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LIST OF ILLUSTRATIONS.

ADVERTISEMENT.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions* take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind: the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*; which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them: without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

PHILOSOPHICAL TRANSACTIONS.

I. *Tidal Friction in the Irish Sea.*

By G. I. TAYLOR, *M.A.*

Communicated by Sir NAPIER SHAW, *F.R.S.*

Received December 4, 1918,—Read March 20, 1919.

THE dissipation of energy in the tides has recently formed the subject of a paper by Mr. R. O. STREET.* In that paper it is assumed that the energy is dissipated by the viscous drag of layers of water which move parallel to the bottom of the sea. The assumption that tidal currents move in laminar motion is so opposed to ordinary observation of the surface of the sea in a tideway that I felt certain, on reading the paper, that if some other method could be found, which did not depend on any special assumptions as to the nature of the motion, it would be found that Mr. STREET'S estimate of the dissipation is very much too small.

This view is strengthened by the consideration that REYNOLDS' criterion† of stability would lead us to expect that eddies would form in any stream of sea-water flowing at a speed of 1 knot or more, when the depth is greater than some quantity of the order of magnitude of 1 or 2 cm. Since the mean depth of the Irish Sea is over 40 fathoms, mathematical considerations alone would lead us to suspect the existence of the eddies, which can in fact be seen marking the surface of the sea in places where the current runs exceptionally strongly, or over a particularly uneven bottom. Several of these places are marked as "ripples" on the chart of the Irish Sea, the sheet of water to which Mr. STREET applied his calculations.

Dissipation of Energy in Tidal Currents.

The mechanism by means of which energy is dissipated in a tidal current by friction on the bottom must be similar to the mechanism by which the energy of a river is dissipated by friction on its bed, and also to the mechanism by which the energy of the wind is dissipated by friction on the ground. The amount of friction in both these cases is known. It can in both cases be expressed by a term of the form F, the skin-friction per square centimetre, which is equal to $K\rho V^2$, where ρ is

* 'Roy. Soc. Proc.,' A, vol. 93, 1917, p. 349.

† See OSBORNE REYNOLDS, "On the Dynamic Theory of Incompressible Viscous Fluid and the Determination of the Criterion," 'Phil. Trans.,' A, 1894, p. 123.

 [Published November 29, 1919.

the density of the fluid, V its velocity, and K is a constant depending on the nature of the surface.

Friction on the Bed of a River.—A very large amount of work has been done on the friction of a river on its bed. The results of these researches have been used to make various empirical formulæ. One of the best known of these is that of BAZIN, which takes the following form

$$rs = \frac{1}{7569}\left(1 - \frac{\gamma}{\sqrt{r}}\right)^2 \mathrm{V}^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where r is the "hydraulic radius" of the channel, *i.e.*, the area divided by the wetted part of the perimeter of the cross-section, s is the slope of the bed, γ is a constant which depends on the nature of the bottom. In this engineering formula metres are used instead of centimetres as the unit of length. In order to find out the relationship between this formula and one of the type

$$\mathrm{F} = \mathrm{K}\rho\mathrm{V}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

one must equate the resistance acting up-stream to the component of the weight of the fluid acting down-stream. This gives

$$\mathrm{F} \times (\text{wetted part of perimeter}) = s\rho g \times (\text{area of cross-section}),$$

or

$$\frac{\mathrm{K}\rho\mathrm{V}^2}{\rho g} = rs. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Comparing this with (1) it will be seen that

$$\mathrm{K} = \frac{g}{7569}\left(1 - \frac{\gamma}{\sqrt{r}}\right)^2.$$

But $g = 9{\cdot}81$ expressed in metre-second units.

Hence

$$\mathrm{K} = 0{\cdot}0013\left(1 + \frac{\gamma}{\sqrt{r}}\right)^2,$$

where K is non-dimensional. In the case of the Irish Sea, to which this formula will be applied, the depth is about 80 metres. In the case of a stream which is very broad compared with its depth, the depth and the hydraulic radius are the same thing. Hence for the Irish Sea $\sqrt{r} = \sqrt{80} = 9$, approximately.

The value of γ depends on the nature of the bottom. For a clean stony, or smooth earth bottom, BAZIN* gives $\gamma = 0{\cdot}85$. Taking this value as being applicable to the Irish Sea,

$$\mathrm{K} = 0{\cdot}0013\left(1 + \frac{0{\cdot}85}{9}\right)^2 = 0{\cdot}0016. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

* See 'Cours d'Hydraulique,' J. GRIALOU, Paris, 1916.

In places where the bottom is uneven or weedy, BAZIN gives 1·7 as the value of γ. Under these circumstances

$$K = 0{\cdot}0013\left(1+\frac{1{\cdot}7}{9}\right)^2 = 0{\cdot}0018. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It will be seen that large changes in the amount of roughness produce only small changes in the amount of friction on the bottom. On looking at BAZIN's formula it will be seen that this is due to the fact that the sea is deep. In order that the roughness of the bottom may have a large effect in slowing down a stream, it is necessary that r should be small. It seems, in fact, that the size of the projections which constitute the roughness or inequality of the bed must be some definite fraction of r in order that their effect may be felt on the stream as a whole. In other words, the direct effect of the projections extends to a distance which is some multiple of the linear dimension of the projections; and if these are small enough compared with the depth, very little difference is made to the total flow of the stream by changing the amount of roughness on the bottom.

This conclusion is important in the present application, because it means that by adopting the values of K given above we shall be able to get a fairly accurate estimate of the friction of the sea on the bottom without knowing the exact nature of the bottom. We may under-estimate the friction, but we are certainly not likely to over-estimate it; for our estimate will not take account of unevennesses, such as boulders and rocks, which are comparable with the depth of the sea, nor will it take account of the increase in K in the shallow areas of estuaries and outlying banks.

Friction of the Wind on the Ground.—It has already been pointed out that the friction of the sea on the sea-bottom is similar to the friction of the wind on the ground. According to the principle of dynamical similarity the flow-patterns of the sea and air will be the same, provided the scale of the projections which constitute the roughness are the same, and provided

$$\frac{v_w}{v_a} = \frac{\mu_w\rho_a}{\mu_a\rho_w}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where ρ_a and ρ_w are the densities of air and sea-water respectively, μ_a and μ_w are their viscosities, and v_a and v_w are their velocities. Using values obtained from physical tables $\mu_w\rho_a\,(\mu_a\rho_w)$ will be found to be equal to $\frac{1}{11}$.

In a previous communication to the Royal Society* the author has shown from meteorological observations that the friction of the wind over the grass land of Salisbury Plain may be expressed by means of the formula $F = 0{\cdot}002\rho_a v_a^2$ over the whole range of velocities tested, *i.e.*, from 6 to 30 miles per hour.

According to the principle of dynamical similarity therefore this same expression may be expected to apply to tidal currents of $\frac{6}{11}$ to $\frac{30}{11}$ miles per hour, *i.e.*, roughly

* 'Roy. Soc. Proc.,' A, vol. 92, p. 196, 1916.

$\frac{1}{2}$ to 3 knots. This is the very range of speed with which we have to deal in tidal measurements. Hence, if we assume that the roughness of the bottom of the sea is about the same as that of the grass land of Salisbury Plain, the formula

$$F = 0\cdot002\rho v^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

for the friction of a tidal stream, of velocity v, on the sea-bottom may be expected to give reasonably accurate results.

It will be noticed that the value of K, 0·002, is very nearly the same as the values 0·0016 and 0·0018 obtained from experiments and observations on the flow of large rivers. It also agrees fairly well with laboratory experiments on the friction of air and water in pipes and with experiments on the friction of flat surfaces in water.

Calculation of the Energy Dissipated by Tidal Friction.—We can now proceed to calculate the amount of energy dissipated by tidal currents in the Irish Sea, the sheet of water which it is proposed to discuss.

The rate of dissipation of energy by friction is equal to the friction multiplied by the relative velocity of the surfaces between which the friction acts. Using the expression $F = K\rho v^2$ for the friction of the current on the bottom, the amount of energy dissipated per square centimetre per second is therefore

$$K\rho v^3.$$

The currents in the Irish Sea vary from place to place, and also with the varying state of the tide. It is necessary therefore to find the average value of $K\rho v^3$ during a tidal period, and then to take the average value of this expression over the whole area considered.

The tidal stream at any time t, after it has attained its maximum velocity, may be taken roughly as $v = V\cos\frac{2\pi t}{T}$, where V is the maximum tidal stream and T is the semi-diurnal tidal period of 12h. 25m.

The average rate of dissipation of energy over each square centimetre of the Irish Sea is therefore equal to the mean value of

$$K\rho V^3 \cos^3\frac{2\pi t}{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The average value of $\cos^3\frac{2\pi t}{T}$ taken without regard to sign is $4/3\pi$.

The average value of V^3 over the Irish Sea could be obtained from tidal measurements. Mr. STREET, in the paper already referred to, has found the average value of V^2 at spring tides over the Irish Sea. His estimate is 5 (knots)^2. This would make $V = 2\frac{1}{4}$ knots. If we assume this as the value of V in (8) we shall not be far from the truth, because the variability of the maximum streams in the Irish Sea is not sufficiently great to give rise to much difference between the square root of the mean square of the velocity and the mean velocity, or between this and the cube root of the mean cube of the velocity. By taking $V = 2\frac{1}{4}$ knots $= 114$ cm. per second in (8) we slightly under-estimate the friction; we shall not in any case over-estimate it.

Using in (8) the value $K = 0{\cdot}002$,* and remembering that ρ, the density of sea-water, is $1{\cdot}03$, it will be found that w, the mean rate of dissipation of energy, per square centimetre per second, in the Irish Sea at spring tides is

$$w = 0{\cdot}002\,(1{\cdot}03)\,(114)^3\left(\frac{4}{3\pi}\right) = 1300 \text{ ergs per square centimetre per second.} \quad (9)$$

Using the least admissible value of K† it will be found that

$$w = 1040 \text{ ergs per square centimetre per second.} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Mr. STREET's estimate, when reduced to C.G.S. units, is 7 ergs per square centimetre per second, which is only $\frac{1}{150}$th of our minimum estimate.‡

Rate at which Energy enters the Irish Sea owing to the Action of External Forces.

The amount of dissipation found by the method just described is so different from that obtained by Mr. STREET, and so much larger than any previous estimate of tidal friction that I have come across, that it seemed worth while to try and verify it, if possible, by some different method. Instead of trying to measure the rate of dissipation at every point of the Irish Sea, I have calculated the rate at which energy enters the Irish Sea through the North and South Channels. To this must be added the rate at which work is done by lunar attraction on the waters of the Irish Sea. The sum of these will give the rate at which the energy of that sea is increasing, *plus* the rate of dissipation of energy. When the average values of these expressions are taken during a complete tidal period it is evident that, since the energy of the Irish Sea does not increase or decrease continually, the average value of the rate of increase in energy is zero. Hence the average rate of dissipation of energy by the tidal currents can be found.

Rate at which Energy flows into the Irish Sea.—The calculation of the rate at which energy flows into the Irish Sea is very simple. Consider the flow of energy across the surface, S (fig. 1), formed by the vertical lines between a closed curve, s, on the surface of the sea and its projection on the bottom.

* See equation (7).

† See equation (4).

‡ Mr. STREET informs me that since publishing the paper already referred to, he has obtained other results which confirm his previous results. He hopes to publish them when circumstances permit.

Let ds be an element of length of the curve, s,

v = velocity of current at any point on s,

θ = angle between element ds and direction of current,

D = depth of bottom below mean sea-level.

h = height of tide above mean sea-level.

ρ = density of sea-water,

g = acceleration due to gravity.

First consider the rate at which energy is communicated to the portion of the sea which, at time, t, was enclosed by the surface, S. Let S′ be the moving surface which encloses this water.

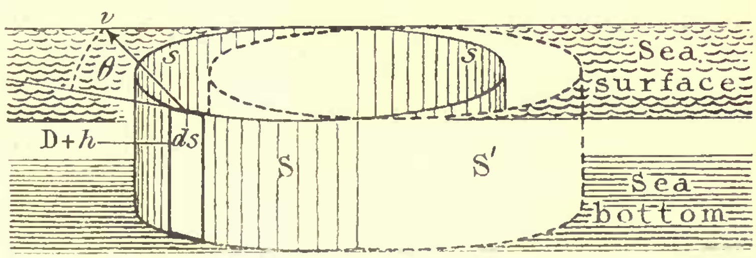

Fig. 1. Diagram showing successive positions of a surface, S, formed by the vertical lines through the curve, s, as that surface moves with the current.

The mean hydrostatic pressure on a vertical strip of S, of height $D+h$ and width ds is $\rho g \frac{1}{2}(D+h)$. Its area is $(D+h)\,ds$. The work done by hydrostatic pressure on the portion of the sea originally enclosed in the surface is therefore

$$\int \rho g \left(\frac{D+h}{2}\right)(v \sin\theta\, dt)\,(D+h)\,ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This then is the amount of energy which has flowed during the time, dt, through the surface, S′, which originally coincided with the fixed surface, S, but which moves with the fluid.

To find the amount of energy which has crossed the original surface, S, during the time, dt, it is necessary to take account of the energy contained in the fluid which has actually crossed the fixed surface, S. The gravitational potential energy of a vertical column of fluid, of height $D+h$ and horizontal cross-section $v \sin\theta\, dt\, ds$, is evidently $\rho g (D+h)\left(\frac{h-D}{2}\right) v \sin\theta\, dt\, ds$, mean sea-level being regarded as the surface of zero potential. The kinetic energy of the same column is

$$\tfrac{1}{2}\rho v^2 (D+h)\, v \sin\theta\, dt\, ds.$$

The amount of energy in the fluid which crosses the element of surface dS during the time dt is the sum of these two. The amount in the fluid which crosses the whole surface, S, is therefore

$$\int \tfrac{1}{2}\rho v \sin\theta \, dt \, \{gh^2 - gD^2 + v^2 (D+h)\} \, ds, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where the integral is taken round the curve, s.* The amount of energy which crosses the surface, S, in time dt is the sum of (11) and (12), that is,

$$\int \tfrac{1}{2}\rho v \sin\theta \, dt \, \{g(D+h)^2 + gh^2 - gD^2 + v^2(D+h)\} \, ds$$

$$= \rho g \, dt \int Dhv \sin\theta \, ds + \int \tfrac{1}{2}\rho v \sin\theta \, dt \, (2gh^2 + Dv^2 + hv^2) \, ds. \quad . \quad . \quad (13)$$

We shall now assume that h is small compared with D. This is true for the Irish Sea where the average maximum rise of tide above the mean sea-level is about 6 feet (one fathom) at spring tides, while the average depth is over 40 fathoms.

It is evident also that since the order of magnitude of v must be that of ch/D, where c is the velocity of a tidal wave in water of depth D (*i.e.*, $c = \sqrt{gD}$),

* It has been suggested to me that a term should be added to allow for the potential energy of the entering water due to the moon's attraction. This appears to be a misapprehension. Potential energy is only a mathematical expression used in finding the work done on matter by certain systems of forces.

The work done in time δt by the moon's attraction on the liquid contained in any surface which is fixed relatively to the earth is

$$-\delta t \iiint \rho \frac{d\Omega}{dt} \, dv, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

where Ω is the potential due to the moon's attraction, dv is an element of volume, and the integration extends throughout the volume. If the linear dimensions of the surface are small, so that Ω does not vary appreciably throughout its volume (this may be taken as true for the Irish Sea), then $\iiint \rho \, dv = M$, the mass of the liquid contained in the surface at any time.

The total work done by the moon in a complete period is $-\int M \frac{d\Omega}{dt} dt = -\int M \, d\Omega$.

Integrating by parts $-\int M \, d\Omega = -[M\Omega] + \int \Omega \, dM$, where $[M\Omega]$ represents the change in the product $M\Omega$ during a complete period. This is evidently equal to 0. Hence

$$-\int M \, d\Omega = \int \Omega \, dM = \int \Omega \frac{dM}{dt} dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

$\frac{dM}{dt}$ is the rate at which water enters the volume and $\Omega \frac{dM}{dt}$ is the potential energy of the entering water.

In calculating the work done by the moon on the waters of the Irish Sea we could therefore use either expression A or expression B, but we must not use *both*.

At a later stage in this paper (see p. 18) the work done by the moon's attraction has been calculated from expression A. The potential energy, due to the moon's attraction, of the entering water has therefore been left out at the present stage.

therefore all the terms in the second integral of (13) are small compared with those of the first.

We shall therefore neglect

$$\int \tfrac{1}{2}\rho v \sin\theta \, dt \, (2gh^2 + Dv^2 + hv^2) \, ds$$

in comparison with

$$\rho g \, dt \int Dhv \sin\theta \, ds. \qquad (14)$$

Taking account of the conservation of energy, this must be equal to the sum of the increase in kinetic energy of the sea included in the area enclosed by s, the energy dissipated during the time dt by tidal friction, and the work done by the moon's attraction during the same time. It has already been pointed out that since there is no continual increase in the kinetic energy of any portion of the sea, the first of these will vanish when we come to consider the mean rate of dissipation of energy over a complete tidal period.

If W is the average rate at which energy is dissipated by tidal friction in the portion of the sea enclosed by s, and W_m is the average rate at which work is done on it by the moon's attraction, it will be seen from (14) that

$$W - W_m = \text{average value of} \left\{ g\rho \int Dhv \sin\theta \, ds \right\}. \qquad (15)$$

Application to the Irish Sea.

In applying this expression to the Irish Sea, it will be necessary to evaluate the integrals across sections of the North and South Channels; and in choosing the exact positions of these sections, it is clear that those parts of the channels must be selected where the greatest number of observations of the rise of tide and the strengths of the currents have been made.

The only observations of tidal currents in the Irish Sea to which I have had access are contained in the Admiralty publication 'Tides and Tidal Streams of the British Islands.'* The observations on the rise and fall of tide are contained in the 'Admiralty Tide Tables' and the 'Irish Coast Pilot.'

Height of the Tide.—The Tide Tables give the time of H.W. at full and change of the moon. They also give the range of tide at spring tides and at neaps. They afford no indication, as a rule, of the height of the tide at the intermediate hours, except when there is some marked peculiarity such as the long-continued H.W. at Poole, or the bore in the Bristol Channel. The principal tidal phenomenon is, however, the semi-diurnal rise and fall of tide, with a period of 12h. 25m., and in a

* First edition, 1909.

large majority of cases this can be represented with sufficient accuracy for most purposes by a term of the form h, the height of the tide above mean sea-level

$$= \mathrm{H} \cos \frac{2\pi}{\mathrm{T}}(t+\mathrm{T}_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where

2H is the range of tide between H.W. and L.W.

T is the tidal period of 12h. 25m.

t is the time measured from the time of the moon's passage over the Greenwich meridian. At full and change of the moon, t is Greenwich mean time.

T_1 is the time of H.W. at full and change of the moon, *i.e.*, the "establishment" of the place in question. We shall henceforth assume that h can be expressed by means of the equation (16).

In evaluating the integral (15) it will be seen that it is necessary to know the height of the tide at all points on the section. Unfortunately nearly all the measurements of rise and fall of tide have been made on the coast. None have been made in the middle of the channel, or at any rate none are recorded in the tables.

At first sight we might expect tidal range to be the same on the two sides of a channel, but this not the case. On the opposite sides of the South Channel, at the entrance to the Irish Sea, for instance, the tidal ranges at spring tides are 4 feet at Arklow on the Irish side and 15 feet at Bardsey Island on the Welsh side. This is not an accidental circumstance connected with particular formations of the coast in the neighbourhood of Bardsey or Arklow; all the tidal ranges in the neighbourhood show the same characteristic. On the Irish coast there is Arklow with a tidal range of 4 feet; Courtown, $3\frac{3}{4}$ feet; Arklow Bank, $4\frac{1}{4}$ feet; and Kilmichael Point, $4\frac{3}{4}$ feet; while on the Welsh coast there are St. Tudwall Road, 14 feet; Port Dynllayn, $12\frac{1}{4}$ feet; Llanddwyn Island, $14\frac{1}{2}$ feet; Bardsey Island, 15 feet; and Holyhead, 16 feet. In evaluating the integral (15), therefore, it is important to know how the tidal range varies from the Welsh to the Irish coasts. In other words, does the level change more rapidly near the Welsh or near the Irish coast, or does the sea at H.W. slope uniformly down, and at L.W. slope uniformly up, from Bardsey to Arklow? In deciding this question, dynamical considerations are of great assistance.

The reason for the difference in the range on the two sides of a channel is well known; it is connected with the "geostrophic" force, due to the earth's rotation, which tends to deflect bodies moving on the earth's surface to the right in the Northern, and to the left in the Southern Hemisphere. The flood stream into the Irish Sea cannot be deflected to the right because of the Welsh coast. The water therefore piles up on that side till the hydrostatic pressure-gradient is sufficient to keep the water moving straight. The same reasoning applies to the ebb stream which piles itself up against the Irish coast. At the particular section from Arklow to Bardsey the flood stream is a maximum at H.W. and the ebb stream a maximum at L.W. Hence the effect of the slope of the sea surface, which is necessary to keep the stream straight against the deflecting force due to the earth's rotation, is to add to H.W. and to subtract from L.W. on the Welsh side, thus increasing the range above the mean range for the section. The effect on the Irish side is exactly the reverse, so that the tidal range is diminished there. Though this explanation is given in general terms it is a simple matter to express the forces and slopes concerned in a quantitative manner.

The application to the present question follows directly. If it can be shown by observation that the tidal currents move straight up and down the channel without being deflected across it, then the slope of the sea surface must everywhere correspond with the velocity of the current. If the current is nearly uniform right across the channel, then the sea will slope down uniformly from one side of the channel to the other. It will be shown later, in discussing the tidal currents, that both these conditions are satisfied. Dynamical considerations therefore enable us to say what the tidal range in mid-channel is, when we know it at either side.

Confidence in the correctness of this view is greatly strengthened by calculating the difference to be expected in the tidal ranges on the two sides of the channel, and showing that it is in close agreement with the observed difference.

The deflecting force due to the earth's rotation which acts on each cubic centimetre of the sea is $2\omega\rho v \sin\lambda$, where ω is the angular velocity of the earth's rotation, and λ is the latitude.

The slope of the surface, in a direction perpendicular to the stream, which will just balance this force, is therefore

$$\frac{2\omega\rho v \sin\lambda}{\rho g}, \quad \text{or} \quad \frac{2\omega v \sin\lambda}{g} \qquad (17)$$

The measured maximum velocity of both the flood and the ebb stream at spring tides across the section, AB, from Arklow to Bardsey* is 3·2 knots,† = 162 cm. per second; $\omega = 0{\cdot}000073$; in latitude 52°, $\sin\lambda = 0{\cdot}79$, $g = 981$. Hence from (17) the slope is $1{\cdot}9 \times 10^{-5}$ radians.

The distance across the channel in a direction perpendicular to the current from Bardsey Island to Arklow, on the Irish coast, is 48 nautical miles = 288,000 feet. Hence the difference in level at time of the maximum current between the sea surface at Bardsey Island and at Arklow should be $1{\cdot}9 \times 10^{-5} \times 2{\cdot}88 \times 10^{5} = 5{\cdot}7$ feet.

Now, as has been mentioned already, the streams in this part of the Irish Sea have their maximum velocities at H.W. and L.W. The curves shown in fig. 2 represent

* See map, fig. 3.

† See p. 12 later.

the velocities of the tidal streams at various states of the tide at various lightships in the neighbourhood of the Arklow-Bardsey section.

On inspecting the curves shown in that figure, it will be seen that the maximum current is at about 3h. before H.W. at Dover. It is H.W. at Dover at 11h. 7m. full and change, and it is H.W. on the Arklow-Bardsey line at 8h. 10m.* full and change. Hence it is H.W. on AB 3h. before H.W. at Dover, *i.e.*, at the time of the maximum tidal current.

Since H.W. coincides with the time of maximum current, the difference in range at spring tides between those at Arklow and those at Bardsey should be $2 \times 5{\cdot}7$ feet $= 11{\cdot}4$ feet. The measured range at Bardsey at spring tides is 15 feet, while that at

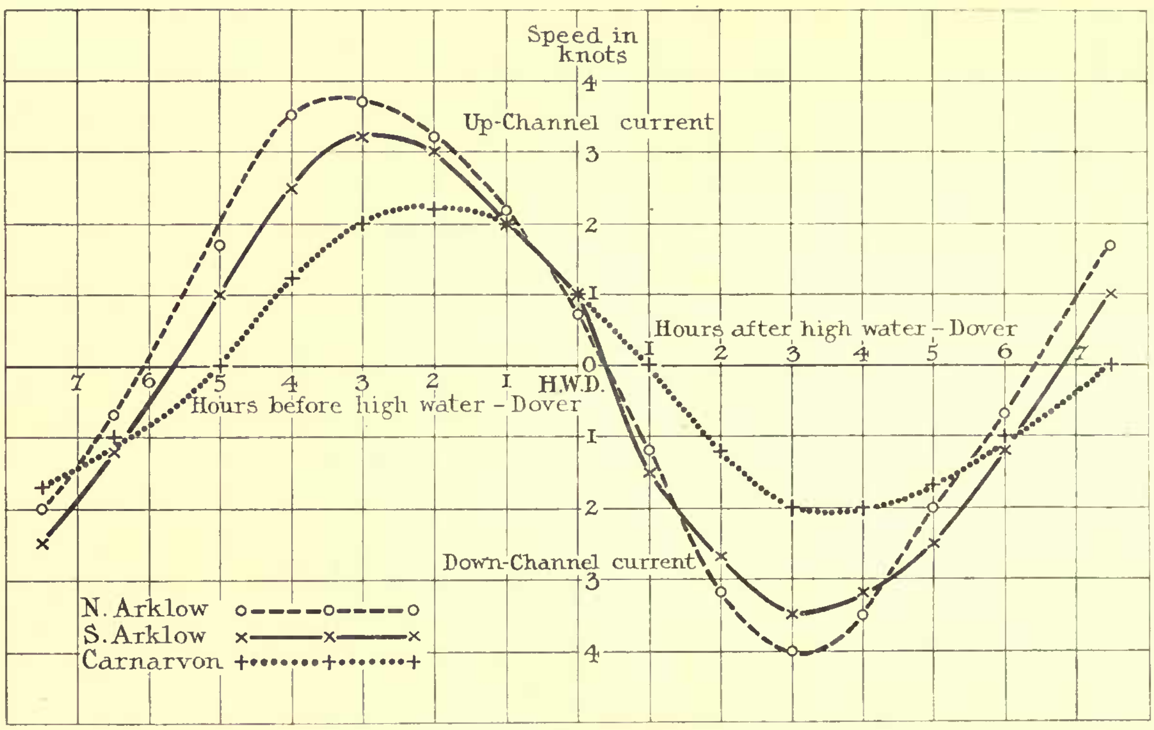

Fig. 2. Curves showing the velocity of the tidal currents at three light ships in the South Channel to the Irish Sea at various states of the tide.

Arklow is 4 feet. The difference, 11 feet, is almost exactly equal to the calculated difference $11\frac{1}{2}$ feet.

The accuracy with which this calculation is verified by observation is very good evidence that the sea actually slopes in the way we should expect from the current measurements, that is to say, uniformly from Bardsey Island to Arklow. It is not necessary, therefore, for our purpose to have actual tidal measurements in mid-channel, though it is to be hoped that these conclusions will some day be tested by observation.

* See p. 15 later.

The considerations just advanced show that if a channel is so narrow that the water is forced to travel straight up and down it, then the difference in level between the water on the two sides may be calculated on the assumption that the sea slopes to an extent which gives rise to a pressure gradient across the channel which is exactly equal to the deflecting force due to the earth's rotation. If the channel is rather wider the central parts of the stream may be able to move across the channel slightly. This would reduce the slope. In the case of the South Channel of the Irish Sea, however, these cross currents are very small, as may be seen by examining the figures given in the table on p. 14, where it is shown that the direction of the current is practically constant during the ebb and during the flood streams. We are therefore justified in assuming that the South Channel is narrow enough to allow us to apply the calculations given above.

Velocity of the Tidal Currents.—We now come to the measurements of tidal currents. These are the principal factors which determine our choice of sections suitable for measuring the flow of energy into the Irish Sea.

South Channel.—In the South Channel the best section is that shown as AB in the map (fig. 3). It runs from Bardsey Island through the south end of Arklow Bank. Along this section tidal measurements have been made at the points marked in the map as S_1, S_2, S_3, S_4.

In the position S_1, 5 miles from Arklow Bank, the maximum velocities of the ebb and flood streams are both 3·6 knots. The direction of the flood stream is N. 32° E., while that of the ebb is S. 26° W.

At S_2, 15 miles from Arklow Bank, the maximum flood stream is N. 35° E. at 3·2 knots, while the maximum ebb stream is S. 32° W. at 3·3 knots.

At S_3, 15 miles from Bardsey Island, the maximum flood stream is N. 25° E., 3·2 knots, while the maximum ebb is S. 28° W., 3·0 knots.

At S_4, 5 miles from Bardsey Island, the maximum flood stream is N. 16° E., 3·0 knots, while the maximum ebb is S. 16° W., 2·3 knots.

It will be seen, therefore, that the maximum current velocity is nearly constant along the section, its average value being 3·2 knots. The direction also varies very little; the average direction of the flood stream being N. 27° E., while that of the ebb S. 26° W. These are practically opposite directions. They will (for simplicity) be assumed to be exactly opposite during the rest of this discussion.

No measurements of the speed and direction of the currents at the points S_1, S_2, S_3, S_4, are given for the intermediate hours; but several such measurements are given for other points in the neighbourhood. I have selected three of these sets which were taken at the nearest points to the section AB. They were made at the South Arklow, North Arklow and Carnarvon Bay light-ships, respectively.

In the accompanying table* are given the velocities and directions of the current

* Taken from 'Tides and Tidal Streams of the British Islands,' first edition, 1909.

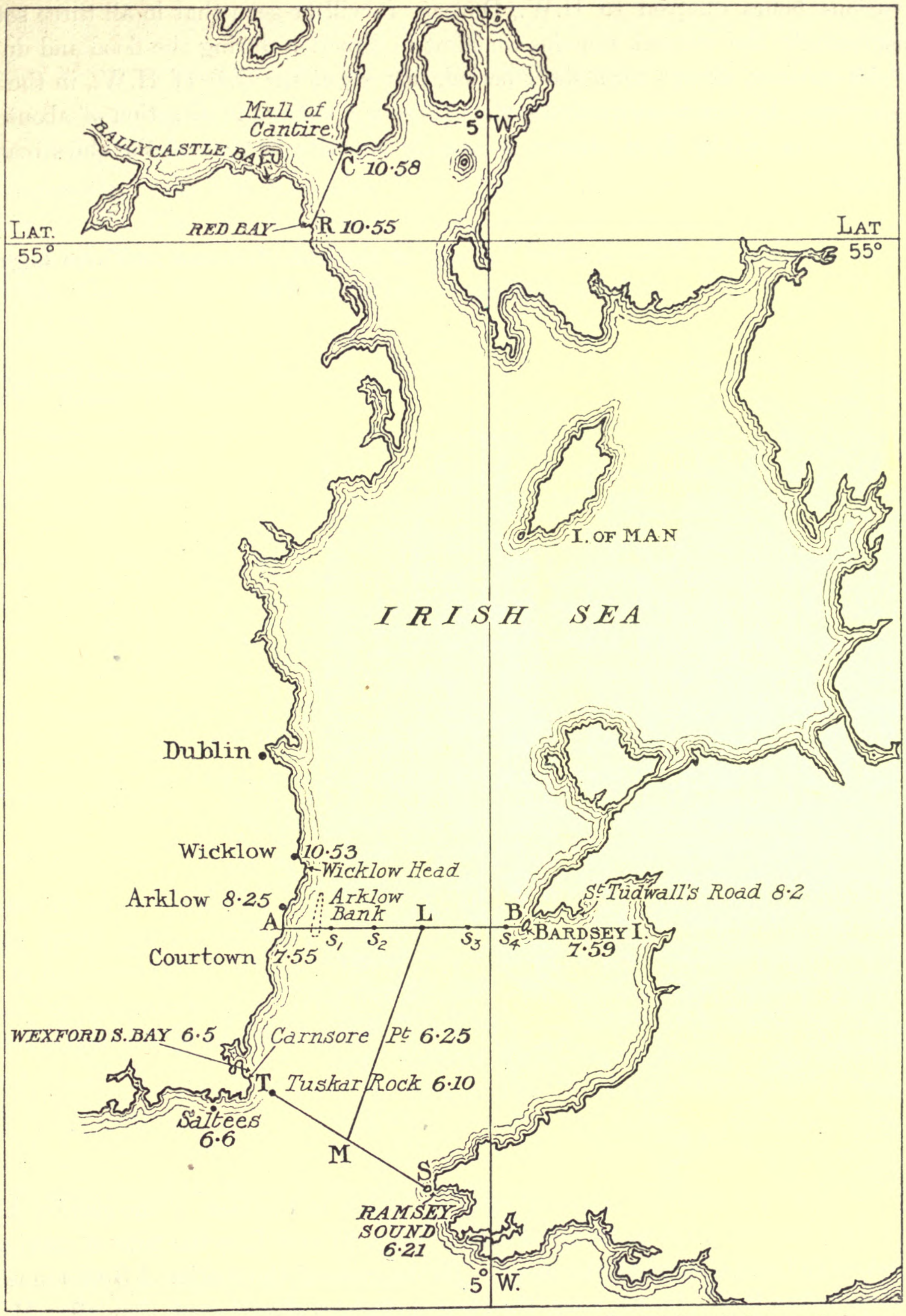

Fig. 3. Map of the Irish Sea. The figures are the times of H.W. at full and change of the moon. Thus Courtown 7·55 means that at Courtown it is H.W. at 7h. 55m. at full and change of the moon.

at various hours referred to H.W., Dover. It will be seen that in all three sets of measurements the current remains constant in direction during the flood and during the ebb streams, except for a short period, just about the time of H.W., in the case of the South Arklow measurements when there is a change in direction of about 10°. The direction of the flood stream is exactly opposite to that of the ebb stream in this region.

TABLE showing the Direction and Velocity of the Tidal Streams at Three Light-Ships at Various States of the Tide.

		North Arklow light vessel.		South Arklow light vessel.		Carnarvon Bay light vessel.	
		Diurnal magnitude.	Rate. Knots.	Diurnal magnitude.	Rate. Knots.	Diurnal magnitude.	Rate. Knots.
Hours before H.W. at Dover	5 .	N. 43° E.	1·7	N. 43° E.	1·0	Slack	—
	4 .	N. 43° E.	3·5	N. 43° E.	2·5	N. 21° E.	1·2
	3 .	N. 43° E.	3·7	N. 43° E.	3·2	N. 21° E.	2·0
	2 .	N. 43° E.	3·2	N. 43° E.	3·0	N. 21° E.	2·2
	1 .	N. 43° E.	2·2	N. 43° E.	2·0	N. 21° E	2·0
H.W. at Dover . .		N. 43° E.	0·7	N. 54° E.	1·0	N. 21° E.	1·0
Hours after H.W. at Dover	1 .	S. 43° W.	1·2	S. 32° W.	1·5	Slack	—
	2 .	S. 43° W.	3·2	S. 43° W.	2·7	S. 21° W.	1·2
	3 .	S. 43° W.	4·0	S. 43° W.	3·5	S. 21° W.	2·0
	4 .	S. 43° W.	3·5	S. 43° W.	3·2	S. 21° W.	2·0
	5 .	S. 43° W.	2·0	S 43° W.	2·5	S. 21° W.	1·7
	6 .	S. 43° W	0·7	S. 43° W.	1·2	S. 21° W.	1·0

The variation of current velocity with the state of the tide is shown in the curves in fig. 2, which are drawn from the measurements recorded in the above table. On inspecting the curves of fig. 2 it will be seen that in the neighbourhood of the section AB the tidal streams can be represented sufficiently accurately for many purposes by a sine curve. We can, therefore, express the current by the mathematical expression

$$v = \mathrm{V} \cos \frac{2\pi}{\mathrm{T}} (t + \mathrm{T}_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where V is the maximum current velocity, which in the case of the section AB is 3·2 knots, t has the same meaning as before (see p. 9), and T_0 is the time of maximum current at full and change of the moon. In the case of the section AB this must be 2h. 45m. before H.W., Dover, since the stream changes direction at 15m. after H.W., Dover. Since H.W., Dover, at full and change of moon is at 11h. 7m., therefore, T_0 = 8h. 20m. approximately.

Height of Tide at Section AB.—Having now chosen the section AB of the South Channel along which we intend to calculate the average value of $\left\{g\rho\int \mathrm{D}hv \sin\theta ds\right\}$ we must return to the discussion of the values of h. In the first place the line AB is practically a co-tidal line, *i.e.*, a line through all points at which it is H.W. simultaneously. On the Irish side it is H.W. at Arklow Bank at 8h. 24m.; at Arklow at 8h. 25m.; at Kilmichael Point, where the section AB strikes the Irish coast, at 8h. 25m.; and at Courtown, about 4 miles south of Kilmichael Point, at 7h. 55m. On the Welsh side it is H.W. at St. Tudwall Road at 8h. 2m., at Bardsey Island at 7h. 59m. The time of H.W. all along the line AB may therefore be taken as 8h. 10m., which is the mean of the times at either end. This will only be a few minutes wrong at either end, and the convenience, in evaluating the integral, of assuming a constant time of H.W. along the section, is very great. The value of T_1 will therefore be taken as 8h. 10m. all along AB.

In the expression $h = \mathrm{H}\cos\frac{2\pi}{\mathrm{T}}(t+\mathrm{T}_1)$ which was adopted to give the height of the tide at any point, the value of H varies from point to point.

It has been shown, however, that H must decrease uniformly from one side of the channel to the other, and that in the case of the section AB, where there is practically no difference between the time of H.W. and the time of the maximum flood stream, the sea at H.W. slopes at an angle $\frac{2\omega v \sin\lambda}{g}$.

If y is the distance of any point from the central line of the channel measured in the direction perpendicular to the current and towards the Irish Coast, then

$$\mathrm{H} = \mathrm{H}_1 - \frac{2\omega v \sin\lambda}{g} y, \qquad (19)$$

where H_1 is half the range of tide in mid-channel. We have already seen that the ranges of tide on the two sides of the channel are 4 and 15 feet; hence $\mathrm{H}_1 = \frac{1}{2}\left(\frac{15+4}{2}\right) = 4\frac{3}{4}$ feet. If s is the distance measured from the central point, L, of AB, then

$$s \sin\theta = y. \qquad (20)$$

Evaluation of the Rate of Transfer of Energy through the South Channel of the Irish Sea.

We are now in a position to evaluate W_{ab}, the average rate at which energy enters the Irish Sea across the section AB,

$$\mathrm{W}_{ab} = \text{average value of} \left\{g\rho\int_{\mathrm{A}}^{\mathrm{B}} \mathrm{D}hv \sin\theta\, ds\right\}. \qquad (21)$$

Substituting in (21) from (18), (19) and (20),

$$W_{ab} = \text{average value of} \left\{ g\rho \int_A^B D \left(H_1 - \frac{2\omega v \sin\lambda}{g} s \sin\theta \right) \right.$$
$$\left. \cos\frac{2\pi}{T}(t+T_1)\, V \cos\frac{2\pi}{T}(t+T_0) \sin\theta\, ds \right\}.$$

Since the only terms which contain t are

$$\cos\frac{2\pi}{T}(t+T_1) \qquad \text{and} \qquad \cos\frac{2\pi}{T}(t+T_0),$$

we can integrate them with respect to t to find the average value of the main integral. Thus the average value of

$$\cos\frac{2\pi}{T}(t+T_1)\cos\frac{2\pi}{T}(t+T_0)$$

is evidently

$$\tfrac{1}{2}\cos\frac{2\pi}{T}(T_1-T_0).$$

Hence taking out all the quantities which are nearly constant across the section

$$W_{ab} = \tfrac{1}{2} g\rho V \sin\theta \cos\frac{2\pi}{T}(T_1-T_0) \int_A^B D\left(H_1 - \frac{2\omega v \sin\lambda}{g} s \sin\theta\right) ds. \quad . \quad . \quad (22)$$

To evaluate this it is only necessary to measure the depths at all points across the section. Actually this is not really necessary, for the depth is nearly uniform across the section AB, the average depth being 37 fathoms.

Under these circumstances, since the origin of s is taken at mid-channel, the value of $\int_A^B D \frac{2\omega v \sin\lambda}{g} s \sin\theta\, ds$ is zero. If the channel had not happened to be nearly uniform in depth, it would have been possible to evaluate this integral from the charted depths across the section. Hence

$$W_{ab} = \tfrac{1}{2} g\rho V \sin\theta \cos\frac{2\pi}{T}(T_1-T_0)\, DH_1L, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where L is the length of AB, 50 nautical miles. The numerical values of the other constituents which occur in (23) are

$g = 981.$

ρ = density of sea water = 1·03.

V = 3·2 knots = 163 cm. per second.

θ = angle between current and direction of AB.

AB runs in a direction N. 86° E., while the current runs in a direction N. 26° E., so that $\theta = 60°$ and $\sin\theta = 0{\cdot}87$.

$$T = 12{\cdot}4\text{h.}, \qquad T_1 = 8\text{h. }10\text{m.}, \qquad T_0 = 8\text{h. }20\text{m.}$$

so that

$$T_1 - T_0 = 10\text{m.} = \tfrac{1}{6}^{\text{th}}\text{h.},$$

and

$$\cos\frac{2\pi}{T}(T_1-T_0) = \cos\left(\frac{180}{6\times 12{\cdot}4}\right)^0 = \cos 2{\cdot}4° = 1{\cdot}0,$$

$$D = 37 \text{ fathoms} = 6800 \text{ cm.}$$

$$H_1 = 4\tfrac{3}{4} \text{ feet} = 145 \text{ cm.}$$

$$L = 50 \text{ nautical miles} = 9{\cdot}1\times 10^6 \text{ cm.}$$

Hence the mean rate at which energy is transmitted across the section AB is

$$\begin{aligned} W_{ab} &= \tfrac{1}{2}\times 981\times 1{\cdot}03\times 163\times 0{\cdot}87\times 1{\cdot}0\times 6800\times 145\times 9{\cdot}1\times 10^6 \\ &= 6{\cdot}4\times 10^{17} \text{ ergs per second.} \qquad (24) \end{aligned}$$

North Channel.—The same method may be applied to the North Channel, but it is at once obvious that practically no energy enters the Irish Sea through this channel. The tidal streams set strongly through the North Channel, running in from 5h. to 11h. and out from 11h. to 5h. at full and change of the moon. The neck between the Mull of Cantyre and the Irish Coast forms a loop in a stationary oscillation. It is H.W. at Mull of Cantyre at 10h. 58m. At Red Bay, on the Irish Coast, it is H.W. at 10h. 55m. The co-tidal line for 10h. 55m., therefore, runs from the Mull of Cantyre to Red Bay, and it is for this reason that it has been chosen for the section RC (see fig. 3) along which the integral for W_{RC} (the energy which flows across RC) will be taken. Since the streams change direction at H.W., Dover, *i.e.*, at 11h. 7m., the phase difference between the tidal stream and the height of the tide is only 12m. of time short of the quarter period, *i.e.*, 87° expressed as an angle.

The maximum current runs through the North Channel at a rate of 4 knots. The rise and fall of tide in the North Channel is very small; at Red Bay it is 4 feet, and at Ballycastle Bay, to the N.W. of Red Bay, it is only 3 feet. At the Mull of Cantyre it is also 4 feet. The equality of the heights of the tide on the two sides of the channel is probably due to the fact that, at the times the stream is running at its maximum speed, when therefore we should expect the maximum difference in level on the two sides of the Channel owing to geostrophic force, the water is at its mean level. At H.W. and L.W. the streams are slack, so that no geostrophic effect is to be anticipated at those times.

In the formula

$$W_{RC} = \rho g\frac{DVH}{2}\cos\theta\cos\frac{2\pi}{T}(T_0-T_1)\times(\text{length of RC}) \qquad (25)$$

for the rate of flow of energy into the Irish Sea, across the section RC, the numerical values of the terms are

$$H = \tfrac{1}{2}\,(4 \text{ feet}) = 61 \text{ cm.}$$

$$\frac{2\pi}{T}(T_0 - T_1) = 87^\circ, \text{ so that } \cos \frac{2\pi}{T}(T_0 - T_1) = 0{\cdot}05$$

$$V = 4 \text{ knots} = 200 \text{ cm. per second.}$$

(Length of RC) $\times \cos\theta$ is evidently equal to the breadth of the North Channel normal to the stream. This is 11 nautical miles, or 2×10^6 cm.

D_1 the mean depth, is about 65 fathoms $= 10^4$ cm. Hence the mean rate at which energy enters the Irish Sea by the North Channel is

$$W_{RC} = \tfrac{1}{2} \times 1{\cdot}03 \times 981 \times 10^4 \times 200 \times 61 \times 2 \times 10^6 \times 0{\cdot}05 = 6{\cdot}2 \times 10^{15} \text{ ergs per second.}$$

This is only $\frac{1}{100}$th of the energy which enters by the South Channel. It is obvious that no high degree of accuracy is aimed at in obtaining this figure. It is merely intended to show that the amount of energy which enters the Irish Sea by the North Channel is quite insignificant compared with the amount which enters by the South Channel. In the work which follows, I shall neglect it altogether, and shall consider merely the South Channel.

Amount of Work Done by the Moon's Attraction on the Waters of the Irish Sea.

The attraction of the moon may be expressed by means of a potential function Ω. Consider the work done by the moon's attraction on the water contained in an element of volume, Δ, which is fixed to the earth's surface. If the element contains water during two complete tidal periods, *i.e.* till it comes back to its original position relative to the moon, no work will be done on it. If on the other hand, the element, Δ, is situated within the space which is filled with water at high-tide and is empty at low-tide, work may be done on the water contained in Δ.

If ρ be the density of sea-water, h the height of the tide above mean sea-level, the work done by the moon's attraction during two complete lunar semi-diurnal tides, on a column of sea of 1 sq. cm. cross-section and stretching from the sea bottom to the surface is evidently

$$m = \int h\rho\, d\Omega, \qquad \qquad (26)$$

the integral extending over all the changes in Ω which occur during the complete cycle. Evidently the total energy communicated by the moon's attraction during two periods is

$$E_M = \iint m\, d\sigma, \qquad \qquad (27)$$

$d\sigma$ being an element of surface, and the integral extending over the whole surface of the Irish Sea included between the two sections AB and RC.

The potential of the moon's attraction on the sea is represented by the function

$$\Omega = \tfrac{3}{2} \gamma \frac{MR^2}{D_m^{\ 3}} (\tfrac{1}{3} - \cos^2 \vartheta)^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where

γ is the constant of gravitation.

M is the mass of the moon.

D_m is the radius of the moon's orbit.

R is the earth's radius.

ϑ is the angle between the line joining the centre of the earth to the moon, and the radius of the earth which passes through the point on the earth's surface which is being considered.

If λ be the latitude of the place (*i.e.*, 52° in the case of the Irish Sea), and if ϕ be the angle through which the earth has turned relative to the radius vector to the moon, since the moon was on the meridian, then by spherical trigonometry,

$$\cos \vartheta = \cos \lambda \cos \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Also, if 2H be the range of tide at the place which is being considered,

$$h = H \cos 2 (\phi + \phi_0), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where ϕ_0 is the phase of the tide at the time when the moon crosses the meridian.

Combining (26), (28), (29), (30), it will be seen that

$$m = -\rho H \tfrac{3}{2} \gamma \frac{MR^2}{D_m^{\ 3}} \cos^2 \lambda \int_0^{2\pi} \cos 2 (\phi + \phi_0)\, d (\cos^2 \phi)$$

$$= -\tfrac{3}{2} \pi \rho H \gamma \frac{MR^2}{D_m^{\ 3}} \sin^2 \phi_0 (\cos^2 \lambda). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

(31) may be written

$$m = -\tfrac{3}{2} \pi \rho H \cos^2 \lambda \sin^2 \phi_0 \left(\frac{\gamma E}{R^2}\right) \left(\frac{M}{E}\right) \left(\frac{R^3}{D_m^{\ 3}}\right) R,$$

where E is the mass of the earth.

$\frac{\gamma E}{R^2}$ is the attraction of the earth at its surface, *i.e.*,

$$\frac{\gamma E}{R^2} = g = 981 \text{ in C.G.S. units.}$$

* See LAMB'S 'Hydrodynamics,' p. 339 (1906 edition).

$\frac{M}{E}$, the ratio of the masses of the moon and the earth, is $\frac{1}{81}$; $\frac{R}{D_m}$, the ratio of the radius of the earth to the radius of the moon's orbit, is $\frac{1}{60}$.

$$\cos^2\lambda = 0{\cdot}38,$$
$$\rho = 1{\cdot}03,$$
$$R = 6{\cdot}4 \times 10^8 \text{ cm.}$$

Hence

$$m = -\tfrac{3}{2} \times \pi \times 1{\cdot}03 \times 0{\cdot}38 \times 981 \times \tfrac{1}{81} \times (\tfrac{1}{60})^3 \times 6{\cdot}4 \times 10^8 \times H \sin^2\phi_0$$
$$= -6{\cdot}6 \times 10^4 H \sin^2\phi_0 \text{ ergs.} \qquad (32)$$

The mean rate at which energy is communicated by lunar attraction is found by dividing this by $8{\cdot}7 \times 10^4$, the number of seconds in two semi-diurnal periods, *i.e.*, in 24h. 50m.

Hence w_M, the mean rate at which work is done by the moon's attraction on each square centimetre of the Irish Sea, is

$$w_M = \frac{-6{\cdot}6 \times 10^4}{8{\cdot}7 \times 10^4} \times (\text{average value of } H \sin^2\phi_0 \text{ over the Irish Sea}).$$

Now the mean value of 2H, the rise and fall of tide in the Irish Sea, is about 14 feet or 420 cm. Hence H may be taken as 210 cm. The average time of H.W. is about $1\frac{1}{2}$h. before the moon's meridian passage. Hence $\phi_0 = +22\frac{1}{2}°$ and $\sin^2\phi_0 = 0{\cdot}7$.

A rough approximation to the average value of $H \sin^2\phi_0$ is therefore $210 \times 0{\cdot}7 = 150$ cm.; hence

$$w_M = \frac{-6{\cdot}6}{8{\cdot}7} \times 150 = -110 \text{ ergs per square centimetre per second.} \qquad (33)$$

It will be noticed that since it is H.W. shortly before the moon's meridian passage, work is done by the tides in the Irish Sea on the moon. This is indicated by the negative sign in (33).

Dissipation of Energy in the Irish Sea.

We have now seen that the rate at which energy flows into the Irish Sea through the North and South Channels is $6{\cdot}4 \times 10^{17}$ ergs per second.

The area of the Irish Sea between the sections AB and RC is 11,600 square nautical miles $= 3{\cdot}9 \times 10^{14}$ sq. cm. Hence energy enters the Irish Sea through the Channels at a rate of $\frac{6{\cdot}4 \times 10^{17}}{3{\cdot}9 \times 10^{14}} = 1640$ ergs per second per square centimetre of its area. Of this energy we have just seen (see equation 33) that 110 ergs per square centimetre per second are used in doing work against the moon's attraction. The remainder $1640-110 = 1530$ ergs per second are dissipated by tidal friction.

It will be remembered that the estimates previously given from a direct consideration of skin friction were 1040* and 1300† ergs per square centimetre per second.

It will be seen that the agreement between the two methods of estimating the energy dissipation due to tidal friction is quite remarkable. The conclusion appears inevitable that the dissipation of energy by tidal friction is very much larger than has previously been supposed.

Proportion of the Tidal Wave which is Absorbed in the Irish Sea.

The large amount of tidal energy which these calculations show to be absorbed in the Irish Sea naturally gives rise to speculations as to how large a proportion of the energy of the tidal wave is absorbed, and how much of it is reflected back to the Atlantic.

It has generally been believed that tidal friction plays only a very small part in tidal phenomena. Reasoning on the lines of the present work, however, the mere fact that it is possible to find a section where the rise and fall of tide is appreciable and where the phases of the tidal current and tidal height are the same, is a proof that an appreciable proportion of the energy of the tidal wave entering through the South Channel is absorbed. We cannot measure the size of the tidal wave which enters the Irish Sea directly, because at all points the effects of the entering and of the emerging wave will be felt simultaneously. It is necessary therefore to disentangle those effects. When this has been done it will be found that apparently complex tidal phenomena in the South Channel are really very simple.

It has often been pointed out that the Irish Sea behaves like a resonator with two open ends. The rise and fall of tide at the two open ends, which are "loops" in the oscillation, is small, while the current is a maximum at these points. In the middle of the Irish Sea in the neighbourhood of the Isle of Man, the currents are small while the rise and fall of tide is large. If the motion of the sea at the two channels is at all similar to a "loop" in a stationary oscillation, it may evidently be analysed into two waves, one going in and the other coming out.

We shall assume that the motion of the water in the South Channel is all backwards and forwards along the channel, as in fact the current observations show it to be. We have seen that the effect of the deflecting force due to the earth's rotation is to increase the tide on one side of the channel and to decrease it on the other, leaving the rise-and-fall of the central part of the channel unaffected. We shall deal first with the tidal phenomena which do not depend on this deflecting force, by considering the motion in the central part of the channel.

* See equation (10).

† See equation (9).

Let us assume that the tidal phenomena in the South Channel can be represented by the superposition of two waves, one of amplitude a going in, and the other of amplitude b, going out. These may be represented mathematically by the formula

$$h = a\cos\frac{2\pi}{\mathrm{T}}\left(t-\frac{x}{c}\right)-b\cos\frac{2\pi}{\mathrm{T}}\left(t+\frac{x}{c}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where the first term represents the wave entering the channel, and the second represents the reflected wave leaving the channel. $2a$ and $2b$ are the ranges of the tides due to the two waves separately; x is the distance measured along the channel in the direction of the Irish Sea; and c is the velocity of a long wave in water of the depth, D, of the channel so that $c = \sqrt{g\mathrm{D}}$.*

Our problem is to analyse the observed tidal phenomena so as to find the values of a and b, and to show that the various characteristic features of the tidal phenomena of the South Channel can be accounted for by considering them as being due to these two waves.

The current due to the entering tidal wave is $a\sqrt{\frac{g}{\mathrm{D}}}\cos\frac{2\pi}{\mathrm{T}}\left(t-\frac{x}{c}\right)$.† The current due to the out-going tidal wave is $b\sqrt{\frac{g}{\mathrm{D}}}\cos\frac{2\pi}{\mathrm{T}}\left(t+\frac{x}{c}\right)$.

They are both positive at $x = 0$ if a and b are both positive, because the original formula (34) assumed for h, gives h as the difference of two terms and not as the sum.

Hence the tidal current, v, is

$$v = a\sqrt{\frac{g}{\mathrm{D}}}\cos\frac{2\pi}{\mathrm{T}}\left(t-\frac{x}{c}\right)+b\sqrt{\frac{g}{\mathrm{D}}}\cos\frac{2\pi}{\mathrm{T}}\left(t+\frac{x}{c}\right).$$

* It has been suggested that the velocity of the waves into and out of the Irish Sea are not equal to $\sqrt{g\mathrm{D}}$, because they are forced waves due to the moon. The moon's attraction, however, does not appear to be capable of exerting sufficient force to alter appreciably the velocity of a free wave of the amplitude with which we are concerned travelling down a channel of a depth of about 37 fathoms.

If f be the horizontal component of the moon's attraction, the maximum possible value of f is $8\cdot57\times10^{-8}g$ (see LAMB'S 'Hydrodynamics,' 4th edition, p. 256).

The maximum value of the horizontal force F, due to the pressure gradient in a free wave of height $2a$ from trough to crest, is $\mathrm{F} = \frac{2\pi}{\sqrt{g\mathrm{D}}\mathrm{T}}$, where T is the tidal period of 12·4h.

It will be seen later that the semi-amplitude of the smaller of the two waves with which we are concerned, *i.e.*, the out-going wave, is 145 cm. Taking $a = 145$, D = 37 fathoms = 6800 cm., it will be found that $\mathrm{F} = 8\times10^{-6}g$.

It appears therefore that f, the horizontal force due to the moon's attraction, is only $\frac{1}{100}$th of F, the force due to the horizontal pressure gradient in a free wave of the height with which we are concerned. The velocity of these waves cannot therefore differ appreciably from that of free waves in the channel.

† This is the well-known formula connecting the current velocity and tidal range in a tidal wave.

The tidal current is a maximum when $x = 0$ and $t = 0$. Its value is then

$$V = (a+b)\sqrt{\frac{g}{D}}. \qquad (35)$$

At the point $x = 0$ the phases of the current and of the height of the tide are the same. In applying these equations to the channels of the Irish Sea, we must choose as origin, $x = 0$, the place where the height of the tide and the current have the same phase. As we have already seen in the South Channel, this point must be very close to the section AB from Arklow Bank to Bårdsey Island, for the phase difference in that region is only 10m. of time. It is hardly worth while in the present investigation to take account of so small a difference as 10m.

At the point $x = 0$, the range of tide, which we have called $2H_1$, is evidently $2(a-b)$. Now we know the values of V, g and D; we can therefore calculate $a+b = V\sqrt{\frac{D}{g}}$ from (35), a and b can therefore be found separately.

Using the values already given for the section AB,

$$H_1 = \tfrac{1}{2} \text{ tidal range} = 4\tfrac{3}{4} \text{ feet} = 145 \text{ cm.}$$

$$D = 37 \text{ fathoms} = 6800 \text{ cm.}$$

$$g = 981.$$

$$V = 3\cdot2 \text{ knots} = 163 \text{ cm. per second,}$$

it will be seen that

$$a+b = 163\sqrt{\frac{6800}{981}} = 430 \text{ cm.} \qquad (36)$$

and

$$a-b = H_1 = 145 \text{ cm.} \qquad (37)$$

Hence, solving (36) and (37) we get for the semi-amplitudes of the in- and outgoing tidal waves,

$$\left.\begin{aligned} a &= 287 \text{ cm.} \\ b &= 143 \text{ cm.} \end{aligned}\right\}$$

and

$$\frac{a}{b} = 2\cdot0. \qquad (38)$$

It appears, therefore, that at spring tides, the tidal wave is reduced almost exactly to half its amplitude during its passage into and out of the Irish Sea. The wave which comes out of the Irish Sea therefore contains only quarter of the energy of the wave which goes in.

The result just obtained does not appear to be open to any theoretical objection, but it is opposed to the generally accepted view that tidal friction has very little effect on the regime of the tides. It is worth while, therefore, to try and confirm it in some other way. With this end in view, we shall discuss the movement of the co-tidal lines in the South Channel.

First, let us consider the theoretical aspects of the case. A co-tidal line is a line at all points of which it is H.W. at the same time. In a progressive wave the co-tidal lines are the positions of the crest of the wave at a series of successive times. The distance apart of the co-tidal lines corresponding with a series of times, separated by intervals of 1h., will be a measure of the velocity of the wave. In drawing a map of co-tidal lines, therefore, one is apt to think that they represent the successive stages of advancement of a progressive tidal wave. This idea is incorrect. In the case of two superposed waves moving in opposite directions, for instance, it will be found that the co-tidal line moves in the same direction as that one of the two waves which has the greater amplitude; but that it does not move at the same speed.

In certain places the line moves faster than the wave, while in others it moves more slowly, and a knowledge of the relationship between the velocity of the co-tidal line and the velocity of the wave will enable us to determine the ratio of the amplitudes of the two waves.

The height of the tide above mean sea-level at any time is

$$h = a \cos\frac{2\pi}{\mathrm{T}}\left(t-\frac{x}{c}\right)-b \cos \frac{2\pi}{\mathrm{T}}\left(t+\frac{x}{c}\right).$$

This may be written in the form

$$h = \mathrm{A} \cos \frac{2\pi}{\mathrm{T}}(t-t_x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

where

$$\mathrm{A} = \sqrt{a^2+b^2-2ab \cos \frac{4\pi x}{c\mathrm{T}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

and

$$\cot \frac{2\pi t_x}{\mathrm{T}} = \frac{a-b}{a+b} \cot \frac{2\pi x}{c\mathrm{T}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

The co-tidal line, therefore, moves in time t_x, through the distance x, from the place where the phases of current and tide are the same, x and t_x are related by the equation (41).

The velocity, V_c, of the co-tidal line is therefore obtained by differentiating (41)

$$\mathrm{V}_c = \frac{dx}{dt_x} = c\,\frac{\left(\frac{a-b}{a+b}\right)^2 \cot^2\frac{2\pi x}{c\mathrm{T}}+1}{\cot^2\frac{2\pi x}{c\mathrm{T}}+1}\left(\frac{a+b}{a-b}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

At the point $x = 0$ where the tidal heights of the two waves oppose, and the tidal streams concur, the velocity of the co-tidal line is therefore a fraction $\frac{a-b}{a+b}$ of the velocity of the tidal wave. On examining the data at our disposal it will be found that they are hardly sufficient to place the co-tidal lines for 6, 7, 8, 9 and 10 hours sufficiently accurately on the map to make an accurate determination of the velocity of the co-tidal line in the neighbourhood of the line AB.

On looking at maps of co-tidal lines for the Irish Sea, however, such as that given in KRUMMEL'S 'Ozeanographie,'* it will be seen that the co-tidal lines for successive hours are crowded together in the neighbourhood of the Arklow-Bardsey line. V_c is therefore a minimum in that region, as we should expect from (42).

Though we cannot measure accurately the velocity of the co-tidal line as it passes the Arklow-Bardsey section, there are two sections of the channel where the positions of single co-tidal lines can be determined with considerable accuracy. We can therefore determine the mean velocity of the co-tidal line between these two sections and can compare this with theory. The line AB, which is practically a co-tidal line for 8h. 10m. is one example. Bardsey Island at its eastern end is separated from the mainland, so that the error due to a shelving shore is lessened. The time of H.W. at Courtown, a few miles south of the point where the western end of AB strikes the Irish coast, is 8h., while the time of H.W. at Arklow Bank and Arklow, both a little north of AB, and therefore a little later in their tides, is 8h. 25m. Greenwich mean time.

The co-tidal line for 8h. 10m. is therefore well determined and is practically coincident with AB. As before we shall take it as being coincident with $x = 0$.

The other co-tidal line referred to is the one for 6h. 15m. It is H.W. at Carnsore Point at 6h. 25m., and at Tuskar Rock, $4\frac{3}{4}$ miles off the Irish coast, at 6h. 10m., while the other end of the line is determined by Ramsey Sound, off the Welsh coast, where it is H.W. at 6h. 21m. The co-tidal line for 6h. 15m. is shown as the line TS in the map, fig. (3).

Let us then apply the formula (41) to find the ratio of the amplitudes of the two tidal waves. The distance between the mid points, M and L of the two co-tidal lines AB and TS, is about 43 nautical miles, so that in (41) $x = -43$ miles. The mean depth of the water between the two sections is 45 fathoms, and the velocity of a long wave in water of this depth is 56 nautical miles per hour. Remembering that T, the period of the semi-diurnal tide is 12h. 25m. or 12·4h., it will be found that

$$\cot\frac{2\pi x}{c\mathrm{T}} = \cot\frac{2\pi(-43)}{56\times 12{\cdot}4} = \cot(-22{\cdot}3^\circ) = -2{\cdot}44. \quad . \quad . \quad . \quad . \quad (43)$$

Also $t_x = 6\text{h. }15\text{m.} - 8\text{h. }10\text{m.} = -1{\cdot}92\text{h.}$ and

$$\cot\frac{2\pi t_x}{\mathrm{T}} = \cot\frac{2\pi(-1{\cdot}92)}{12{\cdot}4} = \cot(-56^\circ) = -0{\cdot}67. \quad . \quad . \quad . \quad . \quad (44)$$

* 'Handbuch der Ozeanographie,' vol. 2, p. 336 (1911 edition).

Hence from (41)

$$\frac{a-b}{a+b}=\frac{\cot\dfrac{2x\pi}{c\mathrm{T}}}{\cot\dfrac{2\pi t_x}{\mathrm{T}}}=\frac{2\cdot44}{0\cdot67},$$

so that

$$\frac{a}{b}=\frac{2\cdot44+0\cdot67}{2\cdot44-0\cdot67}=1\cdot8. \qquad (45)$$

The agreement between this and the previous result $\frac{a}{b}=2\cdot0$,* is remarkable, because they are based on quite different data.

At this point it is worth while to look back at what we have done. We began by assuming that the tides in the South Channel of the Irish Sea can be represented by two tidal waves moving in opposite directions and with the velocity appropriate to the depth of the channel, *i.e.* $\sqrt{g\mathrm{D}}$.

We then used two totally different methods for finding the ratio of the rise and fall of tide due to each of the two waves.

The first of these methods depends on the relationship between the tidal currents, the depth and the rise and fall of tide across the section where the tidal current and tidal height are in the same phase.

The tidal currents have been measured at four points across the section in question. The depths of course are well known and are marked on all charts. The height of the range of the tide has only been measured at points at, and near the ends of, the section. The fact that the tidal current moves backwards and forwards in a straight line and that it is practically uniform across the section, is however a strong reason for believing that the tidal range decreases uniformly from the Welsh to the Irish shore. It is worth pointing out, however, that the mean rise and fall of tide across the section is not a measured quantity; it is a deduction, based on dynamical conceptions it is true, but still a deduction, from the measured amounts of the rise and fall of tide at each end of the section, and the measured tidal currents across it. The ratio of the amplitudes of the in- and out-going tidal waves was found by this method to be 2·0.

The second method of determining the ratio of the amplitude of the in-going tidal wave to the out-going wave depends on the ratio of the rate of movement of the co-tidal line to the velocity of the tidal wave. It was possible to get two well determined positions of the co-tidal line, one at each end of the South Channel. From measurements of the distance between the mid points of these two lines, and the interval of time between the two H.W's., the ratio of the amplitudes of the two waves in mid channel was found to be 1·8, almost exactly the same result as that obtained by the other method. It is worth noticing that this second method

* See equation (38).

involves only measurements of depth and time of H.W. It does not involve measurements of current or tidal range at all.

The remarkable agreement between these two methods of estimating the loss of energy in the tidal wave during its passage into and out of the Irish Sea is strong evidence that three-quarters of the energy of the tidal wave entering by the South Channel is dissipated in the Irish Sea. The main purpose of this paper is therefore accomplished.

To complete the investigation, however, it is worth while to show that certain other tidal phenomena of the South Channel, hitherto apparently not very well understood, are simple consequences of the superposition of two tidal waves of different amplitudes moving in opposite directions.

The first of these is the difference between the velocities of the co-tidal line on the two sides of the channel. This difference is very marked. The distance from Tuskar Rock to the south end of Arklow Bank, traversed by the co-tidal line up the Irish Coast during the interval between 6h. 15m. and 8h. 10m. is only 30 nautical miles, while the distance from Ramsey Island to Bardsey Island traversed by the co-tidal line in the same interval on the Welsh Coast is 59 miles, so that the velocity of the co-tidal line on one side of the channel is about double its velocity on the other. This difference evidently causes the co-tidal line to turn through a large angle, independently of any turning which the wave fronts themselves may experience in passing through the channel owing to a difference between the depths on the two sides.

In the case of the South Channel the depths are practically equal on the two sides, so the fronts of the waves will not turn, though of course they may be very slightly convex, owing to the greater depth in mid-channel.

The direction of the line AB from South Arklow to Bardsey is N. 86° W. The direction of the line TS from Tuskar to Ramsey is S. 57½° W. The angle turned through by the co-tidal line from 6h. 15m. to 8h. 10m. is therefore N. 86° W. *minus* S. 57° W. = 36½°.

As a matter of fact the angle turned through by the co-tidal line is, if anything, rather greater than this, because the true co-tidal line for 8h. 10m. must be a little north of Bardsey and south of South Arklow, while the true co-tidal line for 6h. 15m. must run from a point slightly north of Tuskar Rock to a point slightly south of Ramsey Island. The true angle between the co-tidal lines for 6h. 15m. and 8h. 10m. is therefore slightly greater than 36½°.

Now let us turn to the explanation. It has been pointed out already (see p. 9) that the "geostrophic" or deflecting force due to the earth's rotation increases the rise and fall of tide on the side of the channel which lies on the right-hand side of an observer who faces in the direction in which the tidal stream is running at H.W. In the case of a progressive tidal wave the current at high water is moving in the direction in which the wave is travelling.

The amplitude due to the in-going wave is therefore greater on the Welsh Coast than on the Irish Coast. In the case of the out-going wave, the right-hand side of the channel is the Irish Coast. The amplitude due to the out-going wave is therefore greater on the Irish Coast. The result of this is that the ratio of the amplitudes of the two waves is very much greater on the Welsh Coast than it is on the Irish side of the channel. The consequence is that the rate of travel of the co-tidal line near the point where the two waves oppose, *i.e.*, near the section AB, is much less on the Irish Coast than it is on the Welsh Coast.

This explanation can be verified quantitatively. Let y be the distance of any point from the central line of the South Channel, *i.e.*, from the line LM (fig. 3) joining the mid points of the two sections AB and TS. x and y are then co-ordinates of any point in the South Channel.

Since the tidal currents in the South Channel flow straight backwards and forwards without any appreciable circulatory motion, the increase in the height of the tidal oscillation on the right-hand side of the advancing wave can be expressed approximately in the form

$$h = a\left(1 - 2\frac{\omega y}{c}\sin\lambda\right)\cos\frac{2\pi}{\mathrm{T}}\left(t - \frac{x}{c}\right),*$$

while the out-going wave is

$$b\left(1 + \frac{2\omega y\sin\lambda}{c}\right)\cos\frac{2\pi}{\mathrm{T}}\left(t + \frac{x}{c}\right).$$

The height of the tide at the point (x, y) and at time, t, is therefore given by

$$h = a\left(1 - \frac{2\omega y\sin\lambda}{c}\right)\cos\frac{2\pi}{\mathrm{T}}\left(t - \frac{x}{c}\right) - b\left(1 + \frac{2\omega y\sin\lambda}{c}\right)\cos\frac{2\pi}{\mathrm{T}}\left(t + \frac{x}{c}\right).$$

It is evident that all the analysis given above respecting the rate of travel of the co-tidal line still applies for any fixed value of y provided we use $a\left(1 - \frac{2\omega y\sin\lambda}{c}\right)$ instead of a, and $b\left(1 + \frac{2\omega y\sin\lambda}{c}\right)$ instead of b. The actual values of a and b which we found apply to the middle line, $y = 0$.

Denoting $a\left(1 - \frac{2\omega y\sin\lambda}{c}\right)$ by h_1, and $b\left(1 + \frac{2\omega y\sin\lambda}{c}\right)$ by h_2, where h_1 and h_2 are functions of y, the equation for the co-tidal line for 6h. 15m. is evidently found from (41) by replacing a and b by h_1 and h_2. The equation in question is

$$\cot\frac{2\pi t_x}{\mathrm{T}} = \frac{h_1 - h_2}{h_1 + h_2}\cot\frac{2\pi x}{c\mathrm{T}}. \qquad (46)$$

where t_x is constant but h_1, h_2 and x vary.

* This is evident from the analysis given on p. 10, but reference may also be given to LAMB'S 'Hydrodynamics,' p. 304, 1906 edition, where the expression $\zeta = ae^{-\frac{2\omega y}{c}}\cos\mathrm{K}(cb - x)$ occurs for the height of a long wave in a long rotating canal. The above expression is an obvious modification of this.

For this analysis to be correct, the co-tidal line at $x = 0$, *i.e.*, the line for 8h. 10m. should be perpendicular to the direction in which the wave has been assumed to be moving, *i.e.*, perpendicular to the middle line of the channel. As a matter of fact the angle between the co-tidal line AB, and the central line LM, differs considerably from a right angle. This is no doubt due partly to modifications introduced by the fact that the channel has not got parallel sides, but more probably it is due to the fact that the tidal wave from the Atlantic does not strike the channel in such a way as to allow the co-tidal line for 6h. 15m. to be at such an angle with the direction of the middle line of the channel as to allow it to become perpendicular to the channel (owing to the co-tidal line travelling faster on the Welsh side than on the Irish side) when it has travelled up the channel as far as the line AB.

It is worth while, however, to apply equation (46) to find out what angle the co-tidal line would have turned through, theoretically, in the time from 6h. 15m. to 8h. 10m.

The angle, θ, between the co-tidal line for 6h. 15m. and the co-tidal line for 8h. 10m. should be given by

$$\tan\theta = \frac{dx}{dy} \qquad (47)$$

where $\frac{dx}{dy}$ is obtained by differentiating (46).

Turning now to the figures, we have seen (see equation 44) that

$$\cot\frac{2\pi t_x}{\mathrm{T}} = -0{\cdot}67.$$

Hence (46) becomes

$$\cot\frac{2\pi x}{c\mathrm{T}} = -0{\cdot}67\,\frac{\dfrac{h_1}{h_2}+1}{\dfrac{h_1}{h_2}-1} \qquad (48)$$

But

$$\frac{h_1}{h_2} = \frac{a\left(1-\dfrac{2\omega y\sin\lambda}{c}\right)}{b\left(1+\dfrac{2\omega y\sin\lambda}{c}\right)} \qquad (49)$$

and if we limit ourselves to the consideration of the angle through which the co-tidal line turns during its passage up the central part of the channel, *i.e.*, up the line ML, y may be considered as small. In this case (49) may be written approximately

$$\frac{h_1}{h_2} = \frac{a}{b}\left(1-\frac{4\omega y\sin\lambda}{c}\right). \qquad (50)$$

Differentiating (48) it will be found that

$$-\frac{2\pi}{cT}\operatorname{cosec}^2\frac{2\pi x}{cT}\frac{dx}{d\left(\frac{h_1}{h_2}\right)} = \frac{0{\cdot}67\times 2}{\left(\frac{h_1}{h_2}-1\right)^2} \qquad (51)$$

Since y is small, h_1 and h_2 may be taken as a and b in places where they are not being differentiated. Hence, since $a/b = 1{\cdot}8$*, the right-hand side of (51) is equal to

$$\frac{0{\cdot}67\times 2}{(1{\cdot}8-1)^2} = 2{\cdot}1, \text{ and (51) becomes } \frac{dx}{d\left(\frac{h_1}{h_2}\right)} = -\frac{2{\cdot}1cT}{2\pi\operatorname{cosec}^2\frac{2\pi x}{cT}} \qquad (52)$$

Differentiating (50)

$$\frac{d}{dy}\left(\frac{h_1}{h_2}\right) = -\frac{a}{b}\frac{4\omega\sin\lambda}{c} \qquad (53)$$

Combining (52) and (53) with (47)

$$\tan\theta = \frac{dx}{dy} = \frac{2{\cdot}1cT}{2\pi\operatorname{cosec}^2\frac{2\pi x}{cT}}\times\frac{a}{b}\frac{4\omega\sin\lambda}{c}\Big) \qquad (54)$$

In (54), T = 12·4h., and from (43),

$$\cot\frac{2\pi x}{cT} = -2{\cdot}44,$$

hence

$$\operatorname{cosec}^2\frac{2\pi x}{cT} = 1 + (2{\cdot}44)_2 = 7,$$

also

$$\frac{a}{b} = 1{\cdot}8.$$

ω, the angular velocity of the earth in radians per hour is $\frac{2\pi}{24}$, and $\sin\lambda = 0{\cdot}79$.

Hence

$$\tan\theta = \frac{2{\cdot}1\times 12{\cdot}4\times 1{\cdot}8\times 4}{7\times 2\pi}\times\frac{2\pi}{24}\times 0{\cdot}79 = 0{\cdot}88$$

and

$$\theta = 41^\circ.$$

The agreement between this and the measured angle, $36\frac{1}{2}^\circ$, is quite as good as could be expected.

* See equation (45).

Effect of the Shape of the Coast in Determining the Time of H.W. at Points on the Coast.

This is too complicated a matter to treat quantitatively, there is, however, one point in connection with local peculiarities of the tides of the South Channel of the Irish Sea which can be explained qualitatively by the analysis contained in this paper, and that is the effect of a point of land projecting into the channel in altering the times of H.W. on the two sides of it.

Consider the effect of a point of land on the range of tide due to a tidal wave passing along a channel. Let AB, fig. (4), represent a portion of one side of a channel and let CDE be a point of land projecting into it.

Suppose the tidal wave moves along in the direction from A to B. At H.W. the tidal stream is moving from A to B, it might be expected therefore that, owing to the piling up of the water on side facing the current, the level of the water would be higher at C than at E at H.W., that is to say, the range of tide should be greater

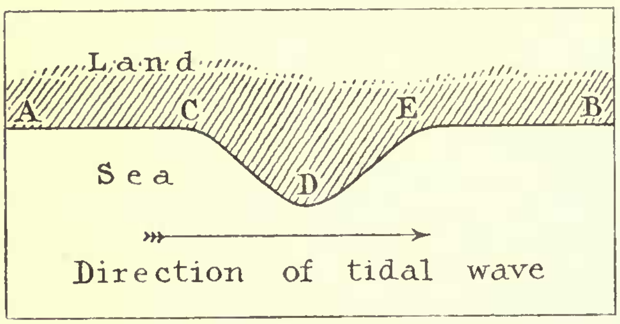

Fig. 4. Effect of a cape on times of H.W. on either side of it.

on the side of the cape which faces the direction from which the tidal wave comes than it would be on the side which faces away from the direction of the tidal wave. This effect does not materially alter the time of H.W. when there is only one tidal wave in the channel. When, however, there are two nearly equal waves, one going up and the other going down, the case is altered.

The time t_x of H.W. at distance x is given by

$$\cot \frac{2\pi t_x}{T} = \frac{a-b}{a+b} \cot \frac{2\pi x}{cT} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Suppose now, that, without altering x, a is decreased while b is increased by the action of some local peculiarity* as it is on the side DE of the cape (fig. 4).

If x is positive and $\frac{2\pi x}{cT} < \frac{\pi}{2}$ then equation (41) shows that a decrease in a and an increase in b will lead to an increase in t_x, the time of H.W.

* This increase must not be so great as to reverse the tides.

Similarly on the down-channel side, CD, of the cape, the time of H.W. will be made earlier by this local peculiarity.

When x is negative and when $-\frac{2\pi x}{c\mathrm{T}} < \frac{\pi}{2}$ a decrease in a and an increase in b leads to an increase in $-t_x$, *i.e.*, to a decrease in the time of H.W. Similarly, on the down-channel face of the cape, H.W. is made late by the cape.

There are two interesting examples of this on the S.E. coast of Ireland. One is at Wicklow Head. This is situated in the region where x is positive. We should, therefore, expect it to be H.W. later on the northern side of the cape than on the southern side. It is H.W. at Wicklow, a few miles N. of the cape at 10·53, 2h. 30m. later than H.W. at Arklow, some 11 miles south of the cape. This effect evidently appears to make the co-tidal line travel very slowly past Wicklow Head.

The other example is that of Greenore Point. In this case x is negative, we should therefore expect the effect of the coast line to be to make the time of H.W. earlier on the north side of the cape than on the southern side.

This effect might, if it were sufficiently great, reverse the direction of travel of the co-tidal line in the neighbourhood of the point. As a matter of fact the effect is great enough to do this. It is H.W. at Saltees, some 10 miles S.W. of Carnsore Point, at 6h. 6m. At Carnsore Point, 4 miles S. of Greenore Point, it is H.W. at 6h. 25m. At Tuskar Rock, 4 miles out from Greenore Point, it is H.W. at 6h. 10m. At Wexford South Bay, on the north side of the Point, it is H.W. at 6h. 5m. After that the time of H.W. gets later as one goes further northwards up the coast. It will be seen that from Carnsore Point to Wexford South Bay, therefore, the direction of travel of the co-tidal line is just reversed. The fact that H.W. at Wexford South Bay, which is well round Greenore Point, is actually earlier than H.W. at Tuskar Rock which is south of Greenore Point, besides being 4 miles out at sea, is remarkable. I do not know whether any explanation has been offered before of how it is that the effect of a cape on the tidal phenomena in its neighbourhood is so very different in different parts of the sea.

Summary of Conclusions. (*Added October 24, 1919.*)

The rate of dissipation of energy at spring tides in the Irish Sea is calculated from the known formulæ for skin-friction of the wind on the ground and the friction of rivers on their beds. The results range from 1040 to 1300 ergs per square centimetre per second. The least of these is 150 times as great as Mr. STREET'S previous estimate of 7 ergs per square centimetre per second.

The rate at which energy flows into the Irish Sea is next calculated from the rise and fall of tide, the strength of the tidal current and their phase difference over two sections taken across the North and South Channels. The rate of dissipation of energy is found to be 1530 ergs per square centimetre per second. This is in good agreement with the previous result.

It is next shown that this absorption of energy is sufficient to reduce the amplitude of the in-coming wave to one-half, so that three-quarters of the energy of the in-coming tidal wave is absorbed.

This absorption of energy explains most of the chief characteristics of the tidal phenomena of the South Channel to the Irish Sea, the velocity of the co-tidal line, which is only about one-third of the velocity of the tidal wave, the angle through which the co-tidal line turns in passing up the channel and the effect of Carnsore Point and Wicklow Head on the times of H.W. to the north and south of them.

II. *Electromagnetic Integrals.*

By Sir G. GREENHILL, *F.R.S.*

(Received April 22,—Read May 9, 1918.)

THESE are the integrals, elliptic integrals (E.I.) for the most part, and of the first, second and third kind (I., II., III., E.I.) arising in the practical problem of the measurement and determination of the electrical units, for their regulation and definition by Act of Parliament in commercial use.

The experiments have been carried out with the ampère balance invented by VIRIAMU JONES, also with the Lorenz apparatus for measuring resistance ('Phil. Mag.,' 1889, 'Phil. Trans.,' 1891, 1913), constructed at the charge of Sir ANDREW NOBLE and the British Association, in use at the National Physical Laboratory (N.P.L.), Teddington.

A description of this current weigher, ampère balance, is given by AYRTON, MATHER and F. E. SMITH in the 'Phil. Trans.,' A, vol. 207, 1907, and the theory is developed, with a description of the accuracy of measurement obtainable, to be recorded in the Act of Parliament. Also of the Lorenz apparatus, by F. E. SMITH, in 'Phil. Trans.,' 1913.

Our object here is to revise and simplify the mathematical treatment required in these calculations and to present the theory in a form adapted for elementary instruction; at the same time, to reconcile the notation and results of the various writers, VIRIAMU JONES, G. M. MINCHIN, and others, and to standardise them in accordance with MAXWELL's 'Electricity and Magnetism' (E. and M.).

1. Starting then on § 703, E. and M., it can be shown that all the results required can be made to originate and grow out of MAXWELL's expression for M, mutual induction of two parallel circular currents on the same axis, in circles of radius $OB = a$ and $NP = A$, a distance $ON = b$ apart, as on fig. 1 and 2 (p. 38), and M is given in his notation by

$$(1) \qquad M = \int_0^{2\pi} 2\pi A a \cos\theta \frac{d\theta}{PQ}, \qquad PQ^2 = A^2 + 2Aa\cos\theta + a^2 + b^2, \qquad AOQ = \theta,$$

and M is expressible by the complete E.I., I. and II.

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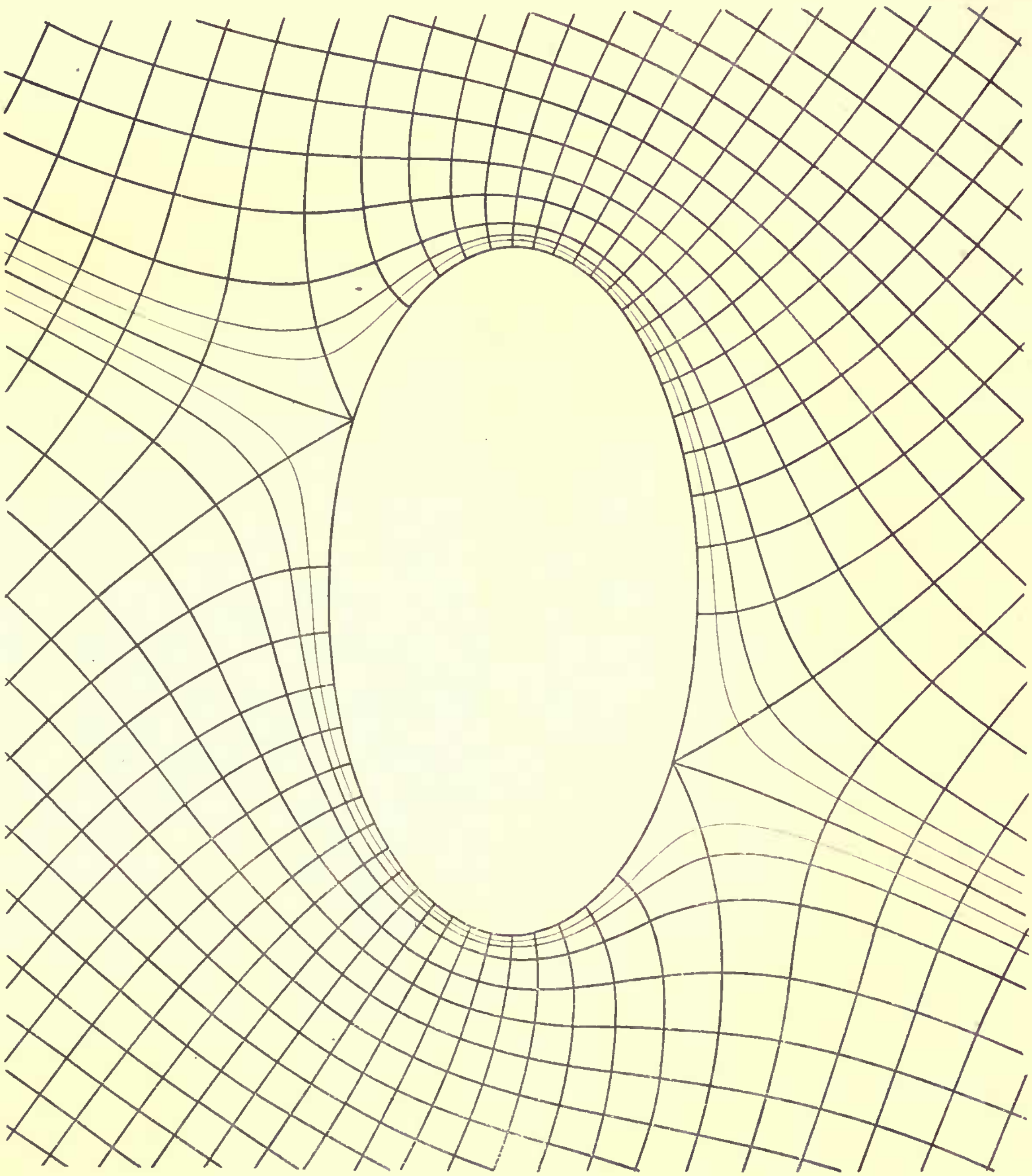

Fig. 3.

A re-drawing has been made and a lantern slide, of MAXWELL'S fig. XVIII, of the curves of constant M, or lines of magnetic force of the circular current round the circle AB perpendicular to the plane, on the diameter joining the foci, in which the confocal co-ordinates were employed of WEIR'S azimuth diagram.

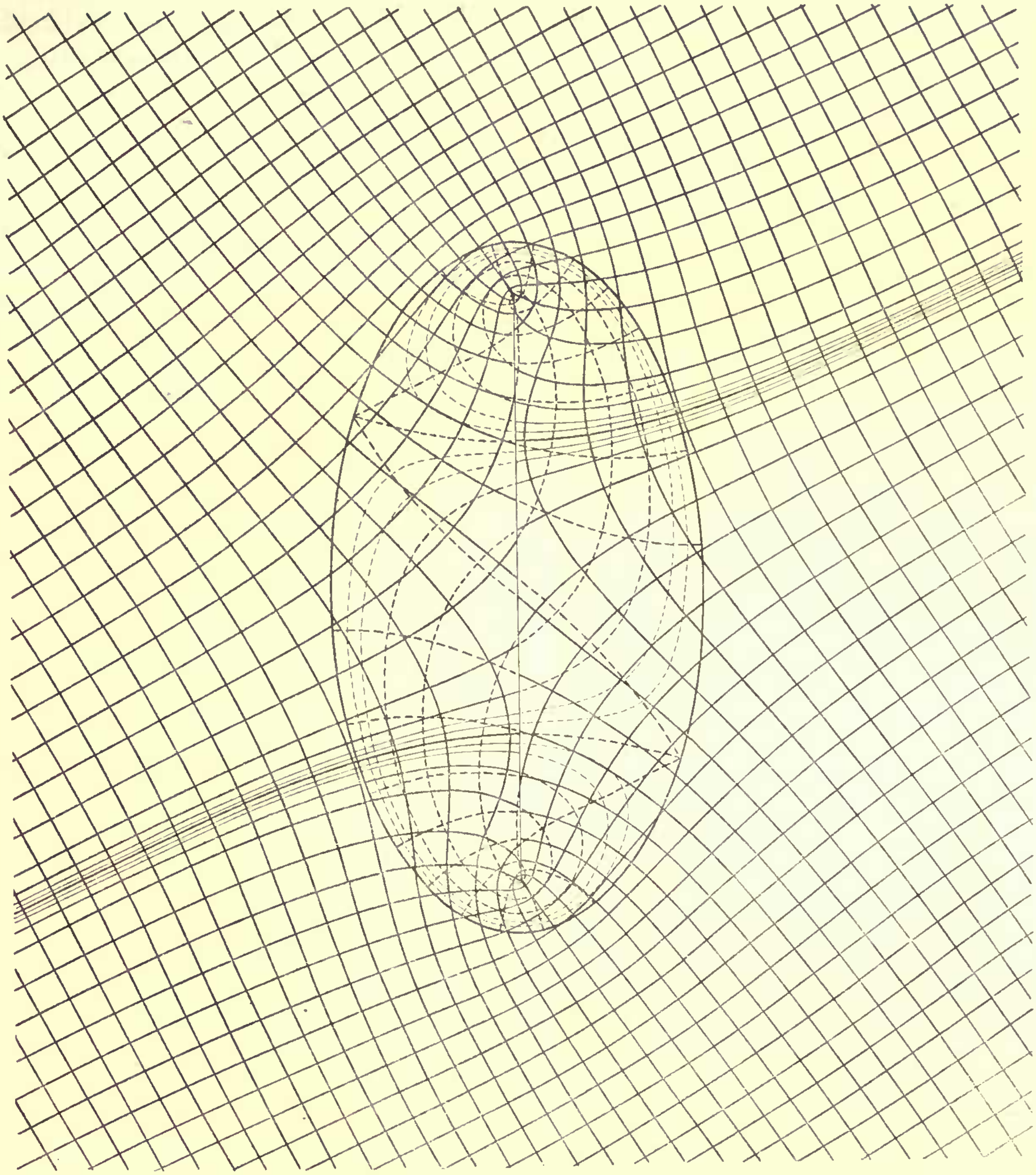

Fig. 4.

This was carried out by Mr. J. W. HICKS for comparison with MAXWELL's figure using the formulas given later in § 12.

The same confocal co-ordinates on the Weir chart were employed by Colonel HIPPISLEY in drawing fig. 3 and 4, and the lantern slides, showing the lines of uniplanar flow past an ellipse; but here the co-ordinate ruling has been rubbed out.

The conformal representation of the mapping connecting $z = x+iy$ and $\zeta = \eta+i\xi$ is given by $z = c \operatorname{ch} \zeta$; and then with $w = \phi+i\psi$, $\gamma = \beta+i\alpha$, the relation $w = \operatorname{ch}(\zeta-\gamma)$ gives the mapping of ϕ, ψ, the velocity and stream function of the motion, so that

$$(2) \qquad \phi = \operatorname{ch}(\eta-\alpha)\cos(\xi-\beta), \qquad \psi = \operatorname{sh}(\eta-\alpha)\sin(\xi-\beta).$$

Fig. 3 gives the ordinary stream motion of the distortion of a uniform current by an elliptic post, and should figure as the typical diagram in a treatise on Hydrodynamics. But fig. 4 is curious in showing the analytical prolongation of the functions

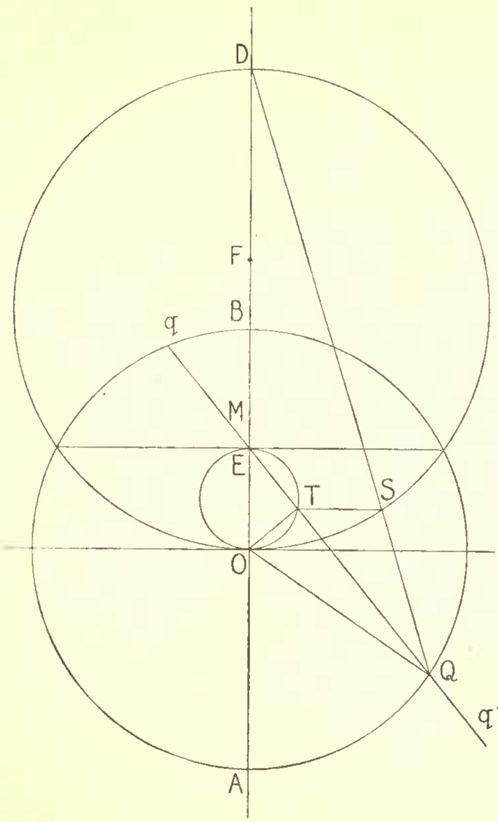

Fig. 1.

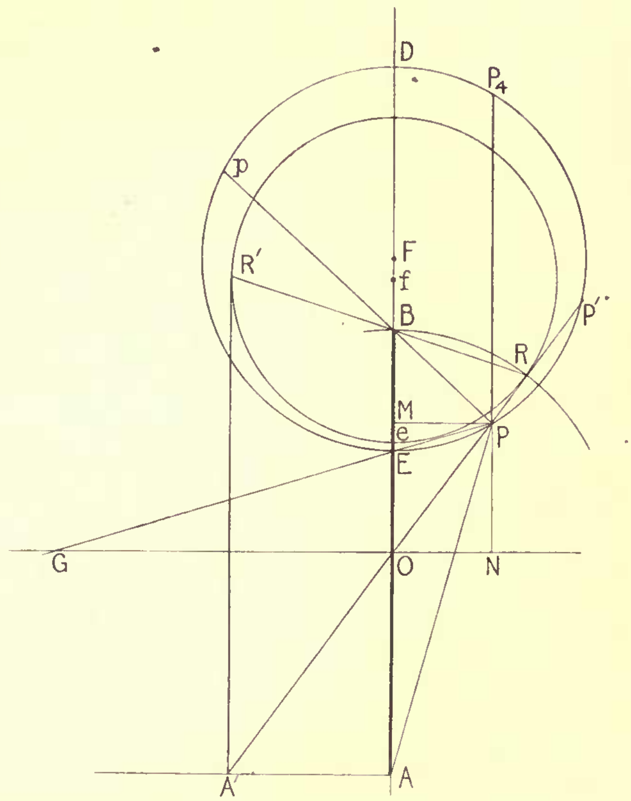

Fig. 2.

for $\eta < \alpha$ on the Riemann sheet, with the cut along the line SS′ joining the foci. It shows the middle stream coming to a waterfall across SS′ and circulating in a whirlpool chamber in the interior of the ellipse $\eta = \alpha$, and then emerging in another stream off to infinity.

The Weir chart would be ideal to employ in plotting some of the curves described by LEGENDRE (F.E., I., p. 411), orbits invented by EULER, 1760, under two centres of gravitation at S, S′, as the variables employed by EULER and LEGENDRE are $\tan \frac{1}{2}\xi$, $\operatorname{th} \frac{1}{2}\eta$, and these are separated in the equations of motion.

2. An integration of M with respect to b will lead immediately to V. JONES's expression for the mutual induction, L, between the circular current PP′ on the diameter 2A, and a uniform current sheet flowing round the cylinder on the diameter $2a$, stretching from the circle AB a length b, up to the circle PP′; drawing out the circle AQB axially, like a concertina.

The current sheet is taken as the equivalent of the close helical winding in the ampère balance of the wire on the cylinder carrying the electric current and forming a solenoidal magnet, of which a constant L gives a line of magnetic force, the one passing through P, these lines circulating through the solenoidal tube and closing again outside.

In the hydrodynamical analogue L would be the stream function (S.F.) of liquid circulating through the tube.

Employing the lemma of the integral calculus, for the line potential of MP at Q,

$$(1)\qquad \int \frac{db}{\mathrm{PQ}} = \mathrm{th}^{-1}\frac{b}{\mathrm{PQ}} = \mathrm{sh}^{-1}\frac{b}{\mathrm{MQ}} = \mathrm{ch}^{-1}\frac{\mathrm{PQ}}{\mathrm{MQ}}, \qquad \mathrm{MQ}^2 = \mathrm{A}^2+2\mathrm{A}a\cos\theta+a^2,$$

$$(2)\qquad \mathrm{L} = \int_0^b \mathrm{M}\,db = \int_0^b\int_0^{2\pi} 2\pi\mathrm{A}a\cos\theta\,\frac{d\theta\,db}{\mathrm{PQ}} = \int_0^{2\pi} 2\pi\mathrm{A}a\cos\theta\,\mathrm{th}^{-1}\frac{b}{\mathrm{PQ}}\,d\theta,$$

and integrating by parts, with the lemma of the differential calculus

$$(3)\qquad \frac{d}{d\theta}\,\mathrm{th}^{-1}\frac{b}{\mathrm{PQ}} = \frac{\mathrm{A}a\sin\theta}{\mathrm{MQ}^2}\cdot\frac{b}{\mathrm{PQ}},$$

$$(4)\qquad \mathrm{L} = 2\pi\mathrm{A}a\left(\sin\theta\,\mathrm{th}^{-1}\frac{b}{\mathrm{PQ}}\right)_0^{2\pi} - \int 2\pi\mathrm{A}a\sin\theta\,\frac{\mathrm{A}a\sin\theta}{\mathrm{MQ}^2}\cdot\frac{b\,d\theta}{\mathrm{PQ}}$$

$$= * - \int 2\pi\,\frac{\mathrm{A}^2a^2\sin^2\theta}{\mathrm{MQ}^2}\cdot\frac{b\,d\theta}{\mathrm{PQ}},$$

the $*$ marking a term which vanishes at the limits, and with

$$(5)\qquad 4\mathrm{A}^2a^2\sin^2\theta = 4\mathrm{A}^2a^2-(\mathrm{MQ}^2-a^2-\mathrm{A}^2)^2$$

$$= -\mathrm{MQ}^2+2(a^2+\mathrm{A}^2)\,\mathrm{MQ}^2-(a^2\mathrm{A}^2)^2,$$

$$(6)\qquad \mathrm{L} = \tfrac{1}{2}\pi\int\left[-\mathrm{MQ}^2+2(a^2+\mathrm{A}^2)-\frac{(a^2-\mathrm{A}^2)^2}{\mathrm{MQ}^2}\right]\frac{b\,d\theta}{\mathrm{PQ}}$$

$$= \tfrac{1}{2}\pi\int\left[a^2+\mathrm{A}^2-2\mathrm{A}a\cos\theta-\frac{(a^2-\mathrm{A}^2)^2}{\mathrm{MQ}^2}\right]\frac{b\,d\theta}{\mathrm{PQ}},$$

introducing the complete elliptic integral, I., II., III.; and this is the expression employed by V. JONES, but obtained by a complicated dissection.

3. In the reduction of these integrals to a standard form, for the purpose of a numerical calculation by use of the tables of LEGENDRE, it is convenient to put $\theta = 2\omega$, $\omega = \mathrm{ABQ}$, in fig. 1, and to introduce a new variable t, and constants τ, t_1, t_2, t_3, in accordance with the notation of WEIERSTRASS, such that, m denoting a homogeneity factor,

(1) $$\mathrm{PQ}^2 = r^2 = m^2(t_1-t), \qquad \mathrm{MQ}^2 = r^2-b^2 = m^2(\tau-t), \qquad \mathrm{PM}^2 = b^2 = m^2(t_1-\tau),$$
$$\mathrm{PA}^2 = r_3^2 = m^2(t_1-t_3), \qquad \mathrm{PB}^2 = r_2^2 = m^2(t_1-t_2),$$
$$\mathrm{MA}^2 = r_3^2-b^2 = (a+\mathrm{A})^2 = m^2(\tau-t_3), \qquad \mathrm{MB}^2 = r_2^2-b^2 = (a-\mathrm{A})^2 = m^2(\tau-t_2),$$
$$\mathrm{PA}^2-\mathrm{PQ}^2 = r_3^2-r^2 = 2\mathrm{A}a(1-\cos\theta) = 4\mathrm{A}a\sin^2\omega = m^2(t-t_3),$$
$$\mathrm{PQ}^2-\mathrm{PB}^2 = r^2-r_2^2 = 2\mathrm{A}a(1+\cos\theta) = 4\mathrm{A}a\cos^2\omega = m^2(t_2-t),$$
$$\mathrm{PA}^2-\mathrm{PB}^2 = r_3^2-r_2^2 = 4\mathrm{A}a = m^2(t_2-t_3),$$

(2) $$\infty > t_1 > \tau > t_2 > t > t_3 > \infty.$$

In the notation of LEGENDRE

(3) $$\mathrm{PQ}^2 = r^2 = r_3^2\cos^2\omega + r_2^2\sin^2\omega = r_3^2\Delta^2(\omega,\gamma), \qquad \gamma' = \frac{r_2}{r_3},$$

(4) $$\frac{d\theta}{\mathrm{PQ}} = \frac{2d\omega}{r_3\Delta\omega}, \qquad \mathrm{P} = \int_0^{2\pi}\frac{a\,d\theta}{\mathrm{PQ}} = \frac{2a}{r_3}\int_0^{\pi}\frac{d\omega}{\Delta\omega} = \frac{4a\mathrm{G}}{r_3},$$

and P is the rim potential of the circle on AB, with

(5) $$\mathrm{G} = \int_0^{\frac{1}{2}\pi}\frac{d\omega}{\Delta(\omega,\gamma)}, \qquad \mathrm{P} = \frac{4a}{\sqrt{(r_2 r_3)}}\mathrm{G}\sqrt{\gamma'},$$

(6) $$\mathrm{Q} = \int_0^{2\pi}\frac{-a\cos\theta\,d\theta}{\mathrm{PQ}} = \int_0^{\pi}(2\sin^2\omega-1)\frac{2a\,d\omega}{r_3\Delta\omega}$$
$$= \frac{4a}{\gamma^2 r_3}\int_0^{\frac{1}{2}\pi}\left[2(1-\Delta^2\omega)-\gamma^2\right]\frac{d\omega}{\Delta\omega}$$
$$= \frac{4a}{\gamma^2 r_3}\left[(2-\gamma^2)\mathrm{G}-2\mathrm{H}\right], \qquad \mathrm{H} = \mathrm{E}(\gamma) = \int_0^{\frac{1}{2}\pi}\Delta(\omega,\gamma)\,d\omega,$$

(7) $$\mathrm{M} = -2\pi\mathrm{QA} = -2\pi r_3\left[(2-\gamma^2)\mathrm{G}-2\mathrm{H}\right],$$

and $\mathrm{Q}\cos\phi$ is the magnetic potential of the circle on AB, with uniform magnetisation parallel to AB.

In the Third Elliptic Integral keep to the Weierstrassian form, with the variable t,

(8) $$m^2\,dt = 2\mathrm{A}a\sin\theta\,d\theta = m^2\sqrt{(t_2-t\,.\,t-t_3)}\,d\theta,$$

(9) $$\frac{d\theta}{\mathrm{PQ}} = \frac{2dt}{m\sqrt{\mathrm{T}}}, \qquad \frac{b\,d\theta}{\mathrm{PQ}} = \frac{2\sqrt{(t_1-\tau)}\,dt}{\sqrt{\mathrm{T}}},$$

(10) $$T = 4 \,.\, t_1 - t \,.\, t_2 - t \,.\, t - t_3, \qquad t_2 > t > t_3,$$

(11) $$\frac{a^2 - A^2}{MQ^2} = \frac{\sqrt{(\tau - t_2 \,.\, \tau - t_3)}}{\tau - t},$$

(12) $$\frac{a^2 - A^2}{MQ^2} \frac{b\,d\theta}{PQ} = \frac{\sqrt{(-U)}}{\tau - t} \cdot \frac{dt}{\sqrt{T}},$$

(13) $$-U = 4 \,.\, t_1 - \tau \,.\, \tau - t_2 \,.\, \tau - t_3, \qquad t_1 > \tau > t_2 > t_3,$$

(14) $$\int_0^{2\pi} \frac{a^2 - A^2}{MQ^2} \frac{b\,d\theta}{PQ} = \int_{t_3}^{t_2} \frac{2\sqrt{(-U)}}{\tau - t} \cdot \frac{dt}{\sqrt{T}},$$

in a standard Weierstrassian form of the III. E.I.

The expression of this III. E.I. when complete, by means of the E.I., I. and II., complete and incomplete, was given by LEGENDRE, 'Fonctions Elliptiques,' chap. 23, and (14) falls under his class (m').

4. But we shall avoid the Legendrian form, and start by making use of the lemma

(1) $$\sqrt{T} \frac{d}{dt} \frac{\frac{1}{2}\sqrt{T}}{\tau - t} - \sqrt{U} \frac{d}{d\tau} \frac{\frac{1}{2}\sqrt{-U}}{\tau - t} = \tau - t,$$

proved immediately by effecting the differentiation.

Integrating, in either order, with respect to the differential elements

$$\frac{dt}{\sqrt{T}} \quad \text{and} \quad \frac{d\tau}{\sqrt{-U}}$$

and between the limits $t_2 > t > t_3$, and t_1, τ of τ,

(2) $$\int_\tau^{t_1} \frac{d\tau}{\sqrt{-U}} \int_{t_3}^{t_2} \frac{dt}{\sqrt{T}} \sqrt{T} \frac{d}{dt}\left(\frac{\frac{1}{2}\sqrt{T}}{\tau - t}\right)$$
$$= \int \frac{d\tau}{\sqrt{-U}} \int \frac{d}{dt}\left(\frac{\frac{1}{2}\sqrt{T}}{\tau - t}\right) dt$$
$$= \int \frac{d\tau}{\sqrt{-U}} \left(\frac{\frac{1}{2}\sqrt{T}}{\tau - t}\right)_{t_2}^{t_3} = 0,$$

(3) $$-\int \frac{dt}{\sqrt{T}} \int_\tau^{t_1} \frac{d\tau}{\sqrt{-U}} \sqrt{-U} \frac{d}{d\tau}\left(\frac{\frac{1}{2}\sqrt{-U}}{\tau - t}\right)$$
$$= -\int \frac{dt}{\sqrt{T}} \int \frac{d}{d\tau}\left(\frac{\frac{1}{2}\sqrt{-U}}{\tau - t}\right) d\tau$$
$$= -\int \frac{dt}{\sqrt{T}} \left(\frac{\frac{1}{2}\sqrt{-U}}{\tau - t}\right)_\tau^{t_1}$$
$$= \int_{t_3}^{t_2} \frac{dt}{\sqrt{T}} \frac{\frac{1}{2}\sqrt{-U}}{\tau - t},$$

as in (14), § 3; so calling it 2B, as a standard type of the III. E.I., it is proved by the lemma (1) that

(4) $$B = \int_\tau^{t_1}\int_{t_3}^{t_2}(\tau - t)\frac{dt}{\sqrt{T}}\frac{d\tau}{\sqrt{-U}},$$

in which the variables are separated, t and τ. Then with

(5) $$G = \int_{t_3}^{t_2}\frac{\sqrt{(t_1-t_3)}\,dt}{\sqrt{T}}, \qquad G' = \int_{t_2}^{t_1}\frac{\sqrt{(t_1-t_3)}\,d\tau}{\sqrt{-U}},$$

to the modulus $\gamma = \sqrt{\dfrac{t_2-t_3}{t_1-t_3}}$, and co-modulus $\gamma' = \sqrt{\dfrac{t_1-t_2}{t_1-t_3}}$, and with e and f to denote fractions, such that

(6) $$eG = \int_{t_3}^{t}\frac{\sqrt{(t_1-t_3)}\,dt}{\sqrt{T}}, \qquad 2fG' = \int_{\tau}^{t_1}\frac{\sqrt{(t_1-t_3)}\,d\tau}{\sqrt{-U}},$$

(7) $$eG = \mathrm{sn}^{-1}\sqrt{\frac{t-t_3}{t_2-t_3}} = \mathrm{cn}^{-1}\sqrt{\frac{t_2-t}{t_2-t_3}} = \mathrm{dn}^{-1}\sqrt{\frac{t_1-t}{t_1-t_3}},$$

(8) $$2fG' = \mathrm{sn}^{-1}\sqrt{\frac{t_1-\tau}{t_1-t_2}} = \mathrm{cn}^{-1}\sqrt{\frac{\tau-t_2}{t_1-t_2}} = \mathrm{dn}^{-1}\sqrt{\frac{\tau-t_3}{t_1-t_3}},$$

(9) $$\iint (t-t_3)\frac{dt}{\sqrt{T}}\frac{d\tau}{\sqrt{-U}} = 2fG'\int\frac{t-t_3}{\sqrt{(t_1-t_3)}}\frac{dt}{\sqrt{T}}$$
$$= 2fG'\int_0^1(1-\mathrm{dn}^2 eG)\,deG = 2fG'(G-H),$$

where H denotes $E(\gamma)$, the complete II. E.I. to modulus γ;

(10) $$\iint (\tau-t_3)\frac{d\tau}{\sqrt{-U}}\frac{dt}{\sqrt{T}} = G\int\frac{\tau-t_3}{\sqrt{(t_1-t_3)}}\frac{d\tau}{\sqrt{-U}}$$
$$= G\int_0^{2f}\mathrm{dn}^2 2fG'\,d2fG' = G(2fH' + \mathrm{zn}\,2fG')$$

with $H' = E(\gamma')$; and then

(11) $$B = G(2fH' + \mathrm{zn}\,2fG') - 2fG'(G-H)$$
$$= 2f(GH' + G'H - GG') + G\,\mathrm{zn}\,2fG'$$
$$= \pi f + G\,\mathrm{zn}\,2fG',$$

by LEGENDRE's relation, and this is the equivalent statement of his equation (m'), expressed in the notation of JACOBI.

Then L is given by the three E.I.'s in the form

(12) $$L = \tfrac{1}{2}\pi P\left(a + \frac{A^2}{a}\right)b + \tfrac{1}{2}Mb - 2\pi B(a^2 - A^2),$$

and so may be said to be expressed in finite terms, that is, by tabulated functions.

5. The various quantities required are shown geometrically on the diagram of fig. 1 and 2. The front aspect is shown in fig. 1 of the circle on AB, and

(1) $$\mathrm{AOQ} = \theta, \qquad \mathrm{ABQ} = \omega, \qquad \frac{\mathrm{EB}}{\mathrm{EA}} = \frac{\mathrm{DB}}{\mathrm{DA}} = \gamma',$$

$$\sin\omega = \sqrt{\frac{t-t_3}{t_2-t_3}} = \mathrm{sn}\, e\mathrm{G}, \qquad \omega = \mathrm{am}\, e\mathrm{G},$$

$$\mathrm{AB}q = \mathrm{AQE} = \mathrm{am}\,(1-e)\,\mathrm{G}, \qquad \mathrm{AQ}q' = \mathrm{am}\,(1+e)\,\mathrm{G},$$

$$\mathrm{EQ} = \mathrm{EA}\,\mathrm{dn}\, e\mathrm{G}, \qquad \mathrm{E}q = \mathrm{EA}\,\mathrm{dn}\,(1-e)\,\mathrm{G}.$$

On fig. 2, where the circle AB is seen edgeways,

(2) $$\frac{\mathrm{PB}}{\mathrm{PA}} = \frac{\mathrm{EB}}{\mathrm{EA}} = \gamma',$$

$$\mathrm{OBP} = \chi, \qquad \sin\chi = \frac{b}{r_2} = \sqrt{\frac{t_1-\tau}{t_1-t_2}} = \mathrm{sn}\, 2f\mathrm{G}', \qquad \chi = \mathrm{am}\, 2f\mathrm{G}',$$

$$\mathrm{OAP} = \chi' = \mathrm{BPF}, \qquad \sin\chi' = \gamma'\sin\chi, \qquad \cos\chi' = \Delta\chi = \mathrm{dn}\, 2f\mathrm{G}',$$

$$\mathrm{P}p = \mathrm{ED}\cos\chi' = \mathrm{ED}\,\mathrm{dn}\, 2f\mathrm{G}'.$$

The circle on ED is orthogonal to the circle on AB when turned round into the same plane as in fig. 1, and in fig. 2 the two circles on AB and ED may be taken to represent the typical electric and magnetic circuit linked together.

6. MAXWELL goes on to show that M is the *stream function* (S.F.) of a (P.F.) *potential function* Ω, such that

(1) $$\frac{d\mathrm{M}}{d\mathrm{A}} = 2\pi\mathrm{A}\frac{d\Omega}{db}, \qquad \frac{d\mathrm{M}}{db} = -2\pi\mathrm{A}\frac{d\Omega}{d\mathrm{A}},$$

(2) $$\frac{d}{d\mathrm{A}}\left(\mathrm{A}\frac{d\Omega}{d\mathrm{A}}\right) + \frac{d}{db}\left(\mathrm{A}\frac{d\Omega}{db}\right) = 0,$$

(3) $$\frac{d}{d\mathrm{A}}\left(\frac{1}{\mathrm{A}}\frac{d\mathrm{M}}{d\mathrm{A}}\right) + \frac{d}{db}\left(\frac{1}{\mathrm{A}}\frac{d\mathrm{M}}{db}\right) = 0,$$

and a line of force along M a constant is at right angles to a surface of constant Ω.

If a return should be made to the usual co-ordinates, it is preferable to employ the ordinary (x, y) of plane geometry, and not the cylindrical or columnar co-ordinates (z, ϖ) or (z, ρ) of some writers, or MAXWELL'S (b, A).

Then these equations (2) and (3) will appear in the familiar form

(4) $$\frac{d}{dx}\left(y\frac{d\Omega}{dx}\right) + \frac{d}{dy}\left(y\frac{d\Omega}{dy}\right) = 0,$$

(5) $$\frac{d}{dx}\left(\frac{1}{y}\frac{d\mathrm{M}}{dx}\right) + \frac{d}{dy}\left(\frac{1}{y}\frac{d\mathrm{M}}{dy}\right) = 0,$$

reducing to ordinary conjugate function relations at a great distance from the axis Ox, where y is large. And with any conjugate system, $x+iy = f(u+iv)$, (dx, dy) are replaced by (du, dv); thus for polar co-ordinates $x+iy = e^{\log r+i\vartheta}$, $du = dr/r$, $dv = d\vartheta$.

If the motion is not symmetrical about the axis Ox, and is not uniaxial, the S.F. does not exist; and in equation (4) an additional term is required for the variation with angle ϕ of azimuth, so that in this general case

(6) $$\frac{d}{dx}\left(y\frac{d\Omega}{dx}\right)+\frac{d}{dy}\left(y\frac{d\Omega}{dy}\right)+\frac{d^2\Omega}{y\,d\phi^2} = 0,$$

expressing the resultant leakage or crowding-convergence of Ω through an element of volume $dx\,.\,dy/y\,d\phi$, when this is zero.

Thus the result in (7), § 3, that $\frac{\mathrm{M}}{2\pi\mathrm{A}}\cos\phi$ is a P.F., $-\mathrm{Q}\cos\phi$, is true for any S.F. M; for changing A into y, and putting $\mathrm{M} = \mathrm{V}y$,

(7) $$\frac{d}{dx}\left(\frac{1}{y}\frac{d\mathrm{M}}{dx}\right)+\frac{d}{dy}\left(\frac{1}{y}\frac{d\mathrm{M}}{dy}\right)$$
$$= \frac{d^2\mathrm{V}}{dx^2}+\frac{d^2\mathrm{V}}{dy^2}+\frac{1}{y}\frac{d\mathrm{V}}{dy}-\frac{\mathrm{V}}{y^2}$$
$$= \frac{1}{y}\left[\frac{d}{dx}\left(y\frac{d\mathrm{V}}{dx}\right)+\frac{d}{dy}\left(y\frac{d\mathrm{V}}{dy}\right)-\frac{\mathrm{V}}{y}\right] = 0,$$

so that $\mathrm{V}\cos\phi$ satisfies equation (6) for Ω.

7. MAXWELL shows further that Ω is the magnetic potential of any sheet bounded by the circle AB with uniform normal magnetization, so that, taking the plane circle AB, Ω is given by the normal component of the surface attraction of the circular disc AB, and so is the solid conical angle subtended at P.

This is true for any boundary AB; for if $d\mathrm{S}$ denotes any elementary area enclosing a point Q, the element of normal attraction, $\frac{d\mathrm{S}}{\mathrm{PQ}^2}\cos\mathrm{QPM}$, is the element of surface of unit sphere with centre at P, cut out of the cone on the base $d\mathrm{S}$.

In MAXWELL's expression for P, surface potential of a spherical segment on the circular base AB, given in the form of a series of zonal harmonics (E. and M., § 694), he proves that

(1) $$\frac{1}{c}\frac{d}{dr}(\mathrm{P}r) = \Omega,$$

but he does not notice that

(2) $$\mathrm{P} = \Omega c+\Omega'\frac{c^2}{r} = \Omega c+\Omega' r',$$

where Ω' is the solid angle or apparent area of the circle AB from the inverse point in the sphere, As the result is independent of the size of the segment, it holds true when the segment is made small, and this can be proved in a couple of lines of elementary geometry, as given in the 'American Journal of Mathematics' (A.J.M.), October, 1917, p. 237. Thence, by summation, the result for P holds in the same form when the spherical segment has any arbitrary boundary not restricted to be circular.

For the analytical expression of Ω the complete elliptic integral of the third kind (E.I. III.) is required. This is not attempted by MAXWELL, and he leaves Ω in the form of a series of zonal harmonics obtained and written down from the axial expansion.

But the chief difficulty in the theory of the ampère balance is the reduction and manipulation of Ω, a multiple-valued function with a cyclic constant 4π for a magnetic circuit through the circle on AB, say round the circle on ED, linked with the electric circuit round AB.

8. The III. E.I. required for Ω will be of the same nature as B which occurs in L (6), § 2, (14), § 3, and to obtain the relation, take MAXWELL's M and differentiate with respect to A, then

(1) $$\mathrm{A}\frac{d\Omega}{db} = \frac{1}{2\pi}\frac{d\mathrm{M}}{d\mathrm{A}} = \int \frac{a\cos\theta\, d\theta}{\mathrm{PQ}} - \int \frac{\mathrm{A}a\cos\theta\,(\mathrm{A}+a\cos\theta)\,d\theta}{\mathrm{PQ}^3}:$$

Making use of the lemma,

(2) $$\int \frac{db}{\mathrm{PQ}^3} = \frac{1}{\mathrm{MQ}^2}\cdot\frac{b}{\mathrm{PQ}},$$

Ω is obtained by an integration with respect to b,

(3) $$\mathrm{A}\Omega = \int a\cos\theta\,\mathrm{th}^{-1}\frac{b}{\mathrm{PQ}}\,d\theta - \int \frac{\mathrm{A}a\cos\theta\,(\mathrm{A}+a\cos\theta)}{\mathrm{MQ}^2}\cdot\frac{b\,d\theta}{\mathrm{PQ}}.$$

Integrating the first of these integrals by parts,

(4) $$\mathrm{A}\Omega = \left(a\sin\theta\,\mathrm{th}^{-1}\frac{b}{\mathrm{PQ}}\right)_0^{2\pi} - \int a\sin\theta\,\frac{\mathrm{A}a\sin\theta}{\mathrm{MQ}^2}\cdot\frac{b\,d\theta}{\mathrm{PQ}} - \int \frac{\mathrm{A}a\cos\theta\,(\mathrm{A}+a\cos\theta)}{\mathrm{MQ}^2}\cdot\frac{bd\,\theta}{\mathrm{PQ}}$$
$$= * - \int \frac{\mathrm{A}a\,(\mathrm{A}\cos\theta+a)}{\mathrm{MQ}^2}\cdot\frac{b\,d\theta}{\mathrm{PQ}},$$

the * marking the place of a term which vanishes at both limits,

(5) $$\Omega = \text{constant} - \int \frac{\mathrm{A}a\cos\theta+a^2}{\mathrm{MQ}^2}\cdot\frac{b\,d\theta}{\mathrm{PQ}}$$
$$= \text{constant} - \Omega\,(\mathrm{MQ}),$$

suppose, where

(6) $$\Omega\,(\mathrm{MQ}) = \int_0^{2\pi} \frac{\mathrm{A}a\cos\theta+a^2}{\mathrm{MQ}^2}\cdot\frac{b\,d\theta}{\mathrm{PQ}}.$$

In MINCHIN's dissection of the circle on AB by lines radiating from M, 'Phil. Mag.,' February, 1894, the solid angle cut out by a complete revolution of PQ about PM at a constant angle is $\left(1-\frac{\mathrm{PM}}{\mathrm{PQ}}\right)2\pi$, so that for an elementary angle $d\eta$,

$$(7) \qquad d\Omega = \left(1-\frac{\mathrm{PM}}{\mathrm{PQ}}\right)d\eta, \quad \text{and} \quad \tfrac{1}{2}\mathrm{MQ}^2\,d\eta = \tfrac{1}{2}\mathrm{MY}\,.\,a\,d\theta,$$

if $\mathrm{MY} = \mathrm{A}\cos\theta + a$ is the perpendicular from M on the tangent at Q; so that

$$(8) \qquad d\Omega = d\eta - \frac{\mathrm{A}a\cos\theta + a^2}{\mathrm{MQ}^2}\cdot\frac{b\,d\theta}{\mathrm{PQ}} = d\eta - d\Omega\,(\mathrm{MQ}).$$

In a complete circuit of the circle on AB, η grows from 0 to 2π, if M is inside the circle on AB $(a > \mathrm{A}, f < \frac{1}{2})$,

$$(9) \qquad \Omega = 2\pi - \Omega\,(\mathrm{MQ}).$$

Replacing $\mathrm{A}a\cos\theta$ by $\frac{1}{2}(\mathrm{MQ}^2 - a^2 - \mathrm{A}^2)$,

$$(10) \qquad \Omega\,(\mathrm{MQ}) = \tfrac{1}{2}\int_0^{2\pi}\frac{a^2-\mathrm{A}^2}{\mathrm{MQ}^2}\cdot\frac{b\,d\theta}{\mathrm{PQ}} + \tfrac{1}{2}\int\frac{b\,d\theta}{\mathrm{PQ}}$$

$$= 2\mathrm{B} + \frac{2b\mathrm{G}}{r_3}$$

$$= 2\pi f + 2\mathrm{G}\,\mathrm{zn}\,2f\mathrm{G}' + 2\mathrm{G}\gamma'\,\mathrm{sn}\,2f\mathrm{G}',$$

$$(11) \qquad \Omega = 2\pi(1-f) - 2\mathrm{G}\,\mathrm{zn}\,2f\mathrm{G}' - 2\mathrm{G}\gamma'\,\mathrm{sn}\,2f\mathrm{G}'.$$

This agrees in making $\Omega = 2\pi$ when P is at E and AB is viewed close up, and $\Omega = 0$ when $f = 1$ and P is at D, where the circle AB is seen edgeways; and then, with this value of B in (12), § 4,

$$(12) \qquad \mathrm{L} = \pi\mathrm{P}ab + \tfrac{1}{2}\mathrm{M}b - \pi(a^2 - \mathrm{A}^2)(2\pi - \Omega).$$

In making the circuit of the circle EPD, and starting from E, where $f = 0$, $\Omega = 2\pi$, then f grows from 0 to 1, and Ω diminishes from 2π to 0 at D. After passing D, f grows from 1 to 2, and Ω is taken negative for the reverse aspect of the circle AB, and on arrival at E again with $f = 2$, $\Omega = -2\pi$.

Thus 4π must be added in crossing AB if P circulates counter-clockwise. But with the clock, the other way round, 4π must be subtracted in crossing AB, just as twelve hours is deducted on the clock in passing through XII o'clock.

9. But proceeding to Ω through

(1) $$\frac{d\Omega}{dA} = -\frac{1}{2\pi A}\frac{dM}{db} = \int a\cos\theta\,\frac{b\,d\theta}{PQ^3},$$

and utilising the integral

(2) $$\int\frac{dA}{PQ^3} = \frac{A+a\cos\theta}{PR^2}\cdot\frac{1}{PQ}, \qquad PR^2 = a^2\sin^2\theta+b^2,$$

PR the perpendicular from P on QQ′ parallel to AB in fig. 1,

(3) $$\Omega = \Omega(PR) = \int\frac{Aa\cos\theta+a^2\cos^2\theta}{PR^2}\cdot\frac{b\,d\theta}{PQ},$$

a new form of the III. E.I., not recognisable in the previous expression in (6), § 8.

We have to make use of the theorems given in the 'Trans. American Math. Society,' 1907 (A.M.S.), connecting the various forms of dissection of Ω in the III. E.I., and here the relation connecting the incomplete integrals in θ of Ω(MQ) and Ω(PR) is

(4) $$\Omega(MQ)+\Omega(PR) = \text{angle between MQP, MQR}$$

$$= \sin^{-1}\frac{QN}{PR}\cdot\frac{PQ}{MQ} = \cos^{-1}\frac{MN}{PR}\cdot\frac{PM}{MQ} = \cot^{-1}\frac{A+a\cos\theta}{a\sin\theta}\cdot\frac{b}{PQ},$$

as is soon verified by the differentiation; and for the complete integrals between 0 and 2π the sum is 2π.

In Ω(PR) the dissection of the circle AB would be in strips QQ′ parallel to AB.

10. Another form for Ω is obtained from the theorem that

(1) $$\Phi+\Omega = 2\pi$$

connects Ω, the area of the spherical curve of the cone on the base AB, and Φ, the perimeter of the curve of the reciprocal cone, both on the unit sphere with centre at the vertex P.

The section SS′ of the reciprocal cone made by the plane AB is the polar reciprocal of the circle AB with respect to the pole M; a conic with focus at M, and

(2) $$SM\,.\,MY = ZM\,.\,MQ = b^2.$$

The projection of the elementary sector PSS′ of the reciprocal cone on the plane AB is

(3) $$\tfrac{1}{2}PS^2\,.\,d\Phi\,.\cos PZM = \tfrac{1}{2}MS^2\,d\theta,$$

(4) $$\frac{d\Phi}{d\theta} = \frac{MS^2}{PS^2}\cdot\frac{PQ}{PM} = \frac{MP^2}{PY^2} = \frac{PQ\,.\,b}{PY^2} = \left(1+\frac{QY^2}{PY^2}\right)\frac{b}{PQ}.$$

In the reduction of this form we employ a new substitution, putting

(5) $$\cos^2 \text{QPY} = \frac{\text{PY}^2}{\text{PQ}^2} = m^2(\sigma - s),$$

where s is the new variable, and σ a constant; and take $s = s_3$ at A where QPY = 0, $\cos^2 \text{QPY} = 1 = m^2(\sigma - s_3)$,

(6) $$\frac{\text{PY}^2}{\text{PQ}^2} = \frac{\sigma - s}{\sigma - s_3}, \qquad \frac{\text{QY}^2}{\text{PQ}^2} = \frac{s - s_3}{\sigma - s_3} = \frac{\text{A}^2 \sin^2\theta}{r^2} = \frac{r_3{}^2 - r^2 \,.\, r^2 - r_2{}^2}{4a^2 r^2},$$

so that $s = \infty$ at $r = 0, \infty$. Then take $s = s_2, s_1$ for $r^2 = \pm r_2 r_3$; this makes

(7) $$\frac{s_2 - s_3}{\sigma - s_3} = \left(\frac{r_3 - r_2}{2a}\right)^2 = \left(\frac{2\text{A}}{r_3 + r_2}\right)^2, \qquad \frac{s_1 - s_3}{\sigma - s_3} = \left(\frac{r_3 + r_2}{2a}\right)^2 = \frac{2\text{A}}{(r_3 - r_2)^2}.$$

With the variable s we are employing the elliptic function has a new modulus κ, obtained by a quadric transformation of the former modulus γ, and associated with the elliptic integral

(8) $$\text{K} = \int_{s_3}^{s_2} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{\text{S}}}, \qquad \text{K}' = \int_{s_2}^{s_1} \frac{\sqrt{(s_1 - s_3)}\, d\sigma}{\sqrt{-\Sigma}},$$

(9) $$\text{S} = 4 \,.\, s_1 - s \,.\, s_2 - s \,.\, s - s_3, \qquad -\Sigma = 4 \,.\, s_1 - \sigma \,.\, \sigma - s_2 \,.\, \sigma - s_3,$$

(10) $$\infty > s_1 > \sigma > s_2 > s > s_3 > -\infty,$$

(11) $$\kappa^2 = \frac{s_2 - s_3}{s_1 - s_3} = \left(\frac{r_3 - r_2}{r_3 + r_2}\right)^2, \qquad \kappa = \frac{1 - \gamma'}{1 + \gamma'},$$

and with fractions e and f, such that

(12) $$2e\text{K} = \int_{s_3}^{s} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{\text{S}}}, \qquad f\text{K}' = \int_{\sigma}^{s_1} \frac{\sqrt{(s_1 - s_3)}\, d\sigma}{\sqrt{-\Sigma}},$$

(13) $$2e\text{K} = \text{sn}^{-1}\sqrt{\frac{s - s_3}{s_2 - s_3}} = \text{cn}^{-1}\sqrt{\frac{s_2 - s}{s_2 - s_3}} = \text{dn}^{-1}\sqrt{\frac{s_1 - s}{s_1 - s_3}},$$

(14) $$f\text{K}' = \text{sn}^{-1}\sqrt{\frac{s_1 - \sigma}{s_1 - s_2}} = \text{cn}^{-1}\sqrt{\frac{\sigma - s_2}{s_1 - s_2}} = \text{dn}^{-1}\sqrt{\frac{\sigma - s_3}{s_1 - s_3}}.$$

Produce PE on fig. 2 to meet Ox in G, and describe the sphere, centre G, passing through the circle on AB. Then P, E are inverse points in this sphere, and

(15) $$\frac{PQ}{EQ} = \frac{PA}{EA} = \frac{PB}{EB} = \frac{PA+PB}{AB} = \frac{r_3+r_2}{2a},$$

(16) $$\mathrm{sn}^2\, 2eK = \frac{s-s_3}{s_2-s_3} = \frac{s-s_3}{\sigma-s_3}\cdot\frac{\sigma-s_3}{s_2-s_3} = \frac{A^2 \sin^2\theta}{r^2}\left(\frac{r_3+r_2}{2A}\right)^2,$$

(17) $$\mathrm{sn}\, 2eK = \frac{r_3+r_2}{2PQ}\sin\theta = \frac{a\sin\theta}{EQ} = \sin\phi, \qquad \phi = \mathrm{am}\, 2eK,$$

where ϕ = AEQ on fig. 1, and

(18) $$\kappa = \frac{OE}{OB} = \frac{OB}{OD} = \frac{BE}{BD}.$$

And with ODQ = ϕ' = OQE on fig. 1,

(19) $$\sin\phi' = \kappa\sin\phi, \qquad \cos\phi' = \Delta\phi = \mathrm{dn}\, 2eK,$$

so that $Qq = AB\cos\phi' = AB\, \mathrm{dn}\, 2eK$.

On fig. 2, and from (7), (14),

(20) $$\mathrm{dn} f K' = \sqrt{\frac{\sigma-s_3}{s_1-s_3}} = \frac{2a}{r_3+r_2} = \frac{EB}{PB} = \frac{Bp}{BD},$$

and with DEp = DPB = ψ,

(21) $$Bp^2 = BD^2\cos^2\psi + BE^2\sin^2\psi = BD^2\Delta^2(\psi, \kappa'),$$

(22) $$\Delta(\psi, \kappa') = \mathrm{dn} f K', \qquad \psi = \mathrm{am} f K',$$

and

$$DEP = DpB = \mathrm{am}(1-f)K'.$$

Thence any formula of the Landen quadric transformation, first and second, can be interpreted geometrically on fig. 1 and 2, and we reconcile the baffling and conflicting notation of previous writers on the subject.

Interpreted dynamically, with e proportional to the time, $t = \frac{1}{2}e\text{T}$ for period T, Q circulates round the circle on AB in fig. 1 with velocity due to the level of F, or proportional to EQ or DQ, and beats the elliptic function to modulus γ, while T circulates round the circle on OE with velocity due to the level of D, or S at the same level will oscillate, beating elliptic functions to modulus κ.

So also for the motion of P round the circle on ED, with f proportional to the time t, and velocity due to the level of O, or proportional to BP or AP, gravity being reversed.

11. Combined into one quadric transformation, the first and second of Landen, from modulus γ to κ, and then κ to γ,

$$(1) \qquad \sqrt{\kappa}\,\text{sn}\,(2e\text{K}+f\text{K}'i) = \frac{\gamma\,\text{sn}\,(e\text{G}+f\text{G}'i)\,\text{cn}\,(e\text{G}+f\text{G}'i)}{\text{dn}\,(e\text{G}+f\text{G}'i)}$$

$$= \gamma\,\text{sn}\,(e\text{G}+f\text{G}'i)\,\text{sn}\,(\text{G}-e\text{G}-f\text{G}'i),$$

$$(2) \qquad \text{dn}\,(2e\text{G}+2f\text{G}'i) = \frac{1-\kappa\,\text{sn}^2\,(2e\text{K}+f\text{K}'i)}{1+\kappa\,\text{sn}^2\,(2e\text{K}+f\text{K}'i)},$$

and then f or e is made zero. And

$$(3) \qquad \text{cn}\,(2e\text{K}+f\text{K}'i) = \frac{\text{dn}^2\,(e\text{G}+f\text{G}'i)-\gamma'}{(1-\gamma')\,\text{dn}\,(e\text{G}+f\text{G}'i)}$$

$$= \frac{\text{dn}\,(e\text{G}+f\text{G}'i)-\text{dn}\,(\text{G}-e\text{G}-f\text{G}'i)}{1-\gamma'},$$

$$(4) \qquad \text{dn}\,(2e\text{K}+f\text{K}'i) = \frac{\text{dn}^2\,(e\text{G}+f\text{G}'i)+\gamma'}{(1+\gamma')\,\text{dn}\,(e\text{G}+f\text{G}'i)}$$

$$= \frac{\text{dn}\,(e\text{G}+f\text{G}'i)+\text{dn}\,(\text{G}-e\text{G}-f\text{G}'i)}{1+\gamma'},$$

$$(5) \qquad \frac{\kappa}{\kappa'}\,\text{cn}\,(2e\text{K}+f\text{K}'i) = \tfrac{1}{2}\frac{\text{dn}\,(e\text{G}+f\text{G}'i)}{\sqrt{\gamma'}} - \tfrac{1}{2}\frac{\text{dn}\,(\text{G}-e\text{G}-f\text{G}'i)}{\sqrt{\gamma'}},$$

$$(6) \qquad \frac{\text{dn}\,(2e\text{K}+f\text{K}'i)}{\kappa'} = \tfrac{1}{2}\frac{\text{dn}\,(e\text{G}+f\text{G}'i)}{\sqrt{\gamma'}} + \tfrac{1}{2}\frac{\text{dn}\,(\text{G}-e\text{G}-f\text{G}'i)}{\sqrt{\gamma'}}.$$

12. By logarithmic differentiation of (6), § 10,

$$(1) \qquad \frac{ds}{s-s_3} = \frac{-r^4+r_2^2r_3^2}{r_3-r^2 \,.\, r^2-r_2^2}\frac{dr^2}{r^2},$$

and with $s = s_2, s_1$, for $r^2 = \pm\, r_2r_3$,

$$(2) \qquad \frac{s_1-s}{\sigma-s_3} = \left(\frac{r^2+r_2r_3}{2ar}\right)^2, \qquad \frac{s_2-s}{\sigma-s_3} = \left(\frac{r^2-r_2r_3}{2ar}\right)^2,$$

$$(3) \qquad \frac{\sqrt{(s_1-s \,.\, s_2-s)}}{\sigma-s_3} = \frac{r^4-r_2^2r_3^2}{4a^2r^2},$$

$$(4) \qquad \sqrt{\frac{s-s_3}{\sigma-s_3}} = \frac{\sqrt{(r_3^2-r^2 \,.\, r^2-r_2^2)}}{2ar},$$

$$(5) \qquad \frac{\sqrt{\mathrm{S}}}{(\sigma-s_3)^3} = \frac{(r^4-r_2^2r_3^2)\sqrt{(r_3^2-r^2 \,.\, r^2-r_2^2)}}{4a^3r^3},$$

$$(6) \qquad \frac{ds}{\sigma-s_3} = (r^4-r_2^2r_3^2)\frac{-2\,dr}{4a^2r^3},$$

$$(7) \qquad \frac{\sqrt{(\sigma-s_3)}\,ds}{\sqrt{\mathrm{S}}} = \frac{-2a\,dr}{\sqrt{(r_3^2-r^2 \,.\, r^2-r_2^2)}}.$$

Then with

$$(8) \qquad \theta = 2\sin^{-1}\sqrt{\frac{r_3^2-r^2}{r_3^2-r_2^2}} = 2\cos^{-1}\sqrt{\frac{r^2-r_2^2}{r_3^2-r_2^2}}, \qquad d\theta = \frac{-2r\,dr}{\sqrt{(r_3^2-r^2 \,.\, r^2-r_2^2)}},$$

$$(9) \qquad d\mathrm{P} = \frac{a\,d\theta}{\mathrm{PQ}} = \frac{-2a\,dr}{\sqrt{(r_3^2-r^2 \,.\, r^2-r_2^2)}} = \frac{\sqrt{(\sigma-s_3)}\,ds}{\sqrt{\mathrm{S}}}$$

$$= \sqrt{\frac{\sigma-s_3}{s_1-s_3}}\,d2e\mathrm{K} = \mathrm{dn}f\,\mathrm{K}'\,d2e\mathrm{K} = \frac{\mathrm{EB}}{\mathrm{PB}}\,d2e\mathrm{K}.$$

Comparing this with the previous form in (4), § 3,

$$(10) \qquad d\mathrm{P} = \frac{2a}{r_3}\frac{d\omega}{\Delta\omega} = \frac{\mathrm{AB}}{\mathrm{AP}}\,de\mathrm{G},$$

$$(11) \qquad \frac{\mathrm{EB}}{\mathrm{PB}}\,2\mathrm{K} = \frac{\mathrm{AB}}{\mathrm{AP}}\,\mathrm{G}, \qquad \frac{2\mathrm{K}}{\mathrm{G}} = \frac{\mathrm{AB}}{\mathrm{EB}}\cdot\frac{\mathrm{PB}}{\mathrm{AP}} = \frac{2\gamma'}{1-\kappa}.$$

$$(12) \qquad (1-\kappa)\,\mathrm{K} = \mathrm{G}\gamma', \qquad \mathrm{K}\kappa' = \mathrm{G}\sqrt{\gamma'}.$$

And similarly

$$(13) \qquad (1-\kappa)\,\mathrm{K}' = 2\mathrm{G}'\gamma', \qquad \frac{\mathrm{K}'}{\mathrm{K}} = 2\,\frac{\mathrm{G}'}{\mathrm{G}},$$

as in the quadric transformation. Thus

$$(14) \qquad \mathrm{P} = 4\mathrm{K}\,\mathrm{dn}f\mathrm{K}' = \frac{8a\mathrm{K}}{r_2+r_3} = \frac{4a\mathrm{K}\kappa'}{\sqrt{(r_2 r_3)}}, \quad \text{or} \quad \frac{4a\mathrm{G}}{r_3} = \frac{4a\mathrm{G}\sqrt{\gamma'}}{\sqrt{(r_2 r_3)}}.$$

On fig. 1, $\theta = \phi+\phi'$,

$$(15) \qquad \cos\theta = \cos\phi\cos\phi' - \sin\phi\sin\phi'$$

$$= \mathrm{cn}\,2e\mathrm{K}\,\mathrm{dn}\,2e\mathrm{K} - \kappa\,\mathrm{sn}^2\,2e\mathrm{K},$$

$$(16) \qquad d\mathrm{Q} = \frac{-a\cos\theta\,d\theta}{\mathrm{PQ}} = \mathrm{dn}f\mathrm{K}'\,(\kappa\,\mathrm{sn}^2\,2e\mathrm{K} - \mathrm{cn}\,2e\mathrm{K}\,\mathrm{dn}\,2e\mathrm{K})\,d2e\mathrm{K},$$

and integrating round AB from $0 < e < 2$, the second term in $d\mathrm{Q}$ vanishes by inspection, and

$$(17) \qquad \mathrm{Q} = 2\,\frac{\mathrm{dn}f\mathrm{K}'}{\kappa}\int_0^2 (1-\mathrm{dn}^2\,2e\mathrm{K})\,d2e\mathrm{K} = 4\,\frac{\mathrm{K}-\mathrm{E}}{\mathrm{dn}\,(1-f)\,\mathrm{K}'}$$

$$= \frac{r_3+r_2}{2\mathrm{A}}\,4\,(\mathrm{K}-\mathrm{E}) = \frac{4a}{\sqrt{(r_2 r_3)}}\,(\mathrm{K}-\mathrm{E})\,\frac{\kappa'}{\kappa},$$

$$(18) \qquad \mathrm{M} = -2\pi\mathrm{QA} = -4\pi\,(r_2+r_3)\,(\mathrm{K}-\mathrm{E}) = -2\pi\sqrt{(r_2 r_3)}\,\frac{\mathrm{K}-\mathrm{E}}{\kappa'}.$$

Thus in the construction of the curves of constant M on the Weir chart, a table was first drawn up from LEGENDRE of E and K for every degree of the modular angle θ, and then of $\mathrm{K}-\mathrm{E}$ and $\dfrac{\mathrm{K}-\mathrm{E}}{\sin\theta}$; and with the hour angle $\alpha = \frac{1}{2}\pi-\xi$, and λ the latitude, such that $\mathrm{ch}\,\eta = \sec\lambda$, and $r_3, r_2 = c\,(\mathrm{ch}\,\eta \pm \cos\xi)$,

$$(19) \qquad \mathrm{N} = -\frac{\mathrm{M}}{8\pi c}\,\mathrm{ch}\,\eta\,(\mathrm{K}-\mathrm{E}), \qquad \cos\lambda = \frac{\mathrm{K}-\mathrm{E}}{\mathrm{N}},$$

$$(20) \qquad \sin\theta = \frac{r_3-r_2}{r_3+r_2} = \frac{\cos\xi}{\mathrm{ch}\,\eta} = \sin\alpha\cos\lambda, \qquad \sin\alpha = \frac{\sin\theta}{\cos\lambda} = \frac{\mathrm{N}}{\dfrac{\mathrm{K}-\mathrm{E}}{\sin\theta}},$$

whence λ, α were calculated for given N, starting from $\lambda = 0$, when $\mathrm{N} = \mathrm{K}-\mathrm{E}$, $\alpha = \theta$

Another method is given by MAXWELL in 'E. and M.,' § 702.

13. Next, for Φ and Ω,

(1)
$$d\Phi = \frac{PQ^2}{PY^2}\cdot\frac{b\,d\theta}{PQ} = \frac{\sigma-s_3}{\sigma-s}\cdot\frac{b}{a}\cdot\frac{\sqrt{(\sigma-s_3)}\,ds}{\sqrt{S}},$$

(2)
$$\frac{s_1-\sigma}{\sigma-s_3} = \left(\frac{r_3+r_2}{2a}\right)^2-1, \qquad \frac{\sigma-s_2}{\sigma-s_3} = 1-\left(\frac{r_3-r_2}{2a}\right)^2, \qquad \frac{\sqrt{(s_1-\sigma\,.\,\sigma-s_2)}}{\sigma-s_3} = \frac{b}{a},$$

(3)
$$d\Phi = \frac{\frac{1}{2}\sqrt{-\Sigma}}{\sigma-s}\,\frac{ds}{\sqrt{S}},$$

(4)
$$\Phi = 4\int_{s_3}^{s_2}\frac{\frac{1}{2}\sqrt{-\Sigma}}{\sigma-s}\,\frac{ds}{\sqrt{S}} = 2\pi f + 4K\,\mathrm{zn}\,fK',$$

in accordance with the previous expression for B, or $B(2fG')$, and

(5)
$$\Omega = 2\pi-\Phi = 2\pi(1-f)-4K\,\mathrm{zn}\,fK'.$$

Comparing this with the previous expression for Ω in (11), § 8, we have the theorem of the quadric transformation of the zeta function

(6)
$$2K\,\mathrm{zn}\,fK' = G\,\mathrm{zn}\,2fG'+G\gamma'\,\mathrm{sn}\,2fG'.$$

This is obtained by taking the quadric transformation formula, obtained from the geometry of fig. 2,

(7)
$$\mathrm{dn}\,fK' = \frac{\mathrm{dn}\,2fG'+\gamma'\,\mathrm{cn}\,2fG'}{1+\gamma'}, \quad\text{or}\quad 2K\,\mathrm{dn}\,fK' = G\,\mathrm{dn}\,2fG'+G\gamma'\,\mathrm{cn}\,2fG',$$

squaring it, and integrating both sides with respect to f.

According as the modulus γ or κ is employed, connected by the quadric transformation, as in MAXWELL'S 'E. and M.,' § 702, we take, to the modulus γ,

(8)
$$P = \frac{4aG}{r_3} = \frac{4aG\sqrt{\gamma'}}{\sqrt{(r_2r_3)}}, \qquad P\frac{b}{a} = 4G\gamma'\,\mathrm{sn}\,2fG',$$

$$M = -2\pi r_3[(2-\gamma^2)G-2H],$$

$$\Omega(f) = \Omega = 2\pi(1-f)-2G\,\mathrm{zn}\,2fG'-2G\gamma'\,\mathrm{sn}\,2fG',$$

$$\Omega(f)+\Omega(1-f) = 2\pi-P\frac{b}{a},$$

$$\mathrm{sn}\,2fG' = \frac{b}{r_2} = \frac{MP}{PB}, \qquad \mathrm{cn}\,2fG' = \frac{a-A}{r_2} = \frac{MB}{PB},$$

$$\mathrm{dn}\,2fG' = \frac{a+A}{r_3} = \frac{Pp}{ED} = \cos BAP = \cos FPp', \qquad \gamma' = \frac{PB}{PA},$$

or, to the modulus κ,

$$(9) \qquad \mathrm{P} = \frac{8a\mathrm{K}}{r_2+r_3} = \frac{4a\mathrm{K}\kappa'}{\sqrt{(r_2 r_3)}}, \qquad \mathrm{P}\frac{b}{a} = 4\mathrm{K}\kappa'^2 \operatorname{sn} f\mathrm{K}' \operatorname{sn}(1-f)\mathrm{K}',$$

$$\mathrm{M} = -2\pi(r_3-r_2)(\mathrm{K}-\mathrm{E}) = -4\pi\sqrt{(r_2 r_3)}(\mathrm{K}-\mathrm{E})\frac{\kappa}{\kappa'},$$

$$\Omega = 2\pi(1-f) - 4\mathrm{K}\operatorname{zn} f\mathrm{K}'.$$

$$\kappa = \frac{r_3-r_2}{r_3+r_2} = \frac{\mathrm{OE}}{\mathrm{OB}} = \frac{\mathrm{OB}}{\mathrm{OD}} = \frac{\mathrm{BE}}{\mathrm{BD}},$$

$$\mathrm{DPB} = \operatorname{am} f\mathrm{K}', \qquad \mathrm{DEP} = \operatorname{am}(1-f)\mathrm{K}',$$

$$\operatorname{dn} f\mathrm{K}' = \frac{\mathrm{B}p}{\mathrm{BD}}.$$

The article in the 'Trans. American Math. Society,' October, 1907 (A.M.S.), may be consulted for an elaborate and detailed discussion of the elliptic function analysis and procedure of former writers, and a numerical calculation is given there for the helix employed originally by VIRIAMU JONES. Measurement of fig. 2 gives κ, ψ, and thence, from LEGENDRE'S tables, K, E, $\mathrm{F}\psi$, and $f = \mathrm{F}\psi/\mathrm{K}'$.

Another numerical application of these formulas can be chosen from the dimensions of the current weigher at the N.P.L., Teddington, described in 'Phil. Trans.,' 1907.

14. Integrate Ω with respect to b to obtain the magnetic potential of the solenoidal current sheet, or of the equivalent close helical winding in the ampère balance.

In these integrations with respect to b the form $\Omega(\mathrm{MQ})$ of the III. E.I. comes in most appropriate as not involving b in MQ, and then

$$(1) \qquad \int \Omega\, db = 2\pi b - \iint \frac{\mathrm{A}a\cos\theta + a^2}{\mathrm{MQ}^2}\frac{b\, d\theta\, db}{\mathrm{PQ}}$$

$$= 2\pi b - \int \frac{\mathrm{A}a\cos\theta + a^2}{\mathrm{MQ}^2}\mathrm{PQ}\, d\theta$$

$$= 2\pi b - \int (\mathrm{A}a\cos\theta + a^2)\left(1 + \frac{b^2}{\mathrm{MQ}^2}\right)\frac{d\theta}{\mathrm{PQ}}$$

$$= 2\pi b - \frac{\mathrm{M}}{2\pi} - \mathrm{P}a - b\Omega(\mathrm{MQ})$$

$$= -\mathrm{P}a + \mathrm{QA} + \Omega b.$$

This solenoidal magnetic potential is the same as that of the cylinder on which the helix is wound, and so is the equivalent of the axial component of the gravitation attraction of the solid cylinder, and this is the difference of the potentials of the end circular plates. In this way we have arrived at the expression for W, the surface potential of the circle on AB, in the form

$$\text{(2)} \qquad W = Pa - QA - \Omega b,$$

$$\text{(3)} \qquad \left(\frac{dW}{da}, \frac{dW}{dA}, \frac{dW}{db}\right) = P, \ -Q, \ -\Omega,$$

in accordance with EULER'S theorem for a homogeneous function, in this case W is a homogeneous function of the first degree in the three variables a, A, b.

At a meeting of the London Mathematical Society, November 11, 1869 ('Proc. L.M.S.,' vol. III, p. 8), Prof. CAYLEY presiding, the Secretary, Mr. JENKINS, read a letter from Mr. CLERK MAXWELL asking the following question: "Can the potential of a uniform circular disc at any point be expressed by means of elliptic integrals?—I am writing out the theory of circular currents in which such quantities occur."

But the result is obvious from the theorem above of a homogeneous function, so that

$$\text{(4)} \qquad W = \frac{dW}{da}a + \frac{dW}{dA}A + \frac{dW}{db}b,$$

in which

$$\text{(5)} \qquad \frac{dW}{da} = P = \int \frac{a\,d\theta}{PQ}, \qquad \frac{dW}{dA} = \int \frac{+a\cos\theta\,d\theta}{PQ} = -Q, \qquad \text{and} \qquad \frac{dW}{db} = -\Omega,$$

for any shape of the disc.

Prof. CAYLEY'S attention was thereby directed to the subject, and he extended the investigation to the elliptic disc ('L.M.S.,' vol. VI).

15. Integrate P with respect to b to obtain the skin P.F. of the curved surface of the cylinder, drawing out the circle on AB like a concertina,

$$\text{(1)} \qquad \int P\,db = \iint \frac{a\,d\theta\,db}{PQ} = \int \text{th}^{-1}\frac{b}{PQ}a\,d\theta = I,$$

suppose, an intractable integral, $\text{th}^{-1}(b/PQ)$ being the potential of the generating line element of length b.

But $\int \cos\theta\,\text{th}^{-1}\frac{b}{PQ}\,d\theta$, as in the expression for L in § 2 (2), is tractable and given in finite terms, while $\int \sin\theta\,\text{th}^{-1}\frac{b}{PQ}\,d\theta$ is non-elliptic, expressed in the variable $\cos\theta$.

The integral I cannot be made to depend on a finite number of elliptic integrals, but requires to be expanded in an infinite series, and so we say it cannot be expressed in finite terms.

Expanded in a series

(2) $$\text{th}^{-1}\frac{b}{\text{PQ}} = \Sigma\frac{1}{2n+1}\left(\frac{b}{\text{PQ}}\right)^{2n+1} = \Sigma\frac{1}{2n+1}\left(\frac{b}{r_3}\right)^{2n+1}\frac{1}{(\Delta\omega)^{2n+1}},$$

where, as before, in (3), § 3,

(3) $$\theta = 2\omega, \qquad r^2 = r_3^{\,2}\cos^2\omega + r_2^{\,2}\sin^2\omega = r_3^{\,2}\,\Delta^2(\omega, \gamma), \qquad \gamma' = \frac{r_2}{r_3},$$

(4) $$\int_0^{2\pi}\text{th}^{-1}\frac{b}{\text{PQ}}\,.\,a\,d\theta = \Sigma\frac{1}{2n+1}\left(\frac{b}{r_3}\right)^{2n+1}\int_0^{\pi}\frac{2a\,d\omega}{(\Delta\omega)^{2n+1}} = 4a\Sigma\frac{1}{2n+1}\left(\frac{b^2}{r_2r_3}\right)^{n+\frac{1}{2}}\text{P}_n(u),$$

where $\text{P}_n(u)$ is the toroidal function, introduced by W. M. HICKS, 'Phil. Trans.,' 1881–4, defined by

(5) $$\text{P}_n(u) = \int_0^{\pi}\frac{d\theta}{(\text{ch}\,u + \text{sh}\,u\cos\theta)^{n+\frac{1}{2}}} = \int_0^{\pi}\left(\frac{\text{EA}\,.\,\text{EB}}{\text{EQ}^2}\right)^{n+\frac{1}{2}}d\vartheta$$

$$= 2\sqrt{\gamma'}\int_0^{\text{G}}\left(\frac{\text{dn}^2 v}{\gamma'}\right)^n dv, \qquad \gamma' = e^{-u},$$

given by the substitution

(6) $$\text{ch}\,u + \text{sh}\,u\cos\theta = \frac{\text{dn}^2 v}{\gamma'} = \frac{\text{EQ}^2}{\text{EA}\,.\,\text{EB}} = \frac{\text{EA}\,.\,\text{EB}}{\text{E}q^2}, \qquad \tfrac{1}{2}\theta = \text{am}(v, \gamma);$$

and P_n satisfies the differential equation

(7) $$\frac{d^2\text{P}}{du^2} + \coth u\frac{d\text{P}}{du} = \frac{d}{d\text{C}}(\text{C}^2-1)\frac{d\text{P}}{d\text{C}} = (n^2-\tfrac{1}{4})\,\text{P},$$

with $\text{C} = \text{ch}\,u$, and the sequence difference equation

(8) $$(2n+1)\,\text{P}_{n+1} - 4n\text{CP}_n + (2n-1)\,\text{P}_{n-1} = 0.$$

Expressed otherwise, with $u = e\text{G}$, $v = \text{G}+2f\text{G}'i$,

(9) $$\text{sn}\,v = \frac{r_3}{a+\text{A}}, \qquad \text{cn}\,v = \frac{ib}{a+\text{A}}, \qquad \text{dn}\,v = \frac{a-\text{A}}{a+\text{A}}, \qquad \theta = 2\,\text{am}\,u,$$

(10) $$\text{th}^{-1}\frac{b}{\text{PQ}} = \text{th}^{-1}\frac{i\,\text{cn}\,v}{\text{sn}\,v\,\text{dn}\,u} = i\tan^{-1}\frac{\text{cn}\,v}{\text{sn}\,v\,\text{dn}\,u} = \tfrac{1}{2}i\,[\text{am}(u-v) - \text{am}(u+v)],$$

(11) $$\text{I} = 2ai\int_0^{\text{G}}[\text{am}(u-v) - \text{am}(u+v)]\,d\,\text{am}\,u.$$

16. The S.F. of the P.F. W is L, so that the S.F. of P is $\frac{dL}{da}$; and in (2), § 2, with

(1) $$\frac{d}{da}\,\text{th}^{-1}\frac{b}{PQ} = -\frac{A\cos\theta + a}{MQ^2}\cdot\frac{b}{PQ},$$

(2) $$\frac{dL}{da} = \int 2\pi A\cos\theta\,\text{th}^{-1}\frac{b}{PQ}\,d\theta - \int 2\pi Aa\cos\theta\,\frac{A\cos\theta+a}{MQ^2}\cdot\frac{b\,d\theta}{PQ},$$

(3) $$a\frac{dL}{da} = L - \int 2\pi\,\frac{A^2a^2\cos^2\theta + Aa^2\cos\theta}{MQ^2}\cdot\frac{b\,d\theta}{PQ}$$

$$= -2\pi\int\frac{A^2a^2+Aa^2\cos\theta}{MQ^2}\frac{b\,d\theta}{PQ}$$

$$= -2\pi a^2\int\frac{MQ^2 - Aa\cos\theta - a^2}{MQ^2}\frac{b\,d\theta}{PQ}$$

$$= -2\pi Pab + 2\pi a^2\Omega\,(MQ),$$

(4) $$\frac{dL}{da} = 2\pi a\,(2\pi-\Omega) - 2\pi Pb = 2\pi a\Omega\,(1-f),$$

is the S.F. of P, the rim potential of the circle AB, and L in (12), § 8, is the S.F. of the circular disc on AB. But then, from (8), § 13, $\Omega\,(1-f)$ is the solid angle of the circle on the radius NP seen from Q on the circle on AB.

The S.F. of I, P.F. of the cylindrical skin, is then given by

(5) $$J = \int\frac{dL}{da}\,db = 2\pi a\int\frac{-MQ^2+Aa\cos\theta+a^2}{MQ^2}\,PQ\,d\theta$$

$$= \pi a\int(-MQ^2+a^2-A^2)\left(1+\frac{b^2}{MQ^2}\right)\frac{d\theta}{PQ}$$

$$= \pi a\int(-MQ^2+a^2-A^2-b^2)\frac{d\theta}{PQ} + \pi ab\int\frac{a^2-A^2}{MQ_2}\cdot\frac{b\,d\theta}{PQ}$$

$$= \pi a\int(-2A^2-b^2-2Aa\cos\theta)\frac{d\theta}{PQ} + 4\pi abB$$

$$= -\pi P\,(2A^2+b^2) - Ma + 2\pi ab\Omega\,(MQ) - \pi Pb^2$$

$$= -2\pi P\,(A^2+b^2) + 2\pi QAa + 2\pi ab\,(2\pi-\Omega),$$

(6) $$\frac{J}{2\pi} - Wa = -P\,(a^2+A^2+b^2) + 2QAa + 2\pi ab,$$

so that J is given in finite terms, while I is intractable, and requires to be given in a series. And generally in these investigations we find the S.F. has the superiority over the P.F. in simplicity of analytical structure. Thus the S.F. at P of the rod AB is PA$-$PB, and of the electrified disc AB is $\sqrt{[AB^2-(PA-PB)^2]}$.

17. The P.F. of the solid cylinder is given by

(1) $$V = \int W\,db,$$

in which

(2) $$\int Pa\,db = \iint \frac{a^2\,d\theta\,db}{PQ} = \int \text{th}^{-1}\frac{b}{PQ}\,a^2\,d\theta = a\text{I},$$

(3) $$\int -QA\,db = \iint Aa\cos\theta\,\frac{d\theta\,db}{PQ} = -\frac{L}{2\pi},$$

(4) $$\int -\Omega b\,db = \int [\Omega\,(MQ)-2\pi]\,b\,db$$

$$= \iint \frac{Aa\cos\theta+a^2}{MQ^2}\,\frac{b^2\,d\theta\,db}{PQ} - \pi b^2,$$

bringing in again the same intractable integral I.

We obtain V otherwise from the integration of

(5) $$\frac{dV}{da} = \text{I}.$$

As a verification we have to prove by differentiation that

(6) $$\frac{1}{2\pi A}\frac{dJ}{dA} = \frac{dI}{db} = P,$$

implied in the integration of (1), § 15, and

(7) $$\frac{dJ}{db} = -2\pi A\frac{dI}{dA} = \frac{dL}{da} = -2\pi Pb+2\pi a\,(2\pi-\Omega) = 2\pi a\Omega\,(1-f),$$

implied in (4), § 16, the expression of the rim S.F. of the circle AB.

And for the P.F. W and its S.F. $\frac{dL}{da}$

(8) $$2\pi A\frac{dW}{dA} = -2\pi QA\,\frac{d^2L}{da\,db},$$

(9) $$-2\pi A\frac{dW}{db} = 2\pi\Omega A = \frac{d^2L}{da\,dA}.$$

18. An integration of L in (6), § 2, with respect to b will give the S.F. of the solid cylinder

(1) $$N = \int L\,db = \tfrac{1}{2}\pi\int\left[a^2+A^2-2Aa\cos\theta-\frac{(a^2-A^2)^2}{MQ^2}\right]PQ\,d\theta,$$

and then $\frac{dN}{da}$ should be J, the S.F. of the cylindrical skin, as a verification. Here

$$(2) \qquad \int PQ\, d\theta = P\frac{A^2+a^2+b^2}{a} - 2QA,$$

$$(3) \qquad \int PQ \cos\theta\, d\theta = \tfrac{2}{3}PA - \tfrac{1}{3}Q\frac{A^2+a^2+b^2}{a},$$

$$(4) \qquad \int \frac{(a^2-A^2)^2}{MQ^2} PQ\, d\theta = \int (a^2-A^2)^2\left(1+\frac{b^2}{MQ^2}\right)\frac{d\theta}{PQ}$$
$$= (a^2-A^2)\left(P\frac{A^2-a^2}{a} + 4Bb\right),$$

$$(5) \qquad N = \tfrac{4}{3}\pi P a A^2 + \tfrac{1}{2}\pi P\frac{b^2}{a}(a^2+A^2) - \tfrac{1}{3}\pi QA(2a^2+2A^2-b^2) - 2\pi Bb(a^2-A^2)$$
$$= \pi Pa(\tfrac{4}{3}A^2+b^2) - \tfrac{1}{3}\pi QA(2a^2+2A^2-b^2) - \pi(a^2-A^2)b\Omega(MQ).$$

In the interior of the solid cylinder of unit density, LAPLACE'S equation (2), § 6, changes to

$$(6) \qquad \frac{d}{dA}\left(A\frac{dV}{dA}\right) + \frac{d}{db}\left(A\frac{dV}{db}\right) + 4\pi A = 0,$$

or with $V' = V + \pi A^2$,

$$(7) \qquad \frac{d}{dA}\left(A\frac{dV'}{dA}\right) + \frac{d}{db}\left(A\frac{dV'}{db}\right) = 0,$$

so that the S.F. N′ is given by

$$(8) \qquad \frac{dN'}{dA} = 2\pi A\frac{dV'}{db} = 2\pi A\frac{dV}{db},$$
$$\frac{dN'}{db} = -2\pi A\frac{dV'}{dA} = -2\pi A\left(\frac{dV}{dA} + 2\pi A\right)$$
$$= -2\pi A\frac{dV}{dA} - 4\pi^2A^2,$$

requiring the subtraction of $4\pi^2A^2$ in the interior volume.

19. With these values of a P.F. and its S.F. the relations must verify in § 6 (1, 2, 3).

Thus for the P.F. V and S.F. N of the solid cylinder,

$$(1) \qquad \frac{1}{2\pi A}\frac{dN}{dA} = \frac{dV}{db} = W,$$

implied in the integration in (1), § 18,

$$(2) \qquad \frac{dN}{db} = -2\pi A\frac{dV}{dA} = L.$$

In making these verifications use must be made of the differentiation formulas given in the 'American Journal of Mathematics' ('A.J.M.'), 1910, p. 392, where D denoting $r_2^2 r_3^2 = (A^2+a^2+b^2)^2 - 4A^2a^2$,

(3) $$\frac{dP}{dA} = -PA\frac{A^2-a^2+b^2}{D} + Qa\frac{A^2-a^2-b^2}{D},$$

(4) $$\frac{dP}{db} = -Pb\frac{A^2+a^2+b^2}{D} + QA\frac{2ab}{D} = \frac{4abr_1E}{D},$$

(5) $$a\frac{dP}{da} = -A\frac{dP}{dA} - b\frac{dP}{db} = P + a\frac{d\Omega}{db},$$

(6) $$\frac{dQA}{dA} = -A\frac{d\Omega}{db} = -PaA\frac{A^2-a^2-b^2}{D} + QA^2\frac{A^2-a^2+b^2}{D},$$

(7) $$\frac{dQA}{db} = A\frac{d\Omega}{dA} = -PaA\frac{2Ab}{D} + QAb\frac{A^2+a^2+b^2}{D},$$

with the check formulas

(8) $$a\frac{dP}{dA} - A\frac{dQ}{dA} - b\frac{d\Omega}{dA} = 0,$$

$$a\frac{dP}{db} - A\frac{dQ}{db} - b\frac{d\Omega}{db} = 0,$$

$$a\frac{dP}{da} - A\frac{dQ}{da} - b\frac{d\Omega}{da} = 0,$$

$$Pa - QA - \Omega b = W,$$

(9) $$\frac{dW}{dA} = -Q, \qquad \frac{dW}{db} = -\Omega, \qquad \frac{dW}{da} = P.$$

Reviewing these calculations it will be noticed that the S.F. again shows generally a marked superiority over the P.F. in its analytical simplicity.

This N in (1), § 18, is the expression which gives the potential energy (P.E.) of the two co-axial helical currents, or their equivalent current sheet solenoids, investigated by V. JONES, 'Roy. Soc. Proc.,' 1897, or the mutual P.E. of the two pairs of equivalent end plates ('A.M.S.,' 1907, § 59).

But as it is the force only between the two currents which is required, and this is given by $dN/db = L$, we need only calculate the end values of L for measurement in the current weigher.

20. As another illustration of the extra analytical simplicity of the S.F., take the calculations of BROMWICH ('L.M.S.,' 1912), where the results are expressed in a series for the attraction and P.F. of a circular disc, the circle on AB, where the surface density is $\sigma = ky^n$, varying as the n^{th} power of the distance y from the centre O.

The ring P.F. of a circular element is

(1) $$dP = \int \frac{\sigma\, dy \,.\, y\, d\theta}{PQ'}, \qquad PQ'^2 = A^2 + 2Ay\cos\theta + y^2 + b^2,$$

and the S.F. is

(2) $$dR = 2\pi\sigma\, dy\, [y\Omega\, (MQ') - P'b].$$

Changing from $\Omega\,(MQ')$ to the form

(3) $$\Omega\,(PZ) = \int \frac{Aa\cos\theta + A^2 + b^2}{PZ^2} \cdot \frac{b\, d\theta}{PQ},$$

$PZ^2 = A^2\sin^2\theta + b^2$, as not involving a or y, PZ the perpendicular on OQ, this form of $\Omega\,(PZ)$ is obtained by the dissection of the circle into the sector elements $\frac{1}{2}a^2\, d\theta$ ('A.M.S.,' p. 506, § 48),

(4) $$R = \int 2\pi\sigma y\, dy \int \frac{Ay\cos\theta + A^2 + b^2}{A^2\sin^2\theta + b^2} \cdot \frac{b\, d\theta}{PQ} - \int 2\pi\sigma b\, dy \int \frac{y\, d\theta}{PQ'}$$

$$= \iint 2\pi\sigma y\, dy \frac{A^2\cos^2\theta}{A^2\sin^2\theta + b^2} \cdot \frac{b\, d\theta}{PQ'} + \iint 2\pi\sigma y\, dy \frac{Ay\cos\theta}{A^2\sin^2\theta + b^2} \cdot \frac{b\, d\theta}{PQ'}$$

$$= \iint 2\pi\sigma y\, dy \frac{A^2\cos^2\theta + Ay\cos\theta}{A^2\sin^2\theta + b^2} \cdot \frac{b\, d\theta}{PQ'},$$

or with $\sigma = ky^n$,

(5) $$R = \int \frac{ky^{n+1}\, dy}{PQ'} \int \frac{2\pi A^2\cos^2\theta}{A^2\sin^2\theta + b^2} \,.\, b\, d\theta + \int \frac{ky^{n+2}\, dy}{PQ'} \int \frac{2\pi A\cos\theta}{A^2\sin^2\theta + b^2} \,.\, b\, d\theta.$$

Here the y integrations are effected by the formula of reduction obtained from the integration of

(6) $$\frac{d}{dy}(y^{n+1}PQ') = [(n+2)\, y^{n+2} + (2n+3)\, Ay^{n+1}\cos\theta + (n+1)\,(A^2+b^2)\, y^n]\, \frac{1}{PQ},$$

and so R can be obtained in finite terms.

But if we attempt the determination of the P.F. the intractable I puts in an appearance when n is odd.

Consider, for example, the flat lens of § 16, 'A.J.M.,' 1919, where $\sigma = k\left(1 - \frac{y^2}{a^2}\right)$; or for $\sigma = k\left(1 - \frac{y^2}{a^2}\right)^{-\frac{1}{2}}$, as in the distribution of electricity in the circular disc.

21. Taking the form in (3), § 20, it can be resolved into

(1) $$\Omega\,(PZ) = -\Omega_1\,(PZ) + \Omega_2\,(PZ),$$

(2) $$\Omega_1 = \tfrac{1}{2}\int \frac{a - \sqrt{(A^2+b^2)}}{\sqrt{(A^2+b^2)} + A\cos\theta} \cdot \frac{b\, d\theta}{PQ}, \qquad \Omega_2 = \tfrac{1}{2}\int \frac{a + \sqrt{(A^2+b^2)}}{\sqrt{(A^2+b^2)} - A\cos\theta} \cdot \frac{b\, d\theta}{PQ},$$

two III. E.I.'s in the form of B in (10), (11), § 4.

To reduce Ω_1 to this standard form, put

(3) $$2a\sqrt{(A^2+b^2)}+2Aa\cos\theta = m^2(\tau_1-t),$$

$$2a\sqrt{(A^2+b^2)}+2Aa = m^2(\tau_1-t_3), \qquad 2a\sqrt{(A^2+b^2)}-2Aa = m^2(\tau_1-t_2),$$

$$a^2+A^2+b^2-2a\sqrt{(A^2+b^2)} = [a-\sqrt{(A^2+b^2)}]^2 = m^2(t_1-\tau_1),$$

(4) $$\Omega_1 = \int_{t_2}^{t_3}\frac{\sqrt{-U_1}}{\tau_1-t}\cdot\frac{dt}{\sqrt{T}};$$

and to reduce Ω_2, put

(5) $$2a\sqrt{(A^2+b^2)}-2Aa\cos\theta = m^2(t-\tau_2)$$

$$2a\sqrt{(A^2+b^2)}-2Aa = m^2(t_3-\tau_2), \qquad 2a\sqrt{(A^2+b^2)}+2Aa = m^2(t_2-\tau_2)$$

$$a^2+A^2+b^2+2a\sqrt{(A^2+b^2)} = [a+\sqrt{(A^2+b^2)}]^2 = m^2(t_1-\tau_2),$$

(6) $$\Omega_2 = \int\frac{\sqrt{-U_2}}{t-\tau_2}\cdot\frac{dt}{\sqrt{T}}.$$

Then the sequence runs

(7) $$\infty > t_1 > \tau_1 > t_2 > t > t_3 > \tau_2 > -\infty,$$

and we take

(8) $$f_1 G' = \int_{\tau_1}^{t_1}\frac{\sqrt{(t_1-t_3)}\,d\tau}{\sqrt{-U}} = \mathrm{sn}^{-1}\sqrt{\frac{t_1-\tau_1}{t_1-t_2}} = \mathrm{cn}^{-1}\sqrt{\frac{\tau_1-t_2}{t_1-t_2}} = \mathrm{dn}^{-1}\sqrt{\frac{\tau_1-t_3}{t_1-t_3}},$$

(9) $$f_2 G' = \int_{-\infty}^{\tau_2}\frac{\sqrt{(t_1-t_3)}\,d\tau}{\sqrt{-U}} = \mathrm{sn}^{-1}\sqrt{\frac{t_1-t_3}{t_1-\tau_2}} = \mathrm{cn}^{-1}\sqrt{\frac{t_3-\tau_2}{t_1-\tau_2}} = \mathrm{dn}^{-1}\sqrt{\frac{t_2-\tau_2}{t_1-\tau_2}},$$

(10) $$\Omega_1 = \pi f_1+2G\,\mathrm{zn}\,f_1 G', \qquad \Omega_2 = \pi f_1+2G\,\mathrm{zs}\,f_2 G',$$

and with $f_2-f_1 = 2f$,

(11) $$\Omega(\mathrm{PZ}) = 2\pi f+2G\,\mathrm{zn}\,2fG'+2G\gamma'\,\mathrm{sn}\,2fG'$$
$$= \Omega(\mathrm{MQ}) = 2\pi-\Omega.$$

Interpreted geometrically on fig. 2, with

(12) $$\mathrm{sn}\,f_1 G' = \sqrt{\frac{t_1-\tau_1}{t_1-t_2}} = \frac{a-\sqrt{(A^2+b^2)}}{r_2},$$

(13) $$\mathrm{sn}\,(1-f_1)\,G' = \sqrt{\frac{t_1-t_3}{t_1-t_2}}\sqrt{\frac{\tau_1-t_2}{\tau_1-t_3}} = \frac{r_3}{r_2}\sqrt{\frac{\sqrt{(A^2+b^2)}-A}{\sqrt{(A^2+b^2)}+A}},$$

(14) $$\mathrm{sn}\,f_2 G' = \sqrt{\frac{t_1-t_3}{t_1-\tau_2}} = \frac{r_3}{a-\sqrt{(A^2+b^2)}},$$

(15) $$\mathrm{sn}\,(1-f_2)\,G' = \sqrt{\frac{t_3-\tau_2}{t_2-\tau_2}} = \sqrt{\frac{\sqrt{(A^2+b^2)}-A}{\sqrt{(A^2+b^2)}+A}}$$
$$= \frac{b}{\sqrt{(A^2+b^2)}+A} = \frac{\sqrt{(A^2+b^2)}-A}{b} = \gamma'\,\mathrm{sn}\,(1-f_1)\,G'.$$

We may drop G′ without ambiguity, and then

(16) $$\gamma' \operatorname{sn} f_1 \operatorname{sn} f_2 = \frac{a-\sqrt{(A^2+b^2)}}{a+\sqrt{(A^2+b^2)}},$$

(17) $$\operatorname{cn} f_1 \operatorname{dn} f_1 = \frac{\sqrt{(\tau_1-t_2 \,.\, \tau_1-t_3)}}{t_1-t_2 \,.\, t_1-t_3} = \frac{2ab}{r_2 r_3},$$

(18) $$\operatorname{cn} f_2 \operatorname{dn} f_2 = \frac{\sqrt{(t_3-\tau_2 \,.\, t_2-\tau_2)}}{t_1-\tau_2} = \frac{2ab}{[a+\sqrt{(A^2+b^2)}]^2},$$

(19) $$\operatorname{sn}(f_2-f_1) = \frac{\operatorname{sn} f_2 \operatorname{cn} f_1 \operatorname{dn} f_1 - \operatorname{sn} f_1 \operatorname{cn} f_2 \operatorname{dn} f_2}{1-\gamma'^2 \operatorname{sn}^2 f_1 \operatorname{sn}^2 f_2} = \frac{b}{r_2} = \operatorname{sn} 2f\mathrm{G}'$$

$$f_2-f_1 = 2f, \qquad \operatorname{cn} 2f\mathrm{G}' = \frac{a-A}{r_2}, \qquad \operatorname{dn} 2f\mathrm{G}' = \frac{a+A}{r_2}.$$

(20) $$\operatorname{sn}(f_2+f_1) = \frac{ab}{r_2\sqrt{(A^2+b^2)}} = \frac{a}{\sqrt{(A^2+b^2)}} \sin \mathrm{BOP} = \frac{\mathrm{OB}}{\mathrm{OP}} \sin \mathrm{BOP}$$
$$= \frac{\mathrm{OP}'}{\mathrm{OB}} \sin \mathrm{BOP} = \sin \mathrm{OBP}',$$

$\mathrm{OBP}' = \operatorname{am}(f_2+f_1)\,\mathrm{G}' = \operatorname{am} 2f'\mathrm{G}'$, suppose.

When OP is produced to cut the circle on AB in R, and the circle on ED again in P′, PP′ will touch the co-axial circle in R; and by the poristic property of these circles with the elliptic function interpretation,

(21) $$\mathrm{OBP}_1 = \operatorname{am} f_1\mathrm{G}', \qquad \mathrm{OBP}_2 = \operatorname{am} f_2\mathrm{G}',$$

if the tangent at the lowest point e, e' of the R circle, and of the other co-axial circle touching P′P″, where EP″ = EP, cuts the circle on ED in P_1 and P_2.

22. Treating Ω(PR) of (3), § 9, in the same way

(1) $$\Omega(\mathrm{PR}) = \Omega_3+\Omega_4-\int \frac{b\,d\theta}{\mathrm{PQ}},$$

(2) $$\Omega_3 = \tfrac{1}{2}\int \frac{\sqrt{(a^2+b^2)}-A}{\sqrt{(a^2+b^2)}+a\cos\theta}\,\frac{b\,d\theta}{\mathrm{PQ}}, \qquad \Omega_4 = \tfrac{1}{2}\int \frac{\sqrt{(a^2+b^2)}+A}{\sqrt{(a^2+b^2)}-a\cos\theta}\,\frac{b\,d\theta}{\mathrm{PQ}},$$

and a similar reduction will give

(3) $$\Omega_3 = \int \frac{\sqrt{-U_3}}{\tau_3-t} \cdot \frac{dt}{\sqrt{T}} = \pi f_3 + 2\mathrm{G} \operatorname{zn} f_3\mathrm{G}',$$

(4) $$\Omega_4 = \int \frac{\sqrt{-U_4}}{t-\tau_4} \cdot \frac{dt}{\sqrt{T}} = \pi f_4 + 2\mathrm{G} \operatorname{zs} f_4\mathrm{G}',$$

(5) $$\infty > t_1 > \tau_3 > t_2 > t > t_3 > \tau_4 > -\infty,$$

(6) $$\operatorname{sn} f_3 G' = \frac{\sqrt{(a^2+b^2)}-A}{r_2}, \qquad \operatorname{sn} f_4 G' = \frac{r_3}{\sqrt{(a^2+b^2)}+A},$$

(7) $$\operatorname{sn}(1-f_4)G' = \sqrt{\frac{\sqrt{(a^2+b^2)}-a}{\sqrt{(a^2+b^2)}+a}} = \frac{b}{\sqrt{(a^2+b^2)}+a} = \gamma' \operatorname{sn}(1-f_3)G',$$

and so on, with $f_3+f_4 = 2f$. Because

(8) $$\gamma' \operatorname{sn} f_3 \operatorname{sn} f_4 = \frac{\sqrt{(a^2+b^2)}-A}{\sqrt{(a^2+b^2)}+A},$$

(9) $$\operatorname{cn} f_3 \operatorname{dn} f_3 = \frac{2Ab}{r_2 r_3}, \qquad \operatorname{cn} f_4 \operatorname{dn} f_4 = \frac{2Ab}{[\sqrt{(a^2+b^2)}+A]^2},$$

(10) $$\operatorname{sn}(f_4+f_3) = \frac{b}{r_2} = \operatorname{sn} 2fG',$$

(11) $$\operatorname{sn}(f_4-f_3) = \frac{Ab}{r_2\sqrt{(a^2+b^2)}} = \frac{A}{\sqrt{(a^2+b^2)}} \sin \mathrm{OBP}$$

Produce NP parallel to AB to cut the circle on ED again in P_4, then

(12) $$\frac{A}{\sqrt{(a^2+b^2)}} = \frac{NP}{NB} = \frac{BP}{BP_4},$$

because P, P_4 are inverse points in the circle, centre N, through B; so that

(13) $$\operatorname{sn}(f_4-f_3) = \frac{BP}{BP_4} \sin BPP_4 = \sin BP_4P,$$

(14) $$BP_4P = \operatorname{am}(f_4-f_3)G', \qquad OBP_4 = \operatorname{am}(2-f_4+f_3)G'.$$

Thence a geometrical construction for $\operatorname{am} f_3 G'$ and $\operatorname{am} f_4 G'$, similar to that above for f_1 and f_2.

The pole of the chord RR′ through B will lie on the line through A perpendicular to AB, at A′ suppose, and the tangent A′R′ will be parallel to AB.

A whole chapter might be written of elliptic function theory, showing in this manner the geometrical interpretation of the various formulas, especially of the quadric transformation, in relation to co-axial circles.

23. Our chief object was to employ a straightforward integration of MAXWELL'S result as a direct road to the analytical results required in ampère-balance current weighing. The check on the arithmetical calculations has been explained and carried out in the 'Transactions of the American Mathematical Society' ('A.M.S.'), 1907, § 56, p. 516.

Considering that the chief analytical and numerical difficulties in these operations arise in the III. E.I. expression of Ω, and that this is cancelled by making

(1) $$A = a, \qquad f = \tfrac{1}{2}, \qquad \Omega = \pi - 2G\gamma' = \pi - 2K(1-\kappa),$$

$\gamma' = \cos \text{APB}$, so that APB is the modular angle,

$$(2) \qquad L = \pi P a b + \tfrac{1}{2} M b, \qquad N = \pi P a \left(\tfrac{4}{3} a^2 + b^2\right) + M \left(\tfrac{2}{3} a^2 - \tfrac{1}{6} b^2\right),$$

involving only the complete E.I. I. and II., given in LEGENDRE'S tables with extreme accuracy, it would appear to be of practical advantage to make all the helical coils of the same diameter.

This would prevent one coil from going inside another, and they would require to be opposed in axial prolongation, as in the Lorenz apparatus at Teddington, described in 'Phil. Trans.,' 1913, by F. E. SMITH.

Here is a question to be decided by practical experience as to the advantage or defects of this suggestion.

The current weigher is designed for legal commercial use, in the definition of the electrical units in an Act of Parliament, and these require to be measured to as many significant figures as possible, warranted by the most careful measurement of skilled observers.

The legal definition must be specified with the same precision of language as we find in the Act of Parliament on Weights and Measures, defining the standard pound and yard, the length of the seconds pendulum with a view of checking and preserving the standard, the volume of the gallon in cubic inches, and other standards of measure in civilised life.

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III. Bakerian Lecture, 1918.—*Experiments on the Artificial Production of Diamond.*

By the Hon. Sir Charles Algernon Parsons, *K.C.B., F.R.S.*

Lecture delivered April 25,—MS. received July 15, 1918.

Contents.

In this paper is given an account of experiments on the artificial production of diamond which I commenced in 1887, and have carried on intermittently till the commencement of the War, when they were interrupted. Although the account is not as full as I could have wished, yet it is hoped that from the description of such experiments as relate to the salient features, followed by a summary of their bearings upon the research, and the conclusions at which we have arrived, together with an Appendix stating briefly the character of about one-third of the total number of experiments, a fair idea may be gathered of this research.

One reason for writing this paper at the present time has been a publication on the same subject by Otto Ruff in 'Zeitschrift für Anorganische Chemie,' vol. 99, pp. 73–104, May 25, 1917, who also referred to the work of Lummer on the apparently molten aspect of the surface of the carbon of the electric arc.

In my paper to the Royal Society in 1888 were described experiments where a carbon rod heated by a current of electricity (fig. 1) was immersed in liquids at pressures up to 2200 atmospheres, and where the liquids, benzene, paraffin, treacle, chloride and bisulphide of carbon, were found to yield deposits of amorphous carbon.

In my paper of 1907 allusion was made to experiments in liquids at a pressure of 4400 atmospheres, and to the distillation of carbon in carbon monoxide and dioxide at this pressure with similar results, also to an attempt to melt carbon at pressures up to 15,000 atmospheres, which produced soft graphite, and an experiment where a carbon crucible, containing iron previously heated and carburized in the electric furnace, was quickly transferred to a steel die, and while molten and during cooling subjected to a pressure of 11,200 atmospheres, the analyses showing less crystalline residue than if the crucible had been cooled in water.

It was also emphasized that the pressure of 11,200 atmospheres must be greater than could be produced in the interior of a spheroidal mass of cast iron when suddenly cooled, and that the inference from these experiments was that mechanical pressure is not the cause of the production of diamond in rapidly cooled iron, as had been

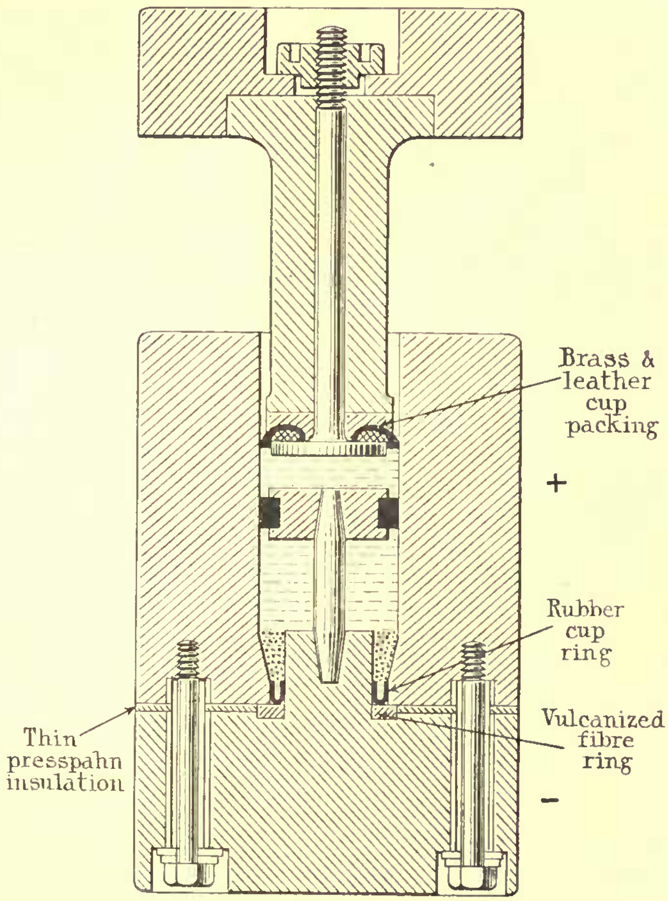

Fig. 1.

supposed by MOISSAN. This conclusion appears to us in the light of our more recent experiments to be one of great importance, and it will be further discussed in this paper.

It may be well to state that, in order to facilitate a clearer view of the bearing of each experiment on the subject, they are not placed always in chronological order. The difficulty of ensuring satisfactory experiments and the elusive character of the analyses must be the excuse for the random character of some of the former. The great majority of the experiments were failures as regards results, but a few have given information that was scarcely anticipated when they were devised.

Several thousand experiments have been made and a much greater number of analyses, generally following the methods of MOISSAN and CROOKES; the more important experiments are described at some length, and in most cases are typical of groups or repetitions of the same experiment with small variations.

The selection has been chiefly determined by their bearing on the general trend of the results of our own work and the work of others.

Those who are familiar with analyses for the detection and isolation of minute particles of diamond will know of the tendency of such particles to float, and to become lost in the frequent washings. To diminish the risk of arriving at erroneous

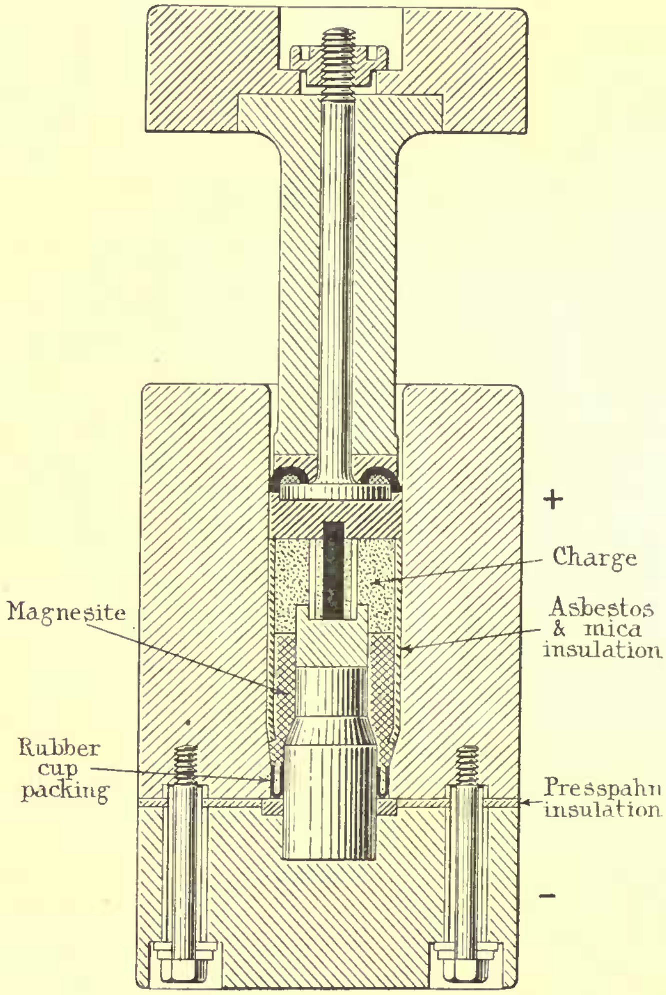

Fig. 2.

conclusions the analyses of the more important experiments have generally been repeated several times.

Experiments under High Pressure.

In the experiments designed to test chemical reactions under high pressure, where the charge was heated by passing an electric current through a central core (fig. 2) small residues of diamond occasionally occurred. A review of these experiments, however, indicates in most cases an association with iron, whether introduced intentionally, or present from the melting of the poles, or from other causes; allusion to this is made in the Appendix.

Experiments Designed to Melt Carbon under Pressure by Resistance Heating.

In the attempts to melt carbon under pressure by this method (fig. 3) heat was applied for a duration of 5 seconds, sufficient in amount to melt the graphite core six times over, with the result of only altering the structure. RICHARD THRELFALL

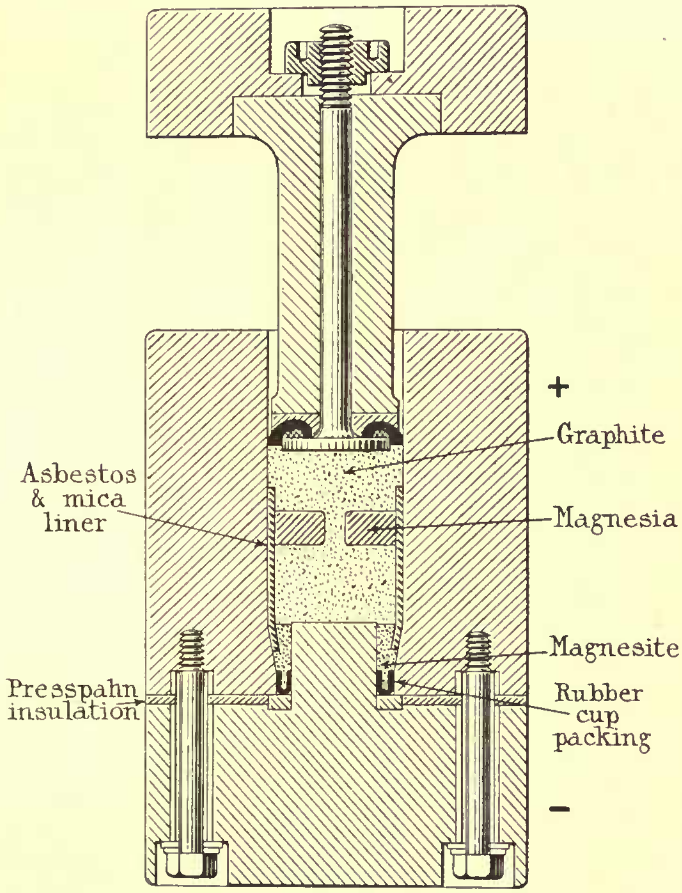

Fig. 3.

independently came to the conclusion from his experiments at about the same time, 1907, that under 100 tons per square inch, graphite, electrically heated, remained graphite.

It appeared, however, desirable further to investigate the possibility of carbon losing its electrical conductivity when approaching its melting point, as alleged by LUDWIG and others, and of thus shunting the current from itself on to the contiguous molten layers of the insulating barrier surrounding it. There had, however, been no indication of this having occurred, even momentarily; the evidence was rather that the graphite core had been vaporized and condensed in the surrounding parts of the charge, yet it was thought well to repeat the experiment with rods of iron and tungsten imbedded in the core, so that should the temperature of volatilization of the metals under a pressure of 12,000 atmospheres exceed that necessary to liquefy carbon under the same pressure, the presence of these metals might produce a different result. No change however occurred, though in one experiment the pressure was raised to 15,000 atmospheres.

Experiments Designed to Melt Carbon under Pressure by the Rapid Compression of Flame.

A different mode of attack was then arranged, which would ensure that carbon should be subjected to an extremely high temperature concurrently with high pressure, obtained by the rapid compression of the hottest possible flame, that of acetylene and oxygen, with a slight excess of the former to provide the carbon.

The arrangement was as follows (figs. 4 and 5):—

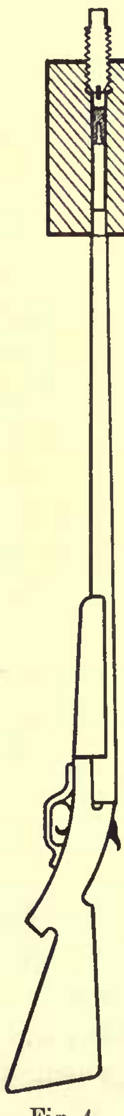

Fig. 4.

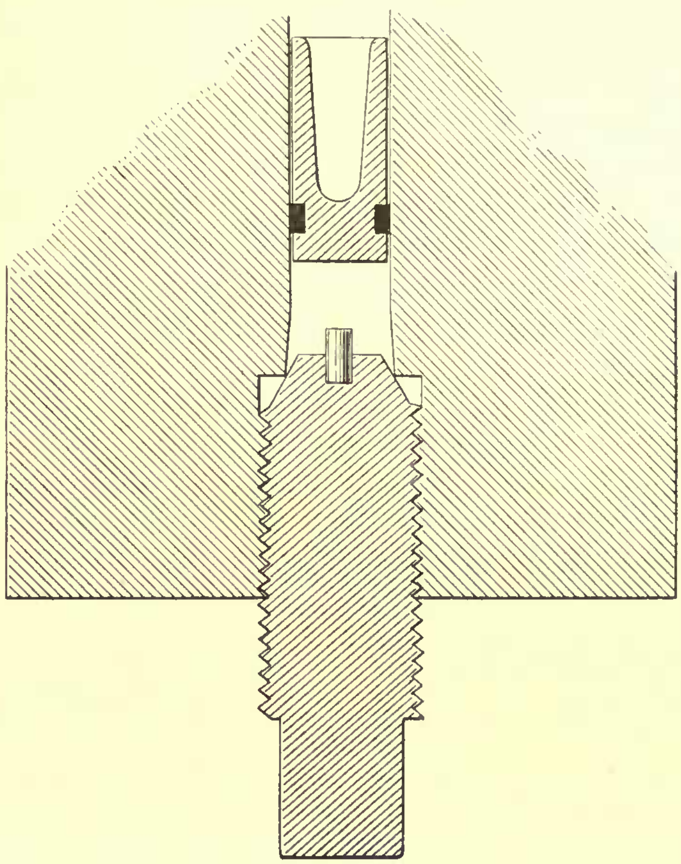

Fig. 5.

A very light piston made of tool steel was carefully fitted to the barrel of a Duck-gun of 0·9-inch bore; the piston was flat in front, lightened out behind, and fitted with a cupped copper gas check ring, the cup facing forward; the total travel of the piston was 36 inches. To the muzzle of the gun was fitted a prolongation of the barrel, formed out of a massive steel block, the joint being gas-tight. The end of the bore in the block was closed by a screwed-in plug made of tempered tool steel, also with a gas-tight collar. A small copper pin projected from the centre of the plug to give a record of the limit of travel of the piston.

The gun was loaded with 2 drachms of black sporting powder, which amount had been calculated from some preliminary trials. The barrel in front of the piston was filled with acetylene and oxygen, with a small excess of acetylene. It was estimated that this mixture would explode when the piston had travelled about half-way along the bore; when fired the piston travelled to within $\frac{1}{8}$-inch of the end, as had been estimated, giving a total compression ratio of 288 to 1.

Result.—The surfaces of the end plug, the fore end of the piston, and the circumference of the bore up to $\frac{3}{8}$-inch from the end of the plug had been fused to a depth of about 0·01-inch and were glass hard, the surface of the copper pin had been vaporized and copper sprayed over the surface of the end plug and piston.

The end plug showed signs of compression, and the bore of the block for $\frac{3}{8}$-inch from the plug was enlarged by 0·023-inch in diameter, both deformations indicating that a pressure of above 15,000 atmospheres had been reached. A little brown carbon was found in the chamber, which was easily destroyed by boiling sulphuric acid and nitre with no residue. There was a small crystalline residue from the melted layer of the end plug, from which was isolated one non-polarizing crystal, probably diamond, but too small to identify with absolute certainty.

Considering the light weight of the piston and the short duration of the exposure to heat, also the small diameter and volume of the end clearance space, the observed effects would seem to indicate that a very abnormal temperature had been reached, many times greater than exists in the chambers of large guns. There was, however, no evidence of any melting and re-crystallization of the free carbon present. In the Appendix is given a calculation from which it seems that the temperature reached was probably above 15,250° C.

Experiments with High Velocity Bullets.

As it seemed desirable to try the effect of still higher pressures, a rifle, 0·303-inch bore, was fitted with a specially strong breech mechanism by RIGBY, capable of withstanding a charge of cordite 90 per cent. in excess of the service charge.

The gun (fig. 6) was fixed in a vertical position on the wall of the armoured press house, with its muzzle 6 inches from a block of steel, in which a hole

0·303-inch diameter had been drilled to a depth somewhat greater than the length of the bullet, and in alignment with the bore of the gun; the trigger was pulled by a string from without. Cylindrical bullets of steel with a copper driving band were used, shorter than the service bullet, and about one-half of the weight, some with cupped noses to entrain material, some with coned noses to match the bottom of the hole in the block. The velocity with 90 per cent. excess charge was estimated to be about 5000 ft./secs.

The substance to be compressed was placed either at the bottom of the hole when the coned-nose bullet was used, or over the mouth of the hole when the cupped-nose

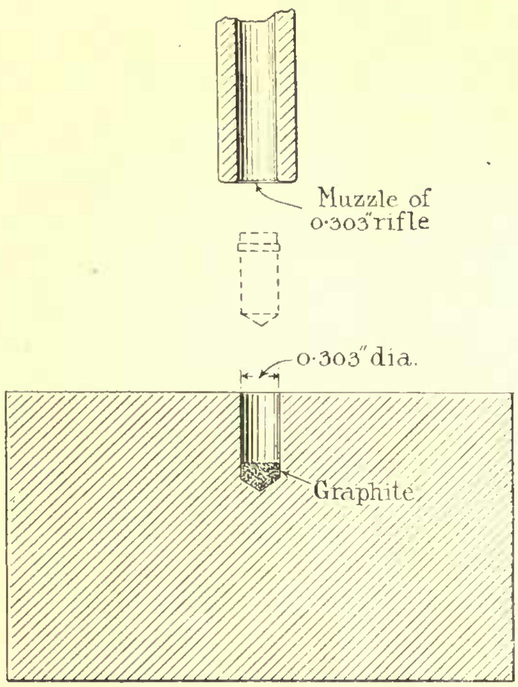

Fig. 6.

bullets were used. Some of the bullets were of mild steel, but those with cupped noses were of tool steel.

The substances placed in the hole are given in the Appendix, and included graphite, sugar carbon, bisulphide of carbon, oils, &c., graphite and sodium nitrate, graphite and fulminate of mercury, finely divided iron and fine carborundum, olivine and graphite. After each shot (fig. 7) the bullet and surrounding steel were drilled out, and the chips and entrained matter analysed.

Several experiments were also made with a bridge of arc-light carbon just over the hole, raised to the limit of incandescence by an electric current, and the shot fired through into the hole at the moment the carbon commenced to vaporize, as observed in a mirror from without. Also an arc between two carbons was arranged just over the hole (fig. 8) and the shot fired through it, as also through a crucible of carbon with a very thin bottom containing a little molten highly carburized iron.

Of all these experiments the only ones that yielded a reasonable amount of residue were one made with graphite wrapped in tissue paper, the bullet, however, in this case having grazed the side of the hole, and thus producing some molten iron by the friction, as also the shots through the incandescent bridge, where again some molten metal would probably occur. The residues were in all cases exceedingly small and not more than would be produced from a small amount of iron melted, carburized and quickly cooled. There was no evidence of any incipient transformation of carbon in bulk into diamond that could be detected by analysis.

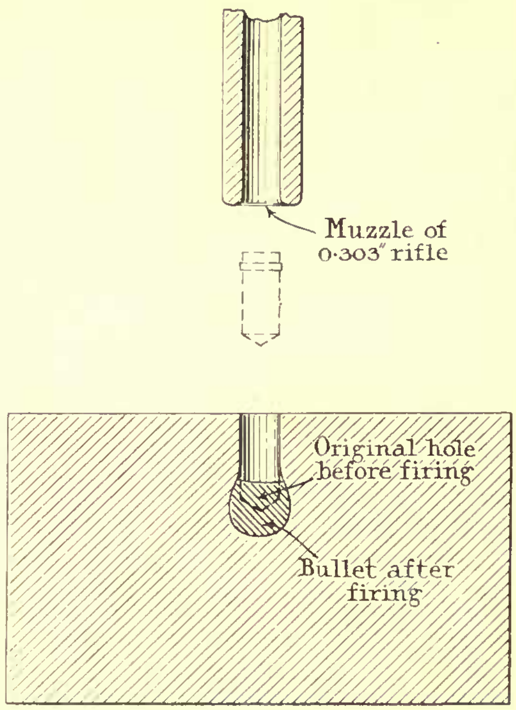

Fig. 7.

A bullet was also fired into a long hole, 0·303-inch in diameter, bored in a steel block and filled with acetylene gas, retained by gold-beater's skin over the mouth, thus repeating the flame experiment (but in this case without oxygen) on a small scale with the intensest pressures available. The residue was nil.

The pressure on impact of a steel bullet fired into a hole in a steel block which it fits is limited by the coefficient of compressibility of the steel, and with a velocity of 5000 ft./secs. is about 2000 tons per sq. inch. Measurements made from a section through the block and bullet (fig. 7) showed that the mean retarding force on the frontal face, after impact till it had come to rest, was about 600 tons per sq. inch.

Several experiments were made by substituting a tungsten-steel block, and a hole tapering gently from 0·303-inch at the mouth to 0·125-inch at the bottom, and using a mild steel bullet, which on entry would be deformed and a greatly increased velocity imparted to the nose. Progressively increased charges were used, and even with relatively small charges the block cracked on the second round. With the 90 per cent. excess charge, the block always split on the first shot, but this probably occurred after impact, and not till the full instantaneous pressure had been exerted, which was estimated to be greater than with the plain hole, probably over 5000 tons.

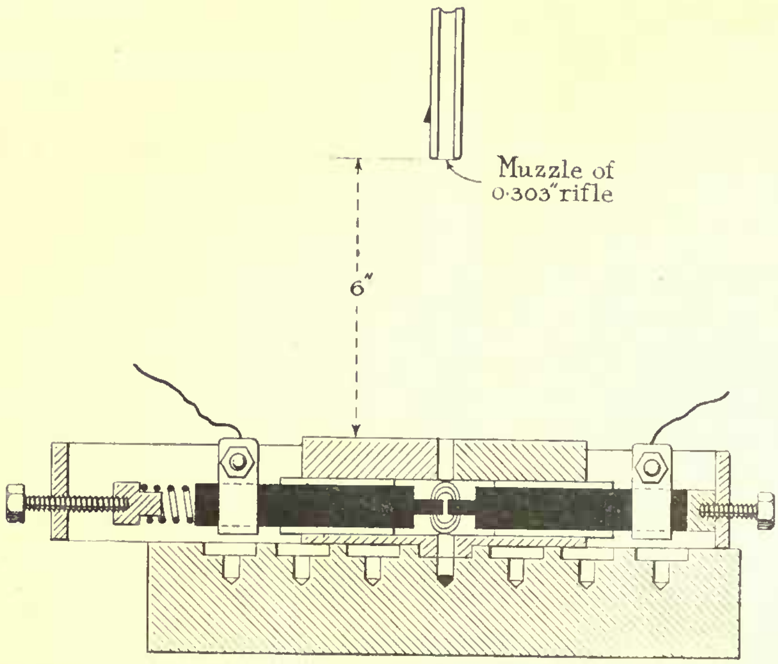

Fig. 8.

Only graphite was placed at the bottom of the hole in these latter experiments, and the analysis yielded nothing.

Experiments on Pressure in Cast Iron when Cooled.

It has been generally assumed that iron rich in carbon expands on setting, and that this supposed property is a contributory cause in the formation of diamond.

Several experiments were made by pouring iron saturated with carbon from the electric furnace through a narrow git into a very massive steel mould, closed at the bottom with a breech screw (fig. 9). When cold, the breech screw was easily removed, and there was no sign of any appreciable pressure having come on the threads. Not being sure that, because of capillarity, the corners of the mould had been quite filled, a steel mandril was, immediately after pouring, forced down the git-hole by a press giving a fluid pressure in the mould of 75 atmospheres. The observed pressure on the breech screw appeared not to have exceeded this pressure. Highly carburized iron, therefore, does not expand with any considerable force on setting.

The reason why a lump of cast iron thrown into a ladle of molten metal first sinks to the bottom and soon rises and floats on the surface is probably that cast iron is about seven times stronger in compression than in tension. Therefore when a sufficiently thick layer of the cold metal has been heated the interior is torn asunder by the expansion of the outer skin, and the specific gravity of the whole mass is diminished. (See Mr. Wrightson's paper "On Iron and Steel at High Temperatures," with discussion, 'Journal of the Iron and Steel Institute,' No. 1 for 1880.)

We may therefore safely conclude that when iron is suddenly cooled, the only compressive bulk pressure that is brought to bear on the interior is that arising from

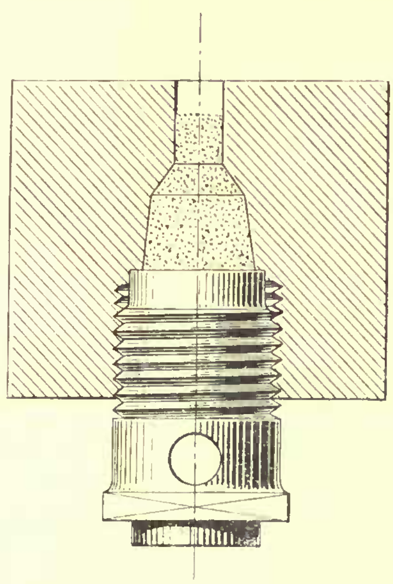

Fig. 9.

the contraction of the outer layers after setting, and with highly carburized iron this can only be small because of the low tensile strength of the metal.

Gases Ejected from Cast Iron on Setting.

As bearing upon the question of the possibility of the occluded gases playing a part, Moissan was the first to observe that spherules or small spheres of iron with cracks and geodes never contained diamond. We have made experiments by pouring highly carburized iron, alloys and mixtures on to iron plates, the cooling taking place from one side only, and under such conditions no diamond results; in fact it only occurs when the ingot or spherule is cooled on all sides nearly simultaneously, so that an envelope of cold metal is formed all over before the centre sets.

Since my paper in 1907, the experiment of heating iron in a carbon crucible and transferring it to a steel die and subjecting it to 11,200 atmospheres pressure has been repeated, and it has been found that if the iron is allowed to set before the pressure is applied the amount of diamond is much greater than if pressed when very hot and molten, and that it is then about the same as when the crucible is cooled in water. The only reason that suggests itself to account for this is, that when pressure is applied while the iron is very hot some of the latter permeates the carbon of the crucible, and because of the greater specific heat and lesser conductibility of the carbon, the iron next to and in the carbon remains molten after the ingot has been cooled by direct contact with the steel cup on the face of the plunger. Thus, when cooling, the occluded gases have a free exit from the ingot, through the molten metal (which is pervious to gas) into the carbon of the crucible, and are not retained in the ingot to the same extent as when it is set and enclosed in an envelope of colder iron impermeable to the gases before pressing.

The experiments of BARADUC MULLER ('Iron and Steel Institute, Carnegie Scholarship Memoirs,' 1914, p. 216), on the extraction of gases from molten steel, showed that steel is permeable to gases down to 600° C.

Other Experiments.

The action of water on carbide of calcium, and of concentrated sulphuric acid on sugar for 6 hours under pressure of 30,000 atmospheres were tried; in both cases amorphous carbon was formed and no diamond.

HANNAY's experiments were repeated, where paraffin and dipple-oil with the alkali metals, especially potassium, were sealed in steel tubes and subjected to a red heat for several hours. The analysis gave no diamonds; in fact it became apparent that when hydrocarbons or water were relied on to produce pressure, the latter could only exist for a short time at the commencement, for when a red heat was reached the hydrogen escaped through the metal, and the oxygen combined with the steel.

We did not analyse the steel tubes themselves. Many experiments were however tried with central heating under the press at 6000 atmospheres, and nothing was obtained of interest with the substances used by HANNAY, unless, as previously mentioned, some iron was present. FRIEDLANDER's experiment was repeated, where a molten globule of olivine, in a reducing flame, or with carbon added, was stated by him to contain minute diamonds. An experiment was made with molten olivine in a carbon crucible in a wind furnace stirred with a carbon rod, with and without an electric current passing between the rod and crucible.

Many experiments were also tried at 6000 atmospheres under the press with central heating with olivine associated with carbon, hydrocarbons, bisulphide of carbon, water, &c., also with blue ground from Kimberley instead of olivine. The results of the analyses were in all cases negative, except occasionally when metallic iron was present. Thus in some cases the olivine or blue ground was partially smelted by the heating carbon rod or by the associated hydrocarbons, &c., when such were added, and iron globules were formed. In these, diamond was occasionally found when cooling was rapid and they were centrally situated in the charge.

Very Quick Cooling.—To test the action of very quick cooling a carbon crucible of 2-inch internal diameter charged with iron, sugar carbon, 2 per cent. silicide of carbon, well boiled by resistance heating under atmospheric pressure and 2 per cent. of iron sulphide added, was quickly placed on asbestos mill-board resting on a steel table frictionally held in the bore of the 4-inch mould, below being placed 2 lbs. of carbon dioxide snow, and the plunger quickly brought down by the press, subjecting the whole to 6000 atmospheres pressure. When taken out the crucible was intact, the contents had divided into a lower portion consisting of a large grained crumbling mass of graphite admixed with granules of very hard iron, in the centre a rounded pillar of white iron equally hard. The cooling seemed to have been unusually rapid.

The experiment was repeated, the crucible being charged with iron, sugar carbon, 5 per cent. manganese, 5 per cent. cobalt, 2 per cent. silicide of carbon, boiled, and 2 per cent. iron sulphide added.

It was also repeated with water instead of carbon dioxide snow. The result of all these experiments was similar to the first. No diamond was found in any part.

An experiment which seemed to give practically instantaneous cooling was as follows:—A small carbon crucible containing iron, with traces of silicon, aluminium, calcium, magnesia and sulphur, was floated on a carbon block on a bath of mercury, all contained in a vessel exhausted to 2 mm. absolute. The crucible was heated by an arc from an upper carbon, the holder passing through a stuffing box. When the crucible was sufficiently hot and the contents carburized, the upper carbon was thrust down, submerging the crucible under the mercury; the cooling was almost explosive and instantaneous—the finely divided iron and graphite on analysis yielded no diamond.

Extremely rapid cooling does not, therefore, seem to be a direct cause in the production of diamond.

Experiments at Atmospheric Pressure.

A convenient method of studying the effect of the association of other elements with iron on a small scale uncontaminated by the vapours of a furnace lining suggested itself, and a series of experiments were made as follows:—A deep iron dish was packed tightly with Acheson graphite with a slight dimple in the centre to hold the ingot, above, graphite was filled in loosely to a depth of half an inch covering the ingot. An arc was struck by a carbon on to the ingot submerged in the loose graphite. When the iron was well boiled the surrounding graphite with the ingot in it was dug out entire and thrown into a bowl of mercury covered with water.

The results showed that, using ordinary mild steel, no diamond ever occurred on analysis, but that a small percentage of silicon is absolutely essential; small percentages of aluminium, magnesium, calcium, one or all are important; sulphur, manganese, and cobalt increase the yield, nickel appeared to be a disadvantage. An alloy of iron and 10 per cent. manganese, 10 per cent. cobalt, and 5 per cent. silicon gave out much gas when cooled slowly, and on quick cooling in water and mercury most of the spherules were burst and shredded.

Finally about 1 to 3 per cent. of the other elements added to iron appeared to give the best results and the spherules were not then burst.

An experiment was made by letting the ingot remain in the bed till it had quite set, hard enough to handle with the iron spoon, and then, cooled in water and mercury, it gave a fair diamond residue.

Experiments on the Conversion of Diamond to Graphite.

A clear octahedral diamond was placed in a small carbon crucible and packed loosely with Acheson graphite and heated for 10 minutes to about 1400° C. The diamond was coated with a firm layer of graphite.

After two prolonged treatments with fuming nitric acid and potassium chlorate, alternating with boiling sulphuric acid and nitre, the opaque coating was removed and there remained a blackish translucent skin. When fractured the interior was unaltered and perfectly transparent.

A piece of bort somewhat laminated, after the same treatment, showed the laminations separated by cracks starting from the outside. Upon breaking, the interior surface of the fissures showed an incipient change to graphite, but less rapid than on the outside surface. There was a sinuous pitting, deepest near the outside and diminishing inwards. The substance of the bort between the fissures was unaltered.

The change of diamond to graphite under the conditions described is gradual, the surrounding gases, carbon monoxide, carbon dioxide, nitrogen, hydrogen, and also vapour of iron (as an impurity in the graphite) singly, or collectively, probably play a part, and further investigation as to this seems to be desirable.

Sir James Dewar, in 1880, heated a diamond in a carbon tube to a temperature of 2000° C., while a flow of pure hydrogen was maintained through the tube. The diamond soon became covered with a coating of graphite ('Proceedings of the Royal Institution').

A clear diamond plunged into molten iron saturated with carbon at about 1400° C. for 5 minutes was deeply pitted. When removed from the iron small globules of iron adhered to the surface and the pits appeared to occur at these spots.

A clear diamond was disintegrated by cathode rays, the temperature by pyrometer being 1890° C., the splinters were quite black and opaque, but after several prolonged treatments with fuming nitric acid and potassium chlorate, alternating with boiling sulphuric acid and nitre, the coating that remained was a dusky grey, but semi-transparent, the gas present being chiefly hydrogen. (Paper by Parsons and Swinton, January 16, 1908, 'Roy. Soc. Proc.,' A, vol. 80.)

In this latter experiment the surface action appeared to be much less in proportion to the incipient change of the under layer to graphite, and the impression is that at 1890° C. the temperature of bulk transformation is being approached, also that carbon monoxide, carbon dioxide, nitrogen, hydrogen, and iron, one or more, act as catalysts in the change of diamond to graphite.

Experiments on the Oxidation of Alloys of Iron when Molten.

Iron was melted in a carbon crucible and highly carburized; when it had somewhat cooled, the other elements were added, in small percentages of aluminium, silicon, calcium, magnesium, manganese, iron sulphide, collectively and in some cases singly; the crucible was then removed from the furnace and superheated steam blown through a carbon tube into the metal; energetic action took place and much heat was evolved; on analysis, after destroying the graphite, a bulky transparent crystalline residue remained.

With aluminium alone the crystals were chiefly crystallized alumina, and with the other elements the spinels and other crystals were produced; all were transparent and colourless, but when chromium was added some rounded crystals occurred resembling pyrope. When submitted to sulphur dioxide and carbon dioxide the result was the same, but less residue was produced. Under the microscope there appeared to be a small proportion of very small crystals like diamond; these burnt in oxygen. When the bulky residue was placed in a test-tube with the double nitrate of silver and thallium, and the density adjusted so that a diamond floated midway between the top and bottom, there collected into its immediate neighbourhood after a time an amount of the small crystals which was estimated to be about 5 per cent. of the total residue.

One prolonged treatment of hydrofluoric acid had no apparent effect on the bulky residue, and it required so many treatments to destroy it that we failed to isolate the very small particles whose size did not exceed $\frac{1}{20}$ mm.; they were probably lost by flotation. These experiments were repeated many times with the same result, but they merit further investigation, with steam under high pressure and conditions favourable to the formation of larger crystals.

Note.—Marsden observed in silver the association of black diamond with crystalline alumina, silicide of carbon, &c., 'Roy. Soc. Proc.,' 1880.

Experiments in Vacuo.

The presence of diamond in some meteorites suggested a series of experiments under various degrees of vacuum up to the highest obtainable.*

* Also an impression suggested itself in 1907 that hydrogen had an adverse effect on the formation of diamond.

It is probable that some meteoric matter may have been melted by collision or ejected into space in a molten state and cooled by radiation, and that under such conditions the absence, or diminution, of occluded gases might be a factor conducive to the crystallization of carbon.

One of the 4-inch diameter pressure moulds (fig. 10) was used in a preliminary experiment as the container. The crucible was turned out of a $1\frac{1}{2}$-inch carbon rod, and so formed on a stem that the electric current heated the bottom and sides equally. The cover was similarly formed and its holder was electrically connected with the container, but free to move vertically and to rest its weight on the crucible,

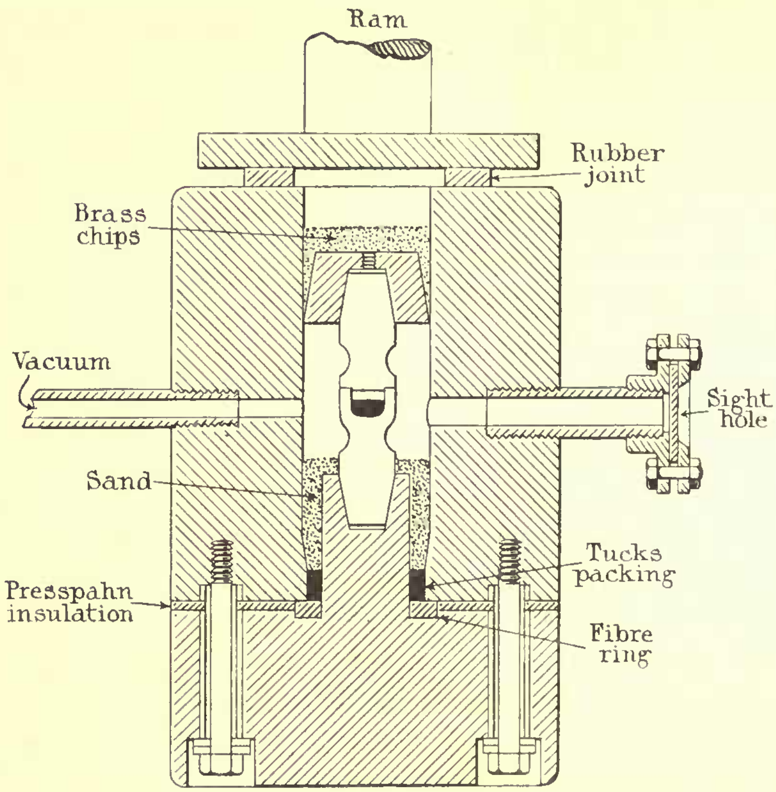

Fig. 10.

electrical connection to the container being made by a layer of brass or iron turnings resting on the holder. A current of 1000 amperes at 16 volts sufficed, and the temperature was observed through a glass window at the side of the container.

The crucible was charged with reduced iron and lampblack. The Geryk pump evacuated the container to $\frac{3}{8}$-inch mercury absolute; current was turned on for 15 seconds, the vacuum fell to 3 inches, when it had risen again to $\frac{3}{8}$-inch current again turned on. This was repeated three or four times, finally current was applied for 30 seconds and the vacuum again fell to 3 inches. The gas was drawn off and collected, it amounted to a total of $\frac{1}{2}$ gallon at atmospheric pressure and consisted of 95 per cent. carbon monoxide, 1 per cent. hydrogen, 2 per cent. hydrocarbon, 2 per cent. nitrogen.

The carbon which formed the crucible and cover contained a large percentage of silica, but the carbon monoxide was produced chiefly by the action of sand (of which there was a thick layer on the bottom of the container to protect the insulating joint from iron spilled from the crucible) on the carbon of the stem of the crucible. About one half of the iron had been evaporated, and there remained an ingot about the size and shape of a broad bean. It contained rather large graphite crystals and was easily broken. The analysis gave the largest residue of diamond in proportion to the amount of iron of any of our experiments, the largest crystals being 0·7 mm. in length.

This experiment was repeated several times with the same result. The time of cooling of the crucible, from switching off the current to the temperature of setting, was 15 seconds, and probably sufficiently rapid to allow of a skin to be formed around the ingot before the centre was solidified, for the configuration of the crucible and cover were such as to ensure nearly equal and simultaneous cooling on all sides of the ingot. At the time, vacuum was erroneously thought to be the chief contributory cause and not the presence of carbon monoxide in large proportion.

High Vacuum Experiments.

The molecular pump not having yet been evolved, a powerful pumping system was arranged, consisting of three steam-jet exhausters in series, the last ejector of the series discharging into a jet condenser with separate air and water pumps, the former assisted by a steam jet. The two steam-jet exhausters nearest to the exhausted chamber were fed with highly superheated steam at 200 lbs. pressure, and the suction pipe to the chamber was 4 inches in diameter—the chamber 2 feet 6 inches diameter—of spherical shape (fig. 11). A vacuum of $\frac{1}{6}$ mm. absolute could be reached.

The crucible was placed on a large block of carbon, resting on the base of the chamber, and forming the bottom pole. The cover was insulated from the chamber, and through an oil-sealed gland passed a 2-inch brass rod, carrying a crown holder, with four 2-inch carbons which rested on the lip of the crucible for resistance heating. An observation window was placed at the apex of a long iron cone, projecting from the side of the cover, which gave a good view of the crucible and its contents. The whole of the chamber was submerged in a tank of water, up to the level of the gland in the cover.

Iron and iron alloys were boiled and allowed to cool slowly by radiation, or were rapidly quenched by admitting water through a large valve from the tank into the vacuum vessel. The iron and carbon vapour from the boilings deposited dust and globules on the cover and sides and bottom of the chamber. A very small diamond residue generally resulted from the small iron globules, and also from the dust, but never anything from the ingot remaining in the crucible.

In several experiments water was admitted, which played directly on the crucible, the upper carbons resting on the rim prevented its upsetting by the force of the water, and still there was no residue. In one experiment the carbons were lifted and the charge flowed out, forming spherules of varying size in the water. There was a very small diamond residue from these spherules.

In one experiment a crucible was filled with iron and carbon and closed by a tight carbon cover, a hole bored in the side of the crucible, a massive block of iron placed

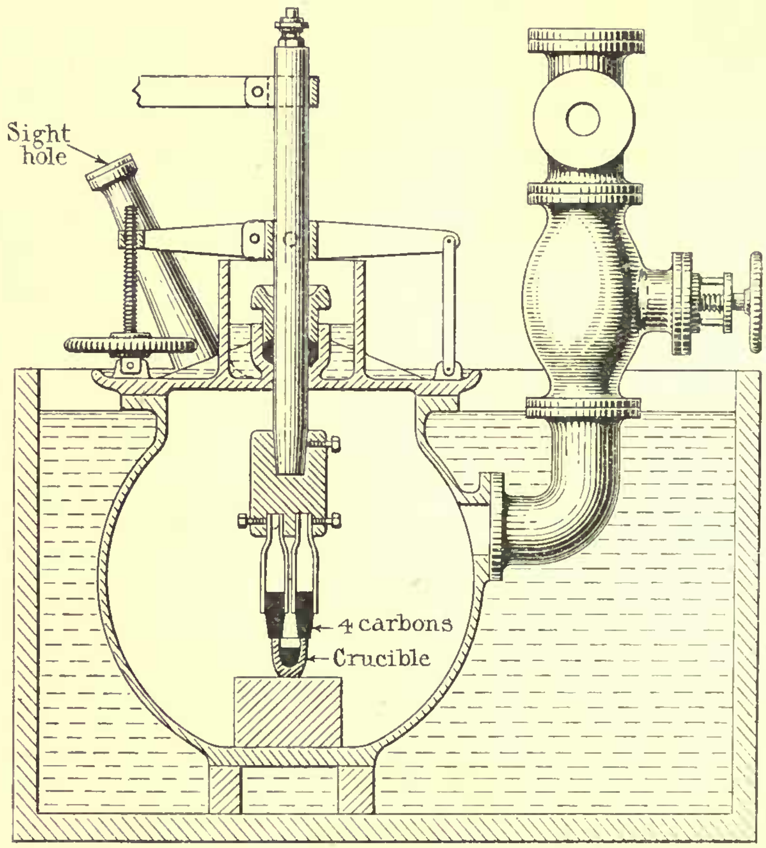

Fig. 11.

close opposite the hole and the crucible boiled, the vacuum being under 1 mm. No crystallised residue was found in the deposit on the iron block from this high velocity jet of vapour of iron and carbon.

In another experiment a powerful electro-magnet was provided with poles to give a concentrated field, and an arc struck between two carbons, arranged to burn within this field and regulated from without by hand. There was an iron block upon which the arc directed by the field could play and condense its carbon vapour. The analysis gave no diamond.

It was thought that the vapour from boiling iron saturated with carbon might, by the action of bisulphide of carbon, cause a crystalline deposit, but all the experiments to this end yielded no results.

Experiments under X-ray Vacuum.

Experiments were made under X-ray vacuum in a new chamber of cast iron with very thick walls to absorb the heat, exhausted through an 8-inch diameter suction by a large molecular pump alongside, in series with a dry, high speed, two stage, pump, 12-inch diameter pistons, and last of the series a 3-inch + 2-inch compound

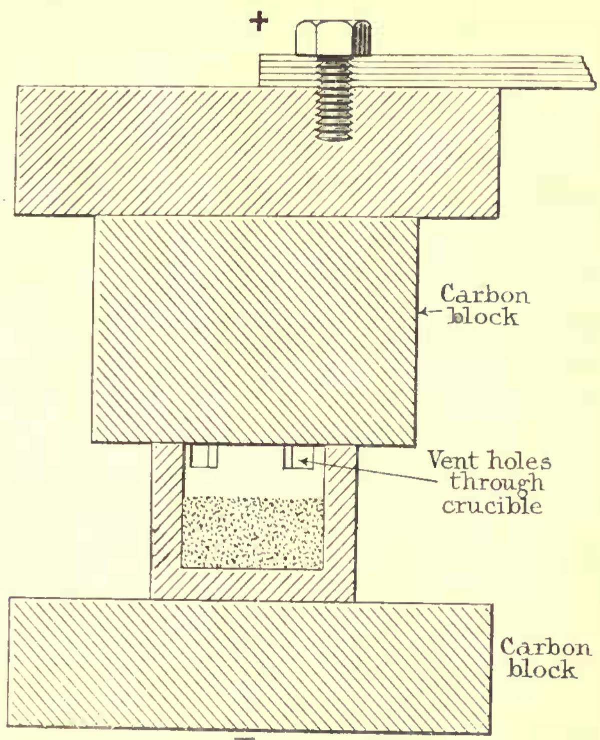

Fig. 12.

Fleuss. The crucible was resistance-heated as before (fig. 12). No diamond was produced in any of these experiments, except in those where iron, sand, and other elements, with or without sulphur, were first heated and well boiled in the carbon crucible at atmospheric pressure, and after cooling transferred to the vacuum furnace and re-heated by resistance under X-ray vacuum; violent ebullition occurred owing to the liberation of occluded gases, and many iron spherules were ejected, which cooled by radiation and conduction where they fell; diamond was found in these, which burnt in oxygen, but no diamond was ever found in the ingot remaining in the crucible.

It occurred to us to try the effect of great mechanical pressure accompanied by heat upon small particles and powders, the interstices being exhausted to a high vacuum.

Several experiments were made in the press under a mass pressure of 3000 atmospheres.

A layer of cast-iron turnings resting on a layer of carborundum grit, the exhaustion being effected through a hole in the side of the mould covered by a perforated steel plate within the layer of grit, heat was applied as usual by a central carbon rod.

Analysis yielded some thin crystal plates from the grit which had lain in the line between the cast iron and the suction outlet at the grid, and also from the layer of grit which had lain against the cast-iron turnings which had become heated but not melted by the central carbon rod.

To ascertain the cause of the occurrence of these plates, experiments were made without bulk pressure on the concentrated action of the gases given off from cast-iron turnings heated up to a good red, and drawn by a high-vacuum pump through carborundum grit placed in a silica tube heated by a gas burner at the centre of its length to dull red. These yielded similar crystal plates.

Control experiments showed that no similar plates existed in the untreated grit.

It was also found that the cast-iron turnings would not produce this effect on a second heating unless they had been subjected to CO at atmospheric pressure for some hours. Carbon monoxide, sulphur dioxide, cyanogen, hydrogen, nitrogen, oxygen, nitric acid gas, chlorine, ammonia, ammonium oxalate vapour, ammonium chloride, acetylene, coal gas, produced no plates.

These plates resemble diamond very closely in appearance and form of crystallization, they do not polarize, and some have triangular markings; they will not, however, burn in oxygen at 900° C., and are completely destroyed by chlorine purified from oxygen and water vapour at 1100° C.; their specific gravity is about 3·2, they are therefore not diamond.

Note.—Recent experiments have shown that carbon monoxide passed over molten iron sulphide and then over carborundum grit below red heat at atmospheric pressure also produces these plates, and that if coal gas is substituted for carbon monoxide no plates are formed. Also that only a few of the grains produce plates.

The composition of the grains is—

Carborundum	36·56
Iron oxide and alumina	44·09
Lime	10·45
Magnesia	5·57

Summary of Experiments and Conclusions.

The experiments have shown that all the hydrocarbons, chlorides of carbon, and oxides of carbon tested, deposit amorphous carbon or graphite on a carbon rod electrically heated at any pressure up to 4400 atmospheres, and in a few experiments up to 6000 atmospheres.

That at 15,000 atmospheres carbon and graphite electrically heated are either directly transformed into soft graphite or are first vaporized and then condensed as such.

While the experiment of rapidly compressing a mixture of acetylene and oxygen and the production of temperatures much in excess of that necessary to vaporize carbon, accompanied by a momentary pressure of about 15,000 atmospheres, confirms the conclusion that the negative results obtained in the attempts to convert graphite into diamond by electrical heating are not due to lack of temperature; on the other hand, the presence of minute crystals in the molten layer of the steel of the end of the barrel subjected to high gaseous pressures of carbon monoxide, carbon dioxide, and hydrogen appears to be connected with the other experiments bearing upon the inclusion of gases in metal as a factor in the production of diamond.

The experiment of firing a high velocity steel bullet with cupped nose through vaporizing carbon into a hole in a block of steel has tested the effect of a momentary pressure of about 300,000 atmospheres on carbon initially near its melting-point, and probably raised by adiabatic compression by another 1000° C.

The fact that only a very few minute crystals resembling diamond were produced (probably from the iron) raises the question as to whether the duration of the pressure is sufficient to start a transformation of graphite to diamond which can be detected by analysis. We have distinct evidence that, with iron as the matrix, the time is sufficient to form very small crystals which can be identified with some certainty, so it therefore seems reasonable to conclude that there was no incipient transformation in bulk, and that however long the pressure of 300,000 atmospheres were applied, it is extremely doubtful if any change would occur.

The pressure of 300,000 atmospheres is between one quarter to one half that obtaining at the centre of the Earth, but vastly greater pressures exist at the centre of the larger stars, and are produced by the collision of large bodies in space; these pressures are many thousands of times greater, and whether they would effect the change it is impossible to predict. On the other hand, a heating effect on large masses of iron might be produced by collisions, and owing to the heat generated by adiabatic compression of the central portions, some of the mass would be melted and subsequently cooled on release of the pressure, so that if heating and cooling under pressure are alone necessary for the production of diamond large stones might result. These considerations, though of interest as bearing upon the presence of diamonds in meteorites and also indicating a possible origin of natural diamond, are of no practical value to us because the pressures required are entirely beyond our reach. There are, however, other considerations arising out of the experiments of MARSDEN, MOISSAN, and CROOKES, as well as our own, which seem to give some hope of solutions of the problem at issue which lie within the means at our disposal.

A repetition has been made of many of the experiments in which diamond is claimed to have been produced. These have given negative results in all cases except where iron has played a part, as for instance when olivine, being partly reduced by carbon or a reducing flame, small spherules of iron are produced and may, if the mass is quickly cooled, be found to contain diamond.

The repetition of MOISSAN's experiments under a variety of conditions and pressures has not only confirmed his results but has thrown, it is hoped, additional light on the causes operating to produce diamond in iron.

The experiments under high pressure in steel moulds, where heating of the charge was effected by a central core through which current was passed, enabled HANNAY's experiments with dipple oil to be tried under much higher pressures, and more thoroughly than is possible with steel tubes in a furnace.

The Appendix gives some indication of the many substances and chemical reactions tested. The results were chiefly negative. The few that were favourable were generally attributable, as has been said, to the presence of iron. It was noticed that the iron seldom contained diamond unless when so situated in the charge as to cause equal cooling on all sides, and it will be remembered that the experiments under atmospheric pressure showed this condition to be essential for the formation of diamond.

In some of the experiments of this group considerable gaseous pressure existed up to 6,000 atmospheres, but it is doubtful if in these the right kind of gas was present or a sufficiency of heating or carburization of the iron occurred. On the whole, therefore, it would appear that all, or nearly all, the chemical reactions as such, under pressures up to 6000 atmospheres, have given negative results.

The experiments on very rapid cooling would seem to dispel the theory that carbon can be caught in a state of transition, and to lead us to the conclusion that quick cooling is not in itself a cause of the occurrence of diamond in rapidly cooled iron.

MOISSAN observed that when the spherules of granulated iron were cracked, or contained geodes, no diamond was ever found in them, and he attributed this to want of mechanical pressure. The experiments we have made not only corroborate this fact, but they tend to show, we think conclusively, that the cracks in the spherules act by allowing a free passage for the occluded gases to escape, and the geodes by providing cavities in which the gases can find lodgment without much gaseous pressure occurring in the metal.* Further, the experiments have shown that iron when it sets does not expand with appreciable force, and that the only compressive forces that are brought to bear on the interior are those arising from the contraction of the outer layers.

Our experiments further show that when a crucible of molten iron is subjected to pressure more than three times as great as can be produced by these contractile forces, the yield of diamond is not increased. On the other hand, when the

* Conversely they may act to allow gases to enter the metal.

conditions of the experiment operate to imprison the occluded gases, then the yield of diamond is about the same as if the crucible had been plunged into water, while if the conditions are such as to allow a free passage through the skin of the ingot, the yield is at once diminished, even though the bulk pressure on the ingot is the same.

The experiment, on compressing acetylene and oxygen, has shown that minute crystals, probably diamond, are produced almost instantaneously in the molten surface of metal exposed on one side to gases consisting of carbon monoxide, carbon dioxide, and hydrogen at very high temperature and at 15,000 atmospheres. Sir WILLIAM CROOKES' experiment described in his lecture before the British Association at Kimberley in 1905 is somewhat analogous; cordite with a little additional carbon was fired in a chamber, the pressure reaching 8,000 atmospheres, a few crystals of diamond were found and isolated; this result CROOKES attributed to the melting of the carbon under the temperature of explosion and crystallization under the pressure on cooling.

Under the conditions of the experiment there would be a considerable amount of the surface of the chamber melted and swept into the products of the charge by the turbulence of the explosion, and the spherules of iron would thus be carburized and cooled while still under heavy pressure.

In the acetylene-oxygen experiment there is a molten surface with reducing gases on one side at high pressure, and on the other metal impervious to gases. In CROOKES' experiment the globules of metal are surrounded by gases at high pressure. In both cases the metal has solidified with the occluded gases imprisoned by the high external gaseous pressure, for we have seen that the pressure of occluded gases in highly carburized iron when quickly cooled cannot exceed about 1000 atmospheres.

The experiments under vacua from 75 mm. up to X-ray vacua have shown generally that as the vacuum is increased the yield of diamond in the crucible is diminished, and that below 2 mm. none has been detected. But when alloys previously boiled at atmospheric pressure are quickly heated up under high vacuum violent ebullition takes place, from the large volume of gases liberated, and some of the contents are ejected into the vacuum chamber before they have had time and sufficient temperature to part with their occluded gases, and diamond occurs in the spherules so ejected.

The gases occluded in cast iron which are given off when heated *in vacuo* have been investigated by H. C. CARPENTER and others, and the relative amounts of the constituents are found to vary widely according to the previous heat treatment and the nature of the gases in contact with the metal while molten and during cooling; they are carbon monoxide and carbon dioxide, hydrogen and nitrogen.

H. C. CARPENTER ('Journal of Iron and Steel Institute,' 1911) states that, when heating up a bar of cast iron *in vacuo* in a silica tube, "After the twenty-fifth heat it was noticed that in the water-cooled areas of the quartz tube a lustrous black ring had formed. On being strongly heated, some of this, evidently carbon, burnt off, leaving a white film, presumably silica. This seems to show that a volatile silico-organic compound, containing carbon, hydrogen, and silicon, was evolved from the iron on heating."

It would appear from our experiments that probably a ferro-silicon carbonyl is given off from the iron, for, as has been said, we observed a corrosive action on carborundum by the gas evolved from iron borings at red heat under a high vacuum, and the same action was produced by gaseous ferro-carbonyl, and also by carbon monoxide, previously passed over molten iron sulphide at atmospheric pressure.

Let us consider what happens in an ingot or spherule when rapidly cooled simultaneously on all sides. It is first surrounded by a thin coat of solidified metal which, below 600° C., is impervious to gases. As the coat thickens layer within layer, more and more gas is ejected by the solidifying metal, and its semi-solidified centre, still pervious to gas, receives the charge. As this process progresses the pressure may rise higher and higher, though there may be a limit to the pressure against which the metal is able to eject gas when setting. All we, however, know is, that the mechanical strength of the ingot or spherule places a limit of about 7000 atmospheres on the gaseous pressure, and, as we have already mentioned in the case of some iron alloys, most of the spherules are split or shredded, with an appearance consistent with this view.

Crookes' microscopical examination of diamonds with polarized light supports this view. In his lecture at Kimberley, in 1905, he states: "I have examined many hundred diamond crystals under polarized light, and with few exceptions all show the presence of internal tension.

"On rotating the polarizer, the black cross most frequently seen revolves round a particular point in the inside of the crystal; on examining this point with a high power we sometimes see a slight flaw, more rarely a minute cavity. The cavity is filled with gas at enormous pressure, and the strain is set up in the stone by the effort of the gas to escape."

It seems therefore probable, or indeed almost certain, from the accumulated evidence, that the chief function of quick cooling in the production of diamond in an ingot or spherule is to bottle up and concentrate into local spots the gases occluded in the metal which, under slow cooling, would partially escape and the remainder become evenly distributed throughout the mass.

As to the condition in which the gases exist within the iron at temperatures above 500° C. little is known, though at 200° C. and at 180 atmospheres Mond has shown that iron penta-carbonyl is formed. The intimate contact between the occluded gases and other elements, metals or carbides, must favour complex interactions as cooling takes place. Such actions might be concentrated by the heat flow across the metal on quick cooling.

It appears probable that concentration of gaseous pressure causes certain reactions which bring about an association of carbon atoms in the tetrahedral form—against their natural tendency to assume the more stable form of graphite.*

The necessity of subjecting the iron to a temperature above 2000° C. before cooling would seem to imply the necessity of carbides of the other metals, such as silicon, magnesium, &c., being present to insure the necessary chemical reactions with the gases at high pressure within the ingot.

In reviewing all our experiments, the greatest percentage of diamond occurred when the atmosphere around the crucible consisted of 95 per cent. carbon monoxide and 1 per cent. hydrogen, 2 per cent. hydrocarbons, 2 per cent. nitrogen, the mean pressure in the vessel being about 1 inch absolute of mercury. The weight of diamond we estimated to be about $1 \div 20,000$ of the weight of the iron. If we, for the moment, assume a volume of carbon monoxide at atmospheric pressure equal to 0·69 that of the iron, the weight of carbon contained in it equals that of the diamond.

For the following reasons it would appear that the formation of diamond in rapidly-cooled iron takes place when it is solid or in a plastic condition, or even at a still lower temperature. The rapid pitting of a diamond in highly carburized iron just above its melting point is so pronounced that the largest diamond hitherto produced artificially would be destroyed in a second or two if the iron matrix were molten. The production of diamond was obtained in an ingot rapidly cooled after it had set sufficiently hard to be handled in a spoon. A similar result was obtained in the case of a crucible placed in the die and subjected to 11,200 atmospheres pressure after the contents had set. MOISSAN found the diamonds to occur in the centre of the ingots both in the case of iron and also of silver.

It has been seen that iron is permeable to carbon monoxide and hydrogen at temperatures above 600° C., and there appears to be no reason why the concentration of the occluded gases should not take place within the mass as effectively at 600° C. as at higher temperatures, provided that they cannot escape. The most probable temperature, however, may be the point of recalescence at 690° C.†

It would appear that the function of the impervious metal coating thrown around the ingot by quick cooling might be better effected by gas of the same composition as that which the metal ejects on cooling, the pressure being sufficient to ensure that the gaseous pressure around the ingot shall be equal to, or greater than could occur on quick cooling. Such a substitution might result in a larger gaseous content and a larger proportion of the ingot being brought into a suitable condition for the formation of diamond, and the yield might thereby be increased. Some gradations

* It also appears that the conditions may operate to the exclusion of some gas or element inimical to the formation of diamond from certain parts of the metal, viz., the graphite liberated and the cooled metal of the outer layers may absorb some gas or element from the inner portion of the ingot and leave none for the central portion.

† These conditions may also operate to exclude some gases from eertain portions of the metal.

of temperature might still be found necessary to concentrate the reactions. It seems however probable that the rate of cooling might be so much prolonged as to obtain much larger crystals and a larger total yield.

The presence of crystals of silica, alumina and magnesia and the spinels and pyrope associated with diamond in rapidly cooled iron alloys, and also when oxidized by steam and some other gases, appears to have a bearing upon the presence of similar crystals usually found in association with diamond, and to be compatible with the conclusions of BONNEY that eclogite is the parent-rock of the diamond in South Africa. It seems probable that both the eclogite and the diamond may have been crystallized nearly simultaneously from an iron alloy.

MOISSAN, after a recital of the geological conditions existing in the South African pipes (see 'Four Electrique,' p. 115), came to the conclusion that diamond was not a vein mineral, but must have been evolved in the midst of a plastic mass; and he concludes that iron at high pressure must have been the matrix. Our experiments, however, seem to show that bulk pressure on the metal does not play a part, but that the previous heat treatment, the impurities in the iron and the condition of the gases within the metal, are the important factors.

It is interesting to note that in the best experiments the yield of diamond in rapidly-cooled iron has reached $1 \div 20,000$ of the weight of iron, whereas the weight of diamond obtained from the blue ground of the South African mines is only $1 \div 5,400,000$. This comparison appears to be confirmed by the relative rarity of microscopic diamonds we have found in the many analyses we have made of blue ground and of the conglomerate from Brazil.

Thus in cooled iron there may be more than 270 times as much diamond as exists in the bulk average of blue ground.

In conclusion, I desire to express my obligations for kind assistance and advice to Sir DUGALD CLERK, Prof. JEANS, Mr. STANLEY COOK, Mr. CAMPBELL SWINTON, and to many other friends, as also to Mr. H. M. DUNCAN.

From 1906 to 1908 inclusive, the late Mr. TREVOR CART assisted me in the arrangement of the experiments and was responsible for most of the analyses until the time of his death.

From January, 1911, to August, 1914, Mr. H. M. DUNCAN acted as my assistant and analyst and has given valuable help in the collection and tabulation of the whole of the work. During the preceding and intervening periods the analyses were made in the laboratory at my house.

APPENDIX.

ABRIDGED SCHEDULE OF EXPERIMENTS.

(A.) UNDER PRESSURES GREATER THAN ATMOSPHERIC.

k.w.m. = kilowatt minutes. Nil in Result column means no diamond formed.
tons = pressure per square inch.

From every experiment several samples were selected representing different parts of the ingot or mixture and analysed separately.

(See figs. 1, 2, and 3.)

Experiment.	Result.
4-inch mould; iron tube core, filled with ferrous oxalate, marble round, iron disc to bring current to top carbon, and graphite on top, 10 tons pressure, 100 k.w.m. total heating, heating $\frac{1}{4}$ minute.	Nil.
2-inch mould; carbon rod core, layer of silicon carbide, then calcium carbide, then carbon, iron nose piece, 80 k.w.m. total heating, 10 tons pressure, heating 2 minutes.	Nil.
4-inch mould; carbon rod core, water round, then marble, 15 tons pressure, 30 k.w.m. total heating, heating $\frac{1}{2}$ minute.	Nil.
4-inch mould; $\frac{7}{8}$-inch carbon rod core surrounded by marble bush-ring with $1\frac{1}{2}$-inch hole and fitting mould on outside, naphthalene and iron filings surrounding rod below ring, anthracene and naphthalene surrounding rod in hole and above to maintain fluid pressure, iron plate to bring in current to top of rod, 10 tons pressure, 70 k.w.m. total heating, soft carbon produced, heating $1\frac{1}{2}$ minutes.	Nil.
Same as last, but oxalic acid and ferrous oxalate instead of naphthalene, with iron filings below marble ring and also between carbon rod and ring. Anthracene above ring as before to maintain fluid pressure, $\frac{1}{2}$ ton gaseous pressure when somewhat cooled and ram released, carbon eaten away, 400 k.w.m., duration 10 minutes.	Nil.
4-inch mould; carbon core, charcoal round, then marble, crushed arc light carbons on top, 25 tons pressure, 60 k.w.m. total heating, heating 34 seconds, maximum current 14,000 amperes, 11 volts.	Nil.
4-inch mould; carbon core, marble round lower part, crushed carbon and perforated iron disc to bring current to top carbon round upper part, 1 lb. carbon dioxide, snow on top covered with graphite and iron chips, 15 tons pressure, 60 k.w.m. total heating, about $\frac{1}{2}$ ton gas pressure when somewhat cooled and ram released, found to contain about 50 per cent. CO_2, 10 per cent. CO, and hydrocarbon burning with luminous flame, heating 2 minutes.	Nil.
2-inch mould; graphite neck, magnesia piece bridge, 100 tons pressure, 4 k.w.m. total heating, little gas produced, heating 25 seconds.	Nil.

Experiment.	Result.
2-inch mould; $\frac{9}{16}$-inch carbon rod, titanium oxide bridge piece, 30 tons pressure, 10 k.w.m. total heating, carbon fused, heating 11 seconds.	Two good crystals.
4-inch mould; graphite neck, marble bridge piece, 20 tons pressure, 60 k.w.m. total heating, heating $\frac{1}{2}$ minute.	Nil.
2-inch mould; graphite rod core, calcium carbide plus 10 per cent. sulphur packed round, 24 tons pressure, $2\frac{1}{2}$ k.w.m. total heating, heating 25 seconds.	Nil.
Ditto plus carbon tetrachloride	Nil.
2-inch mould; carbon core, ferric oxide and 20 per cent. sugar charcoal round, graphite on top, 24 tons pressure, 2 k.w.m. total heating, 1 ton gaseous pressure when somewhat cooled and ram released, heating 5 seconds.	Nil.
4-inch mould; graphite core, charcoal and 10 per cent. arsenic round, 20 tons pressure, 13 k.w.m. heating, heating 10 seconds.	Nil.
4 inch mould; graphite core plus 5 per cent. sulphur, willow charcoal round, 20 tons pressure, 16 k.w.m. total heating, heating 10 seconds.	Nil.
4-inch mould; carbon rod core, sand below around lower pole piece, carbon tetrachloride to give fluid pressure throughout whole, perforated iron disc to bring current to top carbon, 10 tons pressure, 3 k.w.m. total heating, about $\frac{1}{2}$ ton gaseous pressure when somewhat cooled and ram released, soft amorphous carbon and silicide of carbon formed, heating $1\frac{3}{4}$ minutes.	Nil.
2-inch mould; iron core, potassium ferrocyanide round, graphite on top, 40 tons pressure, 1 k.w.m. total heating, heating 4 seconds, cone melted, current interrupted.	Nil.
4-inch mould; carbon rod core surrounded with aluminium carbide, carbon dioxide snow and charcoal, 10 tons pressure, 18 k.w.m. total heating, heating 5 minutes.	Nil.
4-inch mould; carbon rod core, reduced iron and carbon bisulphide round, 10 tons pressure, 20 k.w.m. total heating, heating 3 minutes, action complete.	Nil.
2-inch mould; aluminium rod core, wood charcoal round, 20 tons pressure, 6 k.w.m. total heating, heating 1 minute.	Nil.
2-inch mould; magnesium rod core, carbon round, 20 tons pressure, 6 k.w.m. total heating, heating $1\frac{1}{4}$ minutes.	Nil.
4-inch mould; 1-inch magnalium rod core, carbon round, 20 tons pressure, 40 k.w.m. total heating, heating $1\frac{1}{2}$ minutes.	Nil.
2-inch mould; $\frac{5}{8}$-inch calcium rod core, carbon round, 20 tons pressure, 3 k.w.m. total heating, short circuit producing some iron which would be rapidly cooled, heating 8 seconds.	A few crystals.
4-inch mould; $\frac{5}{8}$-inch sodium rod core, carbon round, 10 tons pressure, 4 k.w.m. total heating, 4 tons of gaseous pressure when ram released, heating 20 seconds.	Nil.
2-inch mould; carbon rod core, sodium chloride round and graphite on top, 20 tons pressure, 1 k.w.m. total heating, heating $\frac{3}{4}$ minute, carbon rod eaten.	Nil.

Experiment.	Result.
2-inch mould; $\frac{3}{4}$-inch carbon core bored and $\frac{3}{16}$-inch iron rod placed inside, sodium chloride packed round, 20 tons pressure, 8 k.w.m. total heating.	Nil.
2-inch mould; aluminium coil core, wood charcoal round, phosphorus at base, 6 tons pressure, 3 k.w.m. total heating, heating $1\frac{1}{2}$ minutes.	Nil.
4-inch mould; 1-inch aluminium rod core, drilled centre $\frac{3}{8}$-inch filled with carborundum charcoal around, 20 tons pressure, 50 k.w.m. total heating, heating 1 minute.	Altered carborundum.
2-inch mould; carbon rod core, phosphorus and carborundum round, graphite on top, 20 tons pressure, 8 k.w.m. total heating, heating 14 seconds.	Nil.
4-inch mould; magnesium and silicon carbide core, magnesia round, graphite top and bottom, 20 tons pressure, 57 k.w.m. total heating, heating 2 minutes.	Nil.
4-inch mould; 1-inch carbon core, boric anhydride round, 20 tons pressure, $22\frac{1}{2}$ k.w.m. total heating, heating $5\frac{1}{2}$ minutes.	Nil.
9-inch mould; 2-inch carbon core surrounded by wood charcoal, poles water jacketed, 10 tons pressure, 400 k.w.m. total heating, heat melted lower pole and made hole into water cavity causing explosion, molten iron shot out on to floor and walls, heating $4\frac{1}{2}$ minutes.	Several good crystals from ejected iron spherules.
4-inch mould; 1-inch carbon rod core, drilled $\frac{3}{8}$-inch hole, filled with calcium carbide, sand top and bottom, 30 tons pressure, 50 k.w.m. total heating, heating 20 minutes.	Nil.
4-inch mould; $\frac{1}{2}$-inch carbon core, sand, carbon silicide, and calcium carbide in layers round it, 30 tons pressure, 40 k.w.m. total heating, heating 8 minutes.	Nil.
Same arrangement of mould, but calcium carbide and carbon silicide only, plus small amount of sulphur, 30 tons pressure, 7 k.w.m. total heating, some iron present from bottom poles, and also CO produced, heating 1 minute.	Several good crystals which burnt in oxygen.
4-inch mould; $\frac{1}{2}$-inch carbon core, sand top and bottom, $\frac{1}{2}$-inch layer of calcium sulphide in middle, 28 tons pressure, 30 k.w.m. total heating, some iron present from bottom pole, also CO produced, heating 20 seconds.	Some good crystals, some burnt in oxygen.
Same, but layer of calcium carbide, sulphur added to sand	Many crystals which burnt in oxygen.
4-inch mould; $\frac{5}{8}$-inch carbon rod core, sand top and bottom, calcium oxide in centre, plus 5 per cent. ferric oxide, 30 tons pressure, 47 k.w.m. total heating, heating 15 minutes.	Nil.
4-inch mould; $\frac{9}{16}$-inch carbon rod core, slaked lime top and bottom, sand in middle, 30 tons pressure, 14 k.w.m. total heating, heat of 10 to 20 kilowatts applied for 30 seconds, four times, with 30 seconds intervals.	Nil.
4-inch mould; slaked lime top and bottom, sand, salt (10 per cent.) in middle, 30 tons pressure, 8 k.w.m. total heating, heating 35 seconds.	Nil.

Experiment.	Result.
4-inch mould; $\frac{9}{16}$-inch carbon core, slaked lime and sodium carbonate on the bottom, silver sand in middle, slaked lime on top, 30 tons pressure, 770 k.w.m. total heating, heating 30 minutes.	Nil.
4-inch mould; $\frac{9}{16}$-inch carbon core, sand and sodium hydroxide 5 per cent., 25 tons pressure, 40 k.w.m. total heating, iron melted from bottom pole, heating 30 minutes.	Some crystals from centre which burnt in oxygen.
Same experiment plus 10 per cent. lampblack, 25 tons pressure, 12 k.w.m. total heating, heating $1\frac{1}{2}$ minutes.	Nil.
4-inch mould; carbon core, containing 5 per cent. lime, sand round, 25 tons pressure, 8 k.w.m. total heating, heating $1\frac{1}{2}$ minutes.	Nil.
4-inch mould; carbon rod core with sodium carbonate, sand round as before, 25 tons pressure, 6 k.w.m. total heating, heating 25 seconds.	Nil.
4-inch mould; carbon core, olivine, graphite, and water round, 10 tons pressure, 12 k.w.m. total heating, charge blew out at base, the pellets would be charged with CO and H from water and carbon, heating 2 minutes.	Pellets of iron shot out, which gave several good crystals.
4-inch mould; iron rod core, olivine, rubber shavings and vaseline round, 10 tons pressure, 9 k.w.m. total heating, heating 45 seconds.	Nil.
4-inch mould; carbon core, pyrope round, Dippel's oil and paraffin poured on top, covered with iron and graphite, 26 tons pressure, 11 k.w.m. total heating, heating 15 seconds.	Nil.
4-inch mould; 1-inch carbon core, graphite and water round, 20 tons pressure, 8 k.w.m. total heating, heating 15 seconds.	Nil.
4-inch mould; 1-inch carbon core, iron oxide, lampblack and water round, 10 tons pressure, 18 k.w.m. total heating, heating $1\frac{1}{2}$ minutes, much gas produced.	One good crystal from iron buttons.
4-inch mould; $\frac{3}{4}$-inch core, magnesite round, iron turnings on top, then carbon bisulphide, 20 tons pressure, 8 k.w.m. total heating, heating $1\frac{1}{2}$ minutes.	Nil.
4-inch mould; 1-inch carbon core, 2-inch paper tube, filled with iron filings, sand round, cast-iron borings on top, carbon and tetrachloride of carbon above, 10 tons pressure, 20 k.w.m. total heating, action very vigorous, solid iron core formed round carbon, and the whole of the sand was permeated with deposited carbon and iron chloride, heating 4 minutes.	Nil.
2-inch mould; $\frac{3}{8}$-inch carbon rod core, 1-inch paper tube filled with carborundum and iron sulphide, sand and sulphur round, 10 tons pressure, 6 k.w.m. total heating, silicon formed, heating 22 seconds.	Nil.
2-inch mould; $\frac{1}{4}$-inch iron rod core, olivine and 5 per cent. carbon round, 100 tous pressure, 4 k.w.m. total heating, heating 1 minute.	Nil.
2-inch mould; $\frac{3}{8}$-inch carbon rod core, $\frac{1}{8}$-inch iron in centre, sodium carbonate round, 100 tons pressure, 4 k.w.m. total heating, heating 10 seconds, charge detonated, mould swelled $\frac{1}{8}$-inch in diameter, charge blew out top and bottom.	One or two crystals from iron.

Carbide mixture = CaC_2 1, SiO_2 2, Al_4C_3 3, FeS 16, Mg 2 parts by weight.

Experiment.	Result.
4-inch mould; iron sulphide core, carbide mixture, and wood charcoal round, 20 tons pressure, 10 k.w.m. total heating, heating 1¼ minutes.	Nil.
4¾-inch mould; iron sulphide core, carborundum, caustic soda, and sulphur round, 20 tons pressure, 30 k.w.m. total heating, heating 3¾ minutes.	Some crystals.
4¾-inch mould; iron sulphide core, carborundum, caustic soda, sodium carbonate and sulphur round, 15 tons pressure, 30 k.w.m. total heating, about ½ ton of gaseous pressure when somewhat cooled and ram released, heating 3 minutes.	Some crystals.
4¾-inch mould; iron sulphide core, carborundum, sodium carbonate and sulphur round, 15 tons pressure, 30 k.w.m. total heating, heating 8 minutes.	Nil.
4¾-inch mould; iron sulphide core, carborundum and caustic soda round, 15 tons pressure, 25 k.w.m. total heating, about 2 tons of gaseous pressure when ram released, heating 6 minutes.	Nil.
4¾-inch mould; iron sulphide core, carborundum, calcium fluoride and sulphur round, 20 tons pressure, 25 k.w.m. total heating, about 1 ton of gaseous pressure when ram released, heating 1½ minutes.	Nil.
4¾-inch mould; carbon core, paper tube round it filled with iron fluoride, silica round, 20 tons pressure, 40 k.w.m. total heating, heating 1 minute.	Nil.
4-inch mould; carbon rod core, 2-inch paper tube filled with carborundum and sodium, outside paper tube packed with marble, graphite on top, 20 tons pressure, 25 k.w.m. total heating, sodium carbide and silicon formed, heating 1 minute 5 seconds.	Nil.
4-inch mould; ¾-inch carbon rod core, 3-inch paper tube containing sodium and carborundum, calcium carbide round, 26 tons pressure, 18 k.w.m. total heating, heating 1 minute.	Nil.
4-inch mould; ¾-inch carbon rod core, iron filings round, carborundum on top, then sodium and mercury covered with iron filings, 26 tons pressure, 80 k.w.m. total heating, heating 1 minute.	Nil.
4-inch mould; ½-inch carbon rod core, 1⅛-inch paper tube containing iron filings, carbon round, sodium on top, 26 tons pressure, 30 k.w.m. total heating, heating 40 seconds.	Nil.
4-inch mould; ½-inch carbon rod core, 1⅛-inch paper tube containing iron, barium oxide and strontium oxide, carborundum round, sodium on top, 26 tons pressure, 50 k.w.m. total heating, heating 2¾ minutes.	Nil.
4-inch mould; filled with marble, pressed, 1½-inch hole drilled in it, ½-inch carbon rod in centre, iron, carbon, carborundum and titanium round, sodium on top, 26 tons pressure, 37 k.w.m. total heating, heating 3¼ minutes.	Nil.
4-inch mould; filled with bauxite pressed to 20 tons, 1½-inch hole drilled in it, ½-inch carbon rod in centre, iron, lampblack, carborundum and bauxite round, sodium on top, 26 tons pressure, 30 k.w.m. total heating, heating 4½ minutes, charge blew out.	Some crystals from ejected iron.

Experiments.	Result.
4-inch mould; packed with dry alumina, $1\frac{1}{4}$-inch hole, $\frac{1}{2}$-inch carbon rod core, iron, lampblack, carborundum, ferro manganese and thorium nitrate round, sodium on top, 26 tons pressure, 20 k.w.m. total heating, heating 1 minute 10 seconds.	Nil.
4-inch mould; packed with carborundum, $1\frac{1}{4}$-inch hole drilled in it, $\frac{1}{2}$-inch carbon rod core, $\frac{3}{4}$-inch paper tube filled with iron and ferro vanadium, carbon and iron round, sodium on top, 26 tons pressure, 20 k.w.m. total heating, heating 1 minute.	Nil.
4-inch mould; packed with anhydrous ferrous oxalate, $\frac{1}{2}$-inch hole drilled, $\frac{1}{4}$-inch iron rod core, sodium round, ferrous oxalate on top, covered with graphite, 26 tons pressure, 25 k.w.m. total heating, heating 3 minutes.	Nil.
4-inch mould; packed with sodium chloride and carborundum, $\frac{5}{8}$-inch carbon rod core, 26 tons pressure, 25 k.w.m. total heating, heating 4 minutes.	Sodium carbide and silicon formed.
4-inch mould; alumina, rouge, magnesium chloride and calcium chloride, pressed; $1\frac{1}{4}$-inch hole, $\frac{5}{8}$-inch carbon rod core, carborundum in hole, 26 tons pressure, 15 k.w.m. total heating, heating $1\frac{1}{4}$ minutes.	Green carborundum produced.
4-inch mould; packed with carborundum, calcium carbide, aluminium carbide, sodium chloride, magnesium chloride, sulphur, iron filings; $\frac{5}{8}$-inch hole drilled, carbon rod put in, carbon tetrachloride poured on top, covered with graphite, 26 tons pressure, 50 k.w.m. total heating, heating 13 minutes, charge blew out, mould dilated.	Some crystals from centre and also from ejected matter.
4-inch mould; packed with alumina, $\frac{1}{2}$-inch hole, $\frac{1}{4}$-inch iron rod core, lead peroxide and carborundum placed in hole and on top, covered with iron filings, 26 tons pressure, 8 k.w.m. total heating, heating 1 minute.	Nil.
4-inch mould; packed with alumina, 1-inch hole, $\frac{1}{4}$-inch iron rod core, rouge and carborundum in hole, 26 tons pressure, 50 k.w.m. total heating, heating 3 minutes.	Some amorphous carbon and silicate of iron.
4-inch mould; packed with carborundum and 4 per cent. sodium carbonate, pressed at 16 tons, 1-inch hole drilled, $\frac{5}{8}$-inch carbon rod core, carborundum and sodium carbonate placed in hole, 26 tons pressure, 15 k.w.m. total heating, heating 3 minutes, formed grey solid which detonated when struck.	Sodium carbide and pale green carborundum.
4-inch mould; $\frac{1}{2}$-inch carbon rod core, iron sulphide and sulphur round, then sodium silicate and lampblack, 26 tons pressure, 50 k.w.m. total heating, heating $2\frac{3}{4}$ minutes.	Nil.
4-inch mould; 1-inch carbon rod core, sodium silicate, alumina, rouge, magnesia, and lime round, 26 tons pressure, 40 k.w.m. total heating, heating $1\frac{1}{2}$ minutes.	Nil.
4-inch mould; packed with artificial pyrope, 1-inch hole, $\frac{1}{2}$-inch carbon rod core, carborundum and sodium peroxide placed in hole, 26 tons pressure, 8 k.w.m. total heating, heating $3\frac{1}{2}$ minutes.	Nil.
4-inch mould; filled with sand, 1-inch hole, $\frac{1}{2}$-inch carbon rod core, filled hole with potassium nitrate and carborundum, 26 tons pressure, 15 k.w.m. total heating, heating 1 minute.	Nil.

Experiment.	Result.
$4\frac{1}{4}$-inch mould; packed with sand, coke, sawdust and salt pressed, $\frac{1}{2}$-inch hole drilled, $\frac{1}{2}$-inch carbon rod core covered with iron filings and graphite, 26 tons pressure, 28 k.w.m. total heating, heating $1\frac{1}{2}$ minutes, about 1 ton of gaseous pressure when somewhat cooled and ram released.	Green carborundum non-polarizing.
Same as the above plus 20 per cent. ferrous oxalate, 5 tons pressure, 30 k.w.m. total heating, about 3 tons of gaseous pressure when somewhat cooled and ram released, heating 1 minute 40 seconds.	Many green and also clear non-polarizing plates.
Same as above plus zinc dust, $\frac{3}{4}$ ton pressure, 28 k.w.m. total heating, heating 3 minutes, about 1 ton of gaseous pressure when somewhat cooled and ram released.	Green and yellow plates which would not burn in oxygen.
$4\frac{1}{4}$-inch mould; three-fourths full of CO_2, snow pressed hard, sand, coke, sawdust and salt put on top, carbon rod core, 15 tons pressure, 9 k.w.m. total heating, heating $5\frac{1}{4}$ minutes.	Nil.
2-inch mould; $\frac{1}{2}$-inch carbon rod core, carborundum and chromium oxide packed round, 20 tons pressure, 12 k.w.m. total heating, heating $\frac{3}{4}$ minute.	Green carborundum and silicon produced.
2-inch mould; $\frac{1}{2}$-inch carbon rod core, ferrous oxalate and magnesium packed round, 20 tons pressure, $\frac{1}{2}$ k.w.m. total heating, a few rounded particles of hard carbon, charge blew out after 2 seconds heating, repeated three times with same result, should be further investigated.	Nil.
2-inch die; melted tin and rubber poured into die, crucible containing molten iron placed on top and pressed to 18 tons.	Nil.
$\frac{3}{8}$-inch diameter air-hardened tungsten steel die; graphite and 7 grains of fulminate under 230 tons pressure, die heated 180° C., fulminate exploded and retained in mould.	Nil.
$\frac{3}{8}$-inch diameter air-hardened tungsten steel die; graphite and 15 per cent. potassium chlorate placed in press, and pressure increased till mixture detonated at about 200 tons per sq. inch.	Nil.
$\frac{3}{8}$-inch diameter air-hardened tungsten steel die; graphite and saltpetre and three fulminate caps, heated with burner to above temperature of detonation of fulminate under 230 tons pressure, nothing escaped.	Nil.
Melted iron in carbon crucible, added bismuth, poured into steel mould and pressed to 20 tons.	Nil.
$1\frac{1}{2}$-inch tempered steel die; sodium in bottle surrounded with mixture of carborundum and lampblack, added water, and pressed at 100 tons for 10 minutes.	Nil.
$1\frac{1}{2}$-inch tempered steel die; lithium in bottle, carborundum and water, 50 tons pressure for 15 minutes.	Nil.
$1\frac{1}{2}$-inch tempered steel die; sodium in bottle, carborundum and water, 40 tons pressure, temperature of die raised to 250° C. by means of gas burners.	Nil.
$1\frac{1}{2}$-inch tempered steel die; potassium in bottle and carbon bisulphide, 60 tons pressure, die heated by gas to 250° C.	Nil.
$1\frac{1}{2}$-inch tempered steel die; ferrous oxalate and sodium, 100 tons pressure for $4\frac{1}{2}$ hours.	Nil.

Experiment.	Result.
$1\frac{1}{2}$-inch tempered steel die; gum arabic and phosphoric oxide, 60 tons pressure, heated by gas for 1 hour to 250° C., black cindery deposit produced.	Nil.
$1\frac{1}{2}$-inch tempered steel die; lead peroxide and carborundum grit mixed with lead fluoride, 50 tons pressure, heated by gas for 1 hour, red leaflets of lead produced.	Nil.
$1\frac{1}{2}$-inch tempered steel die; sodium peroxide, carborundum No. 6 grit and sodium chloride heated 45 minutes to 200° C., 50 tons pressure.	Nil.
$1\frac{1}{2}$-inch tempered steel die; sodium peroxide, carborundum and artificial pyrope, heated by gas $\frac{3}{4}$ hour to 200° C. at 50 tons pressure.	Nil.
$1\frac{1}{2}$-inch tempered steel die; calcium carbide and glass bulbs filled with water, pressure 50 tons, time 30 minutes.	Nil.
Ditto plus a small amount of sodium	Nil.

DUCK-GUN EXPERIMENTS.

(See figs. 4 and 5.)

Experiments.	Result.
Fired piston on to charge of graphite and cotton wool placed at end of barrel which contained air at atmospheric pressure; propellant, 20 grains black powder.	Nil.
Same as above, but barrel filled with oxygen; propellant, 40 grains black powder.	Nil.
Same as above, but barrel filled with oxygen and acetylene, and propellant 57 grains of black powder.	Small crystals in skin of piston and end plug, probably Moissan effect.

Calculation of the Temperature Reached on the Compression of Acetylene and Oxygen Experiment.

By Stanley S. Cook.

The temperature reached may be estimated from the final pressure, which the observed deformation of the block and plug indicates to have been in the neighbourhood of 100 tons per sq. inch. But it must be remembered that there is a change of molecular volume as a result of combustion. Thus the mixture which, as C_2H_2 and 5 (O), has $3\frac{1}{2}$ molecular volume, would on combustion to $2CO_2$ and H_2O have only 3 molecular volumes. The final temperature deduced from the pressure will therefore depend upon the extent to which chemical combination has taken place.

The original mixture being at atmospheric pressure and a temperature of 290° C. absolute, a pressure of 100 tons per sq. inch after compression to $\frac{1}{288}$ of its original volume would indicate a temperature of 15,250° C. If, however, complete combustion has taken place, this same pressure would correspond to a temperature greater in ratio of $3\frac{1}{2}$ to 3, viz., to 17,700° C. The actual temperature must therefore have been something between these two values.

(B.) RIFLE EXPERIMENTS.

(See figs. 6, 7, and 8.)

Experiment.	Result.
0·303 bullets were fired into holes, 0·303-inch diameter (in some cases of tungsten steel air hardened, tapering to $\frac{1}{8}$-inch at the bottom) in steel blocks; in the holes were placed the following substances:—	
Coarse carborundum; No. 6 grit, which after the experiment was found to be crushed to small splinters.	Nil.
Carborundum (very fine grained) and sodium; carborundum unaltered	Nil.
Carborundum and iron; very fine grained carborundum and finely powdered iron, the carborundum appeared slightly whitened, otherwise unaltered.	Nil.
Calcium carbide and sulphur; calcium carbide all destroyed and calcium sulphide formed.	Nil.
Carborundum (very fine grained) and nickel filings; no whitening of the carborundum.	Nil.
Carborundum and sodium peroxide	Sodium silicate and amorphous carbon formed.
Bort; somewhat crushed	No change.
Fired bullet through carbon arc into hole in steel block	Nil.
Fired bullet through carbon rods arcing in a bed of graphite contained in fire clay crucible, above the hole in steel block.	Nil.
Graphite and sodium peroxide	Graphite destroyed.
Fired bullet through white hot iron plate into hole containing carbon rod heated white hot by resistance heating.	Nil.
Fired bullet into hole containing graphite in steel block whilst arcing between top of hole and a carbon rod to produce vapour of iron.	Rounded fragments of graphite.
Calcium carbide and paper (wet); amorphous carbon formed	Nil.
Cotton wool; a little amorphous carbon formed	Nil.
Potassium chlorate and sugar carbon	Nil.
Carbon bisulphide and rubber	Nil.
Carbon bisulphide and sodium bismuthate	Nil.
Carbon bisulphide and potassium chlorate	Nil.
Mineral oil	Nil.
Carbon bisulphide	Nil.
Sodium bismuthate, benzene, and carbon tetrachloride	Nil.
Graphite and paper; bullet struck side of hole in block, producing a small amount of molten iron.	A few crystals.
Through white hot carbon-bridge over hole electrically heated to point of vaporization.	A few crystals.
Graphite and iron	Nil.
Graphite and fulminate caps	Nil.
White hot graphite, contained in small crucible previously heated in arc furnace and placed above hole.	Nil.
Red hot graphite as above	Nil.
Graphite and fulminate in paper	Nil.

Experiment.	Result.
Graphite, naphthalene, and fulminate	Nil.
Through carbon crucible, containing white hot highly carburized iron.	Nil.
Graphite, saltpetre, and fulminate	Nil.
Iron, caps, and carbon bisulphide	Nil.
Sugar carbon	Nil.
Sugar carbon plus two caps	Nil.
Sugar carbon plus two caps plus potassium chlorate	Nil.
Sugar carbon plus two caps plus potassium nitrate	Nil.
Sugar carbon, two caps, potassium chlorate and iron	Nil.
Sugar carbon, sulphur and reduced iron	Nil.
Cotton wool soaked in solution of potassium nitrate and dried	Nil.
Iron filings and fulminate	Nil.
Cotton wool soaked in HF., iron and two caps	Nil.
Cotton wool, iron fluoride and caps	Nil.
Sodium fluoride, sugar carbon and fulminate	Nil.
Sodium fluoride and cotton wool	Nil.
Cotton wool, carbon bisulphide and iron filings	Nil.
Cotton wool, carbon bisulphide and fulminate	Nil.
Gun cotton, carbon bisulphide, fulminate and iron	Nil.
Sodium fluoride, iron, fulminate and sugar carbon	Nil.
Sodium fluoride, iron, fulminate, gun cotton, sugar carbon and iron chloride.	Nil.
Sodium fluoride, iron chloride, gun cotton, and graphite	Nil.
Rouge, reduced iron, sugar carbon and carbon bisulphide	Nil.
Rouge, reduced iron, sugar carbon and sulphur	Nil.
Aluminium, rouge, graphite and chlorate	Nil.
Rouge, iron fluoride, chlorate, graphite and caps	Very small residue; doubtful.
Bullet fired into barrel filled with acetylene and screwed into steel block containing cavity, mouth of barrel closed by gold beater skin.	Nil.

(C) EXPERIMENTS AT ATMOSPHERIC PRESSURE.

All experiments allowed to cool by radiation only unless otherwise stated.

Experiment.	Result.
Aluminium, magnesium, carbon, iron, ferric oxide, olivine, and boric anhydride fused in wind furnace.	Nil.
Aluminium carbide fused in carbon crucible by arc, iron added and then sand, heated for 3 minutes. Ingot of iron obtained and a greenish-blue slag ($Al_2O_3 . SiO_2$) charged with carbon.	Nil.
Aluminium carbide melted and run into molten sand in carbon crucible	Nil.
Aluminium carbide and iron heated in carbon crucible	Nil.
Iron pipe packed with graphite melted in electric furnace and then dropped into sulphuric acid with layer of mercury on bottom.	Nil.
Calcium silicate fused in electric furnace, graphite added and molten iron dropped on to it.	Nil.

Various Moissan experiments when quickly cooled gave good results,

Experiment.	Result.	
Phosphate of iron, covered with olivine in carbon crucible, heated by arc	Nil.	
Olivine, iron and graphite, melted in carbon crucible by electric arc . .	Nil.	
Iron, carbon and chromium, 1 per cent., fused in carbon crucible, cooled in water.	One crystal, 0·5 mm. long, which burnt in oxygen.	
Melted olivine in carbon crucible, added calcium carbide, covered with a layer of sand, fused and allowed to cool slowly.	Nil.	
Fused sodium carbonate and sand, added calcium carbide.	Nil.	
Fused sodium carbonate, silica, sodium chloride, calcium carbide and graphite.	Nil.	
Melted iron in carbon crucible, then forced naphthalene vapour through the melt with arc still on, cooled in water.	A few crystals.	
Carbon bisulphide vapour blown through molten iron charged with carbon, cooled in water.	Doubtful.	
Superheated steam blown through carbon tube into east iron melted by arc in carbon crucible, cooled in water, large crystalline deposit round carbon tube.	About 10 per cent. burnt in oxygen.	
Same experiment repeated many times and carbide mixture* added while cooling before steam applied.	Same result.	
Iron 20 per cent., magnalium 20 per cent., calcium carbide 30 per cent., sand 30 per cent., melted in carbon crucible by arc and then water added.	Same result.	
Iron melted in carbon crucible, aluminium carbide and silica added, stirred and then sulphur dioxide blown through for half an hour, quenched in water.	Same result.	
Steam and benzene blown through carbon tube into silicon carbide and iron melted in carbon crucible by arc.	Nil.	
Poured iron highly charged with carbon through narrow git into massive steel mould closed at bottom by breech screw, no mechanical pressure exerted on breech screw.	—	
Did same with plunger forced down git; no pressure from expansion on setting.	—	Fig. 9.
Passed CO through molten iron in carbon crucible, poured melt into the above mould.	Nil.	
Melted iron, added 5 per cent. zinc and poured into the above mould. .	Nil.	
Poured molten carburized iron into molten zinc	Nil.	
Olivine fused in small carbon crucible, stirred with a graphite rod, and oxycoal gas blowpipe played on the molten surface, covered melt with graphite, and cooled slowly.	Nil.	
Melted olivine and 10 per cent. graphite in carbon crucible, oxycoal gas blown through, then water poured on.	Nil.	
Same as above but "blue ground" in place of olivine, granite-like mass formed.	Nil.	
Iron melted in carbon crucible, bismuth added, cooled in water. . . .	Nil.	
Heated iron and carbon, and plunged yellow diamond into it when near setting point.	Etched and pock marked.	

* *Carbide mixture* = CaC_2 1, SiO_2 2, Al_4C_3 3, FeS 16, Mg 2 parts by weight.

Experiment.	Result.
Same experiment with ferro-titanium	Etching deeper.
Iron melted in carbon crucible, aluminium carbide, iron sulphide and silica added, steamed, quenched in water.	Crystals which burnt in oxygen.
Heated iron, magnesium, strontium oxide and calcium, cooled in water	Nil.
Heated iron, tin, bismuth, antimony, and ferro-vanadium, cooled in water.	Nil.
Heated iron, lead, silver, copper, cooled in water	Nil.
Silica 4, magnesia 5, rouge 1, parts by weight, carborundum and iron melted in electric furnace on graphite bed.	Octahedra (spinels) which would not burn in oxygen.
Melted a round steel file by arcing under oil	Nil.
Heated chromium oxide and carborundum in carbon crucible, cooled in water.	Nil.
Fused calcium carbide and added potassium chlorate, cooled in water	Nil.
Passed hydrogen for 1 hour over carborundum No. 6 grit, heated cherry red, grit unaltered.	Nil.
Passed CO for 1 hour over carborundum No. 6 grit, heated cherry red, grit very little altered.	Nil.
Passed iron pentacarbonyl over carborundum No. 6 grit, heated nearly red for 3 hours, the grit was much etched.	Several plates, thought to be correct at the time, but would not burn in oxygen.
Passed iron pentacarbonyl over magnesium (redness) 1 hour	Nil.
Passed iron pentacarbonyl over magnesium carbonate (redness) ½ hour	Nil.
Passed iron pentacarbonyl over magnesium oxide (redness) 1 hour	Nil.
Passed iron pentacarbonyl over carborundum No. 6 grit and sodium chloride (redness) 1 hour, the grit was etched.	Nil.
Passed iron pentacarbonyl over carborundum No. 6 grit, sodium chloride and iron filings, orange heat 20 minutes.	Nil.
Passed the following over carborundum No. 6 grit contained in a silica tube heated to varying temperatures along its length from just red to white heat for periods varying from 20 minutes to 1½ hours; cyanogen, nitrogen, nitric acid gas mixed with iron pentacarbonyl; chlorine gas; ammonia gas; CO_2 and CO; ammonium chloride vapour; sulphuretted hydrogen; acetylene; coal gas.	All nil.
Passed iron pentacarbonyl over the following substances, heated in a silica tube for varying lengths of time; magnesium silicide; silicon, magnesium and calcium alloy; graphite; ferro silicon; ferro silicon and magnesium alloy; ferro silicon, magnesium powder and sodium chloride; iron sulphide; ferrous carbonate; ferrous ammonium sulphate; sodium; potassium and magnesium sulphates; calcium silicate; magnesium silicate; cadmium; artificial pyrope; ferro manganese; silicon and manganese; chromium; silicon, manganese and carbon alloy; finely divided nickel; sodium silicate; nickel and sand.	All nil.
50 per cent. ferro silicon in silica tube heated to 1300° C., carbon bisulphide vapour passed over it for half an hour.	Nil.
Silicon heated to 1100° C. in silica tube, carbon bisulphide vapour passed over it for half an hour.	Nil.

Experiment.	Result.
Passed CO over iron sulphide melted in salamander crucible in wind furnace. Top of crucible closed by massive iron lid containing on side inlet tube for CO and in centre 1 inch hole with perforated bottom, hole fitted with No. 6 grit.	Many clear plates, some with triangular markings.
Same experiment repeated many times	Would not burn in oxygen.

(D) EXPERIMENTS IN VACUUM.

(See figs. 10 and 11.)

All experiments allowed to cool by radiation only unless otherwise stated.

Experiments.	Result.
4-inch mould; heated reduced iron and lampblack in carbon cup, pressure varied between $\frac{5}{8}$-inch and 3-inch absolute. Cooling by radiation only. Large amount of gas produced containing CO 95 per cent., hydrogen 1 per cent., hydrocarbon 2 per cent., nitrogen 2 per cent. (Fig. 10.)	Large residue of crystals which burnt in oxygen.

High Vacuum Experiments under pressures varying from 10 *mm. to* $\frac{1}{8}$ *mm. absolute.*

Heated iron and 10 per cent. sugar carbon in salamander crucible for half an hour by arc on top, vacuum about 1 mm.	Nil.
Iron, sugar carbon and iron sulphide heated in carbon crucible 15 minutes, then water allowed to enter through tap causing spherules to be ejected, vacuum about 1 mm.	10 per cent. of residue from ejected spherules burnt in oxygen.
As above, but carbon bisulphide instead of water	Nil.
Magnesia, lampblack and tar heated in carbon crucible	Metallic magnesium and magnesium carbide produced.
Alumina and borax fused in wind furnace then heated in carbon crucible in vacuum.	Clear transparent alumina.
Alumina, lampblack and tar heated in carbon crucible	Aluminium carbide formed.
Aluminium carbide, calcium carbide, magnesia, iron, carbon and tar, baked and placed in crucible, resistance heated in vacuum furnace, crucible of sulphur either side to give off sulphur vapour in chamber.	Nil.
Crucible filled with iron and carbon, closed by carbon cover, hole bored in side of crucible and massive iron block placed opposite hole, contents boiled, deposit of carburized iron on block analysed, vacuum about 0·25 mm.	Nil.
Arc deflected by electro-magnet on to iron block, deposit on block analysed, vacuum about 10 mm.	Nil.
Tungsten steel packed round with carbon in carbon crucible and heated to ebullition.	Nil.
Iron, lampblack, Prussian blue heated in carbon crucible by resistance heating.	Nil.
Carbon crucible containing potassium ferrocyanide, lampblack and reduced iron, melted by resistance heating.	Nil.

Experiment.	Result.
Carbon crucible containing sand, sulphur and iron, heated under 1 mm., and coal gas admitted during slow cooling.	Nil.
Iron sulphide, carborundum and magnesia, heated in carbon crucible . .	Nil.
Reduced iron and carborundum, heated in carbon crucible	Nil.
Calcium and carborundum, heated in carbon crucible	Carborundum decomposed to coke-like mass.
Iron, carbon and garnet, heated in carbon crucible	Nil.
Iron, carbon, olivine and iron sulphide, heated carbon in crucible . . .	Nil.
Carborundum, heated white hot in carbon crucible and carbon bisulphide admitted through tap to top of crucible.	Carborundum decomposed.
Reduced iron and lampblack, heated in carbon crucible, sand and sulphur round.	Nil.
Ferrous oxalate, boric acid, sugar carbon and iron, heated in carbon crucible.	Nil.
Sodium carbonate, carborundum and iron, heated for 20 minutes at bright red in iron crucible.	Nil.
Carbon and iron placed in carbon crucible, carborundum grit on top, moderate heat.	Nil.
Carbon and iron chips placed in carbon crucible, massive iron lid with hole bored in it, filled with No. 6 grit, temperature of crucible bright orange, temperature of lid and grit not exceeding dull red heat.	Several crystal plates from top of tube which resembled diamond and burnt in oxygen.
Carbon with reduced iron and rouge in small crucible, same as previous experiment, heated ten minutes bright red.	Nil.
Carbon, iron and carborundum, heated to bright red 50 minutes . . .	Nil.
Calcium and carborundum, heated for 15 minutes to bright red, carborundum No. 6 grit in tube on top.	Nil.
Carborundum, iron and rouge heated 20 minutes to bright red	Doubtful.
Iron, carbon and lime heated 20 minutes to bright red	Nil.
Ferro-vanadium, carborundum, lampblack and iron heated 15 minutes bright orange.	Nil.
Iron pole arcing on iron in crucible rod carborundum on top	Nil.
Carbon pole arcing on iron rod placed in crucible containing iron filings at bottom and carborundum on top.	Several crystal plates very like diamond.
Carbon pole arcing on iron and magnesium, carborundum on top . . .	Nil.
Nickel and carbon heated to orange for 15 minutes, carbon crucible containing No. 6 grit on top.	Nil.
Heated ferrous chloride and carborundum to bright red for 20 minutes, No. 6 grit in tube on top.	Nil.
Gases from heated iron filings in steel bottle passed through silica tube containing carborundum No. 6 grit heated to a bright red.	Several plates as before.
Repeat with same filings gave nothing. To test if due to exhaustion of filings made following experiment:—	
Passed CO over the previously used iron filings, 30 minutes, placed filings in crucible, and heated in vacuum dull red, 20 minutes, carborundum No. 6 grit in hole in massive iron lid on top.	A few plates.

Experiment.	Result.
Heated iron filings and sodium carbonate No. 6 grit in tube on top (bright red) 30 minutes.	Nil.
Heated iron and sodium chloride (bright red) 20 minutes carborundum No. 6 grit in hole in cast-iron lid on top.	Nil.
Heated ferro silicon (15 per cent.) in carbon crucible till most of contents of crucible had volatilized, analysed dust from walls of vacuum pot and ingot remaining in crucible.	Nil.
Heated iron, ferro silicon, ferro manganese, carbon and calcium carbide in carbon crucible.	Nil.
Heated iron, iron sulphide, carbon and calcium carbide, cooled slowly in vapour of carbon bisulphide.	Nil.
Heated iron, carbon, calcium carbide and ferro titanium in atmosphere of carbon bisulphide, pressure $\frac{1}{3}$ mm. absolute.	Nil.
Heated iron, ferro silicon, and ferro titanium in atmosphere of carbon bisulphide, pressure $\frac{1}{3}$ mm. absolute.	Nil.
Heated iron, ferro titanium and ferro silicon	Nil.
Heated iron and carborundum in atmosphere of carbon bisulphide, pressure $\frac{1}{3}$ mm. absolute.	Nil.
Heated reduced iron and lampblack, calcium metal placed near crucible to absorb nitrogen, pressure $\frac{1}{10}$ mm. absolute.	Nil.

EXPERIMENTS IN X-RAY VACUA.

(See fig. 12.)

Experiment.	Result.
Heated cast iron in carbon crucible to vaporising point, cooling by radiation only.	Nil.

The following were all heated to temperature of vaporisation and then allowed to cool by radiation only :—

Heated calcium carbide	Nil.
Heated ferro-silicon	Nil.
Heated calcium carbide and No. 6 grit	Nil.
Heated nickel and No. 6	Nil.
Heated aluminium, magnesium, calcium carbide, and No. 6 grit . . .	Nil.
Heated ferro-silicon, nickel, and No. 6 grit	Nil.
Heated nickel, iron sulphide, and No. 6 grit	Nil.
Heated ferro-titanium and No. 6 grit	Nil.
Heated iron sulphide, zinc, and No. 6 grit	Nil.
Heated ferro-silicon, sand, and lampblack	Doubtful.
Heated reduced iron, sand, and lampblack to point of ebullition at atmospheric pressure, then re-heated under vacuum, much ebullition of gas from metal and many iron spherules thrown out of crucible.	Large yield of rounded particles and plates from ejected metal. These all burnt in oxygen. Contents of crucible yielded nothing.

Experiment.	Result.
Heated samarskite, zircon, iron, carborundum, No. 6 grit, and lampblack	Nil.
Heated reduced iron, sand, and lampblack	Nil.
Heated silicon and carbon	Nil.
Heated olivine, sand, iron, and lampblack	Nil.
Heated magnesia, sand, iron, and lampblack	Nil.
Heated glass, sand, carbon, and iron	Nil.
Heated ferro-silicon, sand, carbon, and iron	Nil.
Heated silicon, iron phosphide, iron, and carbon	Nil.
Heated silicon, manganese, carbon, and iron	Nil.
Heated reduced iron, rouge, sand, and carbon	Nil.
Heated reduced iron, sand, aluminium, and carbon	Nil.

BULK PRESSURE AND VACUUM APPLIED TO A GRANULAR MASS.

Experiment.	Result.
4-inch mould; 1-inch carbon rod with $\frac{1}{4}$-inch iron rod down centre, coarse carborundum round, the vacuum exit tube through side of mould protected by perforated steel plate and drawing the gases from the carborundum, iron turnings on top to bring current into rod, 40 tons pressure, 72 k.w.m. vacuum 0·5 mm. absolute.	Carborundum next to exit tube and layer next iron turnings whitened, and some crystalline plates.

SEALED TUBE EXPERIMENTS.

Experiment.	Result.
Heated sodium, calcium carbide and water in sealed iron tube at 400° C.	Nil.
Heated sodium carbonate and carborundum in sealed iron tube to red heat for 50 minutes, sodium silicate and amorphous carbon formed.	Nil.
As above, but heated for 2 hours 10 minutes at an orange heat, not so much carbon and silicate formed as in previous experiment.	Nil.
Sealed tube containing ferrous oxalate and magnesium powder, heated dull red $\frac{1}{2}$ hour.	Amorphous carbon.
Sealed tube half filled with olivine, graphite, sugar and water, heated to full red for 1 hour.	Nil.

IV. *The Pressure upon the Poles of the Electric Arc.*

By W. G. Duffield, *D.Sc.*, *Professor of Physics, Captain, Royal Air Force*, Thos. H. Burnham, *B.Sc.*, *and* A. H. Davis, *B.Sc.*, *Post-Graduate Students in Physics, University College, Reading.*

Communicated by Prof. O. W. Richardson, *F.R.S.*

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Suspending one pole of a carbon arc and keeping the other pole fixed it was found that there was an apparent repulsion between them. There is in fact a pressure upon each electrode which tends to separate them. The first part of this paper is devoted to the experimental methods of estimating this pressure, the second to a discussion of its origin.

PART I.—EXPERIMENTAL.

Three series of observations have been made, the original observations and preliminary series by Duffield in 1912,* the second series in conjunction with Burnham, and the third in conjunction with Davis. In spite of the very small forces examined the three series agree within reasonable limits. The general form of the apparatus (fig. 1) was the same in each series, though there were important differences in the dispositions of the carbons in different sets of experiments.

A stirrup was suspended by a torsion fibre, or sometimes by two fibres, F, as in the illustration, in this was placed a copper rod, E, to whose extremity was fixed at right angles a short carbon rod, C, which was balanced by a counterpoise, W, at the other end. The arc was formed between this carbon rod and another, D, fixed either as shown in the figure or in some other manner to be described later.

In its zero position the copper rod swung freely between two stops, S, placed close to one end. The sensitivity of the suspension and the long period of swing necessitated some simple means for bringing the rod back to the zero position, and the **V**-grip device illustrated in fig. 2 was ultimately adopted in place of the stops, S ; the adjustment was made by twisting the torsion head until on turning down the **V**-grip the suspended wire remained stationary ; this control also enabled the arc-length to be maintained nearly constant during an experiment. The difference between the

* A paper entitled "The Pressure upon the Poles of a Carbon Arc," was read in title at the British Association Meeting, Australia, 1914.

 [Published December 22, 1919.

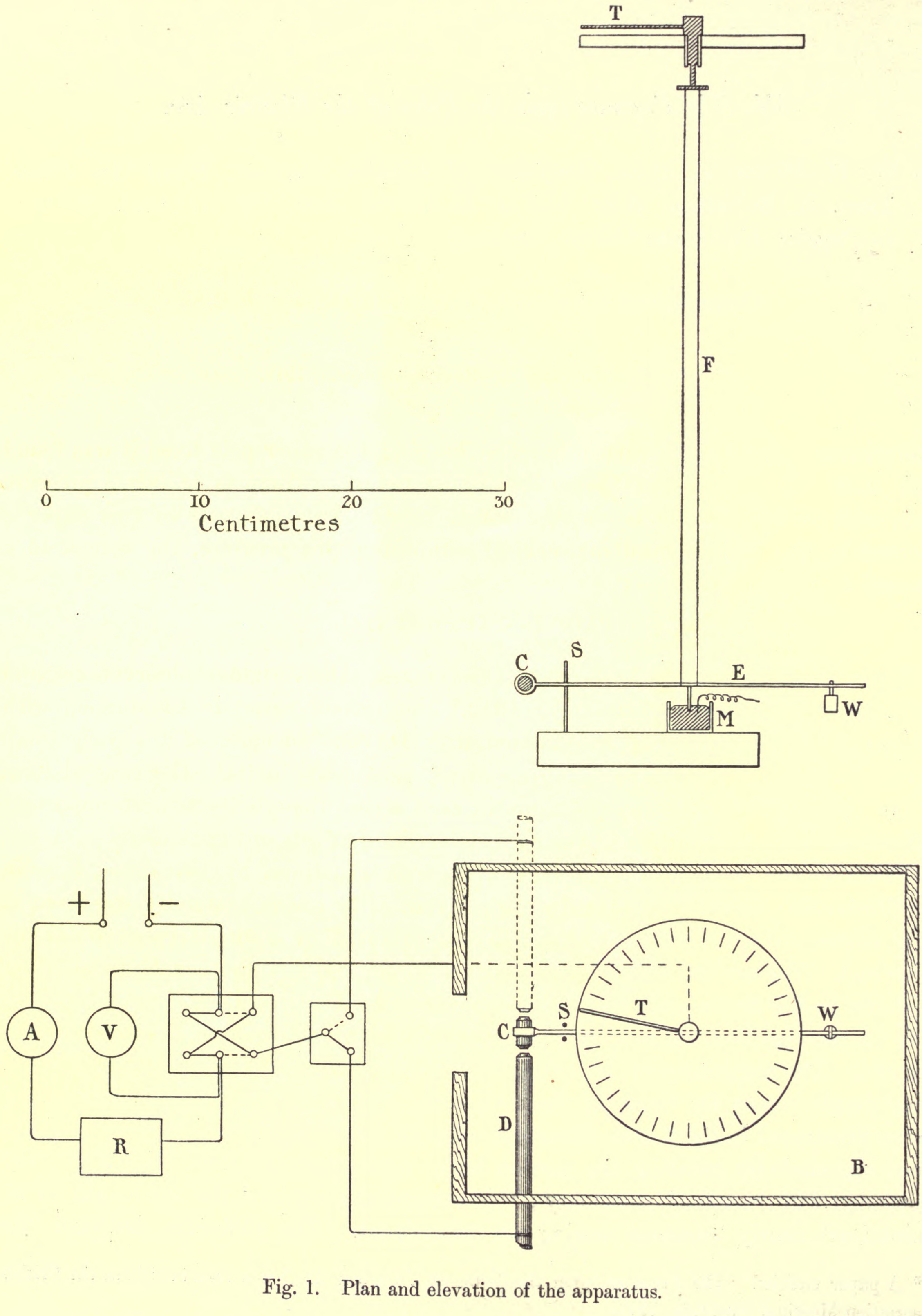

Fig. 1. Plan and elevation of the apparatus.

readings of the torsion head, T, when the current was on and off measures the couple acting upon the suspended copper rod if the constants of the suspension are known.

The movable parts of the apparatus were completely enclosed in a box, B, with a glass top to prevent disturbance from air currents in the room, and appropriate windows and holes were made in it to enable observations to be made. The torsion

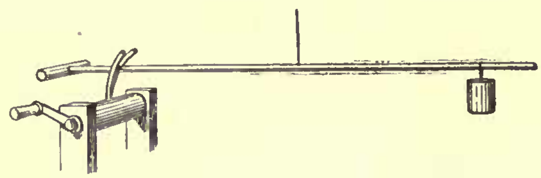

Fig. 2. The **V**-grip.

fibre was enclosed in a vertical tube. A lens focussed an image of the arc upon a screen to facilitate the measurement of the arc-length.

The observed couple is due to :—

(1) A pressure upon the poles due to forces within the arc, including the effect of convexion currents, electrostatic effects, &c. These will be treated as a whole in the first instance, and called the total pressure ;

(2) The interaction between electric currents in the suspended part of the circuit and the earth's magnetic field ;

(3) Interaction between electric currents in the suspended part of the circuit and the currents in the rest of the circuit, called briefly the electromagnetic effect.

Series A. Alternating Current.

Method 1.—The current was conducted to the rod, E, through the mercury trough, M.

Alternating current was used because it at once eliminated the couple due to the action of the earth's magnetic field. The couple due to the electromagnetic effect was estimated by experiments described on p. 124. The values given in the first row of Table I. represent the total pressures upon the pole after allowing for this couple ; the arc length was 3·5 mm. throughout.

Table I.—Alternating Current.

Ampères ∼ . . .		3·5	4·3	5·0	5·2	6·5	7·0	8·0	9·0	9·5	10·0	12·0
		Total pressures in dynes.										
Row 1	Method 1	0·01	0·11	·	0·27	0·40	0·60	0·87	0·94	1·02	1·07	·
Row 2	Method 2	·	·	·	·	·	·	·	1·0	·	·	1·7
Row 3	Method 3	·	·	·	·	·	·	0·76	·	·	·	·
Row 4	Method 4	·	·	0·32	·	·	0·59	·	1·0	·	·	·

Method 2.—The disposition shown in fig. 3 was employed, the fixed carbon rod occupied either position A or B. When in position A the arc was vertical and the couple was caused by the electromagnetic action between the movable part of the apparatus and the rest of the circuit. When placed in position B the arc was horizontal and there was an additional couple occasioned by the pressure upon the pole. The difference between the readings gave the couple to be measured and hence the pressure in dynes. The results for currents of 9 and 12 ampères are recorded in row 2 of Table I.

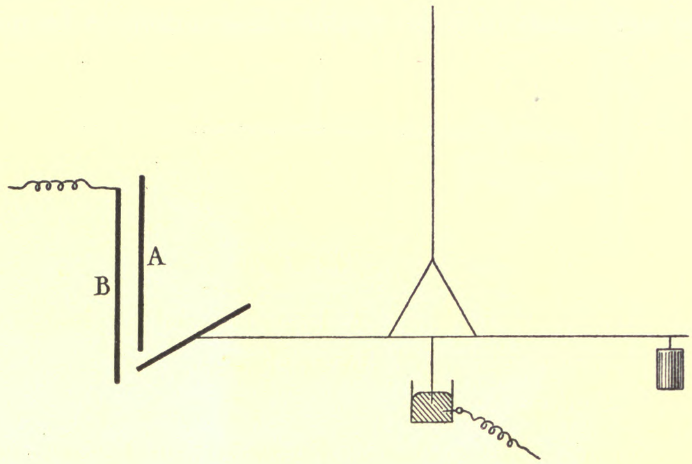

Fig. 3.

Method 3 (*Double Arc*).—The mercury cup was removed and the circuit completed through a second arc shown at C or C′ in fig. 4; as it was vertical it did not add anything to the deflecting couple upon the copper rod. It constituted an extremely flexible electrical joint. As before, A and B were alternative positions for the other carbon, in the latter of which the arc was horizontal and the pressure effective; the differences between A and B measured in a typical experiment are recorded in

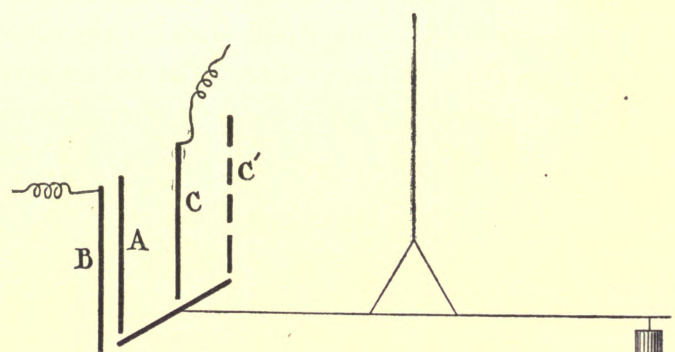

Fig. 4.

Table II., they indicate the nature of the agreement between different readings; when two arcs were used the readings became much more difficult.

TABLE II.—Deflexions. Subsidiary Carbon in Centre. 8 Ampères, Alternating Current.

Zero.	B.	A.	B.	A.	B – A.	Total pressure in dynes.
29°·5	– 5°·0	23°·5	– 3°·5	23°·0	27°·5	0·76

Method 4.—The arrangement was as shown in fig. 5. Carbon rods were fixed axially to the ends of the swinging copper rod, and between these and the sides of two vertical carbons arcs were started. Various dispositions, shown in the same figure, were used, which gave couples which were simple multiples of that due to the

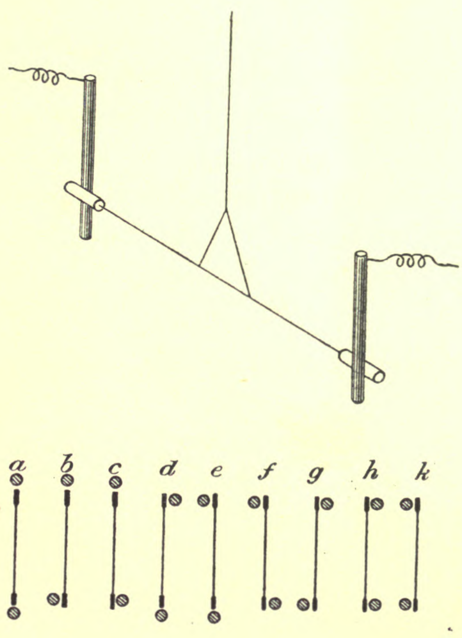

Fig. 5.

pressure upon the pole acting at the end of the swinging arm. For example, for disposition *g* the couple is twice that given by disposition *d*. Save for the small element of current in the arc itself the electromagnetic effects should be eliminated.

Method 4 is a good one save for the effects of convexion currents which are considerable, because they rise over the curved side of the carbon.

TABLE III.—Couples Due to a Single Arc.

Current in ampères.	$a-b$.	$a-c$.	$a-d$.	$a-e$.	$\frac{1}{2}(a-f)$.	$\frac{1}{2}(a-g)$.	a_1-b_1.	a_1-c_1.	Mean.	Total pressure in dynes.
	°	°	°	°	°	°	°	°	°	
5	19	29	26	21	26	17	—	—	23	0·32
7	40	47	27	49	50	37	48	50	43	0·59
9	—	>54	76	71	—	—	—	—	73	1·0

Regarding the experiments upon alternating current arcs as a whole the results (Table I.) show a satisfactory measure of concordance for the various methods.

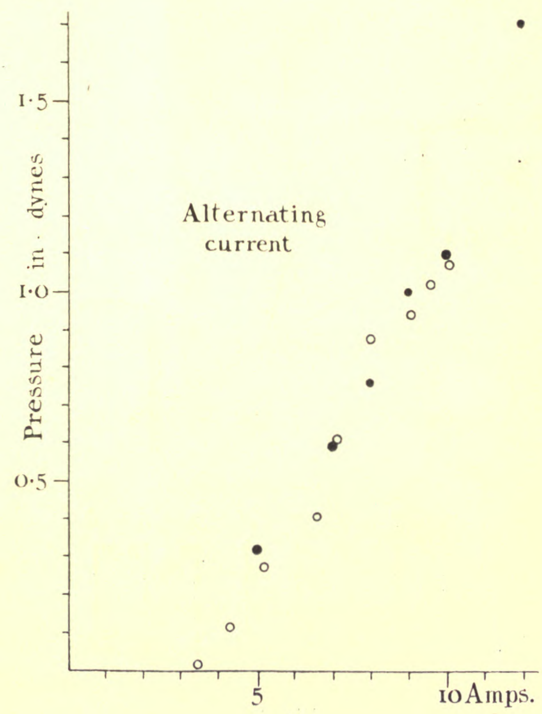

Fig. 6. From Table I.

Method 1 . . . ○. Other methods . . . ●.

Fig. 6 shows the results graphically, open circles representing the observations obtained by method 1, full circles the results of the other three methods.

The Influence of Length of Arc upon the Pressure. Alternating Current.

In the above experiments the arcs were all approximately 3·5 mm. long.

The following experiment was undertaken to determine whether the pressure was increased or diminished by shortening the arc. The disposition was that shown in fig. 7, a current of 9 ampères ∼ was used. The arcs were both horizontal.

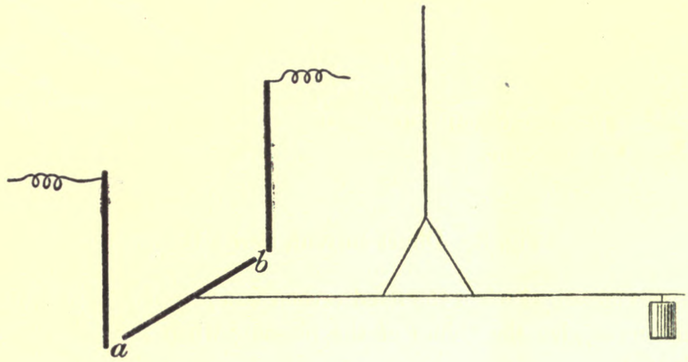

Fig. 7.

When $a < b$ pressure on a greater than pressure on b.

$a > b$,, b ,, ,, a.

Hence for alternating currents the pressure is greater for small arc lengths.

II. Direct Current. Series B.

Observer: Mr. T. H. Burnham.

The disposition of method 1 (fig. 1) was employed. By reversing the direction of the current the pressures upon the anode and cathode were separately determined. In order to eliminate the effect of the earth's magnetic field upon the swinging arm, E, which now carries a current, the fixed pole was placed first on the west and then on the east (dotted position in fig. 1), so that upon one occasion the sum of the pressure upon the pole and the earth's effect was measured, and upon the other their difference. A typical example is shown in fig. 8. The mean of the two curves thus obtained is free from the influence of the earth's magnetism. The values were then corrected as before for the electromagnetic effect due to the rest of the circuit.

The results are given in Tables IV., V., and VI. Discussion of them is reserved until a further method of attack has been described.

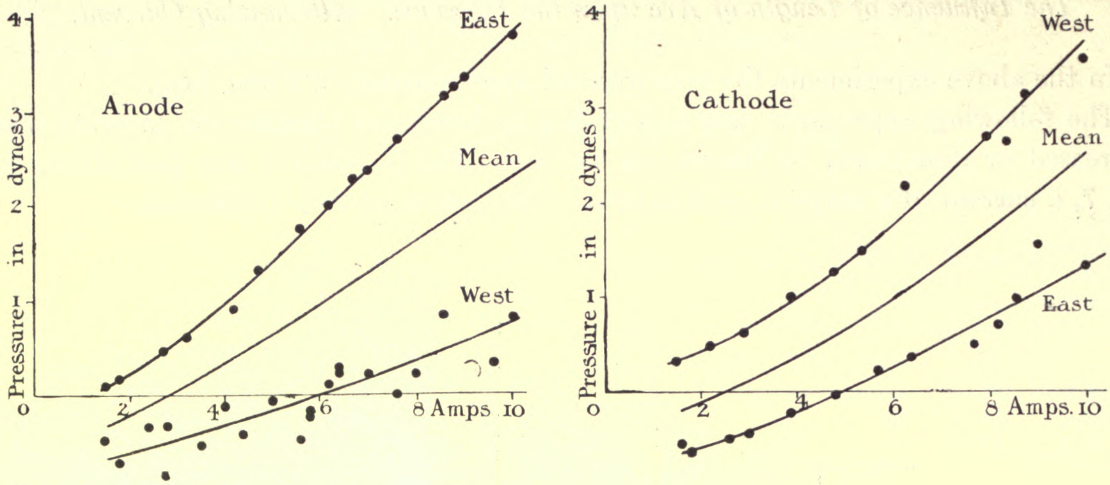

Fig. 8. Direct current, Series B.

Showing the total pressures measured when the fixed carbon was placed East and West of the suspended carbon. The mean curves give the values of the pressure when the effect of the earth's magnetic field has been eliminated.

TABLE IV. (BURNHAM).—Direct Current. Variation of Total Pressure with Current. Arc-Length Constant (3·5 mm.). After Correction for the Earth's Field and Electromagnetic Effects.

Current in ampères.	Total pressure on anode in dynes.	Total pressure on cathode in dynes.	Current in ampères.	Total pressure on anode in dynes.	Total pressure on cathode in dynes.	Current in ampères.	Total pressure on anode in dynes.	Total pressure on cathode in dynes.
1·5	−0·32 (2)	−0·22	4·2	0·13	—	7·0	0·74 (2)	—
1·6	—	−0·19	4·4	0·17	—	7·6	0·86 (2)	—
1·8	−0·30 (2)	−0·17	4·7	0·15	—	7·7	—	0·95
2·2	—	−0·12	4·8	—	0·34 (2)	8·0	0·91	1·05
2·4	−0·19	—	5·0	0·29	—	8·2	—	1·12 (2)
2·6	—	−0·05	5·4	—	0·48	8·4	—	1·10 (2)
2·7	−0·17	—	5·6	0·42 (2)	—	8·6	1·01 (2)	1·19
2·9	−0·15	−0·02	5·7	—	0·53	8·8	1·05	1·17
3·0	—	−0·01	5·8	0·45 (2)	—	9·0	1·08	1·31
3·2	0	—	6·2	0·56 (2)	—	9·6	1·19	—
3·5	0·03	—	6·3	—	0·63	10·0	1·26	1·56 (2)
3·9	—	0·20 (2)	6·4	0·58 (2)	0·66			
4·0	0·09	—	6·7	0·66	—			

The figures in brackets indicate the number of observations where more than one were made.

TABLE V. (BURNHAM).—Direct Current. Variation of Total Pressure with Current and Arc-length. After Correction for Earth's Field and Electromagnetic Effects.

Current in ampères	Total pressure upon anode in dynes.					Total pressure upon cathode in dynes.					
	3	4	5	6	7	2	3	4	5	6	10
Arc-length. mm.											
0·09	−0·02	0·34	0·60	0·71	·	−0·13	0·30	0·72	0·90	1·13	·
1·3	·	·	·	0·70	·	−0·13	0·08	·	·	0·90	·
1·7	0·17	0·34	0·59	0·66	0·74	−0·25	−0·14	0·34	0·60	·	··
2·2		0·34	·	0·70	·	−0·35	·	·	·	0·83	
2·6		0·26	·	·	·	·	−0·19	0·24	0·49	·	
3·0	·	·	·	·	·	·	0·15	·	·	0·73	
3·5	−0·02	0·32	0·57	·	0·74	−0·45	0·13	0·10	0·36	·	1·55
3·8	·	·	·	0·68	·	·	−0·24	·	·	0·63	
5·2	·	·	0·59	0·65	·	·	−0·40	0	0·30	0·55	
6·0	0·11	0·16	0·56	·	·	·	·	·	0·26	·	
6·9	·	·	·	·	·	−0·53	−0·44	−0·14	·	0·43	
8·6	·	0·27	0·53	0·68	0·74	·	−0·45	·	0·20	·	
9·0	·	·	·	·	·	·	·	·	·	0·45	
10·3	0·11	·	·	·	·	·	·	·	·	·	

TABLE VI. (BURNHAM).—Direct Current. Variation of Total Pressure with the Current for two given Arc-lengths.

Current in ampères	Total pressure upon anode in dynes.						Total pressure upon cathode in dynes.							
	2·0	2·8	3·5	5·0	7·0	8·5	2·4	2·8	4·0	5·0	6·0	7·0	8·5	9·0
Arc-length. mm.														
1·7	·	0·22	·	0·45	1·28	1·56	·	0·37	·	0·70	·	1·40	1·82	1·60
7·0	0·11	·	0·13	·	1·15	1·40	0·10	·	0·25	·	0·75	·	1·33	·

DIRECT CURRENT. SERIES C.

Observer: Mr. A. H. DAVIS.

The arrangement shown in fig. 9 was adopted; the moment of inertia of the arm was reduced by moving the balance weight very close to the centre of the swinging arm, and a vane dipping into water was added to damp the oscillations. The double-arc method was adopted because it eliminated the electromagnetic effects upon the movable parts of the circuit due to the earth's field and to the rest of the circuit;

any outstanding effect which might be due to these influences was specially looked for by using two vertical arcs, but it was found to be inappreciable.

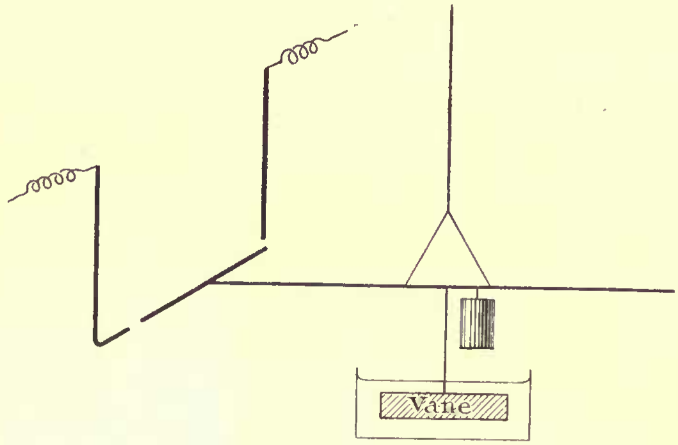

Fig. 9.

Tables VII. and VIII. give the means of a large number of observations obtained by this method.

TABLE VII. (DAVIS).—Direct Current. Variation of Total Pressure with Current and Arc-length.

Current in ampères	Total pressure upon anode in dynes.									Total pressure upon cathode in dynes.								
	2·5	3	4	5	6	7	8	9	10	2·5	3	4	5	6	7	8	9	10
Arc-length. mm.																		
1	0·15	0·26	0·52	0·70	1·16													
2	0·13	0·21	0·51	0·75	0·90	0·98	1·24	1·63	2·07	0·03	0·05	0·18	0·32	0·46	0·85	1·03	1·16	1·75
3 { Set A	·	0·09	0·39	0·37	0·54	0·78	1·17	1·54	1·72		0·08	0·13	0·23	0·39	0·59	0·71	0·74	1·20
3 { Set B	·	0·03	0·24	0·39	0·51	0·61	0·86	1·12	1·58									
5	·	·	0·34	0·65	0·81	0·87	1·21	1·47	1·55			0·23	0·31	0·41	0·59	0·70	0·98	1·15
7			0·52	0·75	0·90	0·90	1·21	1·42	2·02									

TABLE VIII. (DAVIS).—Total Pressures for a Current of 6 Ampères. Variation with Arc-length.

Arc-length in millimetres	Anode.								Cathode.				
	1	2	3	4	5	6	7	8	1	2	3	4	5
Total pressure, set A	0·75	0·62	0·41	0·60	0·67	0·36	0·28	0·35	0·40	0·34	0·43	0·36	0·32
Total pressure, set B	0·95	0·57	0·56	0·52	0·62	0·47	0·50	0·39	·	·	·	·	·

In this table mean values only are given. Fig. 12 shows the individual readings.

Variation of Total Pressure with Arc-length for Constant Current.

This relationship is illustrated by fig. 10, from Table V., for Series B, and by fig. 11, from Table VII., for Series C. Neither set is corrected for convexion-currents.

The Cathode.—The two diagrams agree in showing a rapid drop in the total pressure as the arc-length is increased from very small values; for long arcs the total pressure approaches a constant value which is usually reached at about 10 mm. More weight is attached to the curves of Series B as the experimental method was not so difficult. For very small arcs the curve appears to be asymptotic to the pressure axis, indicating very high values of the reaction for very short arcs.

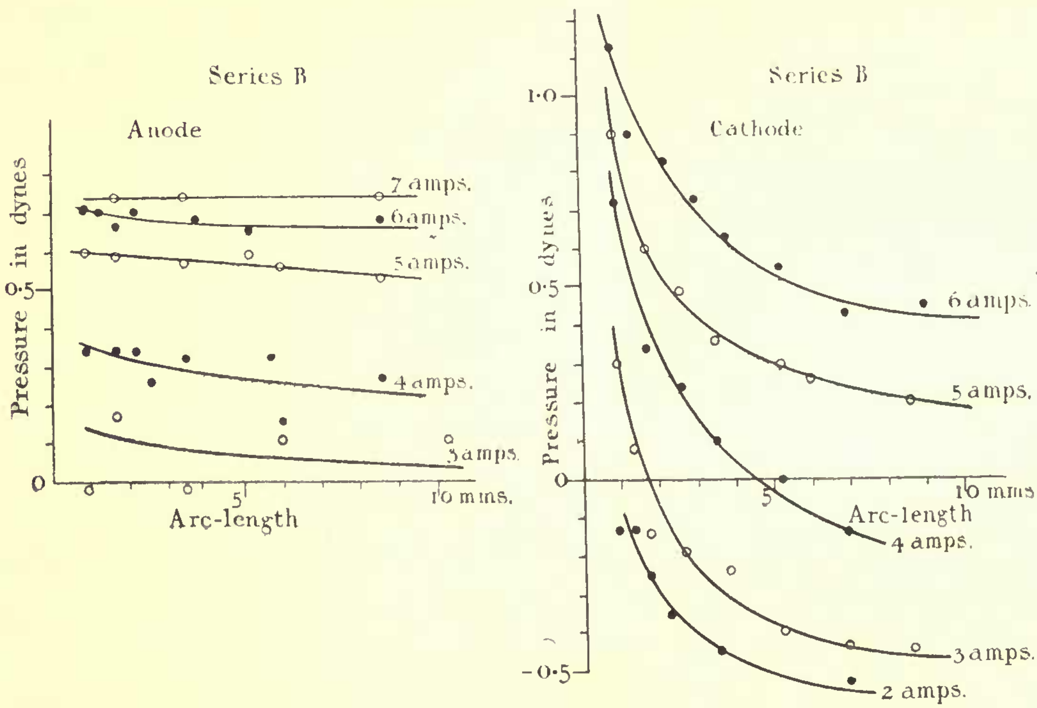

Fig. 10. From Table V.

The Anode.—Fig. 10 differs appreciably from fig. 11 in that the latter shows at first a pronounced fall of total pressure with increasing arc-length, while the former indicates a more constant value. Fig. 11 suggests a minimum value at an arc-length of about 3 mm. for small currents, whereas there is very little indication of this in fig. 10. Two further sets of observations were made with the double-arc method to check this point, and fig. 12, from Table VIII., confirms the accuracy of fig. 10 as far as the constancy of the pressure beyond 3 mm. is concerned, but it also shows that the drop in the value observed for short arcs in Series C is

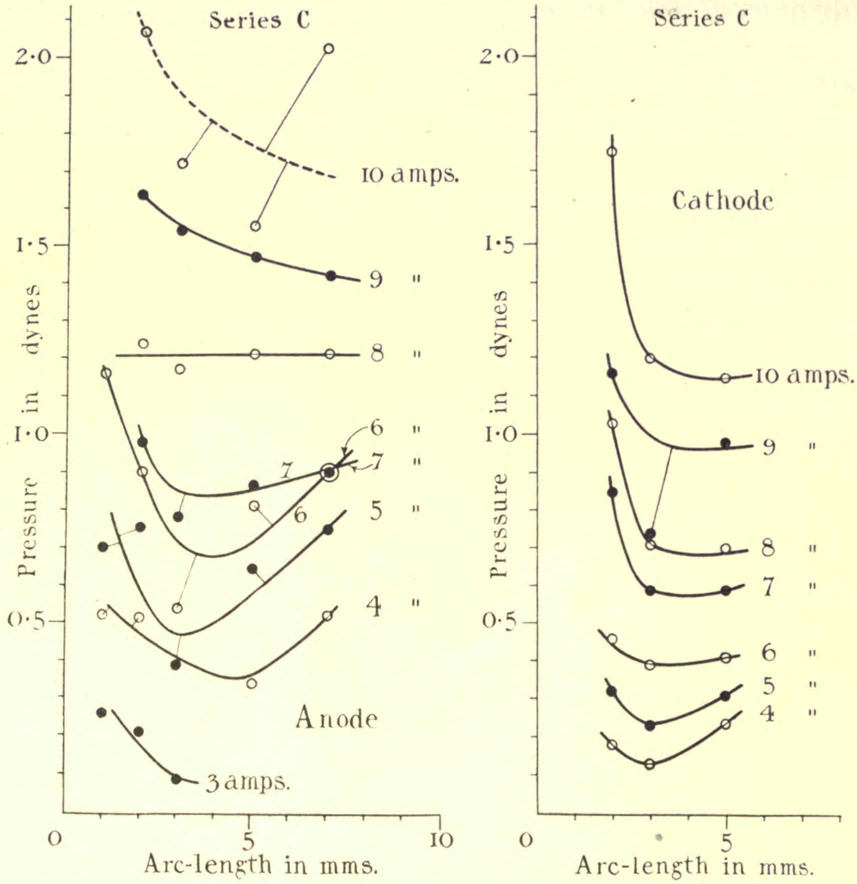

Fig. 11. From Table VII.

probably real; no reason is known for the discrepancy between the two sets of readings in fig. 12.

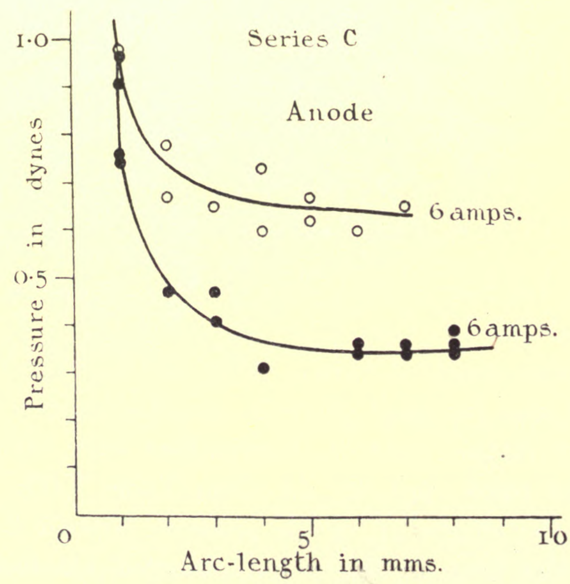

Fig. 12. From Table VIII.

From a comparison of the anode and cathode pressure curves we see that it depends upon the arc-lengths whether one or the other is the greater, but for long arcs there is little doubt that the anode pressure preponderates.

The fact that for direct current the total pressure is greater for short arcs than for long arcs is in agreement with the observation upon alternating current arcs quoted on p. 115.

Variation of Total Pressure with Current for Constant Arc-lengths.

Fig. 13 shows graphically the observations contained in Table IV., Series B, single-arc method, 3·5 mm. arc-length, and fig. 14 depicts the results of Table VII., Series C, double-arc method, 3 mm. and 6 mm. arc-length. The graphs are

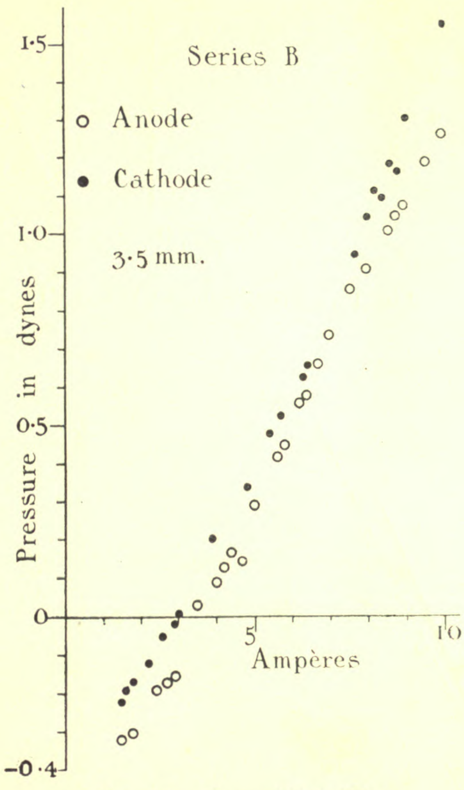

Fig. 13. From Table IV.

not straight lines but appreciably convex to the current axis, indicating that the total pressure increases rather more rapidly than with the first power of the current, not, however, as rapidly as the second power. The curves do not pass through the origin but cut the current axis at about 3 ampères. The approximately linear rate of variation is shown for other lengths of arc in fig. 15 for the anode, and in fig. 16 for the cathode, both single- and double-arc methods. For reasons which will appear in Part II., special attention has been given to arc-lengths of 6 mm. We defer further discussion of these curves until convexion-current effects have been eliminated from the total pressure.

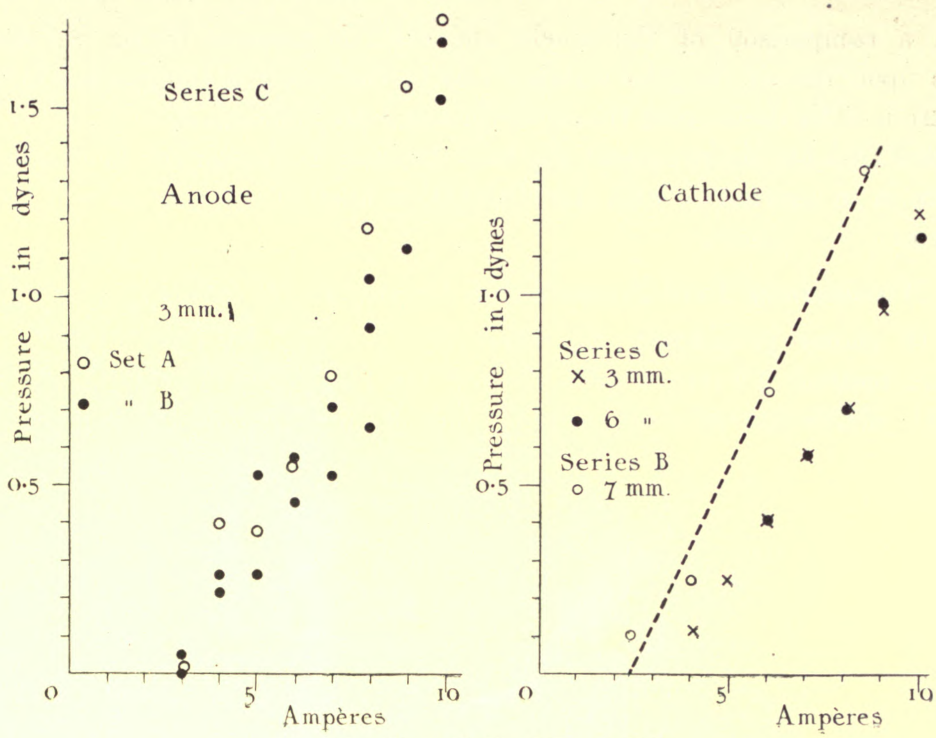

Fig. 14. From Tables VI. and VII.

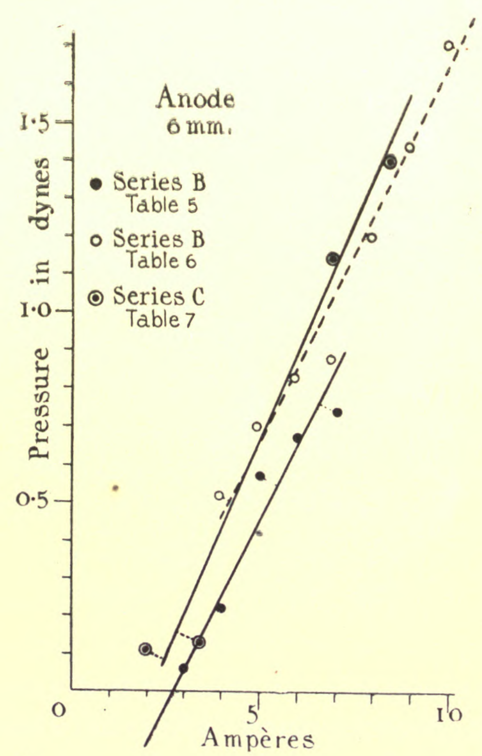

Fig. 15.

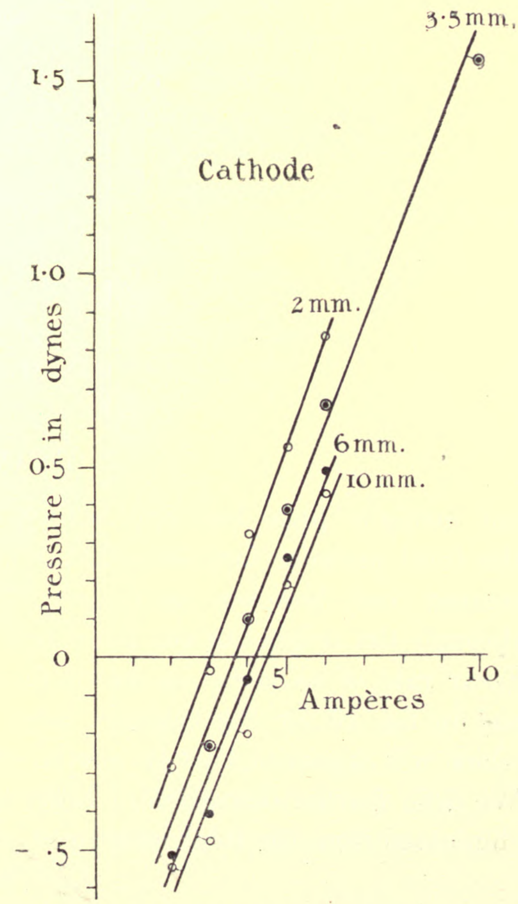

Fig. 16. From Table V.

The Separation of the Pressure from the Electromagnetic Effects.

The evidence contained in the preceding sections proves beyond dispute the existence of a pressure upon the poles of the arc, and shows the general nature of its variation with the current strength. The experiment now to be described provides confirmatory evidence that the electromagnetic effects of the earth's field and of the rest of the circuit have been eliminated, and evaluates these effects.

The current was led to the carbon through a mercury cup as in fig. 1, and the carbon fixed axially at the end of the swinging arm. The dispositions of the poles in the different experiments are depicted in fig. 17. The deflexion may be due to the

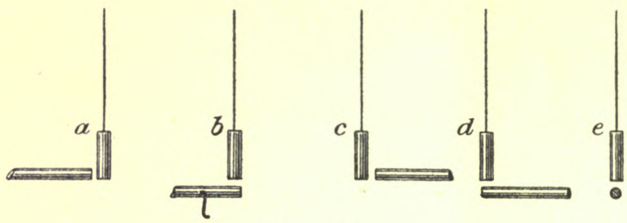

Fig. 17.

couple, V, occasioned by the earth's field, or to the electromagnetic couple, E, occasioned by the rest of the circuit, or to the total pressure upon the poles, P, in which category, as before, we include for the present the influence of air currents, &c.

In the above arrangements the deflexions were as follows:—

TABLE IX.

Arrangement.	Couple due to	8 ampères, 3 mm. Torsion. Set A.	8 ampères, 3 mm. Torsion. Set B.	9 ampères, 5 mm. Torsion.
		°	°	°
a	P + E + V	290	312	− 40
b	E + V	235	290	− 87
c	− P − E + V	90	60	− 323
d	− E + V	120	104	− 275
e	V	175	126	− 188

The value of V is given by e or $\frac{1}{2}(a+c)$ or $\frac{1}{2}(b+d)$,

E ,, ,, $\frac{1}{2}(b-d)$, $b-e$, $e-d$,

P ,, ,, $a-b$, $d-c$,

whence

	At 8 ampères.					At 9 ampères.			
	°	°	°		°	°	°	°	°
V =	175,	190,	178	Set A	Mean 175	V = 188,	181,	181	Mean 183
	126,	186,	197	Set B					
E =	58,	60,	55	Set A	Mean 75	E = 94,	101,	97	Mean 97
	93,	164,	22	Set B					
P =	55,	30		Set A	Mean 38	P = 47,	48		Mean 47·5
	22,	44		Set B					

The couples due to P, E, and V are thus approximately as 1 : 2 : 4 in the actual arrangement employed, but this is accidental. The pressures, P, at 8 and 9 ampères reduce to 0·91 and 1·03 dynes respectively, values in good agreement with those obtained by other methods.

The Electromagnetic Effect.

In order to evaluate the couple upon the suspended arm due to the electromagnetic effect of the rest of the circuit, and particularly the effect of the fixed pole at right angles to it, the apparatus was arranged as in fig. 17, *b*. The current was led to the centre of the copper rod from the mercury trough and from the fixed carbon by a vertical wire. The couple was measured for different lengths of the fixed pole and was found to reach a maximum at about 11 cm. Alternating current obviated any influence of the earth's magnetic field. The effect of altering the current strength with a fixed length of pole was also examined and the couple found to vary, as was expected, with the square of the current. These data enabled the necessary corrections to be made where method No. 1 was adopted (*loc. cit.*).

Electrostatic Effects.

Previous to striking the arc the electrostatic attractions between the poles amounted to 0·125, 0·031, 0·008, 0·006 dynes for arcs of 1, 2, 3, 4, and 5 mm. respectively. But when the arc is struck the distribution of charges within it entirely alters these conditions.

In Part II. is developed a theory of the pressure which does not appear to be seriously affected by electric effects within the vapour. This view is supported by the observation that the pressure depends upon the nature of the poles.

Convexion Currents.

Hitherto we have dealt with the total pressure upon the poles, it remains to consider to what extent it is caused by convexion currents of hot air and vapour rising from the arc. It will appear that convexion currents tend to cause the poles to move towards one another, and that if they could be eliminated the pressure would be higher.

(1) Using alternating current the arrangement shown in fig. 18 was adopted. No current passed through any of the movable parts of the apparatus. An arc was struck between two vertical carbons below the end of the suspended pole. The

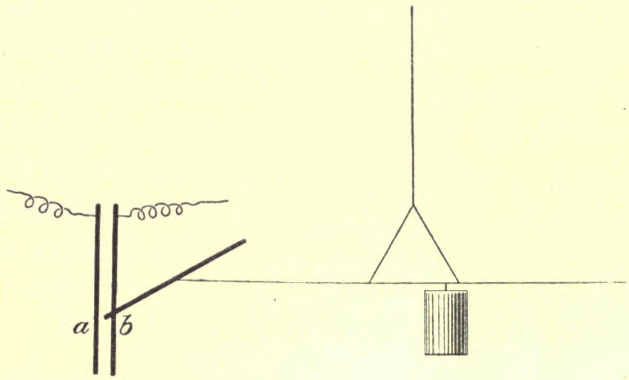

Fig. 18.

torsion head showed a *reduction* of pressure amounting to 0·4 dyne. The effect persisted after the circuit was broken, and until the poles cooled down.

This quantity is to be *added* to the observed pressure upon the poles to give the true reaction.

(2) The apparatus was arranged as in fig. 19 for the double arc method. No current flowed through the main apparatus, but an arc was started 1·5 cm. below the arc gap between two carbon rods placed parallel to the suspended carbon; with a

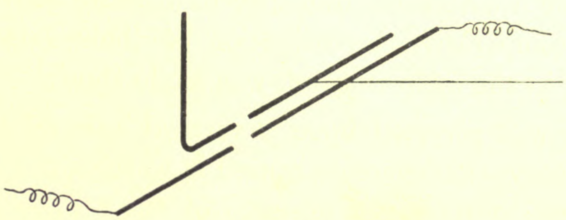

Fig. 19.

direct current of 8 ampères a negative pressure of $-0·32$ dynes was recorded. It is probable that the convexion pressure is actually greater than this, because the arc was necessarily some distance below the suspended poles.

(3) In an experiment with 8 ampères direct current (double-arc method) the suspended pole was shaped as shown in fig. 20, with the object of preventing the end from growing pointed, but this was partially defeated by the deposition of a small point of carbon upon its centre, like a boss upon a shield. The disc on the right-hand side was too large (3 cm. in diameter) to be heated over its whole area by the arc above it, hence the convexion currents due to the two sides did not balance. For a

current of 8 ampères and 3 mm. arc length, a deflexion of 30 degrees was recorded. When the current was turned off and the poles were hot, balance was obtained when the deflexion in the opposite direction was 17 degrees. The deflexion corrected for convexion* is therefore 47 degrees, corresponding to 1·22 dynes. The couple due to convexion is in this experiment approximately 36 per cent. of the total couple upon the poles, and it corresponds to a negative pressure of −0·44 dyne.

It had been hoped that the arrangement of an arc at each end of the suspended carbon (double-arc method, fig. 4), besides eliminating the electromagnetic effects, would obviate the convexion current difficulty, but this was not the case, since DAVIS found that moving the vertical arc from the centre to the end did not necessitate more than a minor change in the current strength necessary to produce a given deflexion. No doubt the currents about the vertical arc were very different from those about the horizontal arc at the tip of the carbon rod.

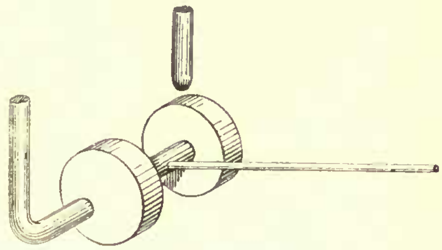

Fig. 20.

The above experiments enable us to fix the lower limit to the effect of the convexion currents rising from a 7–8-ampère arc at −0·5 dynes.

Since it is a matter of considerable difficulty to measure convexion currents in the manner described, a comparative investigation of their value over the range of current strengths used in the main research was made thus :—

A light paper vane was pivotted upon a vertical axis 60 cm. above a horizontal arc, so that the rising convexion currents caused it to turn, the number of revolutions of the vane in 1 minute providing a measure of their velocity. Arcs of constant length and varying current were employed, and the mean curve C in fig. 21 was obtained as the result of a number of experiments, whence it is clear that over the range 2 to 10 ampères there is an increase in the velocity of the convexion currents. For a given curvature of pole the suction upon it is proportional to the square of the velocity of the air moving past it, the curve D has therefore been drawn with ordinates proportional to the square of those of curve C, but on a different scale, it represents, therefore, the pressure effect of the convexion currents. We can approximately fix the scale of curve D from the knowledge that the convexion is approximately 0·5 dynes at 8 ampères, remembering however that this is an underestimate.

* Also radiometer action if it were effective, see p. 130.

The observation that convexion currents from the poles occasion a reduction of the pressure between them is in accord with the experiments of DEWAR referred to elsewhere. He states that "the effect of hot poles upon the registration of the

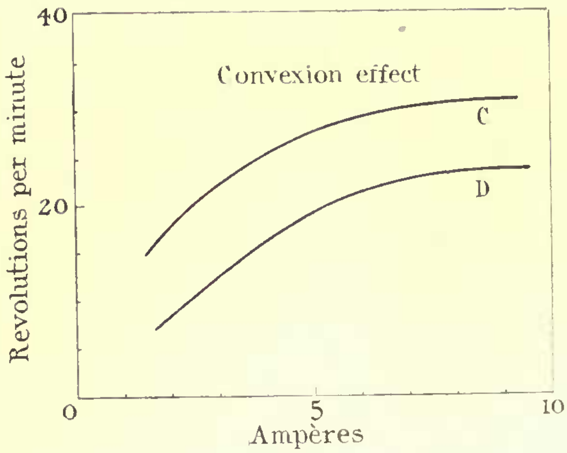

Fig. 21.

manometers was to produce a small negative pressure when the arc was stopped, due to the passage of currents of hot air."

The Elimination of Convexion Effects.

The curve D in fig. 22 is the same curve as in fig. 21, but drawn to the appropriate scale, and shown below the current axis instead of above it. A is the curve representing the observed total pressure upon the anode taken from fig. 14. The corrected pressure is given by the curve A referred to D as datum line, or by the curve B referred to the original axes. It will be seen that correcting for convexion results in a rather more linear curve than that obtained from direct observation, and that the curve now more nearly passes through the origin. Remembering that the effects of convexion currents have been under-estimated, and therefore that the convexion current curve should have been rather lower, we see that it is probable that the corrected curve is linear, and that the reaction varies directly with the current; it is unfortunate that information is very difficult to obtain in the crucial part of the curve where the current is small; there is no evidence that the pressure when corrected for convexion ever becomes negative. It is clear therefore that, without much error, we may take the origin of the observed curve at the point at which the straight portion when produced backwards cuts the vertical or pressure axis. This provides us with the simplest means for correcting for convexion currents, and all previous diagrams should be so treated.

For example, in fig. 15, Series B, instead of 0·45 dynes being the pressure at 5 ampères it is 1·0 dynes, since if produced backwards the curve cuts the vertical axis at $-5{\cdot}5$ dynes.

The curves for Series B have been represented by straight lines wherever possible, but it is obvious that the points are often better represented by curves slightly

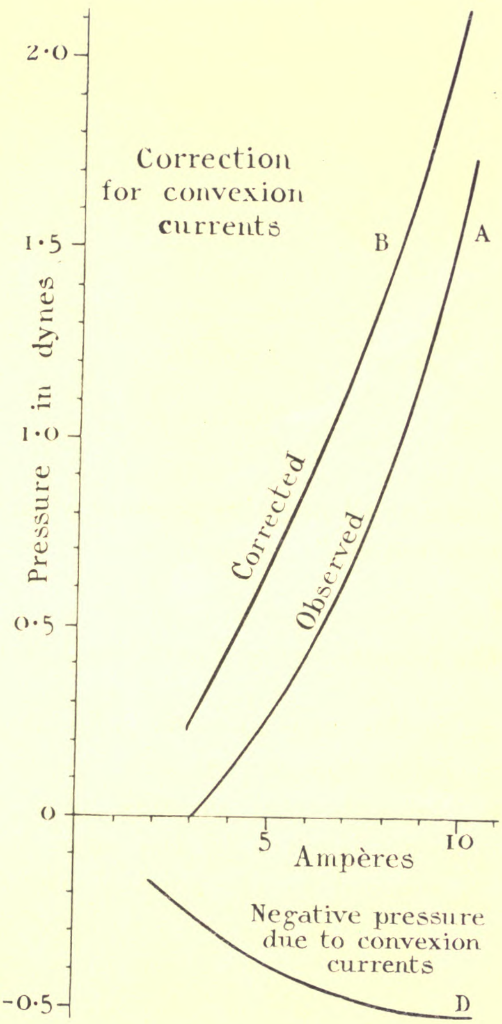

Fig. 22. For "observed" curve, see fig. 14.

convex to the current axis; they are subject to the same corrections as Series C, and they become more nearly straight after allowance is made for convexion currents.

In Table X. are shown the values of the pressures corrected for convexion currents for arcs of 10 ampères and 6 mm. length. We shall make further use of these values in a later section.

TABLE X.—The Pressure upon the Poles of an Arc, 6 mm. long, carrying 10 ampères.

Series.	Figure.	Observed total pressure in dynes.	Convexion correction. Dynes.	Corrected pressure. Dynes.	Mean corrected pressure.	Current carried solely by electrons.		Half-current carried by electrons.	
						e/m.	v.	e/m.	v.
				1. Cathode.					
B	16	1·60	1·10	2·70	2·18	$6 \cdot 4 \times 10^7$ E.M.U.	$2 \cdot 8 \times 10^8$ cm./sec.	$1 \cdot 6 \times 10^7$ E.M.U.	$1 \cdot 4 \times 10^8$ cm./sec.
B	14	1·69*	0·54	2·23					
C	14	1·11	0·50	1·61					
				2. Anode.					
B	15	1·35	0·52	1·87	2·10	$19 \cdot 0 \times 10^7$	$7 \cdot 9 \times 10^8$	$4 \cdot 7 \times 10^7$	$3 \cdot 9 \times 10^8$
B	15	1·79*	0·60	2·42					
C	15	1·68	0·34	2·02					

The values for the observed pressures were taken from the diagrams named in column 2 by drawing a line through the points of observation and noting where it met the vertical axis, and this point taken as the origin. When the graph was not straight the tangent was drawn to touch the curves at 10 ampères and similarly treated.

* Calculated for arc-length of 7 mm.

Previous Investigation.

In 1882 DEWAR† measured the hydrostatic pressure within the arc by using hollow carbons connected to delicate water manometers and found that "during the maintenance of the *steady* arc the manometer connected with the positive pole exhibited a fixed increase of pressure corresponding to 1 to 2 mm. of vertical water pressure in different experiments and under varied conditions. The manometer connected with the negative pole shows no increase of pressure, but rather, on the average, a diminution."

DEWAR gives no data respecting current and arc-length, but we may note that the hydrostatic pressure near the anode is about one hundred times the total pressure upon the poles measured in the present research.

DEWAR suggests that the phenomenon is consistent with "the well-defined boundary of the heated gases acting as if it had a small surface tension," and he suggests as a possible cause "the motions of the gas particles under the conditions of transit of material from pole to pole or a succession of disruptive discharges."

† DEWAR, 'Roy. Soc. Proc.,' xxxiii., 262, 1882.

PART II.—THE ORIGIN OF THE PRESSURE.

By Prof. W. G. DUFFIELD.

In the foregoing pages experimental evidence has been adduced demonstrating (1) the existence of a pressure upon the poles, (2) its variation with arc length, (3) its variation with current strength.

There is reasonably good agreement between the results obtained by different methods in the various series of observations both for direct and alternating current. In view of the delicate nature of the observations, it is satisfactory to find that the rigorous examination given in the section upon the separation of the electromagnetic and pressure effects confirms both qualitatively and quantitatively the existence of the pressure.

Assuming that the corrections for convexion currents have been made with approximate accuracy it remains to attempt to account for the reaction upon the poles.

Radiometer Action and Evaporation.

Before giving the evidence in favour of the reaction being occasioned by electronic projection from the poles, it is necessary to consider whether it is explicable either by radiometer action, produced by the departure with increased velocity of air molecules after impact upon the hot poles, or by the evaporation of carbon atoms.

Against the effect being due to either of these causes we have the experimental evidence that, whereas for a constant current the reaction upon the cathode remains constant or diminishes with increasing arc-length (figs. 10 and 11), the amount of carbon lost from the poles increases rapidly with the arc-length* over the same range. As the amount of carbon consumed depends upon the rate of evaporation or (and) upon the access of air molecules to the poles, it would be expected that, if the reaction depended upon either of these factors, it would increase with the consumption of carbon, which is not the case.

Furthermore, radiometer action is not usually appreciable at atmospheric pressure, though, when the object has been of very small dimensions, it has been observed at about $\frac{1}{3}$ atmosphere. It may be argued that radiometer action is to be expected because the pole face is curved and the intensity of the reaction against the air molecules is not necessarily equal to the reaction upon the poles ; against this we have experimental evidence that, starting with a flat pole face, the reaction upon the pole became perceptibly less as it burnt to the usual curved form.

Experiment 3, p. 125, shows that the hot pole is subject to a *negative* pressure when the current is off, indicating that, if any radiometer effect exists, it is very small and is masked by convexion effects.

Again, if we assume that a pressure can arise from the expulsion of carbon atoms

* "Consumption of Carbon in the Electric Arc," DUFFIELD, 'Roy. Soc. Proc.,' A, vol. 92, p. 122, 1915. See Diagrams I, 2, 4, 5.

in the process of evaporation, we find marked disagreement between observed and calculated values :—

Any reaction due to evaporation should be calculable from a knowledge of the number n of molecules of mass m which leave the pole in 1 second with a velocity of v cm. per second. The product mn can be measured, but the determination of v presents some difficulty. Assuming that in the arc the carbon is at its boiling-point, and that the carbon atoms are in thermal equilibrium with the air into which they are escaping, *i.e.*, the carbon atom possesses the same kinetic energy as is possessed by an oxygen or nitrogen atom at the temperature at which boiling occurs, we have for the velocity of the carbon atom at 0° C., using the fact that the velocity of H_2 at 0° C. $= 18{\cdot}39 \times 10^4$ cm. per second,

$$v = 18{\cdot}39 \times 10^4 \times \sqrt{\tfrac{1}{6}} = 7{\cdot}5 \times 10^4 \text{ cm. per second.}$$

Since the boiling-point of carbon is about 4000° C., the atomic velocity at that temperature $= 7{\cdot}5 \times 10^4 \times \sqrt{\frac{4273}{273}} = 2{\cdot}97 \times 10^5$ cm. per second.

In experiments already quoted the amounts of carbon liberated from the anode and the cathode have been determined under various conditions of arc length and current strength (*loc. cit.*).

In a typical experiment with an arc of 6 mm. length and a current of 10 ampères, 85×10^{-5} grm. of carbon were lost by the cathode in 1 second, a much larger loss being recorded for the anode. Taking the above data the loss of momentum from the cathode per second $= 8{\cdot}5 \times 10^{-4} \times 2{\cdot}97 \times 10^5 = 252$ grm. cm. per second².

On account of the nearly hemispherical curvature of the pole tip for an arc of this length only the components of the momenta along the axis are effective, hence the reaction recorded by the torsion fibre should be one-half of the above pressure—namely, 126 dynes. The observed value of the reaction, after correcting for convexion currents, under the same conditions of current and arc length, is 2·18 dynes, which is not as much as 2 per cent. of the calculated value. It does not appear that we can account for the reaction at the cathode on any simple assumption which regards its cause as molecular or atomic projection.

The Nature of the Particles Projected from the Cathode.

It seemed possible to discover the nature of the particles projected from the cathode from the following considerations :—

If p is the observed pressure corrected for convexion currents, we have, assuming symmetrical projection from a hemispherical pole tip, or random projection from a small area on a flat pole face,

$$2p = mnv, \quad \ldots \ldots \ldots \quad (1)$$

where m is the mass, v the velocity of each particle, and n the number projected per second.

Assuming in the first instance that each particle carries a single electronic charge, e, and that electrons alone are responsible for carrying the current between the poles, the current C in absolute units is given by

$$C = ne. \qquad (2)$$

The potential drop, V, across the pole face is, on this assumption, due to the projection of these particles, whence their kinetic energy is equal to that derived from the source of current supply, and we have

$$VC = \tfrac{1}{2}mnv^2. \qquad (3)$$

From (1) and (3)

$$v = \frac{VC}{p},$$

whence from (1) and (2)

$$\frac{e}{m} = \frac{Cv}{2p} = \frac{V}{2}\left(\frac{C}{p}\right)^2.$$

DUDDELL* has found the values of V at the anode and cathode of an arc 6 mm. long carrying a current of 10 ampères to be 16·7 and 6·1 volts respectively, both being electromotive forces acting *towards* the poles. These are electromotive forces across the pole faces and are distinct from those within the vapour in the arc-gap.

We have already set forth in Table X. the values of p for a similar arc employed in this series of experiments, and, by substitution in the above formulæ, we derive the values of e/m and of v, which are recorded in the final column of the same table; for the cathode the mean value of e/m is $6{\cdot}4 \times 10^7$ E.M.U. The values of e/m for electrons and for hydrogen atoms are $1{\cdot}77 \times 10^7$ and $9{\cdot}58 \times 10^3$ E.M.U. respectively, and, if carbon is quadrivalent, e/m for that element is $3{\cdot}2 \times 10^3$. Even without further refinement of our assumptions, the experimental evidence is overwhelmingly in favour of the projection responsible for the reaction being electronic rather than molecular.

We expect to find electronic projection from the poles of an arc, because the intense heat may occasion thermionic action, and the richness of the arc light in waves of short length is favourable for photoelectric action.

Electronic emission is thus in accord with expectation, but it is at first sight surprising that it should be capable of producing a measurable recoil.

It has been assumed in the foregoing that electrons carry the whole of the current, but the case for an atomic drift of positively charged atoms on to the cathode has already been considered by the writer (see "The Consumption of Carbon in the Electric Arc—No. 1," 'Roy. Soc. Proc.,' A, vol. 92, p. 122 (1915)). If we assume that half the current in the neighbourhood of the cathode is carried by such atoms (which may be supposed to contribute no more to the pressure than do the gas

* DUDDELL, 'Phil. Trans. Roy. Soc.,' A, vol. 203, p. 305, 1904.

molecules which they replace), the value of e/m appropriate to these conditions is $1\cdot6\times10^{7}$ E.M.U. which approximates closely to the value for the electron derived from other methods, namely, $1\cdot77\times10^{7}$ E.M.U.

The corresponding velocity of projection is $1\cdot4\times10^{8}$ cm. per second, which is of the order of magnitude to be expected if the emission is due to photoelectric action, though higher than the velocities measured by LENARD from carbon plates. But in the arc the proximity of the luminous vapour to the poles enables light of very short wave-length to reach them, so a correspondingly high electronic velocity is to be expected. Moreover, the condition of the pole, its high temperature, boiling and intense incandescence, are favourable for the liberation of the corpuscles with the minimum loss of energy, indeed, it may be that it is the undiminished momentum of the electron as it leaves the atom which has been measured. If the arguments are sound the experiments constitute the measurement of quanta by a direct mechanical method.

The kinetic energy of the electron as it leaves the cathode is given by $\frac{1}{2}mv^2$, which, from the data of the present set of experiments, amounts to $8\cdot6\times10^{-12}$ erg. Assuming that this is due to photoelectric action, and taking the radiation constant h as $6\cdot55\times10^{-27}$ erg seconds, we find λ, the mean wave-length of the light emitted by carbon vapour, which may be regarded as effective in promoting the emission, to be $1\cdot22\times10^{-5}$ cm.; this is a reasonable result as it is smaller than the threshold wave-length for soot given by HUGHES as $2\cdot6\times10^{-5}$ cm. We note that the electronic energy is less on emission than the amount $5\cdot5\times10^{-11}$ ergs, which is the minimum required to produce ionization (RUTHERFORD), but in the arc the further fall of potential beyond the negative pole face rapidly increases the velocity and therefore the kinetic energy of the corpuscle.

The Mechanism of the Arc.

It is clear that the balance of evidence favours the conclusion that the particles responsible for the recoil are electrons. It is doubtful if we can press our results much further than this in view of the very small forces to be measured and the complex conditions under which experiments of this nature must be conducted, but the view of the mechanism of the arc which is most favoured by this research (indeed the agreement with it is remarkable, though it may be accidental) is that an oxygen atom arrives at the cathode with two positive charges of electronic magnitude, and that uncharged CO is formed which removes two of the four electrons, which we have already shown to be associated with the departure of each carbon atom from this pole,* and which are derived ultimately from the source of current supply. The oxygen atoms on arrival and departure contribute no more to the pressure than do the air molecules on the other side of the suspended pole. The remaining two electrons

* DUFFIELD, *loc. cit.*

are liberated, and their expulsion involves the recoil which has been measured in the present experimental investigation.

Under these conditions the mechanical effect would be least likely to be disturbed by electric forces within the arc, because the oxygen atom approaching with two positive charges would contribute to the attractive force upon the pole an amount not very different from the repulsive force occasioned by the two receding electronic charges.

In a normal arc the effects at the anode are very complicated, there is electronic projection due to thermionic and photoelectric action, and probably access of electrons and negatively charged atoms which carry the current to it. Nevertheless, the values of e/m obtained by the method already described is of the right order of magnitude, though three times higher than it should be, if the recoil is in this case also to be explained by the projection and impact of electrons and if they bear half the current. If we could accept the view that the momentum of the electron derived from the cathode is handed on through the vapour from atom to atom until it reaches the anode, the discrepancy would be reduced. Elsewhere we have shown that it is possible to reduce the carbon consumed by the anode to almost negligible quantities, it would be interesting to determine the changes in the anode recoil under these circumstances, but the experiments would be of very great difficulty.

The writer tenders the above account of the mechanism of the arc with due appreciation of the assumptions underlying it. As far as the details are concerned, a great deal depends upon the accuracy of DUDDELL'S results, but any reasonable assumption regarding the magnitude of the potential drop across the cathode pole face would lead to a value for e/m which is of the order of magnitude of that of the electron and far removed from that associated with atoms. If instead of assuming random projection, we assumed normal projection from a small area on the cathode, the values of e/m would be four times those given in Table X., and still in accord with their electronic rather than their atomic nature.

The view I have taken of the mechanism of the arc attributes the fall of potential across the negative pole face to electronic projection there, contrary to the theory which regards the electric force as responsible for the extraction of the electron. POLLOCK,* *assuming* electronic projection, took the same view, and from DUDDELL'S work calculated the velocities in different parts of the arc in an important contribution to this subject. The discharge of electrons has frequently been assumed, but I do not think that there has hitherto been any mechanical evidence in its favour.

Such action, photoelectric or thermiônic, as occasions in the arc a discharge of negative electrons from the poles is probably assisted by the chemical interactions between the poles and the surrounding gas. This point has already been discussed in the paper by the writer, to which reference has been made.

* POLLOCK, 'Phil. Mag.,' vol. XVII., p. 361, 1909.

The method of starting an electric arc by forcing a spark between the separated poles possibly depends as much upon the photoelectric action induced as upon the ionization within the spark gap.

Polar Lines in Arc Spectra.

The original experiment upon the pressure upon the poles on which the above research is based was carried out in 1912, and had been undertaken in the expectation of finding a recoil effect. The writer had previously described a series of spectrum lines which made their appearance near the poles of an iron arc, to which the name "polar lines" had been given,* and in the discussion of their origin something in the nature of an explosion upon the surface of the pole was suggested to account for the potential drop with which they appeared to be associated.

It seems now possible to go further than I did in the original paper and state that the explosion results in the liberation of an electron with high speed, and it is further suggested that the polar line is due to the particular type of vibration which is set up at the instant when an electron is expelled from the atom. The other, or median, lines being due to the secondary action when the electron with its ionizing velocity impinges upon another atom. The feeble intensity of the polar line contrasted with the greater intensity of the median line is in accord with this view.

A feature of the occurrence of polar lines is their predilection for regions of the spectrum of short wave-length. In the case of the iron arc they became increasingly numerous as more refrangible parts of the spectrum were reached. It is quite possible that their preference for this range is due to their photoelectric origin, since such action is limited to regions of high frequency; in the rare case (only as far as I know in the iron arc) in which a small group is found in another part of the spectrum the effect may be a resonance one.

Rossi† has observed polar lines in the spectrum of the copper arc, and it appears that 2714 Å.U. is the wave-length of the least refrangible one. Richardson and Compton‡ give 3000 Å.U. or 3090 Å.U. as the longest wave-length capable of producing photoelectric emission from copper. There is thus further evidence of the polar lines being due to photoelectric action. In the case of iron the polar lines were found to be nearly equally strong at the two poles, but with the copper arc Rossi found that unless the current was strong they were confined to the cathode; this suggests that the attraction of the anode for the electron was sufficient to prevent its expulsion.

The reason why an explanation based upon photoelectric rather than thermionic action is offered is that in the experiments already quoted it was found that the

* Duffield, 'Astrophysical Journal,' XXVII., 264, 1908.

† Rossi, 'Astrophysical Journal,' XXXV., 279, 1912.

‡ Richardson and Compton, 'Phil. Mag.,' XXIV., 575, 1912.

loss of an atom of carbon from the cathode was associated with the transfer of four electronic charges between the poles, this favours the ejection of electrons from the atom itself rather than from the pole face considered as a whole, but the writer does not wish to rule the possibility of thermionic emission out of account.

One point which emerges from the present research deserves mention. After the discovery of the recoil, and during the endeavour to find a means of disentangling it from the electromagnetic effect due to the rest of the circuit, it was suggested that the two might be identical, that is to say, that the mutual interactions between various parts of a circuit were occasioned by the mechanical effect of the flow of electrons through it. It seemed possible to find a plausible explanation of the motion of a movable wire in the plane of a circuit on this basis. The experiments described on pp. 123 and 124 showed that the two exist simultaneously, and that the electromagnetic effect under the conditions of the experiment was about twice that observed for the recoil. Moreover, the rates of increase of the two with the current strength were different, a fact which effectively disposed of this idea.

The writer has observed a similar recoil upon the suspended cathode within a highly exhausted vacuum tube, but the mechanical effect has not yet been measured.

The experiments were conducted in the Physics Laboratory of University College, Reading, and valuable assistance was given by Mr. J. S. BURGESS. Mr. DAVIS was in receipt of a Research Grant from the Committee of the Privy Council for Scientific and Industrial Research, to whom the thanks of the authors are accorded.

V. *On Intensity Relations in the Spectrum of Helium.*

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CONTENTS.

(I.) *Introductory.*

THE two most fundamental characteristics of a spectrum line are its wave-length and its intensity, and it is very remarkable that, at the present time, while the former can often be expressed with an accuracy of one part in half a million, the tabulated intensity may frequently be affected by an error even greater than a thousand per cent. Yet for the elucidation of the main problems of astrophysics the relative intensities of spectrum lines may assume an importance scarcely inferior to that of a precise knowledge of their wave-lengths. Although data of the latter kind afford precise evidence of the presence of certain elements, and of the motions of stars and nebulæ in the line of sight, it is to the distribution of energy in the spectrum and to the reproduction of specified conditions in the laboratory that we must look for a further knowledge of the physical and more especially the electrical conditions obtaining in celestial bodies.

The changes which occur in spectra under varying conditions of excitation are often of a very conspicuous character, and the study of "spark" or enhanced lines

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has already led to results of fundamental importance, but the observation of such phenomena depends for its success upon the magnitude of the changes involved, and whereas the appearance of new series of lines under appropriate conditions is often apparent at once, a strictly quantitative determination of the relative intensities of the spectrum lines is necessary for the study of the less conspicuous changes, which may, nevertheless, be of fundamental importance. In particular, the intensity changes occurring under varying conditions in lines belonging to the same or to mathematically related series must be a matter for serious consideration in any theory of radiation which involves a theoretical interpretation of the laws of spectra.

In a recent investigation* we have made quantitative measurements of the intensities of the lines of Helium and Hydrogen, and it was found that under certain conditions of energetic excitation, the relative intensities of the lines were altered, in the sense that there was a transfer of energy from the lower to the higher members of the various series. This phenomenon was found to occur under appropriate circumstances in every series investigated, although the absolute magnitude of the change or transfer is peculiar in each case to the individual series. The principal difficulty encountered in any attempt to obtain an interpretation of such results lies in the absence of any precise knowledge of the conditions of excitation which actually obtain with any specified experimental arrangement. The three cases which we investigated in connection with Helium were the spectrum, from the capillary of a vacuum tube of the Plücker form, produced by the passage of an uncondensed discharge from an induction coil, and alternatively by a condensed discharge with a spark gap in the circuit, together with the spectrum from the bulb produced with a condenser in parallel and with a very small spark gap ; but in each of these cases, our knowledge of the manner in which the atom is excited to luminosity is not sufficiently definite to justify any attempt to correlate theoretically the observed intensity changes.

The variations in the intensity distribution among the lines of a spectrum, produced by the presence of impurities or by the direct admixture of other gases, constitute another field for research, and in this connexion a large number of entirely distinct effects may occur. Quite apart from the emission of band spectra by definite compounds or perhaps elementary molecules, and of such spectra as the water-vapour bands and the ammonia bands, which have been shown recently† by FOWLER to be present in the solar spectrum, there exist such effects as the reduction of intensity of the band spectrum of Helium, produced by the action of certain impurities, and the similar action of Oxygen on the secondary spectrum of Hydrogen. We have also confirmed, in a quantitative sense, the original observation of LIVEING and DEWAR that a transfer of energy from longer to shorter wave-length in the Balmer

* 'Phil. Trans.,' A, 1917, vol. 217 p. 237.

† 'Roy. Soc. Proc.,' A, 1918.

series of Hydrogen is brought about by the admixture of Neon. The importance of the mutual effects of gases on the intensity distribution in their spectra is considerably enhanced by the fact that, in celestial spectra, the radiation from a pure gas is never in question, and if indeed the spectrum of any single element were manifest, its presence would not disprove the presence of other elements which, though not giving rise to perceptible radiations peculiar to themselves, might nevertheless exert an influence on the distribution of intensity in lines due to other elements.

A third and most significant condition which affects the relative intensities of spectrum lines is the pressure of the gas from which they are produced. In Helium, as is well known, this is peculiarly conspicuous; the colour of the discharge, for example, being green at low pressures. The existence of this phenomenon has, in fact, been familiar for many years, and indeed was responsible at one time for the erroneous view that Helium was a mixture of two gases. This misconception was only removed by the demonstration that the effect in question was due to variations of the pressure of the gas in the tube, but there has since been no quantitative investigation of the nature of the changes which are known to occur.

It is thus evident that there are a number of circumstances which modify the distribution of intensity in the spectrum of an element, and that in order to obtain further information it is desirable to investigate the simplest possible cases in which the nature of any changes introduced into the method of excitation of the spectrum can be followed in some detail. Such considerations have been the determining factor in the particular conditions which have been selected for study in the work described in the present communication.

(II.) *The Cathode Glow.*

A source of light in which we already have some definite information with regard to the electrical conditions is obviously presented by the glow around the cathode of a vacuum tube. The radiations obtained from this source in the case of Helium are of especial interest, for they include at once the "arc" lines, the spark line at "4686," and also the band spectrum which FOWLER has shown recently* to be of an unusual type, inasmuch as the heads of the bands are not related by the law of Deslandres appropriate for the usual band spectra, but by the Rydberg formula which had been regarded hitherto as applying exclusively to line series. The presence of the "4686" line in the same source is also interesting, as the appearance of a characteristic "spark" line in company with a band spectrum is perhaps somewhat surprising, although the existence of such a phenomenon shows clearly that, though the conditions necessary for the production of these radiations may be different, they are evidently at the same time not incompatible.

* 'Roy. Soc. Proc.,' A, vol. 91, p. 208, 1915.

In recent years the radiation from the dark space has become of particular interest in view of the fact that it is in this region that the Stark effect—or the electrical resolution of spectrum lines analogous to the magnetic resolution known as the Zeeman effect—is observed.

In this region the quantitative relation between the electric field and the distance from the cathode was first investigated by SCHUSTER,* who expressed his results by the empirical formula

$$V = V_0(1-e^{-kx}),$$

where the potential of the cathode is taken as zero and V_0 is the potential of the cathode glow, V is the potential at distance x from the cathode and k is a constant.

This formula gives the distribution of potential in the dark space, and more recently LO SURDO,† from a series of measurements of the electrical separation of spectrum lines in front of the cathode, has verified that it is a satisfactory first approximation. Investigations in this direction have also been carried out by ASTON‡ and by HARRIS,§ who measured the deflection of a beam of cathode rays passing in a direction perpendicular to the electric field.

We do not discuss these observations in detail. Very recently the distribution of potential in narrow tubes has been investigated somewhat exhaustively by TAKAMINE and YOSHIDA,‖ who found that, under the conditions of their experiments, the relation between the electric field and the distance from the cathode could be represented, within the limits of experimental error, by a parabolic law.

In the work described in the present communication, we are concerned with pressures somewhat greater than have been used by these investigators, and with the cathode glow itself in addition to the dark space, and although a knowledge of the precise distribution of potential, from the cathode to a distance at which there is no longer any perceptible luminosity, would be of value, it is not in the first instance essential to a discussion of our results. For this purpose we may, in fact, merely assume that the electric field falls away rapidly with increasing distance from the cathode without the necessity of postulating any exact law. For it would appear that the average velocities of the electrons at different distances from the cathode (in which the effect of collisions naturally plays an important part) are probably more strictly relevant to a discussion of the results. A visual examination through coloured glasses of the cathode spectrum of the tube used in this investigation at once shows that the term "dark space" is, in fact, a purely relative one, and refers only to the integrated effect on the eye of all the radiations emitted.

* 'Roy. Soc. Proc.,' vol. 47, p. 541, 1890.

† 'Rendiconti R. Accad. Lincei,' vol. 23, 117, 1914.

‡ 'Roy. Soc. Proc.,' vol. 84, p. 526, 1910.

§ 'Phil. Mag.,' vol. 30, 182, 1915.

‖ 'Mem. Kyoto Imp. Univ.,' vol. ii., 6, 1917.

(III.) *The Method of Measurement.*

The method adopted for the determination of the intensities of lines in a spectrum has been described in a previous communication,* in which it was shown that the absolute values of the intensities can be obtained from the "photographic" intensities by the adoption, as a standard, of the radiation from the positive crater of the carbon arc, in which the distribution of intensity along the spectrum can be calculated by PLANCK'S or WIEN'S formula. For the purpose of the present investigation, the photographic intensities afford all the necessary information, and the results exhibited below are accordingly limited to a determination of these values.

The spectrograph consisted of a large single prism constant-deviation instrument by Hilger, with a camera attachment in place of the telescope. Instead of the V-shaped slide for reducing the length of the slit, a brass slide with a rectangular opening was adopted, and in front of this opening was fixed the neutral glass wedge. This consisted of a prism of neutral-tinted glass cemented to a similar prism of colourless glass in such a manner that the combination formed a plane-parallel plate. When light is allowed to fall on to the slit through this wedge, the resulting spectrum is found to consist of lines which are bright at one end, corresponding to the thin end of the wedge, and which fade away in the direction corresponding to the dense end of the wedge, the length of the line on the plate thus depending on its intensity and also on the "density" of the wedge for that particular wave-length.

The spectra under investigation were photographed on Wratten Panchromatic plates, and these were developed with a Hydroquinone and Formaline developer which gives results showing great contrast. From the negatives thus obtained, positives were printed by contact on Paget Half-tone or Paget Slow Lantern plates, which were found to give the best results for this stage of the process. These positives were then intensified with Mercuric Chloride and Ammonia, and enlargements were subsequently made on bromide paper using a Zeiss "Tessar" lens, which, under the conditions of use, gave no measurable amount of distortion of the image. The enlargements were made with the aid of a ruled process screen, which was placed immediately in front of the bromide paper. The resulting enlarged negative image was in this way built up from a number of small dots, one-hundredth of an inch apart. On the enlargement obtained by this method, it is a matter of no difficulty to pick out the last dot visible on each line, and thus to determine with considerable accuracy the relative lengths of the lines composing the spectrum. In the absence of the process of reproduction of the image in dots, this would be a matter of great difficulty, and the results would be subject to considerable personal error.

The plate-holder of the spectrograph was provided with a rack and pinion motion in order to allow of the possibility of photographing a number of spectra on the same plate. The spectra under comparison are thus photographed on adjacent portions of

* *Loc. cit.*

the same plate, ensuring a valid basis for the comparison, and pass simultaneously through all the subsequent stages of the process.

It has been found most convenient to deduce the photographic intensities in the following manner, the theory of which has been given in some detail in a previous paper,* though circumstances have slightly modified the method in the present instance. We define the "density" of the wedge at any point as $-\log_{10}\left(\frac{I_1}{I_0}\right)$ where I_0 and I_1 are respectively the intensities of the incident and transmitted rays. This density is proportional to the length measured from the thin end, and the density at the thick end was denoted by D_λ in the former paper, the suffix λ relating to the particular wave-length in question. The photographic intensity of a line was proportional to the function

$$\log_{10}^{-1}\left(\frac{D_\lambda h_\lambda}{H}\right)$$

where h_λ and H were the heights of the line, and of the wedge, on the enlarged photograph. If h is the height of the wedge on the original plate, and m is the magnification,

$$H = mh.$$

Let $D_\lambda/h = d_\lambda$, the change of density of the wedge per millimetre or its density gradient. Then the photographic intensity of a line of wave-length λ is measured by

$$\log_{10}^{-1}\left(\frac{d_\lambda\, h_\lambda}{m}\right).$$

The height of the line on the enlargement is h_λ and on the original plate before magnification, is h_λ/m. The magnification m can be found at once if the interval between any two lines, such as $\lambda\lambda$ 6678 and 3888, is known both on the original plate and on the magnified photograph.

A precise knowledge of the values of d_λ at various typical points in the region of the spectrum under investigation is required or, in other words, the wedge must be calibrated. The wedge used in the present experiments was of somewhat more convenient dimensions than that employed in our previous investigation, and an improved method of calibrating it has been adopted.

For this purpose, a vacuum tube containing Helium was excited to luminosity by means of the induction coil with a mercury jet interrupter, and the capillary of the tube was brought to a focus of the slit of the spectrograph, with the wedge in position, by means of two convex lenses. The distance of the tube from the slit, and the positions of the lenses, were so adjusted that the distances between the capillary and the first lens, and between the second lens and the slit, were equal respectively

* *Loc. cit.*

to the focal lengths of the lenses. With this arrangement, an exposure was made for a definite time on the plate.

A perforated metal plate was then introduced between the two lenses, and another exposure for the same period of time was made on an adjacent portion of the same plate. This perforated plate consisted of a thin sheet of metal drilled with small holes at regular intervals of about a millimetre. By taking the mean of a number of micrometric determinations of the diameters of these holes and of the distances between them, the effective "density" of the metal plate could be calculated. The difference in the lengths of corresponding lines in the two spectra thus denotes the density step due to the plate, which is equal to the "density" of the perforated metal plate, from which the density step per millimetre length of wedge was calculated. The values thus found by the use of all the stronger Helium lines were plotted on squared paper against the wave-lengths, and a curve was drawn through these points. This curve was quite regular, and of the same type as that shown in the previous paper for another wedge, though obtained now by a different and in some respects better method. Actual values of the density gradient may be found in the tables given in later sections of this communication.

(IV.) *Experimental.*

In the present investigation we have examined the radiation in front of a flat aluminium cathode about 1 inch in diameter, which fitted closely into a cylindrical tube, as in fig. 1. The tubes were highly exhausted by means of a Gaede mercury

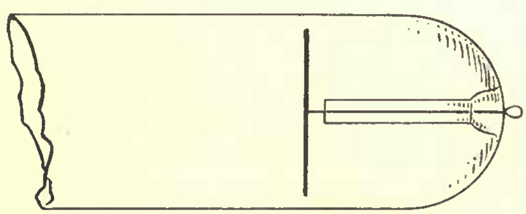

Fig. 1.

pump, and after continuous sparking, connection with the pump was cut off and Helium was introduced by heating a quantity of powdered Thorianite contained in a fused silica bulb, which was connected with the vacuum tubes through a tube containing pieces of caustic potash and a U-tube containing charcoal cooled with liquid air. After sparking for some time, the tubes were sealed off, and were found to contain, in addition to the Helium, a certain amount of Hydrogen and also of Mercury vapour. A great part of the latter disappeared on further sparking, and finally the Mercury spectrum settled down to a constant intensity. The pressure in the tubes was such that the thickness of the dark space was about 1 mm. With electrodes of these dimensions, the tubes could be run with a moderate current for an almost indefinite period without any noticeable change, and there was no trace, on the walls of the tube, of any metallic deposit from the electrodes. We have, in addition, used tubes of the ordinary H pattern for the investigation of the mutual action of Hydrogen and Helium, and of the effect of pressure on the spectrum of Helium, and we are indebted to Sir HERBERT JACKSON, K.B.E., F.R.S., for a tube of the conventional Plücker form which contained Helium in a very high state of purity. The tubes containing Hydrogen and Helium were filled in the same manner as those with flat cathodes, with the exception that the preliminary exhaustion was effected with an oil pump, the tubes being washed out repeatedly during the process of exhaustion with pure Hydrogen. This was admitted by heating in a Bunsen flame a palladium tube which was sealed into a glass tube connected with the apparatus. After the Helium had been admitted, the desired quantity of Hydrogen could be introduced in this way.

In all the experiments recorded, the tubes were excited by means of an induction coil capable of giving a 10-inch spark in air, with a mercury jet interrupter by means of which the discharge could be maintained uniformly over any desired period of time.

(V.) *The Helium Spectrum as a Function of Cathode Distance.*

We now enter upon a discussion of a series of plates of the Helium spectrum, taken at points whose distance from the cathode increased regularly. The tube was filled with Helium containing a little Hydrogen and Mercury, but the exposure given was not in any case sufficient to enable these impurities to appear on the enlarged photographs.

The essential features of the experimental arrangement are sufficiently evident without the necessity of a diagram. The cathode was flat, and was arranged with its length parallel to the plane of the slit of the spectrograph. By moving the vacuum tube in a direction perpendicular to the plane of the cathode, light from any desired region of the tube could be allowed to enter the slit and collimated in the usual manner. It has not been possible to isolate the spectrum of each region with great purity, but the slight overlapping of the effects of consecutive regions, which could not be avoided, does not affect the conclusions subsequently reached.

A series of eight photographs will be discussed. The first relates to the region immediately in front of the cathode, and the others to regions at successive distances of 1 mm. from this region. In the first five photographs the photographic intensities are all directly comparable, as they were all taken on the same plate, with two hours' exposure in each case. The other three were necessarily taken on a different plate, and though directly comparable among themselves, are not necessarily so with regard to the former set. At the same time, serious differences in the behaviour of the two plates are not to be expected, for they were selected from the same batch of plates.

The experimental measures of the heights of the various lines in each case are given in the following table, together with the density gradients of the wedge per millimetre at each wave-length, determined from the graph of density gradient in the manner already described. The table contains all the experimental data from which the ensuing conclusions are drawn. As these conclusions are of a very general nature, it was thought necessary to give the measurements relating to one set of photographs in complete detail.

The photographs are described by Roman numerals.

TABLE I.

λ.	d_λ.	Photograph I. at cathode.	Photograph II. at 1 mm.	Photograph III. at 2 mm.	Photograph IV. at 3 mm.	Photograph V. at 4 mm.	Photograph VI. at 5 mm.	Photograph VII. at 6 mm.	Photograph VIII. at 7 mm.
		h.	h.	h.	h.	h.	h.	h.	h.
6678	0·329	7·7	12·7	12·3	10·8	7·4	5·2	2·9	—
5876	0·396	9·4	13·1	13·6	13·2	11·1	9·5	7·9	3·4
5047	0·414	—	2·3	1·2		—	—	—	—
5015	0·415	8·7	12·2	11·3	8·9	6·0	4·3	2·5	—
4922	0·415	4·0	7·2	6·9	5·4	3·3	1·7	—	—
4713	0·420	4·0	8·0	7·8	6·0	3·9	1·8	—	—
4686	0·421	—	0·6	—	—	—	—	—	—
4472	0·453	10·0	13·4	13·2	12·7	11·2	9·8	8·1	3·9
4437	0·461	—	2·0	—	—	—	—	—	—
4388	0·475	3·3	5·2	5·3	4·8	3·8	1·9	1·1	—
4144	0·569	—	1·8	1·7	just seen	—	—	—	—
4121	0·582	—	2·0	2·0	just seen	—	—	—	—
4026	0·650	3·2	5·4	6·0	5·7	4·2	3·0	1·9	—
3965	0·707	2·0	3·3	3·3	2·1	just seen	—	—	—
3888	0·815	4·9	6·4	6·2	5·2	4·1	3·0	2·4	—

The magnification of the various photographs was not completely identical in all cases. On the original plate, the distance between the lines λλ6678, 3888, was 44·95 mm. If this distance be measured on any individual photograph, its magnification m is deduced by simple division. On photographs I.–V. inclusive, we found $m = 3·181$, and on photographs VI.–VIII., $m = 3·159$.

Some of the Helium bands appear on the intermediate photographs, though absent very close to the cathode, and again at some distance from it. They are, in fact, only shown on the photographs IV.–VII., according to the details set forth in Table II. The wave-lengths are only rough values serving to identify the individual bands, and the values of d_λ are obtained as before from the calibration curve of the wedge. We merely record in the table some of the more conspicuous examples of these bands, as an illustration of their behaviour. We do not propose to discuss them, for they are in reality band heads consisting of a large number of nearly overlapping lines, and the interpretation of the exact meaning of the intensities requires considerations which are not strictly relevant to the present communication.

TABLE II.—Helium Bands.

λ.	d_λ.	Photograph IV. at 3 mm.	Photograph V. at 4 mm.	Photograph VI. at 5 mm.	Photograph VII. at 6 mm.
		h.	h.	h.	h.
4650	0·423	3·1	3·5	2·9	2·5
4546	0·439	just seen	1·2	1·0	just seen
4459	0·456	1·2	1·7	just seen	—
4447	0·459	1·1	1·4	just seen	—
4427	0·464	2·2	2·5	1·5	1·2
4414	0·468	2·6	2·9	2·0	1·8
4399	0·471	2·2	2·5	1·7	1·1
4336	0·490	—	1·1	1·5	0·9

For a line taken twice on the same plate, but coming from different regions of the discharge tube, a direct comparison of its heights in the two cases is of no value as an indication of relative intensities, for the difference in height corresponding to any definite intensity-ratio in the two cases depends very much upon the density gradient of the wedge, whose variations along the spectrum are considerable. We must accordingly, as a preliminary to any discussion, obtain the photographic intensities of the lines in all cases, according to the formula

$$\log_{10}^{-1}(d_\lambda h_\lambda/m),$$

for these, as we have seen, are strictly comparable for the same line on the same plate with two different conditions or regions of excitation. The photographic intensities of the series lines are given in Table III.

The results of calculation of the photographic intensities of the Helium bands, on the photographs which register them, are as given in Table IV.

In general, we may say, in connection with these bands, that although in a qualitative sense they are intensified or weakened together, according to the region from which the spectrum is photographed, this general correspondence is not strictly quantitative, the relative intensities of any band in two regions being dependent to a small extent on the wave-length. In other words, the regions of maximum emission of these bands, which can appear simultaneously with the series spectrum of Helium, are not identical. This question will not be discussed further in this communication, owing to the difficulty, already indicated, of interpreting the exact meaning of the intensity in this case. The table already sufficiently indicates the general nature of the phenomena presented by the band heads in this form of experiment.

TABLE III.—Intensities of Helium Lines.

λ.	Photogragh I. at cathode. — Photographic intensity.	Photograph II. at 1 mm. beyond. — Photographic intensity.	Photograph III. at 2 mm. beyond. — Photographic intensity.	Photograph IV. at 3 mm. beyond. — Photographic intensity.	Photograph V. at 4 mm. beyond. — Photographic intensity.	Photograph VI. at 5 mm. beyond. — Photographic intensity.	Photograph VII. at 6 mm. beyond. — Photographic intensity.	Photograph VIII. at 7 mm. beyond. — Photographic intensity.
6678	6·25	20·6	18·7	13·1	5·82	3·48	2·00	—
5876	14·8	42·8	49·3	44·0	24·1	15·5	9·77	2·67
5047	—	1·99	1·43	—	—	—	—	—
5015	13·65	39·0	29·8	14·5	5·24	3·67	2·13	—
4922	3·33	8·69	7·94	5·07	2·69	1·67	—	—
4713	3·37	11·4	10·7	6·195	3·27	1·73	—	—
4686	—	1·20	—	—	—	—	—	—
4472	26·55	80·9	75·7	64·4	39·4	25·4	14·5	3·62
4437	—	1·95	—	—	—	—	—	—
4388	3·11	5·97	6·18	5·21	3·69	1·93	1·46	—
4144	—	2·10	2·01	—	—	—	—	—
4121	—	2·32	2·32	—	—		—	—
4026	4·51	12·7	16·8	14·6	7·21	4·14	2·46	—
3965	2·78	5·41	5·41	2·93	—	—	—	—
3888	17·99	43·6	38·8	21·5	11·2	5·94	4·16	—

λ 3965 is just seen on V.

TABLE IV.—Intensities of Helium Bands.

λ.	Photograph IV. — Photographic intensity.	Photograph V. — Photographic intensity.	Photograph VI. — Photographic intensity.	Photograph VII. — Photographic intensity.
4650	2·58	2·92	2·44	2·16
4546	just seen	1·47	1·38	just seen
4459	1·49	1·75	—	—
4447	1·44	1·59	—	—
4427	2·13	2·32	1·66	1·50
4414	2·42	2·67	1·98	1·85
4399	2·12	2·34	1·79	1·46
4336	—	1·48	1·71	1·38

*Diffuse Series and Cathode Distance.**—In order to isolate the various phenomena of intensity distribution presented by the spectrum of Helium at different distances

* HICKS has proposed a new arrangement of Helium series, regarding P series as being in fact F series. We have thought it more convenient, however, to retain the older terminology in our discussion throughout, as we deal only with experimental results.

from the cathode, three entirely distinct lines of enquiry must be investigated. These are—

(1) The relative intensities of the successive lines of any one series, as a function of cathode distance.
(2) The relative intensities of corresponding lines of the Principal, Sharp and Diffuse series, either of Helium or of Parhelium, under the same circumstances.
(3) The relative behaviour of the Helium lines (double) and of the Parhelium lines (single) in the case of corresponding members.

The entire phenomena presented can be regarded as the result of a superposition of these three effects, each of which is in itself of considerable interest in connection with any theory of the origin of spectra. Such a general enquiry into one definite spectrum, such as that of Helium, is necessarily somewhat long, but the spectrum of Helium is, in many respects, so typical, and our knowledge of the origin of series is so doubtful, that it is evidently desirable to push the investigation to the extreme limit in this individual case. Only by the definite isolation of the three effects mentioned can further progress in the elucidation of the nature of spectra apparently be made, and quantitative measurements of intensity have not hitherto been sufficiently sensitive to small changes, for the purpose of obtaining definite conclusions on any one of these subjects.

In the present section, we confine ourselves to a discussion of the relative behaviour of successive lines corresponding to increasing term number in a Diffuse series. Two such series are available on the present set of photographs—the doublets characteristic of the Diffuse series of Helium, and the single lines classed generally as Parhelium. The necessary data with regard to these lines—in the case of Helium being the joint effect of the two components of the doublet in each case—are set forth in Tables V. and VI. For the time being, we do not consider the interesting question of the position, with respect to the cathode, of maximum emission of any one line of such a series, but only relative intensities in the series on each photograph, one particular line being arbitrarily chosen as 10 in every case. The results of this computation are as follows:—

TABLE V.—Diffuse Series of Helium.

Intensity λ.	Photograph I. at cathode.	Photograph II. at 1 mm.	Photograph III. at 2 mm.	Photograph IV. at 3 mm.	Photograph V. at 4 mm.	Photograph VI. at 5 mm.	Photograph VII. at 6 mm.	Photograph VIII. at 7 mm.
5876	10	10	10	10	10	10	10	10
4472	18	19	15	15	16	16	15	14
4026	3·0	3·0	3·4	3·3	3·0	2·7	2·5	not seen

In connection with the interpretation of this table, it is necessary to remark, in the first place, that the actual numbers themselves give no information in the absolute sense, or in the relative sense down one column, as to the relative intensities of the three lines in question, for the photographic plate is not equally sensitive in three regions. But the actual changes in the numbers from one column to another give decisive information, since the intensity of $\lambda 5876$ is reduced to a uniform scale. These changes are very small, though quite definite, even taking into consideration the fact that the numbers are derived from an exponential type of formula, and they cannot be regarded as within the error of observation and consequent calculation. To at least a close approximation, however, *the relative intensities in the Diffuse series do not vary with the distance from the cathode.*

The small variations which do occur present no striking regularity, and it is evident that the behaviour of the last three photographs, already stated to be on a different plate to the others, is not appreciably different in these regions, so that we have further justification for the supposition that the two sets of photographs are directly comparable. There is a small amount of evidence in the table, although it is not decisive, that a slight energy transfer to the longer wave-lengths takes place with increasing distance from the cathode, but if it be real, it is yet so small as to be a comparatively unimportant phenomenon. There is no effective transfer of energy along the Diffuse series of Helium with increasing cathode distance.

Small variations in the numbers are to be expected, for it is difficult to maintain complete uniformity in the experimental conditions over a long period, and the various photographs were necessarily taken at different instants. But such variations in the conditions from one photograph to another apply to all the series alike, and from the uniformity of the numbers in Table V. we may assume with confidence that they are small.

According to this conclusion regarding the absence of an energy transfer along the series, it is not difficult to show that the vanishing of $\lambda 4026$ on VIII. is to be expected. For on the basis of 10 for the photographic intensity of $\lambda 5876$, the average value for $\lambda 4026$ is 3·0. The actual photographic intensity of $\lambda 5876$ on VIII. is, from a preceding table (Table III.) 2·67. That of $\lambda 4026$ should therefore be, on this scale,

$$2{\cdot}67 \times 3{\cdot}0/10 = 0{\cdot}80.$$

Accordingly, for this line, if h_λ be its height,

$$10^{d_\lambda h_\lambda / m} = 0{\cdot}80$$

which is less than unity, and therefore h is negative. This signifies that the exposure is insufficient to show the line even on theoretical grounds. In fact, on the scale in Tables III. and IV. the minimum photographic intensity which can be visible is not zero but unity. This particular scale, according to the definition of photographic intensity adopted is the *true scale,* and will be referred to as such in later parts of this communication.

We may now consider Table VI. which shows that the Diffuse series of Parhelium behaves in the present connection in a quite different manner. The arbitrary intensity 10 is ascribed to λ6678 in each case.

TABLE VI.—Diffuse Series of Parhelium.

Intensity λ.	Photograph I. at cathode.	Photograph II. at 1 mm.	Photograph III. at 2 mm.	Photograph IV. at 3 mm.	Photograph V. at 4 mm.	Photograph VI. at 5 mm.	Photograph VII. at 6 mm.	Photograph VIII. at 7 mm.
6678	10	10	10	10	10	10	10	absent
4922	5·3	4·2	4·2	3·9	4·6	4·8	absent	absent
4388	5·0	2·9	3·3	4·0	6·3	5·5	7·3	absent
4144	absent	1·0	1·0	just seen	absent	absent	absent	absent

There is an initial drop of intensity on II. down this series, more pronounced in the third member, which becomes weaker relatively to the second. The fact that λ4144 is not visible on I. is readily interpreted, for on this photograph, the true photographic intensity of λ6678 is 6·25, so that λ4144 would become invisible if its intensity on the arbitrary scale of the last table were less than 16, which it may readily be, by comparison with the remainder of the last table.

After this initial drop, a remarkable enhancement takes place in λ4388, which is not confined to the last three photographs, and which cannot therefore be interpreted as due to a difference in behaviour of the separate plate on which they were taken. The change is of a quite different order of magnitude from any change found in Table V.

On the apparent law followed by the rest of the table, the true intensity of λ4144 on IV. is found to be 1·2, which is in accordance with the fact that it is just visible. The intensities of this line on later photographs, even on the supposition of quite considerable enhancement after the manner of λ4388, are all less than unity, so that its disappearance is to be expected. The theoretical intensity of λ4922 on VII., on the same supposition, cannot exceed about 0·9 on the true scale, which is in accordance with its disappearance from this photograph. It is therefore true in general that the absence of lines in this series on the various photographs presents no disturbing feature.

The general conclusion regarding the Diffuse series of Parhelium is that, after an initial tendency to enhancement of the first member λ6678 at the expense of the others, taking place almost exactly at the extremity of the dark space, there is a subsequent transfer of energy of the series to the higher members, as the distance from the cathode is increased. This phenomenon shows the Diffuse series of Helium and of Parhelium in definite contrast, and is the first clear indication we have obtained of a real difference of behaviour down the two series under the same sets of conditions and excitation. We may in this connection again recall the anomalous behaviour of λ4388 in many celestial spectra.

Sharp Series.—The corresponding data, reduced to an arbitrary scale in each case, relating to the Sharp series of Helium, are contained in Table VII.

TABLE VII.—Sharp Series of Helium. Intensity Ratios.

λ.	Photograph I.	Photograph II.	Photograph III.	Photograph IV.
4713	10	10	10	10
4121	absent	2·0	2·2	just seen

The Sharp series of Helium evidently behaves like the Diffuse series in preserving a constant intensity ratio between consecutive lines, at least for a distance of 3 mm. from the cathode Taking the average ratio as almost precisely 5·1, we deduce, from the true intensities 3·4, 6·2, of λ4713 on photographs I., IV., that the corresponding intensities of λ4121 should be 0·7 and 1·2, of which the second should be just visible and the first invisible. This is in accordance with the observational data in Table VII. Thus the intensity ratio down the Sharp series does not appear to vary with the cathode distance.

The only members of the Sharp series of Parhelium shown on our photographs are λ5047 and λ4437, which only occur together on II. It is not therefore possible to examine the variations in their intensity ratio.

Principal Series.—In the region of the spectrum which we have examined, the only Principal line of Helium is λ3888, so that no conclusion can be drawn in the present enquiry as regards the relative behaviour of the members of this series as the cathode distance is varied. But the Principal series of Parhelium contains two members in this region whose relative intensity on the various photographs is indicated in Table VIII.

TABLE VIII.—Principal Series of Parhelium.

λ.	Photograph I.	Photograph II.	Photograph III.	Photograph IV.	Photograph V.
5015	10	10	10	10	10
3965	2·0	1·4	1·8	2·0	just seen

The initial relative enhancement of the first member, practically at the end of the dark space—a definite feature of the Diffuse series of Parhelium—is shown prominently in this series also. As the distance from the cathode is increased further, this phenomenon disappears, and the second member becomes more intense, in a regular manner, with respect to the first.

This process appears to be continuous, for the true intensity of λ3965 on this scale, on the supposition that the ratio 1·5 of IV. is preserved on V., becomes 1·0 on calculation, which is not sufficient to render it so visible as it actually is on Photograph V. Evidently, therefore, the increase of relative intensity of λ3965 continues, until there is an actual relative enhancement with respect to the first photograph. The Diffuse and Principal series of Parhelium thus behave similarly.

The general conclusions, with which all the results hitherto detailed are in accordance, may be stated as follows :—

As the cathode distance is increased, there is no definite change of relative intensity in the components of any Helium series, with the possible exception of a slight enhancement of the first member in the Diffuse series at a considerable distance from the cathode.

Parhelium, on the other hand, is in striking contrast. Earlier members of its series are enhanced at the expense of later members at the extremity of the dark space. Beyond this point, the phenomenon is gradually reversed, until finally there is a definite enhancement of later members at the expense of those of lower term-number.

This difference of behaviour of the single-line and doublet series must be of importance to any discussion of their origin. From a general point of view, it appears to imply at least that the two sets of series are not produced from the same atoms.

(VI.) *Comparison of Principal, Sharp and Diffuse Series.*

Superposed on the phenomena investigated above is another of considerable interest—the variation in the relative intensities of corresponding members of the three series of Helium or of Parhelium. The conclusions already reached as to the uniformity of behaviour for example in the three series of Helium, render it unnecessary to discuss the validity of this comparison based on corresponding members, for the conclusions to be obtained in this section are not dependent, in consequence, on the particular corresponding members selected for illustration.

In the case of Helium, we shall select the lines λλ5876, 4713, 3888, as representatives of the three series, reducing the first to intensity 10 on a new scale for each photograph. The results are indicated in Table IX.

The intensity of the Sharp series, after a definite increase, again at the end of the dark space, continuously decreases with reference to that of the Diffuse, at first

TABLE IX.—Helium—Comparison of Series.

Series.	λ.	Photograph I.	Photograph II.	Photograph III.	Photograph IV.	Photograph V.	Photograph VI.	Photograph VII.	Photograph VIII.
Diffuse	5876	10	10	10	10	10	10	10	10
Sharp	4713	2·3	2·7	2·2	1·4	1·4	1·1	absent	absent
Principal . . .	3888	12·2	10·2	7·9	4·9	4·65	3·8	4·3	absent
Principal Sharp .	—	5·3	3·8	3·6	3·5	3·3	3·4	—	—

rapidly as the distance increases, but afterwards more slowly. The true intensity of the line λ5876 on VII., VIII. is so small that λ4713 could not appear on these photographs unless this law were suddenly changed in this region, so that its absence presents no difficulty. The Principal series, on the other hand, is not relatively intensified at the end of the dark space, but is already decreasing in intensity. There is evidence of an ultimate reversal at some distance from the cathode (on VII.) but it is not conclusive.

The intensity-ratio of Principal and Sharp series exhibited in the last row of the table, clearly shows that the Sharp series tends to become stronger relatively to the Principal series, rapidly at the end of the dark space, and afterwards very slowly. These phenomena are very definite.

It is perhaps desirable again to point out that the actual numbers in a table of this kind are in no way a representation of the energy emission in the various wave-lengths, owing to the curve of sensibility of the photographic plate. Only the change from one column to another has any significance in the present enquiry.

In the discussion of Parhelium from the same point of view, we may confine attention to the lines λλ6678, 5047, 5015, belonging respectively to the Diffuse, Sharp and Principal series. The intensity of λ6678 is in each case reduced to 10.

TABLE X.—Parhelium. Comparison of Series.

Series.	λ.	Photograph I.	Photograph II.	Photograph III.	Photograph IV.	Photograph V.	Photograph VI.	Photograph VII.	Photograph VIII.
Diffuse	6678	10	10	10	10	10	10	10	10
Sharp	5047	absent	0·96	0·76	absent	—	—	—	—
Principal . . .	5015	21·8	18·9	15·9	11·1	9·0	10·5	10·6	absent

We may consider, in the first place, the cases of absent lines. The true intensity of λ6678 on I. is 6·25, and the absence of λ5047 from this photograph merely indicates that its intensity on the true scale does not exceed unity, and therefore on the present scale does not exceed 10/6·25 or 1·6. Comparison with the remainder of the table shows therefore that its absence is to be expected if the drop of intensity from I. to II. is not of a different order from that found in any other series. The disappearance of the same line from IV. and later photographs is also to be expected, unless a great increase of its relative intensity takes place suddenly at this point.

In the case of Parhelium, the Sharp and Principal series decrease in intensity as compared with the Diffuse series, without the temporary reversal of this phenomenon, at the end of the dark space, found in the case of Helium. The apparent reversal at a considerable distance, found in the case of Helium on one plate of the Principal series and stated not to be decisive, is repeated on two plates in the Principal series of Parhelium (VI., VII.) and now appears to be real. Very considerable exposures, however, would be necessary at greater distances in order to establish the fact that the phenomenon continued to occur. We have preferred, in the experiments recorded in this communication, to confine attention to a series of photographs taken with identical duration of exposure.

It is difficult to draw any conclusions, in the case of Parhelium, with regard to the relative transfer of energy between the Sharp and Principal series, for the former is only visible on two photographs. The only definite difference of behaviour with regard to Helium and Parhelium thus appears to lie in the region at the end of the dark space, where there is a temporary relative diminution of the Diffuse series of Helium, but not of Parhelium.

(VII.) *Comparison of Helium and Parhelium.*

A related problem of some interest is the determination, on some precise basis, of the relative changes which take place in the corresponding doublets (Helium) and single lines (Parhelium) in the spectrum. We have seen in the last section that the relative phenomena of the three series are the same in general in each case, except for a small difference on photograph II. The best standard of comparison is apparently given by the leading lines of the three series in each case.

We accordingly compare $\lambda\lambda$5876, 6678 as the leading lines of the two Diffuse series, $\lambda\lambda$4713, 5047, for the Sharp series, and $\lambda\lambda$3888, 5015, for the Principal series. Intensities in the doublet series are all reduced to 10.

The disappearance of 6678 on VIII. is in accordance with a still further reduction of its intensity on this scale, below 2·05, so that the decrease of relative intensity persists to the extreme photograph. There is a reversal at the end of the dark space on II. in the usual manner, the conditions of emission in this region evidently possessing special features which affect all the lines in the spectrum. Apart from this effect, the Parhelium Diffuse series steadily decreases in intensity, with increase of distance from the cathode, relatively to the Helium Diffuse series. The phenomenon

TABLE XI.—Helium and Parhelium.

Series.	λ.	Photograph I.	Photograph II.	Photograph III.	Photograph IV.	Photograph V.	Photograph VI.	Photograph VII.	Photograph VIII.
Diffuse	5876	10	10	10	10	10	10	10	10
	6678	4·22	4·81	3·80	2·98	2·41	2·25	2·05	—
Sharp	4713	10	10	10	10	10	10	—	—
	5047	—	1·75	1·33	—	—	—	—	—
Principal	3888	10	10	10	10	10	10	10	—
	5015	7·59	9·00	7·69	6·74	4·68	6·18	5·12	—

of the dark space is shown definitely also in the Principal series of Parhelium, which at this point becomes almost as intense as that of Helium. Subsequently the decrease of the Parhelium spectrum is shown definitely until we arrive at VI., where another temporary reversal occurs. Although the photographs VI.–VIII. may not be strictly comparable with the others, the phenomenon appears to be real, for it corresponds to similar effects in previous comparisons of series made in this communication, which are not restricted to special ranges of wave-length in which the plate on which VI.–VIII. were taken might have special properties. Moreover, it does not occur at all in other series, for example the two Diffuse series of the present section, where there is no sudden change in the character of the numbers characterizing λ6678 on passing from V.–VI.

It seems necessary to conclude that there is a region, distant about 5 mm. from the cathode in the present experiment, where, as at the extremity of the dark space—1 mm. from the cathode—the conditions of excitation reach some form of critical point, with a consequent change in the nature of the law of intensity variation of certain lines and series with cathode distance. In particular, there is a tendency for relative enhancement of the Principal series of Parhelium, but not the Diffuse series, at this point.

(VIII.) *Regions of Maximum Emission.*

The regions in the tube from which individual spectral lines are radiated with greatest intensity are of considerable importance in connection with theories of the origin of spectra. The present measurements enable us to obtain some quantitative data with regard to many lines in the spectrum of Helium. We do not attempt to discuss all the lines from this point of view, the exposure being in many cases only sufficient to show some of the lines on one or two photographs, so that no graphical or other method can be used to determine the exact law which connects their intensities with the distances from the cathode, enabling the positions of the maxima to be read off the curve of intensity or calculated by analysis. Moreover, it is sufficient, for a general survey of the question, to discuss only typical lines, in view of the previous tables. For example, in the Diffuse series of Helium, we found that the ratios of intensity of the three members visible, λλ5876, 4472, 4026, remained effectively constant over the whole range of the photographs, so that their maxima must occur at the same place, and the examination of λ5876 is sufficient. The photographs all had the same duration of exposure, and being on the same plate, those numbered I.–V. are strictly comparable even as regards the intensities shown by an individual line, except in so far as variations—already seen in another connection to be very small—may occur owing to the difficulty of maintaining uniform conditions of excitation throughout the exposures of the various photographs. We have, moreover, in the preceding sections found no reason to believe that the other plate, on which VI.–VIII. were taken, is in any important respect different from the first. We shall therefore assume, as a basis, that the sequence of eight photographs can be compared as regards the intensity of an individual line.

The sequence of intensities of λ5876, which we may take as the first example, is from Table I.

$$14{\cdot}8,\quad 42{\cdot}8,\quad 49{\cdot}3,\quad 44{\cdot}0,\quad 24{\cdot}1,\quad 15{\cdot}5,\quad 97{\cdot}7,\quad 2{\cdot}67,$$

and it is at once evident that the seat of maximum emission is at about 2 mm. from the cathode.

Attempts to fit these numbers to an interpolation formula of the type

$$\mathrm{I} = a + bx + cx^2 + \ldots$$

where I is the intensity and x the cathode distance, are not successful. It is in fact evident from the later members of the sequence that the law is partly exponential. The sequence of logarithms of intensity is found to be, to base 10,

$$1{\cdot}17,\quad 1{\cdot}63,\quad 1{\cdot}69,\quad 1{\cdot}64,\quad 1{\cdot}38,\quad 1{\cdot}19,\quad 0{\cdot}99,\quad 0{\cdot}43,$$

and these also, especially when the dark space is included, do not fit well into an interpolation formula of the above type. It is probable that any law, in order to be valid over this wide region, must be somewhat cumbrous. The dark space must, in fact, be left out of consideration in obtaining such a formula, and an example of a three-constant one is

$$\log_{10} \mathrm{I} = a + bx + cx^2,\ a = 1{\cdot}452,\ b = 0{\cdot}235,\ c = 0{\cdot}0575.$$

which gives the second, third, fourth, and sixth numbers accurately and 1·47 for the fifth, whose actual value was found to be 1·38. The formula is not very good, but sufficient for our purpose, and the calculated maximum is at the point

$$x = b/2c = 2{\cdot}05 \text{ mm.}$$

Other formulæ of interpolation which have been used, though not described in this communication, give 2·0 mm. as the position of maximum emission of the Diffuse series of Helium in these experiments.

In the case of other series, the position is not the same for each member, and the best method of studying the phenomena is apparently by means of a graph, as shown in fig. 2. The abscissæ are distances from the cathode, while the ordinates represent photographic intensities of lines at these distances. Graphs for several of the more important lines in the spectrum are given. The general features of these typical curves can at once be seen. The close resemblance between the leading lines of the two Principal series, λλ3888, 5015, is very strikingly different from the behaviour of the curves obtained for lines of the associated series, which in themselves are similar.

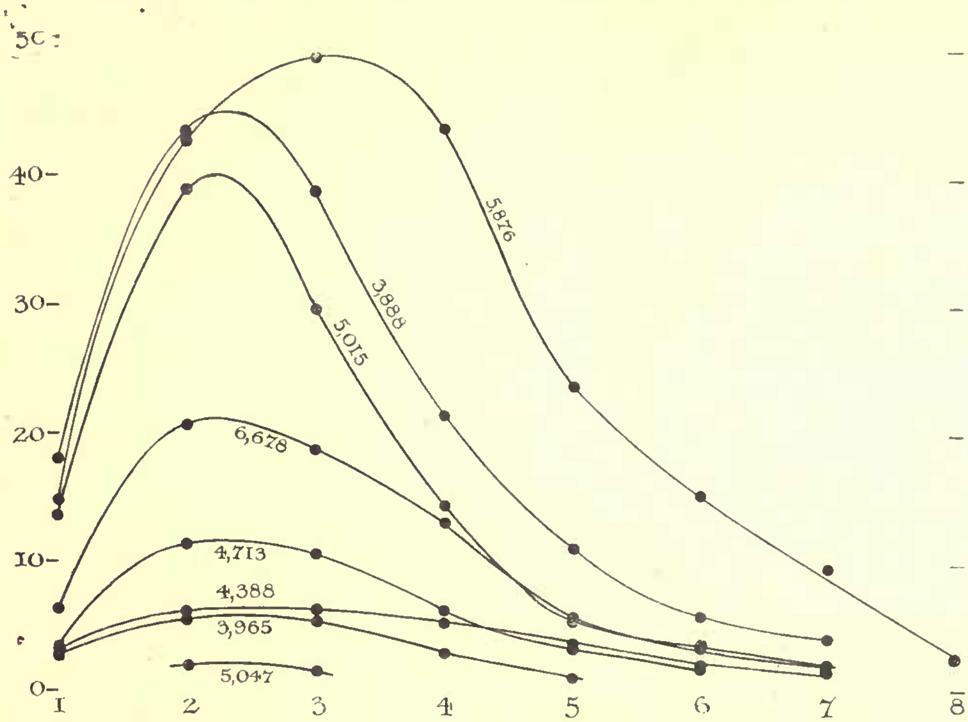

Fig. 2.

A qualitative survey of the variation in distribution of the lines in front of the cathode is given in fig. 3. This plate was obtained by photographing, without the wedge, the glow in front of the cathode, which was in this case perpendicular to the length of the slit. It presents certain remarkable features which have not hitherto been discussed. In order to enhance the effects, the plate has been reproduced by processes introducing excessive photographic contrast which, whilst accentuating the outstanding features, have obliterated fainter lines such as Hα and Hg λ5461, which were clearly visible on the original plate. It will be noticed that whilst the Helium lines start from the cathode and fade away continuously, the Hydrogen lines show a very definite "dark space." The Mercury lines reach their maximum of intensity at a still greater distance from the cathode. Of particular interest is the behaviour of the characteristic "spark" line λ4686, and the band spectrum, of which only the more prominent heads are visible.

It has been known for some time that the band spectrum and λ4686 appear in the

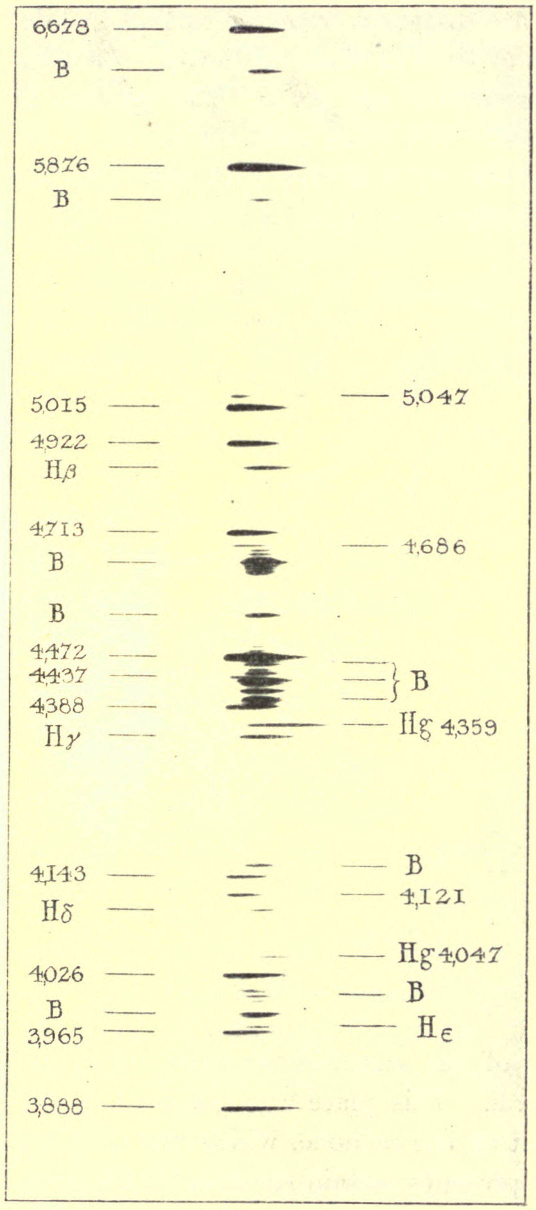

Fig. 3.

glow around the cathode. Whilst confirming this, we see that the band spectrum is localised in a restricted region at some distance from the cathode. On the other hand, λ4686 has its seat of maximum emission in a region where the band spectrum is only just visible, but it is interesting to note that over a definite region λ4686 and the band spectrum appear together. It seems possible that this mode of observation will be of use in the resolution of complex spectra into series.

The somewhat narrow region to which the band spectrum is confined would seem to imply that the conditions of excitation which give rise to this spectrum fall between very restricted limits.

(IX.) *The Spectra of Mixed Gases.*

A considerable amount of qualitative information has already been obtained by various investigators, who have examined the effect, on the spectrum of a gas, produced by impurities, or by a definite mixture with another gas. We have already referred to this work in our introductory section and in a preceding communication, in which we described some strictly quantitative results shown by the spectrum of Hydrogen when this gas is mixed with a certain amount of Neon. The effect on the spectrum of Neon, which may be expected to be in some sense complementary, of the large admixture of Hydrogen was not investigated on account of our lack of knowledge of the series relations in the Neon spectrum. In order to determine in what sense complementary effects occur in the spectra of the two mixed gases, it is necessary to select two gases whose series relations are known, and the present section of this communication details the experimental results obtained by our method of measurement, with a view to the discussion given in later sections.

In our experiments with mixed gases, we have not been able to eliminate as a source of error the possibilities of effects arising from small differences in pressure in the different tubes, but as the effects observed do not correspond to those which would accompany an alteration in pressure, we feel that the observed phenomena may be described as particular to the mixed gases.

We have made experiments on this subject in two cases, which present strikingly dissimilar phenomena. In the first case, the gas consisted mainly of Helium, with only a very small admixture of Hydrogen—sufficiently small, in fact, to justify the statement that practically only a trace of Hydrogen was present. In the second case, a considerable addition of Hydrogen to the Helium was made, so that the tube actually contained a "mixed gas," in the sense that the orders of magnitude of the quantities present were the same. For purposes of comparison, we also examined the spectrum, under like conditions, of the purest Helium which could be obtained, and which we have already mentioned. We shall refer to this as "pure Helium," and to a similar spectrum obtained with the purest available Hydrogen as that of "pure Hydrogen."

Allied to this investigation is another on the spectrum of Helium under very low pressure, and it is convenient to record and reduce the observations relating to this question with those relating to mixed gases, the mode of reduction being identical. In the subsequent discussion, they may also be taken together for the sake of brevity.

It does not appear to be necessary to add any further details of the experimental arrangements in these cases, which have been dealt with in an earlier section in general terms.

The magnification of all the enlargements in these cases was the same, the interval between λλ6678, 3888, being 147 mm. as against 44·95 mm. on the original plate. The magnification is therefore

$$m = 147/44·95 = 3·270.$$

The following notation will be adopted for brevity :—

(*a*) = Spectrum of pure Helium at very low pressure, the resistance of the tube corresponding to about 2 cm. alternative spark gap.

(*b*) = "Ordinary" spectrum of pure Helium from the capillary with 1 mm. dark space.

(*c*) = Spectrum of Helium containing a trace of Hydrogen, just sufficient to show the Hydrogen lines. Taken from the capillary (1 mm. dark space).

(*d*) = Spectrum as in (*c*), but with a larger admixture of Hydrogen.

Table XII. gives the observed heights of the various lines :—

TABLE XII.—Observations of Helium under Various Conditions.

λ.	d_λ.	Photograph (*a*).	Photograph (*b*).	Photograph (*c*).	Photograph (*d*).
		h.	*h*.	*h*.	*h*.
7065	0·233	absent	10·6	10·0	11·3
6678	0·329	5·3	18·0	17·9	16·8
5876	0·396	9·0	21·8	20·3	19·0
5047	0·414	absent	8·3	7·7	10·2
5015	0·415	9·4	16·6	14·8	15·8
4922	0·415	5·0	12·8	12·0	13·2
4713	0·420	6·3	15·5	14·4	16·6
4471	0·453	11·5	19·9	18·1	18·7
4437	0·461	absent	6·7	5·9	7·6
4388	0·475	4·2	10·1	9·3	10·0
4169	0·556	absent	1·2	absent	2·0
4144	0·569	absent	4·9	3·7	5·0
4121	0·582	absent	7·7	6·6	7·9
4026	0·650	4·0	9·2	8·4	9·4
4009	0·664	absent	0·9	absent	1·5
3965	0·707	2·2	6·6	5·3	6·4
3888	0·815	3·6	11·1	10·1	10·1

The next table (Table XIII.) gives the corresponding photographic intensities of the lines, as the results of calculation by the usual formula.

TABLE XIII.—Intensities of Helium Lines under Various Conditions.

λ.	Photograph (*a*).	Photograph (*b*).	Photograph (*c*).	Photograph (*d*).
	I.	I.	I.	I.
7065	absent	5·69	5·16	6·38
6678	3·41	64·7	63·2	49·0
5876	12·30	437	288	200
5047	absent	11·2	9·44	19·55
5015	15·6	128	75·7	101
4922	4·32	42·1	33·3	47·3
4713	6·44	97·9	70·6	135·5
4471	39·18	571·5	322	390
4437	absent	8·81	6·79	11·8
4388	4·07	29·3	22·4	28·4
4169	absent	1·60	absent	2·19
4144	absent	7·13	4·41	7·41
4121	absent	23·4	15·0	25·5
4026	6·24	67·5	46·8	73·8
4009	absent	1·52	absent	2·02
3965	2·99	26·7	14·0	24·2
3888	7·89	585	329	329

Helium Series under Various Conditions of Pressure and of Purity.—We may begin the discussion of the phenomena contained in the last table by a consideration of the Diffuse and Sharp series of Helium. The relative intensities of lines belonging to different series, including the classical example of λ5876 and λ5015 at low pressures, will be considered in a later section.

We shall for the moment confine attention to the question of energy transfer up or down the series, from one line to another of the *same* series, produced by very low pressure—as distinguished from "ordinary" conditions of pressure—or by admixture of a large or small quantity of Hydrogen. In each photograph we reduce the intensity of λ5876, in discussing the case of Helium, to 10 on any necessary scale, with corresponding calculations of the reduced intensities on the same scale, of λ4471 and λ4026. The results are shown, with the corresponding ones obtained in the same manner for the Sharp series—represented by the lines λλ7065, 4713, 4121—in Table XIV. There are three members in each case, and we have accordingly appended also the intensity ratio of the two other members, as they must also be compared with one another as well as with the first member. In the case of the Sharp series it has been more convenient to take λ4713 as the standard instead of λ7065.

In discussing these results we must, of course, take the ordinary spectrum of pure Helium given on photograph (*b*), and refer the others to this as a standard. Inspection of the table reveals the following main characteristics of these spectra :—Low pressure definitely enhances the line λ4471 with respect to λ5876, and at the particular pressure we have adopted, its relative intensity is more than doubled. The importance of this result, from the point of view of the conditions of pressure occurring in nebulæ, is sufficiently evident, for the behaviour of this line in nebulæ as compared with λ5876 is in the same sense. It is not unlikely that further reduction of the pressure may carry the process further, and this question forms an important subject for future investigation.

TABLE XIV.—Diffuse Series and Sharp Series (Helium).

λ.	Reduced photographic intensities.			
	(*a*) Low pressure.	(*b*) Ordinary.	(*c*) Trace of hydrogen.	(*d*) With hydrogen.
5876	10	10	10	10
4471	31·1	13·1	11·2	19·5
4026	5·04	3·69	1·63	3·69
Ratio	0·162	0·189	0·146	0·190
7065	absent	0·58	0·73	0·47
4713	10	10	10	10
4121	absent	2·4	2·12	1·88
Ratio	—	4·16	2·90	4·00

The relative intensity of λ4026 with respect to D_3, on the other hand, is not altered so appreciably. With respect to λ4471 this line is much reduced. It is evident that the phenomenon is not correctly described as a transfer of energy to the members of higher term number in the series, and therefore that the effect of low pressure cannot be classed with the phenomena of enhancement of λ4472 which we recorded in a previous communication. For in these cases, the line λ4026 also participated in the effect to a much greater extent, and even relatively to λ4471.

Our conclusion must be that reduction of pressure in the tube can enhance the line λ4472 to a great degree, but at the same time leaves other members of the series with nearly the same relative intensities. In the case of the Sharp series, the discussion is made rather more difficult by virtue of the disappearance of λ7065 and λ4121 from our plate at low pressure. But we can calculate the limiting intensities they can have. If their "true" intensity was unity in this case, while λ4713 had its true intensity 6·44, they would be visible. On a scale of intensity 10 for λ4713, they become visible if their intensity exceeds the value 10/6·44 or 1·55. We may accordingly assume that it is less.

Comparing this investigation with the fact of the existence of an intensity 2·4 on this scale, for the line λ4121 in the "ordinary" spectrum, as in the table, it is evident that the phenomenon found in the Diffuse series is present here also, and to the same degree. In the case of nebulæ, the Sharp series of Helium is always very weak, but the line λ4713 is well-known. Its behaviour under low pressure is, in the light of these experiments, strictly comparable with that of λ4471, and these lines are respectively the second members of the two series.

We may now take up the consideration of the effects produced by admixture of Hydrogen. There is in this case some quantitative information available in one direction. For in a previous communication, we discussed the effect produced on the spectrum of Hydrogen by the admixture of heavier gases, such as Helium and, more especially in that communication, Neon. It was found that a transfer of energy occurred in the Hydrogen spectrum under these circumstances from the members of lower to those of higher term number, and that, in the quantitative sense, this transfer, which could be measured very accurately, was considerable. We now consider the other side of the problem of inter-action of two gases, from the point of view of the heavier gas. The series arrangement in Neon being unknown, this could not be discussed previously, but the present data for Helium give a basis for discussion.

Passing now to the Diffuse series of Helium, as shown on photographs (*b*) and (*c*), and in Table XIII., the effect of a small quantity of Hydrogen is very marked. On a scale which preserves λ5876 with intensity 10 in each case, the intensity of λ4471 is 19·5 in pure Helium, but only 11·2, or only half as great, when a trace of Hydrogen is inserted. Moreover, λ4026 falls in intensity from 3·69 to 1·63—or less than half. In fact, it falls even relatively to λ4471, so that the result implies a definite energy transfer of considerable amount towards the members of low term number in the series, and more especially towards λ5876. This is precisely the converse phenomenon to that found in Hydrogen itself when mixed with a large quantity of Neon or Helium.

The Sharp series of Helium behaves in the same manner, and to an extent which is nearly equivalent, in the quantitative sense. While λ4713 is retained at 10, the higher member—of lower term number—λ7065 is enhanced from 0·58 to 0·73, in the proportion 3·2. At the same time λ4121 falls from 2·4 to 2·1—a change quite outside the possible limits of experimental error in this mode of measurement. We may therefore state, in general terms, that the effect of a trace of Hydrogen is to throw the energy in the two series much more completely into members of lower term number, so that each is reduced in intensity relatively to any earlier member.

A comparison of photographs (*b*) and (*d*) indicates the effect of a large admixture of Hydrogen. This is quite different, for the Diffuse series shows at once a tendency for transfer of energy in the opposite direction. For on the equivalent reduced scales, λ4472 is enhanced only from 13·1 to 19·5, and λ4026 is unaltered. The phenomenon is therefore in this case not at all defined as a transfer in increasing amounts to the members of higher term numbers. It is apparently the resultant of a combination of this process with the opposite process, resulting in a direct and special enhancement of λ4472 of the same nature as we found with low pressure.

But the important fact for our present enquiry is that the role played by a large quantity of Hydrogen is directly contrary to that played by a small trace, and we may argue that the mechanism of inter-action of the two gases is quite different in the two cases. A definite phenomenon has been quantitatively isolated which demands for its appearance only a spectroscopic "trace" of one of the acting gases.

This reversal of the effect of a trace of Hydrogen, by the admixture of more Hydrogen is, however, interesting in another way, for it introduces us to a striking difference of behaviour between Diffuse and Sharp series. Inspection of Table XIII. indicates that the line λ7065 shows very little tendency to change in relation to λ4713 by the action of this Hydrogen—or at least that the change in the Diffuse series is of quite another order. Moreover, the change among the lines λλ7065, 4713, 4121, though comparatively small, is quite definitely present as a combination of two effects. For λ4121 is reduced relatively to λ4713, as by the effect of the trace of Hydrogen, while λ4713, as against λ7065, is quite definitely enhanced. It seems that the Sharp series under these circumstances is just ceasing to show the first phenomenon, due to the trace of Hydrogen, and commencing to show the second, so that if the quantity of Hydrogen were increased yet further, the second might predominate. In other words, the essential difference between the Diffuse and Sharp series is that in the latter case a more considerable admixture of impurity is needed to produce the effects observed in the Diffuse series. Sharp series are in fact sensitive, to an equal extent with Diffuse series, to the influence of a trace of Hydrogen, but not to a comparable degree to the different mechanism of interaction with large quantities of Hydrogen. We feel no doubt that the available data can be summarised in this way, for the phenomena shown by the Parhelium spectrum follow the same course throughout.

The Principal series of Helium, showing only one member λ3888 is not, of course, capable of test in this manner by the present experiments.

Principal Series under the same Conditions.—The Diffuse series of Parhelium contains five members on some of our plates, and we can therefore make by its use a much more exhaustive test of the conclusions outlined in the preceding section. It is also possible to obtain information relating to Principal series, and this will be our first object in the present section. Since there are only two visible members, in these experiments, the Table (XV.) is very short.

TABLE XV.

λ.	(a).	(b).	(c).	(d).
5015 3965	10 1·92	10 2·08	10 1·85	10 2·4

By comparison of (*a*) and (*b*), we see that the second member of the Principal series is not enhanced by low pressure relatively to the first. It is in fact definitely reduced. This series behaves accordingly in a different manner from the others. It would seem that the phenomenon found in the other case—that of a selective transfer of energy—is not produced.

We may also notice that the selective enhancement at low pressure of the higher members in any one series which, as we have seen, must be regarded as a phenomenon relating to the Diffuse and Sharp series, but more especially to the former, is one which would be expected on theoretical grounds from a theory such as that of BOHR, which regards lines of higher term number as due to the passage of atoms between "stationary states" of relatively large atomic radius—a state of things to be expected with greater frequency under the influence of a considerable reduction of pressure. We may recall, for example, BOHR's explanation of the great extent of visibility of BALMER's series of Hydrogen lines in the solar spectrum. It is noteworthy, in this connection, that the Diffuse series of elements are those in which the Rydberg phase constant is nearly unity, so they accord very closely, in quite general terms, with the present principles underlying BOHR's theory. So far as the present investigation is concerned, the quantitative examination of the alteration of the spectrum of Helium produced by reduction of pressure lends a certain amount of support to BOHR's theory, at the same time, however, implying that the theory in question, if in its general basis correct as regards Diffuse series, does not furnish any interpretation of the origin of Principal series, in which the Rydberg phase is usually widely different from unity. We do not, however, propose to discuss this question further at the present time, as evidence in the other direction can be adduced also. For example, we showed in a previous communication that the Balmer series of Hydrogen lines does not in fact possess the characteristics of a Diffuse series, for the separations of the doublets which compose it are not constant as regards wave number, but are, on the other hand, appropriate to a Principal series. The question of the relation of our results to BOHR's theory must therefore be left unsolved at the present time, and we prefer to summarise the selective effect of low pressure in individual series in the Helium spectrum into the statement that while in the Diffuse and Sharp series, there is an energy transfer to higher term numbers, the effect on the Principal series is in the opposite sense.

We have not, of course, yet examined the effect of low pressure on the Diffuse and Sharp series of Parhelium. This examination will be given briefly after the corresponding tables have been exhibited, and will be found to correspond exactly to the similar effects observed in Helium. Meanwhile, we may complete the discussion of Principal series by a short survey of the change produced by admixture of a light gas such as Hydrogen.

Referring again to Table XIII., photograph (*b*) and (*c*), we find that the influence of a trace of Hydrogen decreases $\lambda 3965$ in intensity relatively to $\lambda 5015$. The original intensity is, however, more than restored by the addition of more Hydrogen. This behaviour is precisely similar to that shown by the Diffuse series of Helium, although the actual changes are of a smaller order of magnitude.

Diffuse and Sharp Series of Parhelium.—We stated at the beginning of the last section that the Diffuse series of Parhelium supplied a peculiarly exhaustive test of the more general applicability of some of our conclusions. The main details regarding the intensities of the lines under the conditions in question are given in Table XVI.

TABLE XVI.—Diffuse Parhelium under Various Conditions.

λ.	(*a*) Low pressure.	(*b*) Ordinary.	(*c*) With trace of hydrogen.	(*d*) Mixture.
6678	10	10	10	10
4922	12·7	6·51	5·27	9·66
4388	11·9	4·53	3·55	5·80
4144	absent	1·10	0·70	1·51
4009	absent	0·23	absent	0·41

The enhancement of λ4922 and λ4388 relatively to λ6678 is at once obvious, by inspection of the table, in the case of the low pressure spectrum. It is in fact even more remarkable than in the corresponding Helium series. Moreover, λ4388 is enhanced relatively to λ4922. The remarks which we made earlier regarding λ4472 in the nebular spectrum apply with greater force to λ4388. The behaviour of these two lines in nebulæ is thus correlated, in the light of these experiments, by the fact that nebulæ are in a state of extremely low pressure—and certainly much lower than in the present investigation, so that the relative enhancement of λ4471 and λ4388 may be expected to be much greater. But the degree to which the phenomenon occurs, even with the present exhaustion of the tube, is sufficiently convincing.

As in the case of the Diffuse series of Helium, this effect again cannot be described as a continuous transfer of energy down the series, for if this were the case, λ4144 would become visible when enhanced to a greater degree than λ4388. It is actually invisible, and calculation shows that this fact implies that its intensity relatively to λ6678 is not more than doubled. We must therefore repeat the former conclusion that the strong enhancements of particular lines at low pressure are peculiar to these lines, and in fact to the three lines λλ4471, 4922, 4388, of the two Diffuse series. Under still lower pressure, λ4922 may be expected to become quite subordinated to λ4388, which is already as strong in our experiments, and in the very low conditions of pressure in nebulæ, λλ4471, 4388, should therefore be the two most prominent Helium lines of the Diffuse series. This is a well-known fact of observation in astrophysics.

We may now take up the consideration of the effect of a trace of Hydrogen. Inspection of the table is almost sufficient to show that the energy-transfer to longer wave-lengths found in Diffuse Helium is repeated in the corresponding series of Parhelium. For λ4922 is reduced from 6·5 to 5·3. λ4388 is even more reduced from 4·5 to 3·5, and λ4144 from 1·1 to 0·7. In fact each is reduced relatively to all its predecessors in the series. We are in this case dealing with a phenomenon of a different type to that caused by variation of pressure, and as suggested in connection with Diffuse Helium, we prefer to restrict the term "energy-transfer" to cases in which the change of intensity of a line is greater than, or less than, the change in all its predecessors in the series.

The effect of a large admixture of Hydrogen is again, as in Helium, directly contrary to the effect of a small trace just discussed. There is an actual enhancement of the members of higher term number in the series.

There remains the necessity of verifying the fact that the Sharp series of Parhelium presents no exceptional features, and of observing from a consideration of Hydrogen lines emitted in the presence of Helium in these experiments—they have been observed in presence of Neon in an earlier communication—the simultaneous effect on the lighter component of the mixture.

For consideration of the Sharp series of Parhelium we have calculated, reducing λ5047 to intensity 10 in each case, the following intensities of the next member, λ4437 :—

λ4437, intensity 7·87 in the ordinary spectrum, 7·2 with a trace of Hydrogen, and 6·03 with more Hydrogen. The reduction of the second member by a trace of Hydrogen is again evident, though not very strongly. A slight further reduction is manifest with more Hydrogen, but the changes are so small that we may conclude, from these data, that Parhelium presents no contradiction to the view that the reversal of energy-transfer in the Sharp series takes place at a later stage of continued admixture of Hydrogen than in the Diffuse series. The present numbers appear to indicate, as did those for the Sharp series of Helium, that the reversal is on the point of taking place.

The Spectrum of Hydrogen.—In the following table (Table XVII.) the results are given for the spectrum of pure Hydrogen, taken in the ordinary way from a capillary

TABLE XVII.—Hydrogen.

λ.	d_λ.	Pure.		Mixed with Helium.		Ratio of photographic intensities.
		h.	Photographic intensity.	*h.*	Photographic intensity.	
H_α	0·343	9·9	10·9	17·0	60·7	5·6
H_β	0·416	7·8	9·82	14·6	72·1	9·2
H_γ	0·490	5·3	6·22	11·1	46·9	7·4
H_δ	0·595	2·0	2·31	5·4	9·60	4·1
H_ϵ	0·705	absent	—	1·3	1·91	—

with a pressure of 1 mm. of dark space, and the spectrum of Hydrogen as shown under the same circumstances in conjunction with that of Helium on photograph (*b*). A comparison of these tables supplies the necessary basis for a determination of the effect of mixture on the spectrum of the lighter gas.

The magnification *m* was in each case 3·270. The transfer of energy towards the higher term numbers is very evident, although in the later members, H_δ, &c., it is not completely established. The effect on the lighter gas is therefore the same as that on the heavier, when the quantities of each present in the mixture are comparable. In fact in a comparable mixture of the two gases there is a tendency in both cases towards relative diminution of the leading lines of series, and towards, in general, a shift of the energy of emission towards the violet.

It is now clear that the mechanism of this effect must be wholly different from that operative when only a small quantity of Hydrogen is present. For in the latter case, the effect of a trace of the lighter gas on the spectrum of the heavier one is to transfer the energy emission of the latter towards the leading members of series, while a trace of the heavier gas transfers the energy emission of the lighter gas away from the leading members.

(X.) *Comparison of Different Series under Low Pressure.*

We have, in earlier sections, discussed the relative behaviour of different lines *of the same series* under various conditions, and have restricted the use of the term "selective" to the enhancement or reduction of any line relatively to other lines of the same series. Phenomena which involve the relative behaviour of *different* series, or corresponding members thereof, are, in many cases, of even greater significance. The classical example is the behaviour of λ5015—belonging to the Principal series of Parhelium—under low pressure, as compared with λ5876, of the Diffuse series of Helium. It is well recognised that λ5015 in particular is essentially a low pressure line. In the following table (Table XVIII.) the four lines, λλ5876, 4472, 5015, 4388, are considered together (i) at a pressure corresponding to 1 mm. dark space and (ii) at low pressure. The intensities are taken from previous tables and reduced in each case to a scale on which the intensity of λ5876 is 10.

TABLE XVIII.

λ.	At 1 mm. dark space.	Low pressure.
5876	10	10
4472	13·1	31·8
5015	2·93	12·7
4388	0·67	3·30

The other three lines are all greatly enhanced with respect to λ5876. They were selected because of the astrophysical interest attaching to these lines. The greatest degree of enhancement occurs in λ4388, which would suggest the conditions obtaining in nebulæ, but there is an almost equivalent enhancement—producing in fact an actual greater intensity—in λ5015. In a previous section of this communication, we concluded that the relative behaviour at low pressure, as regards their own individual series, of λλ4472 and 4388, reproduced the intensity relations found in the nebulæ. But the corresponding effect for λ5015 shows that we have not isolated the conditions which produce the nebular intensity relations in this way. Some further condition, which is capable of producing a diminution of intensity in λ5015—with perhaps a further enhancement of λλ4472 and 4388—must be superposed. A consideration of the results of a previous communication* relating to the effect of a condensed discharge, suggests that the great intensity of excitation obtaining under such conditions may be the necessary co-operative condition. It is at least evident that if we could superpose the effects separately obtained under the two conditions—very low pressure and the condensed discharge—a very close approximation to the actual intensity relations of the Helium spectrum in nebulæ would be realised.

(XI.) *Further Relations of Different Series.*

In the previous section we have considered one special problem of interest to astrophysicists—the reproduction in the laboratory of the intensity distribution among Helium lines as known in nebulæ. It is to a great extent a problem involving mainly the lines whose behaviour is in some degree exceptional. But there are in addition a number of relations which may be described as more normal, and these are of considerable interest in connection at least with laboratory spectra. They may all be studied by relating together only one individual line of each of the series—the first member except in one case, the Sharp series of Helium, where it is more convenient to use λ4713 rather than λ7065. Apart from such exceptional lines as λλ4472, 4388, the general intensity relations of the series are sufficiently defined by those of the particular representatives selected below for comparison. It is of course understood that all the effects now to be discussed are superposed on the various purely selective effects—selective as regards one particular series in each case—discussed in all sections, after the experimental description, except the last, which deals with a purely special problem.

We shall take, in the first place, the available data on the general emission in the three Helium series under various conditions, these series being represented by the lines λ5876 (Diffuse), λ4713 (Sharp) and λ3888 (Principal). The arbitrary intensity of the first is 10 in every case in the table (Table XIX.).

* *Loc. cit.*, p. 259.

Table XIX.—General Emission of Helium Series.

Series.	λ.	Pure Helium.	Low pressure.	Helium and trace of hydrogen.	$He + H_2$.
Diffuse	5876	10	10	10	10
Sharp	4713	2·24	5·24	2·45	6·77
Principal	3888	13·4	6·41	11·42	16·45

It is perhaps desirable to remark again at this point that the numbers in any one column give no indication of the relative emissions of energy in these wave-lengths, the properties of the plate being widely different in the three regions of the spectrum concerned. Comparison is justifiable in rows, but not in columns.

Low pressure produces a considerable enhancement of the Sharp series relatively to the Diffuse series—in fact the intensity is doubled—and a simultaneous reduction of the Principal series of an equivalent magnitude.

A trace of Hydrogen produces only small differences in the relative emission of the three series, as determined by these lines. The Sharp series is slightly enhanced, and the Principal series to a somewhat greater degree in relation to the Diffuse series. This enhancement of the Principal series continues, but not in a very important manner, with the addition of more Hydrogen, whose effect, however, is very striking in the case of the Sharp series, which is increased threefold in intensity, as compared with the Diffuse.

The corresponding table (Table XX.) for Parhelium is as follows :—

Table XX.—General Emission in Parhelium Series.

Series.	λ.	Pure Helium.	Low pressure.	Helium with trace of H_2.	$He + H_2$.
Diffuse	6678	10	10	10	10
Sharp	5047	1·73	absent	1·49	3·98
Principal	5015	19·8	45·9	12·0	20·6

The enhancement of the Sharp series found in the case of Helium, under low pressure, may occur here also even to a somewhat greater extent, without appearing on the plate, and the effect cannot be tested. But the most significant feature is undoubtedly the behaviour of the Principal series characterised by λ5015, which is greatly enhanced in relation to the Diffuse series, in complete contrast to the corresponding effect in Helium.

The effect of a trace of Hydrogen is also directly in contrast in the two cases. In the present case it weakens the Sharp and Principal series relatively to the Diffuse.

The addition of a comparable amount of Hydrogen, however, brings the two sets of series into line, for in this case also the ultimate effect is an enhancement of the Sharp and Principal series in relation to the Diffuse. In the case of the Sharp series, the phenomenon is again very striking.

We find by a comparison which it is not thought necessary to reproduce in detail that when corresponding series of Helium and Parhelium, typified by their first members, are compared, the following conclusions may be drawn:—

The addition of Hydrogen to Helium makes only small differences in the relative radiation in any corresponding pair of lines—one a single line (Parhelium) and the other a doublet (Helium). But low pressure gives an enormous relative strengthening of the Principal series of Parhelium with respect to that of Helium. The actual ratio of relative enhancement in our experiments is about 9·0. This is, of course, one aspect of the well known character of $\lambda 5015$ as a low pressure line.

(XII.) *Discussion and Summary.*

In discussing the results obtained, it may at once be stated that the phenomena which appear to be most important are those relating to the relative intensities of lines at different distances from the cathode, for in this case we are able to define in a general way at least some of the conditions of excitation accompanying these changes. In the experiments with mixtures of Hydrogen and Helium, and at very low pressure, the observed phenomena are quite as definite, but their discussion must necessarily be more of a descriptive than of a rational nature, for the latter conditions give rise to changes in the mode of excitation which in the present state of our knowledge seem to defy any precise specification.

With regard to the changes at different distances from the cathode, it may be stated that the electric field and the average velocity of the electrons decrease with the distance from the cathode, but there is no doubt that we are at every point dealing with a very heterogeneous excitation, and although we may speak of the average velocity, we have no information as to the distribution of velocity of the electrons. Perhaps the most striking phenomenon observed relates to the difference in behaviour between the series of Helium and Parhelium, for, in the former, lines belonging to a series maintain a practically constant intensity ratio at every point, whilst in the latter, the relative intensity of any two lines of the same series varies with the distance from the cathode. In a more general summary the seat of maximum emission in Helium is the same for lines of the same series, and is peculiar to that series, whilst in Parhelium its position is affected by the term-number of the line in the series.

In the qualitative result shown in fig. 3, we see that the Band spectrum is restricted to very narrow limits as regards the conditions of excitation. The quantitative results enable us to define in a similar way the range of conditions through which the different series are most strongly developed.

The phenomena here are very definite. In the case of lines belonging to Principal series the seat of maximum emission is closer to the cathode, and falls away with increasing distance from this point more rapidly than in the case of lines belonging to associated series. The Diffuse series appear to preserve the most uniform intensity over a wide range of conditions.

Whilst it is impossible to discuss these phenomena rationally, their importance in any comprehensive theory of the origin of spectra is evident. The "dark space" is a region in which the integrated effect of all the radiations is small, and the end of the dark space in the same way is a point at which this integrated effect suffers an abrupt change, but it is evident that the true "dark space" is different for different radiations, and there appears to be another point, which in our experiments was about 5 mm. from the cathode, at which another change occurs in certain lines, whilst others do not appear to be affected; but further investigation of this phenomenon is required.

Turning now to the radiation, when the pressure in the discharge tube is very low, we find an entirely different phenomenon. Instead of a *progressive* transfer of energy in the series, there is a *selective* transfer peculiar to certain lines. In particular the lines $\lambda\lambda$4388, 4472 and 5015 are relatively enhanced under these conditions, whilst λ3888 is reduced. Of these lines λ4388 and λ4472 are especially prominent in the spectra of nebulæ, but the simultaneous enhancement of λ5015, which is not found in nebulæ, shows that we have not isolated the conditions for reproducing the intensity relations found in the celestial spectrum, which would, however, be very closely represented by a super-position of the results at low pressure, and those found in a previous investigation when the tube was excited by a highly condensed discharge. No explanation can be offered as to the precise manner in which the excitation is altered at low pressures.

As regards the behaviour of mixtures of Helium and Hydrogen the results have not quite the same quantitative significance, in the sense that there must be small differences in the pressure of the gas in different tubes.

Taking account of this and other sources of error there are still changes which appear to be peculiar to the conditions obtaining in the mixed gases. In mixtures of Hydrogen and Helium, where the partial pressure of each gas is of the same order of magnitude, there is in general a transfer of energy in the spectra of both gases to the lines of higher term-number, in comparison with the distribution of intensity in the spectrum of Helium which was so pure that the Hydrogen spectrum could not be seen. On the other hand, in the presence of what we have called a trace of Hydrogen, the Helium lines are affected in the opposite sense; that is to say, there is a transfer of energy to the members of lower term-number. It is remarkable that the addition of a trace of Hydrogen affects both the Diffuse and the Sharp series to a comparable extent, whilst the inverse effect produced, by a larger quantity of the lighter gas, affects the Diffuse series to a much greater extent than the Sharp series.

Finally, we may refer again to the qualitative results, which show that the seat of maximum emission is widely different for lines of Helium, Hydrogen and Mercury, and can be very strikingly seen in spite of the heterogeneous nature of the excitation in our tubes. An explanation of the apparent distribution of the elements in celestial bodies upon such a basis might be worthy of consideration, but the experimental evidence seems hardly sufficient to justify such an extrapolation.

VI. *The Scattering of Plane Electric Waves by Spheres.*

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INTRODUCTORY NOTE.

THE problem which gives its title to the present paper has been handled by various writers, notably by Lord RAYLEIGH, Sir J. J. THOMSON, and Prof. LOVE. In most cases the solutions have been expressed in a Cartesian form; but it appears to me that a marked simplification is introduced by using spherical polar co-ordinates. The preliminary analysis becomes shorter, and the conclusions are easier to interpret; in fact, the analysis is nearly as simple as in the analogous problem of electrostatics, when an electric field is disturbed by the presence of a dielectric sphere.

To obtain the requisite solutions a new general solution of the electromagnetic equations in Cartesian form is given in § 1, and is then transformed to the spherical polar form; §§ 2, 3 contain a summary of certain analytical results required in the sequel.

§ 4 contains the general solution of the problem of finding the scattered waves when a plane simple harmonic wave strikes a sphere; and in § 5 the solution is applied to the case of a small sphere. These formulæ (all of § 4 and part of § 5) were originally worked out in 1899, but publication was postponed in the hope of completing the problem of the large sphere.

In § 6 the problem of a large sphere is considered by applying to the formulæ of § 4 a method of approximation devised by Prof. H. M. MACDONALD* for dealing with waves incident from a Hertzian oscillator on a conducting sphere. The formulæ of § 6 were worked out early in 1910 and were given in my University lectures at Cambridge in that year.†

At the same time I succeeded in obtaining a different treatment (given in § 7 below) which confirmed the other results, and gave an easier process for dealing with

* 'Phil. Trans. Roy. Soc.,' A, vol. 210, 1910, p. 113. Prof. MACDONALD tells me that he had worked out (at about the same time) results in reference to the problem of § 6; but these have not been published.

† An alternative solution was obtained by Prof. J. W. NICHOLSON at about the same time; his solution starts from Sir J. J. THOMSON'S formulæ. Prof. NICHOLSON'S results originally differed from those of § 6; but on revision agreement was obtained ('Proc. Lond. Math. Soc.,' vol. 9, 1910, p. 67; vol. 11, 1912, p. 277).

points behind the sphere. The method of this section is similar in some respects to one used by Prof. MACDONALD in a later paper.*

The formulæ of §§ 6, 7 have been delayed in publication for two reasons: in the first place I wished to obtain some confirmation from direct numerical calculation. This has now been carried out by Messrs. PROUDMAN, DOODSON and KENNEDY (of Liverpool University).† It appears that the agreement with the formulæ of § 6 is quite close (for $\kappa a = 9, 10$) from $\theta = 0°$ to $90°$, and for the Z-component up to about 120°. The formulæ of § 7 also give good results in a cone of about 10° behind the sphere (that is, from $\theta = 170°$ to $180°$). It is clear, however, that an approximation suitable from $\theta = 90°$ to $170°$ (for Y) and from $\theta = 120°$ to $170°$ (for Z) has still to be obtained. But nevertheless the present approximations proved a valuable auxiliary‡ in checking and testing the numerical work.

[The paper in its original form was presented to the Society on April 13, 1916; owing to the difficulties in regard to labour and paper during the war, I was asked to condense the introductory matter of §§ 1–3. This proved to be impossible until now, on account of pressure of war-work of various kinds. In the present version § 1 has been re-written so as to reduce its bulk; in §§ 2, 3 certain formulæ have been omitted which were not used in the applications of §§ 4–6.

In re-arranging the paper it proved convenient also to number the formulæ differently. The decimal system has now been adopted; here the figure before the decimal point indicates the section of the paper in which the formula occurs. The figures following the decimal point are to be regarded as following the same order as ordinary decimal fractions. Thus (5·21) and (5·22) fall between (5·2) and (5·3), and all these formulæ occur in § 5.—*Added March* 18, 1919.]

§ 1. A GENERAL SOLUTION OF THE FUNDAMENTAL ELECTROMAGNETIC EQUATIONS.§

The fundamental equations of electromagnetic waves may be written

$$\text{(E)}\qquad \frac{K}{c^2}\frac{\partial X}{\partial t} = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}, \qquad \frac{K}{c^2}\frac{\partial Y}{\partial t} = \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x}, \qquad \frac{K}{c^2}\frac{\partial z}{\partial t} = \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}.$$

$$\text{(M)}\qquad -\mu\frac{\partial \alpha}{\partial t} = \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}, \qquad -\mu\frac{\partial \beta}{\partial t} = \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}, \qquad -\mu\frac{\partial \gamma}{\partial t} = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}.$$

* 'Phil. Trans. Roy. Soc.,' A, vol. 212, 1912, p. 299. The two methods are not identical; but they appear to yield equivalent results in all the cases to which they have been applied.

† 'Phil. Trans. Roy. Soc.,' A, vol. 217, 1917, p. 279. The calculation was originally undertaken by Dr. PROUDMAN in consequence of a suggestion made in my lectures of 1912; the work, however, proved to be longer than had been anticipated and was completed by Messrs. DOODSON and KENNEDY.

‡ See the paper last quoted, p. 292 *et seq.*

§ Revised March 18, 1919; see note at the end of the introductory remarks above.

Here (X, Y, Z) denotes the electric force, (α, β, γ) the magnetic force, K is the dielectric constant, μ is the magnetic permeability, and the axes of reference are a Cartesian right-handed system. The units adopted are those of the electromagnetic system, and c is the fundamental constant generally identified with the velocity of radiation in free space; the equations (E) are those derived from AMPÈRE'S law, and the equations (M) are similarly derived from FARADAY'S law, the two together constituting the circuital relations of the electromagnetic field.

It has proved possible to obtain a solution of a very general type, by assuming that

$$(1\cdot1) \qquad X = \frac{\partial P}{\partial x} - xQ, \qquad Y = \frac{\partial P}{\partial y} - yQ, \qquad Z = \frac{\partial P}{\partial z} - zQ;$$

then equations (M) yield

$$(1\cdot2) \quad -\mu\frac{\partial \alpha}{\partial t} = y\frac{\partial Q}{\partial z} - z\frac{\partial Q}{\partial y}, \qquad -\mu\frac{\partial \beta}{\partial t} = z\frac{\partial Q}{\partial x} - x\frac{\partial Q}{\partial z}, \qquad -\mu\frac{\partial \gamma}{\partial t} = x\frac{\partial Q}{\partial y} - y\frac{\partial Q}{\partial x}.$$

Substitute from equations (1·2) in the first equation (E) and we obtain

$$(1\cdot3) \qquad \frac{\mu K}{c^2}\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial x}\left(Q + x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z}\right) - x\Delta^2 Q,$$

where Δ^2 denotes LAPLACE'S operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

On comparing equations (1·1) and (1·3) they will be seen to be consistent provided that

$$(1\cdot4) \qquad \frac{\mu K}{c^2}\frac{\partial^2 P}{\partial t^2} = Q + x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z}$$

and that

$$(1\cdot5) \qquad \frac{\mu K}{c^2}\frac{\partial^2 Q}{\partial t^2} = \Delta^2 Q.$$

Thus Q must satisfy the fundamental wave-equation, which is satisfied by any component of the electric or magnetic forces (X, Y, Z) or (α, β, γ).

For our purpose it is more convenient to express the above solutions in terms of spherical polar co-ordinates r, θ, ϕ; these are supposed to form a right-handed system, when taken in this order, so as to avoid changes of sign in introducing the new co-ordinates. We write here* (R_1, R_2, R_3) for the components of electric force in the directions of r, θ, ϕ respectively; and (H_1, H_2, H_3) for the components of magnetic force.

Equations (1·1) then become

$$(1\cdot11) \qquad R_1 = \frac{\partial P}{\partial r} - rQ, \qquad R_2 = \frac{1}{r}\frac{\partial P}{\partial \theta}, \qquad R_3 = \frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi}.$$

* This is done to avoid confusion with the Cartesian components used in equations (E) and (M); but in the subsequent sections we shall use (X, Y, Z) and (α, β, γ) for the spherical polar components here denoted by (R_1, R_2, R_3) and (H_1, H_2, H_3) respectively.

As regards the transformation of (1·1) to (1·11), it is sufficient to note that the gradient of P has the spherical polar components

$$\frac{\partial P}{\partial r},\ \frac{1}{r}\frac{\partial P}{\partial \theta},\ \frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi},$$

and that $(r, 0, 0)$ corresponds to the Cartesian vector (x, y, z).

To obtain the formulæ corresponding to (1·2) we observe that the vector on the right is equal to the vector-product of the two vectors,

$$(x,\ y,\ z) \qquad \text{and} \qquad \left(\frac{\partial Q}{\partial x},\ \frac{\partial Q}{\partial y},\ \frac{\partial Q}{\partial z}\right);$$

and that these two are represented by

$$(r,\ 0,\ 0) \qquad \text{and} \qquad \left(\frac{\partial Q}{\partial r},\ \frac{1}{r}\frac{\partial Q}{\partial \theta},\ \frac{1}{r\sin\theta}\frac{\partial Q}{\partial \phi}\right).$$

Thus the vector-product has the spherical polar components

$$\left(0, \quad -\frac{1}{\sin\theta}\frac{\partial Q}{\partial \phi}, \quad \frac{\partial Q}{\partial \theta}\right).$$

Consequently equations (1·2) now become

$$(1{\cdot}21) \qquad -\mu\frac{\partial H_1}{\partial t} = 0, \qquad -\mu\frac{\partial H_2}{\partial t} = -\frac{1}{\sin\theta}\frac{\partial Q}{\partial \phi}, \qquad -\mu\frac{\partial H_3}{\partial t} = \frac{\partial Q}{\partial \theta},$$

while (1·4) and (1·5) give

$$(1{\cdot}41) \qquad \frac{\mu K}{c^2}\frac{\partial^2 P}{\partial t^2} = Q + r\frac{\partial Q}{\partial r} = \frac{\partial}{\partial r}(rQ)\cdot$$

$$(1{\cdot}51) \qquad \frac{\mu K}{c^2}\frac{\partial^2 Q}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial Q}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Q}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 Q}{\partial\phi^2}$$

$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}(rQ) \quad + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Q}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 Q}{\partial\phi^2}.$$

A consideration of these formulæ suggests that further simplifications can be obtained by writing

$$(1{\cdot}6) \qquad P = \frac{\partial U}{\partial r}, \qquad rQ = \frac{\mu K}{c^2}\frac{\partial^2 U}{\partial t^2},$$

which together satisfy equation (1·41); and then equation (1·51) leads to the equation for U :—

$$(1{\cdot}7) \qquad \frac{\mu K}{c^2}\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}.$$

Substituting from (1·6) in (1·11) and (1·21) we obtain the final expression for the field in terms of U :—

$$(1\cdot 8)\quad \left\{\begin{array}{ll} R_1 = \dfrac{\partial^2 U}{\partial r^2} - \dfrac{\mu K}{c^2}\dfrac{\partial^2 U}{\partial t^2} & cH_1 = 0 \\[2ex] R_2 = \dfrac{1}{r}\dfrac{\partial^2 U}{\partial\theta\,\partial r} & cH_2 = +\dfrac{1}{\rho}\dfrac{\partial}{\partial\phi}\left(\dfrac{K}{c}\dfrac{\partial U}{\partial t}\right), \qquad \rho = r\sin\theta, \\[2ex] R_3 = \dfrac{1}{\rho}\dfrac{\partial^2 U}{\partial\phi\,\partial r} & cH_3 = -\dfrac{1}{r}\dfrac{\partial}{\partial\theta}\left(\dfrac{K}{c}\dfrac{\partial U}{\partial t}\right). \end{array}\right.$$

In like manner we obtain another set of solutions by making an assumption similar to (1·1) for the components of magnetic force (α, β, γ). This gives the field :—

$$(1\cdot 9)\quad \left\{\begin{array}{ll} R_1 = 0 & cH_1 = \dfrac{\partial^2 V}{\partial r^2} - \dfrac{\mu K}{c^2}\dfrac{\partial^2 V}{\partial t^2} \\[2ex] R_2 = -\dfrac{1}{\rho}\dfrac{\partial}{\partial\phi}\left(\dfrac{\mu}{c}\dfrac{\partial V}{\partial t}\right) & cH_2 = \dfrac{1}{r}\dfrac{\partial^2 V}{\partial\theta\,\partial r}, \qquad \rho = r\sin\theta, \\[2ex] R_3 = +\dfrac{1}{r}\dfrac{\partial}{\partial\theta}\left(\dfrac{\mu}{c}\dfrac{\partial V}{\partial t}\right) & cH_3 = \dfrac{1}{\rho}\dfrac{\partial^2 V}{\partial\phi\,\partial r}, \end{array}\right.$$

where V is a second solution of equation (1·7).

It can be proved* that (1·8) gives the most general field in which the radial magnetic force (H_1) is zero, while (1·9) gives the most general field in which the radial electric force (R_1) is zero. It can also be shown that the field is uniquely determined by the value of R_1 and H_1; and accordingly the most general solution can be obtained by the superposition of (1·8) and (1·9).

§ 2. Further Specialization of the Solution of § 1.

If we superpose the fields (1·8), (1·9), and now utilize (X, Y, Z), (α, β, γ) to denote the spherical polar components of the field, we have the general solution† :—

$$(2\cdot 1)\quad \left\{\begin{array}{l} X = \dfrac{\partial^2 U}{\partial r^2} - \dfrac{\mu K}{c^2}\dfrac{\partial^2 U}{\partial t^2} \\[2ex] Y = \dfrac{1}{r}\dfrac{\partial^2 U}{\partial\theta\,\partial r} - \dfrac{1}{\rho}\dfrac{\partial}{\partial\phi}\left(\dfrac{\mu}{c}\dfrac{\partial V}{\partial t}\right), \qquad \rho = r\sin\theta, \\[2ex] Z = \dfrac{1}{\rho}\dfrac{\partial^2 U}{\partial\phi\,\partial r} + \dfrac{1}{r}\dfrac{\partial}{\partial\theta}\left(\dfrac{\mu}{c}\dfrac{\partial V}{\partial t}\right), \end{array}\right.$$

* See a paper in the 'Philosophical Magazine,' July, 1919 (6th ser., vol. 38), p. 143.

† Originally worked out in 1899, and first published as a question in Part II. of the 'Mathematical Tripos,' 1910.

$$
(2\cdot 2) \qquad \left\{
\begin{aligned}
c\alpha &= \frac{\partial^2 V}{\partial r^2} - \frac{\mu K}{c^2}\frac{\partial^2 V}{\partial t^2} \\
c\beta &= \frac{1}{r}\frac{\partial^2 V}{\partial\theta\,\partial r} + \frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{K}{c}\frac{\partial U}{\partial t}\right), \qquad \rho = r\sin\theta, \\
c\gamma &= \frac{1}{\rho}\frac{\partial^2 V}{\partial\phi\,\partial r} - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{K}{c}\frac{\partial U}{\partial t}\right),
\end{aligned}
\right.
$$

where U, V are any two solutions of the equation

$$
(2\cdot 3) \qquad \frac{\mu K}{c^2}\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}.
$$

A solution of (2·3) which is sufficiently general for the applications in view may be found by assuming that U and V can be expressed as sums of terms of the type

$$
F(r, t) \times Y(\theta, \phi).
$$

It is easy to see that then (2·3) leads to the equation

$$
(2\cdot 4) \qquad \frac{r^2}{F}\left(\frac{\partial^2 F}{\partial r^2} - \frac{\mu K}{c^2}\frac{\partial^2 F}{\partial t^2}\right) = -\frac{1}{Y}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right\},
$$

and since the two sides of equation (2·4) are functions of r, t and of θ, ϕ respectively, it is clear that each side must be a mere constant. If we write the constant in the form $n(n+1)$, it is evident that Y must be a surface-harmonic of order n.

Accordingly in problems (such as those with which we shall be concerned in the sequel) where the *whole* of angular space is considered, the value of n must be a positive integer; for (except when n is an integer) there are no surface-harmonics which are everywhere continuous and single-valued.

Thus we may reduce our solution to the form

$$
(2\cdot 5) \qquad U \text{ or } V = \Sigma F_n(r, t)\, Y_n(\theta, \phi), \qquad n = 0, 1, 2, 3, \ldots,
$$

where F_n is a solution of the equation

$$
(2\cdot 6) \qquad \frac{\partial^2 F_n}{\partial r^2} - \frac{1}{c_1^2}\frac{\partial^2 F_n}{\partial t^2} - \frac{n(n+1)}{r^2}F_n = 0,
$$

and

$$
c_1^2 = c^2/(\mu K).
$$

The general solution of equation (2·6) is well known, and it is given by*

$$(2\cdot7)\qquad \mathrm{F}_n(r,\,t) = r^{n+1}\left(-\frac{1}{r}\frac{\partial}{\partial r}\right)^n\left\{\frac{f(c_1t-r)+g(c_1t+r)}{r}\right\},$$

where the functions f and g are arbitrary.

In the special case of *divergent waves*, the function g can be omitted in (2·7); and, if the region considered includes the origin, then $g(c_1t+r) = -f(c_1t+r)$, so as to make $\mathrm{F}_n(r,\,t)$ continuous at $r = 0$.

It will be convenient to notice that in consequence of equation (2·6) the radial components of force can be written in the simpler forms

$$(2\cdot8)\qquad \mathrm{X} \quad\text{or}\quad c\alpha = \Sigma\,\frac{n(n+1)}{r^2}\,\mathrm{F}_n(r,\,t)\,\mathrm{Y}_n(\theta,\,\phi).$$

It will be noticed that we can at once determine the form (2·5) for U or V when the radial forces have been expressed in the form (2·8); this agrees with the general conclusion stated at the end of § 1, that (in spherical polar co-ordinates) the remaining components of force are completely determined when the two radial components are known.

§ 3. Special Case of Simple Harmonic Waves and the Appropriate Functions.

We assume in future that the waves are simple harmonic, of wave-length $2\pi/\kappa$ in free space; we can then suppose the time to occur only in the form of a time-factor $e^{\iota\kappa ct}$, with the usual convention that finally only the real (or the imaginary) parts of the formulæ will be used.

The functions f, g occurring in equation (2·7) above are then exponentials of the types

$$e^{\iota\kappa_1(c_1t-r)} \qquad\text{and}\qquad e^{\iota\kappa_1(c_1t+r)},$$

where κ_1 is given by

$$\kappa_1c_1 = \kappa c, \qquad\text{or}\qquad \kappa_1 = \kappa\sqrt{(\mu\mathrm{K})}.$$

Thus (if we now suppress the time-factor $e^{\iota\kappa ct}$) the functions given by (2·7) are of the types

$$(3\cdot1)\qquad r^{n+1}\left(-\frac{1}{r}\frac{\partial}{\partial r}\right)^n\frac{e^{-\iota\kappa_1 r}}{r}, \qquad r^{n+1}\left(-\frac{1}{r}\frac{\partial}{\partial r}\right)^n\frac{e^{+\iota\kappa_1 r}}{r}.$$

We shall be concerned with two special types only: (i.) divergent waves; (ii.) waves which are continuous at $r = 0$. The former of these corresponds to the

* See, for instance, Lamb's 'Hydrodynamics,' 1906, art. 295; an alternative method of solution given in § 3 of my paper in the 'Philosophical Magazine,' quoted on p. 179 above; compare also A. E. H. Love ('Phil. Trans. Roy. Soc.,' A, vol. 197, 1901, pp. 9, 10).

first expression in (3·1); while the latter is found by combining the two expressions so as to yield

$$(3·2) \qquad r^{n+1}\left(-\frac{1}{r}\frac{\partial}{\partial r}\right)^n \frac{\sin(\kappa_1 r)}{r}.$$

Using the notation explained in (3·4) and (3·5) below, the standard functions are

$E_n(\kappa_1 r)$ for divergent waves,

and

$S_n(\kappa_1 r)$ for waves within a spherical boundary.

Consequently, for waves inside a spherical boundary, (2·5) and (2·8) can now be replaced by the forms

$$(3·3) \qquad \begin{cases} U \quad \text{or} \quad V = \Sigma S_n(\kappa_1 r)\, Y_n(\theta, \phi), \\ X \quad \text{or} \quad c\alpha = \Sigma \dfrac{n(n+1)}{r^2} S_n(\kappa_1 r)\, Y_n(\theta, \phi), \end{cases}$$

for divergent waves the function $S_n(\kappa_1 r)$ must be replaced by $E_n(\kappa_1 r)$.

Definitions and Properties of the Two Standard Functions $S_n(z)$, $E_n(z)$.

We write for brevity

$$(3·4) \qquad S_n(z) = z^{n+1}\left(-\frac{1}{z}\frac{d}{dz}\right)^n\left(\frac{\sin z}{z}\right)$$

$$= \frac{z^{n+1}}{1\,.\,3\,.\,5\ldots(2n+1)}\left\{1 - \frac{z^2}{2(2n+3)} + \frac{z^4}{2\,.\,4(2n+3)(2n+5)} - \ldots\right\}.$$

In terms of the known Bessel function we can write

$$(3·41) \qquad S_n(z) = \sqrt{\left(\frac{\pi z}{2}\right)}\, J_{n+\frac{1}{2}}(z),$$

and accordingly the function $S_n(z)$ is the same as that denoted by u in one of MACDONALD'S papers.*

In the notation adopted by LAMB,† and those writers who have used LAMB'S solutions as the fundamental forms, we have the identity

$$(3·42) \qquad S_n(z) = z^{n+1}\psi_n(z).$$

* 'Phil. Trans. Roy. Soc.,' A, vol. 210, 1910, p. 113. See in particular p. 115.

† 'Hydrodynamics,' 1906, Art. 287.

Similarly, we write

$$(3\cdot5)\qquad \mathrm{E}_n(z) = z^{n+1}\left(-\frac{1}{z}\frac{d}{dz}\right)^n\left(\frac{e^{-\iota z}}{z}\right)$$
$$= \mathrm{C}_n(z)-\iota\mathrm{S}_n(z),$$

where

$$(3\cdot6)\qquad \mathrm{C}_n(z) = z^{n+1}\left(-\frac{1}{z}\frac{d}{dz}\right)^n\left(\frac{\cos z}{z}\right)$$
$$= \frac{1\,.\,3\ldots(2n-1)}{z^n}\left\{1-\frac{z^2}{2(1-2n)}+\frac{z^4}{2\,.\,4(1-2n)(3-2n)}-\ldots\right\}.$$

In terms of the K_n function (the modified Bessel function used by MACDONALD) we have the relation

$$(3\cdot51)\qquad \mathrm{E}_n(z) = \sqrt{\left(\frac{2z}{\pi}\right)}\, e^{\frac{1}{2}(n+\frac{1}{2})\pi\iota}\mathrm{K}_{n+\frac{1}{2}}(\iota z).$$

Thus

$$\mathrm{E}_n(z) = v-\iota u$$

in terms of the notation used by MACDONALD in the paper last quoted, and in LAMB'S notation

$$(3\cdot7)\qquad \mathrm{E}_n(z) = z^{n+1}\{\Psi_n(z)-\iota\psi_n(z)\} = z^{n+1}f_n(z).$$

In consequence of the equation (2·6) we see that both $\mathrm{S}_n(z)$ and $\mathrm{E}_n(z)$ are solutions of the differential equation

$$(3\cdot8)\qquad \frac{d^2\mathrm{S}_n}{dz^2}+\left\{1-\frac{n(n+1)}{z^2}\right\}\mathrm{S}_n = 0,$$

The functions $\mathrm{S}_n(z)$, $\mathrm{C}_n(z)$ and $|\mathrm{E}_n(z)|$ have been tabulated from $z = 1$ to 10, and for values of n ranging from 0 to 22, by Mr. DOODSON,* and these tables have formed the basis of the numerical calculations mentioned on p. 176 above.†

It will be convenient to collect here the simple relations amongst the functions S_{n-1}, S_n, S_{n+1}, which correspond to the known results for Bessel functions, or to those given by LAMB for the equivalent function $\psi_n(z)$.

Difference Relations for the Functions S_n, E_n.

From (3·4) we see that

$$(3\cdot81)\qquad \mathrm{S}_{n+1}(z) = -z^{n+1}\frac{d}{dz}\left\{\frac{\mathrm{S}_n(z)}{z^{n+1}}\right\} = \frac{n+1}{z}\mathrm{S}_n(z)-\frac{d\mathrm{S}_n}{dz},$$

and by using (3·5) we see that the same relation holds for $\mathrm{E}_n(z)$.

Again, it will be found that

$$\left(\frac{d}{dz}+\frac{n}{z}\right)z^{n+p} = (2n+p)\,z^{n+p-1},$$

* 'British Association Report,' 1914.

† PROUDMAN, DOODSON and KENNEDY, 'Phil. Trans. Roy. Soc.,' A, vol. 217, 1917, p. 279.

and using this in equation (3·4) we deduce that

(3·82)
$$S_{n-1}(z) = \left(\frac{d}{dz} + \frac{n}{z}\right) S_n(z) = \frac{n}{z} S_n(z) + \frac{dS_n}{dz}.$$

Combining (3·81) and (3·82) we have also

(3·83)
$$S_{n-1}(z) + S_{n+1}(z) = \frac{2n+1}{z} S_n(z).$$

The relations (3·82), (3·83) hold equally for $E_n(z)$ and $C_n(z)$, as may be seen from (3·5) and (3·6). As $E_n(z)$, $S_n(z)$ are independent solutions of the equation (3·8), it is evident that

$$E_n(z)\frac{dS_n}{dz} - S_n(z)\frac{dE_n}{dz} = \text{const.}$$

Now when z is small, it is easy to verify from (3·4) and (3·6) that

$$E_n(z)\frac{dS_n}{dz} = \frac{n+1}{2n+1} + O(z^2), \qquad S_n(z)\frac{dE_n}{dz} = -\frac{n}{2n+1} + O(z^2),$$

and accordingly we have

(3·84)
$$E_n(z)\frac{dS_n}{dz} - S_n(z)\frac{dE_n}{dz} = 1.$$

In the discussions of § 6, when n, z are both large, it will be convenient to adopt the following notation :—

(3·85) $|E_n(z)| = R$, $E_n(z) = Re^{-\iota\psi}$, so that $S_n(z) = R\sin\psi$, $C_n(z) = R\cos\psi$.

Substituting from (3·85) in (3·84) we deduce that

(3·86)
$$R^2\frac{d\psi}{dz} = 1.$$

Before leaving these preliminary formulæ it will be convenient to quote the formula for $e^{\iota\kappa z}$ in terms of our standard functions; namely

(3·9)
$$e^{\iota\kappa z} = \sum_{n=0}^{\infty}(2n+1)\,\iota^n\,\frac{S_n(\kappa r)}{\kappa r}\,P_n(\mu),$$

where $z = r\cos\theta = r\mu$ and $P_n(\mu)$ is LEGENDRE'S polynomial of order n.

This result follows at once from the formula given in LAMB'S 'Hydrodynamics,' Art. 291, on using the relation (3·4) between $\psi_n(\kappa r)$ and $S_n(\kappa r)$, already quoted. It is of course evident that an expansion of the type (3·9) might be anticipated, since each side satisfies the wave-equation, is symmetrical about the axis of z, and is continuous at $r = 0$; the determination of the numerical coefficients may be then carried out quickly by comparing the terms in $(\kappa r\mu)^n$ on the two sides of the equation.

§ 4. PLANE ELECTROMAGNETIC WAVES INCIDENT ON A SPHERICAL OBSTACLE.

Suppose that the incident wave-train is travelling along the negative direction of the axis of z (that is, from $\theta = 0$ towards $\theta = \pi$); and that it is polarized in the plane of yz (that is, in the plane $\phi = \frac{1}{2}\pi$). Suppose further that the electric force in the wave-train has unit amplitude; then, in terms of the Cartesian specification, the incident wave is defined by*

$$X = -e^{\iota\kappa(ct+z)}, \qquad c\beta = +e^{\iota\kappa(ct+z)},$$

the remaining components of force being zero.

We must first express this wave in the standard forms of (2·1) and (2·2); we therefore introduce polar co-ordinates, and then proceed to find the radial components of force, which will suffice to determine the functions U, V.

These radial components are given by

$$(4·1) \qquad X = -\sin\theta\cos\phi\, e^{\iota\kappa r\cos\theta}, \qquad c\alpha = +\sin\theta\sin\phi\, e^{\iota\kappa r\cos\theta},$$

where the time-factor $e^{\iota\kappa ct}$ is now omitted.

Now from (3·9) we have the formula

$$e^{\iota\kappa r\cos\theta} = \sum_{n=0}^{\infty}(2n+1)\,\iota^n\frac{S_n(\kappa r)}{\kappa r}P_n(\cos\theta).$$

So, differentiating with regard to θ, we find that

$$(4·2) \qquad \sin\theta e^{\iota\kappa r\cos\theta} = -\sum_{n=1}^{\infty}(2n+1)\,\iota^{n-1}\frac{S_n(\kappa r)}{(\kappa r)^2}P'_n(\cos\theta)\sin\theta.$$

Accordingly, on substituting (4·2) in (4·1), we find that in the incident wave

$$(4·3) \quad \left\{\begin{aligned} U &= -\frac{\sin\theta\cos\phi}{\kappa^2}\sum_{n=1}^{\infty}\frac{(2n+1)}{n(n+1)}\iota^{n-1}S_n(\kappa r)P'_n(\cos\theta), \\ &\text{and} \\ V &= +\frac{\sin\theta\sin\phi}{\kappa^2}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\iota^{n-1}S_n(\kappa r)P'_n(\cos\theta)\end{aligned}\right.$$

by comparing the two formulæ (2·5) and (2·8).

The corresponding waves in the interior of the sphere will be given by the two functions

$$(4·4) \quad \left\{\begin{aligned} U_1 &= -\frac{\sin\theta\cos\phi}{\kappa^2}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\iota^{n-1}B_nS_n(\kappa_1 r)P'_n(\cos\theta), \\ &\text{and} \\ V_1 &= +\frac{\sin\theta\cos\phi}{\kappa^2}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\iota^{n-1}D_nS_n(\kappa_1 r)P'_n(\cos\theta),\end{aligned}\right.$$

where

$$\kappa_1 = \kappa\sqrt{(\mu K)}$$

and K, μ are the fundamental constants of the spherical obstacle.

* It is assumed that in the incident wave we may take $\mu = 1$, $K = 1$, $c_1 = c$, $\kappa_1 = \kappa$.

Similarly the scattered waves will be given by the two functions

$$(4\cdot5)\quad \begin{cases} U_0 = -\dfrac{\sin\theta\cos\phi}{\kappa^2}\displaystyle\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\iota^{n-1}A_nE_n(\kappa r)\,P'_n(\cos\theta) \\ \text{and} \\ V_0 = +\dfrac{\sin\theta\sin\phi}{\kappa^2}\displaystyle\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\iota^{n-1}C_nE_n(\kappa r)\,P'_n(\cos\theta). \end{cases}$$

The boundary conditions are given by the continuity of the tangential components of electric and magnetic force at the sphere $r = a$.

It is evident from the form of equations (2·1), (2·2) that these conditions will be satisfied if we take

$$(4\cdot6)\quad \begin{cases} \dfrac{\partial U_1}{\partial r} = \dfrac{\partial U}{\partial r}+\dfrac{\partial U_0}{\partial r}, \qquad KU_1 = U+U_0 \\ \text{and} \\ \dfrac{\partial V_1}{\partial r} = \dfrac{\partial V}{\partial r}+\dfrac{\partial V_0}{\partial r}, \qquad \mu V_1 = V+V_0. \end{cases}$$

Thus we find that A_n and C_n (the coefficients in the scattered waves) are given by

$$(4\cdot7)\quad \begin{cases} \{S_n(\kappa a)+A_nE_n(\kappa a)\}\dfrac{\kappa_1}{\kappa K}\dfrac{S'_n(\kappa_1 a)}{S_n(\kappa_1 a)} = S'_n(\kappa a)+A_nE'_n(\kappa a) \\ \text{and} \\ \{S_n(\kappa a)+C_nE_n(\kappa a)\}\dfrac{\kappa_1}{\kappa\mu}\dfrac{S'_n(\kappa_1 a)}{S_n(\kappa_1 a)} = S'_n(\kappa a)+C_nE'_n(\kappa a). \end{cases}$$

The special case of a *perfectly conducting sphere* is given by making the tangential electric force zero at the sphere $r = a$; and this condition is satisfied if

$$(4\cdot61)\qquad 0 = \frac{\partial U}{\partial r}+\frac{\partial U_0}{\partial r}, \qquad 0 = V+V_0.$$

Thus we find the simpler formulæ

$$(4\cdot71)\qquad S'_n(\kappa a)+A_nE'_n(\kappa a) = 0 \qquad S_n(\kappa a)+C_nE_n(\kappa a) = 0,$$

which may be regarded as limiting forms of (4·7), when $|K|\rightarrow\infty$ and $\mu\rightarrow 0$.

The formulæ (4·5) and (4·71) lead at once to those quoted by Dr. J. PROUDMAN* in calculating the pressure of radiation due to a plane wave incident on a small conducting sphere.

In all the applications with which we shall be concerned at present the point at which the disturbance is to be calculated will be at a distance large compared with

* 'Monthly Notices of the Royal Astronomical Society,' vol. 73, 1913, p. 535.

the wave-length. Then we can simplify the general formulæ by observing that (3·5) may be replaced by the approximation

$$E_n(\kappa r) = \iota^n e^{-\iota\kappa r}$$

if $1/\kappa r$, $(1/\kappa r)^2$, &c., are neglected. Further, in the final formulæ for the forces, U_0 and V_0 occur only in the two combinations

$$M = \frac{1}{r}\frac{\partial U_0}{\partial r} = -\frac{\iota\kappa U_0}{r}, \qquad N = -\frac{1}{r}\frac{\partial V_0}{\partial r} = \frac{\iota\kappa V_0}{r},$$

where terms of the relative order $1/\kappa r$ have been rejected.

On substituting from (4·5) we find, to the same degree of accuracy,

$$(4\cdot8)\quad \left\{\begin{aligned} M &= +\frac{1}{r}\frac{\partial U_0}{\partial r} = -\frac{\iota\kappa}{r}U_0 = \sin\theta\cos\phi\,\frac{e^{-\iota\kappa r}}{\kappa r}\sum_{n=1}^{\infty}(-1)^n\frac{2n+1}{n(n+1)}A_n P'_n(\cos\theta) \\ &\text{and} \\ N &= -\frac{1}{r}\frac{\partial V_0}{\partial r} = +\frac{\iota\kappa}{r}V_0 = \sin\theta\sin\phi\,\frac{e^{-\iota\kappa r}}{\kappa r}\sum_{n=1}^{\infty}(-1)^n\frac{2n+1}{n(n+1)}C_n P'_n(\cos\theta). \end{aligned}\right.$$

Then, substituting in the general formulæ (2·1) and (2·2), we find that (to our present order of approximation) the radial components of force are zero, and that the transverse components are given by

$$(4\cdot9)\quad \left\{\begin{aligned} Y &= \frac{\partial M}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial N}{\partial\phi} = +c\gamma \\ Z &= \frac{1}{\sin\theta}\frac{\partial M}{\partial\phi} + \frac{\partial N}{\partial\theta} = -c\beta. \end{aligned}\right.$$

Accordingly *the electric and magnetic forces in the scattered waves are at right angles to each other and to the radius, and their magnitudes are related in the same manner as in a plane wave.*

This conclusion might very well have been anticipated; and for the case of *small* obstacles of any shape (with constants K, μ differing but little from unity) the conclusion is contained in a paper by Lord RAYLEIGH.* But I cannot find that it has been noticed for the case of spheres of any size, and of any electrical and magnetic constants.

This may serve to indicate one advantage of the formulæ in spherical polars over those in Cartesian co-ordinates.

The formulæ (4·8), (4·9), with the values of A_n, C_n given by (4·71), were those used by Messrs. PROUDMAN, DOODSON, and KENNEDY in their numerical calculations quoted in the introduction to this paper.

* 'Scientific Papers,' vol. 1, pp. 522–536. For a *small* perfectly conducting sphere the same conclusion is given by Sir J. J. THOMSON, 'Recent Researches,' p. 448.

§5. Spheres Small Compared with the Wave-Length.

The fundamental assumption is that $|\kappa a|$ is small enough to justify us in rejecting all but one or two terms in the power-series for $S_n(\kappa a)$ and $E_n(\kappa a)$.

It has been usual to assume further that $|\kappa_1 a|$ is correspondingly small; but Dr. Proudman has remarked that in the case of a dielectric sphere with a large value of K the second assumption need not follow from the first. It seems worth while therefore to simplify the formulæ (4·7) by expanding in powers of $|\kappa a|$, while retaining the general forms for $S_n(\kappa_1 a)$; it will be seen moreover that the resulting formulæ form a link between the results for dielectric spheres and those for conductors.

Remembering that $S_n(\kappa a)$ is of order $(\kappa a)^{n+1}$ and that $E_n(\kappa a)$ is of order $(\kappa a)^{-n}$, it is easy to see from (4·7) that in general A_n and C_n are both of order $(\kappa a)^{2n+1}$. Thus in the first approximation it will be sufficient to deal only with the coefficients A_1 and C_1; and for these we need the formulæ for $S_1(\kappa_1 a)$ and $S'_1(\kappa_1 a)$. Now from (3·4) we have

$$S_1(z) = z^2\left(-\frac{1}{z}\frac{d}{dz}\right)\left(\frac{\sin z}{z}\right) = \frac{\sin z}{z} - \cos z = \frac{\sin z}{z}(1 - z\cot z),$$

and so

$$S'_1(z) = \sin z\left(1 - \frac{1}{z^2}\right) + \frac{\cos z}{z} = \frac{\sin z}{z^2}(z^2 - 1 + z\cot z).$$

Hence

$$\frac{zS'_1(z)}{S_1(z)} = \frac{z^2 - 1 + z\cot z}{1 - z\cot z} = F(z) - 1,$$

where now

$$F(z) = z^2/(1 - z\cot z).$$

In the second place, for the functions of κa, from (3·4) and (3·6) we find the first approximations

$$S_1(z) = \tfrac{1}{3}z^2, \qquad E_1(z) = 1/z.$$

Substituting, it will be seen that for $n = 1$ the first equation in (4·7) gives

$$\left\{\frac{3A_1}{(\kappa a)^3} + 1\right\}\frac{q}{K} = 2 - \frac{3A_1}{(\kappa a)^3},$$

where

$$q = \frac{\kappa_1 a S'_1(\kappa_1 a)}{S_1(\kappa_1 a)} \doteq F(\kappa_1 a) - 1.$$

After a little reduction the last equation gives

$$A_1 = \tfrac{1}{3}(\kappa a)^3 \frac{2K + 1 - F(\kappa_1 a)}{K - 1 + F(\kappa_1 a)}. \tag{5·1}$$

The second equation in (4·7) gives a similar formula for C_1, with μ taking the place of K. However, in most cases, it is sufficient to write $\mu = 1$, and then the formula simplifies further and becomes

$$C_1 = (\kappa a)^3 \left\{ \frac{1}{F(\kappa_1 a)} - \tfrac{1}{3} \right\}. \tag{5·2}$$

The two formulæ (5·1) and (5·2) are due to Dr. PROUDMAN, who has pointed out that they connect the results found by Lord RAYLEIGH for the case of dielectric spheres, and by Sir J. J. THOMSON for conducting spheres.

To deal with the case of dielectric spheres we do not regard K as large, so that $\kappa_1 a$ may be regarded as small (of the same order as κa); and then the approximations

$$1 - (\kappa_1 a) \cot(\kappa_1 a) = \tfrac{1}{3}(\kappa_1 a)^2, \qquad \text{or} \qquad F(\kappa_1 a) = 3,$$

may be used. This gives, in place of (5·1), (5·2) the simpler forms due to Lord RAYLEIGH*

$$A_1 = \tfrac{2}{3}(\kappa a)^3 \frac{K-1}{K+2}, \qquad C_1 = 0. \tag{5·21}$$

On the other hand, Sir J. J. THOMSON'S case corresponds to the assumption that K is of the form $K_1 - \iota K_2$ where K_2 is very large; then $|\kappa_1 a|$ may be regarded as large, and $\kappa_1 a$ as complex, with a negative imaginary part. Thus approximately† $\cot(\kappa_1 a) = \iota$, and so $|F(\kappa_1 a)| = |\kappa_1 a|$, which (although large) is small compared with $|K| = |\kappa_1 a|^2/(\kappa a)^2$. Hence we find from (5·1) and (5·2) the approximate results,

$$A_1 = \tfrac{2}{3}(\kappa a)^3, \qquad C_1 = -\tfrac{1}{3}(\kappa a)^3 \tag{5·22}$$

as given by Sir J. J. THOMSON.‡ Of course this pair of formulæ follow at once from (4·71), on inserting the approximations for $S_1(\kappa a)$ and $E_1(\kappa a)$ given on p. 188 above.

Dr. PROUDMAN makes the further remark that, under the conditions assumed in (5·1) and (5·2), variations in the wave-length may produce very considerable changes in the magnitudes of A_1 and C_1, on account of the presence in $F(\kappa_1 a)$ of $\cot(\kappa_1 a)$, which may vary very fast. It is of course supposed that the sphere is dielectric, otherwise $\cot(\kappa_1 a)$ could be replaced by ι, as already stated.†

It is worth while to note the simple formulæ for the scattered wave, derived from (4·9); these give, to the present order of approximation

$$\left\{ \begin{aligned} Y &= +c\gamma = \frac{e^{-\iota\kappa r}}{\kappa r}\left(\tfrac{3}{2}C_1 - \tfrac{3}{2}A_1 \cos\theta\right)\cos\phi, \\ Z &= -c\beta = \frac{e^{-\iota\kappa r}}{\kappa r}\left(\tfrac{3}{2}A_1 - \tfrac{3}{2}C_1 \cos\theta\right)\sin\phi\,; \end{aligned} \right. \tag{5·3}$$

* 'Scientific Papers,' vol. 4, p. 321 (106); see also vol. 1, p. 526.

† Provided that the imaginary part of $\kappa_1 a$ exceeds π in numerical value, the error in this approximation is less than one half per cent.

‡ 'Recent Researches,' p. 448.

and so (to this order) the scattered wave is zero in the direction given by

$$(5{\cdot}31) \qquad \phi = 0, \qquad \cos\theta = C_1/A_1,$$

provided that A_1 is numerically greater than C_1. Thus in Lord RAYLEIGH'S case, the direction is given by $\theta = \frac{1}{2}\pi$;* and in Sir J. J. THOMSON'S by $\theta = \frac{2}{3}\pi$.

It may be noted here that, if the sphere has a sufficiently large dielectric constant K, it may happen that A_1 is numerically less than C_1; and then the direction given by (5·31) is no longer real.

Taking K to be real (the case of a conductor having been already considered on p. 189), it is easy to see that $A_1 < C_1$ gives $F(\kappa_1 a) < 1$ (on the assumption $K > 1$). Now the function $F(z)$ steadily decreases from 3 to 0 as z varies from 0 to π; and a rough calculation shows that for $z = \frac{7}{8}\pi$, $F(z)$ is slightly less than unity. Also in order to justify the approximations used for $S_1(\kappa a)$ and $E_1(\kappa a)$, we must suppose that $(\kappa a)^2 \leqq \frac{1}{10}$.

Hence the possibility contemplated may occur if, say,

$$(\tfrac{7}{8}\pi)^2 \leqq (\kappa_1 a)^2 \leqq \pi^2, \qquad \text{and} \qquad (\kappa a)^2 \leqq \tfrac{1}{10},$$

giving

$$K \geqq 75.$$

The direction in which the scattered wave vanishes will be given by

$$(5{\cdot}32) \qquad \phi = \tfrac{1}{2}\pi, \qquad \cos\theta = A_1/C_1 = 2F(\kappa_1 a)/\{3 - F(\kappa_1 a)\},$$

the final formula being simplified by remembering that K is large.

I am not aware that there is any experimental evidence showing traces of this phenomenon; in fact all the evidence shows that $\phi = 0$, $\theta = \frac{1}{2}\pi$ is not far from the truth. Thus the circumstances in actual experiments cannot have been such as to introduce the reversal of magnitude between A_1 and C_1.

(ii.) *Second Approximations.*

We proceed next to find second approximations, assuming that $|K|$ is not large; it will be necessary to retain the second terms in the series (3·4) and (3·6) for S_1 and E_1, but the first terms will suffice for S_2 and E_2.† It is easy to see that then terms of order $(\kappa a)^5$ occur in the coefficients A_1, C_1 and A_2, but that no other coefficients can contain terms of order lower than $(\kappa a)^7$.

Using now the series (3·4) for $S_1(z)$ we have

$$S_1(z) = \tfrac{1}{3}z^2(1 - \tfrac{1}{10}z^2), \qquad S'_1(z) = \tfrac{2}{3}z(1 - \tfrac{1}{5}z^2),$$

retaining the second terms only in each series. Thus we find, to the same order,

$$z\frac{S'_1(z)}{S_1(z)} = 2 - \tfrac{1}{5}z^2.$$

Also (3·6) gives similarly

$$E_1(z) = \frac{1}{z}(1 + \tfrac{1}{2}z^2), \qquad E'_1(z) = -\frac{1}{z^2}(1 - \tfrac{1}{2}z^2).$$

* A closer approximation is worked out on the next page; see formulæ (5·6) below.

† It would be possible, of course, to obtain second approximations to (5·1) and (5·2), but a glance at the formulæ shows that the work is so laborious as to be almost impracticable.

On substituting these results (4·7) becomes

$$\left(\frac{2}{\mathrm{K}}-\frac{p^2}{5}\right)\left\{\frac{p^2}{3}\left(1-\frac{p^2}{10}\right)+\frac{\mathrm{A}_1}{p}(1+\tfrac{1}{2}p^2)\right\}=\frac{2p^2}{3}\left(1-\frac{p^2}{5}\right)-\frac{\mathrm{A}_1}{p}\left(1-\frac{p^2}{2}\right)$$

and

$$\left(2-\frac{\mathrm{K}p^2}{5}\right)\left\{\frac{p^2}{3}\left(1-\frac{p^2}{10}\right)+\frac{\mathrm{C}_1}{p}(1+\tfrac{1}{2}p^2)\right\}=\frac{2p^2}{3}\left(1-\frac{p^2}{5}\right)-\frac{\mathrm{C}_1}{p}\left(1-\frac{p^2}{2}\right),$$

where, for brevity, we have written

$$\kappa a = p, \qquad (\kappa_1 a)^2 = \mathrm{K}p^2.$$

On reducing these equations, the results are

$$(5·4)\quad \begin{cases} \mathrm{A}_1 = \frac{2}{3}p^3\left(\frac{\mathrm{K}-1}{\mathrm{K}+2}\right)\left\{1+\frac{3}{5}\left(\frac{\mathrm{K}-2}{\mathrm{K}+2}\right)p^2\right\} \\ \text{and} \\ \mathrm{C}_1 = \frac{1}{45}(\mathrm{K}-1)p^5. \end{cases}$$

To determine A_2, the series for $\mathrm{S}_2(z)$ and $\mathrm{E}_2(z)$ will be required to the first terms only; these are

$$\mathrm{S}_2(z)=\frac{z^3}{15}, \qquad \mathrm{E}_2(z)=\frac{3}{z^2},$$

and on substituting in (4·7), we find

$$(5·5)\qquad \mathrm{A}_2=\frac{p^5}{15}\left(\frac{\mathrm{K}-1}{2\mathrm{K}+3}\right).$$

The field of the scattered waves is then given by

$$(5·6)\quad \begin{cases} \mathrm{Y}=+c\gamma=\frac{e^{-\iota\kappa r}}{\kappa r}(\frac{3}{2}\mathrm{C}_1-\frac{3}{2}\mathrm{A}_1\cos\theta+\frac{5}{2}\mathrm{A}_2\cos 2\theta)\cos\phi, \\ \mathrm{Z}=-c\beta=\frac{e^{-\iota\kappa r}}{\kappa r}(\frac{3}{2}\mathrm{A}_1-\frac{3}{2}\mathrm{C}_1\cos\theta-\frac{5}{2}\mathrm{A}_2\cos\theta)\sin\phi. \end{cases}$$

The field (5·6) *is accordingly zero* (*to the same degree of approximation*) *in the direction given by*

$$\phi=0, \qquad \cos\theta=(\mathrm{C}_1-\tfrac{5}{3}\mathrm{A}_2)/\mathrm{A}_1=\tfrac{1}{15}\frac{(\mathrm{K}-1)(\mathrm{K}+2)}{2\mathrm{K}+3}(\kappa a)^2.$$

This conclusion is apparently new; but it confirms an approximate result due to Lord RAYLEIGH,* according to which the scattered wave is zero in the direction given by

$$\phi=0, \qquad \cos\theta=\tfrac{1}{25}(\mathrm{K}-1)(\kappa a)^2,$$

when (K−1) *is treated as small.* But on the other hand, our result contradicts

* 'Scientific Papers,' vol. 1, p. 531, formula (61).

a statement made by Prof. LOVE* that there is no direction in which the scattered wave is completely cut out; however, on a closer examination of Prof. LOVE's formulæ, they appear to confirm the present conclusion.

The formulæ in question are (42) and (43) of the paper just quoted, but apparently there is a slip in (43). In the last line of (43) the factor given as 0 should really be $(z^2-y^2)/r^2$; the source of the inaccuracy being apparently in the passage from the formula (39) to (41). On introducing this additional term in the magnetic force, it appears that the electric and magnetic forces are zero in the direction given by†

$$x = 0, \qquad \frac{z}{r} = \frac{(K+2)(K-1)}{15(2K+3)}(\kappa a)^2$$

in Prof. LOVE's notation; of course "zero" means that the forces are really of order $(\kappa a)^7$ at most.

It is not difficult to prove that the formulæ (5·6) agree with those found by Lord RAYLEIGH‡ and Prof. LOVE;§ the method to be adopted is similar to that used in § 4 of my paper in the 'Philosophical Magazine' (quoted on p. 179 above). But it should be observed that in the specification of the incident wave adopted by Lord RAYLEIGH and Prof. LOVE, the electric force is parallel to the axis of y; but here the electric force is parallel to the negative direction of x. Thus if ϕ' denotes the azimuthal angle corresponding to the former specification, it is evident that $\phi' = \frac{1}{2}\pi$ corresponds to $\phi = \pi$; and accordingly we shall have in general the relation

$$\phi - \phi' = \tfrac{1}{2}\pi,$$

because both angles are measured in the right-handed sense about the axis of z.

Lord RAYLEIGH's paper contains tables and graphs from which it is easy to determine the variation of the field with θ; and in order to connect his tables with our formulæ, let us write (5·6) in the form

$$(5\cdot7) \qquad \left\{ \begin{array}{l} Y = +c\gamma = \dfrac{e^{-\iota\kappa r}}{\kappa r} R\cos\phi, \qquad Z = -c\beta = \dfrac{e^{-\iota\kappa r}}{\kappa r} S\sin\phi, \\ \text{where} \\ R = \frac{3}{2}C_1 - \frac{3}{2}A_1\cos\theta + \frac{5}{2}A_2\cos 2\theta, \\ S = \frac{3}{2}A_1 - \frac{3}{2}C_1\cos\theta - \frac{5}{2}A_2\cos\theta. \end{array} \right.$$

Consider, first, points in the plane given by $x = 0$, in Lord RAYLEIGH's notation; this gives $\phi' = \frac{1}{2}\pi$, or $\phi = \pi$. Hence, omitting the factor $e^{-\iota\kappa r}/(\kappa r)$, the electric force is equal to R (in the direction of θ *decreasing*); and accordingly R *is represented by the graph of the Cartesian component* $(yZ - zY)/r$, given by Lord RAYLEIGH.

Secondly, consider the plane $y = 0$; that is $\phi' = 0$, or $\phi = \frac{1}{2}\pi$. Here the electric

* 'Proc. Lond. Math. Soc.,' vol. 30, 1899, p. 318.

† A numerical slip in the first line of each of the formulæ (42) and (43) has to be corrected; the correction was given in the *Errata*, vol. 31, 'Proc. Lond. Math. Soc.'

‡ 'Scientific Papers,' vol. 5, p. 559, (u) and (v).

§ Formulæ (42) and (43) of the paper just quoted (allowing for the corrections just mentioned).

force is equal to S, perpendicular to the plane; thus S *is represented by the graph of the Cartesian component* Y.

In general, the resultant electric force is represented by

$$\frac{e^{-\iota\kappa r}}{\kappa r}\sqrt{(\mathrm{R}^2\cos^2\phi+\mathrm{S}^2\sin^2\phi)}.$$

§ 5. (iii.) *Second Approximations for Conducting Spheres.*

The foregoing algebra needs no alteration beyond replacing $\sqrt{\mathrm{K}}$ by the appropriate complex refractive index associated with the particular metal and wave-length considered. This of course assumes that $|\mathrm{K}|$ is not so large that the convergence of $\mathrm{S}_1(\kappa_1 a)$ becomes too slow to justify the approximation made above; and then the formulæ (5·4) to (5·6) provide the solution. It will be noticed that when K is complex, the equation

$$\cos\theta = \tfrac{1}{15}\frac{(\mathrm{K}-1)(\mathrm{K}+2)}{2\mathrm{K}+3}(\kappa a)^2$$

will not usually give a real value for θ: and so *there is usually no direction in which the scattered wave is zero** (or of order $(\kappa a)^7$).

It may be of interest to note here that experimental work on the scattering of light by fine particles has been carried out with silver particles suspended in water.† The corresponding values of κa seem to vary from $\frac{1}{2}$ to 2, and the value of $\sqrt{\mathrm{K}}$ is taken as $0{\cdot}2-\iota\,(3{\cdot}6)$; thus the approximations in (ii.) are not sufficient to calculate either $\mathrm{S}_n(\kappa a)$ or $\mathrm{S}_n(\kappa_1 a)$ with any accuracy.‡ In actual fact it proved necessary to use Lord RAYLEIGH'S exact formulæ, equivalent to (4·7) above, and to go as far as $n = 4$ in the series.§

§ 6. CASE OF LARGE PERFECTLY CONDUCTING SPHERES.

Before proceeding to the final formulæ, it will be convenient to state certain results given by MACDONALD‖ for the values of the functions $\mathrm{S}_n(z)$, $\mathrm{E}_n(z)$, when both n and z are large.

* For the case in which $\mathrm{K}-1$ is small this conclusion is given by G. W. WALKER, 'Quarterly Journal of Mathematics,' vol. 30, 1899, p. 217. The formulæ given on that page agree with (5·6), when $\mathrm{K}-1$ is small; but the more general formulæ on the preceding page do not agree with (5·6) completely. I have not succeeded in tracing the discrepancy on account of the fact that G. W. WALKER has omitted some of the details of his preliminary calculations.

† E. T. PARIS, 'Phil. Mag.,' vol. 30 (Ser. 6), 1915, p. 459.

‡ To obtain an accuracy of 1 per cent. in $\mathrm{S}_1(z)$ by retaining two terms of the series only, it must be supposed that $|z|$ does not exceed 1·3.

§ E. T. PARIS, *loc. cit.*, p. 472.

‖ 'Phil. Trans.,' vol. 210, A, 1910, p. 134: the formulæ are due to L. LORENZ originally. A very interesting method of deriving the results is given by DEBYE ('Math. Annalen,' Bd. 67, 1909, p. 535).

Provided that $z-(n+\frac{1}{2})$ is of an order higher than $z^{1/3}$, the formulæ are

$$(6\cdot1)\quad \begin{cases} \mathrm{E}_n(z) = \mathrm{R}e^{-\iota\psi}, & \mathrm{S}_n(z) = \mathrm{R}\sin\psi, \\ \text{where} & \\ \mathrm{R}^2 = 1/\sin\alpha, & \psi = z\sin\alpha + \frac{1}{4}\pi - (n+\frac{1}{2})\alpha, \\ \text{and} & \\ \cos\alpha = (n+\frac{1}{2})/z. & \end{cases}$$

We shall need also the corresponding formulæ for $\mathrm{S}'_n(z)$ and $\mathrm{E}'_n(z)$; it will be seen that

$$\frac{\mathrm{E}'_n(z)}{\mathrm{E}_n(z)} = \frac{1}{\mathrm{R}}\frac{d\mathrm{R}}{dz} - \iota\frac{d\psi}{dz} = -\frac{1}{\mathrm{R}^2}\left(\frac{1}{2}\frac{\cos\alpha}{\sin^2\alpha}\frac{d\alpha}{dz} + \iota\right),$$

because (3·86) gives

$$\mathrm{R}^2\frac{d\psi}{dz} = 1.$$

Hence

$$(6\cdot2)\quad \begin{cases} \dfrac{\mathrm{E}'_n(z)}{\mathrm{E}_n(z)} = -\dfrac{1}{\mathrm{R}^2}(\tan\chi + \iota) = -\dfrac{\iota}{\mathrm{R}^2}\dfrac{e^{-\iota\chi}}{\cos\chi}, \\ \text{where*} \\ \tan\chi = \dfrac{1}{2}\dfrac{\cos\alpha}{\sin^2\alpha}\dfrac{d\alpha}{dz} = \dfrac{1}{2}\dfrac{\cos^2\alpha}{z\sin^3\alpha}. \end{cases}$$

Similarly, we find that

$$(6\cdot3)\quad \frac{\mathrm{S}'_n(z)}{\mathrm{S}_n(z)} = \frac{1}{\mathrm{R}}\frac{d\mathrm{R}}{dz} + \cot\psi\frac{d\psi}{dz} = \frac{1}{\mathrm{R}^2}(-\tan\chi + \cot\psi) = \frac{1}{\mathrm{R}^2}\frac{\cos(\psi+\chi)}{\sin\psi\cos\chi}.$$

The formulæ to be used finally are those for A_n and C_n, given in (4·71), thus we take

$$\mathrm{A}_n = -\frac{\mathrm{S}'_n(z)}{\mathrm{E}'_n(z)}, \qquad \mathrm{C}_n = -\frac{\mathrm{S}_n(z)}{\mathrm{E}_n(z)},$$

where z now denotes κa. It follows from (6·1) above that

$$\mathrm{C}_n = -\sin\psi e^{+\iota\psi} = +\tfrac{1}{2}\iota(e^{2\iota\psi}-1),$$

and using (6·2) and (6·3) we see that

$$\mathrm{A}_n = -\iota e^{\iota(\psi+\chi)}\cos(\psi+\chi) = -\tfrac{1}{2}\iota\{e^{2\iota(\psi+\chi)}+1\}.$$

It is now an easy matter to write down an approximation to the functions M and N defined in (4·8), provided that θ is not near to 0 or π. Under these conditions we can take the approximate value

$$\mathrm{P}_n(\cos\theta) = \sqrt{\left(\frac{2}{n\pi\sin\theta}\right)}\cos\{(n+\tfrac{1}{2})\theta - \tfrac{1}{4}\pi\},$$

* Under our conditions α is not near to zero and z is large, so that χ is small.

giving a corresponding approximation

$$P'_n(\cos\theta) = \frac{2n+1}{\sin\theta}\frac{\sin\{(n+\frac{1}{2})\theta-\frac{1}{4}\pi\}}{\sqrt{(2n\pi\sin\theta)}}.$$

Thus

$$\sin\theta\frac{2n+1}{n(n+1)}A_nP'_n(\cos\theta) = -\frac{1}{\sqrt{(2n\pi\sin\theta)}}\{e^{2\iota(\psi+\chi)}+1\}(e^{\iota\omega}-e^{-\iota\omega}),$$

where

$$\omega = (n+\tfrac{1}{2})\theta-\tfrac{1}{4}\pi,$$

and we have replaced $(n+\frac{1}{2})^2/\{n(n+1)\}$ by unity, because n is large.

Thus, on putting $\chi = 0$, we find that (4·8) gives the approximation

$$\text{(6·4)} \qquad M = -\cos\phi\frac{e^{-\iota\kappa r}}{\kappa r}\Sigma\frac{e^{\iota n\pi}}{\sqrt{(2n\pi\sin\theta)}}(e^{2\iota\psi}+1)(e^{\iota\omega}-e^{-\iota\omega}).$$

Similarly, we get the formula

$$\text{(6·5)} \qquad N = +\sin\phi\frac{e^{-\iota\kappa r}}{\kappa r}\Sigma\frac{e^{\iota n\pi}}{\sqrt{(2n\pi\sin\theta)}}(e^{2\iota\psi}-1)(e^{\iota\omega}-e^{-\iota\omega}).$$

With series of this type, the leading part is found by making the index of the exponential stationary (regarded as a function of n). Now in both M and N there is one index only, $\sigma = 2\psi+n\pi-\omega$, which can be stationary: and the condition is

$$\frac{d\sigma}{dn} = 2\frac{d\psi}{dn}+\pi-\theta = 0.$$

Now, from (6·1)

$$\frac{d\psi}{dn} = \{z\cos\alpha-(n+\tfrac{1}{2})\}\frac{d\alpha}{dn}-\alpha = -\alpha,$$

and so the leading terms in (6·4) and (6·5) arise from taking

$$2\alpha = \pi-\theta, \quad \text{or} \quad n+\tfrac{1}{2} = z\sin\tfrac{1}{2}\theta.$$

The corresponding value of the index σ is then

$$\begin{aligned}\sigma_0 = 2\psi+n\pi-\omega &= 2z\sin\alpha+\tfrac{1}{2}\pi-(2n+1)\alpha+n\pi-(n+\tfrac{1}{2})\theta+\tfrac{1}{4}\pi,\\ &= 2z\cos\tfrac{1}{2}\theta+\tfrac{1}{4}\pi.\end{aligned}$$

To determine the form of the index near to this special value of n we take

$$\frac{d^2\sigma}{dn^2} = -2\frac{d\alpha}{dn} = \frac{2}{z\sin\alpha} = \frac{2}{z\cos\frac{1}{2}\theta},$$

and then we find the approximate formulæ

$$\text{(6·51)} \quad \left\{\begin{array}{ll} & \sigma = \sigma_0+\dfrac{(n-n_0)^2}{z\cos\frac{1}{2}\theta}, \\ \text{where} & \\ & n_0+\tfrac{1}{2} = z\sin\tfrac{1}{2}\theta. \end{array}\right.$$

The leading parts of M, N are accordingly given by the approximations

$$M = +\cos\phi\frac{e^{-\iota\kappa r}}{\kappa r}Q, \qquad N = -\sin\phi\frac{e^{-\iota\kappa r}}{\kappa r}Q, \tag{6·6}$$

where

$$Q = \Sigma\frac{e^{\iota\sigma}}{\sqrt{(2n\pi\sin\theta)}},$$

and σ has the value given in (6·51).

The value of Q is approximately equal to the integral

$$\int_{-\infty}^{\infty}\frac{e^{\iota\sigma}\,dx}{\sqrt{(2n_0\pi\sin\theta)}}, \tag{6·61}$$

where

$$n = n_0+x, \qquad \sigma = \sigma_0 + x^2/(z\cos\tfrac{1}{2}\theta).$$

Thus, approximately

$$Q = \frac{e^{\iota\sigma_0}}{\sqrt{(2n_0\pi\sin\theta)}}\sqrt{\left(\frac{\pi z\cos\frac{1}{2}\theta}{-\iota}\right)}$$

$$= \frac{e^{\iota(\sigma_0+\frac{1}{4}\pi)}}{2\sin\frac{1}{2}\theta} = \frac{\iota e^{\iota(2z\cos\frac{1}{2}\theta)}}{2\sin\frac{1}{2}\theta}.$$

Accordingly, to the same degree of approximation, we can take

$$\frac{dQ}{d\theta} = \tfrac{1}{2}ze^{2\iota z\cos\frac{1}{2}\theta}. \tag{6·62}$$

Then the components of force are given, as in (4·9), by

$$\left\{\begin{aligned} Y &= +c\gamma = \frac{\partial M}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial N}{\partial\phi} = \cos\phi\frac{e^{-\iota\kappa r}}{\kappa r}\frac{dQ}{d\theta}, \\ Z &= -c\beta = \frac{1}{\sin\theta}\frac{\partial M}{\partial\phi} + \frac{\partial N}{\partial\theta} = -\sin\phi\frac{e^{-\iota\kappa r}}{\kappa r}\frac{dQ}{d\theta}, \end{aligned}\right. \tag{6·63}$$

where differential coefficients with respect to ϕ are small compared with those with respect to θ, and so have been rejected. Thus, using (6·62), we have the approximations to the forces in the scattered waves

$$\left\{\begin{aligned} Y &= +c\gamma = \cos\phi\,\frac{a}{2r}\,e^{2\iota\kappa a\cos\frac{1}{2}\theta-\iota\kappa r}, \\ Z &= -c\beta = -\sin\phi\,\frac{a}{2r}\,e^{2\iota\kappa a\cos\frac{1}{2}\theta-\iota\kappa r}, \end{aligned}\right. \tag{6·7}$$

assuming that θ is neither near to 0 nor to π.

When θ is small, the approximation to $P_n(\cos\theta)$ must be taken as

$$P_n(\cos\theta) = J_0\{(2n+1)\sin\tfrac{1}{2}\theta\},$$

and so

$$\sin\theta P'_n(\cos\theta) = (n+\tfrac{1}{2})\cos\tfrac{1}{2}\theta\,J_1\{(2n+1)\sin\tfrac{1}{2}\theta\}.$$

Thus we now write

$$\sin\theta\,\frac{2n+1}{n(n+1)}\,\mathrm{A}_n\mathrm{P}'_n(\cos\theta) = -\iota\,(e^{2\iota\psi}+1)\cos\tfrac{1}{2}\theta\,\mathrm{J}_1\{(2n+1)\sin\tfrac{1}{2}\theta\},$$

and, proceeding as before, we are led to the conclusion that α is near to $\frac{1}{2}\pi$; thus the value of n/z is small, in the parts of the series which contribute the principal part of the sum. Then we can replace (6·1) by the approximate formulæ

$$(6{\cdot}8)\quad \begin{cases} \alpha = \frac{1}{2}\pi-(n+\frac{1}{2})/z \\ \text{and} \\ \psi = z+\frac{1}{4}\pi-(n+\frac{1}{2})(\frac{1}{2}\pi)+\frac{1}{2}(n+\frac{1}{2})^2/z. \end{cases}$$

Thus

$$2\psi+n\pi = 2z+(n+\tfrac{1}{2})^2/z.$$

Hence the approximation corresponding to (6·4) is now

$$(6{\cdot}81)\qquad \mathrm{M} = -\iota\cos\tfrac{1}{2}\theta\cos\phi\,\frac{e^{-\iota\kappa r}}{\kappa r}\,\Sigma\, e^{2\iota z+\iota(n+\frac{1}{2})^2/z}\mathrm{J}_1\{2n+1)\sin\tfrac{1}{2}\theta\}.$$

In like manner the value of N is found to differ from (6·81) only in having $+\sin\phi$ as a factor instead of $-\cos\phi$.

In the series (6·81) the value of n may be supposed to vary from 0 to ∞; and so we obtain the principal part of the sum by using the integral

$$(6{\cdot}82)\qquad e^{2\iota z}\int_0^\infty e^{\iota\zeta^2/z}\mathrm{J}_1(2\zeta\sin\tfrac{1}{2}\theta)\,d\zeta;$$

and when $\sin\frac{1}{2}\theta$ is very small the value of (6·82) is approximately equal to

$$e^{2\iota z}(\tfrac{1}{2}\iota z\sin\tfrac{1}{2}\theta).$$

Thus

$$(6{\cdot}83)\quad \begin{cases} \mathrm{M} = +\frac{1}{4}z\sin\theta\cos\phi e^{2\iota z}\dfrac{e^{-\iota\kappa r}}{\kappa r}, \\ \text{and} \\ \mathrm{N} = -\frac{1}{4}z\sin\theta\sin\phi e^{2\iota z}\dfrac{e^{-\iota\kappa r}}{\kappa r}. \end{cases}$$

Accordingly the components of force are now found to be

$$(6{\cdot}9)\quad \begin{cases} \mathrm{Y} = +c\gamma = \dfrac{\partial\mathrm{M}}{\partial\theta}-\dfrac{1}{\sin\theta}\dfrac{\partial\mathrm{N}}{\partial\phi} = \frac{1}{2}z\cos\phi\,\dfrac{e^{2\iota z-\iota\kappa r}}{\kappa r}, \\ \mathrm{Z} = -c\beta = \dfrac{1}{\sin\theta}\dfrac{\partial\mathrm{M}}{\partial\phi}+\dfrac{\partial\mathrm{N}}{\partial\theta} = -\frac{1}{2}z\sin\phi\,\dfrac{e^{2\iota z-\iota\kappa r}}{\kappa r}. \end{cases}$$

These results in (6·9) are precisely the same as would be found by writing $\theta = 0$ in the approximations (6·7), and *accordingly the formulæ* (6·7) *do remain valid right up to the axis* $\theta = 0$.

When θ is nearly equal to π, the calculation on the present lines becomes more difficult,* and we shall accordingly obtain the corresponding approximation by a different process in the next section (§ 7).

It appears that for the special value $\kappa a = 10$, the formulæ (6·7) do give the forces with a fair degree of accuracy up to an angle $\theta = \frac{2}{3}\pi$; the approximation in fact appears to be better than might have been expected. [See p. 176 above.]

§ 7. Alternative Method, applicable to any Conductor whose Dimensions are Large compared with the Wave-length.

It follows at once from Green's theorem that if u, v are solutions of the equations

$$\Delta^2 u + \kappa^2 u = 0, \qquad \Delta^2 v + \kappa^2 v = 0,$$

at points within a closed simple surface S, then

$$\int \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) d\mathrm{S} = 0, \tag{7·1}$$

where the integral is taken over the surface S ($d\nu$ being the element of outward normal), and it is supposed that u, v are both free from singularities in the interior of S.

Similarly if u has no singularities and v is a solution which behaves like $e^{-\iota\kappa\mathrm{R}}/\mathrm{R}$ near a particular point P (R denoting the distance measured from P), we see that

$$\int \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) d\mathrm{S} = -4\pi u_{\mathrm{P}}, \tag{7·11}$$

provided that P is inside the surface S.

Equations of similar forms apply when the space considered is *outside* the surface S; but then the sign of the last equation (7·11) is reversed, giving

$$\int \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) d\mathrm{S} = +4\pi u_{\mathrm{P}}; \tag{7·12}$$

it is then necessary to assume also that at infinity, u, v both correspond to divergent waves (unless it is known that u tends to zero more rapidly than $1/r$).

* Compare Macdonald, *loc. cit.*, pp. 120–122. The cause of the difficulty is to be found in the fact that now the stationary value may be expected to arise from values of n for which α is small. Then n is nearly equal to z, and in all such cases more complicated analysis is inevitable. In fact the approximations to $\mathrm{S}_n(z)$ and $\mathrm{E}_n(z)$ require to be modified by different formulæ corresponding to the cases $n > z$, $n < z$ and to the cases in which $|n-z|$ is of order $z^{1/3}$ or of lower order.

Let us now consider the problem of waves incident from some source (or sources) and reflected from the surface S. Let u denote any Cartesian component of force in the incident wave, and let the point P be outside S. Then (7·1) applies, and so

$$\int \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) d\mathrm{S} = 0,$$

because u has no singularity inside S.

If u' denotes the corresponding component of force in the reflected wave, we have from (7·12)

$$\int \left(u' \frac{\partial v}{\partial \nu} - v \frac{\partial u'}{\partial \nu} \right) d\mathrm{S} = 4\pi u'_{\mathrm{P}},$$

because u' has no singularity outside S (and u' will correspond to a divergent wave at infinity).

By addition we have the result

$$(7\cdot13) \qquad 4\pi u'_{\mathrm{P}} = \int \left\{ (u+u') \frac{\partial v}{\partial \nu} - v \frac{\partial}{\partial \nu} (u+u') \right\} d\mathrm{S}.$$

where v is taken to be $e^{-\iota\kappa\mathrm{R}}/\mathrm{R}$.

Now $u+u' = w$ gives the corresponding component of force in the complete wave; and this accordingly satisfies certain known relations at the surface S (the exact form depending on the physical properties of S). It must, however, be clearly understood that we cannot usually obtain both w and $\frac{\partial w}{\partial \nu}$ by any simple methods, any more than the analogous problems of electrostatics can be solved by a mere appeal to GREEN'S Theorem.

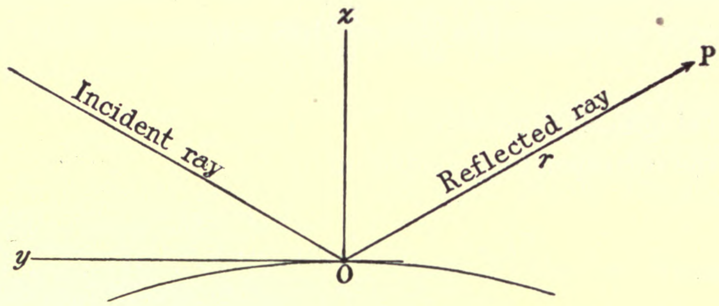
However, we can obtain *an approximate solution, suitable to the problem of short wave-lengths*, by assuming that near the reflecting surface, the character of u' can be determined from that of u by the rules of elementary geometrical optics. Thus we treat the reflected wave as derived from the incident by simple reflexion *in the tangent-plane* at the point of incidence. Making this hypothesis it is an easy matter to construct both w and $\frac{\partial w}{\partial \nu}$, when the form of u is given.

It will be noticed that we shall have $w = 0$, $\frac{\partial w}{\partial \nu} = 0$ at all points within the geometrical shadow; and so the final integral (7·13) extends only over the illuminated side of the surface S.

Suppose now that we consider electric waves incident on a simple convex conducting surface; and take an origin O on the surface such that OP is the reflected ray (in the sense of geometrical optics). Take the plane of incidence as the plane of yz, and the normal at O as the axis of z.

Then in the immediate neighbourhood of O we can represent the incident wave by the components of electric force

$$(A, \; B, \; C)\, e^{\iota\kappa(my+nz)}$$



where $m^2+n^2 = 1$, so that m is the cosine and n the sine of the angle of incidence. We have also the relation

$$Bm+Cn = 0$$

because the electric force is perpendicular to the incident ray.

The corresponding reflected wave has the components

$$(A', \; B', \; C')\, e^{\iota\kappa(my-nz)}$$

where

$$A+A' = 0, \qquad B+B' = 0,$$

and

$$B'm-C'n = 0,$$

so that

$$C'-C = 0\,;$$

these results follow by making the tangential components of force zero on the tangent-plane $z = 0$ (instead of at the surface).

It is now clear that, at the point O, the components of the total force will be

$$(0, \; 0, \; 2C), \tag{7·2}$$

and the normal differential coefficients of the total force will be

$$\iota\kappa n\,(A-A', \; B-B', \; C-C') = 2\iota\kappa n\,(A, \; B, \; 0). \tag{7·21}$$

We shall now insert these values for w and $\dfrac{\partial w}{\partial \nu}$ in the general formula (7·13), including also the factor $e^{\iota\kappa(my+nz)}$, on account of phase-differences at points near to O. Since reflexion takes place only from the immediate neighbourhood of O, the error introduced by this simplification will be small.

Since the co-ordinates of P are $(0, -mr, nr)$, the value of R is given by

$$\begin{aligned} \mathrm{R}^2 &= x^2 + (y+mr)^2 + (z-nr)^2, \\ &= r^2 + 2r(my-nz) + x^2 + y^2 + z^2, \end{aligned}$$

where (x, y, z) is a point on the surface near to O. Thus, when r is very large in comparison with the dimensions of the surface (as we assumed in the previous investigations, §§ 4–6), we can use the approximate formula

$$(7\cdot3) \qquad \mathrm{R} = r + my - nz.$$

Thus we can write in (7·13)

$$v = \frac{e^{-\iota\kappa(r+my-nz)}}{r} = v_0 e^{-\iota\kappa(my-nz)},$$

if v_0 is the value of v at O. Then the most important term in $\frac{\partial v}{\partial \nu}$ is seen to be

$$\frac{\partial v}{\partial z} = \iota\kappa n v_0 e^{-\iota\kappa(my-nz)}.$$

Accordingly the components of electric force in the reflected wave will be given by the approximation

$$(7\cdot4) \qquad \left\{ \begin{aligned} \mathrm{X} &= -\frac{v_0}{4\pi}\int (2\iota\kappa n\mathrm{A})\, e^{2\iota\kappa nz}\, d\mathrm{S}, \\ \mathrm{Y} &= -\frac{v_0}{4\pi}\int (2\iota\kappa n\mathrm{B})\, e^{2\iota\kappa nz}\, d\mathrm{S}, \\ \mathrm{Z} &= +\frac{v_0}{4\pi}\int (2\iota\kappa n\mathrm{C})\, e^{2\iota\kappa nz}\, d\mathrm{S}. \end{aligned} \right.$$

To evaluate the integrals in (7·4) we must write out the equation to the surface in the approximate form

$$2z = -(\alpha x^2 + 2\beta xy + \gamma y^2).$$

Then*

$$\int e^{-\iota\kappa n(\alpha x^2 + 2\beta xy + \gamma y^2)}\, d\mathrm{S} = \frac{\pi}{\iota\kappa n}\frac{1}{\sqrt{(\alpha\gamma-\beta^2)}}.$$

Now $\alpha\gamma-\beta^2$ is the absolute (or Gaussian) curvature of the surface at the point O; and since the surface is supposed convex, we represent this curvature by $1/\rho^2$.

* This is most easily found by taking the integral as $\lim\limits_{\theta \to 0} \int e^{-(\theta+\iota\kappa)n(\alpha x^2+2\beta xy+\gamma y^2)}\, d\mathrm{S}$.

Thus we get

$$(7{\cdot}41) \qquad \iota\kappa n \int e^{2\iota\kappa n z}\, dS = \pi\rho.$$

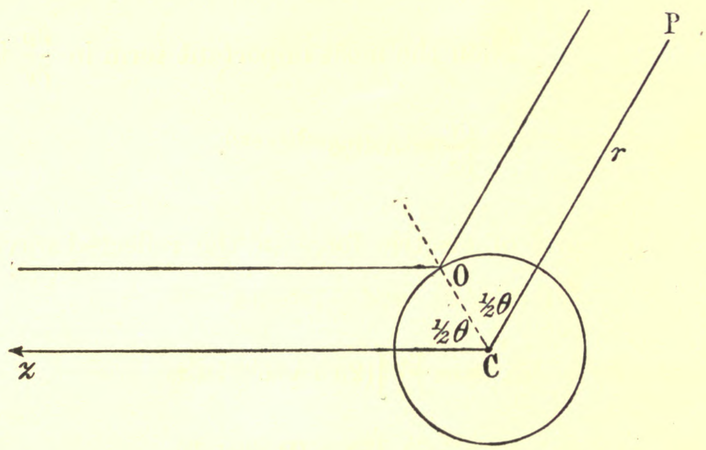
Hence, using (7·4), we see that the principal part of the reflected wave is given by

$$(7{\cdot}5) \qquad (X, Y, Z) = (-A, -B, +C)\frac{\rho}{2r}e^{-\iota\kappa r}.$$

In order to interpret (7·5) for any axes of co-ordinates, we need only notice that $(-A, -B, +C)$ represents a force numerically equal to the force in the incident wave; and that the new force is perpendicular to the reflected ray, arranged in such a way that the tangential components are opposite to those in the incident wave.

We can apply the formula (7·5) to the problem of § 6 at once; clearly $\rho = a$.

The point of incidence corresponding to the scattered wave (θ, ϕ) is given by $(\frac{1}{2}\theta, \phi)$.



Then the incident wave at O has the components of electric force

$$-\cos\phi e^{\iota\kappa a\cos\frac{1}{2}\theta} \text{ in the plane of incidence,}$$

and

$$+\sin\phi e^{\iota\kappa a\cos\frac{1}{2}\theta} \text{ perpendicular to the plane of incidence.}$$

Further, the r of formula (7·5) is measured from O; to compare with § 6, we take r to be the distance CP, measured from the centre of the sphere. Thus we are to replace r in (7·5) by $r - a\cos\frac{1}{2}\theta$. Accordingly the components of force at P, in the reflected wave, are

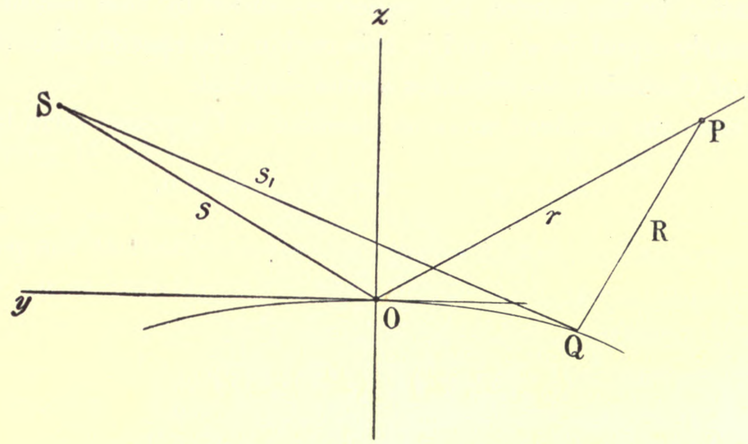
$$+\frac{a}{2r}\cos\phi e^{2\iota\kappa a\cos\frac{1}{2}\theta - \iota\kappa r} \text{ perpendicular to } r \text{ in the plane ZCP,}$$

and

$$-\frac{a}{2r}\sin\phi e^{2\iota\kappa a\cos\frac{1}{2}\theta - \iota\kappa r} \text{ perpendicular to the plane ZCP.}$$

These results agree with (6·7) and (6·9) of § 6 above.

It is easy to modify the general formulæ (7·5) so as to cover the case of waves incident from a point source (say at distance s from the point of incidence).



Then

$$s_1^2 = x^2 + (ms - y)^2 + (ns - z)^2$$
$$= s^2 - 2s(my + nz) + x^2 + y^2 + z^2.$$

Thus with the usual approximation of geometrical optics

$$s_1 = s - (my + nz) + \frac{1}{2s}\{x^2 + (ny - mz)^2\}.$$

Similarly

$$R = r + my - nz + \frac{1}{2r}\{x^2 + (ny + mz)^2\}.$$

Now z is of the second order in comparison with x, y; and so we can write

$$s_1 + R = s + r - n(\alpha x^2 + 2\beta xy + \gamma y^2) + \left(\frac{1}{2r} + \frac{1}{2s}\right)(x^2 + n^2y^2).$$

Thus here we find

$$\int e^{-\iota\kappa(s_1+R)}\, dS = e^{-\iota\kappa(s+r)}\frac{\pi}{\iota\kappa n\sigma}$$

where

$$\sigma^2 = \left\{\tfrac{1}{2}\left(\frac{1}{r} + \frac{1}{s}\right) - n\alpha\right\}\left\{\tfrac{1}{2}\left(\frac{1}{r} + \frac{1}{s}\right) - \frac{\gamma}{n}\right\} - \beta^2$$
$$= \tfrac{1}{4}\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right),$$

r_1, r_2 being the distances of the focal lines (of geometrical optics) from the point of incidence.

Thus now the principal parts of the reflected wave are given by

$$(-A,\ -B,\ +C)\, e^{-\iota\kappa r} \Big/ \left\{\left(1 - \frac{r}{r_1}\right)\left(1 - \frac{r}{r_2}\right)\right\}^{1/2};$$

assuming that r is not close either to r_1, or to r_2.

The results of the foregoing analysis depend on the tacit assumption that n is not zero; and as a consequence the character of the approximations will change when n

is small. That is, near the edge of the shadow, in the ordinary phrase of geometrical optics.

In the application to the sphere, the region excluded by this condition corresponds to values of θ nearly equal to π; and in this region the specification of the scattered wave by means of Cartesian co-ordinates seems simplest.

We consider then the incident wave as specified in Cartesian form by

$$\text{(7·6)} \qquad X = -e^{\iota\kappa z}, \qquad Y = 0, \qquad Z = 0,$$

and consider the approximation to the reflected wave incident at the point (al, am, an) on the sphere for which the direction-cosines of the normal are l, m, n. The expressions will be of the form

$$\text{(7·61)} \qquad (X', Y', Z') = (A', B', C')e^{\iota\kappa\zeta}$$

where (treating the tangent-plane as the reflecting surface)

$$\zeta = z+p\,(a-lx-my-nz)$$

and p is determined by

$$(-lp)^2+(-mp)^2+(1-np)^2 = 1.$$

Thus

$$p = 2n.$$

Further, the resultant of (X, Y, Z) and (X′, Y′, Z′) at the point of incidence must be along the normal; and so

$$\frac{A'-1}{l} = \frac{B'}{m} = \frac{C'}{n}.$$

Also (A′, B′, C′) is perpendicular to the reflected ray; and so

$$A'(-2nl)+B'(-2nm)+C'(1-2n^2) = 0.$$

Hence

$$\text{(7·62)} \quad \left\{ \begin{aligned} &\frac{A'-1}{l} = \frac{B'}{m} = \frac{C'}{n} = \frac{2nl}{n-2n} = -2l, \\ &\text{or} \\ &A' = 1-2l^2, \quad B' = -2lm, \quad C' = -2ln. \end{aligned} \right.$$

The components $X+X'$, $Y+Y'$, $Z+Z'$ at the point of incidence are accordingly equal to

$$\text{(7·63)} \qquad -2\,(l^2, lm, ln)\,e^{\iota\kappa an}.$$

We have still to evaluate the normal differential coefficients, which are found to be

$$\text{(7·64)} \qquad \iota\kappa n\,(-1-A', -B', -C')\,e^{\iota\kappa an} = 2\iota\kappa n\,(l^2-1, lm, ln)\,e^{\iota\kappa an}.$$

The value of R is now seen to be given by

$$R^2 = (x-al)^2+(y-am)^2+(z-an)^2 = r^2-2a\,(lx+my+nz)+a^2.$$

Thus, when we regard a/r as small, we may take

$$(7.65)\quad \begin{cases} \mathrm{R} = r - a\,(l \sin\theta \cos\phi + m \sin\theta \sin\phi + n \cos\theta), \\ \text{and} \\ \dfrac{\partial \mathrm{R}}{\partial \nu} = -(l \sin\theta \cos\phi + m \sin\theta \sin\phi + n \cos\theta). \end{cases}$$

Now, in applying (7·13), $\dfrac{\partial \mathrm{R}}{\partial \nu}$ occurs only in the coefficients and not in the exponential index: thus we can get the first approximation by putting $\theta = \pi$ in $\dfrac{\partial \mathrm{R}}{\partial \nu}$; this gives the value

$$(7.66)\qquad \frac{\partial \mathrm{R}}{\partial \nu} = n.$$

Thus, to our degree of accuracy

$$\frac{1}{4\pi}\left(w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu}\right) = -\frac{v}{4\pi}\left(\iota\kappa w\frac{\partial \mathrm{R}}{\partial \nu} + \frac{\partial w}{\partial \nu}\right),$$

$$= -\frac{v}{4\pi}\left(\iota\kappa n w + \frac{\partial w}{\partial \nu}\right).$$

We can now substitute for w and $\dfrac{\partial w}{\partial \nu}$ the values given by (7·63) and (7·64): it will be seen that the components parallel to y, z give zero (to this order), and that the component parallel to x gives

$$\frac{\iota\kappa n}{2\pi}\,\frac{e^{\iota\kappa(an-\mathrm{R})}}{r}.$$

Accordingly the reflected wave is given by

$$(7.67)\qquad \mathrm{X} = \frac{\iota\kappa}{2\pi r}\int n\,d\mathrm{S}\,e^{\iota\kappa(an-\mathrm{R})},$$

where R is found from (7·65) and the integral extends over the positive hemisphere. For the purpose of integration we write

$$l = \sin\theta' \cos\phi', \qquad m = \sin\theta' \sin\phi', \qquad n = \cos\theta'.$$

Then in (7·67) we have

$$an - \mathrm{R} = -r + a\,\{(1+\cos\theta)\cos\theta' + \sin\theta \sin\theta' \cos(\phi-\phi')\}.$$

The integration of (7·67) with respect to ϕ' can be carried out at once, because

$$(7.68)\qquad \int_0^{2\pi} e^{\iota\kappa a \sin\theta \sin\theta' \cos(\phi-\phi')}\,d\phi' = 2\pi \mathrm{J}_0(\kappa a \sin\theta \sin\theta').$$

Our result accordingly becomes

$$(7\cdot69) \qquad X = \frac{\iota\kappa a^2}{r} e^{-\iota\kappa r} \int_0^{\frac{1}{2}\pi} e^{\iota\kappa a(1+\cos\theta)\cos\theta'} J_0(\kappa a \sin\theta \sin\theta') \sin\theta' \cos\theta' \, d\theta'.$$

An exact evaluation of the integral would be troublesome; but it can be transformed by integration by parts. This process leads to a series of which the first term is

$$(7\cdot7) \qquad X = \frac{\iota\kappa a^2}{r} e^{-\iota\kappa r} \frac{J_1(\kappa a \sin\theta)}{\kappa a \sin\theta}.$$

The formula (7·7) will represent (7·69) sufficiently accurately if $(1+\cos\theta)/\sin\theta = \cot\frac{1}{2}\theta$ is regarded as small; and this is the correct assumption here, since θ is supposed to be nearly equal to π.

In the special cases $\kappa a = 9$, 10 it appears that the approximation (7·7) represents the scattered wave sufficiently well within a cone extending to about 10° from the axis.

VII. *On the Fundamental Formulations of Electrodynamics.*

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1. MODERN electrical theory based on MAXWELL'S concept of an æthereal displacement current, is generally regarded as being sufficiently complete in itself to cover all actions so far revealed to us, if we exclude those intra-atomic phenomena which probably involve some additional but not necessarily inconsistent action in their working. There, still, however exists a good deal of uncertainty as to the actual results of the development of this theory in certain directions, and no account has yet been taken of the great degree of latitude allowed by it in its simplest and most general form. For example, in most presentations of the theory of energy streaming in the electromagnetic field the discussion is given in a way which might lead one to believe that POYNTING'S form* of the theory is the only one conceivable. A single alternative has on one occasion† been suggested, but rather as an improvement on POYNTING'S form than as an indication of its uncertainty. Whilst it cannot be denied that POYNTING'S theory is probably the most appropriate one yet formulated, yet it must be recognised that there are an infinite number of fundamentally different forms each of which is itself perfectly consistent with MAXWELL'S theory as expressed in his differential equations of electromagnetic interaction.

Again, but now we are on a different plane, it has usually been stated that MAXWELL'S theory is not of sufficient generality to cover the cases where there exists the complication of non-linear induction in ferromagnetic media.‡ This view appears to have originated with the idea that the *magnetic force* is the fundamental æthereal vector of the magnetic field, whereas, as a matter of fact, the only consistent view§ of the energy relations of such a field leads to the conclusion that the *magnetic induction* is the true æthereal vector, the magnetic force being an auxiliary vector

* 'Phil. Trans.,' A, vol. 175 (1884).

† MACDONALD, 'Electric Waves,' Chs. IV., V., VIII.

‡ This is the view of practically all Continental writers on this subject.

§ *Cf.* 'Roy. Soc. Proc.,' A, vol. 93, p. 20 (1916).

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derived in the process of averaging the minute current whirls into their effective representation as a distribution of magnetic polarity.*

Further, the expression for the forcive of electromagnetic origin acting on the elements of a polarised medium still seems to be the subject of some doubt. MAXWELL† derives an expression in the magnetic case by statical considerations based on the method of energy, and then seems to regard it as generally valid under all circumstances. Objection has however been taken to MAXWELL'S expression by certain writers who, basing themselves on the presumed analogy between the dielectric and magnetic cases, prefer a form of expression differing from MAXWELL'S by a quantity which vanishes in the statical case considered but which is of fundamental importance in the derived problem of reducing the general forcive of electrodynamic origin to a representation by means of an imposed stress system. It appears in fact that the presence of this extra part in MAXWELL'S expression is effective in securing the ordinary expression for the subsidiary term arising in the induction, which has given rise to the conception of electromagnetic momentum, on account of its being a perfect time differential. In the alternative form of the theory the perfect time differential is not secured so that the idea of electromagnetic momentum is lost.‡ In his edition of MAXWELL'S treatise, J. J. THOMSON adds a note attempting to justify MAXWELL'S form of the expression, but his discussion can easily be shown to be erroneous, for he fails to distinguish between the true and complete currents of the theory, the latter containing a constituent, viz., the rate of change of æthereal displacement, which is not affected by the magnetic part of the complete electromagnetic forcive; nevertheless, the later discussions of the question from the point of view of the theory of electrons have confirmed MAXWELL'S original expression for the magnetic forcive, but they apparently still give the alternative expression for the dielectric case.

It was with the view to clearing up these and certain other difficulties that the present discussion was undertaken, the object aimed at being the formulation of a complete and precise statement of the theory in the only form in which it is logically consistent, then to compare this form with current statements of the theory,§ and finally to exhibit in their true aspects the various derived theories which are included in the general scheme. The original differential theory will be linked up with the subsequent dynamical theories by a discussion in its most general form of the derivation

* *Cf.* LARMOR, 'Roy. Soc. Proc.,' vol. 71 (1903), "On the Mechanical and Thermal Relations of the Energy of Magnetisation."

† 'Treatise,' II., Ch. II.

‡ *Cf.* LEATHEM, 'Roy. Soc. Proc.,' A, vol. 89 (1913), p. 34. In this note Mr. LEATHEM attempts to avoid the discrepancy by adding a new term to the force in the elementary polar theory. His only argument in favour of this force is that it overcomes the difficulty, so that it is not very convincing.

§ A complete statement of the fundamental results of the theory so far as they existed up till 1916 is given in my treatise, 'The Theory of Electricity' (Cambridge, 1918). The present paper may be regarded in some measure as a correction and generalisation of the statement there given.

of the fundamental equations based on the principle of Least Action, in the course of which certain inconclusive aspects of this derivation present themselves for consideration.

2. A complete statement of MAXWELL's theory as originally given and in the form which will include most of the recent extensions depends on certain field vectors which first require consistent and independent definition. These vectors:—

(*a*) E, the *electric force*, defined at any point of the field as the vectorial ratio to a small electric charge of the force acting on it when placed at rest in that position. When the point under consideration is inside the matter in the field there exists the possibility of an additional contribution to this force due to local conditions of the matter in the neighbourhood of the point, but for the present we shall disregard any complication of this kind.

(*b*) C, the complete electric current; in the most general case this consists of several distinct parts. Firstly, there is the differential drift of free ions constituting the true conduction current and the material dielectric displacement current; then there is a part due to the convection of charged and electrically polarised media, and finally the æthereal constituent essential and peculiar to MAXWELL's theory.

It has been definitely established that all but the last constituent of the current are in themselves true movements of electricity, or at least effectively equivalent to such, so that so long as we retain the definite concept of an electrical entity the origin of these different constituents merely remains a matter of kinematics, and they have, in fact, been fully dealt with on this basis.*

(*c*) H, the magnetic force, defined in a similar manner to the electric force, with the aid of the concept of a magnetic pole, but now *without* the possibility of a local contribution due to surrounding magnetic polarity if the point is inside the matter.

It is perhaps as well to emphasise the fact that both the electric and magnetic forces are defined in a theoretical manner which almost excludes the possibility of direct experimental verification. Electromagnetic measurements, particularly as regards the fields inside the matter, are concerned almost entirely with matter in bulk, and it is then only the mechanical or molar parts of these forces that are then under observation, the local parts, if they exist at all, being balanced on the spot by other forces of an origin not at present under review. We have however evidence that these local parts of the forces do exist.

(*d*) B, the magnetic induction, which is defined in the elementary theory in terms of the magnetic force H and the magnetic polarisation intensity I by the relation

$$B = H + 4\pi I,$$

and which is always assumed to be subject to the relation

$$\operatorname{div} B = 0.$$

* *Cf.* my 'Theory of Electricity,' p. 363.

In the dynamical theory, however, this vector B appears as the complete æthereal magnetic force,* the part H being merely the mechanically effective part of this force in the material media. This new conception of the magnetic force, which is supported by such phenomena as the Hall effect, &c., where the deviations in ferro-magnetic media are proportional to the induction,† is not really inconsistent with the previous definition given under (*c*) above. As there explained, although the vector H is defined as the force on unit magnetic pole, there is still the possibility that there may exist in addition a force (4πI) not directly detectable in a mechanical experiment.

(*e*) In addition to these fundamental field vectors there are certain others of a subsidiary nature determining the electric and magnetic conditions in the media. One of these, the magnetic polarisation intensity I, has already been introduced, and there is an electric analogue in P, the dielectric polarisation intensity; these two vectors define respectively the effective resultant magnetic and electric bi-polar moments per unit volume of the media.

3. MAXWELL'S theory may now be expressed in the statement that in the most general case of interaction between the magnetic and electric fields the four fundamental vectors defining these fields are subject to the two differential vector equations

$$\text{Curl H} = \frac{4\pi \text{C}}{c},$$

$$\text{Curl E} = -\frac{1}{c}\frac{d\text{B}}{dt},$$

together with the scalar relation defining the electric charge density ρ

$$\text{div E} = 4\pi\rho.$$

The second vectorial equation practically implies that

$$\text{div B} = 0,$$

so that it is hardly necessary to add this as an additional equation.

For the further development of the theory all that is now necessary‡ is the kinematical specification of the current C in terms of the material vectors and an

* This was first clearly recognised by LARMOR. *Cf.* his papers "On a Dynamical Theory of the Electric and Luminiferous Medium," 'Phil. Trans.,' 1894–1897, or 'Æther and Matter' (Cambridge, 1900).

† The results are interpreted usually as implying that the deviations are proportional to the polarisation intensity in such cases, but this is equivalent to the statement given.

‡ We are not here concerned with the constitutional equations giving the laws of induction and conduction. These form a separate branch of the subject, whose results are irrelevant to the present discussion.

expression for the rate of change of the magnetic force vector at any point of the field. The first of these has the usual form

$$C = \frac{dD}{dt} + \rho\nu_m + C_1,$$

wherein C_1 is the density of the true conduction current, ν_m is the velocity of the material medium at the typical field point, and

$$D = \frac{1}{4\pi} E + P$$

is the total dielectric displacement of MAXWELL'S theory which consists in part of the dielectric polarisation P and in part of an æthereal constituent proportional to the electric force.

The time rate of change of the composite vector D requires careful specification; it consists in the main of the terms

$$\frac{1}{4\pi}\frac{dE}{dt} + \frac{dP}{dt},$$

but when the dielectric media are in motion there is in addition a term arising on account of the convection of the polarisation. This term has been shown* to be equal to

$$\text{Curl}\,[\dot{P}\nu_m],$$

so that

$$\frac{dD}{dt} = \frac{1}{4\pi}\frac{dE}{dt} + \frac{dP}{dt} + \text{Curl}\,[P\nu_m],$$

The equation expressing the rate of change of the magnetic force is analogously

$$\frac{dH}{dt} = \frac{dB}{dt} - 4\pi\frac{dI}{dt} - 4\pi\,\text{Curl}\,[I\nu_m].$$

This latter equation must be specially emphasised as it has apparently never yet been explicitly introduced in the theory, although it is necessary to secure greater consistency in the dynamical theory. The expression $\frac{dI}{dt}$ represents the rate of change of the magnetic polarisation at a fixed point in the field only when the magnetic media as a whole are at rest. When these media are in motion there will be a contribution to this rate due to convection just as in the electric case, and the argument for its exact form may be developed on the same lines. The vectors B and H are, so to speak, attached to the æther, just as were the vectors D and E,† whilst

* *Cf.* my 'Theory of Electricity,' pp. 365–367, or LARMOR, 'Æther and Matter,' Chap. IV.

† The vector H being the composite vector of the magnetic theory is analogous to the vector D of the electric theory; the æthereal vector B is analogous to the æthereal vector E. This is the reverse of the usual convention, but see below § 10.

the vector I, as the vector P, is attached to the matter and moves with it. The last two equations contain, therefore, an explicit expression of the effect of the motion so that they are in a sense more convenient than the equations

$$D = \frac{1}{4\pi}E+P, \qquad H = B-4\pi I,$$

defining merely the values of D and H at any point: they are, of course, ultimately consistent with these equations for, taking the second one as an example, we have

$$\frac{d}{dt}\operatorname{div} H = \frac{d}{dt}\operatorname{div} B-4\pi\frac{d}{dt}\operatorname{div}-4\pi\operatorname{div}\operatorname{Curl}[I\nu_m],$$

or

$$\frac{d}{dt}\operatorname{div}(H-B+4\pi I) = 0.$$

With the possible exception of the equation defining dH/dt it is now generally agreed that the scheme here presented provides a completely effective specification of the kinematical connexions in the electromagnetic field.

To obtain some idea of the effect of these connexions on the dynamical processes operative in the field a further assumption is necessary, and this may take one of several forms which will be reviewed in the sequel. For the present we are concerned merely with these equations as effective representatives of the electromagnetic processes. They are sometimes given another form, by the introduction of a scalar potential ϕ and a vector potential A, these being such that

$$B = \operatorname{Curl} A, \qquad E = -\frac{1}{c}\frac{dA}{dt}-\operatorname{grad}\phi,$$

with the other two equations

$$\operatorname{Curl} H = \frac{4\pi}{c}C, \qquad \operatorname{div} E = 4\pi\rho.$$

The first two of these equations are equivalent to the remaining fundamental equation of MAXWELL'S theory which they replace, but they suffer from the serious disadvantage that the quantities A and ϕ specified in them are not completely defined by the equations as given and require additional data to fix them.

4. We have just noticed that an additional assumption of a dynamical character is necessary to render the Maxwellian electromagnetic scheme completely effective as an electrodynamic theory. The simplest and most direct form of this assumption may be taken to be that expressing the force of electrodynamic origin acting on an arbitrarily moving element of charge, this force being, per unit charge equal to

$$F = E+\frac{1}{c}[\nu B]$$

ν being the velocity of the charge and square brackets being employed as throughout to denote the vector product. This expression, first given in its complete form by LORENTZ and LARMOR, is generally regarded as being exact.*

But there are other forms of the additional assumption which are equally effective and in some respects more general than that just given. We may, for instance, assume definite expressions for the potential and kinetic energies of electromagnetic origin in the system and combine these with the assumption that the dynamical processes operative in the field are governed by the same general laws as are the processes in a similar mechanical system. This form of the argument proves ultimately to be consistent with the first as regards the expression for the effective mechanical force on a moving element of charge, but it has the advantage of being expressed in more general terms, thus carrying with it the possibility of fitting better with any modification that it may subsequently be thought desirable to make in our general conception of the theory. It need not be presumed that this form of the argument is any less general than the first on account of the fact that it apparently involves more than one assumption, for this increase in number is counterbalanced by the fact that the dynamical argument ultimately reduces the two fundamental equations to one, FARADAY'S relation being derived simultaneously as a consequence of AMPÈRE'S.

The general dynamical argument was first formulated by MAXWELL† for the case of the field surrounding a system of linear currents. His analysis was subsequently extended to cover the more general case, firstly by VON HELMHOLTZ‡ and LORENTZ,§ later by LARMOR,|| MACDONALD,¶ ABRAHAM,** and others. In the later investigations the whole subject is regarded from the point of view of the theory of electrons, wherein every manifestation of the field is regarded as arising in the aggregate disposition or motion of electronic charges; even in the former investigations, although they are apparently of a more general character, certain assumptions are involved which render their analyses theoretically effective only under the same restricted circumstances. In each of these cases it is assumed, practically speaking, that the potential energy of the electromagnetic field is represented by the expression

$$W = \frac{1}{8\pi}\int E^2\,dv$$

and the kinetic energy by

$$T = \frac{1}{2c}\int (AC)\,dv,$$

* Relatively modifications are not here under consideration.

† 'Treatise,' vol. 2, Ch. VI., VII.

‡ 'Ann. Phys. Chem.,' vol. 47 (1892), p. 1.

§ 'La Théorie Électromagnétique de MAXWELL' (Leiden, 1892), §§ 55–61.

|| 'Æther and Matter,' § 50.

¶ 'Electric Waves,' Appendix I.

** 'Ann. Phys.,' vol. 10 (1903), p. 105.

both volume integrals being taken throughout the entire field: the derivation of the dynamical and field equations is then accomplished by an application of one or the other of the well-known processes of analytical dynamics. The interpretation of the same results for the case when the kinetic energy is given by the usual expression

$$T = \frac{1}{8\pi} \int B^2 \, dv$$

was given by the present author.*

Whilst the theoretical simplicity of these discussions, which results from their interpretation in terms of the simple electronic hypothesis, is a great point in their favour, it seemed, nevertheless, of theoretical interest at least to attempt to formulate the problem under less restricted conditions, especially in view of the pronounced tendency exhibited in some quarters to deny the adequacy of the Maxwellian theory as a complete microscopic theory. Besides the more general discussion in the form in which it is here presented emphasises certain difficulties inherent in the usual formulations which have not hitherto received adequate attention.

5. The most general dynamical principle which determines the motion of every material system is the Law of Least Action expressible in the usual form

$$\delta \int_{t_1}^{t_2} L \, dt \equiv \delta \int_{t_1}^{t_2} (T - W) \, dt = 0$$

wherein T denotes the kinetic energy and W the potential energy of the system in any configuration and formulated in terms of any co-ordinates that are sufficient to specify the configuration in accordance with its known properties and connexions, and where the variation refers to a fixed time of passage of the system from the initial to the final configuration. This is the ordinary form of HAMILTON'S principle, but it involves in any case a complete knowledge of the constitution of the systems, because, before it can be applied it is necessary to know the exact values of the kinetic and potential energies expressed properly in terms of the co-ordinates and velocities. As however we have frequently to deal with systems whose ultimate constitution is either wholly or partly unknown it is necessary to establish a modified form of the principle allowing for a possible ignorance of the constitution of the systems with which we may have to deal. The modification is fully discussed in most works on analytical dynamics,† and we may here content ourselves by merely presenting the results, interpreting them however in a manner somewhat different from that usually given, in order to throw some light on certain questions which arise in the subsequent application in our present theory. Suppose then that it has been found impracticable to express the Lagrangian function L in terms of the chosen co-ordinates of the system, the typical one of which we may denote by q; but that it is expressed in

* 'Phil. Mag.,' vol. 32 (1916), p. 195.

† *E.g.*, 'Treatise on Analytical Dynamics' (2nd ed., Cambridge, 1918), by E. T. WHITTAKER.

terms of a certain number of variables $x_1, x_2, \ldots x_k$, which are known to be connected with the co-ordinates q and their velocities $\dot{q}$ by a series of relations of the type

$$M_s = 0$$

M_s being a function of the co-ordinates q, the velocities $\dot{q}$, the variables x and the differential coefficients of these latter variables with respect to the time. For the sake of simplicity we shall restrict our statement to the case when the first differentials only appear. The usual method of procedure is to introduce a set of multipliers λ_s, functions of the time, and then to consider the variations of the integral

$$\int_{t_1}^{t_2} (L + \Sigma\lambda_s M_s)\, dt$$

where the q's and x's undergo independent variations. The equations obtained for the vanishing of the variation are of two types. Firstly, there is an equation of the type

$$\frac{d}{dt}\left(\Sigma_s \lambda_s \frac{\partial M_s}{\partial \dot{x}}\right) - \Sigma\lambda_s \frac{\partial M_s}{\partial x_s} - \frac{\partial L}{\partial x} = 0$$

for each variable x: these with the restricting equations will determine the x's and λ's as functions of the co-ordinates q and the time. Then there is an equation of the type

$$\frac{d}{dt}\left(\Sigma_s \lambda_s \frac{\partial M_s}{\partial \dot{q}}\right) - \Sigma\lambda_s \frac{\partial M_s}{\partial q} = 0$$

for the motion in each q co-ordinate.

The latter equations only involve the Lagrangian function L through the quantities λ and x which enter into it, and once these are determined the rest of the solution involves only the restricting conditions. In fact when once these multipliers and variables are determined and regarded as functions of the time only the motion in the q co-ordinates is completely determined by the condition that the integral

$$\int \Sigma\lambda_s M_s\, dt$$

is stationary for independent variations of the co-ordinates q. It may even happen that the relations M involve the co-ordinates q and the variables x in such a way that it is possible to separate M into two terms, one of which is a function explicitly of the q's only and the other of the x's only. In this case the part of the integral required in the above statement is only that part of it involving the q's and this is independent entirely of the co-ordinates x.

This remark has an important bearing on a question which occurs in the sequel, and it shows that the existence of a variational form for the equations of motion does not necessarily imply that the integrand involved is a true Lagrangian function for the system.

6. Now let us apply these principles to our electromagnetic problem. The conditions in the field surrounding a number of bodies are specified in the usual way by the magnetic induction vector B and the electric force vector E, and the part of the Lagrangian function associated with this field may be taken to be

$$\frac{1}{8\pi}\int (B^2-E^2)\,dv,$$

the first term denoting the magnetic or kinetic energy and the second the electric or potential, and the integral is extended over the whole of space. In addition to these energies there will be the energies of the material bodies in the field which will consist in part of the kinetic energies of their organised motions, in part of their potential energy relative to one another or to any extraneous fields of non-electric nature, and in part finally of internal energy of elastic or motional type in the media. The part of the Lagrangian function corresponding to these energies can, in the most general case, be denoted by

$$\int (L_0-W_i)\,dv$$

where L_0 is the Lagrangian function of the organised motions of the media, reckoned per unit volume at each place and assumed to be a function only of the position co-ordinates and velocities, and W_i is the internal energy of all types reckoned as potential energy per unit volume: this latter term will be a function of the electric and magnetic polarisations in the media, but will be assumed not to depend to any appreciable extent on the rates of variation of these conditions, and in so far as some of the internal energy is essentially of kinetic type, it will be in reality a sort of modified Lagrangian function with the energy corresponding to the motional terms converted to potential energy in the usual way. The function L_0 may also be taken to include a part arising from the assumed inertia of any free electrons that may be present.

The motion of the system can now be expressed in the form

$$\delta\int_{t_1}^{t_2} dt \int \left[L_0-W_i+\frac{1}{8\pi}(B^2-E^2)\right] dv$$

and we could conduct the variation directly were it not for the fact that our functions are not all expressed explicitly in terms of the independent co-ordinates of the systems, which are in reality the position co-ordinates of the elements of matter and electricity. As indicated above we can however avoid the use of any such explicit interpretation by the use of undetermined multipliers. In this way the variations of E and B can be temporarily rendered independent of each other and of the actual co-ordinates of the material and electrical elements.

We know that the vectors of the theory are connected with one another and the actual co-ordinates of the system by the equations

$$\operatorname{div} \mathrm{E} = 4\pi\Sigma e - 4\pi \operatorname{div} \mathrm{P}$$

$$\operatorname{Curl} \mathrm{H} = \frac{1}{c}\frac{d\mathrm{E}}{dt} + \frac{4\pi}{c}\frac{d\mathrm{P}}{dt} + \frac{4\pi}{c}\operatorname{Curl}[\mathrm{P}\dot{r}_m] + \frac{4\pi}{c}\Sigma e(\dot{r}_m + \dot{r}_e)$$

$$\frac{d\mathrm{B}}{dt} = \frac{d\mathrm{H}}{dt} + 4\pi\frac{d\mathrm{I}}{dt} + 4\pi\operatorname{Curl}[\mathrm{I}\dot{r}_m].$$

In these equations P is the dielectric and I the magnetic polarisation intensity; $\dot{r}_m$ is the velocity and r_m the position vector of the element of matter and $\dot{r}_e$ and r_e the velocity and position vectors relative to this element of the typical element of free charge (e) over which the sum Σ in the first and second equations is taken *per unit volume* at each place.

In these equations we have purposely refrained from assuming a definite electronic constitution for the dielectric and magnetic polarisations as it was desired to emphasise certain points in connexion with the mechanical forcive which have not yet been adequately dealt with.

We have thus to introduce three undetermined multipliers one scalar ϕ and two vectors A_1, A_2 and it is then the variation of

$$\begin{aligned}\int_{t_1}^{t_2} dt \int dv \Big[&\mathrm{L}_0 - \mathrm{W}_i + \frac{1}{8\pi}(\mathrm{B}^2 - \mathrm{E}^2) + \frac{\phi}{4\pi}(\operatorname{div}\mathrm{E} + 4\pi\operatorname{div}\mathrm{P}) \\ &- \frac{1}{4\pi}\left(\mathrm{A}_1, \operatorname{Curl}\mathrm{H} - \frac{1}{c}\frac{d\mathrm{E}}{dt} - \frac{4\pi}{c}\frac{d\mathrm{P}}{dt} - \frac{4\pi}{c}\operatorname{Curl}[\mathrm{P}\dot{r}_m]\right) \\ &+ \frac{1}{4\pi c}\left(\mathrm{A}_2, \frac{d\mathrm{B}}{dt} - \frac{d\mathrm{H}}{dt} - 4\pi\frac{d\mathrm{I}}{dt} - 4\pi\operatorname{Curl}[\mathrm{I}\dot{r}_m]\right) \\ &- \Sigma e\phi + \frac{1}{c}\Sigma e(\mathrm{A}_1, \dot{r}_m + \dot{r}_e)\Big]\end{aligned}$$

that is to be made null, afterwards determining the forms of the various undetermined functions to satisfy the restrictions which necessitated their introduction. In conducting the variation we can now treat the electric force, displacement and polarisation, the magnetic force induction and polarisation and the position co-ordinates of the electrical and material elements as all independent. We here see the reason for introducing the equation expressing the rate of change of H instead of the equation

$$\mathrm{H} = \mathrm{B} - 4\pi\mathrm{I}$$

determining its value, for this latter equation does not in reality enable us to obtain a relation between the variations of H and the position co-ordinates of the matter, so we could not treat all our variables as independent.

In the main the details of the variational calculation possess no novel features and need not here be elaborated. There are, however, one or two terms which require careful handling, especially when finding the variations due to the alteration of position of the matter.

The variations of terms of the type

$$\int \phi \operatorname{div} \mathrm{P} dv, \quad \frac{1}{c}\int \left(\mathrm{A}_1 \frac{d\mathrm{P}}{dt}\right) dv, \quad \frac{1}{c}\int (\mathrm{A}_1 \operatorname{Curl} [\mathrm{P}\dot{r}_m])\, dv,$$

due to variations in the position co-ordinates of the elements of matter to which the vectors P and r_m are attached should not be performed until the differential operators affecting the P function are eliminated by an integration by parts. If we bear this in mind we shall find that the final result for the variation consists of terms at the time and space limits, which require separate adjustment, together with

$$\begin{aligned}
\int_{t_1}^{t_2} dt \int dv \Bigg[& \delta \mathrm{L}_0 + \frac{1}{4\pi}\left(\delta \mathrm{B},\ \mathrm{B} - \frac{1}{c}\frac{d\mathrm{A}_2}{dt}\right) - \frac{1}{4\pi}\left(\delta \mathrm{E},\ \mathrm{E} + \nabla\phi + \frac{1}{c}\frac{d\mathrm{A}_1}{dt}\right) \\
& - \frac{1}{4\pi}\left(\delta \mathrm{H},\ \operatorname{Curl} \mathrm{A}_1 - \frac{1}{c}\frac{d\mathrm{A}_2}{dt}\right) - \left(\mathrm{P},\ (\delta r_m \nabla)\left(\nabla\phi + \frac{1}{c}\frac{d\mathrm{A}_1}{dt}\right)\right) \\
& - \frac{1}{c}\left(\delta r_m,\ \left[\operatorname{Curl} \mathrm{A}_1,\ \frac{d\mathrm{P}}{dt}\right]\right) - \frac{1}{c}\left(\delta r_m \left[\operatorname{Curl} \frac{d\mathrm{A}_1}{dt},\ \mathrm{P}\right]\right) \\
& - \frac{1}{c}(\delta r_m \nabla)(\operatorname{Curl} \mathrm{A}_1,\ [\mathrm{P}\dot{r}_m]) + \frac{1}{c}\left(\mathrm{I},\ (\delta r_m \nabla)\frac{d\mathrm{A}_2}{dt}\right) \\
& + \frac{1}{c}\left(\delta r_m,\ \left[\operatorname{Curl} \mathrm{A}_2,\ \frac{d\mathrm{I}}{dt}\right]\right) + \frac{1}{c}\left(\delta r_m \left[\operatorname{Curl} \frac{d\mathrm{A}_2}{dt},\ \mathrm{I}\right]\right) \\
& + \frac{1}{c}(\delta r_m \nabla)(\operatorname{Curl} \mathrm{A}_2,\ [\mathrm{I}\dot{r}_m]) \\
& + \Sigma e\left(\delta r_m + \delta r_e,\ -\nabla\phi - \frac{1}{c}\frac{d\mathrm{A}_1}{dt} + \frac{1}{c}[\dot{r}_m + \dot{r}_e,\ \operatorname{Curl} \mathrm{A}_1]\right) \\
+ \delta \mathrm{W}_i + & \left(-\nabla\phi - \frac{1}{c}\frac{d\mathrm{A}_1}{dt} + \frac{1}{c}[\dot{r}_m,\ \operatorname{Curl} \mathrm{A}_1],\ \delta \mathrm{P}\right) \\
& \qquad + \left(\frac{1}{c}\frac{d\mathrm{A}_2}{dt} - \frac{1}{c}[\dot{r}_m,\ \operatorname{Curl} \mathrm{A}_2],\ \delta \mathrm{I}\right)\Bigg],
\end{aligned}$$

where in the terms $(\delta r_m \nabla)(\operatorname{Curl} \mathrm{A}_1, [\mathrm{P}\dot{r}_m])$ and $(\delta r_m \nabla)(\operatorname{Curl} \mathrm{A}_2, [\mathrm{I}\dot{r}_m])$ the vector operator ∇ $\left(\text{whose components are } \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is presumed to affect only the functions A_1 and A_2.

The variations δr_m, δr_e which determine the virtual displacements of the electrical and material elements and the variations δE, δB, δP, ..., can now be considered as all independent and perfectly arbitrary, and hence the coefficient of each must vanish separately in the dynamical variational equation. The coefficients of the latter variations lead to the equations

$$B = \frac{1}{c}\frac{dA_2}{dt}, \qquad E = -\nabla\phi - \frac{1}{c}\frac{dA_1}{dt},$$

$$\text{Curl } A_1 = \frac{1}{c}\frac{dA_2}{dt},$$

from which it follows that the multipliers ϕ and A_1 are the ordinary scalar and vector potentials of the theory so that further

$$\text{Curl } E = -\frac{1}{c}\frac{d}{dt}\text{ Curl } A_1.$$

As regards the vector A_2, this is a new vector potential whose *curl* is required in our subsequent discussions. For this we have

$$\text{Curl } B = \text{Curl } H + 4\pi \text{ Curl } I = \frac{1}{c}\frac{d}{dt}\text{ Curl } A_2,$$

whilst if we use C_e as the total current of true electric flux we have, by AMPÈRE's equation

$$\text{Curl } H = \frac{1}{c}\frac{dE}{dt} + \frac{4\pi}{c}C_e.$$

Thus, if we use

$$C'_e = C_e + c \text{ Curl } I,$$

we have

$$\frac{d}{dt}(\text{Curl } A_2) = \frac{dE}{dt} + 4\pi C'_e.$$

The main part of *Curl* A_2 is therefore represented by the electric force : there is however in addition a local term E_0 depending on the time integral of the current density at the point. We can thus write

$$\text{Curl } A_2 = E + E_0.$$

If we use the values thus determined for the various undetermined multipliers introduced at the outset, the remaining terms of the variation give for the motion of the material and electrical elements equations of the type

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{x}_m}\right) - \frac{\partial L}{\partial x_m} = {} & \left(P\frac{\partial E}{\partial x_m}\right) + \left(I\frac{\partial B}{\partial x_m}\right) + \frac{1}{c}\left([P\dot{r}_m], \frac{\partial B}{\partial x_m}\right) \\ & -\frac{1}{c}\left([I\dot{r}_m], \frac{\partial (E+E_0)}{\partial x_m}\right) - [P, \text{ Curl } E]_x \\ & -[I, \text{ Curl } B]_x - \frac{1}{c}\left[B\frac{dP}{dt}\right]_x + \frac{1}{c}\left[E+E_0, \frac{dI}{dt}\right]_x \\ & +\rho\left(E_x + \frac{1}{c}[\dot{r}_m B]_x\right) + \frac{1}{c}[C_1 B]_x,\end{aligned}$$

for the motion of the matter and of type

$$\frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{x}_m}\right)-\frac{\partial L}{\partial x_m}=e\left(E_x+\frac{1}{c}[\nu B]_x\right)$$

for the electrical elements. In the first of these equations $\rho = \Sigma e$ is the density of the free charge, and $C_1 = \Sigma e\dot{r}_e$ is the density of the true conduction current; in the second equation $r = r_m + r_e$ is the absolute position co-ordinate of the electrical element and $\nu = \dot{r}$ is its resultant velocity.

In addition we have the variational equation for the internal energy which can be left as it stands in the form

$$\delta W_i = \left(E+\frac{1}{c}[\dot{r}_m B],\ \delta P\right)+\left(B-\frac{1}{c}[\dot{r}_m,\ E+E_0],\ \delta I\right).$$

Interpreted in terms of the language of ordinary dynamics these equations imply that

$$(P\nabla)\,E_x+(I\nabla)\,B_x+\frac{1}{c}\left([P\dot{r}_m],\ \frac{\partial B}{\partial x}\right)-\frac{1}{c}\left([I\dot{r}_m],\ \frac{\partial (E+E_0)}{\partial x}\right)$$
$$+\frac{1}{c}\left[\frac{dP}{dt}+C_1,\ B\right]_x-\frac{1}{c}\left[\frac{dI}{dt},\ E+E_0\right]_x+\rho\left(E_x+\frac{1}{c}[\dot{r}_m B]_x\right)$$

is the force per unit volume at each place tending to accelerate the motion of the matter, whilst

$$e\left(E+\frac{1}{c}[\nu B]\right)$$

is the force tending to accelerate the motion of the element of electricity e.

The electrical terms in the first of these results are consistent with those obtained by LARMOR* and others, but the magnetic terms, which are in fact analogous with the electrical terms depending on the dielectric polarisation, are fundamentally different from those obtained by these authors. Further, since

$$E=-\frac{1}{c}\frac{dA_1}{dt}-\nabla\phi,$$

it follows that

$$\text{Curl } E=-\frac{1}{c}\frac{d}{dt}\text{Curl } A_1$$
$$=-\frac{1}{c}\frac{dB'}{dt},$$

so that our equations are in complete agreement with the most general form of MAXWELL's theory.

* 'Æther and Matter,' Chap. VI.

7. If we examine the above analyses closely we shall notice a rather important point bearing on a fundamental question which has already been the subject of some discussion.* If we take the integral in its complete form with the variation carried out and with the values of the various multipliers inserted it can be seen to reduce to the expression

$$\int_{t_1}^{t_2} dt \int dv \Big[\delta L_0 - \delta W_i + (P\,\delta E) + (I\,\delta B) - \frac{1}{c}((\delta r_m \nabla),\ B,\ [P\dot{r}_m]) - (\delta r_m,\ [P\ \mathrm{Curl}\ E])$$
$$- \frac{1}{c}((\delta r_m \nabla)(E+E_0),\ [I\dot{r}_m]) - (\delta r_m\,[I\ \mathrm{Curl}\ B])$$
$$+ \left(\frac{\delta r_m}{c},\ \left[\frac{dP}{dt},\ B\right]\right) - \frac{1}{c}\left(\delta r_m,\ \left[\frac{dI}{dt},\ E+E_0\right]\right)$$
$$+ \left(E + \frac{1}{c}[\dot{r}_m B],\ \delta P\right) + \left(B - \frac{1}{c}[\dot{r}_m,\ E+E_0],\ \delta I\right)$$
$$+ \Sigma e \left(\delta r_m + \delta r_e,\ E + \frac{1}{c}[VB]\right)\Big],$$

in which neither the electric nor magnetic energy contributes an explicit term. This is the first definite indication we have that the modified function with which we may operate to find the equations of motion of the electric and material elements is explicitly independent of the expression for the energy in the æthereal field. We may, in fact, see that, just as in the dynamical problem examined above, the whole circumstances of the motion in the real co-ordinates of the system can be derived by the variational principle, using the integral

$$\int_{t_1}^{t_2} dt \int dv \Big[L_0 - W_i - \phi\ \mathrm{div}\ P - \frac{1}{c}\left(A_1\,\frac{dP}{dt}\right) - \frac{1}{c}(A_1\ \mathrm{Curl}\,[P\dot{r}_m])$$
$$- \frac{1}{c}\left(A_2\,\frac{dI}{dt}\right) - \frac{1}{c}(A_2\ \mathrm{Curl}\,[I\dot{r}_m]) + \Sigma e\phi + \Sigma e\,(A_1,\ \dot{r}_m + \dot{r}_e)\Big]$$

just as we used the Hamiltonian integral, taking in it E, A_1, A_2, ϕ as functions of the time and space co-ordinates only. It is of course possible to establish this directly, for it is easily verified that the difference between the integrand just employed and the previous one, viz.,

$$L_0 - W_i + \frac{1}{8\pi}(B^2 - E^2),$$

involves only complete differentials with respect to the time or space co-ordinates. This difference therefore integrates out to the limits and remains ineffective as regards the general dynamical variational equations, and we can therefore use either integrand indiscriminately.

* *Cf.* 'Phil. Mag.,' vol. 32 (1916), p. 195, where references to previous work are given.

This is the general form of the result obtained by SCHWARZSCHILD* that in the special case when we are concerned only with free electrons moving in an æthereal field free from matter their equations of motion can be derived by the variational principle, using the integral

$$\int_{t_1}^{t_2} dt \left[L_0 - \Sigma e\phi + \Sigma \frac{1}{c}(A\dot{r}) \right],$$

where ϕ, A, the ordinary scalar and vector potentials, are regarded as functions of the time and space variables only.

We now see why it is that consistent formulæ have been obtained by different authors using apparently different expressions to represent the field energies. The results are in fact all explicitly independent of any particular interpretations for these energy expressions.

8. The general dynamical formulation of § 6 agrees with the fact that the material media of the field have an internal constitution which enables them to resist the setting up of the electric and magnetic polarisations by forces $E + \frac{1}{c}[\dot{r}_m B]$ and $B - \frac{1}{c}[\dot{r}_m, E + E_0]$ respectively, and that in consequence of any change in the polarised state of these media their intrinsic energy of elastic or motional type is increased by the amount

$$\int dv \int_1^2 (E' \,\delta P) + (B' \,\delta I),$$

where we have used

$$E' = E + \frac{1}{c}[\dot{r}_m B], \qquad B' = B - \frac{1}{c}[\dot{r}_m, \ E + E_0].$$

Thus in setting up the electric field and its associated dielectric polarisations in the medium the potential energy of the field is increased by the amount

$$\frac{1}{8\pi}\int E^2 \, dv$$

on account of the establishment of the æthereal field together with

$$\int dv \int (E' \,\delta P)$$

on account of the material polarisation, both amounts being reckoned as potential energy.

This gives a total for the field equal to

$$\int dv \int \left\{ \frac{1}{4\pi}(E\,\delta E) + (E\,\delta P) + \frac{1}{c}([\dot{r}_m B]\,\delta P) \right\}.$$

* 'Gött. Nachr. (Math.-phys. Kl.),' 1903, p. 125.

This result is consistent with that generally obtained in these theories. The last term arising on account of the current due to the convection of the polarisations is however probably of kinetic origin.

Of course, in the general case, all the potential energy put into the field cannot be got out of it again in the form of useful mechanical work, or, in other words, it is not all available. The effectively available energy in the present case consists in the main of the part

$$\int dv \left(\frac{E^2}{8\pi} + \int (P\,\delta E)\right).$$

For the magnetic polarisations the results are somewhat different. In this case the kinetic energy of the field is assumed to be

$$\frac{1}{8\pi}\int B^2\,dv,$$

to which we must add the intrinsic kinetic energy involved in the induced magnetic polarisations to obtain the total energy in the field; reckoned as potential energy the intrinsic energy is

$$\int dv \int (B'\,dI),$$

and therefore as kinetic energy it is

$$-\int dv \int (B'\,dI),$$

giving a total for the field equal to

$$\frac{1}{8\pi}\int B^2\,dv - \int dv\,(B'\,dI)$$

$$= \frac{1}{4\pi}\int dv \int (B\,dH) + \frac{1}{c}\int dv \int (E+E_0,\ [dI,\ \dot{r}_m]),$$

a result which is again practically equivalent to that usually given in this theory.

If we treat the convection of the dielectric polarisation as effectively equivalent to a magnetic polarisation of intensity

$$I' = \frac{1}{c}[P\dot{r}_m],$$

and the convection of the magnetic polarisation as effectively equivalent to a dielectric polarisation of intensity

$$P' = -\frac{1}{c}[I\dot{r}_m],$$

and include the energies corresponding to these two parts in with the appropriate totals the formulæ just obtained are somewhat simpler in form. The total kinetic energy will be now

$$\frac{1}{8\pi}\int\left[B^2-8\pi\int(B\,dI)-8\pi\int(B\,dI')\right]dv$$

or

$$\int\frac{dv}{4\pi}\int(B\,dH'),$$

where

$$B = H'+4\pi I+\frac{4\pi}{c}[P\dot{r}_m],$$

and the potential energy is

$$\frac{1}{8\pi}\int\left[E^2+8\pi\int(E\,\delta P)+8\pi\int(E\,\delta P')\right]dv,$$

or is

$$\int\int(E\,dD')\,dv,$$

where

$$D' = \frac{E}{4\pi}+P+P'.$$

This, however, leaves out of account the term

$$\int dv\int(E_0\,\delta P'),$$

which, representing as it does the energy of local actions and reactions, may be assumed to be inoperative as regards general mechanical processes, and may in consequence be rejected altogether.

In connexion with the discussions of the expressions for the internal energy of the material media it may be worth while emphasising the significance of the various terms in the expression for the forcive on the polarised material elements derived in the previous paragraph, especially the way in which the various terms arise and their dependence on the term

$$\text{Curl}\,[P\nu_m]$$

in the complete current. Further, we may notice how the complete expression for the change of intrinsic energy per unit volume, viz.,

$$\left(E+\frac{1}{c}[\nu_m\,B],\,\delta P\right)$$

is derived, the latter term being due to the above-mentioned term in the current.

This last remark points to the possibility of obtaining an elementary deduction of the expression

$$E+\frac{1}{c}[\nu_m, B]$$

for the complete electromotive forcive simply by calculating the rate of change of intrinsic energy of a moving bi-pole, and the calculation has in fact been carried out by LARMOR,* taking however a parallel plate condenser with equal and oppositely charged plates, moving in a uniform magnetic field.

An analogous argument in the magnetic case will give a deduction of the magneto-motive forcive

$$B-\frac{1}{c}[\nu_m, E].$$

9. We have stated that the magnetic energy expressions just obtained are effectively equivalent to those usually derived, whereas as a matter of fact this is true only of the final result; the various formulæ employed in the derivation of this result are not in their usual form but it has been shown elsewhere† that they are consistent with the complete dynamical theory, the more usual formulæ and the various modifications of them which have from time to time been suggested being all inadmissible on this score. A complete discussion of the justification for this last statement is necessarily beyond the scope of the present paper, but it may perhaps serve a useful purpose if a brief outline of some of the more important reasons is given, especially as they have some bearing on points raised elsewhere in the present discussion.

In the first place there is probably little or no difficulty in seeing the fallacy in the usual and simplest form of the theory wherein the expression $\mu H^2/8\pi$ for the magnetic energy density is derived in the statical theory as potential energy and in the dynamical theory as kinetic energy: we need only enquire as to the type of energy represented by the same expression when the field is due in part to rigid magnets and in part to steady currents. The more consistent result is obtained by taking

$$\frac{\mu H^2}{8\pi}-\frac{1}{4\pi}(HB)$$

as the expression for the density in the statical case as this agrees with the opposite sign in the dynamical case and yet gives the same total.

There is however another form of the results first tentatively suggested by HERTZ and HEAVISIDE and subsequently developed in great detail by other writers, more particularly by R. GANS‡ and H. WEBER, wherein the difficulty presents itself in

* 'Proc. Lond. Math. Soc.' (1915).

† *Cf.* my 'Theory of Electricity,' p. 417, or 'Roy. Soc. Proc.,' vol. 93, A, p. 20 (1916).

‡ 'Ann. der Physik,' vol. 13 (1904), p. 634, and 'Encyklopädie der Math. Wissensch.,' vol. v., art. 15, where references to other authors will be found.

another and more involved form. The fundamental basis of this theory is the assumption of a distribution of true magnetic matter of density at any place equal to

$$\rho_m = -\text{div } \mathrm{I}_0 = \frac{1}{4\pi} \text{div } \mathrm{B}$$

wherein I_0 is the density of the permanent magnetic polarity. This magnetic matter is supposed to be distributed continuously throughout the space but so that the amount in any portion of the matter is zero, a condition which is perhaps rather difficult of realisation, as it would make the distribution in any particular portion of the matter dependent on the distribution in all the surrounding portions.

In this theory the magnetic energy is first calculated on analogy with the electrostatic energy; the magnetic induction vector B is regarded as a sort of composite displacement produced by the acting force H, so that the energy per unit volume is

$$\frac{1}{4\pi}\int^{\mathrm{B}} (\mathrm{H}\, d\mathrm{B}).$$

This expression is then verified to be equivalent in the purely statical case to the volume integral

$$\int dv \int^{\rho_m} \phi\, d\rho_m$$

taken over the entire field, the surface integral over the infinite boundary contributing nothing in all regular cases; ϕ is the magnetic potential of the field.

In generalising the theory to the case where the field is due to linear currents the same physical basis is adopted as regards the expression

$$\int dv \int^{\mathrm{B}} (\mathrm{H}\, d\mathrm{B}),$$

which still therefore remains valid, and when there are no permanent magnets about this is easily verified by the usual argument to be equivalent to the summation

$$\frac{1}{c}\Sigma \int^{\mathrm{N}} \mathrm{J}\, d\mathrm{N}$$

over the different current elements, J denoting the typical current strength and N the induction through its circuit. When there are permanent magnets present this expression becomes

$$\int dv \int^{\rho_m} \phi\, d\rho_m + \frac{1}{c}\Sigma \int^{\mathrm{N}} \mathrm{J}\, d\mathrm{N}.$$

It is then shown that the mechanical forcive on the magnetic matter in any one of its co-ordinates is derivable as the appropriate *negative* gradient of this energy function, whilst the force on a current is to be obtained as the *positive* gradient with respect to its position co-ordinate.

Unfortunately all the authors concerned merely talk of magnetic energy without specifying whether it is to be taken as kinetic or potential energy. One might perhaps infer that as the results are interpreted in terms of a static potential function it is implied that all the energies are potential, but the fact that the forces on the currents are derivable as the positive gradients of the function

$$\frac{1}{c}\Sigma\int^{\mathrm{N}}\mathrm{J}\,d\mathrm{N}$$

suggests that this part of the energy at least is kinetic energy. The difficulty of sign is therefore still present.

Even if we confine ourselves to the statical theory the same interpretation is not entirely free from difficulties of another kind. The potential energy in the field is taken to be represented by

$$\int dv\int^{\rho_m}\phi\,d\rho_m,$$

but this expression really represents the total energy in the field ; in the general case the only part of this energy which is mechanically available is

$$\int dv\int^{\phi}\rho_m\,d\phi,$$

and this is properly speaking the potential function from which the mechanical forces acting on the magnetism are to be derived. Of course, when the law of induction is linear the intrinsic energy of the field is equal to the available energy, but even then their natures are fundamentally different and equality in their magnitudes is hardly a sufficient justification for confusing the one with the other.

Apart from this difficulty, however, the next step employed in the development of the theory will cause some trouble. To effect the transformation from the expression

$$\int dv\int^{\rho_m}\phi\,d\rho_m$$

to the equivalent expression

$$-\int dv\int^{\mathrm{I}_0}(\mathrm{H}\,d\mathrm{I}_0)$$

the method of integration by parts is employed. But LARMOR has shown that two expressions of this type being derived the one from the other by the method of integration by parts, really represent fundamentally different distributions of the energy in the field, although the total amounts represented by them are the same. The two expressions cannot therefore be used indiscriminately to determine the stresses around an element of the magnetic matter. It is not, of course, possible at the present stage to say which of the two expressions does represent the true energy distribution, but any examination of the mechanical inter-relations of the different magnetic masses in the field postulates a previous decision as to the proper expression to be taken as representing the available energy of these masses in its normal form, respecting both its total amount and proper distribution. Once this decision has been made it is unsafe to employ the method of integration by parts unless due regard is paid to the surface integrals thereby introduced.

The distribution of energy interpreted in terms of ideal magnetic matter which is properly equivalent to the expression

$$-\int dv \int^{I_0} (\mathrm{H}\, d\mathrm{I}_0)$$

is such that the energy in any volume of the field consists of a distribution throughout it of density at each point equal to

$$-\int \phi \operatorname{div} d\mathrm{I}_0 = -\int \phi\, d\rho_m$$

together with a surface distribution of density

$$\int \phi\, d\mathrm{I}_n$$

over its surface. This corresponds properly to POISSON'S distribution of magnetic matter and emphasises the necessity for the inclusion of surface distributions of magnetic matter.

This explains why it is that the above theory determines properly the forces on the permanent magnets as a whole, but fails to give a consistent account of the internal forces between different parts of the same magnet. At the surface of an ordinary magnet it may quite legitimately be assumed that owing to the existence of a transition layer, the normal component of the magnetisation vanishes there, and consequently the surface integrals applied to the outer surfaces of any such body would also vanish; the two different expressions for the contained energy thus become equivalent.

This is perhaps sufficient to justify a summary rejection of this new interpretation of the energy relations of the magnetic field, as being at most no better than the older one which it presumed to displace. The real trouble in both cases seems to have arisen mainly in an effort to discover an analogy in the relations of the electric and magnetic fields. HERTZ and HEAVISIDE were the first to insist on the existence of this analogy, and practically all the modern writers follow them in this matter, even so far as to regard it as providing sufficient justification for certain fundamental formulæ of the theory. A close scrutiny of the subject will, however, reveal the fact that although the mathematical relations connecting the magnetic force induction and polarisation are to a certain extent similar to those connecting the electric force, total electric displacement and dielectric displacement, the similarity ends with these relations, and the dynamical characteristics of the two types of field are essentially different; and the analogy itself, so far as it exists, seems to be based on erroneous and confused conception of the nature of the magnetic energy as determined by the usual expression of the theory, so that it finds no place in a consistent formulation of the subject, notwithstanding even HEAVISIDE's spirited defence in criticism of LARMOR.

In his treatise ABRAHAM adopts the analogy as a sufficient basis for the discussion of the magnetic theory, but decides that the procedure is not without its difficulties, particularly as regards the ferro-magnetic phenomena; not being able to overcome these he condemns the whole procedure as being inadequate to include a proper account of these matters.

10. We now turn from these discussions to a brief review of the general energy relations of the electromagnetic field. A concise account of these relations so far, that is, as they have been dealt with in existing accounts of the subject, has been given with full references in my treatise (Ch. XIV.), and it will suffice for the present to give the barest outlines of the discussion so far as they may be required.

The fundamental equation expressing in its most concise form the energy principle for the electromagnetic field can be written in the form

$$\frac{dW}{dt}+\frac{dT}{dt}+F+\int S_n\,df = 0,$$

wherein W and T are respectively the potential and kinetic energies inside any volume bounded by the closed surface f in the field, F is the rate of dissipation of electromagnetic energy into energy of other types such, for instance, as results mainly from the inertia of the electric charges constituting the conduction and convection currents; and S is the vector determining the flow of electromagnetic energy outwards over the bounding surface.

In this equation it is generally assumed that our knowledge of the forms of F and W is precise and accurate, and that in fact in agreement with the results of paragraph 8.

$$W = \frac{1}{8\pi}\int E^2\,dv+\int dv\int^{P}(E'\,\delta P)$$

where P is the polarisation intensity of the dielectric media produced by the electromotive force

$$E' = E+\frac{1}{c}\,[V_m B]$$

whilst

$$F = \int (EC_1)\,dv$$

where C_1 represents the part of the total current depending on the motion of the electrons constituting the conduction current and the current due to the convection of electric charges, but excluding the part due to the convection of the polarised media.

It seems difficult to dispute the form of the expression for W, but careful consideration will also convince one that it is probably just as difficult to support it in the most general case, except it be by the results which are derived from it, which certainly seem to be in satisfactory agreement with our knowledge of these things. A similar reservation must be applied also to the expression for F, but there is here an additional point worth noticing. It is not often remarked that the form given tacitly involves an assumption which is derived as an independent result from discussions based on this special form. In fact it involves the definite assumption that no work is done by the magnetic forces during the motion of electric charges. Of course the usual expression for such force as proportional to the vector product of the velocity and magnetic force confirms this assumption, but the derivation of this expression by dynamical methods from results derived from the present discussion is by so much deprived of interest. In fact, if to the assumption that these forces do no work we add the further conditions that they are linear in the magnetic and velocity vectors, it would appear that their form is completely determined, at least to a constant factor, without further considerations either of a dynamical or any other nature.

The form for the expression T is not usually regarded as being sufficiently definite to be used in the present connexion, mainly because it is the more readily convertible into equally simple alternative forms. We have in our previous discussions made certain assumptions which have proved to be equivalent to taking

$$T = \frac{1}{4\pi}\int dv \int^{B} (H\, dB),$$

but this special form will subsequently be proved to be irrelevant to the discussion. It is usually regarded as being most advisable to consider the expression for T as bound up with that for S, the equation connecting the two being that derived from the energy principle by the insertion of the forms chosen for W and F, viz.,

$$\frac{dT}{dt} + \int_f S_n\, df = -\int (EC)\, dv - \int \frac{1}{c}\left([\nu_m B],\ \frac{dP}{dt}\right) dv + \frac{1}{c}\int (E,\ \mathrm{Curl}\,[P\nu_m])\, dv$$

C being now the total current of MAXWELL's theory. This is all we can derive from the energy principle. The various possibilities open to us have been examined in detail before. We may take

$$\mathrm{Curl}\, H = \frac{4\pi}{c} C$$

and then we get POYNTING'S theory in which

$$\mathrm{T} = \frac{1}{4\pi}\int dv\left[\int^{\mathrm{B}}\left(\mathrm{H}-\frac{4\pi}{c}[\mathrm{P}\nu_m],\ d\mathrm{B}\right)-\frac{1}{c}\int^{\mathrm{P}}\left([\nu_m\mathrm{B}]\,d\mathrm{P}\right)\right]$$

and

$$\mathrm{S} = \frac{c}{4\pi}\left[\mathrm{E},\ \mathrm{H}-4\pi[\mathrm{P}\nu_m]\right] = \frac{c}{4\pi}[\mathrm{E},\ \mathrm{H}']$$

where

$$\mathrm{H}' = \mathrm{H}-4\pi[\mathrm{P}\nu_m]$$

is equivalent to the vector H′ introduced in paragraph 8.

This theory has the advantage, in addition to that already discussed at length, that it involves no further dynamical assumption other than those expressed in the special forms chosen for W and F. The Amperean equation used to effect the separation being more in the nature of a kinematical definition of the electric current or magnetic force than of a dynamical relation between the field vectors.

Another form can be obtained by using the equations

$$\mathrm{E} = -\frac{1}{c}\frac{d\mathrm{A}}{dt}-\operatorname{grad}\phi$$

with

$$\operatorname{div}\mathrm{C} = 0$$

we then get

$$\mathrm{T} = \frac{1}{c}\int dv\int^{\mathrm{A}}(\mathrm{C}\,d\mathrm{A})-\frac{1}{c}\int dv\int^{\mathrm{P}}(\nu_m[\mathrm{B}\,d\mathrm{P}])$$
$$-c\int^{\mathrm{A}}(\mathrm{Curl}\,[\mathrm{P}\nu_m],\ d\mathrm{A})$$

with

$$\mathrm{S} = \phi\,(\mathrm{C}-c\,\mathrm{Curl}\,[\mathrm{P}\nu_m]).$$

The special form of this result when the media are at rest has been shown* to be inconsistent with our usual conception of such things as radiation phenomena.

Yet another form of the theory can be obtained by taking

$$\mathrm{T} = -\int dt\int\left[(\mathrm{EC}+\frac{1}{c}\left([\nu_m\mathrm{B}],\ \frac{d\mathrm{P}}{dt}\right)-c\,(\mathrm{E}\ \mathrm{Curl}\,[\mathrm{P}\nu_m])\right]dv$$

and then

$$\mathrm{S} = 0.$$

In such a theory there would be no such thing as radiation.

We can go on multiplying the different forms of this theory indefinitely and each form obtained would in itself be perfectly consistent with the Maxwellian electrodynamic theory. The expressions for S and T in them are of course dependent

* 'Phil. Mag.,' vol. 34 (1917), p. 385.

on each other, being connected by the equation given above, and there will be a relation between the forms for two different theories. In fact if S and T are the forms corresponding to any one mode of separation and if we write

$$S' = S + \frac{dU}{dt}$$

where U is any arbitrary vector function we shall have

$$\frac{dT}{dt} + \int S_n\, df = \frac{dT}{dt} + \int \left(S'_n - \frac{dU_n}{dt}\right) df$$

$$= \frac{dT'}{dt} + \int S'_n\, df$$

where

$$T' = T - \int \operatorname{div} U\, dv.$$

and S′ and T′ are appropriate forms for a new mode of separation. In this way, by assigning convenient values for U, we might tentatively construct a number of interesting formulæ.

The last result also shows why it is that the particular form chosen for the kinetic energy is irrelevant to the general dynamical discussion of paragraph 7. In fact, if, instead of the form T used on that occasion, we had employed the general value derived above

$$T - \int \operatorname{div} U\, dv$$

the part of the variation depending on this energy becomes the time integral of

$$\delta T - \int \operatorname{div} \delta U\, dv,$$

and the latter integral reduces to a surface integral over the infinitely distant boundary and cannot therefore contribute anything in this general variational equation.

Of course, from another point of view, the various forms of the theory here under review, differ merely in assigning different distributions to the magnetic energy in the field, each of these distributions being ultimately consistent with the same proper total for this quantity; and the fact that they all lead to the same dynamical equations, merely verifies a well-known result of analytical dynamics that the particular form of expression for the energies of the system is immaterial to the ultimate dynamical equations for the field inside a continuous medium. Of course the solutions of boundary problems such as are, for example, involved in a specification of the energy flux, depend essentially on the particular form assumed for the energy distribution;

and it has been shown on a previous occasion* that the only form of specification of the energy distribution which is consistent with our usual ideas on these matters is that which makes the density of the magnetic energy equal to $B^2/8\pi$, and as the present discussion shows that our only hope of discrimination lies in that direction, we may assume that the evidence in favour of this special form is conclusive, at least for the present; it is besides the only form in which the most general case is completely representative of the distribution in any volume of the field without requiring the introduction of boundary terms involving surface distributions.

11. We next turn to a consideration of the expression for the forcive of electromagnetic origin acting on the polarised media in the field. We have seen that the mechanical forcive on the dielectrically polarised media is such that its x-component per unit volume at any place is of the form

$$(P\nabla)\,E_x+\frac{1}{c}\left([P\nu_m],\ \frac{\partial B}{\partial x}\right)+\frac{1}{c}\left[\frac{dP}{dt},\ B\right]_x$$

or as it first appears in the analysis

$$\left(P\frac{\partial E}{\partial x}\right)-[P\,\mathrm{Curl}\,E]_x+\frac{1}{c}\left[\frac{dP}{dt},\ B\right]+\frac{1}{c}\left([P\nu_m]\,\frac{\partial B}{\partial x}\right)$$
$$=P\frac{\partial E}{\partial x}+\frac{1}{c}\frac{d}{dt}[PB]+\frac{1}{c}\left([P\nu_m]\,\frac{\partial B}{\partial x}\right).$$

This result is in complete agreement with that derived by LARMOR in the electron theory,† but the present derivation indicates clearly the origin of the different terms in it. The expression

$$\left(P\frac{\partial E}{\partial x}\right)$$

is that corresponding to the expression derived in the statical theory from energy considerations and corresponds to MAXWELL's magnetic expression; the second term, viz.,

$$-[P\,\mathrm{Curl}\,E]_x$$

is one of the terms arising as a result of the convection of the media, and this is the term which is effective in reducing the electric part of the forcive to the form

$$(P\nabla)\,E_x$$

which is the result derived in the elementary theory by regarding the forcive as the resultant of the forces on the elementary bi-poles.

* 'Phil. Mag.,' vol. 34 (1917), p. 385. *Cf.* also 'Phil. Mag.,' vol. 32 (1916), p. 162.

† 'Æther and Matter,' p. 104.

Similar results apply in the magnetic case. The general expression for the forcive on the magnetic media is, per unit volume, equal to

$$\left(\mathrm{I}\frac{\partial \mathrm{B}}{\partial x}\right) - [\mathrm{I}\ \mathrm{Curl}\ \mathrm{B}]_x - \frac{1}{c}\left([\mathrm{I}\nu_m],\ \frac{\partial(\mathrm{E}+\mathrm{E}_0)}{\partial x}\right)$$

$$-\frac{1}{c}\left[\frac{d\mathrm{I}}{dt},\ \mathrm{E}+\mathrm{E}_0\right]_x$$

$$= (\mathrm{I}\nabla)\,\mathrm{B}_x - \frac{1}{c}\left([\mathrm{I}\nu_m]\frac{\partial(\mathrm{E}+\mathrm{E}_0)}{\partial x}\right) - \frac{1}{c}\left[\frac{d\mathrm{I}}{dt},\ \mathrm{E}+\mathrm{E}_0\right]$$

$$= \left(\mathrm{I}\frac{\partial \mathrm{B}}{\partial x}\right) - \frac{1}{c}\frac{d}{dt}[\mathrm{I},\ \mathrm{E}+\mathrm{E}_0] - \frac{1}{c}\left([\mathrm{I}\nu_m],\ \frac{\partial(\mathrm{E}+\mathrm{E}_0)}{\partial x}\right).$$

In these expressions the parts depending on E_0 and I^2, representing as they do forces on the elements of the media determined solely by the conditions in those elements, would be neglected in a mechanical theory.* The expression for the effective forcive thus reduces to

$$\left(\mathrm{I}\frac{\partial \mathrm{H}}{\partial x}\right) - \frac{1}{c}\frac{d}{dt}[\mathrm{IE}] - \frac{1}{c}\left([\mathrm{I}\nu_m],\ \frac{\partial \mathrm{E}}{\partial x}\right)$$

$$= (\mathrm{I}\nabla)\,\mathrm{H} - \frac{1}{c}\left([\mathrm{I}\nu_m]\frac{\partial \mathrm{E}}{\partial x}\right) - \frac{1}{c}\left[\frac{d\mathrm{I}}{dt},\ \mathrm{E}\right].$$

This expression is only equivalent to MAXWELL'S expression in the statical case he considers. It is, however, practically equivalent to that derived by counting the forces on the constituent poles, but even here the general result rather suggests a modified conception of the force on a magnetic pole, this force involving in the general case a term due to the electric force. The question of the existence of forces on a magnetic pole due to its motion in an electric field does not appear to have been investigated on an independent basis, although it is definitely contained in the relations of transformation involved in the theory of relativity, which require the form for this forcive

$$\mathrm{B} - \frac{1}{c}[\nu_m \mathrm{E}].$$

It will however be proved below in the next paragraph that such forces do probably exist and are in fact of precisely the correct type.

It may, of course, be objected that the last term in the equation

$$\frac{d\mathrm{B}}{dt} = \frac{d\mathrm{H}}{dt} + 4\pi\frac{d\mathrm{I}}{dt} + 4\pi\ \mathrm{Curl}\ [\mathrm{I}\nu_m],$$

which is the origin of the discrepancy obtained for the magnetic forcive, does not in reality exist, but yet the other results derived from this equation are almost certainly

* *Cf.* LARMOR, 'Æther and Matter,' p. 98.

indisputable, and it seems difficult to realise a state of affairs where the equation without the last term would be generally true.

The whole question of mechanical forces on polarised media is ultimately bound up with the question of the variation of the intrinsic energy of those media, and the expression

$$\left(I\frac{\partial \mathrm{H}}{\partial x}\right),$$

for the x-component of the forcive per unit volume implies that the internal energy of those media change in a small displacement by

$$(\mathrm{H}\ \delta \mathrm{I})$$

per unit volume. But when the media are in motion the expression for the change $\delta \mathrm{I}$ used in this expression involves a part due to the convection of the polarisation which is more properly concerned with the mechanical forces than with the intrinsic elastic or motional ones, as it would exist if the internal constitution of the media was maintained rigidly constant. It is therefore suggested that the result derived above, that the expression for the rate of change of the intrinsic energy is practically

$$\left(\mathrm{H}-\frac{1}{c}[\nu_m \mathrm{E}],\ \delta \mathrm{I}\right)$$

per unit volume, is the more legitimate form of this expression, as allowance is made in it for the convection, and if this is granted, then the equivalent expression for the mechanical forcive, viz.,

$$(\mathrm{I}\nabla)\,\mathrm{H}_x-\frac{1}{c}\left([\mathrm{I}\nu_m]\frac{\partial \mathrm{E}}{\partial x}\right)-\frac{1}{c}\left[\frac{d\mathrm{I}}{dt},\ \mathrm{E}\right]$$

must be regarded as the only adequate form.

Moreover these two expressions essentially involve the particular form of equation adopted for defining $d\mathrm{H}/dt$, and are the only ones which are capable of fitting in with a general relativity theory.

The results here derived also emphasise the difficulties involved in treating the currents due to the convection of polarised media as effectively equivalent to a polarisation of the opposite kind. If, for example, we had treated the convection current

$$\mathrm{Curl}\ [\mathrm{P}\nu_m]$$

as equivalent to a distribution of magnetic polarity of intensity

$$[\mathrm{P}\nu_m]$$

at each place from the outset we should have been led to an entirely erroneous expression for the forcive on the polarised media, the reason being that the inclusion of this quantity with the magnetism hides its true character, particularly as regards its dependence on the velocity of the medium.

Nevertheless many of the relations of the theory will be considerably simplified if this procedure is adopted.

12. The question of forces on fictitious magnetic poles moving in an electric field is easily resolved by imparting to such poles a substantiality sufficient to allow us to talk of forces on them, and then applying any of the general methods used in this theory. The Lagrangian function of the system is still of the form

$$L+\frac{1}{8\pi}\int(B^2-E^2)\,dv,$$

L being that part of it which is not directly determined by the conditions in the field and which will as usual be assumed to be a function of the positions and velocities of the electric and magnetic elements only. The sequence of changes is then best described by the fact that the action integral

$$\int_{t_1}^{t_2} dt\left[L+\frac{1}{8\pi}\int(B^2-E^2)\,dv\right]$$

taken between fixed time limits is stationary subject to the implied conditions of the field. If we assume generally that there are a number of discrete electric particles as well in the field, these conditions may be written in the form

$$\begin{aligned}\operatorname{div} E-4\pi\Sigma e &= 0,\\ \operatorname{div}(B-H)+4\pi\Sigma m &= 0,\\ \operatorname{Curl} H-\frac{1}{c}\frac{dE}{dt}-\frac{4\pi}{c}\Sigma e\nu_e &= 0,\\ \frac{dB}{dt}-\frac{dH}{dt}-4\pi\Sigma m\nu_m &= 0,\end{aligned}$$

wherein e is the charge of the typical electron and ν_e its velocity, m is the strength of the typical magnetic particle whose velocity is ν_m and the sum Σ in each equation is taken per unit volume at each place over the respective elements indicated in it.

We now introduce four undetermined Lagrangian multiplying functions, two scalar quantities ϕ_1 and ϕ_2, and two vector quantities A_1 and A_2, it is thus the variation of

$$\int dt\left[L+\frac{1}{8\pi}\int dv\left[B^2-E^2+2\phi_1\operatorname{div} E+2\phi_2\operatorname{div}(B-H)\right.\right.$$
$$\left.-2\left(A_1,\operatorname{Curl} H-\frac{1}{c}\frac{dE}{dt}\right)+\frac{2}{c}\left(A_2\frac{dB}{dt}-\frac{dH}{dt}\right)\right]$$
$$\left.-8\pi\Sigma e\phi_1+8\pi\Sigma m\phi_2+\frac{8\pi}{c}\Sigma e(A_1\nu_e)-\Sigma m(A_2\nu_m)\right].$$

that is to be made null, afterwards determining the functions ϕ_1, ϕ_2, A_1, A_2 to satisfy the restrictions which necessitated their introduction. The variation can now be affected in the usual way, and the condition that it vanishes leads to the following equations

$$E+\text{grad}\ \phi_1+\frac{1}{c}\frac{dA_1}{dt}=0,$$

$$B-\text{grad}\ \phi_2-\frac{1}{c}\frac{dA_2}{dt}=0,$$

with

$$\text{Curl}\ A_1-\text{grad}\ \phi_2-\frac{1}{c}\frac{dA_2}{dt}=0,$$

with three equations of each of the types

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_e}\right)-\frac{\partial L}{\partial x_e}=-e\left(\frac{\partial\phi_1}{\partial x_e}+\frac{1}{c}\frac{dA_{1x}}{dt}\right)+\frac{e}{c}[\nu_e\ \text{Curl}\ A_1].$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_m}\right)-\frac{\partial L}{\partial x_m}=m\left(\frac{\partial\phi_2}{\partial x_m}+\frac{1}{c}\frac{dA_{2x}}{dt}\right)-\frac{m}{c}[\nu_m\ \text{Curl}\ A_2].$$

The first and third of these equations show that ϕ_1 and A_1 are the usual scalar and vector potentials; in fact from the third we have

$$\begin{aligned}\text{Curl}\ A_1&=\text{grad}\ \phi_2+\frac{1}{c}\frac{dA_2}{dt}\\&=B,\end{aligned}$$

so that A_1 is the vector potential and then ϕ_1 is the scalar potential.

The fourth equation thus determines the usual expression for the reaction forces on the moving electron; the fifth equation determines similarly the force on the moving magnetic pole in the form

$$m\left(\text{grad}\ \phi_2+\frac{1}{c}\frac{dA_2}{dt}\right)-\frac{m}{c}[\nu_m\ \text{Curl}\ A_2]=m\left(B-\frac{1}{c}[\nu_m,\ \text{Curl}\ A_2]\right).$$

We have yet to determine Curl A_2: we have

$$B=\text{grad}\ \phi_2+\frac{1}{c}\frac{dA_2}{dt},$$

so that

$$\text{Curl}\ B=\frac{1}{c}\frac{d}{dt}\text{Curl}\ A_2.$$

Now

$$\text{Curl}\ B=\frac{4\pi}{c}C_1+4\pi\ \text{Curl}\ I+\frac{1}{c}\frac{dE}{dt}=\frac{4\pi}{c}C_0=\frac{1}{c}\frac{dE}{dt},$$

wherein C_1 is the true current density of electric flux;

$$I = \frac{B-H}{4\pi},$$

is the intensity of magnetic polarity, and

$$C_0 \equiv C_1 + c \text{ Curl } I.$$

It follows that

$$\text{Curl } A_2 = E + 4\pi \int^t C_0 \, dt = E + E_0,$$

say. The vector A_2 is a slightly more general form of the second vector potential introduced in our previous dynamical discussion and its *curl* is identical with the *curl* of that vector.

The main part of Curl A_2 is thus determined by the electric force in the field, and its mechanically effective part is completely represented by this vector; the forcive on the moving magnetic pole is thus to all intents and purposes equal to

$$m\left(B - \frac{1}{c}[\nu_m E]\right)$$

an expression which agrees with that suggested by the relativity transformation.

It must, however, be noted that the local term is necessary in the complete relation defining the vector A_2 for the simpler relation

$$\text{Curl } A_2 = E,$$

carries with it the consequence that

$$\text{div } E = 0$$

at all points of the field, and this is true only of those points where there is no electricity.

The expression for the forcive on the magnetic media is now attainable by regarding it as the resultant of the forces on its contained poles; for the volume v bounded by the closed surface f, it is in fact

$$\begin{aligned} &-\int (\text{div } I)\, B \, dv + \int I_m B \, df \\ &+ \frac{1}{c}\int \left[\frac{dI}{dt} + c \text{ Curl } [I\nu_m],\ E + E_0\right] dv \\ &+ \int [[[I\nu_m]\, n_1],\ E + E_0]\, df. \end{aligned}$$

The second and fourth integrals transform by GREEN's lemma to the volume integrals

$$\begin{aligned} &\int (B(\nabla I) + (I\nabla) B)\, dv - \int [[\text{Curl } [I\nu_m]\, E + E_0] + \text{grad } ([I\nu_m]\, E + E_0) \\ &+ \qquad\qquad -([I\nu_m] \text{ div } (E + E_0))]\, dv, \end{aligned}$$

where in the last term but one the gradient operation only affects the E functions. Now

$$\operatorname{div}(\mathrm{E}+\mathrm{E}_0) = 0,$$

and thus the resultant force may be taken as distributed throughout the volume with intensity at each point equal to

$$(\mathrm{I}\nabla)\,\mathrm{B}+\frac{1}{c}\left[\frac{d\mathrm{I}}{dt},\ \mathrm{E}+\mathrm{E}_0\right]-\operatorname{grad}\left(\,[\mathrm{I}\nu_m],\ \mathrm{E}+\mathrm{E}_0\right)$$

in agreement with the general result derived above. The local terms in I^2 and E_0 may again be presumed to balance out with other forces of a type not at present under review.

13. The two new potentials A_2 and ϕ_2, introduced in the analysis of the last paragraph, are the general forms of the potentials analogous to the ordinary scalar and vector potentials of this theory, and they satisfy similar equations. We have, in fact,

$$\operatorname{Curl} \mathrm{A}_2 = \mathrm{E}+4\pi\int^t \mathrm{C}_0\, dt$$

where C_0 is the total current density of electric flux including the effective representation of the magnetism. Thus

$$\operatorname{Curl}\operatorname{Curl} \mathrm{A}_2 = \operatorname{Curl} \mathrm{E}+4\pi\int^t \operatorname{Curl} \mathrm{C}_0\, dt$$

$$= -\frac{1}{c}\frac{d\mathrm{B}}{dt}+4\pi\int^t \operatorname{Curl} \mathrm{C}_0\, dt.$$

Thus

$$\operatorname{grad}\operatorname{div} \mathrm{A}_2-\nabla^2\mathrm{A}_2 = -\frac{1}{c^2}\frac{d^2\mathrm{A}_2}{dt^2}-\frac{1}{c}\operatorname{grad}\frac{d\phi_2}{dt}+4\pi\int^t \operatorname{Curl} \mathrm{C}_0\, dt$$

whilst since $\operatorname{div} \mathrm{B} = 0$, we have also

$$\nabla^2\phi_2+\frac{1}{c}\frac{d}{dt}\operatorname{div} \mathrm{A}_2 = 0.$$

We may now adopt one of a number of alternatives. The simplest one is got by taking $\phi_2 = 0$, when we also have

$$\operatorname{div} \mathrm{A}_2 = 0$$

with, therefore,

$$\nabla^2\mathrm{A}_2 = \frac{1}{c^2}\frac{d^2\mathrm{A}_2}{dt^2}+4\pi\int^t \operatorname{Curl} \mathrm{C}_0\, dt.$$

The last equation really involves the first, for

$$\nabla^2(\operatorname{div} \mathrm{A}_2) = \frac{1}{c^2}\frac{d^2}{dt^2}(\operatorname{div} \mathrm{A}_2),$$

so that $\operatorname{div} \mathrm{A}_2$ must be zero as it has no singularities.

The vector A_2 chosen in this way is, practically speaking, the æthereal displacement vector employed by LARMOR in his mechanical model of the electric and luminiferous medium. The *curl* of this vector is the electric force, or at least as regards its rate of change, whilst the magnetic induction B, which is proportional to the time rate of change of A_2, appears as the velocity.

We need not, however, take the quantities in this way. We might take

$$\operatorname{div} A_2 = \frac{1}{c}\frac{d\phi_2}{dt} - 4\pi \operatorname{div}\int^t I\,dt,$$

and then we should have

$$\nabla^2\phi_2 = \frac{1}{c^2}\frac{d^2\phi_2}{dt} - \operatorname{div} I$$

with

$$\nabla^2 A'_2 = \frac{1}{c^2}\frac{d^2 A'_2}{dt^2} + 4\pi\frac{dI}{dt} + 4\pi\int^t \operatorname{Curl} C_0\,dt$$

where we have used

$$A'_2 = A_2 - 4\pi\int^t I\,dt.$$

In this case ϕ_2 is the scalar potential of the magnetic distribution, whilst A'_2 belongs to the current distribution. With these differential equations the general values of ϕ_2 and A_2 in regular fields are such that

$$\phi_2 = \frac{4\pi}{c}\int\frac{[\operatorname{div} I]}{r}\,dv$$

whilst

$$\frac{1}{c}\frac{dA_2}{dt} - \frac{4\pi I}{c} = \frac{4\pi}{c^2}\int r^{-1}\left[\operatorname{Curl} C_0 + \frac{d^2 I}{dt^2}\right]dv$$

the square brackets in the integrands denoting that their values are taken at each point for the time $\left(t - \frac{r}{c}\right)$.

These are the most interesting cases of the solutions for ϕ_2 and A_2, but we may construct any number of others. It must be noticed, however, that the equation

$$B = \frac{1}{c}\frac{dA_2}{dt} + \operatorname{grad}\phi_2,$$

does not imply that the vector B is derived from a potential in steady fields, for it is impossible to satisfy the equations with A_2 independent of the time; we may have dA_2/dt constant in time but not A_2. This is the origin of the difficulty in LARMOR'S mechanical model which seems to necessitate the piling up of æthereal displacement in a steady magnetic field.

14. We have determined the complete expression for the forcive per unit volume on the media occupying the electromagnetic field. The next step in the general

theory is to reduce these forces to a representation by means of an applied stress system of ordinary character. This discussion leads in the usual way to the introduction of the concept of electromagnetic momentum.

The actual calculations for the present form of the results are not materially different from those given *in extenso* elsewhere, so that it will again be sufficient to outline the principal stages in the discussion. The method employed is to attempt to express, say, the x component of the force per unit volume in the form

$$\frac{\partial \mathrm{T}_{xx}}{\partial x}+\frac{\partial \mathrm{T}_{xy}}{\partial y}+\frac{\partial \mathrm{T}_{xz}}{\partial z}.$$

Now the total forcive of electrodynamic origin acting on the medium of the field at any place is such that its x component per unit volume is

$$\left(\mathrm{P}-\frac{1}{c}[\mathrm{I}\dot{r}_m], \frac{\partial \mathrm{E}}{\partial x}\right)+\left(\mathrm{I}+\frac{1}{c}[\mathrm{P}\dot{r}_m], \frac{\partial \mathrm{B}}{\partial x}\right)$$
$$-[\mathrm{P}, \mathrm{Curl}\ \mathrm{E}]_x-[\mathrm{I}, \mathrm{Curl}\ \mathrm{B}]_x-\frac{1}{c}\left[\mathrm{B}. \frac{d\mathrm{P}}{dt}\right]+\frac{1}{c}\left[\mathrm{E}, \frac{d\mathrm{I}}{dt}\right]$$
$$+\rho\left(\mathrm{E}_x+\frac{1}{c}[\nu_m \mathrm{B}]\right)+\frac{1}{c}[\mathrm{C}_1 \mathrm{B}]_x.$$

Again writing

$$\mathrm{H}' = \mathrm{H}-\frac{4\pi}{c}[\mathrm{P}\nu_m].$$

it is proved just as in the usual form of the theory that the forcive of which this is the representative component is represented in the main by a stress system in which

$$\mathrm{T}_{xx} = \mathrm{E}_x\mathrm{D}_x-\frac{1}{8\pi}\mathrm{E}^2+\frac{1}{4\pi}\mathrm{B}_x\mathrm{H}'_x-\frac{1}{8\pi}\mathrm{H}'^2$$

and

$$\mathrm{T}_{xy}\mathrm{E}_x\mathrm{D}_y+\frac{1}{4\pi}\mathrm{B}_{xy}\ \mathrm{H}'_x$$

with symmetrical expressions for the other constituents; but with this representation there is an outstanding part of the complete forcive, viz.,

$$-\frac{1}{4c}\frac{d}{dt}[\mathrm{EB}]+\frac{1}{c}\frac{d}{dt}[\mathrm{EI}]-\frac{1}{c}\left([\mathrm{I}\nu_m]. \frac{\partial \mathrm{E}}{\partial x}\right) = -\frac{1}{4\pi c}\frac{d}{dt}[\mathrm{EH}]-\frac{1}{c}\left([\mathrm{I}\dot{\nu}_m], \frac{\partial \mathrm{E}}{\partial x}\right)$$

which cannot be so reduced. The first term in this outstanding part, being a complete differential with respect to the time, is usually taken to represent a part of the complete forcive arising as the kinetic reaction to a rate of change of momentum, and this is the origin of the concept of electromagnetic momentum. This idea is however partly destroyed by the remaining term in the above expression which cannot be developed either as a forcive of ordinary type or as a kinetic reaction to a rate of change of momentum, so that we are rather forced to regard these outstanding terms as pointing to the failure of the ideas from which we set out. This conclusion does not, of course, invalidate the results derived in the simpler electron theory, as the concept of momentum will remain under the simplest conditions as a convenient mathematical expression for the actual result, whatever be its ultimate physical basis.

The present formulation possesses another disadvantage which is apparently not inherent in the simplest presentations of the momentum idea. In the electron theory, as usually developed, the momentum remains as a fundamental quantity and is distributed over the field with the density

$$\frac{1}{4\pi c}[\mathrm{EB}]$$

at each point; this gives it a purely æthereal constitution as the vectors E and B are those which define the conditions in the æthereal field. In the present formulation the vector B is replaced by the vector H which is essentially an auxiliary mechanical vector in the theory; the fundamental nature of the momentum vector is therefore entirely lost. We can, of course, assume that some of the momentum is in reality attached to the matter, and such an assumption has certain points in its favour. The force of electromagnetic origin on the dielectric media for example has an x component which per unit volume is

$$\left(\mathrm{P}\frac{\partial \mathrm{E}}{\partial x}\right)+\frac{1}{c}\left([\mathrm{P}\nu_m],\frac{\partial \mathrm{B}}{\partial x}\right)-[\mathrm{P}\ \mathrm{Curl}\ \mathrm{E}]_x+\frac{1}{c}\left[\frac{d\mathrm{P}}{dt},\mathrm{B}\right],$$

and this may be written in the form

$$\left(\mathrm{P}\frac{\partial \mathrm{E}}{\partial x}\right)+\frac{1}{c}\left([\mathrm{P}\nu_m],\frac{\partial \mathrm{B}}{\partial x}\right)+\frac{1}{c}\frac{d}{dt}[\mathrm{PB}].$$

The first two terms appear as those appropriate to the energy function in the statical theory which would be

$$-(\mathrm{PE})-\frac{1}{c}([\mathrm{P}\nu_m],\mathrm{B}),$$

so that the third might be regarded as a kinetic reaction to a rate of change of momentum, which would be distributed throughout the medium with a density

$$\frac{1}{c}[\mathrm{PB}]$$

at each point.

A similar analysis and analogous results hold for the magnetic media.

There is, too, a relation satisfied by the momentum vector which appears in the simplest form of the theory and to which a fundamental significance is attached by some authors, but which is not satisfied by the results of our present discussion. The vector determining the flux of electromagnetic energy has been seen to be

$$S = \frac{c}{4\pi}[E, H']$$

and that determining the momentum is

$$M = \frac{1}{4\pi c}[E, H].$$

In the absence of magnetic media and convective dielectric polarisations these two expressions satisfy the equation

$$M = \frac{1}{c^2}S.$$

but under the most general circumstances this relation is not satisfied.

We have so far conducted the discussions as though the quantity derived as a momentum is unique and definite, whereas, as a matter of fact, this is far from being the case. We saw that the idea of the momentum arose from certain outstanding terms which remained when attempting to reduce the electromotive forces to a representation by a stress system. Now we can give a number of different forms to this reduction and each one carries with it a different expression for the electromagnetic momentum. We can, for instance, write

$$\begin{aligned}\frac{1}{4\pi c}[EH] &= -\frac{1}{4\pi c^2}\left[\frac{dA}{dt}, H\right] - \frac{1}{4\pi c}[\nabla\phi, H] \\ &= -\frac{1}{4\pi c^2}\left[\frac{dA}{dt}, H\right] - \frac{1}{4\pi c}[\nabla\phi, H] + \frac{1}{c_2}\phi C\end{aligned}$$

and the second term in this expression when differentiated with respect to the time might be included in the stress specification. This would leave a new expression for the electromagnetic momentum which is

$$\frac{1}{c_2}\left(\phi C - \frac{1}{4\pi}\left[\frac{dA}{dt}, H\right]\right)$$

a form which would probably be suitable for use in connexion with a theory in which the radiation phenomena are represented by MACDONALD's form of the theory.

This is not the only alternative to the usual theory for we can construct similarly any number of others. It appears, however, that the usual presentation is probably the simplest possible one, and this is a great advantage in its favour ; but subsequent developments of the theory may require a modification, and then it is as well to remember that there are other forms of the theory perfectly consistent with the general relations of the electromagnetic field, both as regards its general and dynamical aspects.

It is hoped in a future communication to examine in detail some of these alternative expressions for the momentum, but so far the results obtained are not of sufficient interest to warrant their discussion at the present stage.

15. It may now be convenient to summarise the conclusions and results of our discussion. The differential theory of MAXWELL as expressed in the usual way by the equations

$$\text{Curl H} = \frac{4\pi}{c}\left(\frac{d\text{D}}{dt} + \text{C}_1 - \rho\nu\right) \qquad \text{Curl E} = -\frac{1}{c}\frac{d\text{B}}{dt} \qquad \text{div D} = \rho$$

is supplemented by the addition of an equation expressing the rate of change of the magnetic force

$$\frac{d\text{H}}{dt} = \frac{d\text{B}}{dt} - 4\pi\frac{d\text{I}}{dt} - 4\pi \,\text{Curl}\,[\text{I}\nu_m]$$

this equation being analogous to that expressing the rate of change of electric displacement

$$\frac{d\text{D}}{dt} = \frac{1}{4\pi}\frac{d\text{E}}{dt} + \frac{d\text{P}}{dt} + \text{Curl}\,[\text{P}\nu_m].$$

The fundamental dynamical equations are then derived by a variational principle equivalent to the principle of Least Action in dynamics ; in this discussion the assumption of a definite electronic constitution for the dielectric and magnetic polarisations is specially avoided in order to bring out certain points of the theory which have not previously received adequate treatment. In this way, in addition to deriving the equation

$$\text{Curl E} = -\frac{1}{c}\frac{d\text{B}}{dt},$$

it can be proved that the forces on the media occupying the field consist of several distinct parts. There is firstly a part

$$\rho\left(\text{F} + \frac{1}{c}[\nu_m\text{B}]\right)$$

due to the free charges associated with the typical point of the matter and a part

$$\frac{1}{c}[\text{C, B}]$$

due to the true conduction current. Due to the dielectric polarisations there is a part whose x component is

$$(\text{P}\nabla)\,\text{E}_x + \frac{1}{c}\left[\frac{d\text{P}}{dt},\ \text{B}\right]_x - \frac{1}{c}\left([\text{P}\nu_m]\,\frac{\partial \text{B}}{\partial x}\right),$$

and the magnetic polarisations give rise to the analogous terms

$$(\mathrm{IV})\ \mathrm{B}_x - \frac{1}{c}\left[\frac{d\mathrm{I}}{dt}, \mathrm{E}\right] - \frac{1}{c}\left([\mathrm{I}\nu_m], \frac{\partial \mathrm{E}}{\partial x}\right).$$

With the exception of the magnetic terms these results are in general agreement with those usually derived on the basis of the electronic theory, and the discrepancy in the magnetic terms is proved to arise from the inadequacy of the treatment of the magnetic relations in that theory, no allowance being made in it for the convection of the magnetic polarisations.

The results derived from the dynamical theory are then examined in connexion with the usual developments of the theory in regard to radiation phenomena, to the energetic relations of the magnetic media and, finally, to the fundamental problem of the representation of the forces in the field, as an applied stress system and the subsidiary question of electromagnetic momentum. In regard to each of these results are derived which do not differ materially from those usually given, but the slight discrepancies in each case, although probably of little or no practical significance, prove ultimately to be of theoretical importance as helping to justify the fundamental equations on which they are based. The auxiliary conception of electromagnetic momentum is not however completely attained under the most general conditions, although it will still remain to enable us to obtain an effective mode of expressing certain results of the simpler theory; it is probably present in no other capacity in former interpretations of the theory so that this is hardly a disadvantage of the present formulation.

The present theoretical relations require, of course, to be supplemented by the usual empirical laws for the induction of the two polarisations and the conduction current. We have however specially refrained from introducing these relations as it was desired to emphasise the fact that the theory in its complete form is entirely independent of these laws, so that for example it necessarily covers the most complex fields, involving ferromagnetic inductions and polarisations. If we interpret the theory as determining the electrodynamic changes in the system during its transition from one configuration to another even the presence of hysteritic qualities in the inductions will not vitiate its validity. This is, of course, no special advantage attaching to the present form of the theory as it is in reality fundamentally inherent in every interpretation, although it may be hidden by the particular form of expression adopted.

VIII. *The Influence of Molecular Constitution and Temperature on Magnetic Susceptibility.*

Part IV.—Further Applications of the Molecular Field.

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Contents.

(1) Introduction.

The present work is a continuation of that published in 'Phil. Trans., Royal Society,' vol. 214, pp. 109–146, 1914 (Parts I. and II.) and vol. 215, pp. 79–103, 1915 (Part III.).* It will lead to clearness in the following development of the subject if a brief résumé of these papers is given. At the same time, I wish to discuss one or two points in connexion with the views which have been previously advanced and the relation between mechanical and molecular theory.

In Part I. the experimental evidence brought forward has justified the hypothesis of molecular distortion enunciated at the outset. We have thereby been led to regard the molecular configuration of a material medium as a distorted one, and this applies particularly to a substance which is crystalline. The extent of this distortion is small, but is sufficient to account for the observed change of specific susceptibility which occurs on crystallization. Such change will naturally depend upon the particular crystalline symmetry assumed by the substance.

The theoretical treatment given in Part II. is an attempt to account for the phenomena observed by extending the electron theory developed by Langevin so

* For brevity, reference to these researches is given under Parts I., II., and III.

as to include the effects of mutual molecular influences. Such a representation of the facts has led to the recognition of a large molecular forcive, in all crystalline media, depending upon the nature and proximity of the molecules in any particular crystalline grouping. The existence of this intrinsic molecular field can only be inferred indirectly. Although the actual properties of the crystalline state depend upon the operation of this field, yet, except in the case of substances of a ferro-magnetic nature, there is no direct experimental evidence which discloses its very great magnitude.

From a theoretical point of view, there seems to be no doubt that the mutual actions of the molecules are represented by enormous internal forcives in all crystalline media. The usual method of determining the forcive at an internal point of the material medium is to take a cavity whose dimensions are small in comparison with ordinary lengths (*e.g.*, 1 cm.) and yet large compared with molecular dimensions. A convenient designation of the dimensions of the cavity is contained in the phrase "physically small."* In molecular theory, the subdivision of the medium into elements is not valid beyond the limits of physical smallness and only in media which are absolutely continuous may the elements be pushed to limits of "mathematical smallness." In a continuous medium our mathematical functions give us an accurate estimate of the forcives and potentials operating at internal points; in a medium composed of discrete particles these same functions give us only an approximate estimate. A discussion of the nearness of the approximation which can be obtained for material media is of great importance from the point of view of our subject. For the liquid state the question has been considered by Sir JOSEPH LARMOR† who found that the part contributed to the forcive at any internal point by the molecules immediately surrounding that point was, on account of rapid motions and irregular distributions of the axes of the molecules, negligibly small. To quote from LARMOR‡: "The general conclusion may be expressed, in an adaptation of CAUCHY's terminology, by the principle that whenever the integrals in the formulæ for mechanical forces on a material medium cease to be convergent, their principal values must be substituted," and again in the footnote to p. 265,

"This statement (*i.e.*, the above quotation) may be considered to be the mathematical expression of the principle of the mutual compensation of molecular forcives, for which, *cf.*, 'Phil. Trans.', A, 1897, p. 260. The principal value of CAUCHY, as regards the completely defined analytical integrals of Pure Mathematics, would be the value at the centre of a minute spherical cavity. But the quantities which, to avoid periphrasis, have been here called integrals, are really summations of contributions from finite though very small, and complexly constituted, polarised molecules; the distribution of these molecules that occupy our minute cavity is entirely

* LEATHEM, "Volume and Surface Integrals used in Physics," 'Cambridge Monographs,' No. 1, p. 5.

† 'Æther and Matter,' p. 261.

‡ *Loc. cit.*, p. 265.

unknown and may be continually changing, so that the only possible principal value is the one that omits the contribution of neighbouring molecules altogether."

When now we come to the case of crystalline media, the molecules which were removed from our small spherical cavity would affect considerably the value of the forcive at the centre, for there is in this case no averaging out on account of random motions and orientations of the molecules originally occupying the cavity. As we do not know the relative dispositions of the molecules composing the space lattice, or the law of force which is operative between the molecules, it is quite impossible to calculate the value of the intrinsic forcive for a point inside a crystalline medium, and as the method of averages, the application of which is quite satisfactory when the medium is in a fluid state, breaks down in the case of a crystalline medium, it is clear that our only way of progress lies in indirect deduction from experimental facts which record the change of physical properties accompanying the transition from the liquid to the crystalline state. This is the method which has been adopted in the previous portions of this work and it has been shown that the internal forcive at a point of a crystalline medium is extremely large and comparable, if interpreted magnetically, with the molecular field in ferro-magnetic substances (10^7 gauss).

In Part III. it was shown that the potential energy of the molecules forming a crystalline structure was sufficiently large to account for the magnitude of the thermal energy required for fusion. On p. 201 of his "Æther and Matter," LARMOR states: "These various actions (referring to the disturbance of configuration of steady orbits in molecules by action of an applied magnetic field) involve energy terms for each individual molecule, and the sum for all the molecules, if it could be formed, would represent the total energy of the disturbance of the medium. But such a mere aggregate of terms would be of no use for applications to matter in bulk; what we are concerned with there is the mechanical part of the energy, which must be an analytical function of the specification of matter by volume, determined as to mathematical form by the character of the molecular actions, but with coefficients whose values are to be obtained only by direct experiment."

Although for a fluid medium the total energy of the disturbance of the medium, due to the application of a magnetic field, has little significance, yet, in the transition from the liquid to the crystalline state, during which the molecular field becomes operative, the sum total of the energy disturbance of the medium due to the action of this molecular field is representative of the latent thermal energy which is absorbed when the crystalline medium is fused, and has a definite value for each particular substance.

From the point of view of fluids, the intrinsic forcives mutually compensate and the mathematical functions may be treated as analytic, their principal values being taken. In a crystalline structure, however, the functions cannot be treated as analytic. Indirectly we have obtained a measure of the intrinsic forcive in this case with the aid of experimental data. The most we can obtain by direct experiment, however, is a measure of the mean molecular field, which expresses mathematically how the susceptibility of matter *in bulk* depends upon a transition from the liquid to the crystalline state. When thus considering diamagnetic matter in bulk, the large local forcive which has been shown to bind the molecules of the crystalline structure together need not be considered, since for matter *in bulk* its effects are cut out by the mutual compensation of molecular forcives. It is only when we enquire into the *molecular* structure of the crystalline medium, or to changes in this structure, that we pass to the inner limit where the principles of LARMOR and CAUCHY for the fluid state no longer apply. LARMOR remarks : "The result of the integration still however gave us a valid estimate of the effect of the material system *as a whole*, when we bore in mind that the infinite or rather undetermined term entering at the inner limit really represents the part of the result which depends *solely* upon the local molecular configuration ; a part whose actual magnitude could be determined only when that configuration is exactly assigned or known" (*loc. cit.*, p. 125).

It is with this "infinite or rather undetermined term . . . . . . which depends *solely* upon the local molecular configuration" that these researches are mainly concerned. It has been called the local molecular field of the crystalline medium (Part III., p. 83).

(2) ON THE ENERGY AND ULTIMATE TENSILE STRENGTH ASSOCIATED WITH CRYSTALLINE MEDIA OR GELS.

The large intrinsic potential energy associated with a crystalline medium has been discussed in para. 5, Part III., pp. 90–95. It now remains for us to examine the accompanying stresses to see how far the elastic properties of material media may be interpreted in terms of these intrinsic forcives. Consider first the case of a liquid which is gradually cooled in liquid air so that it passes into a glass-hard transparent gel when it arrives at the temperature of the liquid air.

It has been suggested (Part III., p. 81) that the appearance of rigidity in the gel is due to an interlocking of the irregularly shaped molecules (arranged at random) whose thermal agitation is sufficiently reduced. On account of this random orientation of the interlocked molecules the gel will be isotropic. At such a low temperature, also, the molecular motions will be highly constrained so that a particular molecule will present practically the same aspect to the surrounding molecules over a long period. If this is the case, then the local molecular forcive between this and a neighbouring molecule will act in a definite direction and will not be rapidly changing its direction as would be the case with the same molecules at a considerably higher temperature (in the ordinary liquid state). It is clear therefore that between the molecules of the gel at low temperature we shall have a large local forcive in operation, due to the interaction of the magnetic systems or revolving electrons within each molecule, but the direction of the action of this forcive between any pair of molecules will be one of random distribution, as we pass from pair to pair of molecules, although at any given point it is fixed in direction.

On further cooling, the molecules continue to readjust themselves and the rigidity increases until a glass is formed. Still lower temperatures, accompanied by further molecular readjustment due to reduction of amplitude, may result in such a distortion of the lines of force binding the pairs of molecules together, that a completely new pairing of molecules takes place, resulting in spontaneous crystallization accompanied by thermal evolution as the more stable crystalline state is formed.

Whatever may be the nature of the forces which hold the molecules of a liquid together, we have, in addition to these forces, the intrinsic field referred to above, when the substance passes into a rigid gel or crystallizes, and it is due to this intrinsic field that the two latter media show rigidity.

If H_c be this intrinsic field,* I the local intensity of magnetization,† the potential energy per unit volume associated with the gel or crystalline medium will be

$$\tfrac{1}{2} . H_c . I \quad \ldots \ldots \ldots \ldots \quad (1)$$

and this will be over and above any potential energy which the molecules of the liquid possess. This is also a measure of the mechanical stress which binds the molecules of the gel or crystalline medium together and determines their rigidity.

In Part III. we have given reasons for locating the source of the local molecular field within the molecule and we found that in the immediate neighbourhood of a

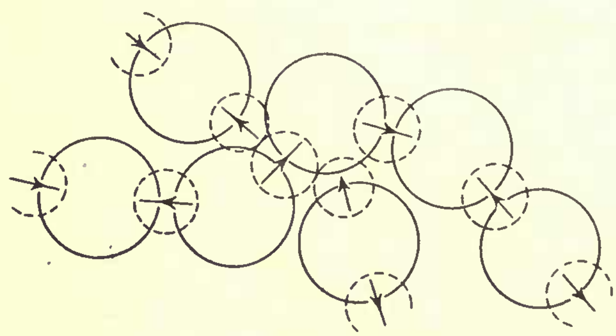

Fig. 1.

molecule the value of this field, as determined from the properties of crystalline media, is of the order 10^7 gauss. However the molecule is orientated, provided that orientation is not variable with time, the local forcive will be of this order of intensity and in some direction determined by the orientations of the two molecules between which it acts. In a gel, as we pass from molecule to molecule, the direction of this stress will be continually changing (fig. 1). Throughout a crystal, on the other hand, its direction will be constant and will in fact be one of the determining factors of a particular form of crystalline symmetry (fig. 1A).

In a gel, the whole collection of molecules is bound together into one homogeneous

* See Part III., p. 86.

† *Loc. cit.*, p. 90.

isotropic mass (*i.e.*, as viewed in bulk) whereas in a crystalline medium the mass will be æolotropic.

In Part III. evidence was brought forward showing that in the case of diamagnetic media the local intensity of magnetization I is of the order 100, while the local molecular field H_c between the molecules is of the order 10^7 gauss. The energy of the molecular configuration of the crystalline medium (over and above that due to

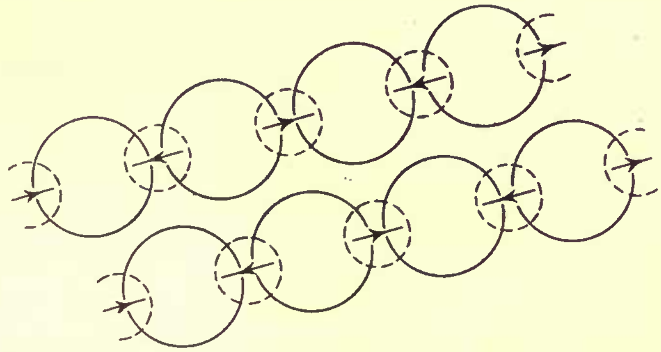

Fig. 1A.

the molecular configuration in the liquid state) or of the gel at low temperatures, will therefore be of the order $\frac{1}{2} . H_c . I = 10^9$ ergs per unit volume* and the internal stress 10^9 dynes per square centimetre, or 1000 atmospheres approximately. For wrought iron the energy per unit volume will be

$$\frac{1}{2} . H_c . I = \frac{1}{2} \times 6 \cdot 5 \times 10^6 \times 1700 = 5 \cdot 5 \times 10^9 \text{ ergs}$$

and the internal stress about 5500 atmospheres.

In nickel the intrinsic pressure is $1 \cdot 4 \times 10^9$ dynes per square centimetre or 1400 atmospheres; in cobalt $4 \cdot 4 \times 10^9$ dynes per square centimetre or 4400 atmospheres, in cast iron 4×10^9 dynes per square centimetre or 4000 atmospheres. These internal stresses are a measure of the forces binding the molecules together and should give an estimate of the ultimate tensile strength or tenacity of the medium. Moreover the tenacities of ferro-magnetic, paramagnetic and diamagnetic media should be roughly of the same order. That this is so is seen from the following values† :—

	Material.	Tenacity (dynes per square centimetre).
Ferro-magnetic	Iron, (wire)	$5 \cdot 0 - 6 \cdot 0 \times 10^9$
	Iron, wrought	$2 \cdot 9 - 4 \cdot 5 \times 10^9$
	Iron, cast	$1 \cdot 2 - 1 \cdot 9 \times 10^9$
	Nickel (wire)	$5 \cdot 3 \times 10^9$
	Mild steel (0·2 per cent. carbon)	$4 \cdot 3 - 4 \cdot 9 \times 10^9$
	High carbon steel	$7 \cdot 0 - 7 \cdot 7 \times 10^9$
	Nickel steel (5 per cent.)	$6 \cdot 2 \times 10^9$

* This, as we have seen in Part III., p. 93, is compatible with the values of the latent heat of fusion of diamagnetic crystalline media. See also *infra*, pp. 253–4, p. 256.

† KAYE and LABY, 'Physical and Chemical Constants,' 1918, p. 28.

	Material.	Tenacity (dynes per square centimetre).
Diamagnetic	Lead	$0{\cdot}16 \times 10^9$
	Zinc	$1{\cdot}1 - 1{\cdot}5 \times 10^9$
	Glass (sometimes paramagnetic)	$0{\cdot}3 - 0{\cdot}9 \times 10^9$
	Quartz	10×10^9
	Copper	$1{\cdot}2 - 2{\cdot}5 \times 10^9$
	Silver (wire)	$2{\cdot}9 \times 10^9$
	Gold (wire)	$2{\cdot}6 \times 10^9$
Paramagnetic	Aluminium	$0{\cdot}6 - 1{\cdot}5 \times 10^9$
	Tin	$0{\cdot}16 - 0{\cdot}38 \times 10^9$
	Glass (sometimes diamagnetic)	$0{\cdot}3 - 0{\cdot}9 \times 10^9$
	Platinum (wire)	$3{\cdot}3 \times 10^9$

We may conclude that the intrinsic stress due to the local molecular field, calculated as above, gives a satisfactory interpretation of the ultimate tensile strength of crystalline media, as observed experimentally, irrespective of the nature of their magnetic property.

The energy associated with the medium, in virtue of this intrinsic pressure of crystallization, is consistent to the right order with the value of the latent heat of fusion of the material.*

The energies per cubic centimetre of various diamagnetic and paramagnetic media, calculated from the latent heats of fusion, are given below :—

1. Organic Compounds Investigated in Parts I. and III.

Substance.	Magnetic nature.	Latent heat.	Density.	Energy (ergs per cubic centimetre).
Benzene	diamagnetic	30 (calories per gramme)	0·88	$1{\cdot}1 \times 10^9$
Xylene	,,	39	—	—
Chlorobenzene	,,	30	1·12	$1{\cdot}4 \times 10^9$
Bromobenzene	,,	20	1·49	$1{\cdot}3 \times 10^9$
Aniline	,,	21	1·02	$0{\cdot}9 \times 10^9$
Acetophenone	,,	33	—	—
Benzophenone	,,	23	1·10	$1{\cdot}0 \times 10^9$
Phenylhydrazine	,,	36	1·10	$1{\cdot}6 \times 10^9$
Pyridine	,,	22	0·985	$0{\cdot}9 \times 10^9$
Nitrobenzene	,,	22	1·19	$1{\cdot}1 \times 10^9$
Naphthalene	,,	35	1·15	$1{\cdot}7 \times 10^9$
Naphthylamine	,,	22	—	—
Acetic acid	,,	44	1·05	$1{\cdot}9 \times 10^9$
Glycerine	,,	42	1·26	$2{\cdot}2 \times 10^9$
Carbon tetra-chloride	,,	4	1·58	$0{\cdot}26 \times 10^9$

* See also Part III., p. 93.

2. ELEMENTS and Inorganic Compounds.

Substance.	Magnetic nature.	Latent heat.	Density.	Energy (ergs per cubic centimetre).
Bismuth	diamagnetic	13 (calories per gramme)	9·8	5·32 × 10^9
Cadmium	,,	14	8·6	5·0 × 10^9
Lead	,,	5	11·4	2·4 × 10^9
Silver	,,	22	10·5	9·7 × 10^9
Zinc	,,	28	7·1	8·3 × 10^9
Phosphorus	,,	5	1·8	0·38 × 10^9
Mercury	,,	3	13·6	1·7 × 10^9
Copper	,,	43	8·9	16·0 × 10^9
Sulphur	,,	9	2·0	0·8 × 10^9
Ice	,,	80	0·9	3·0 × 10^9
Aluminium	paramagnetic	77	2·7	8·7 × 10^9
Tin	,,	14	7·3	4·3 × 10^9
Palladium.	,,	36	11·4	17·0 × 10^9
Platinum	,,	27	21·5	24·0 × 10^9
Potassium	,,	16	0·86	0·56 × 10^9
Iron* (transformation $\beta - \gamma$ at A_3 point) .	—	1·4 (calories per gramme)	7·9	0·46 × 10^9

In the case of iron at the A_3 point, the transformation is from one cubic crystalline form to another, and we should expect the change of internal energy to be smaller than in the general case of actual crystallization from the liquid state.

We can obtain a measure of intrinsic pressures in crystalline media in another way, which depends on extrapolation of the relation connecting the temperature of the freezing point with applied pressure.

If

v_l = volume of 1 gramme of liquid,
v_c = volume of 1 gramme of crystal,
ϑ = temperature of fusion,
p = pressure, in atmospheres,
L = latent heat,

we know that

$$\frac{\partial \vartheta}{\partial p} = \frac{(v_l - v_c) \,.\, \vartheta}{\mathrm{L}} = \frac{\delta \mathrm{V} \,.\, \vartheta}{\mathrm{L}}.$$

If the applied pressure be such that $v_l = v_c$ then $\delta \mathrm{V} = 0$, and if we can determine the pressure π for which this condition exists, we have determined the intrinsic pressure due to the crystalline grouping, for if the latter were greater than, or less than, π there would be a change of volume on crystallization. If $v_l = v_c$ since, as the curve of fusion shows, both ϑ and L are finite, $\frac{\partial \vartheta}{\partial p}$ must be zero, *i.e.*, we have to

* A. E. OXLEY, 'Trans. Faraday Society,' vol. XI., Part 2, February, 1916.

determine the value of the applied pressure corresponding to the maximum of the closed area on the ϑ, p diagram. We cannot expect this method to give us anything but an approximate value of π because the extrapolation beyond laboratory pressures is considerable, but the results of the calculation are suggestive.

The relations between the temperature of fusion (ϑ) and applied pressure (p) in atmospheres, for the substances* here referred to are taken from 'Kristallisieren und Schmelzen' by G. TAMMANN, Leipzig, 1903, p. 204, *et seq.*

For *water* TAMMANN found

$$\vartheta - 22 = 0{\cdot}00438\,.\,(p-2200) - 77 \times 10^{-8}\,.\,(p-2200)^2.$$

Differentiating and equating $\frac{\partial \vartheta}{\partial p}$ to zero,

$$\frac{\partial \vartheta}{\partial p} = 0{\cdot}00438 - 154 \times 10^{-8}\,.\,(p-2200) = 0,$$

or

$$\pi = \frac{0{\cdot}00438 + 154 \times 2200 \times 10^{-8}}{154 \times 10^{-8}}$$

$$\doteqdot 5000 \text{ atmospheres} \doteqdot 5 \times 10^9 \text{ dynes/square centimetre.}$$

Benzophenone—

$$\vartheta = 48{\cdot}11 + 0{\cdot}02757p - 0{\cdot}00000136\,.\,p^2.$$

Differentiating and equating $\frac{\partial \vartheta}{\partial p}$ to zero we find the intrinsic pressure

$$\pi \doteqdot 10{,}000 \text{ atmospheres.}$$

Acetophenone—

$$\vartheta = 19{\cdot}2 + 0{\cdot}0235p - 0{\cdot}00000152\,.\,p^2$$

$$\pi \doteqdot 7700 \text{ atmospheres.}$$

Aniline—

$$\vartheta = -6{\cdot}1 + 0{\cdot}0203p - 0{\cdot}00000112\,.\,p^2$$

$$\pi \doteqdot 9000 \text{ atmospheres.}$$

Nitrobenzene—

$$\vartheta = 5{\cdot}67 + 0{\cdot}02344p - 0{\cdot}00000116\,.\,p^2$$

$$\pi \doteqdot 10{,}000 \text{ atmospheres.}$$

Xylene—

$$\vartheta = 13{\cdot}2 + 0{\cdot}03438p - 0{\cdot}00000171\,.\,p^2$$

$$\pi \doteqdot 10{,}000 \text{ atmospheres.}$$

Benzene—

$$\vartheta = 5{\cdot}43 + 0{\cdot}0283p - 0{\cdot}00000198p^2$$

$$\pi \doteqdot 7100 \text{ atmospheres.}$$

* Most of these substances show a change of diamagnetic susceptibility on crystallization. See Part I., pp. 120–131; Part III., pp. 96–97.

Naphthalene—

$$\vartheta = 79{\cdot}95 + 0{\cdot}03657p - 0{\cdot}00000180p^2$$

$$\pi \doteqdot 10{,}100 \text{ atmospheres.}$$

Carbon tetra-chloride—

$$\vartheta = -23{\cdot}0 + 0{\cdot}0350p - 0{\cdot}00000147p^2$$

$$\pi \doteqdot 11{,}900 \text{ atmospheres.}$$

Ethylene di-bromide—

$$\vartheta = 9{\cdot}85 + 0{\cdot}0252p - 0{\cdot}00000125p^2$$

$$\pi \doteqdot 10{,}000 \text{ atmospheres.}$$

Formic acid—

$$\vartheta = 7{\cdot}75 + 0{\cdot}01276p - 0{\cdot}00000080p^2$$

$$\pi \doteqdot 7980 \text{ atmospheres.}$$

Potassium—

$$\vartheta = 59{\cdot}5 + 0{\cdot}0146\ - 0{\cdot}0000007p^2$$

$$\pi \doteqdot 10{,}000 \text{ atmospheres.}$$

Phosphorus—

$$\vartheta = 43{\cdot}93 + 0{\cdot}0275p - 0{\cdot}00000050p^2$$

$$\pi \doteqdot 27{,}500 \text{ atmospheres.}$$

*Sulphur, rhombic-monoclinic**

$$\vartheta = 95{\cdot}4 + 0{\cdot}03725p + 0{\cdot}00000213p^2$$

$$\pi \doteqdot -8700 \text{ atmospheres.}$$

Solid CO_2—

$$\vartheta = -56{\cdot}8 + 0{\cdot}01999p - 0{\cdot}00000075p^2$$

$$\pi \doteqdot 13{,}300 \text{ atmospheres.}$$

These values, it is true, appear rather high when compared with those found from other considerations, but due importance must be attached to the difficulties of experimental work of this nature and to the fact that the experimental data have been extrapolated over a considerable pressure interval (several thousands of atmospheres).

Comparing these results with those given on pp. 252 to 254 it is considered that a mean value of the intrinsic stress in diamagnetic crystalline media, viz., 2×10^9 dynes per square centimetre is representative of the true order of magnitude of the forcive which binds the molecules in the space lattice of a crystalline medium. This implies that the energy per unit volume of the diamagnetic crystalline medium, in virtue of the crystalline grouping, is comparable with 2×10^9 ergs.

* This transition of sulphur from the rhombic to the monoclinic form is accompanied by thermal absorption. Since $\frac{\partial^2\vartheta}{\partial p^2}$ is positive the transition line for different pressures will be convex to the p-axis.

As the corresponding stresses and energies are of the same order in the ferro-magnetic metals, and, further, since it has been shown (see Part II., pp. 143, 145 and Part III., pp. 84–87) that the local molecular forcive in diamagnetic media is of the same order as that in the ferro-magnetic metals, we may conclude that the local intensities of magnetization in the two types of media are comparable.

Since

$$\tfrac{1}{2} \cdot \mathrm{H}_c \cdot \mathrm{I} \doteqdot 2 \times 10^9 \text{ ergs},$$

and

$$\mathrm{H}_c \doteqdot 10^7 \text{ gauss (see Part III., p. 86)},$$

we find

$$\mathrm{I} \doteqdot 400,$$

and

$$a'_c = \frac{\mathrm{H}_c}{\mathrm{I}} \doteqdot 2{\cdot}5 \times 10^4.$$

In a diamagnetic crystalline medium the local forcives are comparable with those in iron, and, since the latter medium shows hysteresis in a magnetic field, we may enquire whether a similar phenomenon will be shown by diamagnetic media. If the diamagnetic molecules are magnetically unsymmetrical, the application of an external magnetic field will tend to orientate them.* But this will be a differential effect on our conception of a diamagnetic molecule, and thus the tendency of the applied field to produce new molecular groupings will be small. We should therefore expect that hysteresis due to magnetization will be inappreciable in diamagnetic media. In iron, on account of the unbalanced magnetic nature of the molecules or atoms, new groupings are actually produced under fields of moderate intensity and the formation of these implies a loss of energy which is measured by the area of the hysteresis loop.

If, however, we take a diamagnetic copper wire and subject it to mechanical strain, the medium shows mechanical hysteresis. If sufficiently large stresses are employed, a permanent set is produced within the individual crystalline grains, new groupings of the molecules are formed, and a certain amount of energy is dissipated. All media, whether they are ferro-, para-, or diamagnetic, will show mechanical hysteresis. The difference from a magnetic point of view lies merely in the compensated nature of the diamagnetic molecule as compared with the uncompensated nature of the ferro-magnetic molecule, but the local forcives are comparable, so that under mechanical stress the mechanical hysteresis effects will be comparable.

(3) A Comparison of the Elasticities of some Diamagnetic Crystals with those of Crystalline Paramagnetic and Ferro-magnetic Media.

It is well known that the application of an external magnetic field alters the distribution of stress in a mass of iron crystals. On our theory we see how the

* This orientation in a diamagnetic liquid gives rise to the induced magnetic double refraction. See Part III., p. 87.

molecules orientate themselves during crystallization under the influence of the local forcives which are characteristic of the molecular configuration. The influence of these local forcives will produce in the crystalline medium a distribution of internal stress which will in general be different across different planes, and in this way the planes of cleavage can be defined. In the direction where the stress is greatest, we should expect the elastic properties of the crystal to be abnormally high, comparable in fact with the elastic properties of steel.

In other directions we should expect the elastic properties to be less pronounced, and indeed the shearing of crystals, merely by the insertion of a knife blade and the application of small pressure parallel to a plane of cleavage, is evidence of this.

The following values of YOUNG's Modulus of Rigidity for various ferro-magnetic, paramagnetic, and diamagnetic media show that the power to resist distortion is of the same order whatever the magnetic nature of the crystalline medium.

	Substance.	Magnetic nature.	YOUNG's modulus (dynes per square centimetre).	Rigidity (dynes per square centimetre).
*	Iron (0·1 per cent. carbon)	ferro-magnetic	$2 \cdot 13 \times 10^{12}$	$8 \cdot 3 \times 10^{11}$ (calc.)
*	Steel (1 per cent. carbon)	„	$2 \cdot 09 \times 10^{12}$	$8 \cdot 1 \times 10^{11}$
*	Nickel	„	$2 \cdot 02 \times 10^{12}$	$7 \cdot 7 \times 10^{11}$ (calc.)
*	Aluminium	paramagnetic	$7 \cdot 05 \times 10^{11}$	$2 \cdot 67 \times 10^{11}$
*	Tin	„	$5 \cdot 43 \times 10^{11}$	$2 \cdot 0 \times 10^{11}$ (calc.)
*	Glass	(sometimes diamagnetic)	$6 \cdot 5 - 7 \cdot 8 \times 10^{11}$	$2 \cdot 6 \times 10^{11}$
*	Platinum	paramagnetic	$1 \cdot 68 \times 10^{12}$	$6 \cdot 1 \times 10^{11}$
*	Lead	diamagnetic	$1 \cdot 62 \times 10^{11}$	$0 \cdot 56 \times 10^{11}$ (calc.)
*	Zinc	„	$1 \cdot 25 \times 10^{12}$	5×10^{11}
*	Copper	„	$1 \cdot 23 \times 10^{12}$	$4 \cdot 55 \times 10^{11}$
*	Silver	„	$7 \cdot 9 \times 10^{11}$	$2 \cdot 87 \times 10^{11}$
*	Gold	„	$8 \cdot 0 \times 10^{11}$	$2 \cdot 77 \times 10^{11}$
*	Bismuth	„	$3 \cdot 19 \times 10^{11}$	$1 \cdot 2 \times 10^{11}$ (calc.)
*	Quartz (fibre)	„	$5 \cdot 18 \times 10^{11}$	$3 \cdot 0 \times 10^{11}$
			Principal YOUNG's moduli (dynes per square centimetre).	Principal rigidity (dynes per square centimetre).
†	Quartz (crystalline)	diamagnetic	$1 \cdot 00 \times 10^{12}$	$5 \cdot 7 \times 10^{11}$
†	Beryl	paramagnetic	$2 \cdot 06 \times 10^{12}$ $2 \cdot 25 \times 10^{12}$	$6 \cdot 54 \times 10^{11}$ $9 \cdot 6 \times 10^{11}$
†	Topaz	diamagnetic	$2 \cdot 25 \times 10^{12}$ $2 \cdot 83 \times 10^{12}$ $2 \cdot 60 \times 10^{12}$	— — —
†	Rock salt	„	$0 \cdot 41 \times 10^{12}$	$1 \cdot 27 \times 10^{11}$
†	Potassium chloride	„	$0 \cdot 36 \times 10^{12}$	$0 \cdot 64 \times 10^{11}$
†	Fluor spar	„	$1 \cdot 44 \times 10^{12}$	$3 \cdot 4 \times 10^{11}$
†	Pyrites	—	$3 \cdot 46 \times 10^{12}$	$10 \cdot 5 \times 10^{11}$

* KAYE and LABY, 'Physical and Chemical Constants,' p. 27, 1918.

† A. E. H. LOVE, "The Mathematical Theory of Elasticity," 'Camb. Univ. Press,' p. 160. In the cases of beryl and topaz the different values correspond to bars whose lengths are in the directions of the different axes of symmetry.

As LOVE points out, the values of these elastic constants for beryl and topaz are remarkable in that they are greater than the corresponding constants in ordinary steel. The values of the elastic coefficients for most of the other substances in the above table are comparable with the constants for steel, and it is considered that these results give very strong evidence in favour of the large intermolecular forcive operative in diamagnetic crystalline media and confirm the suggestion made in Part II., p. 143, that this local forcive is comparable with that in ferro-magnetic media.

If a crystalline medium be heated, then as long as the crystalline state prevails, rotational vibrations of large amplitude are prevented, so that the specific heat of the crystalline medium is lower than that of the supercooled liquid.* In the latter case, the liquid at low temperatures passes into a rigid gel, and when this is heated, the molecules acquire rotational vibrations gradually until finally the ordinary liquid state is reached, possessing no appreciable rigidity. It is important to note that the molecules are vibrating under a local forcive to which we are ascribing the elastic properties of the medium, and therefore the theory is consistent with the theory of specific heat developed by DEBYE, in which the forces which control the thermal vibrations of the molecules are identical with those which determine the elastic constants of the medium. MADELUNG and SUTHERLAND have similarly suggested that the elastic forces resisting mechanical strain are just those forces which determine the infra-red optical vibrations of the atoms in the solid substance. It has been found possible to calculate the infra-red frequencies from a knowledge of the mechanical properties. In the present researches it has been shown that we can calculate both the optical frequencies† and the mechanical stresses from the local molecular forcive. *Within* the core of the atom the local controlling force may be more intense, and although such an intense forcive would not be directly operative in determining the state of crystallization, yet it might be responsible for determining frequencies on the ultra-violet side comparable with X-ray frequencies. (See *infra*, pp. 273 and 278.)

(4) THE CHANGE OF DENSITY ON CRYSTALLIZATION INTERPRETED AS A MAGNETO-STRICTION EFFECT OF THE MOLECULAR FIELD.

If we subject a liquid to a magnetic field, a change of volume occurs to such an extent that the internal pressure is reduced by an amount equal to the potential energy per unit volume of the magnetic field. This change of internal pressure (see Part III., p. 91) is

$$\tfrac{1}{2} \cdot k_l \cdot \mathrm{H}^2 + \tfrac{1}{2} \cdot \lambda \cdot k_l^2 \cdot \mathrm{H}^2 \qquad (2)$$

where

k_l is the susceptibility of the liquid per unit volume,

λ, a constant equal to 1/3,

and

H, the applied field intensity.

* Part III., p. 94.

† A. E. OXLEY, 'Roy. Soc. Proc.,' A, vol. 95, p. 58, 1918, and Part III., p. 84.

As k_l for diamagnetic media is of the order -7×10^{-7}, the second term in (2) is insignificant in comparison with $\frac{1}{2} . k_l . \mathrm{H}^2$.

If c be the compressibility of the liquid, the change of volume in cubic centimetres per cubic centimetre will be given by

$$\delta v = \frac{1}{2} . c . k_l . \mathrm{H}^2. \qquad (3)$$

This relation has been experimentally verified by QUINCKE.* The compressibility c is of the order 10^{-10}, particular values for different substances being :—

Substance.	$c \times 10^{10}$.
Benzene	0·8
Chlorobenzene	0·7
Toluene	0·8
Xylene	0·7
Water	0·5
Carbon tetrachloride	0·9
Acetic acid	0·4
Carbon bisulphide	0·9

Substance.	$c \times 10^{12}$.
Mercury	3·7
Potassium	31·5
Sodium	15·4
Lead	2·2
Tin	1·7
Bismuth	2·8
Iron	0·4

Since the largest magnetic field at our disposal is 50,000 gauss, the largest value of δv is

$$-\frac{1}{2} \times 10^{-10} \times 7 \times 10^{-7} \times 2{\cdot}5 \times 10^{9} = -8 \times 10^{-8} \doteqdot -10^{-7} \text{ c.c./c.c.}$$

Now we have shown (Part III., p. 90) that the potential energy term corresponding to (2) for a crystalline medium is

$$\frac{1}{2} . k_l . \mathrm{H}^2 + \frac{1}{2} . a'_c . \mathrm{I}^2$$

per unit volume, where a'_c is the constant of the local molecular field and I is the aggregate of the local intensity of magnetization per unit volume. The term $\frac{1}{2} . a'_c . \mathrm{I}^2$ is associated with each cubic centimetre of the crystalline structure whether

* See G. T. WALKER, "Aberration and the Electromagnetic Field," 'Camb. Univ. Press,' pp. 72–83.

H be zero or not. It is really due to the spontaneous *local* intensity of magnetization per unit volume and corresponds to the similar energy term, $\frac{1}{2}NI^2$ (WEISS), due to the spontaneous magnetization in iron. Using the values which have already been assigned to a'_c and I,* viz.,

$$\left.\begin{array}{r} a'_c \doteqdot 2\cdot5 \times 10^4\dagger \\ I \doteqdot 400 \end{array}\right\}, \qquad (4)$$

we see at once that this term is large in comparison with $\frac{1}{2} . k_l . H^2$ and therefore the change of volume to which the potential energy term $\frac{1}{2} . a'_c . I^2$ gives rise will be large in comparison with that which we can produce artificially in a liquid by applying the largest field available in the laboratory.

When a diamagnetic substance crystallizes, the alteration of internal pressure will be $\frac{1}{2}a'_c . I^2$ and therefore the accompanying change of volume in cubic centimetres per cubic centimetre will be

$$\delta V = \tfrac{1}{2} . c . a'_c . I^2,$$

and substituting from (4) we find

$$\delta V = \tfrac{1}{2} . 0\cdot8 \times 10^{-10} \times 16 \times 10^4 \times 2\cdot5 \times 10^4 = 0\cdot16 \text{ c.c./c.c.}$$

to the appropriate order.

Some values of δV in cubic centimetres per cubic centimetre are :—

Substance.	δV.	
Benzene	0·10	‡
Naphthalene	0·14	‡
Benzophenone	0·19	‡
Di-phenylamine	0·10	‡
Formic acid	0·10	‡
Sodium	0·03	‡
Potassium	0·03	‡
Mercury	0·036	‡
Lead	0·03	§
Tin	0·03	§
Bismuth	0·03	§
Iron (at A_3 point)	0·003	§

The values calculated agree as well as could be expected with the experimental determinations, since we know the orders of magnitude only of a'_c and I, for these are

* See p. 257 *supra*.

† A physical explanation of this large value of a'_c and of the corresponding constant N in ferromagnetism is given on p. 267 *infra*.

‡ G. TAMMANN, 'Kristallisieren u. Schmelzen,' Leipzig, 1903, pp. 204 *et seq.*

§ DESCH, 'Metallography,' p. 242.

unknown functions of the molecular structure and space lattice of each substance. It will be noticed that those substances showing a small value of δV have low compressibilities.

The case of iron at the A_3 transformation is particularly interesting. From the curves given by CHARPY and GRENET,* dealing with the expansion of iron and iron-carbon alloys between 200° C. and 1000° C., we can show that the extent of linear contraction which occurs suddenly at the A_3 point (900° C.) is of the order 0·003 centimetre per centimetre. The change of volume will be of this order of magnitude, which is small in comparison with the change of volume accompanying the crystallization of many organic compounds, but is very large compared with the magneto-striction effect which can be induced in either a ferro-magnetic or diamagnetic substance with a field of 50,000 gauss.

Taking into account the small compressibilty of iron,† which is only $0{\cdot}4 \times 10^{-12}$, or about 1/200 that of the liquids above referred to, this change of volume may be interpreted as due to a change of internal energy represented by $\frac{1}{2} . N . I^2$ where N is the constant of the ferro-magnetic field, of the order $0{\cdot}38 \times 10^4$,‡ I the saturation intensity of magnetization, of the order 1760. For we have

$$\begin{aligned}\delta V &= \tfrac{1}{2} . c . N . I^2 \\ &= \tfrac{1}{2} \times 0{\cdot}4 \times 10^{-12} \times 0{\cdot}38 \times 10^4 \times 1{\cdot}76^2 \times 10^6 \\ &= 0{\cdot}002\,(4) \text{ c.c./c.c.}\end{aligned}$$

which is of the order of magnitude found experimentally. The molecular field exists in an unmagnetized piece of iron and is accompanied by the large spontaneous magnetization of that element throughout an individual grain, but as these grains have all types of orientation, the large molecular field and the accompanying spontaneous magnetization are hidden in a piece of iron large enough to contain many grains. The molecular field will nevertheless produce the magneto-striction effect referred to above.§ Let us suppose that such a piece of iron is subjected to an external magnetic field. The molecules of all the grains will tend to come into alignment with the applied field and there will be a new distribution of stress. In an unmagnetized piece of iron, taken as a whole, the stress may be regarded as equal in all directions, but when an external field is applied, this is no longer the case and the iron shows a new magneto-striction effect consisting of an expansion in one direction and a contraction in the other. The extent of the redistribution of stress should be determined by a term of the form $\frac{1}{2} . a'_c . \iota^2$ where a'_c is the constant of the molecular field

* *Loc. cit.*

† RICHARDS, 'Journ. Chem. Soc.,' vol. 99, p. 1201, 1911.

‡ WEISS and BECK, 'Journ. de Phys.,' sér. iv., vol. 7, p. 249, 1908.

§ See *infra*, p. 265. It is assumed here that the molecular field disappears just above the A_3 point, at least in so far as it is effective in causing spontaneous magnetization. This is in accordance with the small paramagnetic susceptibility of iron above A_3.

and ι is the resultant intensity of magnetization induced in the direction of the external field. The compressibilities will now be different along and perpendicular to this direction, and the change of volume should be proportional to ι^2 for the particular value of the applied field. This result agrees with experiment.*

We can see in a general way how the sign of δV on crystallization may sometimes be positive and sometimes negative. Usually the molecular packing in the crystalline state will be closer than that in the liquid state, but it may happen that the configuration of the molecule is such that, when the parts which have the strongest magnetic attraction for each other are in the position of minimum potential energy, the packing is more open than in the liquid state. The appearance of the internal forcive on crystallization will in this case be accompanied by expansion. In the former case the appearance of the internal forcive will be accompanied by contraction.

The region of stability of the crystalline state is represented on the pressure temperature diagram by a closed area (fig. 2).

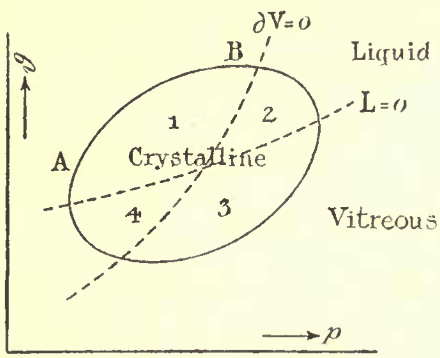

Fig. 2.

This area will in general be divided into four quadrants by the loci of the lines $\delta V = 0$ and $L = 0$, where δV is the change of volume on crystallization and L is the latent heat.

In the four quadrants the following conditions hold :—

Quadrant.	L.	∂V.	$\frac{\partial \vartheta}{\partial p}$.	$\frac{\partial^2 \vartheta}{\partial p^2}$.
1	+	+	+	−
2	+	−	−	−
3	−	−	+	+
4	−	+	−	+

If the melting-point is at some point along the arc AB of the first quadrant, increase of pressure raises the m.p. and δV and L are positive. This corresponds

* NAGAOKA and HONDA, 'Phil. Mag.,' vol. 46, p. 268, 1898.

with the case for benzophenone. In the case of water the diagram is as in fig. 3 and for $p = 1$ atmosphere, δV is negative and L is positive. The m.p. is in this case located on the arc A′B′ in the second quadrant. A similar case is that of pure iron as the temperature is raised through the critical point A_3 (see fig. 4). At A_3, δV is negative and L is positive, while $\frac{\partial \vartheta}{\partial p}$ is negative.

Now

$$\frac{\partial \vartheta}{\partial p} = \frac{(v_\gamma - v_\beta) . \vartheta}{L} = \frac{\delta V . \vartheta}{L}.$$

Therefore $v_\beta > v_\gamma$ which gives a shrinkage in iron on heating through the A_3 point at about 900° C.

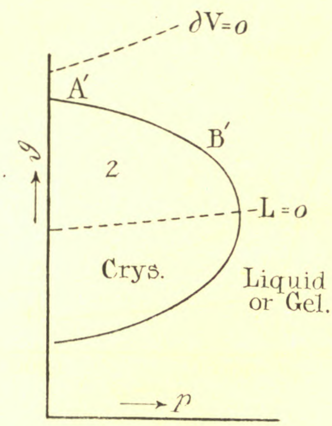

Fig. 3.

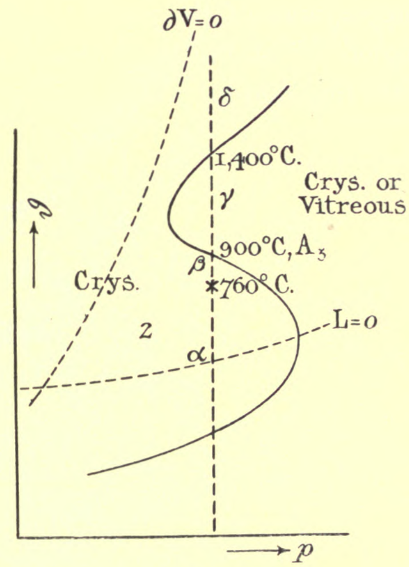

Fig. 4.

The application of the relation

$$\frac{\partial \vartheta}{\partial p} = \frac{(v' - v'') . \vartheta}{L} = \frac{\delta V . \vartheta}{L}$$

is interesting in connexion with some abnormalities of heats of recalescence in ferromagnetic media. The change of volume δV, both expansion and contraction; the absorption or evolution of heat of the amount L, depend solely upon the shape of the region of crystalline stability and its position relatively to the ϑ, p axes.

Thus nickel steel with no carbon and pure cobalt show no recalescence at the magnetic change points. In these cases L is very small and since δV and ϑ are finite, $\frac{\partial \vartheta}{\partial p}$ will be large (fig. 5). In other words, under the pressure of one atmosphere, the path of the crystallization curve cuts the line AB at a steep angle in the neighbourhood of the intersection of the fusion curve with the neutral line $L = 0$.

The apparent discontinuities of the susceptibility temperature curve are suggestive in this connexion (fig. 6). The upper branch of the curve AB for the so-called β-iron is practically continuous with the branch CD for the δ-range above 1400° C. Between B and C there is a break, the branch BE representing β-iron over a somewhat narrow range just below the critical temperature ϑ_c and the branch EF representing paramagnetic γ-iron. The locus BEFC corresponds to a crystalline modification of iron which is more stable over this temperature interval than the crystalline grouping or groupings over the ranges AB and CD.

If this view is correct, the molecular field is operative in iron over a temperature interval from 1400° C. upwards as well as below the critical temperature. The existence of this forcive above 1400° C. implies a crystalline symmetry involving an appreciable mutual action between the molecules consistent with the enhanced susceptibility found by CURIE* and by WEISS and FOEX† over this range. In the

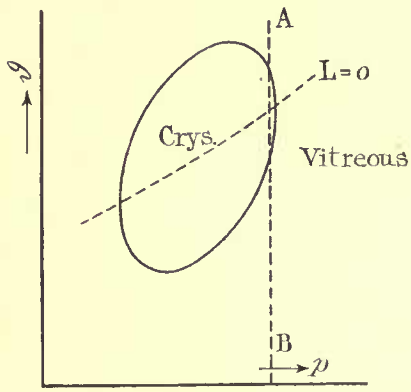

Fig. 5.

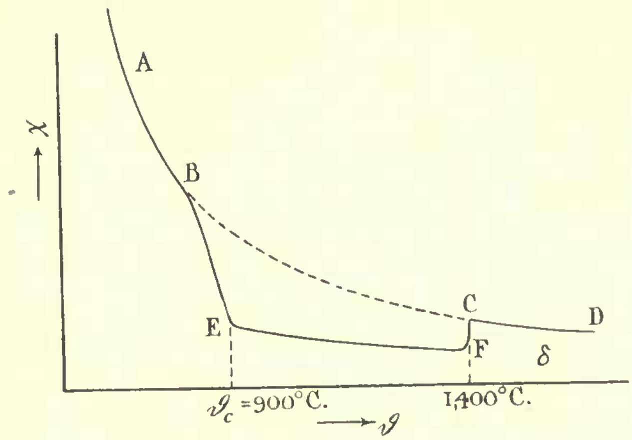

Fig. 6.

intermediate range, between 900° C. and 1400° C., iron shows a paramagnetic quality only. Perhaps we may regard the molecular state in this range as more allied to a gel, consisting of very small interlocked grains, each with relatively few molecules,‡ rather than to a coarse grain crystalline arrangement of the molecules. The orientations of the molecular axes as we pass from one small grain to another will be different, so that each grain is, as it were, surrounded by a surface of vitreous material. As stated on p. 251 the molecular field would then be non-effective in so far as the production of

* 'Annales de Chimie et de Physique,' sér. iv., vol. v., p. 289, 1895.

† 'Archives des Sciences, Genève,' sér. 4, vol. xxxi., p. 88, 1911.

‡ This smallness of grain structure above A_3 in wrought iron or mild steel is consistent with the experiments of J. E. STEAD, 'Iron and Steel Institute,' 1898, No. 1, p. 145. The reverse effect, that of a coarse structure on cooling from above A_3, observed by STEAD and CARPENTER ('Iron and Steel Institute,' No. 11, p. 119, 1913) in the case of thin strips of electrolytic iron, may possibly be attributed to surface forces. See also a paper "On the Part Played by the Amorphous Phase in the Hardening of Steels," by J. C. W. HUMFREY, 'Trans. Faraday Soc.,' May, 1915.

spontaneous magnetization is concerned, but the tenacity might even surpass that of the chaotic β-state since the transition $\beta-\gamma$ is accompanied by shrinkage. Such an increase of tenacity was actually found by ROSENHAIN and HUMFREY.*

The effects of prolonged heating on the plasticity of mild steel are interesting in this connexion. EWING and ROSENHAIN† have shown that the plasticity of a material is caused by slips occurring on cleavage or "gliding" surfaces *within each* of the crystalline grains, although the elementary portions of the crystals retain their primitive form and the crystalline structure of the metal as a whole is preserved. In the case of mild steel, exposure to a temperature of 1200° C. or higher temperatures for several hours may cause the material to lose much of its plasticity, while some specimens of soft iron after prolonged exposure at 700° C. to 800° C. (less than the critical temperature) have been made brittle. These results are consistent with a similarity of molecular configuration for temperatures above 1200° C. and below the critical temperature (about 850° C.). On the other hand an exposure at 900° C. or 1000° C. (*i.e.*, in the region of the paramagnetic state), followed by a slow or fairly rapid cooling, induces considerable plasticity in the material, and this treatment may even be used to remove brittleness originating from heating to the higher or lower ranges of temperature mentioned above. As the plasticity is produced by slipping on cleavage surfaces *within* the crystalline grains, this smallness of the grain structure may, under stress, determine a molecular rotation. EWING and ROSENHAIN ‡ have shown that in some metals, in addition to slips or motions of pure translation, there results a molecular rotation from strain which gives rise to twin-crystals. It is interesting to note that the formation of twin-crystals is common in iron through the γ-range, but has not been observed in the β and α ranges.

As the temperature falls below 1400° C. there is some modification of the crystalline cubic arrangement, resulting in a closer packing of the molecules, and accompanied by an interlocking of the fine grains. Thus iron in the γ-range (1400° C. to 900° C.) will be paramagnetic. At a lower temperature than 900° C., this state is unstable and another modification of the crystalline grouping occurs, accompanied by expansion, thermal evolution, and the appearance of spontaneous magnetization. This latter effect seems inconsistent with a more open packing of the molecules, but an analogy is found in the case of water, where the molecular influence in the liquid just above 0° C. is small compared with that in ice just below freezing point, although the packing of the molecules in the liquid state is closer than that in the crystalline state (see *supra* p. 264). As the temperature is lowered the transformation progresses rapidly until a point B is reached, after which the increase of magnetic quality is somewhat less rapid. On continued cooling, the iron passes into the α-range where the magnetic property is capable of attaining a saturation value.

* 'Roy. Soc. Proc.,' A, vol. 83, 1909; 'Iron and Steel Institute,' No. 1, 1913.

† 'Phil. Trans. Roy. Soc.,' A, vol. 193, p. 279.

‡ See EWING, 'The Strength of Materials,' p. 47, 1906.

(5) Further Discussion of the Nature of the Local Molecular Field in Ferro-magnetic and Diamagnetic Media.

This subject has already been discussed in Part III., pp. 89 and 100, but no interpretation was then given to the magnitudes of the constants N and α'_c of the local ferro-magnetic and diamagnetic fields. The values of N given by Weiss and Beck* are $0{\cdot}38 \times 10^4$ for iron, and $1{\cdot}27 \times 10^4$ for nickel. For diamagnetic crystalline media α'_c is of the order 2×10^4.

Ewing and Low† have shown that in very strong magnetic fields the relation between induction (B) and applied field (H) may be represented by the equation

$$B = H + \text{a constant.} \qquad (1)$$

This constant has the value $4\pi I$ (where I is the saturation intensity) and is equal to 21,360 in wrought iron, 6470 in nickel and 16,300 in cobalt.

In the case of wrought iron

$$B = H + 21360. \qquad (2)$$

Suppose we could apply a field equal to the molecular field, $6{\cdot}53 \times 10^6$ gauss for iron. The limiting value of the permeability μ_L for this field will be from (2)

$$\mu_L = 1 + \frac{21360}{6{\cdot}53 \times 10^6},$$

and the limiting susceptibility per unit volume

$$\chi_L = \frac{\mu_L - 1}{4\pi} = \frac{21360}{4\pi \times 6{\cdot}53 \times 10^6} = 2{\cdot}60 \times 10^{-4},$$

and

$$\frac{1}{\chi_L} = 0{\cdot}38 \times 10^4.$$

This is equal to the value of N, the coefficient of the molecular field, as we should expect.

Similar calculations may be made for nickel and cobalt, the limiting susceptibilities being respectively

$$\chi_L = 0{\cdot}81 \times 10^{-4} \text{ for nickel,}$$

$$\chi_L = 2{\cdot}0 \times 10^{-4} \text{ for cobalt.}$$

Now we may ask the question, why is it that, in spite of the fact that all the molecules are ordered into a definite space lattice under the influence of the respective molecular fields, the materials still show a finite susceptibility to magnetization? The

* 'Journ. de Phys.,' sér. iv., vol. 7, p. 249, 1908.

† 'Phil. Trans. Roy. Soc.,' A, p. 242, 1889.

explanation, I think, is to be found in the finite though small angular oscillations which constitute a portion of the thermal energy of the molecules. The molecules are fixed relative to one another and form a definite space lattice but they are oscillating with small amplitude under the molecular field. This allows them to retain a finite susceptibility. Suppose we could double the molecular field, the limiting susceptibility would become one-half its former value and the saturation intensity of magnetization would be slightly increased. If we could increase the molecular field indefinitely, the susceptibility would get indefinitely small, the product of the two however tending to a finite limit equal to the *true* saturation intensity of magnetization. The amplitude of the molecular oscillations would, under the influence of this indefinitely large forcive, be indefinitely small. This state might be attained in a practical manner by cooling the substance, say in liquid hydrogen, when the limiting susceptibility would become vanishingly small.

As N is the reciprocal of the limiting susceptibility the constant of the molecular field will become indefinitely large. WEISS, however, supposes N to be constant.* The tendency of χ_L to approach a small limiting value as the temperature is lowered is confirmed experimentally for ferro-magnetic substances† and is particularly noticeable in the case of weak magnetic fields. The reduction of the amplitude of vibration of the molecules as the absolute zero is approached merely implies a higher frequency of angular oscillation under the increasing molecular field and does not necessarily imply that the rotational energy becomes vanishingly small. In this case it should be noted that the saturation intensity of magnetization we are considering is smaller than that which would be given by the simple summation of all the magnetic moments of the molecules in unit volume. In other words, the difficulty of producing this latter saturation by an external field becomes increasingly difficult on account of the larger molecular forcive at low temperatures, in agreement with the vanishingly small susceptibility referred to above. At higher temperatures the susceptibility to an external field is far greater; the molecules are, as it were, helped over their difficulties with respect to the molecular field, when the external field is applied, by the increased energy of the rotational oscillations, and having passed this critical point they are held in new combinations. Beyond the critical point the molecular state is chaotic, the molecules being interlocked (*cf.* p. 265), and the external field has sufficient control to produce a paramagnetic effect only.

* Following WEISS, we have taken the molecular field proportional to I. WEISS writes the molecular field NI and assumes N to be constant. This applies with sufficient accuracy in a temperature region just below the critical temperature, but cannot be true over the whole region down to absolute zero, because, as the molecular translational vibrations die down, the molecules approach one another more closely and the molecular field must necessarily increase considerably although I remains practically constant. This increase is accounted for by the increase of the coefficient N, which is the reciprocal of the limiting susceptibility.

† EWING, 'Magnetic Induction in Iron and other Metals,' p. 172 *et seq.*, where curves are given for iron, hard steel, nickel and various nickel steels. See also p. 269 *infra* and EWING, *loc. cit.*, p. 354.

The fall off in the value of χ as the temperature is reduced, may, in part, be explained by the increased value of the molecular field, due to the nearer approach of the molecules. As the local molecular field becomes very big, the induced diamagnetic effect in each molecule will become big, in the same proportion, and this will tend to reduce the value of χ and make this quantity tend to a limiting value. If the molecular field is of the order 10^7 gauss at ordinary temperatures, we have seen that the ratio of the induced diamagnetic moment ΔM to the magnetic moment of the electron orbit M, is of the order 1/100. If the molecular field at low temperatures approaches 10^9 gauss, owing to closer proximity of the magnetic elements in neighbouring molecules, the diamagnetic effect would be comparable with the ferro-magnetic effect.* On our view this does not imply that diamagnetic substances should acquire a large diamagnetic susceptibility at very low temperatures. For taking the molecules in pairs, locally they are paramagnetic and the action of the local molecular field is to reduce this paramagnetic effect so that the local magnetic moment becomes smaller and the susceptibility to an external field tends to zero as in iron.

According to EWING† experiments carried out to test this effect have neither proved nor disproved this theory, probably because the external fields were not sufficiently strong. But during crystallization we are applying unconsciously to each molecular current a magnetic field 500 or 1000 times stronger than the largest field we can apply externally, and probably even greater local intensities are attained at low temperatures, since the interacting magnetic elements in adjacent molecules may be almost touching one another. The mutual induction and temperature effects combine to cause χ to approach the limit zero at the absolute zero or in very powerful external fields.

In diamagnetic media we have seen that the constant of the local molecular field a'_c (which corresponds to N in WEISS's ferro-magnetic field) is of the order $2{\cdot}5 \times 10^4$ and the reciprocal of this, viz., $+4 \times 10^{-5}$, is the order of magnitude of the *local positive limiting susceptibility* of a diamagnetic crystalline medium. (At ordinary temperatures the diamagnetic susceptibility per unit volume is of the order -10^{-6}). The parts of molecules adjacent to one another in a diamagnetic crystalline medium attract in a similar manner to the adjacent parts of molecules of a ferro-magnetic or paramagnetic medium.

In a ferro-magnetic medium, as the temperature is raised, the susceptibility increases up to a certain point just below the critical temperature and then falls off rapidly. The temperature controls the susceptibility in two ways; first, by helping the molecules to overcome the difficulties of orientation, produced by the neighbouring molecules, to a point just below the critical temperature; second, by overdoing this effect and by giving the molecules too much rotational energy, at the critical temperature and above, so that the susceptibility to magnetization falls very rapidly.

* In nickel the molecular field is $6 \cdot 3 \times 10^6$ gauss. An applied field of 10^6 gauss would make a substance as diamagnetic as bismuth have a saturation value equal to that of nickel.

† 'Magnetic Induction in Iron and other Metals,' p. 353.

In a diamagnetic crystalline medium, as the temperature is raised, the local positive susceptibility will obey a similar law, the temperature of fusion now corresponding to the critical temperature in the ferro-magnetic case. Although locally the relation between susceptibility and temperature is the same in the two cases, the effect passes unnoticed in the diamagnetic case because the molecule has a total zero magnetic moment. Nevertheless, the effect of temperature acts in its two antagonistic ways in diamagnetic as well as in ferro-magnetic media. When the temperature is above the melting point, the rotational energy of the molecules annuls the local forcive (liquid state), when it is very low the molecules become interlocked and cannot readjust themselves in a space lattice (gel state). There is an intermediate region of temperature where opportunity is offered for the tendency of self-orientation under the mutual local forcives to display itself, and over this range crystallization may take place. This intermediate temperature range defines the closed region of stability of the crystalline form on the pressure temperature diagram of equilibrium of the crystalline and amorphous states (see p. 263).

A discussion as to how far we may regard it as proved that the local molecular field in crystalline media is of magnetic nature was given in para. 8, of Part III. The conclusion reached was that the molecular field is certainly in part magnetic. It is possible to bring forward further evidence of the truth of this deduction. In some noteworthy researches* published by TYNDALL, as long ago as 1870, it was shown that magnetic properties of crystalline media bear a close relation to molecular aggregation. About 100 different crystals were examined and from the deportment of these, when subjected to a magnetic field, TYNDALL found that "if the arrangement of the component particles of any body be such as to present different degrees of proximity in different directions, then the line of closest proximity, other circumstances being equal, will be that chosen by the respective forces for the exhibition of their greatest energy. If the mass be magnetic this line will stand axial, if diamagnetic, equatorial."†

The exactness of the dependence of magnetic deportment on the position of cleavage planes is remarkably shown in these experiments. Whatever the crystal examined, it was found that the magnetic deportment disclosed accurate information of the planes of cleavage. TYNDALL describes the results of his important experiments in such elegant language that it may be permissible to quote some of them at length. Thus he continues :—" From this point of view, the deportment of the two classes of crystals represented by Iceland spar and carbonate of iron, presents no difficulty. This crystalline form is the same, and as to the arrangement of the particles, what is true of one will be true of the other. Supposing then, the line of closest proximity to coincide with the optic axis; this line, according to the principle expressed, will stand axial or equatorial, according as the mass is magnetic or diamagnetic, which is

* 'On Diamagnetism and Magnecrystallic Action,' 1870.

† *Loc. cit.*, p. 23.

precisely what the experiments with these crystals exhibit. Analogy as we have seen justifies the assumption here made. It will, however, be of interest to enquire, whether any discoverable circumstance connected with crystalline structure exists upon which the difference of proximity depends and knowing which, we can pronounce with tolerable certainty, as to the position which the crystal will take up in the magnetic field.

"The following experiments will perhaps suggest a reply.

"If a prism of sulphate of magnesia be suspended between the poles with its axis horizontal, on exciting the magnet the axis will take up the equatorial position. This is not entirely due to the form of the crystal; for even when its axial dimension is shortest, the axis will assert the equatorial position, thus behaving like a magnetic body, setting its longest dimension from pole to pole.

"Suspended from its end with its axis vertical, the prism will take up a determinate oblique position. When the crystal has come to rest, let that line through the mass which stands exactly equatorial be carefully marked. Lay a knife-edge along this line, and press it in the direction of the axis. The crystal will split before the pressure, disclosing shining surfaces of cleavage. This is the only cleavage the crystal possesses and it stands equatorial. Sulphate of zinc is of the same form as sulphate of magnesia, and its cleavage is discoverable by a process exactly similar to that just described. Both crystals set their planes of cleavage equatorial. Both are diamagnetic.

"Let us now examine a magnetic crystal of similar form. Sulphate of nickel is, perhaps, as good an example as we can choose. Suspended in the magnetic field with its axis horizontal, on exciting the magnet the axis will set itself from pole to pole, and this position will be persisted in, even when the axial dimension is shortest. Suspended from its end, the crystalline prism will take up an oblique position with considerable energy. When the crystal thus suspended has come to rest, mark the line along its end which stands *axial*. Let a knife edge be laid along this line and pressed in a direction parallel to the axis of the prism. The crystal will yield before the edge and discover a perfectly clean plane of cleavage.

"These facts are suggestive. The crystals here experimented with are of the same outward form; each has but one cleavage, and the position of this cleavage with regard to the form of the crystal, is the same in all. The magnetic force, however, at once discovers a difference of action. *The cleavages of the diamagnetic specimens stand equatorial; of the magnetic, axial.*

"A cube cut from a prism of scapolite, the axis of the prism being perpendicular to two of the parallel faces of the cube, suspended in the magnetic field, sets itself with the axis of the prism from pole to pole.

"A cube of beryl of the same dimensions with the axis of the prism from which it is taken also perpendicular to two of the faces, suspended as in the former case, sets itself with the axis equatorial. Both these crystals are magnetic.

"The former experiments showed a dissimilarity of action between magnetic and diamagnetic crystals. In the present instances, both are magnetic, but still there is a difference; the axis of the one prism stands axial, the axis of the other equatorial. With regard to the explanation of this, the following fact is significant. Scapolite cleaves *parallel* to its axis, while beryl cleaves *perpendicular* to its axis; the cleavages in both cases, therefore, stand axial, thus agreeing with sulphate of nickel. The cleavages hence appear to take up a determinate position regardless of outward form, and they seem to exercise a ruling power over the deportment of the crystal.

"A cube of saltpetre, suspended with the crystallographic axis horizontal, sets itself between the poles with this axis equatorial.

"A cube of topaz, suspended with the crystallographic axis horizontal, sets itself with this axis from pole to pole.

"We have here a kind of complementary case to the former. Both these crystals are diamagnetic. Saltpetre cleaves parallel to its axis; topaz perpendicular to its axis. The planes of cleavage, therefore, stand in both cases equatorial, thus agreeing with sulphate of zinc and sulphate of magnesia.

"Where do these facts point? A moment's speculation will perhaps be allowed us here. May we not suppose these crystals to be composed of layers indefinitely thin, laid side by side, within the range of cohesion, which holds them together, but yet not in absolute contact? This seems to be no strained idea; for expansion and contraction by heat and cold compel us to assume that the particles of matter in general do not touch each other; that there are unfilled spaces between them. In such crystals as we have described these spaces may be considered as alternating with the plates which compose the crystal. From this point of view it seems very natural that the magnetic laminæ should set themselves axial, and the diamagnetic equatorial.

"Our fundamental idea is, that crystals of one cleavage are made up of plates indefinitely thin, separated by spaces indefinitely narrow. If, however, we suppose two cleavages existing at right angles to each other, then we must relinquish the notion of plates and substitute that of little parallel bars; for the plates are divided into such by the second cleavage. If we further suppose these bars to be intersected by a cleavage at right angles to their length, then the component crystals will be little cubes, as in the case of rock-salt and others. By thus increasing the cleavages, the original plates may be subdivided indefinitely, the shape of the little component crystal bearing special relation to the position of the planes. It is an inference which follows immediately from our way of viewing the subject, that if the crystal have several planes of cleavage, but all parallel to the same straight line, this line, in the case of magnetic crystals, will stand axial; in the case of diamagnetic, equatorial. It also follows that in the so-called regular crystals, in rock-salt, for instance, the cleavages annul each other, and consequently, no directive power will be exhibited, which is actually the case."

The above quotation from TYNDALL'S work clearly shows how closely allied are the different forces of crystallization in different directions (which forces determine the planes of cleavage) with the magnetic behaviour of the crystallized medium and lead us to suspect that the forces of cohesion are probably of magnetic nature. The fine points are so completely explained by the magnetic deportment that it is difficult to dissociate the crystalline forces from a magnetic origin. If we assume that these forces are of an electrostatic nature, then it must be admitted that the electrostatic axis of the molecule must coincide with the magnetic axis if the action of a magnetic field is to be decisive, as TYNDALL proved it to be, in isolating the planes of cleavage. But if the electrostatic and magnetic symmetries of the molecules are coincident the application of a field of either nature should induce a double refraction of the same kind in a given liquid. This, however, is not true experimentally, the electric induced double refraction in liquid carbon bisulphide being opposite in sign to the magnetic induced double refraction.* Moreover, in crystalline media, the greatest axes of the ellipsoids representing the magnetic and electric properties of the molecule do not in general coincide. We may therefore say that the evidence points to the conclusion that the forcive which holds the molecules together in a crystalline space lattice is magnetic in nature and not electrostatic.†

DRUDE,‡ in his experiments on the relation between valency and dispersion,

* COTTON and MOUTON, 'Comptes Rendus,' vol. 155, p. 1232, December, 1912.

† [*Note added April 26, 1919.*—After the present communication had passed out of my hands, an important paper "On the Origin of Spectral Series" was published by Sir J. J. THOMSON ('Phil. Mag.,' April, 1919). In this a new theory of atomic structure is suggested in which the atomic nucleus and the revolving electrons play similar rôles to those described on p. 274. Within the contour of the atom, according to Prof. THOMSON, the electrostatic forcive due to the nucleus is of a periodic character and determines a series of spherical or approximately spherical surfaces where the electric force vanishes and over which the periodic motion of the boundary electrons is determined solely by the magnetic field of the atom. This magnetic field is supposed to be radial. If this is the case, these intra-atomic fields must be of the order of magnitude 10^8 gauss $\left(\text{as a simple calculation shows, since } \nu = \frac{H\epsilon}{2\pi m}\right)$ to account for the frequencies of the visible spectrum. Still larger intra-atomic fields will exist nearer to the nucleus, of the order 10^9 gauss. These will be sufficient to account for the frequencies of the K series. The infra-red series will be accounted for by fields of the order 10^7 gauss. But this latter value is of the order of the intermolecular magnetic field which has been deduced independently in various ways in the present researches. Moreover, it is to this local field that we have ascribed the rigidity and other properties of crystalline media in general. The frequencies of the infra-red series will, on this view, correspond with the elastic vibrations of the rigid medium in conformity with the quantum theory of specific heats of EINSTEIN and DEBYE as already stated (see Part III., p. 94, and *supra*, p. 259). Reasons have already been given for assigning a magnetic nature to the intermolecular field in crystalline media (see Part III., pp. 101–3, and *supra*, pp. 270–276). This intermolecular magnetic field, which is of the order 10^7 gauss, is suggestive in connexion with Prof. THOMSON's theory, referred to above. On p. 274 (footnote) it was suggested that the forces determining crystalline cohesion are magnetic in nature, the symmetry of the magnetic forces being determined, however, by the electrostatic action of the nucleus. Therefore, in this fundamental sense, the present theory and that of Sir J. J. THOMSON are identical.]

‡ 'Ann. der Phys.,' vol. 14, p. 677 and p. 936, 1904.

suggested that the electron couples constituting the molecule were of two kinds: (1) those of the atoms themselves, the sum of which presumably determine the atomic weight; (2) those of valency which alone are sufficiently free to vibrate synchronously with light waves and hence are particularly concerned in the refraction and dispersion of light.

The valency or boundary electrons are vibrating under the control of the nucleus, but are less firmly held in the system than those near the nucleus. These valency electrons which have periods corresponding with luminous vibrations are affected by an external magnetic field in accordance with the well-known Zeeman and diamagnetic effects. The highly constitutive nature of the magnetic susceptibility is consistent with this view that the origin of the magnetic property is partly located near the molecular boundary. The nuclear electrons in the free atom will determine symmetry of the molecule, and are directly responsible, by their magnetic effect, for the symmetry of the crystalline grouping. This latter will therefore be determined by the nucleus, which controls the nuclear and boundary electrons, in an indirect manner and the distribution of atomic nuclei in accordance with crystalline symmetry as disclosed by X-ray methods is apparent.

W. H. and W. L. BRAGG have shown the difficulty, even in simple cases, of defining the molecular boundaries in a crystalline space lattice, although in some cases this is possible. But to determine by the X-ray method whether in any given crystal any atom has a special relation to a neighbouring atom would be practically impossible. The X-ray effects which they investigate are determined only by the nucleus or core of the atom and the outer electrons of the atom which contribute to its magnetic property, though they are controlled by the nucleus, are probably distorted by the influences of neighbouring magnetic elements. This distortion, which explains a large number of observed phenomena, defines the molecular boundary within the space lattice and determines a definite chemical molecule. These outer regions of the atom or molecule remain undetected by the X-ray experiments.*

* A. E. OXLEY, 'Nature,' No. 4, 1915. The core or electrostatic part of the atom is at a much greater distance from the atomic boundary than are the circular currents which give rise to the magnetic properties. As the intensity of the magnetic field due to a circular current varies inversely as the cube of the distance, and as in a crystalline structure two such circuits may approach so as almost to touch, each electron describing a *small* circle, the local magnetic forcive may be sufficiently large to account for the facts.

The view that the cohesive force in crystalline media is of a magnetic nature was expressed in Part II. of this research, pp. 83–86. It was there stated that in a diamagnetic crystalline medium the molecules are held together by the local magnetic forces due to the revolving electrons. It is possible that each electron is completely bound to its own nucleus by a narrow tube of force, when the molecules would be capable of attracting or repelling one another electromagnetically according to their directions of rotation. The advantage of an electromagnetic cohesive force lies in the fact that by it we can readily see how similar molecules will cling together. Electrostatically such attraction implies an electron transfer which, we know, does not always take place. Reasons are given on p. 277 that the atomic forces which determine the structure of the molecule are in part at least of a magnetic nature. The advantage of

The theory propounded by TYNDALL was called by him the "theory of reciprocal induction," and the direction within the crystalline medium where the molecules had the closest proximity and along which the greatest energy was displayed, he called the "line of elective polarity." This theory is identical with our hypothesis of mutual molecular distortion enunciated at the beginning of Part I. and subsequently confirmed in a variety of ways by other physical phenomena. The direction of closest approach of the molecules, *i.e.*, the line of elective polarity is the line along which the crystal shows the maximum elastic properties. TYNDALL'S explanations of the phenomena he had discovered were prophetic. The diamagnetic forces were known to be so minute that the theory of reciprocal induction appeared incredible, and, as a correspondence between Lord KELVIN and TYNDALL shows,* the former expressed emphatically his view that this theory was quite incapable of accounting for the effects observed. On our modern conception of the magnetic structure of matter, this doubt is dispelled and the smallness of the diamagnetic property is no barrier to the theory of reciprocal induction. The effect of applying pressure to a diamagnetic medium, produced, in the direction of the pressure, an increase in the diamagnetic property. This was attributed by TYNDALL to the mutual actions of the diamagnetic polarities which are so minute that their effects, as then understood, would be of such a small order of magnitude that they could not be detected by experiment. On our view of a diamagnetic molecule, which maintains that such a molecule is paramagnetic locally, the effects observed by TYNDALL can be accounted for quantitatively for the local molecular forcives are comparable with those in para- and ferro-magnetic media.

But even in the case of ferro-magnetic media it is not obvious that the magnetic forces are sufficient to explain the mechanical phenomena unless we realise the localised nature of the forcive. If we take, for example, a crevasse of the usual

assigning a magnetic nature to the forces of valency is clear, for in this way, without admitting an electron transfer between the various atoms forming the molecule, we can secure the necessary attraction, and this by a fixed or directed force which at the same time is compatible with a characteristic orbital frequency such as appears to be necessary to account for ordinary absorption, magnetic rotation, and diamagnetic phenomena. The possibility of a satisfactory interpretation of many problems suggested by stereochemistry, in terms of the magnetic force due to revolving electrons, has been ably expounded by A. L. PARSON ('Smithsonian Miscellaneous Collections,' vol. 65, No. 11, a paper to which, on account of war service, I have only recently had access). Though PARSON'S theory involves new difficulties in connexion with the distribution of positive electricity in the atom, the advantages from a chemical standpoint which he secures by the introduction of magnetic forces of chemical combination cannot be denied. Granting this, it is natural to suppose that the cohesive forces, which hold the molecules in position in a space lattice, will be residual magnetic forces, and that they will closely resemble, in distribution at least, the atomic forces determining the configuration of the molecule. It will be of great interest to see how far such magnetic cohesive forces are capable of interpreting the spacing of molecules in a crystalline lattice in accordance with the distribution disclosed by X-ray analysis. Magnecrystallic action, as we have seen, is explicable in this way.

* Various letters, 'On Diamagnetism and Magnecrystallic Action,' 1870.

conventional dimensions, within an iron crystal where the saturation intensity is I, the mechanical stress is $2\pi I^2$, which is far smaller than the ultimate tensile strength of the iron. Indeed, EWING* remarks "we may, if we please, regard the magnetic molecules as pulling at one another across any imaginary interface, while the stress with which they pull is balanced by thrust in the framework of the iron, but neither the pull nor the thrust is competent to explain the mechanical strains." The above value of the stress, viz., $2\pi I^2$ is obtained by taking a crevasse whose gap, although "physically small," is sufficiently wide to accommodate several molecules in line. If we take a narrower crevasse, approaching "mathematical smallness" in width of gap, we obtain a measure of the forcive acting between the molecules, and this includes the localised forcive NI or $\alpha'_c I$ which is of the order 10^7 gauss. The localised stress across this interface is $\frac{1}{2}NI^2$ or $\frac{1}{2}\alpha'_c I^2$, which we have seen to be of the order 2×10^9 dynes per square centimetre. This is of the same order as the ultimate tensile strength of crystalline media both ferro-magnetic and diamagnetic.

The magnetic resistance of joints is interesting in connexion with the localised nature of the molecular field in iron. It has been shown by Sir J. J. THOMSON and H. F. NEWALL† that the susceptibility of an iron bar is much reduced if it is severed and the two parts put in contact. Later, Sir JAMES EWING and W. LOW‡ investigated this effect in a more exhaustive manner when the joints were carefully trued up and also for rough joints, under varying pressures. They found that for a carefully planed joint a compressive stress of 226 kilogrammes per square centimetre restored almost completely the loss of magnetic property produced by cutting, but that this stress had only a small restorative effect in the case of a rough joint. In the latter case, we may suppose that the number of points of contact between the two parts of the bar is small, in the former that the two portions are in contact over a large percentage of the available area of contact. Under a compressive stress of 226 kilogrammes per square centimetre, it appears that in the trued-up specimens the order of contact of the molecules is the same as in the uncut metal and therefore this stress is a measure of the internal stress within the material. As 226 kilogrammes per square centimetre is equal to $0 \cdot 5 \times 10^9$ dynes per square centimetre, this stress, although lower, is comparable with that calculated on p. 252, and we may regard the width of the resulting crevasse as approaching mathematical smallness, the spheres of influence of the molecules on either side of it overlapping to an extent comparable with the overlap in the interior of the uncut bar (see also Part III., p. 89). But even with the most carefully faced junction there will be irregularities, coarse compared with molecular dimensions, and in such regions the localised nature of the molecular field will determine a finite air gap which would account for the difference of stress mentioned above. Perfectly faced surfaces of soft iron or mild steel (annealed) might be

* 'Magnetic Induction in Iron and other Metals,' p. 254.

† 'Proc. Camb. Phil. Soc.,' 1887.

‡ 'Phil. Mag.,' September, 1888.

expected, under the influence of an external magnetic field, to form a perfect junction, in other words to become welded together.

Taylor Jones* has obtained an induction as high as 74,200 Maxwells in soft iron under strong fields. The tension necessary to pull the surfaces apart in this case will be $\frac{B^2}{8\pi} = \frac{7\cdot4^2 \times 10^8}{8\pi}$, or 2×10^8 dynes/square centimetre, which is about a twenty-fifth of the tensile strength of the material.

(6) On a Magnetic Theory of Chemical Combination.

On the theory of chemical action developed by Sir J. J. Thomson,† the determining feature of an atom from the point of view of chemical combination is the number of positive valency electrons it possesses. These electrons are dragged from their loose attachment to the nucleus, during chemical combination, and pass from one atom to another. The two originally neutral atoms thus become oppositely charged and so attract one another and form, as it were, an electric doublet. Let us look at this problem from the magnetic standpoint. Each electron orbit is equivalent to a small magnetic doublet and it is interesting to enquire how far the magnetic forces of such doublets may represent the force of chemical affinity. Recent work on radio-activity, the wide deflections of β-rays, and the diffraction of X-rays, all point to a localisation of the electrostatic charges in a minute core or nucleus. Round this nucleus, and under its control, the valency electrons (in part responsible for the magnetic properties) rotate. It is conceivable, therefore, that the magnetic forces, in addition to the important role they play in crystallization, may also in part be responsible for the forces of chemical affinity.‡

If, during chemical combination, there is a definite transfer of valency electrons from one atom to another, we should expect to find an abrupt change in the magnetic behaviour of an atom before and after chemical combination. If, on the other hand, there is no such electron transference, we might expect that the atoms would preserve their magnetic properties, which would be more or less of an additive nature. In a remarkable series of investigations, Pascal§ has shown that in a very large number of organic compounds, the molecular susceptibility is the sum of the atomic susceptibilities of the component atoms, provided the molecule contains no peculiarity of molecular configuration—such, for instance, as the ethylene linkage, unsaturated atom, or complex nucleus. Thus if χ_M is the molecular susceptibility, and χ_A the atomic susceptibility of a component atom, we have

$$\chi_M = \Sigma\chi_A + \lambda$$

where the summation extends to all the atoms in the molecule and λ is a positive or

* 'Phil. Mag.,' vol. xli., p. 165, 1896.

† 'The Corpuscular Theory of Matter,' 1907, Chap. VI.

‡ See footnote p. 274. Also W. M. Hicks, 'Roy. Soc. Proc.,' A, vol. 90, p. 356.

§ 'Ann. de Chim. et de Physique,' sér. 8, vol. 19, p. 5, 1910.

negative constant for a certain type of peculiarity of molecular configuration. In normally saturated compounds $\lambda = 0$. The elements carbon, hydrogen, chlorine, bromine and iodine have constant atomic susceptibilities in a large variety of simple and complex organic compounds. This suggests that the origin of the valencies of these elements is also the origin of a definite amount of diamagnetism, under the influence of a definite magnetic field. In other words, a hydrogen atom, in whatever organic compound it is found, has a constant atomic susceptibility equal to $-30{\cdot}5 \times 10^{-7}$, while the carbon atom has a constant atomic susceptibility equal to $-62{\cdot}5 \times 10^{-7}$, and so on.

This result of PASCAL'S, in conjunction with—

(1) The enormous magnitude of the local molecular field in diamagnetic media, and

(2) The conception of diamagnetism as due to an induction effect in oppositely spinning electrons (as developed in Parts I., II. and III.), led me to suspect that the magneton may be a constituent of the diamagnetic hydrogen molecule. The calculation showed* that if there is one electron in each hydrogen atom whose period is equal to

* 'Roy. Soc. Proc.,' A, vol. 95, p. 58, 1918.

At the time this paper was written, I was out of touch with the latest available data concerning the values of AVAGADRO'S constant (N) and the ratio ϵ/m. The calculation was to determine M from the relation

$$M = -\frac{\chi \cdot 4\pi m}{N \cdot n \cdot \epsilon \cdot \tau}.$$

Taking χ = molecular susceptibility of hydrogen,
= $-61{\cdot}0 \times 10^{-7}$ (PASCAL),
$N = 6{\cdot}06 \times 10^{23}$ (MILLIKAN),
$\epsilon/m = 1{\cdot}77 \times 10^{7}$ e.m.u. (BUCHERER),
$n = 2$, the number of electrons per molecule,
$\tau = 2{\cdot}19 \times 10^{-15}$ sec., the period of revolution for the line Hα,

we find on calculation

$M = 16{\cdot}3 \times 10^{-22}$ for the moment of the magneton.

This gives for r, as calculated from $M = \frac{\pi\epsilon r^2}{\tau}$, the value $0{\cdot}85 \times 10^{-8}$ cm.

In this connexion it should be pointed out that, on PLANCK'S theory of quanta of energy, the constant h is consistent with the existence of a unit of magnetism. Assuming, as NICHOLSON and BOHR have done, that the angular momentum of the electron is an integral multiple of $\frac{h}{2\pi}$, CHALMERS showed that the magnetic moment of the electron orbit is

$$M = \frac{\epsilon}{m} \cdot \frac{h}{4\pi}.$$

This gives $M = 92{\cdot}4 \times 10^{-22}$ e.m.u., which is 5 times the experimental value of the moment of the magneton. If we leave aside PLANCK'S theory of energy quanta and adopt instead SOMMERFELD'S theory of quanta of action, LANGEVIN showed that a remarkable relation between h and M exists. He found

$$M = \frac{\epsilon}{m} \cdot \frac{h}{24\pi},$$

when the law of attraction between the nucleus and the electron is the inverse square. This gives for the magnetic moment of the electron orbit $M = 15{\cdot}4 \times 10^{-22}$ e.m.u., a value nearly equal to the most recent experimental value of the moment of the magneton, viz., $18{\cdot}5 \times 10^{-22}$ e.m.u.

that of the line H_α, then the molecular diamagnetic susceptibility of hydrogen $(-61 \cdot 0 \times 10^{-7})$ can be accounted for, and each electron orbit of radius 10^{-8} cm. has a magnetic moment $+16 \cdot 3 \times 10^{-22}$, nearly equal to that of the magneton, $+18 \cdot 5 \times 10^{-22}$.

The atomic susceptibility of carbon in combination is shown by PASCAL to be $-62 \cdot 5 \times 10^{-7}$, and in connexion with the additive law this value is consistent with the experimental values of the molecular susceptibilities. The mean experimental value of the atomic susceptibility of diamond is $-59 \cdot 0 \times 10^{-7}$.* The mean of these values is $-60 \cdot 7 \times 10^{-7}$, which is probably as accurate a value as is available at present. But this value is almost exactly twice that of the atomic susceptibility of hydrogen $-30 \cdot 5 \times 10^{-7}$. Probably therefore the atom of carbon contains two magnetons. As to the period of revolution we have

$$M = \tfrac{1}{2} \cdot \epsilon \cdot \omega r^2 = \frac{\pi \epsilon r^2}{\tau}$$

where

M = moment of orbit,
ϵ = electron charge in e.m.u.,
r = radius of orbit,
τ = period,

and this implies that the period τ for the carbon atom is not equal to that of the line H_α unless $r = 10^{-8}$ for the carbon atom.

The sum of the atomic susceptibilities of the atoms in the group CH_2 is

$$-123 \cdot 5 \times 10^{-7}.$$

Experimentally, PASCAL showed that the difference of molecular susceptibility in a number of compounds whose constitution differed by this group was

$$-118 \cdot 7 \times 10^{-7}.$$

The mean of these two values is $-121 \cdot 1 \times 10^{-7}$ which is probably very near the true value. But this is almost exactly four times the value of the atomic susceptibility of hydrogen, viz., $-30 \cdot 5 \times 10^{-7}$, and in this combination, CH_2, we may say that each hydrogen atom has one magneton and the carbon atom has two magnetons.

An ethylene linkage, according to PASCAL, lowers the diamagnetic molecular susceptibility of the compound by $+57 \times 10^{-7}$, while two or more such linkages lower it by $+110 \times 10^{-7}$. These values are respectively nearly equal to twice and four times the atomic susceptibility of hydrogen.

As the additive law holds in the case of the simpler liquid hydrocarbons, it will hold for all the others which differ only by CH_2 groups, and therefore, taking the values of λ into account, any member of the three homologous series

$$C_n \cdot H_{2n+2}, \qquad C_n \cdot H_{2n}, \qquad C_n \cdot H_{2n-2}$$

* HONDA, 'Ann. der Phys.,' vol. 32, p. 1044, 1910, gives $-59 \cdot 5 \times 10^{-7}$; OWEN, 'Ann. der Phys.,' vol. 37, p. 693, 1912, gives $-58 \cdot 5 \times 10^{-7}$.

may be considered as containing respectively

$$4n+2, \quad 4n-2, \quad 4n-6 \text{ magnetons.}$$

Moreover, PASCAL has shown that the magnitude of χ_{CH_2} is quite independent of the presence of nitrogen, oxygen, or sulphur in the compound, so that, where the additive law holds for such complex molecules, it will hold for the whole series of *additive* compounds formed with the group.

PASCAL also investigated the halogens, fluorine, chlorine, bromine and iodine. The atomic susceptibilities of these elements, as deduced from the molecular susceptibilities of organic compounds in which they are contained, are given in column 2, the approximate number of magnetons per atom is given in column 3.

Substance.	$\chi_A \cdot 10^7$.	n.
Fluorine	$-65 \cdot 5$	2
Chlorine	$-209 \cdot 5$	7
Bromine	$-319 \cdot 2$	10
Iodine	$-465 \cdot 0$	15

It must be admitted, however, that these last four results do not warrant an extension of the magneton theory to these elements. Moreover, although the results for the hydrocarbons mentioned above are very suggestive, yet there remain difficulties, such for example as the values of λ, which for the benzene ring is equal to -15×10^{-7}, the interpretation of which does not fall into line with the magneton view. Further difficulties are met with in the cases of nitrogen and oxygen.

Perhaps these difficulties should be expected, since it has been proved that the additive law breaks down for many atoms, especially as regards the metallic elements. When, in addition, we take into consideration that the mutual disturbances of the electron orbits, in atoms containing a relatively large number of electrons, have been neglected, the agreement is probably as good as could be expected. Thus in the case of the hydrocarbons considered above, compounds which show no electrolytic dissociation and in the formation of which no transfer of electrons takes place from one atom to another on combination, we might expect that the addition of a hydrogen atom or a CH_2 group would add a definite amount of diamagnetism to the compound. But in the case of the metals and some other elements, chemical combination may be accompanied by the transference of electrons, *i.e.*, by a breakdown of the magnetic elements of the atoms. In such cases the additive law could not hold.

Thus iron-carbonyl ($Fe(CO)_5$) and nickel-carbonyl ($Ni(CO)_4$) are diamagnetic*; potassium ferri-cyanide is paramagnetic, while potassium ferro-cyanide is diamagnetic.† It would appear that in these cases the loss of magnetic property of the iron and nickel atoms is due to a transfer of valency electrons, *i.e.*, it involves an electric charging up of the atoms. The behaviour of the oxygen atom in organic compounds, in compounds with chlorine and in metallic oxides, where it acts always as an electronegative element, may possibly be accounted for in the same way. Free oxygen and ozone are strongly paramagnetic, but no semblance of an additive nature of the magnetic property is found in any of the oxygen compounds.

The appearance of strong magnetism in the Heusler alloys and its disappearance in manganese steels, are similar effects, dependent on the formation of intermetallic chemical compounds accompanied by an electron transfer.

There are many paramagnetic substances which possess molecular magnetic moments comparable with, and in some cases much superior to, those shown by ferro-magnetic substances. The apparent feeble susceptibility they possess is due to the fact that, with the largest magnetic field which can be applied, we can never produce anything like a saturation effect. In fact, according to WEISS, the molecule of cobalt chloride, $Co.Cl_2$ contains 25 magnetons, while an atom of cobalt below the critical temperature contains 9 magnetons. We may well ask ourselves—what is the nature of the process by which the addition of a diamagnetic substance, H.Cl to cobalt, produces such a large increase in the number of magnetons per molecule.‡ Assuming the work of WEISS holds good, and there is certainly a very considerable amount of evidence in favour of his theory, we can interpret this result either by supposing that the atom of cobalt really contains more than 9 magnetons,§ or else that the *diamagnetic* acid supplies the additional magnetons when it acts on the cobalt to form the chloride. In either case, it seems that we must admit that a molecule may possess systems of magnetons which, in certain circumstances, are so arranged to counterbalance one another, producing no additional moment of the molecule.‖ These magnetons would contribute nothing to the paramagnetic or ferro-magnetic property of a substance and could not be included in WEISS's theory. The grouping of these "latent" magnetons, according to our extended view, would be perturbed by the union of the cobalt atom with the Cl ion, in a manner similar to that by which an external field reveals the spontaneous magnetization in iron (as interpreted on EWING's theory) by orientating groups which were formerly so constituted as to show no magnetic effect externally. If this is so, then we are only justified in assuming that purely diamagnetic molecules contain groups of magnetons so arranged that the

* A. E. OXLEY, 'Proc. Camb. Phil. Soc.,' vol. 16, p. 102, 1911.

† J. S. TOWNSEND, 'Phil. Trans. Roy. Soc.,' A, vol. 187, p. 547, 1896.

‡ The specific susceptibility of cobalt chloride is 90×10^{-6}, that of hydrochloric acid $-0 \cdot 80 \times 10^{-6}$.

§ Some of which are self-compensated.

‖ *I.e.*, of the molecule as a whole.

molecule possesses no resultant magnetic moment. For example, while iron below the critical temperature possesses, according to WEISS, 11 magnetons to the atom, the molecule of ferric sulphate possesses 30, ferric chloride 28, sodium ferro-pyrophosphate 26, sodium ferrous oxalate 27.* Again, nickel below the critical temperature possesses 3 magnetons to the atom, above the critical temperature either 8 or 9 magnetons to the atom, while a molecule of nickel sulphate contains 16 magnetons. In general the number of magnetons, per molecule of a salt of a ferro-magnetic element, is large compared with the number of magnetons associated with an atom of the pure ferro-magnetic. The fact that cupric salts are paramagnetic while cuprous ones are diamagnetic is interesting from our point of view. Although copper is diamagnetic, yet a molecule of cupric sulphate contains 10 magnetons. It seems as if the large local atomic fields, which have been recognised in diamagnetic and ferro-magnetic molecules, have the power, when the molecules approach so that their fields overlap, to upset the equilibrium of the atoms in combination and redistribute their magnetic elements. This is easily possible when the great intensity of the local molecular field is borne in mind. In most cases a diamagnetic molecule, on account of its symmetry, would, under the influence of such a field, remain diamagnetic, but each orbit would be distorted by the field and the susceptibility of the substance would be slightly modified.

(7) ON SOME ANOMALIES IN THE MAGNETIC ROTATION EFFECT.

Diamagnetic media are in general dextro-gyric. The only exception is titanium chloride which is lævo-gyric.† Paramagnetic media are sometimes dextro-gyric and sometimes lævo-gyric, while the ferro-magnetic elements, iron, nickel and cobalt are all dextro-gyric. At present no theory seems capable of accounting for these anomalies and it is therefore interesting to examine to what extent the local molecular field may cause the effects observed. At one time it was thought that the direction of rotation probably depended merely on the diamagnetic or paramagnetic property of the molecule, but experiment soon disproved this generalisation. VOIGT ‡ suggested that the production of an intense reverse field, when the external field was applied, would account for the effects, but no physical explanation of a possible origin for this intense reverse field was given.

On the views of diamagnetic and paramagnetic polarity developed in these researches, the necessary fields demanded by VOIGT are found in the immediate neighbourhood of the molecular boundary. In diamagnetic liquids the molecules have zero magnetic moments and their axes are distributed at random. The application of

* 'Journal de Phys.,' vol. 1, sér. v., p. 974, 1911.

† For solutions of salts in water it should be noted that VERDET'S constant for the solute is to be regarded as negative if VERDET'S constant for the solvent is less than 0·0130, which is the value of the constant for pure water at 20° C.

‡ See P. ZEEMAN, 'Researches in Magneto Optics,' p. 185.

an external field produces a minute diamagnetic polarity, which differential effect determines the magnitude and sign of the rotation. When a polarised ray passes over such molecules, it will be rotated considerably locally, but only to be rotated in the opposite direction to an almost equal extent in a neighbouring molecule or in another part of the same molecule.

This will also be true if the diamagnetic medium is crystalline. Hence the final rotation will be a relatively small differential effect compatible with the diamagnetic effect of the medium in bulk. We shall omit the exceptional case of titanium chloride for the present and pass on to consider the ferro-magnetic media iron, nickel and cobalt.

In ferro-magnetic media, on account of the continuity of magnetic induction, there is an enormous reverse local field, and if this acts over regions of the molecule containing magnetically active electrons, a large rotation will be produced, which will not be compensated in neighbouring molecules, when the ferro-magnetic material is saturated. Hence we should expect a very large rotation to be produced by such media. This has been confirmed experimentally, in the cases of iron, nickel and cobalt, by KUNDT.

Paramagnetic solutions lie in an intermediate category. The application of an external magnetic field causes a certain amount of molecular orientation depending on the temperature. Such orientation causes a reverse field over a part of the system, but over the neighbouring molecules in combination with the one considered, the field depends upon the difference between the reverse local field and the applied field, which difference may be positive or negative. If, in such regions, there are electrons capable of orientating the polarised beam, the rotation may be dextro- or lævo-gyric, according as the resultant field is opposite to, or in the same direction as, the applied field. This will not explain the exceptional case of titanium chloride unless this molecule possesses some peculiar dissymmetry, whereby, in spite of its diamagnetic nature, it becomes orientated under the applied field. Such orientation is the basis of the magnetic double refraction theory developed by LANGEVIN* and confirmed experimentally in some respects by COTTON and MOUTON† (see Part III., p. 87). An alternative explanation of the behaviour of the titanium chloride molecule may be found in a rotation of the paramagnetic titanium atom relative to the compound molecule.

BECQUEREL has deduced an interesting relation‡ connecting the magnetic rotation (r) with the Zeeman coefficient $\left(Z = \frac{\epsilon}{4\pi m}\right)$, the applied field H, wave-length λ and refractive index μ, viz.,

$$r = -2\pi \cdot \frac{ZH\lambda}{c} \cdot \frac{\partial\mu}{\partial\lambda}, \qquad (1)$$

where c is the velocity of light.

* LANGEVIN, 'Le Radium,' vol. 7, p. 251, 1910.

† 'Ann. de Chim. et de Phys.,' sér. viii., vol. 19, p. 155, 1910.

‡ SCHUSTER, 'Theory of Optics,' p. 307.

We shall assume that this relation applies to all media. On our theory H will be the sum of the applied and local fields and will be considered positive if it is in the direction of the external field. Z we shall take to be negative, *i.e.*, the negative electron is always responsible for the Zeeman effect.

In diamagnetic media, in general, H is positive, $\frac{\partial\mu}{\partial\lambda}$ is negative, and therefore r is negative, or clockwise looked at along the direction of H. The rotation is dextrogyric.

In ferro-magnetic media, H is the resultant of the applied field and a very large reverse molecular field, so that H is very large and negative; $\frac{\partial\mu}{\partial\lambda}$ is positive. Hence again r is negative and the rotation is dextro-gyric as in diamagnetic media.

In paramagnetic solutions, H may be positive or negative over magnetically active atoms, and therefore the sign of r may be positive or negative. Hence some paramagnetic solutions will be dextro- and some lævo-gyric.

In the ferro-magnetic elements the magnitude of the rotation is remarkably high. Thin films of saturated iron show a rotation of the order 260 million times that of carbon bi-sulphide subjected to an external field of one gauss. To test this we may write

$$r_{\text{Fe (saturated)}} = -2\pi \,.\, \frac{Z_{\text{Fe}} \,.\, H_c}{c} \,.\, \lambda \,.\, \frac{\partial\mu_{\text{Fe}}}{\partial\lambda} \text{ for iron,}$$

$$r_{\text{CS}_2\text{ (H = 1 gauss)}} = -2\pi \,.\, \frac{Z_{\text{CS}_2} \,.\, H}{c} \,.\, \lambda \,.\, \frac{\partial\mu_{\text{CS}_2}}{\partial\lambda} \text{ for carbon bisulphide.}$$

where H_c is the reversed molecular field, of the order $-6{\cdot}5\times10^6$ gauss, H = 1 gauss. The Zeeman coefficients Z_{Fe} and Z_{CS_2} we shall take to be of the same order, hence :—

$$\frac{r_{\text{Fe (saturated)}}}{r_{\text{CS}_2\text{ (H = 1 gauss)}}} = +\frac{H_c \,.\, \dfrac{\partial\mu_{\text{Fe}}}{\partial\lambda}}{H \,.\, \dfrac{\partial\mu_{\text{CS}_2}}{\partial\lambda}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

For iron*

$\lambda\times10^8$.	μ.
2570	1·01
4410	1·28
5890 (D)	1·51

Near

$$\lambda = 5890\times10^{-8}, \qquad \partial\mu_{\text{Fe}}/\partial\lambda = +\frac{0{\cdot}23}{1480\times10^{-8}}.$$

* 'Smithsonian Physical Tables,' p. 196.

For carbon bisulphide*

$\lambda \times 10^8$.	μ.
4340 (H_γ)	1·6920
4860 (H_β)	1·6688
5890 (D)	1·6443

Near

$$\lambda = 5890 \times 10^{-8}, \qquad \partial\mu_{CS_2}/\partial\lambda = -\frac{0·0255}{1030 \times 10^{-8}}.$$

From (2)

$$\frac{r_{\text{Fe (saturated)}}}{r_{\text{CS}_2\,(\text{H}=1\text{ gauss})}} = +\frac{6·53 \times 10^6 \times 0·23 \times 1030 \times 10^{-8}}{0·0255 \times 1480 \times 10^{-8}} \doteqdot 40 \times 10^6.$$

For nickel†

$\lambda \times 10^8$.	μ.
2750	1·09
4410	1·16
5890 (D)	1·30

Near

$$\lambda = 5890 \times 10^{-8}, \qquad \frac{\partial\mu_{Ni}}{\partial\lambda} = +\frac{0·14}{1480 \times 10^{-8}},$$

and

$$\frac{r_{\text{Ni (saturated)}}}{r_{\text{CS}_2\,(\text{H}=1\text{ gauss})}} = +\frac{6·35 \times 10^6 \times 0·14 \times 1030 \times 10^{-8}}{0·0255 \times 1480 \times 10^{-8}} \doteqdot 25 \times 10^6.$$

For cobalt‡

$\lambda \times 10^8$.	μ.
2750	1·41
5000	1·93
6500	2·35

Near

$$\lambda = 5890 \times 10^{-8}, \qquad \frac{\partial\mu_{Co}}{\partial\lambda} = +\frac{0·42}{1500 \times 10^{-8}}.$$

Taking the reverse molecular field equal to that in iron

$$\frac{r_{\text{Co (saturated)}}}{r_{\text{CS}_2\,(\text{H}=1\text{ gauss})}} = +\frac{6·53 \times 10^6 \times 0·42 \times 1030 \times 10^{-8}}{0·0255 \times 1500 \times 10^{-8}} \doteqdot 87 \times 10^6.$$

These values, although rather small, are comparable with the enormous ratios of the rotations of saturated ferro-magnetics to diamagnetics, as actually observed

* *Loc. cit.*, p. 192.
† *Loc. cit.*, p. 196.
‡ *Loc. cit.*, p. 195.

experimentally, and point to the importance of the local molecular forcive in magneto-optic phenomena. It should be noted that the value of the reverse field we have taken is that of the *intermolecular* field. Within the atom the reverse field will probably be greater than $-6{\cdot}5 \times 10^6$ gauss. If it is of the order -5×10^7 gauss, the above ratios become 320×10^6 for iron, 200×10^6 for nickel and 700×10^6 for cobalt. Intra-atomic fields of the order 10^8 gauss are required by HUMPHRIES* to explain the pressure shift of spectral lines and by RITZ† in his theory of spectral series (see also Part III., p. 100, and *supra*, p. 273, footnote).

(8) SUMMARY OF CONCLUSIONS.

(I.) The applications of the local molecular forcive, in diamagnetic, paramagnetic and ferro-magnetic media, have in the present research been extended to interpret the ultimate tensile strength of crystalline and vitreous media. It has been shown by EWING and ROSENHAIN that the permanent set which occurs prior to breaking is due to slipping along the cleavage planes *within* the individual crystalline grains. We should therefore expect that the material would be fractured when the applied mechanical stress is equal to that produced internally by the local molecular forcive. The internal stress within the material is shown to be of the order 2×10^9 dynes per square centimetre which is approximately the mean value of the ultimate tensile strengths of crystalline and vitreous media (pp. 250–259).

(II.) As a consequence of this internal stress, the energy per unit volume will be 2×10^9 ergs, and this energy, which is over and above that which exists in the fluid state, should be a measure of the latent heat of fusion per cubic centimetre. This test which was applied in Part III. to test the order of the local forcive, has been extended to a variety of organic and inorganic media, including the metals and is found to accord with the experimental values to the right order (pp. 253–4).

(III.) Since the forces under which the molecules vibrate are those to which we ascribe the elastic properties of crystalline media, the results obtained are consistent with the theory of specific heats developed by DEBYE, in which the specific heat is attributed to purely translational vibration, and it has been shown (Part III.) that, near the fusion point, the rotational energy acquired by the molecules will give a measurable departure from this theory which is actually observed experimentally. As we should expect, it is found that the elastic constants of a variety of ferro-magnetic, diamagnetic and paramagnetic media are of the same order, several diamagnetic and paramagnetic media even surpassing steel in their power to resist distortion (pp. 257–259).

(IV.) Any change of internal pressure will be accompanied by a change of volume defined by the compressibility of the medium and dependent as to sign upon

* 'Astrophysical Journal,' vol. 23, p. 232, 1906; vol. 35, p. 268, 1912.

† 'Ann. der Phys.,' vol. 25, p. 660, 1908.

peculiarities of the molecular configuration. It has been shown that the energy change which occurs on crystallization is compatible with a volume change of the same order of magnitude as that accompanying crystallization, and we may therefore interpret the change of volume on crystallization as a magneto-striction effect of the local molecular forcive. The magneto-striction effect depends on molecular orientation which is proportional to the square of the magnetic forcive (pp. 259–263). In Part III. it was shown that the double refraction of crystalline media can be interpreted as due to the magnetic double refraction effect of the local molecular forcive which orientates the molecules into a crystalline space lattice. This effect is also proportional to the square of the magnetic forcive and the two effects mutually support one another.

(V.) The above results are interesting in connexion with TAMMANN'S theory of the closed region of stability of the crystalline state, as represented on the pressure temperature diagram. TAMMANN'S experimental work gives an alternative method of determining intrinsic pressures, but the results are notably higher than those found in other ways. Possibly this is due to extrapolation over a wide pressure range. The pressure temperature diagrams showing the fusion curve are instructive in dealing with problems relating to thermal evolution (or absorption), and volume changes and possible interpretations of these peculiarities in the ferro-magnetic elements, iron, nickel and cobalt, have been given (pp. 263–266).

(VI.) A physical interpretation has been given of the large values of the coefficients, N and a'_c, of the molecular fields in ferro-magnetic and diamagnetic crystalline media respectively. These coefficients are the reciprocals of the limiting local susceptibilities of the media under field strengths equal to the respective molecular fields. The local susceptibility of a diamagnetic molecule is comparable with that of a ferro-magnetic molecule and the two vary in the same way with field strength and temperature. In diamagnetic media, however, magnetic hysteresis will be inappreciable, since the molecule as a whole possesses a zero magnetic moment. Nevertheless, mechanical hysteresis in diamagnetic media will be of the same order as in ferro-magnetic media (pp. 267–270 and p. 257).

(VII.) From TYNDALL'S experiments on the deportment of paramagnetic and diamagnetic crystals in a magnetic field, the positions of the planes of cleavage can be traced. These results show that the forces responsible for crystalline symmetry are very probably of a magnetic nature. If the forces are of an electrostatic nature, then, since an electric field must disclose the same planes of cleavage, the electric and magnetic symmetries must coincide. This is not the case however. The magnetic forces are partly due to the valency or boundary electrons whose orbits are controlled by the atomic nuclei. The nuclei determine the crystalline symmetry, indirectly, through the medium of the magnetic forces of the electrons. This conclusion is not at variance with the results of X-ray diffraction experiments; the latter determine only the positions of the diffuse diffracting cores and give no indication as to the outer regions of the atom which determine the valency forcives. Hence, in spite of the disclosure of the X-ray methods, it is maintained that within a crystalline medium the molecules, though distorted, are still essentially integral units and that it is possible to imagine a surface enclosing each (pp. 270–277).

(VIII.) The smallness of the nucleus disclosed by recent work on radio-activity suggests that in addition to the part played by the magnetic forces in crystallization, these forces are in part responsible for chemical combination (footnote, p. 274). The theory of chemical combination developed by Sir J. J. THOMSON implies the transference of electrons from one atom to another whereby the atoms become oppositely charged. Such a transference of the valency electrons implies a complete readjustment of the magnetic property and therefore this property could not be of an additive nature. This is borne out by a large amount of experimental evidence.

If the chemical combination takes place without a transference of the valency electrons from one atom to another, which probably happens in many organic compounds in which electrolytic dissociation does not take place, we might expect the magnetic properties to be more or less of an additive nature. PASCAL'S work confirms this view (pp. 277–281).

(IX.) The author has previously shown that the magneton may be a constituent of the diamagnetic hydrogen molecule. It appears that this idea may be extended to carbon and the hydrocarbons in general, where the molecular susceptibility can be directly calculated from the atomic susceptibilities of the component atoms. The results, however, are not so convincing when extended to other elements, but the fact that departures from the additive law occur in such cases leads us to suppose that some disturbing influence has been neglected. WEISS'S work on salts of the ferro-magnetic elements, taken in conjunction with our present conception of diamagnetism, suggests that diamagnetic substances contain magnetons, compensated so as to produce a diamagnetic effect of the medium in bulk. The forces of chemical combination may, however, perturb this state, and by rearranging the magnetic elements give rise to a compound possessing more magnetons than are contained in an unbalanced state in a ferro-magnetic element. Examples of this are given and also of the reverse effect which may equally well arise (pp. 281–282).

(X.) The principle of the continuity of magnetic induction, as applied to the local molecular forcive, suggests a possible interpretation of known anomalies of the magnetic rotational effect. In paramagnetic solutions, the dextro-gyric or lævo-gyric property is attributed to a differential effect of the reversed local field and the applied field over rotationally active electrons, the sign of the effect depending on the direction of the resultant field acting over these electrons.

The very powerful rotational effect of thin films of iron, nickel and cobalt, in comparison with that shown by carbon bisulphide, has also received an interpretation in terms of the local molecular forcive (pp. 282–286).

(XI.) It is considered that the above conclusions when conjoined with those obtained in Parts I., II., III., which bring the theory into line with the magnetic atom fields of RITZ and probably suggest an origin of spectral series, amply justify the importance of the magnetic forcive in crystalline and vitreous media. The magnitude of this local magnetic forcive, first calculated to interpret the change of diamagnetic susceptibility observed on crystallization of a large number of organic compounds experimented upon in Parts I. and III., has been found capable of correlating a number of additional physical phenomena of wide difference of origin. It is hoped to continue these applications to other branches of optics including spectral series and optical activity.

(XII.) [*Added February 28, 1920.*—It has been established that the intermolecular field, in all crystalline media, is of the order 10^7 gauss. The electrons, within the free atom, are controlled by electrostatic, and possibly also by magnetic, forces, whose origin lies in the core. When the atoms are grouped into a definite space lattice, the cohesive force between them is of a magnetic nature, and the rigidity of such media is due to the localised mechanical stress, exerted at definite points across the atomic "surface"; the electrons revolving in small circles in adjacent atoms (p. 274). This mechanical stress, due to the local magnetic forces, is responsible for the change of specific susceptibility, and other properties, on crystallization; and is balanced by the stress due to the distortion of the internal electrostatic configuration of the atoms. In this way, a balance is secured between the electrostatic and magnetic stresses; these stresses predominating alternately, as we pass through the crystalline structure, thus giving rise to a system in equilibrium.

The rotation of electrons in *small* circles, at definite points near the atomic "surface," is suggestive in connexion with the theory of directed valencies required to explain stereo-chemical phenomena (p. 274, footnote).]

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IX. *A Determination of the Deflection of Light by the Sun's Gravitational Field, from Observations made at the Total Eclipse of May* 29, 1919.

By Sir F. W. Dyson, *F.R.S., Astronomer Royal, Prof.* A. S. Eddington, *F.R.S., and Mr.* C. Davidson.

(*Communicated by the Joint Permanent Eclipse Committee.*)

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[Plate 1.]

Contents.

I. Purpose of the Expeditions.

1. The purpose of the expeditions was to determine what effect, if any, is produced by a gravitational field on the path of a ray of light traversing it. Apart from possible surprises, there appeared to be three alternatives, which it was especially desired to discriminate between—

(1) The path is uninfluenced by gravitation.

(2) The energy or mass of light is subject to gravitation in the same way as ordinary matter. If the law of gravitation is strictly the Newtonian law, this leads to an apparent displacement of a star close to the sun's limb amounting to $0''\cdot 87$ outwards.

(3) The course of a ray of light is in accordance with Einstein's generalised relativity theory. This leads to an apparent displacement of a star at the limb amounting to $1''\cdot 75$ outwards.

In either of the last two cases the displacement is inversely proportional to the distance of the star from the sun's centre, the displacement under (3) being just double the displacement under (2).

It may be noted that both (2) and (3) agree in supposing that light is subject to gravitation in precisely the same way as ordinary matter. The difference is that, whereas (2) assumes the Newtonian law, (3) assumes Einstein's new law of gravitation. The slight deviation from the Newtonian law, which on EINSTEIN'S theory causes an excess motion of perihelion of Mercury, becomes magnified as the speed increases, until for the limiting velocity of light it doubles the curvature of the path.

2. The displacement (2) was first suggested by Prof. EINSTEIN* in 1911, his argument being based on the Principle of Equivalence, viz., that a gravitational field is indistinguishable from a spurious field of force produced by an acceleration of the axes of reference. But apart from the validity of the general Principle of Equivalence there were reasons for expecting that the electromagnetic energy of a beam of light would be subject to gravitation, especially when it was proved that the energy of radio-activity contained in uranium was subject to gravitation. In 1915, however, EINSTEIN found that the general Principle of Equivalence necessitates a modification of the Newtonian law of gravitation, and that the new law leads to the displacement (3).

3. The only opportunity of observing these possible deflections is afforded by a ray of light from a star passing near the sun. (The maximum deflection by Jupiter is only $0''\cdot017$.) Evidently, the observation must be made during a total eclipse of the sun.

Immediately after EINSTEIN'S first suggestion, the matter was taken up by Dr. E. FREUNDLICH, who attempted to collect information from eclipse plates already taken; but he did not secure sufficient material. At ensuing eclipses plans were made by various observers for testing the effect, but they failed through cloud or other causes. After EINSTEIN'S second suggestion had appeared, the Lick Observatory expedition attempted to observe the effect at the eclipse of 1918. The final results are not yet published. Some account of a preliminary discussion has been given,† but the eclipse was an unfavourable one, and from the information published the probable accidental error is large, so that the accuracy is insufficient to discriminate between the three alternatives.

4. The results of the observations here described appear to point quite definitely to the third alternative, and confirm EINSTEIN'S generalised relativity theory. As is well-known the theory is also confirmed by the motion of the perihelion of Mercury, which exceeds the Newtonian value by 43″ per century—an amount practically identical with that deduced from EINSTEIN'S theory. On the other hand, his theory predicts a displacement to the red of the Fraunhofer lines on the sun amounting to about $0\cdot008$ A in the violet. According to Dr. ST. JOHN‡ this displacement is not confirmed. If this disagreement is to be taken as final it necessitates considerable modifications of EINSTEIN'S theory, which it is outside our province to discuss. But, whether or not changes are needed in other parts of the theory, it appears now to be established that EINSTEIN'S law of gravitation gives the true deviations from the Newtonian law both for the relatively slow-moving planet Mercury and for the fast-moving waves of light.

It seems clear that the effect here found must be attributed to the sun's gravitational field and not, for example, to refraction by coronal matter. In order to produce the

* 'Annalen der Physik,' vol. XXXV, p. 898.

† 'Observatory,' vol. XLII, p. 298.

‡ 'Astrophysical Journal,' vol. XLVI, p. 249.

observed effect by refraction, the sun must be surrounded by material of refractive index $1 + \cdot 00000414/r$, where r is the distance from the centre in terms of the sun's radius. At a height of one radius above the surface the necessary refractive index $1 \cdot 00000212$ corresponds to that of air at $\frac{1}{140}$ atmosphere, hydrogen at $\frac{1}{60}$ atmosphere, or helium at $\frac{1}{20}$ atmospheric pressure. Clearly a density of this order is out of the question.

II. Preparations for the Expeditions.

5. In March, 1917,* it was pointed out as the result of an examination of the photographs taken with the Greenwich astrographic telescope at the eclipse of 1905 that this instrument was suitable for the photography of the field of stars surrounding the sun in a total eclipse. Attention was also drawn to the importance of observing the eclipse of May 29, 1919, as this afforded a specially favourable opportunity owing to the unusual number of bright stars in the field, such as would not occur again for many years.

With weather conditions as good as those at Sfax in the 1905 eclipse—and these were by no means perfect—it was anticipated that twelve stars would be shown. Their positions are indicated in the diagram on next page, on which is also marked on the same scale the outline of a 16 × 16 cm. plate (used with the astrographic telescopes of 3·43 metres focal length) and a 10 × 8-inch plate (used with a 4-inch lens of 19 feet focal length).

The following table gives the photographic magnitudes and standard co-ordinates of the stars, and the gravitational displacements in x and y calculated on the assumption of a radial displacement $1'' \cdot 75 \frac{r_0}{r}$, where r is the distance from the sun's centre and r_0 the radius of the sun.

Table I.

No.	Names.	Photog. Mag.	Co-ordinates. Unit = 50′.		Gravitational displacement.			
					Sobral.		Principe.	
			x.	y.	x.	y.	x.	y.
		m.			″	″	″	″
1	B.D., 21°, 641	7·0	+0·026	−0·200	−1·31	+0·20	−1·04	+0·09
2	Piazzi, IV, 82	5·8	+1·079	−0·328	+0·85	−0·09	+1·02	−0·16
3	κ^2 Tauri	5·5	+0·348	+0·360	−0·12	+0·87	−0·28	+0·81
4	κ^1 Tauri	4·5	+0·334	+0·472	−0·10	+0·73	−0·21	+0·70
5	Piazzi, IV, 61	6·0	−0·160	−1·107	−0·31	−0·43	−0·31	−0·38
6	υ Tauri	4·5	+0·587	+1·099	+0·04	+0·40	+0·01	+0·41
7	B.D., 20°, 741	7·0	−0·707	−0·864	−0·38	−0·20	−0·35	−0·17
8	B.D., 20°, 740	7·0	−0·727	−1·040	−0·33	−0·22	−0·29	−0·20
9	Piazzi, IV, 53	7·0	−0·483	−1·303	−0·26	−0·30	−0·26	−0·27
10	72 Tauri	5·5	+0·860	+1·321	+0·09	+0·32	+0·07	+0·34
11	66 Tauri	5·5	−1·261	−0·160	−0·32	+0·02	−0·30	+0·01
12	53 Tauri	5·5	−1·311	−0·918	−0·28	−0·10	−0·26	−0·09
13	B.D., 22°, 688	8·0	+0·089	+1·007	−0·17	+0·40	−0·14	+0·39

* 'Monthly Notices, R.A.S.,' LXXVII, p. 445.

It may be noted that No. 1 is lost in the corona on the photographs taken at Sobral. The star, No. 13, of magnitude 8·0, is shown on some of the astrographic plates at Sobral.

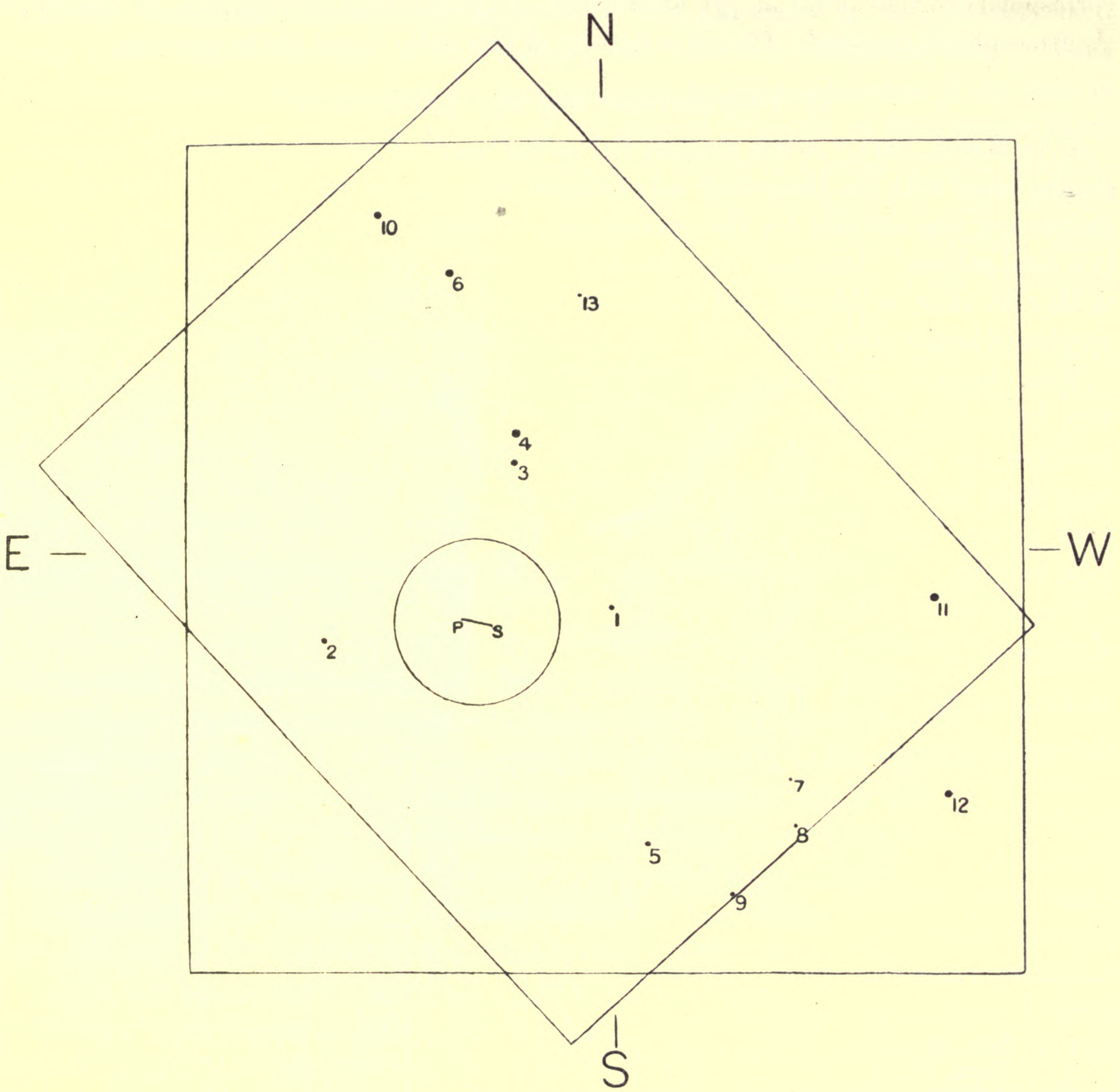


Diagram 1.

6. The track of the eclipse runs from North Brazil across the Atlantic, skirting the African coast near Cape Palmas, passing through the Island of Principe, then across Africa to the western shores of Lake Tanganyika. Enquiry as to the suitable sites and probable weather conditions was kindly made by Mr. HINKS. It appeared that a station in North Brazil, the Island of Principe, and a station on the west of Lake Tanganyika were possible. A station near Cape Palmas did not seem desirable from the meteorological reports though, as the event proved, the eclipse was observed in a cloudless sky

by Prof. BAUER, who was there on an expedition to observe magnetic effects. At the station at Tanganyika it was thought the sun was at too low an altitude for observations of this character, owing to the large displacements which would be caused by refraction.

A circular received from Dr. MORIZE, the director of the Observatory at Rio, stated that Sobral was the most suitable station in North Brazil and gave copious information of the meteorological conditions, mode of access, &c.

7. Acting on this information the Joint Permanent Eclipse Committee at a meeting on November 10, 1917, decided, if possible, to send expeditions to Sobral in North Brazil, and to the island of Principe. Application was made to the Government Grant Committee for £100 for instruments and £1,000 for the expedition, and a sub-committee consisting of Sir F. W. DYSON, Prof. EDDINGTON, Prof. FOWLER and Prof. TURNER was appointed to make arrangements for the expeditions. This sub-committee met in May and June, 1918, and made provisional arrangements for Prof. EDDINGTON and Mr. COTTINGHAM to take the object glass of the Oxford astrographic telescope to Principe, and Mr. DAVIDSON and Father CORTIE to take the object glass of the Greenwich astrographic telescope to Sobral. It was arranged for the clocks and mechanism of the cœlostats to be overhauled by Mr. COTTINGHAM. Preliminary inquiries were also set on foot as to shipping facilities, from which it appeared very doubtful whether the expeditions could be carried through.

Conditions had changed materially in November, 1918, and at a meeting of the sub-committee on November 8, it was arranged to assemble the instruments at Greenwich, and make necessary arrangements with all speed for the observers to leave England by the end of February, 1919. In addition to the astrographic object glasses fed by 16-inch cœlostats, Father CORTIE suggested to the sub-committee the use of the 4-inch telescope of 19-feet focus, which he had used at Hernosand, Sweden, in 1914, in conjunction with an 8-inch cœlostat, the property of the Royal Irish Academy. It was arranged to ask for the loan of these instruments. As Father CORTIE found it impossible to spare the necessary time for the expedition his place was taken by Dr. CROMMELIN of the Royal Observatory.

8. In November, 1918, the only workman available at the Royal Observatory was the mechanic, the carpenter not having been released from military service. In these circumstances Mr. BOWEN, the civil engineer at the Royal Naval College, was consulted. He kindly undertook the construction of frame huts covered with canvas, which could be easily packed and readily put together. These were generally similar to those used in previous expeditions from the Royal Observatory (see 'Monthly Notices,' Vol. LVII., p. 101). He also lent the services of a joiner who worked at the Observatory on the woodwork of the instruments.

It was found possible to obtain steel tubes for the astrographic objectives. These were, for convenience of carriage, made in two sections which could be bolted together. The tubes were provided with flanges at each end, the objective being attached to one of these, and a wooden breech piece to the other. In the breech piece suitable provision was made for the focussing and squaring on of the plates. The plate holders were of a simple construction, permitting the plate to be pushed into contact with three metal tilting screws on the breech piece thus insuring a constancy of focal plane. Eighteen plate-carriers were obtained for each of the astrographic telescopes, made according to a pattern supplied.

With the 4-inch lens Father CORTIE lent the square wooden tube used by him in 1914. This was modified at the breech end to secure greater rigidity and constancy of focus.

It was designed for dark slides carrying 10×8 inch plates, and four of these, carrying eight plates, were lent with the telescope. The desirability of using larger plates was considered, but the time at disposal to make the necessary alterations was insufficient.

The 16-inch cœlostats which had been overhauled by Mr. COTTINGHAM were mounted and tested as far as the unfavourable weather conditions of February, 1919, would permit. The 8-inch cœlostat was constructed for these latitudes. To make it serviceable near the equator a strong wooden wedge was made on which the cœlostat was bolted.

The 8-inch mirror was silvered at the observatory, but owing to lack of facilities for maintaining a uniform temperature approaching 60° F. in the wintry weather of February, the larger mirrors were sent away to be silvered.

Photographic plates, suitably packed in hermetically sealed tin boxes, were obtained from the Ilford and Imperial Companies. The Ilford plates employed were Special Rapid and Empress, and those of the Imperial Company, Special Sensitive, Sovereign and Ordinary.

The instruments were carefully packed and sent to Liverpool a week in advance, with the exception of the objectives. These were packed in cases inside hampers and remained under the personal care of the observers, who embarked on the " Anselm " on March 8.

III. THE EXPEDITION TO SOBRAL.

(*Observers, Dr.* A. C. D. CROMMELIN *and Mr.* C. DAVIDSON.)

9. Sobral is the second town of the State of Ceara, in the north of Brazil. Its geographical co-ordinates are: longitude 2h. 47m. 25s. west; latitude 3° 41′ 33″ south; altitude 230 feet. Its climate is dry and though hot not unhealthy.

The expedition reached Para on the " Anselm " on March 23. There was a choice of proceeding immediately to Sobral or waiting for some weeks. It was considered undesirable to go there before we heard from Dr. MORIZE what arrangements were being made, so we reported our arrival to him by telegram and decided to await his reply. As we had thus some time on our hands we continued the voyage to Manaos in the " Anselm," returning to Para on April 8.

By the courtesy of the Brazilian Government our heavy baggage was passed through the customs without examination and we continued our journey to Sobral, leaving Para on April 24 by the steamer " Fortaleza " and arriving at Camocim on April 29.

Here we were met by Mr. JOHN NICOLAU, who had been instructed to assist us with our baggage through to Sobral. We proceeded from Camocim to Sobral by train on April 30, our baggage following the next day.

We were met at Sobral station by representatives of both the Civil and Ecclesiastical Authorities, headed respectively by Dr. JACOME D'OLIVEIRA, the Prefect, and Mgr. FERREIRA, and conducted to the house which had been placed at our disposal by the kindness of its owner, Col. VICENTE SABOYA, the Deputy for Sobral. We were joined there nine days later by the Washington (Carnegie) Eclipse Commission, consisting of Messrs. DANIEL WISE and ANDREW THOMSON.

We are greatly indebted to Dr. LEOCADIO ARAUJO, of the State Ministry of Agriculture, who had been deputed to interpret for us and to assist us in our preparations. His services were invaluable, and contributed greatly to our success, as also to our well-being during our stay.

10. A convenient site for the eclipse station offered itself just in front of the house; this was the race-course of the Jockey Club, and was provided with a covered grand stand, which we found most convenient for unpacking and storage and in the preparatory work. We laid down a meridian line, after which brick piers were constructed for the cœlostats and for the steel tube of the astrographic telescope. Whilst this was in progress the huts were being erected.

The pier of the small cœlostat was constructed so as to leave a clear space in the middle of one end for the fall of the weight, which was thus below the driving barrel of the clock. By continuing the hole below the foundations of the pier, space was provided for a fall of the weight permitting a run of 25 minutes. In the case of the 16-inch cœlostat, the clock was mounted on the top of a long wooden trunk, nearly 4 feet in length, which was placed on end, and sunk in the earth to a depth of about 2 feet. The weight descended inside the trunk directly from the driving barrel, and had space for a continuous run of over half-an-hour.

The 16-inch cœlostat had free adjustment for all latitudes; but the 8-inch one, constructed for European latitudes, was mounted on a wooden base, inclined at an angle of about 40 degrees, constructed before leaving Greenwich. The clock had to be separated from the cœlostat, mounted on a wooden base and reversed, to adjust to the Southern Hemisphere. It performed very satisfactorily, and no elongation of the star images is shown with 28 seconds' exposure.

To provide for the changing declination of the sun the piers of the astrographic telescope were made with grooves in the top, in which the wooden V-supports of the tube could slide, thus allowing for the change of azimuth.

The tube of the astrographic telescope was circular in section, and could rest in any position in the Vs; for convenience it was adjusted so that the directions of R.A. and declination were parallel to the sides of the plate; this involved a tilt of the plate holders of about 4 degrees to the horizontal.

The 4-inch lens was taken as an auxiliary; we used the square wooden tube, 19 feet in length, originally used by Father CORTIE at Hernosand in 1914, together with the 10 × 8-inch plate carriers. Study of the star-diagram showed that seven stars could be photographed by turning the plate through 45 degrees. The tube was therefore placed on its angle, large wooden V-supports being prepared to fit the tube; these rested on strong wooden trestles.

The focussing was at first done visually on Arcturus, using an eyepiece fitted with a cobalt glass (after the plate supports and object-glass had been adjusted for perpendicularity to the axis). A series of exposures was then made, the focus being varied slightly so as to cover a sufficient range. Examination of these photographs showed at once that there was serious astigmatism due to the figure of the mirror of the 16-inch cœlostat. By inserting an 8-inch stop this was reduced to a large extent, and this stop was henceforth used throughout; but the defect was of such a character that it was clear that it would be necessary to stay at Sobral and obtain comparison plates of the eclipse field in July when the sun had moved away.

The focus of the 4-inch was determined in a similar manner. The images, though superior to those of the astrographic, were not quite perfect, and here again comparison plates in July were necessary. Once the focus had been decided on, the breech end was securely screwed up to avoid any chance of subsequent movement.

A few check plates of the field near Arcturus were taken, but have not been used.

11. The following is a summary of the meteorological conditions during our stay. The barometer record was interesting in that it showed very little change from day to day, in spite of changes in the type of weather; there was, however, a very well marked semi-diurnal variation, with range of about 0·15 inch. The temperature range was fairly uniform, from a maximum of about 97° F. towards 3 p.m. to a minimum of about 75° F. at 5 a.m. The relative humidity (as shown by a hygrograph belonging to the Brazilian Commission) followed the temperature closely, varying from 30 per cent. in the afternoon to 90 per cent. in the early morning.

May is normally the last month of the rainy season at Sobral, but this year the rainfall was very scanty; there were a few afternoon showers, each ushered in by a violent gust of wind; and on May 25 there was very heavy rain, which was welcome for its moistening effect on the ground, the dust hitherto having been troublesome to the clockwork although every care had been taken to protect it. There was a fair amount of cloud in the mornings, but the afternoons and nights were clear in the majority of cases. Mt. Meruoca, 2,700 feet high, about 6 miles to the N.W., was a collector of cloud, its summit being frequently veiled in mist. In spite of its cooler climate, the summit would thus not have been a suitable eclipse station, and, in fact, nothing of the total phase of the eclipse was seen from it.

12. Although water was generally scarce, we were very fortunately situated as we enjoyed an unlimited supply of good water laid on at the house. This was of great benefit in the photographic operations. Ice was unobtainable, but by the use of earthenware water-coolers it was possible to reduce the temperature to about 75°, and by working

only at night or before dawn development of the plates was fairly easy. Formalin was used in every case to harden the films, and thereby minimise the chance of distortion due to the softening of the films by the warm solutions.

We had provided ourselves with two brands of plates, but it had become apparent from photographs taken and developed before the eclipse that one of these brands was unsuitable in the hot climate, and it was decided to use practically only one brand of plates.

In taking the experimental photographs it was noticed that the clocks and cœlostats were very sensitive to wind. We had reason to fear strong gusts about the time of totality, such as had occurred in other eclipses; and as the conditions of our locality seemed to render them specially probable, protective wind screens were erected round the hut openings at every point where it was possible without interfering with the field of view. Happily dead calm prevailed at the critical time. Screens also protected all projecting parts of the telescope tubes from direct sunlight.

The performance of the 16-inch cœlostat was unsatisfactory in respect of driving. There was a clearly marked oscillation of the images on the screen in a period of about 30 seconds. For this reason exposure time was shortened, so as to multiply the number of exposures in the hope that some would be near the stationary points.

13. The morning of the eclipse day was rather more cloudy than the average, and the proportion of cloud was estimated at $\frac{9}{10}$ at the time of first contact, when the sun was invisible; it appeared a few seconds later showing a very small encroachment of the moon, and there were various short intervals of sunshine during the partial phase which enabled us to place the sun's image at its assigned position on the ground glass, and to give a final adjustment to the rates of the driving clocks. As totality approached, the proportion of cloud diminished, and a large clear space reached the sun about one minute before second contact. Warnings were given 58s., 22s. and 12s. before second contact by observing the length of the disappearing crescent on the ground glass. When the crescent disappeared the word "go" was called and a metronome was started by Dr. LEOCADIO, who called out every tenth beat during totality, and the exposure times were recorded in terms of these beats. It beat 320 times in 310 seconds; allowance has been made for this rate in the recorded times. The programme arranged was carried out successfully, 19 plates being exposed in the astrographic telescope with alternate exposures of 5 and 10 seconds, and eight in the 4-inch camera with a uniform exposure of 28 seconds. The region round the sun was free from cloud, except for an interval of about a minute near the middle of totality when it was veiled by thin cloud, which prevented the photography of stars, though the inner corona remained visible to the eye and the plates exposed at this time show it and the large prominence excellently defined. The plates remained in their holders until development, which was carried out in convenient batches during the night hours of the following days, being completed by June 5.

14. No observation of contact times was made, but it is known that these times were

somewhat before those calculated. As the times recorded were reckoned from second contact, it is assumed that this occurred May 28, 23h. 58m. 18s. G.M.T.

The details of the exposures are given in the following tables :—

EXPOSURES with the 13-inch Astrographic Telescope stopped to 8 inches.

Ref. No.	G.M.T. at Commencement of Exposure.	Exposure.	Plate.	Ref. No.	G.M.T. at Commencement of Exposure.	Exposure.	Plate.
	d. h. m. s.	s.			d. h. m. s.	s.	
1	28 23 58 23	5	O.	11	29 0 1 7	5	S.R.
2	37	10	E.	12	22	10	E.
3	57	5	E.	13	36	5	E.
4	59 11	10	S.	14	51	10	S.R.
5	30	5	S.R.	15	2 10	5	S.R.
6	45	10	S.R.	16	25	10	S.R.
7	29 0 0 4	5	S.R.	17	44	5	E.
8	19	10	E.	18	58	10	E.
9	39	5	E.	19	3 18	5	O.
10	53	10	S.R.				

EXPOSURES with the 4-inch Telescope.

Ref. No.	G.M.T. at Commencement of Exposure.	Exposure.	Plate.	Ref. No.	G.M.T. at Commencement of Exposure.	Exposure.	Plate.
	d. h. m. s.	s.			d. h. m. s.	s.	
1	28 23 58 21	28	S.R.	5	29 0 0 56	28	S.R.
2	59 0	28	S.R.	6	1 34	28	S.R.
3	38	28	S.R.	7	2 13	28	S.R.
4	29 0 0 17	28	S.R.	8	52	28	S.R.

In the fourth column the letter O stands for Imperial Ordinary.
E ,, ,, Empress.
S ,, ,, Sovereign.
SR ,, ,, Ilford Special Rapid.

With the astrographic telescope 12 stars are shown on a number of plates, and seven stars on all but three (Nos. 13, 14 and 19). Of the eight plates taken with the 4-inch lens, seven show seven stars, but No. 6, which was taken through cloud, does not show any.

The following table of temperatures, communicated by Dr. MORIZE, and converted into the Fahrenheit scale, shows how slight the fall was during totality, probably owing to the large amount of cloud in the earlier stages which checked the usual daily rise.

G.M.T.	Ther.	G.M.T.	Ther.	G.M.T.	Ther.	G.M.T.	Ther.
d. h. m.	°	d. h. m.	°	d. h. m.	°	d. h. m.	°
28 22 45	82·4	28 23 30	80·6	29 0 15	82·0	29 1 0	83·8
23 0	84·2	45	82·4	30	82·4	15	84·2
15	82·4	29 0 0	80·6	45	83·1	30	84·2

15. On June 7, having completed the development, we left Sobral for Fortaleza, returning on July 9 for the purpose of securing comparison plates of the eclipse field.

Before our departure we dismounted the mirrors and driving clocks which were brought into the house to avoid the exposure to dust. The telescopes and cœlostats were left *in situ.* Before removing the mirrors we marked their positions in their cells so that they could be replaced in exactly the same position.

After our return to Sobral the mirrors and clocks were remounted ; the photography of the eclipse field was commenced on the morning of July 11 (civil). The difficulty of finding the field with the cœlostats was overcome by making a rough hour circle on the heads of the cœlostats out of millimetre paper.

The following is the list of exposures made on the field for comparison with the eclipse photographs :—

Astrographic Telescope.						4-inch Telescope.					
Ref. No.	Date.	G.M.T.	No. of exposures.	Duration.	Altitude.	Ref. No.	Date.	G.M.T.	No. of exposures.	Duration.	Altitude.
		h. m.		s.	°						
11_1	July 10	20 5	3	5	28·9						
11_2		20 16	2	5	31·1			h. m.		s.	°
11_3		20 21	1	5	32·2	14_1	July 13	20 7	2	25	32·4
14_1	July 13	20 13	3	5	33·7	14_2		20 16	2	20	34·3
14_2		20 17	2	5	34·5						
14_3		20 19	2	5	34·9	15_1	July 14	20 17	2	20	35·4
15_1	July 14	20 15	3	5	34·9	15_2		20 22	2	20	36·4
15_2		20 20	2	5	36·1						
15_3		20 23	2	5	36·6	17_1	July 16	20 6	3	15	34·7
17_1	July 16	20 2	4	3	33·8	17_2		20 24	2	15	38·6
17_2		20 15	3	3	36·6						
17_3		20 23	2	3	38·3						
17_4		20 25	2	5	38·8	18_1*	July 17	19 57	3	20	33·6
18_1	July 17	19 50	3	4	32·8	18_2		20 24	2	20	39·2
18_2		20 1	2	4	34·4						
18_3		20 20	3	4	38·6						
18_4		20 25	2	3	39·5						

The reference numbers follow the civil dates.

* The 4-inch plate, No. 18_1, was taken through the glass (see § 17, *infra*) to facilitate the measurement, and is referred to as the scale plate.

Thermometer readings, July 10, 74°·4; July 13, 73°·7; July 14, 71°·9; July 16, 72°·3; July 17, 72°·3.

By July 18 we had obtained a sufficient number of reference photographs. Dismantling of the instruments was commenced, and the packing was completed on July 21. We left Sobral on July 22, leaving the packing cases in the hands of Messrs. NICOLAU and CARNEIRO to be forwarded at the earliest opportunity, and arrived at Greenwich on August 25.

The observers wish to record their obligations to Mr. CHARLES BOOTH and the officers of the "Booth" Line for facilitating their journeys to and from their station at a difficult time.

PHOTOGRAPHS TAKEN WITH THE 4-INCH OBJECT GLASS.

16. These photographs were taken on 10 × 8-inch plates. By suitably mounting the camera it was made possible to obtain seven stars on the photographs, viz., Nos. 2, 3, 4, 5, 6, 10 and 11 of the table in § 5. Of the eight photographs taken during the eclipse seven gave measurable images of these stars, the other plate (No. 6) taken through cloud only showing a picture of the prominences.

Plates of the same field taken under nearly similar conditions as regards altitude were taken on July 14, 15, 17 and 18 (civil date). Of these photographs, the second taken on July 14 with two exposures (referred to as 14_{2a} and 14_{2b}), two photographs taken on July 15 (referred to as 15_1 and 15_2), two on July 17 (17_1 and 17_2), and the second photograph on July 18 (18_2) were measured for comparison with the eclipse plates.

17. The micrometer at the Royal Observatory is not suitable for the direct comparison of plates of this size. It was therefore decided to measure each plate by placing, film to film upon it, another photograph of the same region reversed by being taken through the glass. A photograph for this purpose was taken on July 18. This plate is regarded merely as an intermediary between the eclipse plates and comparison plates and is referred to as the scale plate, being used simply as a scale providing points of reference. In all cases measurement was made through the glass of the scale plate, adjusted on the eclipse or comparison plate which was being measured, so that the separation of the images on the two plates did not exceed one-third of a millimetre. The plates were held together by clips which ensured contact over the whole surface. This method of measurement was found to be very convenient. Each plate was measured in two positions, being reversed through 180 degrees, and the accordance of the result showed that the method of measurement was entirely satisfactory.

The measures, both direct and reversed, were made by two measurers (Mr. DAVIDSON and Mr. FURNER), and the means taken. There was no sensible difference between the measurers, which is satisfactory, as it affords evidence of the similarity of the images on the eclipse and comparison and scale plates.

The value of the micrometer screws (both in R.A. and Decl.) is 6″·25.

18. The results of the measures are as follows :—

TABLE II.—Eclipse Plates—Scale.

No. of Star.	I.		II.		III.		IV.		V.		VII.		VIII.	
	Dx.	Dy.	Dx.	Dy.	Dx.	Dy.	Dx.	Dy.	Dx.	Dy.	Dx.	Dy.	Dx.	Dy.
	r	r	r	r	r	r	r	r	r	r	r	r	r	r
11	−1·411	−0·554	−1·416	−1·324	+0·592	+0·956	+0·563	+1·238	+0·406	+0·970	−1·456	+0·964	−1·285	−1·195
5	−1·048	−0·338	−1·221	−1·312	+0·756	+0·843	+0·683	+1·226	+0·468	+0·861	−1·267	+0·777	−1·152	−1·332
4	−1·216	+0·114	−1·054	−0·944	+0·979	+1·172	+0·849	+1·524	+0·721	+1·167	−1·028	+1·142	−0·927	−0·930
3	−1·237	+0·150	−1·079	−0·862	+0·958	+1·244	+0·861	+1·587	+0·733	+1·234	−1·010	+1·185	−0·897	−0·894
6	−1·342	+0·124	−1·012	−0·932	+1·052	+1·197	+0·894	+1·564	+0·798	+1·130	−0·888	+1·125	−0·838	−0·937
10	−1·289	+0·205	−0·999	−0·948	+1·157	+1·211	+0·934	+1·522	+0·864	+1·119	−0·820	+1·072	−0·768	−0·964
2	−0·789	+0·109	−0·733	−1·019	+1·256	+0·924	+1·177	+1·373	+0·995	+0·935	−0·768	+0·892	−0·585	−1·166
	−1·500*	−0·554	−1·500	−1·324	+0·500	+0·843	+0·500	+1·226	+0·400	+0·861	−1·500	+0·777	−1·300	−1·322

COMPARISON Plates—Scale.

No. of Star.	14_{2a}.		14_{2b}.		15_1.		15_2.		17_1.		17_2.		18_2.	
	Dx.	Dy.	Dx.	Dy.	Dx.	Dy.	Dx.	Dy.	Dx.	Dy.	Dx.	Dy.	Dx.	Dy.
	r	r	r	r	r	r	r	r	r	r	r	r	r	r
11	−0·478	−0·109	+0·967	+1·170	+1·098	+1·228	+0·725	+0·830	−1·073	−1·330	+1·242	−0·302	−1·188	−1·572
5	−0·544	−0·204	+1·013	+1·192	+0·899	+1·232	+0·692	+0·938	−1·072	−1·075	+1·161	−0·224	−1·195	−1·432
4	−0·368	−0·136	+1·030	+1·249	+1·133	+1·086	+0·725	+0·854	−1·296	−1·031	+1·354	−0·281	−1·165	−1·454
3	−0·350	−0·073	+1·044	+1·305	+1·164	+1·114	+0·732	+0·893	−1·278	−1·014	+1·342	−0·261	−1·178	−1·394
6	−0·317	−0·144	+0·980	+1·319	+1·244	+1·012	+0·714	+0·824	−1·375	−1·052	+1·363	−0·390	−1·165	−1·473
10	−0·272	−0·146	+0·997	+1·327	+1·249	+0·960	+0·722	+0·831	−1·424	−1·038	+1·370	−0·423	−1·164	−1·476
2	−0·396	−0·182	+1·102	+1·289	+0·969	+1·052	+0·731	+0·941	−1·236	−0·909	+1·278	−0·328	−1·164	−1·335
	−0·552*	−0·206	+0·967	+1·170	+0·899	+0·960	+0·690	+0·824	−1·424	−1·330	+1·161	−0·423	−1·195	−1·572

* The numbers −1·500, −0·554, &c., given below the line, were taken out to make the values of Dx, Dy small and positive for arithmetical convenience.

19. The values of Dx and Dy were equated to expressions of the form

$$ax + by + c + \alpha \mathrm{E}_x \, (= \mathrm{D}x)$$

and

$$dx + ey + f + \alpha \mathrm{E}_y \, (= \mathrm{D}y),$$

where x, y are the co-ordinates of the stars given in Table I., and E_x, E_y are coefficients of the gravitational displacement.

The quantities c and f are corrections to zero, depending on the setting of the scale plate on the plate measured, a and e are differences of scale value, while b and d depend mainly on the orientation of the two plates. The quantity α denotes the deflection at unit distance (*i.e.*, 50′ from the sun's centre), so that $\alpha \mathrm{E}_x$ and $\alpha \mathrm{E}_y$ are the deflection in R.A. and Decl. respectively of a star whose co-ordinates are x and y.

The left-hand sides of the equation for the seven stars shown are :—

No.	Right Ascension.	Declination.
11	$c-0\cdot160b-1\cdot261a-0\cdot587\alpha$	$f-1\cdot261d-0\cdot160e+0\cdot036\alpha$
5	$c-1\cdot107b-0\cdot160a-0\cdot557\alpha$	$f-0\cdot160d-1\cdot107e-0\cdot789\alpha$
4	$c+0\cdot472b+0\cdot334a-0\cdot186\alpha$	$f+0\cdot334d+0\cdot472e+1\cdot336\alpha$
3	$c+0\cdot360b+0\cdot348a-0\cdot222\alpha$	$f+0\cdot348d+0\cdot360e+1\cdot574\alpha$
6	$c+1\cdot099b+0\cdot587a+0\cdot080\alpha$	$f+0\cdot587d+1\cdot099e+0\cdot726\alpha$
10	$c+1\cdot321b+0\cdot860a+0\cdot158\alpha$	$f+0\cdot860d+1\cdot321e+0\cdot589\alpha$
2	$c-0\cdot328b+1\cdot079a+1\cdot540\alpha$	$f+1\cdot079d-0\cdot328e-0\cdot156\alpha$

20. Normal equations formed from these equations of condition are as follows :—

TABLE III.—Eclipse Plates—Right Ascension.

		I.	II.	III.	IV.	V.	VII.	VIII.
$+7\cdot000c\ +1\cdot657b\ +1\cdot787a\ +0\cdot226\alpha$	=	$+2\cdot159$	$+2\cdot986$	$+3\cdot250$	$+2\cdot461$	$+2\cdot185$	$+3\cdot263$	$+2\cdot648$
$+4\cdot664\ +2\cdot089\ +0\cdot335$	=	$-0\cdot063$	$+0\cdot986$	$+1\cdot320$	$+0\cdot866$	$+1\cdot051$	$+1\cdot464$	$+1\cdot130$
$+4\cdot094\ +2\cdot534$	=	$+1\cdot034$	$+1\cdot689$	$+1\cdot866$	$+1\cdot469$	$+1\cdot480$	$+1\cdot972$	$+1\cdot723$
$+3\cdot142$	=	$+0\cdot712$	$+0\cdot919$	$+0\cdot924$	$+0\cdot860$	$+0\cdot844$	$+0\cdot930$	$+0\cdot973$
$+4\cdot271b\ +1\cdot666a\ +0\cdot281\alpha$	=	$-0\cdot575$	$+0\cdot278$	$+0\cdot550$	$+0\cdot283$	$+0\cdot533$	$+0\cdot691$	$+0\cdot502$
$+3\cdot683\ +2\cdot476$	=	$+0\cdot483$	$+0\cdot928$	$+1\cdot037$	$+0\cdot841$	$+0\cdot923$	$+1\cdot140$	$+1\cdot048$
$+3\cdot135$	=	$+0\cdot643$	$+0\cdot823$	$+0\cdot820$	$+0\cdot781$	$+0\cdot774$	$+0\cdot826$	$+0\cdot888$
$+2\cdot988a\ +2\cdot366\alpha$	=	$+0\cdot707$	$+0\cdot820$	$+0\cdot822$	$+0\cdot731$	$+0\cdot715$	$+0\cdot871$	$+0\cdot852$
$+3\cdot116$	=	$+0\cdot681$	$+0\cdot805$	$+0\cdot784$	$+0\cdot762$	$+0\cdot739$	$+0\cdot780$	$+0\cdot855$
$+1\cdot242\alpha$	=	$+0\cdot121$	$+0\cdot156$	$+0\cdot133$	$+0\cdot183$	$+0\cdot173$	$+0\cdot090$	$+0\cdot180$
α	=	$+0\cdot098$	$+0\cdot126$	$+\ \cdot107$	$+0\cdot148$	$+0\cdot140$	$+0\cdot073$	$+0\cdot145$
a	=	$+0\cdot158$	$+0\cdot174$	$+0\cdot189$	$+0\cdot127$	$+0\cdot128$	$+0\cdot233$	$+0\cdot169$
b	=	$-0\cdot203$	$-0\cdot011$	$+0\cdot048$	$+0\cdot007$	$+0\cdot042$	$+0\cdot066$	$+0\cdot042$

Table IV.—Comparison Plates—Right Ascension.

					14_{2a}.	14_{2b}.	15_1.	15_2.	17_1.	17_2.	18_2.
+7·000*c*	+1·657*b*	+1·787*a*	+0·226α	=	+1·190	+0·364	+1·463	+0·214	+1·214	+0·983	+0·146
	+4·664	+2·089	+0·335	=	+0·700	+0·017	+0·992	+0·078	−0·340	+0·603	+0·083
		+4·094	+2·535	=	+0·638	+0·220	+0·499	+0·073	−0·172	+0·450	+0·085
			+3·142	=	+0·253	+0·159	−0·029	+0·037	−0·164	+0·105	+0·041
	+4·271*b*	+1·666*a*	+0·281α	=	+0·418	−0·069	+0·645	+0·027	−0·627	+0·370	+0·048
		+3·683	+2·476	=	+0·334	+0·127	+0·126	+0·018	−0·481	+0·199	+0·048
			+3·135	=	+0·215	+0·147	−0·076	+0·030	−0·203	+0·074	+0·036
		+2·988*a*	+2·366α	=	+0·172	+0·154	−0·126	+0·007	−0·236	+0·055	+0·029
			+3·116	=	+0·188	+0·152	−0·119	+0·028	−0·162	+0·050	+0·033
			+1·242α	=	+0·052	+0·030	−0·019	+0·022	+0·025	+0·006	+0·010
			α	=	+0·042	+0·024	−0·015	+0·018	+0·020	+0·005	+0·008
			a	=	+0·024	+0·032	−0·030	−0·012	−0·094	+0·014	+0·003
			b	=	+0·086	−0·030	+0·164	+0·012	−0·111	+0·081	+0·010

Table V.—Eclipse Plates—Declination.

+7·000*f*	+1·787*d*	+1·657*e*	+3·316α	=	+3·688	+1·927	+1·646	+1·452	+1·389	+1·718	+1·906
	+4·094	+2·089	+1·840	=	+2·200	+1·168	+0·719	+0·823	+0·555	+0·610	+0·840
		+4·664	+3·694	=	+1·860	+1·159	+1·129	+0·984	+0·874	+1·023	+1·193
			+5·784	=	+2·657	+1·681	+1·535	+1·361	+1·335	+1·545	+1·707
	+3·638*d*	+1·666*e*	+0·994α	=	+1·260	+0·677	+0·299	+0·453	+0·201	+0·172	+0·354
		+4·271	+2·908	=	+0·986	+0·702	+0·739	+0·640	+0·545	+0·616	+0·741
			+4·212	=	+0·909	+0·768	+0·755	+0·673	+0·677	+0·731	+0·804
		+3·508*e*	+2·453α	=	+0·409	+0·392	+0·602	+0·431	+0·453	+0·537	+0·579
			+3·941	=	+0·565	+0·583	+0·673	+0·549	+0·622	+0·684	+0·707
			+2·224α	=	+0·279	+0·309	+0·252	+0·247	+0·305	+0·308	+0·302
			α	=	+0·126	+0·139	+0·114	+0·111	+0·137	+0·139	+0·136
			e	=	+0·029	+0·015	+0·092	+0·045	+0·033	+0·056	+0·070
			d	=	+0·299	+0·141	+0·009	+0·074	+0·003	−0·016	+0·028

Table VI.—Comparison Plates—Declination.

+7·000*f*	+1·787*d*	+1·657*e*	+3·316α	=	+0·446	+0·661	+0·964	+0·343	+1·861	+0·752	+0·868
	+4·094	+2·089	+1·840	=	+0·060	+0·420	−0·156	+0·140	+1·038	+0·041	+0·476
		+4·664	+3·694	=	+0·202	+0·394	−0·203	−0·117	+0·526	−0·110	+0·122
			+5·784	=	+0·380	+0·482	+0·220	+0·044	+1·004	+0·296	+0·419
	+3·638*d*	+1·666*e*	+0·994α	=	−0·054	+0·251	−0·402	+0·053	+0·563	+0·151	+0·255
		+4·271	+2·908	=	+0·096	+0·237	−0·431	−0·198	+0·085	−0·288	−0·084
			+4·212	=	+0·168	+0·169	−0·237	−0·119	+0·122	−0·060	+0·008
		+3·508*e*	+2·453α	=	+0·121	+0·122	−0·247	−0·222	−0·173	−0·219	−0·201
			+3·941	=	+0·183	+0·100	−0·127	−0·133	−0·032	−0·019	−0·062
			+2·224α	=	+0·098	+0·015	+0·046	+0·022	+0·089	+0·134	+0·079
			α	=	+0·044	+0·007	+0·021	+0·010	+0·040	+0·060	+0·036
			e	=	+0·004	+0·030	−0·085	−0·070	−0·077	−0·104	−0·082
			d	=	−0·028	+0·054	−0·077	+0·044	+0·179	−0·010	+0·098

21. The values of α are collected in Table VII :—

TABLE VII.

Right Ascension.		Declination.	
Eclipse — Scale.	Comparison — Scale.	Eclipse — Scale.	Comparison — Scale.
r	r	r	r
$+0\cdot098$	$+0\cdot042$	$+0\cdot126$	$+0\cdot044$
$+0\cdot126$	$+0\cdot024$	$+0\cdot139$	$+0\cdot007$
$+0\cdot107$	$-0\cdot015$	$+0\cdot114$	$+0\cdot021$
$+0\cdot148$	$+0\cdot018$	$+0\cdot111$	$+0\cdot010$
$+0\cdot140$	$+0\cdot020$	$+0\cdot137$	$+0\cdot040$
$+0\cdot073$	$+0\cdot005$	$+0\cdot139$	$+0\cdot060$
$+0\cdot145$	$+0\cdot008$	$+0\cdot136$	$+0\cdot036$
Mean $+0\cdot120$	$+0\cdot015$	$+0\cdot129$	$+0\cdot031$

By subtracting the α of the comparison plates the scale plate is eliminated, and we derive from right ascensions $\alpha = +0^r\cdot105$ and from declinations $\alpha = +0^r\cdot098$.

Reference to the normal equations shows that the declination result is of double the weight of that from the right ascensions.

Thus

$$\alpha = +0^r\cdot100 = +0''\cdot625.$$

This is at a distance $50'$ from the sun's centre. At the time of the eclipse the sun's radius was $15'\cdot8$; thus the deflection at the limb is $1''\cdot98$.

The range in the values of α is attributable to the errors inherent to the star images of the different plates, and cannot be reduced by further measurement. The mean values $+0^r\cdot015$ and $0^r\cdot031$ arise from the errors in the intermediary scale plate.

22. The probable error of the result judging from the accordance of the separate determinations is about 6 per cent. It is desirable to consider carefully the possibility of systematic error. The eclipse and comparison photographs were taken under precisely similar instrumental conditions, but there is the difference that the eclipse photographs were taken on the day of May 29, and the comparison photographs on nights between July 14 and July 18. A very satisfactory feature of the photographs is the essential similarity of the star images on the two sets of photographs.

The satisfactory accordance of the eclipse and comparison plates is shown by a study of the plate constants. The following corrections for differential refraction and aberration are calculated from the times and dates of exposure.

	a.	*e.*	*b.*	*d.*
	r	*r*	*r*	*r*
Eclipse plates	+0·240	+0·168	+0·062	+0·062
Scale plate	+0·423	+0·207	+0·096	+0·096
Comparison 14_{2a}	+0·409	+0·207	+0·091	+0·091
,, 14_{2b}	+0·409	+0·207	+0·091	+0·091
,, 15_1	+0·390	+0·207	+0·087	+0·087
,, 15_2	+0·370	+0·202	+0·087	+0·087
,, 17_1	+0·399	+0·207	+0·091	+0·091
,, 17_2	+0·337	+0·202	+0·077	+0·077
,, 18_2	+0·327	+0·202	+0·072	+0·072

When these are applied to the values of the constants found from the normal equations, we find the following values of the scale of the several photographs and their orientation relative to the scale plate :—

	Scale Value.		Orientation.		Adopted Scale Orientation.	
	From x.	From y.	From x.	From y.		
	r	*r*				*r*
Eclipse I.	−0·025	−0·010	−0·237	−0·265	0·000	−0·251
,, II.	−0·009	−0·024	−0·045	−0·107	0·000	−0·076
,, III.	+0·006	+0·053	+0·014	+0·025	0·000	+0·020
,, IV.	−0·056	+0·006	−0·027	−0·040	0·000	−0·034
,, V.	−0·055	−0·006	+0·008	+0·031	0·000	+0·020
,, VII.	+0·050	+0·017	+0·032	+0·050	0·000	+0·041
,, VIII.	−0·014	+0·031	+0·008	+0·006	0·000	+0·007
Comparison 14_{2a}	+0·010	+0·004	+0·081	+0·033	+0·013	+0·057
,, 14_{2b}	+0·008	+0·030	−0·035	−0·049	+0·013	−0·042
,, 15_1	−0·063	−0·085	+0·155	+0·086	−0·084	+0·120
,, 15_2	−0·065	−0·075	+0·003	−0·035	−0·084	−0·016
,, 17_1	−0·118	−0·077	−0·116	−0·174	−0·084	−0·145
,, 17_2	−0·072	−0·109	+0·062	+0·029	−0·084	+0·046
,, 18_2	−0·093	−0·087	−0·014	−0·074	−0·084	−0·044

The agreement in the scale values obtained from x and y is satisfactory. There appears to be a small difference in the orientations as derived from the two directions in the comparison plates. This is, however, of small importance in the determination of α. There is a difference of scale value from July 15–18 shown in both co-ordinates. For the purpose of exhibiting the gravitational displacements, residuals have been computed using adopted values for the scale and orientation given above, along with the calculated corrections for differential refraction and aberration. This has the advantage of reducing the number of constants employed in the reduction of the plates, and lessens the possibility of masking any discordances, though greater irregularities necessarily appear when four arbitrary constants instead of six are used in the reduction of each plate. The quantities are converted from revolutions to seconds of arc, as the more familiar unit facilitates judgment of the results.

TABLE VIII.—Comparison of the Eclipse and Comparison Photographs with the Scale Plate, after Correction for Differential Refraction and Aberration, Orientation and Change of Scale.

No. of Star.	I.	II.	III.	IV.	V.	VII.	VIII.	Mean.
			ECLIPSE Plates—Right Ascension.					
	″	″	″	″	″	″	″	″
11	−0·18	−0·51	−0·46	−0·07	−0·04	−0·72	−0·43	−0·34
5	−0·45	−0·81	−0·38	−0·58	−0·60	−0·36	−0·62	−0·54
4	+0·08	+0·11	−0·08	−0·11	−0·11	−0·16	−0·18	−0·06
3	−0·23	−0·11	−0·19	−0·05	−0·02	−0·02	−0·01	−0·09
6	−0·14	+0·23	−0·09	−0·11	−0·13	+0·13	−0·08	−0·03
10	+0·17	+0·06	+0·14	−0·18	−0·11	+0·14	−0·01	+0·03
2	+0·75	+1·03	+1·06	+1·09	+1·01	+0·98	+1·30	+1·03
			ECLIPSE PLATES—Declination.					
	″	″	″	″	″	″	″	″
11	0·00	−0·08	−0·03	+0·02	+0·17	+0·16	+0·01	+0·03
5	−0·38	−0·54	−0·61	−0·30	−0·39	−0·73	−0·81	−0·54
4	+1·19	+1·04	+1·03	+0·98	+1·11	+1·19	+1·24	+1·11
3	+1·42	+1·58	+1·50	+1·39	+1·55	+1·49	+1·49	+1·49
6	+0·65	+0·79	+1·01	+0·97	+0·71	+0·95	+1·01	+0·87
10	+0·62	+0·46	+1·03	+0·54	+0·56	+0·58	+0·74	+0·65
2	+0·01	+0·25	−0·40	−0·09	−0·22	−0·14	−0·17	−0·11

	14_{2a}.	14_{2b}.	15_1.	15_2.	17_1.	17_2.	18_2.	Mean.
			COMPARISON Plates—Right Ascension.					
	″	″	″	″	″	″	″	″
11	−0·19	−0·24	−0·23	−0·28	+0·11	−0·19	−0·02	−0·15
5	−0·42	+0·16	−0·36	−0·32	−0·24	−0·33	−0·26	−0·25
4	−0·01	+0·03	−0·01	+0·05	−0·04	+0·23	+0·08	+0·05
3	+0·14	+0·09	+0·28	+0·10	−0·03	+0·21	−0·01	+0·11
6	+0·02	−0·18	+0·26	+0·06	+0·13	+0·03	+0·14	+0·07
10	+0·17	−0·06	+0·20	+0·18	+0·13	−0·02	+0·15	+0·11
2	+0·31	+0·18	−0·16	+0·22	−0·04	+0·08	−0·06	+0·08
			COMPARISON Plates—Declination.					
	″	″	″	″	″	″	″	″
11	−0·07	+0·08	−0·26	−0·04	−0·26	−0·18	−0·16	−0·13
5	−0·23	−0·03	+0·03	0·00	−0·19	+0·03	−0·20	−0·08
4	+0·23	+0·05	+0·29	+0·18	+0·45	+0·53	+0·23	+0·28
3	+0·64	+0·41	+0·42	+0·36	+0·48	+0·60	+0·54	+0·49
6	+0·22	+0·36	+0·33	+0·26	+0·41	+0·21	+0·32	+0·30
10	+0·28	+0·32	+0·31	+0·36	+0·36	+0·15	+0·29	+0·30
2	+0·25	+0·14	+0·18	+0·21	+0·09	−0·03	+0·27	+0·16

Subtracting the results of the comparison plates, so as to eliminate the errors arising from the intermediary scale plate we find for the displacements of the different stars, as compared with those as given by EINSTEIN'S Theory, with value 1″·75 at the sun's limb :—

No. of Star.	Displacement in Right Ascension.		Displacement in Declination.	
	Observed.	Calculated	Observed.	Calculated.
	″	″	″	″
11	−0·19	−0·32	+0·16	+0·02
5	−0·29	−0·31	−0·46	−0·43
4	−0·11	−0·10	+0·83	+0·74
3	−0·20	−0·12	+1·00	+0·87
6	+0·10	+0·04	+0·57	+0·40
10	−0·08	+0·09	+0·35	+0·32
2	+0·95	+0·85	−0·27	−0·09

PHOTOGRAPHS TAKEN WITH THE ASTROGRAPHIC OBJECT GLASS.

23. As stated above these photographs were taken with the astrographic object glass stopped down to 8 inches, mounted in a steel tube and fed by a 16-inch cœlostat. From many years' experience with the object glass at Greenwich it is certain that, when the object glass is mounted in a steel tube, the change of scale over a range of temperature of 10° F. should be insignificant, and the definition should be very good. It was realised that this high standard would not be obtained with the glass used in conjunction with the cœlostat taken to Brazil, but nevertheless the results shown when the plates were developed were very disappointing. The images were diffused and apparently out of focus, although on the night of May 27 the focus was good.* Worse still, this change was temporary, for without any change in the adjustments, the instrument had returned to focus when the comparison plates were taken in July.

These changes must be attributed to the effect of the sun's heat on the mirror, but it is difficult to say whether this caused a real change of scale in the resulting photographs or merely blurred the images.

The photographs were measured in the astrographic duplex micrometer, the eclipse photographs being directly compared with the comparison plates taken in July. All

* The following note made at the time is quoted in full :—" May 30, 3 a.m., four of the astrographic plates were developed, and when dry examined. It was found that there had been a serious change of focus, so that, while the stars were shown, the definition was spoilt. This change of focus can only be attributed to the unequal expansion of the mirror through the sun's heat. The readings of the focussing scale were checked next day, but were found unaltered at 11·0 mm. It seems doubtful whether much can be got from these plates."

the stars shown were measured. They were reduced by the same method as that employed for the "4-inch" photographs. With the exception of plates Nos. 15 and 16, taken through clouds, the stars numbered 3, 4, 5, 6, 10, 11 and 12 are shown on all the plates; the fainter stars 2, 7, 8 and 9 are sometimes shown, but No. 1, which is very near the sun, is always drowned in the corona. These plates were only measured in declination, as the right ascensions were of little weight.

24. In the following table is given the value of a, the constant of the gravitational displacement, as calculated from the measures; the apparent difference of scale e between the eclipse and comparison plates; d the difference of orientation of the plates given by the measures of y, and depending on the adjustment of the plates in the measuring machine.

TABLE IX.

($1^r = 12''\cdot3$).

No. of Eclipse Plate.	Ref. No. of Comparison Plate.	No. of Stars.	Values of d, e, α in Revolutions at 50′ Distance.			α at Sun's Limb in Arc.
			d.	e.	α.	
			r	r	r	″
1	18_4	7	+0·051	+0·089	+0·033	+1·28
2	18_4	11	−0·009	+0·059	+0·025	+0·97
3	18_4	8	−0·074	+0·101	+0·028	+1·09
4	18_4	11	−0·168	+0·091	+0·033	+1·28
5	11_3	10	+0·094	+0·076	+0·025	+0·97
6	11_3	11	+0·186	+0·082	+0·021	+0·82
7	14_3	12	+0·006	+0·119	0·000	0·00
7	18_3	7	−0·054	+0·166	0·000	0·00
8	14_3	10	+0·093	+0·064	+0·021	+0·82
9	17_4	7	−0·096	+0·129	+0·008	+0·31
10	17_4	10	+0·090	+0·045	+0·026	+1·01
11	11_1	10	+0·073	+0·061	+0·032	+1·24
12	11_1	11	−0·009	+0·102	+0·049	+1·91
12	17_2	7	−0·102	+0·114	+0·019	+0·74
15	15_3	6	+0·111	+0·036	+0·018	+0·70
16	15_3	7	−0·002	+0·037	+0·018	+0·70
17	17_2	8	−0·022	+0·109	+0·012	+0·47
18	17_2	7	+0·045	0·000	+0·030	+1·17
Mean				+0·082	+0·022	+0·86

Thus the mean value of a obtained from all the astrographic plates is $0''\cdot86$, a figure considerably less than that obtained from the 4-inch photographs.

25. Reference to the diagram shows that the measurement of displacement depends essentially on the position of the stars Nos. 3 and 4 relative to 5 on one side and 6 and 10 on the other. These are all bright stars, and in this respect their images are

more comparable than are the images of the fainter stars. The measures of these stars are given in the following table :—

No. of Eclipse Plate.	Measured Values of Dy for Stars Nos.— 5	4	3	6	10	No. of Eclipse Plate.	Measured Values of Dy for Stars Nos.— 5	4	3	6	10
	r	r	r	r	r		r	r	r	r	r
1	−0·051	+0·175	+0·169	+0·201	+0·235	9	−0·059	+0·121	+0·109	+0·205	+0·180
2	+0·558	+0·656	+0·724	+0·668	+0·702	10	+0·033	+0·270	+0·188	+0·258	+0·280
3	+0·124	+0·285	+0·286	+0·274	+0·355	11	+0·025	+0·215	+0·210	+0·233	+0·274
4	+0·111	+0·222	+0·247	+0·231	+0·167	12	−0·068	+0·144	+0·124	+0·160	+0·167
5	+0·034	+0·228	+0·232	+0·218	+0·308	15	−0·038	+0·138	+0·107	+0·172	—
6	+0·164	+0·488	+0·478	+0·557	+0·637	16	−0·050	+0·076	+0·046	+0·127	+0·073
7	−0·051	+0·156	+0·162	+0·250	+0·279	17	−0·071	+0·104	+0·081	+0·186	+0·164
8	+0·108	+0·330	+0·314	+0·376	+0·397	18	+0·016	+0·092	+0·109	+0·099	+0·084

The equations given by these stars are

$$-0\cdot160d - 1\cdot107e - 0\cdot789a + f = \mathrm{D}y_5 \qquad (1)$$
$$+0\cdot334d + 0\cdot472e + 1\cdot336a + f = \mathrm{D}y_4 \qquad (2)$$
$$+0\cdot348d + 0\cdot360e + 1\cdot574a + f = \mathrm{D}y_3 \qquad (3)$$
$$+0\cdot587d + 1\cdot099e + 0\cdot726a + f = \mathrm{D}y_6 \qquad (4)$$
$$+0\cdot860d + 1\cdot321e + 0\cdot589a + f = \mathrm{D}y_{10} \qquad (5)$$

The mean of (4) and (5) added to (1) gives

$$+0\cdot564d + 0\cdot103e - 0\cdot131a + 2f = \mathrm{D}y_5 + \tfrac{1}{2}(\mathrm{D}y_6 + \mathrm{D}y_{10}).$$

While the sum of (2) and (3) gives

$$+0\cdot682d + 0\cdot832e + 2\cdot910a + 2f = \mathrm{D}y_3 + \mathrm{D}y_4.$$

Subtracting these we get

$$3\cdot041a + 0\cdot729e + 0\cdot118d = \mathrm{D}y_3 + \mathrm{D}y_4 - \mathrm{D}y_5 - \tfrac{1}{2}(\mathrm{D}y_6 + \mathrm{D}y_{10}).$$

This equation has a small coefficient for e and a very small one for d.

Calculating the quantities on the right-hand side, assuming e to be the same for all the plates, and substituting the values of d from the previous table, we find :—

r		r	
$a + 0\cdot240e = +0\cdot056$	1	$a + 0\cdot240e = +0\cdot035$	9
$a + 0\cdot240e = +0\cdot049$	2	$a + 0\cdot240e = +0\cdot048$	10
$a + 0\cdot240e = +0\cdot047$	3	$a + 0\cdot240e = +0\cdot045$	11
$a + 0\cdot240e = +0\cdot059$	4	$a + 0\cdot240e = +0\cdot059$	12
$a + 0\cdot240e = +0\cdot050$	5	$a + 0\cdot283e = +0\cdot026$	15
$a + 0\cdot240e = +0\cdot059$	6	$a + 0\cdot240e = +0\cdot024$	16
$a + 0\cdot240e = +0\cdot036$	7	$a + 0\cdot240e = +0\cdot028$	17
$a + 0\cdot240e = +0\cdot046$	8	$a + 0\cdot240e = +0\cdot029$	18

In photograph No. 15, star 10 is not shown, and the equation is slightly modified. It may also be noticed that the values are somewhat smaller for Nos. 15 to 18.

The means of the 16 photographs treated in this manner give

$$\alpha + 243e = + 0^r\cdot 0435,$$

or with the value of the scale $0^r\cdot 082$ from the previous table

$$\alpha = + 0^r\cdot 024 = 0''\cdot 93 \text{ at the limb.}$$

It may be noticed that the change of scale arising from differences of refraction and aberration is $0^r\cdot 020$. If this value of e be taken instead of $0^r\cdot 082$ we obtain

$$\alpha = + 0^r\cdot 039 = + 1''\cdot 52 \text{ at the sun's limb.}$$

The equations on p. 311 were also solved by least squares for each plate. There is a considerable range in the deduced values of α, as is to be expected when α and e are determined independently for each plate. The mean result for α is $0''\cdot 99$, or very nearly the same as that already found.

The photographs taken with the astrographic telescope support those obtained by the "4-inch" to the extent that they show considerable outward deflection, but for the reasons already given are of much less weight.

IV. The Expedition to Principe.

(*Observers, Prof.* A. S. Eddington *and Mr.* E. T. Cottingham.)

26. The expedition left Liverpool on the "Anselm" on March 8, and travelled in company with the Sobral expedition as far as Madeira. It was necessary to wait there until April 9, when the journey was continued on the "Portugal," belonging to the Companhia Nacional de Navegação. The expedition landed at the small port of S. Antonio in the Isle of Principe on April 23.

Vice-Admiral Campos Rodrigues and Dr. F. Oom of the National Observatory, Lisbon, had kindly given us introductions, and everything possible was done by those on the island for the success of the work and the comfort of the observers. We were met on board by the Acting Administrator Sr. Vasconcélos, Sr. Carneiro, President of the Association of Planters, and Sr. Grageira, representing the Sociedade d'Agricultura Colonial, who made all necessary arrangements. The Portuguese Government dispensed with any customs examination of the baggage.

27. Principe is a small island belonging to Portugal, situated just north of the equator in the Gulf of Guinea, about 120 miles from the African coast. The extreme length and breadth are about 10 miles and 6 miles. Near the centre mountains rise to a height of 2500 feet, which generally attract heavy masses of cloud. Except for a certain amount of virgin forest, the island is covered with cocoa plantations. The

climate is very moist, but not unhealthy. The vegetation is luxuriant, and the scenery is extremely beautiful. We arrived near the end of the rainy season, but the *gravana*, a dry wind, set in about May 10, and from then onwards no rain fell except on the morning of the eclipse.

We were advised that the prospects of clear sky at the end of May were not very good, but that the best chance was on the north and west of the island. After inspecting two other sites on the property of the Sociedade d'Agricultura Colonial, we fixed on Roça Sundy, the headquarters of Sr. CARNEIRO's chief plantation. We were Sr. CARNEIRO's guests during our whole visit, and used freely his ample resources of labour and material at Sundy. We learnt later that he had postponed a visit to Europe in order to entertain us. We were also greatly indebted to his manager at Sundy, Sr. ATALAYA, with whom we lived for five weeks; his help and attention were invaluable. Mr. WRIGHT and Mr. LEWIS of the Cable Station kindly assisted us as interpreters when necessary.

Sundy is situated in the north-west of the island overlooking the sea at a height of 500 feet, and as far as possible from the cloud-gathering peaks. Our telescope was erected in a small walled enclosure adjoining the house, from which the ground sloped steeply down to the sea in the direction of the sun at eclipse. On the other side it was sheltered by a building. The approximate position was latitude 1° 40′ N., longitude 29m. 32s. E.

28. The baggage was brought to Sundy on April 28 mainly by tram, but with a break of about a kilometre, where it had to be transported through the wood by native carriers. After a week spent on the preparations, we returned to S. Antonio for the week, May 6–13, as it was undesirable to unpack the mirror so early in the damp climate. On our return to Sundy the installation and adjustments were soon completed, and the first check plates were taken on May 16. Meanwhile the gravana had begun, which, although there is no rain, is generally accompanied by increased cloud. There were, however, some days of clear sky, and the nights were usually clear.

The cœlostat was mounted on a stone pier built for the purpose. The clock weight fell into a pit below the clock deep enough to allow a run of 36 minutes without rewinding. Care was taken to use a particular part of the cœlostat-sector, considered to be the most perfect, in photographing the eclipse and the check field. The telescope (Oxford astrographic object-glass, see p. 295) rested on wooden V's near the two ends, the V's being supported on packing-cases; the one at the breech-end could be moved laterally to allow of different declination settings, and was marked with an approximate declination scale. A series of exposures of one second was made on a bright star to test whether there was any shake of the telescope after inserting the plate: no shake was detected even when the exposure was made immediately; but as a safeguard for the eclipse photographs a full second was allowed to elapse before beginning the exposure. The exposure was made by moving a cardboard screen unconnected with the instrument. The telescope pointed slightly downwards, and the tube was turned so as to give the right orientation to the plate, the lines of declination being two or three degrees inclined to the horizontal. A canvas screen was arranged to protect the tube and object-glass from the direct radiation of the sun.

The adjustments call for little comment. In view of the purpose of the observations, it was desirable to adjust the tilt of the object-glass and plate with special care. It was also important that the setting on the field should be nearly exact. The sun appeared on the eclipse day in sufficient time to allow of the setting being made by means of the solar image ; but arrangements had been tested by which the correct field would have been obtained if it had been cloudy up to totality.* The telescope was focussed by trial photographs of stars, and owing to the uniform temperature of the island the focus was unchanged for day observations.

The object-glass was stopped down to 8 inches for the eclipse photographs and for all check and comparison photographs used in the reductions.

29. The days preceding the eclipse were very cloudy. On the morning of May 29 there was a very heavy thunderstorm from about 10 a.m. to 11.30 a.m.—a remarkable occurrence at that time of year. The sun then appeared for a few minutes, but the clouds gathered again. About half-an-hour before totality the crescent sun was glimpsed occasionally, and by 1.55 it could be seen continuously through drifting cloud. The calculated time of totality was from 2h. 13m. 5s. to 2h. 18m. 7s. G.M.T. Exposures were made according to the prepared programme, and 16 plates were obtained. Mr. COTTINGHAM gave the exposures and attended to the driving mechanism, and Prof. EDDINGTON changed the dark slides. It appears from the results that the cloud must have thinned considerably during the last third of totality, and some star images were shown on the later plates. The cloudier plates give very fine photographs of a remarkable prominence which was on the limb of the sun.

A few minutes after totality the sun was in a perfectly clear sky, but the clearance did not last long. It seems likely that the break-up of the clouds was due to the eclipse itself, as it was noticed that the sky usually cleared at sunset.

It had been intended to complete all the measurements of the photographs on the spot ; but owing to a strike of the steamship company it was necessary to return by the first boat, if we were not to be marooned on the island for several months. By the intervention of the Administrator berths, commandeered by the Portuguese Government, were secured for us on the crowded steamer. We left Principe on June 12, and after transhipping at Lisbon, reached Liverpool on July 14.

30. The following is a list of the photographs, including the comparison photographs kindly taken for us by Mr. F. A. BELLAMY at Oxford, before the instrument was dismounted. All the eclipse photographs are given, though only W and X furnished

* The method depended on setting the cross-wires of the theodolite (attached to the cœlostat) on a terrestrial mark, and then starting the clock at a particular instant.

results. Of the other series, only the exposures actually used in the reductions are given.

List of Plates.

Check Field (R.A. 14h. 12m. 47s., Declination +20° 30′)

Ref.	Place.	Date.	Loc. Sid. T.	Exp.	Approx. Z.D.	Bar.	Ther.	Plate.
		1919.	h. m. s.	s.	°	m.	°	
a_1	Oxford	January 16	12 55 10	60	35	29·64	37·0	S.
b_1	,,	January 17	13 10 40	60	34	29·83	35·3	S.
c_1	,,	,,	13 54 55	60	31	29·83	35·3	S.
d_1	,,	,,	14 9 25	60	31	29·83	35·3	S.
e_1	,,	January 23	13 13 30	60	33	30·45	29·0	S.
			G.M.T.					
q_1	Principe	May 22	12 25 40	40	43	29·45	76·5	S.R.
r_1	,,	,,	12 31 20	40	45	29·45	76·5	S.R.
s_2	,,	,,	12 37 50	80	46	29·45	76·5	S.R.
v_1	,,	May 25	12 22 20	40	45	29·45	76·5	S.S.
w_1	,,	,,	12 26 20	40	46	29·45	76·5	S.S.

Notes.

Column 1.—The letter is marked on the original plates (preserved at Cambridge Observatory). The number refers to the exposure, disregarding exposures taken without the 8-inch stop.

Column 2.—The co-ordinates of Oxford Observatory are 5m. 3s. W., 51° 46′ N., and of the site at Principe, 29m. 32s. E., 1° 40′ N.

Column 4.—The mid-instant of the exposure is given. Times for check plates at Principe were only noted roughly. Times for the eclipse plates are deduced from the calculated time of totality, the interval from the end of one exposure to the beginning of the next being assumed uniform.

Column 7.—Readings at Principe were taken with an aneroid recording instrument, and therefore automatically reduced to the latitude of England. The barometer during our visit was practically constant except for a regular semi-diurnal wave of amplitude about 0·05 in.

Column 9.—Brand of Plate : S. = Imperial Sovereign, S.S. = Imperial Special Sensitive, S.R. = Ilford Special Rapid, E. = Ilford Empress. Backed plates were used at Principe.

Eclipse Field (R.A. 4h. 19m. 30s., Declination +21° 43′)

Ref.	Place.	Date.	Loc. Sid. T.	Exp.	Approx. Z.D.	Bar.	Ther.	Plate.
		1919.	h. m. s.	s.	°	m.	°	
D_1	Oxford	January 16	3 58 1	5	30	29·65	39·0	S.
G_1	,,	January 22	4 4 39	5	30	30·30	31·0	S.
H_1	,,	,,	4 34 28	5	30	30·30	31·0	S.
I_2	,,	,,	4 48 46	10	31	30·30	31·0	S.
K_2	,,	February 9	4 45 24	10	30	30·48	24·5	S.
			G.M.T.					
K′	Principe	May 29	2 13 9	5	46	29·45	77·0	S.R.
L	,,	,,	2 13 28	10	46	29·45	77·0	S.R.
M	,,	,,	2 13 46	3	46	29·45	77·0	S.R.
N	,,	,,	2 14 1	5	46	29·45	77·0	E.
O	,,	,,	2 14 20	10	46	29·45	77·0	S.S.
P	,,	,,	2 14 44	15	46	29·45	77·0	S.S.
Q	,,	,,	2 15 6	5	46	29·45	77·0	S.R.
R	,,	,,	2 15 30	20	46	29·45	77·0	S.R.
S	,,	,,	2 15 53	3	46	29·45	77·0	S.S.
T	,,	,,	2 16 13	15	46	29·45	77·0	E.
U	,,	,,	2 16 37	10	46	29·45	77·0	S.R.
V	,,	,,	2 16 56	5	46	29·45	77·0	S.S.
W	,,	,,	2 17 15	10	46	29·45	77·0	S.
X	,,	,,	2 17 33	3	46	29·45	77·0	S.R.
Y	,,	,,	2 17 47	2	46	29·45	77·0	S.R.
Z	,,	,,	2 18 1	2	46	29·45	77·0	S.R.

NOTES.

Columns 1 to 9. See previous page.

The large proportion of Ilford Special Rapid plates used at the eclipse was due to the fact that experience in developing the check plates showed that these suffered less than the others from the high temperature of the water (78° F.). Ice was generally available for the check plates through the kindness of Sr. GRAGEIRA ; but the supply failed after the eclipse, and formalin was used to harden the films. This was unsatisfactory except for the I.S.R. plates, and so plates P, S, T, W were brought home undeveloped. The developing at Principe was done at night, and the drying was accelerated by use of alcohol.

The use of an 8-inch stop in front of the object-glass was suggested to us by Mr. DAVIDSON, who showed that a great improvement of the images resulted ; it was originally intended, however, to use the full aperture for part of totality. Early measures of check plates made at Principe soon convinced us that the results from the full aperture were greatly inferior, and we decided to rely entirely on the 8-inch aperture.

The Check Plates.

31. In addition to the eclipse field, a check field was photographed both at Oxford and at Principe. The field chosen included Arcturus, so that it was easily found with the cœlostat. Its declination was nearly the same as that of the eclipse field, and it was photographed at the same altitude at Principe in order that any systematic error, due to imperfections of the cœlostat mirror or other causes, might affect both sets of plates equally. The primary purpose was thus to check the possibility of systematic error arising from the different conditions of observation at Oxford and Principe, and from possible changes in the object-glass during transit. Unlike the Sobral expedition, we were not able to take comparison photographs of the eclipse field at Principe, because for us the eclipse occurred in the afternoon, and it would be many months before the field could be photographed in the same position in the sky before dawn. The check plates were therefore specially important for us.

As events turned out the check plates were important for another purpose, viz., to determine the difference of scale at Oxford and Principe. As shown in the report of the Sobral expedition, it is not necessary to know the scale of the eclipse photographs, since the reductions can be arranged so as to eliminate the unknown scale. If, however, a trustworthy scale is known and used in the reductions, the equations for the deflection have considerably greater weight, and the result depends on the measurement of a larger displacement. On surveying the meagre material which the clouds permitted us to obtain, it was evident that we must adopt the latter course ; and accordingly the first step was to obtain from the check plates a determination of the scale of the Principe photographs.

32. All the measures were made by Prof. EDDINGTON with the Cambridge measuring machine.* An Oxford and a Principe plate were placed film to film so that the images of corresponding stars nearly coincided—this was possible because the Oxford plates were taken direct, and the Principe plates by reflection in the cœlostat mirror.

The small differences Δx and Δy, in the sense Principe—Oxford, were then measured for each star. Eight settings were made on each image ; for half of them the field was rotated through 180 degrees by the reversion prism. Five pairs of plates were measured, and the measures are given in Table XI.

* 'Monthly Notices, R.A.S.,' vol. LXI, p. 444.

TABLE XI.—Check Plates, Measures.

Star.	Approx. Co-ords.		q_1—a_1.		w_1—b_1.		s_2—c_1.		r_1—d_1.		v_1—e_1.	
	x.	y.	Δx.	Δy.	Δx.	Δy.	Δx.	Δy.	Δx.	Δy.	Δx.	Δy.
1	1·41	20·31	4346	7180	3199	4259	6012	7375	3921	8796	5435	4399
2	5·89	12·74	3865	6405	3394	4129	4922	6132	3039	7440	5978	4170
4	9·46	11·13	3640	5932	3408	4118	4369	5366	2638	6776	5966	4441
5	12·00	6·84	3311	5590	—	—	3831	4752	1938	6156	—	4314
6	12·80	27·33	5415	6561	3192	5140	7689	5925	5379	7580	5032	5794
7	13·75	13·78	4076	5630	3496	4290	4891	4805	3101	6461	5906	4826
8	15·50	24·38	5125	6300	—	—	—	—	—	—	5139	5412
10	20·13	10·49	3965	4940	3679	4505	4656	3568	2866	5370	6398	5229
11	20·81	0·93	2874	4352	3876	3759	2845	2815	1238	4758	7268	4482
12	22·91	6·23	3685	4436	3931	4158	4039	2738	2270	4551	6765	5076
13	26·46	8·96	4222	4288	4045	4326	4724	2232	2720	4120	6836	5561

The unit for x and y is 5 millimetres, which is approximately equal to 5′. The differences Δx, Δy are given in units of the fifth place of decimals $= 0''\cdot003$. The centre of the plate is near $x = 14$, $y = 14$.

Plate-constants were then calculated in the usual way, by the formulæ

$$\Delta x = ax + by + c$$

$$\Delta y = dx + ey + f$$

These were applied, and the residuals $\Delta_1 x$, $\Delta_1 y$ converted into arc are as follows :—

TABLE XII.—Check Plates, Residuals.

Star.	q_1—a_1.		w_1—b_1.		s_2—c_1.		r_1—d_1.		v_1—e_1.		Mean.	
	$\Delta_1 x$.	$\Delta_1 y$.	$\Delta_1 x$.	$\Delta_1 y$.	$\Delta_1 x$.	$\Delta_1 y$.	$\Delta_1 x$.	$\Delta_1 y$.	$\Delta_1 x$.	$\Delta_1 y$.	$\Delta_1 x$.	$\Delta_1 y$.
	″	″	″	″	″	″	″	″	″	″	″	″
1	−0·02	−0·02	+0·29	−0·34	+0·02	−0·07	−0·03	+0·22	+0·49	+0·01	+0·15	−0·04
2	+0·39	+0·15	+0·16	+0·14	+0·69	0·00	+0·69	−0·29	+0·10	−0·23	+0·41	−0·05
4	−0·14	−0·04	−0·16	+0·09	−0·38	−0·12	−0·02	−0·37	−0·54	+0·12	−0·25	−0·06
5	−0·08	+0·35	—	—	+0·25	+0·19	−0·21	−0·21	—	−0·01	−0·01	+0·08
6	−0·06	−0·10	−0·28	+0·27	−0·09	+0·14	−0·10	+0·12	+0·15	+0·49	−0·08	+0·18
7	−0·06	−0·28	−0·10	−0·16	−0·74	−0·09	−0·31	+0·02	−0·39	−0·12	−0·32	−0·13
8	−0·30	+0·34	—	—	—	—	—	—	−0·38	−0·68	−0·34	−0·17
10	−0·02	−0·10	−0·21	+0·52	−0·15	+0·16	+0·08	+0·25	−0·08	+0·34	−0·08	+0·23
11	−0·46	−0·01	−0·13	−0·22	−0·13	+0·11	−0·13	+0·71	+0·30	−0·28	−0·11	+0·06
12	+0·16	−0·14	+0·13	−0·04	+0·19	−0·06	+0·17	−0·09	−0·13	+0·08	+0·10	−0·05
13	+0·59	−0·12	+0·32	−0·26	+0·34	−0·25	−0·13	−0·38	+0·48	+0·28	+0·32	−0·15

The mean residual without regard to sign is $\pm\,0''\cdot21$, from which the probable error of a determination of Δx or Δy is $\pm 0''\cdot22$.

Star 7 is much the brightest. Stars 1, 6, 11, 13 are rather bright. Stars 2, 4, 10, 12 are fainter and more comfortable to measure. Stars 5 and 8 are very faint. Arcturus is on the plates but is much too bright to measure. No measures have been rejected.

The determination of the deflection on the eclipse plates is based on the declinations (y), and the last column of Table XII. shows that on the check plates the y-comparisons are free from any serious systematic error.

Star 7 is of particular interest ; its position near the centre of the field corresponds to that of κ_1, κ_2 Tauri in the eclipse field, from which the greatest deflection is expected. The images (which are not quite round) have the same characteristic shape. Further, the brightness of No. 7 corresponds with, but exaggerates, the brightness of κ_1 Tauri which is the brightest star in the eclipse field. It is therefore a valuable check to find that its systematic error in declination is insignificant compared with the displacement (of the order of 1″) afterwards found for κ_1 and κ_2 Tauri.

The systematic errors in right ascension are larger (probably through imperfect driving of the clock). They may affect the displacement indirectly through the orientation constant, but with much reduced effect. Allowing for this reduction in importance there appears to be nothing to trouble about.

The primary purpose of the check plates is thus fulfilled. They show that photographs of a check field of stars taken at Oxford and Principe show none of the displacements which are exhibited by the photographs of the eclipse field taken under precisely similar instrumental conditions. The inference is that the displacements in the latter case can only be attributed to presence of the eclipsed sun in the field.

33. We turn now to the differences of scale between Oxford and Principe, which are given by the plate-constants a, b, d, e determined from the measures. As determined, these include the effects of differential refraction and aberration. The latter corrections were calculated for each plate by the usual formulæ and applied, so as to determine the corrected plate-constants, a', b', d', e' free from differential refraction and aberration. Due allowance was made for the change in the coefficient of refraction owing to the difference of barometer and temperature (about 40°) between Oxford and Principe. The results are as follows (in units of the fifth place of decimals) :—

TABLE XIII.—Check Plates, Plate-Constants.

Comparison.	Uncorrected.				Corrected.				
	a.	b.	d.	e.	a'.	b'.	d'.	e'.	$b'+d'$.
$q_1 - a_1$	+32·7	+101·0	− 87·8	+58·2	+32·7	+ 98·4	− 90·4	+32·1	+ 8·0
$w_1 - b_1$	+26·2	− 16·0	+ 25·9	+53·6	+30·4	− 22·5	+ 19·4	+31·4	− 3·1
$s_2 - c_1$	+31·5	+192·5	−173·5	+64·8	+35·8	+182·6	−183·4	+42·1	− 0·8
$r_1 - d_1$	+28·2	+165·0	−146·8	+69·8	+32·1	+157·8	−154·0	+45·0	+ 3·8
$v_1 - e_1$	+21·6	− 76·2	+ 70·6	+61·4	+25·2	− 80·5	+ 66·3	+35·7	−14·2
Mean					+31·2	—	—	+37·3	− 1·3

The sign of the results shows that the scale of the photographs is larger at Principe than at Oxford ; in fact the focus must have been set about 1·2 mm. further out (apart from any change of length compensated by expansion of the photographic plates). As the error in focussing was probably not more than 0·5 mm., the greater part of this shift must be due to the focal length of the lens combination increasing with temperature more rapidly than the linear expansion of the glass.

If the only difference were a change of focal length, we should have $a' = e'$. There is a fairly strong indication that e' is greater than a'. This is no doubt due to a change in the definition caused by the cœlostat mirror or by a shift of the object-glass lenses on the journey ; and, as it will presumably affect the eclipse plates in the same way, it is best to adopt the values of a' and e' as determined, rather than to take a mean. In so doing we shall at any rate not exaggerate the displacement, which depends mainly on the y-measures and is reduced by adopting too large a value of e'.*

The difference $b' - d'$ merely gives the relative orientation of the two plates as placed face to face. The sum $b' + d'$ practically vanishes, as it should do. However, for consistency we adopt the small value found.

From the internal discordances of our determination of e' (the most important of these constants) the probable error of the mean is $\pm 2 \cdot 1$. This, as shown later, will cause a probable error of our final determination of the deflection, reduced to the limb of the sun, of amount $\pm 0'' \cdot 14$, affecting all determinations systematically. Errors in the other constants have much smaller influence.

The Eclipse Plates.

34. The eclipse plates from K to S show no star images. After that the cloud lightened somewhat, and some images appear on the remaining plates. The sky was never clear and nothing fainter than $5' \cdot 5$ is shown. The cloud was variable in different parts of the plate, so that the brightness of the images varies erratically and the diffusion is also variable.

In order to obtain results of any weight the stars 4 and 3 (κ_1 and κ_2 Tauri), which theoretically should be strongly displaced, must be shown. They appear on all plates from T to Z, and being near the centre of the field have good images. They are relatively rather faint on plate U, but are bright on the other plates. The appearance of the remaining stars is as follows :—

Plate T. 6 bright ; 10 faint.
Plate U. 6, 10 very bright ; 11 faint.
Plate V. 6 bright ; 10 fair.
Plate W. 5, 6 good ; 10 diffused.
Plate X. 5, 6, 11 good.
Plate Y. 5, 6, 11 faint, diffused ; 12 very faint.
Plate Z. 5, 6, 11 faint, diffused.

* It happens that it is also reduced, but to a less extent, by using too small a value of a'.

The possibility of a determination of deflection practically depends on the appearance of star 5. The relative displacement of 5 and 3 is on EINSTEIN's theory, $1''\cdot 2$ in the y-co-ordinate. Further, the x-measures of 5 are needed for a really good determination of the orientation. Star 11 can scarcely take its place. It is true that the relative displacement is then $0''\cdot 8$; but the orientation affects this with a much larger factor, and the orientation is badly determined in the absence of star 5.

Accordingly plates W and X are the only ones likely to give a trustworthy result. X is somewhat the better plate of the two.* Measures have been made of the faint diffused images on plates Y and Z; but, as might have been expected, they are hopelessly discordant and cannot be reconciled by any adopted value of the deflection.

35. We give the measures of plates X and W in detail. Both comparisons of X were measured at Principe a few days after the eclipse. Plate W, which was not developed until after the return of the expedition, was measured at Cambridge on August 22–23.†

Plate X.

(1) Comparison with Oxford Plate G_1.

The differential refraction for all the eclipse plates is

$$a = -46\cdot 5, \qquad b, d = +8\cdot 2, \qquad e = -27\cdot 0$$

the differential aberration being zero.

For the comparison plate G_1

$$a = -19\cdot 1, \qquad b, d = +0\cdot 7, \qquad e = -28\cdot 3.$$

Hence for $X - G_1$

$$a = -27\cdot 4, \qquad b, d = +7\cdot 5, \qquad e = +1\cdot 3.$$

* Plate X has also the merit of a short exposure, 3s. We should mistrust the x-measures of a long exposure with variable cloud and imperfect guiding, because there is nothing to show that the images of the different stars are formed at the same time.

† Of the comparisons of check plates, $w_1 - b_1$ was measured on August 20, and the others about the end of September. Previous measures had been made at Principe with three earlier check plates taken on the night of May 16; but a slight change of adjustment of tilt was made the following day (thereafter it remained unaltered until the eclipse), and the small change of focus allowed for in the comparisons. These furnished a provisional scale which was used to obtain preliminary results. Afterwards the measurement of check plates was undertaken in a more systematic way, using later plates about which no doubt could arise, and giving the results printed above. No change of any importance was found; the final value for the deflection at the limb was reduced by $0''\cdot 4$ compared with the provisional value, but this was mainly due to the adoption of separate values of a' and e' instead of adopting the mean, and to recalculation of the differential refraction and aberration.

To these must be added the terms representing change of scale, determined from the check plates (Table XIII.), viz.,

$$a = +\,31\cdot2, \qquad b, d = -\,0\cdot6, \qquad e = +\,37\cdot3.$$

Hence the whole difference $X - G_1$ is given by

$$a = +\,3\cdot8, \qquad b, d = +\,6\cdot9, \qquad e = +\,38\cdot6.$$

The first step is to take the measured differences Δx, Δy, and take out the parts $ax + by$, $dx + ey$, due to these terms, leaving the corrected differences $\Delta_1 x$, $\Delta_1 y$.

$\Delta_1 x$ and $\Delta_1 y$ contain (1) the Einstein displacement, if any, and (2) the unknown relative orientation of the plates giving rise to terms of the form, $\Delta x = +\,\theta y$, $\Delta y = -\theta x$. These two parts could be separated by a least-squares solution, but in view of the poor quality of the material it seems better to adopt a method which keeps a better check on possible discordances and shows more clearly what is happening. The Einstein displacement in x is small, and we might perhaps neglect it altogether in determining θ from the x-measures. However, it is clear from preliminary trials that a displacement exists—whether the half or the full Einstein displacement. Hence if we take out three-quarters of the full Einstein displacement ($\frac{3}{4}E_x$) we divide the already slight effect by 4, and at the same time deal fairly between the two hypotheses.* The residuals $\Delta_2 x$ result.

From the equations $\Delta_2 x = c + \theta y$ we determine by least squares the orientation θ, which is found to be $+\,163$. Removing the term $163y$ we obtain the residuals $\Delta_3 x$.

Turning to $\Delta_1 y$, we correct for the orientation by taking out the term $-163x$, leaving $\Delta_3 y$. These values should agree for all the stars, except for the displacement and the accidental error.

Denoting the value of the displacement at 50′ (or 10 réseau-intervals) from the centre of the sun by κ, the y-displacements of the various stars will be $\kappa\alpha_y$, where α_y has the values tabulated below. We can therefore obtain κ by solving by least-squares the equations

$$\Delta_3 y = f + \kappa\alpha_y.$$

The radius of the sun during the eclipse was $15'\cdot78$. Hence the full Einstein displacement of $1''\cdot75$ corresponds to $0''\cdot55$ at 50′ distance, or, in our units of $0''\cdot003$, $\kappa = 184$. It is easily seen that the value is somewhere near this, and it is therefore easier and more instructive to take out $E_y = 184\alpha_y$, and determine the correction to κ from the residuals $\Delta_4 y$. We also remove the mean of $\Delta_4 y$ obtaining the final residuals.

The normal equations corresponding to equations of condition

$$\text{residual} = \delta f + \alpha_y\,\delta\kappa$$

* The smaller the displacement provisionally assumed for x, the larger is the displacement ultimately found from y (see p. 327).

are found to be

$$5\delta f + 2\cdot83\delta\kappa = -\ 1$$

$$2\cdot83\delta f + 4\cdot83\delta\kappa = +64$$

whence

$$3\cdot23\delta\kappa = +64,$$

$$\delta\kappa = +20.$$

An increase of 20 on 184 corresponds to an increase of $0''\cdot19$ on $1''\cdot75$. Hence the resulting deflection at the limb is $1''\cdot94$.

Since the full deflection is indicated we complete the results for x by taking out the remaining $\frac{1}{4}E_x$, obtaining $\Delta_4 x$, and then tabulate the residuals from the mean values —5942.

The successive steps are shown below :—

Star.	x.	Δx.	$3\cdot8x$.	$6\cdot9y$.	$\Delta_1 x$.	$\frac{3}{4}E_x$.	$\Delta_2 x$.	$+163y$.	$\Delta_3 x$.	$\Delta_4 x$.	Resid. (unit = $0''\cdot003$).
11	1·39	—3916	5	86	—4007	—76	—3931	2021	—5952	—5927	+ 15
5	12·40	—5518	47	20	—5585	—79	—5506	478	—5984	—5958	— 16
4	17·34	—2869	66	129	—3064	—54	—3010	3051	—6061	—6043	—101
3	17·48	—2924	66	121	—3111	—69	—3042	2869	—5911	—5888	+ 54
6	19·87	—1568	75	172	—1815	+ 3	—1818	4075	—5893	—5894	+ 48

Star.	y.	Δy.	$6\cdot9x$.	$38\cdot6y$.	$\Delta_1 y$.	$-163x$.	$\Delta_3 y$.	E_y.	$\Delta_4 y$.	α_y.	Resid.
11	12·40	6398	10	479	5909	— 227	6136	+ 6	6130	+0·03	+ 5
5	2·93	4121	86	113	3922	—2021	5943	—127	6070	—0·69	— 55
4	18·72	4512	120	722	3670	—2826	6496	+234	6262	+1·27	+137
3	17·60	4236	121	679	3436	—2849	6285	+272	6013	+1·48	—112
6	24·99	4148	137	965	3046	—3239	6285	+136	6149	+0·74	+ 24

(2) Comparison with Oxford Plate H_1.

The reductions are similar and are given in a rather more condensed form below. The theoretical plate constants are

$$a = +3\cdot8, \qquad b, d = +8\cdot3, \qquad e = +38\cdot6.$$

Star.	Δx.	$\Delta_1 x$.	$\Delta_2 x$.	$+10y$.	$\Delta_3 x$.	$\Delta_4 x$.	Resid.
11	7290	7182	7258	124	7134	7159	+235
5	6751	6680	6759	29	6730	6756	—168
4	7126	6905	6959	187	6772	6790	—134
3	7320	7108	7177	176	7001	7024	+100
6	7429	7147	7144	250	6894	6893	— 31

Star.	Δy.	$\Delta_1 y$.	$-10x$.	$\Delta_3 y$.	E_y.	$\Delta_4 y$.	Resid.
11	1586	1095	− 14	1109	+ 6	1103	+172
5	858	642	−124	766	−127	893	− 38
4	1881	1015	−173	1188	+234	954	+ 23
3	1785	961	−175	1136	+272	864	− 67
6	1909	779	−199	978	+136	842	− 89

The normal equations are

$$5\delta f + 2\cdot83\delta\kappa = +1$$

$$2\cdot83\delta f + 4\cdot83\delta\kappa = -105$$

whence

$$3\cdot23\delta\kappa = -105,$$

$$\delta\kappa = -33.$$

The corresponding deflection at the limb is

$$1''\cdot75 - 0''\cdot31 = 1''\cdot44.$$

Plate W.

Although the exposure was only 10 seconds the images have jumped in R.A., so that the appearance is dumb-bell shaped. They are, however, symmetrical, so that fair measures of x can be made ; the y measures on which the result chiefly depends are unaffected. Star 10 is very diffused in R.A.

(1) Comparison with Oxford Plate D_1.

Theoretical plate-constants

$$a = +4\cdot9, \qquad b, d = +6\cdot5, \qquad e = +39\cdot7.$$

Star.	x.	Δx.	$\Delta_1 x$.	$\frac{3}{4}E_x$.	$\Delta_2 x$.	$+91y$.	$\Delta_3 x$.	$\Delta_4 x$.	Resid.
5	12·40	2450	2370	−79	2449	267	2182	2208	+ 40
4	17·34	3948	3741	−54	3795	1704	2091	2109	− 59
3	17·48	3834	3634	−69	3703	1602	2101	2124	− 44
6	19·87	4525	4266	+ 3	4263	2275	1988	1987	−181
10	22·60	5199	4911	+17	4894	2476	2418	2412	+244

Star.	y.	Δy.	$\Delta_1 y$.	$-91x$.	$\Delta_3 y$.	E_y.	$\Delta_4 y$.	α_y.	Resid.
5	2·93	5320	5123	−1128	6251	−127	6378	−0·69	+ 70
4	18·72	5745	4889	−1578	6467	+234	6233	+1·27	− 75
3	17·60	5911	5098	−1591	6689	+272	6417	+1·48	+109
6	24·99	5628	4507	−1808	6315	+136	6179	+0·74	−129
10	27·21	5616	4389	−2057	6446	+114	6332	+0·62	+ 24

Normal equations

$$5\delta f + 3 \cdot 42 \delta\kappa = -1$$

$$3 \cdot 42 \delta f + 5 \cdot 21 \delta\kappa = -62$$

whence

$$2 \cdot 87 \delta\kappa = -61$$

$$\delta\kappa = -21.$$

Hence deflection at the limb is

$$1'' \cdot 75 - 0'' \cdot 20 = 1'' \cdot 55.$$

(2) Comparison with Oxford Plate I_2.

Theoretical plate constants

$$a = +4 \cdot 0, \qquad b, d = +9 \cdot 1, \qquad e = +38 \cdot 8.$$

Star.	Δx.	$\Delta_1 x$.	$\Delta_2 x$.	$-30y$.	$\Delta_3 x$.	$\Delta_4 x$.	Resid.
5	5050	4973	5052	− 88	5140	5166	+ 46
4	4732	4493	4547	−562	5109	5127	+ 7
3	4622	4392	4461	−528	4989	5012	−108
6	4635	4329	4326	−750	5076	5075	− 45
10	4764	4426	4409	−816	5225	5219	+ 90

Star.	Δy.	$\Delta_1 y$.	$+30x$.	$\Delta_3 y$.	E_y.	$\Delta_4 y$.	Resid.
5	−6824	−7051	372	−7423	−127	−7296	− 15
4	−5751	−6635	520	−7155	+234	−7389	−108
3	−5609	−6451	524	−6975	+272	−7247	+ 34
6	−5425	−6576	596	−7172	+136	−7308	− 27
10	−5109	−6371	678	−7049	+114	−7163	+118

Normal equations

$$5\delta f + 3 \cdot 42 \delta\kappa = +2$$

$$3 \cdot 42 \delta f + 5 \cdot 21 \delta\kappa = -24$$

whence

$$2 \cdot 87 \delta\kappa = -25,$$

$$\delta\kappa = -9.$$

Hence deflection at the limb is

$$1'' \cdot 75 - 0'' \cdot 08 = 1'' \cdot 67.$$

Plate U.

Comparison with Oxford Plate K_2.

Since Plate U shows some good images it has been examined, although owing to the absence of star 8 the weight is small. The measures were made at Principe.

Theoretical plate-constants

$$a = +2\cdot 8, \qquad b, d = +8\cdot 9, \qquad e = +37\cdot 7.$$

Star.	x.	Δx.	$\Delta_1 x$.	$+240y$.	E_x.	$\Delta_4 x$.	Resid.
11	1·39	2905	2791	2976	−101	− 84	−147
4	17·34	4508	4292	4493	− 72	−129	−192
3	17·48	4626	4420	4224	− 92	+288	+225
6	19·87	6270	5992	5998	+ 4	− 10	− 73
10	22·60	7110	6805	6530	+ 23	+252	+189

Star.	x.	Δy.	$\Delta_1 y$.	$-240x$.	E_y.	$\Delta_4 y$.	Resid.
11	12·40	9026	8547	− 334	+ 6	8875	− 94
4	18·72	5846	4986	−4162	+234	8914	− 55
3	17·60	5985	5165	−4195	+272	9089	+120
6	24·99	5458	4339	−4769	+136	8972	+ 3
10	27·21	4911	3684	−5424	+114	8994	+ 25

In this case it is not possible to determine the orientation with sufficient accuracy from the x-measures; the value here applied is an arbitrary preliminary value. We accordingly make a least-squares solution from both x- and y-residuals to determine the correction to the orientation, $\delta\theta$, as well as δc, δf and $\delta\kappa$.

The result is

$$\delta\theta = +2, \qquad \delta\kappa = +121.$$

This gives the deflection

$$2''\cdot 90.$$

The probable error is, however, $\pm\ 0''\cdot 87$, so that the result is practically worthless. Further, it is much more likely to be affected by systematic error than the previous results.

The large probable error is partly due to the large residuals which are greater than in the previous measures; in particular star 3 is unduly faint. If the same accuracy had been obtained, the theoretical weight would have been half that of plates W and X;

but having regard to possible systematic error, probably a quarter weight would more nearly represent the true value.

This determination is ignored in the subsequent discussion.

36. It is easy to calculate the effects of any errors in the adopted scale, orientation, &c., on the final result (deflection at the limb). We give some illustrations.

An error in the adopted scale of y of 10 units (in the fifth place of decimals) would lead to an error $0''\cdot68$ in the result from either plate. Thus the probable error $\pm2\cdot1$ in the determination of e' gives a probable error $\pm0''\cdot14$ in the final result; or, if we adopted the largest (rather discordant) value found for e' instead of the mean, we should reduce the result by $0''\cdot52$.

An error of 10 units in the orientation gives an error in the result of $0''\cdot45$ for plate X, and $0''\cdot22$ for Plate W. It is therefore of less importance, and further it is not likely to be systematic.

Errors in the measurement of x only affect the result through the orientation. For Plate X, a probable error of $\pm0''\cdot20$ in the x-measures would give an error $\pm4\cdot0$ in the orientation, leading to an error $\pm0''\cdot18$ in the result; whereas an error of the same magnitude in the y measures gives directly an error $\pm0''\cdot35$ in the result. For Plate W, the probable error of $\pm0''\cdot20$ in x gives an error $\pm3\cdot5$ in the orientation and $\pm0''\cdot08$ in the result, compared with $\pm0''\cdot38$ for similar inaccuracy in y. It is particularly fortunate that the x-measures are so unimportant for Plate W, because, as already mentioned, the images trailed on that plate.

Finally, it will be remembered that in order not to commit ourselves to the Einstein hypothesis prematurely we neglected the correction $\frac{1}{4}E_x$ in determining the orientation. This will make a difference of $0''\cdot029$ in the results from Plate W and $0''\cdot092$ from Plate X. The effect is that the deduced deflection needs to be decreased, and the mean correction $-0''\cdot06$ should be applied to the mean result obtained, or rather, to make the adopted deflection for x consistent with the deduced value from y, the correction needed is $-0''\cdot04$.

Discussion of the Results.

37. The four determinations from the two eclipse plates are

X − G	$1''\cdot94$
X − H	$1''\cdot44$
W − D	$1''\cdot55$
W − I	$1''\cdot67$

giving a mean of $1''\cdot65$.

They evidently agree with EINSTEIN's predicted value $1''\cdot75$.

The residuals* in the separate comparisons reduced to arc are as follows. They do not appear to show any special peculiarities.

Star.	x residuals.					y residuals.				
	G.	H.	D.	I.	Mean.	G.	H.	D.	I.	Mean.
	″	″	″	″	″	″	″	″	″	″
11	+0·04	+0·70	—	—	—	+0·01	+0·52	—	—	—
5	−0·05	−0·50	+0·12	+0·14	−0·07	−0·16	−0·11	+0·21	−0·04	−0·02
4	−0·30	−0·40	−0·18	+0·02	−0·21	+0·41	+0·07	−0·22	−0·32	−0·02
3	+0·16	+0·30	−0·13	−0·32	0·00	−0·34	−0·20	+0·33	+0·10	−0·03
6	+0·14	−0·09	−0·54	−0·13	−0·16	+0·07	−0·27	−0·39	−0·08	−0·17
10	—	—	+0·73	+0·27	—	—	—	+0·07	+0·35	—

The average y-residual is $\pm 0''\cdot 22$, which gives a probable error for y of $\pm 0''\cdot 21$. It is satisfactory that this agrees so nearly with the probable error ($\pm 0''\cdot 22$) of the check plates, showing that the images are of about the same degree of difficulty and therefore presumably comparable. The probable error of x is $\pm 0''\cdot 25$, but we are not so much concerned with this.

The weight of the determination of $\delta\kappa$ is about 3 (strictly 3·23 for Plate X and 2·87 for Plate W). The probable error of κ is therefore $\pm 0''\cdot 12$, which corresponds to a probable error of $\pm 0''\cdot 38$ in the final values of the deflection.

As the four determinations involve only two eclipse plates and are not wholly independent, and further small accidental errors may arise through inaccurate determination of the orientation, the probable error of our mean result will be about $\pm 0''\cdot 25$. There is further the error of $\pm 0''\cdot 14$ affecting all four results equally, arising from the determination of scale. Taking this into account, and including the small correction $-0''\cdot 04$ previously mentioned, our result may be written

$$1''\cdot 61 \pm 0''\cdot 30.$$

It will be seen that the error deduced in this way from the residuals is considerably larger than at first seemed likely from the accordance of the four results. Nevertheless the accuracy seems sufficient to give a fairly trustworthy confirmation of EINSTEIN'S theory, and to render the half-deflection at least very improbable.

38. It remains to consider the question of systematic error. The results obtained with a similar instrument at Sobral are considered to be largely vitiated by systematic

* The residuals refer to the theoretical deflection $1''\cdot 75$, not the deduced deflections.

errors. What ground then have we—apart from the agreement with the far superior determination with the 4-inch lens at Sobral—for thinking that the present results are more trustworthy ?

At first sight everything is in favour of the Sobral astrographic plates. There are 12 stars shown against 5, and the images though far from perfect are probably superior to the Principe images. The multiplicity of plates is less important, since it is mainly a question of systematic error. Against this must be set the fact that the five stars shown on Plates W and X include all the most essential stars; stars 3 and 5 give the extreme range of deflection, and there is no great gain in including extra stars which play a passive part. Further, the gain of nearly two extra magnitudes at Sobral must have meant over-exposure for the brighter stars, which happen to be the really important ones; and this would tend to accentuate systematic errors, whilst rendering the defects of the images less easily recognised by the measurer. Perhaps, therefore, the cloud was not so unkind to us after all.

Another important difference is made by the use of the extraneous determination of scale for the Principe reductions. Granting its validity, it reduces very considerably both accidental and systematic errors. The weight of the determination from the five stars with known scale is more than 50 per cent. greater than the weight from the 12 stars with unknown scale. Its effect as regards systematic error may be seen as follows. Knowing the scale, the greatest relative deflection to be measured amounts to $1''\cdot 2$ on EINSTEIN'S theory; but if the scale is unknown and must be eliminated, this is reduced to $0\cdot''67$. As we wish to distinguish between the full deflection and the half deflection, we must take half these quantities. Evidently with poor images it is much more hopeful to look for a difference of $0''\cdot 6$ than for $0''\cdot 3$. It is, of course, impossible to assign any precise limit to the possible systematic error in interpretation of the images by the measurer; but we feel fairly confident that the former figure is well outside possibility.

A check against systematic error in our discussion is provided by the check plates, as already shown. Its efficacy depends on the similarity of the images on the check plates and eclipse plates at Principe. Both sets are fainter than the Oxford images with which they are compared, the former owing to the imperfect driving of the cœlostat, which made it impossible to secure longer exposures, the latter owing to cloud. Both sets have a faint wing in declination, but this is separated by a slight gap from the true images, and, at least on the plates measured, the wing can be distinguished and ignored. The images on Plates W and X are not unduly diffused except for No. 10 on Plate W. Difference in quality between the eclipse images and the Principe check images is not noticeable, and is certainly far less than the difference between the latter and the Oxford images; and, seeing that the latter comparison gives no systematic error in y, it seems fair to assume that the comparison of the eclipse plates is free from systematic error.

The writer must confess to a change of view with regard to the desirability of using an extraneous determination of scale. In considering the programme it had seemed too risky a proceeding, and it was thought that a self-contained determination would receive more confidence. But this opinion has been modified by the very special circumstances at Principe; and it is now difficult to see that any valid objection can be brought against the use of the scale.

The temperature at Principe was remarkably uniform and the extreme range probably did not exceed 4° during our visit—including day and night, warm season and cold season. The temperature ranged generally from $77\frac{1}{2}°$ to $79\frac{1}{2}°$ in the rainy season, and about 1° colder in the cool gravana. All the check plates and eclipse plates were taken within a degree of the same temperature, and there was, of course, no perceptible fall of temperature preceding totality. To avoid any alteration of scale in the daytime the telescope tube and object-glass were shaded from direct solar radiation by a canvas screen; but even this was scarcely necessary, for the clouds before totality provided a still more efficient screen, and the feeble rays which penetrated could not have done any mischief. A heating of the mirror by the sun's rays could scarcely have produced a true alteration of scale though it might have done harm by altering the definition; the cloud protected us from any trouble of this kind. At the Oxford end of the comparison the scale is evidently the same for both sets of plates, since they were both taken at night and intermingled as regards date.

It thus appears that the check scale is legitimately applicable to the eclipse plates. But the method may not be so satisfactory at future eclipses, since the particular circumstances at Principe are not likely to be reproduced. As regards other sources of systematic error, our chief guarantee lies in the comparatively large amount of the deflection to be measured, and the test satisfied by the check plates that photographs of another field under similar conditions show no deflections comparable with those here found.

V. General Conclusions.

39. In summarising the results of the two expeditions, the greatest weight must be attached to those obtained with the 4-inch lens at Sobral. From the superiority of the images and the larger scale of the photographs it was recognised that these would prove to be much the most trustworthy. Further, the agreement of the results derived independently from the right ascensions and declinations, and the accordance of the residuals of the individual stars (p. 308) provides a more satisfactory check on the results than was possible for the other instruments.

These plates gave

From declinations	$1''\cdot 94$
From right ascensions	$2''\cdot 06$

The result from declinations is about twice the weight of that from right ascensions, so that the mean result is

$$1''\cdot 98$$

with a probable error of about $\pm 0''\cdot 12$.

The Principe observations were generally interfered with by cloud. The unfavourable circumstances were perhaps partly compensated by the advantage of the extremely uniform temperature of the island. The deflection obtained was

$$1''\cdot 61.$$

The probable error is about $\pm 0''\cdot 30$, so that the result has much less weight than the preceding.

Both of these point to the full deflection $1''\cdot 75$ of EINSTEIN'S generalised relativity theory, the Sobral results definitely, and the Principe results perhaps with some uncertainty. There remain the Sobral astrographic plates which gave the deflection

$$0''\cdot 93$$

discordant by an amount much beyond the limits of its accidental error. For the reasons already described at length not much weight is attached to this determination.

It has been assumed that the displacement is inversely proportional to the distance from the sun's centre, since all theories agree on this, and indeed it seems clear from considerations of dimensions that a displacement, if due to gravitation, must follow this law. From the results with the 4-inch lens, some kind of test of the law is possible though it is necessarily only rough. The evidence is summarised in the following table and diagram, which show the radial displacement of the individual stars (mean from all the plates) plotted against the reciprocal of the distance from the centre. The displacement according to EINSTEIN'S theory is indicated by the heavy line, according to the Newtonian law by the dotted line, and from these observations by the thin line.

RADIAL Displacement of Individual Stars.

Star.	Calculation.	Observation.
	"	"
11	0·32	0·20
10	0·33	0·32
6	0·40	0·56
5	0·53	0·54
4	0·75	0·84
2	0·85	0·97
3	0·88	1·02

Thus the results of the expeditions to Sobral and Principe can leave little doubt that a deflection of light takes place in the neighbourhood of the sun and that it is of the amount demanded by EINSTEIN's generalised theory of relativity, as attributable to the sun's gravitational field. But the observation is of such interest that it will probably be considered desirable to repeat it at future eclipses. The unusually favourable conditions of the 1919 eclipse will not recur, and it will be necessary to photograph fainter stars, and these will probably be at a greater distance from the sun.

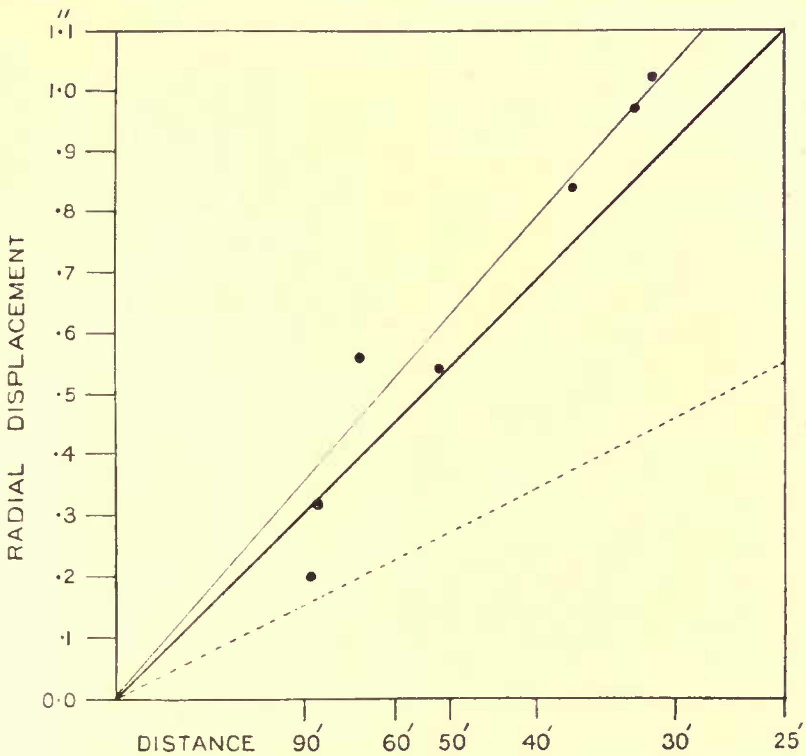
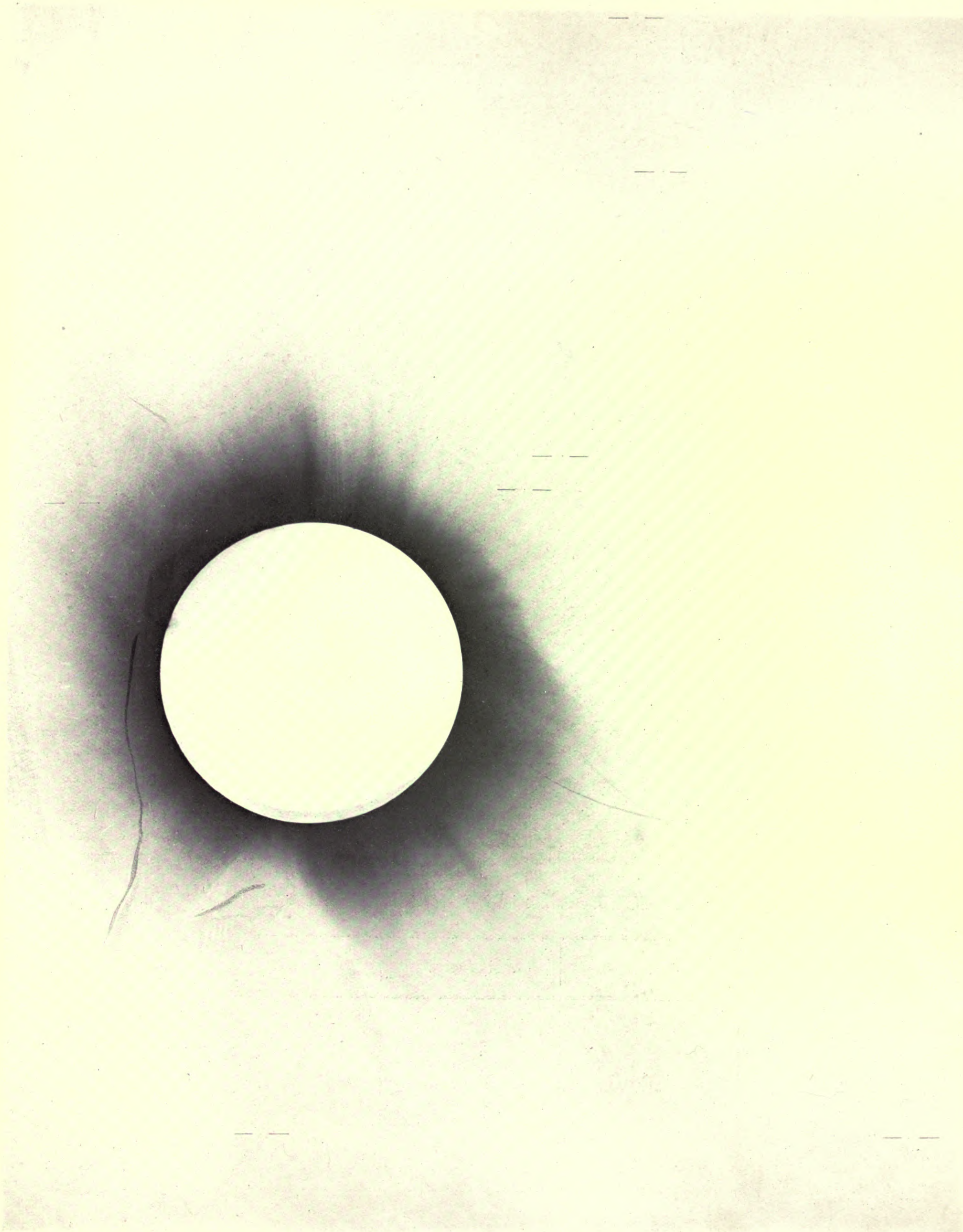


Diagram 2.

This *can* be done with such telescopes as the astrographic with the object-glass stopped down to 8 inches, if photographs of the same high quality are obtained as in regular stellar work. It will probably be best to discard the use of cœlostat mirrors. These are of great convenience for photographs of the corona and spectroscopic observations, but for work of precision of the high order required, it is undesirable to introduce complications, which can be avoided, into the optical train. It would seem that some form of equatorial mounting (such as that employed in the Eclipse Expeditions of the Lick Observatory) is desirable.

In conclusion, it is a pleasure to record the great assistance given to the Expeditions from many quarters. Reference has been made in the course of the paper to some of these. Especial thanks are due to the Brazilian Government for the hospitality and facilities accorded to the observers in Sobral. They were made guests of the



Government, who provided them with transport, accommodation and labour. Dr. MORIZE, Director of the Rio Observatory, acting on behalf of the Brazilian Government, made most complete arrangements for the Expedition, and in this way contributed materially to its success.

On behalf of the Principe Expedition, special thanks are due to Sr. JERONYMO CARNEIRO, who most hospitably entertained the observers and provided for all their requirements, and to Sr. ATALAYA, whose help and friendship were of the greatest service to the observers in their isolated station.

We gratefully acknowledge the loan for more than six months of the astrographic object-glass of the Oxford University Observatory. We are also indebted to Mr. BELLAMY for the check plates he obtained in January and February.

Thanks are due to the Royal Irish Academy for the loan of the 4-inch object-glass and 8-inch cœlostat.

As stated above, the expeditions were arranged by the Joint Permanent Eclipse Committee with funds allocated by the Government Grant Committee.

[In Plate 1 is given a half-tone reproduction of one of the negatives taken with the 4-inch lens at Sobral. This shows the position of the stars, and, as far as possible in a reproduction of this kind, the character of the images, as there has been no retouching.

A number of photographic prints have been made and applications for these from astronomers, who wish to assure themselves of the quality of the photographs, will be considered and as far as possible acceded to.]

X. *A Critical Study of Spectral Series.—Part V. The Spectra of the Monatomic Gases.*

By W. M. Hicks, *F.R.S.*

Received November 14, 1918,—Read January 23, 1919.

[Plates 2–5.]

Contents.

 [Published May 19, 1920.

THE present communication has two objects. Its subject matter is an attempt to obtain some knowledge of the series relations in the spectra of the group of the monatomic rare gases Ne to RaEm, whilst the methods employed will serve to illustrate the fundamental importance as instruments for further research of the new facts brought to light in the previous communications.* The importance of the first object will be generally acknowledged, but it does not yet seem to be realised how definite and exact those new relationships are, even in their as yet undeveloped form, and how powerful an instrument is placed in our hands for the analysis of spectra. It may be well therefore to commence by a brief *résumé* of some of these laws as applied in the succeeding pages. Further, as the establishment of the results obtained must by its nature depend on the numerical comparison of a very large number of lines in all five spectra, and as this evidence must be fully set out to

* "A Critical Study of Spectral Series," Part I., 'Phil. Trans.,' A, vol. 210, p. 57.
„ „ „ Part II. „ „ 212, p. 33.
„ „ „ Part III. „ „ 213, p. 323.
„ „ „ Part IV. „ „ 217, p. 361.
These will be referred to respectively as [I.], [II.], [III.], [IV.].

enable a specialist judgment to be formed on it, the communication has unfortunately become very lengthy. The mass of detail will perhaps be rather dreary to the general reader not specially interested in this line of study. It is apt also to hide by its amount and complexity the general conclusions arrived at. I propose therefore to give a slight general survey of these conclusions before giving the evidence.

As is well known the wave-numbers of series lines depend on four types of sequences $p(m)$, $s(m)$, $d(m)$, $f(m)$, and that in any one series they depend on the differences between one sequent of one type and the successive terms of the sequence of another type. These sequences are all of the form $N/\{\phi(m)\}^2$, where N is RYDBERG'S constant and $\phi(m)$ is of the form m+fraction, the fraction being, as a rule, determinable as a decimal to six significant figures. Our aim is to discover the properties of these functions. The fractional part depends in some way on the order m, although whether it can be considered a definite function of m in the ordinary sense is doubtful.* This fractional part will be referred to as the mantissa, and in dealing with it, it will be regarded as multiplied by 10^6, *i.e.*, as if the decimal point were removed.

The Oun.—It is found† that in each element a constant quantity particular to each element plays a fundamental part in the constitution of the sequences. This is called the oun. The d and f sequences depend in definite ways on multiples of this quantity, whilst it also enters into the constitution of the p and s. Its determination is therefore for each element a matter of the first importance. Denoting its value by δ_1, the quantity $\delta = 4\delta_1$ is of such frequent recurrence that it is useful to treat it as one datum. The oun is accurately proportional to the square of the atomic weight, and is given by $\delta = (361 \cdot 8 \pm \cdot 1)(w/100)^2$, where w denotes the atomic weight.

In the case of doublet or triplet series, the corresponding separations between them are due to different limits whose mantissæ differ by amounts Δ or Δ_1, Δ_2 (say). In all cases these are found to be integral multiples of the oun. For triplets $\Delta_1 : \Delta_2$ is always somewhat greater than 2.

In the case of D series where satellites occur, the separations of the latter are due to differences in their d sequences. The mantissæ of these latter again differ by quantities which are multiples of the oun, and in the case of triplets they appear in normal types to be very close to the ratio 5 : 3.

The d Sequence.—In the normal type the sequent of the extreme satellite has its mantissa a multiple of Δ_2. The only known exceptions are found in Sr, Cd which show the multiple law, Sr in d_{12} and Cd in d_{11} instead of in d_{13}. In both these cases also the Zeeman pattern is abnormal. As the main lines D_{11} (and in triplets D_{12} also) have their mantissæ greater than that of the outer satellite by multiples of the oun, it follows that all the d sequences for the first order have mantissæ multiples of the oun. It is probable that this is true for all orders of m, but the data are not

* 'Astro. J.,' 44, p. 229, see also [III., p. 339].

† [III.], also 'Proc. R. S.,' A, 91 (1915).

sufficiently accurate to prove this, although they obey the rule within error limits. [III., p. 340.]

The f Sequence.—This sequent of the first order also has the multiple of Δ_2. The material at disposal is not so comprehensive as in the case of the D series, for, except in the second group of the periodic table, the F lines occur chiefly in the ultra-red. The proof of the above statement is perhaps, therefore, not so conclusive as in the case of the d sequence. It completely stands the test however in the rare gases. There seems some evidence that F series also show a satellite effect in a small degree —of one or two ouns. In the second group it seems to be a general rule that in many of the low orders ($m = 1, 2\ldots$) the f sequents receive very large displacements from their normal value, so that a normal line is much weaker or is altogether absent and replaced by others separated from it by considerable numbers. This also is found to be the rule in the present case.

Displacement.—Regarding the ordinary doubtlet or triplet series we may consider the second (or third) as displaced from the first by the deduction of a certain number of ouns from the mantissa of the limit; or better perhaps regard the last satellite set as the fundamental one and the others as displaced by the addition of ouns. When such displacements occur in the limit of one line the new one is indicated by writing the displacement on the left. Thus $S_2(m) = (-\Delta_1)S_1(m)$ or $S_1(m) = (\Delta_1)S_2(m) = (\Delta_1+\Delta_2)S_3(m)$. With satellites, on the other hand, the similar effect is produced in the sequence terms. In this case it is entered on the right. Thus $D_{12}(m) = D_{11}(m)(-x\delta_1)$ or $D_{11}(m) = D_{12}(m)(x\delta_1)$; $D_{22}(m) = (-\Delta_1)D_{12}(m) = (-\Delta_1)D_{11}(m)(-x\delta_1)$. Displacements of both kinds are very common in spark spectra and put themselves specially in evidence in the succeeding pages. A normal line may not only show lines displaced from it, but often it appears to be replaced by them, and, in general, when it does not disappear its intensity is abnormally low. This is practically what happens in the D satellites. The D_{13}, D_{23}, D_{33} appear to be the normal lines in which we should expect descending order of intensity; but most of the energy (or the majority of the emitting centres) appears carried over to the more intense and displaced set D_{12}, D_{22}; and, again, most of what should be expected in D_{12} is carried over to become the strongest line D_{11}. Frequently the D_{13} line has disappeared and only the fragment D_{23}, D_{33} of the triplet left. In general, the D_{11} lines of any element are the strongest of the series. But in the present vacuum tube spectra (spark type) we shall find very frequently that the line required for D_{11} is comparatively weak, and in this case there appear other lines related to it by oun displacements chiefly in the limit. As the real existence of these displaced series is a matter of some importance considerable space has been given in the discussion of the X spectrum (p. 399) to its demonstration in the case of two series depending on the limits $(\pm 2\delta_1)D(\infty)$. It seems a peculiarity of these displacement series that a term of one series may be absent but appear in another. Thus $(-2\delta_1)D(m)$ may not be observed, but a $(+2\delta_1)D(m)$ may be and *vice versâ*. The presence of a similar effect in the sequent terms of F series has been referred to above. One good illustration of double displacements fully established is found in the KrS series (p. 349), in which the indications are shown for $m = 1, 2, 3$. A knowledge of the laws governing displacements is much to be desired. Very little is known at present.

Linkages.—Arc spectra are distinguished, as a rule, by the presence of well-defined series, depending on single groups of P.S.D.F. type. In spark and vacuum tube spectra, however, these seem to be weakened, and a very large number of other lines appear which are related to one another by certain constant separations (links) to form congeries of linkages each connected to a series line. These links can be calculated when the values of Δ, or of Δ_1, Δ_2, are known. The evidence for these was given in [IV.]. There appear to be links of several types. Those already discussed are of two types: (1) separations between successive double displacements of Δ_1 on either side of $S_2(\infty)$ or $p_2(1)$; (2) displacements of Δ_1 on either side of $P(\infty)$ or $s(1)$. Of these, use is confined almost entirely in the present communication to one only of type (1) and both of type (2). They are

$$e = (-2\Delta_1)\,p_2(1)-(2\Delta_1)\,p_2(1),$$

or

$$= (-3\Delta_1)\,S_1(\infty)-(\Delta_1)\,S_1(\infty),$$

$$u = s(1)-(\Delta_1)\,s(1), \qquad v = (-\Delta_1)\,s(1)-s(1).$$

These links themselves may also be subject to small displacements by having their sources on, say, $(x\delta_1)\,S_1(\infty)$ instead of $S_1(\infty)$. For the present purpose, however, no use can be made of these.

In [IV.] the prevalence of these separations in a spectrum in excess of their occurrence from mere chance was exhibited in a series of curves with abscissæ = separation and ordinates = number of occurrences within a given small amount on each side. Such occurrency curves are also given here for the e links and for the u, v of Kr in Plate 2. The e links seem to be a normal accompaniment to series lines (often displaced, however, when directly attached to those of low order). A further peculiarity of these linkages is the prevalence of the combination $e \pm u$, or $e \pm v$. They are indicated by writing the letter denoting the link to the left of the line when deducted and to the right when added. Thus, in the example below, $44236 = e.47419$, or $47419 = 44236.e$.

Sounding.—In the following pages the unravelling of the complete series of linkages has not been touched upon, but the e, u, v links have been used for testing the existence of lines outside the observed region, a method we may call sounding. A link thus used may be referred to as a sounder. In this way it is possible to obtain evidence of the existence, or of the exact value, of a calculated line which lies beyond the region observed. It may even serve as evidence for the real existence of a line in the observed region too weak to have been observed, for it was shown in [IV.] that the e link appeared to have a tendency to increase the intensity of one of the two lines to which it was attached. The method may be illustrated by an example from KrS. In Kr, $e = 3183{\cdot}35$. The value of $S_1(5)$ as calculated from the formula is 47419·39, which is in the ultra-violet outside the observed region. But $47419{\cdot}39 - e = 44236{\cdot}04$. This is within the observed region, and as a fact the corresponding line is found at 44237·61 with $d\lambda = -{\cdot}08$ if e is free from error. As an individual case this might be due to a coincidence, but when the same effect occurs with line after line the cumulative effect becomes convincing. To see this it is necessary to get at a glance a survey of all the cases, and for this purpose they are exhibited in sets of diagrams in Plate 3. These diagrams also include links within the observed region in order to show that where the method can be tested it holds. It may be specially noted how the similar arrangement of sounders holds for the same order in the three lines of the same triplet, and how in certain cases the u, v seem alternative. *Cf.* for example XS (1, 3, 4, 7, 8), or the main lines of the three parallel D sets in X, viz. $(2\delta_1)D_{11}$, D_{11}, $(-2\delta_1)D_{11}$, or particularly the prevalence of the $-(e+v)$ combinations in the unobserved lines for KrD. In RaEm these links are too large to be of wide application and in Ne too small to be of use. In RaEm the e link is ,23678 and can reach from the unobserved ultra-red across to the unobserved ultra-violet. In Ne the e link is 196, so that its reach is too small to be useful. As this method of sounding is new and clearly of importance if substantiated, considerable attention has been given to its illustration, but as the details themselves are only necessary for a critical study they have been printed in smaller type and may be omitted on a first reading.

Abnormal D *triplet Separations.*—It has generally been held since RYDBERG's discovery of the satellite systems that the triplet separations for the D and S series are the same. The actual measures did not absolutely prove this, in fact, they indicated small differences, but the accuracy was not sufficient to establish a real difference especially as against a natural bias to expect equality. MEGGERS,* however, has recently placed it beyond doubt that frequently the separations are really different. In Group I., for instance, the separations as measured from D lines are less than those determined from S lines. In the rare gases also this difference appears quite decisively, but here (group 0) the separations as determined from the D lines are, in general, larger than those from S. The key to the explanation is found in the fact that the difference between the two determinations diminishes with increasing order—in other words, that the sequent in the same set of satellites is not the same, and that in a large number of cases the value of $\nu_1+\nu_2$ is the same in both S and D although ν_1, ν_2 themselves are different. It is found to be completely explained by the displacement of one or two ouns between the sequents of the first or second members of a triplet. Sometimes it occurs in the third member. The same explanation accounts for the fact that the F separations are frequently smaller than the

* 'Bur. Stand.,' Washington, No. 312 (1918).

corresponding observed satellite separations. It also accounts for the appearance of F satellites as shown here and in [III., pp. 389–395] in connection with the F lines in the alkaline earths. The matter is considered in detail for Kr (p. 363) and is found to hold for the cases of the other rare gases.

The Atomic Weight.—It is clear that an accurate knowledge of a first f sequent, or of a d sequent which belongs to the satellite involving the Δ_2 multiple, gives the means of determining the oun to in general a unit in the sixth significant figure. For these mantissæ are usually of the order of magnitude of 0·8 and are known to six figures. Hence, if the multiple is known, Δ_2 can itself be determined, and since Δ_2 is a known multiple of the oun (determined by the S multiplet separations), the oun is also known to the same degree of accuracy. Further, as the sources of determining it are often quite independent they serve as tests of the determinativeness of the oun itself to the same degree of accuracy. When Δ_2 is considerable, its value is known sufficiently well for it to determine the multiple, and then this exact integer conversely gives the exact value of Δ_2. In the cases of A and Ne, however, the values of Δ_2 are too small to determine uniquely this multiple directly. The difficulty, however, is surmounted by obtaining successively values with increasing accuracy from other considerations until the final test can be applied. As a fact the Ne oun is amongst the most accurate found. Its determination (p. 461) is specially interesting, and indeed is only possible because the material at disposal depends on interferential measures and large accurately known separations. That of X also is a good determination, and is interesting as depending on a number of quite independent data.

As the oun is proportional to the square of the atomic weight within the limits of error of determination of the latter, it is natural to assume that the relation is exact and that $\delta = q.w^2$, where q is a number between $361{\cdot}8 \pm {\cdot}1$. If this were sustained it would be possible to obtain w with twice the degree of accuracy of the oun and therefore far in advance of any obtainable by chemical methods. In fact the question is raised as to what is actually understood by the atomic weight. Does it refer to the mass of the positive nucleus, or to that and all or a portion of the electrons? The hope might even be entertained of obtaining by this method some knowledge of the number of electrons partaking in the emission of a line if slight changes in the oun could be found. For instance, we shall find in these spectra not a single group of S, D, or F series as in arc spectra, but several independent groups, viz., d and f sequents, depending on different multiples of Δ_2. If these gave slightly different values of the oun it could be explained by a transference of electrons. There is little evidence of such variation, but it might occur, for instance, in the oun as deduced respectively from Δ_1 and Δ_2. As Δ_1 depends alone on the measurement of the ν_1 separation of a triplet it is not susceptible of such exact determination as Δ_2, and, as a fact, a suspicion sometimes arises that such a slight difference may exist, and that δ from ν_1 is somewhat less than from ν_2 [III., p. 333] as also here.

Suppose the atomic weight is W and the number of electrons involved is xW. Then the oun is given by

$$\delta_x = q\,(\mathrm{W}+x\mathrm{W}/1850)^2 = \delta_0\left(1+\frac{x}{925}\right).$$

If another value depends on y electrons

$$\delta_y = \delta_0\left(1+\frac{y}{925}\right),$$

whence

$$x-y = 925\,\frac{\delta_x-\delta_y}{\delta} = 925\,\frac{d\delta}{\delta},$$

which gives the transference. At present these considerations are only of speculative interest, but a numerical illustration is given below (p. 381) in connection with Kr.

The results obtained in this investigation have given the oun with much greater exactness than any value obtained in [III.], even than that of Ag. The value of $q = \delta/w^2$ has been determined [III., p. 404] as near 361·75 with Ag = 107·88. I now believe from later work that the true value is closer to this than I thought at that time, but in any case it is far less accurate than the ouns themselves. While, therefore, we can use the ouns to give extremely accurate values of the ratios of the atomic weight of the gases, the actual values in terms of Ag are not so exact, although more accurate than those obtained by chemical means. This statement of course depends on the supposition of the exact proportionality of oun and square of atomic weight.

The values of δ as obtained later are here collected and the atomic weight deduced from them by taking $q = 361·75$.

	Ne.	A.	Kr.	X.	RaEm.
δ . . .	14·4708 ± ·0006;	57·9209 ± ·002;	249·536 ± ·004;	611·0100 ± ·0017;	1787·024 ± ·05
W . . .	20·0005 ± ·0004;	40·0141 ± ·0006;	83·0543 ± 0006;	129·963 ± ·00018;	222·259 ± ·003
Chemical .	20·2	39·88	82·92	130·2	222 to 222·4

It will be seen that in all cases the spectral determinations are much closer to integral values than the chemical, except in the case of RaEm as estimated from HÖNIGSCHMIDT's value for Ra. In this case, however, the spectral material is defective. It is shown from one of the criteria that a value of the oun = 1785·23 is just possible but improbable, or = 1783·38 almost impossible. These would give respectively w = 222·148 ± and 222·033 ±. It is curious also that from the defective observational work for Ra [III., p. 327] the value of δ from $\nu_1+\nu_2 = 254096 = 137\delta$, whence w = 226·43 is also greater than HÖNIGSCHMIDT's and more in accordance with the value obtained by Mme. CURIE. The value for the Emanation is, however, much more reliable than the above for Ra. If, regarded as a whole, the deviations from the chemical values (RaEm excepted) are greater than chemists will allow possible, it would seem that in this case we are not dealing with precisely the same entity in the two cases.

Special F *Series.*—There appears to be a remarkably stable triplet series of the F type apparent in most of the gases, but more especially evident in X, in which element it was first noticed. Not only are the lines strong and present in a large number of orders, but they appear, at least in X, to be little susceptible to displacements such as are common in other types. The separations are 1864, 829. The occurrency curve for 1864 is shown in Plate 2, fig. 3. In this, in strong contrast to other such curves, it rises to a very high single peak and is practically symmetrical on both sides of the peak. The similar curve for A is shown in Plate 2, fig. 5.

Summation Series.—In the investigation of this XF series a quite new type of series was brought to light. The hitherto recognised series appear as the differences of two terms A−B. The new one has its wave-numbers of the form A+B. In other words, where the old series are difference frequencies the new ones are the corresponding summation frequencies. The notation adopted is to write the corresponding terms in Clarendon type. Thus

$$\mathrm{F}(m) = \mathrm{A} - f(m), \qquad \mathbf{F}(m) = \mathrm{A} + f(m).$$

The list of the lines in X is given on p. 385 up to $m = 30$. For low orders, $m < 3$, the lines are in the ultra-violet and have to be sounded for. Similar summation series coupled with other F series are also common. It probably explains also the crowding of F separations in spectra like that of Cu in short wave regions far beyond the F limit which has always appealed to me as a difficulty. It is possible that summation series may also exist for the P.S.D. series in all elements, but, as a rule, the limits of these are far larger than the $\mathrm{F}(\infty)$, with the consequence that any **P.S.D.** lines must lie very far in the ultra-violet, a fact which explains why such types if existing have not hitherto been recognised. The existence of these summation series is thoroughly established and their importance as bearing on theories of the origin of spectral lines is evident. They would seem difficult to explain on any of the current theories. But apart from this the existence of the type is of great value for quantitative determinations. This is fully dealt with on p. 384 and it need not be recapitulated here. Its importance for this purpose may be realised when it is seen that it forms the starting point in the analysis of the RaEm spectrum, that it settles in a quite definite way a difficulty arising in the evaluation of the oun in Kr, and that it fixes a very accurate value for the limit of the 1864 series in X, thus simultaneously fixing a particular d sequent subject only to observation error in one line.

Groups of D *and* S *Series.*—Not only do we meet with different groups of D series depending on different multiples of Δ_2,* but in the case of Kr there appear to be two

* As an example, see p. 403, in X with groups depending on $70\Delta_2$ and $79\Delta_2$.

sets of lines suitable for S_3—in other words, there are quadruplets. Whilst the two sets S'_3, S_3 give different separations with S_2, and consequently different Δ'_2, Δ_2, they give the same oun, and in connection with them appear two D groups whose outside satellites depend one on a multiple of Δ'_2 and the other on a multiple of Δ_2. It is to be suspected that this is only one example of what may be a common occurrence in spark spectra.

The order of presentation is generally that in which the investigation was taken. The key was found in obtaining the KrS system, a result first rendered possible by the publication of wave-lengths in the ultra-violet by LEWIS (1915). Amongst them the KrS(1) triplet was found. XS, AS come next in order of definiteness. The spectra of RaEm and Ne are more difficult to deal with, the first because of its defectiveness in range, number of lines, and accuracy, and the latter because of the smallness of its oun and its triplet separations. After the S series of Kr, X, A come the D and F series of Kr, X, the spectrum of RaEm, the D and F for A, and, lastly, the whole sets for Ne. Led by possibly a false analogy to He [I., p. 105], in which doublet series appear in the blue spectrum, the blue spectra were chosen for investigation, and the family group being of even order triplets were looked for. In Ne, with a single spectrum of composite character, the results obtained may have some reference to the red type as well as the blue, especially in connection with certain remarkable constant separations found by WATSON and analogous to the RYDBERG constant separations in the red spectrum of A. One is inclined to think that these red spectra consist mainly of lines of the F type. But the red or first spectra are outside the scope of the present communication. Although it is a very lengthy one as it stands only the beginning of an analysis has been made. The aim has been to lay the foundation for the series framework of this family of elements, and little beyond has been done. The linkages, as a whole, have not been isolated, the red spectra not touched upon, and many interesting effects which will require clearing up are passed over without reference. A great field for investigation is open in these and other spectra for any who are willing to give the necessary time and patience. In some few cases the presentation might have been slightly shortened by merely stating the final result and showing how the necessary conditions are satisfied. But not only would this have passed over certain phenomena of special interest, but one of the objects of the present communication would have been missed, viz., to illustrate the power of the new facts to guide a search even when the details are most bewildering. Moreover, the evidence itself is the more striking when developed from step to step than when the result is directly presented as a finished product.

Krypton.—Krypton shows two spectra, without and with capacity, the former in the red region and the latter further towards the blue. We have measures of some of the stronger lines by RUNGE, and a considerable number of weaker lines, not observed by others, by LIVEING and DEWAR, although the latter are only given to the nearest Ångstrom. The most complete and reliable sets of measurements are by BALY* (red spectrum 6456—3502; blue 5871—2418), and LEWIS† (blue 2416—2145), both of about the same degree of accuracy with probable error in the neighbourhood of ·03 A. Of exact measures there are only two by FABRY and BUISSON‡ for lines at 5870.9172, 5570.2908 I.A. In the red spectrum RUNGE has pointed out constant separations of 945, to which PAULSON§ has added three others. The observations of LEWIS gave me the first clue to the KrS set of lines and thus formed the starting point for the present communication, although a great deal of preliminary discussion of material for this group of elements had been previously done, especially in connection with the separations for certain linkages. In the case of Kr a very large number of separations in the neighbourhood of 786 to 788 had been found, connected also with others of 309, indicating groups of triplets having these values for ν_1, ν_2. Amongst LEWIS' lines a set was found with separations in the reverse order, clearly pointing to a first set of $-S(1)$ or $+P(1)$ lines and corresponding sets for other orders were then easily found. It would seem that there are always a considerable number of separations governed by oun displacements in the limits of the first order, and that of these, three seem to be of a more stable value and correspond to normal triplets. For instance, in all these gases we find a very large number of cases where a S_1 or D_1 line is followed by a line with a separation very close to $\frac{1}{2}\nu_1$. They force themselves on attention on account of their value being so close to the half of a number being sought for, and others may be present although they have not been looked for. In the present case two alternative sets of lines for the S_3 series, one with $\nu_2 = 309$ and the other with $\nu_2 = 341$ appear. In the original search the former was taken because it is reproduced in the D series as well. But later certain difficulties in the determination of the oun, combined with the fact that the corresponding oun multiple in Δ_2, although quite definite, is out of step with the march of their values in the other gases, led me to include the second. This gives a multiple quite in step with the others, and also affords the means of obtaining a good approximation to the oun.

The lines are given in the following table, which also embrace a few obtained by sounding, both wave-lengths and wave-numbers are given :—

* 'Trans. Roy. Soc.,' A, vol. 202, p. 183 (1903).

† 'Astro. Journ.,' 43, p. 67 (1915).

‡ 'C.R.,' March 25 (1913).

§ 'Kong. Fys. Sälls. Hand.,' N.F. 25, Nr. 12.

KrS.

	S_1.		S_2.		S'_3.		S_3.
1.	–(10) 2353·95 42468·98	786·52	–(10) 2398·38 41682·46	309·20	–(3) 2416·31 41373·26	340·51	(–1) 2418·13‡ 41341·95
2.	(10) 3778·23 26460·07	786·45	(9) 3669·16 27246·62			341·49	(4) 3623·74 27588·11
3.	(3) 2489·51 40156·61	769·80	(1) 2442·68* 40926·41		‡	‡	
4.	(1) 2216·72 45097·86	774·3	(1) 2179·3† 45872·2 ± 2	311·4	(2) 2164·6 46183·6 ± 2	786·45 + 339·86	(2) 2162·7 46224·17 ± 2
5.	(2) 2259·83.*e* (47421·14)					786·45 + 343·5	(6) 2299·02.*e.u.* (48551·12)
6.	(4) 2362·18.2*e*? (48688·12) or (48698·34) §	788·06	(1) 2159·5.*e* (49476·18)	301·54	(6) 2302·88.2*e* (49777·72)	786·45 + 340·4	(8) 2300·35.2*e* (49825·2)

The first three S_1 lines give for the formula

$$n = 51651·29 - N/\{m + ·093630 - ·014156/m\}^2.$$

The calculated wave-numbers for $m = 4 \ldots 7$ are in order 45095·25, 47419·39, 48695·38, 49470·47. The first gives O–C, $d\lambda = ·12$; the others are outside the observed region but are reached by sounding and give O–C values of –·08, ·12, ·22, the errors including those of the sounders.

Quantities relating to the separation 309 will be denoted by dashed letters. With the limit $51651·29 + \xi$ and separations $786·45 + d\nu_1$, $309·20 + d\nu'_2$, $341·16 + d\nu_2$, || the values of Δ_1, Δ_2 are found to be

$$\begin{aligned} \Delta_1 &= 10969 - ·316\xi + 13·79\, d\nu_1 = 44\,(249·30 + ·314\, d\nu_1 - ·007\xi), \\ \Delta'_2 &= 4244 - ·121\xi + 13·67\, d\nu'_2 = 17\,(249·63 + ·80\, d\nu'_2 - ·007\xi), \\ \Delta_2 &= 4680 - ·121\xi + 13·65\, d\nu_2 = 18\tfrac{3}{4}\,(249·64 + ·73\, d\nu_2 - ·007\xi). \end{aligned}$$

The value of the oun is thus given by $\delta = 249·30 + ·314\, d\nu_2$, $249·63 + ·8\, d'\nu_2$, with the uncertainty usually found from triplet separations [see III., p. 332]. From the limit $= p(1)$ and Δ_1 the values of the a, b, c, d, e links are at once found. The result for $e = p_2(1)(-2\Delta_1) - p_2(1)(2\Delta_2)$ is 3183·35. Since $S_1(1) = p(1) - s(1)$,

* ? $S_2(3)(-9\delta)$, see displaced sets below.

† Is $F_3(2)$, see p. 376.

‡ In region at very end of BALY'S list; 41341 is his last.

§ (6) 2291·26.*e.u.*

|| Obtained as least square value from $m = 1, 2, 3$, supposing λ equally probable.

$s(1) = 94120{\cdot}27 + \xi = \mathrm{N}/\{1{\cdot}079474 - 5{\cdot}73\xi\}^2$. From this the u, v links, viz., $u = s(1) - s(1)(\Delta_1)$, $v = s(1)(-\Delta_1) - s(1)$ can be calculated. The results are

$$u = 1884{\cdot}03 + 2{\cdot}33\, d\nu_1 - {\cdot}02\xi,$$
$$v = 1942{\cdot}44 + 2{\cdot}48\, d\nu_1 - {\cdot}03\xi,$$

with

$$e = 3183{\cdot}34 + 4{\cdot}096\, d\nu_1 - {\cdot}006\xi.$$

The occurrency curves for these are shown in Plate 2 (fig. 1). It will be seen at once that the maximum occurrences appear with sharp peaks in close agreement with the calculated values.

The calculated lines for $m > 4$ are outside the observation region, but their existence can be indicated by using the above links as sounders. At the same time, in order to give more confidence in the application of the method, the similar linkages are examined for the lines which are observed. It is, of course, understood that the existence of a single link is no conclusive evidence of the existence of the unseen line, as it may be a coincidence. The evidence is cumulative and is seen at a glance in the diagram in Plate 3 (fig. 1). The actual data are given here. In the application of this the frequent link modification must be allowed for, as well as the effect of observation errors. Consequently separations deviating by not more than two or three units from the values determined above are admitted. As a fact, this will exclude a number of real link connections as well as include a certain number of pseudo ones. But by limiting the deviations to these small amounts the conclusions drawn will be more reliable.

Sounding Data.

S (1).

(2) 37340		(1) 39799	1882·57	
1944·49				
(1) 39284	3184·49 S_1	(1n) 38497	3185·23 S_2	S_3
1882·11				
(3) 41166		(1) 39741	1940·62	

Here the u, v links form a series inequality in S_1 with 39284 and a corresponding parallel inequality with S_2. The $-e$ link with S_2 is probably spurious, since 38497 will later be shown to be $D_{15}(2)$ (p. 371), and the link as shown is excessive. It is omitted in the diagram. No links are seen with S_3.

S (2).

(1) 21332				1884·03	(1) 29130		(5) 31726
1994·17							1886·73
(2) 23276	3183·57 S_1	(4) 24062	3183·70 S_2	3183·38	(1) 30430	3183·11	(3) 33613
		1940·68					1940·93
(5) 24578	1881·95	(2n) 26003					(2) 35554

Here is present a chain of e links in S_2 so frequent in Ag and Au [IV.]. Also, as in the case of $m = 1$, are found the same modification effect of about 2 in the u, v links. It may be noted that in addition to the above there are lines (1*n*) 26014·23 and (3*n*) 25295·58 respectively 1232·39 below S_2 and 1232·66 above 24062·92. Now 1231·14 + ·30 $d\Delta_2$ is the calculated value of a link e' constituted in the same way as e, but with Δ_2 in place of Δ_1.

S (3).

			(2) 39063	**1879·80**			
(2*n*) 33791	2 × **3182·59** S_1				[S_2]	**3180**	(1) 44123
			(3) 39000	**1942·56**			

The line 40926 entered under S_2 is 769·80 ahead of S_1. The *a* link is 768·98, so that it may be $S_1(3)+a$. It is also numerically exactly the displaced line $S_2(3)(-9\delta)$, also $(\delta)S_2(3)$, but see below. Note that there is a rather abnormal $+e$ link with the normal value of S_2. The calculated value of S_3 is 41252·26 and is close on the limit of the region observed by BALY. Again with this, e' links are in evidence, viz., (2) 42483·95 and (1) 40023·21 respectively 1231·69 ahead and 1229·05 behind [S_3], whilst the former has $\pm v$ links from it, viz., 1942·92 and − 1942·97. They are however not entered on the diagram, since only *e*, *u*, *v* links are there shown.

In the foregoing three orders the links are found in evidence. In the succeeding orders they are used as sounders.

S (4).

(1) 41919	**3178·11** S_1	(1) 42686	**3185·84** $(3\delta_1)S_2$	(3) 43000	**3183·04** $(2\delta_1)S_3$
1943·10					
(1) 39976		(1) 43987	**1884·63**		

The links here are unsatisfactory as well as the lines given for $S_{2,3}$. Numerically the second and third lines are respectively $(3\delta_1)S_2$ and $(2\delta_1)S_3$. They may really be displacements on the sequence terms, but if so the order is too high to make any certain decision. Taking S_1 as correct, the normal value for S_2 and S_3 would be 45884·31 and 46193·51. The former has links **1881·66** back to (1) 44002·65 with a further *v*, **1942·72**, back to (3) 42059·93. No direct link is found for normal S_3. Its $-e$ link should produce a line at 43010·16. This is very close to the mean of the 43000 shown in the table and a line (4) 43019·17, *i.e.*, 43009·87. In other words, these two lines are $(e)(2\delta_1)S_3$ and $(e)(-2\delta_1)S_3$, indicating that the normal S_3 line has been split into two by displacements $\pm 2\delta_1$ on the limit (or $\mp$ equal displacements in sequent). Both linked lines are seen, but only one of the lines themselves, which is possibly due to the fact that they are close on the limits of the observed region. We shall find indisputable evidence of such displaced parallel series as a general phenomenon.

S (5).

(1) 42293		
1884·65		
(2) 44237	**3181·73**	[S_1]
1944·48		
(3) 42352		

The separation $-e$ is given on the value calculated from the formula. In $m = 4$ the observed is 2·6 larger than the calculated, pointing to too small a value for the limit, which is quite possible as the limit determined from lines involving $S(1) = -P(1)$ is never found correctly. An increase of this order makes the $-e$ link in $S_1(5)$ normal. It gives the line in the table with $d\lambda = -{\cdot}08$.

It is most important to get evidence which can carry conviction as to the reality of displaced lines, especially where the displacements occur simultaneously on both limit and sequent. A numerical coincidence in this case can have little weight by itself. In fact, it can have weight only in each particular case provided it is known from other considerations that such displacements are a common and universal rule. For this purpose it is instructive to adduce here some striking evidence afforded by certain sets of lines connected with the S series. The lines in question are arranged in the two following schemes, expressed in wave-numbers :—

$-(1)$ 42844	$-(3)$ 42059	$-(8)$ 41496
123·00	**124·53**	**122·92**
$-(2)$ 42721 **252·37** $S_1(1)$	$-(1)$ 41935 **252·94** $S_2(1)$	$-(1)$ 41625 **252·30** $S'_3(1)$
300·11		
$-(1)$ 42421		

$(1n)$ 26407 **35·23**	(1) 26442	(2) 27175	
17·31	**17·29**	**22·53**	
(10) 26424 **35·31**	$S_1(2)$	(1) 27207 **38·70** $S_2(2)$	(1) 27520 **35·17** $[S'_3(2)]$
299·56	**299·72**	**298·15**	
(2) 26724 **35·47**	(5) 26759	$(2n)$ 27506	

These lines are numerically displaced with reference to the S by amounts represented by the following parallel schemes :—

$S_1(1)(-9\delta)$		$S_2(1)(-9\delta)$			$S'_3(1)(-9\delta)$
$S_1(1)(-6\delta)$,	$S_1(1)$,	$S_2(1)(-6\delta)$,	$S_2(1)$,	$S'_3(1)(-6\delta)$,	$S'_3(1)$
$(-17\frac{1}{4}\delta)S_1(1)(-6\delta)$					

$S_1(2)(-9\delta)$,	$S_1(2)(-3\delta)$,	$S_2(2)(-9\delta)$			
$S_1(2)(-6\delta)$,	$S_1(2)$,	$S_2(2)(-6\delta)$,	$S_2(2)$,	$S'_3(2)(-6\delta)$,	$[S'_3(2)]$
$(-17\frac{1}{4}\delta)S_1(2)(-6\delta)$,	$(-17\frac{1}{4}\delta)S_1(2)$,	$(-17\frac{1}{4}\delta)S_2(2)(-6\delta)$.			

In addition, for $m = 3$, we have seen that in place of normal $S_2(3)$ the displaced $S_2(3)(-9\delta)$ is seen. The parallelism in spite of lacunæ show that the set are definitely related, and the fact that the same displacement on the sequents for $m = 1, 2, 3$ are required to represent the observed separations is specially striking, it being remembered that a displacement on the limit gives constant separation for different orders, whilst one on a sequent gives different for different orders. Here, for instance, 252 in $m = 1$, 35 in $m = 2$, and 16 in $m = 3$, all depend on the same oun multiple, 9δ, displacement in the sequences. Also 123 in $m = 1$ and 17 in $m = 2$ on the same 6δ, whilst the constant displacement 300 is explained by $-17\frac{1}{4}\delta = -69\delta_1$ on the limit.

Xenon.—Xenon also shows two spectra, without capacity in the red region and with it, extending far into the ultra-violet. Practically the only material at disposal is contained in the extensive lists of BALY* (red spectrum 6198—2536, blue 6097—2414) with an accuracy of about ·03 A. This is supplemented by observations by LIVEING and DEWAR,† especially by longer wave-lengths up to 6596, but unfortunately only measured to the nearest unit.

BALY draws attention to the large number of lines apparently common to both Kr and X. The number of lines in the whole spectrum is very large. BALY gives 1376 in the blue spectrum, but perhaps the most noticeable point for our present purpose is the great variability with change in the conditions of excitation. This is very clearly indicated by a comparison of intensities of corresponding lines as observed by L., D. and B. The following are a few examples out of a large number illustrating this. The numbers following a wave-length give the intensities as estimated respectively by BALY and by LIVEING and DEWAR :—

5191	5, 6	4890	5, 3
5188	4, 3	4887	5, 0
5080	7, 2	4884	1, 4
5068	not seen, 5	4883	6, 0
5045	3, 6	4844	10, 10

As between the two spectra also, a fact noticed by L. and D. is of importance. They say "there is one very remarkable change in the xenon spectrum produced by the introduction of a jar into the circuit. Without the jar the xenon gives two bright green rays at about $\lambda 4917$ and $\lambda 4924$, but on putting a jar into the circuit they are replaced by a single, still stronger, line at about 4922. In no other case have we noticed a change so striking." They also state that changes occur with the same kind of discharge as between different tubes. These are clear cases of oun displacements. PAULSON again (*loc. cit.*) gives some constant separations in the first spectrum. The triplet separations observed are about $\nu_1 = 1778$, $\nu_2 = 814$, in due order of magnitude with those for Kr. No line suitable for S (1) comes within the observed region, but there are two lines with W.N. 40375·40, 39561·50 separated by 813·99, which would serve for $S_2(1)$ and $S_3(1)$ and are in a similar position to the KrS lines. They clearly suggest that the $S_1(1)$ line is at $-42153{\cdot}39$, using $\nu_1 = 1777{\cdot}90$ as found from the S_2 set. This is further substantiated by employing the value of the e link, found below to be 7314, as a sounder. It requires a line at about 34839, and such a line is found at (<1) 34836·78 (but see discussion under

* 'Phil. Trans.,' A, vol. 202, p. 183 (1903).

† 'Roy. Soc. Proc.,' vol. 68, p. 389 (1901); 'Coll. Papers,' p. 494.

$S_3(3)$). The observed S sets, as well as others found by sounding, are given in the following table :—

XS.

m.	S_1.		S_2.		S_3.
1.	–[2371·57] [42153·39]	**1777·90**	–(10) 2476·02 40375·49	**813·99**	–(4) 2526·97 39561·50
2.	(4) 3854·44 25936·90	**1777·90**	(5) 3607·17 27714·80	**815·11**	(1) 3503·99 28530·91
3.	(4) 2527·10 39559·46	**1776·69**	(1) 2418·47 41336·15	**814·63**	(<1) 2869·71.*e* (42150·88)
4.	(1*n*) 2689·82.*e* (44480·53)	**1777·44**	(4) 2871·85.*e.u* (46257·97)	**817·68**	(1) 2829·35.*e.v* (47075·65)
5.	(1) 2828·01.*e.u* (46797·58)	**1779·33**	(1) 2944·78.2*e* (48576·91)		
6.	(2) 2452·76.*e* (48072·37)	**1778·33**	(1*n*) 2623·31.*e.v* (49850·70)	**815·34**	(1) 2549·05.*e.u* (50666·04)
7.	(<1) 2921·74.*e.e* $(-\delta_1)$ (48866·68)	**1777·80**	(4) 2777·10.*e.e* (δ_1) (50624·48)	**815·22**	(1) 2715·91.*e.e* $(-\delta_1)$ (51439·70)
8.	(2) 2658·37.*e* $(-\delta_1)$.*v* (49350·39)	**1778·69**	(1) 2538·16.*e.v* (51129·08)	**816·00**	(2*n*) 2468·54.*e.u* (51945·08)
9.	(1) 2850·41.2*e* (49700·78)				
10.	(1) 2616·79.*e* $(-\delta_1)$.*v* $(-\delta_1)$ (49949·59)	**1777·55**	(1) 2701·99.2*e* (51727·14)	**815·61**	(1) 2432·87.*e* $(-\delta_1)$.*u* $(-\delta_1)$ (52542·75)
11.	(1) 2584·04.*e.u* (50134·99)	**1781·25**	(2) 2470·30.*e.u* (51916·24)	**813·57**	(1) 2943·07.2*e.u* (52729·81)
12.	(2) 3202·17.2*e.v* (50276·15)	**1777·00**	(1) 2479·98.*e.v* (52053·14)	**817·59**	(3) 2614·13.2*e* (52870·73)
13.			(2) 2663·43.2*e* (52162·72)	**815·00**	(<1) 2921·74.2*e.u* (52977·72)

The first three lines gives the formula

$$n = 51025·29 - \mathrm{N} \Big/ \left\{ m + ·096726 - \frac{·011826}{m} \right\}^2.$$

From the limit 51025, and using the observed separations given by the S (2) lines, viz., 1777·90, 815·05, the values of Δ_1, Δ_2 are found to be

$$\Delta_1 = 24893 + 13·64\, d\nu_1 - ·72\xi = 40\tfrac{3}{4}\{610·87 + ·334\, d\nu_1 - ·018\xi\},$$

$$\Delta_2 = 10996 + 13·33\, d\nu_2 - ·31\xi = 18\ \{610·89 + ·74\, d\nu_2 - ·017\xi\}.$$

To a first approximation therefore the oun is given by $\delta = 610·88$. The calculated value of the *e* link from the value of Δ_1 is $e = 7314·09 - ·0056\xi + 4·23\, d\nu_1$. The result of examining the lines of the blue spectrum for separations of this magnitude is shown in Plate 2, fig. 2. It is distinguished from those of elements hitherto discussed by showing one definite maximum alone at about 7315·3, although there are indications of the appearance of another peak beyond 7317. The displaced $e(\delta_1)$ link shows a difference of 2·32, so that a second peak might be expected at 7317·62. If the actual value is at 7315·3 it would require $4·23\,d\nu_1 = 1·2$, or $d\nu_1$ about ·30, $d\lambda = ·04$ distributed between the two $S_{1,2}(2)$ lines. This is possible with $\pm·02$ on each line, but probably excessive for an error on one of them. Both values are tested as links below for the observed lines, and the results show that with the exception of $S_1(1)$ the value of e, calculated from the original ν_1, is extraordinarily exact. For this reason, and because the exact position of the peak of the frequency curve depends on several disturbing conditions, the original value $e = 7314·1$ will be used for sounding purposes on lines outside observed regions. Again the first line -42153 and the limit 51025 gives 93178 as the value of $P(\infty)$ or $s(1)$. From this the values of the u, v links are found. The complete set are

Links.	Changes per δ_1 displacement.
$u = 4133·18 - ·049\xi + 2·19\,d\nu_1$	1·77,
$v = 4428·00 - ·061\xi + 2·51\,d\nu_1$	1·84,
$e = 7314·1 - ·0056 + 4·23\,d\nu_1$	2·32.

The results obtained by sounding are shown at a glance in diagram Plate 3, which embraces orders up to $m = 13$. The cumulative weight of the evidence is overwhelming in support of the general application of this method. The existence of a series parallel to the normal S at a distance $-e$ is proved, whilst the presence of other linked lines is rendered extremely probable by succession of similar linking in the same set, and in neighbouring orders. Compare, for instance, the triplets in $m = 1, 3, 4, 7, 8$ and the sets for $m = 8, 9, 10$.

Detailed Discussion.—In the following discussion the starting point for the consideration of each triplet set is—after $m = 3$—the value of $S_1(m)$ calculated from the series formula obtained above. The sounders are indicated to the left of each observed line and the values O−C in $d\lambda$ are given on the right, the observed or O line being regarded as the observed sounded line + the link as given above. The value entered in the table of S lines above is, however, not the line as calculated from the formula, but the most probable value as deduced from sounding. They are indicated below by asterisks. For the first three orders the values of $d\lambda$ obtained by using $e = 7315·3$ are placed to the right of those depending on $e = 7314·1$.

S (1).

[−42153·39]			−40375·49			−39561·50		
$-e(<1)$ 34836	·15,	·07	$-e(5)$ 33061	·01,	−·06	$-e(3)$ 32248	−·07,	−·14
$-2e(4)$ 27523	·13,	−·02	$-2e(1)$ 25747	·00,	−·15	$-e-u(1)$ 28113	·04,	−·07
$-e-u(3)$ 30700	·37,	·11	$-e-u(1)$ 28928	·00,	−·11	$-e+u(3)$ 36379	·11,	·05
						$-e-v(1)$ 27819	·00,	−·12

In the case of S_1 it appears as if the $e = 7315$ is much superior. But, as it happens, the value of $S_3(3)$ as calculated from $S_2(3)$ by ν_2 is 42151·20, and so is very close to $S_1(1)$. It will be shown that for this the 7314 link gives very close values, and the linkage probably belongs to $S_3(3)$. The $-e-u$ is doubtful.

S (2).

25936			27714			28530		
$-e(3)$	18622	·00, −·19	$-v(2)$	23282	·51	$v(2)$	32958	·00, ·17
$-e+u(10)$	22755	·08, ·16	$-v(-2\delta_1)$		·00			
$-2e$	[15441·84]							

In S_1 the intensity of 22755 would suggest that its linkage is a coincidence. L.D. give a line 15447 which may possibly be $(-2e)\,S_1$, for their measures are only to the nearest A.U.

S (3).

39559	41336				[42151]			
See $S_3(1)$	$-e(<1)$	34022	−·03,	−·10	$-e(<1)$	34836†	·02,	−·05
	$-e+u(<1)$	38155	−·03,	−·06	$-2e(4)$	27523	·00,	−·17
	$-e-u(1)$	29892	−·20,	−·31	$-e-u(3)$	30700	·22,	·14
	$-e+v(<1)$	38453	−·16,	−·22	$-e-u(-2\delta_1)$	,,	·01	
	$-2e+u(1)$	30843	−·14,	−·16				

† $S_1(1)$ and $S_3(3)$ are very close. This sounder probably is correct here and does not hold for $S_1(1)$, for which it differs by about 3.

S (4).

[44481·06]				[46258·43]				[47073·48]			
$-e(1n)$	37166	·02	*	$-e(1)$	38940	·16		$-e(1n)$	39763	−·15	
								or (<1)	39754	·23	
$-e-u(3)$	33030	·17		$-2e(1)$	31628	·09		$-e-v(1)$	35333	−·08	*
$-e-u(-2\delta_1)$	,,	·00		$-e-u(4)$	34810	·02	*				
				$-e-v(<1)$	34518	−·08					

The 38940, 39763 are too far out to be dependent on e links. They could be dependent respectively on $e(-2\delta_1)$ and $e(+2\delta_1)$. Or more probably the lines 38940, 39754 may be $S_2(4)(-2\delta).e$, $S_3(4)(-2\delta).e$.

S (5).

[46799·33]				[48577·23]				[49392·28]
$-e(1n)$	39491	·28		$-e(1)$	41257	−·23		
$-e-u(1)$	35350	·08	*	$-e-v(<1)$	36832	·11		
				$-2e(1)$	33948	·01	*	

39491 is too far out for a link. It would give a reading for $S_1 = 46805·45$. It is curious, however, to notice that we have sounders for a set with this value, viz. :—

46805·45	**1779·80**	48585·01	**815·52**	49400·53
$-e(1n)$ 39491		$-e(3)$ 41271·15		$-e-u(1)$ 37593·25

S (6).

[48072·77]				[49850·67]				[56665·97]			
$-e(2)$	40758	·02	*	$-e$	[42536]			$-e$	[43352]		
				$-e-v(1n)$	38108	·00	*	$-e-u(1)$	39218	·00	*

S (7).

[48846·60]	[50624·50]	[51439·80]
$-e$[41532]	$-e$[43310]	$-e$[44125]
$-2e(<1)$ 34216 ·08 *	$-2e(4)$ 35998 $-$·08 *	$-2e(1)$ 36809 ·08 *
$-e-v(2)$ 37105 $-$·02		

There seems clear evidence of displacement here producing a separation of about 2, which is the same as that by δ_1 on e. The $-2e$ soundings give $S_3-S_1 = 1777\cdot90+815\cdot12$, which is correct, but 35998 is 2 in the opposite direction. The lines in the list are therefore deduced from this set, using as sounders $e+e(-\delta_1)$, $e+e(\delta_1)$, $e+e(-\delta_1)$. The values of $d\lambda = \cdot00$ for the set.

S (8).

[49351·72]	[51129·62]	[51944·92]
$-e$[42037]	$-e$[43815]	$-e$[44630]
$-e-v(2)$ 37606 ·14	$-e-v(1)$ 39386 ·02 *	$-e-v(1)$ 40199 ·12
$-e-u(3)$ 37901 ·11	$-e-u(3n)$ 39683 $-$·06	$-e-u(2n)$ 40497 00 *
$-2v(2n)$ 40497 $-$·11		

The linkage separation gives for the $-e-v$ $1780\cdot83+812\cdot41 = \nu_1+\nu_2+\cdot29$.

„ „ „ „ $-e-u$ $1782\cdot19+813\cdot95 = \nu_1+\nu_2+3\cdot19$.

The separations about 1780 are very common and will be discussed more completely under the D series, but here the origin must be a different one. For S (1) the sounder $-e(-\delta_1)-u$ is taken.

The $-2v$ link for S_1 is probably spurious. The line comes under S_3 as well. This is because of the numerical coincidence $\nu_1+\nu_2+2v = 11448\cdot95$ and $e+u = 11447\cdot28$. It appears in $m = 9$, 10 also.

S (9).

[49699·55]	[51477·45]	[52292·75]
$-e$[42385]	$-e$[44163]	$-e$[44978]
$-2e(1)$ 35072 $-$·05 *	$-2e(1)$ 36842 ·24	$-e-u(1)$ 40841 ·16
$-e-v(1)$ 37953 ·18		
$-2v(1)$ 40841 ·10		

Only one reliable value $-2e$ for S_1.

S (10).

[49949·21]	[51727·11]	[52542·41]
$-e$[42635]	$-e$[44313]	$-e$[45228]
$-e-v(1)$ 38203 ·16 *	$-2e(1)$ 36998 ·00 *	$-2e(2)$ 37916 ·17
$-2v(1)$ 41091 ·07		$-e-u(1)$ 41091·56 ·13 *

The $-e-v$ in S_1, $-2e$ in S_2, eu in S_3 give $1781\cdot53+811\cdot70 = \nu_1+\nu_2+\cdot28$ the modified ν_1. For S_1 and S_3 the modified $e(-\delta_1)$, $u(-\delta_1)$, $v(-\delta_1)$ are taken. The line 37916 under S_3 is also $(ev)D_{16}(8)$, considered later.

S (11).

[50134·46]	[51912·36]	[52727·66]
$-e$[42820]	$-e$[44598]	$-e$[45413]
$-e-u(1)$ 38687 $-$·02 *	$-e-u(2)$ 40468 $-$·28	$-2e-u(1)$ 33968 $-$·08
$-2e-v(<1n)$ 31076 ·05		$-2e-v(3)$ 33674 $-$·12

Again the modified $\nu_1 = 1780$ with $eu(S_1)\,euS_2$. These and $(2e,\,u)\,S_3$ give

$$1781 \cdot 25 + 813 \cdot 57 = \nu_1 + \nu_2 - 1 \cdot 37.$$

S (12).

[50275·68]	[52053·58]	[52868·88]
$-e$[42961]	$-e$[44739]	$-e$[45554]
$-2e-v(2)$ 31219 $-\cdot 02$ *	$-e-v(1)$ 40311 $\cdot 01$ *	$-2e(3)$ 38242 $-\cdot 07$ *

S (13).

[50385·80]	[52163·70]	[52979·00]
	$-2e(2)$ 37534 $\cdot 03$	$-2e-u(<1)$ 34216 $\cdot 04$

These higher orders are necessarily close to high orders of the D series, and many are therefore apparently common to both. *E.g.*, 34216 has been adduced as $(2e)\,S_1(7)$ and is also connected with a D line. Also their wave-numbers are now so high that it requires two sounders in series to just reach the limits of the observed region. The later identifications are therefore all doubtful.

Argon.—For the red or non-condensed spark spectrum about 360 lines between 8015 and 2476 together with 16 lines in the ultra-red have been observed. For the blue or condensed spark spectrum the number amounts to about 780 between 6682 and 2050 together with another 40 lines in the extreme ultra-violet between 1886 and 1333. The measures in the red are due to PASCHEN* (ultra-red), KAYSER,† RUNGE and PASCHEN,‡ and EDER and VALENTA,§ and in the blue to KAYSER,† EDER and VALENTA,§ and LYMAN‖ (extreme ultra-violet). In addition we have exact interferential measures in I.A. by MEISSNER¶ for some red lines and measures by BALY** for a few extra lines in the blue spectrum. The red spectrum is noted for the existence of the sets of constant separations discovered by RYDBERG.†† The present communication, however, deals chiefly with the blue spectrum.

The search for the S series in A is more difficult than in the cases of Kr and X. There are an extremely large number of separations of about the same value but clearly distinct. They range round 179 to 188, and, as will be seen later, displacements are very common. The clue is given from the analogous S (1) lines for Kr and X. The only strong triplet lines in the corresponding positions are those given in the following list :—

* 'Ann. d. Phys.,' vol. 27, p. 537 (1908).

† 'Berl. Ber.' (1896), p. 551; 'Astro. Journ.,' vol. 4, p. 1 (1896).

‡ 'Astro. Journ.,' vol. 8.

§ 'Denks. Wien. Akad.,' vol. 64, p. 216 (1896).

‖ 'Astro. Journ.,' vol. 33, p. 107 (1911).

¶ 'Ann. d. Phys.,' vol. 51, p. 95 (1916).

** 'Phil. Trans.,' A, vol. 202, p. 188 (1904).

†† 'Astro. Journ.,' vol. 6, p. 338 (1897).

AS.

m.					
1.	−(5) 2344·4 42642·10	**179·43**	−(3) 2354·3 42462·67	**75·60**	−(1) 2358·5 42387·07
2.	(5) 3765·463 26549·76	**181·64**	(2) 3739·88 26731·40	**74·74**	(9) 3729·450 26806·14
3.	(1) 2484·1 40244·05	**179·10**	(2) 2473·1 40423·15	**70·39** **75·59**	(1) 2468·8 40493·54 (40498·74)
4.	(2212·7) (45179·30)	**179·50**	(2204·0) (45358·80)	**75·60**	(2200·3) (45434·40)
5.			(2098·5) (47681)		
6.	[2049·5] [48776·74]		[2042·2] [48950·34]		[2038·8] [49031·94]

The observation errors for $m = 1$ and 3 are very considerable, since the measures are only given to ·1 A.U. and ·05 A. produces about $dn = {}$·8. Consequently it is possible only to obtain approximate values for ν_1, ν_2. On the other hand, for $m = 2$, where we have very accurate measures, there must be some doubt about the allocation of $S_3(2)$ because its intensity, 9, is so excessive in comparison with the 5, 2 of S_1 and S_2, and the ν_1 separation of 181·64 is greater than observation errors on the lines for $m = 1$, 3 allow. The latter objection, however, can be set aside as it corresponds to the excess ν_1 observed in Kr and X diffuse sets and, as will be found later, in NeS. In these cases $\nu_1+\nu_2$ comes out to be normal. Here, however, the sum is about 1·35 ± too large, and with the S(1) separations the typical $S_3(2)$ would be at 26804·79 or $d\lambda = {}$·18, probably of intensity 1, and so overshadowed by the strong line in the list. As will be seen immediately, the linkages will show that this value is preferable. The linkages will also show the probability of a line at 40498 for $S_3(3)$.

The three first S_1 lines give the formula

$$n = 51731{\cdot}05 - \mathrm{N} \Big/ \left\{m + {\cdot}095901 - \frac{{\cdot}017878}{m}\right\}^2.$$

For the determination of Δ_1, Δ_2, the oun and the links, reliable values of ν_1 and ν_2 are required. We have seen that the values obtained from the observed sets of lines are subject to large observational errors. Nevertheless that the true value of ν_1 is not far from that shown by S (1) is indicated by the fact that there are several accurate separations of about 179·5, *e.g.*,

(6) 20621	**179·31**	(8) 20800	**179·61**	(4) 20980
(3) 26893	**179·35**	(1) 27072		
(1) 31359	**179·62**	(4) 31538		

of which the first is part of a linkage. With the limit $51731{\cdot}05+\xi$ and $\nu_1 = 179{\cdot}50+d\nu_1$, $\nu_2 = 75{\cdot}60+d\nu_2$, the values of Δ_1, Δ_2 are

$$\Delta_1 = 2519-{\cdot}073\xi+14\,d\nu_1 = 43\tfrac{3}{4}(57{\cdot}59+{\cdot}32\,d\nu_1-{\cdot}0016\xi),$$
$$\Delta_2 = 1057-{\cdot}030\xi+14\,d\nu_2 = 18\tfrac{1}{4}(57{\cdot}91+{\cdot}77\,d\nu_2-{\cdot}0016\xi),$$

or

$$\Delta = \Delta_1+\Delta_2 = 62\times 57{\cdot}70.$$

The oun is thus given by $4\delta_1 = \delta = 57{\cdot}7$ with some uncertainty owing to inexactness in the observed triplet separations. Its value calculated direct from the atomic weight 39·9 should be $\delta = 361{\cdot}8\times({\cdot}399)^2 = 57{\cdot}6\pm{\cdot}14$, the uncertainty being due to the uncertainty $\pm{\cdot}05$ in the atomic weight. The value of $s(1) = p(1)-S(1) = 94373{\cdot}10$, from which the u, v links may be calculated. The e, u, v links are found to be

$$e = 719{\cdot}71+4\,d\nu_1,$$
$$u = 439{\cdot}47+2{\cdot}43\,d\nu_1,$$
$$v = 442{\cdot}67+2{\cdot}47\,d\nu_1.$$

The examination of the spectrum gives the occurrency curve shown in Plate 2, fig. 4. It shows a very distinct maximum in the region around 720 but little to show the exact position. The values appear somewhat irregular. If, *e.g.*, the ordinate for 719·6 be drawn, it would be the same as for those at 720·5 and 721. It may be noted that if the maximum is taken at 720·4, ν_1 is ·18 larger = 179·68 and the oun calculated from Δ_1 becomes the same as from Δ. But this is rarely the case in triplets. As, moreover, the link 719·7 when applied to the observed S (1, 2, 3) lines gives better agreement than a link 720·4 we shall use it for the purpose of sounding.

The results of sounding are exhibited in diagram form in Plate 3. The details follow :—

S (1).

42642·10		42462·67		42387·07	
$-e$[41922·39]		$-e$[41742·96]		$-e$(3) 41666	$-{\cdot}06$
$(-7\delta_1)$(2) 41474		$-e-v$(4) 41299	·02	$(-7\delta_1)$(2) 41218	
$-e-u$ 81·86	·06			$-e-v$	$-{\cdot}03$
$(6\delta_1)$(1) 41488		$-2e$(2) 41023	$-{\cdot}02$	$(6\delta_1)$(2) 41231	
$e(-6\delta_1)$(1) 43355	·00	$-2e+u$(4) 41460	·11	$-2e$(1) 40949	$-{\cdot}10$
		$-2e-u$(2) 40585	$-{\cdot}10$		
		$+e$(3) 43183	$-{\cdot}06$		

We find in the three sets clear indications of displacements producing separations of 12 or 13. As they appear in different orders such displacements can only arise in the limit. Two cases occur in m = 1, viz.,

$S_1(1) - e - u$ and $S_3(1) - e - v$. They land between the lines indicated above, where the separations are 13·76 and 13·59, or the same within error limits. Now $13\delta_1$ on $S(\infty)$ produces 13·26, and the four lines in question are $41474{\cdot}51 = (7\delta_1)S_1(1) - e - u$ or $e.u.(7\delta_1)S_1(1)$, $41488{\cdot}27 = e.u.(-6\delta_1)S_1(1)$, $41218{\cdot}08 = e.v.(7\delta_1)S_3(1)$, $41231{\cdot}67 = e.v.(6\delta_1)S_3(1)$. That is, S_3 repeated from S_1.

S (2).

26549·76		26731·40		26806·14	
(-3δ)(4) 25817		$(-6\delta_1)$(1) 27165		$u - 5{\cdot}8$(1) 27239·82	
$-e$ 29·44	·09	u 71·50	−·09		
(2δ)(7) 25841		$(6\delta_1)$(2) 27177		$v - 6{\cdot}2$(1) 27242·58	
$2e - v$(4) 24665	·34	$-u(-7\delta_1)$(3) 26285	−·05	$-e - 12$(3) 26098·42	
$-e - u$(1) 25391	−·17	−(3) 26001		$-2e + u$(1) 25804	·31
		$-e$ 11·98	−·04		
		(3) 22		$-2e + u + v$(3) 26248	·11
		$-2e$(1) 25290	·14	$-2e - v$(1) 24921	
		$-2e - v$(3) 24845	·58		

Several interesting points emerge from the above.

(1) The links to 26806 as S_3 are all bad. On the other hand, if the links are regarded as correct, the last three point back respectively to 26803·97, 05·34, 03·35, or a mean 26804·22. If S_1 is correct and $\nu_1 + \nu_2 = 179{\cdot}50 + 75{\cdot}60$ as assumed S_3 should be 26804·86, the same within the various error limits. This is therefore supporting evidence in favour of the supposition made above that the very strong line (9) 26806 is not itself $S_3(2)$, but that it overshadows the true one, which ought to be a very weak one.

(2) The existence of separations in the neighbourhood of 12 is also very marked. There are lines (4) 26562·08 at 12·22 above S_1, (2) 26744·15 at 12·75 above S_2, and (3) 26098·42 at 12 above the linked line $S_3 + e$. Further, 12, 2×12, appear in connection with the different linked systems, showing the existence of sets of lines depending on displacements in $S(\infty)$.

(3) In S_2 the allotted line is 181·64 ahead of S_1 instead of the normal 179·50. The majority of the links from S_1 are to displaced lines, in which δ_1 gives a separation of about 1·03, so that no direct evidence as to normal value of S_2 is directly available, beyond the fact itself that displacement is very prevalent. The two only direct links, $2e$, $2e + v$, would refer back to lines at 26730·34, 27·77, or a mean of 26729·05, which is 179·3 ahead of S_1 and clearly points to the normal S_2. In other words, the normal S_2 is displaced by $(-2\delta_1)S(\infty)$, but the linked lines refer back to the undisplaced normal lines. With the development of our knowledge of the laws of spectral construction, such facts as these may be expected to be of the greatest importance in settling questions of internal constitution of individual vibrating systems.

S (3).

40244		40423		(40498·74)	
$e + 6$(3) 40969		$e + 7{\cdot}3$(1) 41150		e(2) 41218	·02 *
$e + v$(3) 41407	−·04	$2e + 6{\cdot}4 - v$(3) 41426		$e + v$(2) 41860	·04
$2e$(3) 41681	·07	$2e$(2) 41860	·14	$2e$(3) 41940	−·16
$2e - v$(3) 41243	·01	$-e - u$(1) 39265	−·12	(1) 41488	
				$2e - u$ 500	−·10
$-e + 12$(4) 39536	·02	$-v$(5) 39981	−·08	(1) 41512	
$-2e + u$(1) 39244	−·01	$-2e + u$(3) 39420	·16	(2) 40926	
				u 37	·02
				(1) 40949	

Here again the links to the line observed near normal S_3 are bad, whereas the links treated as good refer back to a line exactly ν_2 ahead of S_2. Further the 12 or 6 separation is again in evidence. Also the links $2e$, $-2e+u$, $-e-u$ refer back to lines differing by about 6 from S_2.

S (4).

[45179·30]		[45358]		[45434·40]	
u(1) 45618	−·02	$e-v$(1) 45635	·00	$-e-u$(1) 44996	−·09
$-v$(4) 44735	·07	$-v$(3) 44913	·12	$-e-u$(1) 44275	−·01
$-2e+v$(2) 44175	−·16				

Here the calculated lines agree remarkably with the sounded. The calculated are therefore adopted.

S (5).

[47501·68]		[47681·18]		[47756·78]	
		v(1) 48126	−·12		
		$u+6$(1) 48126	·00		

The only links apparent are for S_2, again with the 6 displacement. These lines are all close to the limit of observed region.

S (6).

[48776·74]		[48956·34]		[49031·94]	
$-2e+v$(1) 47784	·16	$-2e-12$(1) 47522	·09	$-e+u$(1) 48753	·06
				$-2e-u$(1) 47155	−·10

The lines are now so far out of the observed region that the links are too small to refer back to well observed regions even if such lines are really existent.

The foregoing discussion of the S series affords evidence of the existence of displacements of about 12 (or 6). This requires further consideration as affording material on which additional knowledge may be obtained regarding the laws which such displacements follow. The presence of the same displacements in successive terms of the series points to a modification on the limit—either a pure displacement, a linkage effect, or, as the separation is small, possibly the difference of two links. The further evidence to follow points decisively to the existence of displacement. Whether they are due to displacements by multiples of the oun on the limit is, of course, not so convincing. The numerical relations are very closely represented on this hypothesis, but in the case of argon the δ_1 is so small that it produces in S_1 separations of 1·03 only. In lines whose wave-numbers lie about 40,000 or greater, this produces changes in λ of ·06 and therefore comparable with observation errors. In the case of S (2) only—wave-numbers of order 26700—does it produce $d\lambda$ = ·15. The measurements here are by KAYSER, whose errors are probably < ·02, almost certainly < ·05, and a close agreement between calculated and observed lines will give evidence of some weight.

$m = 1$. In S (1) cases of the displacement associated with the linked lines have been given above. They also exist in connection with the S lines themselves. Near $S_2(1)$ are 42448·25 and 42473·49, both of intensity 1; the first 14·42 above and the second 10·82 below $S_2(1)$, or a difference of $25{\cdot}24 = 2 \times 12{\cdot}62$. Again (3) 42401·45 is 14·38 above $S_3(1)$. Now $3\frac{1}{2}\delta = 14\delta_1$ displacement on S (∞) produces a separation of 14·42, and of $2\frac{1}{2}\delta = 10\delta_1$ one of 10·30. If, therefore, the observation errors are very small the lines in question are respectively $(-14\delta_1)\,S_2(1)$, $(10\delta_1)\,S_2(1)$, and $(-14\delta_1)\,S_3(1)$. There is no corresponding $(-14\delta_1)\,S_1(1)$, but (1) 42623·93 is $(-18\delta_1)\,S_1(1)$ with $d\lambda = {\cdot}00$.

$m = 2$. In S (2) we find direct displacements on the S (2) lines with a whole set of linked lines. They are indicated in the following table :—

					(9) 26806·14	
(4) 26562·08		**182·07**	(3) 26744·15		**75·60**	19·75
					(1) 26833·37	
$-e$ (7) 25841	$-{\cdot}13$		$-e$ (3) 26022		$-e$ (3) 26098	
$-2e$ (5) 25121	$-{\cdot}09$		e (2) 27464	$-{\cdot}12$	$-e+24{\cdot}61$ (3) 26123	
$-3e$ (1) 24399	·30		$-u$ (2) 26305	$-{\cdot}12$		
$-u$ (3) 26123	·06		$-e-u$ (1) 25582			
$-e+v$ (3) 26285	·00					
$-3e+v$ (3) 24845	·00					

The line 26562 has been already adduced. It is $(-2\delta)\,S_1(2)$ with $d\lambda = -{\cdot}02$. The lines 26022, 26098 under $(e)\,S_2$ and $(e)\,S_3$ differ by 75·57 or a normal ν_2, and refer back to lines which would give the true triplet separations with the first line 26542.

$m = 3$. From $S_1(3)$ the two lines (1) 40258·62 and (4) 40273·21 are displaced successively by 14·57 and 14·59, the same separation as in the case of $m = 1$. They are $(-14\delta_1)\,S_1(3)$ and $(-28\delta_1)\,S_1(3)$, with O–C given by $d\lambda = {\cdot}00$ in both cases. The first has a modified e link $-722{\cdot}61$ and the second another of 718·46 to observed lines (4) 39536·01; (6) 40991·67.

The D *and* F *Series.*

It will be found that the rare gases show a large preponderance of sets of lines which have all the characteristics of belonging to satellites of D series, or to parallel multiplet F series. In other words where, as in other elements, the d sequences show two displacements from the main d_{11} sequence when triplets are in question, the rare gases show a large number of such displacements, both in the d and the f sequences, many of these being due to large multiples of Δ_2. No attempt has been made to determine the whole system of these satellite sets, a problem belonging to an intensive study of individual elements, but a sufficient number have been adduced to prove their existence and to exhibit some of their characteristics.

Krypton.—The following table contains lines allocated to the D series for krypton and discussed in the present communication :—

KrD.

$m = 1$	D_{18}	(1) 19116·44	**788·25**	(2) 19904·69	**309·71**	(2) 20213·86
			790·47	(1) 19906·91	**306·95**	
		1860·19				
	D_{17}	(1) 19928·46	**786·03**	(1) 20714·49	**311·84**	(2) 21026·33
			789·67	(3n) 20718·13	**308·20**	
		1048·17				
	D_{16}	(2) 20669·36	**788·40**	(5) 21457·76	**308·82**	(3n) 21766·58
		307·27				
	D_{15}	[20763·25]	**789·40**	(2) 21552·65	**306·33**	(2) 21858·98
		213·38				
	D_{14}	(2n) 20842·89	**788·78**	(1) 21631·67	**307·68**	(4) 21939·35
		133·74				
	D_{13}	(1) 20871·60	**788·75**	(5) 21660·35		
		105·03				
	D_{12}	(<1) 20875·78	**788·28**	(2) 21664·06		
		100·85				
	D_{11}	(6) 20976·63				
$m = 2$	D_{18}	(1) 38037·85	**787·60**	(1) 38825·45	**311·96**	(2) 39137·41 ?
	D_{17}	[38275·82]	**787·44**	(2) 39063·26		?
		[38324·67]	**787·33**	(2) 39112·00	**308·99**	(1) 39420·99
	D_{15}	(1n) 38497·23	**787·16**	(1) 39284·39	**306·87**	(2) 39591·26
	D_{13}	(1) 38531·94		?		
	D_{11}	(5) 38560·46				
$m = 3$	D_{15}	(2) 44408·53	**787·99**	(2) 45196·52	**308·07**	(45504·59)
	D_{11}	(3) 44426·67				
$m = 4$		(47077·38)	**786·19**	(47863·57)	**308·15**	(48171·72)
		(47080·55)				
$m = 5$		(48500·71)	**786·88**	(49287·59)	**305**	[49593·55]
$m = 6$		(49345·98)	**787·97**	(50133·95)	**309·89**	(50443·84)
		[49349·39]				
$m = 7$		(49896·60)				

It will be clearer to consider first the D_{11} lines by themselves, and then the satellite lines in each order. From the first three lines the calculated formula is

$$n = 51655{\cdot}56 - \mathrm{N} \Big/ \left\{m + {\cdot}897262 - \frac{{\cdot}006513}{m}\right\}^2.$$

The limit is within errors of the value found for $S(\infty)$, but in view of what happens in the cases of X, and RaEm to be considered later, it may be noted that the difference $D_\infty - S_\infty = 4{\cdot}27$ is very close to a $-\delta_1$ displacement in $S_1(\infty)$, which would produce a separation of 4·42. [*Note.*—If $D(\infty) = S(\infty)$, ξ here $= -4{\cdot}27 + \xi$

of S (∞). If D (∞) = $(-\delta_1)$ S (∞), ξ here = $\xi-{\cdot}15$ of S (∞).] The lines calculated from the formula for the succeeding lines are all outside the observed region. For $m = 4 \ldots 7$ they are respectively in wave-numbers

47079·52, 48500·56, 49349·39, 49896·60,

The detailed discussion immediately following will show the evidence for their existence by sounding. The existence of a parallel set at a distance $-(e+v)$ is brought to light which show for the above calculated lines values of O−C = ·09, ·00, ·14, −·01.

The agreement is remarkably close and would seem to show that the value of D (∞) used is very close to its true value, about one or two units in excess. The results of the discussion are exhibited as a whole in Plate 3, fig. 4.

As in the previous cases, the linkings with the observed lines m = 1, 2, 3 are given, in order to show that the method is justified where it can be tested.

1.		2.		3.	
(6) 20976		(5) 38560		(3) 44426	
e (4) 24158	·17	e (1) 41743	·04	$-v$ (10) 42483	·01
v (1) 22914	·86	$-v$ (1) 36619	·27	$-2e$ (4) 37061	·13
$v(-\delta_1)$ 22914	·00				
$-e-v$ (3) 15856					

4.		5.		6.	
[47079·52]		[48500·56]		[49349·39]	
$-e$ (7) 43894	·08	$-e$ [45317]		$-e$ [46166]	
$-e-v$ (1) 41951	·09	$-e-v$ (1) 43374	·00	$-e-v$ (2) 44220	·13
$-u$ (2) 45196	·00				

7.	
[49896·60]	
$-e$ [46713]	
$-e-v$ (1) 44771	−·01
$-2e-v$ (3) 41588	·05

In m = 1 negative links lead to lines in the red where observation is defective. LIVEING and DEWAR give a line to the nearest unit which may possibly be a $-e-v$ link. The line 22914 is too far out to be an exact v, but its difference is 4·47, which corresponds to an exact $-\delta_1$ displacement on the limit. In other words, it belongs to the limit as calculated from the S series.

In m = 4...7 the $-e-v$ sounders all show lines in evidence. The $-e$ sounder for m = 5, 6 would show lines just within the boundary of observation, and the absence of corresponding observed lines is therefore explicable. In m = 4 the line 41951 is part of an apparent triplet.

(1) 41951·59 **786·19** (1) 42737·78 **308·51** (2) 43045·93.

The diminished values of ν_1, ν_2 indicate that the second and third lines correspond to D_{22} and D_{33} lines. The value of ν_1 is however sufficiently close to make the difference due to observation error, in which case

41951 would correspond to a D_{12}, leaving the calculated 47079 as a true D_{11} ($d\lambda = 0$), whilst D_{12} is 47077·38. That 47079 exists is also shown by the u sounder. In the table this arrangement is adopted. The line entered for $m = 5$ is the calculated, as it is so close to the deduced. For $m = 6, 7$ they are the deduced from the $-e-v$ sounder. The sounders for D_2, D_3, $m = 4...7$, are indicated in Plate 3, fig. 4.

The Abnormal Satellite Separations.—The separations of the lines suggested for the satellite sets show abnormal values in that they are roughly about 2 greater than ν_1 for the S triplets. The difference is real and not due to errors of observation, and we shall find a corresponding abnormality in the other elements of the group. Taking BALY's maximum error to be $d\lambda = ·05$, the maximum error in n for the D(1) lines will range from ·21 to ·24, or, say, ·45 on a difference of two lines. All the D(1) readings for ν_1 can therefore be the same within observation errors, but cannot possibly agree with that for the S set. Those for ν_2 however, 308·82, 307·68, cannot be the same without allowing errors larger than $d\lambda = ·05$. If they are to be the same the excessive error is probably in 21766, which is nebulous and would require $d\lambda = ·08$. Further, in addition to the lines assigned here to the D series, there are a very large number of other lines showing separations of 788. The question arises. therefore, as to the origin of this abnormality, and it is important to discuss the various possible sources. The formula gives so closely the values of the lines for D_{11} from $m = 1$ to $m = 7$ that there can be little doubt as to the essential correctness of the D_{11} allocation. The limit of the series cannot then be very different from $S(\infty)$.

(1) Is 788 a real separation—*i.e.*, is it produced by a displacement on $S_1(\infty)$ by a larger oun multiple—in this case of $44\frac{1}{4}\delta$ in place of 44δ? If so the separation would be 4·52 greater, or 791 instead of 788, and such an explanation is therefore quite inadmissible.

(2) Is it a b link modified by displacement? If $D(\infty)$ be as found, *i.e.*, $(-\delta_1)S(\infty)$, ν_1 will be increased by ·09 or 786·45 to 786·54—an inappreciable change. To produce a change of 2 in the value of b or ν_1 the limit would have to be $(-5\delta)S_1(\infty)$, which gives a value 88·5 above $S(\infty)$. But, as a fact, the limit found is quite close to $S_1(\infty)$. This explanation is therefore excluded.

(3) Are these displacements on the d sequences? In the normal case the d sequences for a given satellite triplet are the same, but are displaced from one satellite set to another. Is it possible that the sequences suffer displacement in the same triplet also? Take, for example, the satellite set whose first line is 20669. The sequent is $d = 51655-20669 = 30986·20 = N/(1·881350)^2$. A displacement of $-\delta_1$ on the denominator 1·881 increases d by 2·06. If the displacement is on the first line of the triplet it must be $-\delta_1$, if on the second $+\delta_1$, and both give practically the same value 788·51 for the apparent separation in general agreement with the observed value. There is nothing, however, to show whether the displacement should be $-\delta_1$ on the d_1 or $+\delta_1$ on both d_2, d_3, as the observed 308·82 is within our assumed error limit, but it is interesting to note that if the oun for ν_2 be the same as for ν_1 the true value of ν_2 would be very close to the observed. In this connection it is to be

remembered that in the d sequences the oun is not affected by the peculiar triplet modification shown by all elements. In the 20842 set the observed ν_2 is 307·68, but the value of the observed $\nu_1+\nu_2$ is very nearly normal. This means that the third line does not suffer displacement, but only the middle one 21631.

If this explanation is correct, the modifications must diminish with increasing order. For instance, in $d(2)$, δ_1 produces a change in separation of ·56, and the new $\nu_1 = 786·45+·56 = 787·01$ as against 787·16 observed. For $m = 3$ δ_1 produces ·23, but the possible observation errors in n—maximum $dn = 1·0$—are now so great that the observed separation of 787·99 is well within the limits of 786·68. So far then as merely numerical agreement goes this explanation would seem very satisfactory, but the changes required are so small that by themselves they can give little confidence. We shall, however, see later how it explains certain effects in the $F(\infty)$ (p. 368)—which depend on the $d(1)$ sequents—and, further, how it also explains similar modifications in the other elements of this group. Meanwhile further evidence in its favour may be obtained from linkage considerations. Some examples follow.

(*Note.*—The observation errors in the separations should not exceed about ·50.)

(a) The mesh

	786·03	(1) 20714	**311·84**						
(1) 19928				(2) 21026	**787·36**	(4) 21813	**309·47**	(1) 22123
	789·67	(3n) 20718	**308·20**						

Here with 20714 ν_1 is normal, ν_2 abnormal, but $\nu_1+\nu_2 = 1097·87$. Our explanation gives $788·51+309·20 = 1097·71$. Thus on the upper set the first and second have the same d sequent, whilst the third has $(\delta_1)\,d$. In the lower set, on the other hand, 20718 has $(\delta_1)\,d$, the same as for the third.

So also in D_{18}. In the first set d has δ_1 in second and third lines, in second set it has $2\delta_1$ in second line.

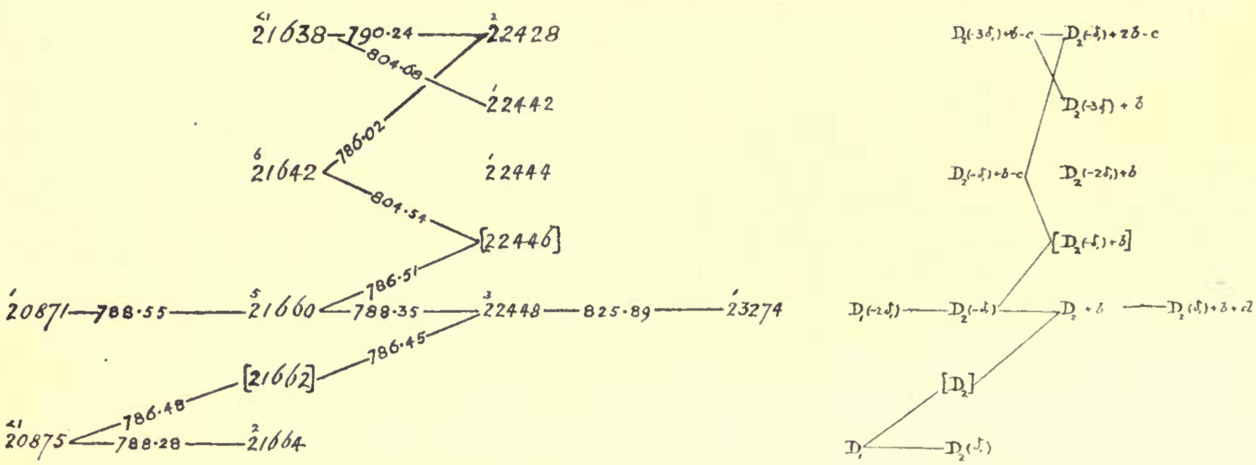
(b)

	—**786·43**	(2) 23390	**788·77**—			
				(3) 24178	**308·77**—	
(4) 22603	**789·15**	(4) 23392	**786·03**—			
	—**311·03**	(1) 22914		**2 × 786·47**		(4) 24487

This is a striking example of persistence of the displacement in linked lines. Of the lines in the second column, the first has the same sequence term as 22603, the next two have the links ν_1, ν_2, but the sequent displaced δ_1. The line 24178 keeps the same displaced sequent as these last, and therefore has a normal ν_1 to 23392, but the abnormal to 23390. So 24487 keeps this same δ_1 displaced sequent and so shows a normal ν_2 to 24178 and $2\nu_1$ to 22914. In other words, if the first line is denoted by X, the above sets may be denoted by the following scheme :—

$$
\begin{array}{llll}
 & X+\nu_1 & & \\
 & & X(\delta_1)+2\nu_1 & \\
X & X(\delta_1)+\nu_1 & & \\
 & X(\delta_1)+\nu_2 & & X(\delta_1)+2\nu_1+\nu_2
\end{array}
$$

(*c*) This example involves the *c*, *d* links whose values are respectively 804·59, 823·29. It belongs to the D(1) system. The lines enclosed in [ ] are hypothetical and introduced to indicate the transitions. The relations are indicated by the accompanying schematic arrangement, starting with 20875 = D_1.



It may be noted that in this fragment of a linkage all the links involved are *p*-links. A similar preponderance of these in linkages connected with D lines is a marked feature also in Ag and Au [IV., p. 389].

The conclusion is to be drawn that there is very considerable evidence in favour of (3) as explaining the origin of these modified separations. It does not, of course, follow that the effect indicated in (1) does not exist amongst the lines of a spectrum. We know also that the effect indicated in (2) is existent, and we shall find clear evidence of it in the existence of lines depending on limits which are displaced S(∞), *e.g.*, in the case of 19928 considered below. Further support to (3) is given by the F separations (p. 370).

The Satellites.—Any allocation of satellites which may be regarded as firmly established is a matter of some difficulty on account of the large number of ν_1, ν_2 separations which enter indirectly as links, and the prevalence of sets depending on displaced limits and displaced *d*-sequences. The lines in the list for $m = 1$ are placed there provisionally and for special discussion, although other lines certainly belong to the system. Consequently, the denotation D_{1n} is to be regarded simply as a means of referring to the different sets considered in this communication only, the *n* giving the ordinal position starting from 20976 as D_{11}.

The criteria that these lines should be possible D satellites with the same D_{11} are that the differences of their mantissæ from that of D_{11} should be multiples of the oun. Should any belong to another group with limit $(y\delta_1)$ D(∞), their mantissæ will be modified, but the qualification test will still hold with reference to the difference between their mantissæ thus modified and that of the original D_{11}, for they must all belong to *d* sequents and so be oun multiples. Nor will the test be affected by the existence of the supposed cause of the abnormal D triplet separations. The value of δ given by both the $S_3(1)$ has been seen to be 249·6 with $\Delta'_2 = 4243·2$, $\Delta_2 = 4780·0$. The maximum observation errors will be taken as ·03 A and the actual O−C error be written $-·03p$. The mantissæ for the various D_1 lines are then

1.	20976	$890749-30·82\xi+4p$,	
2.	20875	$887648-30·66\xi+4p$,	$3101 - ·16\xi+4(p_1-p_2) = 12\frac{1}{2}\delta-19-\ldots$
3.	20871	$887520-30·66\xi+4p$,	$3229 - ·16\xi+4(p_1-p_3) = 13\delta-16-\ldots$
4.	20842	$886640-30·61\xi+4p$,	$4109 - ·21\xi+4(p_1-p_4) = 16\frac{1}{2}\delta-9-\ldots$
5.	[20763]	$884207-30·50\xi+4p$,	$6542 - ·32\xi+4(p_1-p_5) = 26\frac{1}{4}\delta-10-\ldots$
6.	20669	$881350-30·36\xi+4p$,	$9399 - ·46\xi+4(p_1-p_6) = 37\frac{3}{4}\delta-23-\ldots$
7.	19928	$859253-29·30\xi+3p$,	$31496-1·52\xi+4(p_1-p_7) = 126\frac{1}{4}\delta-16-\ldots$
8.	19116	$835908-28·21\xi+3p$,	$54841-2·61\xi+4(p_1-p_8) = 219\frac{3}{4}\delta-8·6-\ldots.$

The test requires that the expressions to the right of the first term in the last column should vanish. The effect of any possible value of ξ is small. Further, as δ is probably known within $\pm·2$, it is only in the two lines, 7, 8, that it can be effective towards satisfying these conditions. It is clear then that the conditions can be satisfied within possible observation errors by Nos. 4, 5, 7, 8, and not by Nos. 2, 3, 6, nor by 8, if δ requires diminution by as much as ·1.

Considering 1, 4, 5, it may be noticed that the separations of 4, 5 from 1 are due to $16\frac{1}{2}\delta$ and $26\frac{1}{4}\delta$. Since $16\frac{1}{2}\times 8 = 132$ and $26\frac{1}{4}\times 5 = 131·25$, these separations are in the normal triplet satellite separation ratio. Moreover, the mantissa of the extreme satellite $884207 = 189\times 4678·3 = 189\Delta_2$ within our present degree of approximation to δ. This again satisfies the normal rule. It is clear, therefore, that these three lines form a normal triplet D(1) set. As to Nos. 7, 8, the ratio of separations is not that of two satellites to 20976 as D_{11}, although near it. The mantissa of (8), however, is $835908 = 197\times 4243·1 = 197 . \Delta'_2$ within our present approximation to δ, and suggests that it is the extreme satellite of another group. Returning to the other lines it remains to see if they satisfy a test with a displaced $D(\infty)$. Now $y\delta_1$ on $D(\infty)$ produces a change of $-4·42y$ and consequent changes in the mantissæ as follows: $135·51y$ in (2), (3), $134·18y$ in (6), $124·7y$ in (8). As $124·80 = 2\delta_1$ it follows that the test for (8) is unaltered and those for 2, 3, 6 become

$$-10·71y-19-\ldots = 0 ; \qquad -10·71y-16-\ldots = 0 ; \qquad -9·38y-16-\ldots = 0 ;$$

whereas in (8) the corresponding condition is not changed, but the mantissa becomes $835924+2y\delta_1 = 197(4243·1+·63y)$. In this case a displacement of one oun in the limit produces the same effect as that of two ouns in the sequent. The test for the others is very closely satisfied by $y = -2$, or the supposition that they belong to a group with a limit $(-2\delta_1)$ D(∞) *or* with a limit $(+4\delta_1)$ D(∞), decreasing by δ the mantissæ of the sequents and requiring

$$\cdot 6 - \ldots = 0\,; \qquad 3\cdot 5 - \ldots = 0\,; \qquad 8\cdot 9 - \ldots\,;$$

which again are easily satisfied within observation errors. With regard, however, to (2), (3), it is not common to find two satellites so close together, and as they both belong to doublet sets we should suspect that they belong to different groups, and in fact the condition is satisfied by regarding the limit of 20875 as displaced by $-\delta_1$ on that of 20871.

We shall consider the conditions more fully later in connection with the more accurate determination of the oun. The preceding results are sufficient to show that (1), (4), (5) are a definite normal D satellite on the basis of $\Delta_2 = 4680 \pm$. That (7) (8) belong to a series probably based on $\Delta'_2 = 4243 \pm 1$, and 2, 3, 6 to a parallel system based on the limit $(-2\delta_1)$ D(∞) or (δ) D(∞).

I omit details about the satellite e. u, v links, but the following points are interesting. The line 19116 affords an example of the two displaced D_2 lines analogous to that given in 3(a) as explaining the source of the abnormal ν_1 separations. In addition the linked line $2e + D_{18}$ shows the same effect with two lines (2) 25380·26, (1) 25382·58. D_{16}, D_{36} are curious as showing successive abnormal u links. Thus

D_{16}

(1) 18780	**1888·61**	(2) 20669	**1888·37**	(4) 22557

D_{36}

(1) 19881	**1885·31**	(3n) 21766	**1885·71**	(3) 23652·29	
					3·69
			1889·40	(3) 23655·98	
					3·31
				(1) 23659·29	

The two lines D_{12}, D_{13} which have been shown to have a relative displacement of $2\delta_1$ show u, v links to a mid-line displaced δ_1 from each. Thus (3) 22815·84 is 1942·15 or $v - \cdot 29$ below their mean; whilst (1) 19778·52 is 1883·69 or $u - \cdot 34$ below the mean of D_{22} and D_{23}.

The satellites for $m = 2$ are more doubtful on account of the large changes in the mantissæ produced by observation errors. The line 38531 has a mantissa difference from that of d_{11} of $12\frac{1}{2}\delta$, exactly the same as in D_{13}(1). It should, however, be one of a doublet with the second line stronger. The others show mantissæ differences of 28δ, 103δ, 124δ, $224\frac{3}{4}\delta$, which, allowing for the usual changes in the second order, clearly point to the sets as indicated in the table. There appears, however, no analogue in $m = 1$ for 38324. In this case it is important to notice that the ν_2

separation 308·99 corresponds to the displaced δ_1 sequent for $m = 2$. It thus supports the explanation adopted for the triplet modification.

For reference the multiples of the oun giving the satellite separations are collected in the following table, where d_1, d_2 refer to the first and second line in a satellite set.

$m = 1.$			$m = 2.$		
$n.$	$d_1.$	$d_2.$	$n.$	$d_1.$	$d_2.$
			3	12½	
2	12	12¼	5	28	28¼
3	12½	12¾	?	103	103¼
4	16½	16¾	7	124	
5	26¼	26½	8	224¾	225
6	37¾	38			
7	126¼	126¼	$m = 3.$		
8	219¾	220			
			4	19½	

KrF.—The F lines form parallel series in which the constant separations depend on the satellite separations of the D series. In other words the limits are the $d(1)$ sequents which form the satellites. Now it has been shown above that the abnormal triplet separations which the D series exhibit is probably due to the fact that the d sequent for a given satellite triplet is not the same for each of the three lines, but that they are subject to a displacement of one or more ouns. For instance where ν_1 is 788 in place of 786 (in round numbers) the difference 2 is due to the fact that d_{2n} is not equal to d_{1n} but is $d_{1n}(\delta_1)$. If the strongest line is to be taken as normal, we should expect the d_{3n}, or d_{2n} to be normal rather than d_{1n} as D_{1n} is always weaker than the other lines of a triplet satellite. In this case $F_n(\infty) = d_{1n}(\delta_1)$ and the F separation $= F_n(\infty) - F_1(\infty) = d_{1n}(\delta_1) - d_{11}$ which is less than the observed satellite separation by about 2. As a fact we do find these diminished separations. In searching for F lines therefore we have to examine the spectrum for wave-lengths longer than d_{11} and showing as multiplets with separations the same as the satellite separations or less. In the particular case of Kr these are 100, 105, 133, 213, 307, 1048, 1860,* and we are to expect series which we will denote by F_n with n from 1 to 8. From this point of view it is unfortunate that allowance has to be made for the rule as to the excessive displacements occurring in the lower orders ($m = 2, 3$) and that a complete multiplet, showing all the above separations, is not to be expected. With $D_{11}(1) = 20976·63$ and $D_1(\infty) = 51655·56 + \xi$, $d_{11} = 30678·93 + \xi = F_1(\infty)$. The mantissa of f is in general large, say, beween ·7 and ·99. Consequently the region in which $F_1(2)$ is to be found is where the wave-number is less than $30768 - N/(2·99)^2$,

* There may, of course, be others depending on D lines other than those considered in the text.

or say, <18491, or $\lambda > 5408$. This region is examined for the separations in question and lists made. It is then found that some depend on the same first line, in which case, they clearly refer to F or related lines. Several sets are found connected by one of the ordinary links, which excludes at least one of them as a direct F line. It is now easy to select a few sets from the lists which seem suitable for the F_1 line. This with the given $F_1\infty$ gives $f(2)$, and then RYDBERG's tables give a rough approximation to F(3). It is then only necessary to examine the lines near these for the separations, in order to find the actual $F_1(3)$. The result of this examination is to show that for the first three orders $m = 2, 3, 4$, the only sets which exist without displacements in any correspond to the separation 301, with F_1 lines (1) 17321·51, (1n) 23353·84, (3) 26057·20. The formula calculated from these gives as the limit 30678·64. This is only ·29 less than d_{11} and is thus in very satisfactory agreement with the rule. Using the value of d_{11} as found from the D series with $D(\infty) = 51655 \cdot 56 + \xi'$ the actual formula is given by

$$n = 30678 \cdot 93 + \xi' - \mathrm{N} \Big/ \left\{m + \cdot 877406 - 577 \cdot 8\xi' - \frac{\cdot 023916 - 363 \cdot 3\xi'}{m}\right\}^2.$$

In this if $D(\infty) = S(\infty)$, $\xi' = \xi - 4 \cdot 27$, but if $D(\infty) = (-\delta_1)\, S(\infty)$, $\xi' = \xi + \cdot 15$.

The mantissa of the first line $F_1(2)$ is

$$865448 - 107 \cdot 26\xi + 16p = 185\,(4678 \cdot 1 - \cdot 580\xi' + \cdot 034p) = 185\Delta_2{}^{*}$$

within error limits. The lines have been selected as showing the given separations. Quite independently they give a formula with the proper limit and with the first sequence mantissa a multiple of Δ_2. The evidence therefore for the correctness of the allocation is incontrovertible.

In the consideration of the notation for the various parallel F series it will be necessary to determine what is to be understood by the normal separations. If the latter are to be decided directly from the $D_{11}(1)$ lines, the separations must be those given above. But a glance at the table of F lines will show that there is a considerable variation from these, and indeed from one another—due as we have seen to the great variability in the oun displacements. Especially is this noticeable in the separations given by the $D_1(1)$ satellites as 213·38 and 307·27. In place of these we find values about 2, 4 or 6 less, corresponding to 1, 2 and 3 oun displacements. The separation 301 is the most frequent displacement from 307·27. Now δ_1 alters the separation by 2·03, and therefore $3\delta_1$ alters 307·27 to 301·18. That these deviations from normal values correspond to real F separations can be seen by their frequent repetition in connection with F lines. See for instance the maps for F, especially F(5), in Plate 4.

* $185\Delta_2 = 204\Delta'_2 + 3\delta_1$.

KrF.

[For brevity F (*m*) is printed as F in each order.]

m = 2.

(1) 17321·51		F_1	(5) 17594·17		$F_1(7\Delta'_2)$
(1) 17622·73	**301·22**	$(3\delta_1)F_6$	(1) 17695·04	**100·87**	$F_2(7\Delta'_2)$
[20504·86]	e	$F_1.e$	(5) 20991·16*	**213·64+e**	$F_5(7\Delta'_2).e$
(<1) 20813·91	**309·05+e**	$(-\delta_1)F_6.e$			
			(1) 19134·54		$(-2\delta_1)F_1(51\Delta'_2)$
(6) 17747·14		$F_1(10\Delta_2)$	(1) 19343·33	**208·89**	$F_5(51\Delta'_2)$
(2) 17952·16	**205·02**	$(\delta)F_5(10\Delta_2)$	(1) 19348·50	**213·96**	$(-2\delta_1)F_5(51\Delta'_2)$
			(5) 20991·16*	**1856·62**	$F_8(51\Delta'_2)$
(2) 17972·78		$F_1(16\Delta'_2+\Delta_2)\equiv X_1$	[22317·89]	e	
(1) 18108·55	**135·77**	$(-\delta_1)X_4$	(3) 22448·65	**130·76+e**	$F_4.e$
(1) 18177·75	**204·97**	$(3\delta_1)X_5$	(4) 22531·66	**213·77+e**	$(-2\delta_1)F_5.e$
(2n) 18282·20	**309·42**	$(-\delta_1)X_6$	(3) 24178·77	**1860·88+e**	$(-2\delta_1)F_8.e$
[21156·13]	e				
(1) 21258·41	**102·28+e**	$X_{23}.e$			
(1) 22203·47	**1047·34+e**	$X_7(-2\delta_1).e$			
(<1) 23013·63	**1857·50+e**	$(\delta_1)X_8.e$			

m = 3.

(1*n*) 23353·84		F_1	(3) 23418·38		$F_1(4\Delta'_2)$
(1) 23659·29	**305·45**	$(d_1)F_6$	(4) 23518·73*	**100·35**	$F_2(4\Delta'_2)$
(4) 25213·56	**1860·72**	F_8	(5) 25278·06	**1059·68**	$F_8(4\Delta'_2)$
(5) 21573·49	**103·68−u**	$u.F_3$			
(7) 24390·12	**213·00+d**	$F_5.d$	(2) 23507·07		$F_1(10\Delta'_2+\delta)$
(4) 23518·73*	**1048·92−u**	$u.F_7$	(1) 23639·82	**132·75**	$F_4(10\Delta'_2+\delta)$
(2) 23340·79		$(6\delta_1)F_1(-\delta)$	(1*n*) 25367·06	**1860·00**	$F_8(10\Delta'_2+\delta)$
(1) 23554·41	**213·62**	$(6\delta_1)F_5(-\delta)$			
(<1) 23349·04		$(2\delta_1)F_1$			
(3) 23655·98	**306·94**	$(2\delta_1)F_6$			

m = 4.			*m* = 5.		
(3) 26057·20		F_1	[27497·70]		F_1
(1*n*) 26157·54	**100·34**	F_2	(9) 27708·28	**210·37**	$(2\delta_1)F_5$
(3) 26189·73	**132·53**	F_4	(7) 29357·89	**1860·19**	F_8
(4) 26358·25	**301·05**	$(3\delta_1)F_6$		———	
(1) 26265·05		F_5-2x	(1) 26727·89		$a.F_1$
[26270·16]	**212·96**	F_5	(7) 26863·38	**135·49**	$a.F_4$
(4) 26275·27		F_5+2x			
(1) 26065·15	**16·05**	F_2+3x	(4) 27482·72	**−15**	F'_1
	———			———	
(2*n*) 26067·66		F_1+4x	(4) 27588·11	**105·39**	F'_3
(2*n*) 26369·36	**304·21**	$(2\delta_1)F_6+4x$	(1) 27784·33	**301·61**	$(3\delta_1)F'_6$
(1) 27924·30	**1859·15**	F_8+3x	(6) 28535·88	**1053·16**	$(-2\delta_1)F'_7$

See also Map F (5).

* At least one a coincidence.

KrF (continued).

$m = 6$.

[28356·47]		[F_1]
(1) 28664·88	308·41	F_6
(2) 26779·99	**307·71 − u**	$u.F_6$
(4) 30218·69	**1862·22**	$(-\delta_1) F_8$
(1) 28361·39	**4**	$(-2\delta_1) F_1$
(1n) 28574·61	**213·22**	$(-2\delta_1) F_5$
(10) 27525·16	**1047·96 − u**	$u.(-2\delta_1) F_7$
(1) 28340·49	**−16**	F'_1
(3) 28444·08	**103·59**	F'_3

$m = 7$.

[28907·8] = u.30791·40		[F_1]
(7n) 29005·44	**97·62**	$(\delta_1) F_2$
(2) 30791·40	u	$F_1.u$
(3) 30999·10	**207·70 + u**	$F_5.u$
(3) 31841·03	**1049·65 + u**	$F_7.u$
(6) 28891·63	**−16**	F'_1
(3) 29198·53	**306·90**	F'_6
(1) 30875·06	**99·24 + u**	$F'_2.u$
(1) 31081·96	**306·14 + u**	$F'_6.u$
(6) 31823·11	**1047·29 + u**	$F'_7.u$
(5) 32635·69	**1859·87 + u**	$F'_8.u$

$m = 8$.

[29286·33]		[F_1]
(3) 29498·44	**212·11**	F_5
(7) 28808·81	**309 − b**	$b.F_6$

$m = 9$.

[29552·20]		[F_1]
(2) 29857·70*	**305·50**	F_6

$m = 10$.

[29728·00]		[F_1]	[29712·00]	[F'_1]
(6) 29823·86	**95·86**	$(2\delta_1) F_2$	(3n) 29845·84	F'_4
(1n) 30036·53	**308·53**	F_6	(2n) 29926·23	F'_5
(2) 29857·70†	**129·70**	$(2\delta_1) F_4$	(1n) 30022·82	$(-\delta_1) F'_6$
(1n) 30777·90‡	**1049·94**	$(-\delta_1) F_7$		

* Or $(2\delta_1) F_4$ (10).

† Or F_6 (9).

‡ Is probably $(-\delta_1) F'_1 (7) u$. It is very diffuse and may be both.

It will be most convenient to deal with the multiplets order by order. We are to expect great displacements in the f sequences in the first two orders—single pairs displaced bodily—if the analogy with the triplet systems of the alkaline earths holds in the rare gases. Also throughout the orders we shall find not only in Kr but in the other gases small oun displacements in the limits, owing to the instability of the d sequences.

$m = 2$. In the first order there appear only one normal pair $F_1(2)$, $F_6(2)$; but ν_5, ν_6, ν_7, occur in a parallel set linked to the normal by the e link.

The following sets occur amongst others:—

(1)

			1049·06	(2) 18744		
(5) 17594	**100·87**	(1) 17695;	**98·37**	(1) 17692·54		
			307·84	(1n) 18002	**97·85**	(2) 18100

Here there is an example of ν_2 or of ν_6, ν_7 appearing as links, omitting 97·85 as a doubtful connection. The ν_2 is the exact D satellite separation. Moreover, the mantissa of 17594 is $895149-110·63\xi$, differing from that of $f(2) = 865448-107·26\xi$ by $29701-3·37\xi+16·5p = 7(4243·00-·48\xi+2·3p) = 7\Delta'_2$. This is a clear displacement by $7\Delta'_2$, for ξ cannot be more than a few units and p is a fraction. It is clear, therefore, that 17594 is $F_1(2)(7\Delta'_2)$ and 17695 is $F_2(2)(7\Delta'_2)$. The next two must therefore be only the linked lines $F_2(2)(7\Delta'_2)+\nu_6$, $+\nu_7$. There is a line (1) 17692·54, separated by 2·5 from 17695, which might suggest the displacement δ_1 in the limit. But 2·5 is too large, and the line itself is really a linked line to D_{12}, viz., $(e)D_{12}(1)$.

(2)

(2) 17972·78 **135·77** (1) 18108·55; **204·97** (1) 18177·75; **309·42** (2n) 18282·20

e

[21154·13] **102·28** (1) 21258·41; **1047·34** (1) 22203·47; **1857·50** (<1) 23013·63

This is a specially interesting displaced set in that it contains all the seven separations, three directly depending on the line 17972 and the remainder on a line linked to it by the e link. The mantissa of 17972 is $937966-115·6\xi$, which is $72518-8·35\xi$ above that of $f(2)$. This is very close to $16\Delta'_2+\Delta_2 = 72553$. The line may be written $F_1(2)(16\Delta'_2+\Delta_2)$. Call this X_1. Then remembering that δ_1 produces 2·03 in the limit and ·54 in the sequent, we see at once that in the others, for $135·77 = 133·74+2·03$ is $(-\delta_1)X_4$ exact; for $204·97 = \nu_5-8$ is $(4\delta_1)X_5$; for $309·42 = \nu_6+2$ is $(-\delta_1)X_6$. The others depend on $e+X_1$. The separation 102·28 is the mean of ν_2, ν_3 which depend on two d sequents differing by $2\delta_1$. It may therefore be written as either $(\delta_1)d_{13}$ or $(-\delta_1)d_{12}$, we will write it $X_{23}+e$. For $1047·34 = \nu_7-1$ the displacement is $-2\delta_1$ in the sequent, or the line is $X_7(-2\delta_1).e$. For $1857·50 = \nu_8-2·6$ and the line is $(\delta_1)X_8.e$.

(3)

$$
\begin{array}{llll}
 & & & (1)\ 19343·33 \\
(1)\ 19134·54 & \left\{\begin{array}{l} \mathbf{213·96} \\ \\ \mathbf{1856·62} \end{array}\right. & & \begin{array}{l} (1)\ 19348·50 \\ \\ (5)\ 20991·16 \end{array} \\
\quad \mathbf{822·11} & & & \\
(1)\ 19956·65 & \mathbf{308·13} & & (1)\ 20264·78
\end{array}
$$

With $30678+\xi$ the mantissa of 19134 is $216805-133·50\xi$ above that of $f(2)$. It is in the neighbourhood of $51\Delta'_2$. The whole set of lines are representable as follows (putting as before $\xi = -1·34$):—

$$
\begin{aligned}
19134 &= (-2\delta_1)F_1(2)(51\Delta'_2-2\delta_1) &&= (-2\delta_1)Y_1(-2\delta_1)\\
19348 &= (-2\delta_1)F_5(2)(51\Delta'_2) &&= (-2\delta_1)Y_5\\
19343 &= F_5(2)(51\Delta'_2) &&= Y_5\\
20991 &= F_8(2)(51\Delta'_2) &&= Y_8\\
19956 &= (-2\delta_1)F_1(2)(51\Delta'_2-2\delta_1)+d &&= (-2\delta_1)Y_1(-2\delta_1).d\\
20264 &= (-2\delta_1)F_6(2)(51\Delta'_2-2\delta_1)+d &&= (-2\delta_1)Y_6(-2\delta_1).d
\end{aligned}
$$

The numerical agreement is very close, and 19956 is a series inequality in which a change of 1 makes the two separations 822, 308 exact d and ν_6 links. On the other hand, as we have already pointed out, 20991 may be a D_{11} line depending on the limit $(7\delta_1) D (\infty)$.

The following points should be noticed in the foregoing allocations :—

(1) The presence of the large displacements in the sequence term by quantities differing from multiples of Δ_2 by one or two oums—in this respect quite analogous to a corresponding effect in the alkaline earths. It seems to point to a kind of satellite effect in the F series analogous to that shown in the D_1, where the main strong line is displaced from the normal satellite depending on a multiple of Δ_2 by the addition of a few ouns. In this case the F satellite is in general too weak to be observed, except possibly the linked line $22203 = X_7 (-2\delta_1) . e$ depending on $f(2)(17\Delta'_2)$.

(2) That where a multiplet line is absent, it frequently appears as a linked e line, but that the linked line never appears directly linked to the F_1.

(3) The line 20991 occurs twice as $F_5 (7\Delta'_2).e$ and as $F_8 (51\Delta'_2)$, also its possible existence as a kind of independent D_{11} line has already been referred to. It cannot of course be all, and at least two of the suppositions must be due to chance. It is also probable that such coincidences may occur in some of the other allocations. The evidence for the general effect is cumulative and not dependent on a single numerical agreement.

The F system of the first order ($m = 2$) have been considered in rather considerable detail in order to establish what appears to be a very general rule that in many groups of elements the configurations producing the normal F lines appear to have been subjected to a sort of explosive effect whereby other configurations producing f sequents displaced by large multiples of Δ_2 are produced. As a natural result the intensities of the normal lines in the spectrum are diminished since the observed intensities must depend on the number of emitting centres as well as the energy emitted by each. We have seen that they are displaced in pairs or sets containing one displaced F_1 line, but no attempt has been made to search for sets not containing the F_1. As we shall see later these displaced sets in the lowest order give a means of obtaining very accurate data for the determination of the value of the oun. In dealing with subsequent orders such a detailed discussion is not called for for this purpose.

$m = 3$. The normal lines are observed for F_1, F_6, F_8, and as illustrating the correctness of the explanation given above for the diminished separation 301 in place of 307 it will be noticed that the normal D value is shown by the line 23349. The other F lines of the set do not directly appear, but as in the case of $m = 2$, they are in evidence as linked lines. Some of the lines linked to F_1 (3), are represented in a map in Plate 5. The denotations of the lines are entered in place of the wave-numbers, which can be reproduced by adding the given separations, and each can be referred to by the column and order in the column in which it occurs. Again we have several examples where small changes (errors or displacements) in lines make all the allied links take their practically exact values, and so give evidence for the reality of each. Thus, $-1·2$ on a1 give c, ν_8; $-1·5$ on a2 give u, ν_7 and the mean of ν_2, ν_3. (or the line $(-\delta_1)\,F_2$ or $(\delta_1)\,F_3 = F_{23}$, say); $-·5$ ($d\lambda$ = ·08) on c4* gives c, ν_8; $-1·5$ on c5 gives d, ν_4, ν_8. These new sets give us representatives of all the lines missing from the direct normal lines. It is noticeable how the f_8 sequent persists.

Certain displaced sets are given in the tables. If 23418 is ·5 less ($d\lambda$ = ·08) the ν_2, ν_8 become exact and it is $F_1\,(3)\,(4\Delta'_2)$. The numerical proofs of these allocations are not given, as these displacements have no importance at present beyond the fact that they exist.

$m = 4$. Direct lines are found for F_n, $n = 1, 2, 4, 6$, whilst 5 appears displaced $\pm 5·01 = 2 \times 2·50$, to observed lines 26265, 26275. There is also a line 26067 ahead of 26065 by 2·51. If this 2·50 be due to some displacement it is probably $2\delta_1$ on the limit and some ouns on the sequence, or all by ouns on the sequence. The order is so high ($m = 4$) that it is not possible to decide, and it is shown in the tables as a difference x = 2·50. Linked lines are shown in the map (Plate 5). Again note that 2·5 on F_1 makes the v, u links exact, and that here again the x appears. Symmetry would seem to indicate that the true $F_1\,(4)$ or 26057 should be about x less. This would diminish the calculated limit of the series to a value nearer that given by the calculated $S\,(\infty)$.

For $m = 5 \ldots 10$ the values for F_1 calculated from the formula are 27498·81, 28357·47, 28909·97, 29286·33. 29554·21, 29730·00. With $\xi = -1·34$ as determined later, we should expect values less than these from about -1 for the first varying to -2 for the last. None of these appear but they have linked lines whilst other of the parallel F sets also appear directly.

$m = 5$. No line has been observed at 27498, but there are lines with it for F_5, F_8, and others for F_{137} by a link$-a$. A value of F_1 27497 is $F_8 - 1860·19$ and reduces it 1·1 as just suggested.† The connections are exhibited in the map (Plate 4). From this order and beyond there appears to be a parallel set at a distance 16 units less. For $m = 5$, this starts from 27482·72 as F_1. As is seen in the map (c9) it has a very large linkage to F lines with similar sets to those connected with the calculated F_1. We may explain its source as a displaced $(2\delta)\,F\,(\infty)$, as the difference of two p-links, $b-c$, or as the direct congeries of F lines depending on 20991 as an independent D_{11} line. In the map the notation depending on the second is adopted. In the list 27482 is written as F'_1 leaving the question of the origin open. The p-links are particularly prevalent. This was found to be the case also in Ag and Au, the only elements in which the linkages have been examined with any thoroughness. In particular the series of successive + and − links from 27497 recalls a similar

* This has been given as a bad $D_{11}.e$. The suggested change makes the e link worse, which increases the improbability of its belonging to the D system.

† The calculated is retained however in the map, as the links show the repetitions more clearly.

arrangement in the AgD (4) linkage shown in the c, d, e columns of the map for AgPiii [IV.]. The series is in fact continued further than is shown in the present map. Starting from 26727 we find $a-c+b-d+a-c+a-c$ (and $+d$) $= 3a+b-(3c+d)$, the actual separations being $770·92-803·607+787·51-821·81+771·07-803·66+770·68-803·16$ (and $+825·35$). Further, it should be noted that each successive pair is a parallel inequality, one in excess and the other in deficit of normal value. It means an increased displacement $2\delta_1$ in each alternate line. But if the observation errors are small, there appear to be indications of simultaneous displacements in the f sequences as well as on the limit. In fact a similar phenomenon is indicated in the two next orders though naturally some elements are wanting. A precisely similar connection is shown by AgS (3), [IV., p. 382], in a still more striking and regular series of changes. The elucidation of the laws governing displacements is of the first importance and should be one of the immediate objects of investigation. For this purpose examples of continuous series of simultaneous and like displacements will be of the utmost value. For this reason maps of certain near lines (Plates 4, 5), are given for all the orders from 3 up to $m = 8$, but no attempt has been made to indicate exact displacements involving unity. The parallel series F′ about 16 below F exist for $m = 5, 6, 7$. The sets connected with F′ (7) all show the displacement unity. In the lists the true lines are entered as 1 less for $m = 6$ and 2 less for 7, 8, 9, 10 (*i.e.*, ξ about -2) than the calculated values. As is seen it makes the observed separations more normal and in so far supports the putting of the limit about 2 less. Later the actual change in the limit is found to be $-1·34$.

KrF. During the work of examining the X spectrum a new type of series, associated with the F series, came to light. Whilst the known F type depends on the differences of two sequences $d(1)-f(m)$, the new type has a series of lines whose frequencies are given by $d(1)+f(m)$. We shall denote the lines of these series by **F**, so that F will denote a difference frequency and **F** a summation.

We have already referred to the general properties of these series in the introduction. Some of the material from the Kr spectrum bearing on the subject are here collected. In the following lists each order is considered by itself. The examination has not been exhaustive so as to involve displaced values, but it is believed all the direct observed lines have been included. A few abnormal ones, with considerable displacement in the f sequence, have also been entered, as they raised questions which require future investigation. The F and **F** lines are arranged in parallel columns. The mean of the two corresponding lines is entered in thick type between them. That for the first corresponds to the fundamental limit. The succeeding ones are given in the form mean of the first + difference, and the difference only (which settles the denomination of the set) is entered. Thus for $m = 2$ the first mean is 30674·77, that for F_6, $\mathbf{F}_6$ is $30976·55 = 30674·77+301·78$ and 301·78 is entered. Also over each line the difference from F_1 or $\mathbf{F}_1$ is entered. Notes on detail are appended below the lists. The evidence is clear as to the existence of a series of the form $A+f(m)$. If the limit were the same for both we should expect the mean to be 30678·93 + ξ instead of 30674·77. The latter is what should be expected if D (∞) = S (∞).

TABLE of F and **F** Lines.

	F.		**F**.
		$m = 2$.	
1.	17321	**30674·77**	(2) 44028·04 = ($3\delta_1$) **F** ?
	53		**148·79**
2. 3.	17376	+ **102·10**	(1) 44176·83
			133·77
4.			(2) 44161·81
	213·27		**209·68**
5.	*e*.20718	+ **211·58**	(8) 43433·13 *c*
	301·22		**302·34**
6.	17622	+ **301·78**	(3) 44330·38
	1033·85		**1052·14**
7.	18355	+ **1043·00**	(1) 45080·18
	1872·65		**1844**
8.	19194	+ **1858·40**	(1) 45872·18

(for the displaced sets see p. 380)

	F.		**F**.
		$m = 3$.	
1.	23353	**30677·27**	(2) 38000·71
		$10\Delta'_2 + \delta$.	
1.	23507	**30674·76**	[37842·45]
2.			
			104·03
3.			(1) 37946·48
	132·75		
4.	23639		
5.			
			306·41
6.			(4) 38148·86
			1048·54
7.			(1) 38890·99
	1860·00		**1859·62**
8.	25367	+ **1859·81**	(1) 39702·07
		$m = 4$ (F′).	
1.	26067	**30677·16**	(6) 35286·68
2.			
3.			
			129·78
4.			(1) 35416·16
			207·17
5.			(6) 35493·85
	301·70		
6.	26369		
			1043·80
7.			(1*n*) 36330·48
	1856·6		**1859·18**
8.	27924	+ **1857·92**	(1) 37145·86

F.		**F**.
	$m = 5$.	
[27498]	**30678·23**	(4) **33857·67**
		214·23
		(1) 34071·90
		(2) 33838·09 = **F**′$_1$
		(1) 34146·82 = **F**′$_5$
	$m = 6$.	
[28357·47]	**30675·60**	[32993·80]
		133·70
		(2*n*) 33127·50
307·41		**303·46**
28664		(1) 33297·26
1860·22		
30218		
	$m = 7$.	
28907	**30674·71**	32441·62 = *e*.35624·97
		33442·54 = *b*.33228·89
F′.		
28891·63		
		1048·78
[F′(7)]	**30677 34**	(1) 33511·28

$m = 2$. To the only two direct F there appear two direct **F** with the same mean 30674·77 and modified separation 301·8. Also there is a direct line to $\mathbf{F}_4$. To the linked $F_4.e$ corresponds a linked $c.\mathbf{F}_4$. In this connection it must be remembered that all the **F** are large, over 44000, and an e link would reach to lines outside the region of observation. Three other sets are included in the list, which involve displaced f sequents. The second pair give a mean $30674 \cdot 77 + 102 \cdot 10$ and belong therefore to the limit midway between F_2 and F_3. But the sequent = half the difference = 13399·95 in place of 13353·26, on the supposition of a common limit. So also the pairs for $F_{7,8}$ show f sequents 13362·41 and 13339·01 on the same supposition.

If the observed 44028 is really $(3\delta_1)\mathbf{F}_1$, where $\mathbf{F}_1$ has the same limit as F_1, and may be called the normal F_1, the mean of the observed F_1 and of $\mathbf{F}_1$ is 30677·80, and should give the true value of the limit subject only to observation errors on the two lines, *i.e.*, within maximum error of 1·1 with $d\lambda = \pm \cdot 05$, and within a probable error much less.

$m = 3$. $\mathbf{F}_1$ corresponds to F_1 with mean $30677 \cdot 27 \pm 1$. There are some cases of displaced $f(3)$ as in $m = 2$. The $10\Delta'_2$ set appear also in **F**, and as they contain several examples they are placed in the list. There are two lines 37836·36 and 38050·15 (separation 213·79) which as F_1 and $\mathbf{F}_5$ give a mean 30671·71. The lines in the list show an unobserved line for $\mathbf{F}_1$, which is the basis for the others, its actual value is taken as $-3\delta_1$ displacement on 37836. The mean is 30674·76 which, on the supposition adopted above, corresponds to $a(3\delta_1)\mathrm{F}(\infty)$. Since $F_3 = (-2\delta_1)F_2$, the line 37946 may be written as normal $\mathbf{F}_2$, giving mean limit $= 30676 \cdot 79 \pm 1$. In the $4\Delta'_2$ set is a line $39799 \cdot 89 = \mathbf{F}_8(4\Delta'_2)$, giving with $F_8(4\Delta'_2)$ a mean $1860 \cdot 19 + 30678 \cdot 78$.

$m = 4$. There appear no direct **F** to the F lines. But they occur in the parallel set F′; but as $\mathbf{F}'_1$, $(2\delta_1)\mathbf{F}'_4$, $(3\delta_1)\mathbf{F}'_5$, $(2\delta_1)\mathbf{F}'_7$, $\mathbf{F}'_8$. The mean is 30677·16.

$m = 5$. Here $\mathbf{F}_1$ and $\mathbf{F}_5$ appear, but as the mean depends only on calculated F_1 it is not reliable. If F_1 be taken from the observed line 26727·89 by the $-a$ link, the F_1 line would be 27496·83, giving mean 30677·25. There are also lines connected with the parallel series F'_1 which has a limit 16 below F_1. $F'_1 = 27482 \cdot 72$ and $\mathbf{F}'_1 = 33838 \cdot 09$, gives mean 30660·40, which is about the proper amount below $\mathrm{F}(\infty) = 30677 \cdot 80$. With this goes 34146·82 as $\mathbf{F}'_5$ with separation 308·73.

$m = 6$. The unobserved lines supposed for the first pair are calculated respectively from the observed $\mathbf{F}_6$, $\mathbf{F}_8$, and $\mathbf{F}_4$. The line 33297 is $(2\delta_1)\mathbf{F}_6$. Corresponding to 33076·55 as $\mathbf{F}'_2$, the mean limit with F'_1 is 30658 in place of 30660. With this might possibly go 34018·93 as $(3\delta_1)\mathbf{F}'_7$.

$m = 7$. $\mathbf{F}_1 + e = 35624 \cdot 97$ gives a mean limit 30674·71. Also with $\mathbf{F}_1 + b = 33228 \cdot 89$ gives a mean limit 30675·12. Also 33511·22 an exact $\mathbf{F}'_7$ with mean 30677·34.

$m = 8$. $\mathbf{F}_1 + e = 35249 \cdot 74$ gives mean 30676·36, but F_1 is uncertain.

$m = 9$. I have not found $\mathbf{F}_1$, but 32015·37 as $\mathbf{F}_5$ gives $\mathbf{F}_1 = 31802 \cdot 00$, which gives mean limit 30677·10.

$m = 10$. No $\mathbf{F}_1$ found, but 31726·38 as $\mathbf{F}_2$ and 31841·05 as $(-\delta_1)\mathbf{F}_5$ gives $\mathbf{F}_1$ the same value 31625·5. This gives a mean limit = 30676·75.

The evidence seems therefore clear for the existence of this type of series.

The Value of the Oun.—For the evaluation of the oun there are at disposal:—

(1) The Δ_1, Δ_2 as determined from the S separations. These have given (p. 346) for a first approximation to δ, the value 249·30 from ν_1 and 249·6 from the two alternative ν'_2, ν_2. The ν's are so ill-determined that these might possibly refer to values giving the same δ. But the fact that the value of e calculated from Δ_1 agrees so closely with the maximum ordinate in the corresponding occurrency curve (Plate 2, fig. 1) shows that Δ_1 must be exceedingly close to the true value, in which case it is

improbable that the ·3 difference in δ could be attributed to a single observation error on each of Δ'_2, Δ_2. As it has been shown in [III., p. 332] that the triplet separation always shows a slight difference in the δ from ν_1, ν_2, it is probable that the same occurs here also. The evidence there given goes to show that the value obtained from $\Delta_1+\Delta_2$ is always closer to the true value. We should expect, therefore, a value between the two values above.

(2) The evidence obtained from the D qualification test.

(3) The D satellites whose mantissæ depend on multiples of Δ_2, viz., 19116 on $197\Delta'_2$, and 20763 on $189\Delta_2$.

(4) The mantissa of $f(2) = 185\Delta_2$.

Before however conditions (3), (4) can be applied it is necessary to obtain if possible a closer approximation to δ from (2). The material for discussion is that given on (p. 366). We shall discuss it on the two bases of $\delta = 249{\cdot}60+x$ and $249{\cdot}30+x$ where x is certainly not greater than ·3. The complete conditions are, using the displaced values $(-2\delta_1)$ D (∞) for (2), (6) and omitting (3) as parallel to (2),

$$
\begin{array}{lrr}
 & 249{\cdot}60+x & 249{\cdot}30+x \\
(2) & -2{\cdot}4+\ {\cdot}16\xi-4(p_1-p_2)+\ 13\tfrac{1}{2}x=0 & -\ 6{\cdot}5+\ldots \qquad\qquad =0, \\
(4) & 9{\cdot}0+\ {\cdot}21\xi-4(p_1-p_4)+\ 16\tfrac{1}{2}x=0 & 4\ \ +\ldots \qquad\qquad =0, \\
(5) & 10{\cdot}0+\ {\cdot}32\xi-4(p_1-p_5)+\ 26\tfrac{1}{4}x=0 & 2{\cdot}1+\ldots \qquad\qquad =0, \\
(6) & -2{\cdot}7+\ {\cdot}46\xi-4(p_1-p_6)+\ 38\tfrac{3}{4}x=0 & -14{\cdot}3+\ldots \qquad\qquad =0, \\
(7) & 16\ \ +1{\cdot}52\xi-4(p_1-p_7)+126\tfrac{1}{4}x=0 & -21{\cdot}8+\ldots \qquad\qquad =0, \\
(8) & 8{\cdot}6+2{\cdot}61\xi-4(p_1-p_8)+219\tfrac{3}{4}x=0 & -56\ \ +\ldots+220x=0.
\end{array}
$$

It is quite clear that the conditions in the first column cannot be satisfied without assuming very large observation errors unless x is negative, nor on the right hand column unless x is positive. In other words, δ must be $< 249{\cdot}60$ and $> 249{\cdot}30$. The first four equations, however, give no indications of amount, as the multiples of x are not sufficient to make the term in x more important than the error terms. In (6, 7, 8) the conditions may be written with $\xi \not> 1$.

$$
\begin{array}{lrl}
(6) & -\ 2{\cdot}7\pm\ 8{\cdot}5\ +\ 38\tfrac{1}{2}x=0 & -14{\cdot}3\ldots, \\
(7) & 16\ \pm\ 9{\cdot}5\ +126\tfrac{1}{4}x=0 & -21{\cdot}8\ldots, \\
(8) & 8{\cdot}6\pm10{\cdot}61+219\tfrac{3}{4}x=0 & -56\ \ldots.
\end{array}
$$

Nos. (6, 7) require x to be about equal and opposite in the two cases, say, $\delta = 249{\cdot}5$. This would make (8) give $-13{\cdot}5+2{\cdot}61\xi-4(p_1-p_8)=0$. This last case offers some difficulties which we will consider later. For a further approximation we will therefore put $\Delta_2 = 4678+x$, $\Delta'_2 = 4241{\cdot}386+{\cdot}907x$ which give $\delta = 249{\cdot}4933+{\cdot}053x$, and $D_{15} = 20763{\cdot}25+dn$. Then, (p. 366)

$$
\begin{aligned}
d_{15} &= 884207-30{\cdot}50\xi'+30{\cdot}50\,dn\pm{\cdot}5 &&= 189\{4678{\cdot}343-{\cdot}161\xi'+{\cdot}161dn\pm{\cdot}002\} \\
f(2) &= 865448-107{\cdot}26\xi'+16\,p\pm{\cdot}5 &&= 185\{4678{\cdot}098-{\cdot}580\xi'+{\cdot}086p\pm{\cdot}002\}.
\end{aligned}
$$

Hence

$$\cdot343-x-\cdot161\xi'+\cdot161dn\pm\cdot002=0,$$

$$\cdot098-x-\cdot580\xi'+\cdot086\,p\ \pm\cdot002=0.$$

Therefore

$$\left.\begin{aligned}\xi' &= -\cdot58-\cdot38dn+\cdot20p\ \pm\cdot009\\ x &= \cdot436+\cdot222dn-\cdot032p\pm\cdot003\end{aligned}\right\}\text{A.}$$

Here dn depends on an extrapolated value from a D_3 line, taken because the ν_2 showed a diminished value corresponding to the usual satellite modification of ν_1. If this is correct dn is the observation error on 21858 and is $\cdot14p$ with $d\lambda=\pm\cdot03$. This is certainly the most *à priori* probable supposition. If on the contrary hypothesis ν_1 is normal D_{11} is $21552\cdot65+\cdot14p-786\cdot45=20766\cdot20$ and $dn=2\cdot95+\cdot14p$. This gives

$$\xi=-1\cdot70\ -\cdot05p+\cdot20p'\pm\cdot009,$$

$$x=\ 1\cdot091+\cdot03p-\cdot03p'\pm\cdot003.$$

Taking now the case of $D_{18}=19116$ we have already seen that a change of $y\delta_1$ on the limit, or $2y\delta_1$ is the sequent produce the same change, so that $(y\delta_1)\,D_{18}\,(2y\delta_1)$ is the same as D_{18}, and the D qualification test remains unchanged, although the mantissa is altered by $2y\delta_1=174\cdot8y$.

Supposing this displacement to take place the mantissa of D_{18} is

$$835908+124\cdot8y-28\cdot21\xi+3p\pm\cdot5=197\,\{4243\cdot188+\cdot6330y-\cdot143\xi+\cdot015p_8\pm\cdot002\}.$$

Here again, as in the case of D_{15}, the question arises whether the value of the normal sequent be taken as that of D_{18} or D_{28} which latter is $\delta_1=62\cdot4$ larger.

The condition then for the exact multiple of Δ'_2 is

$$1\cdot802+\cdot633y-\cdot907x-\cdot143\xi+\cdot015p_8\,(\text{or}+62\cdot4/197)=0.$$

The two cases give

$$\left.\begin{aligned}\xi&=-\ \cdot58, & 1\cdot505+\cdot6330y-\cdot020p_5+\cdot03p'+\cdot015p_8\quad(\text{or}+\cdot316)&=0\\ \xi&=-1\cdot70, & 1\cdot063+\cdot6330y-\cdot03\ p_5+\cdot03p'+\cdot015p_8\quad(\text{or}+\cdot316)&=0\end{aligned}\right\}\text{B.}$$

Neither can be easily satisfied with d_{18}, but with d_{28} the conditions can be satisfied with $y_2=-3,\ -2$, for the two cases respectively.*

The remainders in the two cases are $-\cdot077$ for $\xi=-\cdot58$ and $+\cdot113$ for $\xi=-1\cdot70$.

The list of F lines also give the following which can be used as tests.

(1) (6) $17747\cdot14=F_1(2)\,(10\Delta_2)$, *i.e.*, $f(2)$ mantissa $=195\,\Delta_2$,

(2) (5) $17594\cdot17=F_1(2)\,(7\Delta'_2)$,

(3) (2) $17972\cdot78=F_1(2)\,(16\Delta'_2+\Delta_2)$.

* See, however, final order below.

The mantissa of (1), using $F(\infty) = 30678{\cdot}93+\xi'$ is

$$912207-112{\cdot}60\xi'+11p_1\pm{\cdot}5 = 195\,\{4677{\cdot}986-{\cdot}577\xi'+{\cdot}05p_1\pm{\cdot}002\}.$$

This requires $-{\cdot}014-x-{\cdot}577\xi+{\cdot}05p\pm{\cdot}002 = 0$, or combined with the condition for $f(2)$, $-{\cdot}112+{\cdot}003\xi-{\cdot}086p+{\cdot}05p_1\pm{\cdot}002 = 0$ and can be satisfied within error limits.

The mantissa of (2) $895149-110{\cdot}63\xi'+11p_2$ which differs from that of f (2) by $29701-3{\cdot}37\xi'+11p_2-16p' = 7\,(4243{\cdot}00-{\cdot}48\xi+1{\cdot}6p-2{\cdot}3p')$.

This requires $1{\cdot}61-{\cdot}907x-{\cdot}48\xi+1{\cdot}6p-2{\cdot}3p' = 0$ easily satisfied for both cases within observation errors.

The mantissa of (3) is $937966-115{\cdot}6\xi'+11p_3$, differing from that of f (2) by $72518-8{\cdot}35\xi+11p_3-16p$. With $\xi = -{\cdot}58$ and $-1{\cdot}70$ this becomes 72523·1... and 72532..., or $17\Delta'_2+1\tfrac{3}{4}\delta-23$... and $17\Delta'_2+1\tfrac{3}{4}\delta-24$... on their respective values of D'_2. The amount 23 is perhaps excessive to be covered by the various possible errors but it just comes within. It may be noted that $17\Delta'_2+1\tfrac{3}{4}\delta = 16\Delta'_2+\Delta_2$. These three data do not decisively distinguish between the two cases. This, however, is not to be unexpected because the two arise from a δ_1 displacement in d_{15}, the sequents in this neighbourhood are such that δ_1 on the limit and $2\delta_1$ on the sequent are nearly equivalent, and the multiples involved 185, 189, 195 are too close to produce contrasts. Incidentally, also, the discussion strengthens the allocation of the lines to the displacements given.

The only further test with our present knowledge is to obtain some independent evidence as to the exact value of the limit, and naturally we turn for this to the mean of the F and **F** series. The series however in Kr is not nearly so well developed as in X. As has been already seen there are only three sets of observed pairs ($m = 2, 3, 4$) and these give for $F(\infty)$ respectively values of 30674·77, ... 7·27, ... 7·16. Since a displacement of δ_1 produces a change of 2·03 in $F(\infty)$ the first may be due to the fact that the line taken for **F** (2) is really $(3\delta_1)$ **F** (2), when the true mean would be 30677·81. It is natural to seek further as to the existence of summation lines corresponding to our last three examples. The result shows a most remarkable agreement. The sets are shown in the following list together with those obtained from the normal F and **F**.

m.	F.	$F(\infty)$.	**F.**	
2	(1) 17321·51	30677·82	(44034·13)	$(-3\delta_1)$ (2) 44028·04
3	(1*n*) 23353·84	30677·27	(2) 38000·71	
4	(2*n*) 26067·66	30677·16	(6) 35286·68	
$F_1(2)\,(7\Delta'_2)$	(5) 17594·17	30677·76	(1) 43761·38	$\mathbf{F}_1(2)\,(7\Delta'_2)$
$F_1(2)\,(10\Delta_2)$	(6) 17747·14	30677·73	(1) 43608·33	$\mathbf{F}_1(2)\,(10\Delta_2)$
$F_1(2)\,(16\Delta'_2+\Delta_2)$	(2) 17972·78	30677·70	(1) 43382·63	$\mathbf{F}_1(2)\,(16\Delta'_2+\Delta_2)$

These are remarkably concordant, especially when it is noted that the F (3, 4) are diffuse lines and not so susceptible of exact measurement as the others. The mean limit is 30677·72 and may be taken as practically correct. That calculated from the series, and used in the preceding discussion is $30678{\cdot}93+\xi'$. This, therefore, requires a correction of $-1{\cdot}21$. The equations A would be satisfied by $p = 1$, $dn = 2$, $\xi = -1{\cdot}21$. As however this value of ξ is probably correct within ·1 the best value of Δ_2 is obtainable from $f(2)$, viz.,

$$\begin{aligned}
\Delta_2 &= 4678{\cdot}098 + {\cdot}580\,(1{\cdot}21 \pm {\cdot}1) + {\cdot}086 \times 1 \pm {\cdot}002 \\
&= 4678{\cdot}80 \pm {\cdot}10 \\
\Delta'_2 &= 4242{\cdot}18 \pm 06 \\
\delta &= 249{\cdot}536 \pm {\cdot}005.
\end{aligned}$$

If the difference between δ as found from ν_1, ν_2 be real and depend on electronic changes as hinted at in the introduction, the changes calculate to 73·94 electrons = 74. In other words, the ν_2 would refer to mass of nucleus +37 electrons and ν_1 to mass of nucleus −37 electrons. Is it merely a curious coincidence that the atomic number of Kr is 38, that of H being 1 ; all the electrons acting in one way for ν_1 and in the opposite for ν_2? it being remembered that when S lines are emitttted one electron at least is absent.

Xenon.—The X diffuse system appears to be a most complicated one. As we shall see later there appears to be a congery of series converging to limits which are collaterals of S (∞) = 51025, and connected with these there are again congeries of F series converging to limits collateral to the various d_{1n} sequents or, say, the normal F (∞). These F series further show the existence of satellites—in other words the f sequence is also subject to slight collateral displacements. This renders their disentanglement a very intricate problem not only in itself, but because it renders the region of the spectrum involved very crowded, with lines close together, with the consequence that coincidences occur which may not refer to real relationship. In fact there are cases where the calculated values of supposed lines of different series are the same within observation errors.* This crowding is also increased by the existence of the allied **F** series referred to above. The complete discussion of all these related series should afford valuable material for arriving at a knowledge of displacement laws. Here it will be sufficient to indicate the nature of the problem and to deal with the material so far as to give confidence in the results as to the assignment of series and especially as to accurate determinations of the oun and the various links.

As vacuum tube spectra approximate to the spark type, the difficulty of drawing definite conclusion from the existence of a triplet separation is again enormously increased by the presence of the link relations which these spectra show. In arc spectra the appearance of a ν_1 or ν_2 separation may always with some certainty be ascribed to the fact that the lines in question are directly connected with series terms.

* A case in point is $F_1(5)$ and $F_3(17)$ in the series next considered ; also $F_3(13)$ and $F_2(19)$.

Here however no such certain conclusions can be drawn. They may enter as links. Although their true connection may ultimately be definitely settled the doubt as to whether they give true or false scents renders the task of unravelling most bewildering.

In about the region to be expected we find the set

(1) 19880·72 **1778·42** (5) 21659·14 **815·30** (10) 22474·44

showing from their separations and order of intensities an indubitable satellite set of D_3 type. Below these come a number of indubitable doublets of D_2 type, and then a number of strong lines of D_{11} type. There are also in this region an extremely large number of lines with separations between 1780—1785 corresponding to the enlarged links already found in the KrD spectrum. This portion of the spectrum is set out together with their separations in the following table:—

(2*n*) 17903·44	(3) 19632·44	**1780·84**	(3) 21413·28		
1772·81	(7) 19676·25	**1781·23**	(1) 21457·48		
	(1) 19785·84				
	(3) 19815·84				
	(1) 19829·47				
	(1) 19880·72	**1778·42**	(5) 21659·14	**815·30**	(10) 22474·44
	(1) 19942·53	**1775·45**	(10) 21717·98		
	(1) 19959·64	**1772·60**	(1*n*) 21732·24	**812·77**	(6) 22545·01
(3*n*) 18238·60	(1) 19989·72	**1780·27**	(6*n*) 21769·99		
1778·86	(1) 20017·46	**1783·71**	(10) 21801·17		
1783·06	(1) 20021·66	**1779·51**			
(7) 18266·88	(2) 20029·12				
1774·85	(2) 20041·73	**1782·55**	(1*n*) 21824·28		
		1784·54	(1*n*) 21865·47	**1784·76**	(2) 23650·23
	(4) 20080·93				
		1786·02	(1*n*) 21866·95	**1783·28**	
	(1) 20107				

(1) 20305·60		
(6) 20312·70	**1783·72**	(5) 22096·45
(4) 20320·25	**1776·17**	
(1) 20333·22	**1774·92**	(3*n*) 22108·14
(5) 20443·29		
(5) 20454·88		
(1) 20467·90		
(6) 20470·75		

(7) 20500·13	**1784·68**	(2) 22284·82
1777·11		
(9) 18723·02		
814·13		
(2*n*) 17908·89		
(3) 20529·92	**1780·35**	(7*n*) 22310·27
(8) 20559·08	**1785·64**	(<1*n*) 22344·72
(1) 20581·64		

Before however considering these lines in detail it will be desirable to take here a preliminary discussion which involves a new fact in series relationships, and at the same time will give some reliable data which have a bearing on the present problem of the actual D and F series. The line 20312 and those in its neighbourhood show the following sets of separations, viz. :—

1.	(1) 17615·06	**832·13**	(10) 18447·19	**1865·51**	(6) 20312·70				
2.	(6) 17638·55	**827·92**	(2) 18466·47	**1866·75**	(1) 20333·22				
		1864·74	(5) 19503·29	**829·93**					
3.	(3) 17628·29	**831·70**	(1) 18459·99	**1860·26**	(4) 20320·25	**1859·79**	(1) 22180·04		
4.	(1) 17772·38	**835·41**	(8) 18607·79	**1862·96**	(6) 20470·75				
5.	(3) 19503·29	**829·93**	(1) 20333·22	**830·62**	(2) 21163·84				
				1864·93	(2) 22198·15	**1863·97**	(1) 24062·12	**1862·95**	(<1) 25925·07

In the general survey at the commencement of the investigation a large number of separations of an amount near 1864 was noticed. Suspecting that it indicated the existence of a second type of sharp series, a second smaller separation (for triplets) was looked for, analogous to the $\nu_2 = 815$ of the 1778 set. A further separation was found for a value about 830.* The whole spectrum was searched within the limits 1864 ± 2. The result is shown in the occurrency curve of Plate 2, fig. 3. This curve is unique amongst those hitherto observed in its great height above the pure chance line and also in the steepness of its rise and culmination to a single definite peak. The search brought to light also a very considerable number of long successive chains and of meshes (see *e.g.*, Nos. 2 and 5 above) of the same amount, proving that 1864 enters not only on its original source, as due to a displacement on some fundamental sequence, but also as a link. Now the c link of the normal $\nu_1 = 1778$ is 1872·63 and $c\,(3\delta) = 1865{\cdot}16$, thus suggesting a possible origin, also a corresponding c link for ν_2, *i.e.*, a separation produced by a second Δ_2 displacement is 833·84, or with a modified $c'\,(3\delta) = 830{\cdot}70$. These suggest triplets formed by the same Δ_1, Δ_2 on $p\ (-\Delta_1)$ instead of on p, in other words, series whose limits are $51025 + 1778 = 52803$, 54675·6, 55509·4. But the way in which the separations enter with the suggested D line 20312 indicates that they stand in fundamental relation to it and neither in a linkage relation, nor with the limits named. For in the latter cases it would throw out of gear the whole relation of 20312 with the D set, which some provisional work had seemed to establish. In this work they were considered as part of the D system through 20312 regarded as a D_1 (1) line with 18447, 17615 as satellites. In this case the d_1 sequence is of the order 51025 $-20312 = 30713$. As against this idea is, of course, the greater intensity of 18447, the supposed satellite over that of the D_{11}, and also the fact that no normal ν_1

* As a fact, however, there are several others depending on oun multiples also present, but which at present we need not deal with.

separation occurs with 18447 or ν_1, ν_2 with 17615. But provisionally that was set aside for the moment. If they represented a special D set, the separations ought to reappear in a triplet series of the F type, and in the reverse order. From the sets already excerpted the lines (8) 18607·79, (10) 23915·72, (5) 26365·19 appeared to have all the signs (RYDBERG'S tables) of belonging to one series. The formula calculated from them brought to light a whole long series of observed lines. The limit found was 30724, close to the value already found (30713) as of the order of magnitude to be expected. This so far supported the supposition of the D relation, but there also came to light another result of evident importance in general theory—viz., the **F** series already referred to. The ordinary form of a series is one in which successive lines obey a formula of the type $A-\phi(m)$. In this case we find series associated with it whose successive lines are given by $A+\phi(m)$. This holds for each of the triplet sets, so that the complete series are given by $A\pm\phi m$, $B\pm\phi(m)$, $C\pm\phi(m)$, where $B = A+1864$, $C = A+1864+830$. Quite apart from the importance of this fact in the theory of spectral series the phenomenon is of special use in calculating the various constants on which the series depends. For instance the sum of the wave-numbers of two corresponding lines gives 2A, 2B, 2C, thus determining the values of the limits quite independently of the nature of the series formula used. Moreover, the displacements which so frequently occur in the F and D series in the sequence term introduces uncertainties. This happens in two ways. First through the modified ν_1 values in which it is not always possible to say whether the displacement is produced in the D_1 or the D_2 line. Secondly because the typical line in any order is often wanting and only appears with a very large displacement of multiples of Δ_2 on the sequence term. This effect, however, provided it occurs for both sets (F, **F**), does not influence the values of A, B, C thus determined. Cases in point are the Kr sets $F(2)(7\Delta'_2)$, $\mathbf{F}(2)(10\Delta_2)$. $\mathbf{F}_1(2)(16\Delta'_2+\Delta_2)$ given on p. 380. In consequence it is possible to determine the separations $B-A$, $C-B$ independently of satellite or other displacements. That such sequence displacements occur in these 1864 series is shown by separations which deviate from the normal by more than observation errors.

But, further, the difference of two corresponding F and **F** lines, say $\mathbf{F}_1-F_1$, $\mathbf{F}_2-F_2$, $F_3-\mathbf{F}_3$, should each give $2f(m)$, if as is the normal rule the sequence term is the same for each line of a triplet. When however—as we have seen in Kr, and shall find even more markedly in X—there are displacements in $f(m)$ for successive lines in a triplet, these differences will not be the same, and the observed separations will vary from the normal values. For instance, suppose $f(m)$ becomes $f(m)-x$ for the second set, and $f(m)-y$ for the third. The lines are $A\pm f(m)$, $B\pm(f(m)-x)$.... The values of A, B, C calculated from the sums are not affected, and the real values of the separations given by $A-B$, $C-B$ are not affected although the observed separations are $\nu+x$, $\nu'+y$ and $\nu-x$, $\nu'-y$. In some cases we shall find evidence from close lines with different x or y—but the results are quite definite. If, however, in the corresponding terms of the F and **F** lines the f are different, then the value calculated from their sum shows a change from the normal limit. The effect shows itself at once and the interpretation is less certain. It is possible that where such an affect appears it may not be real, but due to the existence of the two close lines just referred to, of which one in each set is too faint to have been observed. Thus if the displacements are x_1, x_2 instead of finding $B-f+x_1$, $B-f+x_2$, $B+f-x_1$, $B+f-x_2$, the 2nd and 3rd, or the 1st and 4th may not have been observed and we should be led to a wrong conclusion by taking, say, the 1st and 4th as corresponding lines. There are cases of this kind and also where one only is absent—*i.e.*, we find one close doublet for one of the F or **F** lines.

The lines composing the series are given in the table below. The limit calculated from the first three F_1 lines was found to be $30724{\cdot}28 \pm 1{\cdot}80$, the uncertainty being due to supposed maximum observation errors of $\pm{\cdot}05$A in each line. The later discussion of the $\frac{1}{2}(F+\mathbf{F})$ rule will show that the limit should be very close to 30725·26 with an error probably $<{\cdot}3$. The formula was recalculated with this limit by supposing the three standard lines to be in error by $-{\cdot}02$, $+{\cdot}02$, $-{\cdot}02$, *i.e.*, by half their supposed maximum possible errors. The formulæ for the F_1 and $\mathbf{F}_1$ series then become

$$n = 30725{\cdot}26 \mp N \Big/ \left\{m + 1{\cdot}022746 - \frac{{\cdot}028705}{m}\right\}^2.$$

List of F and **F** Lines.

In each order the first line of numbers gives the F set, the second the **F** set. Between these are entered the mean values of the F and **F** which give the corresponding limits. When the values are deduced by methods explained in the notes they are enclosed in ( ), when calculated and not observed in [ ].

$m = 1$	(3010·35)	**1864·64**	(4875·09)	**830**	(5705·00)
	30725·77		**32590·35**		**33420·35**
	(58441·20)	**1864·50**	(60305·70)	**830**	(61135·70)
$m = 2$	(8) 18607·79	**1862·96**	(6) 20470·75	**829·46**	(<1) 21300·21
	30726·05		**32589·09**		
	(4) 31102·16.*e.v**	**1863·17**	(<1) 32965·33.*e.v*		
$m = 3$	(10) 23915·72	**1863·92**	(8) 25779·64	**829·69**	(1) 26609·33
	30725·15		**32589·28**		**33418·35**
	(2) 37534·58	**1864·35**	(δ_1) (2) 39404·36	**828·44**	(1) 40227·37
$m = 4$	(5) 26365·19	**1864·61**	*u*.(1) 32362·98†	**829·67**	(29059·47)
	30724·58		**32589·96**		**33420·07**
	(35083·97)	**1866·16**	(36950·13)	**831·54**	(37781·67)
$m = 5$	(5) 27696·15	**1865·51**	(1) 29561·66	**828·72**	(1) 26257·20.*u*
	30725·48		**32589·38**		**33419·32**
	(1) 33754·81	**1862·29**	(<1) 35617·10	**831·17**	($-\delta_1$) (1*n*) 36442·62

* *More probably* $\mathbf{F}_1(16)$.

† *Or* $F_2(21)$.

List of F and **F** Lines (continued).

$m = 6$	$(2\delta_1)$ (1) 28508·55	**1864·13**	$(-\delta_1)$ (<1) 30357·31	**829·19**	$(-2\delta_1)$ (<1) 31180·63
	30725·29		**32589·86**		**33417·27**
	(1) 32951·97	**1863·81** **1865·76**	(3) 34815·79 (5) 34817·73	**826·83**	(1) 35642·62
$m = 7$	(1) 29019·32	**1864·99**	(5) 30884·31	**832·82**	(5) 31717·13
	30724·37		**32588·57**		**33418·13**
	(2) 32429·42	**1863·42**	(1) 34292·84	**826·30**	(2*n*) 35119·14
$m = 8$	[29377·00]	[**1865·39**]	(1) 31242·39	**829·13**	$(-\delta)$ (1) 32048·92
	30725·60		**32590·19**		**33418·91**
	(1) 32074·21	**1863·78**	(2) 33937·99	**828·31**	(δ) (2) 34788·90
$m = 9$	(1) 29629·10	**1865·33**	(31494·43)	**831·94**	(2) 32326·37
	30724·64		**32589·98**		**33418·72**
	(2) 31820·18*	**1865·36**	$(-2\delta_1)$ (3) 33674·68	**825·54**	(4) 34511·08
$m = 10$	$(-\delta)$ (1) 29802·26	**1863·53**	$(-\delta_1)$ (3) 31680·16	**828·83**	$(2\delta_1)$ (4) 32525·72
	30725·20		**32589·74**		**33418·29**
	(2) 31628·35	**1865·54**	(4) 33493·89	**830·27**	(5*n*) 34324·16
$m = 11$	(29967·00)	**1863·04**	$(-2\delta_1)$ (2) 31820·04*	**832·46**	$(-2\delta_1)$ (3) 32652·20
	30725·18		**32590·25**		**33420·21**
	(5) 31483·37	**1866·08**	(2) 33349·45	**827·08**	(<1) 34176·93
$m = 12$	$(2\delta_1)$ (6) 30091·12	**1864·59**	$(-2\delta_1)$ (2*n*) 31934·91	**830·81**	$(-2\delta_1)$ (1) 32787·85
	30725·25		**32589·48**		**33419·31**
	(1) 31369·31	**1863·88**	$(-2\delta_1)$ (3) 33222·37†	**828·85**	(1) 34062·04
$m = 13$	$(-2\delta_1)$ (1) 30157·45	**1862·96**	(1) 32030·34	**831·00**	(3) 32861·34‡
	30725·48		**32589·18**		**33419·61**
	$(-2\delta_1)$ (3) 31273·55	**1864·64**	(1) 33148·03§	**829·86**	(4) 33977·89
$m = 14$	(4) 30239·16	**1862·57**	(δ_1) (<1) 32107·16		
	30724·58		**32588·06**		
	$(2\delta_1)$ 31219·95	**1864·39**	$(-3\delta_1)$ (5) 33059·49		
$m = 15$	[30297·01]	[**1865·42**]	$(-\delta_1)$ (2*n*) 32166·86	**828·06**	$(-\delta_1)$ (2) 32994·92
	[**30725·13**]				**33418·33**
	$(-\delta_1)$ (4) 31148·28		**1864 + 829·46**		$(-\delta_1)$ (1) 33841·74
$m = 16$	(1) 30348·83	**1864·97**	(<1) 32213·80		
	30725·49		**32589·56**		
	(4) 31102·16	**1863·17**	(<1) 32265·33		(δ_1) (5) 33799·54
$m = 17$	$(-\delta_1)$ (<1) 30382·49	**1861·03**	(3) 32248·49 ?		
	30724·89				
	$(-3\delta_1)$ (<1) 31047·46		**1864 + 828·48**		(1) 33754·81
$m = 18$	(1) 30421·95	**1863·31**	$(-\delta_1)$ (1) 32279·83	**829·68**	$(-2\delta_1)$ (<1) 33103·59 and $(2\delta_1)$ (<1) 33126·29
	30725·06		**32588·13**		**33418·27**
	(δ_1) (1) 31033·15	**1862·42**	(δ) (1) 32912·71		$(-\delta_1)$ (3) 33715·67 and (δ_1) (3) 33724·54

* *See* F_2 (11), *is more probably* $(-2\delta_1)$ (1) 31810·86.

† *This line is numerically* $(2\delta_1)$ $\mathbf{F}_2$ (12) *and* $(-2\delta_1)$ F_3 (22).

‡ F_3 (13) *and* $\mathbf{F}_2$ (19) *have same value.*

§ $\mathbf{F}_2$ (13) *and* F_3 (19) *have same value.*

LIST of F and **F** Lines (continued).

$m = 19$	$(-2\delta_1)$ (<1) 30444·37	**1861·79**	(1*n*) 32316·13	**831·90**	(1) 33148·03†
	30725·42		**32588·93**		**33420·61**
	(2) 30996·51	**1864·83**	(3) 32861·34*	**831·84**	(3) 33693·18
$m = 20$	$(-2\delta_1)$ (1) 30466·53	**1864·01**	(5) 32340·48	**830·69**	(δ_1) (2) 33165·52
	30525·60		**32589·36**		**33418·76**
	$(-5\delta_1)$ (1) 30949·88‡	**1863·52**	(2) 32838·25	**828·73**	(1) 33666·98
$m = 21$	(δ_1) (3) 30503·24	**1864·71**	(1) 32362·98	**828·68**	$(-\delta_1)$ (<1) 33186·10 and
					(δ_1) (<1) 33197·22
	30724·07		**32589·61**		**33418·10**
	(1) 30949·88	**1866·37**	(3) 32816·25	**828·29**	(2) 33644·54
$m = 22$	$(-\delta_1)$ (3) 30511·25	**1865·32**	$(3\delta_1)$ (<1) 32397·90§	**829·46**	$(2\delta_1)$ (3) 33222·37‖
	30725·36		**32588·60**		**33421·74**
	(5) 30933·51	**1862·19**	(1) 32795·70	**830·97**	(δ_1) (1) 33632·32
$m = 23$	[30534·18]	**1863·72**	(<1) 32397·90§		
	30725·57 + $\frac{1}{2}\xi$		**32590·16**		
	(<1) 30916·97	**1685·45**	(δ_1) (1) 32787·85		
$m = 24$	$(+2\delta_1)$ (3) 30559·83				
	30725·25				
	$(-\delta_1)$ (4) 30895·10 and	**1864·66**	(1) 32765·29§		
	(δ_1) (1) 30906·17				
$m = 25$	$(-3\delta_1)$ (<1) 30576·46¶	**1864·57**	(1) 32426·15	**829·45**	(δ) (3) 33278·20††
	30725·68		**32590·24**		
	$(-\delta_1)$ (5) 30884·81**	**1864·55**	$(2\delta_1)$ (1) 32765·29§		
$m = 26$	(<1) 30576·46	**1864·16**	$(-2\delta_1)$ (2) 32429·42§§	**829·83**	(1) 33270·45
			and $(2\delta_1)$ (3) 32451·83		
	30725·37		**32589·73**		**33420·50**
	(δ_1) (1) 30879·26	**1864·55**	$(-2\delta_1)$ (4) 32727·98	**831·70**	$(2\delta_1)$ (<1) 33581·84
$m = 27$	$(-\delta_1)$ (5) 30580·76		**1863·50 + 828·94**		(3) 33278·20
	30725·87				
	$(-\delta_1)$ (6) 30861·05		**1864·5 + 829·46**		$(-\delta_1)$ (6) 33553·33
$m = 28$	(1) 30595·63	**1864·11**	(4) 32459·74	**830·43**	$(2\delta_1)$ (3) 33301·47
			32589·87		
	[ ]		(2) 32719·41	**828·43**	$(-3\delta_1)$ (2) 33530·95‡‡
$m = 29$	(1) 30607·90		**1865·65 + 837·92**		(3) 33301·47
	30725·67		**32590·00**		**33419·00**
	(1) 30843·45	**1863·01**	(3) 32706·46	**830·14**	(2) 33530·95‡‡
$m = 30$	$(2\delta_1)$ (4) 30621·49		**1864 + 829·36**		(3) 33304·91
	[30839·22]	[1861·57]	(<1) 32700·79	**830·16**	(2) 33530·95‡‡

* $\mathbf{F}_3$ (13) *and* $\mathbf{F}_2$ (19) *have same value.*

† $\mathbf{F}_2$ (13) *and* F_3 (19) *have same value.*

‡ *The line is* $\mathbf{F}_1$ (21); $\mathbf{F}_1$ (20) *may be* (4) 30976·73.

§ *Coincidences.*

‖ *This line is numerically* $(2\delta_1)$ $\mathbf{F}_2$ (12) *and* $(-2\delta_1)$ F_3 (22).

¶ *This line is also* F_1 (26).

** *This line is* F_2 (7).

†† *This line is* F_3 (27).

‡‡ *Too close to settle.*

§§ *This line is also* $\mathbf{F}_2$ (7).

The lines of the series seem to be exceptionally numerous. The results of the examination up to $m = 30$ are given in the table and the notes thereto. There are certain lacunæ—especially for $m = 4$. In these cases however corresponding displaced sets are in general observed, and naturally with large values of m this effect is more frequent. In certain cases where a set is absent a parallel set is observed linked to the normal type. This is the case for instance in $m = 4$.

The question naturally arises whether lines exist for $m = 1$. If so the formula gives a triplet with the first line at $n = 3142$, far in the ultra-red. In other spectra these values extrapolated for $m = 1$ differ considerably, often by several hundreds, from the correct ones. We can only conclude that if there are sets based on $m = 1$ they must be such that F_1 (1) must be in the neighbourhood of 3100. The matter can only be settled therefore by other considerations which must depend—with our present knowledge at least—either on sounding or on the presence of combination lines in the observed region. The evidence for such a triplet is given below in the notes to the list of lines. The value of F_1 (1) found is 3010·35 corresponding to a wave-length *in vacuo* of 33218·7 A.U. The mantissa of $3010+dn$ with the limit $30725{\cdot}26+\xi$ is $989285+35{\cdot}9\,(dn-\xi) = 90\,\{10998{\cdot}8-{\cdot}4\xi+{\cdot}4dn\}-611 = 90\Delta_2-\delta$. The uncertainty in Δ_2 as found from ν_2 is too large to settle the exact value of this with so large a multiple as 90, but the fact as it stands that the mantissa differs from a multiple of Δ_2 by only a few oums is what is to be expected if the series belongs to the F type, and so far certainly supports the more direct evidence given below for the existence of the set depending on $m = 1$. With the value of Δ_2 found below $dn = -1{\cdot}5$.

A glance at the list will show that the separations observed in the second and third orders of F are less than the normal values. This points to a satellite effect. The values of ν_1 are 1862·96, 1863·92 which show deficits of 1·54, ·58 from the true value as indicated by the occurrency curve. Now a displacement by one oun produces a change of 1·25 in $m = 2$, and ·50 in $m = 3$. The deviation is then completely explained by the supposition of the existence of the satellite effect depending on δ_1. The ν_2 show similar deficits, which may possibly be due to observation errors. We should expect to find a similar effect (not necessarily the same multiple) in the order $m = 1$. In this order the oun produces a change of 4·25.

For sounders and for link evidence the data have been restricted to e, u, v links only. If we may judge from the examples of Ag and Au, the F and D linkages show a preponderance of the a, b, c, d links, and no doubt fuller evidence might have been adduced by using them, but it was necessary to set limits to the work, as well as to this communication which is long as it stands. But as examples we may give some d links belonging to the orders 2, 3 of F_1. The value of d is 1973·94. For $m = 2$ the lines (1) 20581·64, **1864·85** (3) 22446·49, are 1973·85, above F_1 and $F_1+\nu_1$, or the real F_2. The two lines (5) 23268·21, (2) 82·67 are respectively $\mp 6\delta_1$ displacements of 23275·44 which is 1875·23 (or c) above the observed F_3 or 1873·15 above the normal value $F_1+\nu_1+\nu_2$. For $m = 3$ we find

			(2) 18466·47				
			7313·17				
(1) 28341·62 **4425·90**	F_1	**1863·92**	F_2	**829·69**	F_3	**4426·43**	(2) 31035·76
(4) 28047·34 **4131·62**							
	1975·86		**1974·70**		**1970·10**		
	(4) 25891·58	**1862·76**	(2) 27754·34	**825·09**	(4) 28579·43	**1864·94** →	

Here F_1, 25891, 27754 form a series inequality with $d+2$, ν_1-2; F_1 forms a parallel inequality with 28341 and 28047 with $u-2$, $v-2$; 18466, F_2, 27754 a series inequality with d, e; and F_3 a parallel inequality with $v-2$, $d-2$.

As a rule in the list only displaced lines are dealt with and only occasionally linked ones. The evidence is strong for the existence of possible series with limits 30725 $(\pm x\delta_1)$—especially $x=2$—and that for each order most of the energy goes to a line with one of these limits, not necessarily the same for different orders nor for the same triplet set, nor for the same F or **F** type.

Notes.—A displacement δ_1 produces 4·97 on $F_1(\infty)$. 5·43 on $F_2(\infty)$, and 5·65 on $F_3(\infty)$.

$m=1$. The shortest sounder to reach from the calculated F (1) to the observed region is $2e$, giving values in the neighbourhood of 17700. In this neighbourhood we find a triplet (No. 2 of the sets on p. 383)

(6) 17638·55 **1864·64** (3) 19503·29 **829·93** (1) 20333·22,

which in separation and order of intensity corresponds precisely to the values required. With the normal value of e they indicate lines with wave-numbers at 3010·35, 4875·09, 5705·02. In connection with the question of satellite effect it may be worth noting that lines in the neighbourhood of the above may be arranged with them thus

(6) 17638	**1864·64**	(3) 19503	**829·93**	(1) 20333
		12·52		
....		(3) 19515·81		
25·53				
(5) 17664·07				

To complete the set there should be a line at 17651·31, but it was not observed by BALY. A sequence displacement in $f(1)$ of $3\delta_1$ produces a change of 12·76 and of $6\delta_1$ of 25·53. These are practically exact and correspond to the diffuse satellite arrangement with this difference, that the triplet appears as a main set in which the first line is strongest. They depend on $3\delta_1$, $6\delta_1$ displacements, so that the mantissa of the doublet set, that of 19515, is exceedingly close to $90\Delta_2$, viz., $90\times 10997·17$.

With $F_1(1) = 3010$ should go $\mathbf{F}_1(1) = 58440·17$. This is outside the observed region on the other side, in the ultra-violet, and requires even larger sounders than F (1). The treble link $3e = 21942·30$ requires a line at 36497·87. There is no line here but (1) 36493·82 is less by 5, *i.e.* = (δ_1) **F** (1). We can arrange lines to go with it as a satellite set, viz.,

[36498·90]	**1864·50**	[38363·40]	**830**	(1) 39193·40
		17·64		**17·66**
....		(2) 38345·76	**829·98**	(2) 39175·74
25·45		**26·31**		
(3) 36473·45	**1863·64**	(1) 38337·09		

Here we find the same $6\delta_1$ for $\mathbf{F}_1$ and $\mathbf{F}_2$, whilst in F we had it only for F_1. The set with separation 17 show no F_1, and again is numerically an exact oun multiple displacement, viz., δ which gives 17·0. The line found close to the expected—$(\delta_1)\,\mathbf{F}_1$ above—forms part of a chain

(1) 32765·29 **1864·47** (3) 34629·76 **1864·06** (1) 36493·82

It may be the representative corresponding to the satellite displacement δ_1 observed in F for $m = 2, 3$.

Using links $2e+u = 18761·38$ and $2e+v = 19056·2$ we should find lines at 39678·79 and 39383·97 with the corresponding lines to $\mathbf{F}_2$, $\mathbf{F}_3$ beyond the observed region. As a fact, we find lines at $(3n)$ 39683·95 and (1) 39386·98. The first corresponding to a $-\delta_1$ displacement should give $2e.v.\mathbf{F}_1(1) = 39679·70$. The second would differ from the same displacement by 3 and is therefore inadmissible. It should be noticed that both the $3e$ and $2e+u$ sounders give lines larger by unity than the expected value. This may, of course, be due to combined errors in the sounders and observation errors, but the **F** are entered in the list as if the sounders are correct. As none of the F (1) or **F** (1) lines can be observed, their means can have no weight for an accurate determination of the limit.

$m = 2$. The **F** lines are in the ultra-violet, and should be, with the limit chosen, at 42842·73, 44707·23. With sounder $e+v = 11742·10$ we find the set (4) 31102·16 **1863·17** (<1) 32965·33 with a small separation corresponding to that for F_1. It gives $\mathbf{F}_1 = 42844·26$ and $\mathbf{F}_2 = 44707·43$. But the set appears to be really $\mathbf{F}_1$ (16), $\mathbf{F}_2$ (16).

$m = 3$. A displacement δ_1 in $F_2(\infty)$ produces 5·42. The $\mathbf{F}_2$ line is not observed but (2) 39404 is practically an exact $(-\delta_1)\,\mathbf{F}_2$. In further illustration of this we find (2) 37431·03 linked to it by $-d$ and 5·61 (*i.e.* another δ_1) ahead in the remainder of the triplet (1) 37436·64 **828·13** $(1n)$ 38264·77. The lines in question are $(-\delta_1)\,\mathbf{F}_2$; $d.(-2\delta_1)\,\mathbf{F}_2$; $d.(-2\delta_1)\,\mathbf{F}_3$.

$m = 4$. $\mathbf{F}_2$ is (1) $32362·98 - u = 28229·80$ which, as referred to above, is also (1) $26257·20 + 1972·60$ close to d link. For F_3, $(-\delta_1)$ (4) $29053·89 = 29059·54$ or $(2\delta_1)$ $(1n)$ $29071·03 = 29059·73$ or $(3\delta_1)$ (1) $29076·10 = 29059·15$. Take the mean 29059·47 as $\mathbf{F}_3$. Several other lines in this neighbourhood show indications of displacement by multiples of δ_1, *i.e.* that the energy proper to $\mathbf{F}_3$ has gone into a number of collaterals. None of the direct **F** are observed but there appears a parallel set displaced δ, viz., (1) 35064·09 **1864·34** (1) 36928·45 with (1) 37742·12 due to a further $3\delta_1$ displacement. The **F** lines are entered as due to these.

Both separations are about 1·6 too large and further $\mathbf{F}_1$ is about the same amount too small. Now $2\delta_1$ on the sequence term produces a change of 1·40. This would indicate that with the δ displacement on the limit concomitant displacements of $2\delta_1$ occur in the sequence so that the observed lines are $(\delta)\,\mathbf{F}_1\,(-2\delta_1)$, $(\delta)\,\mathbf{F}_2$, $(7\delta_1)\,\mathbf{F}_3\,(2\delta_1)$. The whole set again is remarkable for connection with parallel sets separated by the normal $\nu_1 = 1778$.

$m = 5$. There are observed lines for $F_{1\cdot 2}$, F_3 is entered as depending on a u link. F_2 is too large by about one unit, but $F_3 - F_1$ is normal. This order affords good evidence for the existence of the displaced sets. Consider the following system of lines:—

		1863·57	(3) 29540·09	**830·59**	
(δ) F	(1) 27676·52		**1·84**		$(1n)$ 30370·68
		1865·41	(2) 29541·93	**828·75**	
	19·63				**11·81**
$(2\delta_1)$ F					(<1) 30382·49
F	(5) 27696·15	**1865·51**	(1) 29561·66		
	15·51				**22·26**
$(-3\delta_1)$ F	(1) 27711·66	**1863·73**	(3) 29575·39	**829·36**	(5) 30404·75
$(-\delta)$ F	(5) 27714·80	**1869·68**	(2) 29584·48		

Here 1·84 is an exact 3δ displacement in the sequent. The mesh shows series inequalities with the $\nu_1 + \nu_2$ = normal values. The two lines in question are clearly $\pm 6\delta_1$ displacements on a normal line $(\delta)\,F_2$. On $F_1(\infty)$, δ gives 19·88, and $3\delta_1$ 14·91; on F_3, $2\delta_1$ gives 11·30 and δ 22·60. These show how closely all the conditions of the allocations are satisfied. Further, it shows how 29561 has its excess value and that the normal sequent should be the same as in $(\delta)\,F_2$, *i.e.*, 9 less, thus making its separation with F = 1864·61. The same sequent change is shown by $\mathbf{F}_2$. Both give it as $(6\delta_1)f$.

$m = 6$. Again with an even order the F lines do not appear, but there are apparent also a congery of displaced lines analogous to that in F (5). The lines given in the list give wave-numbers 28498·61, 30362·74, 31191·93. On the contrary, **F** lines are observed although $\mathbf{F}_3$ has probably been displaced. These also show evidence of displaced sets, *e.g.*, $(1n)$ 32972·50 **1864·28** (<1) 34836·78 is 20·53 ahead of $\mathbf{F}_1$, and δ on the limit gives 19·88.

$m = 7$. The values of $F_1(\infty)$, $F_2(\infty)$ as deduced from the means are clearly too small. $F_1(7)$ is very close to the calculated value, so that if any error has been made it is probably due to the **F** which should be about 1·8 larger, and suggests a close doublet, *i.e.* a small sequence displacement as in the preceding sets. As supporting this there is a line (1) 32426·15 which as $(\delta_1)\,F_1$ would give 32431·12 making with F_1 the limit 30725·22. This corresponds clearly to the normal value. A similar displacement is also found in F_1 in the line (4) 29024·43, which as $(-\delta_1)\,F_1$ gives 29019·46 for F_1. It should be noted that the energy of F_1 has passed chiefly to the displaced line, whilst in F_2 most of its energy remains with it and a fraction passes to the displaced line. This probably means that only a small number of the normal F_2 configurations are broken up, whilst most of the F_1 are. F_3 as (5) 31717·13 gives ν_2 too large. This line and (4) 31705·47 are separated by 11·66 or a $2\delta_1$ displacement, so that there is a concomitant sequence displacement. A similar effect is shown in $\mathbf{F}_3$ with two lines $(2n)$ 35119·14, (2) 35126·05. The lines entered appear correct for they give the normal limit, but their half difference shows a displacement in the $f(7)$ sequent. The normal line would appear to be given by $(-2\delta_1)\,\mathbf{F}_3 = 35136\cdot05$ making $\mathbf{F}_3 = 35124\cdot78$ with $\nu_1 = 831\cdot94$.

$m = 8$. No line is found for the calculated 29377·00, or 77·23 if we allow the same O – C as for $m = 7$. The lines (3) 29368·41 as $(2\delta_1)\,F_1$ and (1) 29403·29 as $(-5\delta_1)$ F give respectively 29378·34 and 78·45, which are larger than should be expected. The calculated value has been taken as correct. Also the lines (1) 32048·92 as $(5\delta_1)\,\mathbf{F}_1$ and (4) 32098·20 as $(-5\delta_1)\,\mathbf{F}_1$ give respectively 32073·76 and 73·36 or a mean of 73·56. But 32048 is also $(\delta)\,F_3$. This is not a mere list coincidence. As a fact $(\delta)\,\mathbf{F}_3$ and $(5\delta_1)\,\mathbf{F}_1$ are very nearly equal, and if both existed would show as a double line too close to have been resolved. The second line has a separation 1864·23 to (2) 33962·43, and its deduced $\mathbf{F}_1$ makes with the calculated F_1 the limit 30725·18 very close to the definitive value found below. For $\mathbf{F}_2$ 33937·99 is supported by $(-2\delta_1)\,F_2 =$ (1) 33948·71 giving $F_2 =$ 33937·85, but the $F_2\,(\infty)$ is large and 31242 shows a separation with F_1 of 1865·39 also large. Also (1) 31247·07 as $(-\delta_1)\,F_2$ gives $F_2 =$ 31241·64 which makes $F_2\,(\infty) =$ 32589·82 and the separation from $F_1 =$ 1864·64 both improved. On the other hand $(-2\delta_1)\,F_2 =$ (4) 31253·32 gives $F_2 =$ 31242·46 precisely the line observed. These small differences depend partly on observation errors and sequence or satellite displacements. In the case of F_3 and $\mathbf{F}_3$ the equally and oppositely displaced lines give the same mean as the lines calculated from them.

$m = 9$. There seem considerable displacements in the sequences here. The calculated values for the first lines are 29632·80 and 31817·72. They are not observed, but the corresponding F_3, $\mathbf{F}_3$ lines are. There are two near observed lines (1) 29629·10 and (2) 31820·18 which give the mean 30724·64, which is small, but (1) 31810·86 $= (2\delta_1)\,\mathbf{F}_1$ would give 31820·79 and the limit 30724·94, close to the normal value. 31820·18 is then $(2\delta_1)\,F_2\,(11)$. With (5) 31483·37 as $(2\delta_1)\,F_2$ and (1n) 31505·49 as $(-2\delta_1)\,F_2$ we get respectively $F_2 =$ 31494·23 and 94·63. The mean is entered, and a similar $-2\delta_1$ displacement gives $\mathbf{F}_2$ as entered. The normal third lines are observed. Probably the $\mathbf{F}_3$ having the same sequent as the $\mathbf{F}_2$ adopted should be that given by $(-2\delta_1)\,\mathbf{F}_3 =$ (4) 34527·64 or $\mathbf{F}_3 =$ 34516·37 with $\nu_2 =$ 830·83.

$m = 10$. The allocation seems satisfactory. The limits also are very close to the correct, but the different triplet separations show that the successive sequents suffer displacement, but the same in each F, **F**.

$m = 11$. The calculated F_1 is 29966·21. With (1) 29982·13 $= (-\delta_1)\,F_1$, $F_1 =$ 29967·22. Moreover the last has links $e =$ 7314·34 to (3) 22652·88 and $u =$ 4133·20 to (1n) 25854·02 in very striking agreement. The value as calculated with normal e is entered. With the lines as entered it is seen that the means of the corresponding separations for the two series are both normal, although the individuals are abnormal. This shows that both corresponding lines have the same limit, and the same sequent, but that the latter shows a displacement from the normal value for the F_2 set. This is supported also by the fact that there are a number of close lines to $\mathbf{F}_2$. For instance, (2n) 33332·22 and (2) 33330·00 as $(3\delta_1)\,\mathbf{F}_2$ give respectively 33348·51 and 6·29 for $\mathbf{F}_2$. They are probably all $\mathbf{F}_2$ lines showing sequence displacements. The first gives the triplet separations 1865·14, 828·42 and limit 32589·77, the second 1862·92, 830·64 and limit 32588·66. In other words, the first gives $\mathbf{F}_2$, with same sequent as in F_1 and $\mathbf{F}_1$, the second as in F_2.

$m = 12$. Note the good agreement—the same $(-2\delta_1)$ displaced limit for F_2. $\mathbf{F}_2$ and same $2\delta_1$ for $\mathbf{F}_1$ and F_3.

$m = 13$. The two displaced sets give $F_1 =$ 30167·38, $\mathbf{F}_1 =$ 31283·59. The calculated $F_1 =$ 30166·40.

$m = 14$ *to* 30. It is remarkable how the series seems to persist to high order. It may be said that this is only apparently so, because in this region the spectrum is so crowded with lines that it is necessarily possible to select sets near the calculated values. But in truth the reason of the crowding is because of the series. The F and **F** lines crowd up together on either side of the three limits, and at the same time there are different sets of limits depending on the $2\delta_1$ displacements. The spectrum has not been examined beyond $m = 30$, and from 14 to 30 the list indicates an allocation without further specification. There is, however, much evidence not adduced here to indicate actual cases where sequence displacement occurs. The calculated values for F_1 from $m =$ 14 to 30 are 30238·21, 30297·01, 30345·73, 30386·58, 30421·17, 30450·68, 30476·20, 30498·14, 30517·35, 30534·18, 30549·11, 30562·32, 30574·08, 30584·61, 30594·07, 30602·60, 30610·32. The deviation from the calculated values for $F_1\,(29)$ and $\mathbf{F}_1\,(29)$, which, however, gives the correct limit, shows that the sequent $f\,(29)$ receives a large displacement value, so large indeed as to totally alter its mantissa. The set must be doubtful. The whole set are more likely to be $(\delta_1)\,\mathrm{F}(30)$ and $(-\delta_1)\,\mathrm{F}(30)$. In fact, in this neighbourhood, the difference of two successive orders in a series is comparable with the change produced by a δ_1 displacement in the limit, and so introduces some uncertainty in allocation. It will be noticed also that, in a few cases, the same line is adduced to fit two cases, which can only happen if a line happens to be a close doublet, an unlikely supposition to happen often.

We are now in a position to determine the limits with considerable accuracy. Taking the average means where they are deduced from actual or displaced actual values, we find the limits come to 30725·340, 32589·443, 33419·079. In the first attack on the problem values of unobserved lines were deduced from observed linked lines. The corresponding mean values for the limits then found had for the last digits 5·292, 9·161, 8·918, very close to those determined from the displacements. The individual deviations from the mean are quite small for $\mathrm{F}_1(\infty)$, considerably smaller than for the others. It is, therefore, the more reliable. The mean deviation in magnitude is ·28 and the maximum is $-$·97 in $m=7$. We may take it therefore that the true value of the F_1 limit is 30725·30 within a few decimals. The separations given by the deduced limits above are 1864·10 and 829·64.

These very accurate values afford a means of testing as to their source. If the limit were known to be a single number there could be no doubt as to its belonging to the d sequences, or as to the series being of the F type. But there is just the possibility that it may be a composite number, comprising one or more links—say, p or s terms—and that the separations may be due to oun displacements in one of them. The suspicion that this may be the case is aroused by the fact that the triplet 17615, 18447, 20312, which would be the origin of the d or $\mathrm{F}(\infty)$ term, and in which therefore the first two lines should behave as satellites do not show complete sets with the separation 1778, 815, as they should do if normal satellites. Moreover, the intensity order with the middle line much more intense than the other is not normal. There is no test for the composite nature of 30725, but if it be really so, the most probable source would be $p=\mathrm{S}(\infty)$, or some near collateral of this. We will therefore test this as $51025{\cdot}26+\xi$, where ξ may be considerable, so as to include near collaterals, and also test 30725 as a d sequent. We will take the latter first.

At the start it may be noted that it is an argument in favour of 30725 being directly the source, that displacements by small multiples of the oun have fitted in so remarkably closely and frequently in the formation of the list of lines above.

Taking then the limits as $30725{\cdot}30+\xi$, $32589{\cdot}40+\xi+d\nu_1$, $33419{\cdot}04+\xi+d\nu_1+d\nu_2$ the denominators are found to be

$$\begin{array}{ll} 1{\cdot}889322-30{\cdot}74\xi & \\ & 54831-2{\cdot}60\xi+28{\cdot}14d\nu_1 \\ 1{\cdot}834491-28{\cdot}14\,(\xi+d\nu_1) & \\ & 22914-1{\cdot}14\xi-1{\cdot}1d\nu_1+27d\nu_2 \\ 1{\cdot}811577-27\,(\xi+d\nu_1)-27d\nu_2 & \end{array}$$

In these ξ cannot be greater than a few decimals and will produce no effect on the differences of the mantissæ, which may themselves be affected with errors ± 1 due to using 7-figure logs.

The differences may be represented as follows :—

$$5\,(10996{\cdot}8+5{\cdot}6d\nu_1-{\cdot}52\xi\pm{\cdot}2)-\delta_1 = 5\Delta_2-\delta_1$$
$$2\,(10999{\cdot}0-{\cdot}5d\nu_1+13{\cdot}5d\nu_2\pm{\cdot}5)+6\delta_1 = 2\Delta_2+6\delta_1,$$

where to a first approximation we know already Δ_2 is 10998 ± 1. The sum

$$= 7\,\{10997{\cdot}4-{\cdot}53\xi\pm{\cdot}1+4\,(d\nu_1+d\nu_2)\}+5\delta_1.$$

Quite small changes in $d\nu_1$, $d\nu_2$ can therefore make the connections with Δ_2 exact, and since the multiples 5, 2, 7 are so small any small errors in the approximate value of Δ_2 can have no effect. If we use the ν_1 as found from the occurrency curve 1864·5 $d\nu_1 = {\cdot}4$, the number in the first bracket is 10999.

The agreement with both is so close to the relation indicated that it speaks strongly in support of the D origin of the limit, and the outstanding small differences may be left for the present.

If now the other supposition be tested, viz., that the separations arise from a S (∞) source $= 51025{\cdot}29+\xi$, the three denominators are found to be

$$\begin{array}{ll} 1{\cdot}466091-14{\cdot}36\xi & \\ & 26067-{\cdot}75\xi+13{\cdot}61d\nu_1 \\ 1{\cdot}440024-13{\cdot}61\,(\xi+d\nu_1) & \\ & 11160-{\cdot}31\xi-{\cdot}31d\nu_1+13{\cdot}30d\nu_2 \\ 1{\cdot}428884-13{\cdot}30\,(\xi+d\nu_1)-13{\cdot}30d\nu_2, & \end{array}$$

and in the 20312 set the S (∞) must enter as a negative quantity since the separations are there in inverse order. In this case ξ may be considerable. The differences may be expressed as follows :—

$$2\Delta_2+6\tfrac{3}{4}\delta-53-{\cdot}75\xi+13{\cdot}61d\nu_1\,;\;\; \Delta_2+\delta_1+9-{\cdot}31\xi+13{\cdot}3d\nu_2.$$

No permissible values of ν_1, ν_2 can make these both multiples of the oun. If it is possible to do so by a proper choice of ξ the latter must satisfy $53+{\cdot}75\xi = 153m$, $-9+{\cdot}31\xi = 153n$, where m, n are integers and 153 is the value of the oun. This requires $22 = 47{\cdot}4m-115n$. A suitable solution is $m = -2$, $n = -1$, which requires $\xi = 480$, or, say, series limit $= (11\tfrac{1}{4}\delta)$ S (∞). This method of explanation looks then improbable especially when taken with the more natural one above. It may be concluded with some confidence that the series in question is of the F type depending on D series for limits as in the usual way.

In the suggested lines for $m = 1$, found by sounding with $2e$ the mantissa of $f(1)$ was found to be $90\,\{10998{\cdot}8-{\cdot}4\xi+{\cdot}4dn\}-\delta$. ξ is small and the term involving it may be omitted. The error dn in 3010 may, however, amount to a few units because the lines on which it was based were assumed to depend on the limit 30725, whereas there is the possibility that they might belong to one of the parallel series found in the F sets depending on $(x\delta_1)$ (30725). In case, however, of dn being small and the mantissa involving the term in δ, we might expect still to find lines depending on the $90\Delta_2$, as the presence of the δ suggests satellites. To get $90\Delta_2$, requires a $+\delta$ displacement which decreases $f(1)$ by 17·02. In other words lines with wave-numbers 17·02 larger for F and less for **F**. We do not find this completely, but the following sets are observed, already given in the notes to the list under $m = 1$.

$2e.F_1$	$2e.F_2$	$2e.F_3$
(6) 17638·55	(3) 19503·29	(1) 20333·22
	12·52	
....	(3) 19515·81	
25·53		
(5) 17664·07		

$3e.F_1$	$3e.F_2$	$3e.F_3$
[36498·90]	[38363·40]	(1) 39193·40
5·08		
(1) 36493·82		
	17·64	**17·66**
....	(2) 38345·76	(2) 39175·74
25·45	**26·31**	
(3) 36473·45	(1) 38337.09	

In which permissible observation errors are $dn = \pm{\cdot}7$. As has been seen the 36493 corresponds to a δ_1 displacement in the limit. The others to $3\delta_1$, $6\delta_1$, and δ in the sequent $f(1)$. The lines 38345, 39175 consequently have their sequent mantissa exactly $90\Delta_2$.

Further it was found that $(3n)$ 39683·95 is $2e.v.(-\delta_1)$ $\mathbf{F}_1(1)$. The next preceding line to this is (1) 39666·49 or 17·46 behind it, again showing the required δ displacement and having the $90\Delta_2$ mantissa.

If it be granted that the series is of the F type, the limit must be a d-sequent. Consequently the mantissa of $30725{\cdot}30+\xi$ must be a multiple of the oun. Its mantissa is $889322-30{\cdot}74\xi = 81\ (10998{\cdot}13-{\cdot}38\xi)-10\delta_1 = 81\Delta_2-10\delta_1$ with great exactness. Let the true value of Δ_2 be $10998{\cdot}20+x$. Then if the relation is exact $81x+30{\cdot}74\xi+5{\cdot}7 = 0$ or $\xi = -2{\cdot}63x-{\cdot}18$, $x = -{\cdot}38\xi-{\cdot}07$. Now we know that ξ must be a small fraction, certainly $< \pm{\cdot}5$. Hence x must lie between $\pm{\cdot}2$ and $\Delta_2 = 10998{\cdot}20 \pm {\cdot}20$. We should, therefore, expect this value for Δ_2 except possibly where electronic changes of atomic weight came in, as has been suggested above. If then the 1864 separations depend on exact $5\Delta_2-\delta_1$ and $2\Delta_2+6\delta_1$ we get as closer approximations $5{\cdot}6d\nu_1 = 1{\cdot}40$ or $d\nu_1 = {\cdot}25$ and ${\cdot}80+13{\cdot}5d\nu_2 = 0$ or $d\nu_2 = -{\cdot}05$, in other words $\nu_1 = 1864{\cdot}35$, $\nu_2 = 829{\cdot}59$ when the limit is 30725. When this limit is displaced by $y\delta_1$ these change by ${\cdot}45y$, ${\cdot}22y$.

Further, the conditions for $f(1)$ require $\cdot 67-x-\cdot 4\xi+\cdot 4dn = 0$, or $dn = -1\cdot 8-\cdot 056\xi$. If the sounder $2e$ was exactly normal this dn must be due to observation errors in 17638 of $d\lambda = \cdot 5$, but the value of e is also subject to some small uncertainty. In any case the result shows that the reference line does not depend on a displaced 30725, for if so dn would be at least 4·97.

Returning now to the discussion of the D series let us consider first this second group of clearly analogous series of lines :—

1.	(1) 19942·53	**1775·45**	(10) 21717·98	4.	(10) 20636·30	**1767·09**	(20) 22403·39	**813·66**→(<1)
2.	(1) 19989·72	**1780·27**	(6n) 21769·99	5.	(2) 20688·96	**1785·54**	(10) 22474·44	
3.	(1) 20017·46	**1783·71**	(10) 21801·17	6.	(1) 20859·23	**1784·57**	(7) 22643·80	
	(1) 20021·66	**1779·51**		7.	(4) 20962·07	**1780·09**	(10) 22742·16	**813·80**→(1)

They all, with the doubtful exception of 4, 7 have the appearance of belonging to first, or doublet, satellite sets, in which the second line is always the stronger. The 1780 separations are clearly associated with the now well recognised mid-triplet abnormality. That it is not itself a normal separation is indicated by No. 3 in which the 1779·51 also occurs.

In (1) the separation 1775·45 is $\nu_1-2\cdot 45$. It differs from the displaced $(\delta)\,\nu_1$ by ·29 which is within error limits. In this case the limit would be $(\delta)\,D(\infty)$ which is 42·48 less than $D(\infty)$ and $= 50982\cdot 81$. With this limit the mantissa of 19942 comes to $879711 = 80\times 10998\cdot 3-\delta_1$ or $80\Delta_2-\delta_1$ within error limits. This is the typical form for the second satellite set of a triplet D series, but modified by the δ_1 displacement, so common in this group of elements, though here it appears in an apparently first satellite set instead of the second. We note at present that taking account of the small corrections, and writing as before $\Delta_2 = 10998\cdot 2+x$ its true value is $80\Delta_2-\delta_1+8-30\cdot 28\xi+30dn-80x$. The observation error dn is $< \cdot 2$ in this region and ξ is probably <1.

In (2) 19989 is 47·19 above (1). The change due to the displacement δ in the limit is 42·55, whilst δ_1 in the sequent gives 5·05 suggesting that the limit of (2) is the normal $D(\infty)$, with sequent $80\Delta_2$. With this limit the mantissa is found to be $879853 = 80\times 10998\cdot 16$, or with small corrections $80\Delta_2-3\cdot 2-30\cdot 29\xi+30dn-80x$.

In (3) we have the modified 1783·71 with the clearly real separation 1779·51 or $\nu_1+1\cdot 61$. Now the displacement due to -3δ on the ν is 1·60 which is practically exact. This gives a limit 31·88 larger or 51057·17, and the mantissa becomes 879855, the same as for (2) and $= 80\Delta_2-1\cdot 4-30\cdot 29\xi+30dn-80x$. The line 20017·46 is 4·20 behind the other. A displacement of δ_1 in the sequent term produces 5·05. Thus 20017 is very close to a line with mantissa $= 80\Delta_2-\delta_1$, but the difference ·85, corresponding to $d\lambda = \cdot 21$, is too great to render the relation exact.

Nos. 4 to 7, although much further towards the violet should not be put aside. They all show the exceptional separations.

In (4) the separation is $1767{\cdot}09 = \nu_1 - 10{\cdot}81$. If it corresponds to a real ν_1 the limit will be $(5\delta)\,\mathrm{D}(\infty)$ which reduces ν_1 by $10{\cdot}7$ and makes the limit about 212 less. The mantissa would be $82\Delta_2 + 7\frac{1}{2}\delta + 71$. This difference (71) from an exact multiple of δ_1 shows this to be impossible.

The discussion of the 1864 series has definitely shown it to be of the F type, and has given the limit within very small errors. This limit is one of the $d(1)$ sequents of the diffuse series. Its mantissa was found to be $81\Delta_2 - 2\frac{1}{2}\delta$ or $80\Delta_2 + 15\frac{1}{2}\delta$. It gives one firm starting point for the discovery of the D series. The results just obtained indicate the lines which through their dependence on multiples of Δ_2 give the origin term of the diffuse sequence. They point, as we have seen, to the existence of several displaced, or parallel, sets of diffuse series. It is possible to show definitely that these exist, even if there be some uncertainty as to the lines occupying the position of $\mathrm{D}_{11}(1)$. In the normal case with a single diffuse series, the $\mathrm{D}_{11}(1)$ is always the strongest line of the series. Also in the normal type the D limit is the same as that of S—here $51025{\cdot}29 + \xi$. When however displacement occurs, the energy of a single line is dispersed amongst several others, and a line corresponding to the normal may be weak, or even too faint to have been observed. As a matter of experience also it is found that the lines of low order ($m = 1, 2 \ldots$) are subject to these displacements in a much greater degree than those for higher orders of m. Now there are a number of lines, which by their position and absence of ν_1 separations to stronger lines have the appearance of being $\mathrm{D}_{11}(1)$ lines. If they are, their mantissæ must differ from multiples of Δ_2 (in the present case $80\Delta_2$) by multiples of the oun. The fact that they may do so does not of course prove that they are D_{11} lines. If they do not do so it proves that they are not. They may however in the latter case belong to a displaced series, satisfying the multiple law when the proper displaced limit $(y\delta_1)\,\mathrm{S}(\infty)$ is employed. This gives us a method of testing as to what displacement a given line may correspond. If our calculus were already fully established the next step would be to apply this test to the above lines. But in reality we are testing our calculus to see if it can be firmly established, and our immediate aim must be to obtain independent evidence for the existence of parallel series. For this immediate purpose it will only be necessary to apply the test to two lines, the general question being postponed for the present.

In the first attempt at arranging the D_{11} series the strong lines (8) $20559{\cdot}08$ and (10) $38366{\cdot}36$ were taken for $m = 1, 2$, and the formula calculated with the limit $\mathrm{D}(\infty) = \mathrm{S}(\infty)$. As will be seen immediately, this gave satisfactory agreement with sounded observed lines up to $m = 15$; and as a matter of fact this series was used to test for the parallel sets displaced $(\pm 2\delta_1)\,\mathrm{S}(\infty)$ on either side of it. Now the formula constants for a set only vary slightly if the wave-number of the line chosen for $m = 1$ is changed considerably. This therefore did not prove definitely that 20559 is the correct $\mathrm{D}_{11}(1)$, and as a fact it does not satisfy the multiple test. Its mantissa is $897337 - 31{\cdot}14\xi$. That of the line 19989, which is shown above to be the origin of the normal D set is $879853 - 30{\cdot}29\xi$. The difference is $17484 = 28\frac{1}{2}\delta + 71$. This is as far as it can be from being a true multiple, and it is quite impossible to explain this by any observation errors in the two lines. For instance our maximum admitted error can only change the mantissa by 6. If the test is valid therefore 20559 cannot be $D_{11}(1)$. Yet it has all the appearance of such a line. Is it a displaced one? Suppose it corresponds to $y\delta_1$. Now a displacement of δ_1 on the limit changes its value by 10·62, so that $(y\delta_1) D(\infty) = 51025{\cdot}29 + \xi - 10{\cdot}26y$. If p denote the ratio of actual observation error to the maximum permissible, *i.e.*, $O = d\lambda = {\cdot}05p$, the mantissa of 20559 with the new limit is $897337 + 330{\cdot}7y + 6p - 31{\cdot}14\xi$. The denominator of 19989 is $80\Delta_2 = 879856 + 80x$. Also $330{\cdot}7 = \frac{1}{2}\delta + 25{\cdot}7$. Hence the mantissa of 20559 is

$$\begin{aligned} & 80\Delta_2 - 80x + 17481 + 2y\delta_1 + 25{\cdot}7y + 6p - 31{\cdot}14\xi \\ & = 81\Delta_2 + 10\tfrac{1}{2}\delta + 2y\delta_1 + 68 - 81x + 25{\cdot}7y + 6p - 31{\cdot}14\xi \end{aligned}$$

in which x is small (about $\pm{\cdot}2$). Also from the consideration of 19989 above

$$-3{\cdot}2 - 80x + 6p' - 30{\cdot}29\xi = 0.$$

Eliminating x

$$71 + 25{\cdot}7y + 6(p - p') - {\cdot}47\xi = M(153)$$

in which $p, p' < 1$ and ξ cannot exceed about 2.

It is not possible to satisfy this with $y = 0, \pm 1$, or ± 2.

With

$$\begin{aligned} y &= 3 \quad -6 + 6(p - p') - {\cdot}47\xi = 0, \\ y &= -3 \quad -5 + 6(p - p') - {\cdot}47\xi = 0. \end{aligned}$$

If, then, 20559 be a D_{11} line it belongs to one of $(\pm 3\delta_1) D(\infty)$, and is definitely excluded as a possible normal D_{11}.

The next line of higher frequency is the weak line (1) 20581·64. Its mantissa is $898040 - 31{\cdot}17\xi = 81\Delta_2 + 11\tfrac{3}{4}\delta + 7 - 81x + 6{\cdot}5p - 31{\cdot}17\xi = 81\Delta_2 - 11\tfrac{3}{4}\delta$ within error limits. This therefore passes the D_{11} test. If it is the actual D_{11} its weak intensity is due to the numerous displacements for $m = 1$. If it be taken as $D_{11}(1)$ with the previously mentioned (10) 38366·36, and the limit $D(\infty)$, the series formula is found to be

$$n = 51025{\cdot}29 - N \Big/ \left\{ m - {\cdot}988854 - \frac{090814}{m} \right\}^2.$$

The lines after $m = 2$ lie in the violet outside the observed region. To test them therefore recourse must be had to sounding, only the *e.u.v.* links have been used for this purpose. The results are given in the middle column of the subjoined table and exhibited in diagram (Plate 3). Details are given in the notes following the table. Lines were calculated down to $m = 15$ and tested. The result may be regarded as conclusive in establishing the series, as well as increasing confidence in the method of sounding—a confidence which reposes not on a single coincidence, but on the recurrence of a large number of successive ones. As will be seen the agreement between the calculated values and those found by sounding is remarkably close. The only doubtful case may be that for $m = 1$. If 20559 be taken for this the formula is only slightly changed and the agreement almost as good, the calculated value for $m = 3$ giving O−C $d\lambda$ = ·05 instead of ·00. It is excluded because it cannot belong to a diffuse series in which the limit is S (∞).

TABLE of Parallel XD Series.

After $m = 2$ the wave-numbers at the head of each set are those calculated from the formula and are not, as usual, enclosed in [ ]. The links are entered as attached to the respective observed lines. On the right of each is entered the O−C ($d\lambda$) calculated as if the error were on the observed line. They would be less—often considerably less if calculated on the series line itself.

m.	$(2\delta_1)$ D (∞).		D (∞).		$(-2\delta_1)$ D (∞).	
1	[20550·58], = $(-2\delta_1)$	31·06	(1) 20581·64	31·06	[20610·70], = $(+2\delta_1)$	
	e.(2) 27864·16	·06			e.(6) 27926·53	−·22
	u.(6) 24683·37	·06			u.(<1) 24749·35 $(-\delta_1)$	−·11
	v.(2) 24977·40	−·18			v.(5) 25036·99	·26
2	(2) 38345·76*	20·60	(10) 38366·36	21·20	[38387·61]	
	(1) 31033·15 (δ_1).e	−·02			(<1) 34251·50.u	·02
	(5) 33915·20 $(-2\delta_1)$.v	·00			(2) 33062·43.v	−·02
	or				v.(5) 35499·89.e	·12
	[38345·11]	21·25			(26940·63).e.u	−·05
	(<1) 34212·13.u	−·01			*or* [38390·24] = $(2\delta_1)$	
	(4) 33908·99 $(6\delta_1)$.v	−·09			(<1) 31076·51.e	−·07
3	44004·50 = $-\delta_1$	21·94	44026·48	21·76	44048·20	
	(<1) 36690·52.e	−·01	(1) 36712·34.e	0	u.(1) 40867·04.e	·01
	(1) 32262·65.e.v	−·02	(2) 39598·95.v	−·02	(2) 32304·23 $(-2\delta_1)$.e.v	·03
4	46557·99	20·24	46578·23	21·43	46599·66	
	(3) 34815·79.e.v	·00	[39264·13.e]		(1) 31971·46.2e	·00
			(<1) 34836·78.e.v	−·05	(1) 27836·93.2e.u	·17
			(35130·95).e.u	0		
5	47927·39	21·39	47948·78	21·45	47970·23	
	(<1) 28871·19.2e.v	·00	(<1) 33322·45.2e	−·18	(2) 36523·41.e.u	−·03
	(3) 33301·47.2e ?	−·20	(3) 36209·04.e.v	−·24	(3) 36229·78.e.v	−·13
			(1) 29189·92.2e.u	−·29		
6	48748·78	21·35	48770·13	21·35	48791·48	
	(1) 37300·91.e.u	·04	(<1) 34140·88.2e	·09	(4) 34163·39.2e	·09
	(3n) 29692·79.2e.v	−·02	(1) 29710·88.2e.v′	−·06	v.(3) 38589·92.2e	·09
			(2) 37320·68.e.u	·15		
			(5) 30005·34.2e.u′	−·01		
7	49280·00	21·25	49301·25	21·25	49322·50	
	(3) 37829·35.e $(-\delta_1)$.u	·00	(1) 34870·17.2e	·24	u.(1) 30559·83.2e	·14
	(2) 37534·58.e $(-\delta_1)$.v	·00	(3) 37849·82.e.u	·29		
	v.(1) 39082·95.2e	−·21	(<1) 37554·87.e.v	·32		
			(<1n) 30444·37.2e.v	·07		

* This is also $3e.\mathbf{F}_2(1)(\delta)$ the $90\Delta_2$ linked $\mathbf{F}_2$. It is probably not $(2\delta_1)$ D (2) but hides it.

TABLE of Parallel XD Series (continued).

m.	$(2\delta_1)$ D (∞).		D (∞).		$(-2\delta_1)$ D (∞).	
8	49643·23	**21·25**	49664·48	**21·25**	49685·73	
	(3) 37901·76.*e.v*	−·04	(<1) 35037·31.2*e*	−·08	*u*.(1) 39193·40.2*e* ?	−·19
	(<1) 35011·06		(1) 38217·68.*e.u*	−·03		
	15·47.2*e*	−·03				
	(4*n*) 35019·89		(1) 30607·90.2*e.v*	·04		
	(1) 30879·26					
	81·78.2*e.u*	·00				
	(1) 30884·31					
	(2) 30588·24.2*e.v*	−·13				
9	49902·63	**21·25**	49923·88	**21·25**	49945·31	
	(1) 38159·20.*e.v*	·09	(35295·68).2*e*	00	(1) 38203·51.*e.v*	−·03
	(3) 31139·94.2*e.u*	·13			*v*.(1) 39745·63.2*e*	−·04
10	50094·30	**21·25**	50115·55	**21·25**	50136·80	
	u.(2) 39598·95.2*e*	·02			(1) 38689·81.*e.u*	−·03
	(2) 31035·76.2*e.v*	·24	(1) 38666·62.*e.u*	·11	(<1*n*) 31076·84.2*e* (−δ_1).*v*	·07
	or *e.e* (δ_1).*v*	·0	(31355·28).2*e.u*			
11	50239·95	**21·25**	50261·20	**21·25**	50282·45	
	(3) 38791·71.*e.u*	·06	(1) 35637·28.2*e* (δ_1)	·03	(1*n*) 38835·86.*e.u*	−·04
	(4) 35610·00.2*e*	·14	(1) 38521·55.*e.v*	·16	*v*.(3) 40082·56.2*e*	·00
	u.(1) 39745·63.2*e*	−·04				
	(3) 31479·21.2*e.u*	−·07				
	(<1) 31180·63.*e.e* (−δ_1).*v*	·09			(3) 31224·82.*e.e* (−δ_1).*v*	·01
12	50353·24	**21·25**	50374·49	**21·25**	50395·74	
	u.(1) 38906·95.*e*	−·06	(2) 35745·69.2*e*	·05	(5) 35767·81.2*e*	−·02
	(1) 31592·28.2*e.u*	−·04	(1) 38632·72.*e.v*	−·02	(1) 31636·86.*e.e* (δ_1).*u*	−·02
13	50443·04	**21·25**	50464·29	**21·25**	50485·54	
	(3) 31680·16.2*e.u.*	·15	(1) 35836·52.2*e*	−·03	(<1) 38935·20	
					38·00.*e.u.*	·01
	(5) 31384·86.*e.e* (−δ_1).*v*	−·03	(1*n*) 38720·67.*e.v*	·10	(1) 38940·81	
			u.(2) 39969·62.2*e*	·02	(6) 31725·89.2*e.u* (δ_1)	·00
					(<1) 31431·51.2*e.v* (δ_1)	−·03
14	50515·50	**21·25**	50536·75	**21·25**	50558·00	
	(3) 38970·66.*e* (−δ_1).*u*	·02	(3) 31479·21.2*e.v*	·13	(1) 39107·56.*e* (−δ_1).*v*	·05
	(1) 35889·37.2*e* ?	−·16			(1) 35929·09.2*e*	·06
	(1*n*) 31755·50.2*e.u*	−·13			*u*.(1) 40063·94.2*e*	−·06
15	50574·68	**21·25**	50595·93	**21·25**	50617·18	
	(1*n*) 38835·86.*e* (δ_1).*v*	−·06	(2) 35963·72.2*e*	·30	(3) 35985·33.2*e* (−δ_1)	−·07
	(1) 35949·76.*e* (δ_1).*e*	−·07	*or* 2*e*(−δ_1)	−·05	(1*n*) 31755·50.2*e.u*	·03
	u.(3) 40082·56.*e* (δ_1).*e*	−·02	(1) 38852·76.*e.v*	·06	(<1) 31559·98.2*e.v*	·10
	(5) 31717·13.*e* (δ_1).*e.u*	−·05				

Notes on Table of D (∞).—*m* = 3. The *e* linked line has separation 1780·29 to (4*n*) 38492. *m* = 4. The *e* linked line is not observed, but it would be separated 1781·55 from (1*n*) 41045. 35130. There is no line to this, but it seems split into two as indicated in the scheme

(2) 35126·05					(1) 37720·05
	31·05	**1776·95**	(3) 36908·00	**816·17**	24·17
(2) 35136·05					(1) 37728·30

In which it may be noticed that the sum of the separations is 2593·12, the normal value. The order m is too large to definitely settle the sequence displacements if in them. They would be close to $\pm 18\delta_1$ for the first and $\pm 14\delta_1$ for the third. But if the connection is real a better explanation might be modification of the links, $\pm \delta_1$ on $e.u$ give the exact numerical agreement for the two D_{33} lines, and $e(\pm\delta_1)$, $u(\pm 2\delta_1)$ would give 5·86 where 6 is observed.

$m = 5$. 33322 has separations **1782·14** (4), **811·06** (4), the sum being normal.

$m = 6$. 34140 has separations 1782·92 to (2) 35923. Further, there is 30005 **1780·45** (1) 31785 **813·4** (1) 32599, in which the sum of the separations is closely normal. The $2e.u$ and $2e.v$ linked lines differ by 3·41 and 3·05 from the calculated, but are correct if the links $u' = u(-2\delta_1)$ and $v' = v(-2\delta_1)$ are taken. They are inserted since the corresponding normal links occur in other lines and the amounts are exact.

$m = 7$. The $2e$ linked 34870 has triplet separations **1784·44** (3), **813·73** (3), also the $2e.v$ has **1778·14** to intensity (<1). The $e.u$, $e.v$ linked lines are inserted although their difference as they stand are so considerable, because they all are exact if the links involved are taken as displaced ($-\delta_1$).

$m = 9$. There is no observed $2e$ linked line, but it seems to have split up into two, thus

(<1) 25293·16	**1778·20**	(4) 37071·36
	95·65	
(<1) 25298·14		

The two observed lines are numerically $D_{11}(9)(\pm\Delta_2)$.

$m = 10$. The $2e.u$ line is split up into two (<1) 31350·91, (<1) 31359·66, of which the former shows **1778·79** (4), **811·75** (5).

It should be noticed how many of the e and $2e$ linked lines introduce the modified triplet separations.

Notes on Table of $(\pm 2\delta_1)$D.—$m = 1$. The displacements on the limits would give 20560·40 and 20602·89. There is the already noted 20559 near the first discarded for $D(\infty)$ because it does not pass the multiple test. It serves better for $(-2\delta_1)$D, but would require at least an observation error $d\lambda = -\cdot 1$ which we have regarded as excessive. There are no observed lines connected by e, u, v links to either, nor near them. Those given in the table are, however, very clear. They make the sequence term displaced $2\delta_1$ from that of the D series, viz., $-2\delta_1$ for $+2\delta_1$ on limit, and $+2\delta_1$ for $-2\delta_1$ on limit, *i.e.*, interchange of $\pm 2\delta_1$ for $\mp 2\delta_1$ on limit. In the third series the e and v linked lines differ respectively by $e + 1\cdot 74$ and $v - 1\cdot 70$. They form therefore a parallel inequality, and are good evidence in spite of the considerable difference 1·7.

$m = 2$. The limit separation should give for the first series 38345·11, and it apparently exists although possibly it belongs to a series commencing with 20559. There is clear evidence of a set corresponding to $m = 1$, shown by sounding and giving a sequence displacement of $-6\delta_1$. In the third series the line depending on the limit change alone would be 38387·56. This gives links $u + 2\cdot 85$ and $v - 2\cdot 87$ with the lines indicated or a parallel inequality. They are explained by $\pm 2\delta_1$ displacements in the sequent.

$m = 3$. Here δ_1 as a displacement in the sequent produces a separation of ·5. The sequent displacement in the first series is therefore $-\delta_1$ and $+\delta_1$ in the third. The line 32304 however shows $-2\delta_1$.

$m = 7$. Modified links $e(-\delta_1)$ are introduced. This is supported by the fact that the two lines given differ respectively by 2·37, 2·32 from values given by normal e, whilst the modification of e by δ_1 produces 2·32. The double example and exact difference give weight to the suggestion.

$m = 10, 11$. Similarly the modified e makes exact agreement, and they enter in a corresponding way and in both series.

We can now use this series to test the question as to the existence of parallel series depending on $(\pm 2\delta_1)$ D (∞). This does not mean that the sequences must be the same in each. In fact it is to be expected that there may be a concomitant change in them also, but they can only differ by a few multiples of the oun. The important point to notice is that for large values of m, the effects of such differences become negligible and the observed separations from the standard D should approximate to $\pm 21\cdot 25$, which is that due to $2\delta_1$ on the limit. It will not be necessary to go into such a full discussion as in the standard series with D (∞), except for the first three sets, where the evidence for changed sequence terms with displaced limits is conclusive and important. The lists and sounders are given in the same table as for the standard series, in the first and third columns respectively. The considerations adduced enable us to feel that the ground is safe in recognising that parallel series depending on displacements on the limit D (∞) = S (∞) really exist. The evidence does not depend on a single numerical coincidence of a line, or of a line found by sounding, but on the fact that these numerical coincidences appear for so many sounding links in all the 15 sets tested. This is not affected by the high probability that several of the sounded lines are chance coincidences. This result then gives more confidence to the method partially applied above, in the application of the law that the D sequences must have mantissæ which differ by multiples of the oun from multiples of Δ_2, or in other words must be themselves multiples of the oun. This method consisted in testing certain lines to see whether by using the displaced limits $(y\delta_1)$ D (∞)—now seen to really exist—the above relation holds.

The evidence seems to show that the typical lines—D (∞) = S (∞)—for $m = 1$ have been much affected by displacement effects, and that consequently the intensities of the normal lines themselves are much diminished. Although this is some disadvantage, it will be well to attempt here to get some insight into the complete satellite system for the first two orders.

We have seen that 19989 belongs to this normal set with a mantissa $= 80\Delta_2$ and that 20581 satisfies the condition necessary for a D_1 line with this. The difference of their mantissa (see below) is $29\frac{1}{4}\delta$. If they are of the D_{12}, D_{11} types, as is indicated by the fact that the first belongs to a doublet and the second stands by itself, a triplet satellite set should be expected whose first line D_{13} is separated from the D_{12} by about three-fifths that of D_{12} from D_{11}. Its mantissa should therefore be about $18\delta = \Delta_2$ less. This would mean a line about 19623 forming the first line of a triplet. No line is observed here. There are, however, lines at (1) 19602·66 and (3) 19632·44 of which 19602 passes the suitability test for a normal D line, and the other does not. The mantissa of 19602 is $79\Delta_2 - \delta$, *i.e.*, 19δ behind that of 19989 and rather too large. On the other hand the problematic 19623 may be too weak, in which case the corresponding D_2, D_3 lines which should be stronger might be observable. The D_2 line should be about 21400. We do find this, in fact, with triplets of a kind. The whole set of these lines can then be arranged as follows :—

(1) 19602·66	**1774·45**	(3n) 21377·11	**821·04**	(2) 22198·15
δ				
[19623·05]	**1777·90**	(2) 21400·95	**809·53**	(2) 22210·48
18δ				
(1) 19989·72	**1780·27**	(6n) 21769·99		
29$\frac{3}{4}\delta$				
(1) 20581·64				

in which it may be noticed that in the first the sum of the separations is the same as 1780·19+815·30, *i.e.*, the modified $\nu_1+\nu_2$, and in the second $809·53 = 815·20-5·67$ whilst a δ_1 displacement on the sequent produces a change of 5·14. With the limit $51025·29+\xi$ the mantissæ of the d sequents are—$d\lambda = ·05p$—

$$19602, \quad 868240-29·73\xi+6p = 79\times 10998·2-618+6p-29·73\xi$$
$$19623, \quad 868846-29·75\xi+6p' = 79\times 10998·2-12+6p'-29·75\xi$$
$$19989, \quad 879853-30·29\xi+6p = 80\times 10998·2-3+6p-30·29\xi$$
$$20581, \quad 898040-31·17\xi+6·5p = 80\times 10998·2+18184+6·5p-31·17\xi$$

in which p lies between ± 1 and p' depends on error of extrapolated line, and may be >1. Writing as before $10998·2 = \Delta_2-x$ these become respectively

$$79\Delta_2-\delta-7+6p-79x-29·73\xi$$
$$79\Delta_2 \;-12+6p'-79x-29·75\xi$$
$$80\Delta_2 \quad -3+6p-80x-30·29\xi$$
$$80\Delta_2+29\tfrac{3}{4}\delta+7+6·5p-80x-31·17\xi$$

The multiple rule requires that the last four terms in each expression must vanish. This is clearly possible for small values of p, say $<\frac{1}{2}$, and a single relation between x and ξ, say $8x = -3\xi$.

As a further test of the reality of the extrapolated line 19623 linked lines may be sought for. There is none for $+e$, but lines are found close for u, v, $e\pm v$, viz.,

(6) 23754·27	**1778·40**	(3) with $u-2·1$	19623·19
($<1n$) 24053·37		,, $v+2·1$	19623·16
(2) 24509·92		,, $e-v$	19623·82
(2) 31362·71		,, $e+v$	19620·66

Taking 19623 as D_{13}, the satellite differences are $29\frac{3}{4}\delta$ and 18δ. Since $18\times 5 = 90$ and $29\frac{3}{4}\times 3 = 89·25$, these separations are very closely in the ratio 5 : 3 in accordance with the rule for the known triplet series in other groups. The triplet set 19602 appears somewhat anomalous. The middle line appears to have the modification so common in the middle set of a triplet, but in the opposite direction to the usual one, *i.e.*, the sequent to the second line is $d(-\delta_1)$ instead of $d(\delta_1)$.

The preceding considerations have shown the existence of a complete set of D (1) satellites. But further, the D suitability test shows that 20305·60 may be a D_1 line with normal limit and that with it may go two sets of extrapolated triplets in which the mantissa of the first is a multiple of Δ_2. They are

$$\begin{array}{llllll} [16013\cdot45] & \left\{\begin{array}{l}\mathbf{1777\cdot90}\\ \mathbf{1784\cdot83}\end{array}\right. & \begin{array}{l}(1)\ 17791\cdot35\\ (1)\ 17798\cdot28\end{array} & \left.\begin{array}{r}\mathbf{816\cdot44}\\ \mathbf{809\cdot51}\end{array}\right\} & (8)\ 18607\cdot79 \\ [17725\cdot39] & \mathbf{1777\cdot90} & (3)\ 19503\cdot29 & \mathbf{816\cdot96} & (4)\ 20320\cdot25 \\ (1)\ 20305\cdot60 & & & & \end{array}$$

The first set, however, involves for the D_{33} the line 18607, which belongs to the 1864 F set discussed above, and is stronger than we should expect. Provisionally we will suppose the true D_{33} is hidden by the F line. The line 20320 has been already adduced (p. 383) as showing connection with the 1864 separations in a similar way to 20312. The mantissæ of the D_1 lines are (p' indicating extrapolated lines), $769890 = 70 \times 10998\cdot2 + 16 + 3\cdot2p' - 25\cdot2\xi = 70\Delta_2 + 16 + 3\cdot2p' - 70x - 25\cdot2\xi$, $814815 + 4\cdot2p' - 27\cdot25\xi$, and $889493 + 6p - 30\cdot75\xi$. They differ successively by 44925 $= 4 \times 10998\cdot2 + 1\frac{1}{2}\delta + 16 + 4\cdot2\,(p'_2 - p'_1) - 2\cdot0\xi = 73\frac{1}{2}\delta + 16 + 4\cdot2\,(p'_2 - p'_1 - 4x - 2\cdot0\xi$ and $74679 = 122\frac{1}{4}\delta - 17 + 6p - 4\cdot2p'_2 - 7x - 3\cdot5\xi$. The conditions are satisfied within error limits that the mantissa of the extreme satellite is $70\Delta_2$, and that the differences for the satellites are due to $122\frac{1}{4}\delta$ and $73\frac{1}{2}\delta$. Since $122\frac{1}{4} \times 3 = 366\cdot7$ and $73\frac{1}{2} \times 5 = 367\cdot2$, the normal ratio of satellite separations is again reproduced. The evidence is clear, therefore, for two groups of normal diffuse series depending on $79\Delta_2$ and $70\Delta_2$ respectively.

The fact that 20320 is connected with 1864 in the same way as 20312, that the ν_2 separation is not good, and that we should expect a doublet here in place of a triplet rather point to the supposition that it is not a member of the set. For the D_{13} line the sequent is $51025\cdot29 - 16013\cdot45 = 35011\cdot84$, on this the oun displacement produces a change of 6·04 which accounts for the modified $\nu_1 = 1784\cdot83$ in the usual way. The mantissæ of the two D_{11} groups 20581, 20320 differ by 14δ.

It will be sufficient here to attempt the allocation of the corresponding satellites for $m = 2$ only. They should be at about the same oun multiple distance from the D_{11} line as for $m = 1$. The $D_{11}(2)$ has been taken as 38366 with denominator 1·943447. Taking the first group, the satellites should have denominators about $29\frac{3}{4}\delta$ and $47\frac{3}{4}\delta$ less than this. These would correspond to 38208·61 and 38111·69. With regard to the first the lines in this neighbourhood are (1) 38199·58, (1) 38203·51, (1) 38217·68, of which the first and third are respectively 9·03 less and 9·07 greater than 38208, and suggest equal displacements on either side. They are found to correspond to $31\frac{1}{2}\delta$, 28δ from D_{11}, or $\pm 7\delta_1$ on either side of 38208, with errors

$d\lambda = \pm \cdot 02$, whilst 38203 corresponds to $30\frac{3}{4}\delta$ with $d\lambda = \cdot 01$. They should form portions of doublets with lines at 39977·48, 39995·58. None have been observed, but it must be remembered that these are close to the end of the observed region where only the stronger lines would be seen. Sounding with $-(e+u)$ we find lines (1) 28530·91, (3) 28548·26 respectively, $e+u-\cdot 71$ and $e+u+\cdot 04$ behind the expected D_{22} lines showing no lines at ν_2 ahead, which should be visible if they existed. We may, therefore, conclude the two lines in question belong to a doublet set. With regard to the suspected D_{13} line (38111) we find in this neighbourhood

(1) 38134·61	**1780·23**	[39915·84]	**815·20**	(1*n*) 40731·04
(1) 38129·37	**1778·21**	(1) 39907·58	**810·36**	(1) 40717·94
(1*n*) 38108·60				

Calculation shows that the first lines of these sets give with $D_{11}(2)$ separations depending on displacements $43\frac{1}{2}\delta\,(d\lambda = 0)$, $44\frac{1}{2}\delta\,(d\lambda = \cdot 01)$, $48\frac{1}{4}\delta\,(d\lambda = -\cdot 03)$. All that can be said is that these lines may be the required satellite sets. As, however, a displacement of $\frac{1}{2}\delta_1$ only produces a change of about ·05 in λ it is not possible to get any certainty. It is further possible that some of the set may belong to parallel series. For instance $38108 = (2\delta_1)\ 38129$. The first two are, however, so close $d\lambda = \cdot 00$ and ·01 that they are entered as D_{13} lines.

For the second group 20305 is a 14δ displacement from 20581. We should expect the corresponding $m = 2$ line about the same displacement from 38366. This is satisfied by the line (1) 38292·32 giving 14δ with $d\lambda = -\cdot 01$. There is also a doublet (1) 38285·86, **1778·08**, (1) 40063·94 with normal separation and displaced $15\frac{1}{4}\delta$ with $d\lambda = \cdot 00$. The first D (1) is displaced $122\frac{1}{2}\delta$ from its D_{11} line 20305 and the second by an extra $73\frac{1}{2}\delta$. If the $D_{11}(2)$ line is taken as 38285 the calculated line with $122\frac{1}{2}\delta$ is 37611·59. There is no line here, but there is a doublet, which with this line may be written

$$\left.\begin{array}{ll} (2)\ 37606{\cdot}15 & \mathbf{1780{\cdot}83} \\ [37611{\cdot}59] & \mathbf{1775{\cdot}39} \end{array}\right\}\ (1)\ 39386{\cdot}98$$

The mean of these two separations is 1778·11 or the normal ν_1. We have here a parallel inequality due to $2\delta_1$ displacement in 39386 and an extra $2\delta_1$ in 37606, or 37606 is $123\frac{1}{2}\delta$ ahead of D_{11}.

Again calculating the $73\frac{1}{2}\delta$ displacement on 30706 the line should be 37174·40. It is not observed but there is a triplet close to it showing a similar inequality to the former. It is

$$\left.\begin{array}{ll} (1)\ 37159{\cdot}39 & \mathbf{1774{\cdot}38} \\ (1n)\ 37166{\cdot}43 & \mathbf{1781{\cdot}42} \end{array}\right\}\ (1)\ 38940{\cdot}81 \quad \mathbf{813{\cdot}67} \quad (<1)\ 39754{\cdot}48$$

The mean of the two first separations is 1777·90 or exact ν_1 and we have an exact parallel inequality. The inequality is due to two successive $1\frac{1}{4}\delta$ displacements, and 37159 is exactly $2\frac{1}{2}\delta$ extra on the calculated 37174, or 76δ from the first satellite.

The first two orders of the two groups are represented in the following scheme in which the satellite separations are given as sequent displacements from d_{11}:—

The $79\Delta_2$ group. The $70\Delta_2$ group.

$m = 1.$

The $79\Delta_2$ group				The $70\Delta_2$ group		
[19623·05]	(2) 21400·95	(2) 22210·48		[16013·45]	(1) 17791·35	(8) 18607
$47\frac{3}{4}\delta$		48δ		$195\frac{3}{4}\delta$		
(1) 19989·72	(6n) 21769·99			[17725·39]	(3) 19503·29	
$29\frac{3}{4}\delta$				$122\frac{1}{4}\delta$		
(1) 20581·64			14δ	(1) 20305·60		

$m = 2.$

The $79\Delta_2$ group				The $70\Delta_2$ group		
(1) 38129·37	(1) 39907·58	(1) 40717·94		(1) 37159·39	(1) 38940·81	(<1) 39754·48
$44\frac{1}{2}\delta$	$44\frac{1}{2}\delta$	$46\frac{1}{4}\delta$		$199\frac{1}{2}\delta$		
				(1n) 37166·43		
				197δ	$198\frac{1}{4}\delta$	$198\frac{3}{4}\delta$
(1) 38199·58						
$31\frac{1}{2}\delta$				(2) 37606·15	(1) 39386·98	
(1) 38217·68	(3) 28548·26.e.u			$123\frac{1}{2}\delta$	$122\frac{1}{2}\delta$	
28δ	28δ					
(10) 38366·36			14δ	(1) 38285·86		

Without dealing with the whole of the material at disposal we will illustrate its application by considering in more detail the portion of the spectrum given on p. 382 in which the majority of the lines undoubtedly belong to D (1) systems. It is to be noticed that the effectiveness of the method in the present case depends on the facts, (1) that the observation errors do not exceed $d\lambda = {\cdot}05$, and (2) that with $m = 1$ it is consequently possible to determine the values of the mantissæ to within 6 units in the sixth significant figures, whilst a displacement of one oun in the sequent produces a change in λ of the order 1·2, or twenty-four times the maximum observation error. The limit 51025 being supposed displaced by $y\delta_1$ becomes $51025{\cdot}29 - 10{\cdot}62y + \xi$. The mantissæ of the sequences are then calculated with this limit, and expressed in terms of Δ_2, δ_1, x and p where $\Delta_2 = 10998{\cdot}2 + x$, and $-p$ is the ratio of the observation error to the maximum ($d\lambda = {\cdot}05$). The series more fully discussed above is definitely taken as depending on the limit $y = 0$. In other words the mantissa of 19989 is exactly $80\Delta_2$ which condition requires, writing q for its p,

$$3{\cdot}2 + 30{\cdot}29\xi - 6q + 80x = 0,$$

and gives a relation between ξ and x. The term in x in each mantissa is then eliminated by means of it. There can be little doubt about the allocation of 19889, but even should it be in error the doubt does not affect the argument as to the relative displacements of the different lines, as $y\delta_1$ denotes the displacements relative to 19889. The final results are given in the following table :—

1	19880	$80\Delta_2+2y\delta_1-5\frac{1}{2}\delta+66+14y+\cdot2\xi+6(q-p)$	**−4,**	11	**−5,**	−4
2	19942	$80\Delta_2-\delta_1$ with $y=4$	**4,**	11		
3	19959	$80\Delta_2+2y\delta_1-\delta-61+16y+\cdot05\xi+$,,	**4,**	3		
4	19989	$80\Delta_2$ with $y=0$	**0**			
5	20017	$80\Delta_2+2y\delta_1+1\frac{1}{4}\delta+77+17y-\cdot05\xi+$,,	**−4,**	9	**4,**	−8
6	20021	,, $+1\frac{1}{2}\delta+52+17y-\cdot05\xi+$,,	**−3,**	1	**6**	
7	20029	,, $+2\delta-31+17y-\cdot05\xi+$,,	**2,**	3		
8	20041	,, $+2\frac{1}{2}\delta+50+17y-\cdot07\xi+$,,	**−3,**	−1	**6,**	0
9	20080	,, $+6\frac{1}{2}\delta-2+18y-\cdot11\xi+$,,	**0,**	−2		
10	20107	,, $+10\delta-33+18y-\cdot17\xi+$,,	**2,**	3		
11	20305	,, $+15\frac{3}{4}\delta+14+21y-\cdot46\xi+$,,	**0,**	14	**−1,**	−7
12	20312	,, $+16\delta+82+21y-\cdot47\xi+$,,	**−4,**	−2	**3,**	−8
13	20320	,, $+16\frac{1}{2}\delta+9+21y-\cdot47\xi+$,,	**0,**	9		
14	20333	,, $+17\frac{1}{4}\delta-50+22y-\cdot50\xi+$,,	**2,**	6	**−5,**	−7
15	20443	$81\Delta_2+2y\delta_1+4\frac{3}{4}\delta-11+24y-\cdot30\xi+$,,	**0,**	−11	**−6,**	−2
16	20454	,, $+5\frac{1}{4}\delta+41+24y-\cdot30\xi+$,,	**−2,**	−7	**5,**	8
17	20467	,, $+6\delta-12+24y-\cdot45\xi+$,,	**0,**	−12	**−6,**	−3
18	20470	,, $+6\delta+76+24y-\cdot45\xi+$,,	**−3,**	4	**3,**	−5
19	20500	,, $+7\frac{1}{2}\delta+71+26y-\cdot48\xi+$,,	**−3,**	−7	**3,**	−4
20	20529	,, $+9\delta+80+26y-\cdot48\xi+$,,	**−3,**	2	**3,**	5
21	20559	,, $+10\frac{1}{2}\delta+71+26y-\cdot48\xi+$,,	**−3,**	−7	**3,**	−4
22	20581	,, $+11\frac{3}{4}\delta+10+26y-\cdot51\xi+$,,	**0,**	10		
23	20596	,, $+12\frac{1}{2}\delta+11+26y+\cdot53\xi+$,,	**0,**	11		
24	20636	,, $+14\frac{1}{2}\delta+36+26y-\cdot62\xi+$,,	**−1,**	13	**4,**	−13

Of the above (14, 18) must be set aside at once: (14) because it belongs to the triplet set linked by $2e$ to the parallel set F (1), and (18) because it is F_2 (2). It may also be noted that neither have the prevalent separation 1780 to lines of higher frequency. Of the numbers on the right of the list, those in thick type give the values of y which bring the outstanding differences to the corresponding number in ordinary type. These differences must be due to errors either in ξ or observation. Since ξ is small, it is seen that they must be capable of annulment by the observation errors $6(q-p)$, and must, therefore, at the maximum be <12. The smallness of ξ can be seen from the following considerations which connect it with the corresponding ξ (say ξ') for the 1864 F series. The limit of the F is $30725\cdot30+\xi'$ with mantissa $889322-30\cdot74\xi'$. It is a $d(1)$ sequent. If 19989 is a D line with $y=0$, its sequent is $51025\cdot29+\xi'-19989\cdot72=32035\cdot57+\xi'$. The mantissa of this is $879853-30\cdot29\xi'+6q$. Both being d sequents must differ by a multiple of the oun. Their difference is $9469+30\cdot29\xi-30\cdot74\xi'-6q$ and $15\frac{1}{2}\delta=9470$. Hence

$$30\cdot29\xi-30\cdot74\xi'=1+6q,$$

or,

$$\xi=1\cdot015\xi'+\cdot03+\cdot2q.$$

Thus $\xi=\xi'\pm\cdot3$. Since the 1864 limit is determined as the mean of F and **F** series its value is subject to a very small uncertainty and ξ' will be a small fraction.

Consequently in the above list the value of ξ has no importance in settling the order of the displacements.

The whole of the foregoing argument is based on the constancy of Δ_2 for all series. This matter has been referred to in the introduction in which the question of what is to be understood by the atomic weight was brought up. The accuracy in the determination of the oun is rendered so great by the constitution of the d and f sequences, that the mass of the electrons connected with the nucleus affect it. It may be interesting to illustrate the considerations there adduced by a concrete example. The example we will take is (1), as the result may possibly throw some light on a difficulty which will appear later. The line 19880 is seen to require a displacement of $-\delta$. Let us determine the transfer of electrons in order to produce a change by one oun. Suppose this transfer changes x to $x+x'$. This means that the mantissa as represented in the list must be diminished by $80x'$, by putting $y = -1$.

Hence

$$x' = \frac{52}{80} = \cdot 65. \qquad d\delta = \cdot 036.$$

The change in the number of electrons (see p. 342) $= 925 \times \cdot 036/611 \times 130 = 1\cdot 97 = 2$. The addition of eight electrons to the mass acting in our standard case, would render 19980 a possible D line with limit $(-\delta_1)\,\mathrm{D}(\infty)$ instead of a possible one with $(-\delta)\,\mathrm{D}(\infty)$.

The preceding treatment of the material is only a first step towards unravelling the intricacies of these D systems. An exhaustive treatment is here impossible, and would involve the consideration of other data—the triplet separations, linkages, similarities of arrangement, dependence on F series and so on. All that can be done is to give a few illustrative instances and to bring into prominence certain problems whose solution in the future may be of extreme importance.

(1) The line 20021 is given as requiring the displacement $-3\delta_1$ in the limit, and in this case the mantissa is $80\Delta_2$. This displacement increases the limit 51025 by 31·88, and the resulting ν_1 should be greater by $3 \times \cdot 535 = 1\cdot 60$ and $= 1779\cdot 50$. As a fact the line (see p. 382) forms a doublet with separation 1779·51, and with intensities 1, 10, in the proper order for a D_{12} set. All the tests support each other. Again in the doublets

(7) 20500·13	**1784·68**	(2) 22284·82
(8) 20559·08	**1785·64**	($<1n$) 22344·72

the same displacements (or of $+3\delta_1$) are indicated. If we suppose these modified values of ν_1 produced in the same way as in Kr by a relative displacement in the sequent, the oun in this case alters the separation by 4·91. The modified ν_1 therefore becomes $1779\cdot 50 + 4\cdot 91 = 1784\cdot 41$, or within error limits of the observed value for 20500, but too small for 20559. That for 20559 corresponds to an extra displacement of $-2\delta_1$ on the limit, or $-5\delta_1$ in all, which is quite inadmissible on the qualifying test.

The mantissæ of both lines differ by an exact 3δ. This is the natural conclusion, viz., same limit, sequents differ by 3δ, but then the value of 1785 remains unexplained, and we should not expect to find two D_{11} lines of the same group so close together. A possible explanation is to allot 20559 to the alternative displacement of $3\delta_1$ which gives $\nu_1 = 1776{\cdot}30$ and to take $2\delta_1$ in the sequent of the second line. This would give a modified separation $1776{\cdot}30+2\times 4{\cdot}91 = 1786{\cdot}12$ or ·48 greater than that observed. It may be noted that as they stand the four lines (18 to 21 of list) requiring $-3\delta_1$ displacement have their sequence mantissæ equally spaced by $1\frac{1}{2}\delta$. The line 20041 also comes into the system with an exact δ sequence displacement from 20021.

In an analogous position to the 16013 line in the D (∞) set appears the $(-3\delta_1)$ D (∞) set

[16044·05] **1785·37** (1) 17829·41 **810·13** (2) 18639·54

in which the 16044 is extrapolated from 18639 by the $\nu_1+\nu_2 = 2595{\cdot}5$. As the oun displacement on the sequent here produces 6·06, the modified ν_1 should be $1779{\cdot}50 +6{\cdot}06 = 1785{\cdot}56$, and the modified $\nu_2 = 816{\cdot}02-6{\cdot}06 = 809{\cdot}96$ which agree with the observed. We may regard the 16044 as subject to the observation errors of 18639, say $d\lambda$. The mantissa of 16044 is $769863 = 70\ (10998{\cdot}04-{\cdot}36d\lambda)$, or $70\Delta_2$ as in the case of D (∞).

With the examples here given the arguments from the capability test and the observed separation agree. Complete sets for $(-3\delta_1)$ D (∞) series are obtainable but are not here adduced.

(2) The line of 19942 has already been considered on (p. 396) as a $(4\delta_1)$ D (∞). It again is a case where the capability test and ν_1 separation both point to the limit displacement of $4\delta_1$ or δ. The list also gives 19959 as requiring the same displacement. But it is $u.F_4(3)$ of the series below and shows a forward link 1864 in analogy with these F series.

(3) The line 20312 is clearly of special importance. It gives the source of the 1864 separations. As is seen the capability test reqires $-4\delta_1$ but there are many difficulties in the way of properly placing it. It forms part of a strong doublet (6) 20312 **1783·72** (5) 22096 in which the intensities are not in normal order. If 1783·72 is the usual modified ν_1, the sequence oun displacement is here 4·98 and the true $\nu_1 = 1783{\cdot}72-4{\cdot}178 = 1778{\cdot}74$. This differs only by ·13 from that proper to $(-2\delta_1)$ D (∞), instead of to the $-4\delta_1$. The 1864 separations are greater than in the F series, being 1865·51, 832·13 against those found from the F above 1865·10, 829·64. The latter were based on the limit 30725·30 on which a displacement of δ_1 produces ·45 in 1864 and ·22 in 829. There is clearly here some triplet modification as the ratio of the separations is not correct. With 1865·51 should go 830·15 or with the given sum 1866·87, 830·77. Now $-3\delta_1$ on the limit produces 1865·45 and increase the limit itself to 30740·21. This should be the d sequent for 20312. If so the D limit should

be $30740 \cdot 21 + 20312 \cdot 70 = 51052 \cdot 91$ which is 27·62 above the normal and has no reference to the oun displacement. If 20312·70 corresponds to the $-4\delta_1$ indicated, the limit is 51067·79 and the d sequent is the difference, or 30755·09. This is the source of the 1864 separations. It is 22·79 greater than the limit 30725·30 of the F series considered in connection with this series and which gave 1864·10 and 829·64. This limit corresponds to a sequent $6\delta_1$ less displacement or 29·82 greater value which is practically exact. This present sequent will therefore increase 1864·10 by $6 \times \cdot 45$ to 1866·70 and 829·64 by $6 \times \cdot 22$ to 830·96. The new sum is 2697·66, as against the observed 2697·44 in remarkably close agreement. This then supports the $-4\delta_1$ indication of the capability test. A $-3\delta_1$ displacement would make the sum = 2697·00—which though a worse agreement may yet be within observation errors. The same $-3\delta_1$ would make the modified $\nu_1 = 1779 \cdot 50 + 4 \cdot 97 = 1784 \cdot 47$, or ·72 greater than the observed. One is almost tempted to suspect here an error greater than the ordinary observation error. An error of ·36, $d\lambda = -\cdot 09$, would make all three tests agree in allotting 20312 to the $-3\delta_1$ set, and would bring it into the group of lines considered in Case 1.

It should be noted that this 20312 line is the line with wave-length 4922 referred to by LIVEING and DEWAR for its peculiar behaviour (see p. 350).

(4) The separations of the triplet 19880 are 1778·42, 815·30. They suggest the separation $-\delta_1$ to which belong 1778·43, 815·42. The capability test gives $-3\delta_1$ and the two are incompatible. In the D series given below the calculated limit from the first three lines give the limit as 51045·37 or 20·08 above the normal. With the uncertainty in a limit found in this way this is a displacement of $-2\delta_1$ which gives 21·25. Further there seems some evidence to show that this line with 20021, 20041, 20312, 20500, 20559 belong to one D (1) group of lines. The evidence consists in the existence of parallel F sets showing displacements equal to the separations of these lines. In other words these lines are D sets with the same limit. All these tests mutually exclude each other. How can their indications be reconciled? I suggest

(1) The limit for the line is $(-\delta_1)$ D(∞) and the observed ν_1, ν_2 are thus explained.
(2) The capability test is met by the transfer of six electrons indicated above.
(3) It is not the first line of the series in question.
(4) The F separations will be found to offer a natural explanation.

In what has preceded an attempt has been made to allocate normal D series, but they are clearly not the strongest sets. In my first attack on the D, F problem the procedure adopted was to take the 19880 triplet as a clear satellite set, and attempt by the application of RYDBERG's table to find a series for this satellite series. The three sets found were :—

(1) 19880	**1778·42**	(5) 21659·14	**815·30**	(10) 22474·44
[37888·59]	**1777·90**	(1) 39666·49	**814·42**	(2) 40480·91
[43801·13]	**1777·74**	[45578·87]	**818·18**	[46397·05]

The second set is near the end of the observed region, and the fainter D_1 line is extrapolated by the normal ν_1, this will introduce a small possible error. The third set is wholly outside the observed region and was obtained by sounding. The data are as follows, the lines used being regarded as linked by e to the first triplet 36487, ...

[43801·13]	$d\lambda$	[45578·87]	$d\lambda$	[46397·05]	
(1) 36487·03.e	·00	(1n) 38264·77.e	·00	(1) 39082·95.e	·00
(1) 29175·18$e.e$ (δ_1)	·00	(1) 30949·88.2e	·08	(<1) 31768·93.2e	−·01
(1) 39372·71.v	−·02	(1) 41155·14 $v(2\delta_1)$	·03	(<1) 34659·40.$e.v(2\delta_1)$	−·07
(1) 39666·49.u	·09	(1) 33841·74.$e.v(2\delta_1)$*	·12		
(3) 32351·67.$e.u(-\delta_1)$	·10	(1) 34126·78.$e.u(-2\delta_1)$*	−·12		
(<1) 24749·35.$e.e(\delta_1).v(\delta_1)$	−·04	(1) 26814·18.2$e.u(2\delta_1)$	·03		

The formula obtained from these is

$$n = 51045{\cdot}37 - \mathrm{N}\Big/\left\{m + {\cdot}898460 - \frac{{\cdot}022504}{m}\right\}^2.$$

This was tested to $m = 13$, with good agreement with the exception of $m = 4$. All are in the ultra-violet and require sounding. As the evidence of the efficacy of sounding already adduced may be regarded as convincing, there is no advantage in giving further details of the results, especially as no additional conclusions are based on this formula.

XF.—We are to look for parallel F sets with the same separations as those of the D satellites. Starting with the triplet 19880 as an undoubted D satellite triplet it was then attempted to determine the others, by picking out those lines of larger wave-length whose mantissæ differed from that of 19880 by oun multiples. The data are contained implicitly in the list on p. 407, independently of the form into which the mantissæ are there thrown. It is clear that the above condition is satisfied by all those which show unsatisfied remainders of the same magnitude as that of 19880—*i.e.*, 66—within error limits of, say, ± 6. The selected lines were 19880, 20021, 20041, 20312, 20500, 20559. We have shown above that the second, third, fifth and sixth of these satisfy the capability test for $-3\delta_1$ displacements on the limits, and have also found some evidence that 20312 also belongs to this set, although perhaps with a somewhat excessive observation error. It has also appeared that 19880 cannot belong to this set, but it has so happened that the satellite separations from it reproduce themselves in the F series sufficiently well as to serve for identification. We shall, therefore, use this allocation. The selected lines give separations from 19880 respectively of 140·94, 161·01, 431·98, 619·41, 679·36. In these 19880 was treated as if it were a fifth satellite to 20559 and the notation adopted of F_6, ... F_1 series. Such series were found, and in what follows I shall chiefly confine consideration to them leaving aside for the present purpose, with one or two exceptions, the very numerous other groups which exist. It would not be advisable to suggest a definite notation

* Parallel inequality.

for these groups until the whole system of D and F series is placed on a secure and comprehensive foundation. The above notation, therefore, is only to be regarded as one which in this communication serves to identify certain of those groups here more specially discussed.

The material was treated by the same method as in Kr, and search made for the above separations in the region where F (2) lines should be expected. From the lines thus obtained the actual F lines were sought for, bearing in mind the large sequence displacements so common in the low orders. In only three cases—those for $F_{2, 3, 6}$—were suitable undisplaced lines found for $m = 2$, and of these only F_3, F_6 gave observed lines for $m = 3$. They were—separations from F_6—

m.	F_6.		F_3.		F_2.
2.	(8) 18812·55	**432·46**	(8) 18380·09	**619·82**	(1) 18192·73
3.	(1) 24253·30	**432·49**	(<1) 23820·81		
5.	(2) 28108·52	**432·00**	(1) 127676·52		

For $m = 4$, F_3 showed no undisplaced line, but in good position for $m = 5$ there were lines for F_6, F_3. From the three lines for F_3 the following formula was found

$$30740{\cdot}17+\xi-\mathrm{N}\Big/\left\{m+{\cdot}986181-622\xi-\frac{{\cdot}014730-1003\xi}{m}\right\}^2.$$

We can at once apply two tests to this as to fulfilling the conditions for F series. The limit 30740 must be a d-sequent, *i.e.*, must differ by oun multiple displacement from some other known d-sequent, and the first f-sequent must depend directly on a multiple of Δ_2. We already have a very accurate d-sequent found as the limit of the 1864 series, viz., $30725{\cdot}30+\xi'$. On this a δ_1 displacement produces a change of 4·97 so that $-3\delta_1$ produces $30725{\cdot}30+\xi'+3\times 4{\cdot}97 = 30740{\cdot}21+\xi'$, and this condition is accurately satisfied by $\xi_1 = \xi'+{\cdot}04$ where ξ_1 belongs to the present case. For the second test the mantissa of $f(2)$ is

$$978816-121\xi_1 = 89\,(10997{\cdot}93-1{\cdot}36\xi_1) = 89\,(\Delta_2-{\cdot}27-{\cdot}36\xi_1-x)$$

which again is seen to satisfy the test exactly. The satisfaction of these two conditions must give full confidence as to the correctness of the allocation of the F_3 lines.

From this formula lines were calculated from $m = 4$ to 27 and the other sets allocated by their corresponding separations. They are not reproduced here, however, beyond $m = 15$, partly because they are only of importance in a systematic arrangement of the X spectrum, and partly because the lines to be identified are so close that it becomes a matter of extreme difficulty to allocate them correctly. For instance, the calculated values of F_2 (14) and F_1 (15) are 30064·38 and 64·16; of $F_4(12)$ and $F_3(16)$ are 30360·47 and 60·29 with many other examples. With the high orders successive lines become close, and with the large number of separate series involved the observed spectrum should be expected to be crowded, as indeed in this region it is.

The results are given in the annexed table, with notes up to $m = 7$. Numbers in brackets refer to values obtained—when none have been directly observed—by connection with linkages or displacement. It may be recalled that a link from an observed line to an expected but unseen one leads to the inference that the unobserved really exists, whereas a displaced line, when the displacement is on the limit, gives evidence only for the value of the sequent $f(m)$, and when the displacement is on the f-sequent, is evidence to that effect alone.

The true $F_3(\infty)$ is $30740{\cdot}17+\xi$ where ξ is small. The other limits will depend on oun displacements from this. Estimated from 20312—the source of $F_3(\infty)$—these displacements, expressed as multiples of δ_1 are $-21\frac{1}{2}$, $-14\frac{1}{2}$, $-13\frac{1}{2}$, $9\frac{1}{2}$, $12\frac{1}{2}$. Their values can, therefore, be calculated with exactness relatively to $F_3(\infty)$. They are

F_1.	F_2.	F_3.	F_4.	F_5.	F_6.
$30493{\cdot}11+1{\cdot}012\xi$	$30552{\cdot}13+1{\cdot}01\xi$	$30740{\cdot}17+\xi$	$31010{\cdot}41+{\cdot}99\xi$	$31030{\cdot}58+{\cdot}98\xi$	$31172{\cdot}24+{\cdot}98\xi$
4·910	4·927	4·964	5·030	5·043	5·090
59·02	247·06	517·30	537·47	679·13	

The numbers below the limits give respectively the changes produced in them by the displacement of one oun. The numbers in the last line give the calculated accurate separations of the corresponding F lines from F_1.

For the first order, $m = 2$, considerable displacements are to be expected. Only normal lines for $F_{2,\,3,\,6}$ are observed. The set (5) 18998·40, (3) 19515·81, (7) 19676·25 give close normal separations 517·41, 677·85. Now the limit of F_1 is 30493·11 and the denominator of 18998 calculated from this is $3{\cdot}088906-134{\cdot}3\xi$, or a mantissa $= 1088906-134{\cdot}3\xi+24p = 99\,(10998{\cdot}14-1{\cdot}35\xi+{\cdot}24p) = 99\Delta_2$. The normal $f(2)$ sequent is $89\Delta_2$. There is, therefore, a displacement of $10\Delta_2$ in the sequence term. Further the defect in the separation 677·85 from the normal is 1·28 whilst a δ_1 displacement on the sequent produces 1·11. The lines in question, therefore, are $F_1(2)(10\Delta_2)$, $F_4(2)(10\Delta_2)$, $F_6(2)(10\Delta_2-\delta_1)$.

To find a representative for F_5 we may test (1) 18332·41, **290·35**, (3) 18622·76 and (1) 18018·31, **289·60**, (6) 18307·91 in which the separations refer to that of F_3, F_5, viz., 290·40. The mantissæ difference for the first set is 5740, and the nearest oun multiple $9\frac{1}{2}\delta = 5804$ is outside error limits. That of the second is 42758 and $4\Delta_2-2\delta = 70\delta = 42770$. An observation error of $d\lambda = {\cdot}03$ would make this exact. The lines in question may therefore be $F_3(2)(70\delta)$, $F_5(2)(70\delta)$. So also it may be shown that 18466·47 is $F_2(2)(-2\Delta_2)$.

With the F difference-series occur also the **F** summation type. As their existence is a new fact of great importance the evidence available up to $m = 10$ is given. The results are embodied with those of F in the table.

TABLE of the $F_n.F_n$ Lines

	n = 1.	ν_2.	n = 2.	ν_3.	n = 3.
Limit	30493·11	**59·02**	30552·13	**247·06**	**30740·17**
m. 2	[18133·42] **93·63** (42853·85) (1n) 2581.u (5) 5262 18998·40 }	**59·31** **57·68** F (10Δ_2)	(1) 5495 18192·73 **52·13** (42911·53) (δ_1) (1) 2808.e	**246·67** **247·91**	(8) 5439 18380·09 **40·92** (43101·76) ? (<1) 3187.$e.v$
3	(3) 4240 23576·11 **92·72** 37409·33 (3) 2672·35	**59·21** **59·01**	v.(3) 3562 (23635·32) **51·83** 37468·34 (3) 2668·14	**244·70** **251·29**	(<1) 4196 23820·81 **40·74** (37660·62) (– 3δ_1) (1) 2655·57
4	(2n) 4617.v (26078·04) **92·12** 34906·21 (<1) 2864	**59·25** **57·98**	(– 2δ_1) (1) 3826 (26137·29) **50·74** (34964·19) (3δ_1) (1) 2858	**247·86** **247·92**	v.(3) 3250 (26325·90) **40·01** (35154·13) (– 2δ_1) (3) 2844
5	(2) 3644 27430·33 **91·83** 33553·33 (6) 2979	**58·49** **60·86**	(– δ_1) (2) 3636 (27488·82) **51·50** (33614·19) (δ_1) (3) 2973	**246·19** **251·18**	(1) 3612 27676·52 **40·51** (33804·51) (– δ_1) (5) 2957
6	(1) 4197.v (28242·73) **94·09** (32745·46) (1) 3530·40.v	**58·23** **59·26**	(δ_1) 3531·93 (28300·96) **52·49** (32804·72) (– δ_1) (1n) 3047	**247·06** **246·90**	(1) 3509 28489·79 **41·07** (32992·35) (– δ) (1n) 3031
7	(1) 3474 28773·67 **93·06** (32212·45)	**58·57** **59·96**	(5) 3467 28832·24 **52·33** (32272·41)† (– 2δ_1) (1) 3098	**245·65** **247·29**	(1) 3445 29019·32 **39·53** 32459·74 (4) 3079

* F_6 (7) *and* F_2 (9) *are coincident.*

from $n = 1$ to $n = 6$.

ν_3.	$n = 4$.	ν_5.	$n = 5$.	ν_6.	$n = 6$.
517·30	31010·41	**537·47**	31030·58	**679·13**	31172·24
					(8) 5314·15
				679·13	18812·55
					72·34
517·86	(43371·71)	**536·27**	(43390·12)	**678·29**	(43532·14)
	(4) 2772.*e*		(1*n*) 2546.*u*		(δ_1) (<1) 2759.*e*
	(3) 5122				(7) 5080
517·41	19515·81			**677·85**	19676·25
	(1) 5008.*u*		(5) 4145		(1) 4122
516·71	(24092·82)	**537·71**	24113·82	**677·19**	24253·30
	10·58		**31·52**		**71·22**
519·02	37928·35	**538·89**	37949·22	**679·81**	38089·14
	(1) 2635		(3) 2634		(1) 2624
	u.(5) 3253		(1) 3756		(1) 3736
517·35	(26595·39)	**537·59**	26615·63	**680·03**	26758·07
	08·22				**72·37**
515·84	35421·05			**680·47**	35586·68
	(1) 2822				(3) 2809
	(5) 3576		(1) 3574		(2) 3556
519·81	27950·14	**537·32**	27967·65	**678·19**	28108·52
	10·50		**30·57**		**72·33**
517·53	(34070·86)	**540·17**	34093·50	**682·82**	34236·15
	($3\delta_1$) (4) 2932·72		(3) 2932·27		(3) 2920
	(δ_1) (1) 3475		(<1*n*) 4056.*u*		(<1) 4033.*u*
517·55	(28760·28)	**537·14**	(28779·87)	**678·89**	(28921·62)
	10·33		**30·41**		**71·78**
514·93	(33260·39)	**535·50**	33380·96	**676·36**	(33421·95)
	($2\delta_1$) (1) 3004·81		(2) 2994		(δ_1) (1) 2990·74
	(1*n*) 3413		($2\delta_1$) (<1*n*) 3409		($-\delta_1$) (2) 3394
514·91	29288·58	**536·94**	(29310·61)	**678·95**	(29452·62)*
	10·79		**30·34**		**72·48**
520·56	(32733·01)	**537·63**	(32750·08)	**679·90**	(32892·35)
	($-\delta_1$) (4) 3054		($-2\delta_1$) (2) 3494.*u*		(δ) (1) 3037

† Coincident values.

TABLE of the $\mathrm{F}_n.\mathrm{F}_n$ Lines

	n = 1.	ν_2.	n = 2.	ν_3.	n = 3.
m. 8	(1n) 3998.u 29134·56 92·83 (31851·09) ($-\delta_1$) (1) 3139‡	58·74 59·46	(1) 4569.e 29193·30 51·92 31910·55 (1) 3132	246·56 247·11	e.(2) 3070.u 29381·12 39·66 32098·20 (4) 3114
9	($2\delta_1$) (1) 3400 (29393·35) 92·82 31592·28 (1) 3164·43	59·15 59·01	($-\delta_1$) (2) 3394 (29452·50)* 51·90 (31651·29) (δ_1) (4) 3632.u	247·58 248·87	e.(δ_1) 3045.u (29640·93) 41·04 (31841·15) δ_1 (1) 3139‡
10	(2) 3379 29584·48 92·58 (31400·68) ($-2\delta_1$) (3) 3184·74	60·21 59·04	(3) 3918.u (29644·69) 52·20 (31459·72) (δ_1) (2) 3177	248·12 247·61	(δ_1) (1n) 3350 (29832·60) 40·44 (31648·29) ($-2\delta_1$) (2) 3160
11	(1) 3362 29727·58	59·41	(1) 3942.v (29786·96)	247·35	(δ) (5) 3332** (29974·93)
12	(3) 3349 29843·17	58·88	v.(5) 2912 (29902·05)	246·94	v.(<1) 2896 (30090·11)
13 F ($2\Delta_2$) or	(1) 3339 29934·65 (δ_1) (1) 3339·37 29932·42	60·16 59·57	(5) 3332 29994·81 ($-2\delta_1$) (1) 3334 29991·99	247·01 247·34	(<1) 3312 30181·66 ($-2\delta_1$) (1n) 3313 30179·76
14	(5) 3331 30005·34		[30064·38]	247·17	($2\delta_1$) (2) 3303 30252·51
15	[30064·16]	59·04	(1) 3318†† 30123·20	248·14	(<1) 3298 30312·30

* F_6 (7) *and* F_2 (9) *are coincident.*

† Coincident values.

‡ May be either.

§ *Coincident with* ($-2\delta_1$) F_1 (12) *of* 1864 *series.*

from n = 1 to n = 6 (continued).

ν_3.	n = 4.	ν_5.	n = 5.	ν_6.	n = 6.
517·37 **516·92**	[29651·93] **09·97** (32368·01) $(-\delta_1)$ (1) 3089	**536·65** **537·09**	[29671·21] **29·70** (32388·18) (δ_1) (3) 3537.*u*	**677·80** **689·72**	$(-2\delta_1)$ (1) 3354 (29812·36) **71·58** (32530·81) $(-\delta_1)$ (4) 3073
516·71 **518·30**	*v*.(<1) 2911·38 (29910·06) **10·32** (32110·58) (δ_1) (3) 3112	**537·23** **538·45**	$(2\delta_1)$ (2) 3339·00 (29930·58) **30·65** (32130·73) $(-3\delta_1)$ (3) 3112	**679·52** **680·55**	*u*.(<1) 2922·62 (30072·87) **72·87** (32272·83)† $(-2\delta_1)$ (1) 3098
516·70 **519·93**	$(2\delta_1)$ (6) 3322§ (30101·18) **10·90** (31920·61) $(-2\delta_1)$ (1) 3132¶	**538·72** **537·00**	(1) 3318‖ 30123·20 **30·44** (31937·68) (2) 3595.*u*	**678·20** **680·16**	(2) 3303‖ 30262·68 **71·76** (32080·84) (δ_1) (1) 3614.*v*
516·61	$(-\delta_1)$ (4) 3306 (30244·19)	**535·10**	(2) 3303†† 30262·68	**677·17**	(5) 3288 30404·75
517·73	*v*.(2) 2873 (30360·90)	**539·32**	(<1) 3290 30382·49	**678·26**	$(-\delta_1)$ (3) 3276 (30521·43)
516·79 **516·98**	$(3\delta_1)$ (1) 3281 (30451·44) $(-\delta_1)$ (<1*n*) 3283 30449·40	**538·38** **538·01**	(4) 3280·66 30473·03 (<1) 3280·94 30470·43	**678·54** **678·89**	*v*.(δ_1) (2*n*) 2852 30613·19 $(2\delta_1)$ (4) 3264 30611·31
	[30522·68]	**538·53**	(3) 3273 30543·87	**679·30**	(1) 3258 30684·64
517·33	$(-\delta_1)$ (<1) 3269 (30581·49)	**537·17**	(δ) (4) 3264 (30601·33)	**677·56**	$(2\delta_1)$ (3) 3250 30743·72

‖ F_6 (10) and F_5 (11) are coincident.

¶ This line is F_2 (8).

** 3332 is F_2 (13).

†† F_5 (10) and F_2 (15) coincident.

In this table under each order the first line gives the wave-length of the observed line to the last Ångstrom, its intensity, and, where necessary, the displacement or linkage to be applied. The second line gives the wave-numbers of the F lines and the thick type the separations from the F_1 line adopted. Up to $m = 10$ the fourth line gives in the same way the wave-numbers of the **F** lines and the fifth the corresponding wave-lengths. In the third line the numbers give the mean limit $\frac{1}{2}(F+\mathbf{F})$, but only the last four significant figures are entered, the complete calculated values being given at the head of the table.

Notes to Table.—$m = 2$. For $\mathbf{F}_3$ in addition to that given there are (-2δ) (1) 38632·72.v = 43101·20, $(3\delta_1)$ (1) 38687·71.v = 01·53, (δ) (1) 38988·33.u = 01·27, (δ_1) (<1) 38973·59.u = **01·71**.

$m = 3$. The linked F_2 agrees with $(3\delta_1)$ (2) 23650·79 = ...35·79. The linked F_4 with $(3\delta_1)$ (1) 24077·41 and $(-6\delta_1)$ (1) 24062·12 both of which give the same value ...92·30.

$m = 4$. The linked F_1 agrees with $(-6\delta_1)$ (1) 26048·86 = ...78·32. For F_2, $(2\delta_1)$ (1) 26147·89 = ...38·06 is closer to the calculated value ...37·95. F_4 has a link v = 4428·62. For F_6, $(-2\delta_1)$ (3) 26744·54 = ...54·72 is only ·26 greater than the calculated value. Most of the observed lines of this order are one or two units larger than the calculated. $\mathbf{F}_3$ is also given by $(3\delta_1)$ (<1) 37675·75 = 60·66. For $\mathbf{F}_4$, $(-\delta_1)$ (2) 35417·16 = ...22·19 gives separation correct. For $\mathbf{F}_6$ $(-\delta_1)$ (1) 35580·98 = ...86·07 gives much closer separation.

$m = 5$. For F_2, $(2\delta_1)$ 27498·89 = ...88·03. For F_4, e.35261·18 = 27947·18 and $(3\delta_1)$ 27963·59 = ...48·50 both give better separations. F_5 shows a series inequality with $-(u+3·28)$ and $u-2·93$. For F_6 also (δ_1) 28113·58 = ...08·49 and (δ) 28133·98 = ...13·62. $\mathbf{F}_1$ is a strong observed line which makes the mean limit ...91·83 too small and some of the other separations too large. $(-2\delta_1)$ 33544·55 = ...54·37 or $(-5\delta_1)$ 33530·95 = ...55·50 are better. The latter makes mean limit .. 92·92 practically exact, and the separations 58·70, 249·00, 538, 680·65 all much improved.

$m = 6$. All the F are in good agreement with the calculated except for F_2. For this e.35617·10 = 28303·00, but too large. Also 28335·60, 28305·20 differ by 30·40 and $6\delta_1$ gives 29·58. Near $\mathbf{F}_1$ 32727·98, 32762·39 differ by 34·41 and $7\delta_1$ gives 34·37.

$-3\delta_1$ on the first or δ on the second give 32742·71 a better line for **F** as it makes the limit sum = 92·77 and gives better separations with $\mathbf{F}_{3,4,5}$. There are clearly two sets with probable displacement in the f sequent. With the $\mathbf{F}_1$ in the table would go better $(-6\delta_1)$ 33232·22 = ...62·39 for $\mathbf{F}_4$ and $(-\delta_1)$ 33378·17 = ...83·21 for $\mathbf{F}_5$. The linked $\mathbf{F}_6$ agrees with (δ_1) 33427·04 = ...21·95.

$m = 7$. This presents several interesting points bearing on general theory. We may consider F_3 as correctly allocated since it differs only ·55 ($d\lambda$ = ·06) from the value calculated from the formula, but it is coincident with F_1 (7) of the 1864 series. Judging from the separations which are too small (except F_6) the observed F_1 is from 1 to 2 too large. This F = 28773·67 would seem to give some insight into the connection between sequent displacements and concomitant limit displacements or linkage attachments. Thus this line has relations with displaced limits with the two lines 28788·83 = $(-3\delta_1)$ $\bar{\text{F}}$ + ·43 and 28733·40 = (2δ) F − ·99 very close, but scarcely sufficiently so to exclude the probability of small sequent displacements. Further, it is linked forwards and backwards with all the three links e u.v as shown in the following scheme on the left. The 24638, 32912 form with F_1 an exact series inequality. Now a series inequality indicates that in the successive lines each is displaced from the preceding by the same amount, in this case about $15\delta_1$. The whole set may then be arranged as indicated in the right-hand scheme where X = 28771·80 = 28773·67 $(-k)$ and k denotes the displacement (? $15\delta_1$). The k may be the same for the different links within observation errors,* but probably not. If we take X as normal F_1 the other

* To make exact it would require the following:—7δ gives 1·84; 4δ, 1·05; 8δ, 2·10; 9δ, 2·36; 3δ, ·79.

separations become normal also, but as they stand F_1 and **F** give the true mean limit. In other words they have the same sequent, or the sequent is modified in the same way for both. The normal $\mathbf{F}_1$ should be as much greater as the normal F_1 is smaller, and we do find such indications. The value given for $\mathbf{F}_1$ was found as the mean of two observed lines 32211·10 and 32213·80, supposed as $\pm$ displacements from the true line. If 32213·80 be taken as normal $\mathbf{F}_1$ it gives as mean limit with X ...92·80. The whole set of F and **F** in this order affords good illustrations of the remarks on p as to the general properties of these series. All the F lines show link connections. Thus F_2, $-(u-{\cdot}1)$, $u-{\cdot}1$; F_3, $+u-{\cdot}73$; F_4, $-(u-{\cdot}32)$, $u-2{\cdot}33$, $v-{\cdot}9$; F_5, $+v+1{\cdot}7$; F_6, $+u+{\cdot}2$.

	21456, 21462			e.X, e.X $(2k)$	
	e − ·29 ± 2·38				
24638		32912	u.X		X $(2k).u$
u + 1·87		u + 1·86			
	F_1			X (k)	
v − ·70		v + 2·06			
24346		33203	v.X (k)		X $(2k).v$
	e − 44·0				
	36083			X $(-k).e$	

In the preceding table the wave-numbers where deduced are entered as depending on one given line. But in nearly all cases these are substantiated also by links or other displaced limit lines. Although in an exhaustive treatment of the spectrum all these must be considered, this is not required for the illustration of general theory, and the notes do not go beyond $m = 7$. The number of the associated lines is remarkable. The proof of the existence of these lines depending on other d sequents as limits is so important, as a matter of general theory, that a large number have been adduced up to $m = 7$. To go beyond would be to overload the present communication with detail. The prevalence of the $u.v.$ links over e may be noted in the F series.

In dealing with the 1864 series the mean of the common F and **F** limits for different orders was treated as giving the true value within a very small possible error of about $\xi = {\cdot}3$. It may be interesting to see how in the present cases the corresponding averages deviate from the values which have been established on the 1864 basis. The mean values found from the tables and expressed in terms of differences from the adopted ones are ·26, ·24, −·33, ·06, ·05, ·15 and justify the supposition of the small limit of error in ξ. The mean separations as found from the same 10 F lines deviate from the correct values by ·02, ·86, ·05, ·06, ·11. The large deviation ·86 is due chiefly to the uncertainty of the displacement in $F_3(3)$.

The preceding discussion is sufficient to show the excessive number and complication of the lines belonging to the F systems in this neighbourhood and that the complete problem has only been touched upon. As however the allocation of the extrapolated lines 16013 and 16044 as extreme satellites of D series based respectively on $D(\infty) = S(\infty)$ and $(-3\delta_1)S(\infty)$ is important, it will be well to consider shortly if corresponding F series can be indicated. The two lines referred to give the same d sequence, depending on a denominator $1+70\Delta_2$. This makes $d(1) = 35012{\cdot}30$.

Taking this as F (∞) and using the $f(m)$ sequences as given by the formula (p. 412) it is possible to calculate the positions of the lines in question. The results are given down to $m = 8$.

m.	F.		35012·30.	**F.**	
	Calculated.	Observed.		Observed.	Calculated.
2	22652·26	...(3) 52·88	11·98	(...71·08)	47372·34
3	28092·94	(...93·54)	11·85	(...30·16)	41931·66
4	30594·43	...(1) 95·63	13·50	(...31·36)	39430·17
5	31948·38	(...47·00)	12·02	(...77·05)	38076·22
6	32763·63	...(1) 62·39	12·76	...(1n) 63·13	37260·97
7	32292·00	(...92·05)	12·14	(.. 32·23)	36732·00
8	33653·80	...(3) 53·04			36370·80

[*Note.*—δ_1 displacement on F (∞) gives a change 6·045.]

2. **F** requires sounding. (2) 35923·80.$e.v$ = 47371·08.
3. ($-5\delta_1$) (3) 28063·32 = ...93·54; **F** requires sounding. (δ_1) (1) 37508·25.v = 41930·21.
4. ($-2\delta_1$) (2) 39419·27 = ...31·36.
5. ($-2\delta_1$) (2n) 31934·91 = ...47·00; ($2\delta_1$) (1) 38089·14 = ...77·05.
6. For **F**. (1n) 37263·13; also ($-6\delta_1$) (<1) 37226·34 = ...62·88, or the same.
7. v.(1) 37720·05 = ...92·05; also ($-3\delta_1$) (4) 33274·11 = ...92·24; (2) 32304·23.v = 36732·23.

The mean limit is 35012·37. The set form an additional test that the extrapolated lines 16013 really exist. It is curious to note that the even orders of F only show directly observed lines, whilst the odd show displaced limit lines. The **F** lines are far to the violet end and come into the observed region only when weakened by high order. The F lines are all linked to lines of higher frequency by the 1864 link, also for $m = 4$, 6 to lines of lower frequency. The same tendency is shown in **F** to lower frequency, any such to higher frequency lead to unobserved regions. This fact is important as showing that at least here the 1864 separation enters in the link relation, and not as a direct displacement on the limit.

To the $F_3(\infty)$ limit corresponds a 1864 triplet series parallel to that originally considered. I have been able to follow it up in the same way as the foregoing as far as $m = 26$ at least. It accentuates the evidence for the displaced sets but as that is sufficiently supported by the results already discussed it would seem unnecessary to overburden the present communication with additional detail. Whenever the actual relations of the various displaced lines to one another are the subject of discussion these details will be of the first importance. A knowledge of these relations should be expected to throw a flood of light on the constitution of spectra, but this new question cannot be taken up here. It may be noted, however, that the first F and **F** lines of the triplets up to $m = 10$ are given in the table under F_3.

RITZ *Combinations.*—The results obtained enable us to test for additional series, associated with the name of RITZ who first pointed out their existence. They are represented by the expressions $n = p(1) - f(m)$, and $s(1) - f(m)$. Here $p(1) = S(\infty) = 51025{\cdot}30$ and $f(m)$ is given by the formula on (p. 412). $s(1) = P(\infty) = 93178{\cdot}69$ and should give a parallel series 42153·39 ahead, and therefore in the extreme ultra-violet. The results are for the sets p_1, p_2, p_3.

m.	Calc. $p_1 - f(m)$.	$d\lambda$.	Observed.				
2.	38665·21	−·07	(1) 38666·56	**1784·06**	(2) 40450·62	**806·91**	(1) 41257·53
3.	44105·93	·18	(44102·34)	**1783·08**	(45885·42)	**815·66**	(46701·08)
4.	46607·42	·03	(46606·52)	**1784·28**	(48390·80)	**807·29**	(49198·09)
5.	47961·81	·06	(47960·42)	**1777·90**	+	**813·68**	(50552·00)

Notes.—$m = 2$. 1784·06+806·91 = 1777·90+813·07

$m = 3$. There is some ambiguity as to $F_3(3)$, which gives the calculated value. The deduced ultra-violet lines are from the observed triplet by the e link.

(7) 36788·24 **1783·08** (2) 38571·32 **805·66** (1) 39386·98.

Also

$e+v+(1)$ 32362·98 **1777·90** (<1) 34140·88 give 44105·08, 45882·98.

$u+(2)$ 39969·62 = 44102·8 with the ν_1, ν_2 lines in the ultra-violet, but an extra v sounder gives $u+v+(2)$ 37320·68 **813·93** (1) 38134·61 = 45881·86, 46695·79, the former being 1779·06 ahead of 44102·80.

$m = 4$. $e+(1)$ 39292·42 **1784·28** (1) 41076·70 = 46606·52, 48390·80 with $2e+(4)$ 34569·89 = 49198·09 which is 1777·90+813·67 above 40606.

$m = 5$. $2e+(2u)$ 33332·22 **1779·90+813·68** (2) 35923·80 = 47960·42, 50552·00; also $e+u+(1)$ 38292·32 **815·34** (1) 39107·56 = 49739·60, 50554·94; the former being 1777·79 above the calculated first line of 47961·81.

The Value of the Xenon Oun.—In the case of xenon the triplet disturbance appears to be small. The result has been obtained that $\Delta_1 = 24893 = 40\frac{3}{4}\delta$ and $\Delta_2 = 10999 = 18\delta$, giving respectively for δ the values 610·87, 611·05 or 610·94 from $\Delta_1+\Delta_2$. In the preceding 611 has been adopted for δ which is sufficiently close except where very large multiples of δ are in question. To obtain more accurate values of Δ_2 recourse must be had to the F and D sequences. Incidentally they have already been touched upon, but we are now in a position by a discussion of the whole material to attain a greater exactness.

Xenon, as has been seen, shows numbers of parallel displaced groups of D and F series, each of which gives data for the determination of Δ_2. Unfortunately those lines which might be expected to give the most accurate values are in the ultra-red beyond the observed region, and the process of extrapolation by the ν_1 separation, or by links, leaves a considerable margin of uncertainty owing to the 1780 modification of ν_1, and the well established, but not yet thoroughly understood, modifications of the e links, although in X this latter is not so marked as in Ag and Au. The material at disposal is—

From the D *Series.*

1.	$[16013 \cdot 45] = 17791 \cdot 35 - \nu_1$	$= 51025 \cdot 29 + \xi - d(1 + 70\Delta_2)$ . . .	(p. 404),
2.	$[16044 \cdot 24] = 17829 \cdot 41 - \nu_1 - \nu_2$	$= (-3\delta_1)\ 51025 \cdot 29 + \xi - d(1 + 70\Delta_2)$	(p. 409),
3.	$19602 \cdot 66$	$= 51025 \cdot 29 + \xi - d(1 + 79\Delta_2 - \delta)$	(p. 403),
4.	$[19623 \cdot 05] = 21400 \cdot 95 - \nu_1$	$=$,, $-d(1 + 79\Delta_2)$.	(p. 403),
5.	$19942 \cdot 53$	$= (\delta)($,, $) - d(1 + 80\Delta_2 - \delta_1)$	(p. 396),
6.	$19989 \cdot 72$	$= ($,, $) - d(1 + 80\Delta_2)$.	(p. 396),
7.	$20021 \cdot 66$	$= (-3\delta_1)($,, $) - d(1 + 80\Delta_2)$.	(p. 396).

From the F *Series.*

8.	$[3010 \cdot 35] = 17638 \cdot 55 - 2e$	$= 30725 \cdot 26 + \xi' - f(1 + 90\Delta_2)$ . . .	(p. 388),
9.	$18380 \cdot 09 = F_3(2)$	$= 30740 \cdot 17 + \xi_1 - f(2 + 89\Delta_2)$ . . .	(p. 412),
10.	$30725 \cdot 26 + \xi' = d(1)$ sequent.		

The relations between the ξ, ξ', ξ_1 have been already determined. $\xi = 1 \cdot 015\xi' + \cdot 03 + \cdot 2q$ from the fact that (10) and the sequents of (6) are both d sequents (p. 407). Here q is the proportion of maximum error in (6). The $\cdot 03$ may be supposed merged in this and $\xi = 1 \cdot 015\xi'$. Also $\xi_1 = \xi' + \cdot 04$ from the fact that (10) and $30740 \cdot 17 + \xi_1$ are both d sequences (p. 412). Within the accuracy attainable for our present purpose we may treat the ξ's as all equal. This was certainly not to be expected, for the limit S (∞), obtained by formula constants from S lines, is in general uncertain to a few units, whilst the ξ_1 and ξ', depending on limits found from $\frac{1}{2}$ (F and **F**) should be very small.

The mantissæ conditions of the above give ($d\lambda = \cdot 04p$)

*1.	$769890 + 2 \cdot 5p_1 - 25 \cdot 27\xi = 70\Delta_2$	$\Delta_2 = 10998 \cdot 43 - \cdot 36\xi + \cdot 036p_1$
*2.	$769867 + 2 \cdot 5p_2 - 25 \cdot 27\xi = 70\Delta_2$	$= 10998 \cdot 10 - \cdot 36\xi + \cdot 036p_2$
3.	$868240 + 4 \cdot 6p_3 - 29 \cdot 73\xi = 79\Delta_2 - \delta$	$= 10998 \cdot 16 - \cdot 37\xi + \cdot 06p_3$
*4.	$868846 + 5 \cdot 5p_4 - 29 \cdot 75\xi = 79\Delta_2$	$= 10998 \cdot 05 - \cdot 37\xi + \cdot 07p_4$
5.	$879711 + 4 \cdot 8p_5 - 30 \cdot 29\xi = 80\Delta_2 - \delta_1$	$= 10998 \cdot 29 - \cdot 37\xi + \cdot 06p_5$
6.	$879853 + 4 \cdot 8p_6 - 30 \cdot 29\xi = 80\Delta_2$	$= 10998 \cdot 16 - \cdot 37\xi + \cdot 06p_6$
7.	$879855 + 4 \cdot 8p_7 - 30 \cdot 29\xi = 80\Delta_2$	$= 10998 \cdot 18 - \cdot 37\xi + \cdot 06p_7$
*8.	$989285 + 35dn - 35 \cdot 4\xi = 90\Delta_2 - \delta$	$= 10998 \cdot 84 - \cdot 39\xi + \cdot 41dn$
9.	$978842 + 16 \cdot 4p_9 - 120\xi = 89\Delta_2$	$= 10998 \cdot 22 - 1 \cdot 35\xi + \cdot 18p_9$
10.	$889326 + \quad - 30 \cdot 28\xi = 80\Delta_2 + 15\frac{1}{2}\delta$	$= 10998 \cdot 19 - \cdot 37\xi$

leaving out for the moment the extrapolated lines, indicated by *, and weighting No. 9 with three times the possible error of the others, the mean value of $\Delta_2 = 10998{\cdot}198 - {\cdot}37\xi$, the same as from No. 10 alone which is exact. They all, with the exception of No. 5, satisfy this within observation errors $< d\lambda = {\cdot}04$. No. 5 requires that the observation error shall be ·06A and the true wave-number 19942·47 in place of ... 2·53. With this the ν_1 separation = 1775·69 and is brought into practically the exact $(\delta)\,\nu_1$ value (see p. 396) required, which is 1775·76. The outstanding ·07 ($d\lambda$ = ·017) would be attached to the strong second line of the doublet (10) 21717. This is in very striking support of the general argument. We have already seen good grounds for putting ξ a small fraction of the order ·25. To determine it with greater exactness a corresponding mantissa differing from the above by considerable multiples is necessary: *e.g.* with mantissa of order ·5 the coefficient of ξ is $15{\cdot}4\xi$. The differences equated to Δ_2 multiples would then give an equation to find ξ in which the error term would have little effect. We get this different ξ coefficient in No. 9, but it is due to an order 2 in which the effect of an error is multiplied to the same extent. The extrapolated lines do not help us as their limits of error are too large. On the contrary the argument enables us to determine their values more correctly: *e.g.* in No. (2) the error is dependent as the line 21400 from which the line is extrapolated. To make the multiple correct requires $p = 2{\cdot}7$, $dn = {\cdot}42$. This reduces the observed $\nu_2 = 809{\cdot}53$ to $809{\cdot}11$. It is supposed modified by a δ_1 shift on the sequent which here produces a change of 6·3 pointing to an original $\nu_2 = 809{\cdot}11 + 6{\cdot}03 = 815{\cdot}14$, practically exact. Applying the method to Nos. 3, 9 gives

$$
\begin{aligned}
{\cdot}98\xi &= {\cdot}06 + {\cdot}18p_3 - {\cdot}06p_3 \\
\xi &= {\cdot}06 + {\cdot}18p_9 - {\cdot}06p_3 = 06 \pm {\cdot}24
\end{aligned}
$$

with

$$
\Delta_2 = 10998{\cdot}14 - {\cdot}064p_9 + {\cdot}043p_3 = 10998{\cdot}14 \pm {\cdot}10
$$

But the preferable choice is to use the fact that (10) is the limit to (9), the same value of ξ must enter, and the result depends only on the observation error.

The result is now

$$
{\cdot}98\xi = {\cdot}03 + {\cdot}18p_9, \qquad \xi = {\cdot}03 + {\cdot}18p_9, \qquad \Delta_2 = 10998{\cdot}187 - {\cdot}06p_9.
$$

Thus with maximum error $d\lambda = {\cdot}04$ maximum uncertainty in Δ_2 is $\pm{\cdot}06$, but the line (9) is a good one for measures and the probable error will not exceed ·02. Hence as the definitive value $\Delta_2 = 10998{\cdot}18$ is probably within ·03 and certainly within ·06. Hence

$$
\Delta_2 = 10998{\cdot}187 \pm {\cdot}03, \qquad \delta = 611{\cdot}0104 \pm {\cdot}0017.
$$

The value of δ obtained from the ν_1 displacement $= 610{\cdot}87 + {\cdot}76d\nu_1 - {\cdot}04\xi$, to make these the same requires $d\nu_1 = {\cdot}19$, $\nu_1 = 1778{\cdot}09$. This is possible though not probable. We cannot say definitely here therefore as in Kr that the triplet modification produces a slight difference in the deduced δ. Here the difference ·14 would correspond to a difference in mass of 27 electrons.

Radium Emanation.—The emanation does not apparently produce the two spectra exhibited by Kr and X. The measurements in the spectrum are very scanty compared with those in the latter. We have early rough determinations by RAMSAY and associates.* More accurate and complete by RUTHERFORD and ROYD†, and later by WATSON.‡ In order to diminish the absorption by the electrodes RAMSAY also used copper instead of Pt electrodes and found a number of new lines, the majority of which have not been seen by succeeding observers. They have generally been explained as due to contamination by xenon as they lie close to X lines within their errors of observation. At first sight this explanation would seem to be very natural, but RAMSAY was confident that there was no such contamination. I am inclined to suspect that the opinion that these lines belong to X is too hasty. As is well known BALY found quite a large number of lines in Kr and X coincident within his observation errors, which indeed were much smaller than those in any measures yet made in RaEm. Now as a fact those suspected lines of RAMSAY and CAMERON'S are also very close to these Kr lines. A strong argument also is this. There are a number of strong lines undoubtedly belonging to the RaEm spectrum, and observed by both RUTHERFORD and ROYD and by WATSON, which are also near strong X lines, yet separated so far from them, that if RAMSAY and CAMERON had had X in their tube they must have seen them and RaEm lines as double, one due to X and the other to RaEm. Compare for instance the following lines :—

RaEm.			X.	Kr.
C. and R.	R. and R.	W.		
(5) 4681	(10) 4680·92	(9) 4681·01	(5) 4683·76	(4) 4680·57
(10) 4626·5	(8) 4625·58	(10) 4625·66	(15) 4624·46	
(10) 4605	(4) 4604·46	(8) 4604·58	(10) 4603·21	
(3) 4578·5	(7) 4577·77	(8) 4578·0	(6) 4577·36	(6) 4577·40
(8) 4463·5	(7) 4459·3	(10) 4460·0	(20) 4462·38	(1) 4463·88
(3) 4189	(4) 4187·97	(5) 4188·2	(10) 4193·25	
(6) 4114	(6) 4114·62	(6) 4114·71	(7) 4116·25	(1) 4113·90

I therefore included these RAMSAY and CAMERON lines in the purview, with the result that a considerable number were found to fall in with places in which they are

* RAMSAY and SODDY, 'Roy. Soc. Proc.,' vol. 73, p. 346 (1904); RAMSAY and COLLIE, *ibid.*, vol. 73, p. 470; CAMERON and RAMSAY, *ibid.*, A, vol. 81, p. 210 (1908).

† RUTHERFORD and ROYDS, 'Phil. Mag.' (6), vol. 16, p. 313 (1908): ROYDS, *ibid.*, vol. 17, p. 202 (1909).

‡ H. E. WATSON, 'Roy. Soc. Proc.,' A, vol. 83, p. 50 (1909).

required by known spectrum laws. It must be remembered that the appearance of the Em lines varies very much in relative intensity with different observers (*cf.*, for instance 4604, 4460 above) that some appear early and then disappear, that others come in after the emanation has stood for a few days, and further that the copper electrodes, which extended the useful duration of the tubes, would probably have some effect on the nature of the emitting sources in the gas. To account for this, the suggestion might be thrown out that the activity of the emanation would by itself ionize the molecules of the gas, and that especially the α-rays would ionize in a different and more drastic way than the ordinary cathode or vacuum tube ionization. That with time the γ-rays from the active deposit might ionize in again a different way and produce again new lines. One would expect that the self-effect—as it may be called—is so drastic that it destroys those configurations which should give the red spectrum analogous to that in the other gases. It is a fact, as I hope to show, that the spectrum, so much as there is of it, is decidedly of the jar, or blue kind.

The degree of accuracy of the observations is not of the best. ROYD claims an accuracy of 0·1A. The spectrum was obtained by a concave grating of 1 metre radius and extended from 5084 to 3005, with some additional lines by a prism spectrograph, subject to errors of ·5A. WATSON'S lines extended from 7057 to 3867 with several new lines. His degree of accuracy is probably about the same as that of ROYD. In the following we shall treat the maximum errors as ·2A except where lines are only given to the nearest unit.

The extent of the spectrum observed is too restricted to expect to find more than the S (2) and D (1) lines, and even in the case of S (2) the S_2 (2) and S_3 (2) may be in the violet where only glass apparatus was used. Further, there is the added disadvantage that the links are so large that they can stretch from the unobserved ultra-red to the unobserved ultra-violet, and consequently can only act as sounders for lines so far in the ultra-violet that an e link lands within the visible red. The F lines should be expected to lie wholly in the observed region, and this must be the chief guide in the unravelling of the series relations.

As a preliminary and definite starting point, we have the value of the oun as calculated from the atomic weight. But here also there is some uncertainty. The value of HÖNIGSCHMIDTS' determination of the atomic weight of Ra, 225·97 is now generally accepted as close to the real value, as against the earlier value of 226·4. This makes the atomic weight of the emanation to be 222 to 222·4. These two give values of $\delta = 361{\cdot}80w^2$ as between 1783·1 and 1789·5, with the probability that it is close to 1783. The uncertainty in the value of the constant 361·80 will not affect this.

An examination of the spectrum for constant separations shows a large number of triplets with ν_1 in the region 5371 to 5383 and ν_2 at 2671 and less. Further, the higher values appear in sets which show inverted order of intensities. This suggests that the lines belong to D satellite systems and that the separations about 5383, 2671 belong to the modified ν_1, ν_2 which D satellites have already shown in Kr and X, whilst 5371, 2641 or thereabouts belong to the normal separations depending on displacements in the $S(\infty)$ alone. Moreover, another very frequent separation is 5631, connected with other sets as triplets with a ν_2 in the neighbourhood of 2800. This at once suggests the analogue of the 1864 F series of X.

A first quite definite starting point, from the material at disposal, is found by a search for lines of the F and **F** type, or the twin $A \pm B$ sets. The limit A belongs to a D sequence, which from analogy with Kr and X should be expected to be of the order $n = 30000$. Now in the observed spectrum there is a long gap between 27671 and 32031, within which such limit must be. That no lines should be found near this limit is to be expected. If, however, such double sets exist we should expect to find sets of lines with exactly the same separations on either side of this gap. Unfortunately there are only four lines on the violet side, but one such set is found. They are

(1) 26669·23		(0) 33259·05	
	897·13		896·91
(2) 27566·36		(0) 32362·14	

The corresponding limit should be the mean of either of the two corresponding lines, viz., 29964·14 or ...4·25, say 29964·20 ± ·05. The possible observed errors in these lines are not large and the practically exact equality of the two separations is strong evidence of the reality of the suspected connection. But any doubt on this point must be removed when it is noted that by RYDBERG's tables, the separation 897 is that due to two denominators 5·78, 6·78, whilst if the denominators are calculated using 29964 as limit the same values are found. The two results are quite independent. By a further use of RYDBERG's tables it is possible to find approximate positions for other lines of the F system, the **F** being quite beyond the observed region in the ultra-violet. Such lines are found at (5) 15846, (4) 22310·4, (8) 25171 for $m = 2, 3, 4$. Again, connected with 15846, are (3) 21488, (6) 24296·9 giving separations 5642, 2809. The lines 22310, 21487 are due to C. R.'s copper electrodes and are subject to considerable possible errors $d\lambda = 1\text{A}$, $dn = 5$, or even more. These strikingly correspond to the 1864 sets in XF.

The limit 29964·20 must be very accurate and subject only to any systematic errors in R. and R.'s measurements. This is shown by the exactness of the observed separations in sets so far removed from one another as 26669 and 32362. Using this limit with the lines 15846, 26669·23 for $m = 2$ and 5 the calculated formula is

$$n = 29964 \cdot 20 - \mathrm{N} \Big/ \left\{ m + \cdot 757457 + \frac{\cdot 059446}{m} \right\}^2 .$$

The two lines used may be regarded as having possible errors $dn = 2$ and ·7 respectively and any consequent errors in the constants will scarcely affect the calculated frequencies for $m > 2$. These for $m = 3, 4, 6$ are 22277·28, 25148·60, 27569·40. The last gives $d\lambda =$ ·40. No lines correspond to the others, but 22211 observed by C. and R. with Cu electrodes and (15) 25107·14 (R.), (9)...6·64 (W.) are respectively 66·28 and 41·56, (41·96) less. Now a δ_1 displacement on the limit gives 13·96, so that these correspond within error limits to a $(5\delta_1)\,\mathrm{F}(\infty)$, $d\lambda = -$·5, and $(3\delta_1)\,\mathrm{F}(\infty)$, $d\lambda = -$·06, and stand in some analogy to what has been seen to happen in X. The intensity of the second line however would seem very great for $m = 4$. In support of the C.R. line with its large possible error is (2) 22196·92 (Royd) at 13·08 behind it or an extra δ_1. It would make $22196 = (6\delta_1)\,\mathrm{F}$, and consequently F = 22280·68, *i.e.*, $d\lambda = -$·7. If the 5640 lines exist, they all lie outside the region of observation except in the case of the first line 15846 where the corresponding triplet set is found (see above). However, anticipating the value of the links obtained later and using them as sounders, viz., $e =$ 23678·3, $u =$ 11191·8, $v =$ 13680·6 with some uncertainties we can test for their existence. Taking the observed lines 22196·92, 22211 we expect lines about 27838, 27858, only to be observed if strong. There are none, but there are (3) 14166 (W.) **2816** 16982·0 (C.R.) which with the v link give 27846·6 ± 1·9 + dv, 30662·6 ± 3 + dv, separation = 2816 ± 5. These suggest the triplet set (δ_1) F, viz.,

22196·9	**5649·7**			
22210·9	**5635 + 1·9p + du**	27846·6 + 1·9p + dv	**2816 + 3p′ − 1·9p**	30662·6 + 3p' + dv

The results of sounding are indicated in the following, where, since e, u, v are only approximately known, their values are supposed corrected by de, du, dv. The letters after wave-lengths refer to observers:—

$m = 3$. $(5\delta_1)$ F.

Outside		(3) 14166 (W.).v		16982 (C.R.).v
22196·9	**5649·7**			
22210·9	**5635 + 1·9p + dv**	27846·6 + 1·9p + dv	**2816 + 3p′ − 1·9p**	30662·6 + 3p + dv

With the calculated value of F_1 the first separation is 3·5 larger. The first line is δ_1 displacement on 22210. If the former does not belong to the system, we may take C.R.'s direct line which has possible error of 4 or 5.

$m = 4$. $(3\delta_1)$ F.

1.	Outside		$(2\delta_1)$ 19583 (C.R.).u		$(2\delta_1)$ 22398 (C.R.).u
	25107·14	**5640·6 + 4p + du**	(30747·8) + 4p + du	**2814·2 + 5p′ − 4p**	(33562) + 5p + du
2.	v.(0) 15136 (W.).e		v.(0) 20784·31 (W.).e		v.(0) 23598·2 (W.).e
			$v.(-\delta_1)$ (1) 20799 (C.R.).e		
	(25134·3)	**5648·31 + de − dv**	(30782·6)	**2814·9**	(33596·5)
3.	[25148·60]	**5648**	v.(1) 20799.e	**2816·0**	u.(1) 21126·0.e

No. (2) is close to $(-2\delta_1)$ 25107 or (δ_1) F_1. It should be noted that the C.R. copper line is a $(-\delta_1)$ displacement on that observed by WATSON, and that the latter was only observed after the emanation had been standing two days. No. 3 gives the calculated values.

$m = 5$.

1.	Outside		(1) 21126·0 (W.).u		23948 (C.R.).u
	(1) 26669·23	**5648·6 + du**	32317·8	**2822 + 6p**	35139 + $6p$
2.			(2) 18641·4 (W.).v		(6) 21458·86 (W.).v, (6·56) (R.R.)
	26669·23	**5652·8 + dv**	32322·0	**2817·4**	35139·46
3.			u.(1) 19842·4 (W.).e		u.(4) 22636 (C.R.).e
	26669·23	**5659·7 + de − du**	32328·9	**2793·6 + 6p**	35122·5

These would seem to show in Nos. 2, 3 sequence displacements; 2δ would here produce a change of 4. In (2) it is displaced in the second and continued in the third. In (3) it is displaced only in the second. 21126 acts as a sounded line for $F_3(4)$ and $F_2(5)$. Both cannot be real or the line is double.

$m = 6$.

1.			(1) 22015 (C.R.).u
	(2) 27566·36	**5640·4 + 6p + du**	33206·8 + $6p$
2.			(3) 19527·8 (W.).v
	27566·36	**5642 + dv**	33208·4

There is a C.R. copper line (2) 16386 ± 3 which with the u link gives 27577 ± 3 and may be $(-\delta_1)$ F (6).

The two last observed lines have been included in order to show that sounding is justified by them. The calculated line for $m = 7$ is 28155·7, with error probably within 3 or 4 units. There is 16982 (C.R.) which with u sounder gives $28173{\cdot}8 \pm 3$ and may well correspond to (δ_1) $F_1(7)$. Also 22636 (C.R.).u gives 33827·8 which is 5654 ahead and may be (δ_1) $F_2(7)$. But this line with an $e-u$ link also gives $F_3(6)$. It should be noticed how the lines observed by CAMERON and RAMSAY with copper electrodes come in to fill parallel and displaced lines where they seem called for.

The corresponding **F** lines for the orders above the observed ones are in the ultra-violet, but evidence for their existence is given by sounding. In what follows, the value of $\mathbf{F}_1$ calculated from F_1 is enclosed in square brackets.

$m = 2$. $[44082{\cdot}4 \pm 2.]$

1.	(δ_1) (6) 21712·09 (R.).$2u$		$(3\delta_1)$ (0) 27389·61 (R.).$2u$		
	44081·7 + $2du$	**5649·6**	49731·3 + $2du$		
2.	(3) 16725·1 (W.).$2v$				(8) 25170·10 (W.).$2v$?
	44086·3 + $2dv$	**8445 = 5645 + 2814 − 13·96**			52531·3 + $2dv$

In No. 1 the sounded line $\mathbf{F}_2$ is on the verge of the observed region. In No. 2 the sounded is about 4 too large, and there appears a δ_1 displacement on the last. The u, v links themselves are too short to reach.

$m = 3$. [37650·6 from Calculated F, 37717 from Observed $(5\delta_1)$ F.]

(3) 23973·69.v		(0) 15944 (W.).$2v$		(2) 18739 (C.R.).$2v+4p$
37654·29+dv	**5651**+dv	43305·2+$2dv$	**2795**+4p	46100·2+$2dv$

Here the reproduced line refers to the normal line calculated from the formula. The v link is too short for the second and third lines, $2v$ reaches it, but $2v$ on the first would require a reference line in the ultra-red. Also 15944 is possibly $(-7\delta_1)$ F_1 (2).

$m = 4$. [34779·80.]

(3) 23584 (C.R.).u (δ_1) (0) 23598·2 (W.).u		$(-2\delta_1)$ (3) 16725·1 (W.).e
34776·0	**5655·2**+de−du	40431·2

Again note a C.R. copper line supported by a W. and R.R. displaced δ_1 line

$m = 5$. [33259·17.]

		(2) 25213·7 (W.).v		(0) 18024 (W.).e
33259·05	**5635·25**+dv	38894·3	**2808·0**+de−dv	41702·3

Here appears the frequent 5635. It is 5649−14, that is, there is a limit displacement of δ_1 in the second line and an extra one in the third, making the second separation 2816.

$m = 6$.

The same links cannot serve as sounders for all three lines.

For $\mathbf{F}_1$ is (2) 211725·4 (W.).u = 32364·34+du as against calculated 32362·14.
For $\mathbf{F}_2$ is v.(2) 16816·5 (W.).$e.u$ = 38004·8+$de+du-dv$.
For $\mathbf{F}_3$ (4) 18448 (C.R.).$2u$ = 40831·6+$2du$.

These give separations 5640·5, 2826·8, where the sum has the normal value. In $(-\delta_1)$ (3) 23973·69 (W.).u = 35178·45 which is 2816·2 ahead of $\mathbf{F}_1$ we have the completion of a mesh with the other three lines.

The foregoing discussion has shown: (1) that this F set belongs to the 1864 type discovered in X; (2) that the usual displacements are present and that the calculated value per oun—13·96—satisfies all the numerical relations formed; and (3) that the lines observed with copper electrodes by CAMERON and RAMSAY seem to belong specially to parallel series to this set. We could feel complete confidence in the allocation of the lines were it not that the α constant in the formula is positive, and that the line $m = 6$ is not reproduced more closely. The absence of direct representatives for $m = 3$, 4 is not surprising as their limit displaced values are certainly observed and the change is in full agreement with what takes place in the other elements.

We shall assume in what follows that the preceding allocation is correct, in other words the limit is $29964{\cdot}20+\xi$, the line for $m = 2$ is $15846+2{\cdot}5p$, and that the series belongs to the F type. In that case the mantissa of the limit and of the sequent are both multiples of the oun. These mantissæ are respectively $913165-31{\cdot}92\xi$ and $787174+246p-98{\cdot}7\xi$. As both are oun multiples, so must be their difference. This difference is

$$\left.\begin{aligned} 125991-246p+68\xi &= 70\tfrac{1}{2}(1787{\cdot}10-3{\cdot}49p+{\cdot}97\xi) \\ &= 70\tfrac{3}{4}(1780{\cdot}75-3{\cdot}49p+{\cdot}97\xi) \end{aligned}\right\} \quad (1)$$

In which if WATSON is correct to nearest unit p is equally probable between $\pm{\cdot}5$.

It is very unfortunate that here we have to deal with two uncertainties not generally met with, viz., on the one hand the uncertainty as to the real value of the atomic weight, and on the other the magnitude of the possible observation error in the fundamental wave-length, which WATSON has only measured to the nearest Ångström. If this had been ·1, *i.e.*, $p = {\cdot}1$, the above result would show that since δ lies between 1789 and 1783, the multiple must be $70\frac{1}{2}$ without any doubt, and consequently δ in the neighbourhood of 1787. The value of ξ is so small, that its term will not affect our present reasoning. We have, however, to allow for this uncertainty and a value of $p = -{\cdot}7$ makes the second multiple $= 70\frac{3}{4}\times 1783{\cdot}2$ with a possible δ. In this case the first multiple gives $70\frac{1}{2}\times 1789{\cdot}55$, or δ just on the improbable limit and it might be excluded. The result, therefore, is

Equally possible, $p<{\cdot}5>-{\cdot}5$, multiple $= 70\frac{1}{2}$ and δ between $1785{\cdot}3$ and 1788.
Improbable, but perhaps possible, $p = -{\cdot}7$, multiple may be $70\frac{3}{4}$ and $\delta = 1783{\cdot}1$.
Very improbable, $p = 1$, multiple $70\frac{1}{2}$ and $\delta = 1783{\cdot}6$.

But also the limit and sequent mantissæ must also be oun multiples, now

$$\left.\begin{aligned} 913165-31{\cdot}92\xi &= 512\,(1783{\cdot}52-{\cdot}062\xi) \\ &= \dots \\ &= 511\,(1787{\cdot}016-{\cdot}062\xi) \end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned} 787174+246p-98{\cdot}7\xi &= 441\tfrac{1}{2}(1782{\cdot}95+{\cdot}55p-{\cdot}22\xi) \\ &= \dots \\ &= 440\tfrac{1}{2}(1787{\cdot}00+{\cdot}55p-{\cdot}22\xi) \end{aligned}\right\} \quad (3)$$

It might occur to the reader that the last should be a multiple of Δ_2. But if the series is the analogue of the 1864XF, to which the foregoing argument has pointed, it should have a line of order $m = 1$ (n about $= 3260$). This should show $\mathrm{M}(\Delta_2)$.

The limit condition is independent of p and can only be modified by ξ in the second decimal place. It gives quite definitely six possible values for δ, viz., 1783·52, 1784·39, 1785·26, 1786·13, 1787·01, 1787·87 with multiples 512 diminishing by $\frac{1}{4}$ to $510\frac{3}{4}$. Of these the following can be satisfied by the mantissa difference condition (1)

> 1783·5 by $p = -·8$, multiple $70\frac{3}{4}$, not probable.
> 1783·5 by $p = 1$, multiple $70\frac{1}{2}$, very improbable.
> The last four by $p > -·25 < ·5$, multiple $70\frac{1}{2}$, equally probable.
> The others by $p > ·5 < 1$, multiple $70\frac{1}{2}$, improbable.

If WATSON'S readings are really to the nearest unit, $p = \pm·5$. This probable consideration would largely reduce the limits of uncertainty. It would in conditions (1) exclude the second and with multiple $70\frac{1}{2}$ give 1787·0 with $p = 0$, 1786·1 with $p = ·3$, 1785·3 with $p = ·5$. Conditions (3) would then of these give 1787 with $p = 0$, multiple $440\frac{1}{2}$, 1786·16 with $440\frac{3}{4}$, 1785·27 with 441. All of these have equal probability, but they exclude the 1783 based on HÖNIGSCHMIDTS' atomic weight. The lowest value 1785·3 would make the atomic weight $= 222·15 \pm ·02$, and that of $Ra = 226·15$ as compared with HÖNIGSCHMIDTS' 225·97.

Before passing from this series it will be important to get as close an estimate as possible of the two separations. Regarded as oun displacements on the limits they should give some further data for the determination of the oun—or, vice versa. The separations given in the sounding operations above are here collected. Errors from WATSON'S, or BALY'S observations will not amount to more than a few decimals at the outside.

F.			F.		
1.	$\left.\begin{matrix}5649\cdot7\\ 35\cdot7\end{matrix}\right\} + 1\cdot9p + dv$	$2816 + 3p' - 1\cdot9p$	9.	$5649\cdot6$	
			10.	$50\cdot9 + dv$	$2795 + 4p$
2.	$40\cdot6 + 4p + du$	$14\cdot2 + 5p' - 4p$	11.	$55\cdot2 + de - du$	
3.	$48\cdot31 + de - dv$	$14\cdot9$	12.	$35\cdot25 + dv$	$2802\cdot1 + de - dv$
		$16\cdot0 - du + dv$	13.	$36\cdot6$	$33\cdot0$
4.	$48\cdot6 + du$	$22 + 6p$			
5.	$52\cdot8 + dv$	$17\cdot4$			
6.	$59\cdot7 + de - du$	$793\cdot6 + 6p$			
7.	$40\cdot4 + 6p + dv$				
8.	$42\cdot0 + dv$				

The ν_1 cluster round 5649, 40, 35 and the ν_2 around 2816.

In ν_1 the 49 and 35 differ by 14 and are clearly due to a δ_1 displacement in the limit. We will consider the exceptions in order. In (2) there is an uncertainty $4p$. If $p = -1$ we get, with other errors close to 35, in this case $\nu_2 = 18·2 + 5p'$ and a small error in p' brings it to the 16 neighbourhood. In (4) the uncertainty $6p$ reduces again ν_2 to the 16. In (5) ordinary errors bring ν_1 to the 49 value. In (6) modification of 10·7 in the middle line brings ν_1 to 49 and $p = ·25$ brings ν_2 to 16 or $\nu_1 + \nu_2 =$ normal. In (7) the error allows 35. No. 8 does not seem amenable and may not therefore be a real connection. In (10) ν_1 is a normal 49 with error 2 and ν_2 with $p = 1$ is $00 = 14-14$ or the oun displacement in the last line. In (12) also the oun displacement in the last line makes $\nu_2 = 16$. No. 13 can be explained by the oun displacement on the middle line alone. They can all then with the exception of one be explained by $\nu_1 = 5635$ or 5649, and $\nu_2 = 2816$ within limits of about ± 1. The consideration of their source is postponed until the D sequents have been touched upon (p. 441).

An examination of the spectral list shows values of ν_1 varying between 5361 and 5388 with ν_2 from 2639 to 2685. In many cases two lines are separated by about 5371 and a line approximately midway with separations $2680\pm$. Now in the other gases the appearance of such $\frac{1}{2}\nu_1$ separations is so common as to be almost the rule. This lends weight on the one side to the assumption that $5371\pm$ is ν_1 and on the other that 2680 is not ν_2 but corresponds to $\frac{1}{2}\nu_1$. Values therefore in the neighbourhood of 2686 may be left out of account in the search for ν_2 which is always less than $\frac{1}{2}\nu_1$.

There are a considerable number of separations in the neighbourhood of $5380\pm$ which again suggest the analogous modified ν_1 related to the D satellites in Kr and X. In those cases the explanation was adopted that at least in the main they arose from displacements of small multiples of the oun in the sequence term. No exact value of these displacements can be obtained until the value of the limit itself is known, but it is possible to arrive at an approximate estimate by employment of a value of the limit which may be a few hundreds wrong. Such an estimate will be of great value in guiding our search.

The values of $S(\infty)$ for A, Kr, X are respectively 51731, 51651, 51025. From analogy, that for Ra, Em would be in the neighbourhood of 50500, say $50500+\xi$. With ν_1 in the neighbourhood of 5371 ± 1 calculation shows that a displacement of δ_1 will produce a change in the limit of $30{\cdot}86+{\cdot}0011\xi$ and in the ν_1 of $4{\cdot}72$. The D sequence terms vary with each satellite set. One such sequence set has already been found in the F series already discussed, viz., 29964. Here the oun displacements produces a change in n of $13{\cdot}96$. But higher values than this for $d(1)$ sequents are to be allowed for up to even 31000. For 30500 the oun displacement produces a change of $14{\cdot}88$. It is safe therefore to take the oun displacement on the sequent as producing a change varying from 14 to 15.

If now the $5370\pm$ separations be analysed it will be found that three lie at $71{\cdot}4\pm{\cdot}4$, eight at $74{\cdot}2\pm{\cdot}6$, five at $80\pm{\cdot}5$, six at $84{\cdot}2\pm 1$ and a group at 88 with none between. There is a clear majority about 74 with a set at 88, or about 14 ahead (*i.e.*, $-\delta_1$ on the sequent) and a few about 61, or 14 behind. Whilst the $71{\cdot}7$ and 78 sets suggest the δ_1 displacements in the limits, definitely seen to exist in Kr and X.

For the ν_2, the oun changes on the sequent are the same and on the limit are about half the previous, say, about 2 or 3. In this case are found a majority near 2649 with some about 14 on either side, and another few about 2652.

The material at disposal then goes to show that the S separations are near 5371, 2649 for some definite limit and 5374, 2652 for another whose limit mantissa is one oun less.

Arguing from analogy with the successive spectra of A, Kr, X we should expect to find in the observed region only lines corresponding to D (1) and S (2). The D (2) lines would be considerably shorter than the last observed line $n = 33259$. The S (1) would be in reversed order with $S_1(1)$ near -42100 or $S_3(1)$ near -34000, the first absorbed by the glass apparatus used. We should expect D (1) lines up to the longest observed ($n = 14166$) with $D_{11}(1)$ lines down to at least $n = 21000$, and showing the same kind of modified separations as in previous cases, and taken account of above. $S_1(2)$ should be about 25600 with $S_2(2)$ about 31270, just on the observed boundary and $S_3(2)$ quite beyond about 33900. This absence of S separations is the reason why it was so difficult above to obtain accurate values of ν_1, ν_2 for a definite limit. By themselves therefore the material is hopelessly inadequate to determine the S (∞) limit, the values of Δ_1, Δ_2, or of the various links. We have only five possible—or three probable—choices for δ to 5 significant figures, and also the value of one $d(1)$ sequent correct to a few decimals, with estimates of the F and S triplet separations. The only method of attack then seems to be an indirect one, to tabulate the sets of lines giving the triplet separations, to try to distinguish between those related to D and S systems, to obtain as close a value as possible of the ν_1, ν_2, to determine some of the satellite separations, and from these last to attempt to find the corresponding F series with the same constant separations. These F series ought in each set to consist of several orders (m) at least, as the F (∞) all lie in the observed although badly observed region. The observed separations and the values of ν_1, ν_2 combined with the approximate value of the oun may enable a determination of the important constants S (∞), Δ_1, Δ_2 to be arrived at.

We shall take then $5371+d\nu_1$ $2649+d\nu_2$ and a set about 3 larger for the values of ν_1, ν_2, the two sets belonging to two limits, relatively displaced by an oun, and both giving the same values of Δ_1, Δ_2. In the above the $d\nu_1$, $d\nu_2$ will probably not be greater numerically than 1. The calculations will be made with the 5371 set and the conditions applied $d\nu_1 = \pm 1$, &c., and $d\nu_1 = x \pm 1$, $d\nu_2 = y \pm 1$ where x, y are the changes produced by the oun displacement. We start the first approximation by taking $S(\infty) = 50500+\xi$ where ξ may amount to several hundreds and in which x, y are of the order 4·7, 2·2. The denominators of the S_1, S_2, S_3 limits with their differences calculated from these are

$$
\begin{array}{ll}
1·473697-14·591\xi & \\
 & 72624-2·053\xi+12·54\, d\nu_1 \\
1·401073-12·538\,(\xi+d\nu_1) & \\
 & 32076-·839\,(\xi+d\nu_1)+11·70\, d\nu_2 \\
1·368995-11·699\,(\xi+d\nu_1+d\nu_2) &
\end{array}
$$

These differences may be written

$$72624 = 40\tfrac{3}{4}\,(1782{\cdot}18 - {\cdot}050\xi + {\cdot}31 d\nu_1) \qquad 32076 = 18\ \ (1782{\cdot}00 - {\cdot}047\xi - {\cdot}047 d\nu_1 + {\cdot}65 d\nu_2)$$
$$= 40\tfrac{1}{2}\,(1793{\cdot}18 \ldots \quad ,, \quad) \qquad = 17\tfrac{3}{4}\,(1807{\cdot}1 \ldots \quad ,, \quad)$$

The first consequence to be drawn from these numbers is that as δ is to be of the same order of magnitude as calculated from Δ_1 and Δ_2 it can only take place in the first arrangement since the co-efficients of ξ (the only variable capable of large values) are practically equal. In other words $\Delta_1 = 40\tfrac{3}{4}\delta$, $\Delta_2 = 18\delta$, the same multiples as in X. This is one quite definite conclusion of importance.

But further we know that δ must be one of the following sets 1783·5, 84·4, 85·3, 86·1, 87·0, of which the first two though perhaps possible are improbable and the last three equally probable. We will state the results for the first, third, and fifth, the others being intermediate. The values of $d\nu_1$, $d\nu_2$ being uncertain to about unity, the values of ξ can only be estimated to the same order. Taking Δ_1, in order to produce the required values of δ, ξ must have the respective values of -27, -64, -100 within a few units, and the numbers inside the brackets for Δ_1, Δ_2, become

$$1783{\cdot}53 - {\cdot}050\xi + {\cdot}31 d\nu_1 \qquad 1783{\cdot}27 - {\cdot}047\xi - {\cdot}047 d\nu_1 + {\cdot}65 d\nu_2$$
$$1785{\cdot}38 - \ldots \quad ,, \qquad 1785{\cdot}00 \ldots \quad ,,$$
$$1787{\cdot}18 - \ldots \quad ,, \qquad 1786{\cdot}7 \ldots \quad ,,$$

where now ξ cannot exceed a few units. The respective limits become 50473, 50436, 50400. One of these is close to a S limit. There is another corresponding to the $-\delta_1$ displacement about 31 ahead and giving $\nu_1 = 5375$ about. We already have an accurate value of one d sequence, viz., the F $(\infty) = 29964{\cdot}20$. Combined with the limit this should give a D line $= \mathrm{S}(\infty) - 29964{\cdot}20$, *i.e.*, near 20509, 20472, or 20435, or the 31 higher values. We only find the following possible lines, viz. (2) 20515 and (4) 20438·8 (W.), (0) 20446·38 (R.R.), 20473, the first and fourth being C.R.'s copper electrode lines. The first is one of a chain of 2680 separations, or $\tfrac{1}{2}\nu_1$ stretching back to 15136 and by 2689 to the strong line (10) 23204·6 (R.) or 7·57 (W). It has none of the signs of a D line. The line 20438 is one of the strongest in this neighbourhood has no ν_1 separation link, but 5625·8 back to 14813 ± 2 (W) to which also 20446 has the (δ_1) 5649 link, viz.,5633. Again the 14813 shows a D modified ν_1, ν_2 set, viz.,

(0) 14813 (W)	**5382·71**	(4) 20195·71 (W)	**2672·41**	(5) 22868·12 (W),
	5385·91	(0) 20198·91 (R)	**2669·16**	(1) ·07 (R).

Hence here we have all the signs of a D_{11} line in 20438 with the above as a satellite set, at the F separation (56...) behind. Now it is a very striking fact that the measures of W. and of R.R. for these lines not only differ by amounts larger than can be attributed to observation errors, but also that their intensities are not comparable. Where W. gives 20438·8, with intensity 4, R.R. give 20446·38 with intensity (0), and a similar effect is shown in the second line of the above triplet (R.R. have not observed so far in the red as 14813). The differences 5·6, 3·2 are comparable with the differences shown by the ν_1 sets roughly estimated above, say 5371, 5374, and would seem to be due to the same effect, although this would not go with the explanation there suggested. Further R.R.'s 20446·38 is $5633·4 \pm 2$ ahead of 14813 and is therefore one exact determined value of the F_2 separation. We are justified therefore in taking 20438 as a D_{11} line belonging to the $S(\infty) = 50400 + \xi$. With $d = 29964·20 + \xi'$ the calculated D line is therefore $20435·80 + \xi - \xi' = 20438·8 + ·4p$

$$\xi = 3·0 + ·4p + \xi'$$

where ξ' is small. Hence $S(\infty) = 50403·00 + \xi$ where ξ is small and equal to $\xi' \pm ·4$. A reliable value of this limit has thus been obtained. The measure of its reliability is that of two assumptions (1) that the 5649 (or 34) series studied at the beginning of the discussion of this element is an F type analogous to the 1864 of X, and (2) that 20438* is a D_{11} line. With this, $S(\infty) = 50403·00 + \xi$ and $\nu_1 = 5371 + d\nu_1$, $\nu_2 = 2649 + d\nu_2$ the denominators and values of Δ_1, Δ_2, calculated directly are 1·475114, 1·402290, 1·370130 and

$$\Delta_1 = 72824 - 2·06\xi + 12·571 d\nu_1 \qquad = 40\tfrac{3}{4}\{1787·09 - ·050\xi + ·31 d\nu_1\}$$
$$\Delta_2 = 32160 - ·84(\xi + d\nu_1) + 11·726 d\nu_2 = 18\ \{1786·66 - ·046(\xi + d\nu_1) + ·651 d\nu_2\}.$$

The δ as determined from the Δ_2 has always been the same as those found from the D and F mantissæ, here $1787·015 - ·062(\xi - ·4p)$. Hence $d\nu_2 = ·53$ correct to the first decimal place and $\nu_2 = 2649·53 + d\nu_2$ where $d\nu_2 < ·1$ and $\Delta_2 = 32166$. If the δ from Δ_1 is the same $d\nu_1 = -·3$ and $\nu_1 = 5370·7$, $\Delta_1 = 72820$. Wherever it is different it has always been slightly less than that from Δ_2, so that we may regard 5370·7 as a maximum estimate for ν_1. It may be a few decimals smaller, but we have no direct means from observed lines to determine it more closely.

A similar treatment with the four equally probable values of δ are appended. Also changes in $d\nu_1$, if δ from ν_1 is the same as from ν_2. This will at least give maximum values of ν_1. The cases are given for the limit used above and also in the last two columns for that if 20446 is the D_{11} line, that is the limit about 7 larger.

	$d\nu_1$	$d\nu_2$	$d\nu_1$	$d\nu_2$
$1787·87 - ·062\xi$	2·5	1·7	3·5	2·5
1787·01 –	– ·3	·5	·9	1
1786·13 –	–3	– ·8	–2	– ·3
1785·26	–6	–2·1	–5	–2

* Possibly not so certain. 20446 gives one of the normal 5633 to 14818. Its weak intensity may be due, as in the other gases, to the presence of displacements and so may be D_{11}. It would make $S(\infty) = 50410$.

The limit 50403 and $\delta = 1787{\cdot}01$ require the smallest changes and seems preferable, although limit 50410 and $\delta = 1786$ is perhaps possible. Both 1787·87 and 1785·26 would appear excluded. The numerical work to follow will be based on the limit 50403, $\nu_1 = 5370{\cdot}7 + d\nu_1$, $\nu_2 = 2649{\cdot}5 + d\nu_2$, $\mathrm{S}(\infty) = 50403{\cdot}00 + \xi$. These make $\Delta_1 = 72820$, $\Delta_2 = 32166$.

With these more exact values of $\mathrm{S}(\infty)$ and ν_1 it is now possible to obtain the important constants, the p-links. They are found to be

$$a = 4630{\cdot}72 - 4{\cdot}11x + {\cdot}010\xi + {\cdot}74d\nu_1 \qquad c = 6277{\cdot}21 - 6{\cdot}16x - {\cdot}020\xi + 1{\cdot}347d\nu_1$$

$$b = 5370{\cdot}70 - 5{\cdot}01x \qquad d = 7399{\cdot}60 - 7{\cdot}67x - {\cdot}046\xi + 1{\cdot}822d\nu_1$$

$$e = 23678{\cdot}42 - 22{\cdot}96x - {\cdot}056\xi + 4{\cdot}912d\nu_1$$

where the terms in x denotes the changes for a displacement of $x\delta_1$ in $\mathrm{S}(\infty)$.

To determine the u, v links requires a knowledge of $\mathrm{S}_1(1)$. It is impossible to obtain really definite information on this point from direct observations. In default of this the following considerations will give some indications, and will serve to illustrate how the laws of relationship already determined can give clues indirectly. In the cases of A, Kr, X the lines for $\mathrm{S}_1(1)$ are respectively -42642, -42469, -42153. The corresponding line for RaEm should therefore be looked for about -41800, with $s(1) = \mathrm{P}(\infty)$ about $50403 + 41800$, or, say 92200. The e link unfortunately is too large to test for the lines for it would reach back to lines about 11103 **2649·5** 12751·5 **5370·7** 18122·2, of which the last only would be in the observed region. Consequently no evidence can be obtained with reference to it from the ν_1, ν_2 separations. There is a strong line at (8) 17909·9. It would give $\mathrm{S}_1(1) = 41586$, smaller than seems likely. Also it has a 5637 link not usually connected, so far as we yet know, with S lines. It however is the only strong line in the neighbourhood, and it will be probably wiser to conclude that the S (1) line has no linked e line to it. The above value for $\mathrm{P}(\infty)$ gives a v link about 13660, and this link on the supposed S (1) should produce a line about 28140. We find the set

(2) 20163·7 **2645** (4) 22808·68 [28183·7]

in which the last is extrapolated by the normal $\nu_1 + \nu_2$ from the first.

It falls in the large gap where no lines have been observed and the spectroscopic apparatus was defective. If this be really the $\mathrm{S}(1) - v$ set, it is possible to calculate what the u, v links are. Thus $\mathrm{S}_1(1) = -(28183{\cdot}7 + v)$, $s(1) = \mathrm{P}(\infty) = 78586{\cdot}7 + v$, and v is produced by the $-\Delta_1$ displacement in the denominator of this. As v by a rough determination is in the neighbourhood of 13660, it can be put $= 13660 + x$ where x is not very large and can be determined by the condition that $13660 + x$ is the change produced in $92246+x$ by a decrease of $\Delta_1 = 72820$ in its denominator. The result is that

$$P(\infty) = S(1) = 92266 = N/\{1\cdot090264\}^2$$
$$u = 11191\cdot8$$
$$v = 13680\cdot0$$
$$S_1(1) = -41864\cdot3$$

The set of S (1) lines would then be

$$-41864\cdot3, \qquad -36493\cdot6 \text{ or } 89\cdot7,^* \qquad -33844\cdot1.$$

On the two latter u, $e-u$, $e-v$ links as sounders should give the following lines, below which are placed the nearest observed,

$-u$	25301·8 or 297·3	22652·3
	(2) 25299·1 ± ·7 (W.)	(4) 22636 ± 6 (C.R.)
$-(e-u)$	24007·1 or 3·2	21357·6
	(20) 23993·82 (R.) 4·45 (W.)	(10) 21357·43 (R.) 6·57 (W.)
$-(e-v)$	26495·3	23846
	...	(2) 23841·9 (W.)

They are not satisfactory sounded lines, but are given as material at disposal only, $u.S_2$ and $e.S_3.u$ are really good.

There is a striking triplet which has all the aspect of being a S (1) set except that of position. It is

(2) 18611·7 (W.)	**2648·9**	(5) 21260·21 (W.)		
(0) ... 09·94 (R.)	**2652·3**	(3) 21262·25 (R.)	**5371·3**	(10) 26633·64 (R.)

Its separations give support to the values obtained above. It is a parallel set to our adopted S (1), separated from it by 15230·7. I have not been able to recognise any arrangement of links which give this value although $2b+a$, and $e-v+b$ are close to it.†

* Sounded from observed line.

† There is the possibility to be kept in mind that it may be a real and independent S set of lines, not analogous to the set considered in the previous elements. If so, it must have the same limit and $s(1) = 77036\cdot64 = N/\{1\cdot193177\}^2$. Now the s sequence depends on the atomic volume, although the exact relation is not known. In the next group, the alkalies, the denominator is of the form $\cdot987+kv$, where v is a number proportional to the atomic volume. If here the group constant be $\cdot99$, the denominators of the two types of s sequences can be expressed as $\cdot99+kv$ and $\cdot99+2kv$. In other words, this new type may depend on twice the atomic volume.

$S_1(2)$ should be a strong line in the neighbourhood of 25900, with $S_2(2)$, $S_3(2)$ in the ultra-violet. There are three strong lines in this region, viz. (8) 25262·73, (10) 25769·9, (10) 26633·64. The last is that which has been seen to be a parallel to $S_1(1)$. The first line gives a mantissa less than that for $S_1(1)$, and therefore would make the α constant positive, in opposition to all experience for S series. We are left therefore with 25769·9±3, but this is a C.R. copper line and so far dubious. If, however, it be taken as $S_1(2)$ the formula for the S series is

$$n = 50403 - \mathrm{N} \Big/ \left\{m + \cdot 129855 - \frac{\cdot 039596}{m}\right\}^2.$$

The value of $S_1(3)$ calculated from this is 39112·08. Sounding with $-v$ gives 25432·08. WATSON gives a line at (0) 25432·3±1 and this is corroborative so far as it goes, but the formula is out of step with those for A, Kr, X, and does not give confidence. If, however, lower intensities are admitted it is possible to obtain lines which fall in excellently with all the conditions, and moreover indicate other displaced sets. We find the lines, to which extrapolated ν_1 lines are added

(3) 25416·89 (R.)	ν_1	[30788·59]	(0) 25432·35 (W.)	ν_1	[30803·05]
(2) 25425·51 (W.)	ν_1	[30796·21]	(0) 25457·0 (W.)	ν_1	[30827·7]
			(1) 25453·1 (R.)		

If the extrapolated lines are sounded for by $-v$, the 30803·05 should give a line at 17123·05. This was seen by W. at (0) 17123·0, the nearest to this being (1) 17150·93 (R.) which might possibly be within error limits of v.[30827·7]. If, with this indication of 25432·35, 30803·0 as part of a triplet set suitable for S (2) we calculate the formula with the previously alloted $S_1(1)$ and $S(\infty)$, it comes to

$$x = 50403 - \mathrm{N} \Big/ \left\{m + \cdot 101230 - \frac{\cdot 010966}{m}\right\}^2$$

which is in close analogy with that for the other elements. For $m = 3$ it gives 38972·54. A v-sounder requires a line at 25292·54. ROYDS has observed (3) 25292·12, and again in this neighbourhood we find some close lines observed by only one of W. or R., viz., (8) 25262·73 (W. and R.),* (2) 25299·1 (W.) again evidencing the presence of displacements. For $m = 3$ the oun produces a change in the sequent of 3·30, so that 25299·1 which is 6·66 above $S_1(3)$ is $S_1(3)(2\delta_1)$ exactly, and 25262·73 which is 29·39 behind is $S_1(3)(-9\delta_1)$. The lines calculated from the formula for $m = 3 \ldots 8$ are given in the accompanying list, which also gives the values as sounded

* R. gives intensity 8, W. gives 3, although as a rule W.'s intensities are higher than R.'s—again pointing to changed displacements with changed conditions of excitation.

from an observed line. Thus in $m = 3$, the observed is entered as $v.(3)$ 38972·12, whereas the actually observed is v less.

m.	Calculated.	Observed.	$d\lambda$.
1		$-\{(2)\ 20163{\cdot}7\ (\text{W.})+\nu_1+\nu_2\}.v$	*
2		(0) 25432·35 (W.)	*
3	38972·54	$v.(3)$ 38972·12 (R.)	·02
4	43873·80	$e.(4)$ 43874·02 (W.)	−·01
5	46184·76	see Note	·07
6	47454·96		
7	48227·14	$u.v.(1)$ 48227·00 (W.)	·00
8	48731·22	see Note	

Note.—For $m = 3$ the e sounder is too large, and for $m = 4$ the v is too small to act.

The agreement is so good that the chosen low intensity lines for $S_1(1)$ are justified. The low intensity may be explained by the supposition that the energy of the normal line has been partly transferred to other displaced ones, in the way indicated above for $m = 3$ and as shown also by higher orders. For $m = 4$ together with 20195—the line sounded from—W. only observes (2) 20163·7, and R. only (0) 20133·0. These differ successively from the first by $30\pm$. For $m = 4$ the oun displacement produces 1·42 on the sequent and the normal 30·5 on the limit. They are therefore possibly $e.(\delta_1)\,S_1(4)$ and $e.(2\delta_1)\,S_1(4)$. In $m = 5$ the e sounder requires 22506, not observed by either, but there is an isolated group, 22516·9 by both, (5) 22535·5 (W.), (3) 22540·4 (R.), of which 22535 is 30 ahead of the required line and 22540 is 35, so that the former is $(-\delta_1)$ on the limit. The latter, however, sounds to 46218·71 for $(-\delta_1)\,S_1(5)$ and gives $S_1(5) = 46183{\cdot}25$. The $d\lambda =$ ·07 of the list is based on this. For $m = 6$ the e requires 23776·65 with observed (0) 23760·9 (W.) and (10) 23783·75. In $m = 8$ the $u+v$ link = 24871·8 requires 23859·5. The only observed are (2) 23841·9 (W.) and (4) 23871·3 (W.) or (5) 70·00 (R.), about equally displaced on either side of the normal line.

Some space has been devoted to the consideration of the S series as on it depends the determination of the important u, v links as well as the limit for the D series. The latter is perhaps in general the more important as it affords with the F, the means of obtaining accurate determinations of the oun. In the present case, however, the measures have such large possible observation errors that they do not add to the accuracy already found in the foregoing discussion. A few points only will therefore be here referred to. Both the D satellites for $m = 1$ and the F lines for $m = 2$ with related lines form the majority of the lines in the long wave end of the spectrum ($n < 20000$). One clue as to a distinction between the two sets may be found in the fact that where the 5640 link occurs on a D line it links backwards, while in the F it links forward. Consequently where a line has a link forward it is probably not a D line, but is either a F line, or is attached to a D line (as in the 1864 set in X attached to the D group near 20312). The D-qualification test with $D(\infty) = S(\infty)$ can be applied to these suspected D lines, taking the 20438 as a definite D_{11} line. That is the mantissa difference of the lines from that of 20438 must be an oun multiple. A very large number are found to satisfy this test. It must be remembered that as the oun is so large as 447 there can be no doubt as to the satisfaction of this test or not, even for the largest allowable observation errors, nor to the actual oun multiple when it is satisfied; on the contrary the possible observation errors are so great, that the observations do not enable us to increase the accuracy of the oun itself as already found. By combining a large number of cases it would no doubt be possible to diminish the probable error of its value, but the heavy work would not be justified at present, especially as there should be hope of better measures in the immediate future. At the same time the existence of the D lines may be illustrated by the following two sets for $m = 1$.

			18357 (C.R.)		(1) 21036·25
			5671 ± 4		**5633·0**
$20\frac{1}{4}\delta$	(2) 18641·4 (W.)	**5387·12**	(1) 24028·52 (W.)	**2640·71**	(1) 26669·23
			19583 (Ram.)		
			5630		
	(1) 19842·4 (W.)	**5371·3**	(2) 25213·7 (W.)		
34δ	(2) 16062 (W.)		(1) 21456·56 (R.)		
	5650		**5633·21**		
	(6) 21712·09 (R.)	**5377·68**	(2) 27089·77 (R.)		

			(0) 16445·61		(0) 19086·1 (W.)
			5652·4		**5662**
	(3) 16725·1 (W.)	**5372·9**	(4) 22098 (C.R.)	**2650·6**	(2) 24748·6 (W.)
$18\frac{1}{4}\delta$	(1) 17493·05 (R.)	**5375·2**	(1) 22868·07 (W.)		
			(1) 18357 (C.R.)		
			5636		
	(0) 18609·94 (R.)	**5383·9**	(10) 23993·82 (R.)		
$31\frac{1}{4}\delta$	(0) 14813 (W.)				
	5625·8				
	(4) 20438·8 (W.)				

In these it should be noted:—

(1) The satellite separations are in the usual ratio 5 : 3, for in the first $20\frac{1}{4} \times 5 = 101{\cdot}25$, $34 \times 3 = 102$ or for D_{23} $20\frac{1}{2} \times 5 = 102{\cdot}5$, $33\frac{3}{4} \times 3 = 101{\cdot}25$. In the second $18\frac{1}{4} \times 5 = 91{\cdot}25$, $31\frac{1}{4} \times 3 = 93{\cdot}75$. Both sets have the same undisplaced limit = 50403, and the two corresponding D_{11} lines are displaced $23\frac{1}{2}\delta$ in their sequences.

(2) The 21712 which acts as a D_{11} line shows the modified ν_1 separation to a second line but of much smaller intensity, in close analogy with what has been seen in X.

(3) The mantissa of $16725{\cdot}1 \pm {\cdot}5$ is $804601 - 26{\cdot}8\xi + 13{\cdot}4p = 25\,(32166{\cdot}16 - 1{\cdot}07\xi + {\cdot}53p) + \delta_1 = 25\Delta_2 + \delta_1$. The second line of the triplet 22098 is one of the copper

C.R. lines. Behind this is a line (2) 22079·34 W. ($d\lambda = \pm\cdot05$) corresponding to the $-\delta_1$ displacement in the sequent—*i.e.*, the modified D separation. With S (∞) its mantissa $= 804161 - 26\cdot77\xi - 26\cdot77 d\nu_1 + 5\cdot35p$

$$= 25\,(32166\cdot44 - 1\cdot071\xi - 1\cdot07 d\nu_1 + \cdot21p).$$

If this be combined with the mantissa of $29964 = 913165 = 511\delta$ giving $\Delta_2 = 32166\cdot27 - 1\cdot116\xi$, there results the equation

$$\cdot14 + \cdot045\xi + \cdot21p - 1\cdot07 d\nu_1 = 0.$$

This can be satisfied by $\xi = 0$, $d\nu_1 = 0$, and p a fraction. It does not therefore help to a closer determination. With good measures it should be practicable to find $d\nu_1$ within ·05 and p a small fraction. This equation would then give the small correction for ξ and so increase considerably the degree of accuracy of Δ_2 and δ. The particular point however gained is that here is found one of the fundamental d sequences depending on pure multiples of Δ_2.

(4) They all show -5640 links when this link lands in the observed region except 22868. Where the measures are reliable they congregate round a value $5633\pm$. The 22079 of the last paragraph is 5633·73 above the 16445. This is a further justification of 22079 belonging to a $-\delta_1$ displaced sequent.

With 20438 as D_1 (1), RYDBERG's table gives D (2) as in the neighbourhood 37486 ±100. A u sounder gives the region 23806. There is a line (10) 23783·75 (W.) which if linked in this way gives $D_{11}(2) = 37463\cdot75$. The two lines $m = 1, 2$, and S (∞) give the formula

$$n = 50403 - \mathrm{N}\Big/\left\{m + \cdot909601 + \frac{\cdot003564}{m}\right\}^2$$

with $D(3) = 43242\cdot02$. The e-sounder requires 19563. The only line in the neighbourhood is $\lambda = 5105$ by RAMSAY, who says his measurement is very rough. If we allow $d\lambda = 5$A, the wave-number is 19583 ± 20, and it *may* be the line sought for. There seem also other D_{11} groups as in X. One instance is adduced in the next paragraph.

I end the discussion of the RaEm spectrum by a consideration of the source of the 5640 separation. In X we found the conditions satisfied by oun displacements on the F (∞) of $5\Delta_2 - \delta_1$, and $2\Delta_2 + 6\delta_1$. But here the values of the separations themselves seem very indeterminate. The values as arranged on p. 431 would seem to point to 5633, 5649 with ν_2 about 2800 and 2820. The limit of the particular F series on which our whole discussion of RaEm is based is 29964·20. As a fact, however, this limit can only generate in the proper neighbourhood separations of 5646, 2806, and the displacements are $5\Delta_2 - 6\delta_1$, $2\Delta_2 + 2\delta_1$. The dependence on these displacements so analogous to those in X is evidence that the links in question (5640, 1864) are of analogous type. We have already seen in X that these separations occur as links, as well as direct displacements on sequences present in the wave-number. Let us determine the F limit required to give a separation in the correct neighbourhood with a displacement $5\Delta_2$. The conditions are that not only must the $5\Delta_2$ give the ν_1 but since it must belong to a $d(1)$ sequent, the mantissa of the limit itself shall also be an oun multiple. The result is that the limit should be within close limits of 29617 and that then the ν_1 will be close to 5648 one of the most probable values found on p. 431. But this $29617 = F(\infty) = d(1)$ should belong to a D line $50403-27917 = 20786$. Now there is an observed line (0) 20784·31 (W.) with another at (7) 20750·34 (W.) or (7) ...52·70 (R.) behind it. Taking 20784·31 its sequent is 29618·69 and mantissa 924298 or 11133 above that of 29964·20. Now $6\frac{1}{4}\delta = 11169$ and the difference 36 requires a change in wave-number $dn = 1{\cdot}10$ or $d\lambda = -{\cdot}25$A. The line was only read to ·1A so that this error is permissible. The $F(\infty)$ is then 29617·57. The $-5\Delta_2$ displacement on this gives 35266·16 or a separation 5648·59, and on this it will be found that an extra $2\Delta_2+5\delta_1$ gives $\nu_2 = 2810{\cdot}97 = 2811$ say. Further a change of δ_1 on $5\Delta_2$ or $2\Delta_2+5\delta_1$ produces changes in ν_1, ν_2 respectively of 17·88 and 20·00. This accounts for the concomitant value of 5630·71, and 2831 also observed is due to $5\Delta_1-\delta_1$, and $2\Delta_2+6\delta_1$ precisely those in X.

The Value of the Oun.—The data for the evaluation of the oun are :—

(1) The triplet separations and $S(\infty)$. The ν_1, ν_2 are not determinable directly from S lines and consequently no exact values can be obtained from observation. The general argument on p. 433 is in favour of 5371, 2649, or about 3 larger according to limit. These are strongly supported by the set of lines on p. 437, clearly a linked S (1) or an independent S (1) set showing also the displaced S_2, S_3 by W. and R. respectively. These values point to $\delta = 1787{\cdot}0$.

(2) From the F and **F** series giving definite $F(\infty) = 29964{\cdot}20$ and $f(1)$; p. 430. The definite result is a limitation of δ to one of six alternative values, of which four between 1785·26 ... 87·87 are equally probable, the ambiguity being due to imperfect measurement. This argument is independent of the values of ν_1, ν_2.

(3) The mantissa of the D_{13} line 16725 is $25\Delta_2+\delta_1$ and that of the D_{23} line 22079·34 is $25\Delta_2$. This gives

$$\Delta_2 = 32166{\cdot}44-1{\cdot}07\xi-1{\cdot}07d\nu_1+{\cdot}21p,$$

$$\delta = 1787{\cdot}024-{\cdot}059\xi-{\cdot}059d\nu_1+{\cdot}012p.$$

Here probably

$$\xi<{\cdot}5, \qquad d\nu_1<{\cdot}1, \qquad p<{\cdot}5, \text{ or, say,}$$

$$\delta = 1787{\cdot}024\pm{\cdot}05.$$

These three determinations are all independent.

Argon.—There are a large number of strong lines in the visible spectrum evidently connected with D(1) and F(2) lines. The oun is so small that the qualifying test for D lines is not so definite as in the other gases, although this is to some extent remedied by the fact that the measures on the whole are good with a possible maximum error of ·02A. We shall therefore make no attempt to discuss the F and D series with the same fullness as in the other cases. The groups selected are certainly not the only ones and possibly may not be the most important ones, but they will be sufficient to give data for the determination of the oun to about the same degree of accuracy as for the other gases of this family. Take for the first group.

	$m = 1.$			$m = 3$	
(1) 23782·51 (K.)	**179·92**	(6) 23962·43 (E.V.)	(1) 44827·37)	**175·51**	(1) 45002·88
$71\frac{1}{2}\delta$			$39\frac{1}{4}\delta$		
(2) 23899·08 (K.)			(3) 44835·41		

	$m = 2$	
(3) 39357·06	**178·95**	(4) 39536·01
$171\frac{3}{4}\delta$		
(3) 39420·68		
$39\frac{1}{4}\delta$		
(5) 39439·33		

The lines for $m = 2, 3$ are by EDER and VALENTA. Again the low intensity for $D_{11}(1)$ is to be noticed, but we have here some indication of the source of this peculiarity. EDER and VALENTA give the line at 23899·83 of intensity 9, and state that it appears only with a very strong condenser discharge. The difference in the two measures ($d\lambda$ = ·13A) may possibly be due to observation although it is greater than is the rule between the measures of K. and of E.V. As an oun displacement in the sequent produces a change dn = ·4, the difference may be due to $2\delta_1$ displacement. If so, no error will be introduced in the succeeding considerations by overlooking this, treating it as due to observation error and using KAYSER'S measurements, subject to a smaller possible error. For in either case the dependence of the sequent mantissa on the multiple law is not affected and although the multiple itself is different, at the same time the difference in the two readings will not modify the formula constants to an extent to appreciably affect the calculated lines for $m > 2$.

Using the limit $D(\infty) = S(\infty) = 51731·03$ and the first two D_{11} lines the formula found is

$$n = 51731·03 - \mathrm{N}\Big/\left\{m + 989074 - \frac{·003976}{m}\right\}^2.$$

This gives for $m = 3, 4, 5$, the lines $n = 44834{\cdot}16$, $47323{\cdot}04$, $48672{\cdot}56$. The O−C for the first is $d\lambda = -{\cdot}06$ whilst the others should be weak and are near to the limits of observations. The links are so small that as sounders they produce lines also close to the observed region. For $m = 1$, the difference of the sequent mantissæ of 23899 and 23782 (D_{12}) is 4144 and $4\Delta_2 - 6\delta_1 = 71\frac{1}{2}\delta = 4141{\cdot}2$, or exact with $d\lambda = {\cdot}013$ on two lines. It will be shown later (p. 447) that D_{12} depends on the Δ_2 multiple. The series is, therefore, probably a real doublet series with this for the outer satellites. For $m = 2$ the D_{13} set with a displacement of about $171\frac{3}{4}\delta$ is inserted since it recalls the order of magnitude in Kr and X, whilst the two D_{12} satellites for $m = 2$ and 3 are inserted because they show the same displacement of $39\frac{1}{4}\delta$. An extrapolated line on an observed one [23611·5], **179·5**, (2) 23791·0 gives a displacement 175δ for $m = 1$ corresponding to that for $m = 2$. It may also be noted that in Argon the third lines of the triplets appear to be missing in this D series.

The following arrangement would seem to indicate the existence of a set parallel to the above :—

23899·08	**20·74**	(3) 23919·82	**188·73**	(1) 24108·55
39439·33	**20·65**	[39459·98]	**171·60**	(3) 39631·58
44835·41	**20·65**	[44856·06]	**177·22**	(4) 45033·28
[47323·01]	**20·66**	(47343·67)	**178·97**	(1) 47522·64

The [ ] in the second column are calculated from the first column by adding 20·65. The line in ( ) is determined as the mean of the linked lines $u.$(1) 47784·02 = ... 44·55 and (1) 46183·62.$u.e$ = ... 42·80. As 5δ on the limit produces a separation of 20·65, the lines in the second column would correspond to the parallel displaced series $(-5\delta)\,D_{11}$. The separations of the last column increase with the order to a normal value of ν_1 and point to a constant displacement in the sequent.

With the above D set should be associated a doublet F series whose $F_1(\infty) = d_{11}(1) = 51731{\cdot}03 - 23899{\cdot}08 = 27831{\cdot}95$ and separation that of the D satellite or 116·5. In searching for this allowance will have to be made for the prevalence of displacement in the lower orders. As a fact, however, this only appears for $m = 3$ in F_1, although in F_2 there are considerable signs of it. The following table with succeeding notes contains the data, including also those for the corresponding **F** series ; also for certain lines which show a separation about 150, and which may probably be the analogue of the 1864 X series.

We shall denote this last set by dashing the letters.

The formula obtained from the two first F_1 lines and the given limit is

$$n = 27831{\cdot}95 - N\Big/\left\{m + {\cdot}757701 + \frac{{\cdot}006544}{m}\right\}^2.$$

The O−C between observed (or deduced from observed) and calculated are given under the heading $d\lambda$. As before ( ) refers to deduced lines [ ] to calculated.

m.	F.	$d\lambda$.	27831·95.	F.
1	−(4) 7404·33 (P) **115·65** −(4) 7288·68 (P) ...	*		[63066] ... ...
2	(1) 13444·52 (P) **116·12** (1) 13560·64 (P) **150·91** (1) 13595·43 (P)	*	*31·42*	(42218·33) **115·89** (42334·22) ...
3	[20073·40] **116·85** (20190·25) **151·05** (4) 20224·41	?	*31·14*	(35588·89) **114·3 ± 2** (35703·2 ± 2) ...
4	[89·50] (10) 22991·57	− ·39	*30·13*	(32668·69) **116·31** (32785·0) **150·50** (2) 32819·19
5	[24·49] (5) 24521·94 **115·31** (24637·25) ...	·42	[*30·88*] *29·60*	(1) 31137·27 **116·57** (31253·84) **150·29** (31287·56)
6	(2) 25429·28 **116·54** (25545·82) **152·84** (1) 25582·12	·17	32·12	(1) 30234·96 **116·85** (4) 30351·81 ...
7	(26010·63) **116·44** (26127·07) ...	− ·19	33·29	(3) 29655·76 ... ...
8	(4) 26402·01 **114·27** (26516·28) **147·75** (5) 26549·76	− ·07	30·99	(29259·97) **118·82** (29378·79) ...
9	(1) 26679·62 ... **153·75** (1) 26833·37	− ·01	34·96	(3) 28990·30 **119·00** (29109·30) **150·51** (1) 29140·81

m.	F.	$d\lambda$.	27831·95.	F.
10	(5) 26885·78	−·29	34·10	(1) 28782·42 ... **148·23** (1) 28930·65

Notes.—Displacements are in evidence except for F_1. A desired wave-length can be obtained very closely since a δ_1 displacement produces a change of only ·42. Consequently in deductions by displacement the results have little weight, and in fact would have none at all were it not that we now know as a fact that such exist. Failing such likely displacements, recourse has been had in a few cases to linked lines and here their evidence has weight.

$m = 2$. For $\mathbf{F}_1$, (-16δ) (1) 42192·13 = 18·21; (9δ) (1) 42233·12 = 18·45; mean taken. For $\mathbf{F}_2$ (-5δ) (1) 42326·07 = 34·22.

$m = 3$. There is no observed F_1, but (3) 19351·20.*e* = 20070·91 or with E.V. 71·65;

$$F_1 \text{ (3) } (7\Delta_2) = 20103\cdot 85 \text{ and } 20105\cdot 52 \text{ is observed by E.V. } d\lambda = -\cdot 42.$$

For F_2, $(-38\delta_1)$ (4) 20174·83 (E.V.) = 90·22; $(20\delta_1)$ (2) 20198·39 = 90·29 mean taken.

For $\mathbf{F}_1$, (-5δ) (1) 35580·73 = 88·88, also $\mathbf{F}_2$, (-5δ) (1) 35695·03 E.V. ± 2 = 703·18 ± 2.

$m = 4$. The observed line for F_1 is much stronger than should be expected; also its O − C is in the opposite direction to that of the others. It may be displaced by Δ_2 on the sequent and so intensified. The Δ_2 would produce O − C = ·00 and the observed = F_1 (4) (Δ_2), or it may hide the real F_1. $\mathbf{F}_1$ is inserted as 32819 − 150·50. It only indicates that 32819 is the correct value of $\mathbf{F}'$. $\mathbf{F}_2$ is given as (5δ) (1) 32793·11.

$m = 5$. F_1 shows a link *e* = 719·81 to (3) 25241·75. F_2 is entered as (18δ) (4) 24665·59 = 37·25, but has no weight. $\mathbf{F}_1$ = $(15\tfrac{1}{2}\delta)$ (1) 31279·10.

$m = 6$. F_2 is the mean of $(-6\delta_1)$ (2) 25536·15 = 45·87 and (7δ) (1) 25557·12 = 45·78. The calculated value of F_1 gives a better 151·7 to F'_1.

$m = 7$. The calculated F_1 is 26009·30. The deduced value is (1) 25290·92.*e* = 26010·73, which again has an apparent approximate *e* link = 720·77 forward to an E.V. line at (2) 26731·40. This is however a coincidence as the last line is S_2 (2). In connection with the linked line 26010·73 may be taken the pair of lines (-6δ) (3) 26001·11 = 10·89 and $(7\tfrac{1}{2}\delta)$ (3) 26022·85 = 10·63 whose mean agrees precisely with the former. For F_2 a similar split with two may be observed $(-2\tfrac{1}{2}\delta)$ (3) 26123·03 = 27·07 and (2δ) (1) 26130·61 = 27·30.

$m = 8$. F_2 is (5) 25796·57.*e*; also (-2δ) (2) 26512·75 = 16·05. It may be noticed that the separations to F_1 and F' are both about 2·3 too small, or $1\tfrac{1}{2}\delta$ on the limit. $\mathbf{F}_1$ is (2) 28540·26.*e* and $\mathbf{F}_2$ is (1) 28659·08.*e*. These linked lines in this order are therefore reliable.

$m = 9$. $\mathbf{F}_2$ is (1) 28389·59.*e*.

It should especially be noted how with increasing order the normal calculated lines appear as observed with good intensities. As in the other instances it suggests itself that this is due to the diffusion of the energy of the lower order lines into the formation of numerous displaced ones. The usual accurate determination of the limit as the value of $\tfrac{1}{2}(F_1+\mathbf{F}_1)$ is not here applicable as, for the reason given above, the determination of the actual displaced lines is unreliable. These values are printed in italics. The more reliable results point to a limit higher than that calculated in S (∞), with ξ about +2.

The Value of the Oun.—The preceding results for the D and F series afford material for the more exact determination of the oun. The D_{11} line 23899 must have a mantissa which is a multiple of the oun, though not necessarily of Δ_2. The mantissa of the $F(1) = -7404$ should be a multiple of Δ_2. PASCHEN'S estimate for the maximum error of the $F(1)$ is 1A, and KAYSER'S for the $D(1)$ is ·02A. These mantissæ are, introducing the actual errors as $d\lambda = p_1$, $\cdot 02\,p_2$

$$985098 + 4{\cdot}06p_2 - 35{\cdot}66\xi$$

$$764245 + 13{\cdot}7p_1 - 25{\cdot}03\xi = 723\,\{1057{\cdot}047 + {\cdot}0190p_1 - {\cdot}0346\xi\}.$$

In the case of $F(1)$, given that its mantissa is a multiple of Δ_2 the multiple must be 723 for 724 or 725 would alter Δ_2 by 1·5, whereas we already have its value as close to 1057·0. The mantissa of d_{11} is so large that by itself it is not possible to determine the oun multiple. But the difference of the mantissæ of d_{11} and f is much smaller and also an oun multiple. It is

$$\begin{aligned}&220853 - 13{\cdot}7p_1 + 4{\cdot}06p_2 - 10{\cdot}66\xi \\ &\quad = 209\,\{1057{\cdot}047 + {\cdot}0190p_1 - {\cdot}0346\xi\} - 5\delta_1 + 2 - 17{\cdot}7p_1 + 4{\cdot}06p_2 - 3{\cdot}67\xi.\end{aligned}$$

Even this is too large to settle the multiple on account of the observation errors and uncertainty in ξ. It may be an oun more or less. The following consideration, however, will give some indication on this point. The mantissa of the satellite 23782 as found above was $4\Delta_2 - 6\delta_1$ above that of D_{11}. Consequently its complete mantissa is $209\Delta_2 - 5\delta_1 \pm \delta_1 - (4\Delta_2 - 6\delta_1) = 205\Delta_2 + \delta_1 \pm \delta_1$ above that of $f(1)$ or $= 928\Delta_2$ or $+\delta_1$ or $+2\delta_1$. Now this is so close to the Δ_2 multiple as to point to the fact that this satellite is the fundamental one, in which the rule is a multiple of Δ_2 for the mantissa when there is no relative displacement with the second or third of the triplet satellite set. But here the observed ν_1 is $179{\cdot}92 = 179{\cdot}50 + {\cdot}42$ and ·41 is the change produced by an oun displacement. In other words the mantissæ of the D_{12}, D_{22} lines are either $M(\Delta_2)$, $M(\Delta_2) + \delta_1$, or $M(\Delta_2) - \delta_1$, $M(\Delta_2)$. But it cannot be $M(\Delta_2) - \delta_1$. Hence if the multiple law holds here it must be in the D_{12} satellite, the $-\delta_1$ must be taken above, or the difference is $209\Delta_2 - 6\delta_1$ and $16{\cdot}4 - 17{\cdot}7p_1 + 4{\cdot}06p_2 - 3{\cdot}69\xi = 0$. This is easily satisfied by moderate values of the p's and ξ. Further, the mantissa of 23782 is $928\Delta_2$.*

This makes the mantissa of $D_{11} = 932\Delta_2 - 6\delta_1$, whence

$$932\Delta_2 = 985098 + 6\delta_1 + \ldots = 985184{\cdot}9 + 4p_2 - 35{\cdot}66\xi$$

$$\Delta_2 = 1057{\cdot}064 + {\cdot}004p_2 - {\cdot}0382\xi.$$

* It may be noted in passing that the satellite separation is 116·57, but is reproduced in the F series as 115·65 (P.) for $m = 1$, 116·12 (R.) for $m = 2$. The latter, more reliable, is ·45 less than the corresponding satellite separation. This general effect is therefore completely explained by the F limit for the second series being d_{22} in place of d_{12}.

The D_1 satellite gives

$$928\Delta_2 = 980956 + p_3 - 35{\cdot}44\xi, \quad \text{obs. error} = {\cdot}02p_3$$
$$\Delta_2 = 1057{\cdot}064 + {\cdot}04p_3 - 0382\xi.$$

Also the $f(1)$ gives

$$\Delta_2 = 1057{\cdot}047 + {\cdot}0190p_1 - {\cdot}0346\xi.$$

The value of ξ in $f(1)$ is not precisely the same as in the others as it involves the observed error in D_{11}. If ξ is the value for $d_{11}(1)$ that for $f(1)$ is $\xi - {\cdot}12p_2\,(d\lambda = {\cdot}02p_2)$ and

$$\Delta_2 = 1057{\cdot}047 + {\cdot}0190p_1 + {\cdot}004p_2 - {\cdot}0346\xi.$$

Thus p_1 is of the order ·9 and the best value is

$$\Delta_2 = 1057{\cdot}064 - {\cdot}0382\xi \pm {\cdot}004.$$

The value of ξ is probably a very small positive quantity. The O−C for $D_{11}(4)$ is $dn = 1{\cdot}25$ pointing to a positive ξ. The data for the accurate determination from $\frac{1}{2}(F+\mathbf{F})$ are defective. Of directly observed lines those for $m = 6$ appear reliable as the separations are good. This gives the limit as 27832·12 or $\xi = {\cdot}17$. Failing any better determination we put $\xi = {\cdot}17 + \xi$

$$\Delta_2 = 1057{\cdot}057 - {\cdot}0382\xi \pm {\cdot}004. \qquad \delta = 57{\cdot}9209 - {\cdot}0021\xi \pm {\cdot}0002.$$

Probably

$$\xi \not> 1 \qquad \text{and} \qquad \delta = 57{\cdot}921 \pm {\cdot}002.$$

The manner in which so many independent conditions fit in to give this value is very remarkable, and should give great weight to the accuracy of the above value. It must not, however, by itself be regarded as a definitive value free from all possible doubt without further support from independent groups of D or F series. This is not easy to obtain. In the first place the spectrum of A seems to approximate more to the doublet sets of He, and we lose the advantage of dealing with triplets. Again, it is clear from the way in which the lines are crowded in the region below $n = 2300$ and in which modified separations in the neighbourhood of 180 preponderate that a large number of displaced groups must exist, and displaced groups have generally shown a fragmentary quality. It is difficult, however, to deal with these in the same way as in Kr, X, or RaEm, because the oun is so small that the separations produced by one oun displacement are comparable with observation errors or uncertainties in the limits.*

* I had prepared the details of an additional D group with corresponding F and **F** series depending on a displaced D(∞). As however, while supporting the above value of the oun, it does not decrease its possible error, no present object is to be gained by printing it.

The separation 150 has been referred to as probably the analogue of the 1864 F separation in X. The reasons for this supposition are based on its magnitude and its occurrency curve. This separation in Kr, X and RaEm has been explicable as due to a displacement close to $5\Delta_2$ on the mantissa of a $d(1)$ sequent. In the present case the example taken is not the most important F series, but its limit will be near that to which the separation is due. The displacement required in it to produce a separation 150 is very close to $5\Delta_2$, in fact $5\Delta_2$ produces 149. The separation in question then is caused by displacement of the normal Δ_2-multiple for this series. The second reason is based on the form of the occurrency curve, which shows the same sharply defined single peaked curve as in X. It is represented in Plate 2, fig. 5.

It may be interesting to note that all STARK'S A^{+++} lines, *i.e.*, whose sources have lost three electrons, all show e links except one. They are

			(3) 23674	**719·23**	(3) 24393	
			(3) 23696			
			(2) 24053	**719·22**	(4) 24772	
721·90	(1) 23639	**719·46**	(7) 24359			
		? F_1 (5).	(5) 24521	**719·81**	(3) 25241	**720·24**.

Neon.—The principal sources* for measurements in the spectrum of Neon are observations by LIVEING and DEWAR, BALY, and WATSON. These have been supplemented by interferential measures by PRIEST, MEGGERS, and MEISSNER. ROSSI has added a few lines down to 2352. Through the kindness of Mr. W. F. MEGGERS I have also had the advantage of using an as yet unpublished list of very accurately measured lines made by himself and Messrs. BURNS and MERRIL at the Bureau of Standards in Washington.† The lines by LIVEING and DEWAR are only roughly measured, but as in the case of the other rare gases comprise many not observed by others. BALY'S list extends from 6717 to 3037, WATSON'S from 7245 to 2736 and contains a considerably larger number of lines. Both in BALY'S and WATSON'S lists considerable gaps appear with only a few strong lines, especially between 4250 and 3500. These are filled by a number of weak lines observed by LIVEING and DEWAR. These latter are very important for the complete discussion of the Ne spectrum as they represent the scattered displacements and linked lines of low order series lines which normally should occur as single strong lines but which are here wanting.

* LIVEING and DEWAR, 'Roy. Soc. Proc.,' vol. 67, p. 467 (1900); E. C. C. BALY, 'Phil. Trans.,' A, vol. 202, p. 183 (1903); H. E. WATSON, 'Roy. Soc. Proc.,' A, vol. 81, p. 181 (1908); J. G. PRIEST, 'Bull. Bur. Standards' (U.S.A.), vol. 8, p. 2; W. F. MEGGERS, 'Bull. Bur. Standards,' vol. 12, p. 198 (1915); K. W. MEISSNER, 'Ann. d. Phys.,' vol 51, p. 115 (1916); R. ROSSI, 'Phil. Mag.,' vol. 26, p. 981, (1913).

† Referred to below as B.M.M.

Unfortunately L.D.'s measures are only given to 1A, and thereby their value is greatly diminished as they become merely indicatory and cannot serve as quantitative data. The accuracy of BALY and of WATSON is good and probably about the same. PRIEST claims an accuracy with probable error $<$ ·0005A, MEISSNER with error not $>$ ·0015, but the accuracy of an interferometer measure depends very largely on the nature of the individual line. MEGGERS' results are exceptionally valuable in that he gives interferometer measures of a number of lines of small wave-length 3701 to 3370 where the S (2) and some of the higher order F lines occur. ROSSI has succeeded in allotting lines to series.

Neon affords an apparent exception to the rule amongst the rare gases of different spectra, according as they are developed with or without condenser in the tube discharge. On the other hand its spectrum would appear to be a composite one of the typical "red" and "blue" spectra. It undoubtedly has a portion analogous to the "blue" as will be seen by the results obtained below, completely analogous to those found in this communication for the other gases, which refer to their "blue" spectra. On the other hand in some remarkable sets of accurately equal separations discovered by WATSON* it shows a relation to the analogous well-known constant separations observed by RYDBERG in Argon. Further, it is specially rich in lines in the red region. In the list of lines observed at the Bureau of Standards referred to above there are 225 between 8783 and 5689. Since in each periodic group of elements the number of lines as a whole increases very rapidly with the atomic weight, the excess of red lines in Ne is even comparatively greater than the actual number shows. The majority of these lines are weak, but they almost all fall into a few definite linkages in which the links are the constant separations discovered by WATSON. Some of these special linkages again are connected together by the p and s links, especially the *e.u.v.* They belong to the F type of order $m = 2$, and should afford most valuable information as to the way in which parallel and displaced lines are related. I hope to return to this question on a later occasion, and only refer to them in the present discussion incidentally as they afford some evidence for the determination of the value of the oun.

The wave-numbers of observed lines published stretch from 13251 to 36536. From analogy with the spectra of the other gases we must not therefore expect to find more than one order in each of the S and D series. Nor, with its small atomic weight will the *e.u.v* links be large enough to act as efficient sounders. On the other hand the whole of any F series ($m = 1$ excepted) should lie within the above limits. It is therefore clear that the attack on the problem must be made first on this series. One datum at least is at our disposal in the magnitude of the oun. Taking the atomic weight at $20·0 \pm ·01$ the calculated value of δ is $14·47 \pm ·01$. This value of 14·47 may therefore be treated as exact to one or two units in the last digit.

* 'Proc. Camb. Phil. Soc.,' vol. 16, p. 130 (1911); 'Astro. Journ.,' vol. 33, p. 399 (1911).

The F (2) line should be a strong one in the neighbourhood of 17000. There are a number in this region. The lines in the following list are found to give a good series and is doubtless *one* F series. In this case we possess the great advantage of very accurate interferometer measure of the 1st, 3rd, and 4th lines in international units. These are used to determine the formulæ constants. The wave-lengths are given as measured, the wave-numbers are all in Rowland units.

NeF.

m.	λ.	n.	O.	O – C.
2	(10) 5852·48 I.	17081·46	·00	*
3	(0) 4157·74 R.	24044·88	·05	– ·05
4	(5) 3700·89 I.	27009·47	·00 ?	*
5	(5) 3501·22 I.	28552·37	·00	*
6 {	3393 R.	29464·2	1·00	– ·66
	(3417.e) R.	(29454·2)	1·00	(·47)
7	3329 R.	30030·57	1·00	·58

Notes to Table.—For the first line PRIEST gives $\lambda = 5852·4862$, MEISSNER ·4875, and MEGGERS ·488. These all give the same wave-number to the second decimal place. The second is a weak line by WATSON not observed by BALY. BALY gives a line, intensity 4, $n = 24039·45$ not observed by WATSON. We have here a concrete observational example of the facility with which a normal F line of low order can split up into displaced lines by slightly different excitations. In this case the mantissæ difference in the sequents is 159 and $11\delta = 159·1$, so that 24039 is F (3) (-11δ). The third line is a strong line observed by both WATSON and BALY, but the measure used is deduced from an interferentially measured line $n = 28439·801$ by deducting the WATSON link separation 1429·429 (both accurate). WATSON's measure is 27009·95. The line for $m = 5$ is by MEGGERS, but WATSON gives the same n. The remainder of the series comes in one of the gaps referred to above in which only L.D. have observed. They give lines which may serve for $m = 6, 7$. Also for $m = 6$ there appears a linked line at $29257·2+e = 29454·2$ (using the value of e found below). The calculated wave-numbers for $m = 8, 9, 10$ are 30422·6, 30703·3, 30906·5. No lines are observed between 30203 and 30722. The last two have lines by L.D. near them at $30722·7\pm 9$, $30922·3\pm 9$.

The formula as found from $m = 2, 4, 5$ is

$$n = 31850·19-\mathrm{N}\Big/\left\{m+·794726-\frac{·139254}{m}\right\}^2.$$

The calculated value for F (1) is $\lambda = 12241·97$ *in vacuo*.

There can be little doubt about these lines forming a series, but there is so far no independent evidence that it is of the F type beyond its analogous position in the spectrum to that of the other rare gases. If it is of this type we should expect to find a number of parallel series as well as of the associated **F** type. It is then necessary to test for these conditions.

A considerable number of parallel series can be arranged all weak for $m = 3$ as in the original series. Some of these are given in the following list in which the numbers in clarendon below the wave-numbers give the separations from the corresponding F lines. Those to the right of the vertical line are added. The others are deducted.

(1) 16768 $\begin{cases} 2\cdot67\ (0)\ldots71 \\ 5\cdot56\ (0)\ldots74 \end{cases}$ 312·73	(8) 16816 **265·36**	(2) 16831 **249·59**	(2) 16845 **235·84**	(4) 16889 **191·63**
23623 **315·2 ± 3**	—	(0) 23798 **246·69**	(1) 23812 **232·58**	(2) 23854 **191·29**
26695 **314·6 ± 3**	*26745* **264 ± 3**	*26766* **244 ± 3**	(1) 26776 **233·33**	*26816* **193 ± 6**
	u.(9) 28396 **262·83**	(1) 28302 **250·28**	*28216.u* **228 ± 6**	—
	(3) 29196‡ **261·7**	—	—	*29257* **200 ± 9**

(5) 16925 **155·86**	(1) 16948 **113·00**	(5) 16996 **85·23**	(5) 17222→ **141·51**	(7) 17342 **261·22**
—	—	—	*24183* **138 ± 6**	*24312* **267 ± 6**
26860 **150 ± 6**	(26892)* **117 ±**	(2) 26922 **87·10**	(4) 27148→ **138·83**	*27285* **275 ± 6**
(9) 28396 **156·33**	(5) 28438 **113·61**	(4) 28474 **87·57**	*e*.(5) 28888 **138·68**	(28819)† **267 ± 6**
(3) 29196.*u*‡ **155·0**	*29343* **115 ± 9**	*29369* **89 ± 9**	(1) 29592 **134·15**	*29727* **268 ± 9**
			30175 **139 ± 9**	*30194.u* **265 ± 9**

The wave-numbers in italics are by L.D. and subject to probable errors $d\lambda = \cdot 5$ and possible $d\lambda = 1$. For values of links see below.

* *u.26997·6* = 26891·0; *266948.e* = 26891·8.

† *u.28927* = 28820·3; *28711.u* = 28817·7, mean taken.

‡ One at least must be a coincidence, *or* the two series can simply be linked by *u* or *v*.

Still more striking, and as will be seen later, important, are parallel series with the separations discovered by WATSON. In addition are found also others at 1932 behind F (2). They are

(5) 15149·24	**1932·22**	F (2)	**1429·38**	(6) 18510·84	**422·33**	(1) 18933·34
22103	**1941·7 ± 5**	F (3)		see Note*		
25087	**1922·9 ± 6**	F (4)	**1429·43**	(6) 28438·85	**417·44**	(4) 28856·29
(3) 26628·53	**1924 ±**	F (5)	**1426 ± 8**	*29976·5*	**424**	*30203.e*†

* F (3) has a link 423 to *24467* ± 5 and then 1432 to *25899* ± 5 suggesting a mesh in which the required line is wanting. Here the 1932 separation goes better with the strong displaced F (3) 24039·45, giving 1936 ± 5. The mesh should be

F (3),	24044	423 ± 6	*24467*	1432 ±	*25899*
		1429·42	[25474·30]	425 ± 6	

† Note that 1426 + 424 = 1429·4 + 420·6 ± 9.

Any lines of the **F** type up to $m = 4$ will unfortunately lie in the ultra-violet beyond the observed region. **F** (4) should be 36690, and the largest frequency observed by WATSON is (1) 36536. The others should be weak and in a region where glass apparatus would only allow strong lines to be registered. **F** (5) should be at 35148·01 + 2ξ but is not seen. The line 35259·2 is about a v link ahead, in fact v.35259·2 = 35152·4. It may, however, be noticed that 36536 above is just 154 behind the expected **F** (4), so that it is the **F** line corresponding to the parallel F set above with the separation 156 (say F′). In the same series is also found **F**′ (6) = (2) 34087·1 corresponding to the F′ (6) = 29196.u. These are of value in that it gives the means of determining the limit with great exactness. Denote the parallel series by F′. For $m = 2$ using B.M.M.'s measure for F′ (2) the separation is 17081·46 ± 0 − 16925·43 ± ·05 = 156·03 ± ·05. For $m = 5$ both lines have been measured interferentially and the separation is 28553·342 − 28397·167 = 156·175, correct to the second decimal place. The two separations differ by more than the allowable observation error, and is possibly due to the common change in sequent for series with different limits. In these cases in the separation with the larger m, this effect is very small. Consequently we are justified in taking the separation as 156·17 ± 0. For $m = 4$ we have F (4) and **F**′ (4) but only a L.D. line for F′ (4). The separation, however, gives its exact value as 27009·47 − 156·17 = 26853·30. **F**′ (4) is 36536·62 ± ·66 ($d\lambda$ = ·05). The mean gives the limit for the F′ series as 31694·96 ± ·33 and consequently for F as 31851·13 ± ·33, *i.e.*, ξ = ·94 ± ·33.

But further in the neighbourhood of calculated **F** (6) = [34242] are found also (1) 34336·06, (3) 33918·08 respectively 94 ahead and 323·9 behind it. In analogy also are found (5) 17176·34, (3) 16757·91 respectively about the same amounts ahead of and behind F (2), but no other corresponding F (m) lines appear. We are justified in taking the first two lines as really in the relation indicated, and are thus enabled to arrive at the value of F (6). Using B.M.M. measures the separations given by the F (2) lines are their differences from 17081·46 or 94·88, 323·55 with very small errors ±·05. These, therefore, give for **F** (6) 34336·06±·25−94·88 = 34221·18±·25 and 33918·08±·5+323·55 = 34221·63. Both are therefore within error limits of their mean 34221·40. With this for **F** (6), and the limit corrected as above to 31851·13 the value of F (6) is 29460·96. With the value calculated from the formula O−C = −·23.

As illustrating the way in which D and F sequents are subject to displacements, it is interesting to notice that although the separations 94, 323 do not appear as directly dependent on F lines after $m = 2$, they nevertheless occur in the neighbourhood. It will be sufficient here only to refer in detail to the case of F (4) = 27009·47. At about 50 ahead of this there is a line (6) 27060·60 (W.). With this there is the following scheme :—

26766·3 (L.D.) **96·6** 26863·4 (L.D.) **197·2** (0) 27060·60 (W.) **321·4** 27382 (L.D.)

As the L.D. lines are subject, even if correct to the nearest A.U., to equally probable errors between $d\lambda = \pm$·5, or here to $dn = \pm 4$, these separations correspond to the 94·88, and 323·55, whilst 197·2 is the link e. The displacement of 27060 from F (4) may then be in the F sequent or the limit—probably the former, or a, b or c link.

The existence of the parallel sets, the indications of the associated **F** types, and the nature of the displacements all point to the conclusion that our original series is of the F type. Consequently the limit F (∞) = 31851·13 is a $d(1)$ sequent, but there is nothing as yet to show whether it belongs to a d_{11} or a satellite set. If, however, we can find a D (1) set the value of D (∞) can be obtained with sufficient accuracy to obtain the values of Δ_1, Δ_2 and the e link. The further consideration of the F series will therefore be postponed until this further information has been obtained.

It has already been remarked that only the S (2) and D (1) lines in any S or D group can be expected within the observed region. A superficial inspection of the list of lines shows a very large number of separations in the neighbourhood of 46 to 49 and about 20, clearly related to ν_1, ν_2 values as they show the normal ratio $\nu_1 : \nu_2$, and are in step with those of the other gases. In about the region in which the D lines should be expected BALY gives the strong set (4) 21200·90 **49·24** (4) 21250·14 with an equally strong line at (4) 21230·11 which might be a D_{11} line to the doublet D_{12} set. WATSON, however, gives other strong lines as well, including a triplet. These give

(1) 21156·77 (W.)	**48·18**	(1) 21204·95 (W.)	**19·71**	(4) 21224·66 (B., W.)
44·13		**45·19**		
(4) 21200·90 (B.)	**49·24**	(4) 21250·14 (B.)		
(5) ·81 (W.)	**49·36**	(5) ·19 (W.)		
29·21				
(5) 21230·11 (B., W.)				

We are justified by its form in taking this as a D(1) set for a preliminary trial, although the satellite separations are not in the usual ratio of 5 : 3. It is not so clear that the particular $F(\infty)$ just obtained is a d sequent belonging to it, nor, if so, whether it is a d_{11} or a satellite sequent. The latter point, however, will have little effect for our immediate object—the attainment $\Delta_1 . \Delta_2 . e$—as the differences are small and may be included in an undetermined ξ. The probability is that as this seems the only prominent D triplet set, it belongs to the normal group and that our $F(\infty)$ belongs to it, although on this point something will have to be said later. We shall take on trial that $F(\infty) = d_{11}$. In this case

$$D_1(\infty) = 21230{\cdot}11 + 31851{\cdot}13 = 53081{\cdot}24 + \xi$$

This should also be $S_1(\infty)$ and should give the separations by oun multiples in the denominator of the sequent. The three S or D limits would then be

$$53081{\cdot}24 + \xi \quad \mathbf{49{\cdot}24} \quad 53130{\cdot}48 + d\nu_1 + \xi \quad \mathbf{19{\cdot}71} \quad 53150{\cdot}19 + d\nu_1 + d\nu_2 + \xi$$

where ξ *may* be considerable, owing to uncertainty as to 31851 being of d_{11} or satellite type. The mantissæ of these are

$$437428 - 13{\cdot}540\xi, \qquad 436752 - 13{\cdot}521\xi, \qquad 436485 - 13{\cdot}514\xi$$

which give as differences

$$\Delta_1 = 666 - {\cdot}019\xi + 13{\cdot}5d\nu_1 = 46\,(14{\cdot}477 + {\cdot}29d\nu_1 - {\cdot}0004\xi)$$
$$\Delta_2 = 267 - {\cdot}007\,(\xi + d\nu_1) + 13{\cdot}5d\nu_2 = 18\tfrac{1}{2}\,(14{\cdot}432 + {\cdot}73d\nu_2 - {\cdot}0004\xi),$$

in which it must be remembered that calculations with seven-figure logarithms give uncertainties of a unit in the last digit. The direct calculation of δ from the atomic weight has already given $\delta = 14{\cdot}47 \pm {\cdot}01$. This gives $46\delta = 665{\cdot}6 \pm {\cdot}46$, $18\frac{1}{2}\delta = 267{\cdot}69 \pm {\cdot}18$. Thus these limits give without any doubt the true oun multiples in Δ_1, Δ_2, and the calculated Δ_2 then gives a closer value of $267{\cdot}7 \pm {\cdot}2$. No possible change in ξ can affect these results. Further, these multiples are quite in step with the march in the other gases. The remarkably close agreement sustains the allocation of the $F(\infty)$ to the d sequence of this set, although not necessarily to d_{11}.

As a further test the satellite separations should be due to oun displacements in the sequences. These separations are $29{\cdot}21$ and $45{\cdot}19$, taking the latter because 21224 is a good measure (B. and W. agree) and the observed $\nu_2 = 19{\cdot}71$ agrees so closely with the Δ_2 value and the ν_1 is subject to the very common triplet modification. These separations require displacements in the sequence mantissæ of $851 - {\cdot}040\xi + 29d\nu$,* and $1313 - {\cdot}060\xi + 29d\nu$. Now $59\delta = 853{\cdot}7 \pm 1{\cdot}2$, $90\frac{3}{4}\delta = 1313{\cdot}1 \pm 1{\cdot}8$. This is sufficient to give the satellite multiples, but as the oun is so small, $\delta_1 = 3{\cdot}62$, the close agreement cannot serve as evidence one way or the other as to the satellite nature of the doublet

* WATSON's value of the separation $29{\cdot}30$ makes this $2{\cdot}6$ larger $= 853{\cdot}6$.

and triplet lines in question. The points against the allocation of these lines to the normal D group are, (1) the satellite separations are not in the usual ratio 5 : 3 (they are in fact close to the ratio 3 : 5), (2) there is no appearance of satellite lines corresponding to the parallel F lines noted above, and (3) the limit 53081 is somewhat larger than we should expect from the march of the limits in the other gases.

Taking it, however, as the limit, it is possible with the given value of Δ_1 to calculate the link e. The result is $197+4d\nu_1$. The occurrency curve as found from B.'s and W.'s observations is given in Plate 2, fig. 6. The dotted line is the result when the numerous rough measures by L.D. are included. As is seen the maxima occur at 195·6 and 198, the calculated, at a minimum. The peaks look as if analogous to corresponding peaks found in other elements, but the analogy is doubtful. In the other elements these (much larger ones) are produced by the prevalence of displacements by a few ouns, and by Δ_1 operating on $(x\delta)$ S (∞). Such changes here would be very small and the corresponding effects are really shown by the flattened tops of the peaks. The peaks themselves are due in all probability to another cause to be considered shortly (p. 458).

In the region in which the S triplets should be expected are found (all interferential measures)

(6) 28787·86 **49·81** (5) 28837·67 **18·61** (4) 28856·28

with intensities in the proper order, although the ν_1, ν_2 are slightly different from those obtained in the D set. The interferential measures of MEGGERS and of B.M.M. differ considerably (·008) and are not so reliable therefore as usual. BALY and WATSON agree in giving the separations as 49·72, 18·64. The third line, however, has already appeared in the quite definite relation 1429+417 ahead of F(4). It cannot be S (2), but the latter may be a weak line close to it.

There is also another doublet set

(10)	27818·89	**47·75**	(1) 27866·64 ⎫
		49·93	(1) 68·82 ⎭
(8)	·73	**48·07**	(1) 6·80

The suggested S_2 line is abnormally weak. There are, however, here a number of other weak lines which have the appearance of displaced debris. If the first set form a S group, the second would belong to a group with smaller limit, which is also indicated by the smaller triplet separations. The limit 53081·24 was obtained from apparently the only stable D set, whilst the first set are apparently the only stable strong S group. It is natural therefore to take its limit as the same. If so, the limit for the second, being 969·03 (or ·35 $d\nu$) less, is 52112·20. The two mantissæ are $437420-13{\cdot}54\xi$ and $450723-13{\cdot}92\xi+13{\cdot}92d\nu$. Their difference is

$$13303-{\cdot}38\xi+13{\cdot}92d\nu = 20\,(665{\cdot}15-{\cdot}019\xi+{\cdot}7d\nu) = 20\Delta_1.$$

Furthermore the observed separation 47·75 requires, on this new limit, a displacement $664+13·9\,d\nu_1$. With $d\nu_1 = ·11$ we have Δ_1 the same as before, *i.e.*, the 47·86 is the proper ν_1 separation of the new limit. The ν_2 proper to this limit should be 19·15 giving $S_3 = 27885·90$. This is not seen, but the u-linked line is possibly given by L.D.'s line $27995·5 \pm 4 - 106·8 = 27988·7 \pm 4$. The actual S_2 line appears split up into the additional displaced lines, each (1) 27858·26, (1) 27868·82, (1) 27873·63 from observed S_2. As a parallel u-link to the first we find also L.D., $27964·2 \pm 4 - 106·8 = 27857·4 \pm 4$ for actual 58·26. The oun is too small to decide whether the displacements are produced in the limit or the sequent.

The limit for this new set, 52112·20, is more in step with the progression of the other gases, viz. :—

Ne.	A.	Kr.	X.	RaEm.
52112	51731	51651	51025	50403

It is possible to get an estimate of the $u.v$ links although there is no means apparent at present of getting the exact value of $s(1)$. Either of our S groups gives the same value for $s(2)$, viz., $53081·24 - 28787·86$ or $52112·20 - 27818·89 = 24293·3$. The denominator of this is 2·1248, so that the $\mu+\alpha/2$ of $s(2)$ is ·1248. We can get an estimate of the value of $s(1)$ from the law that $(\alpha+\frac{1}{2}\Delta_1)/(\mu+\frac{1}{2}\Delta_1)$ is about ·2 in the other periodic groups. The corresponding values in this group are A, ·189 ; Kr, ·198 ; X, ·222. If the ratio ·18 is taken for Ne $\mu+\alpha$ comes to about ·1139 with $s(1)$ about 89000. These are only rough estimates, but the values of $u.v$ will only alter slowly with considerable changes in $s(1)$. The values calculated from this $89000+\xi$ with $\Delta_1 = 666$ are

$$u = 106·78 + ·0017\xi, \qquad v = 106·86 + ·0018\xi.$$

This shows that although the value of $s(1)$ may be extremely rough, those of $u.v$ may be relied on within a few decimals.

One is inclined to regard the weak S set as that which is analogous to those determined in the other gases, and which certainly belong to the blue spectrum, and that in Neon, which shows only one spectrum, it is composite, and the blue not strongly developed. The question naturally arises whether there is any evidence of an unstable or weak D set with the same limits, *i.e.*, about 969 behind the former D. There is a set in this neighbourhood

(0) 20173·82 **42·86** (2) 20216·68

of the right order of inverted intensities and a small ν_1, possibly a case of the common D triplet modification. With the limit 52112·20 the mantissa of 20173 is 853094, which is 373 greater than that of the old d. This is within error limits of $26\delta = 376$ This may be explained by the supposition that the d sequence is not affected by the limit and that the present pair run parallel to one 26δ above the old D_{13}. If this existed it should be 21142·64, 21191·98, 21211·69. There are lines for the last two but none for the first. They are

[21142·74] **[49·24]** (0) 21190·61 **20·10** (6) 21210·71.

There is thus considerable evidence for the existence of parallel S groups as well as of parallel D ones. Each set has its corresponding ν_1, ν_2 separations, but with the same oun multiples for Δ_1, Δ_2. We should consequently expect to find the presence of corresponding $a \dots e$ links. With the S limit 52112 the e link is found to be 191·49. Now the occurrency curve for e gives a maximum between 195·6 and 196 pointing to a S limit as basis about 650 less than 53081 or, say, 52430. This makes $e = 195{\cdot}63$, $\nu_1 = 48{\cdot}9$. Do we find evidence for S and D sets about this amount less in wave-number than the old? For the S we are landed in a region which forms a gap in B.'s, or W.'s observations but which contains a number of lines by L.D. Amongst these we find

$$\begin{array}{llcc} & \mathrm{S}_1. & & \mathrm{S}_2. \\ \left.\begin{array}{r}\mathbf{1429}\\ \mathbf{195{\cdot}7}\\ \mathbf{105{\cdot}9}\end{array}\right\} & 27964 \pm 4 & \mathbf{47} & 28011 \pm 4 \\ & \quad\mathbf{213} & & \\ & 28177 \pm 4 & \mathbf{40} & 28217 \pm 4 \end{array}$$

The suggested S_1, S_2 fall into the proper positions, the link 1429 is one of WATSON's constants, 195·7 is the e link proper to the limit, 105·9 is u or v, and 213 is $2u$ or $2v$.

For the D sets we find amongst W.'s lines

$$\begin{array}{lcl} (0)\ 20551{\cdot}35 & \mathbf{49{\cdot}62} & (2)\ 20600{\cdot}97 \\ \quad\ \ \mathbf{157{\cdot}51} & & \\ (3)\ 20708{\cdot}86 & & \end{array}$$

The 20551 is 601 below $D_{13} = 21156$, which when the variability of the satellites is considered may be taken as the analogue of 610 for the S set. If it is corrected* by $dn = {\cdot}7$ ($d\lambda = -{\cdot}17$) the ν_1 becomes the correct value 48·9, and it is then also separated from 20708 as a D_{11} line by 156·8, which at once suggests the origin of the 156 parallel F and **F** sets previously brought to light. If this relation be real, the old $F(\infty)$ 31851 is a d satellite sequent, belonging to 20551·35 or ...2·0 and the limit is $20552{\cdot}0 + 31851{\cdot}13 = 52403{\cdot}13$. Not only is the 156 separation found, but there are a large number of lines in the neighbourhood which give, possibly within the error limits, the other separations indicated by the traces of parallel F series adduced at the beginning. Also as indicating a D region we find large repetitions of e and b links

* Not necessarily error, probably the usual D displacement on sequent.

characteristic of D linkages. As referring to the F series we find, starting from 20551—which corresponds to 31851—

				(3) 20556·84		
			44·13			
(0) 20551·35	**49·62**	(2) 20600·97				
		(0) ... 2·02		**85·81**		
114·84				(0) 20642·61		
(5) 20666·19						
157·51						
D_{11} (3) 20708·86						
194·42						
(2) 20745·77						
232·02						
(3) 20783·37						
249·22						
(1) 20800·57						
	269·39			**312·66**		
		[20820·74]	[**48·78**]	(2) 20869·52		
					3·04	
		275·00		(0) ... 72·56		**5·61**
					2·57	
		(1) 20826·35	**48·78**	(4) ... 75·13		

There is also (1) 20542·57 at 166·29 behind the D_{11} line 20708·86 which seems related to Rossi's series referred to later.

The application of the D qualification test is unfortunately here nugatory on account of the smallness of the oun. It is, however, striking that so many of the F separations put themselves in evidence. This agreement adds considerable weight to the allocation. The evidence is also practically convincing when the Watson separations are brought into evidence as is done a few paragraphs below. Support is also given in the doublet system shown in connection with 20826, which show the same small displacements as are exhibited in the first of the parallel F sets adduced above. In this case to 20869 as a D_2 would correspond a [20820] as a D_1 269·39 ahead of 20551, whereas the F is 312·73 behind the corresponding F_1 line (17081). If the two sets correspond, *i.e.*, the d sequent and the $F(\infty)$ for this set are the same, the D group must belong not to 20551 but to a parallel D group in which the limit is $312·73-269·39 = 43·34$ behind the normal $D(\infty)$. Now this is the separation of 20556 behind 20600. In fact the separation of 20556 and 20869 is 312·66. This further indicates that 20556 is a D_2 line in the position shown in the above table, in which the corresponding D_1 line is too weak to have been seen. This also gives another F separation—85—with 20642. The change 44·13 on the D limit requires a displacement on the mantissa of 611 and $42\frac{1}{4}\delta = 611·3$, which is exact. The ν_1 corresponding to this new limit, with $\Delta_1 = 666$, is 48·30. The difference between this and the observed 48·78 is just within our assumed possible errors on two lines.

Again, the double line 20600·97, 02·02 apparently reproduces itself in the F group, as MEISSNER says 17081 shows also a weak component. It is also in evidence in another relation as we shall see immediately.

The existence of parallel F series with WATSON'S separations and also 1932 has already been pointed out, viz.,

$$\bullet\ \mathbf{1932}\quad \mathrm{F}(1)\quad \mathbf{1429}\ \bullet\ \mathbf{422}\ \bullet$$

If these depend like the others on D series, the same separations should be found in the reverse order. They do not appear with D_1, but they are seen with the stronger line $\mathrm{D}_2 = 20600$, with which also the linkage relation enters. Thus

$$\mathbf{1428{\cdot}46}\quad (8)\ 18753\quad \mathbf{419{\cdot}13}\quad (1)\ 19172\left\{\begin{array}{ll} \mathbf{1428{\cdot}68}\quad (2)\ 20600{\cdot}97 & \left\{\begin{array}{l}\mathbf{1925\pm 3}\quad \mathit{22526}\\ \mathbf{1429{\cdot}45}\end{array}\right. \\ \mathbf{1429{\cdot}73}\quad (0)\ 20602{\cdot}02 & \quad\ \mathbf{1428{\cdot}40}\end{array}\right.\left.\begin{array}{l}\\ \\ \end{array}\right\}(4)\ 22030$$

in which there is also a mesh between 19172 and 22030 and a link back from 18753. There may also be a forward **1932** linked line if L.D. has $d\lambda = -1{\cdot}1$. It is striking that a mesh is repeated in the F(1) set. Thus

$$\begin{array}{ccccc} & \mathbf{1429{\cdot}38} & 18510{\cdot}84 & \mathbf{422{\cdot}50} & \\ 17081{\cdot}46 & & & & 18933{\cdot}34 \\ & \mathbf{1430{\cdot}24} & 18511{\cdot}70 & \mathbf{421{\cdot}64} & \end{array}$$

The separation of the 20600 lines is practically repeated in the 18510.

The parallel D set, with $\mathrm{D}_2 = 20556$, also shows traces of the separations, with the mid lines not seen, in analogy with the weak mid one just considered. Thus

$$(7)\ 18709\qquad \mathbf{417{\cdot}86}\qquad [19127]\qquad \mathbf{1429{\cdot}42}\qquad (3)\ 20556{\cdot}84$$

The absence of the corresponding lines connected to 20551 may be due to the scattering of the lines by displacement. The wanting lines should be [19122·07], [18704·6], the former corresponding to the weak 19172 of the second D group. Now we find strong lines corresponding to both these at the same separations ahead and suitable for the same displacements, viz.,

$$\begin{array}{rcr} [19122{\cdot}07] & \qquad & (1)\ 19172{\cdot}29 \\ \mathbf{20{\cdot}79} & & \mathbf{20{\cdot}27} \\ (5)\ 19142{\cdot}86 & & (4)\ 19192{\cdot}56 \end{array}$$

These numerous F and D relations render it certain that the sets of lines adduced belong respectively to sets of F series and the D series. Moreover, it suggests that the source of the 1429, 417 separations is the d sequent or $F(\infty) = 31851{\cdot}13$.

A possible supposition is that their source should be in the $S(\infty)$ limit. If so we should expect it to appear strongly in the S lines, and so in the $S_1(2)$ lines considered above. As a fact, however, there is no sign of such in either of the S groups adduced, except a very dubious one 1425 between two L.D. lines each of which has an ambiguity ± 4. It takes place between $S_1 = 27964$ and 26539 backwards, so that if its source were here it would be a positive displacement on $S(\infty)$ or a negative one on $s(2)$, both unusual. The strongest argument for its source being in the 31851 is that the separations in question show themselves in *all orders* of $F(m)$—in other words, occur in the limit $F(\infty)$.

The Value of the Oun.—It has already been found that the value of the oun calculated from the chemist's atomic weight is $14{\cdot}47 \pm {\cdot}01$ and that the oun multiple for Δ_2 is $18\frac{1}{2}$ or $\Delta_2^* = 267{\cdot}70 \pm {\cdot}02$. This is too small—or the inexactness too large—to obtain a more accurate value as in the other cases directly from the F or D mantissæ. It is, however, possible to arrive at an extremely accurate estimate by proceeding step by step with successive approximations, and for this purpose the F separations are clearly at disposal. The wave-lengths of many of the $F(1)$ lines are very accurately known (B.M.M. will be used), they are so large that the dn are small multiples of $d\lambda$, and being of order $m = 1$, an oun displacement will produce a comparatively large change in n. In spite, therefore, of the smallness of the oun it is possible to get some definite information. The reliability of the information will depend on two assumptions—

(1) That the lines employed are F lines parallel to the series $F(1) = 17081$.
(2) That no displacements occur in the f sequents themselves.

If the assumption (2) is not satisfied the series in question will not show constant separations from the corresponding $F_1(m)$ lines, but will converge or diverge with increasing order. The lines we shall make use of have been measured probably up to a few thousandths of an Ångstrom, and the accuracy is greater than one in the seventh digit in the value of n. Moreover, in calculating with seven-figure logarithms, in which also we have to do with differences between two numbers, errors amounting to unity or more are liable to enter. Consequently where these very accurate numbers occur nine-figure logarithms have been used. As the wave-lengths are given in I.A. the calculations have been made on that basis. The limit $31851{\cdot}1300$ R $= 31852{\cdot}1816$ I. $N = 109678{\cdot}6$. Put $\Delta_2 = 267{\cdot}70 + x$, therefore $\delta = 14{\cdot}4703 + {\cdot}054x$ where at present x lies between $\pm{\cdot}2$. It should be noted that in the d sequences, the satellite displacements are not in general multiples of δ, but of $\delta_1 = \frac{1}{4}\delta$. The correct value of a wave-length will be taken as the observer's value $-{\cdot}001 \times p$, so that $dn = + \ldots$.

Amongst the sets of F series given (p. 452) two, in addition to F (1) itself have been measured to the required degree of accuracy. They are those showing separations of 85 and 265. Under these conditions

$$\text{Mantissa of } 31852{\cdot}1816+\xi = 855630{\cdot}30-29{\cdot}130\xi$$

$$\text{Wave-numbers of } 5852{\cdot}4870 = 17082{\cdot}0220-{\cdot}0029p_1$$

$$\qquad 85{\cdot}4080-{\cdot}0029\,(p_1-p_2)$$

$$5881{\cdot}8958 = 16996{\cdot}6140-{\cdot}0029p_2$$

$$\qquad 265{\cdot}3518-{\cdot}0029\,(p_1-p_3)$$

$$5944{\cdot}8344 = 16816{\cdot}6702-{\cdot}0029p_3$$

(1) *Separation* = 85·41.—This is the same within error limits as occurs in the D series, but the corresponding F series shows ν increasing to 87·57 W or 87·41 B at $m = 5$, which means additional displacements either in F (∞) or the sequences. As the 85 agrees in both the F (1) and D series this will not happen in the F (∞), and the separation 85·41 will be due only to the actual separation in 31851. The limit of the F series in question is therefore $31852{\cdot}4170-85{\cdot}4080 = 31765{\cdot}8336+\xi$.

$$\text{Its mantissa} = 858133{\cdot}15-{\cdot}085\,(p_1-p_2)-29{\cdot}247\xi$$

$$\text{Difference from } F_1 = 2492{\cdot}85\ldots$$

Now

$$9\Delta_2+5\tfrac{3}{4}\,\delta = 2492{\cdot}504+9{\cdot}311x, \qquad x < {\cdot}2$$

Hence

$$9{\cdot}311x = {\cdot}35-{\cdot}085\,(p_1-p_2)-{\cdot}117\xi$$

$$x = {\cdot}038-{\cdot}009\,(p_1-p_2)-{\cdot}0126\xi$$

The important point to notice is that with our preliminary limit of uncertainty ($x < {\cdot}2$), the oun multiple cannot be any other than that used, so that the second approximation is quite definite. It has already been seen in the discussion of the F_1 series that ξ is probably within $\pm{\cdot}33$ also p_1-p_2 will not numerically be greater than 4. Hence $x = {\cdot}038\pm 036\pm 0042 = {\cdot}038\pm{\cdot}04$

$$\Delta_2 = 267{\cdot}738\pm{\cdot}04$$

$$\delta = 14{\cdot}4705\pm{\cdot}0009$$

(2) *Separation* = 265·3518.—Limit $= 31586{\cdot}8298+\xi+{\cdot}0029\,(p_1-p_3)$

$$\text{Mantissa} = 863408{\cdot}31-{\cdot}085\,(p_1-p_3)-29{\cdot}498\xi$$

$$\text{Difference from } F_1 = 7778{\cdot}01-{\cdot}085\,(p_1-p_3)-{\cdot}368\xi$$

Now

$$29\Delta_2+\delta = 7777{\cdot}77+29{\cdot}05x$$

therefore

$$29{\cdot}05x = {\cdot}23-{\cdot}085\,(p_1-p_3)-{\cdot}368\xi$$

$$x = {\cdot}008-{\cdot}0029\,(p_1-p_3)-{\cdot}0126\xi = {\cdot}008\pm{\cdot}016$$

$$\Delta_2 = 267{\cdot}708\pm 016$$

In comparing these values it must be remembered that ξ and p_1 enter in both. Equating the two, ξ disappears and

$$\cdot030-\cdot006p_1+009p_2-\cdot003p_3 = 0$$

This is easily satisfied by possible observation errors—*e.g.*, $p_1 = -p_2 = p_3 = 1\cdot4$ say,

$$\Delta_2 = 267\cdot713-\cdot0126\xi\pm\cdot01$$

(3) WATSON'S *Separations.*—When the strong lines giving these separations are taken, the exactness of the equality of the separations obtained from them is most remarkable. Using the interferentially measured lines by PRIEST, MEISSNER, and MEGGERS, with 9-figure logarithms the following values are found in I.A. for the means

1429·4292	8	·0065	·0048
417·4533	7	·0120	·0064
1070·075	1		

The last enters as a component of 1429, viz., 1429 = 1070+359.

The second column gives the number of the observations used, the third the maximum deviation of a single observation from the mean, and the last the root mean square of all the deviations. Using these the mantissæ of the following numbers have to be found with $\xi+d\nu$ added—

31852·1816 **1429·4290** 33281·6106 **417·453** 33700·0636, **1070·075** 32922·2530

They are

$$\begin{array}{ll}
855630\cdot30-29\cdot130\xi & \\
 & \mathbf{40286\cdot52-1\cdot857}\xi\mathbf{+27}d\nu_1 \\
815343\cdot78-27\cdot273\,(\xi+d\nu_1) & \\
 & \mathbf{11278\cdot99-\cdot505}\xi\mathbf{+27}d\nu_2 \\
804064\cdot79-26\cdot768\,(\xi+d\nu_1+d\nu_2) & \\
825224\cdot33-27\cdot721\,(\xi+d\nu_3) & \mathbf{9980\cdot55-\cdot448}\xi\mathbf{-27}\,(d\nu_3-d\nu_1)
\end{array}$$

In this particular case, $F_1 = 17082$, and F_1+1429 have both been observed interferentially and the separation is $17082\cdot0240-18511\cdot4499 = 1429\cdot4259$, or $d\nu_1 = -\cdot0031$. The 417 separation is altered by some displacement to $422\cdot52\pm\cdot17$ and is therefore not directly applicable. The calculations have been carried out on the basis of the values obtained on the averages. It must be remembered, however, that when displacements enter in a sequent they frequently occur on values of the sequent which have already received a small displacement, in which case the separations themselves receive small changes. Too much weight must therefore not be given to the 417 case here, in which its actual value for the particular set is not obtainable. The $d\nu_2$, however, is certainly very small.

The discussion of the two F separations has given $\Delta_2 = 267{\cdot}713 - {\cdot}0126\xi \pm {\cdot}01$, $\delta = 14{\cdot}4710 - {\cdot}00068\xi \pm {\cdot}0005$. Let x denote the correction required on this value of Δ_2. Then

$$36\Delta_2 + 16\tfrac{3}{4}\delta = 9880{\cdot}057 - {\cdot}4653\xi + 36{\cdot}90x$$
$$42\Delta_2 + 2\tfrac{1}{2}\delta = 11280{\cdot}123 - {\cdot}5306\xi + 42{\cdot}13x$$
$$150\Delta_2 + 9\delta = 40287{\cdot}19 - 1{\cdot}896\xi + 150{\cdot}48x$$

Supposing that the true multiples of the oun are y greater, and putting $d\nu_1 = -{\cdot}0031 + d\nu_1$

$$36{\cdot}90x - {\cdot}017\xi + 27\,(d\nu_3 - d\nu_1) - {\cdot}40 + 3{\cdot}62y_3 = 0$$
$$42{\cdot}13x - {\cdot}025\xi - 27d\nu_2 + 1{\cdot}13 + 3{\cdot}62y_2 = 0$$
$$150{\cdot}48x - {\cdot}040\xi - 27d\nu_1 + {\cdot}75 + 3{\cdot}62y_1 = 0$$

or

$$x = {\cdot}010 + {\cdot}00046\xi - {\cdot}73d\nu_3 - {\cdot}097y_3$$
$$x = -{\cdot}027 + {\cdot}00057\xi + {\cdot}64d\nu_2 - {\cdot}086y_2$$
$$x = -{\cdot}0050 + {\cdot}00026\xi + {\cdot}18d\nu_1 - {\cdot}024y_1$$

In these ξ cannot be more than a few units, $d\nu < {\cdot}02$ and $x < {\cdot}01$. This can only happen if all the $y = 0$. Thus again there is the very important fact that the oun multiples are quite definite and are those used in the actual calculations. ξ is not large enough to affect the limits of accuracy in x. The separation 1070 is not so well determined as the others and $d\nu_3$ may well be $> {\cdot}01$. Thus the first and third can easily give the same values of x, but the second would require $d\nu_2$ of the order ·03, inadmissible if the ν_2 were accurately determined. But as a fact the average ν_2 as we have seen does not enter in the line here considered and it may be so large as to alter the multiple. The second may therefore be considered as not at disposal, and the third then gives very close limits, viz., with $d\nu_1 \not> {\cdot}01$

$$\Delta_2 = 267{\cdot}708 - {\cdot}0124\xi \pm {\cdot}002$$
$$\delta = 14{\cdot}4708 - {\cdot}0007\xi \pm {\cdot}0001$$

the same value, though with closer limits of accuracy, as was obtained from the 265 separation. With maximum $\xi = {\cdot}33$, $\delta = 14{\cdot}4708 + {\cdot}0003$.

But further, in addition to WATSON's separations, we have found affixed to the F series, another $= -1932$, and this must be tested. The linked line is given by B.M.M. as 6598·953 I.A. Still using 9-figure logarithims, the wave-number is 15149·7338 giving the separation 1932·2902, and corresponding to a limit 31852·1816 $-1932{\cdot}2902 = 29919{\cdot}8914 + \xi - d\nu$. The mantissa of this is $914613{\cdot}17 - 31{\cdot}997\,(\xi - d\nu)$. The displacement on 31852 is therefore $58982{\cdot}87 - 2{\cdot}867\xi + 32d\nu$. With the above values of Δ_2, δ, $220\Delta_2 + 6\delta = 58982{\cdot}58 - 2{\cdot}728\xi \pm {\cdot}44$. This again is an exact agreement.

The foregoing does not give Δ_2 with the desired definiteness unless the value of ξ is determined.. The reason is that it has been based on displacements on the same limit.

An independent datum is necessary to fix the value of ξ, which at present we know cannot be greater than a few units. In the other gases this is obtainable by the conditions that the mantissa of $f(1)$ is a multiple of Δ_2 and that of $F(\infty)$ of δ_1. The value of $F(1)$ is known with great exactness, probably to within less than ·001A. PRIEST gives $\lambda = 5852 \cdot 4862$, MEISSNER ·4875, B.M.M. ·488. They all agree within ·001 of ·487 so that the wave-number of $F(1)$ is $17082 \cdot 0242 + \cdot 0029p$ with p within ± 1. But here we have to answer the question whether our first F line 17082 is really the first of the series. If this F series—a very marked and definite one—belongs to the 1864XF type it has its first line for $m = 1$ in the ultra-red (calculated $n = -8168 \cdot 62 \lambda = 12242$) and far beyond the reach of any sounders. This uncertainty will therefore have to be kept in mind. The mantissa of 17082 must in any case be a multiple of δ_1, and may be of Δ_2 (if it is the first F line). In addition the D set 20551 may belong to that D satellite set with $M(\Delta_2)$, or since the ν_1 separation is modified, differ from a multiple of Δ_2 by a few ouns only. We have then

$$\text{Mantissa } F(\infty) = 855630 \cdot 30 - 29 \cdot 130\xi = M(\delta_1) \text{ and possibly } M(\Delta_2)$$
$$\text{,,} \quad f(1) = 725012 \cdot 36 - 92 \cdot 257\xi - \cdot 2675p = M(\delta_1) \text{ and possibly } M(\Delta_2).$$

The term in p will not affect our immediate purpose and may be omitted. There is evidence for $\xi < \cdot 33$. Also $\Delta_2 = 267 \cdot 708 - 0124\xi + x$ with $x < \cdot 002$. We find

$$2708\Delta_2 = 724953 \cdot 26 - 33 \cdot 58\xi + 2708x, \qquad 2708x < 5 \cdot 4$$
$$3196\Delta_2 = 855594 \cdot 77 - 39 \cdot 63\xi + 3196x, \qquad 3196x < 6 \cdot 4$$

Hence

$$\text{Mantissa of } F(\infty) = 3196\Delta_2 + 35 \cdot 5 + 10 \cdot 5\xi - 3196x$$
$$\text{,,} \quad f \quad = 2708\Delta_2 + 59 \cdot 10 - 58 \cdot 68\xi - 2708x.$$

A first definite result is that 20551 is not a D line of the $M(\Delta_2)$ type. Its mantissa must differ from such by at least 7 or 8 ouns. The coefficient of ξ is so large that the actual oun multiple cannot be uniquely decided. Further f cannot follow the Δ_2 multiple law unless ξ be of the order 1, or three times our estimated limit of variation. We cannot say that it is impossible. If however this F line is the first of the series then ξ must be of this magnitude. It is seen that no further definite and certain information can be obtained. It will however be of some interest to follow out the assumption that its mantissa is a $M(\Delta_2)$. In this case put $\xi = 1 + \xi$ where now ξ is small, and

$$\cdot 42 - 2708x - 58 \cdot 68\xi - \cdot 2675p = 0$$

The uncertainties in ξ are now $\cdot 42 \pm 5 \cdot 4 - 59\xi \pm \cdot 26 = 0$, or $\xi < \cdot 1$.

$$\Delta_2 = 267 \cdot 6957 - \cdot 0124\xi + x.$$

In conclusion some reference is necessary to the series due to ROSSI.* These

* *Loc. cit.*

consist of two doublet series each with separation of order 167 and a third singlet series. The two sets of doublet series appear to converge to the same limits, in the same way as the satellites of a D series. The first lines given by him are of the order $m = 3$, but amongst the B.M.M. as also the MEISSNER lines those for $m = 2$ are also seen. Further, there appear close strong companions to the second lines of doublets in the first series, and in this they recall the behaviour of the F lines in the alkaline earths* and so suggest that they are really lines of the F type, but belonging to a different f sequence from that discussed above. If so the limit must behave as a displaced value from the 31852. The sets are, in ROWLAND units:—

m.					
2			**165·44**	(8) 11932·92	
					1·73
	†	(4) 11767·48	**167·17**	(0) 4·65	
		107·63			
	†	(2) 11875·11	**168·29**	(4) 12043·40	
3				(8) 17342·68	
					1·06
	†	(5) 17176·58	**167·16**	(1) 3·74	
		46·39			
	†	(5) 17222·97	**168·21**	(5) 17391·18	
4	†	(6) 19677·64	**166·58**	(6) 19844·22	
		23·92			
	†	(4) 19701·56	**167·86**	(5) 19869·42	
5	·04	(5) 21033·94	**166·87**	(5) 21200·81	
		14·08			
	–·02	(4) 21048·02	**167·59**	(3) 21215·61	
6	·04	(2) 21850·62	**167·08**	(3) 22017·70	
		10·17			
	–·25	(1) 21860·79	**169·63**	(4) 22030·42	
7	·06	(3) 22380·18	**167·18**	(2) 22547·36	
		9·91			
	–·01	(0) 22386·29‡			
8	·07	(0) 22742·78	**167·31**	(0) 22910·09	
9	·20	[23168–167]		23168 (L.D.)	

* [III., p. 383, *seq.*]

† Lines used in calculating formulæ.

‡ This line was not included by ROSSI.

The formulæ calculated from the first three lines of each series give

$$n = 24104{\cdot}67 - \mathrm{N}/\{m + {\cdot}973110 + {\cdot}016932/m\}^2$$
$$n = 24104{\cdot}48 - \mathrm{N}/\{m + {\cdot}987215 + {\cdot}014948/m\}^2$$

The O−C values for these are given in the above list before the wave-numbers. They are larger than we should expect the observation errors to be. If, however, the limit be reduced by ·40 and μ, α calculated from the first two, the O−C for $m = 4$ is $-{\cdot}04$ and zero for all the others, $m = 5 \ldots 8$. For $m = 9$, using L.D. $23168 \pm 3 - 167$ as the observed for the first of the pair the O−C is ·2. The agreement for all is therefore exceedingly close except for $m = 4$. The calculated wave-number for this is 19677·50 which makes the separation 166·72 and more in step with the others. The uncertainty in the limit 24104·67 must therefore be very small.

The doublet separations show a tendency to converge with increasing order, but this is clearly due to the fact that the constant separation must be taken between the strong first line and the weak second. After $m = 3$ the weak is not seen and the observed separation is not the true one but that between the first and the second strong one. The separation in the second series is somewhat larger than in the first, that for the first being 167·17 and for the second $168{\cdot}25 \pm 04$ (mean for $m = 2, 3$). These require limit mantissæ changes of 7354 in the first where $308\tfrac{1}{4}\delta = 7354{\cdot}1$, and of $47{\cdot}79 \pm 1{\cdot}76$ extra for the other where $3\delta = 46{\cdot}41$. The separation 1·73 between the four satellites in the first doublet must be due to displacement in the sequent. It requires a mantissa change of 209 and $14\tfrac{1}{2}\delta = 209{\cdot}8$.

These considerations point strongly to the conclusion that the series are of the F type. Fortunately, owing to the fact that the first line of the first series has been very accurately measured by B.M.M., as $\lambda = 8495{\cdot}380$ I.A., it is possible to test if the mantissa of $f(2)$ is $\mathrm{M}(\Delta_2)$. Taking the observation error as $-{\cdot}001p$, the wave-number is $11767{\cdot}8680 + {\cdot}0014p$. Its mantissa with limit $24105{\cdot}0731 + \xi$ (the I measure of 4·2700 R.) is

$$981622{\cdot}48 - 120{\cdot}87\xi + {\cdot}17p = 3667\,\{267{\cdot}6910 - {\cdot}03296\xi + {\cdot}00005p\} = 3667\Delta_2$$

within the uncertainty of ξ.

The first line of the second series has not been measured so exactly. Its $\lambda = 8418{\cdot}38 - {\cdot}02p$; $n = 11875{\cdot}50 + {\cdot}028p$; mantissa $= 13092 - 1{\cdot}56\xi + 3{\cdot}52p$ larger than the other. Now $49\Delta_2 - 7\delta_1 = 13091{\cdot}8$. Hence if the two limits are the same 24104·89 (*i.e.*, put $\xi = -{\cdot}19$), the mantissa is larger by $49\Delta_2 - 7\delta_1 + {\cdot}51 + 3{\cdot}42p$. It is satisfied by $p = -{\cdot}14$ or $d\lambda = {\cdot}002$. But it is possible that these F are due to independent groups, *i.e.*, that the sequence of the second series also depends on a whole multiple of Δ_2. This cannot happen unless the two limits are different, which in fact seems to be the general rule. If the limit is displaced by $y\delta_1$, ξ is $3{\cdot}617y/44{\cdot}25 = {\cdot}082y$, and the mantissa difference from $49\Delta_2$ is now

$-7\delta_1+\cdot51+10\cdot04y+3\cdot42p = 0$ or $y = 2\cdot4-\cdot34p$. In other words $y = 2$, $p = 1$. This makes the limit of the second series less than that of the first by ·17, in agreement with the relative values as actually found by direct calculation from the observed lines. Although therefore the limits suggest equality, there is good evidence that the observed difference, though small, is real, and is due to a displacement of two ouns. The sequence depends on an origin of $3667+49 = 3716\Delta_2$.

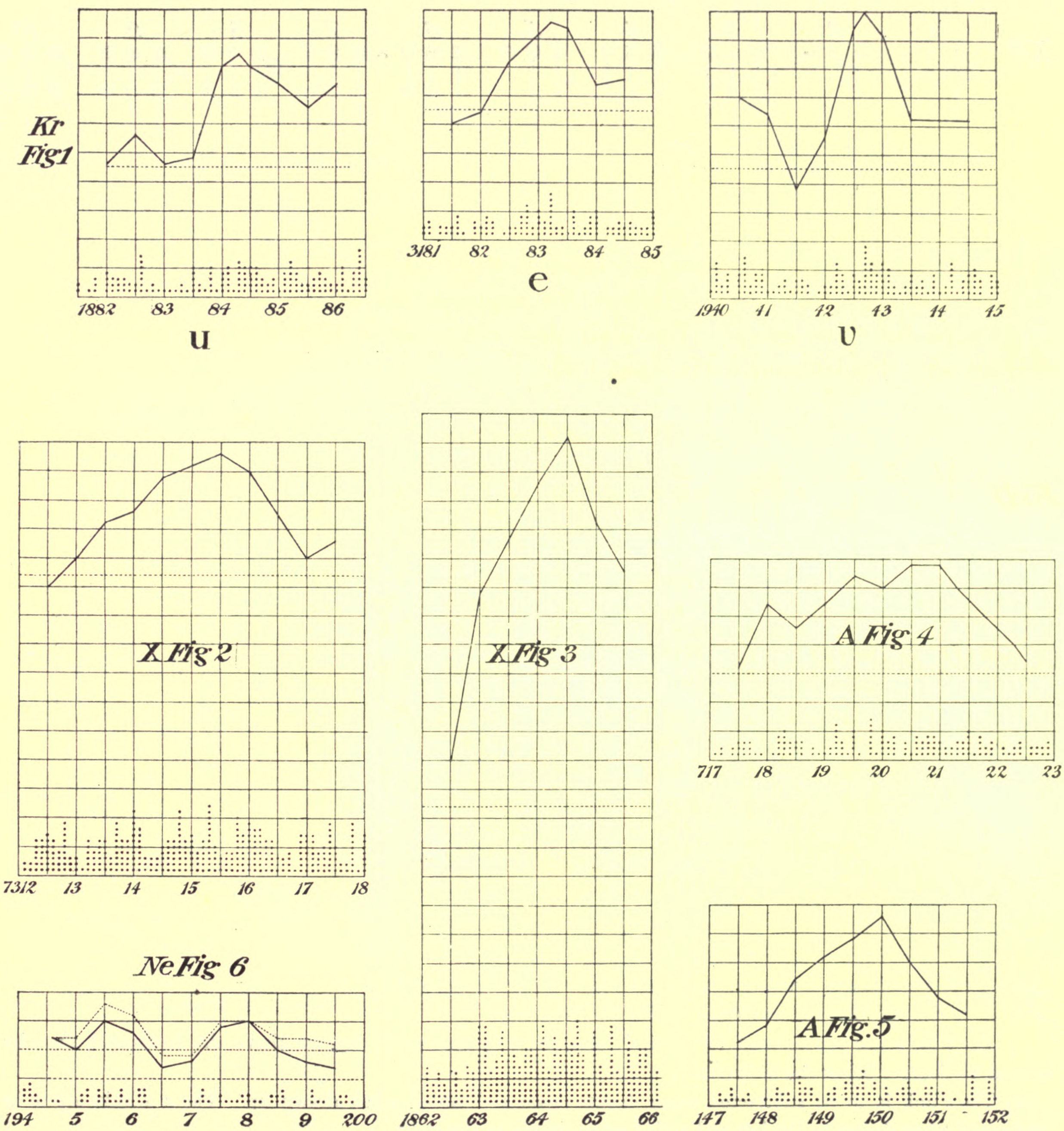
Rossi's third series is (6) 28788·08, (2) 34087·11, (1) 36536·62, (1) 37869·0, the last being observed by himself. The first three give

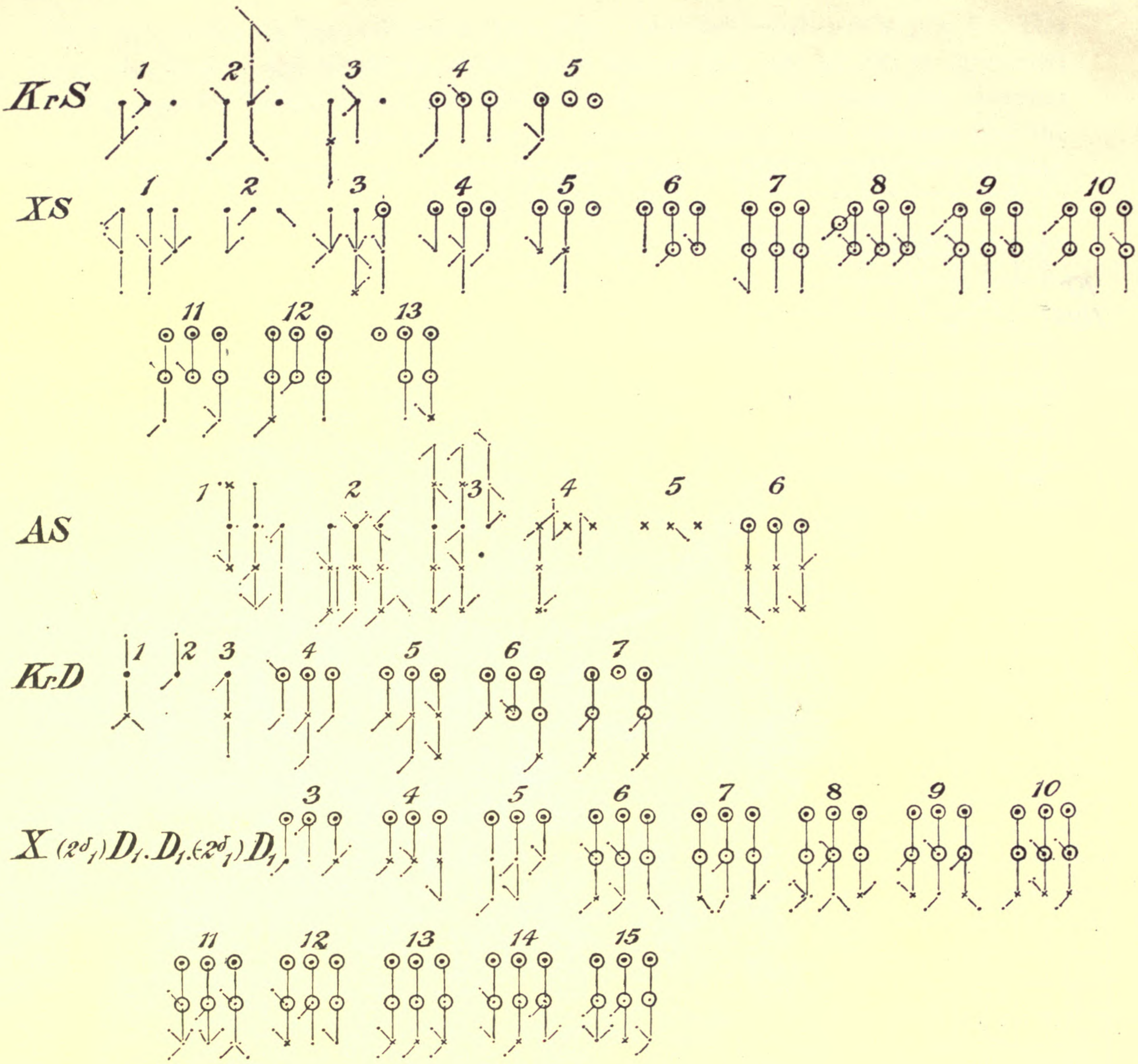
$$n = 40896\cdot73-\mathrm{N}/\{m+\cdot024118-\cdot043614/m\}^2$$

with $\mathrm{O}-\mathrm{C} = -\cdot36$ for the last, probably excessive. The first line is the line adopted as one of the S_1 (2) group above, the third is **F** (4). I feel some doubt, therefore, as to these forming a real series, although the sequence has all the appearance of the $s(m)$ type for these gases. The limit does not seem to have any relation to the doublet set. Its denominator is about 1·63.

TABLE of Constants.

	Ne.	A.	Kr.	X.	RaEm.
ν_1	49·24	179·50	786·45	1777·90	5370·7
ν_2	19·71	75·60	{ 309·20 341·16 }	815·05	2649·53
Δ_1	666	2519	10969	24893	72820
Δ_2	267·708	1057·057	{ 4242·18 4678·80 }	10998·14	32166·44
δ	14·47013	57·9209	249·536	611·0100	1787·024
Δ_1/δ	46	40¾	44	40¾	40¾
Δ_2/δ	18½	18¼	{ 17 18¾ }	18	18
e	191·5	719·71	3183·34	7314·1	23678·4
u	106·78 ?	439·47	1884·03	4133·18	11191·8
v	106·86 ?	442·67	1942·44	4428·00	13680·0
$\mathrm{S}(\infty)$	{ 52112 52403 }	51731·05	51651·29	51025·29	50403·00


Kr
Fig 1
1882
83
84
85
86
u
3181
82
83
84
85
e
1940
41
42
43
44
45
v
X Fig 2
7312
13
14
15
16
17
18
X Fig 3
1862
63
64
65
66
A Fig 4
717
18
19
20
21
22
23
A Fig 5
147
148
149
150
151
152
Ne Fig 6
194
5
6
7
8
9
200



EXPLANATION OF DIAGRAM.

(1) A dot represents a wave-number; a large dot the wave-numbers being sounded for.

(2) A dot to the side of a line denotes a displaced line.

(3) A circle round a dot denotes that it is in an unobserved region.

(4) A × denotes that the wave-number has not been seen.

(5) The e links are represented by vertical lines, the u by lines at 45 degrees above the horizontal, the v by lines at 45 degrees below the horizontal.

(6) The e link is to be subtracted when drawn down and added when drawn up. The u, v links are to be subtracted when drawn to the left and added when drawn to the right.

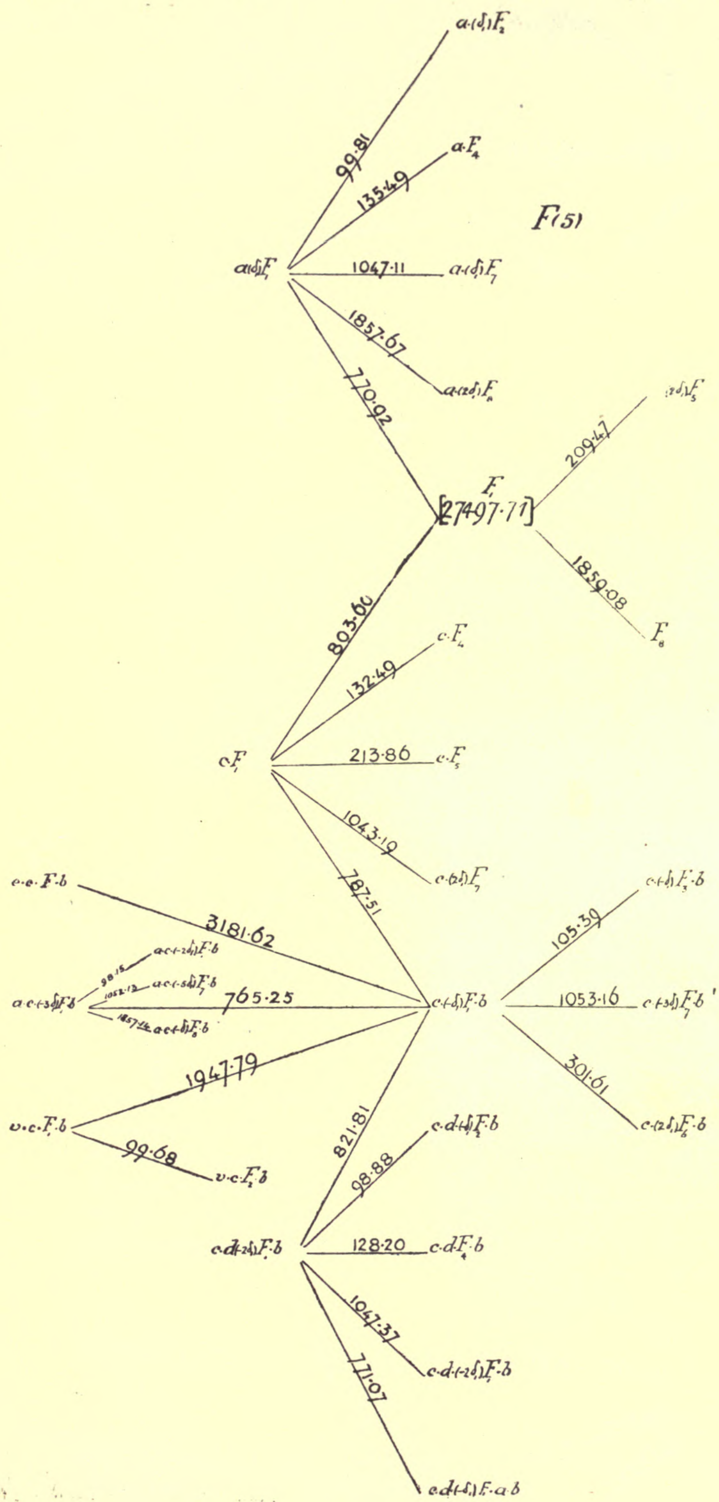
E.g., XS_2 (3) is seen; also the lines linked to it by $-e$, $-e-u$, $-e+u$, $-e+v$; that linked by $-2e$ is not seen, but by $-2e+u$ is.

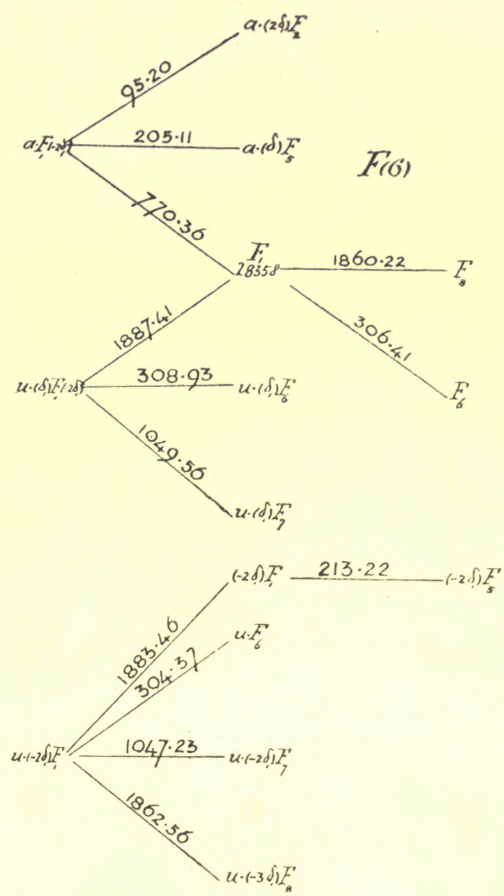
XS_3 (3) is out of the observed region, but the lines linked to it by $-e$, $-e-u$, $-2e$ are in the observed region and have been seen.

AS contains several examples of displaced lines.

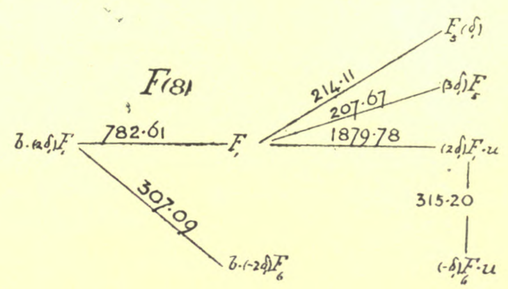
468 b

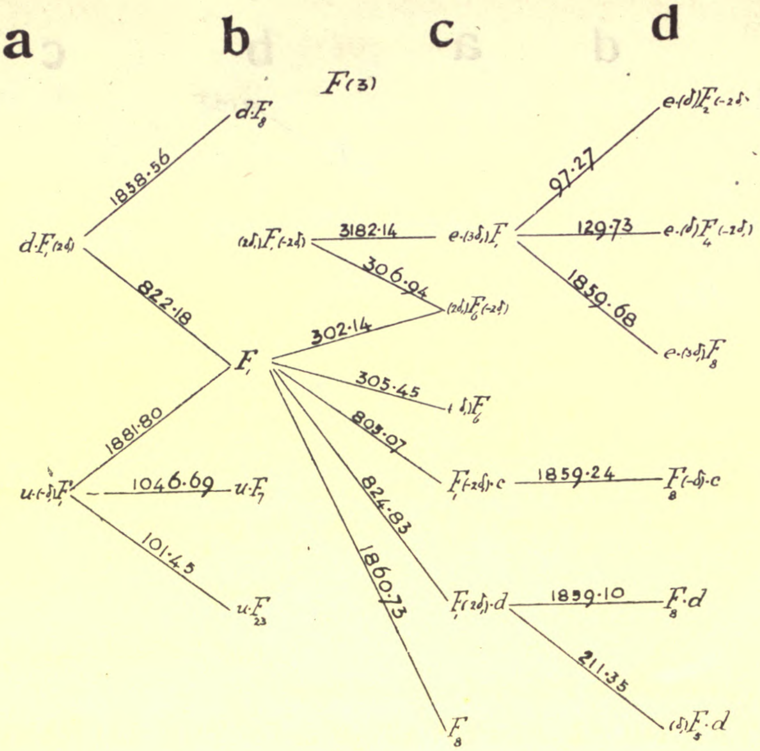
a b c d a b c

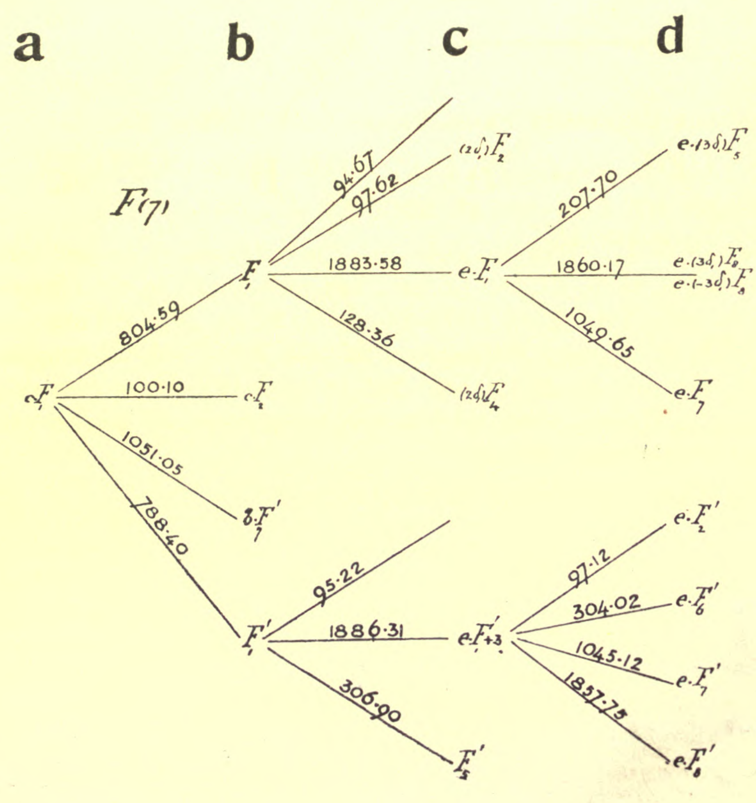


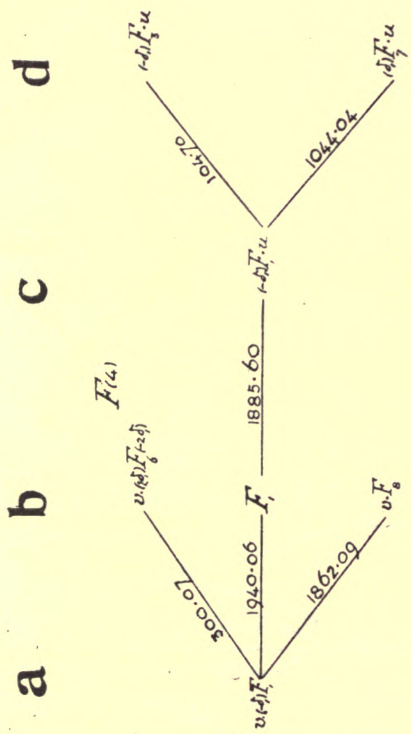


a b c




a
b
c
d
F(3)
1858·56
822·18
1881·80
1046·69
101·45
3182·14
306·94
302·14
305·45
803·07
824·83
1860·73
97·27
129·73
1859·68
1859·24
1859·10
211·35


a
b
c
d
F(7)
804·59
100·10
1051·05
788·40
94·67
97·62
1883·58
128·36
207·70
1860·17
1049·65
95·22
1886·31
306·90
97·12
304·02
1045·12
1857·75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b
c
d
F(4)
300·07
1940·06
1862·09
1885·60
1041·70
1044·04

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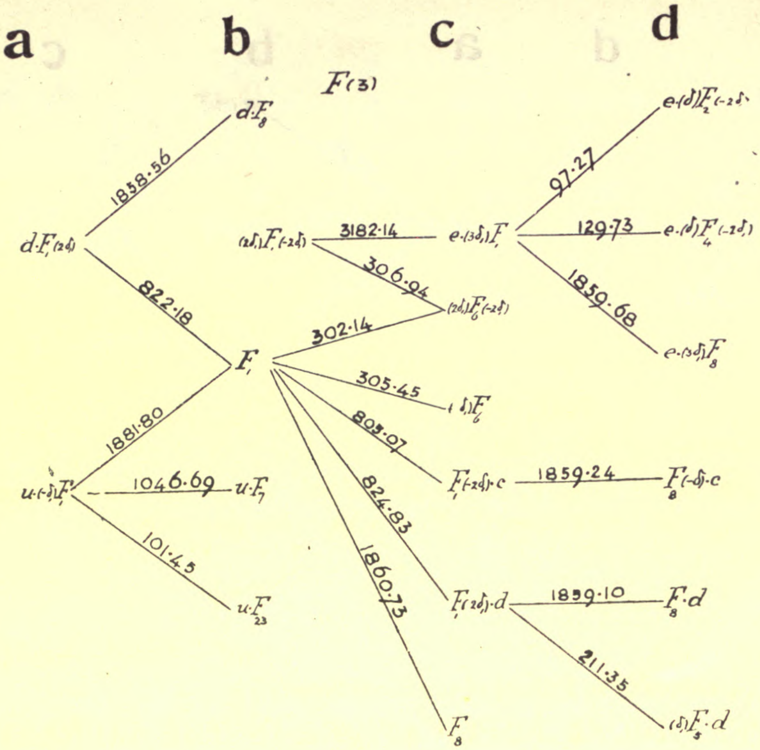
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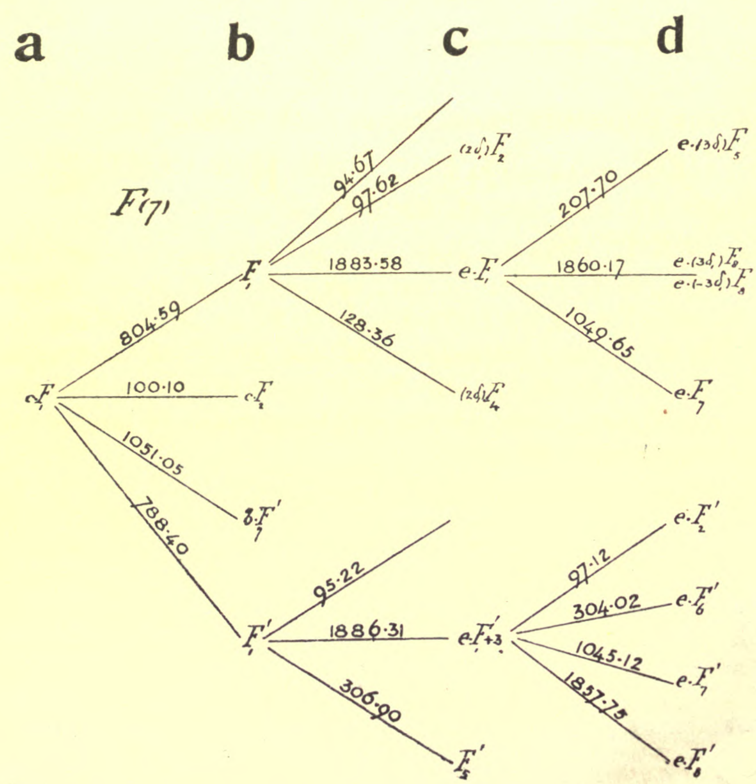
211·35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1883·58
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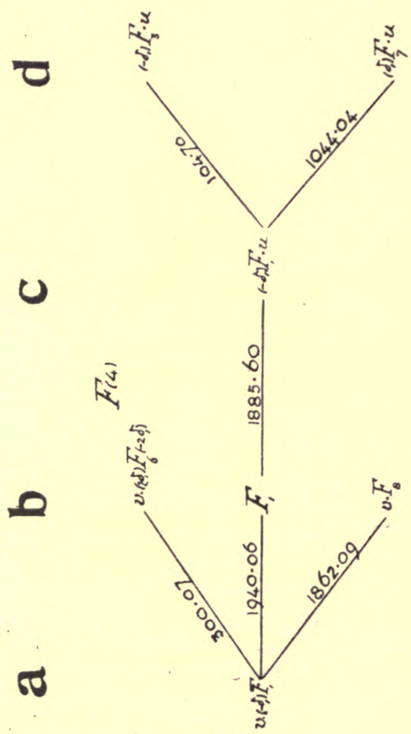
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b
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104·70
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