, solutions were used which were ten times as dilute, and which did not cause a precipitate in the cells, it was found by comparative experiments with *Spirogyra* filaments in ditch water or in distilled water, that growth was retarded by antipyrine and by caffeine, and that fewer nuclear and cell divisions occurred.

I made some experiments with a one percent antipyrine solution and with a $\frac{1}{10}$ percent and a one percent solution of caffeine, in order to see whether a daily sojourn of 10 minutes in these solutions was harmful to *Spirogyra*, grown in ditch water. A period of 10 minutes was selected because it is sufficient for an examination of the tannin content. The result of these experiments was, that it could not be ascertained with certainty whether the procedure employed was harmful to the *Spirogyra*. Sometimes the growth of the controls in ditch water was the stronger, sometimes that of the filaments which had been periodically treated with antipyrine and caffeine solutions. It is not improbable that the differences observed depended

largely on the nature of the cells under investigation. I surmise this because the growth of cells in normal and equal nutrient solutions also showed differences. We may deduce from the results that in general a short daily stay in the various solutions has at most a slight influence on the growth and the vital processes of *Spirogyra*. The above method of investigation of the tannin content may therefore be strongly recommended, especially when it is desired to examine the same cells repeatedly at intervals, without harming them.

As far as I have been able to ascertain, antipyrine- and caffeine solutions have not yet been employed as microchemical tannin reagents. For the sake of completeness I point out, however, that such solutions have already been used by botanists in microchemical investigation, namely by LOEW and BOKORNY¹⁾, to demonstrate the presence of non-organized active protein in the living cell. The above-mentioned reagents are supposed to separate this in the shape of small globules, called by these authors proteosomes. This is therefore an explanation of the phenomenon produced by antipyrine or caffeine in the living cell, which is totally different from that given by myself. As a result of my investigations described above, I adhere to my opinion that antipyrine- and caffeine solutions are valuable tannin reagents, and suppose that LOEW and BOKORNY have given an inaccurate explanation of the phenomenon which they observed.

In the historical survey I pointed out, that, as regards the physiological significance of the tannins, there is a great difference of opinion among investigators, and that in the opinion of various botanists, there is but little, which may be regarded as sufficiently proved, so that we are here face to face with a problem, which has in no way been solved. As was stated above the view that tannins might serve in the formation of cell walls has received little support and met with much opposition. With the aid of the method I have worked out, I have now been able to bring to light facts concerning *Spirogyra*, which indicate that tannin plays an important part in the formation of cell walls, and that during this process tannin is used up, so that it very probably serves as building material. Below I will mention some observations which relate to this. They refer in the first place to the conjugation.

Cells which showed a tendency to conjugate, I found to be richly provided with tannin. I could make out, that the tannin content diminished during conjugation and in the adult zygospores which were filled with reserve material, I could only occasionally observe

¹⁾ O. LOEW and TH. BOKORNY, Versuche über aktives Eiweiss für Vorlesung und Praktikum, Biologisches Centralblatt, 1891, XI, p. 5.

a feeble tannin reaction with ferric chloride. It does not result from this observation what is the fate of the tannin, but when the conjugation is followed in greater detail, it is found that there is good ground for supposing, that at least a portion of the tannin serves as plastic material for the cell wall. Conjugation is a process which proceeds in such a way as to allow us to expect that its study in connexion with the point of investigation referred to will furnish us with important data, for conjugation does not start simultaneously in all cells. Some cells are in advance of others; in a smaller or larger number of cells there is evidently a tendency to conjugate, but the conjugation does not succeed, and other cells again do not show a trace of the conjugation process. Whereas the conjugating cells form much reserve material as starch and fat, those which do not conjugate are apparently very poor in contents and they finally perish. The above mentioned differences seem to be determined by accidental circumstances such as the coming into touch with cells of other filaments, the proximity of such cells and the position of the cells with regard to each other. They may even be observed with material which before conjugation consists exclusively of healthy normal cells.

The point of interest in connexion with the tannin problem is the possibility of comparing, in conjugating *Spirogyra* filaments, cells which a short time before were quite equal and afterwards show more or less important differences, induced by accidental and rather superficial circumstances. It is of interest to trace in these various cells what happens to the tannin content. This was investigated with the caffeine- and antipyrine solutions I have recommended, and it was striking to note, how differences in the development of the cell wall corresponded to the quantity of tannin present in the cells. Thus I could ascertain, that in cells where the lateral protrusion and mutual fusion had taken place, the tannin content was always appreciably smaller than in cells which only showed the first beginnings of the lateral protrusion. These two kinds of cells only differed as regards cell wall and tannin content; for the rest they still agreed perfectly. They were distributed promiscuously over the filaments, as is usual in conjugation. These facts seem to me to prove that there is a connexion between formation of the cell wall and the tannin content, and the supposition, that tannin serves as plastic material for the cell wall is very plausible.

Furthermore there is a remarkable increase in the tannin content of those cells which have not had an opportunity of conjugating or in which the process was interrupted at an early stage; these cells

degenerate and are generally described as having a poor cell content. These cells continue to produce tannin for some time and since the tannin in them is not used up in the formation of cell walls or reserve material, the tannin content increases and on the death of these cells a considerable quantity of plastic material in the form of tannin is lost.

The loss of tannin in nature, e. g. in the fall of leaves in autumn, has repeatedly been used as an argument for the view that tannin cannot be a plastic material and does not take part in metabolism. I cannot share this view and do not think the waste of quantities of a substance, which certain plants require for their development, to be at all strange, and certainly not a proof that it cannot serve as plastic material in the development of the plant. How often do things in nature fail to attain their end and how many are not wasted without being able to fulfil their purpose! Moreover, it seems to me desirable that the plant should have an excess of plastic material at its disposal, in order that development may never at any time be hindered for want of it. The fact that in the autumn the stem is unable to take up all the tannin from the leaves, or all that remains in the leaves from former abundance, hardly proves that tannin cannot serve to build up the tissues. Still less need we wonder at the waste of tannin in *Spirogyra*, for evidently it is here not the intention of nature that it should be wasted. Nature ensures a sufficient supply of tannin in *Spirogyra*, because this substance is required in development, as for instance in conjugation and spore-formation. The occasional failure to conjugate, as a result of which then much tannin is lost, does not prove that it is a waste product and not a plastic material.

A second series of observations, which show that tannin plays a part in the formation of the cell wall, relate to the formation of transverse walls. On investigating *Spirogyra* filaments containing cells undergoing division, it at once struck me that the tannin content of these cells is somewhat smaller than that of other cells, not undergoing division. The difference was not large and perhaps, even escapes detection by some of the tannin reagents which have been used hitherto, such as ferric salts and potassium bichromate, but with antipyrine- and caffeine solutions the existence of a difference in the tannin content could be established with certainty. Not only was it clear that the precipitate with antipyrine- or with caffeine solution was somewhat less in the cells undergoing division than in the others, but on treatment of the filaments with these solutions, it was also found, that the precipitate appeared somewhat later in the cells

in process of division and that on transferring them to distilled water or to ditch water the precipitate also disappeared somewhat sooner. For the sake of completeness I further mention, that no difference could be traced between the tannin content of cells in which the nuclear and cell division had just started, and the tannin content of cells not undergoing division, but the tannin content was found to have diminished, when the process of nuclear and cellular division was at its height or could be considered at an end.

These results show, that a connexion must be looked for between the diminution of tannin content and the process of nuclear and cellular division. This process really consists of two processes, going on simultaneously, and therefore the question arose, which of the two exerted its influence on the tannin content. With reference to this question I carried out some experiments.

As has already been stated, the growth of the cells and the division of cell and nucleus is stopped in a one percent antipyrine solution or in a 0.1 percent caffeine solution. I therefore studied the effect of these solutions on the formation of transverse walls and on karyokinesis, when the dividing cells and those showing the very earliest signs of the process of nuclear and of cell division, were placed in these solutions for some time. Filaments, in which such cells occurred, were left for $1\frac{1}{2}$ hours in the above mentioned solutions, and were then examined next day with regard to the division of cell and nucleus. The transverse walls, in process of formation, had been disturbed in their development, and therefore in these cases the cell was incompletely divided. The result in the cells which were on the point of dividing, when placed in the antipyrine- or caffeine solution, was more interesting; often in these cells no trace of a transverse wall could be found next day. The process of cell division had been completely suppressed.

The process of nuclear division was however quite different. In all the cells where it was going on, or where it was about to begin, it had continued to the end and two normal daughter nuclei always resulted, which were generally situated a little apart in the axis of the cell.

It follows from these experiments, that a temporary fixation of the tannin by antipyrine or caffeine prevents the formation of transverse walls, but does not directly affect nuclear division. On the strength of this result I feel justified in assuming that there must be a connexion between the diminution of the tannin content, referred to above, and the formation of transverse walls. Both abolition of transverse wall formation through fixation of tannin and the dimi-

nition of the tannin content during the formation of transverse walls, point to the tannin being necessary for, and used up in the formation of transverse walls.

In order to obtain still greater certainty with regard to this conclusion, the influence of antipyrine and caffeine on the formation of transverse walls in *Cladophora* was investigated. With ferric chloride, osmic acid, and antipyrine I did not obtain tannin reactions in *Cladophora* and I therefore was interested in knowing how, for instance, the formation of transverse walls would be affected by transferring to a one percent antipyrine solution. I found that transverse walls, which were just beginning to be formed, continued to grow until they were completed. This was even the case if the specimens were left in antipyrine solution during the whole of the process of cell division. This result still further strengthens my view that in *Spirogyra* the interruption or prevention of transverse wall formation is wholly due to the fixation of the tannin. For in *Cladophora*, where no tannin can be used in the formation of transverse walls, a one percent solution of antipyrine does not bring about this disturbance. The only ready explanation which, in my opinion, can be given of the results obtained in the conjugation and transverse wall formation, is this, that the tannin serves as plastic material in the building up of the cell wall.

I wish to add a few results to those already mentioned, which point to a connexion between the tannin content and growth of cell wall. In *Spirogyra* filaments cells are sometimes observed, which, judging from the position of the transverse walls, are distinguished from the others by increased turgor. These cells are generally also distinguished by a larger starch content. On closer examination it is found that the growth of these cells is less than that of the others, or that growth has completely come to a standstill. These symptoms indicate a pathological condition, for generally I was able to ascertain that the above-mentioned cells did not divide further and died off. I cannot give the reason for this condition, but it is remarkable that the tannin content of these cells as revealed by antipyrine or caffeine solution, is larger, and often much larger, than that of the other cells. Once more it is found, as in the case of cells in which conjugation failed, that a cessation of growth is accompanied by an increase in the tannin content.

As was shown by the investigations of GERASSIMOFF¹⁾ and of

¹⁾ J. J. GERASSIMOW, Ueber den Einfluss des Kerns auf das Wachstum der Zelle, Separat-Abdruck aus Bull. d. I. Soc. Imp. des Nat. de Moscou, 1901, No. 1 en 2, p. 193. Zur Physiologie der Zelle, Separat-Abdruck aus Bull. d. I. Soc. Imp. des Nat. de Moscou, 1904, No. 1, p. 7.

myself¹⁾, the growth of cells without nuclei is very slight and gradually stops completely. In anuclear cells with chromatophores and in those without chromatophores, the two kinds being obtained by centrifuging the cells before or during karyokinesis, the tannin content after a time becomes very considerable, as shown by examination with caffeine- and antipyrine solutions. In the absence of a nucleus growth stops, and as a result the consumption of tannin must have fallen off or has stopped altogether. Its production is however continued for some time; hence the increase of the tannin content in cells without nucleus. In this case also there is cessation of growth and an increase in the tannin content.

The results obtained with non-growing nucleated and with non-nucleated cells, agree with those which I obtained with cells conjugating and undergoing division, but are of less importance for the explanation of the physiological significance of tannin, because non-growing nucleated cells must be considered diseased, and those without nuclei are very abnormal. The results obtained with conjugating cells and with cells undergoing division, I consider on the other hand of great importance for the explanation of the physiological meaning of tannin, which in my opinion must be regarded in *Spirogyra* as a substance which serves in the formation of the cell walls. The tannin is here not a reserve-material, however; it belongs to the soluble substances which the plant continually requires for its development. It disappears and gives way to reserve-materials, when the plant forms zygospores and passes into the resting condition. Hence I have arrived at a result, which agrees with the conclusions published by WIGAND nearly half a century ago, but which militates against the view of later investigators, such as SACHS, KRAUS and others. For the sake of clearness I must add, that I do not at all claim that tannin is the only substance, which is used in the formation of the cell wall of *Spirogyra*, nor do I wish to argue that the only physiological significance of tannin is its use as a plastic material.

This paper is a preliminary one. It is my intention to report at some future time more fully on the physiological significance of tannin in *Spirogyra*, and to illustrate with tables the conclusions relating to the comparative experiments on the growth of *Spirogyra* filaments under various conditions, i. e. in antipyrine- and caffeine solutions, in ditch water etc. At the same time various points of investigation, relating to the tannin problem, and not mentioned in this paper, will be dealt with.

¹⁾ G. VAN WISSELINGH. Over wandvorming bij kernlooze cellen. Reprint from Bot. Jaarb. Dodonaea, Vol. 13, 1904, p. 5 and 6. Zur Physiologie der Spirogyrazelle, Beihefte zum Botan. Centralblatt, Bd. XXIV, Abt. 1, p. 170.

Physiology. — “*The Camera silenta*”) of the *Physiological Laboratory at Utrecht*”. By Prof. H. ZWAARDEMAKER.

(Communicated in the meeting of February 26, 1910.)

The extension of the means of communication calls forth nearly everywhere to a higher or lower degree the disadvantages connected with the continual presence of noise. Therefore we want in many instances apartments free from sound, and that at first in those cases in which the continuous existence of disturbing sounds forms an insuperable impediment. Such cases present themselves:

a. in acoustic experiments when the observations have to take place in the proximity of the minimum perceptible;

b. in public consulting rooms for diseases in the ear where through the coming and going of patients the required silence never reigns, and more frequent visits render every minute investigation well nigh impossible, consequently cause also uncertainty of diagnosis, of advice and of decision in case of examination:

c. in modernly built hospitals, which with their smooth walls, naked floors, construction of stone and iron, etc. show a kind of strong resonance, and which, through their many technical ‘institutions’ can never be quiet; the consequence is the impracticableness of a really efficient percussive and auscultatory examination.

Since 1904 a camera silenta ($2.28 \times 2.28 \times 2.20$ M.) has been used for the purpose mentioned under *a* in the Physiological Laboratory at Utrecht²⁾ and also since that time my advice has repeatedly been asked in the building of new laboratories, polyclinics and hospitals in this country and elsewhere. In connection with this I venture here to pronounce the conviction that an apartment free from sound, intended for one of the three above mentioned purposes, will have to satisfy three conditions in order to preclude disappointment. These conditions are:

1. The inner surface of the apartment has to be covered with

¹⁾ *Silentus*, adj. occurring in GELLIIUS, in a fragment from LAEVIUS used by “*loca*”, is, on account of its shortness, preferable to *silentiosus*.

²⁾ *Ned. Tijdschr. v. Geneesk.* 1905, Part 1, p. 571. *Zeitschr. f. Ohrenheilk.* Bd. 54, p. 247.

a material that does not reverberate sound; for if this is neglected, not only the involuntary sounds that are made by us, will have a disturbing influence, but we shall also be hindered by the small remainder of sound that might still be left on account of incompleteness in the construction; the resonance of the space that is shut off will itself seize definite parts of the small quantity of noise that arises or penetrates into it and make them audible in a higher degree.

2. The isolation must be brought about by a double wall, with interstices of air of such a trifling thickness that resonance of audible tones is quite out of the question and moreover no other contact is left between the two walls than of a few narrow lead-contacts.

3. The isolation of the outer wall of the building and of its bottom has to be as complete as possible: the first isolation has to take place through a purposely constructed secondary apartment.

The first condition is fulfilled in our laboratory by means of a covering of horsehair some centimeters thick (*trichopièse*), as it is used in telephone-cells. Thanks are due to Dr. BULTJIS of Gent for making me acquainted with this material, which, moreover, procures an excellent isolation of sound.

The second condition is satisfied at Utrecht by making use, in fastening the *trichopièse*, of a wall of porous stone and by constructing outside it a second wall, consisting of corkstone of German manufacture. Plates of peatmoss from Klazienaveen in the province of Drente would have answered the purpose even better.

The third condition requires the exclusive use of lead-contacts. Especially the bottom has to be well provided for. At Utrecht faults have been made in this respect, which could only partly be made up for by the subsequent addition of an extra-covering.

Taking the above-named chief conditions for granted, we shall have to answer the question, whether an apartment free from sound will have to be constructed underground, on a level with the ground or on a higher floor. My answer is decidedly on a higher floor, for the conduction of the sound coming from the bottom is the obstacle which it is most difficult to overcome. An efficient isolation of the bottom can much more easily be brought about on a higher floor than on a foundation. In the first case the only thing one has to do is to provide lead-contacts with the stone beams, which in their turn are not directly connected with the bottom, whilst in the second case, under the most favourable circumstances, short columns con-

sisting of many strata can be made use of, which, however, have a constant direct communication with the ground.

As to the different tones, the most difficult thing appears to be to keep away the low tones. Inaudible vibrations of very slow periodicity are even not at all excluded in our camera silenta, so that a sensitive microphone, conducted to a gold-thread string-galvanometer does not appear to subside, not even when at a complete adaptation of the organ of hearing not a trace of sound is to be observed. (This does not disturb acoustically, but a somewhat faster periodicity would have been a hindrance).

Besides an apartment free from sound ought to have porous walls, for if perfectly impermeable walls are chosen, it will appear that in case of long experiments a ventilation is necessary, which in its turn would require the supply of ventilation-channels, consequently of sound-leaks. For double-door and double-window (the latter in my opinion hygienically indispensable) as a matter of course apparatus are wanted which require much care and a lasting control. When acoustic experiments are made, the supply of sound should come from sound-sources placed outside the apartment, right through a leaden stopper, that the principle that the two walls of the double wall should have none but a lead-contact, is not discounted¹⁾. Electric light, telephone, supply of air for organ-pipes and sirens through a narrow leaden tube and the necessary conducting-wire to the galvanometer offer no technical difficulties.

An accidental additional advantage of an acoustic apartment with a double wall, double door and double window, duly separated from the outer-walls of the building by means of by-apartments, is this, that it forms a calorimeter. The camera silenta at Utrecht remains without an inhabitant of a constant temperature to within 2 decigrades. By covering the trichopiëse-walls with some meters of extremely fine brasswire (0,1 mm.), a bolometer may be made with a Wheatstone bridge and galvanometer placed in a by-apartment, by which bolometer the rise of temperature that the space undergoes through an inhabitant, may be measured. The production of heat which this causes is determined empirically (D'ARSONVAL). As a respiration-calorimeter, however, the sound-free apartment is not to be used. This is impossible because the walls are porous, and if this is given up, it is no longer free from sound for longer experiments.

A number of investigations may take place in the camera silenta.

¹⁾ The leaden stoppers are 5 cm. thick and possess a central bore, at its narrowest point being 0.4 cm. wide; comp. Onderz. Physiol. Lab. Utrecht (5) VI. p. 138.

Those which have been made in the last six years, are, it is true, not so numerous and extensive as I should wish, but an enumeration with a list of the publications may follow here in order to serve as an example of what is to be reached in a sound-free apartment.

1. The sensation of stillness may be experimented on; unless a perforation of the tympanum exists, a kind of buzzing may be observed, in which at a closer analysis a soft rustling as of the wind in the tops of the trees, accompanied by a high-toned whistling ($\pm g^a$) may be distinguished; persons in whom this physiological ear-buzzing is indistinct, perceive a feeling of oppression¹).

2. The influence of the adaptation may be traced; then appears among others a gradual diminution of the physiological tinnitus aurium, which after a 3 hours' stay in the sound-free apartment has entirely disappeared (BORTOLOTTI), whilst at the same time the feeling of oppression, if existing, gradually increases (MINKEMA); from this one might be inclined to derive that the physiological ear-buzzing, entirely or partly, possesses the character of an after-image²).

3. The phenomenon of accommodation, discovered by HENSEN, may be more closely studied, by conveying to a person standing outside the camera silenta through bone-conduction the tone of a tuning-fork, which then from the person's ear is conducted into the apartment through an auditory tube; whenever a metronome placed outside the apartment is ticking, the sample-person accommodates and the observer bears a strengthened sound (QUIX).

4. From the shortest exposition-time the smallest observable number of sound-vibrations may be derived in the tone of a tuning-fork or that of an organ, conducted to it from the outside; according to BODE this number seems to vary in the scale in a typical manner (DE GROOT³) and VAN MENS).

1) For my ear the physiological ear-buzzing can be suppressed: *a.* by the ticking of a watch; *b.* by the sound of a tuning-fork of the *c'*-pitch and a sound-force of 68.10^{-3} Erg. per cm.^2 and per sec. (Erg. d. Physiol. 1905 p. 452).

2) According to BORTOLOTTI the buzzing returns directly, after one has left the camera for a moment and then returns.

3) H. DE GROOT, Ztschr. f. Sinnesphysiol. Bd. 44 p. 18 and Onderz. Physiol. Lab. (5) X p. 137.

5. The minimum perceptible during the unity of time may be fixed by the scale (MINKEMA ¹⁾).

6. The limit of distinction may be traced and the typical variation it undergoes in the scale (DENIK ²⁾).

7. The sensation of a report, observed by HENSEN at a sudden intonation or interruption of siren-tones, may be demonstrated in tones of different origin and pitch, with the aid of a sudden opening or closing of a telephone-contact or a sudden opening or closing of a particularly constructed lead cock.

8. The spreading of the sound round a tuning-fork with the situation of the well-known interference-planes may be accurately traced, without making the mistakes that must necessarily arise in apartments with echoing walls.

9. The action of the winding mollusc-shells as to their resonance for buzzes may be proved directly.

10. The sound-extinguishing action of different means of isolation may be traced with perfect security; for reports by dropping steel balls on a steel plate ³⁾ (fall-phonometer of ZORN), for tones by electrically touching purely tuned bells; in both cases the instrument put in a small non-resonant space; the walls of this space are covered with the materials that are to be examined, and, on the one side the energy with which the bells are touched, and on the other the distance at which the sound is heard, is defined; the completest isolation with an equal thickness of the walls is got in the case of trichopiëse, then follows the peatmoss-plate from Klazienaveen, then the corkstone; other materials that we examined had a considerably smaller sound-extinction.

¹⁾ H. F. MINKEMA, Onderz. Physiol. Lab. (5) VI. p. 134.

²⁾ Meeting of this Academy 3 Nov. 1905.

³⁾ In order to prevent resonance the steel plate has to be soldered upon a heavy piece of lead.

Mathematics. — “On pairs of points which are associated with respect to a plane cubic.” By Prof. JAN DE VRIES.

(Communicated in the meeting of February 26, 1910).

1. By the symbolical equation

$$a_x^3 = 0$$

a plane cubic c^3 is represented. If the points X , Y , and Z are connected by the relation

$$a_x a_y a_z = 0,$$

each of them lies on the (mixed) polar line of the other two, and every two of those points are harmonically separated by the polar-conic of the third point; they form a polar triangle of c^3 .

Let us look more closely at the case that the three points lie in one line l ; then Z is the point of intersection of l with the polar line of X and Y .

It is evident that the triplets X, Y, Z lying on l form a cubic involution I_2^3 of order two having the points of intersection P, Q, R of c^3 with l as threefold elements.

According to a well known property of the I_2^3 we find that P, Q and R form at the same time a group of the I_2^3 . This is indeed directly to be seen; for, the polar conic of P intersects l in P and in the point H , which is harmonically separated by Q and R from P ; the polar line of Q with respect to that conic therefore passes through R .

To I_2^3 belongs a neutral pair, U, V forming with each point of l a triplet and therefore having l as polar line. The polar conics of the points lying on l form a pencil; two of those conics u^2 and v^2 touch l in the points V and U .

We shall call U and V *associated points*.

Evidently each point U is associated with two points V , viz. with the points which have the polar line and the polar conic of U in common. The associated pairs are thus arranged in an involutory correspondence (2, 2).

If l becomes tangent to c^3 , then in the point of contact L two threefold elements of the I_2^3 unite themselves with the two neutral points U, V . For, all polar conics whose poles lie on l pass through L and one of those curves touches l in L . So c^3 is curve of coincidence of the (2, 2) correspondence.

2. We shall see whether there are points U , for which the corresponding points V form again an associated pair, so that there is a triplet of points which are two by two associated. If we take the three points as vertices of a triangle of reference, their polar conics will be represented by:

$$a_1 x_1^2 + b_1 x_2 x_3 = 0, \quad a_2 x_2^2 + b_2 x_1 x_3 = 0, \quad a_3 x_3^2 + b_3 x_1 x_2 = 0$$

for, each of those points has the connecting line of the other two as tangential chord with respect to its polar conic.

If $u = 0$ is the equation of c^3 , the three polar conics are also represented by

$$\frac{du}{dx_1} = 0, \quad \frac{du}{dx_2} = 0, \quad \frac{du}{dx_3} = 0.$$

From this ensues in the first place that the coefficients b_1, b_2, b_3 must be equal. Farther on it is directly evident that the equation of c^3 is:

$$a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 + 3 b x_1 x_2 x_3 = 0.$$

The triangle of coordinates is therefore a triangle of inflection, i. e. a triangle of which each side contains three points of inflection of c^3 . There being four triangles of inflection, the (2,2)-correspondence of the associated points contains *four involutory triplets*.

3. We shall now determine the locus of the associated pairs, collinear with a given point D .

In the first place D is a node of the locus; the points D' and D'' associated with D are the points of intersection of the polar conic d^2 of D with the polar line d of D . The locus is therefore a nodal biquadratic curve d^4 .

The tangents out of D to c^3 are at the same time tangents to d^4 , for in their tangential points two associated points continually coincide.

So d^2 is the conic of BERTINI of d^4 . For an arbitrary nodal c^4 this conic contains besides the six points of contact of the tangents out of the node, the points of intersection of c^4 with the line connecting the two tangential points of the node, and the tangents in those "fundamental points" to the conic concur in the node¹⁾.

The curve d^4 is a special curve, because its fundamental points coincide with the tangential points D' and D'' , so that these are at the same time the points of contact of a double tangent.

¹⁾ See my paper "La quartique nodale" (Archives Teyler, t. IX, p. 263).

4. It is easy to find the equation of d^4 .

The polar conic of Z with respect to $a_x^3 = 0$ is represented by $a_z a_x^2 = 0$, the polar line by $a_z a_x = 0$. For the tangents out of Z to that conic we have thus

$$a_z a_x^2 b_z^3 = a_z^2 a_x b_z^2 b_x.$$

If these contain the given point Y , then Z is a point of the curve d^4 belonging to Y . So it has, as equation (in current coordinates z):

$$a_y^2 a_z b_z^3 = a_y b_y a_z^2 b_z^2.$$

From this it is again evident, that the polar line of D is double tangent, and that it touches d^4 in the tangential points D' and D'' . For, by combination with $a_y^2 a_z = 0$ we find $a_y a_z^2 \cdot b_y b_z^2 = 0$. The same is obtained by combination with $b_z^3 = 0$; by this is confirmed that d^4 is touched in its points of intersection with the polar conic of D by c^3 and the polar line d .

Out of

$$a_y^2 a_z b_z^3 - a_y b_y a_z^2 b_z^2 = b_y^2 b_z a_z^3 - b_y a_y b_z^2 a_z^2$$

follows that the equation of d^4 can be transformed into

$$\frac{1}{2}(a_y a_z b_z^2 - b_y b_z a_z^2)(a_y b_z - a_z b_y) = 0,$$

so also into

$$(a_y b_z - a_z b_y)^2 a_z b_z = 0.$$

Now

$$a_y b_z - a_z b_y = (a_1 b_2)(y_1 z_2) + (a_2 b_3)(y_2 z_3) + (a_3 b_1)(y_3 z_1).$$

If thus we represent the coordinates of the line YZ by ξ_k , the above equation passes into

$$(ab\xi)^2 a_z b_z = 0.$$

This equation expresses that the polar conic of Z is touched by the line YZ^1 .

5. At the same time is evident from this that the line (ξ) cuts its poloconica in two points. This is more closely confirmed by the observation that the poloconica of (ξ) is the locus of the points whose polar conics touch (ξ) , from which ensues that it intersects (ξ) in two associated points.

The curve d^4 can therefore be generated by determining the points of intersection of each of the lines s through D with the conjugate poloconica σ . The poloconica describes there a system with index 2. For, when σ passes through any point X the polar conic of X is touched by s . And as two lines s satisfy that condition, X lies on two curves σ . This generation of d^4 with the aid of a system of

¹⁾ CLEBSCH, Leçons sur la géométrie, t. II, p. 278.

conics with index 2 and a pencil projective to it is characteristic for the nodal biquadratic curve¹⁾.

6. Each nodal biquadratic curve d^4 of which the nodal tangents pass through the points of contact D and D' of a double tangent d is related in the way mentioned above to a c^3 .

The polar curve d^2 of D has in D the tangents t' and t'' in common with d^4 and it intersects it in the points of contact R of the six tangents concurring in D . Of the 16 points which d^4 has in common with the system of d^2 and d six lie in D , four in D' and D'' , six in the points R . The tangents t' and t'' contain eight of those points; so the remaining eight lie in a conic (curve of BERTINI).

This conic d^2 unites the six points R to the points D' and D'' .

Let us now regard the pencil determined by d^4 and the conic d^2 counted twice; one consisting of the double tangent d and a cubic c^3 belongs to it. From this ensues that d^4 is touched by c^3 in the points of contact R of the tangents drawn out of D to d^4 .

As d^2 passes through the points R , it is the polar conic of D with respect to c^3 ; because D' and D'' are the fundamental points, so that DD' and DD'' are touched by d^2 in D' and D'' , d is the polar line of D with respect to d^2 and of c^3 . So d^4 is the locus of the points associated with respect to c^3 and collinear with D .

7. If d^4 is represented by

$$x_1^2 x_2^2 + x_1 x_2 x_3^2 - c_x^3 x_3 = 0,$$

where

$$c_x^3 = (c_1 x_1 + c_2 x_2)^3$$

then t' , t'' , and d are indicated by $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$, and d^2 by $2x_1 x_2 x_3 - c_x^3 = 0$. From

$$(c_1^2 x_2^2 + x_1 x_2 x_3^2 - c_x^3 x_3) - x_3 (2x_1 x_2 x_3 - c_x^3) = 0$$

then follows for d^2 the equation

$$x_1 x_2 - x_3^2 = 0,$$

and from

$$(x_1 x_2 - x_3^2)^2 - (x_1^2 x_2^2 + x_1 x_2 x_3^2 - c_x^3 x_3) = 0$$

we find for c^3

$$c_x^3 - 3x_1 x_2 x_3 + x_3^3 = 0.$$

For the polar conic η^2 of Y with respect to c^3 follows from this

$$c_y c_x^2 - y_1 x_2 x_3 - y_2 x_1 x_3 + y_3 (x_2^2 - x_1 x_2) = 0,$$

for the polar line of D with respect to η^2

$$2y_3 x_3 - y_1 x_2 - y_2 x_1 = 0,$$

¹⁾ BOBЕК, Denkschriften der Akad. in Wien, Bd. 53, S. 119.

thus for the tangents out of D to the polar conic η^2

$$4y_3 \{c_y c_x^2 - y_1 x_2 x_3 - y_2 x_1 x_3 + y_3 (x_3^2 - x_1 x_2)\} = (2y_3 x_3 - y_1 x_2 - y_2 x_1)^2.$$

When one of these tangents passes through Y we have

$$4y_3 (c_y^3 - 3y_1 y_2 y_3 + y_3^3) = (2y_3^2 - 2y_1 y_2)^2,$$

or

$$y_1^2 y_2^2 + y_1 y_2 y_3^2 - c_y^3 y_3 = 0.$$

From this is again evident that the curve indicated by this equation is the locus of the points associated with respect to c^3 and collinear with D .

This special nodal d^4 is characterized by the property according to which it is touched by a cubic in the six points whose tangents concur in the node. For, when considering the pencil which is determined by d^4 with the conic of BERTINI counted twice it is immediately evident that the remaining points of d^4 lying on this conic are points of contact of a double tangent, which must then also lie on the nodal tangents.

8. We shall now see into what a line (ξ) is transformed by the correspondence of the associated points. To that end we eliminate y_k out of the three equations

$$\xi_y = 0, a_y a_x^2 = 0 \text{ and } a_y^2 a_x = 0.$$

Out of the first two we find

$$y_1 : y_2 : y_3 = (a_2 \xi_3) a_x^2 : (a_3 \xi_1) a_x^2 : (a_1 \xi_2) a_x^2.$$

Substitution in the third then produces

$$(ab\xi)(ac\xi) a_x b_x^2 c_x^2 = 0.$$

A line ξ is thus transformed into a curve ξ^5 of order five. This could be foreseen, for the two associated points lying on ξ pass in the transformation into each other, whilst the three points of intersection of ξ and c^3 correspond to themselves.

When the point U describes the line ξ , its polar line u envelops the poloconica ξ^2 , whilst its polar conic u^2 describes a pencil. From this ensues that ξ^5 is generated by a pencil of conics and a pencil of rays of index 2 projectively related to it. Consequently ξ^5 has nodes in the four basepoints of that pencil and the points associated with U form the pairs of the fundamental involution of pairs appearing on ξ^5 ¹⁾. In connection with this ξ^5 is touched by the poloconica ξ^2 in five points (l. c. p. 48).

¹⁾ See my paper: "Ueber Curven fünfter Ordnung mit vier Doppelpunkten" (Sitz. Akad. Wien, Bd. 104, S. 47).

Mathematics. — “*On continuous vector distributions on surfaces*”.
(2nd communication)¹⁾. By Dr. L. E. J. BROUWER. (Communi-
cated by Prof. D. J. KORTEWEG).

(Communicated in the meeting of February 26, 1910).

§ 1.

The tangent curves to a finite, uniformly continuous vector distribution with a finite²⁾ number of singular points in a singly connected inner domain of a closed curve.

Let γ be the domain under consideration, then we can represent it on a sphere, so we can immediately formulate on account of the property deduced in the first communication (see there page 855):

THEOREM 1. *A tangent curve, which does not indefinitely approach a point zero, is either a simple closed curve, or its pursuing as well as its recurring branch shows one of the following characters: 1st. stopping at a point of the boundary of γ ; 2nd. spirally converging to a simple closed tangent curve; 3rd. entering into a simple closed tangent curve.*

We now shall farther investigate the form (in the sense of analysis situs) of a tangent curve r , of which we assume, that at least one of the two branches (e. g. the pursuing branch) approaches indefinitely one or more points zero, i. e. singular points of the vector distribution.

We start the tangent curve in a point A_0 (not a point zero) and we pursue that curve in the following way: By β_ε we understand a distance with the property that in two points lying inside the same geodetic circle described with a radius β_ε , and possessing both a distance $> \varepsilon$ from the points zero, the vectors certainly make an angle $< \frac{1}{8} \pi$ with each other. We farther choose a fundamental series of decreasing quantities $\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots$ converging to 0, and of corresponding decreasing distances $\beta_{\varepsilon_1}, \beta_{\varepsilon_2}, \dots$, which all we suppose, if α is the distance of A_0 from the points zero, to be smaller than $\alpha - \varepsilon_1$.

We then prove in the manner indicated in the first communication p. 852, that, when pursuing r from A_0 , a point B_0 is reached, possessing a distance β_{ε_1} from A_0 ; we call the arc $A_0 B_0$ a β_{ε_1} -arc. According to our supposition there now exists a finite number n_1 in

¹⁾ For the first communication see these Proceedings Vol. XI 2, p. 850.

²⁾ This restriction we shall drop in a following communication.

such a way, that after having completed n_1 β_{ε_1} -arcs, but not yet n_1+1 β_{ε_1} -arcs, we reach a point A_1 , where for the first time we have approached the points zero as far as a distance ε_1 . Then again there is a finite number n_2 in such a way that, having completed from A_1 n_2 , but not yet n_2+1 β_{ε_2} -arcs, we reach a point A_2 , where for the first time we have approached the points zero as far as a distance ε_2 . From there we pursue r with β_{ε_3} -arcs and continue this process indefinitely.

If we understand by $m(\varepsilon_n)$ the maximum distance from the points zero, which r reaches when being pursued *after* having for the first time approached the points zero as far as a distance ε_n , then a first possibility is, that $m(\varepsilon_n)$ converges with ε_n to zero.

In that case the pursuing branch converges to one single point zero and it is *an arc of simple curve, stopping at that point zero.*

We now suppose the second possibility, that $m(\varepsilon_n)$ surpasses for each ε_n a certain finite quantity e . Then we can effect (by eventually omitting a finite number of terms of the series of ε_n 's), that each $\varepsilon_n < \frac{1}{2}e$ and each $\beta_{\varepsilon_n} < \frac{1}{2}e$.

On the pursuing branch then certainly two points P_1 and Q_1 can be indicated both at a distance e from the points zero, and separated on r by at least one point at a distance ε_1 from the points zero, whilst the distance between P_1 and Q_1 is $\leq \frac{1}{4}\beta_{\varepsilon_1}$. Let P_1S and Q_1U be pursuing β_{ε_1} -arcs, and P_1R and Q_1T recurring β_{ε_1} -arcs.

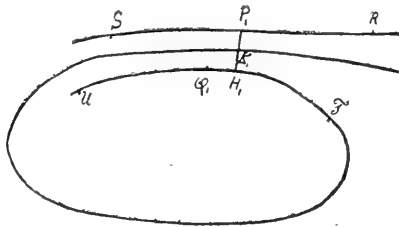


Fig. 1.

Let H_1 be a point of TU , having from P_1 the smallest possible distance, then H_1 cannot coincide with T or U , so that the geodetic arc P_1H_1 is in H_1 normal to the vector direction, and the vector directions in all points of that geodetic arc, forming with

each other an angle $< \frac{1}{8} \pi$, are directed to the same side of the geodetic arc $P_1 H_1$.

Let K_1 be the last point of intersection of the arc $P_1 H_1$ of r with the geodetic arc $P_1 H_1$. Then the arc $K_1 H_1$ of r and the geodetic arc $K_1 H_1$ form a simple closed curve, and we prove in the manner indicated on page 853 of our first communication, that either the pursuing branch of r from H_1 lies in the inner domain, and the recurring branch from K_1 in the outer domain, or the pursuing branch from H_1 in the outer domain, and the recurring branch from K_1 in the inner domain.

Let us first assume that the pursuing branch lies in the *inner domain*, then certainly two points P_2 and Q_2 can be chosen on it, both at a distance e from the points zero and separated on r by at least one point at a distance ε_2 from the points zero, whilst the distance between P_2 and Q_2 is $\leq \frac{1}{4} \beta_{\varepsilon_2}$. With the aid of those two points we construct in the same way as above now a simple closed curve, consisting of an arc $K_2 H_2$ of r and a geodetic arc $K_2 H_2$, in whose inner domain lies the pursuing branch of r from H_2 .

Going on in this way we construct a fundamental series of closed curves u_1, u_2, u_3, \dots lying inside each other. If there is a domain or set of domains G , common to all the inner domains of these curves (which, as we shall presently show, is really the case) then the boundary of G can only be formed by points belonging to none of the curves u_1, u_2, u_3, \dots but being limit points of fundamental series of points lying on those curves.

We assume $q > p$, and B to be a point of u_q having a distance $> 3 \varepsilon_p$ and $> 3 \beta_{\varepsilon_p}$ from the points zero. Let C be the first point when recurring from B , and D the first point when pursuing from B , which reaches a distance $\frac{1}{2} \beta_{\varepsilon_p}$ from B , then we shall assume for a moment that there exists on u_p , but not on the arc CD , a point S lying at a distance $\leq \frac{1}{4} \beta_{\varepsilon_p}$ from B , and we shall show that this assumption leads to an absurdity.

Let SV be a recurring $\frac{1}{2} \beta_{\varepsilon_p}$ -arc and SW a pursuing $\frac{1}{2} \beta_{\varepsilon_p}$ -arc on u_q , then the arcs CD and VW can have no point in common,

and the geodetic arc $K_q H_q$, belonging to u_q , has either no point in common with VW , or none with CD .

In the first case we determine on VW a point M , having from B a distance as small as possible. The geodetic arc BM is then in M normal to VW , and has a last point of intersection N with CD , so that the geodetic arc NM forms with one of the arcs NM of u_q , not containing e.g. the point C , a closed curve; u_q , taken with a certain sense of circuit, would at M enter one of the two domains determined by this closed curve, to leave it no more; further C would lie outside that domain; thus u_q would never be able to reach C , with which the absurdity of our assumption has been proved.

In the second case we determine on CD a point M having from S a distance as small as possible, and on the geodetic arc SM the last point of intersection N with VW . The further reasoning remains analogous to the one just followed: the parts of the arcs VW and CD are only interchanged.

Let now B_ω be the only limit point of a certain fundamental series of points B_1, B_2, B_3, \dots , lying respectively on u_1, u_2, u_3, \dots . We assume that B_ω is not a point zero; it has then for a suitably selected p a distance $> 4 \varepsilon_p$ and $> 4 \beta_{\varepsilon_p}$ from the points zero.

Let further each m_k be $> p$ and let $B_{m_1}, B_{m_2}, B_{m_3}, \dots$ be a fundamental series contained in the series just mentioned, whose points have all from B_ω a distance $< \frac{1}{8} \varepsilon_p$ and $< \frac{1}{8} \beta_{\varepsilon_p}$.

If then further on the different u_{m_k} $B_{m_k} D_{m_k}$ are pursuing, $B_{m_k} C_{m_k}$ recurring $\frac{1}{2} \beta_{\varepsilon_p}$ -arcs, we prove by the reasoning followed in the first communication p. 854, that there exists a series $C_{n_1} D_{n_1}, C_{n_2} D_{n_2}, C_{n_3} D_{n_3}, \dots$ converging uniformly to an arc $C_\omega D_\omega$ of a tangent curve u_ω in such a way, that all arcs $C_{n_k} D_{n_k}$ lie on the same side of $C_\omega D_\omega$.

If we describe round B_ω a geodetic circle with radius $\frac{1}{8} \beta_{\varepsilon_p}$, then it cuts from $C_\omega D_\omega$ an arc FI containing B_ω ; this arc divides its inner domain into two regions, into one of which, to be called g , neither the arcs $C_{n_k} D_{n_k}$, nor any other parts of the curves u_{n_k} can penetrate, as they would get there a distance $\leq \frac{1}{4} \beta_{\varepsilon_p}$ from B_{n_k} .

As further the region g cannot lie outside all curves u_{n_k} , it must lie inside all curves u_{n_k} .

So there is certainly a domain or a set of domains G , common to all the inner domains of the curves u_k , and to the boundary of G belong all points of the limit set λ of the u_k 's, which are not points zero, thus also all points of λ , which are points zero, as the latter are limit points of the former ones. So the boundary of G is identical to the limit set of the u_k 's, is therefore coherent and identical to its *outer circumference*, whilst abroad from the points zero it consists of tangent curves to the vector distribution, which on account of the existence of the domain g can show nowhere in a non-singular point the character mentioned in theorem 1 sub 3.

We shall now show that a tangent curve r' belonging to the boundary of G cannot have the property of r , that its pursuing or recurring branch converges spirally to the boundary of a domain or set of domains G' .

We should then namely be able to form, in the same way as was done above and in the first communication for r , also for r' a closed curve u'_k consisting of a geodetic arc $\leq \frac{1}{4} \beta_{\varepsilon_k}$ and an arc ρ' of r' , joining the same two points K' and H' . And there would exist arcs of r which would converge uniformly to ρ' from the same side, e.g. from the inner side of u'_k . But when pursuing such an arc ψ of r situated in sufficient vicinity of ρ' , we should never be able to return between ψ and ρ' .

As furthermore in the case considered here, that the pursuing branch of r lies in the inner domain of u_k , it is also excluded, that r' reaches the boundary of G , only *one* form remains possible for r' , namely *that of an arc of simple curve, starting from a point zero, and stopping at a point zero.* (For the rest these two end points can very well be identical).

Of such tangent curves there can be in the boundary of G at most two, which possess the same end points, when these end points are different; but there can be an infinite number, which are closed in the same point zero. Of these however there are only a finite number, of which the extent surpasses an arbitrarily assumed finite limit. For, each of these contributes to G a domain with an area, which surpasses a certain finite value.

The curves r' whose extent surpasses a certain finite limit are run along by a u_k of sufficient high index in the same order, as they succeed each other on the outer circumference of G . From this ensues *that for all curves r' the pursuing sense belongs to the same sense of circuit of the outer circumference of G .*

If the pursuing branch of r lies in the *outer domain* of u_1 , the preceding holds with slight modifications. A point of the limit set of the u_k 's now necessarily bounds a region belonging to γ , and lying outside all u_k 's, only then when it is not a point of the boundary of γ . The *inner circumference*, to which r now converges spirally *on the inner side*, consists here again of arcs of simple curve, which are tangent curves to the vector distribution, but these tangent curves can lie entirely or partially in the boundary of γ .

However they have all again a pursuing sense belonging to the same sense of circuit of the circumference.

We now agree about the following: When a pursuing branch of a tangent curve reaches a point zero, we continue it, if possible, along a pursuing branch, starting from that point zero, and not meeting the former within a certain finite distance; but if such a continuation is impossible, we stop the branch at that point zero, and so we do likewise when the branch has entered into a closed curve or has approximated spirally a circumference. Then we can resume the preceding reasonings as follows:

THEOREM 2. *A tangent curve is either a simple closed curve, or save its ends it is an arc of simple curve, of which the pursuing as well as the recurring branch shows one of the following characters: 1st. stopping at a point of the boundary of γ ; 2nd. stopping at a point zero; 3rd. entering into a simple closed tangent curve; 4th. spirally converging to a circumference, consisting of one or more simple closed tangent curves.*

From this ensues in particular:

THEOREM 3. *A tangent curve cannot return into indefinite vicinity of one of its points, after having reached a finite distance from it, unless it be to close itself in that point.*

That the last theorem is not a matter of course, is evident from the fact that it does not hold for an annular surface. On this it is easy to construct tangent curves of the form pointed out by LORENTZ (Enz. der Math. Wiss. V 2, p. 120, 121).

We finally notice that the vector distribution considered in this §, does not possess of necessity a singular point (as is the case on the sphere). This is proved directly, by considering in the inner domain of a circle, situated in a Euclidean plane, a vector everywhere constant.

§ 2.

The structure of the field in the vicinity of a non-singular point.

To classify the singular points we shall surround each of them

with a domain which we shall cover entirely with tangent curves not crossing each other and we shall investigate the different ways in which that covering takes place in different cases. For the sake of more completeness and as an introduction we first do the same for a non-singular point.

Let P be the point under consideration, RS an arc of tangent curve r containing P , UV an arc containing P of an orthogonal curve of the vector distribution. We draw through U and V tangent curves α_0 and α_1 , and through R and S orthogonal curves γ and δ , and we let the four points $R, S, U,$ and V converge together to P . Before they have reached P , a moment comes when $\alpha_0, \alpha_1, \gamma,$ and δ form a curvilinear rectangle, inside which lies P , and inside which lies no point zero of the vector distribution, thus inside which on account of the first communication no closed tangent curve can be drawn.

We shall cover this curvilinear rectangle with tangent curves not crossing each other.

We number α_0 with 0, r with $\frac{1}{2}$, α_1 with 1. Let $Q_{\frac{1}{4}}$ be a point inside or on the rectangle $A_0 B_0 S R$ (fig. 2) having from α_0 and r

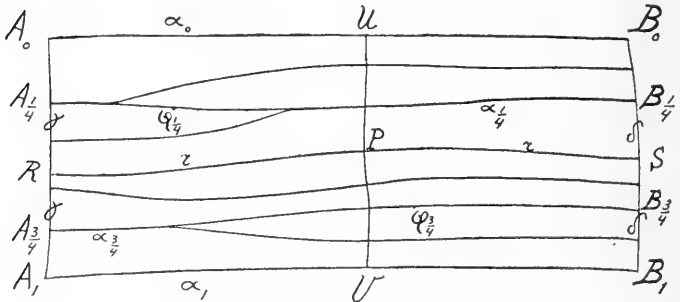


Fig. 2. Non-singular point.

a distance as large as possible. We draw through $Q_{\frac{1}{4}}$ a tangent curve $\alpha_{\frac{1}{4}}$, about which we agree, that, if it meets α_0 or r , we shall continue it, by pursuing or recurring α_0 or r , until we come upon γ or δ .

Then $\alpha_{\frac{1}{4}}$ is a tangent curve joining two points $A_{\frac{1}{4}}$ and $B_{\frac{1}{4}}$ of γ and δ between α_0 and r . In the same way we construct inside

the rectangle $A_1 B_1 S R$ a tangent curve $\alpha_{\frac{3}{4}}$, joining two points $A_{\frac{3}{4}}$ and $B_{\frac{3}{4}}$ of γ and σ between r and α_1 . The rectangle $A_0 B_0 B_1 A_1$ is then divided into four regions. In these we choose in the way described above successively the points $Q_{\frac{1}{8}}, Q_{\frac{3}{8}}, Q_{\frac{5}{8}}, Q_{\frac{7}{8}}$, draw through $Q_{\frac{1}{8}}$ a tangent curve $\alpha_{\frac{1}{8}}$ joining two points $A_{\frac{1}{8}}$ and $B_{\frac{1}{8}}$ of γ and σ , and we deal analogously with the other three points.

Going on in this manner we construct for each fraction $\frac{a}{2^n} < 1$ a tangent curve $\alpha_{\frac{a}{2^n}}$ joining two points of γ and σ ; two of these curves chosen arbitrarily can coincide *partially*, but they cannot cross each other.

All these tangent curves must now cover everywhere densely the inner domain of the rectangle $A_0 B_0 B_1 A_1$. For, if they left there open a domain G , then a domain G_n bounded by two tangent curves with indices $\frac{a}{2^n}$ and $\frac{a+1}{2^n}$ would converge to G . For n sufficiently great however the point $Q_{\frac{2a+1}{2^{n+1}}}$ would then lie inside G , thus in contradiction to the supposition also a tangent curve $\alpha_{\frac{2a+1}{2^{n+1}}}$ would pass through G .

From this ensues, that, if we add the limit elements of the tangent curves $\alpha_{\frac{a}{2^n}}$, which are likewise tangent curves, the inner domain of the rectangle $A_0 B_0 B_1 A_1$ is entirely covered, and further there is for each real number between 0 and 1 *one* and not more than *one* of these tangent curves having that number as its index.

§ 3.

The structure of the field in the vicinity of an isolated singular point. First principal case.

We surround the point zero P , supposed isolated, with a simple closed curve c , inside which lies no further point zero. And we assume as a first principal case that c can be chosen in such a way that inside c no simple closed tangent curve exists, inside which P lies. On account of the first communication there can exist inside c neither a simple closed tangent curve, outside which P lies. We now distinguish 2 cases:

a. There exists inside c a simple closed tangent curve q through P . We can then choose c smaller, so that it meets q , thus containing in its inner domain a tangent curve q_1 which (in its pursuing direction) runs from P to c , and another q_2 running from c to P , and we further look for such tangent curves inside c which cross neither q_1 nor q_2 . Of the possible kinds of tangent curves mentioned at the conclusion of §1 we shall agree about those, which enter into a closed tangent curve, to continue them along that tangent curve until they reach either P or c , and to stop there. Spirally converging to an inner circumference cannot appear, as the other end of such a tangent curve would be separated from P as well as from c , and so would determine a closed tangent curve, outside which P would be lying, which is impossible. Neither can appear spirally converging to an outer circumference, as P would have to lie in that outer circumference and the spiral would necessarily have to cross q_1 and q_2 .

b. There exists inside c no simple closed tangent curve through P . Then inside c there exists no simple closed tangent curve at all, so that again spirally converging is excluded.

In any case, if we agree not to continue a tangent curve, when it reaches P or c , we can distinguish the tangent curves inside c , and not crossing q_1 and q_2 if the latter exist, into three categories:

- 1st. *Closed curves, containing P but not reaching c .*
- 2nd. *Arcs of curve, joining two points of c , but not containing P .*
- 3rd. *Arcs of curve which run from P to a point of c (positive curves of the third kind) or from a point of c to P (negative curves of the third kind).*

Of this third kind there must certainly exist tangent curves. For otherwise the closed sets determined by the curves of the first, and by those of the second kind would cover the whole inner domain of c , thus would certainly possess a point in common; through this point however a curve of the third kind would pass.

So we can commence by constructing one curve of the third kind and we choose eventually q_1 for it. *If possible*, we then draw a second curve of the third kind not crossing the first and we choose eventually q_2 for it. Into each of the two sectors, determined in this way inside c , we introduce if possible again a curve of the third kind, not crossing the already existing ones, and chosen in such a way that it reaches a distance as great as possible from the two curves of the third kind, which bound the sector, whilst, if the new curve terminates somewhere on one of the curves bounding the sector, we further follow the latter curve. In each of the sectors, determined after that in the inner domain of c , we repeat if possible this

insertion, and we continue this process as often as possible, eventually to an indefinite number of insertions.

If in this manner we have obtained an infinite number of tangent curves of the third kind, they determine limit elements which each are either again a tangent curve of the third kind, or contain such a curve as a part. And in particular a fundamental series of positive respectively negative curves of the third kind determines in its limit elements again positive respectively negative curves of the third kind.

After addition of these limit curves of the third kind we are, however, quite sure that no new curves of the third kind not crossing the existing ones can be inserted. This is evident from a reasoning analogous to that followed in § 2. The whole of the curves of the third kind, obtained now, we shall call a *system of base curves of the vicinity of P*.

An arbitrary positive base curve and an arbitrary negative one enclose inside c a sector, of which the area cannot fall below a certain finite limit. For otherwise we should have a fundamental series of positive base curves, and a fundamental series of negative ones, possessing the same base curve as a limit element, which is impossible, as that limit base curve would have to be positive as well as negative.

So the inner domain of c is divided into a finite number of sectors which can be brought under the two following categories:

First category. Sectors bounded by a positive and a negative base curve, between which lie no further base curves. The areas of these sectors surpass a certain finite limit.

Second category. Sectors bounded by two positive (respectively two negative) base curves and containing only positive (respectively negative) base curves. A sector of this category can reduce itself in special cases to a single base curve.

We shall first treat a sector of the *first category* and to that end we first notice that *outside* a curve of the second kind lying in it (i. e. between that curve and c) lie only curves of the second kind, and *inside* a curve of the first kind lying in it only curves of the first kind.

If we draw in the sector a well-ordered series, continued as far as possible, of curves of the second kind enclosing each other, then it converges either to a curve of the second kind, or to two curves of the third kind and between them a finite or denumerable set of curves of the first kind, *not* enclosing each other, and *not* approaching c indefinitely.

If we can construct an infinite number of such series not enclosing

each other, then there are among them which cut from the sector an area as small as one likes, and at the same time the maximum distance, which such a series reaches from c , decreases under each finite limit.

And analogously, if we draw in the sector a well-ordered series, continued as far as possible, of curves of the first kind enclosing each other, it converges either to a curve of the first kind, or to two curves of the third kind and between them a finite or denumerable set of curves of the second kind, *not* enclosing each other, and *not* approaching P indefinitely.

If we can construct an infinite number of such series not enclosing each other, then there are among them which enclose an area as small as one likes, and at the same time the maximum distance, which such a series reaches from P , decreases under each finite limit.

From this ensues that for the sectors of the first category we have to distinguish two cases:

First case. There are curves of the second kind in indefinite vicinity of P . Then the domain of the curves of the second kind is bounded by the two base curves which bound the sector, and a finite or denumerable number of curves of the first kind, *not* enclosing each other, and *not* approaching c indefinitely, in whose inner domains, which we call the *leaves* of the sector, can lie only curves of the first kind.

The region outside the leaves can be covered as follows with curves of the second kind not crossing each other: we first construct one which reaches a distance as great as possible from c and the boundary of the leaves; in this way two new regions are determined, in each of which we repeat this insertion. This process we continue indefinitely, and finally we add the limit curves. That then the region outside the leaves is entirely covered, is evident from the reasoning followed in § 2.

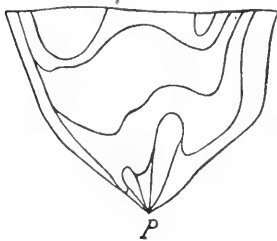


Fig. 3. Hyperbolic sector.

Second case. There are no curves of the second kind in indefinite vicinity of P . Then the domains covered by these curves are cut off from

And in the same way we fill each of the leaves with curves of the first kind not crossing each other. The whole of the tangent curves filling the sector finally gets the form indicated in fig. 3. The sectors being in the discussed first case we shall call *hyperbolic sectors*.

the sector by a finite or denumerable number of curves of the second kind, *not* enclosing each other, and *not* approaching P indefinitely. These domains we take from the sector (consequently modify an arc of c), and there remains a new sector, bounded by the same base curves as the old one, but consisting of one *leaf* inside which lie only curves of the first kind. This leaf we can fill with curves of the first kind not crossing each other (see fig. 4).

These sectors of the second case, which are reduced to a single leaf, we shall call *elliptic sectors*.

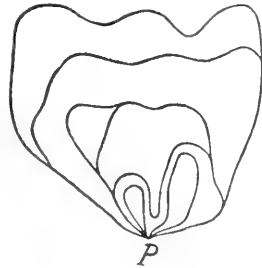


Fig. 4. Elliptic sector.

We now pass to the discussion of a sector of the *second category*, of which, to fix our ideas, we assume, that it is bounded by two positive base curves.

Let us consider the set of points lying in the sector or on its boundary, through which curves of the second kind not crossing the base curves can be drawn. This set of points cannot approach P indefinitely, as otherwise it would give rise to a negative curve of the third kind not crossing the base curves, which is excluded.

In the same way as for the elliptic sectors we destroy the regions covered by this set of points, and there remains a sector of the second category bounded by a modified arc of c , inside which no curves of the second kind not crossing the base curves can be drawn.

In the modified sector we now consider the set of points, through which curves of the first kind not crossing the base curves can be drawn, and it is clear that this set of points cannot indefinitely approach the just now modified curve c . The regions covered by it are therefore bounded by a finite or denumerable number of curves

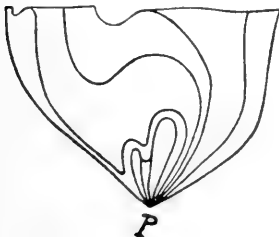


Fig. 5. Parabolic sector.

of the first kind, *not* enclosing each other, *not* indefinitely approaching c , and each enclosing a domain which forms a *leaf*, not differing from those appearing in the hyperbolic sectors.

By the method applied above already several times the region outside the leaves can be filled with curves of the third kind (for instance we can choose for them the system of base curves

present already in the sector), and finally each of the leaves with curves of the first kind (see fig. 5).

The sectors of the second category we shall call *positive* (resp. *negative*) *parabolic sectors*.

In special cases the whole inner domain of c can reduce itself to a single positive (resp. negative) parabolic sector. A point zero where this occurs we shall call a *source point* resp. *vanishing point*.

§ 4.

The structure of the field in the vicinity of an isolated singular point. Second principal case.

In this case any vicinity of P contains a simple closed tangent curve inside which P lies. We can then construct a fundamental series c, c', c'', \dots of simple closed tangent curves converging to P , of which each following one lies inside each preceding one, and we can fill in the following way the inner domain of c with tangent curves not crossing each other.

In each annular domain between two curves $c^{(n)}$ and $c^{(n+1)}$ we choose a point having from the boundary of that domain a distance as great as possible and we lay through it a tangent curve situated in the annular domain. According to § 1 it is either closed or it gives rise to two closed curves, situated in the annular domain with its boundary, into which it terminates or to which it converges spirally, and which we draw likewise. (These closed tangent curves can entirely or partially coincide with $c^{(n)}$ or $c^{(n+1)}$). So the annular domain is either made singly connected or it is divided into two or three (annular or singly connected) new domains.

In each of these we again choose a point having from the boundary a distance as great as possible and we lay through it again a tangent curve. A singly connected domain is certainly divided by it into two singly connected domains; on an annular domain it has the effect just now mentioned.

We repeat this process indefinitely. For each domain it can

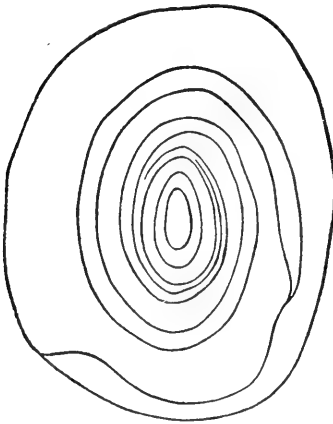


Fig. 6. Rotation point.

happen only once that it undergoes no division; after that namely it becomes singly connected, so is divided at each new insertion of a tangent curve (see fig. 6).

We finally add the limit curves, and we prove in the same way as in § 2 that then through each point of the inner domain of c passes a tangent curve.

A point zero being in the second principal case we shall call a *rotation point*.

So we can say:

THEOREM 4. *An isolated singular point is either a rotation point, or a vicinity of it can be divided into a finite number of hyperbolic, elliptic, and parabolic sectors.*

The filling of a vicinity of a non-singular point in § 2 furnishes in this terminology two hyperbolic and two parabolic sectors.

We must add the observation that in the most general case, where neither in a singular, nor in a non-singular point the tangent curve is determined, sometimes by a modified method of construction, the structure of the first principal case can be given to a vicinity of a point zero being in the second principal case.

Even the form of the sector division of the first principal case is then not necessarily unequivocally determined. Out of the reasonings of the following § we can, however, deduce that, if modifications are possible in the form of the sector division, the difference of the number of elliptic sectors and the number of hyperbolic sectors always remains the same.

§ 5.

The reduction of an isolated singular point.

For what follows it is desirable to represent the domain γ on a Euclidean plane, and farther to substitute for the curve c a simple closed curve c' emerging nowhere from c , containing likewise P in its inner domain, and consisting of arcs of tangent curves and of orthogonal curves. In the second principal case this is already attained, and in the first principal case we have to modify in a suitable way only those arcs of c which bound the hyperbolic and the parabolic sectors.

In a hyperbolic sector we effect this by choosing a point on each of the two bounding base curves, and by drawing from those points H and K inside into the sector orthogonal arcs not intersecting one another. Then there is certainly an arc of a curve of the second kind

joining a point B of one of these orthogonal arcs with a point C of the other, and we bound the modified sector by the orthogonal arcs HB and CK and the tangent arc BC .

If a parabolic sector is bounded by the base curves k and k' , it is always possible to choose between them a finite number of base curves k_1, k_2, \dots, k_n in such a way, that each k_ν and $k_{\nu+1}$ can be connected, inside the sector but outside the leaves lying in it, by an orthogonal arc. By those orthogonal arcs and the arcs of base curves joining their endpoints we bound the modified sector. The simple closed curve c' obtained in this way has a direction of tangents varying everywhere continuously, with the exception of a finite number of rectangular bends. To a definite sense of circuit of c' , which we shall call the positive one, corresponds in each point of c' a definite tangent vector, and for a full circuit of c' that tangent vector describes a positive angle 2π .

We shall now consider two successive parabolic sectors, π_1 and π_2 , of which (for the positive sense of circuit) the first is positive, therefore the second negative, and we suppose them to be separated by a hyperbolic sector \mathcal{F} . On the orthogonal arcs belonging to the boundary of π_1 the given vector then forms with the tangent vector an angle $\left(2n - \frac{1}{2}\right)\pi$ (measured in the positive sense), on the orthogonal arcs belonging to the boundary of π_2 an angle $\left(2n + \frac{1}{2}\right)\pi$.

The transition takes place along the tangent arc belonging to the boundary of \mathcal{F} , by a negative rotation over an angle π of the given vector with respect to the tangent vector.

The same remains the case if we suppose π_1 to be negative, π_2 to be positive.

But if we suppose \mathcal{F} to be an elliptic sector, then the transition under discussion takes place along the tangent arc bounding \mathcal{F} , by a positive rotation over an angle π of the given vector with respect to the tangent vector.

As now the total angle, which the given vector describes for a full circuit of c' , is equal to the total angle which the tangent vector describes plus the total angle which the given vector describes with respect to the tangent vector, the former angle is equal to $\pi(2 + n_1 - n_2)$, where n_1 represents the number of elliptic sectors, n_2 the number of hyperbolic ones.

Let further j be an arbitrary simple closed curve enveloping P , but enveloping no other singular point, then we can transform c' into j by continuous modification in such a way, that at every moment

P , but no other singular point, is enveloped by the modified curve. If we consider for each of the intermediary curves the total angle which the given vector describes by a positive circuit, then on one hand it can only have continuous modifications and on the other hand it must remain a multiple of 2π . Thus it remains unchanged, and we can formulate:

THEOREM 5. *The total angle which, by a circuit of a simple closed curve enveloping only one point zero, the vector describes in the sense of that circuit, is equal to $\pi(2 + n_1 - n_2)$, where n_1 represents the number of elliptic sectors, n_2 the number of hyperbolic ones, which appear when a vicinity of the point zero is covered with tangent curves not crossing each other.*

In particular for source points, vanishing points and rotation points this angle is equal to $+2\pi$.

We now surround P with a simple closed curve z which can be supposed as small as one likes, and we leave the vector distribution outside z and on z unchanged, but inside z we construct a modified distribution in the following way:

Let us first suppose that for a positive circuit of z the vector describes a positive angle $2n\pi$. From an arbitrary point Q inside z we draw to z n arcs of simple curve $\beta_1, \beta_2, \dots, \beta_n$, not cutting each other and determining in this order a positive sense of circuit. Let us call ${}_p z$ the arc of z lying between β_p and β_{p+1} , and G_p the domain bounded by $\beta_p, {}_p z$ and β_{p+1} . Along β_1 we bring an arbitrary continuous vector distribution becoming nowhere zero and passing on z into the original one. Then along β_2 such a one passing on z and in Q into the already existing vectors, that along the boundary of G_1 positively described the vector turns a positive angle 2π . Then along β_3 such a one passing on z and in Q into the existing vectors, that along the boundary of G_2 positively described the vector turns a positive angle 2π , etc.

As the angle described by the vector in a positive circuit of z is equal to the sum of the angles described in positive circuits of the boundaries of the domains G_1, G_2, \dots, G_n , it is finally evident, that also for a positive circuit of G_n the vector describes a positive angle 2π .

In each of the domains G_p with boundary z_p we choose a simple closed curve c_p not meeting z_p , of which in a suitable system of coordinates the equation can be written in the form $x_p^2 + y_p^2 = r^2$.

Inside and on c_p we introduce a finite continuous vector distribution vanishing only in the point $(o, o)_p$, which is directed along the lines

$y'_p = \alpha$ and from the point $(\alpha, \alpha)_p$. This vector describes along c_p a positive angle 2π , just as the existing one along z_p . If then according to SCHOENFLIES we fill the annular domain between z_p and c_p with simple closed curves enveloping each other and as functions of a cyclic parameter passing continuously into each other, then we can thereby at the same time make the vector distribution along z_p pass continuously into that along c_p , and in this way give to the annular domain between z_p and c_p a finite continuous vector distribution vanishing nowhere. Inside z_p we have now obtained a finite continuous vector distribution, having but *one* point zero, namely the point $(\alpha, \alpha)_p$, and that a source point of very simple structure, which we shall call a *radiating point*.

And the inner domain of z is covered with a finite continuous vector distribution passing on z into the original one and possessing inside z , instead of the original point zero P , n radiating points.

Let us furthermore suppose that for a positive circuit of z the vector describes a negative angle $2n\pi$. In an analogous way as above we then divide the inner domain of z into n regions G'_p with boundaries z_p , and we bring along each of these boundaries such a vector distribution, that for a positive circuit of z_p the vector describes a negative angle 2π .

The curves c_p are introduced again as above, but inside and on c_p we introduce a finite continuous vector distribution vanishing only in the point $(0,0)_p$, which is directed along the lines $x_p y_p = \alpha$. For a positive circuit this vector describes along c_p a negative angle 2π , just as the existing vector along z_p .

So the annular domain between z_p and c_p can be filled up in an analogous way as just now with a finite continuous vector distribution vanishing nowhere, and the whole distribution inside z_p possesses then only *one* point zero, namely the point $(0,0)_p$, having four hyperbolic sectors of very simple form (the four separating parabolic sectors are each reduced to a single line), which structure we characterize by the name of *reflexion point*.

After this the inner domain of z is covered with a finite continuous vector distribution passing on z into the original one and possessing inside z , instead of the original point zero P , n reflexion points.

Let us finally suppose that for a circuit of z the total angle described by the vector is zero. We can then choose inside z such a simple closed curve c , that in a suitable system of coordinates its equation can be written in the form $x^2 + y^2 = r^2$. Inside and on c we introduce a finite continuous vector distribution vanishing nowhere,

which is directed along the lines $y = a$. The total angle described by this vector along c is zero, just as the one described by the existing vector along z . The annular domain between z and c can thus be filled up as in the two preceding cases with such a finite continuous vector distribution, that the whole distribution inside z is now free of points zero.

So we can formulate:

THEOREM 6. *A finite continuous vector distribution with a finite number of points zero can be transformed, by modifications as small as one likes inside vicinities of the points zero which can be chosen as small as one likes, into a new finite continuous vector distribution which has as points zero only a finite number of radiating points, and a finite number of reflexion points.*

In particular those points zero about which the angle, described by the vector for a positive circuit, is positive, are broken up into radiating points; those about which this angle is negative, are broken up into reflexion points; whilst those for which it is zero, vanish.

In a following communication we shall extend this theorem to distributions with an infinite (denumerable or continuous) number of points zero.

§ 6.

Remarks on the tangent curves and singular points on a sphere.

If we have on a sphere a finite continuous vector distribution with a finite number of singular points, then the reasonings of § 1 lead with small modifications to:

THEOREM 7. *A tangent curve to a finite continuous vector distribution with a finite number of singular points on a sphere is either a simple closed curve, or save its ends it is an arc of simple curve, of which the pursuing as well as the recurring branch either stops at a point zero, or enters into a simple closed tangent curve, or converges spirally to a circumference consisting of one or more simple closed tangent curves.*

From this ensues that also on a sphere a tangent curve cannot return into indefinite vicinity of one of its points, after having reached a finite distance from it, unless it be, to close itself in that point.

Out of the reasoning of § 1 we can deduce farthermore without difficulty that a fundamental series of closed tangent curves with the property that of the two domains determined by one of them,

one contains no points of the preceding, the other no points of the following curves, converges either to a single singular point, or to the outer circumference, consisting of simple closed tangent curves, of a domain or set of domains.

Let now an arbitrary finite continuous vector distribution on a sphere be given. On account of § 5 we reduce it by means of indefinitely small modifications to a "reduced distribution", possessing as singular points only radiating points and reflexion points, and we investigate the tangent curves of that reduced distribution.

A closed tangent curve can possess no radiating points, but reflexion points it can possess (its tangent direction shows there a rectangular bend).

On the other hand a tangent curve can only stop at a radiating point.

We now consider an arbitrary tangent curve: according to theorem 7 it is either an arc of simple curve joining two radiating points, or it gives rise to a simple closed tangent curve j_0 , which divides the sphere into two domains G and G' .

Then on j_0 no radiating point can lie, but we shall prove, that in G as well as in G' there must lie one.

If namely there were no radiating point in G , we could consider within G a new tangent curve, and as this would not be able to stop in G , it would on account of theorem 2 give rise to a new simple closed tangent curve j_1 enclosing a domain G_1 being a part of G . Within G_1 we could again consider an arbitrary tangent curve, and in this way we should arrive at a simple closed tangent curve j_2 enclosing a domain G_2 being a part of G_1 .

Continuing this process indefinitely we construct a fundamental series of closed tangent curves $j_0, j_1, j_2, j_3, \dots$, which cannot converge to a single singular point, as neither a radiating point nor a reflexion point contains closed tangent curves in an indefinitely small vicinity. On account of the remark made at the beginning of this § there must thus be at least *one* domain G_ω , bounded by a simple closed tangent curve j_ω , and contained in each of the domains G_1, G_2, G_3, \dots .

Within G_ω we could again construct a closed tangent curve $j_{\omega+1}$ bounding a domain $G_{\omega+1}$ being a part of G_ω , and we could continue this process to *any* index of the second class of numbers, which on the other hand is impossible, as the set of domains $G - G_1, G_1 - G_2, \dots, G_\omega - G_{\omega+1}, \dots, G_x - G_{x+1}, \dots$ must remain denumerable.

So we finally formulate:

THEOREM 8. *A reduced distribution on a sphere possesses at least two radiating points.*

Mathematics. — “*The oscillations about a position of equilibrium where a simple linear relation exists between the frequencies of the principal vibrations.*” (Second part). By H. J. E. BETH.
(Communicated by Prof. D. J. KORTEWEG.)

(Communicated in the meeting of February 26, 1910).

$$S = 4. \text{ } ^1)$$

§ 14. In this case the ordinary expansions in series hold as long as $\frac{g}{n_1}$ is great with respect to $\left(\frac{h}{R_1}\right)^2$ (see page 7 of the paper by Prof. KORTEWEG, mentioned above). The difficulty arises as soon as $\frac{g}{n_1}$ has fallen to the order $\left(\frac{h}{R_1}\right)^2$. The calculations not getting simpler with the absence of a residue of relation, we shall immediately assume a residue of relation of order h^2 .

When the relation

$$n_2 + g = 3n_1$$

exists and we proceed to investigate with a view to this which terms in (2) (page 620 of these Proceedings) become disturbing in the sense indicated in § 3, we easily see that no terms of order h^2 appear among the disturbing ones. So when determining the first approximation we may omit the terms of order h^3 in the equation of the surface, which terms agree with the just mentioned terms of order h^2 . It then becomes

$$z = \frac{1}{g} (c_1 x^2 + c_2 y^2 + e_1 x^4 + e_2 x^2 y + e_3 x^2 y^2 + e_4 x y^3 + e_5 y^4);$$

for we need not take for the first approximation in the equations of movement any terms of higher order than h^3 .

The abridged equations of motion, containing only terms of order h , still run as follows:

$$\left. \begin{aligned} \ddot{x} + 2c_1 x &= 0, \\ \ddot{y} + 2c_2 y &= 0. \end{aligned} \right\}$$

Now

$$n_1 = \sqrt{2c_1} \quad , \quad n_2 = \sqrt{2c_2}$$

are the frequencies of the principal vibrations.

¹⁾ For the case $S = 3$ see 1st part, pages 619—635 of these Proceedings.

So

$$2e_3 = (3n_1 - \varrho)^2.$$

We change the abridged equations into:

$$\begin{cases} \ddot{x} + n_1^2 x = 0, \\ \ddot{y} + 9n_1^2 y = 0; \end{cases}$$

but then we must admit into the function R a term:

$$3n_1 \varrho y^2.$$

The canonical solution of the abridged equations is:

$$\begin{aligned} x &= \frac{\sqrt{\alpha_1}}{n_1} \cos(n_1 t + 2n_1 \beta_1), \\ y &= \frac{\sqrt{\alpha_2}}{3n_1} \cos(3n_1 t + 6n_1 \beta_2). \end{aligned}$$

To find which functions the α 's and β 's are of t , we must investigate which form the function R now assumes.

§ 15. As the disturbing terms in the equations of motion are of order h^3 we shall find that $\dot{\alpha}_1$, $\dot{\alpha}_2$, $\dot{\beta}_1$, and $\dot{\beta}_2$ can never exceed order h . Of this we may make use to simplify the terms of order h^3 containing \dot{x} , \dot{y} , \dot{x}^2 , and \dot{y}^2 . We may namely replace in those terms:

$$\begin{aligned} \dot{x}^2 &\text{ by } \alpha_1 - n_1^2 x^2, \\ \dot{y}^2 &\text{ ,, } \alpha_2 - 9n_1^2 y^2, \\ \dot{x} &\text{ ,, } -n_1^2 x \end{aligned}$$

and

$$\dot{y} \text{ ,, } -9n_1^2 y.$$

Then the equations become:

$$\begin{aligned} \ddot{x} + n_1^2 x + 4e_1 x^3 + 3e_2 x^2 y + 2e_3 x y^2 + e_4 y^3 + \\ + \frac{n_1^4}{g^2} (\alpha_1 + 9\alpha_2) x - \frac{2n_1^6}{g^2} (x^2 + 81 y^2) x = 0, \\ \ddot{y} + 9n_1^2 y - 6n_1 \varrho y + e_2 x^3 + 2e_3 x^2 y + 3e_4 x y^2 + 4e_5 y^3 + \\ + \frac{9n_1^4}{g^2} (\alpha_1 + 9\alpha_2) y - \frac{18n_1^6}{g^2} (x^2 + 81 y^2) y = 0. \end{aligned}$$

Now the terms of order h^3 are all disturbing except $e_4 y^3$ in the first and $3e_4 x y^2$ in the second equation; so these may be omitted.

The terms $3e_2 x^2 y$ in the first and $e_2 x^3$ in the second equation owe their disturbing property to the supposed relation.

The remaining terms are always disturbing, also when no relation exists.

To transform the equations to such a form that the disturbing terms may be regarded as derivatives of one and the same function resp. to x and y , let us consider the term with xy^2 in the first and that with x^2y in the second equation. If we substitute the expressions found above as first approximation for x and y in these terms, after the development of the products and powers of the cosines among others terms will appear, differing only in coefficient from the expressions indicated for x and y ; the remaining terms which appear are not disturbing. From this ensues that we may replace :

$$\text{in the first equation: } xy^2 \text{ by } \frac{1}{18} \frac{\alpha_2}{n_1^2} x,$$

$$\text{in the second equation: } x^2y \text{ by } \frac{1}{2} \frac{\alpha_1}{n_1^2} y.$$

Accordingly the equations may be written :

$$\left. \begin{aligned} x + n_1^2 x + \left(4e_1 - \frac{2n_1^6}{g^2} \right) x^3 + 3e_2 x^2 y + \left(\frac{e_3}{9n_1^2} \alpha_2 + \frac{n_1^4}{g^2} \alpha_1 \right) x &= 0, \\ \ddot{y} + 9n_1^2 y + \left(4e_5 - \frac{1458n_1^6}{g^2} \right) y^3 + e_2 x^3 + \\ &+ \left(-6n_1 g + \frac{e_3}{n_1^2} \alpha_1 + \frac{81n_1^4}{g^2} \alpha_2 \right) y = 0. \end{aligned} \right\}$$

We thus see that they take the form of :

$$\left. \begin{aligned} \ddot{x} + n_1^2 x - \frac{\partial R}{\partial x} &= 0, \\ \ddot{y} + 9n_1^2 y - \frac{\partial R}{\partial y} &= 0; \end{aligned} \right\}$$

where :

$$\begin{aligned} -R &= \left(e_1 - \frac{n_1^6}{2g^2} \right) x^4 + \left(e_5 - \frac{729n_1^6}{2g^2} \right) y^4 + \frac{1}{2} \left(\frac{e_3}{9n_1^2} \alpha_2 + \frac{n_1^4}{g^2} \alpha_1 \right) x^2 + \\ &+ \frac{1}{2} \left(-6n_1 g - \frac{e_3}{n_1^2} \alpha_1 + \frac{81n_1^4}{g^2} \alpha_2 \right) y^2 - e_2 x^3 y. \end{aligned}$$

§ 16. We must now write R as function of the α 's and β 's by substituting for x and y , in the expressions obtained, the expressions by which they are represented at first approximation, and by retaining only those terms in which t does not appear explicitly. Thus we arrive at :

$$-R = \frac{1}{2} a \alpha_1^2 + b \alpha_1 \alpha_2 + \frac{1}{2} c \alpha_2^2 + \frac{1}{2} q' h^2 \alpha_2 + m_1 \alpha_1^2 \alpha_2^2 \cos \varphi,$$

where :

$$a = \frac{3e_1}{4n_1^4} + \frac{n_1^2}{8g^2},$$

$$b = \frac{e_3}{18n_1^4},$$

$$c = \frac{e_5}{108n_1^4} + \frac{9n_1^2}{8g^2},$$

$$m_1 = \frac{e_2}{24n_1^4},$$

$$Q = -\frac{Q}{6n_1 h^2},$$

$$\varphi = 6n_1 (\beta_1 - \beta_2).$$

The system of equations giving the time-variability of the α 's and β 's, is now :

$$\left. \begin{aligned} \frac{d\alpha_1}{dt} &= 2Nm_1 \alpha_1^{\frac{3}{2}} \alpha_2^{\frac{1}{2}} \sin \varphi, \\ \frac{d\alpha_2}{dt} &= -2Nm_1 \alpha_1^{\frac{3}{2}} \alpha_2^{\frac{1}{2}} \sin \varphi, \\ \frac{d\beta_1}{dt} &= a\alpha_1 + b\alpha_2 + \frac{3}{2} m_1 \alpha_1^{\frac{1}{2}} \alpha_2^{\frac{1}{2}} \cos \varphi, \\ \frac{d\beta_2}{dt} &= b\alpha_1 + c\alpha_2 + Q' h^2 + \frac{1}{2} m_1 \alpha_1^{\frac{3}{2}} \alpha_2^{-\frac{1}{2}} \cos \varphi; \end{aligned} \right\} \quad (17)$$

where N is put instead of $3n_1$.

From this system it appears at once that :

$$\frac{d\alpha_1}{dt} + \frac{d\alpha_2}{dt} = 0,$$

therefore

$$\alpha_1 + \alpha_2 = \text{constant}.$$

So we put :

$$\alpha_1 = R_0^2 n_1^2 h^2 \xi, \quad \alpha_2 = R_0^2 n_1^2 h^2 (1 - \xi).$$

Furthermore according to § 4 :

$$\frac{1}{2} a\alpha_1^2 + b\alpha_1 \alpha_2 + \frac{1}{2} c\alpha_2^2 + Q' h^2 \alpha_2 + m_1 \alpha_1^{\frac{3}{2}} \alpha_2^{\frac{1}{2}} \cos \varphi = \text{constant}$$

is an integral of the system.

By introduction of ξ this integral takes the form of :

$$\xi \sqrt{\xi} (1 - \xi) \cos \varphi = p\xi^2 + q\xi + r, \quad \dots \dots (18)$$

where :

$$p = \frac{1}{m_1} \left(-\frac{1}{2} a + b - \frac{1}{2} c \right),$$

$$q = \frac{1}{m_1} \left(-b + c + \frac{Q'}{n_1^2 R_0^2} \right),$$

$$r = \frac{1}{m_1} \left(-\frac{1}{2} c - \frac{Q'}{n_1^2 R_0^2} - \frac{C}{n_1^4 R_0^4} \right),$$

where C represents a constant, dependent on the initial state.

The first equation of (17) becomes by the introduction of ξ :

$$\frac{d\xi}{dt} = \frac{2}{9} m_1 R_0^2 N^3 h^2 \cdot \xi \sqrt{\xi(1-\xi)} \cdot \sin \varphi. \quad (19)$$

By eliminating φ between (18) and (19) we arrive at :

$$\frac{d\xi}{\sqrt{\xi^3(1-\xi) - (p\xi^2 + q\xi + r)^2}} = \pm \frac{2}{9} m_1 R_0^2 N^3 h^2 \cdot dt.$$

Let

$$f(\xi) = \xi^3(1-\xi) - (p\xi^2 + q\xi + r)^2,$$

then $f(\xi) > 0$ for the initial value of ξ , but $f(\xi) < 0$ for $\xi = 0$ and $\xi = 1$; so $f(\xi)$ becomes zero for two values ξ_1 and ξ_2 lying between 0 and 1.

So ξ will generally vary periodically between two limits. It may be expressed in the time with the aid of elliptic functions, after which β_1 , β_2 , x , and y are also known as functions of the time.

For the extreme values zero and one of the modulus α of the elliptic functions ($\alpha = \sqrt{\frac{(\beta^2 - \alpha)(\xi_2 - \xi_1)}{(\beta - \xi_2)(\alpha - \xi_1)}}$, when the equation $f(\xi) = 0$ has two real roots α and β besides ξ_1 and ξ_2) we get special cases.

Osculating curves.

§ 17. At first approximation we have found :

$$x = \frac{\sqrt{a_1}}{n_1} \cos(n_1 t + 2n_1 \beta_1),$$

$$y = \frac{\sqrt{a_2}}{3n_1} \cos(3n_1 t + 6n_1 \beta_2),$$

where the α 's and β 's slowly vary with the time.

By introduction of ξ and φ and by change of the origin of time we find that we may determine the equation of an osculating curve by eliminating t between

$$x = R_0 h \sqrt{\zeta} \cos n_1 t$$

and

$$y = \frac{1}{3} R_0 h \sqrt{1-\zeta} \cos(3n_1 t - \varphi).$$

For ζ and φ we must substitute the values, which these quantities have at the moment for which we wish to know the osculating curve.

The osculating curves are Lissajous curves answering to the value $\frac{1}{3}$ for the ratio of the periods of the vibrations. They are described

in the rectangles having as sides $2 R_0 h \sqrt{\zeta}$ and $\frac{2}{3} R_0 h \sqrt{1-\zeta}$.

As ζ varies between two limits the rectangles in which the curves are described lie between two extremes. The vertices lie on the circumference of an ellipse having $2 R_0 h$ and $\frac{2}{3} R_0 h$ as lengths of axes.

The shape of the curve described in a definite rectangle is still dependent on the value of φ , i. e. on the value of the difference in phase at the moment of the greatest deviation to the right.

To an arbitrary value of φ the wellknown Lissajous curve with two nodes of fig. 8 answers. For $\varphi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ the curve is symmetrical in respect to the axes; the nodes lie in the X -axis on either side of O at distances $\frac{1}{2} R_0 h$ (fig. 9). For $\varphi = 0$ or π we get a curve, which is described in both directions alternately and which passes through O (fig. 10).

In fig. 11 we find some of those osculating curves represented for a definite case of motion: *two* belonging to $\varphi = \pi$; *two* for $\varphi = \frac{\pi}{2}$, and *one* for an arbitrary value of $\varphi \left(> \frac{\pi}{2} \right)$.

Out of (19) follows that $\frac{d\zeta}{dt} = 0$ for $\sin \varphi = 0$. In the extreme rectangles the curves are described which we have for $\varphi = 0$ or π . Now a number of different cases are possible, of which we get a clear representation by representing equation (18) in polar coordinates. In fig. 12 some of the curves obtained in this way are represented, where φ is taken as polar angle, $\sqrt{1-\zeta}$ as radius vector. The different shapes of the curves correspond to the roots of the equation:

$$\zeta^3 (1 - \zeta) - (p\zeta^2 + q\zeta + r)^2 = 0. \quad . \quad . \quad . \quad (20)$$

The cases are:

1. The curve indicated in the figure by --- keeps to the right or to the left of O_1 ; φ changes between two limits; the limits are equal and opposite; the positive is smaller than $\frac{\pi}{2}$. For the extreme values of ζ we find φ either both times 0 or both times π .

2. The curve — — — intersects the straight line $\varphi = \frac{\pi}{2}$ at two points above O_1 and at 2 points below O_1 . For the extreme values of ζ we again find φ either both times 0 or both times π .

3. The curve consists of two closed parts (a continuous line in the figure), which surround O_1 . Now φ assumes all values. For the extreme values of ζ $\varphi = 0$ one time $\varphi = 0$ and $\varphi = \pi$ the other.

The transition case between 2 and 3 is represented by — . . . —.

Fig. 11 relates to the 2nd case; for the two extreme values of ζ we find $\varphi = \pi$.

Special cases.

§ 18. These occur for the extreme values of the modulus z of the elliptic functions; two roots of equation (20) have coincided.

1. $z = 1$. The elliptic functions pass into hyperbolic ones. The geometrical representation just now discussed of the relation between ζ and φ and already mentioned as transition case between the second and third cases has a node situated on the axis of the angles. The form of motion approaches asymptotically to a form of motion belonging to $\varphi = 0$ or $\varphi = \pi$.

2. $z = 0$. The elliptic functions pass into goniometrical ones. The curve of fig. 12 becomes an isolated point C (special case belonging to the 1st case of § 17 as limiting case) or it consists of an isolated point and a closed curve (special case belonging to the 3rd case of § 17 as limiting case). If the initial value of ζ coincides with the twofold root of (20) we find that ζ remains constant; φ is continually 0 or π . Thus the same curve is continually described.

Arbitrary mechanism with 2 degrees of freedom for which $S = 4$.

§ 19. In the case that $n_2 = 3n_1 + \varrho$ the terms of order h^2 can give no disturbing terms in the equations of motion.

So we may write:

$$U = \frac{1}{2} n_1^2 q_1^2 + \frac{1}{2} n_2^2 q_2^2 + U_4,$$

where U_4 represents a homogeneous function of degree 4 in q_1 and q_2 . Furthermore we find:

$$T = \frac{1}{2} \dot{q}_1^2 + \frac{1}{2} \dot{q}_2^2 + \frac{1}{2} P_1 \dot{q}_1^2 + P_2 \dot{q}_1 \dot{q}_2 + \frac{1}{2} P_3 \dot{q}_2^2;$$

where:

$$P_1 = a_1 q_1^2 + a_2 q_1 q_2 + a_3 q_2^2,$$

$$P_2 = b_1 q_1^2 + b_2 q_1 q_2 + b_3 q_2^2,$$

$$P_3 = c_1 q_1^2 + c_2 q_1 q_2 + c_3 q_2^2,$$

the a 's, b 's, and c 's being constants.

The equations of LAGRANGE become:

$$\begin{aligned} \ddot{q}_1 + n_1^2 q_1 &= -P_1 \ddot{q}_1 - P_2 \ddot{q}_2 - \frac{1}{2} \frac{\partial P_1}{\partial q_1} \dot{q}_1^2 - \frac{\partial P_1}{\partial q_2} \dot{q}_1 \dot{q}_2 + \\ &+ \left(\frac{1}{2} \frac{\partial P_3}{\partial q_1} - \frac{\partial P_2}{\partial q_2} \right) \dot{q}_2^2 - \frac{\partial U_4}{\partial q_1}, \\ \ddot{q}_2 + n_2^2 q_2 &= -P_2 \ddot{q}_1 - P_3 \ddot{q}_2 + \\ &+ \left(\frac{1}{2} \frac{\partial P_1}{\partial q_2} - \frac{\partial P_2}{\partial q_1} \right) \dot{q}_1^2 - \frac{\partial P_3}{\partial q_1} \dot{q}_1 \dot{q}_2 - \frac{1}{2} \frac{\partial P_3}{\partial q_2} \dot{q}_2^2 - \frac{\partial U_4}{\partial q_2}. \end{aligned}$$

In the same way as was done in § 15 we may replace \ddot{q}_1 , \ddot{q}_2 , \dot{q}_1^2 , and \dot{q}_2^2 in the terms of order h^3 by others.

Now in the first equation a term $-a_2 q_1 \dot{q}_1 \dot{q}_2$ appears which we must consider separately (in the second equation also there are terms containing $q_1 \dot{q}_2$, but these are not disturbing).

We introduce for this a new variable q_1' in such a way that:

$$q_1' = q_1 + \frac{1}{4} a_2 q_1^2 q_2.$$

Then we find:

$$\dot{q}_1' = \dot{q}_1 + \frac{1}{4} a_2 q_1^2 \dot{q}_2 + \frac{1}{2} a_2 q_2 (q_1 \ddot{q}_1 + \dot{q}_1^2) + a_2 q_1 \dot{q}_1 \dot{q}_2,$$

where \ddot{q}_1 and \ddot{q}_2 in the terms of order h^3 may again be simplified.

Of the terms now appearing in the equations of motion the following are disturbing: in the first equation those with $h^2 q_1$, q_1^3 , $q_1^2 q_2$ and $q_1 q_2^2$, in the second those with $h^2 q_2$, q_1^3 , q_2^3 and $q_1^2 q_2$. Now just as in § 15 the terms with $q_1 q_2^2$ in the first equation, those with $q_1^2 q_2$ in the second equation may still be simplified.

If we perform these calculations the result proves that the terms of order h^3 to be inserted in the equations may be put in this form:

$$P h^3 q_1 + c q_1^2 q_2 + c q_1^3 \text{ in the first equation.}$$

$$Q h^2 q_2 + d q_1^3 + d q_2^3 \text{ ,, ,, second ,,}$$

Here P and Q are homogeneous quadratic functions of $\sqrt{a_1}$ and $\sqrt{a_2}$; and

$$e = \frac{1}{4} (3n_1^2 - n_2^2) a_2 + b_1 n_2^2 - 3l,$$

$$f = -\frac{1}{2} n_1^2 a_2 + 3b_1 n_1^2 - l.$$

(The terms $-3l$ in e and $-l$ in f originate from the term $lq_1^2 q_2$ appearing in U_4).

In the terms of higher order we may substitute $3n_1$ for n_2 in the coefficients. We then find:

$$e = 3 \left\{ \left(-\frac{1}{2} a_2 + 3b_1 \right) n_1^2 - l \right\},$$

$$f = \left(-\frac{1}{2} a_2 + 3b_1 \right) n_1^2 - l.$$

So we find that

$$e = 3f.$$

We may now write the equations of motion:

$$\left. \begin{aligned} \ddot{q}_1 + n_1^2 q_1 &= \frac{\partial R}{\partial q_1}, \\ \ddot{q}_2 + n_2^2 q_2 &= \frac{\partial R}{\partial q_2}; \end{aligned} \right\}$$

where

$$R = \frac{1}{2} Ph^2 q_1^2 + \frac{1}{2} Qh^2 q_2^2 + f q_1^2 q_2 + \frac{1}{4} c q_1^4 + \frac{1}{4} d q_2^4.$$

So they get the same form as for the simple mechanism so that in case $S=4$ also the horizontal projection of the point moving over the surface may be regarded as representative point for an arbitrary mechanism with 2 degrees of freedom.

$$S = 2.$$

§ 20. So we suppose that the relation exists:

$$n_1 = n_2 + \varrho,$$

where $\frac{\varrho}{n_1}$ is of order $\left(\frac{h}{R_1}\right)^2$. However, as we have already seen in the cases $S=3$ and $S=4$ in which way such a residue of relation may be taken into account by inserting in the function R a term with ϱa_2 , we restrict ourselves here to the case that the residue of relation is zero, therefore:

$$n_2 = n_1 \equiv n.$$

For the surface the lowest point is an umbilical point. To this belongs as special case the surface of revolution with the Z -axis as axis of revolution, which case is treated by Prof. KORTEWEG at the close of his treatise quoted before.

Omitting the terms of higher order than h^4 , because in the equations of motion we admit no terms of higher order than h^3 , and omitting the terms of order h^3 , because in the equations of motion no terms of order h^2 can be disturbing, we may write the equation of the surface:

$$z = \frac{1}{g} (c_1 x^2 + c_2 y^2 + e_1 x^4 + e_2 x^3 y + e_3 x^2 y^2 + e_4 x y^3 + e_5 y^4),$$

where we avail ourselves of the fact, that by means of a rotation of the system of coordinates round the Z -axis the coefficients of xy^3 and $x^3 y$ may be rendered equal.

The solution at first approximation is:

$$x = \frac{\sqrt{\alpha_1}}{n} \cos (nt + 2n\beta_1),$$

$$y = \frac{\sqrt{\alpha_2}}{n} \cos (nt + 2n\beta_2);$$

where $n = \sqrt{2c_1} = \sqrt{2c_2}$.

§ 21. Let us now pass to the simplification of the equations of motion. Corresponding to what was said in § 15 for the case $S=4$ we may here replace in the terms of order h^3 of the equations of motion:

$$\begin{array}{l} \dot{x}^2 \text{ by } \alpha_1 - n^2 x^2, \\ \dot{y}^2 \text{ ,, } \alpha_2 - n^2 y^2, \\ \ddot{x} \text{ ,, } -n^2 x, \\ \ddot{y} \text{ ,, } -n^2 y. \end{array}$$

The equations become:

$$\left. \begin{array}{l} \ddot{x} + n^2 x + 4e_1 x^3 + 3e_2 x^2 y + 2e_3 x y^2 + e_4 y^3 + \frac{n^4}{g^2} (\alpha_1 + \alpha_2) x - \frac{2n^6}{g^2} (x^2 + y^2) x = 0, \\ \ddot{y} + n^2 y + e_5 x^3 + 2e_3 x^2 y + 3e_2 x y^2 + 4e_4 y^3 + \frac{n^4}{g^2} (\alpha_1 + \alpha_2) y - \frac{2n^6}{g^2} (x^2 + y^2) y = 0. \end{array} \right\}$$

Here we may omit no terms, for all the terms of order h^3 are disturbing. The equations may be written as follows:

$$\left. \begin{array}{l} \ddot{x} + n^2 x - \frac{\partial R}{\partial x} = 0, \\ \ddot{y} + n^2 y - \frac{\partial R}{\partial y} = 0; \end{array} \right\}$$

where we must take

$$- R \equiv e_1 x^4 + e_2 x^3 y + e_3 x^2 y^2 + e_4 x y^3 + e_5 y^4 + \\ + \frac{n^4}{2g^2} (\alpha_1 + \alpha_2) (x^2 + y^2) - \frac{n^6}{2g^2} (x^2 + y^2)^2.$$

§ 22. If we substitute in the function R for x and y the expressions assumed at first approximation and if we retain only those terms not containing t explicitly, we arrive at

$$- R \equiv \frac{1}{2} a \alpha_1^2 + b \alpha_1 \alpha_2 + \frac{1}{2} c \alpha_2^2 + f \alpha_1 \alpha_2 \sin^2 \varphi + f_1 (\alpha_1 + \alpha_2) \sqrt{\alpha_1 \alpha_2} \cos \varphi,$$

where

$$a = \frac{3e_1}{4n^4} + \frac{n^2}{8g^2} \\ b = \frac{3e_3}{8n^4} + \frac{n^2}{8g^2}, \\ c = \frac{3e_5}{4n^4} + \frac{n^2}{8g^2}, \\ f = -\frac{e_3}{4n^4} + \frac{n^2}{4g^2}, \\ f_1 = \frac{3e_2}{8n^4}, \\ \varphi = 2n(\beta_1 - \beta_2).$$

The system of differential equations indicating the time-variability of the α 's and β 's becomes:

$$\left. \begin{aligned} \frac{d\alpha_1}{dt} &= -4nf\alpha_1\alpha_2 \sin \varphi \cos \varphi + 2nf_1(\alpha_1 + \alpha_2) \sqrt{\alpha_1\alpha_2} \cdot \sin \varphi \\ \frac{d\alpha_2}{dt} &= +4nf\alpha_1\alpha_2 \sin \varphi \cos \varphi - 2nf_1(\alpha_1 + \alpha_2) \sqrt{\alpha_1\alpha_2} \cdot \sin \varphi, \\ \frac{d\beta_1}{dt} &= a\alpha_1 + b\alpha_2 + f\alpha_2 \sin^2 \varphi + \frac{1}{2}f_1 \left(3\sqrt{\alpha_1\alpha_2} + \alpha_2 \sqrt{\frac{\alpha_2}{\alpha_1}} \right) \cos \varphi, \\ \frac{d\beta_2}{dt} &= b\alpha_1 + c\alpha_2 + f\alpha_1 \sin^2 \varphi + \frac{1}{2}f_1 \left(\alpha_1 \sqrt{\frac{\alpha_1}{\alpha_2}} + 3\sqrt{\alpha_1\alpha_2} \right) \cos \varphi. \end{aligned} \right\} (21)$$

It appears at once from the system that:

$$\frac{d\alpha_1}{dt} + \frac{d\alpha_2}{dt} = 0,$$

so

$$\alpha_1 + \alpha_2 = \text{constant}.$$

Another integral is according to § 4:

$$\frac{1}{2} a\alpha_1^2 + b\alpha_1\alpha_2 + \frac{1}{2} c\alpha_2^2 + f\alpha_1\alpha_2 \sin^2 \varphi + f_1(\alpha_1 + \alpha_2) \sqrt{\alpha_1\alpha_2} \cos \varphi = \text{const.}$$

§ 25. The results become very intricate for the general case. This is evidently a consequence of the circumstance, that in the function R appear $\cos \varphi$ and $\sin^2 \varphi$ or in other words $\cos \varphi$ and $\cos 2\varphi$. The problem is considerably simplified if we suppose $f_1 = 0$, thus $e_2 = 0$, which means, that we suppose the planes XZ and YZ to be planes of symmetry for the surface.

Let us again introduce ζ , so that

$$\alpha_1 = R_0^2 h^2 \zeta, \quad \alpha_2 = R_0^2 h^2 (1 - \zeta),$$

then the last integral may be written in the form:

$$\sqrt{\zeta(1-\zeta)} \cos \varphi = p \zeta^2 + q \zeta + r,$$

so that we can perform again all integrations in finite form, and x and y may then be found as functions of the time.

Osculating curves.

§ 24. We return to the general case and shall proceed to investigate what becomes of the osculating curves. They are ellipses whose equations are found by eliminating t between

$$x = \frac{\sqrt{\alpha_1}}{n} \cos (nt + 2n\beta_1)$$

and

$$y = \frac{\sqrt{\alpha_2}}{n} \cos (nt + 2n\beta_2).$$

By changing the origin of time we see that for a definite osculating curve we can also find the equation by elimination of t between

$$x = \frac{\sqrt{\alpha_1}}{n} \cos nt$$

and

$$y = \frac{\sqrt{\alpha_2}}{n} \cos (nt - \varphi),$$

so φ represents the difference in phase.

When φ has an arbitrary value, the ellipse has an arbitrary shape and position.

If $\varphi = 0$ or π a straight line is described passing through O .

If $\varphi = \frac{\pi}{2}$ the axes of the ellipse lie along the axes of coordinates.

The ellipses are described in rectangles having their sides parallel to the axes and whose vertices, as is evident from

$$\alpha_1 + \alpha_2 = \text{constant},$$

lie on the circumference of a circle.

To investigate the change in shape and position we may write down the well-known relations which may serve for the calculation of the axes of the ellipse and the angle of inclination of the long axis with the X -axis. If Ah and Bh are half the larger and half the smaller axis and if θ is the angle in view, then these relations become:

$$\frac{1}{A^2 h^2} + \frac{1}{B^2 h^2} = \frac{n^2 R_0^2 h^2}{\alpha_1 \alpha_2 \sin^2 \varphi}, \dots \dots \dots (1)$$

$$\frac{1}{A^2 B^2 h^4} = \frac{n^4}{\alpha_1 \alpha_2 \sin^2 \varphi}, \dots \dots \dots (2)$$

$$\tan 2\theta = \frac{2\sqrt{\alpha_1 \alpha_2}}{\alpha_2 - \alpha_1} \cdot \cos \varphi. \dots \dots \dots (3)$$

From (1) and (2) we now deduce at once: The sum of the squares of the axes of the ellipse is constant.

§ 25. From what we have just found we can easily prove that in case the surface is a surface of revolution the osculating ellipse has an invariable shape.

Then namely we find:

$$-R = \frac{1}{2} a (\alpha_1 + \alpha_2)^2 + f \alpha_1 \alpha_2 \sin^2 \varphi,$$

where:

$$a = \frac{3e_1}{4n^4} + \frac{n^2}{8g^2},$$

$$f = -\frac{e_1}{2n^4} + \frac{n^2}{4g^2}.$$

As

$$\frac{1}{2} a (\alpha_1 + \alpha_2)^2 + f \alpha_1 \alpha_2 \sin^2 \varphi = \text{constant},$$

and also

$$\alpha_1 + \alpha_2 = \text{constant},$$

we find

$$\alpha_1 \alpha_2 \sin^2 \varphi = \text{constant}.$$

From (2) it then follows that

$$ABh^2 = \text{constant},$$

from which in connection with the close of § 24 we may conclude that

$$Ah = \text{const.}, \quad Bh = \text{const.},$$

and so our proposition is proved.

If in further consideration of the case of the surface of revolution we wish to see in what way θ varies, we have to write down the differential equations giving the variability of the α 's and β 's. They now become :

$$\left. \begin{aligned} \frac{d\alpha_1}{dt} &= -4nf\alpha_1\alpha_2\sin\varphi\cos\varphi, \\ \frac{d\alpha_2}{dt} &= 4nf\alpha_1\alpha_2\sin\varphi\cos\varphi, \\ \frac{d\beta_1}{dt} &= a(\alpha_1+\alpha_2) + f\alpha_2\sin^2\varphi, \\ \frac{d\beta_2}{dt} &= a(\alpha_1+\alpha_2) + f\alpha_1\sin^2\varphi. \end{aligned} \right\}$$

We see that in $\frac{d\beta_1}{dt}$ and $\frac{d\beta_2}{dt}$ an equal constant term $a(\alpha_1+\alpha_2) = aR_0^2h^2$ appears. This means that the frequency n is modified by an amount of $2naR_0^2h^2$.

When we now differentiate according to t the relation

$$tg\ 2\theta = \frac{2\sqrt{\alpha_1\alpha_2}}{\alpha_2-\alpha_1} \cos\varphi$$

we may arrive after some reduction at :

$$\frac{d\theta}{dt} = -2fn^2ABh^2,$$

from which it is evident, that the ellipse revolves with a constant angular velocity.

These results agree quantitatively with those found by Prof. KORTEWEG.

§ 26. The change in shape and position of the osculating curve does not seem to become simple for the general case $n_2 = n_1$.

Let us therefore restrict ourselves to the case $e_2 = 0$; then the XZ -plane and the YZ -plane are planes of symmetry for the surface.

The first equation of (21) now becomes

$$\frac{d\alpha_1}{dt} = -4nf\alpha_1\alpha_2\sin\varphi\cos\varphi.$$

Or by introduction of ζ :

$$\frac{d\zeta}{dt} = -4nfR_0^2h^2\zeta(1-\zeta)\sin\varphi\cos\varphi.$$

The relation between ζ and φ becomes :

$$\cos^2 \varphi = \frac{p\check{\zeta}^2 + q\check{\zeta} + r}{\check{\zeta}(1-\check{\zeta})} \dots \dots \dots (22)$$

Here $\check{\zeta}$ again varies periodically between a greater and a smaller value. Now however $\frac{d\check{\zeta}}{dt}$ may become equal to 0 for $\sin \varphi = 0$ and for $\cos \varphi = 0$. Thus barring special cases there are 3 general cases :

1st. For the extreme values of ζ $\cos \varphi = 0$. Then in the extreme rectangles ellipses are described with the axes along the X -axis and Y -axis (fig. 13).

2nd. For the extreme values of ζ $\sin \varphi = 0$. In the extreme rectangles straight lines are described (fig. 14).

3rd. For one of the extreme values of ζ $\sin \varphi = 0$, for the other $\cos \varphi = 0$. (fig. 15).

Special cases.

§ 27. These we have again for the extreme values of the modulus α (α has the same form as in § 16) of the elliptic functions ; which occurs when 2 roots of the equation :

$$f(\check{\zeta}) = (p\check{\zeta}^2 + q\check{\zeta} + r)\{\check{\zeta}(1-\check{\zeta}) - (p\check{\zeta}^2 + q\check{\zeta} + r)\} = 0$$

have coincided.

The special case corresponding to B of § 9 and the second of § 18 occurs here in two ways. We refer to the cases in which the same straight line is continually described (continually $\sin \varphi = 0$; when the surface is surface of revolution, this form of motion is possible in every meridian) and that continually the same ellipse is described ($\cos \varphi = 0$; this becomes for the surface of revolution the uniform motion in a parallel circle).

The special case corresponding to A of § 9 and to the first of § 18 exists here too. The form of motion approaches asymptotically the motion in a definite ellipse.

Envelope of the osculating curves.

§ 28. Two cases may be indicated, in which the envelope assumes a simple shape.

1. For $p = -1$, $q = 1$ in (22) (the case of a surface of revolution), the envelope has degenerated into two concentric circles.

2. For $p = 0$ and $q = 0$ in (22) the envelope has degenerated into two pairs of parallel lines, enclosing a rectangle.

Arbitrary mechanism with 2 degrees of freedom for which $S = 2$.

§ 29. The equations of LAGRANGE get quite the same form here

as for $S=4$. In the terms of order h^2 we may in the same way substitute other terms for the terms \ddot{q}_1 , \ddot{q}_2 , \dot{q}_1^2 , and \dot{q}_2^2 .

Then we have to reduce the terms $-\frac{\partial P_1}{\partial q_2} \dot{q}_1 \dot{q}_2$ and $-\frac{\partial P_2}{\partial q_1} \dot{q}_1 \dot{q}_2$. To this end we introduce q'_1 and q'_2 in such a way, that

$$q'_1 = q_1 + \frac{1}{4} a_2 q_1^2 q_2 + \frac{1}{2} a_3 q_1 q_2^2,$$

$$q'_2 = q_2 + \frac{1}{4} c_1 q_1 q_2^2 + \frac{1}{2} c_2 q_1^2 q_2.$$

After these reductions it is evident that the terms of order h^3 in the first equation assume the form:

$$\left\{ -a_1 \frac{a_1}{n^2} + (a_3 - b_2 + c_1) \frac{a_2}{n^2} \right\} q_1 + \left(\frac{1}{2} c_2 - 2b_3 \right) \frac{a_2}{n^2} q_2 +$$

$$+ 2a_1 q_1^3 + \left(\frac{1}{2} a_2 + b_1 \right) q_1^2 q_2 - (a_3 - b_2 + c_1) q_1 q_2^2 - \left(\frac{1}{2} c_2 - 3b_3 \right) q_2^3.$$

We now substitute $\frac{4}{3} q_2^3$ for $B^2 h^2 q_2$. This is allowed, because substituting $q_2 = Bh \cos(nt + \lambda)$ in $\frac{4}{3} q_2^3$, we obtain besides a term $B^2 h^2 q_2$ terms which are non-disturbing.

We wish to investigate whether the disturbing terms in the two equations are again derivatives of the same function. For this we need not consider the terms with q_1 and q_1^3 , in the first equation and those with q_2 and q_2^3 in the second. The remaining terms become in the first equation:

$$\left(\frac{1}{2} a_2 + b_1 \right) q_1^2 q_2 - (a_3 - 2b_2 + c_1) q_1 q_2^2 + \frac{1}{3} \left(b_3 + \frac{1}{2} c_2 \right) q_2^3.$$

In the second:

$$\frac{1}{3} \left(\frac{1}{2} a_2 + b_1 \right) q_1^3 - (a_3 - 2b_2 + c_1) q_1^2 q_2 + \left(b_3 + \frac{1}{2} c_2 \right) q_1 q_2^2$$

So finally we find that the disturbing terms are derivatives of the same function R ; so the equations become:

$$\ddot{q}_1 + n^2 q_1 - \frac{\partial R}{\partial q_1} = 0,$$

$$\ddot{q}_2 + n^2 q_2 - \frac{\partial R}{\partial q_2} = 0,$$

where

$$R = Ph^2 q_1^2 + Qh^2 q_2^2 + U_4,$$

when P and Q are homogeneous quadratic functions of $\sqrt{a_1}$ and $\sqrt{a_2}$ and when U_4 is a homogeneous function of order four of q_1 and q_2 . The results found for the simple mechanism hold therefore for an arbitrary mechanism with two degrees of freedom.

Mathematics. — “*The cubic involution of the first rank in the plane.*” By Dr. W. VAN DER WOUDE. (Communicated by Prof. P. H. SCHOUTE.)

1. If V is a plane it is in different ways possible to arrange the points of V in groups of three in such a way, that an arbitrary point forms a part of only one group. If P_1 is a point of V there must exist between the coordinates of P_1 and those of the other points of the group, to which P_1 belongs, some relations by which those other points are entirely determined. It is however possible that P_1 can be chosen in such a way that one of these relations is identically satisfied; in that case P_1 forms part of an infinite number of groups.

We now start from the following definition:

The points of a plane V form a cubic involution of the first rank, when they are conjugate to each other in groups of three in such a way that (with the exception of some definite points) each point forms a part of only one group.

A triangle of which the vertices belong to a selfsame group we call an *involution triangle*; each point which is a vertex of more than one, therefore of an infinite number of involution triangles, we call a *singular point* of the involution; each point coinciding with one of its conjugate points is called a *double point*. If one of the sides of an involution triangle rotates around a fixed point, then the third vertex of this triangle will describe a right line or a curve; *we shall restrict ourselves in this investigation to the case, that one vertex of an involution triangle describes a right line, when the opposite side rotates around a fixed point.*

2. When the points of a plane V form a cubic involution of the first rank which satisfies the just mentioned condition and which we shall furtheron indicate by (i_3) , we can conjugate projectively to each point of V the connecting line of its conjugate points. Each vertex of an involution triangle and its opposite side are pole and polar line with respect to a same conic, which in future we shall always call γ_2 ; each involution triangle is a polar triangle of γ_2 . It is clear that reversely not every polar triangle of γ_2 is an involution triangle of (i_3) ; for each point of V is a vertex of an infinite number of polar triangles of γ_2 , but of only *one* involution triangle. If however S is a singular point of the involution, then S must be a vertex of an infinite number of involution triangles, thus each polar triangle of γ_2 having S as vertex is at the same time an

involution triangle. If we assume a point G of the conic γ_2 as a vertex of an involution triangle, then one of the other vertices must coincide with G , so G is a double point of the involution; γ_2 , the locus of these double points, is the *double curve* of the involution.

Each line l whose pole with respect to γ_2 is no singular point of the involution is a side of only one involution triangle, namely of that triangle having the pole of l as vertex. On the other hand each line whose pole is a singular point is a side of an infinite number of involution triangles all having that point as vertex. From this ensues that also the lines of V form a cubic involution (i_3) of the first rank; the polar lines of the singular points of (i_3) are the singular lines of (i_3), the tangents of γ_2 are its double lines and γ_2 is its double curve. Both involutions are with respect to γ_2 polarly related.

The involution triangles of γ_2 are all polar triangles of a selfsame conic γ_2 , which is at the same time the double curve of (i_3). The lines of V form an involution (i_3) which is with respect to γ_2 the polar figure of (i_3). Each polar triangle of γ_2 having a singular point of the involution as vertex is at the same time an involution triangle.

3. We make a point describe a line a_1 and we ask after the locus of its conjugate points. If we draw through A_1 , the pole of a_1 with respect to γ_2 , an arbitrary line p_1 , then P_1 , the pole of p_1 , lies on a_1 , whilst the two points conjugate to P_1 lie on p_1 ; these two points lie also on the locus under discussion. Moreover A_1 itself is conjugated to two points of a_1 , so that A_1 is a double point of this curve and each line through A_1 cuts this curve in a double point and two points more. Hence we find:

If one of the vertices of an involution triangle describes a line a_1 , then the two others describe a curve α^4 of order four with a node in A_1 , the pole of a_1 with respect to γ_2 . As a_1 cuts all singular lines, all singular points lie on α^4 .

A few properties of this curve α^4 may still be given here:

1. Let A_2 and A_3 be the points conjugated to A_1 , then the polar line of A_3 with respect to γ_2 — that is the line A_1A_2 — must cut α^4 in A_1 and in the points forming with A_3 an involution triangle. These two points are A_1 and A_2 . So will α^4 be touched in A_1 by the lines A_1A_2 and A_1A_3 ; A_2 and A_3 are points of intersection of a_1 and α^4 .

2. Besides in A_2 and A_3 the curve α^4 will be intersected in two points more by a_1 ; these points are at the same time the points of intersection of a_1 and γ_2 .

3. Besides in these last points α^4 will still be cut by γ_2 in 6 points more, the tangents in these 6 points to α^4 must pass through A_1 . From this ensues that α^4 is of the tenth class, by which the PLÜCKER numbers of α^4 are entirely determined ($n=4, m=10, d=1$). This holds, for it is easy to investigate that α^4 cannot possess a double point differing from A_1 .

4. If a vertex of an involution triangle describes a line, on which lies a singular point, the curve described by the two other vertices degenerates into the polar line of that singular point and a curve which must be of order three. If a vertex of an involution triangle describes a singular line s , then one of the other two vertices will be a fixed point, namely the pole of s and the other point will describe s itself and as many other lines as there are singular points on s . As both points together must describe a curve of order four, three singular points will lie on s . In like manner each singular point is point of intersection of three singular lines.

If now again a_1 is an arbitrary line and if α^4 has the same signification as above, then the curve α^4 will cut a line b_1 four times; from this ensues that four times a point of a_1 and a point of b_1 are vertices of a selfsame involution triangle. These vertices we call P_1, Q_1, R_1, S_1 and P_2, Q_2, R_2, S_2 , whilst the third vertices of these triangles may be represented by P_3, Q_3, R_3, S_3 ; farthermore T_1 is the point of intersection of a_1 and b_1 and T_2 and T_3 are the two points forming with T_1 an involution triangle.

If now a point describes the line b_1 , then its conjugate points describe a curve β^4 of order four; α^4 and β^4 have 16 points of intersection. These are:

1. the two points T_2 and T_3 ;
2. the four points P_2, Q_2, R_2, S_2 ;
3. ten points more having the property that to each of them two, so an infinite number of pairs of points, are conjugated and which are thus the singular points. Therefore:

The involution (i_3) has 10 singular points; their polar lines are the 10 singular lines of (i_3) .

These singular elements have such a position that on each of these lines three of these points lie and that in each of the points three of the lines intersect each other; so they form a configuration $(10_3, 10_3)$.

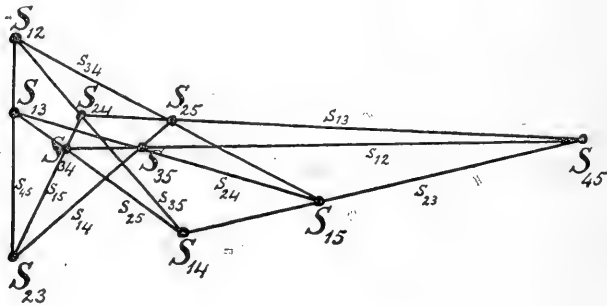
If s_{12} is a singular line and S_{12} its pole with respect to γ_2 , then there are besides S_{12} still 6 singular points not lying on s_{12} . If S_{13} is one of these points and s_{13} the polar line of S_{13} , then the point of intersection of s_{12} and s_{13} is at the same time the pole of S_{12}, S_{13} .

This point forms an involution triangle with S_{12} and with another point of s_{12} and an *other* one with S_{13} and with a point of s_{13} (an "other one", as S_{12} and S_{13} , which do not lie on each other's polar line cannot be vertices of a selfsame involution triangle); so the point of intersection of s_{12} and s_{13} is also a singular point and $S_{12}S_{13}$ a singular line.

Each line connecting two singular points not lying on each other's polar line is a singular line; each point which is the point of intersection of two singular lines not passing through each other's pole is a singular point.

On s_{12} , the polar line of S_{12} , lie 3 singular points; the remaining 6 are connected with S_{12} by 3 singular lines. So each line connecting S_{12} with one of the singular points on s_{12} is not a singular line, as only 3 of these lines pass through S_{12} .

We can indicate the position of the singular points by the following diagram, where the indices have been chosen in such a way that always the points S_{ik} , S_{kl} and S_{il} lie on a selfsame line, that the lines s_{ik} , s_{kl} and s_{il} intersect each other in a selfsame point, and that the point S_{ik} and the line s_{ik} are each other's pole and polar line with respect to γ_2 .



5. We make a point describe a conic α_2 and an other point a line b_1 the two points which are conjugated to the former describe a curve α^n , those which are conjugated to the latter a curve β^4 . As β^4 and α_2 intersect each other in 8 points, b_1 and α^n must have 8 points in common, so α^n is a curve of order eight; we shall call it in future α^8 . As α_2 intersects all singular lines twice, α^8 will have in each of the 10 singular points a node.

If α_2 is described around an involution triangle, then α^8 has also double points in the vertices of this triangle. As all involution

triangles are at the same time polar triangles of a selfsame conic γ_2 , we can describe a conic around each pair of involution triangles; if a conic β_2 is described around two of these triangles, then the curve β^8 conjugate to it will have 6 nodes in its circumference. Also the remaining points of intersection of β_2 and β^8 are easily indicated; they are the four points of intersection of β_2 and γ_2 .

We know moreover that a conic described around an involution triangle and through two of the vertices of an other involution triangle must also contain the third vertex of the latter.

6. It is also clear, that we can easily construct conics described around three involution triangles; to that end we make a conic, pass through the vertices of an arbitrary involution triangle and through two singular points not lying on each other's polar line; for this we choose S_{1_2} and S_{1_3} . As α_2 is described around a polar triangle of γ_2 , it is described around an infinite number of these triangles; further *each* polar triangle of γ_2 having one of the singular points as vertex is at the same time an involution triangle, so that α_2 is described around three involution triangles.

Now the curve α^8 will have in the circumference of α_2 nine nodes; so it must degenerate and α_2 must be one of the parts into which it breaks up. If P_1 is an arbitrary point of α_2 then always one of the two points P_2 and P_3 forming with P_1 an involution triangle will also lie on α_2 , so also the third vertex lies on α_2 (5). If now we let P_1 describe the conic α_2 , then P_2 and P_3 will describe the same curve; every time however that P_1 coincides with one of the singular points on α_2 , P_2 and P_3 will be bound to no other condition, than that they must lie on the polar line of that point and must form with P_1 a polar triangle of γ_2 . So the parts into which α^8 degenerates are:

1. the conic α_2 to be counted double;
2. as many lines as there are singular points lying on α_2 .

From this ensues that besides S_{1_2} and S_{1_3} 2 more singular points lie on α_2 .

This last we can prove still in another way; we construct a second conic β_2 , described around an involution triangle $Q_1 Q_2 Q_3$ and through S_{1_2} and S_{1_3} ; it will cut α_2 in two points more, which being both the vertices of two, i.e. of an infinite number of involution triangles, are therefore singular points. If we construct another conic σ_2 described around a triangle of involution $R_1 R_2 R_3$ and through S_{1_2} and S_{1_3} , then this must still cut α_2 in two singular points; these

must be the same as the points of intersection of α_2 and β_2 , because on α_2 no more than four singular points can lie.

So all conics passing through S_{12} and S_{13} and farthermore described around *one*, hence around an infinite number of involution triangles will form a pencil; the two other base points of this pencil are also singular points. We determine these first: if we choose as β_2 the pair of lines S_{24} and S_{35} and as σ_2 the pair S_{34} and S_{25} , it is evident that S_{14} and S_{15} are the discussed base points. Therefore: *If the 10 singular points, hence also the double curve γ_2 , of the involution are known, we can generate the involution triangles in this way:*

We can construct five different pencils of conics of which each conic is described around an infinite number of polartriangles of γ_2 , which are then at the same time the involution triangles in view; the base points of these pencils consist of the sets of points $(S_{12}, S_{13}, S_{14}, S_{15})$, $(S_{12}, S_{23}, S_{24}, S_{25})$, $(S_{13}, S_{23}, S_{24}, S_{35})$, $(S_{14}, S_{24}, S_{34}, S_{45})$ and $(S_{15}, S_{25}, S_{35}, S_{45})$.

These pencils we shall call in future respectively (B_1) , (B_2) , (B_3) , (B_4) and (B_5) .

If α_1 and α_2 are two conics, the first taken arbitrarily out of (B_1) , the second arbitrarily out of (B_2) , these two will have four points of intersection, viz. S_{12} and the vertices of an involution triangle. Now it can happen in two different ways that 2 of these points of intersection coincide: 1. S_{12} can be at the same time a vertex of the involution triangle 2. one of these vertices can lie on the double curve γ_2 . In each of these two cases α_1 and α_2 will have only three different points in common, but they will touch each other moreover in one of these points.

7. Out of these 5 pencils we choose one — e.g. (B_1) — arbitrarily; an arbitrary conic σ_2 out of (B_1) is described around an infinite number of involution triangles whose vertices form in its circumference an involution of order three. The latter has four double points in the points of intersection of σ_2 with γ_2 , the double curve of the involution (i_2) . Inversely the conics of the pencil (B_1) determine an involution of order four on γ_2 ; the latter has 6 double points in the points in which γ_2 is touched by a conic out of (B_1) . In each of these points three points have thus coincided, forming together a group of (i_3) .

The involution (i_2) has 6 triple points; in each of the points γ_2 is touched by a conic out of each of the pencils (B_1) , (B_2) , (B_3) , (B_4) , and (B_5) .

8. A point whose conjugate points coincide we call a *branch point*, the locus of these points the *branch curve*. If we let a point G' describe the conic γ_2 , then the curve of order eight, generated by the points conjugate to G' , must degenerate into 2 parts, of which one is γ_2 itself and the other the branch curve. From this ensues that the latter is of order six and possesses nodes in the 10 singular points; so it is rational as it should be, as it corresponds point for point to a conic.

Also in an other way we can easily deduce the order of the branch curve; if a point describes a line a_1 , then the conjugate points describe a curve a' having with γ_2 eight points of intersection, of which two coincide with the points of intersection of a_1 and γ_2 , whilst the others point to 6 points of intersection of a_1 with the branch curve.

If G_{12} is a point of the double curve γ_2 and g the tangent in that point to γ_2 , then g will intersect the branch curve in 6 points of which one G_3 forms with the double point G_{12} a group of conjugate points; so in the triple points of the involution γ_2 and the branch curve will have to touch each other.

The branch curve is a rational curve of order six, having double points in the singular points and touching the double curve in the triple points of the involution.

Observation. A rational curve of order six has 10 double points; of which however only 8 can be taken arbitrarily¹⁾; from the preceding follows however that 10 points determining a Cf (10₃, 10₃) can always be double points of a rational curve of order six.

In an other form C. F. GEISER (see his paper quoted in the following number) makes the same observation.

9. We shall now apply the preceding to some problems out of Threedimensional Geometry. To that end we regard the pencil (B) of twisted cubics which can be brought through 5 fixed points P_1, P_2, P_3, P_4 , and P_5 . These determine on an arbitrary plane V a cubic involution of rank one; the lines $P_i P_j$ cut V in the singular points S_{ij} , the planes $P_k P_l P_m$ cut V in the singular lines s_{ij} of the involution. Through an arbitrary point of V passes only one curve out of this pencil, through a singular point S_{ij} however pass an infinite number of curves, which have all degenerated into the fixed line $P_i P_j$ and a variable conic; these conics form a pencil with P_k, P_l, P_m and the point of intersection of $P_i P_j$ with the plane $P_k P_l P_m$ as base points. Each double point of the involution in V is now a point, in which a twisted curve out of the pencil (B)

¹⁾ SALMON-FIEDLER: Höhere ebene Kurven, Zweite Auflage, p. 42.

touches the plane V ; the third point of intersection of this curve with V is a point of the branch curve forming with the point of contact a group of mutually conjugate points of the involution. A triple point of the involution is a point, in which a twisted curve out of (B) is osculated by V . From this ensues:

1. *All twisted cubics passing through 5 given points and touching a given plane V form a surface F^{10} of order ten, which touches V in a conic and cuts V moreover according to a rational curve of order six.*

2. *There are 6 twisted cubics passing through five given points and having a given plane as osculating plane.*

As a special case of this last theorem we have still: *through five given points pass six twisted parabolae.*

Through the pencil (B) of twisted cubics with P_1, P_2, P_3, P_4 and P_5 as base points a plane V is cut according to a cubic involution of the first rank. If α is a curve out of this pencil cutting V in A_1, A_2 and A_3 , then α is projected out of A_1 by a cone cutting V according to the lines $A_1 A_2$ and $A_1 A_3$. If however a curve γ out of (B) touches a plane V in a point G_1 , and if moreover it cuts V in a point G_2 , then γ is projected out of G_1 by a cone cutting V according to $G_1 G_2$ and the tangent in G_1 to γ ; γ is projected out of G_2 by a cone touching V according to $G_2 G_1$. We have seen that $G_1 G_2$ must lie on the double curve and G_2 on the branch curve of the involution, whilst $G_2 G_1$ touches the former; if therefore a quadratic cone is to pass through the base points of the pencil (B) and to touch V moreover, then its vertex must lie on the branch curve and the tangent with V must touch the double curve.

The number of quadratic cones passing through five given points and touching a given plane is singly infinite; the tangents envelope a conic. The vertices of the cones form a rational curve of order six.¹⁾

The tangential planes of all these cones whose number is ∞^2 envelope a surface of which we wish to determine the class and which for the present we will call Φ_n . If K_2 is one of these cones and G_2 its vertex, then through a line l drawn in V through G_2 one more tangential plane to K_2 will pass; as l has with the branch curve 6 points of intersection, it lies still in 6 tangential planes of Φ_n except in V . Furthermore V is a trope of Φ_n (that is a tangential plane touching (γ_2) in the points of a conic) to be counted double; the surface Φ_n is therefore of class eight.

The tangential planes of these cones envelope a surface of class eight¹⁾

¹⁾ C. F. GEISER: "Über Systeme von Kegeln zweiten Grades".

We finally put the question how many twisted circles can be brought through five points where we understand by a twisted circle a twisted cubic cutting the isotropic circle in two points. All twisted cubics through these five points describe on the plane at infinity an involution; if now a point describes the isotropic circle, its conjugate points will describe a curve of order eight having with this circle sixteen points in common; four of these points are at the same time double points of the involution, whilst the other lie two by two on a same twisted circle.

So through five given points pass ten twisted circles, of which four touch the plane at infinity.

Mathematics. — “On the surfaces the asymptotic lines of which can be determined by quadratures”. By J. BRUIN. (Communicated by Prof. HK. DE VRIES).

In a paper entitled as above A. BUHL (*Nouv. Ann. de Math.*, 4^e série, vol. 8, page 433, vol. 9, page 337, *Rev. sem.* XVII 2, page 62, XVIII 1, page 58) discusses the surfaces given by the parameter representation

$$\begin{aligned}x &= r \cos \theta, \\y &= r \sin \theta, \\ \phi(z) &= a \theta + F(r),\end{aligned}$$

in which x, y, z refer to a rectangular system of coordinates, so that z, θ , and r are the so-called cylindric coordinates; these are the only ones which are used in the course of the investigation.

BUHL now gives the differential equation of the asymptotic lines of $\phi(z) = a \theta + F(r)$ with θ and r as independent variables as well as with z and θ . It is then evident that this equation embraces many special cases, where the determination of the asymptotic lines comes to quadratures.

We can put the question more in general: which are the surfaces of one of the forms $z = \varphi(r, \theta)$, or $\theta = f(r, z)$, or $r = f(z, \theta)$, whose asymptotic lines can be determined by quadratures?

Starting from the differential equation of the asymptotic lines

$$D du^2 + 2 D' du dv + D'' dv^2 = 0$$

(BIANCHI-LUKAT, “Vorlesungen über Differentialgeometrie”, page 109), where D, D' and D'' have the values, to be found on page 87 of the quoted work, we find for the differential equation in r and θ of the asymptotic lines of $z = \varphi(r, \theta)$:

$$r \frac{\partial^2 z}{\partial r^2} dr^2 + 2 \left(r \frac{\partial^2 z}{\partial r \partial \theta} - \frac{\partial z}{\partial \theta} \right) dr d\theta + \left(r \frac{\partial^2 z}{\partial \theta^2} + r^2 \frac{\partial z}{\partial r} \right) d\theta^2 = 0. \quad (1)$$

This equation gives rise to quadratures in the following cases:

- a. $z = a\theta + f(r)$, BUHL, l. c., vol. 8, p. 439, comp. also TISSERAND, "Rec. compl. d'exercices", p. 426.
- b. $l(z) = a\theta + f(r)$, BUHL, l. c. p. 440.
- c. $z = Ar \sin(\theta + a) + a\theta + F(r)$,
- d. $z = r f_1(\theta) + f_2(\theta)$,
- e. $z = r^k \sin(\theta \sqrt{k + c})$,
- f. $z = r^{-k} e^{A \sqrt{k + c}}$,
- g. $z = -\theta^2 + 2l(r)$.

2. The differential equation of the asymptotic lines of $\theta = f(r, z)$ in r and z we find by eliminating θ between (1) and $\theta = f(r, z)$.

We find:

$$\left\{ r \frac{\partial^2 f}{\partial r^2} + 2 \frac{\partial f}{\partial r} + r^2 \left(\frac{\partial f}{\partial r} \right)^2 \right\} dr^2 + 2 \left\{ r \frac{\partial^2 f}{\partial z \partial r} + r^2 \left(\frac{\partial f}{\partial r} \right)^2 \frac{\partial f}{\partial z} + \frac{\partial f}{\partial z} \right\} dr dz + r \left\{ \frac{\partial^2 f}{\partial z^2} + r \frac{\partial f}{\partial r} \left(\frac{\partial f}{\partial z} \right)^2 \right\} dz^2 = 0.$$

This equation gives rise to quadratures in the following cases:

- a. $\theta = l(r) + f(z)$,
- b. $\theta = \text{arc cos } \frac{k}{r} + f(z)$, Buhl, l. c., vol. 9, p. 343,

besides a few others mentioned above.

3. In an analogous way we find the differential equation of the asymptotic lines in z and θ of $r = f(z, \theta)$.

It runs:

$$\frac{\partial^2 f}{\partial z^2} dz^2 + 2 \left\{ \frac{\partial^2 f}{\partial z \partial \theta} - \frac{1}{f} \frac{\partial f}{\partial \theta} \frac{\partial f}{\partial z} \right\} dz d\theta + \left\{ \frac{\partial^2 f}{\partial \theta^2} - f - \frac{2}{f} \left(\frac{\partial f}{\partial \theta} \right)^2 \right\} d\theta^2 = 0.$$

Besides in the above mentioned cases this equation gives rise to quadratures for $r = f_1(z) f_2(\theta)$, surfaces of JAMET (Ann. de l'école norm. sup., 1887, Suppl., page 50 etc.; further: PICARD, "Traité d'analyse" I, 2nd ed., page 433).

The classes of surfaces found above are not strictly separated; some even are to be regarded as subclasses of others. They can be ranked according to the most general types to be found among them, whilst others fall under these types, namely as follows:

$$z = Ar \sin(\theta + \alpha) + a\theta + F(r)$$

$$z = a\theta + F(r)$$

$$z = r f_1(\theta) + f_2(\theta)$$

$$r = f_1'(z) f_2'(\theta)$$

$$l(r) = \theta + j(z)$$

$$z = r^k \sin(\theta \sqrt{k} + c)$$

$$z = r^{-k} e^{a \sqrt{k} + c}$$

$$z = -\theta^2 + 2l(r)$$

$$l(z) = a\theta + j(r)$$

$$\theta = \arccos \frac{k}{r} + j'(z).$$

II. Let us discuss one of the above mentioned classes more closely, viz.

$$z = r f_1'(\theta) + f_2'(\theta)^2. \dots \dots \dots (2)$$

It is the general equation of the scrolls with the z-axis as directrix. Of these scrolls we can find the striction line in the following way.

BIANCHI (l.c. p. 223) deduces that the curvature $K = \frac{1}{r_1 r_2}$, for which

in another place was found $K = \frac{DD' - D'^2}{EG - F^2}$, is larger in the central point than in all other points of a generatrix. If we make up K for (2) we find

$$K = \frac{-(f_2')^2}{[r^2 \{1 + (f_1')^2 + (f_1')^2\} + 2r f_1' f_2' + (f_2')^2]^2}$$

Along a generatrix θ is constant; there only the denominator of the expression for K changes. If we determine the value of r for which K becomes maximal we find

$$r = - \frac{f_1' f_2'}{1 + (f_1')^2 + (f_1')^2} \dots \dots \dots (3)$$

So this is the (r, θ) projection of the striction-line.

This equation can be found in an other way, too. We have the property that the tangential plane in the central point of a generatrix is normal to the tangential plane in the point at infinity of that generatrix. We now determine for $f' = -z + r f_1'(\theta) + f_2'(\theta)$ the values of $\frac{\partial f'}{\partial x}, \frac{\partial f'}{\partial y}, \frac{\partial f'}{\partial z}$ in an arbitrary point p and in the point at infinity on the same generatrix. If then p is to be central point the sum of the products $\left(\frac{\partial f'}{\partial x}\right)_p \cdot \left(\frac{\partial f'}{\partial x}\right)_\infty$ must be equal to 0. This gives again the equation (3).

1) In future we shall write f_1 and f_2 for $f_1(\theta)$ and $f_2(\theta)$.

Let us now consider in how far we can find the surface (2) for a given striction line.

a. Let $r = \varphi(\theta)$ be the given projection of the striction line.

This furnishes, regarding (3), the relation between f_1' and f_2' :

$$\frac{f_1' f_2'}{1 + (f_1')^2 + (f_1'')^2} = \varphi$$

$$f_2' = \frac{-\varphi \{1 + (f_1')^2 + (f_1'')^2\}}{f_1'}$$

Thus: The surface $z = r f_1'(\theta) + \int \frac{-\varphi \{1 + (f_1')^2 + (f_1'')^2\}}{f_1'} d\theta$, where f_1' is an arbitrary function of θ , has as (r, θ) -projection of the striction line the curve $r = \varphi(\theta)$.

b. Let now be given that the striction line must be a plane curve lying in the plane $z = Ax + By + C$ or $z = r(A \cos \theta + B \sin \theta) + C$.

By substituting this value for z in (2) we must get (3). This furnishes between f_1' and f_2' the relation

$$f_1' \frac{f_2' - C}{A \cos \theta - B \sin \theta} = \frac{f_2' f_1'}{1 + (f_1')^2 + (f_1'')^2}$$

or

$$\frac{f_2'}{f_2' - C} = \frac{1 + (f_1')^2 + (f_1'')^2}{f_1' (f_1' - A \cos \theta - B \sin \theta)}$$

So the surface $z = r f_1'(\theta) + e^{\int \frac{1 + (f_1')^2 + (f_1'')^2}{f_1' (f_1' - A \cos \theta - B \sin \theta)} d\theta} + C$, in which f_1' is an arbitrary function of θ , has a plane striction line lying in $z = Ax + By + C$.

c. The most general problem here is: what is the surface (2) for which $r = \varphi(\theta)$, $z = \psi(\theta)$ is striction line?

To solve this we substitute these values for r and z in (2) and (3) and we obtain then two equations with f_1' and f_2' as unknown quantities. If we eliminate between these two f_2' , we retain f_1' as only unknown quantity in the equation:

$$\varphi + \varphi(f_1')^2 + f_1' \psi' - \varphi' f_1' f_1' = 0.$$

If f_1' is solved out of this we can find f_2' out of:

$$\psi = \varphi f_1' + f_2'.$$

We can find the solution in explicit form for the special case $\psi = \text{const.}$, for which constant we take 0. We then find:

$$\varphi \{1 + (f_1')^2\} - \varphi' f_1' f_1' = 0,$$

from which ensues

$$\pm f_1' = \sqrt{e^{\int \frac{\varphi'}{\varphi} d\theta} - 1},$$

whilst further

$$f'_2 = -q f'_1.$$

The result is therefore:

The surface

$$\pm z = (r - q) \sqrt[2]{\int \frac{z}{r} dy - 1}$$

has the plane striction line $r = q, z = 0$.

The formulae deduced above hold for the surfaces (2). As was noticed the general types mentioned at the conclusion of I are not strictly separated, however, so that there are still amongst them scrolls with a right directrix, to which then the above formulae are applicable.

Examples of this are:

Of the type $z = Ar \sin(\theta + \alpha) + a\theta + F(r)$

the surface $z = Ar \sin(\theta + \alpha) + a\theta + br$.

Of the type $l(z) = a\theta + f'(r)$

the surface $l(z) = a\theta + l(r + p)$;

these have the z -axis as directrix.

Of the type $r = f'_1(z) f'_2(\theta)$ the scrolls

$$x'(z + 1) = z^n y, \text{ or } r = (tg \theta)^{1/n-1} \sec \theta \left(\frac{z^n}{z+1} \right)^{1/n-1}$$

and

$$x^n = (y - cx)z^n \text{ or } r = \sec \theta (tg \theta - c)^{\frac{1}{n-1}} z^{\frac{n}{n-1}}$$

have still the y -axis as directrix.

Physics. — “A new theory of the phenomenon allotropy.” By Prof. A. SMITS. (Communicated by Prof. A. F. HOLLEMAN.)

Introduction.

In two short communications inserted in the “Chemisch Weekblad” 7, 79 and 155 (1910) I have already sketched the main lines of the theory, an extension and experimental confirmation of which follow here.

Before passing on to this I may, however, be allowed to give the gist of this theory in a few words.

In the investigation of the phenomenon tautomerism it has been possible to show by means of the process of solidification that the liquid phases of tautomeric substance; are composed of two kinds of molecules.

Besides, however, by deposition of different solid substances the complexity of a liquid phase can also be shown in another way, and investigations in this direction have led to the result that it may be considered as the rule that the liquid phase of a substance is built up of different kinds of molecules (ions included).

BANCROFT¹⁾ and BAKHUIS ROOZEBOOM²⁾ have pointed out that when a substance behaves as a unary substance, this is accounted for by the fact that the setting in of inner equilibrium takes place so rapidly in the homogeneous phase that the inner equilibrium if disturbed, is immediately restored by the appearance or disappearance of a new phase; the melting-point, boiling-point, critical-point etc. of a substance which behaves as a unary one, does not relate then to a *single* kind of molecules, but to an equilibrium between *different* kinds of molecules.

BANCROFT'S pupils, viz. CARVETH, SOCH, and CAMERON³⁾ have investigated different tautomeric substances; it then appeared that it may be pretty easily shown in some cases that under certain circumstances the existence of two kinds of molecules in the liquid phase may lead to a binary behaviour, for when the liquid cooled *rapidly*, the inner equilibrium could not follow the temperature, and the mass solidified at a temperature which differed from the unary stable melting-point, for then a point was realized of one of the melting-point lines of the pseudo-binary system $A + B$, which for the examined substances always showed a eutectic point.

As is evident we find the unary stable melting-point where the curve for the inner liquid-equilibrium meets one of the melting-point curves of the pseudo-binary system.

Now it is remarkable, as I already wrote, that nobody has observed what surprising results are arrived at when it is assumed, what is undoubtedly true, that not only mixed crystals are always formed in a greater or less degree, but that moreover the *inner* equilibrium, which exists in the liquid phase, continues to exist in the solid phase.

Starting from this supposition we get the relation between *heterogeneous* and *homogeneous* allotropy, indicated in Fig. 1, from which it appears that the phenomenon of enantiotropy means *unmixing* in the solid state, which phenomenon appears when the curves for

¹⁾ Journ. Phys. Chem. **2**, 143 (1898).

²⁾ Zeitschr. f. phys. Chem. **23**, 289, (1899).

³⁾ Journ. Phys. Chem. **2**, 159 (1898).

ibid. **2**, 364 (1898).

ibid. **2**, 409 „

the stable and metastable solid equilibria s_2g and s_1n meet the mixed crystal lines ep and dm of the pseudo-binary system.

In case of monotropy these meetings between the unary and the pseudo-binary system do not take place *under* but *above* the unary melting-point temperature, and this is the reason that in this case the second line for the solid inner equilibria everywhere indicates metastable states.

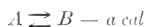
I started from GIBBS' principle of equilibrium, which states that with constant temperature and pressure a number of substances arranges itself in such a way that the thermodynamic potential is a minimum; and then I showed how sharply the relation between the pseudo-binary and the unary system can be defined also in this way, when we bear in mind that a state of inner equilibrium must always lie in the minimum of a potential line.

Further the case was considered that the three-phase temperature lies between the melting points of the substances A and B . After having discussed the phenomena of enantiotropy and monotropy also for this case, I finally pointed out that when A and B are miscible in all proportions in the solid state, *heterogeneous allotropy* is excluded, and only homogeneous allotropy can occur, *unless* unmixing occurs in the pseudo-binary system at lower temperature.

Discussion of the curves of inner equilibrium.

After this introduction which seemed indispensable to me to make the reader acquainted with the main facts, I will consider fig. 1 a little more closely and discuss the curves of inner equilibrium.

It follows from the course of the curve $S'_3S'_1$, that it is assumed here that over the corresponding range of temperature the equilibrium shifts towards B with increase of temperature, and so that



or in words that the transformation from the left to the right is endothermic.

With application of the equation:

$$\frac{d \ln K}{dt} = \frac{Q}{RT^2}$$

we know therefore that Q is positive in the assumed case.

Neglecting the external work we can split up Q into two differential heats of mixing, one of which has the negative sign, because it is a case of unmixing, and further into a heat of transformation, so:

$$Q = -(Q_m)_A + Q_r + (Q_m)_B$$

$(Q_m)_A =$ differential heat of mixing of A
 $(Q_m)_B =$ " " " " " B
 $Q_r =$ mol. heat of transformation.

It is of importance to point out here that as $(Q_m)_A$ and $(Q_m)_B$ have a different sign, the possibility exists that Q has another sign than Q_r ; this might e.g. occur when Q_r was very small, and then we should have the special case that e.g. when Q was negative and Q_r positive, the equilibrium shifted from B to A with rise of temperature, whereas the transformation of A into B is endothermic in itself; this, however, will only rarely occur.

If we drop this question for the present, it is noteworthy that in the point S'_1 unmixing occurs, another solid phase S'_2 appearing by the side of S'_1 . Two cases may be distinguished here.

Generally the newly-formed solid phase S'_2 will possess another form of crystal than S'_1 , but it is possible that the two solid phases are isomorphous, for as is known, also isomorphous substances can show partial miscibility; if this latter, the simplest case occurs, the heat of transformation will be the sum of a heat of unmixing, a heat of transformation, and a heat of mixing¹⁾, another thermal quantity being added to this, viz. that which accompanies the change of crystalline form, when S'_1 and S'_2 are not isomorphous.

If we now follow the inner equilibria *above* the transition-point, it is to be expected that the curves S'_2, q for the solid-, and l_2, k for the liquid inner equilibria will have the same direction as S'_2, S'_1 , as is also assumed in fig. 1.

Soch, however, has found in his investigation of *benzile-orthocarbonic acid* that the curve of the inner liquid equilibrium meets the melting-point curve of the modification with the highest melting-point viz. B , and runs to the A -side for higher temperatures. Further he found that at 65° A passes into B , and combining these two facts, he arrives at the conclusion that the thermal sign of the transformation



must have been reversed between the point of transition and the unary melting-point (137°).

When the pseudo-binary T, v -figure for this substance agrees with fig. 1, which is still an open question, we must of course come to the same result also going by this theory, but I will point out here that this conclusion is not yet imperative at this moment, because though it is not probable, the possibility exists that the mixed crystal

¹⁾ I shall discuss this and the before-mentioned splitting up more fully later on.

curve dm of the pseudo-binary system has the same direction as ep ; in this case the three curves of equilibrium S'_3, S'_1, S'_2, S_2 and L_2k might still have the same direction, and so the sign of Q need not be reversed.

If we look upon the question of the reversal of the thermal sign from a general point of view, the following may already be remarked. When A and B are *isomers*, as for benzile-orthocarbonic acid, a reversal of the sign of Q seems possible, because Q_r is probably small in this case¹⁾. If, however, we have to do with the phenomenon polymerism, we may expect with great probability that Q_r will always predominate, and that the curves for the inner solid and liquid equilibria will always run in such a way that the equilibrium shifts towards the side of the less complex substance with rise of temperature.

This leads us at the same time to the question what the T_x -figure will be for the case that the substance B is a *polymer* of A , and that a transition point exists.

Fig. 2 shows that when the pseudo-binary system possesses a eutectic point, the curve for the inner liquid equilibria must meet the melting-point curve of the less complex substance, because only in these circumstances all the curves for the inner equilibria can run to the A -side with rise of temperature.

Yet this figure will not appear to be quite correct either, in my opinion, as a supposition is implied in it, which is highly improbable.

When B is a polymer of A , and the pseudo-binary system possesses a eutectic point, this means that there are liquids (ac) which contain more polymer than the coexisting solid phases (ad), and this is very improbable, so much so that we may disregard this figure altogether, in spite of HOLLMANN's²⁾ assertion that he has found a eutectic point for the system acetaldehyde-paraldehyde. Probably this assertion of HOLLMANN's rests on not quite reliable observations, for my assistant, Mr. DE LEEUW, who tested the said assertion at my request, has not found it confirmed.

So for the case that B is a polymer of A and the two substances are *not* miscible in all proportions in the solid state, we must conclude to the existence of a T_x -figure as indicated by fig. 3, in which the

¹⁾ In consequence of the considerable displacement of the inner equilibrium at the transition temperature it is possible, that while Q_r predominates *below* this temperature, *above* it the reverse takes place.

Q_r , too, can reverse its sign, but this seems less probable to me.

²⁾ Zeitschr. f. phys. Chem. **43**, 129 (1903).

three-phase-temperature lies *between* the melting-points of the pseudo-components.

Now on this assumption, the solid phase possesses everywhere more of the polymer B than the coexisting liquid phase, and if in the unary system a transition point occurs, the course of the curves of inner equilibrium must be as indicated by kl_2 , S_2S_2' and $S_1'S_1$.

If the curve kl_2 met the melting-point line of B , monotropy alone would be possible, as for enantiotropy reversal of the thermal sign would have to take place in this case, which is very improbable here.

Experimental confirmation.

It is clear that this theory requires that every substance which shows a transition point, must consist of two different kinds of molecules, which are in equilibrium at every temperature.

So if we consider the substance HgJ_2 , the *red* modification of which passes into the *yellow* one at 127° , we must assume two different kinds of molecules, the former of which gives rise to the formation of *red*, and the other to that of *yellow* HgJ_2 .

The investigation of this substance, which was carried out in collaboration with Mr. S. C. BOKHORST chem. cand. has led to a very remarkable result.

That it would appear that working quickly, the substance would betray its binary character, was expected, but that we should find here that case which I already mentioned, but considered as an exception, was highly surprising.

For the sake of clearness the observed phenomena will be discussed here in connection with the schematic fig. 4, in which α means yellow and β red HgJ_2 .

At 127° the red phase passes into the yellow one, which new phase remains intensely yellow up to about 180° ; on further heating we observed that this phase assumed a red colour, at first hardly perceptibly, but then more and more pronounced, and that it becomes a dark red liquid at the melting-point temperature 255° , 4.

This phenomenon, which also appeared with very slow rise of the temperature, was studied in different ways with the naked eye and by means of the microscope, when it appeared that this change of colour takes place *continuously*, and is not owing to a second transition-point.

This continuous change of colour between comparatively narrow limits of temperature made it therefore probable that above the point of transition the curve for the solid inner equilibria at first

runs vertically upwards, after which it bends sharply to the red side, and meets the mixed-crystal curve of the pseudo-binary system near the axis of the red modification.

As therefore, this inner equilibrium curve appears to traverse the T - x -figure over a large concentration range, this pointed already to a region of partial-miscibility in the pseudo-binary figure, which was closed at the top, and so also to a continuous mixed crystal curve acb .

In order to test this supposition more closely, the following experiments were made with HgJ_2 , either in thin-walled narrow capillaries or in so-called alcaloid tubes; it was, namely, quite immaterial which of these were taken, for in either case the experiment yielded the same result.

In these tubes the HgJ_2 was heated in a melting-apparatus up to a certain temperature *above* the transition-point, and then all at once transferred to an oil-bath of lower temperature, but always *above* the transition point.

The considerations which led us to these experiments, were the following.

If it is possible to make the cooling take place so rapidly that the inner equilibrium cannot keep pace with the temperature, the pseudo-binary character must appear, and entering the region of partial-miscibility the substance must split up into two phases.

Suppose that we start from the inner equilibrium p and that we cool this suddenly, in which not the curve of equilibrium, but the curve pS_3 is followed; then the red phase S_4 will appear by the side of the yellow phase S_3 and will have to be clearly visible.

This three-phase system will be strongly metastable, so that it is not to be expected that it will be very permanent; on the contrary, we may confidently predict that this state will very soon change into the only stable equilibrium which must lie on the curve SS_1 .

If we now start from the inner equilibrium q , which lies on the right of the critical mixing-point K , the mixing-curve can be reached in S_5 , and by the side of phase S_5 , the phase S_6 must be found, which has a lighter colour.

As appears from the subjoined table (p. 770) not only these phenomena could be observed with great clearness, but moreover it was ascertained by these preliminary experiments that the mixing-point K must lie above 147° 1).

Though it follows from these experiments that above the transition temperature the T , x -figure of the system HgJ_2 would be as indicated

1) This investigation is continued in different directions.

Temp. HgJ_2	Suddenly cooled down to the temp.	Remarks
200°	130°	No unmixing as yet.
205°	"	Unmixing, <i>red</i> phase appears, but has disappeared again after a few seconds and the whole mass is again yellow.
207°	"	"
210°	"	"
212°	"	"
215°	"	"
225°	"	Unmixing, but now <i>yellow</i> phase appears and after a few seconds everything is yellow.
230°	"	The same phenomenon, and still more pronounced.
.		
212°	140°	Unmixing <i>red</i> phase appears etc.
212°	145°	"
212°	147°	No unmixing is to be observed.

here, the question what the rest of the figure, i.e. under the transition point, would look like, remained unanswered. The answer to this question cannot yet be given in this communication, because the equilibrium sets in exceedingly slowly at temperatures under 100°. ¹⁾

So the dotted curves under the transition temperature do not represent anything but a supposition. For the end in view here, however, the want of certainty *below* the transition temperature is of minor importance, as the phenomena observed at higher temperatures furnish a convincing proof for the validity of the theory.

Before I leave the substance HgJ_2 and proceed to another subject, I will only point out, that if the equilibria are considered not at constant pressure, but at the variable vapour-pressure, also the vapour-curves should be inserted in the T, x -figure, which lie on the side

¹⁾ If a tube with red HgJ_2 is immersed in liquid air, the colour becomes indeed much lighter viz. orange, but this change of colour has nothing to do with a displacement of the equilibrium.

If a mixture of *yellow* and *red* HgJ_2 is taken, and this is cooled down to -190° , the yellow colour changes into white, and the red into orange-yellow. When heated to the temperature of the room the heterogeneous mass is found to be entirely unchanged compared with the initial state.

of yellow HgI_2 , because the *yellow* phase is always deposited from the vapour.

If we now consider the question whether the literature mentions results in support of this theory, the answer is affirmative. These are chiefly the results obtained in the investigation of *sulphur*¹⁾ and that of *phosphorus*²⁾.

In the system sulphur we have two different crystalline modifications, and besides them a third modification S_{II} , which has not yet been obtained in crystalline form.

Considered in the light of this theory we must therefore assume *three* different kinds of molecules, and *sulphur* being known as a substance which is very slow, we can assume with great probability that *sulphur* is not pseudo-binary, but pseudo-ternary, i.e. will behave as a ternary system.

This, however, be only remarked in passing, as these considerations are of no further importance for what follows.

If we now direct our attention to the T,x -figure of the system S_{II} and rhombic sulphur S_R (Fig. 5), it is noteworthy that by extrapolation $110^\circ,6$ has been found for the unary melting-point, and $112^\circ,8$ for the melting-point of pure rhombic sulphur.

It further appeared, however, that when from rhombic S was started from, where the equilibrium had set in at 90° , a melting-point was found at $110^\circ,9$, the melting-point amounting to $111^\circ,4$ when the inner equilibrium had set in at $\pm 65^\circ$.

These are results which support the theory given here, for they point to the fact that we have to do here with a curve SS_1 for the solid inner equilibrium, which runs to the left with rise of temperature. For this curve shows that as we, working quickly, start from an inner equilibrium established at *lower* temperature, this phase will begin to melt at a *higher* temperature, which was also observed here.

The curve for the inner liquid equilibrium, too, runs to the left, so that the two curves of equilibrium have the same direction.

Though the sulphur can furnish further proofs, we now proceed to the phosphorus.

As COHEN and OLIE already mentioned, investigations of TROOST and HAUTEFEUILLE, LEMOINE, HITTORF, and themselves point to the

1) KRUYT, Z. f. phys. Chem. **64**, 513 (1908).

2) COHEN and OLIE, Chem. weekblad **6**, 821 (1909).

fact that for phosphorus we have to do with solid inner equilibria between white and violet phosphorus.

If we consider the following results of the determinations of the specific gravity:

spec. grav. of red P obtained at	$550^\circ = 2,25$
" " " " " " " "	$450^\circ = 2,28$
" " " " " " " "	$357^\circ = 2,22$
" " " " " " " "	$255^\circ = 2,20$
" " " " " " " "	$215^\circ = 2,19$

we should, in view of the fact that the spec. grav. of white $P = 1,82$, and that of violet P may be put at about $2,34$, come to the conclusion that the curve for the inner solid equilibria runs to the violet side with rise of temperature to 450° .

As it, however, followed from the experiments of COHEN and OLIE, that when red P was reduced from a higher to a lower temperature, the spec. grav. in general was *not* lowered, it is clear that they have not investigated states of equilibrium, and that we, therefore, cannot draw conclusions about the course of the curve of the inner equilibria from the above results. As to the existence of the inner equilibria, however, this is no longer doubtful.

So if we start from this, and if we then think of the phenomenon observed by CHAPMAN ¹⁾ that red P when melting, gives a colourless liquid i.e. a liquid which perfectly resembles melted yellow P , a T, x -figure may be constructed in main lines for the pseudo-binary and the unary system, in which, however, the existence of a eutectic point is still an open question.

It has been assumed in fig. 6 that red $P(3P)$ is a polymer of white $P(\alpha P)$, and therefore no eutectic point is drawn. In this figure the phenomenon observed by CHAPMAN has been illustrated, for heated to the melting-point, the red solid phase will pass into a liquid l_2 , which lies entirely on the side of the white P . We see further from this diagram that melted yellow P has about the same composition as melted red P , and that melted yellow P means undercooled liquid red P .

Applications.

Besides the phenomena mentioned here there are others which seen in the light of this theory find a plausible explanation. I allude

¹⁾ Journ. Chem. Soc. 75, 743 (1899).

here to the phenomena of retardation *for so far they only appear when we work rapidly*¹⁾.

If we consider first of all the phenomenon of *undercooling* and *superheating* of the solid, for so far as they are only observed with quick change of temperature, fig. 7 gives a satisfactory explanation.

Starting from the inner liquid equilibrium p , not the curve of equilibrium pl_2 , but another curve e.g. pl_3 will be followed with rapid cooling, and when we get beyond l_3 the state is not only unarily, but also pseudo-binarily *metastable*.

Let us assume for simplicity that in the pseudo-binary system no retardation worth mentioning appears, then the substance will solidify at l_3 and the solid substance S_3 is deposited.

Now this two-phase equilibrium is metastable to a high degree in the pseudo-binary system.

In the unary system equilibrium between liquid and solid substance can only exist under constant pressure at one temperature, and now it is the rule that a metastable state like that of the system $l_3 + S_3$ is at once destroyed. Thus we see e.g. that a supersaturate solution in contact with the substance which this solution must deposit to pass to the stable condition, generally immediately deposits this substance.

So the metastable two-phase equilibrium $l_3 + S_3$ is changed into the stable state $l_2 + S_2$, and this being a process which generates heat, the temperature rises to the unary melting-point.

Starting from the solid inner equilibrium q we get just the reverse, because then the substance melts at too high a temperature if quickly heated, as has already been observed for rhombic sulphur.

If now the curves of inner equilibrium run as in fig. 8, the liquid can solidify too early if cooled too rapidly, the solid substance can melt too early if heated too rapidly, and then the result is that for a *perfectly* pure substance there is a range of temperature over which the solidification and the melting extends, which probably often occurs for organic substances, in which the equilibrium sets in so slowly.

With regard to the phenomena of retardation at the transition point I need only refer to fig. 9, which will now be clear without further elucidation.

It is further hardly necessary to remark that when a substance is not *bi-*, but *tri-*, or *polymolecular*, the phenomena discussed here remain essentially the same.

¹⁾ The peculiar phenomena, which will also appear for more complicated systems, as e.g. $Fe + C$ when we work quickly, will have to be accounted for in the same way.

In conclusion I want to point out that this theory gives the first plausible explanation of the *metastability of the metals*.

In this it is viz. noteworthy that the cooling of the solidified masses proceeds in such a way that the inner solid equilibrium can certainly not follow the temperature, and this is one of the reasons why the metals, as we generally have them, are nearly always in metastable state. We must further bear in mind that if we have a metal which is in inner equilibrium, and it is subjected to some mechanical operation, a necessary consequence of this will be that the metal becomes metastable, because in stable state a change of pressure is generally attended with a shifting of the inner equilibrium, which, however, in consequence of the inner resistance does not appear at all, or on account of the slight velocity of transformation will take place only after a very long time.

The above mentioned circumstances account at the same time for the fact that it hardly ever occurs that two pieces of the same metal are perfectly identical, for this could only occur when the inner state, stable or metastable, was perfectly the same.

Just as so many others the metastable states discussed here can be changed into the stable state by different influences, as increase of temperature, vibration, contact with the stable state etc., in which the transformation which takes place, manifests itself in a recrystallisation.¹⁾

Amsterdam, March 1910. Anorg. Chem. Lab. of the University.

E R R A T A.

In the Proceedings of the Meetings of Jan. and Febr. 1910.

- p. 652 line 9 and p. 677 line 5 from the bottom, p. 654 line 17 from the top: for 11 read 659.
- p. 669, 672, 674 for 20.2 read 20.3.
- p. 670 etc. for carrier read holder.
line 9 and 19 from the top: for modulus read constant.
line 5 from the bottom: for corresponding read in agreement with.
- p. 672 line 16 from the bottom: for dilation read dilatation.
- p. 673 for 14.3 read 14.0.

¹⁾ It is to be expected that this metastability will not be met with only for metals and metal-alloys, but also for other substances, which have been obtained by rapid cooling and solidification of melted masses.

(April 28, 1910).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS OF THE MEETING
of Friday April 29, 1910.

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Mathematics. — “On polar figures with respect to a plane cubic curve. By Prof. JAN DE VRIES.

(Communicated in the meeting of March 26, 1910).

1. If a plane cubic curve γ^3 is represented symbolically by $a_x^3=0$ then $a_x a_y a_w = 0$ represents the polar line p_{xy} of the points X and Y , i.e. the polar line of X with respect to the polar conic π_y of Y and at the same time the polar line of Y with respect to the polar conic π_x of X .

The three polar lines p_{xy} , p_{xz} , and p_{yz} will concur in one point W when the three conditions are satisfied

$$a_x a_y a_w = 0, \quad a_x a_z a_w = 0, \quad a_y a_z a_w = 0 \dots \dots (1)$$

By elimination of the coordinates w_k we find out of it

$$(abc) a_x a_y b_x b_z c_y c_z = 0 \dots \dots (2)$$

So to two given points X, Y belongs a conic γ_{xy}^2 as locus of the point Z ; it passes also through X and Y , for when Z and X coincide, we find

$$(abc) a_x a_y b_x^2 c_y c_x = (cba) c_x c_y b_x^2 a_y a_x = - (abc) a_x a_y b_x^2 c_y c_x = 0.$$

As we can substitute $(abc) a_x a_y c_x c_z b_y b_z = 0$ for (2), thus also $(abc) a_x a_y b_z c_z (b_x c_y - b_y c_x) = 0$, we can also represent γ_{xy}^2 by $(abc) a_x a_y b_z c_z (bc\xi) = 0$, where ξ_k are the coordinates of the line XY . Consequently (2) can be replaced by

$$(bc\xi) (bc\eta) b_z c_z = 0 \dots \dots (3)$$

From this ensues that the conic γ_{xy}^2 is the *poloconica* $\pi_{\xi\eta}$ of the lines ξ and η .

So the poloconica of two lines is the locus of the points Z which with relation to the points of intersection X, Y of this conic with one of the given lines are in such a position that the polar lines p_{xz} and p_{yz} concur on the other one of the given lines, which is then at the same time polar line of X and Y .

2. If Z and W are the points of intersection of $\pi_{\xi\eta}$ with η , it follows out of the symmetry of (3) in connection with the equations (1), that the four points X, Y, Z, W form a closed group, so that each side of the quadrangle determined by them is the polar line of the vertices not lying on it, therefore a *polar quadrangle* (REYE).

Out of our considerations ensues that a polar quadrangle is determined by two of its vertices, but also by two of its opposite sides. In the last case the vertices are determined by the poloconica of the given lines; in the former case we can use the poloconica belonging to the polar line of the given points and their connecting line.

Out of $a_x a_y a_w = 0$ and $a_x a_y a_z = 0$ follows

$$a_x a_y (\lambda a_w + \mu a_z) = 0.$$

Here λ and μ can be determined in such a way that $\lambda a_w + \mu a_z = a_u$ relates to the point of intersection U of XY with WZ .

As $a_u a_x^2 = 0$ indicates the polar conic π_u of U we find that X and Y according to the relation $a_x a_y a_u = 0$ lie harmonically with respect to π_u . In an analogous way ensues from $a_w a_z a_y = 0$ and $a_w a_z a_x = 0$ the relation $a_w a_z a_u = 0$, according to which W and Z are also separated harmonically by π_u .

But then also the points $T \equiv (XZ, YW)$ and $V \equiv (XW, YZ)$ are conjugated with respect to π_u , i.e. we have $a_u a_v a_t = 0$. Now U, V, T are the *diagonal points* of the complete quadrangle $XYZW$, so that it is proved that *the diagonal triangle of a polar quadrangle is always a polar triangle*¹⁾.

3. When the conic γ_{xy}^2 degenerates we can take for Z each point on the line XY . To trace for this the condition, we put $z_k = \lambda x_k + \mu y_k$; from (2) follows

$$(abc) a_x a_y b_z c_y (\lambda b_x + \mu b_y) (\lambda c_x + \mu c_y) = 0,$$

so

$$\lambda^2 (abc) a_x a_y b_x^2 c_x c_y + \lambda \mu (abc) a_x a_y b_x^2 c_y^2 + \mu^2 (abc) a_x a_y b_x b_y c_x c_y + \mu^2 (abc) a_x a_y b_x b_y c_y^2 = 0.$$

By exchanging two of the symbolic factors a, b, c , we see that three of these terms are identically zero; so we have

$$\lambda \mu (abc) a_x a_y b_x^2 c_y^2 = 0.$$

For an arbitrary choice of X and Y this equation furnishes only $\lambda = 0$ and $\mu = 0$, thus the points X and Y . It furnishes *each* point of XY , as soon as

$$(abc) a_x a_y b_x^2 c_y^2 = 0 \dots \dots \dots (4)$$

When X, Y , and Z are collinear, the polar line p_x of X and the polar lines p_{xy}, p_{xz} concur in *one* point; for these three lines are the polar lines of X, Y, Z with respect to the polar conic π_x . If now (4) is satisfied, then also p_{yz} passes through that point, hence, the six polar lines $p_x, p_y, p_z, p_{xy}, p_{yz}, p_{zx}$ concur in a point W . But when p_x, p_y and p_z are concurrent, the poloconica of $\xi \equiv XYZ$, degenerates and ξ is tangent of the *Cayleyana*.

From this ensues that for given Y the equation (4) will represent

¹⁾ Mentioned without proof by CAPORALI (Transunti d. R. A. dei Lincei 1877, p. 236).

three right lines, namely the three tangents which we can draw out of Y to the *Cayleyana*.

This can be confirmed as follows. Let Z be a point of the locus of X , which is determined by (4) and X a second point of that locus lying on YZ , so that we have $a_x = \lambda a_y + \mu a_z$. Out of (4) then follows

$$(abc) a_y c_y^2 (\lambda a_y + \mu a_z) (\lambda b_y + \mu b_z)^2 = 0.$$

By exchanging a and c we see at once that

$$(abc) a_y^2 c_y^2 (\lambda b_y + \mu b_z)^2$$

vanishes identically. Analogously we find that $(abc) a_y c_y^2 a_z b_y^2$ and $(abc) a_y c_y^2 a_z b_y b_z$ vanish identically. As finally the form $(abc) a_y c_y^2 a_z b_z^2$ is zero because Z lies on the locus indicated by (4) the above relation is satisfied by all points of YZ , so the locus consists of three lines through Y .

4. That the line $\xi \equiv XY$ is tangent to the *Cayleyana* as soon as (4) is satisfied, can be confirmed by reducing (4) to the tangential equation of that curve. In the first place we find out of

$$(abc) a_x a_y b_x^2 c_y^2 = 0 \quad \text{and} \quad (acb) a_x a_y c_x^2 b_y^2 = 0$$

the relation

$$(abc) a_x a_y (b_x c_y + b_y c_x) (b_x c_y - b_y c_x) = 0.$$

The last factor can be replaced by $(bc\xi)$ where ξ_k indicate the coordinates of XY . After that the equation can be broken up into two terms, which pass into each other when b and c are exchanged. So we can replace it by

$$(abc) a_x a_y b_x c_y (bc\xi) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

farthermore it is evident from

$$(abc) a_x a_y b_x^2 c_y^2 = 0 \quad \text{and} \quad (cba) c_x a_y b_x^2 a_y^2 = 0,$$

that at the same time is satisfied

$$(abc) b_x^2 a_y c_y (ac\xi) = 0,$$

so also

$$(bac) a_x^2 b_y c_y (bc\xi) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

By combining (5) and (6) we find

$$(abc) a_x c_y (bc\xi) (ab\xi) = 0.$$

So

$$(abc) c_x a_y (bc\xi) (ab\xi) = 0.$$

Out of the last two relations follows finally

$$(abc) (ac\xi) (bc\zeta) (ab\zeta) = 0. (7)$$

This tangential equation really represents the *Cayleyana*¹⁾.

So we have found that *the six polar lines* $p_x, p_y, p_z, p_{xy}, p_{yz}, p_{zx}$ *concur in one point when the points* X, Y, Z *lie on a tangent of the* *Cayleyana*.

5. When p_x, p_y, p_z are concurrent we have

$$(abc) a_x^2 b_y^2 c_z^2 = 0. (8)$$

This equation gives thus the relation between the coordinates of three points lying on one and the same polar conic.

For an arbitrary choice of X and Y this equation is satisfied except by X and Y by no point of the line XY . If it is to be satisfied by $z_k = \lambda x_k + \mu y_k$ we must have

$$(abc) a_x^2 b_y^2 (\lambda c_x + \mu c_y)^2 = 0,$$

therefore

$$\lambda \mu (abc) a_x^2 b_y^2 c_x c_y = 0.$$

This is satisfied for each value of $\lambda : \mu$ when the relation (4) is satisfied, so when X, Y, Z lie on a tangent of the *Cayleyana*.

Now in general the polar lines p_{xy}, p_{xz}, p_{yz} form a triangle inscribed in the triangle $p_x p_y p_z$ (see § 3). If (4) is satisfied then p_{xy}, p_{xz}, p_{yz} are concurrent; but then their point of intersection must be at the same time point of intersection of p_x, p_y, p_z .

If X, Y, Z are three collinear points of the cubic, then p_{yz}, p_{zx} and p_{xy} pass successively through X, Y , and Z .

For, from $a_x^3 = 0, a_y^3 = 0$ and $(\lambda a_x + \mu a_y)^3 = 0$ follows that the point Z is indicated by $a_x a_y (\lambda a_x + \mu a_y) = 0$. So we have $a_x a_y a_z = 0$, so Z lies on the polar line p_{xy} .

If moreover X, Y, Z lie on a tangent of the *Cayleyana*, then p_{yz}, p_{zx}, p_{xy} must coincide with the tangents p_x, p_y, p_z in X, Y, Z .

6. For p_x, p_y , and p_{xy} to be concurrent, there must be a point W for which we have $a_x^2 a_w = 0, b_y^2 b_w = 0$, and $c_x c_y c_w = 0$.

But then $(abc) a_x^2 b_y^2 c_x c_y = 0$.

For arbitrarily chosen Y the locus of X becomes a figure of the third order, passing through Y , because we have $(abc) a_y^2 b_y^2 c_y^2 \equiv 0$. But by taking notice of (4) we see that this figure consists of three tangents of the *Cayleyana*. Out of

$$a_x^2 a_w = 0, a_y^2 a_w = 0 \text{ and } a_x a_y a_w = 0$$

¹⁾ See e.g. CLEBSCH, Leçons sur la géométrie, II, p. 284.

follows indeed

$$(\lambda a_x + \mu a_y)^2 a_z = 0;$$

i.e. if Z lies on XY , then p_z will pass through the point of intersection W of p_x, p_y and p_{xy} , which bears then at the same time p_{yz} and p_{xz} .

So the three lines p_x, p_y , and p_{xy} concur only then in one point when X and Y are united by a tangent of the *Cayleyana*. Their point of intersection bears then also all the polar lines and mixed polar lines belonging to the points of those lines.

The lines p_x, p_{xy} , and p_{xz} will be concurring, when

$$(abc) a_x^2 b_x b_y c_x c_z = 0$$

is satisfied, thus also

$$(abc) a_x^2 c_x c_y b_x b_z = 0,$$

hence also

$$(abc) a_x^2 b_x c_x (b_y c_z - b_z c_y) = 0.$$

If we put

$$y_k z_l - y_l z_k = \check{\xi}_m,$$

we have the condition

$$(abc) a_x^2 b_x c_x (bc \check{\xi}) = 0.$$

As this can also be written in the forms

$$(abc) a_x b_x^2 c_x (ac \check{\xi}) = 0 \quad \text{and} \quad (abc) a_x b_x c_x^2 (ab \check{\xi}) = 0$$

and as out of

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_x \\ b_1 & b_2 & b_3 & b_x \\ c_1 & c_2 & c_3 & c_x \\ \check{\xi}_1 & \check{\xi}_2 & \check{\xi}_3 & \check{\xi}_r \end{vmatrix} = 0$$

follows the relation

$$(abc) \check{\xi}_r = a_x (bc \check{\xi}) + b_x (ca \check{\xi}) + c_x (ab \check{\xi})$$

the above condition can be replaced by

$$(abc)^2 a_x b_x c_x \check{\xi}_r = 0.$$

With arbitrary position of X this is satisfied by $\check{\xi}_r = 0$, i.e. when X, Y , and Z are collinear (see § 3).

If however

$$(abc)^2 a_x b_x c_x = 0,$$

so that X lies on the *Hessian*, then X, Y , and Z are quite arbitrary. This was to be foreseen, now namely π_x is a pair of lines, so that the lines p_x, p_{xy} , and p_{xz} concur in the node of π_x .

Physiology. — “*Contributions to the study of serum-anaphylaxis.*”
 (From the “Institut für Infektionskrankheiten” at Berlin). By
 Dr. J. G. SLEESWIJK. (Communicated by Prof. C. H. H. SPRONCK).
 (4th communication.)

(Communicated in the meeting of March 26, 1910).

During the first months of last year I had an opportunity, in three communications¹⁾, to make the results of my investigations about serumanaphylaxis known to the Academy. Since that time the literature about this subject has not inconsiderably increased. It is not my intention at all here to go in for a discussion of this. Only let it be allowed to me (and I even consider myself obliged to do this before the Academy) to treat here in a few words of these publications, in which my own investigations either directly or indirectly were discussed, and to add the results of a number of further experiments. May it be presupposed that in general the facts communicated by me must be acknowledged as correct.

Principally we have to pay attention to three points:

1st. the part of the red corpuscles of the guinea-pig with respect to horse-serum in the phenomenon of TH. SMITH, 2nd. the problem of the alexine-fixation and of the haemolysis in the anaphylactic shock, and 3rd the application of the specific hypersensibility for proteins in medicina forensis.

Last year I explained why I was of opinion that the sensitizing principle of the first injection and the toxic substance of the second administering of serum must be considered identical and that only quantitative differences are met with here. In the meantime BESREDKA²⁾ has changed his mind and taken this standpoint. Now the logical consequence of my observation that horse-serum by treatment with the blood of a guinea-pig can be deprived of its poisonousness for animals made sensitive, was therefore that with this at the same time the sensitizing substance is fixed. LEVADITI and RAYCHMAN, who could also really prove what I mentioned last, do not refrain, therefore, in this connection, from referring to my communication.³⁾ Also SALUS corroborated my observation concerning the depoisoning action of the red corpuscles on horse-serum.⁴⁾

The problem of the complement fixation has of late attracted attention to a high degree. I remind of my first communication in which I said already, “that a sensitized guinea-pig, which reacts

1) These Proceedings: January, February, and March 1909.

2) Ann. de l'Inst. Pasteur, Oct. 1909.

3) C. R. Soc. de Biol. T. 67, 1909, p. 1078.

4) Wiener Klin. Woch. 1909, no. 48.

upon the second serum-administration with symptoms of intoxication some time after that injection produces a serum that is exceedingly poor in haemolytic alexine." By the side of this I have proved that the serum of hypersensitive animals in not a single combination with horse-serum gives a precipitate, nor is it able to fix complement. A remarkable incongruity therefore, and by which the anaphylactic state is distinguished from the phase of immunity, at which praecipitines are formed, and at which in vivo as well as in the test-tube alexine is fixed. I point out that here I came in conflict with NICOLLE and ABT¹⁾, who had found that the serum of sensitized guinea-pigs does fix complement with horse-serum in vitro. For FRIEDBERGER, who considers anaphylaxis as a peculiar form of the immunity for proteins, at which the praecipitines only for a trifling part have passed into circulation, but principally have remained fixed (sessile) to the cells, the fixation of complement was a welcome phenomenon. He took for granted (evidently without any further control) that the communication of NICOLLE and ABT was correct, whilst, on the other hand, he could confirm by his own investigation my observation about the loss of alexine during the anaphylactic shock.²⁾ Yet even here such quantitative differences came to light that at first sight my observations seemed to show a shade of incorrectness. I had said for example, that the maximum of complement-loss is reached after about half an hour, whereas FRIEDBERGER found this to be* the case already within five minutes. Therefore I am compelled to enter into this somewhat more closely.

FRIEDBERGER has evidently not asked himself where the cause may lie of our diverging results, quantitative as they may only be. He speaks only in passing of "Differenzen in der angewandten Anaphylaxie-technik." But here lies the *cardo quaestionis*, and what I presumed already, appeared to me on closer investigation to be reality. I had namely administered to my animals the toxic serum-injection in not too great a dose in their abdomen; the reaction is then less violent and has a slower course, so that — through the investigation of blood-samples taken from the animals consecutively at different times of the anaphylactic shock — I could in general fix the course of the complement-curve. FRIEDBERGER, however, injected his hypersensitive animals intravenously: the reaction then goes so quickly and is so violent, that the guinea-pigs usually die within few minutes. And at the same time also the complement-loss has soon reached its maximum. Now in WASSERMANN's laboratory I have

¹⁾ Ann. de l'Inst. Pasteur 1908.

²⁾ Zeitschr. f. Imm.forsch. Bd. III, H. 6.

been able to fix these differences in a series of exact quantitative complement-titrations. From these it appeared among others that a few minutes after intravenous injection of 0.2 cm³ of horse-serum the complement-quantity of the testanimal-serum may have decreased nearly as strongly as at intraperitoneal injection of 3 cm³ after half an hour.

My investigations and those of FRIEDBERGER, accordingly, do not contradict each other; they complete each other. Therefore I cannot see in FRIEDBERGER's results anything but an essential corroboration of my observations. The same thing holds good for the haemolysis, which in the anaphylactic shock shows itself in the test-animals. FRIEDBERGER corroborates also this fact, just as POELS¹⁾ does. My contention, therefore, that BESREDKA with his exclusivistic opinion that it is only the elements of the central nervous system which are to be brought to hypersensibility, is wrong, finds satisfactory support in this haemolysis which has been proved in more than one hand.

Now as to the alexine-fixation of the anaphylactic serum with its antigen *in vitro*, I think I can maintain my negative results over-against NICOLLE and ABT. In many series with mixtures of falling quantities of horse-serum with rising quantities of anaphylactic guineapig-serum I could not observe anywhere a specific retardment of the haemolysis. Not even though I stuck accurately to the quantitative proportions, as NICOLLE and ABT have mentioned. In the meantime, thanks to the necessary controlling experiments, I came to the following conclusions. Even normal guinea-pig serum (inactivated) often retard the haemolysis, quite independent of the presence of horse-serum; nay, we sometimes meet a normal serum which has a stronger fixing power than an anaphylactic serum which has also been examined²⁾. Therefore I abide by my former contention that there is here no question of a specific complement-fixation in the test-tube. This incongruity of the alexine-fixation *in vitro* and *in vivo*, to which I drew attention already a year ago, was the other day corroborated with certainty by MICHAËLIS in the meeting of the "Physiologische Gesellschaft" at Berlin (21 January 1910). Also FRIEDBERGER seems after all to share this opinion (*Zeitschr. f. Imm.forsch.* Bd. IV, H. 5). It now seems to me that the labile state of physiological equilibrium, in which the hypersensitive organism finds itself, is biologically characterized by the incongruity referred to just now.

¹⁾ Handelingen v. h. Nederl. Natuur- en Geneesk. Congr. te Utrecht, April 1909.

²⁾ Ampler details I hope to publish elsewhere.

Not long ago T_{SURUN}²⁾ tried to reduce the signification of the alexine fixation in vivo in anaphylaxis. Thus already in normal guinea-pigs normal serum from dog or rabbit would cause loss of complement. This certainly does not hold good for horse serum, as had already appeared from my former controlling experiments; indeed, about this T_{SURUN} does not speak. But moreover this investigator has worked with corpuscles sensitized not strongly enough and with insufficient dilutions of complement, so that his results do not deserve a very great confidence. Leaving this out of consideration, he found, just as I did before, that the intoxication-phenomena and the loss of complement need not run parallel, from which I drew the conclusion that these two are not directly dependent upon each other, but have a common cause.

I now come to the third point that I wish to treat of here, viz. the application of anaphylaxis in the practice of medicina forensis. Evidently this application was so clear that about at the same time and independent of my communication, similar results were made known by THOMSEN, UHLENHUTH, and H. PFEIFFER. The last lays stress upon the strong fall in the temperature of the body during the anaphylactic shock as a resource for the diagnosis.

Concerning the technique of this investigation the following may still be mentioned. If a blood spot has to be identified, and if guinea-pigs are treated intraperitoneally, several cm³ of serum are necessary for each animal that is to be examined. If the animals are treated intravenously (in the jugularis), much smaller quantities of serum are wanted, but then an operation in the neck has to take place, which, however, after some practice for this purpose offers no objection. It has now appeared to me that also with young rabbits of about 1 K.G. the experiment can be very well made, because here both the sensitizing and the trying injection can be easily made in the earveua. A small dried up blood-spot is dissolved in 1 cm³. of physiological salt-solution and injected in such an animal; after a fortnight 1 cm³ of the suspected kinds of blood is injected also intravenously. To rabbits, which had thus been previously treated with extracts from human blood-spots, I have administered on consecutive days serum from goat, horse, cow, and guinea-pig, without the animals reacting in the least. Lastly 1 cm³ of human serum caused them within a few minutes to answer with spasms and paralyzes, with respiratory disturbances, incontinentia urinae et alvi, etc. The anaphylactic reaction will, in my opinion, in the practice

¹⁾ Zeitschr. f. Imm. forsch Bd. IV, H. 5.

of *medicina forensis* henceforth maintain its position by the side of the precipitation *in vitro* as a valuable method.

Among the many questions that show themselves in the study of our subject, there was also the following: what happens to the injected horse-serum during the anaphylactic shock? If there were only a minimal quantity free and unchanged in the circulation of the intoxicated animal, it ought to be possible that with its blood a normal animal could be sensitized. Now it has appeared to me that this is never crowned with success. From this it may be inferred that all the antigen taken up in the blood-circulation is at once fixed by the cells of the hypersensitive organism, resp. deprived of its specific character at the same time.

Mathematics. — “*On the structure of perfect sets of points*”. By Dr. L. E. J. BROUWER. (Communicated by Prof. KORTEWEG).

(Communicated in the meeting of March 26, 1909).

§ 1.

Sets of points and sets of pieces.

The sets of points discussed in the following lines are supposed to be lying within a finite domain of a S_{p_n} .

By a *piece* of a closed set of points μ we understand a *single point or closed coherent set of points*, belonging to μ , and not contained in another closed coherent set of points belonging to μ .

We can regard as elements of μ its pieces as well as its points, in other words we can consider μ on one hand as a *set of points*, on the other hand as a *set of pieces*.

Let us choose among the pieces of μ a fundamental series S_1, S_2, S_3, \dots , then to μ belong one or more pieces ${}_1S_n, {}_2S_n, \dots$ with the property that ${}_nS_n$ lies entirely within a for indefinitely increasing n indefinitely decreasing distance ε_n from one of the pieces ${}_2S_n$. These parts ${}_2S_n$ we shall call the *limiting pieces* of the fundamental series S_1, S_2, S_3, \dots .

As thus the set μ possesses to each of its fundamental series of pieces at least one limiting piece, *a closed set of points is likewise closed as set of pieces*.

By an *isolated piece* of μ we understand a piece having from its rest set in μ a finite distance, in other words a piece, the rest set of which is closed.

THEOREM 1. *Each piece of μ is either a limiting piece, or an isolated piece.*

Let namely S be a non-isolated piece, then there exists in μ a fundamental series of points t_1, t_2, t_3, \dots not belonging to S , converging to a single point t of S . If t_1 lies on S_1 , then S_1 has a certain distance ε_1 from S . There is then certainly a point t'_2 of the fundamental series possessing a distance $< \varepsilon_1$ from S , lying therefore not on S_1 but on another piece S_2 . Let ε_2 be the distance of S_2 from S , then there is certainly a point t'_3 of the fundamental series possessing a distance $< \varepsilon_2$ from S , lying thus neither on S_1 , nor on S_2 , but on a third piece S_3 . Continuing in this manner we determine a fundamental series of pieces S_1, S_2, S_3, \dots , containing consecutively the points t_1, t'_2, t'_3, \dots converging to t . So the pieces S_1, S_2, S_3, \dots converge to a single limiting piece which can be no other than S .

By a *perfect set of pieces* we understand a closed set, of which each piece is a limiting piece.

A perfect set of pieces is also perfect as set of points; but the inverse does not hold. For, a perfect set of points can very well contain isolated pieces.

We shall say that two sets of pieces *possess the same geometric type of order*, when they can be brought piece by piece into such a one-one correspondence, that to a limiting piece of a fundamental series in one set corresponds a limiting piece of the corresponding fundamental series in the other set. So in general a closed set considered as a set of pieces possesses not the same geometric type of order as when considered as a set of points.

A closed set we shall call *punctual*, when it does not contain a coherent part, in other words when all its pieces are points.

§ 2.

Cantor's fundamental theorem and its extensions.

The fundamental theorem of the theory of sets of points runs as follows:

If we destroy in a closed set an isolated point, in the rest set again an isolated point, and so on transitively, this process leads after a denumerable number of steps to an end.

The discoverers of this theorem, CANTOR ¹⁾ and BENDIXSON ²⁾ proved

¹⁾ Mathem. Annalen 23, p. 459—471.

²⁾ Acta Mathematica 2, p. 419—427.

it with the aid of the notion of the *second transfinite cardinal* Ω , which is however not recognised by all mathematicians. LINDELÖF¹⁾ gave a proof independent of this notion, where, however, the process of destruction itself remaining non-considered, the result is more or less obtained by surprise.

Only for linear sets there have been given proofs of the fundamental theorem, which at the same time follow the process of destruction and are independent of Ω ²⁾.

The rest set which remains after completion of the process of destruction and which we may call the *Cantor residue*, is after CANTOR³⁾ a *perfect set of points*, however of the most general kind, thus *in general not a perfect set of pieces*.

An extension of the fundamental theorem, enunciated by SCHOENFLIES⁴⁾ and proved by me⁵⁾, can be formulated as follows:

If we destroy in a closed set an isolated piece, in the rest set again an isolated piece, and so on transfinitely, this process leads after a denumerable number of steps to an end.

My proof given formerly for this theorem was a generalisation of LINDELÖF's method, but at the same time I announced a proof which follows the process of destruction, and which I give now here; in it is contained a proof of the fundamental theorem, which in simplicity surpasses by far the existing ones, is independent of Ω , and follows the process of destruction:

By means of $S\rho_{n-1}$'s belonging to an orthogonal system of directions we divide the $S\rho_n$ into n -dimensional cubes with edge a , each of these cubes into 2^n cubes with edge $\frac{1}{2} a$, each of the latter into 2^n cubes with edge $\frac{1}{4} a$, etc.

All cubes constructed in this way form together a denumerable set of cubes K .

Let now μ be the given closed set, then K possesses as a part a likewise denumerable set K_1 consisting of those cubes which contain in their interior or on their boundary points of μ .

1) Acta Mathematica 29, p. 183—190.

2) SCHOENFLIES, Bericht über die Mengenlehre I, p. 80, 81; Gött. Nachr. 1903, p. 21—31; HARDY, Mess. of Mathematics 33, p. 67—69; YOUNG, Proceedings of the London Math. Soc. (2) 1, p. 230—246.

3) l. c. p. 465.

4) Mathem. Annalen 59; the proof given there p. 141—145, and Bericht über die Mengenlehre II, p. 131—135 does not hold.

5) Mathem. Annalen 68, p. 429.

To each destruction of an isolated point or isolated piece in μ now answers a destruction of at least *one*⁴⁾ cube in K_1 ; but of the latter destructions only a denumerable number is possible, thus also of the former, with which CANTOR'S theorem and SCHOENFLIES'S theorem are proved both together.

Let us call the rest set, which remains after destruction of all isolable pieces, the *Schoenflies residue*, then on the ground of theorem 1 we can formulate:

THEOREM 2. *A Schoenflies residue is a perfect set of pieces.*

§ 3.

The structure of perfect sets of pieces.

Let S_1 and S_2 be two pieces of a perfect set of pieces μ . Let it be possible to place a finite number of pieces of μ into a row having S_1 as its first element and S_2 as its last element in such a way, that the distance between two consecutive pieces of that row is *smaller than* a . Then we say, that S_2 belongs to the a -group of S_1 .

If S_2 and S_3 both belong to the a -group of S_1 , then S_3 belongs also to the a -group of S_2 , so that μ breaks up into a certain number of " a -groups". This number is finite, because the distance of two different a -groups cannot be smaller than a .

If $a_1 < a_2$, and if an a_1 -group and an a_2 -group of μ are given, then these are either entirely separated or the a_1 -group is contained in the a_2 -group.

If two pieces S_1 and S_2 of μ are given, then there is a certain maximum value of a , for which S_1 and S_2 lie in different a -groups of μ . That value we shall call the *separating bound of S_1 and S_2 in μ* , and we shall represent it by $\sigma_\mu(S_1, S_2)$.

If farther on we represent the *distance of S_1 and S_2* by $\alpha(S_1, S_2)$, then $\sigma_\mu(S_1, S_2)$ converges with $\alpha(S_1, S_2)$ to zero, but also inversely $\alpha(S_1, S_2)$ with $\sigma_\mu(S_1, S_2)$. For otherwise convergency of $\sigma_\mu(S_1, S_2)$ to zero would involve the existence of a coherent part of μ , in which two different pieces of μ were contained, which is impossible.

The maximum value of a for which μ breaks up into different a -groups we shall call the *width of dispersion of μ* , and shall represent it by $\sigma(\mu)$. This width of dispersion of μ is at the same time the greatest value which $\sigma_\mu(S_1, S_2)$ can reach for two pieces S_1 and S_2 of μ .

⁴⁾ Even of an infinite number.

The maximum value of a , for which μ breaks up into *at least* n different a -groups we shall call the n -partite width of dispersion of μ , and shall represent it by $\delta_n(\mu)$. Clearly $\delta_n(\mu)$ is $\leq \delta(\mu)$.

For μ exists furthermore a series of increasing positive integers $n_1(\mu), n_2(\mu), n_3(\mu), \dots$ in such a way that $\delta_n(\mu)$ for n between $n_{k-1}(\mu)$ and $n_k(\mu)$ is equal to $\delta_{n_k(\nu)}(\mu)$. This quantity $\delta_{n_k(\nu)}(\mu)$ we call the k^{th} width of dispersion of μ and as such we represent it by $\delta^{(k)}(\mu)$.

We now assert that it is always possible to break up μ into m_1 perfect sets of pieces μ_1, \dots, μ_{m_1} so as to have $\delta(\mu_h) \leq \delta_{m_1}(\mu)$ and $\alpha(\mu_{h_1}, \mu_{h_2}) \geq \delta_{m_1}(\mu)$.

Let namely be $\delta_{m_1}(\mu) = \delta^{(k)}(\mu)$; we can then obtain the required number m_1 by composing each μ_h of a certain number of $\delta^{(k)}(\mu)$ -groups belonging to a same $\delta^{(k-1)}(\mu)$ -group. We are then also sure of having satisfied the condition $\alpha(\mu_{h_1}, \mu_{h_2}) \geq \delta_{m_1}(\mu)$.

Fartheron we can place the $\delta^{(k)}(\mu)$ -groups of a same $\delta^{(k-1)}(\mu)$ -group into such a row that the distance between two consecutive ones is equal to $\delta^{(k)}(\mu)$. If we take care that each μ_h consists of a non-interrupted segment of such a row, then the condition $\delta(\mu_h) < \delta_{m_1}(\mu)$ is also satisfied.

Let us now break up in the same way each μ_h into m_2 perfect sets of pieces $\mu_{h_1}, \dots, \mu_{h_{m_2}}$ in such a way that $\delta(\mu_{hi}) < \delta_{m_2}(\mu_h)$ and $\alpha(\mu_{hi_1}, \mu_{hi_2}) > \delta_{m_2}(\mu_h)$, and let us continue this process indefinitely.

If then we represent by r_ν an arbitrary row of r indices, then we shall always find

$$\delta_{m_\nu}(\mu_{r_\nu-1}) \leq \delta_{m_1+m_2+\dots+m_\nu+1-\nu}(\mu) \cdot \cdot \cdot \cdot \quad (A)$$

As μ is a perfect set of pieces, the width of dispersion $\delta(\mu_{r_\nu})$ can converge to zero only for indefinite increase of ν ; out of the formula (A) follows, however, that for indefinite increase of r that convergency to zero always takes place and, indeed, uniformly for all r^{th} elements of decomposition together.

At the same time the separating bound of every two pieces lying in one and the same r^{th} element of decomposition converges uniformly to zero; so these elements of decomposition converge themselves uniformly each to a single piece.

If finally a variable pair of pieces of μ is given, then their distance can converge to zero only when the order of the smallest element of decomposition, in which both are contained, increases indefinitely.

The simplest mode in which this process of decomposition can be

executed is by taking all m_k 's equal to 2. If then we represent the two elements of decomposition of the first order by μ_0 and μ_2 , those of the second order by $\mu_{00}, \mu_{02}, \mu_{20}, \mu_{22}$, and so on, then in this way the different pieces of μ are brought into a one-one correspondence with the different fundamental series consisting of figures 0 and 2. And two pieces converge to each other then, and only then, when the commencing segment which is common to their fundamental series, increases indefinitely.

Let us consider on the other hand, in the linear continuum of real numbers between 0 and 1, the perfect punctual set π of those numbers which can be represented in the triadic system by an infinite number of figures 0 and 2. The geometric type of order of π we shall represent by ξ .

Two numbers of π converge to each other then and only then, when the commencing segment which is common to their series of figures, increases indefinitely.

So, if we realize such a *one-one* correspondence between the pieces of μ and the numbers of π , that for each piece of μ the series of indices is equal to the series of figures of the corresponding number of π , then to a limiting piece of a fundamental series of pieces of μ corresponds a limiting number of the corresponding series of numbers in π , so that we can formulate:

THEOREM 3. *Each perfect set of pieces possesses the geometric type of order ξ .*

For the case that the set under discussion is *punctual and lies in a plane*, this theorem ensues immediately from the following well-known property:

Through each plane closed punctual set we can bring an arc of simple curve.

Combining SCHOENFLIES'S theorem mentioned in § 2 with theorem 3 we can say:

THEOREM 4. *Each closed set consists of two sets of pieces; one of them possesses, if it does not vanish, the geometric type of order ξ , and the other is denumerable.*

§ 4.

The groups which transform the geometric type of order ξ in itself.

Just as spaces admit of groups of continuous one-one transformations, whose geometric types of order ¹⁾ are again spaces, namely

¹⁾ In this special case formerly called by me "Parameter Mannigfaltigkeiten" Comp. Mathem. Annalen 67, p. 247.

the finite continuous groups of LIE, the geometric type of order ζ admits of groups of continuous one-one transformations, which possess likewise the geometric type of order ζ .

In order to construct such groups we start from a decomposition according to § 3 of the set μ into m_1 "parts of the first order" $\mu_1, \mu_2, \dots, \mu_{m_1}$, of each of these parts of the first order into m_2 "parts of the second order" $\mu_{h1}, \mu_{h2}, \mu_{h3}, \dots, \mu_{hm_2}$, etc.

The parts of the first order we submit to an arbitrary transitive substitution group of m_1 elements, of which we represent the order by p_1 , and which we represent itself by g_1 .

After this we submit the parts of the second order to a transitive substitution group g_2 of $m_1 m_2$ elements which possesses the parts of the first order as systems of imprimitivity and g_1 as substitution group of those systems into each other. We can then represent the order of g_2 by $p_1 p_2$.

The simplest way to construct such a group g_2 , is to choose it as the direct product of g_1 and a substitution group γ_2 , which of the parts of the second order leaves the first index unchanged and transforms the second index according to a single transitive substitution group of m_2 elements.

We then submit the parts of the third order to a transitive substitution group g_3 of $m_1 m_2 m_3$ elements which possesses the parts of the second order as systems of imprimitivity and g_2 as substitution group of those systems into each other. We can represent the order of g_3 by $p_1 p_2 p_3$.

In this way we construct a fundamental series of substitution groups g_1, g_2, g_3, \dots .

Let τ_1 be an arbitrary substitution of g_1 ; τ_2 a substitution of g_2 having on the first index of the parts of the second order the same influence as τ_1 ; τ_3 a substitution of g_3 having on the first two indices of the parts of the third order the same influence as τ_2 ; and so on.

The whole of the substitutions τ_n then determines a substitution of the different fundamental series of indices into each other, in other words a transformation τ of the pieces of μ into each other.

This transformation is in the first place a one-one transformation; for, two different pieces of μ lie in two different parts of a certain, e.g. of the r^{th} order, and these are transformed by τ into again two different parts of the r^{th} order.

If fartheron S_1, S_2, S_3, \dots is a fundamental series of pieces, possessing S_ω as its only limiting piece, then, if $\lambda(n)$ is the lowest possible order with the property that S_n and S_ω lie in different parts of that order, $\lambda(n)$ must increase indefinitely with n .

So by the transformation τ the fundamental series passes into a new fundamental series having as its only limiting piece the piece into which \mathcal{L}_0 passes by τ .

As a set of pieces μ is thus *continuously* transformed by τ .

Let $\tau'_1, \tau'_2, \tau'_3, \dots$ be a series of substitutions satisfying the same conditions as the series $\tau_1, \tau_2, \tau_3, \dots$. If then $\tau_1 \tau'_1 = \tau''_1$; $\tau_2 \tau'_2 = \tau''_2$; etc., then the series $\tau''_1, \tau''_2, \tau''_3, \dots$ likewise satisfies the same conditions.

If furthermore τ' and τ'' are defined analogously to τ , then $\tau \tau'$ is equal to τ'' .

So the transformations satisfying the conditions put for τ form a group, which we shall represent by g .

To investigate the geometric type of order of this group, we decompose in the way indicated in § 3 a perfect set of pieces q into p_1 parts of the first order q_1, q_2, \dots, q_{p_1} ; each of these into p_2 parts of the second order $q_{h1}, q_{h2}, \dots, q_{hp_2}$; and so on.

The p_1 substitutions of g_1 we bring into a one-one correspondence to the parts of the first order of q . Then the $p_1 p_2$ substitutions of g_2 into such a one-one correspondence to the parts of the second order of q , that, if a substitution of g_2 and a substitution of g_1 have the same influence on the first index of the parts of the second order of μ , the part of the second order of q corresponding to the former lies in the part of the first order of q corresponding to the latter.

In like manner we bring the $p_1 p_2 p_3$ substitutions of g_3 into such a one-one correspondence to the parts of the third order of q , that, if a substitution of g_3 and a substitution of g_2 have the same influence on the first two indices of the parts of the third order of μ , the part of the third order of q corresponding to the former lies in the part of the second order of q corresponding to the latter; and so on.

The parts of q corresponding to a series $\tau_1, \tau_2, \tau_3, \dots$ then converge to a single piece of q , which we let answer to the transformation τ deduced from the series. Then also inversely to each piece of q answers a transformation τ , and the correspondence attained in this manner is a *one-one correspondence*.

Furthermore two transformations τ and τ' converge to each other then and only then, when their generating series $\tau_1, \tau_2, \tau_3, \dots$ and $\tau'_1, \tau'_2, \tau'_3, \dots$ have an indefinitely increasing commencing segment in common, in other words when the corresponding pieces of q converge to each other. So the correspondence between the transformations τ and the pieces of q is *continuous*.

The transformations τ , in other words the transformations of the group g , have thus been brought into a continuous one-one correspondence to the pieces of q , so that g possesses the geometric type of order ζ .

If now we adjoin to each substitution group g_n a finite group g'_n of continuous one-one transformations of μ as a set of pieces in itself, transforming of the pieces of μ the first n indices according to g_n , but leaving unchanged all their other indices, then the fundamental series of the groups g'_1, g'_2, g'_3, \dots converges uniformly to the group g .

The set whose elements are the groups g of the geometric type of order ξ constructable in the indicated manner possesses the cardinal number of the continuum. For, already the set of those series m_1, m_2, m_3, \dots , which consist of prime numbers, possesses this cardinal number, and any two different series of this set give rise to different groups g .

We can sum up the preceding as follows:

THEOREM 5. *The geometric type of order ξ allows of an infinite number of groups consisting of a geometric type of order ξ of continuous one-one transformations and being uniformly approximated by a fundamental series of groups consisting each of a finite number of continuous one-one transformations.*

If in particular we consider those groups g for which each g_n is chosen in the way described at the commencement of this § as the direct product of g_{n-1} and a group γ_n , we can formulate in particular:

THEOREM 6. *The geometric of order ξ allows of an infinite number of groups consisting of a geometric type of order ξ of continuous one-one transformations and being uniformly convergent direct products each of a fundamental series of finite groups of continuous one-one transformations.*

§ 5.

The sham-addition in the geometric type of order ξ .

Let us choose the factor groups indicated in theorem 6 as simply as possible, namely g_1 as the group of cyclic displacements corresponding to a certain cyclic arrangement of the first indices, and likewise each γ_n as the group of cyclic displacements corresponding to a certain cyclic arrangement of the n^{th} indices; g is then commutative, and transitive in such a way that a transformation of g is determined uniformly by the position which it gives to one of the elements of μ .

Let us further choose an arbitrary piece of μ as *piece zero*. Let us represent this piece by S_0 , and the transformation, which transfers S_0 into S_x and is thereby determined, by " $\mp S_x$ ". That the

piece S_3 is transferred by this transformation into S_7 , we shall express by the formula

$$S_3 \hat{+} S_x = S_7,$$

which operation is associative and commutative.

Let us finally choose, in order to make the resemblance to ordinary ciphering as complete as possible, all m_n 's equal to 10, let us take for each system of n^{th} indices the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in this order, and let us give to the piece zero only indices 0.

The different pieces of μ we can then represent biuniformly by the different infinite decimal fractions lying between 0 and 1, in such a way, however, that *finite* decimal fractions do not appear and that $\cdot 30$ is not equal to $\cdot 29$, whilst each group γ_n consists of the different ways in which one can add the same number to all n^{th} decimals, modulo 10.

Now according to the above we understand by $\cdot 5473\dots \hat{+} \cdot 9566\dots$ the decimal fraction, into which $\cdot 5473\dots$ is transferred by the transformation which transfers $\cdot 0$ into $\cdot 9566\dots$, or, what comes to the same, the decimal fraction, into which $\cdot 9566\dots$ is transferred by the transformation which transfers $\cdot 0$ into $\cdot 5473\dots$.

We shall call the operation furnishing this result, on the ground of its associativity and commutativity, the "*sham-addition*" of $\cdot 9566\dots$ to $\cdot 5473\dots$; it takes place just as ordinary addition, with this difference that in each decimal position the surplus beyond 10 is neglected, thus that different decimal positions do not influence each other. So we have:

$$\cdot 5473\dots \hat{+} \cdot 9566\dots = \cdot 4939\dots$$

Let us understand analogously by $\cdot 5473\dots \hat{-} \cdot 9566\dots$ the decimal fraction, into which $\cdot 5473\dots$ is transferred by the transformation which transfers $\cdot 9566\dots$ into $\cdot 0$, and let us call the operation furnishing this decimal fraction the "*sham-subtraction*" of $\cdot 9566\dots$ from $\cdot 5473\dots$; then this sham-subtraction is performed in the same way as ordinary subtraction with this difference, that "borrowing" does not take place at the cost of the preceding decimal positions, so that here again different decimal positions do not influence each other. So we have:

$$\cdot 5473\dots \hat{-} \cdot 9566\dots = \cdot 6917\dots$$

By operating only with a finite number, great enough, of consecutive figures directly behind the decimal sign, sham-addition and sham-subtraction furnish in the type of order ζ a result agreeing with the exact one up to any desired degree of accuracy. In this too they behave like ordinary addition and subtraction of real numbers.

Microbiology. — “*Emulsion laevulan, the product of the action of viscosaccharase on cane sugar*”. By Prof. M. W. BEIJERINCK.

In the proceedings of the Academy of 9 February 1910 an enzyme was described which produces from cane sugar and raffinose a viscous matter incapable of diffusion. My further investigations, made conjointly with Mr. D. C. J. MINKMAN, proved that this substance is closely related to the laevulan of LIPPMANN¹⁾ but not identic with it.

Our emulsion laevulan originates in watery nutrient solutions in quite the same way as in the agarplates, so that these solutions change into a milkwhite emulsion; the liquid between the suspending laevulan droplets opalises very strongly. In hot water the substance is fairly soluble and the specific rotation of the polarised light, which, on account of the opalisation can only approximately be determined, is about

$$\alpha_D = -80^\circ$$

whilst LIPPMANN gives for his laevulan

$$\alpha_D = -221^\circ.$$

On account of this considerable difference in its rotating power, a new name, e.g. “sinistran”, might seem desirable. But the word laevulan having a collective meaning to which also the more and the less soluble forms of our substance may be brought, we shall here use the general denomination, the more so as it is sure that the laevulan of the literature, like ours, consists of the cell-wall substance of bacteria.

Besides by LIPPMANN the formation of laevulan by bacteria has also been observed by MAASSEN²⁾, who does not, however, describe the appearance of the emulsion, so that in this case, too, a modification of our emulsion laevulan seems to be produced. The here concerned microbe is a sporulating fermentation organism, called by MAASSEN *Semiclostridium commune*, but not yet found by us.

Preparation and properties of emulsion laevulan.

We were first of opinion that emulsion laevulan might be best prepared by using *Bacillus emulsionis*, for we had stated that this species does not decompose the once formed laevulan, whilst *B. megatherium* and *B. mesentericus*, which likewise produce emulsion

¹⁾ Chemie der Zuckerarten 3^{te} Aufl. 1904. Pag. 906, 1312.

²⁾ Arbeiten aus dem Kaiserl. Gesundheitsamte. Biol. Abt. Bd. 5, p. 2, 1905.

laevulan, attack this substance and use it as food as soon as the cane sugar fails. We have, however, found that with some precaution it is much easier, especially with *B. mesentericus*, to produce large quantities of laevulan, than with *B. emulsionis*; this reposes on the circumstance that the former species, particularly at high temperatures, about 40°, possesses a very strong vegetative power, whilst the latter always grows slowly and has a relatively low temperature optimum, below or near 30° C.

Hence we used for the preparation of laevulan the common hay bacterium, which is the form of *B. mesentericus* obtained by accumulation methods, such as the method of potato slices and that of malt solutions. But this form is so common in our surroundings and so well adapted to the life in cane sugar solutions of for the rest different composition, that these, after pasteurisation or short boiling and when kept warm, of themselves produce laevulan by the development of the spontaneous spores of the hay bacillus. Such solutions then turn milky and slimy by the formation of the microscopic laevulan emulsion.

For the experiments were used large ERLÉNMEYER-flasks with 500 cm³ of a medium of the composition: tapwater, 20% canesugar, 0.05% KNO₃, and 0.05% K₂HPO₄, cultivated at ± 27° C.

This liquid inoculated with *B. mesentericus* very soon obtains the said milky appearance. The same emulsion which to the colonies of *B. mesentericus* and *B. emulsionis* on cane sugar agarplates gives so peculiar a character, is now in large quantity produced in the culture liquid, saturated besides with laevulan in true solution, which causes the strong and characteristic opalisation, not known to us to such a degree in any other substance. Besides, at the bottom of the flasks a thick transparent slime layer is slowly formed, which also proved to consist of laevulan, wherein, however, the bacterial bodies themselves are accumulated, whilst the liquid above it is poor in bacteria but abounds in viscosaccharase and laevulan emulsion. The acid formation in this solution is slight but not absent.

The laevulan may be precipitated with alcohol for which 50% in the solution is sufficient. Only at a much greater alcohol concentration other substances of the liquid also precipitate. By dissolving in boiling water and again precipitating the further purification is easy. After drying and pulverising a snowwhite nearly tasteless powder results.

From a flask as the above which at first contained 100 G. of cane sugar, 8 G. of pure dry laevulan were obtained after 7 days cultivation, there still being in the liquid 20 G. of invert- and 70 G. of cane sugar; the slime at the bottom not being collected.

From another flask quite alike to the preceding, which also contained 100 G. of cane sugar, were gained 15 G. of laevulan after 17 days, 45 G. of cane sugar and 35 G. of invert sugar still being present.

The slime adhering to the bottom, consisting of *B. mesentericus* with thick cell-walls of laevulan, was used for a new culture for which a solution of 2 % of cane sugar, 0.05 % KNO_3 and 0.05 % K_2HPO_4 was used. After 18 days were obtained 2.25 G. from the 10 G. of original cane sugar, accordingly 22.5 % of laevulan was earned.

Pure laevulan is somewhat soluble in cold water, much better in boiling; all solutions opalise very strongly. It does not reduce FEHLING'S coppersolution; only after prolonged boiling a feeble reduction is observed. It is incapable of alcoholic and lactic acid fermentation, but by butyric acid ferments, in absence of air, it gets into as strong a fermentation as cane sugar, whereby hydrogen, carbonic and volatile acid result.

A number of bacteria can feed on it when growing with access of air. *Azotobacter chroococcum* can use it under fixation of free nitrogen and formation of some acid.

By a treatment with acids, especially when warm, it changes readily into laevulose and so becomes fit for alcoholic and lactic acid fermentation. After the inversion, by heating with resoreine and strong hydrochloric acid, the red colour appears, characteristic of laevulose, whilst with oreine and hydrochloric acid the violet colour, indicating pentose, is completely absent. When distilled and treated with sulphuric acid no perceptible quantity of furfural can be detected.

As said, the specific rotation, which cannot be exactly determined on account of the strong opalisation is

$$\alpha_D = -80^\circ,$$

and after hydrolysis

$$\alpha_D = -70^\circ.$$

After prolonged heating with acid in the autoclave at 120° the rotation lowered even to

$$\alpha_D = -64^\circ.$$

That of pure laevulose is

$$\alpha_D = -92^\circ.$$

There is some probability that this diminution is due to destruction of part of the laevulose.

As we had found that the slime at the bottom of the flask is less soluble than that obtained by alcohol from the emulsionated liquid above it, we prepared laevulan from this slime also by separate experiments, for we supposed that dextran might occur therein, which is much less soluble in water than laevulan. However, it was found that the laevulan obtained in this way gives no other rotation after inversion than the emulsion laevulan, from which it does not differ. Hence it is sure that hay bacteria produce no dextran at all, but that their cell-wall consists of various modifications of laevulan of different solubility.

Not only in media of the above composition *B. mesentericus* produces laevulan, this happens quite as easily in a yeast decoction with 2 to 20 % of cane sugar, addition of chalk proving favourable. The temperature of cultivation may also vary and even rise to 40° C., but then care should be taken that the laevulan itself be not attacked by the producer.

From the preceding it may be concluded that the large lumps of slime so easily formed on cane sugar agar-plates by *B. mesentericus* and the other emulsionating species consist as well of laevulan as the emulsion which occurs round the colonies of this species in the agar. Hence, it can neither be doubted that the slime of these colonies, which does not diffuse in the agar, is produced by viscosaccharase from cane sugar, and that this enzyme only partly gets out of the bacterial body proper, the cell-wall included. Evidently in the cell-wall itself the enzyme forms new laevulan by converting the cane sugar, with which both cell-wall and agar-plate are imbibed.

The production of cell-wall substance in consequence of the action of an enzyme, which in my former communication was called probable, must now, as regards laevulan, be considered as proved.

Dextran and the dextran bacteria, which we have likewise studied, shall later be treated more thoroughly. For the moment it may be observed that by this substance the polarised light is strongly rotated to the right; we found

$$\alpha_D = + 132^\circ,$$

whilst in the literature by various authors is given for dextran

$$\alpha_D = + 199^\circ \text{ to } 230^\circ.$$

Quite like laevulan it results exclusively from cane sugar. So laevulan as well as dextran are produced by microbes, neither from laevulose, glucose, or any other sugar, but solely from cane sugar and raffinose. The slimy cell-wall substances formed by other microbes from glucose, laevulose and maltose, are of a different nature.

Physics. — “*Researches on the magnetization of liquid and solid oxygen.*” By H. KAMERLINGH ONNES and ALBERT PERRIER. Communication N°. 116 from the Physical Laboratory, Leiden.

§ 1. *Introduction.* It is scarcely necessary to remark that the investigation of the magnetic properties of oxygen at low temperatures has long occupied a position on the programme of the cryogenic laboratory, or that this has been considered one of the most important items on the programme since the investigation of both liquid and solid oxygen has been made possible by the perfecting of the methods¹⁾ of obtaining detailed series of measurements at constant temperatures in the region of liquid hydrogen. Indeed, while the strongly magnetic properties of oxygen of themselves select it from all other substances as especially suitable for the study of para-magnetism, we can in the meantime for no other substance obtain the *magnetic equation of state*²⁾, which gives a representation of the magnetic properties of a substance in the three states of aggregation at as many successive temperatures and pressures as possible.

The investigation of oxygen at very low temperatures and at pressures that can easily be realised was expected to give at once results of much importance.

CURIE³⁾ found for gaseous oxygen between 20° C. and 450° C. that the *specific susceptibility* (magnetization per gram for $H=1$) was inversely proportional to the absolute temperature, and FLEMING and DEWAR⁴⁾ concluded from their latest measurement of the susceptibility of liquid oxygen at its boiling point that CURIE'S law was obeyed down to -183° C.

Does the specific susceptibility continue to increase so strongly at low temperatures or does it approach a limiting value? Is oxygen in the solid state ferro-magnetic? Does the magnetization finally at extremely low temperatures perhaps begin to decrease and disappear completely at the absolute zero?⁵⁾

1) H. KAMERLINGH ONNES, These Proc. Sept. 1906, Comm. from the Leyden labor. no. 94f (1906).

2) H. KAMERLINGH ONNES, Commun. from the Leyden labor. Suppl. no. 9 p. 28.

3) P. CURIE, Ann. chim. phys. (7) 5 (1895) p. 289.

4) FLEMING and DEWAR Proc. Royal Soc. London 63, p. 311, 1898.

5) It has since appeared that the magnetization of ferro-magnetic substances does not yet give any justification when the temperature is lowered to the melting point of hydrogen for the assumption that the electrons whose motion causes magnetization are frozen fast to the atoms and that therefore this disappearance at the absolute zero may be expected. (P. WEISS and H. KAMERLINGH ONNES, These Proc. Jan./Febr. 1910, Comm. from the Leyden Labor no. 114 p. 9).

These are questions which, considering the possibility of obtaining important contributions to the knowledge of the influence of density upon susceptibility by high pressures in the region where the gaseous state of aggregation changes continuously into the liquid make it a very attractive work to realise, even from a purely experimental point of view, the representation to which we have just referred.

The work was commenced though only when Prof. WEISS extended his magnetical researches to very low temperatures and the measurements on the magnetization of ferro-magnetic and cognate substances at very low temperatures, which were communicated to the February Meeting¹⁾, were undertaken. With that investigation which was carried out at the same time, the present one is very closely related, and for part of them we made use of the same appliances. In our present investigation we have also in various ways made use of Prof. WEISS's method²⁾ of determining the magnetization by means of the maximum couple exerted by a magnetic field of variable direction upon an ellipsoid of the experimental substance, a method which had been entirely successful in the other research. We must also express the great advantage we derived from the continued support given us by Prof. WEISS, and we take this opportunity of gratefully acknowledging our indebtedness to him.

The change with temperature of the specific susceptibility of oxygen, the investigation of which was our first object, is of particular importance seeing that CURIE's law follows from LANGEVIN's kinetic theory of magnetism³⁾. It was soon apparent to us that this law was not valid for oxygen, as was thought, down to -183° C., but that it would have to be replaced by another. According to the important paper of DU BOIS and HONDA communicated to the January Meeting — our experiments had already been completed at that time — various elements were found for which CURIE's law did not hold at temperatures above 0° C. This at once increases the importance of the further investigation of oxygen, for which over a definite region of temperature CURIE's law is valid, while over another region it obeys a second law, viz.: *that of inverse proportionality to the square root of the absolute temperature*. The results concerning this law and also concerning the probability of a sudden change in the value of the specific susceptibility on solidification will be discussed in § 5.

¹⁾ P. WEISS and H. KAMERLINGH ONNES. These Proc. Jan./Febr. 1910. Comm. fr. the Leyden labor. no. 114 (1910).

²⁾ P. WEISS. Journ. de phys. 4e série t. VI, p. 661; 1907.

³⁾ LANGEVIN. Ann. chim. phys. (8) 5, p. 70; 1905.

We have been occupied with another question besides the change of specific susceptibility with temperature, which was suggested both by the experimental results obtained by FLEMING and DEWAR and by the theories of LANGEVIN and WEISS.

In the experiments of the first-named there appears sufficient evidence for the conclusion that there is a decided diminution of the susceptibility as the strength of the field increases (the diminution is of the order of 10% in a field of 2500 gauss). Now, according to the theory of LANGEVIN para-magnetic substances must, it is true, exhibit this phenomenon, but calculation from his formulae limits the magnitude of this change to less than 0.1% in the case of liquid oxygen at its boiling point. Should a higher value than this be obtained, then one would be led to assume the existence of a WEISS molecular field¹⁾. We arranged our experiments so that the liquid and the solid oxygen could be subjected to a field of 16000 gauss, a field very much stronger (about six times) than that used by FLEMING and DEWAR, so that we might expect the phenomenon which appeared in the course of their experiments to be exhibited to a much greater degree in ours even at the same temperatures as were used by them. If what was observed by FLEMING and DEWAR could really be ascribed to the beginning of saturation then the theory would further lead us to expect that as the temperature sank the change would strongly increase (becoming infinite at $T=0$), and that in our experiments with liquid hydrogen it would become very striking. We have, however, observed only small deviations, which we shall discuss further in § 5.

As regards the experimental methods employed by us in our investigation, two completely different schemes were adopted: on the one hand was measured the magnetic attraction exerted upon a column of the liquid, and on the other the maximum couple exerted by a homogeneous field upon an ellipsoid. The second method is more especially suitable for comparative measurements and can also be used for frozen oxygen; the first can be used only for the liquid phase, but on the other hand it makes very trustworthy absolute measurements possible; we have therefore adopted it as the basis of our other measurements. In the carrying-out of each method further precautions are still desirable, so that while we are busy pushing on the investigation, we propose at the same time to repeat it in part in order to increase the accuracy of the results obtained by taking such further precautions as have appeared possible in the course of the work.

¹⁾ WEISS, L'hyp. du champ moléc. loc. cit.

Liquid oxygen I.

§ 2. *Method of the magnetic rise.* As mentioned above, we have rendered the method of the magnetic rise employed by QUINCKE, du Bois and other observers suitable for use at low temperatures.

One limb of a vertical O-shaped tube, the upper portion of which contains the gaseous, and the lower the liquid phase of the experimental liquefied gas was placed between the poles of a magnet whose field was horizontal.

Let H be the field, (H' the field in the other limb is supposed to be so small that $(H'/H)^2$ is negligible), g the acceleration due to gravity, z the difference in height of the levels of the liquid under the influence of H , ρ and ρ_0 the densities of the liquid and of the gaseous phases respectively, K and K_0 their respective volume susceptibilities, then

$$(K - K_0) H^2 = 2z(\rho - \rho_0)g \quad \dots \quad (1)$$

or, by introducing the absolute specific susceptibility χ

$$(\chi\rho - \chi_0\rho_0) H^2 = 2z(\rho - \rho_0)g.$$

If $\chi = \chi_0$ then the equation becomes simply

$$\chi = \frac{2zg}{H^2}, \quad \dots \quad (2)$$

which is the formula we have used for our calculations.¹⁾

So there are striking advantages offered by this method particularly for an absolute measurement, on account of its applicability to the case of a liquid in equilibrium with its own vapour. There are only two magnitudes to be determined, the distance z , which can be measured very accurately with a cathetometer, and the field H ; nor have we to know the density of the liquid in order to be able to find the specific susceptibility.

Magnetic rise apparatus. It is a very easy matter to cause an ordinary liquid to ascend under the influence of magnetic attraction, but the experiment is attended by serious difficulties when one has to deal with a liquefied gas. Boiling must be completely avoided, and care must be taken that the vaporization is unnoticeable. The first precaution is necessary because the motion of the liquid or of its surface would render adjustment quite impossible, and the second

¹⁾ In § 5 we shall give the reasons why we think that $\chi = \chi_0$, and should it be possible that this is not the case there is still the greatest probability that $\chi_0 < 1.5 \chi$; in the most unfavourable case at the boiling-point the correction remains below 0.002 in value, while at lower temperatures it is quite negligible on account of the small value of ρ_0 .

is necessary that the total quantity of liquid may not appreciably alter during the measurement of one rise. Moreover magnetic action itself increases the difficulties; it is easy to see that it can occasion the formation of gas-bubbles which divide the column of liquid into two parts, so that the one portion remains suspended between the poles, while the other falls back again. In that case measurement of the ascent is out of the question.

Starting from the thermodynamic potential it appears that in every case the relation

$$2g \eta \rho > K(H^2 - H_\eta^2)$$

must hold, where H is the field at the surface of the liquid, and H_η the field at a distance η below the surface of the liquid. These conditions shew that there is a limit to the intensity of the fields in which measurements may be made, for they necessitate a range of extended fields (in this case in a vertical direction). Conical pole-pieces are thus as a matter of fact barred.

After several preliminary experiments an apparatus was constructed, the most important part of which consisted of two concentric double-walled vacuum tubes, with which we already succeeded in obtaining rather successful measurements. The walls of the double vacuum tube were not silvered, so that we were able to watch how the liquid behaved during the experiments. From the experience thus acquired the improved apparatus which we shall now proceed to describe was designed and constructed.

It will be seen that the construction of the apparatus lays a very heavy tax upon the art of the glass-blower¹⁾. As before, the chief part consisted of two independent U-shaped vacuum tubes, the one fitting inside the other. The double walls of each tube are completely silvered on the vacuum side, except in the case of the inner tube, where the distance which the liquid ascends is left free, and in the outer where a sufficient length is left unsilvered to leave a strip of a few millimeters breadth through which the level of the liquid can be read. One of the tubes completely surrounds that portion of the other which contains liquid; this we call the protecting tube. The narrowest portion M_4 (fig. 2) is placed between the poles of the electro-magnet. The narrow limb of the inner tube must of course be perfectly cylindrical. The other limb is enlarged and serves as a reservoir. In order to be able to apply equation (2) all care was taken that the temperature of the liquid and vapour up to a height

¹⁾ The double vacuum tube was prepared by Mr. KESSELRING, Laboratory glass-blower, and the remainder by Mr. FLIM, technical assistant at the Laboratory.

somewhat greater than that reached by the column of liquid was everywhere the same both in the wide and in the narrow tube; and further care was taken that where the temperature of the vapour above the liquid in the upper parts of the apparatus changes to ordinary temperature it was as far as possible the same at the same height in the two limbs of the O-shaped space. With this end in view the liquid in the inner tube was, by means of the magnetic field, repeatedly moved up and down under constant vapour pressure, until we might assume that in this tube equilibrium was sufficiently well attained. To make this equilibrium possible the inner tube is surrounded with liquid at the same temperature as that which the liquid in it must attain. In the outer or protecting tube the liquid is kept constantly in motion by means of a stirrer consisting of a brass ring S_1 that can be moved up and down; it is possible to do this and still keep the space closed by utilising the flexible rubber tube S_2 . The vaporization in the inner tube is thus very small (between 0.5 and 1 litre of gas measured under normal atmospheric pressure escapes per hour).

Notwithstanding all these precautions temperature differences must still be encountered. In the liquid, in which the convection currents maintaining heat-equilibrium can be followed by the small particles which they carry along with them, these temperature differences must have been very small. In the gas layer in the upper portion of the O-shaped space there must indeed have been considerable differences; but on account of the small density of the gas, these have but small influence upon the difference of level in the two limbs, and, moreover, that influence may be almost entirely neglected seeing that the observations are simply comparative measurements with and without the magnetic field. Now, care has been taken that the temperature over the distance that the liquid rises can vary but slightly, while in the upper portions of the tube practically the same state of affairs is maintained during both observations. We have therefore omitted the correction that should still have to be applied for possible temperature differences.

Comparing the positions of the liquid in the narrow cylindrical tube with and without the magnetic field also reduces the correction for capillarity to the insignificant differences in form of the menisci, and this correction, too, we have omitted.

The inner and the outer tubes are closed independently of each other by means of the German-silver caps P_1 , P_2 , Q_1 , Q_2 (fig. 1); the junction is made air-tight by the rubber sleeves M_0 , N_0 , which at the same time unite the two tubes firmly together. Liquid oxygen

is introduced into the protecting tube through the small tube P_5 , and into the inner tube through Q_5 . The two tubes P_3 and Q_3 lead the vaporized oxygen through the valves P_8 and Q_8 (fig. 2) to two gasometers. Two manometers P_6 and Q_6 , the latter of which is provided with an indicator Q_7 , so that small vapour pressures may be read off accurately, serve at the same time as safety valves. It is not necessary that the oxygen in the protecting tube should be as pure as that in the inner tube; for the latter, with which the observations were made, very pure oxygen was used.

A double sliding movement R allowed an easy adjustment of the apparatus each time, so that the meniscus in the measuring tube just reached the desired point in the field between the poles, usually in the axis of the pole pieces.

Course of a series of measurements. The field is brought to the desired strength and by means of R the meniscus is made to rise to the desired point, which is read off on a small scale. Then the meniscus is moved up and down several times while care is taken that the field slowly increases. In this way the temperature is made everywhere the same and the walls of the tube are wetted. While the field has the desired value the position of the meniscus is read off; then a reading is made while the field is off; after the meniscus has been three times allowed to rise somewhat higher than the desired position, another reading is made while the field is on; once more a reading is made with the field off and so on several times. In this manner the error arising from vaporization of the liquid during the adjustment of the cathetometer is eliminated¹⁾. It is not essential to know the position of the level in the other limb of the tube: so as to be able to take account of this, we ascertained the ratio of the cross-sections of the two limbs of the tube.

We have further made sure that the residual magnetism exerted no appreciable influence upon the position of the meniscus after the current was cut off. For this purpose a feeble current was sent through the coils in the direction opposite to that which had just been broken. Had the residual field exerted any appreciable influence we should have seen first a further sinking of the level, and then a rise as the current was slowly increased. This has not been observed.

We used the same electro-magnet as was used for the cryogenic investigation of the ferro-magnetic metals²⁾, to which we must refer

¹⁾ To control the position of the meniscus without the magnetic field, we measured the quantity of gas vaporized (cf. preceding page).

²⁾ P. WEISS and H. KAMERLINGH ONNES, l. c.

for details regarding its construction. It was only necessary to replace the conical pole-pieces by cylinders with flat ends. Their distance apart was micrometrically adjusted to 25 mm. and controlled with an accurate callipers. We may here remark that between the measurement of the ascent and that of the field, the pole-pieces remained clamped tight to the cores, so the adjustment of the distance could give rise to no error.

Since in the subsequent calculation the strength of the field is involved to the second power, and since we are concerned with an absolute measurement, we endeavoured to make our measurement of the field strength as trustworthy as possible with our present appliances. With this end in view we measured the strength of an arbitrarily chosen standard field by two different processes, and we compared the strengths of the fields used in our experiments with this standard by successively withdrawing the same coil attached to a ballistic galvanometer from the standard field and from the various fields which we desired to measure.

The standard field was set up with the same flat pole-pieces at a distance of 9 mm. apart, and with a current of 5 amp. All precautions were taken to ensure the demagnetization of the magnetic cycle beforehand. This field was first measured by means of CORROU'S magnetic balance¹). As is well known this method consists of equilibrating weights of a total mass m against the ponderomotive power of the field H on a straight portion of length l of a conductor through which a current flows of intensity i ; then we get

$$H = \frac{m g}{i l} \cdot 10.$$

For the degree of accuracy, however, which we wish to reach, several corrections must be taken into account. In the first place the various parts of the balance were accurately calibrated. The length l of the current element was determined micrometrically and on the dividing engine, and so also was the distance between the arcs of the balance which distance ought to be the same throughout seeing that the arcs must be accurately concentric. The very small deviations from this were allowed for by means of a ballistic investigation of the topography of the field. The balance arms of the weights and of the current element were measured with the cathetometer. The topographical study of the field also gave us the

¹) For this method of measuring the field and for the magnetic balance see: P. WEISS and A. CORROU, Le phénomène de ZEEMAN pour les trois raies bleues du zinc, Bull. Séances Soc. franç. de phys. 1907, p. 140, also J. de phys. 1907.

correction necessary for the force exerted upon the second straight element of the balance (i. e. that outside the pole-gap). The sum total of these positive and negative corrections came to some units per thousand.

The greatest care had to be devoted to the absolute value of i , which was measured by means of an accurate ammeter by SIEMENS and HALSKE. This was calibrated in absolute ampères by comparing on the potentiometer the potential difference between the terminals of an international ohm (or for the stronger currents of 0,1 Ω) with the electromotive force of a WESTON cadmium cell. For the requisite accuracy of the measurements the influence of neighbouring instruments or currents upon the ammeter, or of its position in the earth's field were by no means negligible; we got rid of almost all these irregularities by a suitable adjustment of the distances and of the positions of the rheostats, and we eliminated further possible remaining errors by so connecting *all the conductors* that the currents *in all* except the ammeters could be reversed *at the same time*. Finally we always used the ammeters in the same position with respect to the earth's field as that in which they had been calibrated.

When all calculations and corrections had been completed it was found that the strength of the standard field was 9857 gauss according to this method.

The *second method* by which the value of the same field was found consisted of the sudden withdrawal from between the poles of the magnet of a coil of wire of which the area encircled by the current was known. The change thus caused in the number of induction lines embraced by the coil was compared by means of a ballistic galvanometer with the number of induction lines embraced by a solenoid the dimensions of which were accurately known.

The coil consisted of 19 turns of silk-insulated wire, 0.25 mm. thick, wound round a cylinder of ebonite, 20 mm. in diameter. The dimensions were obtained by various measurements with the micrometer screw and the dividing engine, and were repeatedly controlled. At the same time a control coil was constructed by winding bare copper wire in a helical groove cut in the curved surface of a cylinder of ebonite; the area encircled by the current was then measured for this control coil by the same methods and with the same precautions as were adopted in the case of the first. The ratio of the two was in agreement with the ratio of the deflections of the ballistic galvanometer which were obtained by connecting the two coils in series with the galvanometer and then withdrawing them successively from an unchanged magnetic field. We may further

say that we had previously verified the absence of magnetic properties from the ebonite by means of an apparatus after CURIE in which we utilised the attraction in a non-uniform field.

For the measurement of the field there were placed in circuit with the galvanometer the coil on the ebonite cylinder, a manganin resistance to regulate the sensitivity, a secondary coil of 500 turns fitting round the standard solenoid, and finally, an electromagnetic arrangement which could be used as a damper if desired. We also allowed for the very small deviations from the law of proportionality between the deflections of the galvanometer and the quantities of electricity, which had been determined for the galvanometer (one of the DEPRez-D'ARSONVAL type) by a previous investigation. The solenoid was constructed with the greatest accuracy by winding bare copper wire on a core of white marble¹⁾.

The standardisation of the galvanometer was made by reversing the current in the solenoid; the observations made by withdrawing the coil from the field always took place between two standardisations of the galvanometer; there was, however, no change in the galvanometer constant to be observed. The corrections and precautions necessary in obtaining the strengths of the current are the same as in the case of the balance, and have already been described. The final result of this ballistic method is

9845 gauss.

The relative difference between this and the value given by COTTON'S balance is therefore 0.0012; and this can be neglected especially when one remembers that almost every one of the numerous measurements necessitated by the one method as much as by the other, beginning with the adjustment of the field by means of the ammeter, is accurate only to 0.0005. It may be useful to comment here upon a particular point that increases the difficulty of obtaining this agreement and therefore enables us to rely more upon the correctness of the numbers which we have obtained. The equation for COTTON'S balance involves the strength of the current in the *denominator*, while this magnitude in calculating according to the ballistic method occurs in the *numerator*; a systematic error there-

¹⁾ For the dimensions and the description of the solenoid and galvanometer see: P. WEISS, Mesure de l'intensité d'aimantation à saturation en valeur absolue, Arch. Sc. phys. et nat. February 1910, J. de phys. May 1910.

fore in the *absolute* number of ampères would, of necessity, occasion a relative difference *twice as great* between the values of the field obtained by the two methods (the same ammeter was used with the balance and with the solenoid).

We have given the ballistic method a somewhat greater weight than the other on account of the smaller number of corrections it involved, and thus we have finally taken as the value of the standard field

9850 gauss.

Once this standard field was definitely fixed all other measurements could be rapidly made by the ballistic method described above.

For the conical pole-pieces which are employed in experiments according to the maximum couple method, and which give much more powerful but much less uniform fields, we used a coil of 7 to 8 mm. diameter accurately centred on the axis of the pole-pieces. In this case direct comparison with the standard field just mentioned was not possible since the flat polepieces had to be screwed off to make room for the conical poles. To meet this case the area of the small coil encircled by the current was determined once and for all by withdrawing it from the standard field *before* the flat pole-pieces were removed, and comparing the change thus brought about in the number of the induction lines with those of the solenoid by means of the ballistic galvanometer.

All the measurements that we have given up to the present refer to the field in the centre of the space between the poles. For the few exceptional values of the field, and, consequently, of the ascent of the liquid oxygen for which it was necessary to cause it to rise pretty far above the axis of the pole-pieces, the field was determined at those points by simple ballistic comparison with the fields on the axis, and we made use of the cathetometer to adjust the position of the small coil.

Results of observations and calculations.

Series of observations with the apparatus with unsilvered walls.

T A B L E Ia. $t = -183^{\circ}.0$ C. ¹⁾

Position of meniscus.	Obs. rise z' in cm.	diff. in height with reservoir z in cm.	H in gauss.	$\frac{z}{H^2} \cdot 10^7$
Level of axis	4.032	4.061	2980	4.494
"	4.646	4.690	3727	4.216
"	4.656	4.701	3727	4.224
axis + 2.44 cm.	3.024	3.110	5182	4.158
axis	3.498	3.289	5205	4.214
"	4.46	4.278	5848	4.251
axis + 2.44	4.90	5.050	6570	4.468
axis	5.124	5.270	6600	4.210
axis + 2.44	7.87	8.094	8075	4.242
axis + 2.44	9.20	9.462	9043	4.458*

The difference in height z was obtained from the observed ascent from $z = z' (1 + 0.0285)$. The observation ²⁾ was very difficult and is little reliable.

The deviation in the case of the observation in a field of 5182 gauss is probably due to a mistake of 1 in the number of whole millimeters which were read off, but of this we are not certain.

Deviations from proportionality with H^2 are considerable but by no means systematic. If we take the mean of all the measurements with the exception of the last in which special difficulties were encountered we reach the value

$$\frac{z}{H^2} = 1.209 \cdot 10^{-7},$$

and for the specific susceptibility with $g = 981.3$ for Leiden

$$\chi_{90^{\circ}.1 \text{ K.}} = 237.3 \cdot 10^{-6}.$$

¹⁾ The boiling point of oxygen according to H. KAMERLINGH ONNES and G. BRAAK These Proc. Oct '08. Comm. fr. th. Leyden labor. N^o. 107a § 6.

T A B L E I b. $t = -201^{\circ}.75$ C.

Position of meniscus.	Obs. rise z' in cm.	diff. in height with reservoir z in cm.	H in gauss.	$\frac{z}{H^2} \cdot 10^7$
Level of axis	1.192	1.222	2980	1.376
"	1.893	1.944	3727	1.399
"	1.881	1.935	3727	1.393
"	3.643	3.747	5205	1.383
"	4.623	4.752	5848	1.389
"	5.91	6.078	6600	1.395
axis + 2.5 cm.	7.376	7.586	7421	1.378
axis + 2.5 cm.	8.715	8.963	8069	1.372

$$\text{mean. } \frac{z}{H^2} = 1,386 \cdot 10^{-7}$$

whence it follows that $\chi_{71^{\circ}.25 \text{ K.}} = 272,0 \cdot 10^{-6}$.

Finally, at $-209^{\circ}.2$ C. a single observation was made. The rise was 6.115 cm. in a field of 6600 gauss, which with the correction for the sinking in the reservoir gives

$$\frac{z}{H^2} = 1.444 \cdot 10^{-7} \text{ and } \chi_{63^{\circ}.9 \text{ K.}} = 283.4 \cdot 10^{-6}.$$

We shall now give the series of observations made with the silver-walled apparatus which we have already described.

T A B L E II a. $t = -183^{\circ}.0$ C.

Position of meniscus.	Obs. rise z in cm.	diff. in height with reservoir z in cm.	H in gauss.	$\frac{z}{H^2} \cdot 10^7$
axis	1.060	1.090	2980	1.227
"	1.669	1.715	3727	1.235
"	3.169	3.258	5183	1.213
"	3.220	3.310	5198	1.225
"	4.035	4.148	5807	1.230
"	4.093	4.208	5848	1.230
"	4.101	4.216	5848	1.233
"	5.119	5.262	6578	1.216
axis + 2.5 cm.	7.750	7.967	8075	1.224
"	8.950	9.201	8659	1.227
axis + 3.5 cm.	9.226	9.484	8808	1.222
"	9.266	9.525	8808	1.228

For this apparatus $z = z'(1 + 0,0280)$.

The mean value of $\frac{z}{H^2}$ is $1,226 \cdot 10^{-7}$, whence it follows that

$$\chi_{90^{\circ}.1 \text{ K.}} = 240.6 \cdot 10^{-6}.$$

T A B L E IIb.

 $t = -201^{\circ}.75$

Position of meniscus	Obs. rise z' in cm.	diff. in height with reservoir z in cm.	H in gauss	$\frac{z}{H^2} \cdot 10^7$
axis	1.195	1.228	2980	1.383
"	1.879	1.922	3727	1.391
"	3.625	3.726	5205	1.375
"	4.567	4.695	5848	1.373
axis + 3.5 cm.	5.461	5.614	6399	1.371
axis + 2.5 cm.	5.832	5.995	6567	1.390
axis	5.852	6.016	6600	1.381
axis + 3.5 cm.	6.463	6.644	6986	1.365
axis + 2.5 cm.	6.899	7.092	7169	1.380
axis + 3.5 cm.	8.207	8.437	7863	1.365
axis + 2.5 cm.	8.654	8.892	8069	1.366
axis + 3.5 cm.	8.988	9.240	8212	1.370
axis	8.913	9.162	8212	1.358

Mean of all observations is 1.375 whence it follows that

$$\chi_{71^{\circ}.35 \text{ K.}} = 269.9 \cdot 10^{-6}.$$

T A B L E IIc.

 $t = -208^{\circ}.2 \text{ C.}$

Position of meniscus	Obs. rise z' in cm.	diff. in height with reservoir z in cm.	H in gauss	$\frac{z}{H^2} \cdot 10^7$
axis	1.277	1.313	2980	1.478
"	1.996	2.052	3727	1.477
"	3.813	3.920	5205	1.447
"	4.841	4.977	5848	1.461
axis + 2.5 cm.	6.012	6.180	6567	1.433
axis	6.094	6.264	6600	1.438
"	6.113	6.284	6600	1.443
axis + 2.5 cm.	7.146	7.346	7169	1.429
axis + 3.5 cm.	8.579	8.819	7863	1.426

Mean of all observations 1.448, whence it follows that

$$\chi_{61^{\circ}.9 \text{ K.}} = 284.2 \cdot 10^{-6}.$$

Finally for finding the specific susceptibility the density of oxygen was found from the formula¹⁾

$$\rho = 1.2489 - 0.00481 (T - 68).$$

From table II we obtain

$$K_{90^{\circ}.1 \text{ K.}} = 275.2 \cdot 10^{-6}$$

$$K_{71^{\circ}.35 \text{ K.}} = 332.8 \cdot 10^{-6}$$

$$K_{61^{\circ}.9 \text{ K.}} = 359.0 \cdot 10^{-6}.$$

Table III gives $\chi\sqrt{T}$ for each of the temperatures and for each of the series.

Series with the first apparatus			Series with the improved apparatus		
<i>T</i>	$\chi \cdot 10^6$	$\chi\sqrt{T} \cdot 10^3$	<i>T</i>	$\chi \cdot 10^6$	$\chi\sqrt{T} \cdot 10^3$
90.1	237.3	2.25	90.1	240.6	2.283
71.35	272.0	2.20	71.35	269.9	2.279
63.9	283.4	2.26	64.9	284.2	2.280
mean		2.27			2.284

There is no systematic change to be noticed in the product $\chi\sqrt{T}$; the greatest deviation from the mean is 1% with the first apparatus, and only $\frac{1}{4}$ % with the second; moreover the deviations in the two series at corresponding temperatures are in opposite directions. Hence within the limits of accuracy of the observations the specific susceptibility can be represented by the formula

$$\chi = \frac{2284}{\sqrt{T}} \cdot 10^{-6}.$$

In the comparative measurements which we shall describe in the sequel we shall find the same law, at least as far as its form is regarded. For the discussion of this point we refer to § 5.

The differences between the various values of the ratio $\frac{\chi}{H^2}$ are greater than we should be led to expect from the accuracy obtained

¹⁾ BALY and DONNAN. J. Chem. Soc. 81 (1902) p. 907.

(0,05%) in the measurements with the cathetometer of the displacements of the level, and from the accuracy of the measurements of the field-strengths, of which a discussion is given above. It is certain that the cause of these deviations must arise from a source other than the measurement of these two data, though we cannot with certainty indicate what this may be.

We may in the meantime remark that, at least in the case of the first series, the unsteadiness of the apparatus in the vertical direction in the not quite homogeneous field, and the slight inconstancy of the temperature have certainly been contributory causes of these deviations, since the second apparatus which was improved exclusively in these directions gave much more regular results. This remark, however, does not seem to account sufficiently for certain appreciable changes that occurred without any noticeable corresponding irregularity in the pressure or in the convection current of the liquid, while there was also no noticeable change in the shape of the meniscus.

Liquid oxygen II.

§ 3. *Measurements by the method of the maximum couple exerted upon an ellipsoid.* Further comparative measurements for liquid oxygen at various temperatures were obtained by means of the method of the maximum couple exerted by a uniform field upon an ellipsoid. This method has already been described and discussed in connection with the research on ferro-magnetic substances¹⁾; it will be sufficient to discuss the modifications which were found to be necessary owing to the particular circumstances under which the method had to be applied to the present research.

In the first place on account of the small value of the susceptibility it was necessary to make the couple to be measured as large as possible; with this end in view we chose an *oblate* ellipsoid of revolution, instead of a prolate; its axis of revolution was placed horizontal in a field which could turn round a vertical axis.

The ratio that is taken between the axes is not a matter of indifference; for a given major axis the couple, which is proportional to $(N_1 - N_2)r$, is a maximum for a ratio of the major to the minor axis that is only slightly smaller than 3; we have therefore taken this value of the ratio for the construction of the ellipsoids.

We used the same electromagnet as served for the measurements made by WEISS and KAMERLINGH ONNES (*loc. cit.*). Two pairs of pole-

¹⁾ P. WEISS, *J. de phys.* (4) 6 (1907) p. 655. P. WEISS and H. KAMERLINGH ONNES, *Comm. N° 114 These Proc.* Jan/Febr 1910.

pieces were used; first the cylindrical pole-pieces with quite flat end surfaces that had been used for the measurement of the magnetic rise, and then truncated conical pole-pieces the end surfaces of which (slightly concave, see in this connection p. 818) were 4 cm. in diameter, and the side surfaces of which were connected by convex surfaces of revolution to the cylinders that formed the cores; these were 9 cm. in diameter. These pole-pieces were constructed to give the strongest possible field when the distance between the poles was taken to be 20 m.m. By this means a field of about 16000 gauss was obtained.

Our observations were made with an ellipsoid that was diamagnetic with respect to the surrounding medium — a solid silver ellipsoid immersed in a bath of liquid oxygen. The ellipsoid was turned by the “Société genevoise pour la construction d’instruments de physique” from a block of very pure MERCK silver. A preliminary experiment showed that it was very slightly diamagnetic with respect to air, and that this was quite negligible with respect to the liquid oxygen. The axes were measured microscopically on the dividing engine; this gave major axis = 1.0973 cm. and axis of revolution = 0.3654 cm.

Furthermore, two intermediate ordinates parallel to the axis of revolution were measured on the dividing engine, and they were found to be 2% greater than the corresponding ordinates of a perfect ellipse with the same axes. This deviation from ellipsoidal shape was confirmed by a direct determination of the volume from the weight and the density, which gave

$$0.2329 \text{ c.c.}$$

while calculation from the dimensions of the axes gave

$$0.2308 \text{ c.c.}$$

In the calculations we made use of the value 0.2329.

The cryogenic apparatus, essentially the same as that used by WEISS and KAMERLINGH ONNES is shown in Pl. I fig. 3. Once more we see the cover *B*, the adjusting tube *f'*, and the holder *b'*. The cover with its various parts: the cap with the stuffing-box *D*, glass tube *C*, window with plane parallel glass plate *C*₂, the system *BG* for adjusting the whole apparatus, the tension rods *B*₄ for supporting the DEWAR tube, the helium thermometer *θ*, the little screens to protect the upper portions of the apparatus from cooling, etc. is just the same as before. The DEWAR tube is of the same shape, but the lower portion is of greater diameter. The only difference between the adjusting tube *f'* and that which was used in the other investigations is that the lower portion *f*'₀ is of greater diameter.

The *holder* and the *torsion spring* are, on the other hand, completely altered. On account of the smallness of the couple to be measured all foreign magnetic actions had to be eliminated as carefully as possible. Preliminary experiments showed us that a metallic holder could not be used, not only on account of the traces of para- or ferro-magnetic impurities that are never absent from workable metals but also on account of the difficulty of keeping the surface sufficiently clean; this difficulty was encountered repeatedly in the silver ellipsoid that we used in our experiments, and it is probable that the constant contact of the hands with iron tools plays a part in causing it. Glass seemed to be by far the most suitable material both on account of the absence of inherent magnetization and of the fact that the surface on account of its smoothness can be kept quite clean. The holder which we finally adopted was made completely of glass: it consists of a tube b' 5 mm. in diameter that at b'_3 is drawn out to a narrow but thickwalled stem, 0.7 mm. in diameter. To this stem the silver ellipsoid was attached; for this purpose a hole of sufficient width to fit was bored along one of its greater diameters and the ellipsoid was then fixed at the desired height by means of a little wax that completely filled the narrow space between the glass and the metal. The tube was then pumped free from air and sealed off, so that the liquefaction of the air that it would otherwise contain would be prevented. The flat mirror for measuring the angle of torsion and the oil-damper were also attached to the holder.

The torsion springs. On account of the smallness of the couples to be measured (the constants of the springs were of the order of 1200 c.g.s. while those used for the investigation of the ferro-magnetic substances were some tens of thousands) it was found more suitable to use a straight instead of a helical spring. We took a strip of phosphor bronze about 5.5 cm. long (l') and 0.2×0.01 sq. cm. in cross-section. The upper end was soldered to a spiral spring of three turns made from a much thicker strip than the other; the greatest dimension of this strip was horizontal so that in this way it fulfilled its purpose of being elastic to tension while taking no part in torsion; its presence is essential to prevent the breaking of the thin glass stem or of the platinum-iridium stretching wire that is soldered to the stem. This stretching wire is made from a platinum-iridium wire of 0.1 mm. diameter, which was rolled very thin so as to make its torsion constant extremely small without diminishing to any great degree its resistance to breakage. The stretching wire is fused at b'_4 to the lower end of the glass stem, and at its other extremity it carries a knob c' which is held fast in a ring f'_6 .

The mounting of the apparatus took place with the same precautions regarding the centring of the whole, the tension of the springs, etc. and by a method similar to that which has been described in the research upon the ferro-magnetic metals.

The course of the observations is very simple once everything has been set up in position. First, those azimuths of the electromagnet are tentatively determined for which the couple in both directions is a maximum. It was sufficient to do these experiments two or three times with suitably chosen fields, since the azimuth changes but very little with the field, and for other values of the field one can without danger have recourse to interpolation. After that the series of observations took place in the following manner: Before making a measurement with any particular current this was reversed a certain number of times so as to obtain a well-defined field; we had not here to deal with a value of the saturation-magnetization, which changes but slowly with the field, but in our case the couple was proportional to the square of the field, so that inaccurate values of the field that might be obtained notwithstanding the fact that the iron of the electromagnet was extremely soft would make their influence very strongly felt in our results. Then the electromagnet was adjusted to one of the determined azimuths, the torsion angle was read off for the two directions of the field, the current broken, the electromagnet turned to the opposite azimuth, and so on several times. At the end of a series a measurement with one of the first fields was repeated as a control.

Sources of error, difficulties, corrections, and controls.

1. *Inhomogeneity of the magnetic field.* As will be seen from the following discussion this source of error is by far the most important in our case and is indeed the only one that need be taken into account. If we assume that the field near the centre of the pole-gap may be represented by an expression of the form

$$H = H_0 + \frac{r^2}{2} \left(\frac{\partial^2 H}{\partial x^2} \right)_0 (\cos^2 \theta - \frac{1}{2} \sin^2 \theta). \quad (3)$$

where H_0 is the field in the centre, and r and θ polar coordinates of a point in the pole-gap with respect to the centre as origin¹⁾. Let us now replace the ellipsoid by a vertical disc whose diameter is equal to the major axis of the ellipsoid; by taking the expression for the energy of the magnetized disc in the field and differentiating it with respect to the angle between the disc and the lines of force, we obtain for the couple caused by the inhomogeneity of the field:

¹⁾ Cf. P. WEISS and H. KAMERLINGH ONNES l.c.

$$M' = \frac{3}{16} r^2 I \left(\frac{\partial^2 H}{\partial r^2} \right)_0 \sin \varphi \cos \varphi. \quad \dots \quad (4)$$

(r = radius of the disc).

The ratio $\frac{M'}{M}$ of this couple for an angle of 45° to the fundamental couple is

$$\frac{M'}{M} = \frac{3}{16} \cdot \frac{\left(\frac{\partial^2 H}{\partial r^2} \right)_0}{(N_2 - N_1) I} \cdot r^2.$$

If then we suppose that the relative change of the field in the space occupied by the ellipsoid is of the order of 1 in 1000, the formula given above shews us that although the disturbing couple is a little smaller than the chief couple, the two are *of the same order of magnitude*. Hence we see the great influence that this source of error can have in the investigation of weakly magnetic substances. (With ferro-magnetic bodies it is quite negligible: see the previous paper).

We have accordingly devoted the greatest attention to this source of error. The *conical* pole-pieces were made slightly concave, during which process we every time determined the inhomogeneity of the field by means of a ballistic galvanometer and a small coil that was slightly displaced. We ascertained that the change in the field in a space of about 1 c.c. was certainly less than 1 in 2000. We have not had time to pursue this investigation further, and, besides, we should have to obtain a much more sensitive ballistic galvanometer. But it will be seen that the homogeneity of the field was sufficient for the comparative measurements we proposed to make. We may further remark that all these precautions refer exclusively to the conical pole-pieces; the experiments with the cylindrical pole-pieces were nearly free from these sources of error.

We allow for these disturbing couples in the following way: Assuming that $\left(\frac{\partial^2 H}{\partial r^2} \right)_0 = \lambda H$ the expression for the couple due to inhomogeneity given above becomes ($\varphi = 45^\circ$):

$$\frac{3}{32} r^2 \lambda I H$$

or

$$\frac{3}{32} r^2 \lambda K H^2,$$

which we shall represent by

$$\beta K H^2.$$

If α is the angle of torsion of the holder and C the constant of the spring, then

$$C\alpha = \frac{v}{2} (N_1 - N_2) K^2 H^2 + \beta K H^2. \quad (5)$$

Thus just as if there were no correction for inhomogeneity the second side of the equation remains always proportional to the square of the field. Even without knowing that correction, if β is itself a constant we should be able to deduce from the observations whether K is a function of the field or not. We see, however, that the constancy of β requires that of λ , i.e. that the field must remain homothetic no matter how great it should be. Now this is not the case as can be seen from the quotients $\frac{2\theta}{H^2}$ in tables V, VII, and VIII. Table V shows first an increase, then the quotient reaches a maximum and diminishes considerably; tables VII and VIII shew a change in exactly the opposite direction; this is just what one would expect if β were variable and K constant, for the tables refer to two practically identical bodies, of which the one is dia- and the other para-magnetic. Now in either case the fundamental couple (uniform field) is in the same direction while the couple due to inhomogeneity changes sign with the susceptibility; should, therefore, the correction in the one case first increase and then decrease, it must in the other case first decrease and then increase. We shall return to this point in § 4.

Since this determination aims only at relative measurements, we have once and for all taken as the value of the susceptibility of oxygen at -183°C . the value that was given by the improved apparatus for measuring the magnetic rise. With the help of this value we have calculated the values of β for each field from equation 5): (see tables V and VI). These values fall pretty well on a curve of means. Finally the susceptibility at the lower temperatures is calculated by means of the value of β as a function of the field given by this curve. We shall take the opportunity of the corresponding series of observations to make some remarks upon the influence of the inhomogeneity for each of the three pole-gaps that were used.

2. *The inconstancy of the magnetization as a function of the azimuth.* The general expression for the couple in a uniform field

$$(N_1 - N_2) I^2 v \sin \varphi \cos \varphi$$

only reaches its maximum value just at $\varphi = 45^\circ$, and consequently $\sin \varphi \cos \varphi = \frac{1}{2}$ since I remains constant during the torsion. Here again we see a fundamental difference between the application of this method to the investigation of saturation magnetization and to that of a body of constant susceptibility. It is clear that in the first case the condition $I = \text{constant}$ is, as it were, fulfilled by definition.

In our case the deviation from this is by no means *a priori* negligible: the two limiting values of I^2 ($q = 0^\circ$ and $q = 90^\circ$) differ in our case by 0.3 ‰, and since I^2 always changes between these two limits in the same direction the error caused thereby when $\sin q \cos q = \frac{1}{2}$ is less than 0.1 ‰.

In contrast with the two foregoing sources of error, the reaction of the magnetized ellipsoid upon the *distribution of magnetism over the surface of the pole-pieces* can clearly have no effect in the case of a body of small susceptibility while on the other hand, it had to be taken into account in the case of the ferromagnetic bodies. Indeed, with oxygen we have to deal with a magnetization that in the strongest fields of the electromagnet reaches a value of only a few units (in the case of iron it was 1700!).

3. *Influence of the holder.* In this connection we may notice two actions that may go together. In the first place there is the inherent magnetism of the stem, and then there is also an action analogous to that which we wish to measure, for if the stem is not a perfect body of revolution, it is acted upon in the liquid oxygen just as if it were a *supplementary ellipsoid*. We investigated these two sources of error in a blank experiment in liquid oxygen in which the silver ellipsoid was removed, and the surface of the glass was carefully freed from all traces of wax. From this we obtained a maximum of only 1 to 2 ‰ which need not be taken into account.

4. *The concentration of the oxygen.* The oxygen in the bath contained a little nitrogen, the concentration of which constantly decreased during the experiment owing to its faster vaporization. So as to be able to allow for this we analysed the gas at the beginning and at the end of each series of observations. The mean concentration was 1.25 ‰ at the beginning and 0.35 ‰ at the end (at the moment that the DEWAR vessel was almost empty). We allowed for this concentration as far as possible: in this respect there remains an uncertainty of about 0.3 ‰.

5. *Calibration of the suspension springs.* The main torsion spring described above was calibrated outside the apparatus by observing the time of oscillation of a system suspended from it with and without the addition of a known moment of inertia. For the latter we used a bronze ring of rectangular meridian cross section, the diameters and height of which were measured with the cathetometer. Calculation gave the moment of inertia as

$$582.09 \text{ c.g.s.}$$

Care was taken that the spring was subjected to the same tension during the calibration as it experienced while in the apparatus (by attaching suitable weights to it by a torsion-less wire).

For the constant of the spring we found 1184.5 e.g.s.

The platinum-iridium stretching wire gives a torsion couple as well as the spring; the correction for this was determined by the same method as was used in the analogous case by WEISS and KAMERLINGH ONNES (loc. cit.) and it was found to be 0.0152 times the constant of the spring. The difference between the values of the constant at 18° C. and at -190° C. is smaller than the errors of observation. The calculations were therefore carried out with the constant $1184.5 (1 + 0.0152) = 1202.5$.

6. *Oscillations.* The silver ellipsoid should be protected sufficiently from the influence of oscillations arising from external causes by the occurrence of intensive FOUCAULT currents, but the occurrence of these currents, which were unusually strong gave rise to great difficulties in the observations. In the first place the holder was extremely slow to reach its position of equilibrium. Further, the smallest change in the current flowing through the electromagnet occasioned a sudden kick in the whole moveable apparatus, an immediate result of the oblique position of the ellipsoid with respect to the lines of force. Hence the regulation of the current had to be done with the greatest care. We retained the oil-damper but removed the fixed partitions, for the capillary action of these gave rise to couples that, although small, were still not negligible.

Results of the observations.

TABLE IVa.
Cylindrical pole-pieces 21 mm. apart.
 $t = -183^{\circ}.0$ C.

H gauss	double deflection 2δ cm. of the scale	2δ $H^2 \cdot 10^7$	$K \cdot 10^6$
2250	0.37	0.731	277
4537	1.41	0.685	268.8
6676	3.21	0.7200	275.3
8339	5.10	0.7335	278.1
9387	6.44	0.7307	277.5
10120	7.45	0.7274	277.0
10685	8.26	0.7234	276.4
11130	8.99	0.7258	271.6
11440	9.38	0.7167	274.8
11765	9.90	0.7155	274.6

Mean $K_{90^{\circ}.1K.} = 275.6$

TABLE IVb.
Cylindrical pole-pieces 21 mm. apart.
 $t = -201^{\circ}.1$ C.

H gauss	double deflection 2δ cm. of the scale	$\frac{2\delta}{H^2} \cdot 10^7$	$K \cdot 10^6$
2250	0.50	0.988	324
4537	2.10	1.020	328.0
6676	4.66	1.041	331.1
8339	7.36	1.058	334.0
9387	9.17	1.042	331.4
10120	10.63	1.038	330.8
10685	11.81	1.034	330.2
11130	12.70	1.025	328.6
11440	13.41	1.024	328.4
11765	14.14	1.022	328.0

Mean $K_{72^{\circ}.0K.} = 330.0$.

With it being corrected for lack of uniformity in the field the means give the following values:

$$K_{90^{\circ}.1K.} = \frac{275.6}{1.143} \cdot 10^{-6} = 241.1 \cdot 10^{-6}$$

$$K_{72^{\circ}.0K.} = \frac{330.0}{1.230} \cdot 10^{-6} = 268.3 \cdot 10^{-6}.$$

The corresponding results obtained by the method of the magnetic rise were

$$240.6 \cdot 10^{-6} \text{ and } 269.3 \cdot 10^{-6}.$$

The differences between the results as obtained by the two methods are scarcely 0.4%. This gives us great confidence in the ellipsoid method even for this particularly difficult determination, and it shews that the method is also suitable for absolute measurements if only the necessary care is taken to ensure the uniformity of the field and the correctness of the shape of the ellipsoid.

We must remember that there was a great number of absolute

measurements whose results had to be used (axes and volume of the ellipsoid, constants of the springs, magnetic field, density of the liquid oxygen) and also that the shape of the ellipsoid was not perfect. On the other hand we must remark that the application of the correction for the non-uniformity of the field might conceivably have diminished the correspondence between the results obtained by the two methods. We have, however, both theoretical and experimental grounds for the assumption that this correction remains within the limits of accuracy of not more than 0.5% in the case of cylindrical pole-pieces with flat end surfaces 90 mm. in diameter and at a distance of 21 mm. apart.

T A B L E Va.			
Conical pole-pieces 20 mm. apart.			
$t = -183^{\circ}.0 \text{ C.}$			
(To determine β we assumed $\%_{90^{\circ}.1} = 240.6 \cdot 10^{-6}$).			
H gauss.	Double de- flection 2δ cm. of the scale.	$\frac{2\delta}{H^2} \cdot 10^7$	$10^6 \beta$
3685	1.27	0.935	58.5
4615	1.96	0.920	54.7
6944	4.55	0.9437	60.8
9205	7.96	0.9400	59.7
11280	11.90	0.9348	58.5
12835	15.44	0.9374	59.1
14015	18.26	0.9295	57.0
14900	20.19	0.9098	51.9
15585	21.73	0.8945	48.1
16120	22.87	0.8802	44.5

A graph of β as function of H was made, which was used for the following table.

T A B L E Vb.
Conical pole-pieces 20 mm. apart.
 $t = -208^{\circ} \text{C.}$

H gauss.	Double deflection $\frac{2\delta}{\text{cm.}}$ of the scale.	$\frac{2\delta}{H^2} \cdot 10^7$	$\beta \cdot 10^6$	$K \cdot 10^6$
2296	0.79	1.498	56.0	[357]
4615	2.75	1.291	57.8	[328]
6944	6.82	1.414	58.5	344.2
9265	12.15	1.435	59.8	346.3
11280	18.25	1.434	59.7	346.2
12835	23.67	1.437	59.2	346.9
14015	27.89	1.420	56.5	346.1
14900	30.84	1.389	52.4	345.4
15585	33.24	1.368	48.0	345.0
16120	35.44	1.363	43.8	347.3

The mean with the exception of the two values placed between brackets is 345.9 and it gives

$$\chi_{64^{\circ} \text{K.}} = 275,0 \cdot 10^{-6}$$

while the method of the magnetic rise gave

$$\chi_{64^{\circ} \text{K.}} = 283,5 \cdot 10^{-6}$$

The difference is 3%; but in this connection we must remember that the correction for non-uniformity is about 16%, and that the temperature of the liquid becomes very uncertain at the pressure of 11 mm. under which the liquid boils at this temperature.

Finally, we now give two series of measurements which were made with other pole-gaps so as to obtain other deviations in the uniformity of the field. They were hastily made and under unfavourable circumstances, since oscillations and disturbances caused by the running of machines in the neighbourhood interfered with the observations. We give them more as examples of how the method of calculation followed still leads to good results even when the couples due to non-uniformity of the field are extremely large (28% of the chief couple).

T A B L E VIa.
Conical pole-pieces 18.2 mm. apart.
 $t = -183^{\circ}.0$ C.
(To determine β we assumed $\chi_{90^{\circ}.1} = 240.6 \cdot 10^{-6}$).

H gauss	double de- flection 2δ cm. of the scale	$\frac{2\delta}{H^2} \cdot 10^7$	$\beta \cdot 10^6$
5013	3.34	1.328	159.2
7547	7.31	1.283	147.4
9993	12.64	1.246	137.9
12165	18.33	1.238	136.2
13760	22.39	1.183	121.7
14900	26.26	1.182	121.5
15750	28.83	1.162	116.6
17005	35.58	1.230	133.9

β was again graphed as a function of H , which led to the correction for K in the following table.

T A B L E VIb. Conical pole-pieces 18.2 mm. apart $t = -208^{\circ}.2$ C.				
H gauss	double de- flection 2δ cm. of the scale	$\frac{2\delta}{H^2} \cdot 10^7$	$\beta \cdot 10^6$	$K \cdot 10^6$
5013	46.8	1.861	152	341
7547	40.71	1.880	146.5	348
9993	48.88	1.862	137.5	351
12165	27.92	1.885	130.7	357
13760	35.35	1.868	125.5	359
14900	41.36	1.861	123.0	360
15750	45.67	1.840	122.5	357
17005	51.81	1.791	127.5	347

The mean 353 gives $\chi = \frac{K}{1.255} = 279.5.10^{-6}$ a value that is not much smaller than $283.5.10^{-6}$, which was obtained by the method of the magnetic rise.

Solid oxygen.

§ 4. *Ellipsoid of solid oxygen.* In this case observations had to be made directly upon an ellipsoid of oxygen. The oxygen therefore had to be frozen in a mould of approximately the same form and dimensions as the solid silver ellipsoid described above. This new condition necessitated the following experimental arrangement.

The cover and the DEWAR tube are the same as for liquid oxygen, with the exception of the cap *D*. The adjusting tube is also the same, but it is so arranged that it can be moved as a whole up or down, while the whole apparatus remains closed and in its place. With this end in view it is attached to the tube *m*, which moves through the stuffing-box *D''*₂; this corresponds to *D*₂ of the liquid oxygen apparatus, but in this case the wide glass tube *C*₁ is lengthened by a rigid brass tube *M* that serves to give sufficient play to the vertical movement of the whole adjusting tube. The former stem *k* had to be lengthened by the same amount (*L''*₁ *L''*₂), and is contained in the tube *m*.

The holder is also a glass tube *b''*; it is not however closed, but at *b''*₃ it changes into a very much narrower tube (0.5 mm.) that ends at *b''*₁ in a glass ellipsoid *a''*. To this ellipsoid there is fused a solid stem *b''*₅ that connects it with the stretching wire. The oxygen gets to the ellipsoid through the holding tube which it enters at *b''*₆. A rubber tube *n* (*d* = 3 mm.) admits the gas from outside; it is attached to the inlet tube *n*₄ that passes through the cover and is soldered to it. With this arrangement it is easy to cause the oxygen to solidify inside the ellipsoid. When the apparatus is ready for use the adjusting tube is pulled upwards by the cap Δ till the glass ellipsoid reaches the unsilvered part of the vacuum glass. The vacuum glass is then filled with liquid hydrogen. While the ellipsoid is still connected with a reservoir of oxygen, the adjusting tube with the ellipsoid is slowly pushed downwards until it does not quite touch the liquid hydrogen but is in its vapour. The oxygen is then seen to condense slowly, and, if the operation is carefully performed, the whole ellipsoid and supply tube are seen to fill with liquid oxygen. The tube being lowered still further, vapour is reached that is sufficiently cold to cause the oxygen to solidify. On account of the large

contraction of the oxygen on solidification it is seldom that one does not see some empty space in the ellipsoid; the operation must then be repeated several times, since the oxygen that is still liquid at this temperature has a pretty great viscosity and flows with difficulty from the tube; we shall return to this point later. When the ellipsoid is completely filled with solid oxygen the adjusting tube may be lowered right down. A mark is made beforehand, so that the ellipsoid may be accurately adjusted to the centre of the gap when the silvered tube is again in its place.

Errors, corrections, auxiliary measurements.

1. *Couples due to inhomogeneity.* As will presently appear, we made measurements not only in liquid hydrogen (solid oxygen), but also keeping everything else the same, at two temperatures in a bath of liquid oxygen (i.e. with the same ellipsoid of liquid oxygen). Since the susceptibility of the liquid oxygen was known, we had therefore two measurements of the couples due to inhomogeneity as a function of the field; they are given in Table VII. As a result of the somewhat smaller dimensions of the ellipsoid, these corrections are comparatively much less important.

2. *Purity of the oxygen.* The oxygen was freed from nitrogen by vaporizing a large quantity of impure liquid oxygen under reduced pressure.

3. *Density of the solid oxygen.* We have already mentioned the difficulty of completely filling the ellipsoid with solid oxygen. On account of the opaqueness of the oxygen that has already solidified one cannot with certainty assert that this condition has been fulfilled¹⁾.

Since the specific susceptibility is determined from a known volume this error would have immediate effect upon the result. We tried to eliminate this error as well as possible by determining the density with the same ellipsoid by filling it with solid oxygen under the same circumstances as those obtaining in the experiments and then measuring the quantity of gas formed from it on vaporization. We may assume that the small cavities that may form are pretty much the same in the various cases. Indeed, from two similar measurements the density measured in this way was found to vary by only about 1%. By taking as the mean density that determined by these experiments, the eventual presence of cavities is allowed for. In this way we obtained

$$\rho = 1.41.$$

The *absolute* values of the couples due to inhomogeneity of the

¹⁾ When there is an empty space of a few mm³, however, it can be seen quite well.

field are not modified by a cavity formed in the vertical axis, as was usually the case, for it is clearly those portions towards the surface of the ellipsoid that are the chief contributors to them. On the other hand, they might obtain a greater *relative* influence, but as the observations shew, the sum of the corrections arising from this cause is so small that they may be regarded as independent of the susceptibility within the limits of accuracy of the experiments. In that case this difficulty completely disappears.

4. *Dimensions of the ellipsoid.* The internal volume was obtained by filling the ellipsoid with mercury and weighing it. It was 0.1812 c.c. The change of volume under atmospheric pressure was found to be of no account by pumping the space above the mercury free from air and observing the position of the mercury in the capillary.

The external axes were measured directly. Then the thickness of the glass at ten different points was determined by focussing a microscope on the image of the outer surface formed on the mercury with which the ellipsoid was filled. It changed but slightly from place to place. The mean was taken and twice that value was subtracted from the external measurements. The results were:

1.044 cm.

and

0.335 cm.

Calculating the volume from these figures we get 0.1925 c.c. which is about 6% greater than the true volume as directly determined. This is accounted for by the special shape of the meridian section which curves somewhat too strongly at the outer ends. For calculating the coefficients of demagnetization we took a mean ellipsoid with the same major axis and the minor axis small enough to give the real volume¹⁾. The data for the calculation were therefore:

1.044 cm.

and

0.3173 cm.

5. *Opposing couple.* The suspension spring and the stretching wire were the same as were used for the liquid oxygen. We must, however, allow for the rubber supply tube for the oxygen. This (which was chosen as thin as possible) modified both the zero and the constants of the total opposing couple, as soon as the pressure

¹⁾ It is clearly not quite right to do this; there are, however, experimental data to support this method of correcting: V. QUITTNER (Dis. Zürich 1908, also Arch. sc. phys. et nat. Genève, Sept.—Nov. 1908) found that this method of treatment was sufficiently accurate even for discs, bodies that deviate far more from an ellipsoid than those we used.

difference between the inside and the outside of the tube appreciably altered (on account of the change in shape of the tube). In all our experiments, therefore, we took care that there was a constant pressure difference of 70 mm. between the pressure inside the cover and that inside the holder (the latter was the smaller of the two). We got a very sensitive indication of the constancy of this difference not only from the manometers but also from the zero position of the holder. Experiments carried out outside the apparatus shewed that the constant of the total couple changed about 10% between the complete flattening of the rubber tube by the atmospheric pressure and equality between the pressures on both sides. This corresponds to a deflection on the reading scale of *more than a metre*. If we assume rough proportionality we find that a displacement of 1 cm. would indicate a change in the opposing couple of only 0.1%. The zero was kept constant to a few millimetres.

The calibration was made under circumstances exactly the same as in the experiments (pressure difference, etc.).

The total constant with the addition of that of the stretching wire was

$$1503 + 18 = 1521 \text{ cgs.}$$

Results.

TABLE VII.
Calculation of the corrections for non-uniformity from observations made in a bath of liquid nitrogen.
Conical pole-pieces 20 mm. apart.

H gauss	t = -195°.6 C.			t = -210°.0 C.			$\beta \cdot 10^4$ (mean). Double weight given 195°.6
	2r cm.	$\frac{2z}{H^2} 10^7$	$\beta \cdot 10^4$	2r cm.	$\frac{2z}{H^2} 10^7$	$\beta \cdot 10^4$	
4615	1.18	0.554	-0.137	1.50	0.746	-0.199	-0.158
6944	2.69	5577	427	3.53	0.7322	232	162
9205	4.73	5580	126	6.21	7330	231	160
11280	7.08	5560	132	9.23	7251	250	171
12835	9.17	5564	131	12.10	7341	228	163
14015	11.14	5670	100	14.49	7378	219	140
14000	12.90	5812	060	16.94	7613	162	094
15585	14.29	5884	039	—	—	—	—
16120	15.67	6031	003	20.21	7781	121	040

It can be seen that the values obtained for β are not the same at the two temperatures. Meanwhile it has to be applied here only as a correction for the susceptibility of solid oxygen which at the most is 3%. A difference of temperature of 1° C. in the bath under reduced pressure gives more than half the difference between the two values, whence we have given the determination under reduced pressure only half the weight accorded the measurement at ordinary pressure.

The uncertainty of the mean has less than 1% influence upon the value of the susceptibility of solid oxygen. The curve for β as well as its sign correspond with what were found for the silver ellipsoid.

T A B L E VIII.
Susceptibility of solid oxygen.
 $t = -252^{\circ}.8$
(bath of liquid hydrogen boiling under atmospheric pressure).

H gauss	2σ cm. of the scale	$\frac{2\sigma}{H^2} \cdot 10^7$	$K \cdot 10^6$ uncorrected	$K \cdot 10^6$ corrected according to tab. VII.
2296	0.89	1.69	519	533
4615	3.57	1.676	518.3	532.9
6944	7.92	1.642	512.3	527.3
9205	14.07	1.660	515.0	530.1
11280	21.14	1.661	515.2	530.4
12835	27.92	1.684	518.7	532.8
14015	32.96	1.678	517.6	529.0
14900	37.38	1.683	518.4	527.5
15585	40.77	1.678	517.6	523.7
16120	44.05	1.696	520.0	523.6

Mean 529.0

whence it follows that $\chi_{20^{\circ}.3\text{K.}} = \frac{529.0}{1.41} \cdot 10^{-6} = 375.2 \cdot 10^{-6}$.

T A B L E IX.
Susceptibility of solid oxygen.
t = -258.9°
 (bath of liquid hydrogen under 70 mm. vapour pressure).

<i>H</i> gauss	2 <i>c</i> cm. of the scale	2 <i>c</i> <i>H</i> ² · 10 ⁷	<i>K</i> · 10 ⁶ uncorrected	<i>K</i> · 10 ⁶ corrected
2296	4.49	2.257	600.5	614.5
4615	4.80	2.253	600.1	614.6
6944	10.86	2.252	600.0	615.0
9205	19.24	2.270	602.2	617.1
11280	28.13	2.210	594.3	609.2
12835	37.09	2.250	599.6	613.7
14015	45.05	2.293	604.7	616.7
14900	51.24	2.306	606.8	615.9
15585	55.73	2.293	605.3	614.4
16120	60.20	2.317	608.5	612.1

From the mean 614.0 follows $\chi_{sol. 14^{\circ}.2K.} = \frac{614.0}{1.41} = 435.6 \cdot 10^{-6}$. The products into \sqrt{T} are

$$\begin{aligned} - 252.^{\circ}8 & \quad 375.2 \cdot 10^{-6} \sqrt{20.3} = 1690 \cdot 10^{-6} \\ - 258.^{\circ}9 & \quad 435.6 \cdot 10^{-6} \sqrt{14.2} = 1641 \cdot 10^{-6}. \end{aligned}$$

Hence we can represent the two observations pretty well by

$$\chi_{sol.} = \frac{1690}{\sqrt{T}} \cdot 10^{-6},$$

which is adjusted to the measurement at the higher temperature.

The deviation from this ratio for the lower temperature, however, is somewhat greater than the errors of observation.

§ 5. *Summary and conclusion.* As regards the dependence of specific susceptibility upon temperature our most reliable determination gives

$$\chi_{liq. 90^{\circ}.1 K.} = 240.6 \cdot 10^{-6}.$$

CURIE found $\chi = \frac{33700}{T} \cdot 10^{-6}$ between 20° C. and 450° C. whence it

would follow that for $T = 90^{\circ}.1$ K. $\chi = 374.10^{-6}$ a number that differs essentially from ours¹⁾.

There is therefore no possibility of extrapolating CURIE's law to the liquid phase of oxygen. This was also the conclusion reached by FLEMING and DEWAR in their first treatment of the question, but after more careful experiments they rejected their former result²⁾.

The results obtained from the two magnetic rise apparatus at lower temperatures can, within the limits of experimental error, be expressed by a very simple law: *the specific susceptibility is inversely proportional to the square root of the absolute temperature*. From the observations obtained with the more reliable apparatus we deduce the formula

$$\chi_{liq.T} = \frac{2284}{\sqrt{T}} \cdot 10^{-6}$$

which holds to within 5%₀. None of the results obtained by the method of the maximum couple are in conflict with those deduced from the formula.

The results with solid oxygen approximately follow the relation

$$\chi_{sol.T} = \frac{1690}{\sqrt{T}} \cdot 10^{-6}.$$

At the lowest temperatures there is a small deviation indicating a smaller increase at lower temperatures; it is, however, so small that we may still accept the formula given as approximately correct for the solid state of aggregation below the melting point of oxygen and down to $14^{\circ}.2$ K.

Further experiments at more numerous temperatures must show exactly how far these deductions hold for the liquid and solid states. They shew (see fig. 5) that there is a jump in the value of χ at the melting point, since

$$\chi_{liq.T_m} = 1,3 \chi_{sol.T_m}$$

¹⁾ R. HENNIG's (1893) result should give

$$\chi = \frac{27600}{T} 10^{-6} \text{ and } \chi_{90^{\circ}.1 \text{ K.}} = 307 \cdot 10^{-6}.$$

²⁾ FLEMING and DEWAR's results: 1st paper (1896) $\chi_{90^{\circ}.1 \text{ K.}} = 200 \cdot 10^{-6}$; 2nd paper (1898) $287 \cdot 10^{-6}$, mean $243.5 \cdot 10^{-6}$ pretty much the same as our result. According to the mean of the result of FARADAY and BECQUEREL the specific susceptibility for oxygen at 0° is $91 \cdot 10^{-6}$; this gives by extrapolation from CURIE's law $\chi_{90^{\circ}.1 \text{ K.}} = 299 \cdot 10^{-6}$. The English savants used this number in their second research for the comparison of the susceptibility of liquid oxygen with that of the gas.

We hope to answer the question if this jump really exists by special experiments arranged for the purpose; we may, in the meantime, consider that it does probably exist. What CURIE found in the transformation of γ iron to δ iron is analogous to the sudden change which we here assume to exist while the form of the law remains unaltered, and which can occur at the melting point or at a point of transformation to an allotropic modification. WEISS¹⁾ has shown that this can be accounted for on the assumption that at this particular point di-atomic iron changes into tri-atomic.

On the other hand we consider it probable that the law according to which the specific susceptibility increases with the temperature, viz: inverse proportionality to the square root of the absolute temperature at lower temperatures, gradually transforms into that of inverse proportionality (CURIE'S LAW) at higher temperatures, and that each of these laws, therefore, may be but approximative to the same function over different ranges of values of the independent variable T .

The supposition that the change of specific susceptibility with density is of no importance lies at the bottom of the assumption of the gradual transformation of CURIE'S LAW into that of T^{-1} . If, on the other hand, we assume that this change is of importance, that e. g. when the internal pressure is considerable the molecules under its influence undergo not only a compression but also a lessening of their magnetic moments, then a region of great molecular compressibility in which the specific susceptibility should change both with the temperature and with the density should exist between the gaseous phase in which the specific susceptibility would be pretty well independent of the pressure, and the liquid phase at lower temperatures, in which the molecules would not be appreciably affected by an additional external pressure on account of their already great internal pressure, and in which, therefore, the specific susceptibility would also be pretty well independent of the pressure. As regards the difference between the magnetic moment of the elementary magnets in the condition of saturated liquid and vapour and that at normal or smaller density at the same temperatures, it is to be expected according to that representation, that this difference will change with temperature in consequence of the change of density with temperature.

The assumption can also be made that complex molecules are formed in the liquid state, and that these diminish the intensity of the elementary magnets; in that case changes in susceptibility of

¹⁾ P. WEISS, loc. cit.

mixtures of liquid oxygen with non-magnetic gases should obey the thermodynamic laws that govern the number of such complexes. But all this must be established by further experiments which we hope to complete; in the meantime the most probable assumption is the old one that the specific susceptibility is independent of the pressure.

As regards the question as to whether the specific susceptibility at lower temperatures still follows the law of inverse proportionality to the root of the absolute temperature, if the ferro-magnetism with a very low-lying CURIE point according to WEISS's theory of corresponding magnetic states does not exist, then the change to a still slower increase with decreasing temperature and the approximation to a limiting value is, perhaps, more probable.

The law of $T^{-\frac{1}{2}}$ at once gives rise to the question if instead of the LANGEVIN elementary magnets whose intensity is independent of the temperature, we should assume that their intensity varies directly as \sqrt{T} ; that is, that we should assume the existence of elementary currents or electrons moving in their paths with speeds proportional to (and, therefore, determined by) the speeds of molecular heat motions. In other words, while LANGEVIN's theory already supposes that the planes in which the electrons move follow the motions of the molecules, but that the areas described in those planes are still independent of heat motion, we should now assume that the electrons undergo the influence of heat motion at their motion *in* their paths, and, if the radius of their path has also become invariable, revolve while remaining in the same position with respect to the atom; they would be electrons that are frozen fast to the atom, an assumption that has already been made to explain other phenomena.

This addition to LANGEVIN's theory, however, does not lead to a specific susceptibility proportional to $T^{-\frac{1}{2}}$ as one at first sight would be inclined to think, but to a constant specific susceptibility.

To substantiate that addition it will probably be necessary to proceed to still lower temperatures than those of our experiments. It seems at present that it is not impossible that then the law χ proportional to $T^{-\frac{1}{2}}$ changes to $\chi = \text{const.}$: our observations on solid oxygen seem to indicate a change in this direction. The assumption to which this is equivalent: viz, that the magnetic motions of the electrons cease at the absolute zero, and to which our experiments seem to lead, is much more satisfactory than that the magnetic motions of the electrons still persevere even at the absolute zero.

The second question to which we devoted attention — the dependence of susceptibility upon field strength requires no detailed treat-

ment. The method of the magnetic rise seemed in some instances to give a decrease of the order of 1% in a field of 8000 gauss, while the method of the maximum couple gave with the cylindrical pole-pieces up to 12000 gauss only a very small systematic deviation and with the conical pole-pieces (16000 gauss) the deviation was scarcely appreciable.

The solid oxygen ellipsoid with which a much lower temperature was reached seemed to give a small decrease at 16000 gauss; it is possible, however, that a greater deviation is obscured by the correction for the non-uniformity of the field. We consider, however, that, assuming that the experiments were accurate to within 1% the change of the susceptibility with the field up to 16000 gauss remains within the limits of experimental error. This is in agreement with the theory of LANGEVIN, if this, notwithstanding the deviation from CURIE'S law, is still applied.

Physics. — "*The magneto-optic KERR-Effect in ferromagnetic compounds and alloys*". By STANISLAW LORIA. (Communication from the Bosscha-Laboratory).

It has been shewn by KAZ¹⁾, RIGHI²⁾, KUNDT³⁾, SISSINGH⁴⁾, ZEEMAN⁵⁾ and also by KERR⁶⁾ himself that the phenomenon discovered by the last named in 1876 depends not only on the orientation of the reflecting surface with respect to the magnetic vectors, but also (in a somewhat complicated manner) on the angle of incidence and the position of the plane of polarization of the incident beam. In the simplest and by far the most important case of almost normal incidence of light polarized perpendicularly or parallel to the plane of incidence, the reflected light in general is elliptically polarized according to RIGHI⁷⁾; the rotation of the major axis of the ellipse depends on the magnetisation and the wave-length.

According to the measurements made by DU BOIS⁸⁾ it is in every case proportional to the former; as regards the variation with the

¹⁾ P. C. KAZ, Diss., Amsterdam 1884.

²⁾ A. RIGHI, Ann. de Chim. et Phys. (6) 4 p. 433, 1885.

³⁾ A. KUNDT, Wied. Ann. 23 p. 228, 1884; 27 p. 199, 1886.

⁴⁾ R. SISSINGH, Arch. Néerl. (1) 27 p. 173, 1894.

⁵⁾ P. ZEEMAN, Leiden Comm. no. 15, 1895; no. 29, 1896. Arch. Néerl. 27 p. 252 1894.

⁶⁾ J. KERR, Phil. Mag. (5) 3 p. 339, 1877. Phil. Mag. (5) 5 p. 161, 1878.

⁷⁾ A. RIGHI, Ann. de Chim. et Phys. (1) 9 p. 120, 1886.

⁸⁾ H. DU BOIS, Wied. Ann. 39 p. 25, 1890.

latter, the rotatory dispersion, according to the same author, shews certain regularities. For iron, cobalt, and nickel the rotations visually observed were always *negative*; for iron the dispersion-curve seems to indicate a numerical minimum in the ultraviolet and thence ascends from violet towards red; in the case of cobalt the minimum occurs between blue and green, and for nickel in the yellow. These numerical minima of negative rotation may be considered *algebraic* maxima, their wave-length increasing as the metal's position in the periodic system advances. For magnetite the observed rotations were in every case *positive*, though the curve appeared directed towards negative values beyond the blue; a distinct maximum occurred in the yellow, corresponding to the above algebraic maxima.

More recently INGERSOLL¹⁾ has contributed important papers relative to this subject; he was able to supplement DU BOIS' curves in the infra-red up to about 3μ . According to this author the complete rotatory dispersion-curves thus obtained shew a marked resemblance to a typical dispersion-curve in the region of an exceedingly broad band of resonance-absorption. The particular cases of nickel and magnetite are notable, for the rotation appears to vanish between 1 and 1.5μ and then to change in sign.

Further progress in this subject was difficult in view of the fact, that as yet the only ferromagnetic substances suitable for a study of the KERR-Effect were the four above-mentioned bodies. Several attempts to study with reflected and transmitted light the magneto-optic phenomena connected with the KERR-Effect were made with partially transparent films of metals prepared electrolytically, after the manner of KUNDT, or by cathodic discharge. Although the latest investigations²⁾ on the optical properties of these films of magnetic metals brought to light further interesting but confusing results, yet the conditions in the films can obviously depend on their structure and on their mode of preparation in a very complex way. As a reflecting surface such a film is certainly inferior from a physical point of view to a mirror polished on a compact and massive block of metal.

An attempt to add to the number of substances which exhibit the KERR-Effect was thus of some interest. I entertained some hopes in this respect, since several chemists of late have synthetically prepared new ferromagnetic substances. In the first place, a number of combinations of different oxides with iron oxide, so called metaferrites,

¹⁾ L. R. INGERSOLL, Phil. Mag. (6) **11** p. 41, 1906 & **18** p. 74, 1909.

²⁾ C. A. SKINNER & A. Q. TOOL, Phil. Mag. (6) **16** p. 833, 1908. H. BEHRENS Inaug. Diss. Münster i. W. 1908. L. R. INGERSOLL, loc. cit.

prepared by HILPERT¹⁾), presented an interesting field of research. In all these cases, the chemical structure resembles that of ferroferrite (ferroso-ferric oxide), in that the iron sesquioxide plays the acidic part, thus imparting ferromagnetic properties to the compound. Of this class of substances however, only cupriferrite and calciumferrite could be obtained in a state suitable for my experiments. Secondly, certain alloys of more or less ferromagnetic metals, and in particular those of nickel-iron²⁾, together with the well-known ternary HEUSLER alloy, and WEDEKIND'S³⁾ binary manganese-antimony alloy present considerable interest. So far as I am aware, the magneto-optic properties of these alloys have been only partially investigated, the only account of similar experiments, which I have come across, being INGERSOLL'S communication previously referred to and a Russian paper by TOKMATSCHEW⁴⁾, who described experiments with HEUSLER'S alloy.

I have studied the magneto-optic properties of the above mentioned bodies and also those of the well-known magnetic chromic oxide. Below an account of the preliminary results of my research is given.

Experimental Arrangement. Solar rays were exclusively used; they passed through a direct-vision monochromatic illuminator⁵⁾, with divergence 1:4, thus furnishing light of great intensity. The rays passed (Fig. 1) through a lens (L), a total reflecting prism (P), a LIPPICH'S arrangement of two halfshade Nicols (N₁, N₂) and falling nearly normal on a mirror between the two poles of an electromagnet, were reflected, finally passing through an analyser (N₃) and a telescope of fourfold magnifying power. The dimensions of the lenses, of the diaphragms, of the width and angle of the conical bores in the cores and poles were all calculated beforehand, particular care being taken to maintain maximum brightness, a uniform field of view, and also the avoidance of all unnecessary reflections⁶⁾.

The observations were carried out with nearly normal incidence. RIGHI⁷⁾ found, that up to an angle of incidence of 15° there was

1) S. HILPERT, Ber. deutsch. Chem. Ges. **42** p. 2248, 1909. Verh. deutsch. Phys. Ges. **11** p. 293, 1909.

2) CH. ED. GUILLAUME, Les aciers au nickel, Paris 1898.

3) E. WEDEKIND, Ztschr. f. phys. Chem. **66** p. 614, 1909. K. HONDA, Ann. d. Phys. **32**, 1910.

4) S. TOKMATSCHEW, Journ. d. russ. phys.-chem. Ges., **42** (phys. T.) p. 15, 1910.

5) H. DU BOIS, Verh. d. D. Phys. Ges. **11** p. 708, 1909.

6) A description of the analyser and polariser mentioned is given by H. DU BOIS, Wied. Ann. **46**, p. 545, 1892.

7) A. RIGHI, Ann. de chim. et de phys. (1) **9** pp. 120, 132, 1886.

scarcely any variation of the effect. However, in my experiments the angle between the incident and reflected beams was only 2° or 3° . The incident light was polarized horizontally in the plane of incidence. From RIGHI's observations it is known that even a normal incident beam of linearly polarized light when reflected from a magnetized mirror becomes elliptically polarized, the ellipticity however being only slight; ZEEMAN¹⁾ later measured this ellipticity in the case of iron and cobalt. Up to the present the evaluation of the ellipticity in my experiments has not been attempted; I considered that the slight reflecting power of some of my mirrors would not warrant such an attempt, and in addition it must be borne in mind, that the rotations themselves are small. Moreover the ellipticity, if any, must be nearly inappreciable, for by employing the best of my mirrors and by carefully avoiding diffused light, I have never been confronted with any difficulties, while the extinction of light in each half of the field of view was satisfactory. Even when the rotations are very small it is possible by means of the half-shade arrangement to observe and to measure them with sufficient accuracy. Hence it was thought unnecessary to use the method of multiple reflections, thereby avoiding new complications and further sources of error. The azimuth of the analyser was determined by means of a vertical scale seen through a combination of mirrors.

For the production of the magnetic field a small DU BOIS semicircular electromagnet of resistance 9Ω was employed. To avoid the danger of sparking with reversal of current about 60Ω were shunted across its terminals. The field was determined by means of a standardised thin glass-plate silvered at the back, which could be placed immediately in front of the mirror. The light ($\lambda = 589 \mu\mu$), being reflected by the mirror as described above, suffers a double magnetic rotation in the glass. The ensuing very slight double rotation of the light in its passage to and fro through the magnetized air could be computed from the data of SIERTSEMA²⁾, but proved quite negligible. Indeed, by using a silver-mirror, it was found that the rotation lies within the limits of experimental error. All the measurements were made with "polar" magnetization and at ordinary temperature.

Test-Specimens. The following substances were experimented upon: *Cupriferrite* ($\text{Cu O} \cdot \text{Fe}_2 \text{O}_3$), *Calciumferrite* ($\text{Ca O} \cdot \text{Fe}_2 \text{O}_3$), *Magnetite* (Ferroferrite) ($\text{Fe O} \cdot \text{Fe}_2 \text{O}_3$), *Ilmenite* ($\text{Ti}_2 \text{O}_3 \cdot \text{Fe}_2 \text{O}_3$), ferro-magnetic *chromic oxide* ($\text{Cr}_2 \text{O}_3$), "*Invar*" (36 Ni, 64 Fe), the HEUSLER

¹⁾ P. ZEEMAN, *Lciden-Comm.* No 15, 1895.

²⁾ L. H. SIERTSEMA, *Versl. Kon. Akad. Wet. Amsterdam* 7 p. 289, 1899.

alloy (26 Mn, 13 Al, 61 Cu). The first two were kindly prepared by Dr. HILPERT in the metallurgical Laboratory of the "Technische Hochschule" in Charlottenburg; the natural magnetite is from the collection of the Bosscha-Laboratory, and is the same specimen, possessing a polished octahedral surface, which was formerly examined by DE BOIS¹⁾. A very fine-formed crystal of ilmenite was kindly lent by Prof. LIEBISCH. The HEUSLER alloy was supplied by the DE HAËN chemical factory in Seelze; its interior was full of bubbles, but its surface was capable of polish and supplied a very good mirror. The "invar" contained about 36% Nickel and came from France (Société de Commentry-Fourchambault). For the chromic oxide I am indebted to Dr. KOPPEL. I desire to express my obligations to all the above mentioned gentlemen.

Throughout this paper I shall denote as usual by: \mathfrak{H} , the field intensity in kilogausses, \mathfrak{J} the magnetization, \mathfrak{J}_m its saturation value, ε single rotation of the plane of polarization in minutes, K , KERR's constant. In the tables, the column under N shows the number of readings in each series of measurements, which depended upon the polish of the mirror and the variable brightness, λ denotes the wave-length in $\mu\mu$, Δ the direct scale-reading in mm. of the double rotation produced by reversal of the current. The average values of the single rotations are given in the fourth column and in the fifth and sixth the average errors in minutes and in percentages.

The sense of the rotation is referred as usual to that of the magnetizing current; e.g. in the case of iron the "polar" KERR rotation is negative.

Results. The results obtained with the various substances were as follows:

1. *Cupriferrite*. Measurements were made on two mirrors of this material with similar results. The relation between the rotation and the wave-length in a field of 10,2 kgs. is shown numerically in Table 1 and graphically in Fig. 2. The dispersion-curve exhibits a type which has not been observed hitherto in the visible spectrum.

In the violet the rotation is *positive*, a maximum occurring in the blue; with increasing wave-length the rotation gradually decreases and in the neighbourhood of 587 $\mu\mu$ goes through zero, becoming *negative* for longer wave-lengths. Between 640 and 670 $\mu\mu$ a rather flat minimum is exhibited, the curve then gradually proceeding upwards. The rotations are small throughout, the maximum value not being

¹⁾ H. DE BOIS, Wied. Ann. **39**, p. 25, 1890.

T A B L E 1.

$\varepsilon = \text{funct } (\lambda)$		Cupriferrite		$\mathfrak{H} = 10.25 \text{ Kgs.}$
N	$\lambda (\mu\mu)$	$\Delta (\text{mm})$	$\varepsilon (\text{Minutes})$	$\partial\varepsilon$
19	436	+ 8.3	+ 1.31'	$\pm 0.04' = 3.5\%$
48	477	+ 11.1	+ 1.75'	$\pm 0.05' = 3$ "
43	539	+ 8.1	+ 1.28'	$\pm 0.04' = 3$ "
40	574	+ 2.6	+ 0.41'	$\pm 0.02' = 6$ "
52	599	- 2.3	- 0.36'	$\pm 0.03' = 8$ "
45	637	- 6.0	- 0.95'	$\pm 0.01' = 1$ "
51	688	- 4.9	- 0.78'	$\pm 0.03' = 4$ "

greater than +1,75', but they still admitted of exact measurement. The above-mentioned change of sign is analogous to that found by INGERSOLL in the infra-red and presents a characteristic and theoretically important phenomenon.

The relation between the rotation and the field was also investigated, and the results are shewn in Table 2 and Fig. 3. For low values of the field the two are proportional to each other, the rotation

T A B L E 2.

$\varepsilon = \text{funct } (\mathfrak{H})$		Cupriferrite		$\lambda = 477 \mu\mu$
N	$\mathfrak{H} (\text{kgs})$	$\Delta (\text{mm})$	$\varepsilon (\text{Minutes})$	$\partial\varepsilon$
40	0.93	+ 5.3	+ 0.85'	$\pm 0.02' = 2\%$
22	2.25	+ 8.4	+ 1.34'	$\pm 0.04' = 3$ "
59	4.47	+ 9.7	+ 1.56'	$\pm 0.03' = 2$ "
31	7.19	+ 10.2	+ 1.63'	$\pm 0.03' = 2$ "
20	9.32	+ 10.4	+ 1.66'	$\pm 0.03' = 2$ "
48	10.15	+ 11.1	+ 1.75'	$\pm 0.05' = 3$ "

afterwards assuming a maximum value, which remained nearly constant for further increase of the field. Considering the form of the curve $\varepsilon = \text{funct } (\mathfrak{H})$ and accepting the results previously found

by DU BOIS in the case of iron, nickel, and cobalt, we may assert the proportionality between ε and γ with great probability. Bearing in mind this fact we are able to determine from purely magneto-optic measurements the order of magnitude or at least an inferior limit of maximum magnetization. As DU BOIS¹⁾ has shewn in the case of an unlimited homogeneous plane disc, the magnetization of which is uniform and normal, the abscissa of the point of intersection of the straight line $\varepsilon = K\gamma = K^2D/4\tau$ and of the asymptote $\varepsilon = \text{const.}$ has the value $4\tau \gamma_m$.

Accordingly $\gamma_m \geq 140$ e.g.s. in the case of cupriferrite. The small inclination of the upper part of the curve in Fig. 3 may be explained by the fact that for irregularly formed specimens the real conditions do not correspond to those in the ideal case mentioned above. However this inevitable difference can only produce a decrease in the apparent value of γ_m so that an inferior limiting value is really determined; small fissures, cavities, and impurities in the reflecting surface are particularly capable of exerting such an influence.

2. *Magnetite.* The dispersion of the KERR-Effect is shown in Table 3 and Fig. 4 (continuous line). If we compare this curve with the

T A B L E 3.

$\varepsilon = \text{funct } (\nu)$		Magnetite		$\delta = 11.56 \text{ Kgs}$
N	γ (°)	z (mm)	ε (Minutes)	δz
30	436	- 24.0	- 3.81'	$\pm 0.03' = 0.9\%$
25	442	- 19.9	- 3.15'	$\pm 0.05' = 1.5$ "
15	453	- 9.6	- 1.52'	$\pm 0.03' = 2$ "
30	464	0	0	-
40	477	+ 6.7	+ 1.06'	$\pm 0.03' = 2$ "
26	510	+ 19.4	+ 3.07'	$\pm 0.02' = 0.6$ "
25	539	+ 24.3	+ 3.84'	$\pm 0.02' = 0.5$ "
30	574	+ 28.2	+ 4.45'	$\pm 0.02' = 0.4$ "
30	599	+ 24.9	+ 3.94'	$\pm 0.02' = 0.5$ "
31	637	+ 21.0	+ 3.32'	$\pm 0.04' = 1$ "
30	688	+ 16.0	+ 2.50'	$\pm 0.07' = 3$ "

1) H. DU BOIS, Wied. Ann. 31 p. 965, 1887; Phil. Mag. (5) 29 p. 301, 1890.

previous one given by DU BOIS (dotted line), which he obtained with the same specimen (a holoëdric regular crystal, possessing a natural octahedral surface) we see, that with the exception of a displacement throughout the whole range of wave-lengths amounting to about 10 to 30 $\mu\mu$ — which is explained by the fact that 20 years ago only an imperfect method of spectral decomposition was available — the curves are in agreement in the region between 486 and 671 $\mu\mu$. The rotation attains a maximum value of 4.45' in the yellow and decreases rapidly with decreasing wave-length. DU BOIS¹⁾, who was unable to proceed further than the blue on account of insufficient intensity of light, observed that the rotation probably vanished in the blue; he also considered that a change of sign possibly might occur in the ultraviolet. I have located this zero-point in the visible part of the violet at 464 $\mu\mu$. For smaller wave-lengths the rotation has rather a large *negative* value, which seems to approach a minimum. Unfortunately it was impossible to carry the investigation beyond 436 $\mu\mu$ since the light at that point becomes too feeble.

At all events, the existing observations establish satisfactorily the fact that the dispersion-curve obtained with natural crystalline magnetite ($\text{FeO} \cdot \text{F}_2\text{O}_3$) is of the same type as that obtained above with cupriferrite. Without entering into theoretical considerations it may be seen at once that in both cases the curve passes through a maximum, goes through zero and probably also through a minimum. Experiments are being carried out to see whether the course of these curves depends on the optical constants of the substances investigated, viz. their ordinary absorption- and dispersion-curves.

In the same way as in the case of cupriferrite the relation between the rotation and the field was also investigated. The results are shown in Table 4 and Fig. 5. They give $\chi_m \geq 358$ C. G. S., which agrees with that obtained by DU BOIS²⁾ (350).

The magnetic properties of magnetite crystals have been recently investigated by QUITTNER³⁾, adopting WEISS' methods. From his measurements it follows that the component magnetization parallel to the field, which in this case is alone of interest, reaches a saturation value of about 475 C. G. S.; this subject and the cause of the discrepancy ought to be investigated in greater detail. One remark, however, may be made at once. In many cases the natural magnetite slightly departs from the simple structural formula ($\text{FeO}, \text{Fe}_2\text{O}_3$);

¹⁾ H. DU BOIS l. c. p. 38.

²⁾ H. DU BOIS, Phil. Mag. (5) 29, p. 301, 1890.

³⁾ P. WEISS, Journ. de Phys. (3) 5 p. 435, 1896 and (4) 9, p. 373, 1910, V. QUITTNER, Dissertation, Zürich, 1908.

T A B L E 4.

$\varepsilon = \text{funct } (\delta)$		Magnetite		$\lambda = 574 \mu\mu$
N	δ (Kgs)	Δ (mm)	ε (Minutes)	$\delta\varepsilon$
15	2.10	+ 12.9	+ 2.07'	$\pm 0.04' = 2 \text{ } \frac{0}{10}$
15	3.40	+ 21.3	+ 3.37'	$\pm 0.05' = 1.5 \text{ } "$
15	5.87	+ 28.7	+ 4.54'	$\pm 0.07' = 1.5 \text{ } "$
15	8.87	+ 28.0	+ 4.43'	$\pm 0.05' = 1 \text{ } "$
15	10.82	+ 28.9	+ 4.57'	$\pm 0.06' = 1 \text{ } "$
30	11.56	+ 28.2	+ 4.45'	$\pm 0.02' = 0.4 \text{ } "$

also QUITNER has established the great diversity of samples by measuring their variable densities. It is difficult to foretell the influence of all this on the magneto-optic properties.

3. *Other ferromagnetic compounds.* The distinct analogy in the dispersion for substances of similar chemical structure as e.g. cupri- and ferroferrite in contradistinction to iron, nickel, and cobalt suggests whether the properties of other ferromagnetic ferrites and oxides are not similar. The investigation of *calciumferrite* was in this respect of importance. This substance is very feebly magnetic and brittle. A small piece was surrounded by the easily fusible Wood alloy and then thoroughly polished. No KERR-Effect however was observed although the mirror was sufficiently good. The effect, if it exists, must be smaller than 0,35'. A similar result was obtained with *ilmenite*¹⁾. The light was reflected from the base of the crystal as well as from a plane parallel to the principal axis, but in no case could a rotation be detected. ($< 0,3'$). It was also impossible to detect any rotation with *chromic oxide* Cr_2O_3 , which without doubt is ferromagnetic. The following alloys were tested:

4. *Nickel-iron* with 36% nickel, so called "Invar", known to possess a very small coefficient of expansion, is strongly magnetic and distinctly shows the KERR-Effect. The rotation is exclusively *negative* in the region of the spectrum investigated, and there is only a slight variation with wave-length. (Table 5, Fig. 6). The dispersion-curve lies considerably below the zero-line; with increasing wave-length

¹⁾ See B. BAVINK, Magn. Influenz in Krystallen, Göttinger Dissertation 1904.

T A B L E 5.

$\epsilon = \text{funct}(\rho)$		"Invar"		$\delta = 13.30 \text{ Kgs.}$
N	$\lambda(\rho\rho)$	$\Delta(\text{mm})$	$\epsilon(\text{Minutes})$	δz
15	436	- 74.4	- 11.78'	$\pm 0.05' = 0.4\%$
15	477	- 78.8	- 12.48'	$\pm 0.06' = 0.5$ "
15	539	- 83.5	- 13.22'	$\pm 0.06' = 0.4$ "
20	574	- 86.3	- 13.66'	$\pm 0.03' = 0.2$ "
15	599	- 86.8	- 13.74'	$\pm 0.05' = 0.3$ "
15	637	- 86.7	- 13.72'	$\pm 0.07' = 0.5$ "
15	688	- 86.2	- 13.54'	$\pm 0.06' = 0.4$ "

it proceeds slowly downwards, passes through a flat numerical maximum in the orange, after which the rotation decreases very slowly. The relation between rotation and magnetization, as in the cases above, exhibits distinct proportionality and we have $\lambda_m \geq 530$ (Table 6, Fig. 7).

T A B L E 6.

$\epsilon = \text{funct}(\delta)$		"Invar"		$\lambda = 574 \rho\rho$
N	$\delta(\text{kgs})$	$\Delta(\text{mm})$	$\epsilon(\text{Minutes})$	δz
31	0.54	- 6.5	- 4.02'	$\pm 0.02' = 2 \%$
15	1.80	-23.2	- 3.67'	$\pm 0.02' = 0.5$ "
15	3.20	-39.0	- 6.17'	$\pm 0.03' = 0.5$ "
15	6.32	-69.7	-11.03'	$\pm 0.05' = 0.4$ "
15	10.37	-84.5	-13.36'	$\pm 0.03' = 0.2$ "
15	12.60	-86.6	-13.71'	$\pm 0.02' = 0.1$ "
20	13.30	-86.3	-13.66'	$\pm 0.03' = 0.2$ "
15	14.51	-86.2	-13.65'	$\pm 0.03' = 0.2$ "

It would be interesting to study the magneto-optic behaviour of the nearly non-magnetic nickel-iron alloy, which contains 25 percent nickel.

5. *The HEUSLER alloy*, supposed to contain 61% Cu, 26% Mn and 13% Al is rather strongly magnetisable. Different portions of two well-polished mirrors were carefully examined in various parts of the spectrum but proved to be magneto-optically ineffective. It is of course possible that the KERR-Effect might be less than 0,3' in this case. Quite recently there appeared a communication by TOKMATSCHEW recording similar experiments on the HEUSLER alloy No. 32 (58,9 Cu, 26,5 Mn, 14,6 Al). From theoretical considerations the author arrives at the conclusion of the probability of an effect capable of measurement occurring in the neighbourhood of 450 μ . I have carried out a series of readings at this wave-length but no rotation could be observed. INGERSOLL also failed to notice any measurable effect either in the visible spectrum or in the infra-red.

The discussion of the theoretical signification of the above partially positive and partially negative results I reserve for a future occasion; further experiments are in preparation, and the determination of the purely optical properties of the investigated substances is already in progress.

ERRATA.

In the Proceedings of the Meetings of Jan. and Febr. 1910:

p. 672 Table III for 5050 read 8050.

p. 675 Table VII for 102.58 read 102.85.

p. 676 Table VIII for 71.75 read 71.95.



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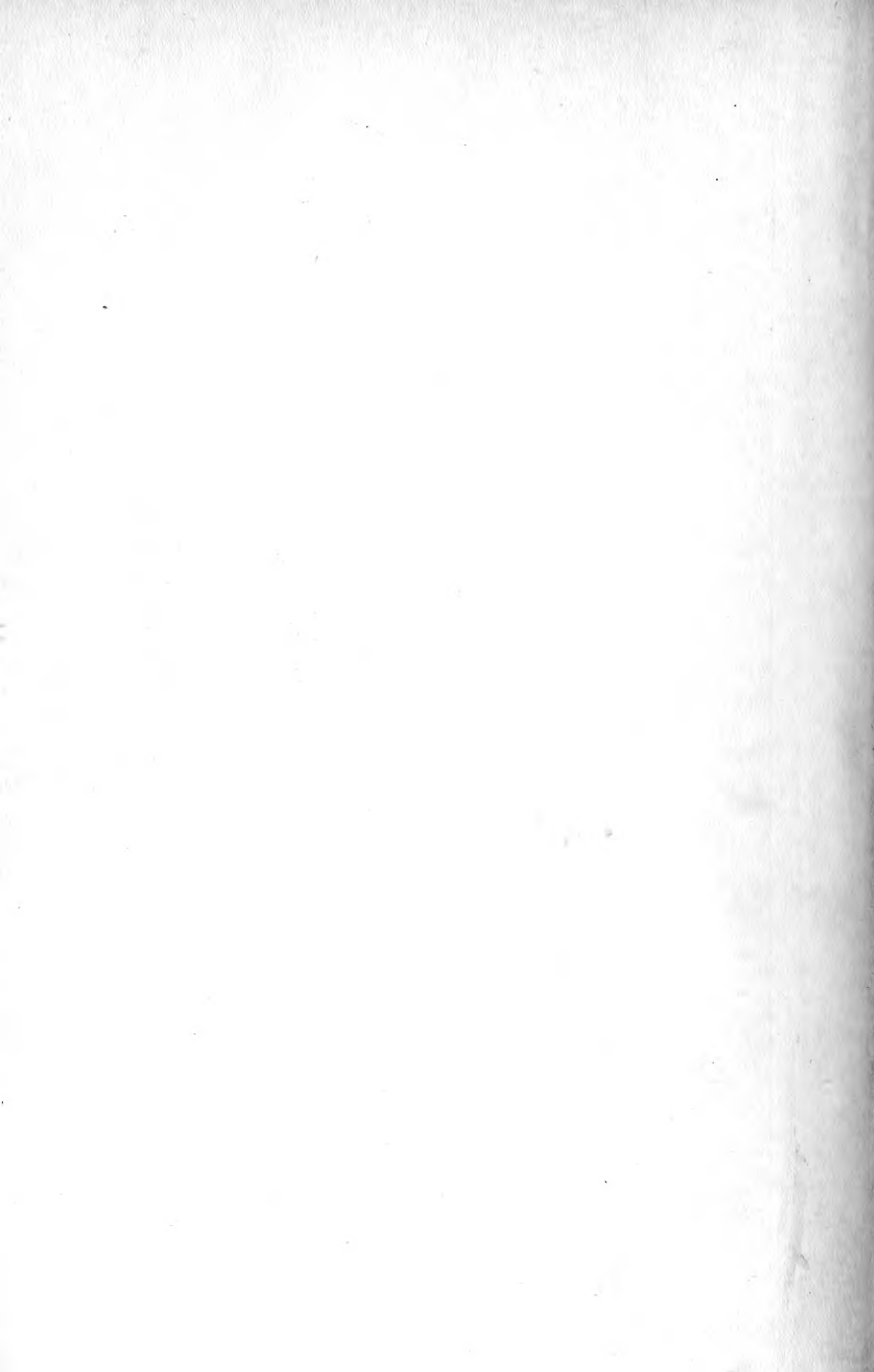
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