THE PRINCIPLES OF FLUXIONS:

DESIGNED FOR THE USE OF

STUDENTS

IN

THE UNIVERSITY.

BY THE

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THE FIRST AMERICAN EDITION.

CORRECTED AND ENLARGED.

PHILADELPHIA,

PUBLISHED BY KIMBER AND CONRAD,

NO. 93, MARKET STREET.

T. & G. Palmer, printers.

1812.

PREFACE.

IN offering to the public a revised edition of Vince's Fluxions, the correction of typographical errors is the only *alteration* which the editor has ventured to make: of these, a considerable number has been detected. The subjoined annotations were designed to elucidate the principles of the science, and therefore relate chiefly to the fundamental propositions; and although the adept may recognize, in these remarks, some repetition of the reasoning in the text, yet, to the student who is just entering upon the subject, it is hoped, they may prove a useful appendage.

CONTENTS.

THE

PRINCIPLES OF FLUXIONS.

SECTION I.

DEFINITIONS.

ARTICLE 1. Every quantity is here considered as generated by motion; a line by the motion of a point; a surface by the motion of a line; a solid by the motion of a surface*.

2. The quantity thus generated is called the *fluent*, or *flowing* quantity.

3. The *velocities* with which flowing quantities increase or decrease at any point of time, are called the *fluxions* of those quantities at that instant.

Cor. 1. As the velocities are in proportion to the increments or decrements *uniformly* generated in a given time, such increments or decrements will represent the fluxions†.

* Sir I. Newton, in the introduction to his *Quadrature of Curves*, observes, that "these geneses really take place in the nature of things, and are daily seen in the motion of bodies. And after this manner the ancients, by drawing moveable right lines along immoveable right lines, taught the genesis of rectangles."

† This is agreeable to Sir I. Newton's ideas on the subject. He says, "I sought a method of determining quantities from the velocities of the motions or increments with which they are generated; and calling these velocities of the motions or increments, *fluxions*, and the generated quantities *fluents*, I fell by degrees upon the method of fluxions."—Introd. to *Quad. Curves.*

Cor. 2. Hence, as any given time may be assumed, the fluxion is not an *absolute* but a *relative* quantity. When we have several cotemporary fluxions, we may assume one fluxion what we please, and thence determine the values of the others. Thus, if x and y increase uniformly, and if x increase by p in the time that y increases by q, then the cotemporary increments of x and y will be p and q, $2\,p$ and $2\,q$, $3\,p$ and $3\,q$, &c. hence, if p be assumed the fluxion of x, the fluxion of y will be q; if the former fluxion be $2\,p$, the latter will be $2\,q$, &c. &c.

Cor. 3. A *constant* quantity has no fluxion.

4. The first letters, a, b, c, &c. of the alphabet are usually put for constant quantities, and the last, v, w, x, y, z, for variable ones; and they are to be thus understood, unless the contrary be expressed.

5. The fluxion of a simple quantity, as x, is expressed by placing a point over it, thus $\dot{x}$.

To find the FLUXIONS of QUANTITIES.

Prop. I.

If two quantities increase or decrease uniformly, the increments or decrements generated in a given time, will be as their fluxions.

6. This appears from Art. 3. Cor. 1.

Prop. II.

If one quantity increase uniformly, and another of the same kind increase with an accelerated or retarded velocity, and two increments be assumed which are generated in the same time; if those increments be diminished till they vanish, that ratio to which they approach as their limit, is the ratio of the fluxions of those quantities.

7. Let the line FK be described with an uniform velocity, and AZ with an accelerated velocity, and let the increments G*s*, P*m* be generated in the same time; let also P*v* be the increment that would have

been generated in the same time, if the velocity at P had been continued uniform; then by Prop. I. the fluxions of FK, AZ, at the points G and P, will be represented by G*s* and P*v*. Let V be the velocity at P, or the velocity with which P*v* is described, and let *r* be the increase of velocity from P to *m*; then the velocity at *m* will be $V+r$, and *vm* is the increment which is described in consequence of the increase *r* of velocity since the describing point left P. Now let $V+w$ be the *uniform* velocity with which P*m* would be described in the same time that P*v* and P*m* are described, as before mentioned; then it is manifest, that this uniform velocity must be between the velocities at P and *m*, that is, $V+w$ is greater than V and less than $V+r$, or *w* is greater than *o* and less than *r*. Also, since the spaces described in the same time are as the velocities, $V : V+w :: Pv : Pm$*. Now

* If we diminish the times in which these increments are described; then as the points *v* and *m* approach to P, P*v* will continue to be described with the uniform velocity V; but *r* will be diminished, and by diminishing the time till it becomes indefinitely small, *r* will become indefinitely small; but *vm* is described in consequence of this increase *r* of velocity; hence, when *r* becomes indefinitely small in respect to V, the space *vm* must become indefinitely small in respect to P*v*; therefore the ratio of P*v* : P*m* is, in that state, indefinitely near to a ratio of equality; but it is manifest that it never can become accurately a ratio of equality, because *vm* will not vanish until P*v* and P*m* vanish; consequently the ratio of the actual increments G*s* : P*m* can never accurately express the ratio of the fluxions, that ratio being expressed by the ratio of G*s* : P*v*. We are therefore to consider, to what ratio

in *every* state of these increments, $V : V+w :: Pv : Pm$; and by continually diminishing the time, and consequently the increments, we diminish r and w, but V remains constant; it is manifest therefore that the ratio of $V : V+w$, and consequently that of $Pv : Pm$, continually approaches towards a ratio of equality, agreeably to what is shown in the note; and when the time, and consequently the increments, become actually $=0$, then $r=0$; consequently $w=0$; therefore the *limit* of the ratio of $Pv : Pm$ becomes that of $V : V$, a ratio of equality*. Hence, the *limit* of the ratio of $Gs : Pm$ is the same as the *limit* of the ratio of $Gs : Pv$, or it is $Gs : Pv$, that ratio being constant; that is, *the limiting ratio of the increments is the ratio of the fluxions.*

The same is manifestly true for the limiting ratio of the decrements of two quantities; for, conceiving the describing points to move backwards, the decrements sG, mP in this case become the same as the increments in the other; consequently their *limiting* ratio will express the ratio of the fluxions at G and P, or the rate at which FG, AP are, at that instant, decreasing.

Hence, the *limiting* ratio of the increments or decrements of two quantities which are *both* generated by variable velocities, will be the ratio of their fluxions. And as the velocities with which these two lines increase or decrease, may be made to agree with the rate of increase or decrease of any two quantities which may be compared together, the proposition must be true for quantities of any kind.

Cor. As the *limiting* ratio of the increments is the

$Pv : Pm$ approaches as its *limit*, when we make the time in which the increments are described, and consequently the increments themselves, vanish.

* By keeping the ratio of the vanishing quantities thus expressed by finite quantities, it removes the obscurity which may arise when we consider the quantities themselves; this is agreeable to the reasoning of Sir I. Newton in his *Principia*, Lib. I. Sect. i. Lem. 7, 8, 9.

ratio of the fluxions, it is manifest that when the increments are in an increasing or decreasing state, the fluxions will be increasing or decreasing.

8. It has been said, that when the increments are actually vanished, it is absurd to talk of any ratio between them. It is true; but we speak not here of any ratio then existing between the quantities, but of that ratio to which they have approached as their *limit;* and that ratio still remains. Thus, let the increments of two quantities be denoted by ax^2+mx and bx^2+nx; then the *limit* of their ratio, when $x=0$, is $m:n$; for in every state of these quantities, $ax^2+mx : bx^2+nx :: ax+m : bx+n ::$ (when $x=0$) $m:n$. As the quantities therefore approach to nothing, the ratio approaches to that of $m:n$ as its *limit.* Hence, if $m=n$, the *limit* of this ratio is a ratio of equality. We must therefore be careful to distinguish between the ratio of two evanescent quantities, and the *limit* of their ratio; the former ratio never arriving at the latter, as the quantities vanish at the instant that such a circumstance is about to take place.

Prop. III.

If the fluxion of x be denoted by $\dot{x}$, the fluxion of ax will be $a\dot{x}$.

9. For if x increase uniformly, ax will also increase uniformly, and a times as fast; hence, by Prop. I. the fluxion of the latter will be a times that of the former, or it will be $a\dot{x}$.

Cor. Hence, in taking the fluxion of a variable quantity multiplied into a constant one, the constant multiplier is retained.

Prop. IV.

The fluxion of $x\pm a$ is $\dot{x}$.

10. For a being constant, and only connected to

x by the signs + or —, it does not affect the *increase* or *decrease* of the quantity; therefore the fluxion is the same as the fluxion of x, or it is $\dot{x}$.

Cor. Hence, constant quantities connected to variable ones by the signs + or —, disappear when the fluxions are taken.

PROP. V.

Given ($\dot{x}$) the fluxion of x, to find the fluxion of x^n, n being a whole number.

11. Let x increase uniformly by v and become $x+v$, then will x^n become $\overline{x+v}|^n$; but (*Algebra*, Art. 232.) $\overline{x+v}|^n = x^n + nx^{n-1}v + n.\frac{n-1}{2}x^{n-2}v^2 + \&c.$ and if from this quantity we take x^n, there remains $nx^{n-1}v + n.\frac{n-1}{2}x^{n-2}v^2 + \&c.$ for the cotemporary increment of x^n; but although x increases uniformly by v, x^n does not increase uniformly; for if in the increment of x^n we substitute 1, 2, 3, &c. for v, and take the differences of the results, these differences will not be equal; hence, to get the ratio of the fluxion of x to the fluxion of x^n we must, according to Prop. 2. take the *limiting* ratio of the increments. Now the increment of x : the increment of x^n :: v : $nx^{n-1}v + n.\frac{n-1}{2}x^{n-2}v^2 + \&c.$ :: 1 : $nx^{n-1} + n.\frac{n-1}{2}x^{n-2}v + \&c.$ and to get the *limiting* ratio of these increments, we must make $v=0$, in which case the ratio becomes $1 : nx^{n-1}$, which therefore expresses the ratio of the fluxion of $\dot{x}$ to the fluxion of x^n; but $\dot{x}$ denotes the fluxion of x, therefore $nx^{n-1}\dot{x}$ represents the cotemporary fluxion of x^n.

If $n=0$, $x^n=1$ a constant quantity; therefore by Art. 3. Cor. 3. it has no fluxion.

Prop. VI.

To find the fluxion of $x^{\frac{n}{m}}$, *m and n being any whole numbers.*

12. Put $y=x^{\frac{n}{m}}$, then $y^m=x^n$; hence, by taking the fluxions, $my^{m-1}\dot{y}=nx^{n-1}\dot{x}$, $\therefore\ \dot{y}=\frac{nx^{n-1}\dot{x}}{my^{m-1}}=$ (by substituting for y its value in terms of x) $\frac{nx^{n-1}\dot{x}}{mx^{\frac{nm-n}{m}}}=$ $\frac{nx^{n-1}\dot{x}}{mx^{n-\frac{n}{m}}}=\frac{n}{m}\times x^{\frac{n}{m}-1}\dot{x}$.

Cor. Let the root be a compound quantity as a^m+x^m, to find the fluxion of $\overline{a^m+x^m}|^{\frac{1}{n}}$. Put $y=\overline{a^m+x^m}|^{\frac{1}{n}}$, then $y^n=a^m+x^m$, and $ny^{n-1}\dot{y}=mx^{m-1}\dot{x}$; hence, $\dot{y}=\frac{mx^{m-1}\dot{x}}{ny^{n-1}}=\frac{mx^{m-1}\dot{x}}{n\times\overline{a^m+x^m}|^{\frac{n-1}{n}}}=\frac{1}{n}\times\overline{a^m+x^m}|^{\frac{1-n}{n}}\times$ $mx^{m-1}\dot{x}=\frac{1}{n}\times\overline{a^m+x^m}|^{\frac{1}{n}-1}\times mx^{m-1}\dot{x}$.

13. Hence it appears, that whether the root be a simple or a compound quantity, the fluxion of any power thereof is found by the following

RULE:

Multiply by the index, diminish the index by unity, and multiply by the fluxion of the root.

EXAMPLES.

Ex. 1. The fluxion of x^9 is $9x^8\dot{x}$.
Ex. 2. The fluxion of $3y^5$ is $15y^4\dot{y}$.
Ex. 3. The fluxion of $\frac{3}{2}y^{\frac{4}{7}}$ is $\frac{12}{14}y^{-\frac{3}{7}}\dot{y}=\frac{6\dot{y}}{7y^{\frac{3}{7}}}$.

Ex. 4. The fluxion of $\frac{5}{9}x^{\frac{7}{11}}$ is $\frac{35}{99}x^{-\frac{4}{11}}\dot{x}=\frac{35\dot{x}}{99x^{\frac{4}{11}}}$.

Ex. 5. The fluxion of $\frac{4}{7}x^{\frac{11}{9}}$ is $\frac{44}{63}x^{\frac{2}{9}}\dot{x}$.

Ex. 6. What is the fluxion of $\overline{a^2+x^2}|^3$?

Here the root is a^2+x^2, and its fluxion $2x\dot{x}$; hence, the fluxion required is $3\times\overline{a^2+x^2}|^2\times 2x\dot{x}=\overline{a^2+x^2}|^2\times 6x\dot{x}$.

Ex. 7. What is the fluxion of $\sqrt{a^2+x^2}$, or of $\overline{a^2+x^2}|^{\frac{1}{2}}$?

Here the root is a^2+x^2, and its fluxion $2x\dot{x}$; hence, the fluxion is $\frac{1}{2}\times\overline{a^2+x^2}|^{-\frac{1}{2}}\times 2x\dot{x}=\frac{x\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}}$.

Ex. 8. What is the fluxion of $\overline{x^2+y^2}|^{\frac{3}{2}}$?

Here the root is x^2+y^2, and its fluxion $2x\dot{x}+2y\dot{y}$; hence, the fluxion required is $\frac{3}{2}\times\overline{x^2+y^2}|^{\frac{1}{2}}\times\overline{2x\dot{x}+2y\dot{y}}$ $=3\times\overline{x^2+y^2}|^{\frac{1}{2}}\times\overline{x\dot{x}+y\dot{y}}$.

Ex. 9. What is the fluxion of $\overline{x+y}|^2$?

Here the root is $x+y$, and its fluxion $\dot{x}+\dot{y}$; hence, the fluxion required is $2\times\overline{x+y}\times\overline{\dot{x}+\dot{y}}$.

Ex. 10. What is the fluxion of $\overline{a^5+x^5}|^{\frac{1}{2}}$?

Here the root is a^5+x^5, and its fluxion $5x^4\dot{x}$; hence, the fluxion required is $\frac{1}{2}\times\overline{a^5+x^5}|^{-\frac{1}{2}}\times 5x^4\dot{x}=\frac{5x^4\dot{x}}{2\times\overline{a^5+x^5}|^{\frac{1}{2}}}$

Ex. 11. What is the fluxion of $\frac{1}{\overline{a^2+x^2}|^{\frac{5}{9}}}$?

This quantity becomes $\overline{a^2+x^2}|^{-\frac{5}{9}}$, and the root is a^2+x^2, whose fluxion is $2x\dot{x}$; hence, the fluxion required is $-\frac{5}{9}\times\overline{a^2+x^2}|^{-\frac{14}{9}}\times 2x\dot{x}=\frac{-10x\dot{x}}{9\times\overline{a^2+x^2}|^{\frac{14}{9}}}$. In like manner, bring any quantity from the denominator up to the numerator, by changing the sign of the index, and then proceed by the rule.

Ex. 12. What is the fluxion of $\overline{ax^2+by^3+cz^4}|^{\frac{7}{3}}$?

Here the root is $ax^2+by^3+cz^4$, and its fluxion $2ax\dot{x}+3by^2\dot{y}+4cz^3\dot{z}$; hence, the fluxion required is $\frac{7}{3}\times\overline{ax^2+by^3+cz^4}|^{\frac{4}{3}}\times\overline{2ax\dot{x}+3by^2\dot{y}+4cz^3\dot{z}}$.

13. What is the fluxion of $\sqrt{x^2+\sqrt{a^2+y^2}}$?

Put $z=\sqrt{x^2+\sqrt{a^2+y^2}}$, then $z^2=x^2+\sqrt{a^2+y^2}$; now the fluxion of $\sqrt{a^2+y^2}$, or of $\overline{a^2+y^2}|^{\frac{1}{2}}$, is $\frac{1}{2}\times$ $\overline{a^2+y^2}|^{-\frac{1}{2}}\times 2y\dot{y}=\overline{a^2+y^2}|^{-\frac{1}{2}}\times y\dot{y}$; hence, $2z\dot{z}=2x\dot{x}$ $+\overline{a^2+y^2}|^{-\frac{1}{2}}\times y\dot{y}$, therefore $\dot{z}=\frac{2x\dot{x}+\overline{a^2+y^2}|^{-\frac{1}{2}}\times y\dot{y}}{2z}=$ $\frac{2x\dot{x}+\overline{a^2+y^2}|^{-\frac{1}{2}}\times y\dot{y}}{2\sqrt{x^2+\sqrt{a^2+y^2}}}$.

Prop. VII.

To find the fluxion of a product xy.

14. The fluxion of $\overline{x+y}|^2$, by the last rule, is $2\times\overline{x+y}\times\overline{\dot{x}+\dot{y}}=2x\dot{x}+2x\dot{y}+2y\dot{x}+2y\dot{y}$; also, $\overline{x+y}|^2$ $=x^2+2xy+y^2$, whose fluxion is $2x\dot{x}+$ *the fluxion of* $2xy+2y\dot{y}$; make these two values of the fluxion of $\overline{x+y}|^2$ equal to each other, omit the first and last terms which are common to both, and we have *the fluxion of* $2xy=2x\dot{y}+2y\dot{x}$; hence, the fluxion of xy is $x\dot{y}+y\dot{x}$.

Otherwise thus. If we suppose x constant, the fluxion of xy is $x\dot{y}$ by Prop. 3; and if we suppose y constant, the fluxion is $y\dot{x}$; hence, if neither be constant, the fluxion is $x\dot{y}+y\dot{x}$.

Cor. Hence, we may find the fluxion of xyz. For if $v=xyz$, and $w=xy$, then $v=wz$, and $\dot{v}=w\dot{z}+$

$z\dot{w}$; but $w = xy$, $\therefore \dot{w} = x\dot{y} + y\dot{x}$; substitute these values for w and $\dot{w}$, and we get $\dot{v} = xy\dot{z} + zx\dot{y} + zy\dot{x}$.

15. In like manner we proceed for any number of factors; hence, the fluxion of the product of any number of quantities is found by the following

RULE:

Multiply the fluxion of each quantity into the product of all the rest, and the sum of all the products is the fluxion required.

EXAMPLES.

Ex. 1. The fluxion of x^2y^3 is $x^2 \times 3y^2\dot{y} + y^3 \times 2x\dot{x}$ $= 3x^2y^2\dot{y} + 2y^3x\dot{x}$.

Ex. 2. The fluxion of $y^{\frac{7}{2}}x^{\frac{5}{3}}z$ is $x^{\frac{5}{3}}z \times \frac{7}{2}y^{\frac{5}{2}}\dot{y} + y^{\frac{7}{2}}z \times \frac{5}{3}x^{\frac{2}{3}}\dot{x} + y^{\frac{7}{2}}x^{\frac{5}{3}}\dot{z} = \frac{7}{2}x^{\frac{5}{3}}zy^{\frac{5}{2}}\dot{y} + \frac{5}{3}y^{\frac{7}{2}}zx^{\frac{2}{3}}\dot{x} + y^{\frac{7}{2}}x^{\frac{5}{3}}\dot{z}$.

Ex. 3. The fluxion of $w^mx^ny^rz^s$ is $mx^ny^rz^sw^{m-1}\dot{w} + nw^my^rz^sx^{n-1}\dot{x} + rw^mx^nz^sy^{r-1}\dot{y} + sw^mx^ny^rz^{s-1}\dot{z}$.

Ex. 4. To find the fluxion of $x^2 \times \overline{a^4+y^4}\big|^{\frac{3}{2}}$.

By the last rule, the fluxion of $\overline{a^4+y^4}\big|^{\frac{3}{2}}$ is $\frac{3}{2} \times \overline{a^4+y^4}\big|^{\frac{1}{2}} \times 4y^3\dot{y} = 6 \times \overline{a^4+y^4}\big|^{\frac{1}{2}} \times y^3\dot{y}$; hence, the fluxion required is $x^2 \times 6 \times \overline{a^4+y^4}\big|^{\frac{1}{2}} \times y^3\dot{y} + \overline{a^4+y^4}\big|^{\frac{3}{2}} \times 2x\dot{x}$.

Ex. 5. To find the fluxion of $\sqrt{a^2+x^2} \times \sqrt{b^2+y^2}$.

Find the fluxion of each part by the last rule, and the fluxion required is $\sqrt{a^2+x^2} \times \frac{y\dot{y}}{\sqrt{b^2+y^2}} + \sqrt{b^2+y^2} \times \frac{x\dot{x}}{\sqrt{a^2+x^2}}$.

16. It appears from this Prop. that the fluxion of xy consists of two parts, $x\dot{y}$ and $y\dot{x}$, the former part arising from the increase of y by $\dot{y}$, and the latter from the increase of x by $\dot{x}$; but if x should decrease whilst y increases, then the fluxion, expressing the *increase* of xy upon the whole, will be $x\dot{y} - y\dot{x}$, being the increase *minus* the decrease. Hence, to express

the rate at which any quantity *increases*, the fluxion of the parts which increase must be written with the sign +, and those which decrease with the sign —*. Now the increasing quantity is considered as positive; but if a negative quantity increase in magnitude, it must be considered as a decreasing quantity, and its fluxion will be negative. In like manner, a negative quantity decreasing in magnitude must be considered as an increasing quantity, and its fluxion will be positive. If therefore the fluxions of increasing quantities be written with the sign +, and of decreasing with —, whenever the fluxion of any quantity is positive, it shows that quantity to be in an increasing state; and, when negative, to be in a decreasing state. In like manner, if $x^2+y^2=$ a constant quantity, then if x decrease and y increase, the fluxion is $-2x\dot{x}+2y\dot{y}=0$.

Prop. VIII.

To find the fluxion of a fraction $\frac{x}{y}$.

17. Put $z=\frac{x}{y}$, then $zy=x$, and $z\dot{y}+y\dot{z}=\dot{x}$ (Art. 14.); $\therefore\ \dot{z}=\frac{\dot{x}-z\dot{y}}{y}=\frac{\dot{x}-\frac{x}{y}\times\dot{y}}{y}=\frac{y\dot{x}-x\dot{y}}{y^2}$. Hence we find the fluxion of a fraction by the following

RULE:

From the fluxion of the numerator multiplied into the denominator, subtract the fluxion of the denominator multiplied into the numerator, and divide by the square of the denominator.

EXAMPLES.

Ex. 1. The fluxion of $\frac{x^2}{y^3}$ is $\frac{2y^3x\dot{x}-3x^2y^2\dot{y}}{y^6}=\frac{2yx\dot{x}-3x^2\dot{y}}{y^4}$.

* Hence it appears, that when a quantity passes through a maximum or minimum, the fluxion on each side has a different sign.

Ex. 2. The flux. of $\frac{x+y}{z^3}$ is $\frac{z^3 \times \overline{\dot{x}+\dot{y}} - \overline{x+y} \times 3z^2\dot{z}}{z^6}$ $= \frac{z \times \overline{\dot{x}+\dot{y}} - \overline{x+y} \times 3\dot{z}}{z^4}$.

Ex. 3. The flux. of $\frac{xy}{z^2}$ is $\frac{z^2 \times \overline{x\dot{y}+y\dot{x}} - xy \times 2z\dot{z}}{z^4}$ $= \frac{z \times \overline{x\dot{y}+y\dot{x}} - 2xy\dot{z}}{z^3}$.

Ex. 4. The fluxion of $\frac{a}{x}$ is $\frac{-a\dot{x}}{x^2}$; for a being constant, the fluxion of the numerator is nothing, and therefore the fluxion of the numerator multiplied into the denominator is nothing; in this case, therefore, the fluxion of the fraction is *minus* the fluxion of the denominator multiplied into the numerator, divided by the square of the denominator.

Ex. 5. The fluxion of $\frac{1}{x^n}$ is $\frac{-nx^{n-1}\dot{x}}{x^{2n}} = -\frac{n\dot{x}}{x^{n+1}} = -nx^{-n-1}\dot{x}$; or the fluxion of $x^{-n} = -nx^{-n-1}\dot{x}$; when therefore the index of a quantity is negative, the fluxion is found by the same rule (Art. 13.) as when the index is positive.

Ex. 6. The fluxion of $\frac{\sqrt{a^2+x^2}}{\sqrt{b^2+y^2}}$ is

$$\frac{\overline{a^2+x^2}|^{-\frac{1}{2}} \times x\dot{x} \times \sqrt{b^2+y^2} - \overline{b^2+y^2}|^{-\frac{1}{2}} \times y\dot{y} \times \sqrt{a^2+x^2}}{b^2+y^2}$$

$$= \frac{x\dot{x}}{\sqrt{a^2+x^2} \times \sqrt{b^2+y^2}} - \frac{\sqrt{a^2+x^2} \times y\dot{y}}{\overline{b^2+y^2}|^{\frac{3}{2}}}.$$

The putting of a quantity into fluxions is called the *direct* method of fluxions.

SCHOLIUM.

18. In questions of a *geometrical* and *philosophical* nature, where we want to get the relation of the fluents from the fluxions, and in others where we want to find whether quantities are positive or negative from the relation of them to their fluxions, it is necessary to pay regard to the *signs* of the fluxions, as explained in Art. 16. But in putting equations into fluxions, as in the problems de Maximis et Minimis, although one variable quantity may increase at the same time that another decreases, yet we may write the fluxion of each positive; for, by writing it so in each equation, in order to obtain the same fluxion from the different equations, the result will not be altered. In these, and such like cases, we may therefore make the fluxion of each quantity positive. We may further observe, that when any fluxion becomes negative according to the above rule, the quantity which expresses its value becomes negative. For instance, if $r =$ the radius of a circle, $x =$ the versed sine, $y =$ the right sine of an arc, then $y^2 = 2rx - x^2$, and $\dot{y} = \frac{r\dot{x} - x\dot{x}}{y}$; now, for the first quadrant, x and y increase, and each fluxion is positive; and the value of $\dot{y}$ is positive, x being less than r; but in the second quadrant, y decreases and its fluxion becomes negative, and its value becomes negative, x being greater than r. This circumstance is similar to the case of a quantity passing through o and changing its sign, for $\dot{y} = o$ at the end of the quadrant.

19. When we compare the fluxions of two quantities, by comparing the increments that would be *uniformly* generated in a given time, the quantities have been supposed to be homogeneous, there being no relation between those which are not homogeneous; yet if, of two heterogeneous quantities, the *numerical value* of one be expressed in terms of the other, it is

manifest that there will be no impropriety in expressing the fluxion of one in terms of the fluxion of the other. If one side of a right-angled parallelogram be represented by 6 and the other by 9, we say, $6\times 9=54$, the area; our numerical operation is perfectly correct, but no one ever imagined that the units represented by 54 are homogeneous to the units represented by 6 and 9; if 6 and 9 represent inches in *length*, 54 will represent so many *square* inches, or so many *square* areas, the side of each of which is 1 inch in length. Or if a and x represent the two sides, the area of the parallelogram will actually be ax, referring that quantity to its proper units; although, therefore, there is no relation between the area and either of its sides, yet it is expressed in terms of the sides. And if a be constant and x variable, the fluxion of the area will be $a\dot{x}$ by Prop. 3; if therefore ($\dot{x}$) the fluxion of the abscissa x be 1 inch in *length*, the corresponding fluxion of the area will be a square inches; if $\dot{x}$ be 2 inches in *length*, the fluxion of the area will be $2a$ square inches. And in general, when we consider any two quantities which are not homogeneous, although their fluxions, which are expressed by their increments *uniformly* generated in a given time, can have no relation to each other, if we carry our ideas no further than the increments themselves; yet when we consider the numerical values of these fluxions, the analytical expression for one may be comprised in terms of the other without any impropriety, and our conclusions will be perfectly just and correct, in the sense in which the units of the respective quantities are understood, notwithstanding the fluxions themselves may be heterogeneous. SIR I. NEWTON, in his *Quadrature of Curves*, in finding the area of a curve, describes a parallelogram on the abscissa (x), the other side (a) of which is constant; and then he compares the fluxion of the area of this parallelogram with the fluxion of the area of the curve, they being homogeneous quantities; and the fluxion of the area of the parallelogram being $a\dot{x}$, he gets the fluxion of the area of the curve. From what has been said above, when we reduce these matters to calculation, there appears to be no absolute necessity for this; but it is more scientific to make the comparison between homogeneous quantities, than between those which are not homogeneous, and therefore the former method is always to be preferred in cases where it can be applied, notwithstanding the conclusions which are otherwise deduced are perfectly true and satisfactory.

20. The ingenious and justly celebrated author of the *Analyst* has endeavoured to show, that the principles of fluxions, as delivered by its author, are not founded upon reasoning strictly logical and conclusive. He lays this down as a Lemma: "If you make any supposition, and, in virtue thereof, deduce any consequence; if you destroy that supposition, every consequence before deduced must be destroyed and rejected, so as from thence forward to be no more supplied or applied in the demonstration." This, he thinks, is so plain as to need no proof. It may perhaps be admitted to be true, when we want to deduce the *absolute* value of a quantity which is to be obtained in virtue of a supposition; but it is not true when we want to obtain the *relative* values of quantities. He seems not to have properly attended to the meaning of the term *limiting* ratio, but went upon the term *ultimate* ratio, assuming equality where it was never intended, thereby totally misunderstanding the subject; and this led him to disregard the connection which there must necessarily be between the two terms x, y, which constitute a ratio, and the two terms m, n, which express the ratio to which x, y approach as their limit, when you diminish them sine limite, called the *limit* of the ratio; for every one must see, that if you make x and y vanish, they must approach to some ratio as their limit; but we do not say (as writers who do not understand the subject would make us say) when x and y become $= 0$, that $o : o :: m : n$; such is the assertion

of those only who are ignorant of the subject. Now it is agreed, that, by diminishing the increments you approach to the ratio of the velocities which the quantities had at the points from whence the increments began to be generated, and that by making them become indefinitely small, you arrive at a ratio indefinitely near to that of the velocities at those points. Let therefore x and y be two increments generated by two flowing quantities in the same time; then as their limit $m : n$ must depend altogether upon x and y, that *limit* is obtained upon the supposition of the existence of the increments; but the *limit* is a certain determinate invariable ratio, totally independent of the *magnitude* of the terms of the ratio, or of the increments, as appears by Art. 8. When we therefore deduce the *limit* by making the increments vanish, the *effect* of the prior existence of the terms x, y of the ratio still remains in the terms m, n, which express the *limit* of the ratio. If the *existence* of the terms m, n, which express the *limit* of the ratio, depended upon the *existence* of the terms themselves x, y of the ratio, the supposition which makes the latter vanish would necessarily make the former also vanish, and then no conclusion could be deduced by making the terms of the ratio vanish; but as that is not the case, the *limit*, which is obtained by making the terms become equal to nothing, contains an effect, after the increments are actually vanished, which depends upon their having existed. The *limiting* ratio is (as expressed by *Maclaurin*) "the term or limit from which the variable ratio of the increments proceeds, or sets out, to increase or decrease." The lemma, therefore, of the author, however true it may be under some circumstances, cannot be applied against the reasoning upon which the Principles of Fluxions are founded. The author admits the conclusions to be true. He says, "I have no controversy about your conclusions, but only about your logic; and it must be remembered, that I am not concerned about the truth of your theorems, but only about the way of coming at them." The above observations show, not only that our conclusions are true, but that they are deduced by steps which are perfectly satisfactory, and strictly logical. It was unfortunate for Science, that neither the ingenious author of the *Analyst*, nor his opponents, had any clear ideas of the subject they disputed upon; the controversy however called forth *Robins* and *Maclaurin*, who showed in the most satisfactory manner, that the grounds of fluxions, according to the ideas of its great author, were defensible, and the investigations founded upon the strictest principles of reasoning.

SECTION II.

On the MAXIMA and MINIMA of QUANTITIES.

Prop. IX.

TO determine the value of a quantity, when it becomes a maximum or minimum.

21. If a quantity first increase and then decrease, at the end of its increase it becomes a maximum; and if it first decrease and then increase, at the end of its decrease it becomes a minimum. And as the fluxion of a quantity is the rate of its increase or decrease (Art. 3.), when it becomes a maximum or minimum its fluxion must be $= 0$, the quantity having, at that point of time, no further increase or decrease.

22. If any quantity be a maximum or minimum, any power or root of that quantity must then, evidently, be a maximum or minimum. For the power or root of a quantity will increase or decrease as long as the quantity itself increases or decreases, and no longer.

Any constant multiple, or part of a quantity which is a maximum or minimum, must also be a maximum or minimum. For the multiple, or part of a quantity, will increase or decrease as long as the quantity itself increases or decreases, and no longer; therefore when its fluxion is made $= 0$, the constant multiplier may be neglected.

EXAMPLES.

Ex. 1. *To divide a given number* a *into two parts,* x, y, *so that* $x^m y^n$ *may be a maximum.*

Since $x+y=a$, and $x^m y^n=$ max. the fluxion of each $=0$, the former, because it is constant, and the latter, because it is a maximum; $\therefore \dot{x}+\dot{y}=0$, and $my^n x^{m-1}\dot{x} + nx^m y^{n-1}\dot{y}=0$; hence, $\dot{x}=-\dot{y}$, and $\dot{x}=-\dfrac{nx^m y^{n-1}\dot{y}}{my^n x^{m-1}} = -\dfrac{nx\dot{y}}{my}$; therefore $-\dot{y}=-\dfrac{nx\dot{y}}{my}$; or, $my=nx$, and $m:n::x:y$. Now $y=\dfrac{nx}{m}$; $\therefore x+\dfrac{nx}{m}=a$, consequently $x=\dfrac{ma}{m+n}$, and $y\left(=\dfrac{nx}{m}\right)=\dfrac{na}{m+n}$.

If $m=n$, the two parts are equal.

Cor. Hence, to divide a quantity a into three parts, x, y, z, so that xyz may be a max. the parts must be equal. For suppose x to remain constant, and y, z to vary; the product yz, and consequently xyz. will be greatest when $y=z$. Or if y remain constant, the product xz, and consequently yxz, will be greatest when $x=z$. Thus it appears that the parts must be equal. And in like manner it may be shown, that whatever be the number of parts, they will be equal.

Ex. 2. *Given* $x+y+z=a$, *and* xy^2z^3 *a maximum, to find* x, y, z.

As x, y, z must have some certain determinate values to answer these conditions, let us suppose such a value of y to remain constant, whilst x and z vary till they answer the conditions, and then $\dot{x}+\dot{z}=0$ and $z^3\dot{x}+3xz^2\dot{z}=0$; hence, $\dot{x}=-\dot{z}=-\dfrac{3xz^2\dot{z}}{z^3}=-\dfrac{3x\dot{z}}{z}$, $\therefore z=3x$. Now let us suppose the value of z to remain constant, and x and y to vary, so as to satisfy the conditions; then $\dot{x}+\dot{y}=0$, $y^2\dot{x}+2xy\dot{y}=0$; hence,

$\dot{x} = -\dot{y} = -\frac{2xy\dot{y}}{y^2} = -\frac{2x\dot{y}}{y}$, $\therefore y = 2x$; substitute in the given equation, these values of y and z in terms of x, and $x + 2x + 3x = a$, or $6x = a$; hence, $x = \frac{1}{6}a$; $\therefore y = \frac{1}{3}a$; $z = \frac{1}{2}a$. In like manner, whatever be the number of unknown quantities, make any one of them variable with each of the rest, and the values of each in terms of that one quantity will be obtained; and by substituting the values of each in terms of that one, in the given equation, you will get the value of that quantity, and thence the values of the others.

Ex. 3. *To find when* y *is a max. in* $\overline{x^3+y^3}|^2 = a^4x^2$.

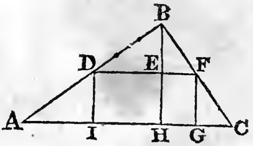
Take the fluxions of both sides, and $2 \times \overline{3x^2\dot{x}+3y^2\dot{y}} \times \overline{x^3+y^3} = 2a^4x\dot{x}$; but when y is a maximum, $\dot{y} = 0$; hence, $6x^2\dot{x} \times \overline{x^3+y^3} = 2a^4x\dot{x}$, $\therefore \overline{x^3+y^3} = \frac{a^4}{3x}$, and $\overline{x^3+y^3}|^2 = \frac{a^8}{9x^2}$; therefore $a^4x^2 = \frac{a^8}{9x^2}$, and $x^4 = \frac{a^4}{9}$, or $x = \frac{a}{\sqrt{3}}$ hence, $y^3 (= a^2x - x^3) = \frac{a^3}{\sqrt{3}} - \frac{a^3}{3^{\frac{3}{2}}} = a^3 \times \overline{\frac{1}{\sqrt{3}} - \frac{1}{3^{\frac{3}{2}}}}$ $= a^3 \times \frac{2}{3\sqrt{3}}$; $\therefore y = a\sqrt[3]{\frac{2}{3\sqrt{3}}}$.

Otherwise. As $y^3 = a^2x - x^3$, $\therefore 3y^2\dot{y} = a^2\dot{x} - 3x^2\dot{x} = 0$, because $\dot{y} = 0$, $\therefore x = \frac{a}{\sqrt{3}}$.

Ex. 4. *To inscribe the greatest parallelogram* DFGI *in a given triangle* ABC.

Draw BH ⊥ AC; put AC $= a$, BH $= b$, BE $= x$, then EH $= b - x$; and by sim. Δs. $b : a :: x :$ DF $= \frac{ax}{b}$; hence, the area DFGI $= \frac{ax}{b} \times \overline{b-x} =$ max. or $x \times \overline{b-x}$

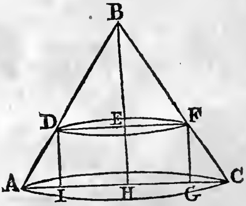
$= bx - x^2 = \text{max.} \therefore b\dot{x} - 2x\dot{x} = 0$; hence, $x = \frac{1}{2}b$;



therefore EH $= \frac{1}{2}$BH.

Ex. 5. *Let* ABC *represent a cone*, AC *the diameter of the base; to inscribe in it the greatest cylinder* DFGI.

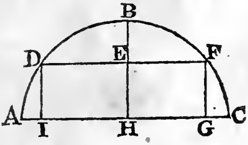
Put $p = ,78539$ &c. then (the same notation remaining) it will appear when we come to treat on the



method of finding the areas of curves, that $\frac{pa^2x^2}{b^2} =$ the area of the end DEF of the cylinder; hence, the content of the cylinder $= \frac{pa^2x^2}{b^2} \times \overline{b-x} = \text{max.}$ or $x^2 \times \overline{b-x} = bx^2 - x^3 = \text{max.} \therefore 2bx\dot{x} - 3x^2\dot{x} = 0$; hence, $x = \frac{2}{3}b$; therefore EH$=\frac{1}{3}$BH.

Ex. 6. *To inscribe the greatest parallelogram* DFGI *in a given parabola* ABC.

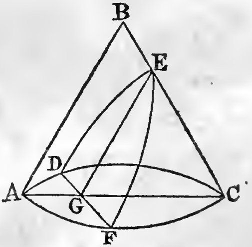
Put BH $= a$, $p =$ the parameter, $x =$ BE; then by the property of the parabola, $DE^2=px$, $\therefore$ $DE=p^{\frac{1}{2}}x^{\frac{1}{2}}$,



and $DF=2p^{\frac{1}{2}}x^{\frac{1}{2}}$; hence, the area $DFGI=2p^{\frac{1}{2}}x^{\frac{1}{2}}\times\overline{a-x}$ $=\text{max. or } x^{\frac{1}{2}}\times\overline{a-x}=ax^{\frac{1}{2}}-x^{\frac{3}{2}}=\text{max.}$ $\therefore \frac{1}{2}ax^{-\frac{1}{2}}\dot{x}-\frac{3}{2}x^{\frac{1}{2}}\dot{x}=0$; hence, $\frac{a}{x^{\frac{1}{2}}}=3x^{\frac{1}{2}}$, or $a=3x$, $\therefore$ $x=\frac{1}{3}a$; consequently $EH=\frac{2}{3}BH$.

Ex. 7. *To cut the greatest parabola* DEF *from a given cone* ABC.

Let AGC be that diameter of the base which is $\perp$ to DGF; now EG is parallel to AB; put $AC=a$, AB



$=b$, $CG=x$, then $AG=a-x$; and by the property of the circle, $DG=\sqrt{ax-x^2}$, $\therefore$ $DF=2\sqrt{ax-x^2}$; also, by sim. Δs, $a:b::x:GE=\frac{bx}{a}$; hence, we have the area of the parabola $=\frac{2}{3}\times\frac{bx}{a}\times 2\sqrt{ax-x^2}=\text{max.}$

hence, $x\sqrt{ax-x^2}=$ max. or $x^2\times\overline{ax-x^2}=ax^3-x^4=$ max. $\therefore 3ax^2\dot{x}-4x^3\dot{x}=0$, and $3a=4x$, $\therefore x=\frac{3}{4}a$.

Ex. 8. *To divide a given arc* A *into two parts, such that the* m^th^ *power of the sine of one part, multiplied into the* n^th^ *power of the sine of the other, may be a maximum.*

Let P and Q be the two parts, x and y their sines, radius being unity; then $x^m\times y^n=$ maximum; hence $my^nx^{m-1}\dot{x}+nx^my^{n-1}\dot{y}=0$, and $my\dot{x}=-nx\dot{y}$. Now (Art. 46.) $\dot{P}=\frac{\dot{x}}{\sqrt{1-x^2}}$, $\dot{Q}=\frac{\dot{y}}{\sqrt{1-y^2}}$; and as P + Q $=$A, $\dot{P}+\dot{Q}=0$, $\therefore\dot{P}=-\dot{Q}$, or $\frac{\dot{y}}{\sqrt{1-y^2}}=\frac{-\dot{x}}{\sqrt{1-x^2}}$; multiply this equation by the equation $my\dot{x}=-nx\dot{y}$, and $m\times\frac{y}{\sqrt{1-y^2}}=n\times\frac{x}{\sqrt{1-x^2}}$, or $m\times$tan. Q$=n\times$tan. P, $\therefore m:n::$ tan. P : tan. Q, and $m+n:m-n::$ tan. P+tan. Q : tan. P—tan. Q :: (Trig. Art. 113.) sin. (P+Q) : sin. (P—Q) : : sin. A : sin. (P—Q)$=\frac{m-n}{m+n}\times$sin. A; hence we know the sine of the difference of the two parts of the arc; therefore we know the difference P—Q of the arcs themselves; and knowing the sum P + Q, or A, we know the two parts P and Q.

Ex. 9. *To determine at what angle the wind must strike against the sails of a mill, so that the effect to put it in motion may be the greatest possible.*

Put $x=$ the cosine of the angle, then $1-x^2=$ the square of the sine, radius being unity; hence (by the Principles of Hydrostatics), the effect is as $x\times\overline{1-x^2}$ $=x-x^3$, which is to be maximum; $\therefore\dot{x}-3x^2\dot{x}=0$; hence, $x=\sqrt{\frac{1}{3}}$ the cosine of 54° 44′.

Ex. 10. *Given two elastic bodies* A *and* C, *to find an intermediate body* x, *so that the motion communicated from* A *to* C *through* x, *may be a maximum.*

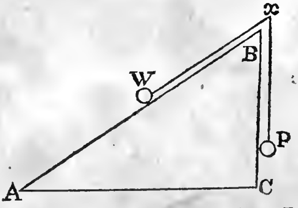
Put $a =$ the given velocity of A, $w =$ the velocity communicated to x, and z the velocity communicated to C; then (by Mechanics),

$$A + x : 2A :: a : w$$
$$x + C : 2x :: w : z$$

$\therefore$ comp. $Ax + x^2 + AC + Cx : 4Ax :: a : z$, or, $A + x + \frac{AC}{x} + C : 4A :: a : z$; now, as the two middle terms are constant, the last term varies inversely as the first; and as the last is to be a maximum, the first must be a minimum; therefore its fluxion $\dot{x} - \frac{AC\dot{x}}{x^2} = 0$; hence, $x^2 = AC$, and $A : x :: x : C$.

Ex. 11. *Given the altitude* BC *of an inclined plane* AB, *to find its length, so that a weight* P *acting upon another* W *in a line parallel to the plane, may draw it up through* AB *in the least time.*

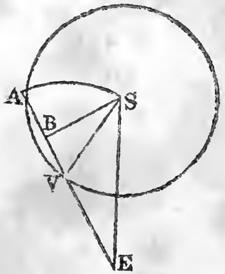
Put $a = BC$, $x = AB$; then (by Mechanics) the accelerating force of W down BA is $\frac{aW}{x}$; hence the mov-



ing force of the two bodies is $P - \frac{aW}{x} = \frac{Px - aW}{x}$; therefore the accelerating force $= \frac{Px - aW}{\overline{P + W} \times x}$; and the time of describing AB varies as $\sqrt{\frac{AB}{\text{ac. for.}}}$, or as $\sqrt{\frac{\overline{P+W} \times x^2}{Px-aW}} = \text{min.}$ or $\frac{x^2}{Px-aW} = \text{min.}$ ∴ $\frac{2x\dot{x} \times \overline{Px-aW} - P\dot{x} \times x^2}{\overline{Px-aW}|^2} = 0$; but when a fraction vanishes, its numerator $= 0$; hence, $2Px^2\dot{x} - 2aWx\dot{x} - Px^2\dot{x} = 0$, or $Px^2 = 2aWx$, ∴ $x = \frac{2aW}{P}$.

Ex. 12. *To find the position of the planet Venus, when it gives the greatest quantity of light to the Earth, the orbits being supposed to be circles with the Sun in their common centre.*

Let S be the Sun, E the Earth, V Venus, produce EV, on which let fall the ⊥ SB, and with the centre V describe the circular arc SA. Put a=SE, b=SV=



AV, x = EV, y =BV, then AB $= b - y$ the versed sine of the angle SVA; and (by the Principles of *Astronomy*) the quantity of light received at the Earth from Venus varies as $\frac{b-y}{x^2} = \frac{b}{x^2} - \frac{y}{x^2} = \text{max.}$ Now (Euc. B. II. p. 12.) $a^2 = b^2 + x^2 + 2xy$, ∴ $y = \frac{a^2-b^2-x^2}{2x} =$ (if $m^2=a^2-b^2$) $\frac{m^2-x^2}{2x}$; hence, the quantity of light varies as $\frac{b}{x^2} - \frac{m^2-x^2}{2x^3} = \frac{2bx-m^2+x^2}{2x^3}$, which is therefore a maximum; hence, its fluxion $\frac{\overline{2b\dot{x}+2x\dot{x}} \times 2x^3 - 6x^2\dot{x} \times \overline{2bx-m^2+x^2}}{4x^6} = 0$, or its numerator $4bx^3\dot{x}+4x^4\dot{x}-12bx^3\dot{x}+6m^2x^2\dot{x}-6x^4\dot{x}=0$, or by dividing by $2x^2\dot{x}$, and uniting the like terms, we have $-x^2-4bx+3m^2=0$, $\therefore x^2+4bx=3m^2=3a^2-3b^2$, a quadratic, from which $x=-2b+\sqrt{b^2+3a^2}$. Hence, we know the three sides of the triangle ESV, to find the angle E of elongation. Now if $a=1$, $b=0,72333$ according to Dr. Halley; hence, $x=0,43046$, and the angle SEV=39° 44′ the elongation of Venus from the Sun when she is brightest. Also, the angle ESV = 22° 21′; but the angle ESV = 43° 40′ at the planet's greatest elongation; hence, Venus is brightest between her inferior conjunction and her greatest elongation.

For the planet *Mercury*, $b=0,3171$, and $x=1,00058$, and the angle SEV = 22° 19′ the elongation of Mercury when brightest. Also, the angle ESV=78° 56′; but the angle ESV = 67° 13′,5 at the time of the planet's greatest elongation; hence, Mercury is brightest between its greatest elongation and superior conjunction.

In questions of a *geometrical* and *philosophical* nature, there are frequently restrictions which do not enter into the analytical expression. In the analytical expression, considered simply as such, the unknown quantity may be assumed of any value, and therefore it may be taken *without* the limits to which it is confined by the question. When its fluxion is therefore made equal to nothing, that equation may contain, besides the roots which are applicable to the question, others which are not applicable; and if none of the roots be applicable, it shows that the maximum or minimum of the expression does not lie within the limit of the unknown quantity, as confined by the question; in which case, the roots deduced from making the fluxion of the equation = 0, can be of no use. In the present instance, the expression is $\frac{2bx - m^2 + x^2}{2x^3}$ (A) for the quantity of light; and putting its fluxion $= 0$, we get $x = -2b \pm \sqrt{b^2+3a^2}$; but it is only the root $x = -2b + \sqrt{b^2+3a^2}$ which is applicable to the question, as this is a value of x which lies within the limits of the question; and it gives the expression (A) a maximum. The other root $x = -2b - \sqrt{b^2+3a^2}$ being negative, which x never can be, cannot be applicable to the question; but it nevertheless gives the value of (A) when a minimum. But although when we make $(\dot{A}) = 0$, the roots of the equation do not give the points in the orbit where the light is a minimum, that is, the superior and inferior conjunctions; yet if we suppose x to be confined to the limits of the question, or to represent EV, and V to move round in the circumference of the circle, in the two conjunctions $\dot{x} = 0$, and we still have $(\dot{A}) = 0$ for those points. The equation therefore $(\dot{A}) = 0$ is, under the above restrictions, true for those points, because $\dot{x} = 0$, and not because the roots give those points. Whilst, in general, a maximum or minimum of (A) lie within the value of x as restrained by the question, the roots of $(\dot{A}) = 0$ will give those points; otherwise, not; and the maximum or minimum in the question must in the latter case be sought for, by considering, when the quantity which is to be a maximum or minimum, ceases to increase or decrease, according to the restrictions of the unknown quantity. In the present instance, it is when $\dot{x} = 0$, or in the two conjunctions; for had (A) decreased and then increased between the maximum of light and either conjunction, there would have been a root of $(\dot{A}) = 0$ which would have shown the point where the light was a minimum; but as there is no such root, it shows that (A) must decrease till the planet comes into each conjunction; and as (A) then increases again by the same steps by which it decreased, the light at those points must have been a minimum. These observations appear to be of some importance, as they tend to remove difficulties which might otherwise arise in the maxima and minima of quantities which are under certain restrictions; for it might naturally be asked, in the present question for instance, why do not the equation $(\dot{A}) = 0$ give three roots, one producing a maximum and the other two the minima of light, there actually being such points in one synodic revolution of the planet?

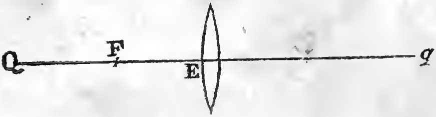
For a *superior* planet, the maximum of light is evidently when the planet is in opposition, the whole face being then illuminated, and the planet is at its nearest distance. Now to find whether the quantity of light becomes a *minimum* in going from opposition to conjunction, we still have $x = -2b \pm \sqrt{b^2+3a^2}$. Now as a is less than b, $b^2 + 3a^2$ is less than $4b^2$, and $\sqrt{b^2+3a^2}$ is less than $2b$; hence, x ($= -2b + \sqrt{b^2+3a^2}$) is negative; and the other root is manifestly negative; which not being possible for x, it appears that there is no *minimum* of light in going from opposition to conjunction, but that the quantity of light continually decreases through that part of the orbit. The expression (A) does not pass through its maximum and minimum in opposition and conjunction, for the reason before given, and therefore the roots of $(\dot{A}) = 0$, cannot give those points.

If $b = a$, $x = 0$, and V coincides with E.

Ex. 13. *Let* Q *be an object placed beyond the principal focus* F *of a convex lens, to find its position, when its distance* Qq *from its image* q, *is the least possible.*

Put $QF = x$, $FE = a$; then (by the Principles of

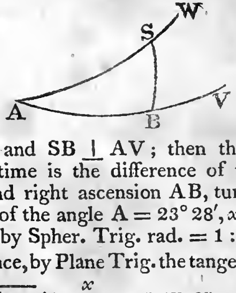
Optics) $x : x+a :: x+a : Qq = \frac{\overline{x+a}^2}{x} =$ a min. hence,



its fluxion $\frac{2\dot{x} \times \overline{x+a} \times x - \dot{x} \times \overline{x+a}^2}{x^2} = 0$, and by assuming the numerator $= 0$, and dividing by $x+a$, we have $2x\dot{x} - x\dot{x} - a\dot{x} = 0$, or $x - a = 0$, $\therefore x = a$.

Ex. 14. *To find the Sun's place in the ecliptic, when that part of the equation of time which arises from the obliquity of the ecliptic, is a maximum.*

Let AV be the equator, AW the ecliptic, S the

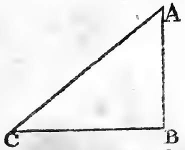


Sun's place, and SB $\perp$ AV; then this part of the equation of time is the difference of the Sun's longitude AS and right ascension AB, turned into time. Put $s =$ cos. of the angle A $= 23° 28'$, $x =$ the tangent of AS; then by Spher. Trig. rad. $= 1 : s :: x :$ tan. of AB$=sx$; hence, by Plane Trig. the tangent of $\overline{\text{AS}-\text{AB}}$ $= \frac{x-sx}{1+sx^2} = \overline{1-s} \times \frac{x}{1+sx^2} =$ max. or $\frac{x}{1+sx^2} =$ max. $\therefore$ its fluxion $\frac{\dot{x} \times \overline{1+sx^2} - 2sx\dot{x} \times x}{\overline{1+sx^2}^2} = 0$; hence, the numerator $\dot{x} + sx^2\dot{x} - 2sx^2\dot{x} = 0$, $\therefore 1 - sx^2 = 0$, and $x = \sqrt{\frac{1}{s}} = 1{,}04416$, the tan. of 46° 14′ the Sun's long. when this part of the equation of time is a maximum.

If we retain 1^2 in the denominator for the square of radius, as the trigonometrical theorem gives it, then $1-sx^2=0$ becomes $1^2-sx^2=0$, and $sx^2=1^2=\overline{\text{rad.}}]^2$; that is, tan. AS × tan. AB$=\overline{\text{rad.}}]^2$; but tan. AS × cot. AS $=\overline{\text{rad.}}]^2$; therefore tan. AB=cot. AS; hence, AS+AB $=90°$.

Ex. 15. *Given the base* CB *of an inclined plane* AC, *to find its altitude* BA, *when the time of the descent of a body down the plane is the least possible.*

Put $a=$ CB, $x=$ BA, then $\sqrt{a^2+x^2}=$ AC; and (by Mechanics) the time down AC varies as $\frac{\sqrt{a^2+x^2}}{\sqrt{x}}$, which is therefore a minimum, or $\frac{a^2+x^2}{x}$ is a mini-



mum; hence, $\frac{2x\dot{x}\times x-\dot{x}\times \overline{a^2+x^2}}{x^2}=0$, or its numerator $2x^2\dot{x}-a^2\dot{x}-x^2\dot{x}=0$, therefore $x^2=a^2$, and $x=a$.

Ex. 16. *Given the base* CB, *to find the perpendicular* BA, *such that a body descending from* A *to* B, *and then describing* BC *with the velocity acquired, the time through* AB *and* BC *may be the least possible.*

Put $m=16\frac{1}{12}$ feet, $a=$ CB, $x=$ BA; then (by Mechanics) the time down AB $=\sqrt{\frac{x}{m}}$; also, with the velocity acquired at B continued uniform, the body would describe 2AB, or $2x$, in the same time; hence, as the space described with an uniform velocity is as the time, $2x:a::\sqrt{\frac{x}{m}}:\frac{a}{2x}\times\sqrt{\frac{x}{m}}=\frac{1}{2}\ a\times\sqrt{\frac{1}{mx}}$

the time of describing BC; hence, the whole time $=\sqrt{\frac{x}{m}}+\frac{1}{2}a\sqrt{\frac{1}{mx}}=x^{\frac{1}{2}}\times\sqrt{\frac{1}{m}}+\frac{1}{2}ax^{-\frac{1}{2}}\times\sqrt{\frac{1}{m}}=$ a minimum, or $x^{\frac{1}{2}}+\frac{1}{2}ax^{-\frac{1}{2}}=$ min. $\therefore \frac{1}{2}x^{-\frac{1}{2}}\dot{x}-\frac{1}{4}ax^{-\frac{3}{2}}\dot{x}=0$, or $x^{-\frac{1}{2}}=\frac{1}{2}ax^{-\frac{3}{2}}$; hence, $x=\frac{1}{2}a$.

Ex. 17. *Given the base* CB *of an inclined place* AC, *to find its altitude* BA, *such that the horizontal velocity of a body at* C *after descending down* AC, *may be the greatest possible.*

Put $a=$ CB, $x=$ BA, then CA $=\sqrt{a^2+x^2}$; now (by Mechanics) the velocity at C is as $\sqrt{x}$, and by the resolution of motion, $\sqrt{a^2+x^2} : a :: \sqrt{x} : \frac{a\sqrt{x}}{\sqrt{a^2+x^2}}$, which is as the velocity at C in the direction BC, which is to be a maximum; or $\frac{x}{a^2+x^2}=$ a maximum; $\therefore \frac{\dot{x}\times\overline{a^2+x^2}-2x\dot{x}\times x}{\overline{a^2+x^2}^2}=0$, or the numerator $a^2\dot{x}+x^2\dot{x}-2x^2\dot{x}=0$; hence, $x=a$.

Ex. 18. *Given the solidity of the cone, to find the base and height, when the time of its vibration shall be a minimum, supposing the point of suspension to be the vertex.*

Put $y=$ radius of the base, $x=$ the altitude, $p=$ 3,14159 &c. then $\frac{1}{3}pxy^2=s$; and (Ex. 8. Prop. 30) $\frac{4x^2+y^2}{5x}=$ the distance from the point of suspension to the centre of oscillation $=$ minimum. But $y^2=\frac{s}{\frac{1}{3}px}$

$=(\text{if } \frac{s}{\frac{1}{3}p} = 2a)\frac{2a}{x}$; hence, $\frac{4x^2 + \frac{2a}{x}}{5x} = \frac{4x^3 + 2a}{5x^2} = \text{min.}$ and $\frac{12x^2\dot{x} \times 5x^2 - 10x\dot{x} \times \overline{4x^3 + 2a}}{25x^4} = 0$; hence, $x = a^{\frac{1}{3}}$; therefore $y = \frac{\overline{2a}|^{\frac{1}{2}}}{\sqrt{x}} = \sqrt{2} \times a^{\frac{1}{3}}$; consequently $x : y :: 1 : \sqrt{2}$.

Ex. 19. *To find when* (A) $x^3 - 18x^2 + 96x - 20$ *becomes a maximum or minimum.*

Assume the fluxion $= 0$, and $3x^2\dot{x} - 36x\dot{x} + 96\dot{x} = 3\dot{x} \times \overline{x^2 - 12x + 32} = 0$; hence, $x = 4$ or 8. Now to determine which value gives the maximum and which the minimum, find whether the value of the fluxion, just before it becomes$=0$, be *positive* or *negative;* if *positive*, the succeeding root gives a *maximum;* if *negative*, a *minimum;* for whilst a quantity increases its fluxion is positive; but when it decreases its fluxion becomes negative, by Art. 16. Now as $3\dot{x} \times \overline{x-4} \times \overline{x-8} = 3\dot{x} \times \overline{x^2 - 12x + 32}$; when x is less than 4, each factor being negative, the value of the fluxion is positive, therefore the root 4 gives (A) $x^3 - 18x^2 + 96x - 20$, a maximum; and as, when x increases from 4 to 8, one factor is positive and the other negative, the fluxion is negative, therefore the root 8 gives (A) a minimum. When we say that by making $x=4$ it gives (A) a maximum, we mean that (A) first increases till x becomes 4 and then it decreases, and not that it is then the greatest possible; for by increasing x after it exceeds 8, the value of (A) increases *sine limite.* And in like manner, (A) decreases whilst x increases from 4 to 8, and then it increases, and therefore when $x = 8$, (A) is said to be a minimum, not that it is then the least possible, for by decreasing x below 4, (A) will decrease *sine limite.*

We have here supposed x to increase; if we suppose x to decrease, and first assume it greater than 8, then as x decreases till it becomes 8, each factor $x-4$, $x-8$ being positive, the product is positive, and therefore it might appear that the root 8 ought to give a maximum; but as x is a *decreasing* quantity, its fluxion $(\dot{x})$ is negative by Art. 16; hence, $3\dot{x} \times \overline{x-4} \times \overline{x-8}$ is negative till x becomes 8, and therefore this root gives (A) a minimum; and whilst x decreases from 8 to 4, $3\dot{x} \times \overline{x-4} \times \overline{x-8}$ is positive, and therefore 4 gives (A) a maximum, agreeable to what was before determined. This instance shows the necessity of attending to the signs of the fluxions of increasing and decreasing quantities, without which we might have determined (A) to have been a maximum when it is a minimum, and a minimum when it is a maximum; for it is merely arbitrary whether we suppose x to increase or decrease.

When all the roots of the fluxional equation are impossible, as no possible value of x can make the equation $= 0$, it shows that by increasing x, the given quantity increases or decreases sine limite, therefore it admits of no maximum or minimum.

It may happen that the fluxion may be $= 0$, and yet the quantity (A) may not be a maximum or minimum, which takes place when two of the roots of the fluxional equation are *equal*, because in that case, the sign of the fluxion is the same both before and after the equation becomes $= 0$ from the substitution of one of the equal roots. For let the given quantity be $x^4 - 16x^3 + 90x^2 - 216x$, whose fluxion is $4x^3\dot{x} - 48x^2\dot{x} + 180x\dot{x} - 216\dot{x} = 4\dot{x} \times \overline{x^3 - 12x^2 + 45x - 54} = 4\dot{x} \times \overline{x-3} \times \overline{x-3} \times \overline{x-6}$. Now just before $x = 3$, this fluxion is negative, and just after $x = 3$, it is also negative; therefore as the fluxion continues negative whilst x passes through 3, that root does not give (A) a minimum; but as the fluxion passes from negative to positive whilst x passes from less than 6 to more than 6, the root 6 gives (A) a minimum, its fluxion after that time being positive shows that (A) then begins to increase.

Let the fluxional equation have three equal roots, as in $\dot{x} \times \overline{x-a} \times \overline{x-a} \times \overline{x-a} \times \overline{x-b}$, and let a be less than b. Then it is manifest, that when x is less than a, this fluxion is positive, and when x passes through a and lies between a and b, the fluxion is negative; therefore $x=a$ gives (A) a maximum. Hence it is manifest, that, in general, when the fluxional equation has an *even* number of equal roots, one of those roots gives (A) neither a maximum nor minimum; but when it has an *odd* number, that root gives (A) either a maximum or minimum. If the reader wish to see any thing further on this point, he may consult LYONS's Fluxions, p. 91.

Ex. 20. *To find the value and position of the greatest and least ordinates of a curve, whose equation is* $y=x^3-px^2+qx-r$, x *being the abscissa and* y *the ordinate.*

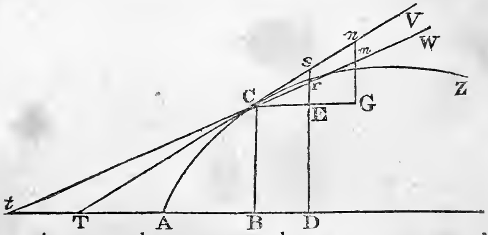
Take the fluxion, and $\dot{y}=3x^2\dot{x}-2px\dot{x}+q\dot{x}$; but when y becomes a max. or min. $\dot{y}=0$; hence, $3x^2\dot{x}-2px\dot{x}+q\dot{x}=0$; consequently $x=\frac{p}{3}\pm\sqrt{\frac{p^2}{9}-\frac{q}{3}}$, the values of the abscissa corresponding to the required ordinates; and if these values of x be respectively substituted into the given equation, the values of the ordinates themselves will be known. Which of the values of x gives the ordinate a maximum and which a minimum, may be found by Ex. 19. If $p=18$, $q=60$, $r=10$, then $x=2$ and 10, the two abscissæ; which substituted for x in the given equation, give 46 and -210 for the two ordinates, the latter of which being negative, shows that the curve at that point lies below the abscissa.

To draw TANGENTS to CURVES.

Prop. X.

Let the curve ACZ *be described by the extremity of the ordinate* BC, *which moves parallel to itself and varies in its length; to draw a tangent to the curve at any point* C.

23. Let TCV be the required tangent; draw any other ordinate D*r* and produce it to *s*; draw also CE parallel to BD; join C*r* and produce it to *t* and W; produce also CE to any point G, and draw G*mn* parallel to E*s*. Now let D*rs* move up to BC, then by the motion of *r*, the line W*r*C*t* will revolve about C, and when *r* coincides with C, it ceases to cut the curve between C and Z, and it does not cut it between C and A, for to cut CA, C*t* must fall below CT, and consequently CW must lie above CV, or *r* must have passed *s*, which it cannot have done, as *r* has been continually approaching to *s* and only now coincides with it; therefore when *r* comes to C, the line W*t*,



ceasing to cut the curve, must become a tangent, and consequently WC*t* will then coincide with VCT. Now whilst the abscissa AB by increasing becomes AD, the ordinate BC becomes D*r*; hence, the increment of the ordinate BC is E*r*; and, by similar triangles, the increment CE of the abscissa : the cotemporary increment E*r* of the ordinate :: CG : G*m*. But when *r* arrives at C, WC coincides with VC, and consequently m must coincide with n; hence, the *limiting* ratio of the increment CE of the abscissa to the increment Er of the ordinate, is that of the finite lines CG : Gn, which (by sim. trian.) is the ratio of CE : Es, taking DEs in any situation before its coincidence with BC; hence, by Proposition 2, if CE represent the fluxion of the abscissa, Es will represent the cotemporary fluxion of the ordinate. Put $AB=x$, $BC=y$, then $BD=CE=\dot{x}$, $Es=\dot{y}$; and as BC is parallel to Es, and TB to CE, the angle TCB = CsE, and CTB=sCE, consequently the triangles TBC, CEs are similar; hence, $\dot{y}$ (Es) : $\dot{x}$ (CE) : : y (CB) : $BT=\frac{y\dot{x}}{\dot{y}}$; therefore set off $BT=\frac{y\dot{x}}{\dot{y}}$, join T and C, and TC will be a tangent to the curve at C. If y decrease whilst x increases, then $\dot{y}$ becomes negative by Art. 16. and consequently $\frac{y\dot{x}}{\dot{y}}$, or BT, becomes negative, which shows that T lies on the other side of B. See *Algebra*, Art. 474.

Def. The line BT is called the *subtangent*.

EXAMPLES.

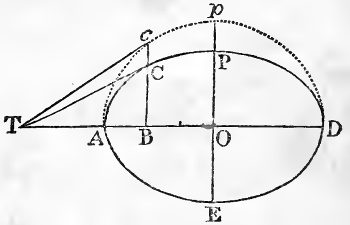
Ex. 1. *Let the curve* AC *be a* parabola, *that is, a curve whose abscissa varies as any direct power of the ordinate; to draw a tangent at the point* C.

The equation expressing the relation between x and y is $ax=y^n$, for then $x : y^n :: 1 : a$, a constant ratio. Take the fluxion of both sides of the equation, and we have $a\dot{x}=ny^{n-1}\dot{y}$; hence, $\frac{\dot{x}}{\dot{y}}=\frac{ny^{n-1}}{a}$, $\therefore BT=\frac{y\dot{x}}{\dot{y}}=\frac{ny^n}{a}$ $=nx$, because $\frac{y^n}{a}=x$.

If $n=2$, it is the common parabola, and $BT=2x$.

Ex. 2. *To draw a tangent to the* ellipse ACPDE, *at any point* C.

Let AD and PE be the two axes; put AO$=a$, PO $=b$, AB$=x$, BC$=y$, then BD$=2a-x$; and by the property of the ellipse, $a^2 : b^2 :: \overline{2a-x} \times x : y^2 = \frac{b^2}{a^2} \times \overline{2ax-x^2}$; take the fluxions, and $\frac{b^2}{a^2} \times \overline{2a\dot{x}-2x\dot{x}} = 2y\dot{y}$; multiply both sides by $\frac{a^2}{b^2}$, divide by 2 which is common, and also by $a-x$, and $\dot{x} = \frac{a^2}{b^2} \times \frac{y\dot{y}}{a-x}$, $\therefore \frac{\dot{x}}{\dot{y}} =$

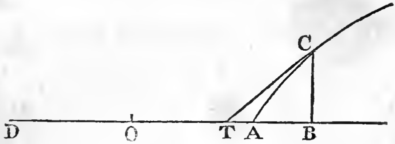


$\frac{a^2}{b^2} \times \frac{y}{a-x}$; hence, BT$= \frac{y\dot{x}}{\dot{y}} = \frac{a^2}{b^2} \times \frac{y^2}{a-x} = \frac{2ax-x^2}{a-x}$, by substituting $\frac{b^2}{a^2} \times \overline{2ax-x^2}$ for y^2.

As this value of TB is independent of b, or PO, if we take pO$=$AO, so that ApD may be a circle, and produce BC to c, cT will be a tangent to the circle. If B be between O and D, so that whilst x increases y decreases, then $\dot{y}$ becomes negative by Art. 16. and consequently $\frac{y\dot{x}}{\dot{y}}$ is negative, which shows that the subtangent BT lies the other way from B.

Ex. 3. *To draw a tangent to the* hyperbola AC, *whose major axis is* AD.

Bisect AD in O; put AO$=a$, the semi-axis minor $=b$, AB$=x$, BC$=y$; then by the property of the hyperbola, $a^2 : b^2 :: \overline{2a+x} \times x : y^2 = \frac{b^2}{a^2} \times \overline{2ax+x^2}$, which is the same equation as for the ellipse, except that



the sign of x^2 is here positive; ∴ BT$=\frac{2ax+x^2}{a+x}$.

Ex. 4. *To draw a tangent to the* Cissoid *of* Diocles, *whose equation is* $y^2 = \frac{x^3}{a-x}$ (*Alg.* Art. 496).

Take the fluxion, and $2y\dot{y} = \frac{3x^2\dot{x} \times \overline{a-x} + x^3\dot{x}}{\overline{a-x}^2} = \frac{3ax^2\dot{x} - 2x^3\dot{x}}{\overline{a-x}^2}$; hence, $\frac{\dot{x}}{\dot{y}} = \frac{2y \times \overline{a-x}^2}{3ax^2 - 2x^3}$; ∴ BT$=\frac{y\dot{x}}{\dot{y}} = \frac{2y^2 \times \overline{a-x}^2}{3ax^2 - 2x^3} = \frac{2x^3}{a-x} \times \frac{\overline{a-x}^2}{3ax^2 - 2x^3} = \frac{2x \times \overline{a-x}}{3a-2x}$.

Ex. 5. *To draw a tangent to the* catenary *curve.*

The equation of this curve is $z^2 = 2ax + x^2$ (Prop. 118); hence, $z\dot{z} = a\dot{x} + x\dot{x}$, and $\dot{z} = \frac{a+x}{z} \times \dot{x}$; but $\dot{y}^2 = \dot{z}^2 - \dot{x}^2$ (Prop. 24) $= \frac{\overline{a+x}^2}{z^2} \times \dot{x}^2 - \dot{x}^2 = \frac{\overline{a+x}^2 - z^2}{z^2} \times \dot{x}^2 = \frac{a^2\dot{x}^2}{z^2}$, and $\dot{y} = \frac{a\dot{x}}{z}$; hence, BT $= \frac{y\dot{x}}{\dot{y}} = \frac{zy}{a} = \frac{y\sqrt{2ax+x^2}}{a}$.

Ex. 6. *To draw a tangent to the* logarithmic *curve.*

Here the equation is $a^x = y$ (Art. 109.); and if A and Y be the hyp. logs. of a and y; then $x\text{A}=\text{Y}$; hence, $\text{A}\dot{x}=\dot{\text{Y}}=\frac{\dot{y}}{y}$(Art. 45.), therefore $\text{BT}=\frac{y\dot{x}}{\dot{y}}=\frac{1}{\text{A}}$.

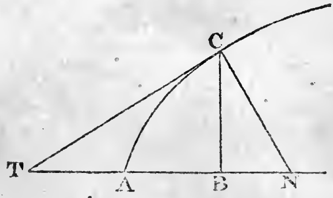
Ex. 7. *To draw a tangent to the curve whose equation is* $x^x = y$.

If X and Y be the hyp. logs. of x and y, we have $x\text{X} = \text{Y}$, and $x\dot{\text{X}} + \text{X}\dot{x} = \dot{\text{Y}}$; but (Art. 45.) $\dot{\text{X}}=\frac{\dot{x}}{x}$ and $\dot{\text{Y}}=\frac{\dot{y}}{y}$; therefore $\dot{x}+\text{X}\dot{x}=\frac{\dot{y}}{y}$, or $y\dot{x}+y\text{X}\dot{x}$ $=\dot{y}$; hence, $\text{BT}=\frac{y\dot{x}}{\dot{y}}=\frac{y\dot{x}}{y\dot{x}+y\text{X}\dot{x}}=\frac{1}{1+\text{X}}$.

Ex. 8. *To draw a tangent to an* hyperbola *between the asymptotes.*

Here $xy = a^2$, therefore $x\dot{y} + y\dot{x} = 0$, and $y\dot{x}$ $= -x\dot{y}$; hence $\text{BT}=\frac{y\dot{x}}{\dot{y}}=-x$, which being negative shows that T lies on the other side of the ordinate in respect to the abscissa.

24. Draw CN perpendicular to the tangent, and it is called the *normal*, and NB the *sub-normal.* Now the triangles TBC, NBC are similar; hence, $\frac{y\dot{x}}{\dot{y}}$ (TB) : y (BC) : : y : $\text{BN}=\frac{y\dot{y}}{\dot{x}}$ the *sub-normal.* Also



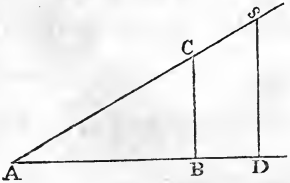
$CN^2 = y^2 + \frac{y^2\dot{y}^2}{\dot{x}^2} = y^2 \times \overline{1 + \frac{\dot{y}^2}{\dot{x}^2}} = y^2 \times \frac{\dot{x}^2+\dot{y}^2}{\dot{x}^2}$; hence, CN $= y \times \frac{\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{x}}$ the *normal.*

Ex. Let the curve be a parabola.

Here $ax = y^n$; $\therefore a\dot{x} = ny^{n-1}\dot{y}$, and $\frac{\dot{x}}{\dot{y}} = \frac{ny^{n-1}}{a}$, $\therefore BN = \frac{y\dot{y}}{\dot{x}} = \frac{a}{ny^{n-2}}$. In the common parabola, where $n = 2$, $BN = \frac{a}{2}$, a being the latus rectum. Also, $CN = \sqrt{y^2 + \frac{1}{4}a^2}$.

25. If two quantities begin together and increase uniformly, one by x and the other by mx, m being constant, then, by the composition of Ratios, the quantities generated will be in the ratio of $x : mx$, or as $1 : m$, a constant ratio.

26. If BC move parallel to itself, and AB and BC increase uniformly, the locus of the point C is a straight line. For let BC come into the posi-



tion Ds; then as AB and BC begin together and increase uniformly, they have always a constant ratio to each other, by Art. 25; therefore AB : BC :: AD : Ds, which is the property of similar triangles; hence, ACs is a straight line. Also, as BC is parallel to Ds, AB : AC :: BD : Cs; but AB : AC in a constant ratio; if therefore BD the increment of the base be constant, the cotemporary increment C*s* of the hypothenuse must be constant, or if the former increase uniformly, the latter will increase uniformly. Hence, the two *uniform* motions of C, one in a direction parallel to AB arising from the motion of BC, and the other in the direction BC, generate an *uniform* motion in a *right* line AC.

27. The fluxion of the curve line AC, cotemporary with CE, E*s* (figure to Art. 23) the fluxions of the abscissa and ordinate, is the space that would be described by the point C with its motion continued uniform for the time in which CE, E*s* are described. Now the motion of C arises from two motions, one by which it is carried parallel to AB by the motion of BC, and the other by which it is carried in the direction BC by the increase of BC; and (Art. 26) the uniform motion of C is determined by making these two motions become uniform; but when these two motions become *uniform*, they are represented by CE and E*s*, by Art. 23, and these two *uniform* motions produce a cotemporary *uniform* motion C*s*, by Art. 26; hence, by Prop. 1, C*s* will represent the cotemporary fluxion of the curve line at the point C.

To draw ASYMPTOTES to CURVES.

DEFINITION.

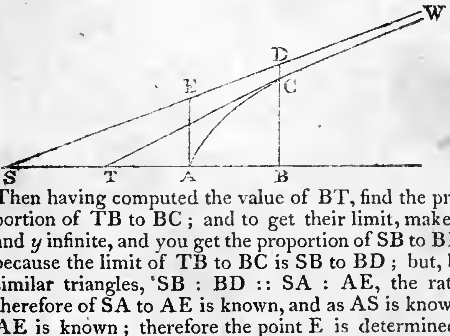
28. If a right line, intersecting the axis of a curve at a finite distance, continually approach to the curve, and arrive nearer to it than by any assignable distance, but indefinitely produced never meets it, it is called an *Asymptote.*

Prop. XI.

To draw an asymptote to a curve.

29. Let SDW be an asymptote to the curve AC; then, by the definition, we may consider the asymptote SW as the limit to which the tangent approaches, when the abscissa AB is increased sine limite. Draw AE parallel to the ordinate BC produced to D, and let TC be a tangent to the curve at C.

Put $AB = x$, $BC = y$; then by Art. 23. $BT = \frac{y\dot{x}}{\dot{y}}$; hence, $AT = \frac{y\dot{x}}{\dot{y}} - x$. From the equation of the curve, find the value of this quantity when x and y are infinite, and if it be then finite, the curve admits of an asymptote SW, and the value of AS is obtained.



Then having computed the value of BT, find the proportion of TB to BC; and to get their limit, make x and y infinite, and you get the proportion of SB to BD, because the limit of TB to BC is SB to BD; but, by similar triangles, SB : BD :: SA : AE, the ratio therefore of SA to AE is known, and as AS is known, AE is known; therefore the point E is determined; draw SE, and produce it indefinitely, and it will be the asymptote.

EXAMPLES.

Ex. 1. *Let* AC *be the common* hyperbola.

Here, by Ex. 3. Art. 23. $BT = \frac{2ax+x^2}{a+x}$, therefore $AT=\frac{2ax+x^2}{a+x}-x=\frac{ax}{a+x}$, the limit of which, when x is infinite, is $\frac{ax}{x}=a=AS$; hence, S is the centre of the hyperbola. Now $BC=\frac{b}{a}\times\sqrt{2ax+x^2}$, and $BT=\frac{2ax+x^2}{a+x}$; hence, $BT : BC :: \frac{2ax+x^2}{a+x} : \frac{b}{a}\times\sqrt{2ax+x^3}$, the limit of which (when x becomes infinite) is as $x : \frac{b}{a}\times x :: a : b ::$ BS : BD :: AS : AE; but $AS=a$, ∴ $AE=b$; hence, draw AE parallel to BC, and take it $=b$, join SE, and produce it indefinitely, and it will be the asymptote.

Ex. 2. *Let the equation of the curve be* $y^3=ax^2+x^3$.

Here $3y^2\dot{y} = 2ax\dot{x} + 3x^2\dot{x}$, and $BT = \frac{y\dot{x}}{\dot{y}} = \frac{3y^3}{2ax+3x^2}=\frac{3ax^2+3x}{2ax+3x^2}$; also, $BC=y=\sqrt[3]{ax^2+x^3}$; hence, $BT : BC :: \frac{3ax^2+3x^3}{2ax+3x^2} : \sqrt[3]{ax^2+x^3}$, the limit of which (when x becomes infinite) is $x : x ::$ BS : BD :: AS : AE; ∴ AS = AE. But $AT=\frac{3ax^2+3x^3}{2ax+3x^2}-x=\frac{ax^2}{2ax+3x^2}$, the limit of which (when x becomes infinite) is $\frac{a}{3}=AS$; hence, $AE=\frac{a}{3}$; take therefore $AS=\frac{a}{3}$, and $AE=\frac{a}{3}$, join SE, and produce it indefinitely, and it will be the asymptote.

To draw TANGENTS to SPIRALS.

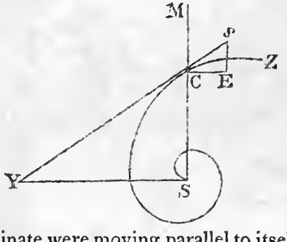
DEFINITION.

30. If an indefinite right line SM revolve about S, and a point C move in it continually from S, it will describe a curve called a *spiral;* S is called the centre, and SC its ordinate.

Prop. XII.

To draw a tangent to any point C *of a spiral.*

31. Let YC*s* be a tangent to the spiral at C, and SY perpendicular to SC ; draw CE perpendicular, and E*s* parallel to SM. Now the describing point C has two motions, one in the direction SM, and the other perpendicular to it, arising from the motion of SM about S. The describing point C is therefore under the very same circumstances as in Art. 23. upon supposition that CE is there perpendicular to the ordinate CB ; the fluxions therefore must be represented here in like manner as they were there ; the fluxions at the point C in the directions CE, CM, and C*s*, depend (Art. 3.) entirely upon the velocities of the describing point C in those directions, without any regard to what may take place afterwards from the further motion of MS about M ; the fluxions therefore will be just the same

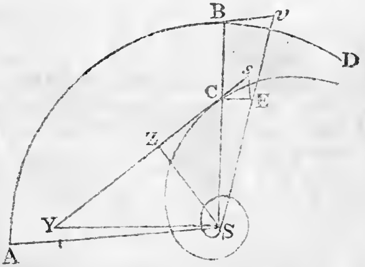


as if the ordinate were moving parallel to itself, and the describing point C had the same two motions given to it: hence, by Art. 27. Cs is the fluxion of the curve, and by Art. 23. Es is the fluxion of the ordinate, and CE the fluxion in the direction perpendicular to SC. Put SC$=y$, then E$s=\dot{y}$; and, by similar triangles, ECs, CSY, E$s(\dot{y})$: CE :: CS (y) : SY$=\frac{y\times \text{CE}}{\dot{y}}$.

Cor. If the point C have no motion in the direction SM, the curve described will be a circle, and Es becoming $=0$, the cotemporary fluxion of a circular arc whose radius SC revolves with the same angular velocity, will be CE.

32. With any radius SA describe the circle ABD, produce SC to B, and SE to v meeting Bv a tangent to the circle; and suppose the angle ASC to vary as SC^m.

Put AS$=r$, SC$=y$, AB$=x$, B$v=\dot{x}$ cotemporary with the fluxions CE, Es; for the velocity of C perpendicular to SC : velocity of B perpendicular to SB :: SC : SB; then as x is the measure of the angle ASC, let us suppose that when x becomes$=r$, y becomes t; then $x : r :: y^m : t^m$, $\therefore \frac{ry^m}{t^m}=x$, and $\frac{mry^{m-1}\dot{y}}{t^m}=\dot{x}=Bv$; and



by similar triangles SBv, SCE, $r : y :: \frac{mry^{m-1}\dot{y}}{t^m}$: CE $=$ $\frac{my^m\dot{y}}{t^m}$; hence, by Art. 31. SY$\left(=\frac{y\times \text{CE}}{\dot{y}}\right)=\frac{my^{m+1}}{t^m}$

Cor. If SZ be perpendicular to CY, we have, by sim. triangles, YSC, SCZ, CY : CS :: CS : CZ= $\frac{CS^2}{CY} = y^2 \div \sqrt{y^2 + \frac{m^2 y^{2m+2}}{t^{2m}}} = \frac{t^m y}{\sqrt{t^{2m} + m^2 y^{2m}}}$. Also, CY : SY :: CS : SZ $= \frac{SY \times CS}{CY} = \frac{m y^{m+1}}{\sqrt{t^{2m} + m^2 y^{2m}}}$.

EXAMPLES.

Ex. 1. *Let the curve be the spiral of* Archimedes.

Here $m=1$, and SY $= \frac{y^2}{t}$; hence, CY $= \sqrt{\frac{y^4}{t^2} + y^2}$ $= \frac{y\sqrt{y^2+t^2}}{t}$; therefore CZ$= \frac{ty}{\sqrt{y^2+t^2}}$. Hence also, SZ$= \frac{y^2}{\sqrt{y^2+t^2}}$.

Ex. 2. *Let the curve be the* reciprocal *spiral.*

Here $m = -1$, and SY$= -t$, a constant quantity.

Ex. 3. *Let the spiral be the* lituus.

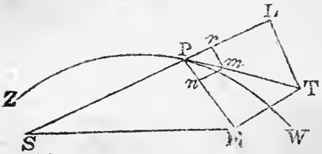
Here $m = -2$, and SY$= -\frac{2t^2}{y}$.

Ex. 4. *Let the curve be the* logarithmic *spiral.*

This curve is generated by the *uniform* angular motion of SC about S, whilst C recedes from S with a velocity proportional to SC ; hence, sE, the fluxion of SC, varies as SC ; but as the angle CSE is always the same in the same time, SC will vary as CE ; hence, CE : Es ($\dot{y}$) :: a : 1, a constant ratio, ∴ $\frac{CE}{\dot{y}} = a$, and SY$= \frac{y \times CE}{\dot{y}} = ay$; consequently SY : SC :: ay : y :: a : 1, a constant ratio ; hence, the triangle SCY continues always similar to itself, and therefore the angle SCY is constant, and is known from the ratio of $a : 1$.

PROP. XIII.

To draw a tangent to a curve ZPW, *the nature of which is expressed in terms of* SP, HP, *drawn from two given points* S, H.



33. Let PT be the tangent at P, produce SP, and taking Pm to express the fluxion of the curve, if mr be drawn perpendicular to PL, and mn to HP, then (Art. 31.) Pr and Pn express the cotemporary fluxions of SP, HP. Draw HT perpendicular to HP, meeting the tangent PT at T, and draw TL perpendicular to PL; then the figure PHTL is similar to Pnmr, and Pr : Pn :: PL : PH; if therefore PH represent the fluxion of PH, PL will represent the cotemporary fluxion of SP; putting therefore SP$=x$, HP$=y$, we have the following rule:

Put the equation of the curve into fluxions; assume $\dot{y}=y$, and find $\dot{x}$; take PL$=\dot{x}$, and perpendicular to PL draw LT, meeting a perpendicular HT to HP, in T, and join PT, and it will be a tangent.

Ex. 1. Let ZPW be an ellipse, whose foci are S and H, and major axis a; then $x+y=a$, and $\dot{x}+\dot{y}=0$, and assuming $\dot{y}=y$ (Art. 3. Cor. 2.), we have $\dot{x}=-y$; take therefore PL=PH, draw LT perpendicular to PL, and HT to HP, and PT is the tangent.

Ex. 2. Let $x^m y^n=a$ a constant quantity; then $my^n x^{m-1}\dot{x}+nx^m y^{n-1}\dot{y}=0$, and assuming $\dot{y}=y$, we get $\dot{x}=\frac{-nx}{m}$; take therefore PL $=\frac{nx}{m}$, draw LT perpendicular to PL, meeting HT perpendicular to HP in T, and PT is the tangent.

Ex. 3. Let $x^m + y^n = a$ a constant quantity; then $mx^{m-1}\dot{x} + ny^{n-1}\dot{y} = 0$, and assuming $\dot{y} = y$, we get $\dot{x} = \frac{-ny^n}{mx^{m-1}}$; take therefore PL $= \frac{ny^n}{mx^{m-1}}$, draw LT perpendicular to PL, meeting HT perpendicular to HP in T, and PT is the tangent.

Ex. 4. Let $x : y :: a : b$ a given ratio; then $x = \frac{ay}{b}$, and $\dot{x} = \frac{a\dot{y}}{b} =$ (by assuming $\dot{y} = y$) $\frac{ay}{b} = x$; hence, PL$=x$; take therefore PL=PS, draw LT perpendicular to PL, meeting HT perpendicular to HP in T, and PT is the tangent. This curve is a *circle.*

On the BINOMIAL THEOREM.

Prop. XIV.

To express the value of $\overline{a \pm x}|^n$ *by a series.*

34. The square of $1 + x$ is $1 + 2x + x^2$; the cube is $1 + 3x + 3x^2 + x^3$, &c. hence it appears, that the coefficients do not depend upon the value of x, but upon the index of the power; therefore if x be diminished and at last vanish, it will make no alteration in the coefficients. And as by the continual multiplication of $1+x$, we manifestly get a quantity with all the powers of x regularly ascending, let us assume $\overline{1+x}|^n = 1 + ax + bx^2 + cx^3 + dx^4 +$&c. Now to determine the values of a, b, c, d, &c. take the fluxion of both sides of this equation, omitting $\dot{x}$ as it will be common to every term; then take the fluxion of the resulting equation, and so on continually, and we get the following equations.

$$n \times \overline{1+x}\rceil^{n-1} = a+2bx+3cx^2+4dx^3+\&c.$$
$$n.\overline{n-1} \times \overline{1+x}\rceil^{n-2} = 2b+2.3cx+3.4dx^2+\&c.$$
$$n.\overline{n-1}.\overline{n-2} \times \overline{1+x}\rceil^{n-3} = 2.3c+2.3.4dx+\&c.$$
$$\&c. \qquad \&c.$$

Now make $x = 0$, and from the first equation, $n=a$; from the second, $n.\overline{n-1} = 2b$; from the third, $n.\overline{n-1}.\overline{n-2} = 2.3c$, &c. hence, $a = n$; $b = n.\frac{n-1}{2}$; $c=n.\frac{n-1}{2}.\frac{n-2}{3}$, &c. where the law of continuation is manifest. Hence, $\overline{1+x}\rceil^n = 1 + nx + n.\frac{n-1}{2}x^2 + n.\frac{n-1}{2}.\frac{n-2}{3}x^3+\&c.$ Now if n be a whole positive number, it is manifest that this series will terminate, for we must come to the coefficient $n.\frac{n-1}{2} \ldots . \frac{n-n}{n+1} = 0$. But the above investigation holds, whether n be a whole number or fraction, positive or negative. If n be a negative whole number, the series will never terminate, because the numerators n, $n-1$, $n-2$, &c. become then $-n$, $-n-1$, $-n-2$, &c. and therefore can never become $= 0$. Also, if n be a fraction, it is manifest that n, $n-1$, $n-2$, &c. can never become $= 0$, because a fraction can never be destroyed by the subtraction of a whole number from it. Hence, the series will always run on ad infinitum, unless n be a whole positive number. If the binomial be $1-x$, then x becoming negative, the odd powers of x will be negative and the even powers will be positive; hence, $\overline{1-x}\rceil^n = 1 - nx + n.\frac{n-1}{2}x^2 - n.\frac{n-1}{2}.\frac{n-2}{3}x^3+\&c.$

35. Hence, we may expand $\overline{a+x}\rceil^n$. For as

$a+x=a\times\overline{1+\frac{x}{a}}$, $\therefore\ \overline{a+x}|^n=a^n\times\overline{1+\frac{x}{a}}|^n$ =(by writing $\frac{x}{a}$ for x in the series in the last article) $a^n\times$ $\overline{1+n.\frac{x}{a}+n.\frac{n-1}{2}.\frac{x^2}{a^2}+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{x^3}{a^3}+\&c.}=a^n+na^{n-1}x+n.\frac{n-1}{2}a^{n-2}x^2+n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}x^3+\&c.$

For the different cases where the series converges or diverges, or becomes = 0, see Dr. WARING's *Med. Anal.* p. 415.

The principal use of this rule is to extract the roots of binomials; for if n be a fraction, the series gives that root of the binomial which the fraction expresses.

EXAMPLES.

Ex. 1. *What is the square root of* a^2+z^2, *or the value of* $\overline{a^2+z^2}|^{\frac{1}{2}}$ *in a series?*

By the Elements of Algebra, Art. 250. $\overline{a^2+z^2}|^{\frac{1}{2}}=a\times\overline{1+\frac{z^2}{a^2}}|^{\frac{1}{2}}$; compare $\overline{1+\frac{z^2}{a^2}}|^{\frac{1}{2}}$ with $\overline{1+x}|^n$, and we have $\frac{z^2}{a^2}=x$, $\frac{1}{2}=n$; hence, by substitution $a\times\overline{1+\frac{z^2}{a^2}}|^{\frac{1}{2}}=$ $a\times\overline{1+\frac{1}{2}.\frac{z^2}{a^2}+\frac{1}{2}.\frac{\frac{1}{2}-1}{2}.\frac{z^4}{a^4}+\frac{1}{2}.\frac{\frac{1}{2}-1}{2}.\frac{\frac{1}{2}-2}{3}.\frac{z^6}{a^6}+\&c.}=a+\frac{z^2}{2a}-\frac{z^4}{8a^3}+\frac{z^6}{16a^5}-\&c.$

Ex. 2. *What is the fourth root of* $1-x$, *or the value of* $\overline{1-x}|^{\frac{1}{4}}$ *in a series?*

Here $n=\frac{1}{4}$, and $\overline{1-x}|^{\frac{1}{4}}=1-\frac{1}{4}x+\frac{1}{4}.\frac{\frac{1}{4}-1}{2}x^2-\frac{1}{4}.\frac{\frac{1}{4}-1}{2}.\frac{\frac{1}{4}-2}{3}x^3+\&c.=1-\frac{1}{4}x-\frac{3}{4.8}x^2-\frac{3.7}{4.8.12}x^3-\&c.$

Ex. 3. *What is the cube root of* a—z, *or the value of* $\overline{a-z}\,|^{\frac{1}{3}}$ *in a series ?*

First, $\overline{a-z}\,|^{\frac{1}{3}}=a^{\frac{1}{3}}\times\overline{1-\frac{z}{a}}\,|^{\frac{1}{3}}$; and comparing $\overline{1-\frac{z}{a}}\,|^{\frac{1}{3}}$ with $\overline{1-x}\,|^{n}$, we have $\frac{z}{a}=x$, $n=\frac{1}{3}$; hence, $a^{\frac{1}{3}}\times\overline{1-\frac{z}{a}}\,|^{\frac{1}{3}}=$

$$a^{\frac{1}{3}}\times\overline{1-\frac{1}{3}\cdot\frac{z}{a}+\frac{1}{3}\cdot\frac{\frac{1}{3}-1}{2}\cdot\frac{z^2}{a^2}-\frac{1}{3}\cdot\frac{\frac{1}{3}-1}{2}\cdot\frac{\frac{1}{3}-2}{3}\cdot\frac{z^3}{a^3}+\&c.}=$$

$$a^{\frac{1}{3}}-\frac{z}{3a^{\frac{2}{3}}}-\frac{z^2}{9a^{\frac{5}{3}}}-\frac{5z^3}{81a^{\frac{8}{3}}}-\&c.$$

Ex. 4. *What is the value of* $\frac{1}{\sqrt{az-z^2}}$ *in an infinite series ?*

First, $\frac{1}{\sqrt{az-z^2}}=\frac{1}{a^{\frac{1}{2}}z^{\frac{1}{2}}\times\sqrt{1-\frac{z}{a}}}=\frac{1}{a^{\frac{1}{2}}z^{\frac{1}{2}}}\times\overline{1-\frac{z}{a}}\,|^{-\frac{1}{2}}$;

and comparing $\overline{1-\frac{z}{a}}\,|^{-\frac{1}{2}}$ with $\overline{1-x}\,|^{n}$, we have $\frac{z}{a}=x$,

$n=-\frac{1}{2}$; hence, $\frac{1}{a^{\frac{1}{2}}z^{\frac{1}{2}}}\times\overline{1-\frac{z}{a}}\,|^{-\frac{1}{2}}=\frac{1}{a^{\frac{1}{2}}z^{\frac{1}{2}}}\times$

$$\overline{1-\frac{-1}{2}\cdot\frac{z}{a}+\frac{-1}{2}\cdot\frac{-\frac{1}{2}-1}{2}\cdot\frac{z^2}{a^2}-\frac{-1}{2}\cdot\frac{-\frac{1}{2}-1}{2}\cdot\frac{-\frac{1}{2}-2}{3}\cdot\frac{z^3}{a^3}+}$$

$$\&c.=\frac{1}{a^{\frac{1}{2}}z^{\frac{1}{2}}}+\frac{z^{\frac{1}{2}}}{2a^{\frac{3}{2}}}+\frac{3z^{\frac{3}{2}}}{8a^{\frac{5}{2}}}+\frac{5z^{\frac{5}{2}}}{16a^{\frac{7}{2}}}+\&c.$$

Ex. 5. *To resolve* $\frac{1}{a^2+2ax+x^2}$ *into an infinite series.*

This quantity is $\frac{1}{\overline{a+x}\,|^{2}}=\overline{a+x}\,|^{-2}$; which compared with $\overline{a+x}\,|^{n}$, gives $n=-2$; hence, $\overline{a+x}\,|^{-2}=a^{-2}$ —

$$2a^{-3}x - 2.\frac{-2-1}{2}.a^{-4}x^2 - 2.\frac{-2-1}{2}.\frac{-2-2}{3}.a^{-5}x^3 - \&c. = \frac{1}{a^2} - \frac{2x}{a^3} + \frac{3x^2}{a^4} - \frac{4x^3}{a^5} + \&c.$$

Ex. 6. *What is the value of* $\frac{1}{2az+z^2}$ *in an infinite series?*

This quantity is equal to $\frac{1}{2az \times \overline{1+\frac{z}{2a}}} = \frac{1}{2az} \times \overline{1+\frac{z}{2a}}\Big]^{-1}$; and by comparing $\overline{1+\frac{z}{2a}}\Big]^{-1}$ with $\overline{1+x}\Big]^{n}$, we have $x = \frac{z}{2a}$, $n = -1$; hence, $\frac{1}{2az} \times \overline{1+\frac{z}{2a}}\Big]^{-1} = \frac{1}{2az} \times \overline{1 - 1.\frac{z}{2a} - 1.\frac{-1-1}{2}.\frac{z^2}{4a^2} - \&c.} = \frac{1}{2az} - \frac{1}{4a^2} + \frac{z}{8a^3} - \&c.$

In like manner we must proceed in the expansion and division of all binomial quantities.

The value of $\overline{1+x}\big]^n$ has been assumed $= 1 + ax + bx^2 + cx^3 + \&c.$ and applied in all cases, whether n be a whole number or a fraction; if n be a whole number, it is manifest from the observation in Art. 34, that this must be the form of the series; but if n be a fraction, it is not so obvious that we may assume the same series; the legality of the assumption however in that case may be thus shown. Let $n =$ any fraction $\frac{r}{s}$, r and s being whole numbers. Now the value of $\overline{1+x}\big]^r$ is expressed by $1 + ax + bx^2 + cx^3 + \&c.$ but $\overline{1+x}\big]^r$ is the s^{th} power of $\overline{1+x}\big]^{\frac{r}{s}}$; therefore such a series must be assumed for $\overline{1+x}|^{\frac{r}{s}}$, that the s^{th} power thereof may give a series of the form $1+ax+bx^2+cx^3+$&c. Now *any* power of the series $1+px+qx^2+rx^3+$&c. will give a series $1+ax+bx^2+cx^3+$&c. therefore we must assume a series of that form, where the powers of x regularly ascend, to represent the value of $\overline{1+x}|^{\frac{r}{s}}$.

SECTION III.

ON THE METHOD OF FINDING FLUENTS.

36. THE business of the *direct* method of fluxions is to find the fluxion from the fluent; to find the fluent from the fluxion is sometimes called the *inverse* method of fluxions. It is not difficult to put any quantity into fluxions, there being direct rules for that purpose; but there are no direct general rules for finding a fluent from a fluxion; and very often it is impossible to do it, except by an approximation by an infinite series, as the fluxion may be such as could not arise from putting any fluent into fluxions. We cannot therefore lay down rules for finding the fluents of any other fluxions than those whose forms show them to have been derived from some fluent.

PROP. XV.

To find the fluent of any power of a simple quantity multiplied by the fluxion of that quantity.

37. The fluxion of x^3 is $3x^2\dot{x}$, therefore we know that the fluent of $3x^2\dot{x}$ is x^3, and it is deduced from the fluxion, by the converse of the rule for putting x^3 into fluxions. In general, the fluxion of x^n is (Art. 12.) $nx^{n-1}\dot{x}$; therefore the fluent of $nx^{n-1}\dot{x}$ must

be x^n, and this fluent is deduced from the fluxion by the following

RULE:

Add unity to the index, divide by the index so increased, and also by the fluxion of the root.

EXAMPLES.

Ex. 1. The fluent of $7x^6\dot{x}$ is x^7.

Ex. 2. The fluent of $x^9\dot{x}$ is $\frac{x^{10}}{10}$.

Ex. 3. The fluent of $5x^3\dot{x}$ is $\frac{5x^4}{4}$.

Ex. 4. The fluent of $\frac{7}{9}x^{\frac{5}{3}}\dot{x}$ is $\frac{3}{8} \times \frac{7}{9} \times x^{\frac{8}{3}} = \frac{7}{24}x^{\frac{8}{3}}$.

Ex. 5. The fluent of $\frac{6\dot{x}}{x^9}$ or $6x^{-9}\dot{x}$ is $\frac{6x^{-8}}{-8} = -\frac{3}{4x^8}$.

Ex. 6. The fluent of $\frac{3\dot{y}}{y^{\frac{3}{5}}}$ or $3y^{-\frac{3}{5}}\dot{y}$ is $\frac{5}{2} \times 3y^{\frac{2}{5}} = \frac{15}{2}y^{\frac{2}{5}}$.

38. If $n = 0$, or the index of x be -1, the fluxion is $\frac{\dot{x}}{x}$; but this fluxion cannot be generated by x^0, because (by the Principles of Algebra) $x^0 = 1$, a constant quantity; hence, the fluent of $\frac{\dot{x}}{x}$ cannot be found by this rule.

PROP. XVI.

To find the fluent of a binomial quantity (one part of which is constant and the other part variable) raised to a power where the term without the vinculum is the fluxion of the variable term under the vinculum, or in a given ratio to it.

39. The fluxion of $\overline{a^r + x^r}\big]^n$ is (Cor. Art. 12) $n \times \overline{a^r + x^r}\big]^{n-1} \times r x^{r-1}\dot{x}$, which is found by the same rule as the fluxion of x^n. Every complete fluxion therefore of this kind must necessarily have the index of the variable quantity without the vinculum, less by unity than the index under the vinculum. Hence, every quantity so circumstanced may have its fluent found by the above rule.

If $r = 1$, then $r - 1 = 0$, and $x^0 = 1$; therefore the fluxion becomes $n \times \overline{a + x}\big]^{n-1} \times \dot{x}$.

EXAMPLES.

Ex. 1. *What is the fluent of* $\overline{a + x}\big]^6 \times \dot{x}$?

Here the fluxion of the root $a + x$ is $\dot{x}$; hence, the fluent is $\dfrac{\overline{a + x}\big]^7 \times \dot{x}}{7\dot{x}} = \dfrac{\overline{a + x}\big]^7}{7}$.

Ex. 2. *What is the fluent of* $\overline{a^2 + x^2}\big]^{\frac{1}{2}} \times x\dot{x}$?

Here the fluxion of the root $a^2 + x^2$ is $2x\dot{x}$; hence, the fluent is $\dfrac{\overline{a^2 + x^2}\big]^{\frac{3}{2}} \times x\dot{x}}{\frac{3}{2} \times 2x\dot{x}} = \dfrac{\overline{a^2 + x^2}\big]^{\frac{3}{2}}}{3}$.

Ex. 3. *What is the fluent of* $\overline{a^4 - x^4}\big]^{\frac{5}{3}} \times 3x^3\dot{x}$?

Here the fluxion of the root $a^4 - x^4$ is $-4x^3\dot{x}$; hence, the fluent is $\dfrac{\overline{a^4 - x^4}\big]^{\frac{8}{3}} \times 3x^3\dot{x}}{\frac{8}{3} \times -4x^3\dot{x}} = -\dfrac{9 \times \overline{a^4 - x^4}\big]^{\frac{8}{3}}}{32}$.

Ex. 4. *What is the fluent of* $\dfrac{x^8\dot{x}}{\overline{a^9 + 6x^9}\big]^{\frac{1}{2}}}$?

This quantity is $= \overline{a^9 + 6x^9}\big]^{-\frac{1}{2}} \times x^8\dot{x}$; and the fluxion of the root $a^9 + 6x^9$ is $54x^8\dot{x}$; therefore the fluent is $\dfrac{\overline{a^9 + 6x^9}\big]^{\frac{1}{2}} \times x^8\dot{x}}{\frac{1}{2} \times 54x^8\dot{x}} = \dfrac{\overline{a^9 + 6x^9}\big]^{\frac{1}{2}}}{27}$.

Quantities which at first do not stand under this form, may frequently be reduced to it.

Ex. 5. *What is the fluent of* $\frac{a\dot{x}}{\overline{a^2+x^2}|^{\frac{3}{2}}}$?

First, $a^2+x^2=\overline{\frac{a^2}{x^2}+1}\times x^2=\overline{a^2x^{-2}+1}\times x^2$; therefore $\overline{a^2+x^2}|^{\frac{3}{2}}=\overline{a^2x^{-2}+1}|^{\frac{3}{2}}\times x^3$; hence, $\frac{a\dot{x}}{\overline{a^2+x^2}|^{\frac{3}{2}}}=$ $\frac{a\dot{x}}{\overline{a^2x^{-2}+1}|^{\frac{3}{2}}\times x^3}=\overline{a^2x^{-2}+1}|^{-\frac{3}{2}}\times ax^{-3}\dot{x}$, where the index of x without is less by unity than that under the vinculum; hence the fluent is $\frac{\overline{a^2x^{-2}+1}|^{-\frac{1}{2}}\times ax^{-3}\dot{x}}{-\frac{1}{2}\times-2a^2x^{-3}\dot{x}}=$ $\frac{1}{\overline{a^2x^{-2}+1}|^{\frac{1}{2}}\times a}=\frac{x}{\overline{a^2+x^2}|^{\frac{1}{2}}\times a}$.

40. If both quantities under the vinculum be variable, and the quantity without be the fluxion of the quantity under the vinculum, or in a constant ratio to it, the fluent may be found by this rule. Thus, the fluent of $\overline{a^2y^2+y^4}|^{\frac{1}{2}}\times\overline{2a^2y\dot{y}+4y^3\dot{y}}$ is $\frac{2}{3}\times\overline{a^2y^2+y^4}|^{\frac{3}{2}}$; but these cases seldom occur.

Prop. XVII.

To find the fluent of $\overline{a+cz^n}|^m\times dz^{rn-1}\dot{z}$, *where the index of z without the vinculum increased by unity, is some multiple of the index of z under the vinculum.*

41. Put $a+cz^n=x$, then $z^n=\frac{x-a}{c}$, $\therefore$ $z^{rn}=\frac{\overline{x-a}|^r}{c^r}$, take its fluxion, and $rnz^{rn-1}\dot{z}=\frac{r\times\overline{x-a}|^{r-1}\dot{x}}{c^r}$, $\therefore$ $z^{rn-1}\dot{z}=\frac{1}{nc^r}\times\overline{x-a}|^{r-1}\times\dot{x}$; hence (putting $r-1=s$), $dz^{rn-1}\dot{z}=\frac{d}{nc^r}\times\overline{x-a}|^s\times\dot{x}=$ (by expanding $\overline{x-a}|^s$)

$\frac{d}{nc^r} \times \dot{x} \times \overline{x^s - sax^{s-1} + s \cdot \frac{s-1}{2} a^2 x^{s-2} - \&c.}$ substitute this quantity for $dz^{rn-1}\dot{z}$, and x^m for $\overline{a + cz^n}|^m$, and the given fluxion is transformed to $\frac{d}{nc^r} \times$

$x^m \dot{x} \times \overline{x^s - sax^{s-1} + s \cdot \frac{s-1}{2} a^2 x^{s-2} - \&c.} = \frac{d}{nc^r} \times$

$\overline{x^{m+s}\dot{x} - sax^{m+s-1}\dot{x} + s \cdot \frac{s-1}{2} a^2 x^{m+s-2}\dot{x} - \&c.}$ the fluent of each of which terms is found by the Rule in Art. 37. hence, the fluent required is $\frac{d}{nc^r} \times$

$$\overline{\frac{x^{m+s+1}}{m+s+1} - \frac{sax^{m+s}}{m+s} + \frac{s \cdot \frac{s-1}{2} \cdot a^2 x^{m+s-1}}{m+s-1} - \&c.}$$

Now let us consider when the fluent of the given fluxion can be expressed in finite terms.

1st. If r, and consequently s, be a whole *positive* number, the series arising from the expansion of $\overline{x-a}|^s$ will terminate, and the fluent can always be found if m be a *positive* whole number, or a *positive* or *negative* fraction.

2dly. If r be a *positive* whole number, and m a *negative* whole number, greater in magnitude than $s+1$, or r, the fluent can always be found. But if m be a *negative* whole number equal to or less in magnitude than r, the denominator of one of the terms must become $= 0$, in which case the fluent of that term fails; for in the fluxion it was of this form $x^{-1}\dot{x}$, which by Art. 38. admits of no fluent by the rule here given; it may, however, be found by logarithms, as will be explained in Art. 45.

3dly. The given fluxion, by reduction, becomes $\overline{az^{-n}+c}|^m \times dz^{m+r \times n-1}\dot{z}$; hence, if m and r be both fractions, but such that $m+r$ may be a whole *negative* number, the fluent can always be found. This will appear, by transforming the fluxion as before; and the series will always terminate; nor can any of the denominators of the terms of the fluent become equal to nothing, so as to make the fluent of such term fail, as it is here taken.

To find FLUENTS by LOGARITHMS.

42. The property of logarithms, or their relation to natural numbers, as has been already explained in Algebra, is this, that as the natural numbers increase in geometric progression, their logarithms increase in arithmetic progression.

43. Let a increase till it becomes b, c, ... m, n, o, &c. and suppose $a : b :: b : c ::$ &c. $:: m : n ::$ &c. then $a : m :: a—b : m—n$; now $a—b$ is the increment of a, and $m—n$ is the increment of m; hence, $a : m ::$ the increment of a : the increment of m; and as this is true in every state of the increments, if we make them vanish, we have $a : m$ in the *limiting* ratio of the increment of a : the increment of m, that is, as the fluxion of a : the fluxion of m, by Art. 7.

44. Let y be any number, and x its logarithm; then if x increase uniformly, or if $\dot{x}$ be constant, y will increase in geometric progression, therefore, by the last article, y varies as $\dot{y}$, and $\frac{y}{\dot{y}}$ is constant; hence, $\frac{y\dot{x}}{\dot{y}}$ is constant; put therefore $\frac{y\dot{x}}{\dot{y}} = \text{M}$, and we have $\dot{x} = \text{M} \times \frac{\dot{y}}{y}$; that is, the fluxion of any logarithm is equal to a *constant* quantity multiplied into the fluxion of the number divided by the number. The quantity M is

called the *modulus* of the system, and may be assumed of any value.

If $M = 1$, the logarithms are called *hyperbolic*, because the same logarithms may be deduced from the hyperbola, as will appear hereafter. In this case $\dot{x} = \frac{\dot{y}}{y}$.

Prop. XVIII.

To find the fluent of a fluxion, which is the fluxion of any quantity ($\dot{y}$) divided by that quantity (y), or in a given ratio to it.

45. Put $x =$ the hyperbolic logarithm of y; then by Art. 44. $\frac{\dot{y}}{y} = \dot{x}$, and the fluent of $\frac{\dot{y}}{y}$ * is x. And as y, although here a simple quantity, may represent any compound quantity whatever, and $\dot{y}$ its fluxion, we have the following

RULE:

When any fluxional expression appears to be the fluxion of a quantity divided by the quantity itself, its fluent is the hyperbolic logarithm of that quantity.

EXAMPLES.

Ex. 1. The fluent of $\frac{\dot{x}}{x \pm a}$ is the h. l. (hyperbolic logarithm) of $\overline{x \pm a}$.

Ex. 2. The fluent of $\frac{2x\dot{x}}{a^2 + x^2}$ is the h. l. $\overline{a^2 + x^2}$.

Ex. 3. The fluent of $\frac{nx^{n-1}\dot{x}}{a^n + x^n}$ is the h. l. $\overline{a^n + x^n}$.

These fluents are obvious, the given fluxion being manifestly the fluxion of the quantity divided by the quantity, for the numerator is the fluxion of the denominator.

* If $x =$ hyp. log. $-y$, then $\dot{x} = \frac{\dot{y}}{y}$; the fluent therefore of $\frac{\dot{y}}{y}$ is h. l. $\pm y$; but the negative value belongs to another system.

Ex. 4. The fluent of $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$ is the h. l. of $\overline{x + \sqrt{x^2 \pm a^2}}$.

For, put $x^2 \pm a^2 = v^2$, then $x\dot{x} = v\dot{v}$, $\therefore x : v :: \dot{v} : \dot{x}$, and $x+v : v :: \dot{x}+\dot{v} : \dot{x}$; hence, $\frac{\dot{x}+\dot{v}}{x+v} = \frac{\dot{x}}{v} = \frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$; therefore the fluent of $\frac{\dot{x}+\dot{v}}{x+v}$, or of $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$, is the h. l. $\overline{x+v} =$ h. l. $\overline{x + \sqrt{x^2 \pm a^2}}$.

Ex. 5. The fluent of $\frac{\dot{x}}{\sqrt{x^2 \pm 2ax}}$ is the h. l. $\overline{x \pm a + \sqrt{x^2 \pm 2ax}}$.

For, put $\sqrt{x^2 \pm 2ax} = y$, then $x^2 \pm 2ax + a^2 = y^2 + a^2$, and $x \pm a = \sqrt{y^2 + a^2}$; hence, $\dot{x} = \frac{y\dot{y}}{\sqrt{y^2+a^2}}$, consequently $\frac{\dot{x}}{\sqrt{x^2 \pm 2ax}} = \frac{\dot{y}}{\sqrt{y^2+a^2}}$, whose fluent, by the last example, is h. l. $\overline{y + \sqrt{y^2+a^2}} =$ h. l. $\overline{x \pm a + \sqrt{x^2 \pm 2ax}}$.

Ex. 6. The fluent of $\frac{2a\dot{x}}{a^2 - x^2}$ is the h. l. $\frac{a+x}{a-x}$.

For $\frac{2a\dot{x}}{a^2-x^2} = \frac{\dot{x}}{a+x} - \frac{-\dot{x}}{a-x}$, whose fluent is the h. l. $\overline{a+x}$ — h. l. $\overline{a-x} =$ h. l. $\frac{a+x}{a-x}$, as shown in the Algebra, Art. 388. In like manner the fluent of $\frac{2a\dot{x}}{x^2-a^2}$ is h. l. $\frac{x-a}{x+a}$.

Ex. 7. The fluent of $\frac{2a\dot{x}}{x\sqrt{a^2+x^2}}$ is the h. l. $\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}$.

For, put $\sqrt{a^2+x^2}=y$, then $a^2+x^2=y^2$, therefore $x\dot{x} = y\dot{y}$, and $\frac{2a\dot{x}}{xy}=\frac{2a\dot{y}}{x^2}$; that is, $\frac{2a\dot{x}}{x\sqrt{a^2+x^2}}=\frac{2a\dot{y}}{y^2-a^2}$, whose fluent, by the last example, is h. l. $\frac{y-a}{y+a}$ = h. l. $\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}$. In like manner, the fluent of $\frac{2a\dot{x}}{x\sqrt{a^2-x^2}}$ is h. l. $\frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}$.

Ex. 8. The fluent of $\frac{x^{-2}\dot{x}}{\sqrt{b^2+x^{-2}}}$ is — h. l. $\frac{1+\sqrt{1+b^2x^2}}{x}$.

For, put $\frac{1}{x}=y$, then $x^{-2}\dot{x}=-\dot{y}$; hence, the fluxion becomes $\frac{-\dot{y}}{\sqrt{b^2+y^2}}$, whose fluent is (by Example 4) — h. l. $y+\sqrt{b^2+y^2}$ = —h. l. $\frac{1}{x}+\sqrt{b^2+\frac{1}{x^2}}$ = —h. l. $\frac{1+\sqrt{1+b^2x^2}}{x}$.

These are the most useful forms of fluxions whose fluents may be found by a table of hyperbolic logarithms; which table may be supplied, by multiplying the logarithm found from the common tables by 2,30258509, which will give the corresponding hyperbolic logarithm.

Ex. The fluent of $\frac{\dot{x}}{1+x}$ is the h. l. of $\overline{1+x}$; if $x=1$, the fluent is the h. l. of $2=0,693147$; if $x=4$, the fluent is the h. l. of $5=1,6094379$.

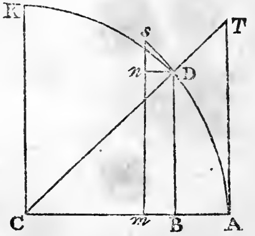
To find FLUENTS by CIRCULAR ARCS.

Prop. XIX.

The length of a circular arc for every degree, minute, and second, to radius $=1$, *being given, to find from thence certain fluents.*

46. Let AD be a circular arc whose centre is C, AT its tangent, DB its sine; draw ms parallel to BD meeting the tangent Ds in s, and Dn parallel to Bm.

Put $CD=a$, $AB=x$, $BD=y$, $AD=z$, $AT=t$, $CT=s$; then by Art. 23. $Ds=\dot{z}$, $Dn=\dot{x}$, $ns=\dot{y}$. Now the triangles CBD, snD are similar, for they are right-angled at B and n, and the angle sDn=CDB, because nDC is the complement of each. Hence, $y:a::\dot{x}:\dot{z}$



$=\frac{a\dot{x}}{y}$; but $y=\sqrt{CD^2-CB^2}=\sqrt{a^2-\overline{a-x}|^2}=$

$\sqrt{2ax-x^2}$; $\therefore \dot{z} = \frac{a\dot{x}}{\sqrt{2ax-x^2}}$. Also, $\sqrt{a^2-y^2}$ (BC) $: a :: \dot{y} : \dot{z} = \frac{a\dot{y}}{\sqrt{a^2-y^2}}$. Again by sim. triangles CAT, CBD, s (CT) : a (CA) :: a (CD) : CB $= \frac{a^2}{s}$, $\therefore$ AB $= a - \frac{a^2}{s}$, whose fluxion Bm or D$n = \frac{a^2\dot{s}}{s^2}$; hence, from the sim. trian. Dsn, CAT, $\sqrt{s^2-a^2}$ (AT) $: s :: \frac{a^2\dot{s}}{s^2} : \dot{z} = \frac{a^2\dot{s}}{s\sqrt{s^2-a^2}}$. Lastly, $\sqrt{s^2-a^2} = t$, $\therefore \frac{s\dot{s}}{\sqrt{s^2-a^2}} = \dot{t}$, and $\dot{z} \left(= \frac{a^2\dot{s}}{s\sqrt{s^2-a^2}}\right) = \frac{a^2\dot{t}}{s^2} = \frac{a^2\dot{t}}{a^2+t^2}$. Hence, the fluxion of the arc AD, or $\dot{z}$, is expressed under four different forms in terms of the right sine, versed sine, tangent, and secant; consequently the fluent of each of these fluxions will be expressed by z. Hence

1st Fluent of $\frac{a\dot{y}}{\sqrt{a^2-y^2}}$ is a cir. arc whose rad. is a and sine y.

2d Fluent of $\frac{a\dot{x}}{\sqrt{2ax-x^2}}$ is a cir. arc whose rad. is a and versed sine x.

3d Fluent of $\frac{a^2\dot{t}}{a^2+t^2}$ is a cir. arc whose rad. is a and tangent t.

4th Fluent of $\frac{a^2\dot{s}}{s\sqrt{s^2-a^2}}$ is a cir. arc whose rad. is a and secant s.

Now, by a table exhibiting the length of circular arcs for all degrees, &c. of the quadrant to radius unity, if these arcs be multiplied by a we shall have their lengths to the radius a. Hence, for example, what is the fluent of $\frac{a\dot{y}}{\sqrt{a^2 - y^2}}$, when y is the sine of $30°$? The length of an arc of $30°$ to radius 1, is 0,5235987: hence, the length of the arc to radius a, is $a \times 0,5235987$, the fluent required. Thus, the fluents of all fluxions under any of these forms may be found.

47. A fluent can have but one fluxion, but a fluxion may have an infinite number of fluents; thus, the fluent of $\dot{x}$ is x, or $x \pm a$, whatever be the value of the constant part a. By Prop. 4. in taking the fluxion of a binomial, the constant part goes out, and therefore, when the fluent is taken back again, that constant part does not appear. Now to determine, in any particular case, what this constant part is to be, or whether any such quantity is to be annexed, consider whether the fluent first taken becomes equal to nothing, or of a known value, at the time it ought; if it do, it requires no constant quantity to be added; if it do not, such a quantity must be annexed to it, as will make it become equal to nothing, or to its proper value. This is called the *correction* of a fluent.

48. Although the fluxion of a quantity be *relative*, that is, if $\dot{x}$ denote the fluxion of x, then will $nx^{n-1}\dot{x}$ be the fluxion of x^n, where $\dot{x}$ may be assumed of any magnitude, yet the fluents are not at all affected by varying $\dot{x}$, the fluents of these quantities $\dot{x}$ and $nx^{n-1}\dot{x}$ being x and x^n, whatever be the value of $\dot{x}$. Hence, of whatever magnitude we assume the fluxion of any quantity, the fluent will always give the quantity generated. In the following Problems, therefore, the fluxion of the area, solid, curve line, or surface, may be assumed of any magnitude, and the fluent, corrected if necessary, will give the quantity which has been generated.

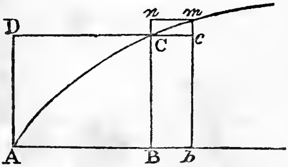
SECTION IV.

To find the AREAS of CURVES.

Prop. XX.

To find the area ABC *of any curve, whose ordinate* BC *is perpendicular to the abscissa* AB.

49. Let ABC be any curvilinear area generated by the uniform motion of the ordinate BC; on AB, BC describe the parallelogram ABCD, and conceive this to have been generated by



the same uniform motion of a line equal and parallel to AD; draw *bm* parallel to BC, and complete the parallelogram B*bmn*, and produce DC to *c*. Then AD being constant whilst BC varies, the next increment of the parallelogram is BC*cb*, and the cotemporary increment of the area ABC is BC*mb*; hence, the ratio of the increment BC*cb* of the parallelogram to the cotemporary increment BC*mb* of the area ABC, is always nearer to a ratio of equality than BC*cb* : B*nmb*, or nearer than BC : *bm*; now, let *bm* move up to, and coincide with BC, in order to obtain the *limiting* ratio of the increments, and we get the *limiting* ratio of BC : *bm*, a ratio of equality; hence, a fortiori, the *limiting* ratio of the increment BC*cb* of the parallelogram, to the cotemporary increment BC*mb* of the area ABC, is a ratio of equality; therefore by Prop. 2. the fluxion of the parallelogram ABCD is equal to the fluxion of the area ABC; but BC*cb* being the increment of the parallelogram *uniformly* generated, will represent its fluxion, by Prop. 1. hence, the fluxion of the area of the curve ABC will be represented by BC*cb*, the cotemporary fluxion of the abscissa AB being B*b*. If therefore $AB=x$, $BC=y$, $Bb=\dot{x}$, and A = the area ABC, then will $\dot{A}=BCcb=y\dot{x}$; the fluent of which, corrected if necessary, gives A.

Cor. Hence, the fluxion of any area generated by the motion of a straight line in a direction perpendicular to itself, is as the length of the generating line and its velocity conjointly. And as a curve line, moving in a direction perpendicular to itself, must describe the same area as a straight line of the same length moving with the same velocity, the fluxion of the surface generated by a curve line, so moving, must be as its length and velocity conjointly.

EXAMPLES.

Ex. 1. *Let* AC *be any* parabola; *to find its area.*

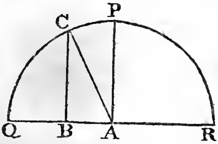
Here $ax=y^n$; hence, $a\dot{x}=ny^{n-1}\dot{y}$, and $\dot{x}=\frac{ny^{n-1}\dot{y}}{a}$, $\therefore y\dot{x}=\frac{ny^n\dot{y}}{a}=\dot{A}$, whose fluent (Art. 37) $A=\frac{ny^{n+1}}{n+1\times a}+C$ (C being the correction if necessary) $=\frac{n}{n+1}\times\frac{y^n}{a}\times y+C=\frac{n}{n+1}\times xy+C$; now when $A=0$, $x=0$, $\therefore C=0$; hence, $A=\frac{n}{n+1}\times xy$.

If $n=2$, it becomes the common parabola, and the area $=\frac{2}{3}xy$.

If $n=1$*, the figure becomes a triangle, and the area $=\frac{1}{2}xy$.

Ex. 2. *To find the area of a* circle, *whose radius is unity.*

Let A be the centre of the circle; draw BC, AP,



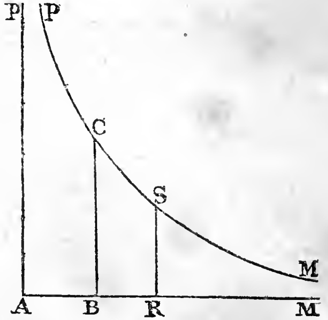
perpendicular to QR, and join AC. Put AC=1, AB $=x$, BC$=y$; then $x^2+y^2=1$, $\therefore y=\overline{1-x^2}|^{\frac{1}{2}}=1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}-$&c. (Art. 34.); $\therefore \dot{A}=y\dot{x}=\dot{x}-\frac{x^2\dot{x}}{2}-\frac{x^4\dot{x}}{8}-\frac{x^6\dot{x}}{16}-\frac{5x^8\dot{x}}{128}-$&c. the fluxion of the area BAPC whose fluent is $A=x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{112}-\frac{5x^9}{1152}-$&c. + C; now when $x=0$, $A=0$, $\therefore C=0$; hence, $A=x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{112}-\frac{5x^9}{1152}-$ &c. Now if the arc PC= 30°, $x=\frac{1}{2}$; and the area ABCP = 0,5—0,0208333—0,0007812—0,0000698—0,0000085—0,0000012—&c.=0,4783055. But as $x=\frac{1}{2}$, $y=\sqrt{\frac{3}{4}}$; therefore the area of the triangle ACB $=\frac{1}{4}\times\sqrt{\frac{3}{4}}=$ 0,2165063, which subtracted from 0,4783055 leaves 0,2617992 the area of the sector ACP; which multiplied by 12 gives 3,14159 &c. = the area of the whole circle.

* If $n=1$, $ax=y$, and $x:y::1:a$, that is, in a constant ratio, which is the case when AC is a straight line, because the triangle ABC continues always similar to itself

Cor. If $r =$ radius of any circle, $a =$ its area ; then, since circles vary as the squares of their radii, $1^2 : r^2 :: 3{,}14159$ &c. : $a = 3{,}14159$ &c. $\times r^2$. If $d =$ the diameter, then $r = \frac{d}{2}$, and $r^2 = \frac{d^2}{4}$; hence, $a = 3{,}14159$ &c. $\times \frac{d^2}{4} = 0{,}78539$ &c. $\times d^2$.

Ex. 3. *To find the area of an* hyperbola *between the asymptotes* AP, AM, *and the curve* MP.

Put $AB = x$, $BC = y$; then $y = \frac{1}{x^n}$, and the fluxion of the area $APCB = y\dot{x} = \frac{\dot{x}}{x^n} = x^{-n}\dot{x} = \dot{A}$, whose fluent



is $A = \frac{x^{1-n}}{1-n} + C$.

Case 1. If n be *less* than unity, when $A = 0$, $x = 0$, $\therefore \frac{x^{1-n}}{1-n} = 0$; hence, $C = 0$; therefore the area APCB (infinite in extent) $= \frac{x^{1-n}}{1-n}$, a finite quantity when x is finite.

Case 2. If n be *greater* than unity, the index $1-n$ being negative, x must come into the denominator, and the fluent will become $A = \frac{1}{1-n \times x^{n-1}} + C = -\frac{1}{n-1 \times x^{n-1}} + C$; now when $A=0$, $x=0$, consequently $C = \frac{1}{n-1 \times x^{n-1}}$ is *infinite*, because the denominator becomes $=0$; therefore the area $APCB = \frac{1}{1-n \times x^{n-1}} + C$ is *infinite*. Whenever there is a negative index, the quantity must always be transferred from the numerator to the denominator, or the contrary, before its value in any particular case can be found.

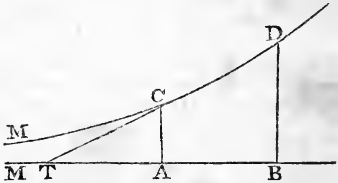
Case 3. In respect to the area BCM, as this area decreases by the same quantity that ABCP increases, it will have the same fluxion, only with a contrary sign, by Art. 16. hence, the fluent will be the same with the sign changed, that is $BCM = \frac{x^{1-n}}{n-1} + C$. If n be *greater* than unity, $BCM = \frac{1}{n-1 \,.\, x^{n-1}} + C$; and when x is infinite, $BCM=0$; hence, $0 = \frac{1}{n-1 \,.\, x^{n-1}} + C$, and therefore $C = -\frac{1}{n-1 \,.\, x^{n-1}} = 0$, x being infinite; consequently $BCM = \frac{1}{n-1 \,.\, x^{n-1}}$.

Case 4. If n be *less* than unity, and x become infinite, $C = \frac{x^{1-n}}{1-n}$ an *infinite* quantity; hence, the area $BCM = \frac{x^{1-n}}{n-1} + C$ is *infinite*.

Case 5. If $n = 1$, this fluent fails (Article 38.) and the hyperbola becomes the common hyperbola. Let AB = BC = 1, BR = x, RS = y, then AR $= 1 + x$, and $y = \frac{1}{1+x}$, therefore the fluxion of the area BCSR $= \frac{\dot{x}}{1+x}$, whose fluent, by Art. 45. is the h. l. $\overline{1+x}$, which wants no correction, because when $x = 0$, the area BCRS $= 0$, and the fluent becomes the h. l. 1, which $= 0$. Hence it appears, that any area BCSR is the h. l. of the abscissa AR, and that the whole area BCM is infinite. The *modulus* is here unity.

Ex. 4. *Let* MCD *be the* logarithmic *curve; to find its area.*

The property of the logarithmic curve is this, that if the abscissa AB increase in arithmetical progression, the ordinate BD will increase in geometrical progres-

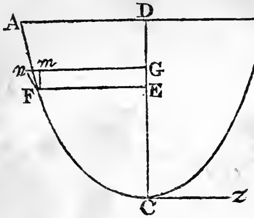


sion; ∴ if $x = $ AB, $y = $ BD, $a = $ AC, then (Art. 44.) $M = \frac{y\dot{x}}{\dot{y}}$, which (by Article 23.) is the subtangent AT; hence, $\dot{A} = y\dot{x} = M\dot{y}$, whose fluent is $A = My + C$; but when $y = a$, $A = 0$, ∴ $0 = Ma + C$, and $C = -Ma$; consequently ABDC $= My - Ma = $ AT $\times \overline{\text{BD} - \text{AC}}$. Hence, the whole area DMB $= $ AT $\times$ BD, because at an infinite distance AC $= 0$.

Ex. 5. To find the area of the catenary *curve* ACB.

Put CE $= x$, EF $= y$, CF $= z$; then $z^2 = 2ax + x^2$ (Prop. 118.), and $z\dot{z} = a\dot{x} + x\dot{x}$; hence, $z^2\dot{z}^2 = \overline{a+x}^2 \times \dot{x}^2$; but $z^2 = 2ax + x^2 = \overline{a+x}^2 - a^2$, and $\dot{x}^2 = \dot{z}^2 - \dot{y}^2$

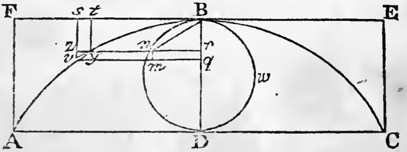
(Prop. 24.); hence $\overline{a+x}^2\times\dot{z}^2-a^2\dot{z}^2=\overline{a+x}^2\times\overline{\dot{z}^2-\dot{y}^2}$, or $a^2\dot{z}^2=\overline{a+x}^2\times\dot{y}^2$, and $a\dot{z}=\overline{a+x}\times\dot{y}=a\dot{y}+x\dot{y}$; hence, $x\dot{y}=a\dot{z}-a\dot{y}$; but *flux.* $xy=x\dot{y}+y\dot{x}$; therefore $x\dot{y}=$ *flux.* $xy-y\dot{x}$; hence, *flux.* $xy-y\dot{x}=a\dot{z}-a\dot{y}$, and $\dot{A}=y\dot{x}=$ *flux.* $xy-a\dot{z}+a\dot{y}$; therefore $A=xy-az+ay+C$; but when $x=0$, then $y=0$, $z=0$, and $A=0$; there-



fore $C=0$; hence, $A=xy-az+ay=\overline{a+x}\times y-a\sqrt{2ax+x^2}$, the area CEF.

Ex. 6. *To find the area of the* cycloid ABC.

Let BD be the axis, on which describe the circle $BnDw$, draw $rnyz \perp$ BD, and yv a tangent at y; and draw yt, $vs \perp$ FB, and vmq parallel to yr, and mn to qr, and join Bn. Now, by the property of the cycloid, the triangles Brn, yzv are similar; hence, Br, or ty, : rn : : zv, or rq, : zy, ∴ $rn\times rq=ty\times zy$, or $\square\, nrqm=\square\, styz$, that is (Art. 49.) the fluxion of the circular area $Bnr=$ the fluxion of the area Bty; and as these areas begin together at B, and their cotemporary fluxions are always equal, the quantities



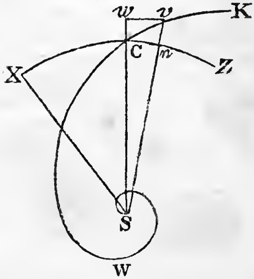
generated are equal; hence, the area $Bty=$ the circular area Bnr; bring therefore yr down to AD, and we have the whole area BFA = the semicircle B*n*D; hence, BFA + BEC = the whole circle B*n*D*w*. Now the parallelogram AFEC = AC × BD = (from the nature of the cycloid) circum. B*n*D*w*B × BD = (by Art. 51. Ex. 3.) four times the area of the whole circle; hence, ABC = three times the whole circle.

To find the AREAS of SPIRALS.

Prop. XXI.

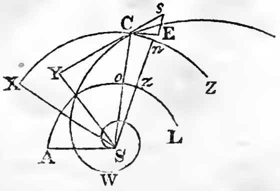
To find the area SWC *of a spiral.*

50. Let SWCK be a spiral, generated by the uniform angular motion of SC about S; SC any ordinate; with the centre S describe the circular arc XCZ; draw any other ordinate S*v*, and with the centre S describe the circular arc *vw* meeting SC produced in *w*. Now



conceive the sector SXC to have been generated by the uniform angular motion of its radius about S, at the same time that the area SWC of the spiral was generated by the same uniform angular motion of SC about S. Then SX being constant whilst SC varies, the increment of the sector SXC is the sector SC*n*, and the cotemporary increment of the area SWC of the spiral is SCv; hence, the ratio of the increment SCn of the sector SXC to the cotemporary increment SCv of the area SWC, is always nearer to a ratio of equality than SCn : Swv, or nearer than $SC^2 : Sv^2$*; now let Sv move up to and coincide with SC, in order to obtain the *limiting* ratio of the increments, and we get the *limiting* ratio of $SC^2 : Sv^2$, a ratio of equality; hence, a fortiori, the *limiting* ratio of the increment SCn to the increment SCv, is a ratio of equality; therefore by Prop. 2. the fluxion of the area of the sector SXC is equal to the fluxion of the area SWC of the spiral; but SCn being the increment of the sector SXC *uniformly* generated, will represent its fluxion, by Prop. 1. hence, the fluxion of the area SWC of the spiral will be represented by SCn.

51. Put SC $= y$, the length of the curve SWC $= z$, XC $= x$, C$n = \dot{x}$, A = the area SWC; then the sector SC$n = \frac{y\dot{x}}{2} = \dot{A}$, whose fluent is the area SWC. Let sCY be a tangent at C, and SY perpendicular to CY; draw CE ⊥ SC, and sE parallel to SC; and with the centre S, and any radius SA, describe a circular arc AL. Put SA $= a$, A$o = w$, $oz = \dot{w}$, CY $= t$, SY $= r$. Then by Art. 31. C$s = \dot{z}$, sE $= \dot{y}$, CE $= \dot{x}$; and as the tri-



angles CEs, CSY are similar, $t : r :: \dot{y} : \dot{x} = \frac{r\dot{y}}{t}$; hence,

* That similar sectors are as the squares of their radii, appears from Euclid, B. XII. p. 2. and B. VI. p. 33.

$SCn = \frac{ry\dot{y}}{2t} = \dot{A}$. Also, by similar sectors Soz, SCn, $a : y :: \dot{w} : \dot{x} = \frac{y\dot{w}}{a}$; therefore $SCn = \frac{y^2\dot{w}}{2a} = \dot{A}$. These different expressions of the fluxion of the area, are to be used as may be convenient.

EXAMPLES.

Ex. 1. *Let* SWC *be the* logarithmic *spiral; to find its area.*

Here $r : t$ in a constant ratio, as $m : n$; hence, $\dot{A} = \frac{ry\dot{y}}{2t} = \frac{my\dot{y}}{2n}$, whose fluent is $A = \frac{my^2}{4n} + C$; but when $y=0$, $A = 0$, $\therefore C = 0$; consequently $A = \frac{my^2}{4n}$.

Ex. 2. *Let* SWC *be the spiral of* Archimedes; *to find its area.*

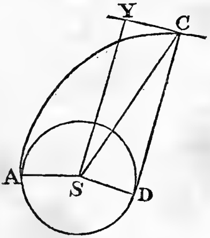
Here $y : w :: m : n$, or in a constant ratio; $\therefore \dot{w} = \frac{n\dot{y}}{m}$, consequently $\dot{A} = \frac{y^2\dot{w}}{2a} = \frac{ny^2\dot{y}}{2ma}$, whose fluent is $A = \frac{ny^3}{6ma} + C$; but when $y=0$, $A=0$, $\therefore C=0$; hence, $A = \frac{ny^3}{6ma}$.

Ex. 3. *Let the spiral be a* circle; *to find its area.*

Here y is constant, and the fluent of $\dot{A} = \frac{y\dot{x}}{2}$ is $A = \frac{yx}{2}$ the area of the sector whose arc is x; hence, if $x =$ the circumference c, the area of the circle $= \frac{cy}{2}$.

Ex. 4. *Let* AC *be the* involute *of the* circle AD, *described by the extremity* C *of a string unwinding itself from the circle; to find its area.*

It is manifest that DC must be perpendicular to the curve, or to its tangent CY, and as SD is also $\perp$ to CD, and SY to CY, SDCY is a parallelogram, and



SD = CY = t; hence, SY = $r = \sqrt{y^2 - t^2}$; $\therefore \dot{A} = \frac{ry\dot{y}}{2t}$ $= \frac{\overline{y^2 - t^2}|^{\frac{1}{2}} \times y\dot{y}}{2t}$, whose fluent, by Art. 39. is $A = \frac{\overline{y^2 - t^2}|^{\frac{3}{2}}}{6t} + C$; but when y (SC) becomes t (SA), then A, or SAC, is $=0$, and $y^2 - t^2 = 0$; hence, $C = 0$; $\therefore$ SAC $= \frac{\overline{y^2 - t^2}|^{\frac{3}{2}}}{6t} = \frac{DC^3}{6SD}$.

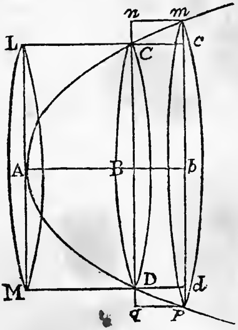
To find the CONTENTS of SOLIDS.

Prop. XXII.

To find the content of a solid generated by the rotation of a curve about its axis, or by the motion of a plane parallel to itself.

52. Let the solid ACD be conceived to be gene-

rated by the uniform motion of the circle CD, beginning at A and increasing in magnitude, having its plane always perpendicular to AB, and its centre in that line. Circumscribe this solid by the cylinder MLCD, conceived also to be generated at the same time by the same uniform motion of a circle. Then AL being constant whilst BC varies, let the circle CD move on to *mp*, and the solid C*mp*D generated, will be the increment of ACD ; suppose also the circle CD to move on to *cd* in the same time without increasing, and it will generate CD*dc* the cotemporary increment of the cylinder ; produce CD to *n* and *q*, meeting *mn* and *pq* drawn parallel to B*b*. Then the ratio of the incre-



ment CD*dc* of the cylinder to the cotemporary increment CD*pm* of the solid ACD, is always nearer to a ratio of equality, than the cylinder CD*dc* : the cylinder *mnqp*, or nearer than $BC^2 : bm^2$. Now let the circle *mp* move up to and coincide with CD, in order to obtain the *limiting* ratio of the increments, and we get the *limiting* ratio of $BC^2 : bm^2$, a ratio of equality; hence, a fortiori, the *limiting* ratio of the increment CD*dc* of the cylinder, to the cotemporary increment CD*pm* of the solid ACD, is a ratio of equality ; therefore by Prop. 2. the fluxion of the cylinder MLCD is equal to the fluxion of the solid ACD; but CDdc being the increment of the cylinder *uniformly* generated, will represent its fluxion, by Prop. 1.; hence, the fluxion of the solid ACD will be represented by CDdc, the cotemporary fluxion of AB being Bb. Put therefore x = AB, y = BC, $\dot{x}$ = Bb, S = the solid ACD, p=3,14159 &c. then (Art. 49. Ex. 2. Cor.) py^2=the area of the circle CBD; hence, the cylinder CDdc = $py^2\dot{x} = \dot{S}$; therefore S=the fluent of $py^2\dot{x}$, corrected if necessary.

The same reasoning will manifestly hold, if the generating plane be any other figure, and continue always parallel to itself. The fluxion therefore of a solid thus generated, will be always expressed by the area of the generating plane and its velocity conjointly.

EXAMPLES.

Ex. 1. *Let* ACD *be a solid generated by the revolution of any* parabola *about its axis.*

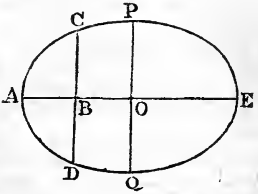
Here $ax=y^n$; hence, $\dot{x}=\frac{ny^{n-1}\dot{y}}{a}$, $\therefore \dot{S} = py^2\dot{x} = \frac{npy^{n+1}\dot{y}}{a}$, whose fluent is $S=\frac{npy^{n+2}}{n+2 \times a}+C=\frac{n}{n+2}\times py^2 \times \frac{y^n}{a} + C = \frac{n}{n+2}\times py^2x + C$; but when x=0, S=0, $\therefore$ C=0: hence, $S = \frac{n}{n+2} \times py^2x$.

If n=2, the solid becomes the common paraboloid, and its content$=\frac{1}{2}py^2x=\frac{1}{2}$ cylinder LCDM.

If n=1, the curve becomes a straight line, and the solid a cone, and its content $= \frac{1}{3}py^2x = \frac{1}{3}$ cylinder LCDM.

Ex. 2. *Let* APEQ *be a solid generated by the revolution of an* ellipse APEQ *about its axis* AE.

Put AB=x, BC=y, AO=a, PO=b; then by the property of the ellipse, $a^2 : b^2 :: 2ax - x^2 : y^2 = \frac{b^2}{a^2} \times$



$\overline{2ax - x^2}$; hence, $\dot{S} = py^2\dot{x} = \frac{pb^2}{a^2} \times \overline{2ax\dot{x} - x^2\dot{x}}$, whose fluent is $S = \frac{pb^2}{a^2} \times \overline{ax^2 - \frac{1}{3}x^3} + C$; but when $x = 0$, $S = 0$, $\therefore C = 0$; hence, $S = \frac{pb^2}{a^2} \times \overline{ax^2 - \frac{1}{3}x^3}$, which is the solid content of ACD; and to get the whole solid, we must make AB equal to AE, or make $x = 2a$; hence, the whole solid $= \frac{pb^2}{a^2} \times \overline{4a^3 - \frac{8}{3}a^3} = \frac{4pb^2a}{3}$. If the ellipse revolve about PQ instead of AE, then, as the same property of the curve holds for each axis, the solid will be $\frac{4pa^2b}{3}$; hence, the solid generated about AE : solid about PQ :: $\frac{4pb^2a}{3} : \frac{4pa^2b}{3} :: b : a ::$ PQ : AE.

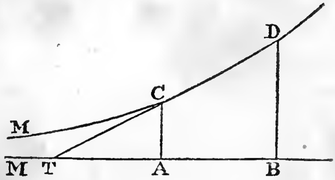
If $b = a$, the ellipse APEQ becomes a circle, and the solid a *sphere*, and the content becomes $= \frac{4pb^3}{3} = 4{,}18879b^3$. Now the content of a cylinder circumscribing the sphere = the area of its end multiplied by its length = (as the radius of the end $= b$, and length $= 2b$) $pb^2 \times 2b = 2pb^3$; hence, the sphere : cylinder $:: \frac{4}{3} : 2 :: 2 : 3$.

Ex. 3. *To find the content of the solid generated by the revolution of the* cissoid of Diocles *about its axis.*

The equation of this curve is $y^2 = \frac{x^3}{a-x}$ (Alg. Art. 426.); hence $\dot{S} = py^2\dot{x} = \frac{px^3\dot{x}}{a-x} = \frac{px^3\dot{x}}{-x+a} =$ (by division) $-px^2\dot{x} - pax\dot{x} - pa^2\dot{x} + \frac{pa^3\dot{x}}{a-x}$; now the fluent of all the terms, except the last, is found by Art. 37. and the fluent of the last, by Art. 45.; hence, the fluent is $S = -\frac{1}{3}px^3 - \frac{1}{2}pax^2 - pa^2x + pa^3 \times -\text{h. l. } \overline{a-x} + C$; now when $x=0$, $S=0$, $\therefore pa^3 \times -\text{h. l. } a + C = 0$, and $C = pa^3 \times \text{h. l. } a$; hence, $S = -\frac{1}{3}px^3 - \frac{1}{2}pax^2 - pa^2x + pa^3 \times -\text{h. l. } \overline{a-x} + pa^3 \times \text{h. l. } a = -\frac{1}{3}px^3 - \frac{1}{2}pax^2 - pa^2x + pa^3 \times \text{h. l. } \frac{a}{a-x}$; because $\text{h. l. } a - \text{h. l. } \overline{a-x} = \text{h. l. } \frac{a}{a-x}$, by the nature of logarithms, as explained in the Algebra, Art. 388.

Ex. 4. *To find the content of the solid generated by the* logarithmic *curve* ABDC *revolving about* AB.

Here $y\dot{x} = M\dot{y}$, by Art. 49. Ex. 4. $\therefore \dot{S} = py^2\dot{x} = Mpy\dot{y}$, whose fluent is $S = \frac{Mpy^2}{2} + C$; but when $y=a$, $S=0$,



$\therefore o = \frac{Mpa^2}{2} + C$, and $C = -\frac{Mpa^2}{2}$; hence, $S = \frac{Mp}{2} \times \overline{y^2 - a^2}$.

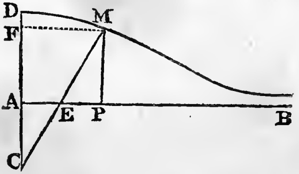
If $AC=a=0$, then $S=\frac{Mpy^2}{2}=$ the whole solid corresponding to the abscissa BM.

Ex. 5. *Let the* catenary *curve revolve about its axis; to find the content of the solid generated.*

By Prop. 118, $z^2=2ax+x^2$, and therefore $z\dot{z}=a\dot{x}+x\dot{x}$; and by the same Prop. $z\dot{y}=a\dot{x}$. Now $\dot{S}=py^2\dot{x}$; assume therefore $S=py^2x+w$, and we have $\dot{S}=py^2\dot{x}+2pxy\dot{y}+\dot{w}$, and as $\dot{S}=py^2\dot{x}$, we have $\dot{w}=-2pxy\dot{y}=$ $\left(\text{as } \dot{y}=\frac{a\dot{x}}{z}\right)-2pay\times\frac{x\dot{x}}{z}=$ (as $x\dot{x}=z\dot{z}-a\dot{x}$) $-2pay\times\overline{\dot{z}-\frac{a\dot{x}}{z}}=-2pay\times\overline{\dot{z}-\dot{y}}=2pay\dot{y}-2pay\dot{z}$; assume $w=pay^2-2payz+v$, then $\dot{w}=2pay\dot{y}-2pay\dot{z}-2paz\dot{y}+\dot{v}$; and as $\dot{w}=2pay\dot{y}-2pay\dot{z}$, we have $\dot{v}=2paz\dot{y}=2pa^2\dot{x}$, therefore $v=2pa^2x$; hence, $S=py^2x+pay^2-2payz+2pa^2x+C$; but when $x=0$, then $y=0$, $z=0$, and $S=0$, therefore $C=0$; consequently $S=py^2x+pay^2-2payz+2pa^2x$.

Ex. 6. *Let the conchoid* DM *of* Nicomedes *revolve about the axis* DA; *to find the content of the solid generated by* DMF.

By the Algebra, Art. 497. if $CA=a$, $AD=EM=b$, $AP=x$, $PM=y$, then $x^2=\frac{\overline{a+y}^2\times\overline{b^2-y^2}}{y^2}$; also,



$px^2=$ the area of the circle generated by FM, and as

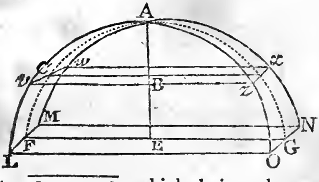
M

$FD = b - y$, $\dot{FD} = -\dot{y}$; hence $\dot{S} = -px^2\dot{y} = -p\dot{y} \times \frac{\overline{a+y}^2}{y^2} \times \overline{b^2 - y^2} = p \times \overline{a+y}^2 \times \dot{y} - pa^2b^2y^{-2}\dot{y} - pb^2\dot{y} - \frac{2pab^2\dot{y}}{y}$, therefore $S = \frac{p}{3} \times \overline{a+y}^3 + \frac{pa^2b^2}{y} - pb^2y - 2pab^2 \times$ h. l. $y + C$; now when $y = b$, $S = 0$, and the equation becomes $0 = \frac{p}{3} \times \overline{a+b}^3 + pa^2b - pb^3 - 2pab^2 \times$ h. l. $b + C$, therefore $C = -\frac{p}{3} \times \overline{a+b}^3 - pa^2b + pb^3 + 2pab^2 \times$ h. l. b; hence, $S = \frac{p}{3} \times \overline{a+y}^3 - \frac{p}{3} \times \overline{a+b}^3 + \frac{pa^2b^2}{y} - pa^2b - pb^2y + pb^3 + 2pab^2 \times$ h. l. $\frac{b}{y}$ = the solid generated by DMF.

The solid generated by the whole curve is infinite, as appears by making $y = 0$.

Ex. 7. Let LAO *be a solid called a* Groin, *generated by a variable square* vwxz *moving parallel to itself; and let the section* FAG *through the middle of the opposite sides be a semicircle.*

Put $a = AE$, $x = AB$, $y = BC$; then, by the property of the circle, $y = \sqrt{2ax - x^2}$, therefore the side of the square $vwxz = 2\sqrt{2ax - x^2}$; hence, the area



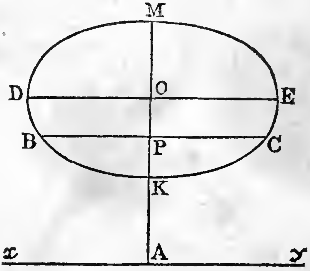
$vwxz = 4 \times \overline{2ax - x^2}$, which being the generating plane, it answers to py^2 in the other cases, and therefore $\dot{S}=4\times\overline{2ax\dot{x}-x^2\dot{x}}$, whose fluent is $S=4ax^2-\frac{4}{3}x^3+C$; but when $x=0$, $S=0$, ∴ $C=0$; hence, $S=4ax^2-\frac{4}{3}x^3$, the solid $Avwxz$; and if we make $x=a$, $S=\frac{8a^3}{3}$, the whole solid ALN.

If the section FAG be any other figure; or if the two sections through the two opposite sides be of different figures, the content may be found in like manner. But the solid content of bodies may also be found by the following proposition.

Prop. XXIII.

Let DMEK *be any curve revolving about an axis* xy; *then the solid generated is equal to the circumference described by the centre of gravity multiplied into the area of the figure.*

53. Let O be the centre of gravity; draw MOKA perpendicular to xy, and BPC, DOE, parallel to xy. Put $AP=x$, $BC=y$, $AO=a$, $p=3{,}14159$ &c. Now



(Art. 58.) $\frac{\text{flu. } yx\dot{x}}{\text{flu. } y\dot{x}}=a$, ∴ flu. $yx\dot{x}=$ flu. $y\dot{x}\times a=$ area DKEM $\times a$. But the surface generated by BC $=2pyx$, and therefore the fluxion of the solid $=2pyx\dot{x}$; and the solid $=2p\times$ flu. $yx\dot{x}=2pa\times$ area DMEK $=$ the circumference described by the centre of gravity $\times$ area of the figure.

Ex. 1. Let DMEK be a circle, then the solid will represent the ring of an anchor; now in this case, if r=OM the radius, the area DMEK$=pr^2$; hence, the solid$=2p^2ar^2$.

Ex. 2. Let MDK be the given area, and let it be the common parabola, then if G be the centre of gravity it lies in the axis DO, and therefore a=its distance from xy; also the area $=\frac{2}{3}$DO $\times$ MK; hence, the solid $=$ $2pa \times \frac{2}{3}$DO $\times$ MK $=\frac{4}{3}pa \times$ DO $\times$ MK.

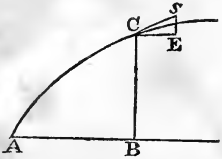
Ex. 3. Let MD, DK be straight lines, or MDK a triangle; then the area $=\frac{1}{2}$DO $\times$ MK; hence, the solid $=pa \times$ DO $\times$ MK.

To find the LENGTHS of CURVES.

Prop. XXIV.

To find the length of a curve line AC, *whose ordinate* BC *is perpendicular to the abscissa* AB.

54. Put AB$=x$, BC$=y$, AC$=z$; then if Cs be a tangent to the curve, CE ⊥ BC, and sE ⊥ CE, we have, by Art. 27. CE$=\dot{x}$, sE$=\dot{y}$, C$s=\dot{z}$; and by



Euclid, B. I. p. 47. $\dot{z}^2=\dot{x}^2+\dot{y}^2$, $\therefore \dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}$, and $z=$ the fluent of $\sqrt{\dot{x}^2+\dot{y}^2}$; corrected if necessary.

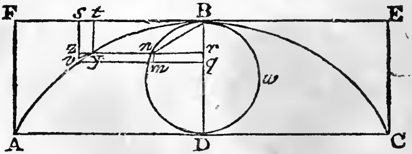
EXAMPLES.

Ex. 1. *Let* AC *be a* semi-cubical parabola, *whose equation is* $ax^2=y^3$; *to find its length.*

Here $x=\frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, $\therefore \dot{x}=\frac{3y^{\frac{1}{2}}\dot{y}}{2a^{\frac{1}{2}}}$, $\therefore \dot{x}^2=\frac{9y\dot{y}^2}{4a}$; hence, $\dot{z}^2=$ $\frac{9y\dot{y}^2}{4a}+\dot{y}^2=\overline{\frac{9y}{4a}+1}\times\dot{y}^2=\frac{9y+4a}{4a}\times\dot{y}^2, \therefore \dot{z}=\frac{\overline{9y+4a}|^{\frac{1}{2}}\times\dot{y}}{2a^{\frac{1}{2}}}$, whose fluent, by Art. 39. is $z=\frac{\overline{9y+4a}|^{\frac{3}{2}}}{27a^{\frac{1}{2}}}+C$; now when $y=0$, $z=0$, in which case, this equation becomes $0=\frac{8a}{27}+C$, $\therefore C=-\frac{8a}{27}$; hence, $z=\frac{\overline{9y+4a}|^{\frac{3}{2}}}{27a^{\frac{1}{2}}}$ $-\frac{8a}{27}$.

Ex. 2. *Let* ByA *be a* cycloid ; *to find its length.*

Put BD$=a$, B$r=x$, B$y=z$, $yv=\dot{z}$, $vz=rq=\dot{x}$; then by the prop. of the circle, Br : Bn :: Bn : BD, $\therefore$ Bn^2=BD $\times$ B$r=ax$, and B$n=a^{\frac{1}{2}}x^{\frac{1}{2}}$; and by the prop.



of the cycloid, x (Br) : $a^{\frac{1}{2}}x^{\frac{1}{2}}$ (Bn) :: $\dot{x}$ (vz) : $\dot{z}$ (vy)$=$ $\frac{a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}}{x}=a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}$; hence, $z=2a^{\frac{1}{2}}x^{\frac{1}{2}}+C$; but when $x=0$, $z=0$, $\therefore C=0$; consequently $z=2a^{\frac{1}{2}}x^{\frac{1}{2}}=2Bn$.

Ex. 3. *Let* AC *be the common* parabola ; *to find its length.*

Here $ax=y^2$, $\therefore \dot{x}=\frac{2y\dot{y}}{a}=$ (if $\frac{a}{2}=b$) $\frac{y\dot{y}}{b}$; hence, $\dot{z}^2=$ $\frac{y^2\dot{y}^2}{b^2}+\dot{y}^2=\frac{y^2+b^2}{b^2}\times\dot{y}^2$, $\therefore \dot{z}=\frac{\overline{y^2+b^2}|^{\frac{1}{2}}\times\dot{y}}{b}=$ (by mul-

tiplying numerator and denominator by $y \times \overline{y^2+b^2}\rceil^{\frac{1}{2}}$)
$\frac{1}{b} \times \frac{y^3\dot{y}+b^2y\dot{y}}{\overline{y^4+b^2y^2}\rceil^{\frac{1}{2}}} = \frac{1}{2b} \times \frac{2y^3\dot{y}+2b^2y\dot{y}}{\overline{y^4+b^2y^2}\rceil^{\frac{1}{2}}} = \frac{1}{2b} \times \frac{2y^3\dot{y}+b^2y\dot{y}}{\overline{y^4+b^2y^2}\rceil^{\frac{1}{2}}}$
$+\frac{1}{2b} \times \frac{b^2y\dot{y}}{\overline{y^4+b^2y^2}\rceil^{\frac{1}{2}}}$ = (by dividing the num. and den. of the last term by y) $\frac{1}{2b} \times \overline{y^4+b^2y^2}\rceil^{-\frac{1}{2}} \times \overline{2y^3\dot{y}+b^2y\dot{y}} + \frac{1}{2}b \times \frac{\dot{y}}{\overline{y^2+b^2}\rceil^{\frac{1}{2}}}$; now the fluent of the first term is $\frac{1}{2b} \times \overline{y^4+b^2y^2}\rceil^{\frac{1}{2}}$, by Art. 40. and the fluent of the last term is $\frac{1}{2}b \times$ h. l. $\overline{y+\overline{y^2+b^2}\rceil^{\frac{1}{2}}}$, by Art. 45. Ex. 4. hence, $z = \frac{1}{2b} \times \overline{y^4+b^2y^2}\rceil^{\frac{1}{2}} + \frac{1}{2}b \times$ h. l. $\overline{y+\overline{y^2+b^2}\rceil^{\frac{1}{2}}} + C$; now when $y=0$, $z=0$, in which case, the equation is $0 = \frac{1}{2}b \times$ h. l. $b + C$; hence, $C = -\frac{1}{2}b$. h. l. b; therefore

$$z = \frac{1}{2b} \times \overline{y^4+b^2y^2}\rceil^{\frac{1}{2}} + \tfrac{1}{2}b \times \text{h. l. } \frac{y+\overline{y^2+b^2}\rceil^{\frac{1}{2}}}{b}.$$

Ex. 4. *To find the length* CD *of any part of the* logarithmic *curve.* (See Fig. pag. 80.)

Put AC $=a$, AB $=x$, BD $=y$, CD $=z$; then $\frac{M\dot{y}}{y} = \dot{x}$ (Art. 49. Ex. 4.), $\therefore\ \dot{z} = \sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{\frac{M^2\dot{y}^2}{y^2}+\dot{y}^2}$
$= \frac{\dot{y}\sqrt{M^2+y^2}}{y}$ = (by multiplying the numerator and denominator by $\sqrt{M^2+y^2}$) $\frac{\dot{y} \times \overline{M^2+y^2}}{y\sqrt{M^2+y^2}} = \frac{y\dot{y}}{\sqrt{M^2+y^2}}$
$+ \frac{M^2\dot{y}}{y\sqrt{M^2+y^2}} = \frac{y\dot{y}}{\sqrt{M^2+y^2}} + \frac{M^2\, y^{-2}\dot{y}}{\sqrt{1+M^2\, y^{-2}}}$;

hence (by Prop. 16. and Prop. 18. Ex. 8.), $z = \sqrt{M^2+y^2} - M \times \text{h. l.} \frac{M+\sqrt{M^2+y^2}}{My} + C$; but when $z=0$, $y=b$, and we have $0=\sqrt{M^2+b^2} - M \times \text{h. l.} \frac{M+\sqrt{M^2+b^2}}{Mb} + C$; hence, $C = -\sqrt{M^2+b^2} + M \times \text{h. l.} \frac{M+\sqrt{M^2+b^2}}{Mb}$; therefore $z = \sqrt{M^2+y^2} - \sqrt{M^2+b^2} + M \times \text{h. l.} \frac{M+\sqrt{M^2+b^2}}{Mb} - M \times \text{h. l.} \frac{M+\sqrt{M^2+y^2}}{My} = \sqrt{M^2+y^2} - \sqrt{M^2+b^2} + M \times \text{h. l.} \frac{My+y\sqrt{M^2+b^2}}{Mb+b\sqrt{M^2+y^2}}$.

Ex. 5. *To find the length of a* circular *arc.*

By Art. 46. $\dot{z} = \frac{a^2\dot{t}}{a^2+t^2} =$ (by division) $\dot{t} - \frac{t^2\dot{t}}{a^2} + \frac{t^4\dot{t}}{a^4} - \frac{t^6\dot{t}}{a^6} +$ &c. hence, $z = t - \frac{t^3}{3a^2} + \frac{t^5}{5a^4} - \frac{t^7}{7a^6} +$ &c. $+C$; but when $t=0$, $z=0$, therefore $C=0$; hence, $z = t - \frac{t^3}{3a^2} + \frac{t^5}{5a^4} - \frac{t^7}{7a^6} +$ &c. Now if $a=1$, and z be an arc of 30°, then $t = \sqrt{\tfrac{1}{3}} = 0{,}5773502$, which being substituted for t, if we take 12 terms of this series, we get $z = 0{,}5235987$, the length of an arc of 30°; which multiplied by 12 gives 6,2831804 for the length of the circumference of a circle whose radius is unity.

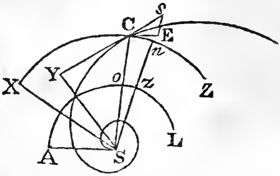
If we take the arc $z = 45°$, then will $t = a$; hence, $z = a \times \overline{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \text{\&c.}}$

TO FIND THE LENGTHS OF SPIRALS.

PROP. XXV.

To find the length of a spiral SC.

55. Let the ordinate SC $= y$, the curve SC $= z$, CY $= w$; then, by Art. 31. C$s = \dot{z}$, E$s = \dot{y}$; and by



sim. triangles, $w : y :: \dot{y} : \dot{z} = \frac{y\dot{y}}{w}$, and z = the fluent of $\frac{y\dot{y}}{w}$, corrected if necessary.

EXAMPLES.

Ex. 1. *Let* SC *be the* logarithmic *spiral; to find its length.*

Here $w:y::m:n$, a constant ratio; hence, $w = \frac{my}{n}$, $\therefore \dot{z} = \frac{n\dot{y}}{m}$, and $z = \frac{ny}{m} + C$; but when $y=0$, $z=0$, $\therefore C=0$; consequently $z = \frac{ny}{m} = \frac{y^2}{w}$; therefore CY : CS :: CS : the length of the curve.

Ex. 2. *Let it be the spiral of* Archimedes; *to find its length.*

By Art. 32. Ex. 1. $w = \frac{ty}{\sqrt{y^2+t^2}}$; hence, $\dot{z} = \frac{\dot{y}\sqrt{y^2+t^2}}{t}$, which is the same as the fluxion of the length of the parabolic arc, Art. 54. Ex. 3. $\therefore z = \frac{1}{2t} \times \overline{y^4 + t^2 y^2}\,\rceil^{\frac{1}{2}} + \frac{1}{2}t$ $\times$ h. l. $\frac{y + \sqrt{y^2 + t^2}}{t}$.

Ex. 3. *Let* AC *be the* involute *of a* circle ; *to find its length.*

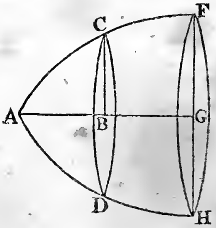
Here w is constant, by Art. 51. Ex. 4., hence, $z = \frac{y^2}{2w} + C$; but when $z = 0$, $y = w$, $\therefore 0 = \frac{w^2}{2w} + C$, and $C = -\frac{w^2}{2w}$; hence, $z = \frac{y^2 - w^2}{2w} = \frac{SY^2}{2SA}$.

To find the SURFACES of SOLIDS.

Prop. XXVI.

To find the surface of a solid generated by the rotation of a curve about its axis, or by the motion of a plane parallel to itself.

56. Conceiving the solid AFH to be generated as in Art. 52. by the circle CD, the surface may be considered as generated by the periphery of that circle; the fluxion, therefore, of the surface will be the periphery of the circle multiplied by the velocity with



which it flows, by Cor. Art. 49. But the velocity with which any point C of the periphery flows, is the velocity with which AC increases at the point C, or it is $\dot{z}$, putting AC$=z$. Hence, if we put AB$=x$, BC$=y$, $p = 6{,}28318$, &c. the circumference of a circle whose radius$=1$(Art. 54. Ex.5.), S$=$the surface ACD; then $1 : y :: p : py$, the circumference of the circle CD; therefore $\dot{S} = py\dot{z}$, the fluxion of the surface; consequently the fluent of $py\dot{z}$, corrected if necessary, will be the surface.

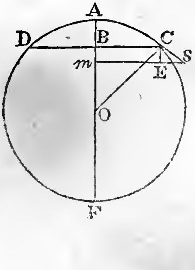
The method of finding the fluxion of the surface of a solid may be further illustrated thus.

Let ACF be protended into a straight line, and let an ordinate perpendicular to it, and always equal to the periphery of the circle CD, move from A to F with the same velocity as the point C, upon the solid, moves; then it is manifest, that the area generated by this ordinate must always be equal to the area generated by the periphery of the circle, the generating lines and their velocities being always equal, and both moving in directions perpendicular to themselves; hence, the fluxion of the surface ACD$=$the fluxion of the area of this curve$=$(by Art. 49.) the ordinate multiplied by the fluxion of the abscissa$=$the periphery of the circle CD multiplied by the fluxion of the curve AC.

EXAMPLES.

Ex. 1. *Let* ADFC *be a* sphere *whose centre is* O; *to find its surface.*

Let Cs be a tangent at C, sEm parallel to BC, and



CE to Bm; then if AB$=x$, BC$=y$, AC$=z$, by Art. 23. C$s=\dot{z}$, CE$=\dot{x}$; and by similar triangles CEs, CBO, $\dot{z}:\dot{x}::a:y$, $\therefore y\dot{z}=a\dot{x}$; hence, $\dot{S}=py\dot{z}=pa\dot{x}$, the fluxion of the surface DAC, whose fluent $S=pax+C$; but when $x=0$, $S=0$, $\therefore C=0$; hence, $S=pax$ the surface DAC. If we make AB equal to AE, or $x=2a$, we have $2pa^2$ for the whole surface of the sphere. Now if we conceive ADFC to be a great circle of the sphere, its area $=\frac{1}{2}pa^2$, by Art. 49. Ex. 2. Cor. Hence, the whole surface of a sphere is equal to four times the area of a great circle of that sphere.

Cor. As the surface DAC$=pax$, it varies as x.

Ex. 2. *Let the solid* AFH *be generated by the common* parabola; *to find its surface.*

Here $ax=y^2$; hence, $\dot{x}=\dfrac{2y\dot{y}}{a}$, and $\dot{x}^2=\dfrac{4y^2\dot{y}^2}{a^2}$; $\therefore$ (Prop. 24.) $\dot{z}^2=\dot{x}^2+\dot{y}^2=\dfrac{4y^2\dot{y}^2}{a^2}+\dot{y}^2=\overline{\dfrac{4y^2}{a^2}+1}\times\dot{y}^2=\dfrac{4y^2+a^2}{a^2}\times\dot{y}^2$, and $\dot{z}=\dfrac{\overline{4y^2+a^2}|^{\frac{1}{2}}\times\dot{y}}{a}$; hence, $\dot{S}=py\dot{z}=\dfrac{p\times\overline{4y^2+a^2}|^{\frac{1}{2}}\times y\dot{y}}{a}$, whose fluent, by Art. 39. is $S=\dfrac{p\times\overline{4y^2+a^2}|^{\frac{3}{2}}}{12a}+C$; now when $y=0$, $S=0$, in which case, the equation becomes $0=\dfrac{pa^2}{12}+C$; hence, $C=-\dfrac{pa^2}{12}$; therefore $S=\dfrac{p\times\overline{4y^2+a^2}|^{\frac{3}{2}}}{12a}-\dfrac{pa^2}{12}$.

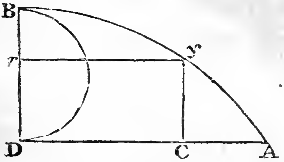
Ex. 3. *Let* ALN *be a* groin, *as in Art.* 52. *Ex.* 7. *to find its surface.*

Put AB$=x$, BC$=y$, AC$=z$; and we have (Art. 46.)

$\dot{z} = \frac{a\dot{x}}{\sqrt{2ax-x^2}}$; also, $vw = 2BC = 2\sqrt{2ax-x^2}$; now vw is the line generating one of the four surfaces; hence, $8\sqrt{2ax-x^2}$ answers to py in the other cases; therefore if S be the surface Avx, $\dot{S} = 8a\dot{x}$, and $S = 8ax+C$; but when $x=0$, $S=0$, ∴ $C=0$; consequently $S = 8ax$; and when $x=a$, $S=8a^2$.

Ex. 4. *To find the surface generated by the revolution of the* cycloidal curve BA *about its base* DA.

Put $By=z$, $Br=x$, $rD=yC=y$, $BD=a$; then, by Art. 54. Ex. 2. $\dot{z}=a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}$; ∴ $\dot{S}=py\dot{z}=pya^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}=$



$p\times\overline{a-x}\times a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}=pa^{\frac{3}{2}}x^{-\frac{1}{2}}\dot{x}-pa^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$; hence, $S= 2pa^{\frac{3}{2}}x^{\frac{1}{2}}-\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}+C$; but when $x=0$, $S=0$, ∴ $C=0$; hence, $S=2pa^{\frac{3}{2}}x^{\frac{1}{2}}-\frac{2}{3}pa^{\frac{1}{2}}x^{\frac{3}{2}}$, the surface generated by By; and when $x=a$, we have $S=\frac{4pa^2}{3}$, the whole surface generated by BA.

Ex. 5. *To find the surface of the solid generated by any part* CD *of the* logarithmic *curve revolving about its axis* AB.

By Prop. 24. Ex. 4. $\dot{z}=\frac{\dot{y}\sqrt{M^2+y^2}}{y}$, therefore $\dot{S} = py\dot{z}=p\dot{y}\sqrt{M^2+y^2}$, which fluxion is the same as that for the value of $\dot{z}$ in Prop. 24. Ex. 3. (the constant multiplier and divisor excepted); therefore $S=\frac{p}{2}\times\sqrt{y^4+M^2y^2}+\frac{pM^2}{2}\times$ h. l. $\overline{y+\sqrt{M^2+y^2}}$

$+ C$; but when $y = a$, $S = 0$; hence, $0 = \frac{p}{2} \times \sqrt{a^4+M^2a^2} + \frac{pM^2}{2} \times \text{h. l. } \overline{a + \sqrt{M^2 + a^2}} + C$, and $C = -\frac{p}{2} \times \sqrt{a^4+M^2a^2} - \frac{pM^2}{2} \times \text{h. l. } \overline{a + \sqrt{M^2+a^2}}$; therefore $S = \frac{p}{2} \times \sqrt{y^4+M^2y^2} - \frac{p}{2} \times \sqrt{a^4 + M^2a^2}$ $+ \frac{pM^2}{2} \times \text{h. l. } \frac{y + \sqrt{M^2+y^2}}{a + \sqrt{M^2 + a^2}}$.

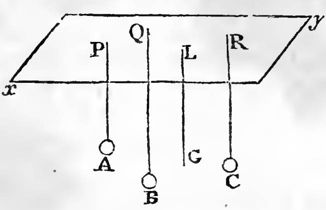
Ex. 6. *To find the surface of the solid generated by the* catenary *curve revolving about its axis.*

By Prop. 118. we have $z^2 = 2ax + x^2$; hence, $a^2 + 2ax + x^2 = a^2 + z^2$, and $a + x = \sqrt{a^2 + z^2}$; therefore $\dot{x} = \frac{z\dot{z}}{\sqrt{a^2 + z^2}}$, and $\dot{y} = \sqrt{\dot{z}^2 - \dot{x}^2} = \frac{a\dot{z}}{\sqrt{a^2 + z^2}}$. Now $\dot{S} = py\dot{z}$; assume $S = pyz - w$, then $\dot{S} = py\dot{z} + pz\dot{y} - \dot{w}$, and as $\dot{S} = py\dot{z}$, we have $\dot{w} = pz\dot{y} = \frac{paz\dot{z}}{\sqrt{a^2 + z^2}}$, whose fluent is $w = pa\sqrt{a^2 + z^2}$ (Prop. 16.); hence, $S = pyz - pa\sqrt{a^2 + z^2} + C = pyz - pa^2 - pax + C$, but when $x = 0$, $y = 0$, and $S = 0$, therefore $C - pa^2 = 0$, and $C = pa^2$; hence, $S = pyz - pax$ the surface generated by the curve CF revolving about the axis CE.

SECTION V.

ON THE CENTRE OF GRAVITY.

57. IF there be any number of bodies A, B, C,



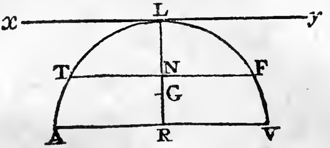
and G be their centre of gravity; and to any plane xy, perpendiculars AP, BQ, CR, GL be let fall, then (*Mechanics*, Art. 173.) $LG = \frac{A \times AP + B \times BQ + C \times CR}{A + B + C}$.

PROP. XXVII.

To find the centre of gravity of a body, considered as an area, solid, surface of a solid, or curve line.

58. Let ALV be any curve, RL the axis, in which the centre of gravity must lie; for as it bisects every ordinate TF in N, the parts on each side LR will always balance each other, and therefore the body will balance itself upon LR; consequently the centre of gravity must be somewhere in that line. Put $LN = x$, $TN = y$, $TL = z$, and draw xy parallel to TF; then if we conceive this body to be made up

of an indefinite number of corpuscles, and multiply



each corpuscle by its distance from xy, the sum of all the products divided by the sum of all the corpuscles, or by the whole body, will give LG by Art. 57. Now to get the sum of all these products, we must first get the fluxion of the sum, and the fluent will be the sum itself. Put $\dot{s}$ for the fluxion of the body at the distance x from xy, then will $x\dot{s}$ be the fluxion of the sum of all the products; also, $\dot{s}$ is the fluxion of the sum of all the corpuscles; therefore by Art. 57. LG$=\frac{\text{flu. } x\dot{s}}{\text{flu. } \dot{s}}$.

1st. If the body be an *area*, then $\dot{s}=2y\dot{x}$ by Art. 49; hence, $LG=\frac{\text{flu. } 2yx\dot{x}}{\text{flu. } 2y\dot{x}}=\frac{\text{flu. } yx\dot{x}}{\text{flu. } y\dot{x}}$.

2nd. If the body be a *solid*, then $py^2\dot{x}=\dot{s}$ by Art. 52; hence, $LG=\frac{\text{flu. } py^2x\dot{x}}{\text{flu. } py^2\dot{x}}=\frac{\text{flu. } y^2x\dot{x}}{\text{flu. } y^2\dot{x}}$.

3rd. If the body be the *surface of a solid*, then $\dot{s}=py\dot{z}$ by Art. 56; hence, $LG=\frac{\text{flu. } pyx\dot{z}}{\text{flu. } py\dot{z}}=\frac{\text{flu. } yx\dot{z}}{\text{flu. } y\dot{z}}$.

4th. If the body be a *curve line* FT, then $\dot{s}=2\dot{z}$; hence, $LG=\frac{\text{flu. } 2x\dot{z}}{\text{flu. } 2\dot{z}}=\frac{\text{flu. } x\dot{z}}{\text{flu. } \dot{z}}=\frac{\text{flu. } x\dot{z}}{z}$.

EXAMPLES.

Ex. 1. *Let $y=ax^n$ be the equation to any* parabola; *to find its centre of gravity.*

As $y=ax^n$, $\therefore yx\dot{x}=ax^{n+1}\dot{x}$, whose fluent is $\frac{ax^{n+2}}{n+2}$;

also, $y\dot{x}=ax^n\dot{x}$, whose fluent is $\frac{ax^{n+1}}{n+1}$; hence, (Art. 58.) $LG=\frac{ax^{n+2}}{n+2}\times\frac{n+1}{ax^{n+1}}=\frac{n+1}{n+2}\times x$.

If $n=\frac{1}{2}$, then $y=ax^{\frac{1}{2}}$, $\therefore y^2=a^2x$, which is the common parabola; hence, $LG=\frac{3}{5}x$.

If $n=1$, then $y=ax$, and the figure is a triangle; hence, $LG=\frac{2}{3}x$.

Ex. 2. *Let* $y=ax^n$; *to find the centre of gravity of the solid generated by the revolution of this curve about its axis.*

As $y^2=a^2x^{2n}$, $\therefore y^2x\dot{x}=a^2x^{2n+1}\dot{x}$, whose fluent is $\frac{a^2x^{2n+2}}{2n+2}$; also, $y^2\dot{x}=a^2x^{2n}\dot{x}$, whose fluent is $\frac{a^2x^{2n+1}}{2n+1}$; hence, by Article 58. $LG=\frac{a^2x^{2n+2}}{2n+2}\times\frac{2n+1}{a^2x^{2n+1}}=\frac{2n+1}{2n+2}\times x$.

If $n=\frac{1}{2}$, the solid becomes a paraboloid, and $LG=\frac{2}{3}x$.

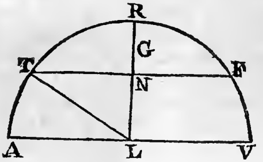
If $n=1$, the solid becomes a cone, and $LG=\frac{3}{4}x$.

Ex. 3. *Let* ALV *be a* hemispheroid; *to find its centre of gravity.*

Put $LR=a$, $AR=b$; then $a^2 : b^2 :: 2ax-x^2 : y^2=\frac{b^2}{a^2}\times\overline{2ax-x^2}$; hence, $y^2x\dot{x}=\frac{b^2}{a^2}\times\overline{2ax^2\dot{x}-x^3\dot{x}}$, whose fluent is $\frac{b^2}{a^2}\times\overline{\frac{2}{3}ax^3-\frac{1}{4}x^4}$; also, $y^2\dot{x}=\frac{b^2}{a^2}\times\overline{2ax\dot{x}-x^2\dot{x}}$, whose fluent is $\frac{b^2}{a^2}\times\overline{ax^2-\frac{1}{3}x^3}$; hence, by Art. 58. $LG=\frac{\frac{2}{3}ax^3-\frac{1}{4}x^4}{ax^2-\frac{1}{3}x^3}$; and when $x=a$, $LG=\frac{\frac{2}{3}a^4-\frac{1}{4}a^4}{a^3-\frac{1}{3}a^3}=\frac{5a}{8}$ for the whole solid. As this is independent of b, if $b=a$, LG remains the same, and the solid becomes a *hemisphere.*

Ex. 4. *Let* ARV *be a* semicircle; *to find its centre of gravity.*

Put LN$=x$, TN$=y$, TL$=r$; then $x^2+y^2=r^2$; hence,



$x\dot{x}+y\dot{y}=0$, $\therefore$ $yx\dot{x}=-y^2\dot{y}$, whose fluent is $-\frac{1}{3}y^3+$ C, which must vanish when TF coincides with AV, or $y=r$; therefore put r for y, and $-\frac{1}{3}r^3+C=0$, $\therefore$ $C=\frac{1}{3}r^3$; hence, the correct fluent of $yx\dot{x}$ is $\frac{1}{3}r^3-\frac{1}{3}y^3$; also, the fluent of $y\dot{x}$ is (Art. 49.) the area ATNL; hence, by Art. 58. $LG=\frac{1}{3}\times\frac{r^3-y^3}{ATNL}$; and when $y=0$, $LG=\frac{r^3}{3ARL}$ for the semicircle.

Ex. 5. *To find the centre of gravity of the* arc ARV.

Put LN$=x$, NT$=y$, RT$=z$; then (Art. 46.), $\dot{z}:\dot{y}::r:x$, therefore $x\dot{z}=r\dot{y}$, whose fluent is ry; hence, by Art. 58. $LG=\frac{ry}{z}$; and when $y=r$, $LG=\frac{r^2}{RA}$.

Ex. 6. *To find the centre of gravity of the* surface ARV *of a* hemisphere.

Put $x=$ RN, $y=$TN, $z=$ RT, and $a=$TL; then (Art. 46.) we have $\dot{z}:\dot{x}::a:y$, therefore $y\dot{z}=a\dot{x}$; hence, $yx\dot{z}=ax\dot{x}$, whose fluent is $\frac{1}{2}ax^2$; also, the fluent of $y\dot{z}$, or $a\dot{x}$, is ax; hence, by Art. 58. $RG=\frac{\frac{1}{2}ax^2}{ax}=\frac{1}{2}x$; and when $x=$RL$=r$, then $RG=\frac{1}{2}r$ for the hemisphere.

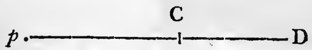
O

On the CENTRE of GYRATION.

DEFINITION.

59. The centre of gyration is that point of a body revolving about an axis, into which if the whole quantity of matter were collected, the same moving force would generate the same angular velocity in the body.

60. Let a body p revolve about C, and let a force act at D to oppose its motion. Then the momentum of p varies as $p\times$ its velocity, or as $p\times p\mathrm{C}$, which we may consider as a power acting at p in opposition to



the force at D; but this power acting at the distance $p\mathrm{C}$ from the centre of motion, its effect to oppose a force at D must (by the property of the lever) be as $p\times p\mathrm{C}\times p\mathrm{C}=p\times p\mathrm{C}^2$. This effect of p to persevere in its motion, or, which is the same, to prevent any change in its motion, is called its *inertia*.

Prop. XXVIII.

To find the centre of gyration *of a body.*

61. Let a body be conceived to be made up of the particles A, B, C, &c. whose distances from the axis are a, b, c, &c. and let x be the distance of the centre of gyration from the axis, then by Art. 59. the inertia of A, B, C, &c. will be as $\mathrm{A}\times a^2$, $\mathrm{B}\times b^2$, $\mathrm{C}\times c^2$, &c. and the inertia of all the matter at the distance x will be as $\overline{\mathrm{A}+\mathrm{B}+\mathrm{C}+\&c.}\times x^2$; now, as the moving force is the same in both cases, the inertia must be the same when the same angular velocity is generated; hence, $\overline{\mathrm{A}+\mathrm{B}+\mathrm{C}+\&c.}\times x^2=\mathrm{A}\times a^2+\mathrm{B}\times b^2+\mathrm{C}\times c^2+\&c.$ therefore $x=\sqrt{\dfrac{\mathrm{A}\times a^2+\mathrm{B}\times b^2+\mathrm{C}\times c^2+\&c.}{\mathrm{A}+\mathrm{B}+\mathrm{C}+\&c.}}$; that is,

if $\dot{s}$ be the fluxion of the body at the distance z from the axis, $x = \sqrt{\frac{\text{flu. } z^2\dot{s}}{s}}$.

EXAMPLES.

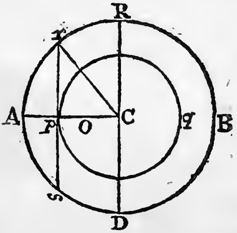
Ex. 1. *Let the* straight line CA *revolve about* C; *to find* O *the centre of gyration.*

Put z=Cp, then s=z, and $\dot{s}$=$\dot{z}$, ∴ $z^2\dot{s}=z^2\dot{z}$, whose

fluent is $\frac{1}{3}z^3$ = (when z = CA) $\frac{1}{3}\text{CA}^3$; hence, CO = $\sqrt{\frac{1}{3}\text{CA}^2} = \text{CA}\sqrt{\frac{1}{3}}$.

Ex. 2. *Let a* circle AB *revolve in its own plane about its centre* C; *to find* O *its centre of gyration.*

Put p=6,28318, &c. the circumference of a circle whose radius = 1, z = Cp; then the circumference $pq = pz$, and $pz\dot{z} = \dot{s}$; hence, the fluent of $z^2\dot{s}$, or of



$pz^3\dot{z}$, is $\frac{1}{4}pz^4$ = (when z = CA = r) $\frac{1}{4}pr^4$. Also, the area of the circle $= \frac{1}{2}pr^2$; hence, CO $= \sqrt{\frac{1}{2}r^2} = r\sqrt{\frac{1}{2}}$.

Cor. The same must be true for a *cylinder* revolving about its axis, it being true for every section parallel to the end.

Ex. 3. *Let* RADB *be a* sphere *revolving about the diameter* RD; *to find* O *its centre of gyration.*

Draw CA ⊥ and *spr* parallel to RD; put $Cr=r$, $Cp=z$, then $pr=\sqrt{r^2-z^2}$; and if $p=6,28318$, &c. the surface of the cylinder generated by *sr* revolving about RD, is $pz\times2\sqrt{r^2-z^2}$; hence, $\dot{s}=2pz\dot{z}\sqrt{r^2-z^2}$, and $z^2\dot{s}=2pz^3\dot{z}\sqrt{r^2-z^2}$. Now to find this fluent, put $r^2-z^2=y^2$, then $z^2=r^2-y^2$, and $z^4=r^4-2r^2y^2+y^4$, $\therefore z^3\dot{z}=-r^2y\dot{y}+y^3\dot{y}$; hence, $2pz^3\dot{z}\sqrt{r^2-z^2}=2p\times\overline{-r^2y^2\dot{y}+y^4\dot{y}}$, whose fluent is $2p\times\overline{-\frac{1}{3}r^2y^3+\frac{1}{5}y^5}$, and when $z=0$, this fluent ought to vanish, but y is then $=r$, and the fluent becomes $2p\times\overline{-\frac{2}{15}r^5}$; hence, the correct fluent is $2p\times\overline{\frac{2}{15}r^5-\frac{1}{3}r^2y^3+\frac{1}{5}y^5}$; and the whole fluent when $z=r$ (in which case $y=0$) will be $\frac{4}{15}pr^5$. Now the content of the sphere $=\frac{2}{3}pr^3$; hence, CO $=\sqrt{\frac{2}{5}r^2}=r\sqrt{\frac{2}{5}}$.

ON THE CENTRE OF PERCUSSION.

DEFINITION.

62. The centre of percussion is that point in the axis * of a vibrating or revolving body, which striking against an immoveable obstacle, the whole motion,

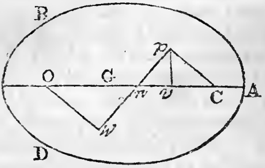
* The axis is here understood to be a right line drawn through the centre of gravity of the body, perpendicular to the axis about which the body revolves.

estimated in the *plane* of the body's motion, shall be destroyed.

PROP. XXIX.

To find the centre of percussion *of a body.*

63. Let ABD be a plane passing through the centre of gravity G of the body, and perpendicular to the axis of suspension which passes through C; and conceive the whole body to be projected upon this plane in lines perpendicular to it, or parallel to the axis; then as each particle is thus kept at the same distance from the axis, the effect, from the rotatory motion about the axis, will not be altered, nor will the centre of gravity be changed. Let O be the centre of percussion, and draw pnw perpendicular to pC, and Ow perpendicular to pw; also pv perpendicular to Cn. As the



velocity of any particle $p \propto p$C, the momentum of p in the direction $pw \propto p \times p$C, it being as the velocity and quantity of matter conjointly; and by the property of the lever, the efficacy of this force to turn the body about O is as $p \times p\text{C} \times \text{O}w =$ (because $\text{O}n : \text{O}w :: p\text{C} : v\text{C}$) $p \times v\text{C} \times \text{O}n = p \times v\text{C} \times \overline{\text{CO} - \text{C}n} = p \times v\text{C} \times \text{CO} - p \times v\text{C} \times \text{C}n =$ (as $\text{C}n : \text{C}p :: \text{C}p : v\text{C}$) $p \times v\text{C} \times \text{CO} - p \times \text{C}p^2$. Now that the efficacy of all the particles to turn the body about O may be $= 0$, we must make the *sum of all the quantities* $p \times v\text{C} \times \text{CO}$ — *sum of all the quantities* $p \times \text{C}p^2 = 0$; hence, CO =

$$\frac{\textit{sum of all the } p \times \text{C}p^2}{\textit{sum of all the } p \times v\text{C}} = \frac{\textit{sum of all the } p \times \text{C}p^2}{\textit{body} \times \text{CG}},$$ these two denominators being equal from the property of the centre of gravity (Art. 57.)

Although the body, by striking at O, may have no tendency to move in the *plane* of its previous motion; and this only is included in the common definition which we here follow, yet it may have a tendency to revolve about AO. If therefore we were to define the centre of percussion, to be that point where the *whole* motion would be destroyed, we must find the plane parallel to ABD, such that the sum of all the forces to turn the body about the line joining the centre of percussion and the axis of vibration in that plane, is also $= 0$. But this is a problem not fit for an elementary treatise.—See the *Hydrostatics*, third edit. Prob. To find the Centre of Pressure.

As the force acting at O destroys the motion, let us suppose a force to act at O and to generate the motion back again; then it is manifest, that the body would *begin* to return under all the same circumstances in which its motion ceased; that is, it would *begin* its motion by revolving about C. In this case, C is called the centre of *spontaneous* rotation; making therefore the point at which a force acts upon a body that can move freely, the centre of percussion, the centre of spontaneous rotation coincides with the centre of rotation corresponding to that centre of percussion.

On the CENTRE of OSCILLATION.

DEFINITION.

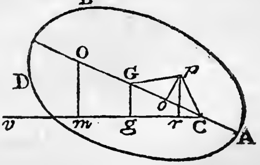
64. The centre of oscillation is that point in the axis of a vibrating body, at which, if a particle were suspended from the axis of motion, it would vibrate in the same time the body does.

Prop. XXX.

To find the centre of oscillation *of a body.*

65. Let ABD be a body projected upon a plane

perpendicular to the axis of rotation, as in Art. 63. the axis passing through C and supposed to be parallel to the horizon; and let G be the centre of gravity, O the centre of oscillation; draw Cv parallel to the horizon, Om, Gg, pr perpendicular to it. Then by the property of the lever, the force of gravity to turn the particle p about C $\propto p \times Cr$; hence, the force of gravity to turn the whole body about C $\propto$ the *sum of all*



the $p \times Cr$. Also, the force of gravity to turn a single particle O at O about C $\propto O \times Cm$. Now by Art. 60. the inertia of $p \propto p \times Cp^2$, and therefore the inertia of the whole body $\propto$ the *sum of all the* $p \times Cp^2$. Also, the inertia of $O \propto O \times OC^2$. Now that the acceleration of the body about C may be equal to that of the particle O, the moving forces must be in proportion to the inertiæ; because, if the powers to produce motion be as the powers to oppose it, the acceleration must be the same. Hence, *sum of all* $p \times Cr : O \times Cm :: $ *sum of all* $p \times Cp^2 : O \times OC^2$, therefore $OC = \frac{\textit{sum of all } p \times Cp^2 \times Cm}{\textit{sum of all } p \times Cr \times OC} = \frac{\textit{sum of all } p \times Cp^2}{\textit{body} \times CG}$, because (by sim. triangles) $Cm : CO :: Cg : CG$, and therefore $\frac{Cm}{CO} = \frac{Cg}{CG}$, and by the property of the centre of gravity, *sum of all* $p \times Cr = \textit{body} \times Cg$. Hence, the centre of oscillation is the same as the centre of percussion. Or if s be the body, x the distance of $\dot{s}$ from the axis of suspension, then $CO = \frac{\text{flu. } x^2\dot{s}}{\text{flu. } x\dot{s}} = \frac{\text{flu. } x^2\dot{s}}{s \times CG}$.

66. Join pG; and draw Po perpendicular to CG; then $Cp^2 = CG^2 + Gp^2 - 2CG \times Go$, therefore $p \times Cp^2 = p \times CG^2 + p \times Gp^2 - 2CG \times p \times Go$, and the *sum of all* $p \times Cp^2 =$ *sum of all* $p \times CG^2 +$ *sum of all* $p \times Gp^2 - 2CG \times$ *sum of all* $p \times Go$; but the *sum of all* $p \times Go = 0$, from the property of the centre of gravity; and the *sum of all* $p \times CG^2 =$ *body* $\times CG^2$; hence, *sum of all* $p \times Cp^2 =$ *body* $\times CG^2 +$ *sum of all* $p \times Gp^2$; consequently

$$CO = \frac{body \times CG^2 + sum\ of\ all\ p \times Gp^2}{body \times CG} = CG + \frac{sum\ of\ all\ p \times Gp^2}{body \times CG};$$

hence, $GO = \frac{sum\ of\ all\ p \times Gp^2}{body \times CG}$. Now as the numerator is constant, GO varies inversely as CG; hence, if we find GO for any one value of CG, we shall know every other value of GO from that of CG. Hence also, if O be the centre of suspension, C will become the centre of oscillation; for as $GO \times GC$ is constant, if C be changed to O, O must be changed to C.

Cor. If x be the distance from C to the centre of gyration; then by Art. 61. $x^2 s =$ *sum of all* $p \times Cp^2$; and by Art. 65. $CO \times s \times CG =$ *sum of all* $p \times Cp^2$; hence, $x^2 = CO \times CG$, and $CG : x :: x : CO$.

EXAMPLES.

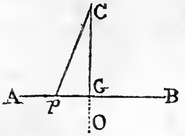
Ex. 1. *Let* CD *be a* straight line *suspended at* C; *to find the centre* O *of oscillation.*

Put $x = Cp$; then the fluent of $x^2 \dot{s} =$ flu. $x^2 \dot{x} = \frac{1}{3} x^3$

$=$ (when $x=$CD) $\frac{1}{3}$CD^3. Also, *body* $\times$ CG $=$ CD $\times$ $\frac{1}{2}$CD$=\frac{1}{2}$CD^2; hence, CO$=\frac{2}{3}$CD.

Ex. 2. *Let the* line AB *vibrate* lengthways *in a vertical plane about* C, *which is equidistant from* A *and* B; *to find its centre* O *of oscillation.*

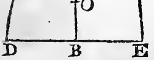
Draw CG perpendicular to AB; and put CG $=a$, G$p=x$; then pC$^2=a^2+x^2$; and the fluent of C$p^2\times\dot{s}$ $=$fluent of $a^2\dot{x}+x^2\dot{x}=a^2x+\frac{1}{3}x^3$ $=$ (when $x=$ AG) a^2 $\times$ AG $+\frac{1}{3}$AG^3; hence, for the whole line AB, it be-



comes $2a^2\times$ AG$+\frac{2}{3}$AG^3. Also, *body*$\times$CG$=a\times$AB $=a\times 2$AG; hence, CO $=\dfrac{2a^2\times \text{AG}+\frac{2}{3}\text{AG}^3}{a\times 2\text{AG}}=a+\dfrac{\text{AG}^2}{3a}$.

Ex. 3. *Let* DAE *be any* parabola *vibrating* flatways, *or about an axis passing through* C *parallel to* PMN; *to find the centre* O *of oscillation.*

Put AC$=d$, AM$=x$, PM$=y$, then $ax^n=y$; hence, $2y\dot{x}=2ax^n\dot{x}=\dot{s}$; and the fluent of CM$^2\times\dot{s}$, or



$2.\overline{d+x}^2\times ax^n\dot{x}$, or $2d^2ax^n\dot{x}+4dax^{n+1}\dot{x}+2ax^{n+2}\dot{x}$, is $\dfrac{2d^2ax^{n+1}}{n+1}+\dfrac{4dax^{n+2}}{n+2}+\dfrac{2ax^{n+3}}{n+3}$, which vanishes when

$x=0$, and therefore it wants no correction. Also, the fluent of $CM \times \dot{s}$, or $\overline{d+x} \times 2ax^n\dot{x}$ is $\frac{2dax^{n+1}}{n+1} + \frac{2ax^{n+2}}{n+2}$; hence, if the former be divided by the latter, we get (by reduction) CO=

$$\frac{\overline{n+2} \cdot \overline{n+3} \cdot d^2 + \overline{n+1} \cdot \overline{n+3} \cdot 2dx + \overline{n+1} \cdot \overline{n+2} \cdot x^2}{\overline{n+2} \cdot \overline{n+3} \cdot d + \overline{n+1} \cdot \overline{n+3} \cdot x}$$

If $d=0$, and $n=1$, the figure becomes a triangle, and $AO=\frac{3}{4}x$.

If $n=\frac{1}{2}$, it becomes the common parabola, and AO $=\frac{5}{7}x$.

Ex. 4. *Let the* parabola *vibrate* edgeways, *and let it be suspended at* A; *to find the centre of oscillation.*

By Ex. 2. the sum of the products of each particle of the line PN into the square of its distance from A, is $2x^2 \times y + \frac{2}{3}y^3 = 2x^2 \times ax^n + \frac{2}{3}a^3x^{3n}$; hence, $2ax^{n+2}\dot{x} + \frac{2}{3}a^3x^{3n}\dot{x}$ is the fluxion of the sum of the products for the whole body; whose fluent is $\frac{2ax^{n+3}}{n+3} + \frac{2a^3x^{3n+1}}{3.\overline{3n+1}}$.

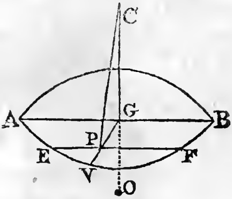
Also, the fluent of $AM \times \dot{s}$ is the same as before, d being now $= 0$; hence, $AO = \frac{\overline{n+2} \cdot x}{n+3} + \frac{a^2 \cdot \overline{n+2} \cdot x^{2n-1}}{3 \cdot \overline{3n+1}}$.

If $n = \frac{1}{2}$, it is the common parabola, and $AO = \frac{5x}{7} + \frac{a^2}{3}$.

If $n=1$, $AO=\frac{3x}{4} + \frac{a^2x}{4}$ for a triangle; and if $a=1$, $AO=x$.

Ex. 5. *Let* CG *be perpendicular to the plane of the circle* ABV, *and let the* circle *vibrate about an axis passing through* C *and parallel to* AB; *to find the centre* O *of oscillation.*

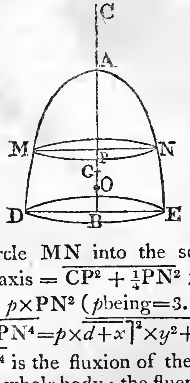
Draw GPV perpendicular to AB, and EF parallel to AB. Put $AG=r$, $CG=a$, $GP=x$, then $CP^2=a^2+x^2$, $PE=\sqrt{r^2-x^2}$, and $EF=2\sqrt{r^2-x^2}$; hence,



$EF\times CP^2=\overline{a^2+x^2}\times 2\sqrt{r^2-x^2}$, which multiplied by $\dot{x}$ gives $\overline{a^2\dot{x}+x^2\dot{x}}\times 2\sqrt{r^2-x^2}$ for the fluxion of the sum of the products of each particle of the area ABFE multiplied into the square of its distance from the axis of vibration. Now to find the fluent, we have the fluent of $a^2\times 2\sqrt{r^2-x^2}\times\dot{x}=a^2\times$ area ABFE by Art. 49. and when $x=r$, the fluent $=a^2\times$ AVB; and as the same is true for the other semicircle, the whole fluent is $a^2\times$circle AEB. The fluent of the second part, $2x^2\dot{x}\sqrt{r^2-x^2}$, may be found thus. Let $\dot{x}\sqrt{r^2-x^2}=\dot{A}$, $x^2\dot{x}\sqrt{r^2-x^2}=\dot{B}$, and $x\times\overline{r^2-x^2}|^{\frac{3}{2}}=P$; then by taking the fluxion of the last, we have $\dot{P}=\dot{x}\times\overline{r^2-x^2}|^{\frac{3}{2}}-3x^2\dot{x}\sqrt{r^2-x^2}=\dot{x}\times\overline{r^2-x^2}\times\sqrt{r^2-x^2}-3x^2\dot{x}\sqrt{r^2-x^2}=r^2\dot{x}\sqrt{r^2-x^2}-4x^2\dot{x}\sqrt{r^2-x^2}$, that is, $\dot{P}=r^2\dot{A}-4\dot{B}$, hence, (by taking the fluents) $P=r^2A-4B$, and $B=\frac{r^2A-P}{4}$; therefore the fluent of $2x^2\dot{x}\sqrt{r^2-x^2}$ is $\frac{r^2A-P}{2}$; but when $x=r$, $P=0$; and the fluent becomes $\frac{r^2A}{2}=\frac{r^2}{8}\times$ circle AEB, because $A=\frac{1}{4}$ of the circle when $x=r$; and for both semicircles it becomes $\frac{r^2}{4} \times$ circle ; hence, the whole fluent is $\overline{a^2 + \frac{1}{4}r^2}$ $\times$ circle, which is the sum of the products of each particle of the circle $\times$ the square of its distance from the axis of vibration. Also, $a \times$ circle = the denominator for the value of CO ; hence, by dividing the former by the latter, we get $CO = a + \frac{r^2}{4a}$.

Ex. 6. *Let the solid formed by the rotation of any curve* DAE *about its axis* AB, *vibrate about* C *in* BA *produced; to find the centre* O *of oscillation.*

By Ex. 5. the sum of the products of each par-



ticle of the circle MN into the square of its distance from the axis = $\overline{CP^2 + \frac{1}{4}PN^2} \times$ circle MN = $\overline{CP^2 + \frac{1}{4}PN^2} \times p \times PN^2$ (p being=3.14159 &c.)$=p \times \overline{CP^2 \times PN^2 + \frac{1}{4}PN^4} = p \times \overline{d+x}|^2 \times y^2 + \frac{1}{4}y^4$; hence, $p\dot{x} \times \overline{d+x}^2 \times y^2 + \frac{1}{4}y^4$ is the fluxion of the sum of all such products for the whole body ; the fluent of which divided by CG $\times$ *body*, gives CO.

Ex. 7. *Let the solid be a* paraboloid ; *to find the centre of oscillation.*

Here $ax = y^2$; hence, $p\dot{x} \times \overline{\overline{d+x}^2 \times y^2 + \frac{1}{4}y^4}$ is equal to $p\dot{x} \times \overline{\overline{d+x}^2 \times ax + \frac{1}{4}a^2x^2}$, whose fluent is $\frac{1}{2}pad^2x^2 + \frac{2}{3}padx^3 + \frac{1}{4}pax^4 + \frac{1}{12}pa^2x^3$; also, (Art. 53. Ex. 1.), the body $= \frac{1}{2}pax^2$; and (Art. 58. Ex. 2.) $AG = \frac{2}{3}x$; $\therefore$ $CG = d + \frac{2}{3}x$; hence, CG $\times$ *body* $= \frac{1}{2}padx^2 + \frac{1}{3}pax^3$;

dividing therefore the above fluent of this quantity, we have $CO=\dfrac{6d^2+8dx+3x^2+ax}{6d+4x}$.

If C coincide with A, $d=0$, and $CO=\dfrac{3x+a}{4}$.

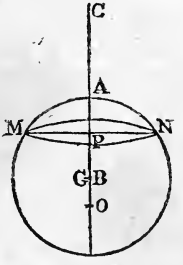
Ex. 8. *Let the solid be a* cone ; *to find the centre of oscillation.*

Put $AB=a$, $BD=b$; then $a : b :: x : y=\dfrac{bx}{a}=$ $\left(\text{if } m=\dfrac{b}{a}\right)$ mx; hence, $p\dot{x}\times\overline{d+x}^2\times\overline{y^2+\frac{1}{4}y^4}=p\dot{x}\times\overline{d+x}^2\times\overline{m^2x^2+\frac{1}{4}m^4x^4}$, whose fluent is $\frac{1}{3}pd^2m^2x^3+\frac{1}{2}pdm^2x^4+\frac{1}{5}pm^2x^5+\frac{1}{20}pm^4x^5$; also, (Art. 52. Ex. 1.) the body$=\frac{1}{3}pm^2x^3$; and (Art. 58. Ex. 2.) $AG=\frac{3}{4}x$, $\therefore CG=d+\frac{3}{4}x$; hence, $CO=\dfrac{20d^2+30dx+12x^2+3m^2x^2}{20d+15x}=\dfrac{20d^2+30da+12a^2+3b^2}{20d+15a}$ for the whole cone, when $x=a$, and $mx=y=b$.

If the cone be suspended at the vertex, then $d=0$, and $CO=\dfrac{4a^2+b^2}{5a}$.

Ex. 9. *Let the body be a* sphere ; *to find the centre* O *of oscillation,* C *being the point of suspension.*

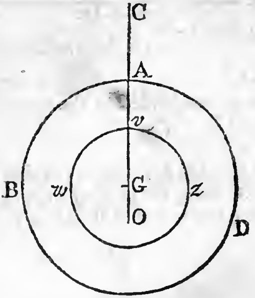
Let B be the centre ; then if $BA=r$, $y^2=2rx-x^2$.



In this case, it will be most convenient to apply the rule in Art. 66. that is, to get the value of CO when C coincides with A, and thence to deduce its value in any other case. Now when C coincides with A, $d=0$, and the expression becomes $p\dot{x} \times \overline{x^2y^2 + \frac{1}{4}y^4} = p \times \overline{r^2x^2\dot{x} + rx^3\dot{x} - \frac{3}{4}x^4\dot{x}}$, whose fluent is $\frac{1}{3}pr^2x^3 + \frac{1}{4}prx^4 - \frac{3}{20}px^5$; and when $x = 2r$ it becomes $\frac{28}{15}pr^5$ for the whole sphere. Also, the *body* × CG (G coinciding with B) $=\frac{4}{3}pr^3 \times r = \frac{4}{3}pr^4$; therefore AO$=1\frac{2}{5}r$; consequently BO $=\frac{2}{5}r$. Hence, (Art. 66.) if $d=$CB, $d : r :: \frac{2}{5}r : \frac{2r^2}{5d} =$ BO when the point of suspension is at C; therefore CO $= d + \frac{2r^2}{5d}$.

Ex. 10. *Let the body be a* circle, *and the axis of vibration pass through* C *perpendicular to its plane.*

Put GA$=r$, CG$=d$, GO$=x$, and $p=6{,}283$ &c. then



$px=$the circumference vwz, and the fluxion of the sum of all the particles multiplied into the square of their distances from G$=px \times x^2 \times \dot{x}$, whose fluent, when $x = r$, is $\frac{pr^4}{4}$; and the area of the circle $\times\, d = \frac{pr^2}{2} \times d$;

hence, (Art. 66.) CO $= d + \frac{r^2}{2d}$.

If C coincide with A, then CO $= \frac{3}{2}r$.

Cor. Hence, the same must be true for a *cylinder*, whose axis is parallel to the axis of vibration.

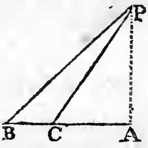
SECTION VI.

On the ATTRACTIONS of BODIES.

Prop. XXXI.

To determine the attraction of a corpuscle P *towards* a right line BA, *in the direction* PA *perpendicular to* AB, *supposing the attraction to each particle of the line to vary inversely as the square of the distance.*

67. Put PA $= a$, AC $= x$, then PC$^2 = a^2 + x^2$, and therefore the attraction of P towards a particle at C is as $\frac{1}{a^2+x^2}$; and by the resolution of forces $\sqrt{a^2+x^2} : a$



$:: \frac{1}{a^2+x^2} : \frac{a}{\overline{a^2+x^2}|^{\frac{3}{2}}}$ = the attraction in the direction PA;

hence, $\frac{a\dot{x}}{\overline{a^2+x^2}|^{\frac{3}{2}}}$ is the fluxion of the whole force, whose fluent (Art. 39. Ex. 5.) is $\frac{x}{\overline{a^2+x^2}|^{\frac{1}{2}} \times a}$, which wants no correction, for when $x=0$, the fluent $=0$; and when $x=$ AB, it becomes $\frac{AB}{PB \times PA}$ for the whole attraction in the direction PA.

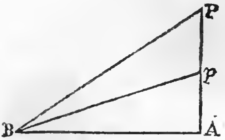
In like manner we find the whole attraction in the direction AB; for $\sqrt{a^2+x^2} : x :: \frac{1}{a^2+x^2} : \frac{x}{\overline{a^2+x^2}|^{\frac{3}{2}}}$, and the fluxion of the force is $\frac{x\dot{x}}{\overline{a^2+x^2}|^{\frac{3}{2}}}$, whose fluent (Art. 39.) is $-\frac{1}{\overline{a^2+x^2}|^{\frac{1}{2}}}$, which wants a correction, for when $x=0$, it becomes $-\frac{1}{a}$; hence, the correct fluent is $\frac{1}{a}-\frac{1}{\overline{a^2+x^2}|^{\frac{1}{2}}}$, and when $x=$ AB, it becomes $\frac{1}{\text{PA}}-\frac{1}{\text{PB}}=\frac{\text{PB}-\text{PA}}{\text{PB}\times\text{PA}}$ for the whole attraction in the direction AB.

Hence, the attraction in the direction PA : the attraction in the direction AB :: AB : PB—PA; take therefore AC = PB—PA, and join PC, and that will be the direction in which the corpuscle P will *begin* to move.

Prop. XXXII.

If the line PA *be perpendicular to the line* BA; *to find the attraction of* PA *to* BA, *upon the same law of force.*

68. Put $a=$AB, $x=$Ap; then (Art. 67.) the attraction of a corpuscle at p to AB$=\frac{a}{x\sqrt{a^2+x^2}}$; hence, $\frac{a\dot{x}}{x\sqrt{a^2+x^2}}$



is the fluxion of the attraction required; whose fluent

(Art. 45. Ex. 7.) is $\frac{1}{2}$h. l. $\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x}+a}$; now when $x=0$,

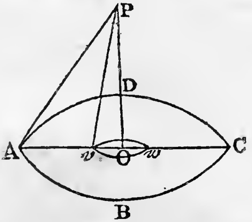
this becomes $\frac{1}{2}$h. l. $\frac{a-a}{a+a}=\frac{1}{2}$h. l. $\frac{o}{2a}$; hence, the fluent

corrected is $\frac{1}{2}$h. l. $\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}-\frac{1}{2}$h. l. $\frac{o}{2a}=$ (when $x=$ AP) $\frac{1}{2}$h. l. $\frac{\text{BP}-\text{AB}}{\text{AB}+\text{BP}}-\frac{1}{2}$h. l. $\frac{o}{2\text{AB}}$, an infinite quantity.

PROP. XXXIII.

Let O *be the centre of a circle* ABCD, *and a corpuscle* P *be situated in the line* OP *perpendicular to its plane; to find the attraction of* P *to the* circle, *supposing the attractive force of* P *to every particle of the circle to vary as the* n^{th} *power of the distance.*

69. Put PO $=a$, P$v=x$, $p=3{,}14159$, &c. then O$v^2=x^2-a^2$, and by Art. 49. $p\times\overline{x^2-a^2}=$ the area of the circle vw; hence, $2px\dot{x}$ is the fluxion of the area at the distance Ov from the centre; and by the resolution of forces, $x:a::x^n$ (the attraction of P toward v) : ax^{n-1} the attraction of P to a corpuscle at



v in the direction PO; hence, the fluxion of the attraction of P towards the circle is as $2px\dot{x}\times ax^{n-1}=2pax^n\dot{x}$, or as $ax^n\dot{x}$, whose fluent is $\frac{ax^{n+1}}{n+1}$; but when $x=a$, O$v=0$, and consequently the attraction vanishes;

but in this case, the fluent is $\frac{a^{n+2}}{n+1}$; therefore the fluent corrected becomes $\frac{ax^{n+1}}{n+1} - \frac{a^{n+2}}{n+1}$; and when x=PA (neglecting the constant denominator) it becomes PO $\times \text{PA}^{n+1} - \text{PO}^{n+2}$, which is as the whole attraction towards the circle.

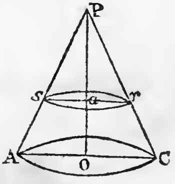
If $n = -2$, it becomes $1 - \frac{\text{PO}}{\text{PA}}$, the denominator neglected being now $= -1$.

If n be a negative number greater than 1, and the radius AO become infinite, so that PA becomes infinite, then PA being in the denominator, the first term $\text{PO} \times \text{PA}^{n+1} = 0$, and the attraction is as PO^{n+2}. Hence, if $n = -2$, the attraction becomes unity; therefore the attraction is the same at all distances PO.

Prop. XXXIV.

Let the attractive force of a corpuscle at P *to each particle vary inversely as the square of the distance; to find the attraction of* P *to the* cone PAC.

70. By the last article, the attraction of P to the circle sr is as $1 - \frac{\text{P}a}{\text{P}s} = 1 - \frac{\text{PO}}{\text{PA}}$; the attraction therefore to every section sr is the same; hence, the attraction



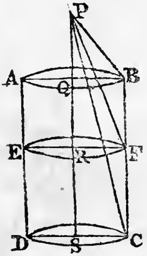
to the whole cone is as $\overline{1 - \frac{\text{PO}}{\text{PA}}} \times$ number of sections, or as $\overline{1 - \frac{\text{PO}}{\text{PA}}} \times \text{PO}$, or as $\text{PO} - \frac{\text{PO}^2}{\text{PA}}$.

Hence, for *similar* cones, $\frac{PO}{PA}$ being constant, the attraction varies as the length.

Prop. XXXV.

If a corpuscle be situated at P *in the axis* SQ *of a cylinder, to find its attraction to the* cylinder, *supposing the attractive force to each particle to vary inversely as the square of the distance.*

71. Put RF$=a$, PR$=x$, then PF$=\sqrt{a^2+x^2}$; and by Art. 69. the attraction of P towards the circle EF is as $1-\frac{x}{\sqrt{a^2+x^2}}$; hence, the fluxion of the attractive force is as $\dot{x}-\frac{x\dot{x}}{\sqrt{a^2+x^2}}$, whose fluent is $x-\sqrt{a^2+x^2}$ (Art. 39.); now when $x=$ PQ, this fluent



becomes PQ—PB, and when $x=$ PS, it becomes PS—PC; and as we want the attraction of P to the solid between these two values of x, their difference SQ + PB — PC is as the attraction required.

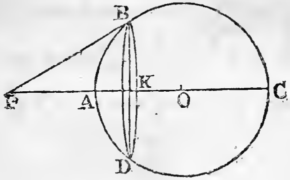
If the length be infinite, then PC = PS; therefore SQ — PC = SQ—PS = —PQ, and the attraction becomes as PB—PQ.

If the diameter AB be infinite, then PC = PB; hence, the attraction becomes as SQ.

Prop. XXXVI.

To find the attraction of a corpuscle P *to a* sphere, *when the attraction to each particle varies inversely as the square of the distance.*

72. Let PAC be perpendicular to BD; put the radius AO $= a$, OP $= b$, AP $= b - a = c$, PK $= y$, and let PB $= c + x$, then AK $= y - c$, CK $= 2a - y + c$, ∴ $\overline{y-c} \times \overline{2a-y+c} = BK^2 = BP^2 - PK^2 = \overline{c+x}^2 - y^2$; hence, $y = \frac{2ac + 2c^2 + 2cx + x^2}{2a + 2c} =$ (as $b = a + c$)



$\frac{2bc + 2cx + x^2}{2b}$; therefore the attraction of P to the circle BD is (Art. 69.) as $1 - \frac{2bc + 2cx + x^2}{2b \times \overline{c+x}}$, or as $\frac{2ax - x^2}{b \times \overline{c+x}}$; also, $\dot{y} = \frac{c\dot{x} + x\dot{x}}{b}$; hence, the fluxion of the attraction to the sphere is as $\frac{2ax\dot{x} - x^2\dot{x}}{b^2}$, whose fluent is $\frac{ax^2 - \frac{1}{3}x^3}{b^2}$, the attraction to ABD, for the fluent wants no correction, as it becomes $= 0$ when ABD $= 0$; and when $x = 2a$, it is $\frac{4a^3}{3b^2}$ the attraction to the whole sphere; which therefore varies as $\frac{a^3}{b^2}$.

If the density d of the sphere should vary, then the attraction will vary as $\frac{da^3}{b^2}$.

If the corpuscle be at the surface of the sphere, then $a = b$, and the attraction varies as da.

Since the quantity of matter m varies as da^3, the attraction varies as $\frac{m}{b^2}$. Now if the sphere were evanescent in magnitude, with the same quantity of matter, the attraction would be the same, it being independent of a. Hence, the attraction of a corpuscle to a sphere is just the same as if all the matter of the sphere were collected into its centre.

SECTION VII.

On SECOND, THIRD, &c. FLUXIONS.

Prop. XXXVII.

TO explain under what circumstances a quantity may have several orders of fluxions.

73. The fluxion of a quantity being the uniform increase or decrease of that quantity in a given time, every quantity which increases or decreases must have a fluxion. Hence, if the fluxion of any quantity be not constant, it must have some certain rate of increase or decrease, which rate of increase or decrease will therefore be the fluxion of that fluxion, or the second fluxion of the original flowing quantity. Also, if this second fluxion be not always the same, it must have a rate of variation, that rate therefore will be the fluxion of the second fluxion, or the third fluxion of the original quantity; and so on*. Thus a quantity will have a successive order of fluxions till some one fluxion becomes constant, and then by Art. 3. it will have no more. Thus, let x increase uniformly; then the fluxion of x^2 is $2x\dot{x}$; now $\dot{x}$ is constant, but x itself increases, therefore $2x\dot{x}$ increases in proportion to the increase of x; the fluxion therefore of x^2 is not constant. Hence, considering x as the variable part of $2x\dot{x}$, its fluxion by Art. 9. is $2\dot{x}\dot{x} = 2\dot{x}^2$, which is

* The fluxion of $\dot{x}$ is denoted thus, $\ddot{x}$; the fluxion of $\ddot{x}$ is denoted thus, $\dddot{x}$; and so on.

the second fluxion of x^2. But if we suppose x not to increase uniformly, then $2x\dot{x}$ will have both x and $\dot{x}$ variable; hence, by Art. 15. the fluxion of $2x\dot{x}$ will be $2\dot{x}\dot{x} + 2x\ddot{x}$, or $2\dot{x}^2 + 2x\ddot{x}$, which therefore is the second fluxion of x^2. But if we should here suppose neither $\dot{x}$ nor $\ddot{x}$ to be constant, then this second fluxion would be variable. Now the fluxion of $2\dot{x}^2$ is found by Art. 13. considering here $\dot{x}$ as the root, and therefore the fluxion of the root is $\ddot{x}$; hence, the fluxion of $2\dot{x}^2$ is $4\dot{x}\ddot{x}$; also, the fluxion of $2x\ddot{x}$ is found by Art. 15. to be $2\dot{x}\ddot{x}+2x\dddot{x}$, both x and $\ddot{x}$ being variable; therefore the fluxion of $2\dot{x}^2 + 2x\ddot{x}$, or the third fluxion of x^2, is $4\dot{x}\ddot{x} + 2\dot{x}\ddot{x} + 2x\dddot{x} = 6\dot{x}\ddot{x} + 2x\dddot{x}$. In like manner we may find the successive orders of fluxions of any quantity.

74. If x increase uniformly, or if $\dot{x}$ be constant, x^n will have n fluxions, and no more, n being an affirmative whole number. For the first fluxion is $nx^{n-1}\dot{x}$; and x only being variable, its fluxion is $n \,.\, \overline{n-1} \,.\, x^{n-2}\dot{x}^2$; and the fluxion of this is $n \,.\, \overline{n-1} \,.\, \overline{n-2} \,.\, x^{n-3}\dot{x}^3$, &c. when therefore we have taken the fluxion n times, the index of x becomes $=0$, and $x^0=1$; hence, the fluxion then becomes $n \,.\, \overline{n-1} \ldots 2.1.\dot{x}^n$, which being a constant quantity, it has no further fluxion.

75. The first fluxion of $x^3 + ay^2$ is $3x^2\dot{x} + 2ay\dot{y}$; and if $\dot{x}$ and $\dot{y}$ be both variable, its fluxion is $6x\dot{x}^2 + 3x^2\ddot{x} + 2a\dot{y}^2 + 2ay\ddot{y}$; but if $\dot{x}$ be constant, then $\ddot{x} = 0$; therefore the second fluxion becomes $6x\dot{x}^2 + 2a\dot{y}^2 + 2ay\ddot{y}$; and if $\dot{y}$ be constant, the second fluxion is $6x\dot{x}^2 + 3x^2\ddot{x} + 2a\dot{y}^2$.

76. The first fluxion of x^ny^m, by Art. 15. is $ny^mx^{n-1}\dot{x}+mx^ny^{m-1}\dot{y}$; and if both $\dot{x}$ and $\dot{y}$ be variable, we are to consider each of these quantities as composed of three variable factors, and then the fluxion, by the same Art. will be $n \,.\, mx^{n-1}y^{m-1}\dot{y}\dot{x} + n \,.\, \overline{n-1} \,.\, y^mx^{n-2}\dot{x}^2+ny^mx^{n-1}\ddot{x}+m \,.\, \overline{m-1}.x^ny^{m-2}\dot{y}^2+mny^{m-1}x^{n-1}\dot{x}\dot{y}+mx^ny^{m-1}\ddot{y}$.

On the POINT of CONTRARY FLEXURE of a CURVE.

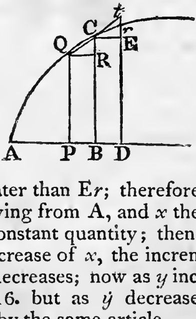
DEFINITION.

77. If a curve be concave in one part and convex in another, the point where the concave part ends and the convex begins, is the point of contrary flexure.

Prop. XXXVIII.

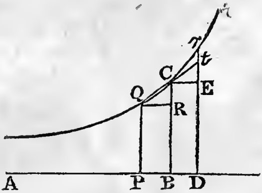
To find the point of contrary flexure of a curve.

78. Let PQ, BC, D*r*, be three equidistant ordinates, and the curve *concave* to the axis; and draw QR, CE parallel to AD, and join QC, and produce it to meet D*r* in *t*. Then the triangles QRC, CE*t*, being similar, and QR = CE, therefore CR = *t*E, and



hence CR is greater than E*r*; therefore if y represent the ordinate, moving from A, and x the abscissa, and PB=BD $=\dot{x}$ a constant quantity; then corresponding to the uniform increase of x, the increment of y, and consequently $\dot{y}$, decreases; now as y increases, $\dot{y}$ is positive by Art. 16. but as $\dot{y}$ decreases, its fluxion, or $\ddot{y}$, is negative by the same article.

If the curve be *convex* to the axis, and the ordinate move from A, then the increment of y, and therefore $\dot{y}$, increases; and as y increases, $\dot{y}$ is positive; and as $\dot{y}$ increases, its fluxion, or $\ddot{y}$, is positive. Therefore when the curve is *concave* to the axis, $\ddot{y}$ is *negative*; when *convex*, $\ddot{y}$ is *positive*, $\dot{x}$ being constant. Hence, at the



point of contrary flexure, $\ddot{y}$ changes its sign; but a quantity may change its sign, either by passing through 0, or *infinity;* hence, at the point of contrary flexure, $\ddot{y}=0$, or *infinity*. What we here mean by infinity is only in respect to its value at any other time, that term being relative; and in this case we are to understand that $\ddot{y}$ is indefinitely greater at that time than at any other. If we conceive a line to be drawn from A parallel to BC, and consider it as an abscissa to the curve, and draw lines from it to Q, C, r, parallel to AD; then the former abscissæ AP, AB, AD, become equal to the ordinates, and the ordinates PQ, BC, Dr become equal to the abscissæ; if therefore $\dot{y}$ be made constant, $\ddot{x}=0$, or *infinity*, at the point of contrary flexure. Hence, we have the following

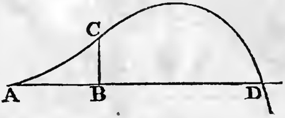
RULE:

Put the equation of the curve into fluxions; make $\dot{x}$ or $\dot{y}$ constant and take the fluxion of the equation again, and get the value of $\ddot{y}$ or $\ddot{x}$, and put it=0, or infinity; from which find the value of x, which gives the abscissa corresponding to the point of contrary flexure. And to determine for any value of x, whether the curve be concave or convex, substitute that value for x into the expression for $\ddot{y}$, the $\dot{x}$ being supposed constant, and if it come out positive, *the curve is* convex *to the axis; if* negative, *it is* concave.

EXAMPLES.

Ex. 1. *Let the equation of the curve be* $y=3x+18x^2-2x^3$.

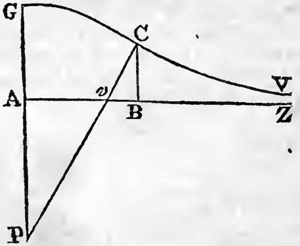
Here $\dot{y}=3\dot{x}+36x\dot{x}-6x^2\dot{x}$, and $\ddot{y}=36\dot{x}^2-12x\dot{x}^2=$ (if $\dot{x}=1$) $36-12x$. Now make $36-12x=0$, and $x=3$; take therefore AB=3, and draw the ordinate BC, and C is the point of contrary flexure. If x be between



0 and 3, $36-12x$ is positive, therefore the part AC of the curve is convex to AB; but when x is greater than 3, $36-12x$ is negative, and therefore the curve is concave towards the axis.

Ex. 2. *Let* GCV *be a curve of such a nature, that if* GA (*which is perpendicular to* AB) *be produced to any point* P, *and* PC *be drawn to any point of the curve*, vC *shall always be equal to* AG.

Put AB $=x$, BC $=y$, PA $=a$, AG $=b$; then by



sim. trian. PAv, BCv, a (PA) : $x-\sqrt{b^2-y^2}$ (AB—Bv) :: y (BC) : $\sqrt{b^2-y^2}$ (Bv); hence, $xy=\overline{a+y}\times\sqrt{b^2-y^2}$; take the fluxion, and $y\dot{x}+x\dot{y}=\dot{y}\sqrt{b^2-y^2}-\frac{ay\dot{y}+y^2\dot{y}}{\sqrt{b^2-y^2}}$; substitute for x its value, and we get $\dot{x}=-\frac{y^3+b^2a}{y^2\sqrt{b^2-y^2}}\times\dot{y}$; now make $\dot{y}$ constant, and we have $\ddot{x} = \frac{2b^4a - b^2y^3 - 3b^2ay^2}{\overline{b^2y^3 - y^5} \times \sqrt{b^2 - y^2}} \times \dot{y}^2$, which put $= 0$, in which case the numerator$=0$; hence, $y^3 + 3ay^2 = 2b^2a$; from whence y may be found, and then x, which will give the point of contrary flexure. This curve is the *Conchoid of Nicomedes.*

Ex. 3. *Let the equation of the curve be* $y = 180x^2 - 110x^3 + 30x^4 - 3x^5$.

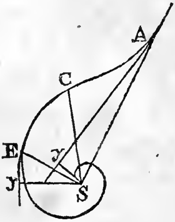
Here $\dot{y} = 360x\dot{x} - 330x^2\dot{x} + 120x^3\dot{x} - 15x^4\dot{x}$, and $\ddot{y} = 360\dot{x}^2 - 660x\dot{x}^2 + 360x^2\dot{x}^2 - 60x^3\dot{x}^2 = 0$, or $-x^3 + 6x^2 - 11x + 6 = 0$, whose simple factors are $1-x$, $2-x$, $3-x$, and the roots are 1, 2, 3, the abscissæ corresponding to the points of contrary flexure, of which therefore there are three. As $-x^3 + 6x^2 - 11x + 6 = \overline{1-x} \times \overline{2-x} \times \overline{3-x}$, when x is less than 1, this quantity is positive, and therefore the curve is convex to the axis; when x is between 1 and 2, it is negative, and the curve is concave; when x is between 2 and 3, it is positive, and the curve is convex; when x is greater than 3, it is negative, and the curve will then continue concave.

79. If by making $\ddot{y} = 0$, the equation has 2 equal roots, then $\ddot{y}$ passes through 0 without changing its sign; in this case therefore, the point found is not a point of contrary flexure. And this will always be the case, when the equation has an *even* number of *equal* roots.

If the Reader wish to see any thing further upon this subject, he may consult Mr. Lyons's *Fluxions*, page 136.

80. To find the point C of contrary flexure of a *Spiral*, it is manifest, that as long as the point A approaches to C, the perpendicular Sy upon the tangent must increase; and after A has passed through C to B, the perpendicular will then decrease; therefore at the point C it is a maximum; hence, if we make the

fluxion of the perpendicular = 0, it will give the point



of contrary flexure.

Ex. Let the spiral be that in Article 32.

Here $Sy = \frac{my^{m+1}}{\sqrt{t^{2m} + m^2y^{2m}}}$; hence, $2Sy \times \dot{\overline{Sy}} = \frac{2m^4y^{4m+1}\dot{y} + \overline{2m+2} \times m^2t^{2m}y^{2m+1}\dot{y}}{t^{2m} + m^2y^{2m}}$; but $\dot{\overline{Sy}} = 0$, therefore $2m^4y^{4m+1} + \overline{2m+2} \times m^2t^{2m}y^{2m+1} = 0$; hence, $y^{2m} = -\frac{\overline{m+1} \times t^{2m}}{m^2}$, and $y = \overline{-\frac{m+1}{m^2}}\Big|^{\frac{1}{2m}} \times t$. Assuming therefore m a whole number, $2m$ must be an even number, and therefore y is impossible, except m be a negative number greater than 1, in which case the quantity under the radical sign becomes positive.

For the *Lituus*, $m = -2$, and $y = \overline{\frac{1}{4}}\Big|^{-\frac{1}{4}} \times t = \overline{4}\big|^{\frac{1}{4}} \times t = \sqrt{2} \times t$.

If $m=1$, it is the spiral of *Archimedes*, and y is impossible, therefore it has no contrary flexure.

If $m = -1$, it is the *reciprocal* spiral, and y is impossible, therefore it has no contrary flexure.

SECTION VIII.

ON THE MOTION OF BODIES ATTRACTED TO A CENTRE OF FORCE.

PROP. XXXIX.

TO find the time and velocity of a body descending or ascending in a non-resisting medium, in a right line to or from a centre of force; supposing the force to vary as any power of the distance from the centre.

81. Let v be the velocity of the body at any time, x the corresponding space, either that described, or to be described, $m = 16\frac{1}{12}$ feet, $t =$ the time, F the force compared with the force of gravity on the earth's surface, which we will represent by unity; then $v\dot{v} = \pm 2mF\dot{x}$, the sign being $+$ when v and x increase together, and $-$ when v increases as x decreases. For by *Mechanics*, $\dot{v} \propto F \times \dot{t}$, and $\dot{t} \propto \frac{\dot{x}}{v}$; hence, $\dot{v} \propto F \times \frac{\dot{x}}{v}$, and $v\dot{v} \propto F \times \dot{x}$, that is, $v\dot{v}$ is to $F\dot{x}$ in some constant ratio; let $v\dot{v} = dF\dot{x}$. Now when a body falls upon the earth's surface, $v^2 = 4mx$ by *Mechanics*, x being the space described; hence, $v\dot{v} = 2m\dot{x}$; but if x be the space to be described, and a the whole space, then $v^2 = 4m \times \overline{a-x}$, and $v\dot{v} = -2m\dot{x}$; hence, $v\dot{v} = \pm 2m\dot{x}$; but in this case, $F=1$; therefore $d = \pm 2m$; hence, $v\dot{v} = \pm 2mF\dot{x}$. Also, the velocity of a body moving uniformly is measured by the space described in $1''$; therefore to find the time corresponding to the space $\pm \dot{x}$, we

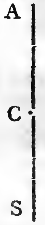
have $v : \pm \dot{x} :: 1'' : \dot{t} = \pm \frac{\dot{x}}{v}$, because v is the velocity with which $\dot{x}$ is described in the time $\dot{t}$, and when the velocity is uniform, the space is as the time.

Cor. If the force of gravity on the earth's surface be represented by $2m$, then $d = 1$, and $v\dot{v} = \pm F\dot{x}$.

Prop. XL.

Let a body begin to fall from any point A *towards the centre of force* S ; *to find the velocity at any point* C, *and the time of describing* AC.

82. Put $a = SA$, $x = SC$, $v =$ velocity at C, and let



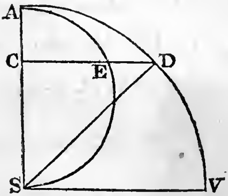
the force vary as x^n, and at any distance c from S, let e represent the force compared with the force of gravity on the earth's surface, or unity; then $c^n : x^n :: e : \frac{e}{c^n} \times x^n = \left(\text{if } d = \frac{e}{c^n}\right) dx^n$, the force at the distance x; hence, $v\dot{v} = -2mdx^n\dot{x}$, and $\frac{v^2}{2} = -\frac{2md}{n+1} \times x^{n+1} + C$; but when $v = 0$, $x = a$, and $0 = -\frac{2md}{n+1} \times a^{n+1} + C$, $\therefore C = \frac{2md}{n+1} \times a^{n+1}$; consequently $\frac{v^2}{2} = \frac{2md}{n+1} \times \overline{a^{n+1} - x^{n+1}}$, and $v = \sqrt{\frac{4md}{n+1}} \times \sqrt{a^{n+1} - x^{n+1}}$. Hence, $\dot{t} = -\frac{\dot{x}}{v} = -$

$\frac{\dot{x}}{\sqrt{\frac{4md}{n+1}} \times \sqrt{a^{n+1} - x^{n+1}}}$, whose fluent gives t; but this can be found only in particular cases.

EXAMPLES.

Ex. 1. If $n = 0$, then $x^n = 1$, and the force is constant, and $v = \sqrt{4md} \times \sqrt{a - x}$. Also, $\dot{t} = \frac{-\dot{x}}{\sqrt{4md} \times \sqrt{a-x}} = \frac{1}{\sqrt{4md}} \times \overline{a-x}\rceil^{-\frac{1}{2}} \times -\dot{x}$, whose fluent (Art. 39.) is $t = \frac{2}{\sqrt{4md}} \times \overline{a-x}\rceil^{\frac{1}{2}} + C$; but when $t=0$, $x=a$, $\therefore C=0$; hence, $t = \frac{2}{\sqrt{4md}} \times \overline{a-x}\rceil^{\frac{1}{2}}$.

Ex. 2. If $n = 1$, then $v = \sqrt{2md} \times \sqrt{a^2 - x^2} = \sqrt{2md} \times CD$, if upon SA a quadrant be described, and the ordinate CD be erected ⊥ to AS. Also, $\dot{t} = \sqrt{\frac{1}{2md}} \times \frac{-\dot{x}}{\sqrt{a^2 - x^2}}$; but if z = AD, then



(Art. 46.) $\dot{z} : -\dot{x} :: a : \sqrt{a^2 - x^2}$, $\therefore \frac{-\dot{x}}{\sqrt{a^2 - x^2}} = \frac{\dot{z}}{a}$; hence, $\dot{t} = \sqrt{\frac{1}{2md}} \times \frac{\dot{z}}{a}$, whose fluent (which wants no correction, because when $t=0$, $z=0$) is $t = \sqrt{\frac{1}{2md}} \times$

$\frac{z}{a}$, the time through AC; hence, if $p=1,57079$ (which is $\frac{1}{4}$ of the circumference of a circle whose radius$=1$), we have $\sqrt{\frac{1}{2md}} \times p$ for the whole time through AS, because here $z=AV=pa$. Hence, from whatever distance the body falls, the whole time of descent will be the same, it being independent of AS.

Ex. 3. If $n=-2$, $v=\sqrt{4md} \times \sqrt{x^{-1}-a^{-1}} = \sqrt{4md} \times \sqrt{\frac{a-x}{ax}}$. Also, $\dot{t} = -\frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times \frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{a-x}} = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times \frac{-x\dot{x}}{\sqrt{ax-x^2}} = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times \overline{\frac{\frac{1}{2}a\dot{x}-x\dot{x}}{\sqrt{ax-x^2}} - \frac{\frac{1}{2}a\dot{x}}{\sqrt{ax-x^2}}}$, whose fluent (Art. 40. and 46.) is $t = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times$ ($\sqrt{ax-x^2}$—a cir. arc, whose rad. $=\frac{1}{2}a$ and versed sine x) $+ C =$ (if upon AS we describe a semicircle) $\frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times (CE-SE) + C$; but when $t=0$, this becomes $0 = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times -$ arc SEA $+ C$, $\therefore C = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times$ arc SEA; consequently $t = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times$ (CE$+$arc AE). Hence, the whole time to S$= \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times$ arc AES.

Ex. 4. If $n=-3$, $v=\sqrt{2md} \times \sqrt{x^{-2}-a^{-2}} = \sqrt{2md} \times \frac{\sqrt{a^2-x^2}}{ax}$. Also, $\dot{t} = \frac{1}{\sqrt{2md}} \times \frac{-ax\dot{x}}{\sqrt{a^2-x^2}}$, and therefore $t = \frac{1}{\sqrt{2md}} \times a\sqrt{a^2-x^2} = \frac{1}{\sqrt{2md}} \times$ AS

$\times$ CD, which wants no correction, because when $t=0$, $CD=0$, and both sides vanish together. Hence, the whole time of descent to S $= \frac{1}{\sqrt{2md}} \times AS^2$.

Ex. 5. If $e=1$, $c=r$, the radius of the Earth, $n=-2$, and a be taken any distance from the Earth's centre greater than r, then $d = r^2$, and $v = \sqrt{4mr^2} \times \sqrt{\frac{a-x}{ax}} = r\sqrt{4m} \times \sqrt{\frac{a-x}{ax}}$ the velocity acquired in falling from any distance a from the centre through $a-x$; and when $x = r$, $v = r\sqrt{4m} \times \sqrt{\frac{a-r}{ar}} = \sqrt{4mr} \times \sqrt{\frac{a-r}{a}}$ the velocity acquired in falling through the space $a-r$ to the Earth's surface. If a be infinite, $v = \sqrt{4mr}$ the velocity which a body would acquire in falling from an infinite distance.

Ex. 6. If $n=1$, and $a=r$, then $d=\frac{1}{r}$; hence, $v = \sqrt{\frac{2m}{r} \times \overline{r^2-x^2}}$; and when $x=0$, $v=\sqrt{2mr}$, which is the velocity a body acquires in falling from the surface of the Earth to the centre, because within the Earth's surface the force varies directly as the distance. Also, by Ex. 2. $t=p \times \sqrt{\frac{1}{2md}} = p\sqrt{\frac{r}{2m}}$. Hence, by Sir I. Newton's *Principia, Lib.* 1. *p.* 38. *Cor.* 1. the time in which a body would revolve about the Earth at its surface $= 4p \times \sqrt{\frac{r}{2m}} = p\sqrt{\frac{8r}{m}}$. This is the time in seconds; also, $4pr$ is the circumference of the Earth; hence, $p\sqrt{\frac{8r}{m}} : 1'' :: 4pr : \sqrt{2rm}$ the

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velocity of a body revolving about the Earth in a circle at its surface, the velocity being always measured by the space described uniformly in 1″. We must take r in feet, m being in feet. Hence it appears, that the velocity of a body falling from the surface of the Earth to its centre, is equal to the velocity of a body revolving at the Earth's surface.

Cor. 1. From hence we may find how far a body must fall above the Earth's surface to acquire the velocity in a circle at the surface, supposing $n = -2$; for then, by the two last examples, $\sqrt{4mr} \times \sqrt{\frac{a-r}{a}} = \sqrt{2mr}$; hence, $a = 2r$, and $a - r = r$ the space fallen through.

Cor. 2. Let s be the space a body must fall through by the constant force of gravity at the Earth's surface to acquire the velocity $\sqrt{2rm}$ in a circle; then, by *Mechanics*, $v^2 = 4ms = 2rm$; hence, $s = \frac{1}{2}r$; and the same is true for any circle.

Ex. 7. If instead of supposing the body to fall from a state of rest at A, it be projected with a velocity b, then when $x = a$, $v = b$; therefore (Art. 82.) $\frac{b^2}{2} = -\frac{2md}{n+1} \times a^{n+1} + C$; hence, $C = \frac{b^2}{2} + \frac{2md}{n+1} \times a^{n+1}$; consequently $v = \sqrt{b^2 + \frac{4md}{n+1} \times \overline{a^{n+1} - x^{n+1}}}$. Now to find to what height the body will ascend if it be projected upwards, we must put $v = 0$, and then $b^2 + \frac{4md}{n+1} \times \overline{a^{n+1} - x^{n+1}} = 0$; hence, $x = \left| \overline{\frac{n+1}{4md} \times b^2 + a^{n+1}} \right|^{\frac{1}{n+1}}$, the greatest distance from the centre of force to which the body ascends. If we assume $v\dot{v} = \pm F\dot{x}$, we get $v = \sqrt{b^2 + \frac{2}{n+1} \times \overline{a^{n+1} - x^{n+1}}}$. Here, when $v = 0$,

$x = \overline{\frac{n+1}{2} \times b^2 + a^{n+1}}\Big|^{\frac{1}{n+1}}$ the greatest distance from the centre to which the body can rise; and this never can become infinite as long as the index $\frac{1}{n+1}$ is positive, or as long as n is greater than -1. But when n is less than -1, the index becomes negative, and therefore x is equal to unity divided by $\overline{\frac{n+1}{2} \times b^2 + a^{n+1}}\Big|^{\frac{-1}{n+1}}$, which will be finite or infinite according as $\frac{n+1}{2} \times b^2 + a^{n+1}$ is positive, or nothing; and if that quantity becomes negative, x becomes negative or impossible, which, as that can never happen, it shows that the supposition of $v=0$ was impossible; that is, the velocity will not be all destroyed when x becomes infinite. If $x = 0$, $v = \sqrt{b^2 + \frac{2}{n+1} \times a^{n+1}}$ the velocity at the centre of force when the body is projected downwards. If $b=0$, or the body fall from a state of rest, $v = \sqrt{\frac{2}{n+1} \times a^{n+1}}$.

If $n=0$, $v=\sqrt{2a}$. If $n=1$, $v=a$. If n be a greater negative quantity than -1, v comes out impossible, the meaning of which is, that the velocity is greater than can be expressed, even by an infinite quantity.

Ex. 8. If $n = -1$, this fluent fails (Art. 38.) for then $v\dot{v} = -2md \times \frac{\dot{x}}{x}$, whose fluent is $\frac{v^2}{2} = -2md \times$ h. l. $x + C$, and when $v = b$, $x = a$, and the fluent becomes $\frac{b^2}{2} = -2md \times$ h. l. $a + C$, and $C = \frac{b^2}{2} + 2md \times$ h. l. a;

therefore $\frac{v^2}{2} = \frac{b^2}{2} + 2md \times$ (h. l. a—h. l. x), and $v^2 = b^2 + 4md \times$ h. l. $\frac{a}{x}$; hence, $v = \sqrt{b^2 + 4md \times \text{h. l. } \frac{a}{x}}$.

On the MOTION of BODIES in RESISTING MEDIUMS.

83. Let a cylinder move in a fluid in the direction of its axis, with the velocity d, and suppose the resistance to be equal to the weight of a column of fluid whose base is equal to the end of the cylinder, and altitude k; and let the resistance of a globe of the same diameter as the cylinder, and moving with same velocity, be to the resistance of the cylinder, as b to 1; and put $p = 0{,}78539$ &c. $h =$ the diameter of the globe, $m = 16\frac{1}{12}$ feet, and let the density of the globe : the density of the fluid :: n : 1. Now the magnitude of the globe is $\frac{2}{3}ph^3$, and the magnitude of a column of fluid equal to the resistance of the cylinder is ph^2k; therefore the magnitude of a column of fluid equivalent to the resistance of the globe is pbh^2k. Hence, the magnitude of the globe : magnitude of a column of fluid whose weight = the resistance of the globe :: $\frac{2}{3}ph^3 : pbh^2k :: 1 : \frac{3bk}{2h}$; therefore their quantities of matter are as $n : \frac{3bk}{2h}$, or as $1 : \frac{3bk}{2nh}$.

Hence, if the weight of the globe, or its gravity, be denoted by unity, $\frac{3bk}{2nh}$ will represent its resistance moving with the velocity d. Hence, the resistance of the cylinder is $\frac{3k}{2nh}$.

Prop. XLI.

Let a globe be projected in a resisting medium, as in the last article, and let the resistance be as the c[th] *power of the velocity; to find the velocity* v, *time* t, *and space* x *described, any one in terms of the other.*

84. Let d be the velocity of projection, and r the resistance corresponding to the velocity d, compared with the force of gravity represented by unity; then $r = \frac{3bk}{2nh}$ by the last. Art. Hence, $d^c : v^c :: r : \frac{r}{d^c} \times v^c$ the resistance corresponding to the velocity v; therefore (Art. 81.) $v\dot{v} = -\frac{2mr}{d^c} \times v^c \dot{x} = \left(\text{if } \frac{1}{e} = \frac{2mr}{d^c}\right)$ $-\frac{1}{e} v^c \dot{x}$; hence, $\dot{x} = -ev^{1-c}\dot{v}$, consequently $x = -\frac{e}{2-c} \times v^{2-c} + C$; but when $x=0$, $v=d$, and the equation becomes $0 = -\frac{e}{2-c} \times d^{2-c} + C$; hence $C = \frac{e}{2-c} \times d^{2-c}$; therefore $x = \frac{e}{2-c} \times \overline{d^{2-c} - v^{2-c}}$.

Hence, when $v = 0$, and c is less than 2, $x = \frac{e}{2-c} \times d^{2-c}$, the whole space described before the velocity is all destroyed.

If $c = 2$, the fluent fails; for then $\dot{x} = -\frac{e\dot{v}}{v}$, and $x = e \times \text{h. l. } \frac{d}{v} = \left(\text{because } e = \frac{nhd^2}{3mbk}\right) \frac{nhd^2}{3mbk} \times \text{h. l. } \frac{d}{v}$. Hence, when $v=0$, x becomes infinite, therefore the velocity will never be destroyed.

If c be greater than 2, $2-c$ is negative, and by making $v = 0$, x becomes infinite, which shows that the velocity will never be all destroyed.

Also (Art. 81.), $\dot{t} = \frac{\dot{x}}{v} = -ev^{-c}\dot{v}$, and $t = -\frac{e}{1-c} \times v^{1-c} + C$; but when $t = 0$, $v = d$; hence, $C = \frac{e}{1-c} \times d^{1-c}$; therefore $t = \frac{e}{1-c} \times \overline{d^{1-c} - v^{1-c}}$.

Hence, when $v = 0$, and c is less than 1, $t = \frac{e}{1-c} \times d^{1-c}$, the time of describing the whole space.

If $c = 1$, the fluent fails; for then $\dot{t} = \frac{-e\dot{v}}{v}$, whose fluent corrected is $t = e \times \text{h. l.} \frac{d}{v}$. Hence, when $v=0$, t becomes infinite. But it appears from above, that, in this case, the space is finite; hence, the body is an infinite time in describing a finite space, and which space is *ed*.

If c be greater than 1, then $1 - c$ is negative, and when $v = 0$, t becomes infinite; but the space will still be finite whilst c is less than 2. When c is equal to, or greater than 2, both the space and time will be infinite.

As $v = \overline{d^{2-c} - \frac{2-c}{e} \times x}\Big|^{\frac{1}{2-c}}$, substitute this quantity for v, and it gives $t = \frac{e}{1-c} \times$
$\overline{d^{1-c} - \overline{d^{2-c} - \frac{2-c}{e} \times x}\Big|^{\frac{1-c}{2-c}}}$, showing the relation between t and x, except in the cases where the fluents fail.

Prop. XLII.

Let a body be projected in a resisting medium directly to or from a centre of force, and be attracted by a constant force towards that centre; to find the space, time, and velocity.

85. Let F be the force compared with gravity which is represented by unity, and retain the notation in Art. 84. Now when the body *descends*, the whole accelerative force = F — the resistance; and when it *ascends*, the retarding force = F + the resistance; that is, in the *former* case the force $= F - \frac{r}{d^c} \times v^c$; and in the *latter*, it $= F + \frac{r}{d^c} \times v^c$. Hence (Art. 81.), $v\dot{v} = \pm 2m \times \overline{F \mp \frac{r}{d^c} \times v^c} \times \dot{x}$, the *upper* signs being used when the body *descends*, and the *lower* when it *ascends*; hence, (if $\frac{r}{d^c} = e$) $\dot{x} = \frac{1}{2m} \times \frac{\pm v\dot{v}}{F \mp ev^c}$.

If $c = 2$, $\dot{x} = \frac{1}{2m} \times \frac{\pm v\dot{v}}{F \mp ev^2}$, whose fluent (Art. 45.) is $x = \frac{1}{2m} \times \frac{1}{2e} \times -$ h. l. $\overline{F \mp ev^2} + C$; but when $x = 0$, $v = d$, and the fluent becomes $0 = \frac{1}{4me} \times -$ h. l. $\overline{F \mp ed^2} + C$; hence, $C = \frac{1}{4me} \times$ h. l. $\overline{F \mp ed^2}$; consequently $x = \frac{1}{4me} \times$ h. l. $\frac{F \mp ed^2}{F \mp ev^2}$. Hence, we may find v in terms of x; for $4mex =$ h. l. $\frac{F \mp ed^2}{F \mp ev^2}$; therefore put $w =$ the number whose h. l. is

$4mex$, and then $w = \frac{F \mp ed^2}{F \mp ev^2}$; hence, $v = \sqrt{\frac{F \mp ed^2 - wF}{\mp we}}$.

86. If the body *ascend*, and $v = 0$, $x = \frac{1}{4me} \times$ h. l. $\frac{F + ed^2}{F}$ the distance to which it ascends.

87. Let the body *descend*. Now when $F = \frac{rv^2}{d^2}$, the resistance becomes equal to the accelerating force; hence, $v^2 = \frac{F \times d^2}{r}$, and $v = d\sqrt{\frac{F}{r}}$, the greatest velocity the body can acquire; for when the resistance becomes equal to the attractive force, there can be no further acceleration.

88. If $d = 0$, $x = \frac{1}{4me} \times$ h. l. $\frac{F}{F \mp ev^2}$.

89. Also (Art. 81.), $\dot{t} = \frac{\dot{x}}{v} = \frac{1}{2m} \times \frac{\pm \dot{v}}{F \mp ev^2}$; hence, when the body *descends*, $\dot{t} = \frac{1}{2me} \times \frac{\dot{v}}{\frac{F}{e} - v^2}$, whose fluent (Art. 45.), (putting $\frac{F}{e} = a^2$) is $t = \frac{1}{4mae} \times$ h. l. $\frac{a+v}{a-v} + C$; but when $t = 0$, $v = d$, and we get $0 = \frac{1}{4mae} \times$ h. l. $\frac{a+d}{a-d} + C$; hence, $C = -\frac{1}{4mae} \times$ h. l. $\frac{a+d}{a-d}$; consequently $t = \frac{1}{4mae} \times \overline{\text{h. l. } \frac{a+v}{a-v} - \text{h. l. } \frac{a+d}{a-d}}$. Hence, if we substitute the value of v in terms of x, we shall get t in

terms of x. If the body fall from a state of rest, $t = \frac{1}{4mae} \times \text{h. l. } \frac{a+v}{a-v}$.

90. When the body *ascends*, $\dot{t} = \frac{1}{2m} \times \frac{-\dot{v}}{F + ev^2} = \frac{1}{2me} \times \frac{-\dot{v}}{a^2 + v^2}$, whose fluent (Art. 46.) is $t = \frac{1}{2mea} \times -M + C$, M being a circular arc whose radius is 1, and tangent $\frac{v}{a}$; but when $t = 0$, $v = d$; put therefore N = the arc whose tangent is $\frac{d}{a}$, and we get $t = \frac{1}{2mea} \times \overline{N - M}$. For the whole ascent, $v = 0$, $\therefore$ M $= 0$; hence, $t = \frac{1}{2mea} \times N$.

91. If we apply these expressions to the descent of a globe in resisting mediums upon the earth's surface, then as unity represents the force of gravity, that is, the force when a body falls in vacuo, we must find the value of F when a body descends in the medium. Let the density of the body : the density of the medium :: $n : 1$; then if $w =$ the weight of the body in vacuo, we have, by *Hydrostatics*, w : weight lost when in the fluid :: $n : 1$; hence, $w : w -$ weight lost, or weight in the fluid, :: $n : n-1$, therefore the weight in the fluid $= w \times \frac{n-1}{n} =$ (if $w = 1$ the force of gravity) $\frac{n-1}{n}$ which is the gravity of the body in the fluid, or the force with which it endeavours to descend; this therefore is the value of F. Also, $c = 2$.

92. By Art. 83. $r = \frac{3bk}{2nh}$; hence (Art. 87.) v ($= d$

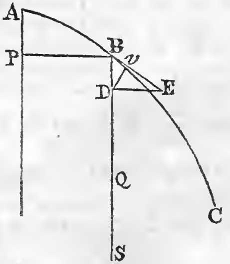
$\sqrt{\frac{F}{r}} = a) = d\sqrt{\frac{2.\overline{n-1}.h}{3bk}}$, the greatest velocity the body can acquire by falling in the fluid. Also, $t = \frac{1}{4mae} \times \left(\text{h. l. } \frac{a+v}{a-v} - \text{h. l. } \frac{a+d}{a-d}\right)$; and when $v = a$, t becomes infinite; therefore the body never can acquire its greatest velocity.

93. The greatest height to which a body can ascend when projected upwards, is (Art. 86.) $\frac{1}{4me} \times \text{h. l.} \frac{F + ed^2}{F} = \frac{nd^2h}{6mbk} \times \text{h. l.} \left(1 + \frac{3bk}{2.\overline{n-1}.h}\right)$.

Prop. XLIII.

To determine the resistance of a medium, by which a body may describe any curve about a centre of force, the force to the centre being given.

94. Let ABC be the given curve, S the centre of force, and F the force of the body at B towards it, the force of gravity being unity; draw DE perpendicular to BS, meeting the tangent BE; and Dv perpendicular to BE. Put $AB = z$, $BS = w$, $BD = -\dot{w}$,



$BE = \dot{z}$, $v =$ the velocity in the curve at B, and $s = BQ = \frac{1}{2}$ the chord of the circle of curvature at B passing through S, $m = 16\frac{1}{12}$ feet. Now it is well known, that a body, whether it moves in a resisting medium, or not, must fall down $\frac{1}{2}s$ by the constant force F to acquire the velocity in the curve; for the resistance causes no deviation from the tangent, but only retards the motion of the body, so that it may preserve its proper proportion corresponding to the force; hence, by *Mechanics*, $v^2 = 4mF \times \frac{1}{2}s = 2mFs$; therefore $\dot{v} = m \times \frac{F\dot{s} + s\dot{F}}{\sqrt{2mFs}}$ the *whole* fluxion of velocity in the direction BE. But, by *Mechanics*, the velocity V which the force F continuing constant for any time t, would generate in the direction BS, is $2mFt$, $\therefore \dot{V} = 2mF\dot{t} = \left(\text{because } \dot{t} = \frac{BE}{v} = \frac{\dot{z}}{v}\right)\ m \times \frac{2F\dot{z}}{\sqrt{2mFs}}$ the fluxion of the velocity in the direction BS, arising from the force F; hence, BD : Bv (: : BE$=\dot{z}$: BD$=-\dot{w}$) $:: m \times \frac{2F\dot{z}}{\sqrt{2mFs}} : m \times \frac{-2F\dot{w}}{\sqrt{2mFs}}$ the fluxion of velocity in the direction BE arising from the force F; from which if we take the whole fluxion $m \times \frac{F\dot{s} + s\dot{F}}{\sqrt{2mFs}}$, there will remain $-m \times \frac{F\dot{s} + s\dot{F} + 2F\dot{w}}{\sqrt{2mFs}}$ which is the fluxion of velocity arising from the resistance, in the time that the force F would generate the fluxion of velocity $m \times \frac{2F\dot{z}}{\sqrt{2mFs}}$; but the fluxion of velocity generated or destroyed in the same time is as the force; hence, the resistance : force F : : $m \times \frac{F\dot{s} + s\dot{F} + 2F\dot{w}}{\sqrt{2mFs}} : m \times \frac{2F\dot{z}}{\sqrt{2mFs}} :: \frac{F\dot{s} + s\dot{F} + 2F\dot{w}}{2F\dot{z}} : 1$, omitting the sign — before the first term, as it only signifies the force to be retarding.

95. When the centre S is at an infinite distance, and the force F becomes constant, and acts in parallel lines, then $\dot{F}=0$, and the resistance : force F :: $\frac{\dot{s}+2\dot{w}}{2\dot{z}}$: 1. But if we draw AP parallel to BS, and PB perpendicular to it, and put AP$=x$, then $\dot{w}=-\dot{x}$; hence, the resistance : force F :: $\frac{\dot{s}-2\dot{x}}{2\dot{z}}$: 1. Or to obtain this proportion in terms of the abscissa and curve, put $y=$PB; then by Art. 54. $\dot{z}^2=\dot{x}^2+\dot{y}^2$; and by Art. 97. $s=\frac{\dot{z}^2}{\ddot{x}}=\frac{\dot{x}^2+\dot{y}^2}{\ddot{x}}$; therefore if we suppose $\dot{y}$ constant, we shall have $\dot{s}=\frac{2\dot{x}\ddot{x}^2-\overline{\dot{x}^2+\dot{y}^2}\times\dddot{x}}{\ddot{x}^2}=\frac{2\dot{x}\ddot{x}^2-\dot{z}^2\dddot{x}}{\ddot{x}^2}$; hence, $\frac{\dot{s}-2\dot{x}}{2\dot{z}}=-\frac{\dot{z}\dddot{x}}{2\ddot{x}^2}$; therefore the resistance : force F :: $\frac{\dot{z}\dddot{x}}{2\ddot{x}^2}$: 1.

EXAMPLES.

Ex. 1. *Let the curve be a* parabola, *and the force be constant, and act in lines parallel to* AP.

Put $x=$AP, $y=$PB, then $ax=y^n$, $\therefore a\dot{x}=ny^{n-1}\dot{y}$, and ($\dot{y}$ being constant) $a\ddot{x}=n.\overline{n-1}.y^{n-2}\dot{y}^2$; also, $\dddot{x}=\frac{n.\overline{n-1}.\overline{n-2}}{a}\times y^{n-3}\dot{y}^3$, and $\dot{z}=\frac{\overline{n^2y^{2n-2}+a^2}|^{\frac{1}{2}}\times\dot{y}}{a}$; hence, $\frac{\dot{z}\dddot{x}}{2\ddot{x}^2}=\frac{n-2}{2.n.\overline{n-1}}\times\frac{\overline{n^2y^{2n-2}+a^2}|^{\frac{1}{2}}}{y^{n-1}}$, the resistance.

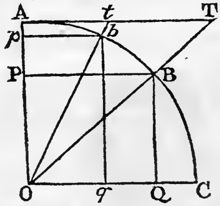
If $n=2$, the resistance becomes $=0$.

If n be less than 2, but greater than 1, the resistance becomes negative; the medium therefore must propel the body, not retard it.

If $n=1$, the medium becomes an infinite propelling one, and the body moves in a right line.

Ex. 2. *Let* ABC *be a* quadrant of a circle, *and the force be constant and act parallel to* AO.

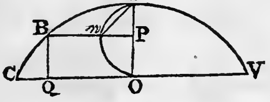
Put AO $= a$, AP $= x$, AB $= z$, then BQ$=s=a-x$, and $\dot{s}=-\dot{x}$; hence, $\frac{\dot{s}-2\dot{x}}{2\dot{z}} = -\frac{3\dot{x}}{2\dot{z}} = \frac{3PB}{2OB}$ = the resistance, gravity being unity. Hence, at A the resistance $=0$. When $3PB=2BO$, or radius : sine of AB :: 3 : 2, the resistance = gravity; and at C, the resistance : gravity :: 3 : 2. Also, the velocity is as $\sqrt{BQ}$. Hence, also, the resistance at B $\propto$ PB. Now if we suppose the resistance to vary as the density of the me-



dium × the square of the velocity, then the density varies as the resistance directly and square of the velocity inversely, or as $\frac{PB}{BQ} = \frac{PB}{PO} = \frac{AT}{AO}$; hence, the density at B varies as the tangent of AB. All this agrees with what Sir I. Newton has proved in his *Principia*, *Lib.* 2. *Sec.* 2. *Pr.* 10.

Ex. 3. *Let* CAV *be a* cycloid, *and the force be constant and act perpendicular to the base* CV.

Here BQ $= \frac{1}{2}s$, and if AO $= a$, $\frac{1}{2}s = a - x$, therefore



$\frac{1}{2}\dot{s} = -\dot{x}$, and $\dot{s} = -2\dot{x}$; also, $\dot{z} = \frac{a^{\frac{1}{2}}\dot{x}}{x^{\frac{1}{2}}}$ (Art. 54. Ex. 2.);

hence, $\frac{\dot{s}-2\dot{x}}{2\dot{z}} = \frac{-2x^{\frac{1}{2}}}{a^{\frac{1}{2}}}$ = (because x : An : : An : AO $=a$) $\frac{2\text{A}n}{\text{AO}}$ the resistance, gravity being unity. Also, the velocity varies as $\sqrt{\text{BQ}}$.

Ex. 4. *Let the force tend to a centre* S, *and vary as* w^n, *and the curve be the* logarithmic spiral.

As $\text{F}=w^n$, $\dot{\text{F}}=nw^{n-1}\dot{w}$; also, $s=w$, ∴ $\dot{s}=\dot{w}$; hence, the resistance $= \frac{w^n\dot{w} + nw^n\dot{w} + 2w^n\dot{w}}{2w^n\dot{z}} = \frac{n+3}{2} \times \frac{\dot{w}}{\dot{z}} =$ (as $\dot{w}$: $\dot{z}$ in some constant ratio c : d) $\frac{n+3}{2} \times \frac{c}{d}$, the force tending to S being unity.

If $n = -3$, the resistance $= 0$.

If $n+3$ be negative, the medium must propel the body.

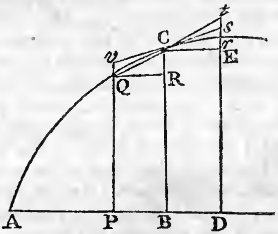
Also, $v = \sqrt{2m\text{F}s} = \sqrt{2m} \times w^{\frac{n+1}{2}}$. Now the resistance being to the force F, as $\frac{n+3}{2} \times \frac{c}{d}$ to 1, if F be represented by its true value w^n, the resistance will become $\frac{n+3}{2} \times \frac{c}{d} \times w^n$; and since the density of the medium varies as the resistance directly and the square of the velocity inversely, the density varies as $\frac{w^n}{w^{n+1}}$, or as $\frac{1}{w}$. Hence, if the density of the medium vary inversely as the distance, the body may describe the logarithmic spiral, whatever be the value of n; agreeable to what Sir I. Newton has proved in his *Principia*, *Lib.* 2. *Sec.* 4. *Prop.* 16. If $n = -2$, $\text{F} = \frac{1}{w^2}$, or F varies as the square of the density, as he has also proved in *Prop.* 15.

ON THE RADIUS OF CURVATURE.

PROP. XLIV.

To find the second fluxion of the ordinate of a curve.

96. Let PQ, BC, Dr be three equidistant ordinates, draw QR, CE parallel to AB, and let vCs be a tangent at C, meeting PQ, Dr in v and s; join QC, and

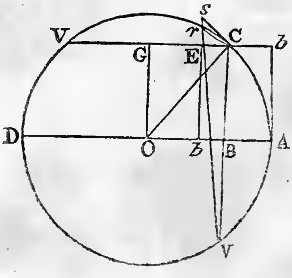


produce it to meet Ds in t. Now as PB = BD, the increment of the abscissa is constant, therefore (Art. 3. Cor. 1.) PB or BD will represent the fluxion of the abscissa, which is also constant. Now the cotemporary increments of the ordinates are RC, Er; but the triangles QRC, CEt are similar, and QR = CE, therefore RC=Et; consequently the cotemporary increments of the ordinates are Et, Er, and their difference is rt; but as the limit of the increment or decrement of the ordinate is the fluxion of the ordinate (Art. 7.), therefore the limit of rt, the difference between two successive increments of the ordinate, or the limit of the increment of the increment, will be the fluxion of the fluxion of the ordinate, or the second fluxion of the ordinate. Now as the triangles CvQ, Cst are similar, and QC=Ct, therefore Qv=st; and as Qv, sr depend upon the curvature of CQ. Cr, if Q and r be brought up to C, so as to get the measure of the curvature at C from each side, it is manifest that the *limit* of Qv to sr must be a ratio of equality; hence, the *limiting* ratio of $rs : st$ is that of equality; consequently the *limiting* ratio of $rt : 2rs$ is a ratio of equality. Hence, if we take $2rs$ in two different parts of the curve and make them vanish, their *limiting* ratio expresses the ratio of the second fluxions of the ordinates. Moreover, rt expresses the difference between the two successive increments of the ordinates, cotemporary with Er which expresses the difference of the two ordinates themselves; therefore by taking the *limit*, so that the latter increment may become the fluxion of the ordinate, the former becomes the fluxion of the fluxion of the ordinate, or the second fluxion of the ordinate; hence, whilst the *limit* of rt, or $2rs$, expresses the *second* fluxion of the ordinate, the limit of Er will express its *first* fluxion; but (Art. 23.) the limit of Er is Es the fluxion of the ordinate, CE and Cs expressing the cotemporary fluxions of the abscissa and curve (Art. 27.); therefore the *limits* of $2rs$, Cs and CE, express the cotemporary second fluxion of the ordinate, the fluxion of the curve AC, and the fluxion of the abscissa AB. In like manner it appears, if the curve be a spiral.

Prop. XLV.

To find the radius of a circle in terms of the fluxions of its abscissa, ordinate, and curve.

97. Let ACrDV be a circle, O the centre, CBV

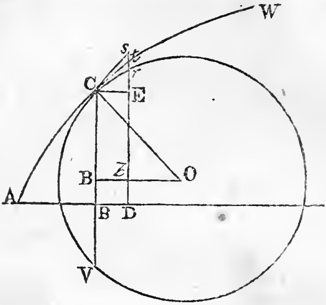


perpendicular to AD, brs parallel to CB, Cs a tangent at C, and join rC, rV. Put AB $= x$, BC $= y$, AC $= z$, and OC $= a$, then $Cs = \dot{z}$, CE $= \dot{x}$, $Es = \dot{y}$. Now the triangles Crs, CVr are similar, for the angle $srC =$ alter. ang. rCV, and the angle $sCr =$ angle CVr in the alternate segment; hence, $sr : r\text{C} :: r\text{C} : \text{CV} = 2\text{CB}$; but by Art. 23. it appears that the *limiting* ratio of rC : sC is a ratio of equality; therefore the *limiting* ratio of $sr : r$C is $sr : s$C, or (Art. 96.) $-\frac{1}{2}\ddot{y} : \dot{z}$, the sign $-$ being prefixed, for the reason in Art. 78. the curve being concave to the axis; hence, $-\frac{1}{2}\ddot{y} : \dot{z} :: \dot{z} : 2\text{BC}$, $\therefore \text{BC} = \frac{\dot{z}^2}{-\ddot{y}}$; and by similar triangles CEs, CBO, $\dot{x} : \dot{z} :: \frac{\dot{z}^2}{-\ddot{y}} : \text{CO} = \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, $\dot{x}$ being constant. If Ab be perpendicular to AO, and bC to Ab; then considering Ab as the abscissa and bC the ordinate, we have, for the same reason, $\text{CO} = \frac{\dot{z}^3}{\dot{y}\ddot{x}}$, $\dot{y}$ being constant, and $\ddot{x}$ positive (Art. 78.), the curve being convex to the axis. Lastly, by similar triangles OBC, CEs, $\dot{x} : \dot{z} :: y : r = \frac{y\dot{z}}{\dot{x}}$, and if we make $\dot{z}$ constant, we have $\frac{\dot{x}\dot{y}\dot{z} - y\ddot{x}\dot{z}}{\dot{x}^2} = 0$; hence, $y = \frac{\dot{x}\dot{y}}{\ddot{x}}$; and by the same proportion, $\dot{x} : \dot{z} :: y \left(\frac{\dot{x}\dot{y}}{\ddot{x}}\right) : r = \frac{\dot{y}\dot{z}}{\ddot{x}}$. Thus we get the radius under three circumstances, when $\dot{x}$ is constant, when $\dot{y}$ is constant, and when $\dot{z}$ is constant.

DEFINITION.

98. Let ACW be any curve, AB the abscissa, BC the ordinate, Cs a tangent at C, and let O be the centre of a circle touching the curve in C, and draw OB′ parallel to AB, and DbErts parallel to BC, cutting the curve in t and the circle in r; then if, by bringing

Ds up to BC, the *limiting* ratio of $sr : st$ be a ratio of



equality, the circle is said to be a *circle of curvature to the curve.*

PROP. XLVI.

To find the radius OC *of the circle of curvature to the curve* AC *at the point* C.

99. Whether we regard the curve AC or the circle, CE, Es, Cs will be the first fluxions of the abscissa, ordinate, and curve; for (Art. 23.) these fluxions depend entirely upon the position of the tangent, which is common to both; and by the Def. (Art. 98.) the *limiting* ratio of $sr : st$ being a ratio of equality, the second fluxions of the ordinates are equal (Art. 96.); hence, the second fluxion of the ordinate is the same, whether we regard the curve or circle. Now in the *circle*, if x, y, and z represent the abscissa, ordinate, and curve, CO $= \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$ (Art. 97.), $\dot{x}$ being constant; hence, in the *curve* AW, if x, y, and z represent the abscissa AB, ordinate BC, and curve AC, the radius of curvature CO $= \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$. For the same reason, CO $= \frac{\dot{z}^3}{\dot{y}\ddot{x}}$, when $\dot{y}$ is constant; and CO $= \frac{\dot{y}\dot{z}}{\ddot{x}}$, when $\dot{z}$ is constant.

When we make $\dot{x}$, $\dot{y}$, or $\dot{z}$ constant, it will simplify the operation, if we substitute unity for them.

EXAMPLES.

Ex. 1. *Let* AC *be the common* parabola; *to find the radius of curvature.*

Here $ax = y^2$, $\therefore y = a^{\frac{1}{2}}x^{\frac{1}{2}}$, and $\dot{y} = \frac{1}{2}a^{\frac{1}{2}}x^{-\frac{1}{2}}$, $\dot{x}$ being constant, and $=1$; hence, $\ddot{y} = -\frac{1}{4}a^{\frac{1}{2}}x^{-\frac{3}{2}} = -\frac{a^{\frac{1}{2}}}{4x^{\frac{3}{2}}}$; also,

$\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{1 + \frac{a}{4x}} = \frac{1}{2}\sqrt{\frac{4x+a}{x}}$; therefore CO$=$

$\frac{\dot{z}^3}{-\dot{x}\ddot{y}} = \frac{\overline{4x+a}\big]^{\frac{3}{2}}}{2\sqrt{a}}$.

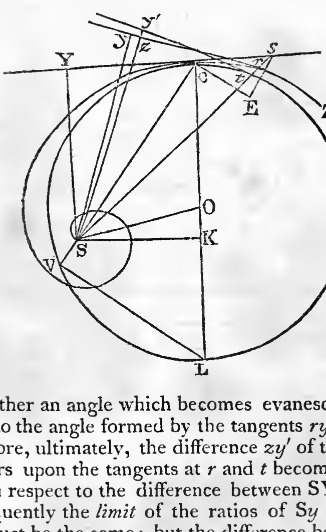
When $x = 0$, CO $= \frac{1}{2}a$, the radius of curvature at the vertex.

Ex. 2. *Let it be the* logarithmic *curve; to find the radius of curvature.*

By Art. 44. $\dot{y} = \frac{y\dot{x}}{m} =$ (if $\dot{x}$ be supposed constant and $=1$) $\frac{y}{m}$, $\therefore \ddot{y} = \frac{\dot{y}}{m}$, and $-\dot{x}\ddot{y} = -\frac{\dot{y}}{m} = \frac{y}{m^2}$; also, $\dot{z} =$ $\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{1 + \frac{y^2}{m^2}} = \frac{\overline{m^2+y^2}\big]^{\frac{1}{2}}}{m}$; hence, CO$= \frac{\dot{z}^3}{-\dot{x}\ddot{y}} =$ $\frac{\overline{m^2+y^2}\big]^{\frac{3}{2}}}{-my}$, which being negative, shows that the centre O lies on the other side of the curve, the curve being concave the other way.

To find the RADIUS of CURVATURE to SPIRALS.

100. Let CO be the radius of the circle of curvature to the spiral SCZ at C, and draw S*trs* meeting the tangent YC in *s*; then by the Definition (Art. 98.), the *limiting* ratio of *sr* : *st* is a ratio of equality; consequently *rt* ultimately vanishes in respect to *sr* or *st*. Hence, the tangents *ry*, *ty'* will ultimately form with



each other an angle which becomes evanescent in respect to the angle formed by the tangents *ry* and *s*CY; therefore, ultimately, the difference *zy'* of the perpendiculars upon the tangents at *r* and *t* becomes evanescent in respect to the difference between SY and S*y*; consequently the *limit* of the ratios of S*y* and S*y'* to SY, must be the same; but the difference between SY and S*y'*, SY and S*y*, or the increment of SY in each case, is ultimately the fluxion of SY in each case; hence, the fluxion of the perpendicular to a tangent to the curve, and to the circle of curvature, is the same.

Prop. XLVII.

To find the radius OC *of the circle of curvature to the spiral at point* C.

101. Put $SC=y$, draw SK perpendicular to CO, and let $SY=CK=v$, $CO=r$; and considering the point C as describing the circle, the points S and O being fixed, SO is constant; now $OS^2=OC^2+CS^2-2OC \times CK = r^2+y^2-2rv$, whose fluxion therefore is $=0$, or $2y\dot{y}-2r\dot{v}=0$, r being constant; hence, $r=\frac{y\dot{y}}{\dot{v}}$. Now if we consider y and v in reference to the spiral instead of the circle, $\dot{y}$, or sE, will be the same for each, by Art. 31. because sE depends only upon the position of the tangent; and (Art. 100.) $\dot{v}$ is the same for the circle and spiral; hence, if we consider the point C as describing the spiral, we shall still have $r=\frac{y\dot{y}}{\dot{v}}$.

Cor. By similar triangles, $y : v :: \frac{2y\dot{y}}{\dot{v}}$ (CL) : $CV=\frac{2v\dot{y}}{\dot{v}}$.

EXAMPLES.

Ex. 1. *Let it be the* logarithmic *spiral; to find the radius of curvature.*

Here $y : v :: m : n$, a constant ratio; hence, $v=\frac{ny}{m}$, and $\dot{v}=\frac{n\dot{y}}{m}$; therefore $CO=y\dot{y}\times\frac{m}{n\dot{y}}=\frac{my}{n}$.

Hence, the chord CV of the circle of curvature passing through S, $=\frac{2v\dot{y}}{\dot{v}}=2y=2SC$.

Ex. 2. *Let it be the spiral of* Archimedes; *to find the radius of curvature.*

By Art. 32. $v=\frac{y^2}{\sqrt{y^2+t^2}}$; hence, $\dot{v}=2y\dot{y}\times\overline{y^2+t^2}|^{-\frac{1}{2}}$

$$-y\dot{y}\times\overline{y^2+t^2}\big]^{-\frac{3}{2}}\times y^2=\frac{2y\dot{y}}{\overline{y^2+t^2}\big]^{\frac{1}{2}}}-\frac{y^3\dot{y}}{\overline{y^2+t^2}\big]^{\frac{3}{2}}}=$$

$$\frac{2y\dot{y}\times\overline{y^2+t^2}-y^3\dot{y}}{\overline{y^2+t^2}\big]^{\frac{3}{2}}}=\frac{y^3\dot{y}+2t^2y\dot{y}}{\overline{y^2+t^2}\big]^{\frac{3}{2}}};$$ therefore $CO=y\dot{y}\times$

$$\frac{\overline{y^2+t^2}\big]^{\frac{3}{2}}}{y^3\dot{y}+2t^2y\dot{y}}=\frac{\overline{y^2+t^2}\big]^{\frac{3}{2}}}{y^2+2t^2}.$$

102. The same expression for the radius of curvature will do for all curves, where the relation between SY and SC is known.

For example, let the curve be a *parabola*, S the focus, and $a=\frac{1}{4}$ of the principal latus rectum; then $y=\frac{v^2}{a}$, and $y^2=\frac{v^4}{a^2}$, $\therefore\ y\dot{y}=\frac{2v^3\dot{v}}{a^2}$; hence, $CO=\frac{y\dot{y}}{\dot{v}}=\frac{2v^3}{a^2}$.

Also, $CV=\frac{2v\dot{y}}{\dot{v}}=4y=4CS$.

SECTION IX.

ON LOGARITHMS.

PROP. XLVIII.

GIVEN a number, to find its logarithm.

103. Let $1 + x$ be the number, y its logarithm, and m the modulus; then (Art. 44.) $\dot{y} = \frac{m\dot{x}}{1 + x} = m \times \overline{\dot{x} - x\dot{x} + x^2\dot{x} - x^3\dot{x} + \&c.}$ hence, by taking the fluents, $y = m \times \overline{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \&c.}$ which wants no correction, because when $x = 0$, y vanishes as it ought, for then the number becomes 1, whose log. $= 0$. Now this series will converge quicker the smaller x is. If $x = 1$, $y = m \times \overline{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \&c.} =$ the log. of 2. If $m = 1$, $y = 1 - \frac{1}{2} + \frac{1}{3} - \&c.$ the h. l. of 2. Hence, as we are at liberty to assume m what we please, we may, to the same number, have as many different systems of logarithms as we please.

104. But to find a series which shall converge quicker, let the given number be $\frac{1 + x}{1 - x}$; then (Art. 44.) $\dot{y} = 2m \times \frac{\dot{x}}{1 - x^2} = 2m \times \overline{\dot{x} + x^2\dot{x} + x^4\dot{x} + \&c.}$ whose fluent is $y = 2m \times \overline{x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \&c.}$ If $m = 1$, we get $y = 2 \times \overline{x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \&c.}$ for the hyp. log. of $\frac{1 + x}{1 - x}$. Let $x = \frac{1}{3}$, and then the number becomes 2;

hence,

x	$=$	0,33333333
$\frac{1}{3}x^3$	$=$	1234567
$\frac{1}{5}x^5$	$=$	82307
$\frac{1}{7}x^7$	$=$	6532
$\frac{1}{9}x^9$	$=$	564
$\frac{1}{11}x^{11}$	$=$	51
		0,34657354
		2
		0,69314708

This h. l. of 2 is true to 6 places; the true value to 7 places being 0,6931472; and it would have required at least 100000 terms of the series in Art. 103. to have given the value with the same degree of accuracy.

105. The common log. of 2 is 0,3010300. Now these different values depend on the different values of m, and in the former case $m = 1$; hence, 0,6931472 : 0,3010300 :: 1 : m in the latter case $=$,43429448 the modulus of the common system. Hence, if any common log. be *divided* by this modulus, it gives the corresponding hyp. log. Or if any hyp. log. be *multiplied* by it, it gives the corresponding common logarithm. For the various methods which have been invented to calculate logarithms, the reader is referred to Dr. HUTTON's very excellent Introduction to his Tables of Logarithms, and to Mr. MASERES's *Scriptores Logarithmici.*

106. By Art. 42. a set of quantities A^0, A^1, A^2, A^3, A^4, &c. in geometric progression will have their logarithms in arithmetic progression; hence, the indices 0, 1, 2, 3, 4, &c. may represent the respective logarithms. Now in the common system of logarithms, $A = 10$; hence, the logarithms of $10^0, 10^1, 10^2, 10^3, 10^4$, &c. or of 1, 10, 100, 1000, 10000, &c. are 0, 1, 2, 3, 4, &c. And if between 10^0 and 10^1, we insert an indefinite number of geometric means, as 10^n, 10^{2n}, 10^{3n}, &c. n being indefinitely small, then some of these means must necessarily make up all the intermediate numbers between 1 and 10, as 2, 3, 4, 5, 6, 7, 8, 9, or at least be indefinitely near to them; the

indices therefore of such means must be the logarithms of these numbers; for instance, if $10^{rn}=2$, then $rn=$ log. of 2; if $10^{sn}=7$, then $sn=$log. of 7; and so for any other number.

DEFINITION.

107. The *measure of a ratio* 1 : N is the number of times which any other assumed ratio 1 : A must be taken to make that ratio. Thus, if $N=A^2$, the measure of the ratio of 1 : A^2 is 2, that ratio containing 2 ratios of 1 : A.

108. The ratio of 1 : A^2, 1 : A^3, 1 : A^4, &c. contain 2, 3, 4, &c. ratios of 1 : A; hence, the indices of A express the number of ratios of 1 : A which that ratio contains; for instance, 1 : A^4 contains 4 ratios of 1 : A; hence, 4 is the measure of the ratio 1 : A^4; also, the measure of the ratio of 1 : A^m is m, that ratio containing m ratios of 1 : A. Now if we put A=10, then the measure of the ratio of 1 : 10^m is m; but by article 106, m is the logarithm of 10^m; hence, the logarithm of any number is the measure of the ratio of that number to unity. In this sense, logarithms are called the measures of ratios, the logarithm of any number N showing how many ratios of 1 : 10 are necessary to make the ratio of 1 : N.

Hence, every ratio 1 : N has some certain measure in every system; now that ratio whose measure is m, the modulus of the system, is called the *Modular Ratio* by Mr. COTES.

109. If $x=y^n$, then by taking the logarithms of both sides (*Trig.* Art. 6), log. $x=n\times$ log. y; hence, if we have any equation of this form, log. $x = n \times$ log. y, then will $x=y^n$. If y be constant and n variable, the curve denoted by this equation is called the *logarithmic* curve.

LEMMA.

110. If $\left\{\begin{matrix} A+Bx+Cx^2+Dx^3+\&c. \\ a+bx+cx^2+dx^3+\&c. \end{matrix}\right\}=0$, or $\overline{A+a}+\overline{B+b}\times x+\overline{C+c}\times x^2+\overline{D+d}\times x^3+\&c.=0$, whatever be the value of x; then must $A+a=0$, $B+b=0$, $C+c=0$, &c. For as we may take x of any value, let $x=0$, and then $A+a=0$; hence, the remaining part, $\overline{B+b}\times x+\overline{C+c}\times x^2+\overline{D+d}\times x^3+\&c.=0$, and dividing by x, $\overline{B+b}+\overline{C+c}\times x+\overline{D+d}\times x^2+\&c.=0$; let $x=0$, and then $B+b=0$; and thus we may proceed for all the coefficients. Or we may consider it thus: The equation cannot become $=0$, but when its roots are substituted for x; the equation therefore cannot vanish for *every* value of x you may assume, unless you make each term vanish, independent of x, by making each coefficient $=0$.

PROP. XLIX.

Given a logarithm, to find its number.

111. Let $1+x$ be any number and y its logarithm, then $\dot{y}=\frac{m\dot{x}}{1+x}$; hence, $\dot{y}+x\dot{y}=m\dot{x}$, and $\dot{y}+x\dot{y}-m\dot{x}=0$. Assume $x=ay+by^2+cy^3+\&c.$ then $\dot{x}=a\dot{y}+2by\dot{y}+3cy^2\dot{y}+\&c.$ substitute these values of x and $\dot{x}$ into $\dot{y}+x\dot{y}-m\dot{x}=0$, and we have,

$\left.\begin{matrix} \dot{y}+ & ay\dot{y}+ & by^2\dot{y}+\&c. \\ -ma\dot{y} & -2mby\dot{y} & -3mcy^2\dot{y}-\&c. \end{matrix}\right\}=0$; hence, (Art. 110.) $1-ma=0$, $a-2mb=0$, $b-3mc=0$, &c. therefore $a=\frac{1}{m}$; $b=\frac{a}{2m}=\frac{1}{2m^2}$; $c=\frac{b}{3m}=\frac{1}{2.3m^3}$; &c. hence, $x=\frac{y}{m}+\frac{y^2}{2m^2}+\frac{y^3}{2.3m^3}+\&c.$ consequently $1+x=1+\frac{y}{m}+\frac{y^2}{2m^2}+\frac{y^3}{2.3m^3}+\&c.$ the number whose logarithm is y.

If $m=1$, then $1+x=1+y+\frac{y^2}{2}+\frac{y^3}{2.3}+$ &c. is the number whose h. l. is y.

Prop. L.

To find the modular ratio.

112. By Art. 108. every logarithm is the measure of the ratio of its corresponding number to 1; hence, y is the measure of the ratio of $1+\frac{y}{m}+\frac{y^2}{2m^2}+\frac{y^3}{2.3m^3}$ + &c. to 1; now (Art. 108.) the modular ratio is that ratio of which the modulus is the measure; hence, if we make $m=y$, m will become the measure of the above ratio, and the ratio will become the modular ratio; making therefore $m=y$, the ratio becomes $1+1+\frac{1}{2}+\frac{1}{2.3}$ + &c. to 1 for the modular ratio, which is therefore the same for every system, it being independent both of m and y.

SECTION X.

On the FLUXIONS of EXPONENTIALS.

DEFINITION.

113. A QUANTITY is called an *exponential*, when its index is variable.

Prop. LI.

To find the fluxion of x^y.

114. Put $x^y = z$, and let X = h. l. x, Z = h. l. z; then by the nature of logarithms, $y\text{X}=\text{Z}$, therefore $y\dot{\text{X}} + \text{X}\dot{y} = \dot{\text{Z}}$; but by Art. 45. $\dot{\text{X}} = \frac{\dot{x}}{x}$, and $\dot{\text{Z}} = \frac{\dot{z}}{z}$; hence, $\frac{y\dot{x}}{x} + \text{X}\dot{y} = \frac{\dot{z}}{z}$, consequently $\dot{z} = \frac{zy\dot{x}}{x} + z\text{X}\dot{y} = yx^{y-1}\dot{x} + \text{X}x^y\dot{y}$.

If x be constant, then $\dot{x}=0$, and $\dot{z}=\text{X}x^y\dot{y}$.

If y be constant, $\dot{y}=0$, and $\dot{z}=yx^{y-1}\dot{x}$, as in Art. 11.

Prop. LII.

To find the fluxion of x^{y^z}.

115. Put $x^{y^z} = w$, and let $x^y = v$, then $v^z = w$; hence, if V = h. l. v, we have (Art. 114.) $\dot{w} = zv^{z-1}\dot{v} + \text{V}v^z\dot{z}$; but $v = x^y$, and $\dot{v} = \overline{yx^{y-1}\dot{x} + \text{X}x^y\dot{y}}$; hence, by substitution, $\dot{w} = zx^{y^{z-1}} \times \overline{yx^{y-1}\dot{x} + \text{X}x^y\dot{y}} + \text{V}x^{y^z}\dot{z}$

$= zyx^{y^{z-1}} \times x^{y-1}\dot{x} + zXx^{y^{z-1}} \times x^{y}\dot{y} + Vx^{y^z}\dot{z}$. If any one of the quantities x, y, z become constant, its fluxion $= 0$, and the term vanishes where that fluxion enters. In like manner, we may find the fluxion, whatever be the number of quantities. The meaning of this notation is, the z power of x^y, not the y^z power of x. If this latter had been the meaning of the notation, we must have put $y^z = v$, instead of $x^y = v$.

On the FLUENTS of QUANTITIES.

Prop. LIII.

To find the fluent of $\frac{z^{\frac{1}{2}n-1}\dot{z}}{a^n+z^n} = \dot{F}$.

116. Put $a^n = b^2$, $z^n = x^2$, then $z^{\frac{1}{2}n} = x$, $\therefore \frac{n}{2} \times z^{\frac{1}{2}n-1}\dot{z} = \dot{x}$, and $z^{\frac{1}{2}n-1}\dot{z} = \frac{2}{n} \times \dot{x}$; hence, $\dot{F} = \frac{2}{n} \times \frac{\dot{x}}{b^2+x^2} = \frac{2}{nb^2} \times \frac{b^2\dot{x}}{b^2+x^2}$; consequently (Art. 46.) $F = \frac{2}{nb^2} \times$ cir. arc, whose rad. $= b$, tan. $= x$.

Prop. LIV.

To find the fluent of $\frac{z^{\frac{1}{2}n-1}\dot{z}}{a^n-z^n} = \dot{F}$.

117. By the same substitution, $\dot{F} = \frac{2}{n} \times \frac{\dot{x}}{b^2-x^2} = \frac{1}{nb} \times \frac{2b\dot{x}}{b^2-x^2}$; hence (Art. 45.), $F = \frac{1}{nb} \times$ h. l. $\frac{b+x}{b-x}$.

PROP. LV.

Let $\dot{F} = \frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{a^n + z^n}}$, *to find* F.

118. By the same substitution, $\dot{F} = \frac{2}{n} \times \frac{\dot{x}}{\sqrt{b^2 + x^2}}$; hence (Art. 45.), $F = \frac{2}{n} \times \text{h. l. } \overline{x + \sqrt{b^2 + x^2}}$.

PROP. LVI.

Let $\dot{F} = \frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{a^n - z^n}}$, *to find* F.

119. By the same substitution, $\dot{F} = \frac{2}{n} \times \frac{\dot{x}}{\sqrt{b^2 - x^2}} = \frac{2}{nb} \times \frac{b\dot{x}}{\sqrt{b^2 - x^2}}$; hence (Art. 46.), $F = \frac{2}{nb} \times$ cir. arc, rad. $= b$, sine $= x$.

PROP. LVII.

Let $\dot{F} = \frac{\dot{z}}{\sqrt{az^2 + bz + c}}$, *to find* F.

120. $\dot{F} = \frac{1}{\sqrt{a}} \times \frac{\dot{z}}{\sqrt{z^2 + \frac{b}{a} \times z + \frac{c}{a}}}$; put $z + \frac{b}{2a} = x$, then $z^2 + \frac{b}{a}z + \frac{b^2}{4a^2} = x^2$; hence, $z^2 + \frac{b}{a}z + \frac{c}{a} = x^2 - \frac{b^2}{4a^2} + \frac{c}{a} =$ (by putting $\frac{c}{a} - \frac{b^2}{4a^2} = d^2$) $x^2 + d^2$; also, $\dot{z} = \dot{x}$; hence, $\dot{F} = \frac{1}{\sqrt{a}} \times \frac{\dot{x}}{\sqrt{x^2 + d^2}}$; and (Art. 45.) $F = \frac{1}{\sqrt{a}} \times \text{h. l. } \overline{x + \sqrt{x^2 + d^2}}$.

Prop. LVIII.

Let $\dot{F} = \frac{x^{n-1}\dot{x}}{\sqrt{ax^{2n} + bx^{n} + c}}$, *to find* F.

121. Put $x^n = z$, then $x^{n-1}\dot{x} = \frac{1}{n} \times \dot{z}$; also, $x^{2n} = z^2$; hence, $\dot{F} = \frac{1}{n} \times \frac{\dot{z}}{\sqrt{az^2 + bz + c}}$, whose fluent is given in the last article.

Prop. LIX.

Let $\dot{F} = \frac{z^r \dot{z}}{\sqrt{az^2 + bz + c}}$, *to find* F.

122. Let $x = \frac{b}{2a} + z$, then $z^2 + \frac{b}{a}z + \frac{c}{a} =$ (by Prop. 57.) $x^2 + d^2$; also, $z^{r+1} = \overline{x - \frac{b}{2a}}|^{r+1}$, and $z^r \dot{z} = \overline{x - \frac{b}{2a}}|^{r} \times \dot{x}$; hence $\dot{F} = \frac{1}{\sqrt{a}} \times \frac{\overline{x - \frac{b}{2a}}|^{r} \times \dot{x}}{\sqrt{x^2 + d^2}}$; expand the numerator, and taking the terms separately, the fluents of those terms where the index of x in the numerator is *odd* are found by Art. 41.; and where they are *even* by Art. 127.

Prop. LX.

Let $\dot{F} = \frac{x^{rn-1}\dot{x}}{\sqrt{ax^{2n} + bx^{n} + c}}$, *to find* F.

123. Put $x^n = y$, then $x^{rn} = y^r$, and $x^{rn-1}\dot{x} = \frac{y^{r-1}\dot{y}}{n}$; hence, $\dot{F} = \frac{1}{n} \times \frac{y^{r-1}\dot{y}}{\sqrt{ay^2 + by + c}}$, whose fluent is found by Prop. 59.

Prop. LXI.

Let $\dot{F} = \frac{x^2\dot{x}}{\sqrt{a^2+x^2}}$, *to find* F.

124. Assume $\dot{v} = \frac{\dot{x}}{\sqrt{a^2+x^2}}$, then (Art. 45.) v=h. l. $\overline{x+\sqrt{a^2+x^2}}$; put $w = \sqrt{a^2x^2+x^4}$, then $\dot{w} = \frac{a^2x\dot{x}+2x^3\dot{x}}{\sqrt{a^2x^2+x^4}} = \frac{a^2\dot{x}}{\sqrt{a^2+x^2}} + \frac{2x^2\dot{x}}{\sqrt{a^2+x^2}} = a^2\dot{v} + 2\dot{F}$; hence, $\dot{F} = \frac{1}{2}\dot{w} - \frac{1}{2}a^2\dot{v}$, and $F = \frac{1}{2}w - \frac{1}{2}a^2v$. Call this P.

Prop. LXII.

Let $\dot{F} = \frac{x^2\dot{x}}{\sqrt{a^2-x^2}}$, *to find* F.

125. Assume $\dot{v} = \frac{a\dot{x}}{\sqrt{a^2-x^2}}$, then (Art. 46.) v = cir. arc, rad. $= a$, sin. $= x$; put $w = \sqrt{a^2x^2-x^4}$, then $\dot{w} = \frac{a^2x\dot{x}-2x^3\dot{x}}{\sqrt{a^2x^2-x^4}} = \frac{a^2\dot{x}}{\sqrt{a^2-x^2}} - \frac{2x^2\dot{x}}{\sqrt{a^2-x^2}} = a\dot{v} - 2\dot{F}$; hence, $\dot{F} = \frac{1}{2}a\dot{v} - \frac{1}{2}\dot{w}$, and $F = \frac{1}{2}av - \frac{1}{2}w$. Call this Q.

Prop. LXIII.

Let $\dot{F} = \frac{x^4\dot{x}}{\sqrt{a^2+x^2}}$, *to find* F.

126. Assume $v = \sqrt{a^2x^6+x^8}$, then $\dot{v} = \frac{3a^2x^5\dot{x}+4x^7\dot{x}}{\sqrt{a^2x^6+x^8}} = \frac{3a^2x^2\dot{x}}{\sqrt{a^2+x^2}} + \frac{4x^4\dot{x}}{\sqrt{a^2+x^2}}$ = (Art. 124.) $3a^2\dot{P} + 4\dot{F}$; hence, $\dot{F} = \frac{1}{4}\dot{v} - \frac{3a^2}{4}\dot{P}$, and $F = \frac{1}{4}v - \frac{3a^2}{4}P$.

Prop. LXIV.

Let $\dot{F} = \frac{x^4\dot{x}}{\sqrt{a^2 - x^2}}$, *to find* F.

127. Assume $v = \sqrt{a^2x^6 - x^8}$, then $\dot{v} = \frac{3a^2x^5\dot{x} - 4x^7\dot{x}}{\sqrt{a^2x^6 - x^8}}$ $= \frac{3a^2x^2\dot{x}}{\sqrt{a^2 - x^2}} - \frac{4x^4\dot{x}}{\sqrt{a^2 - x^2}} =$ (Art. 125.) $3a^2\dot{Q} - 4\dot{F}$; hence, $\dot{F} = \frac{3a^2}{4}\dot{Q} - \frac{1}{4}\dot{v}$, and $F = \frac{3a^2}{4}Q - \frac{1}{4}v$.

In this manner you may continue the fluents when the numerators are $x^6\dot{x}$, $x^8\dot{x}$, $x^{10}\dot{x}$, &c. by assuming $v = \sqrt{a^2x^{10} \pm x^{12}}$, $\sqrt{a^2x^{14} \pm x^{16}}$, $\sqrt{a^2x^{18} \pm x^{20}}$, &c. respectively, and by taking the fluxion, you will, in like manner, get $\dot{v}$ in terms of the given fluxion and of the next inferior fluxion.

Prop. LXV.

Let $\dot{F} = x^n\dot{x}\sqrt{a^2 \pm x^2}$, n *being an even number, to find* F.

128. Multiply and divide the fluxion by $\sqrt{a^2 \pm x^2}$, and $\dot{F} = \frac{a^2x^n\dot{x} \pm x^{n+2}\dot{x}}{\sqrt{a^2 \pm x^2}}$; hence, as the indices of x in the numerator are even numbers, the fluents of $\frac{a^2x^n\dot{x}}{\sqrt{a^2 \pm x^2}}$, and $\frac{x^{n+2}\dot{x}}{\sqrt{a^2 \pm x^2}}$, may each be found by the method directed in the last article.

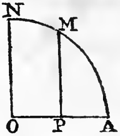
If n be an odd number, F may be found by Art. 41.

Prop. LXVI.

Let $\dot{F} = \dot{x}\sqrt{2ax - x^2}$, *to find* F.

129. Let the radius AO $= a$, AP $= x$, then the sine PM $= \sqrt{2ax - x^2}$, therefore $\dot{F} = \dot{x}\sqrt{2ax - x^2} =$

(Art. 49.) the fluxion of the area AMP; hence, F =



the area APM.

Prop. LXVII.

Let $\dot{F}=x\dot{x}\sqrt{2ax-x^2}$, *to find* F.

130. Assume $w=\frac{1}{3}\times\overline{2ax-x^2}\big]^{\frac{3}{2}}$, then $\dot{w}=\overline{a\dot{x}-x\dot{x}}\times\sqrt{2ax-x^2}=a\dot{x}\sqrt{2ax-x^2}-\dot{F}$; hence, $\dot{F}=a\dot{x}\sqrt{2ax-x^2}-\dot{w}$, and $F=a\times$area APM $-\,w$.

Prop. LXVIII.

Let $\dot{F}=\frac{x\dot{x}}{\sqrt{2ax-x^2}}$, *to find* F.

131. Assume $w=\sqrt{2ax-x^2}$, then $\dot{w}=\frac{a\dot{x}-x\dot{x}}{\sqrt{2ax-x^2}}=\frac{a\dot{x}}{\sqrt{2ax-x^2}}-\frac{x\dot{x}}{\sqrt{2ax-x^2}}=\frac{a\dot{x}}{\sqrt{2ax-x^2}}-\dot{F}$; hence, $\dot{F}=\frac{a\dot{x}}{\sqrt{2ax-x^2}}-\dot{w}$, and (Art. 46.) $F=z-w$, z being a cir. arc, rad. $=a$, versed sine $=x$.

Prop. LXIX.

Let $\dot{F}=\frac{x^m\dot{x}}{x-a}$, *to find* F.

132. Divide the num. by the den. till the index of x in the remainder $=0$, and the remainder will then be $a^m\dot{x}$; hence, $\dot{F}=x^{m-1}\dot{x}+ax^{m-2}\dot{x}+a^2x^{m-3}\dot{x}+$ &c. $+a^m\times\frac{\dot{x}}{x-a}$; therefore (Art. 37. and 45.) $F=\frac{x^m}{m}+\frac{ax^{m-1}}{m-1}+\frac{a^2x^{m-2}}{m-2}+$ &c. $+a^m\times$h. l. $\overline{x-a}$. Here m must

be a whole positive number, otherwise the index of x cannot become $= 0$. If the denominator be $x+a$, the terms will be alternately $+$ and $-$.

Prop. LXX.

Let $\dot{F} = \frac{z^{rm-1}\dot{z}}{a+bz^m}$, *to find* F.

133. $$bz^m+a)\;z^{rm-1}\dot{z}\;\left(\frac{1}{b}\times z^{rm-m-1}\dot{z}-\frac{a}{b^2}\times z^{rm-2m-1}\dot{z}+\&c.\right.$$

$$\underline{z^{rm-1}\dot{z}+\frac{a}{b}\times z^{rm-m-1}\dot{z}}$$

$$-\frac{a}{b}\times z^{rm-m-1}\dot{z}$$

$$\underline{-\frac{a}{b}\times z^{rm-m-1}\dot{z}-\frac{a^2}{b^2}\times z^{rm-2m-1}\dot{z}}$$

$$\frac{a^2}{b^2}\times z^{rm-2m-1}\dot{z}$$

&c. &c.

continue this division till the index of z in the remainder becomes $m-1$, and the remainder will be $\pm\frac{a^{r-1}}{b^{r-1}}\times z^{m-1}\dot{z}$; hence, $\dot{F}=\frac{1}{b}\times z^{rm-m-1}\dot{z}-\frac{a}{b^2}\times z^{rm-2m-1}\dot{z}+\&c.\pm\frac{a^{r-1}}{b^{r-1}}\times\frac{z^{m-1}\dot{z}}{a+bz^m}$; now the last term $=\pm\frac{a^{r-1}}{mb^r}\times\frac{mbz^{m-1}\dot{z}}{a+bz^m}$; hence (Art. 37. and 45.), $F=\frac{1}{b}\times\frac{z^{rm-m}}{rm-m}-\frac{a}{b^2}\times\frac{z^{rm-2m}}{rm-2m}+\&c.\pm\frac{a^{r-1}}{mb^r}\times \text{h. l. }\overline{a+bz^m}$. Here, r must be a whole positive number, otherwise the index of z can never become $m-1$.

LEMMA.

Let $\frac{1}{x^n-px^{n-1}+\&c.}=\frac{K}{x-a}+\frac{L}{x-b}+\frac{M}{x-c}+\&c.$ to find K, L, M, &c. where a, b, c, &c. are the roots of $x^n-px^{n-1}+\&c.=0$.

134. Reduce the fractions to a common denominator, and it will be the same as the denominator on the left, and consequently the sum of the numerators $=1$; hence, $K\times\overline{x-b}\times\overline{x-c}\times\&c.+L\times\overline{x-a}\times\overline{x-c}\times\&c.+M\times\overline{x-a}\times\overline{x-b}\times\&c.+\&c.=1$; now as this is true let x be what it will, make $x=a$, and then $K\times\overline{a-b}\times\overline{a-c}\times\&c.=1$ $\therefore$ $K=\frac{1}{\overline{a-b}\times\overline{a-c}\times\&c.}$ Make $x=b$, and then $L\times\overline{b-a}\times\overline{b-c}\times\&c.=1$, $\therefore$ $L=\frac{1}{\overline{b-a}\times\overline{b-c}\times\&c.}$. In like manner we get the other numerators.

If $\frac{1}{(e+fz^m)\times(g+hz^m)}=\frac{K}{f\left(z^m+\frac{e}{f}\right)}+\frac{L}{h\left(z^m+\frac{g}{h}\right)}$, then in the same manner it apppears, that $K=\frac{f}{fg-he}$ and $L=\frac{h}{he-fg}$.

Prop. LXXI.

Let $\dot{F}=\frac{x^m\dot{x}}{x^n-px^{n-1}+\&c.}$, *to find* F, m *being a whole positive number.*

135. Let $\frac{1}{x^n-px^{n-1}+\&c.}=\frac{K}{x-a}+\frac{L}{x-b}+\&c.$ then K, L, &c. are known by the last article; hence, $\frac{x^m\dot{x}}{x^n-px^{n-1}+\&c.}=\frac{Kx^m\dot{x}}{x-a}+\frac{Lx^m\dot{x}}{x-b}+\&c.$ Now (Art. 132.) the fluent of $\frac{Kx^m\dot{x}}{x-a}$ is $\frac{Kx^m}{m}+\frac{Kax^{m-1}}{m-1}+\&c.+Ka^m\times \text{h. l.}\,\overline{x-a}$; in like manner, the fluents of all the other quantities are found, the sum of all which is F. Now the sum of all these quantities $=\overline{K+L+\&c.}\times\frac{x^m}{m}+\overline{Ka+Lb+\&c.}\times\frac{x^{m-1}}{m-1}+\&c.+Ka^m\times\text{h. l.}\,\overline{x-a}+Lb^m\times\text{h. l.}\,\overline{x-b}+\&c.$ But by Dr. Waring's *Med. Alg. last edit. in the Addenda*, $K+L+\&c.=0$; $Ka+Lb+\&c.=0$; &c. through all those terms, when m is less than n; in this case therefore $F=Ka^m\times\text{h. l.}\,\overline{x-a}+Lb^m\times\text{h. l.}\,\overline{x-b}+\&c.$ If m be equal to or greater than n, the coefficients of the first $n-1$ terms will become $=0$.

136. If m be less than n, the quantity $\frac{x^m\dot{x}}{x^n-px^{n-1}+\&c.}$ may be resolved into $\frac{K'\dot{x}}{x-a}+\frac{L'\dot{x}}{x-b}+\frac{M'\dot{x}}{x-c}+\&c.$ for in this case $K'\times\overline{x-b}\times\overline{x-c}\times\&c.+L'\times\overline{x-a}\times\overline{x-c}\times$

&c.+&c. $= x^m$; hence, if $x=a$, $K' = \frac{a^m}{\overline{a-b} \times \overline{a-c} \times \&c.}$; if $x=b$, $L' = \frac{b^m}{\overline{b-a} \times \overline{b-c} \times \&c.}$; &c. The reason why m must be less than n is this: The quantity $K' \times \overline{x-b} \times \overline{x-c} \times$ &c.$+L' \times \overline{x-a} \times \overline{x-c} \times$ &c.+&c.$-x^m=0$; and that this may be *always* true, the coefficients of the like powers of x must be assumed$=0$ (Art. 110.), and by such an assumption you would deduce the same values of K', L', &c. as above. Now the product of each of the quantities into which K', L', &c. are multiplied, is of $n-1$ dimensions in terms of x, there being $n-1$ factors; hence, if m be greater than $n-1$, there is only *one* term in which x is of m dimensions, therefore this term can never be made to vanish, generally with the rest. But if m be equal to or less than $n-1$, then this term x^m will come in with others having the same power, and the whole coefficient may be made$=0$.

But the denominators may be otherwise expressed; for as $\overline{x-a} \times \overline{x-b} \times$ &c.$=x^n-px^{n-1}+$&c. by taking the fluxion we have $\dot{x} \times \overline{x-b} \times \overline{x-c} \times$ &c. $+ \dot{x} \times \overline{x-a} \times \overline{x-c} \times$ &c.+&c. $=nx^{n-1}\dot{x} - \overline{n-1} \cdot px^{n-2}\dot{x}+$ &c. hence, if $x=a$, we have $\overline{a-b} \times \overline{a-c} \times$ &c.$=na^{n-1} - \overline{n-1} \cdot pa^{n-2}+$&c. If $x=b$, then $\overline{b-a} \times \overline{b-c} \times$&c. $=nb^{n-1} - \overline{n-1} \cdot pb^{n-2}+$ &c. and so on for the rest; hence, take the fluxion of the given equation, omitting $\dot{x}$, and write a, b, c, &c. for x, and we get the denominators.

Hence, when m is less than n, the fluent of $\frac{x^m\dot{x}}{x^n-px^{n-1}+\&c.}$ is $K' \times$h. l. $\overline{x-a}+L' \times$h. l. $\overline{x-b}$ + &c. which agrees with the conclusion in Art. 135. because $K'=Ka^m$, $L'=Lb^m$, &c.

137. If two roots a, b, be equal, one of the quantities must have a quadratic divisor $\overline{x-a}^2$. For example: Let $\frac{1}{x^3-px^2+qx-r} = \frac{Lx+M}{\overline{x-a}^2} + \frac{N}{x-c}$: then reducing the two quantities on the right to the same denominator, and making the numerators equal, we get $Lx^2 - Lcx + Mx - Mc + Nx^2 - 2Nax + Na^2 - 1 = 0$; hence (Art. 110.), making $L + N = 0$, $M - Lc - 2Na = 0$, $-Mc + Na^2 - 1 = 0$, we have, $L = -N$, $M = \frac{Na^2-1}{c}$; consequently $\frac{Na^2-1}{c} + Nc - 2Na = 0$; therefore $N = \frac{1}{\overline{a-c}|^2}$; $L = \frac{1}{-\overline{c-a}|^2}$; $M = \frac{2a-c}{\overline{a-c}|^2}$. Hence, the fluent of $\frac{\dot{x}}{x^3-px^2+qx-r}$, or $\frac{Lx\dot{x}+M\dot{x}}{\overline{x-a}|^2} + \frac{N\dot{x}}{x-c}$ may be thus found. Put $x - a = z$, then $x = z + a$, and $\dot{x} = \dot{z}$; hence, $\frac{Lx\dot{x}+M\dot{x}}{\overline{x-a}|^2} = \frac{Lz\dot{z}+La\dot{z}+M\dot{z}}{z^2} =$ (if $La + M = b$) $\frac{L\dot{z}}{z} + \frac{b\dot{z}}{z^2}$, whose fluent (Art. 45. and 37.) is $L \times \text{h. l. } z - \frac{b}{z} = L \times \text{h. l. } \overline{x-a} - \frac{b}{x-a}$; and the fluent of $\frac{N\dot{x}}{x-c}$ is $N \times \text{h. l. } \overline{x-c}$.

138. If two of the roots be impossible, those two binomial fractions must be incorporated into one. Thus, let $\frac{1}{x^3-px^2+qx-r} = \frac{L}{x-a} + \frac{M}{x-b} + \frac{N}{x-c}$, and suppose a and b to be impossible; then $\frac{L}{x-a} + \frac{M}{x-b} =$

$\frac{\overline{L+M} \times x - \overline{Lb+Ma}}{x^2 - \overline{a+b} \times x + ab}$, and the impossible quantities vanish, as will appear by substituting $m+n\sqrt{-1}$ for a, and $m-n\sqrt{-1}$ for b.

Prop. LXXII.

Let $\dot{F} = \frac{cx\dot{x}+d\dot{x}}{x^2-px+q}$, *to find* F.

139. Put $x - \frac{1}{2}p = z$, then $x = z + \frac{1}{2}p$, and $\dot{x} = \dot{z}$; hence, $d\dot{x} = d\dot{z}$, and $cx\dot{x} = cz\dot{z} + \frac{1}{2}pc\dot{z}$, $\therefore cx\dot{x} + d\dot{x} = cz\dot{z} + \overline{\frac{1}{2}pc + d} \times \dot{z} =$ (if $\frac{1}{2}pc + d = e$) $cz\dot{z} + e\dot{z}$; also, $x^2 - px + \frac{1}{4}p^2 = z^2$; hence, $x^2 - px + q = z^2 + q - \frac{1}{4}p^2 =$ (if $q - \frac{1}{4}p^2 = a^2$) $z^2 \pm a^2$, according as a^2 is positive or negative, or according as the two values of x are impossible or possible. Hence, $\dot{F} = \frac{cz\dot{z} + e\dot{z}}{z^2 \pm a^2} = \frac{cz\dot{z}}{x^2 \pm a^2} + \frac{e\dot{z}}{z^2 \pm a^2}$. Now (Art. 45.) the fluent of $\frac{cz\dot{z}}{z^2 \pm a^2}$ is $\frac{1}{2}c \times$ h. l. $\overline{z^2 \pm a^2}$. Also, taking $+a^2$, $\frac{e\dot{z}}{z^2 + a^2} = \frac{e}{a^2} \times \frac{a^2\dot{z}}{z^2 + a^2}$, whose fluent (Art. 46.) is $\frac{e}{a^2} \times$ cir. arc, rad. $= a$, tan. $= z$. But taking $-a^2$, $\frac{e\dot{z}}{z^2 - a^2} = \frac{e}{2a} \times \frac{2a\dot{z}}{z^2 - a^2}$, whose fluent (Art. 45.) is $\frac{e}{2a} \times$ h. l. $\frac{z-a}{z+a}$; call the fluent of this second part B, and $F = \frac{1}{2}c \times$ h. l. $\overline{z^2 \pm a^2} + B$. Call this fluent Q.

Prop. LXXIII.

Let $\dot{F} = \frac{x^m \dot{x}}{x^2 - px + q}$, *to find* F.

140. If the roots of $x^2 - px + q = 0$ be both possible,

then (Art. 134.) resolve $\frac{1}{x^2-px+q}$ into $\frac{K}{x-a}+\frac{L}{x-b}$; and $\dot{F}=\frac{Kx^m\dot{x}}{x-a}+\frac{Lx^m\dot{x}}{x-b}$, whose fluents are found by Art. 135. But if the roots be impossible, divide $x^m\dot{x}$ by x^2-px+q until the remainder becomes $cx\dot{x}+d\dot{x}$, c and d being put for the coefficients which arise from the division, and let the quotient be $x^{m-2}\dot{x}+ax^{m-3}\dot{x}+bx^{m-4}\dot{x}+$ &c. where $a=p$, $b=p^2-q$ &c.; hence, $\dot{F}=x^{m-2}\dot{x}+ax^{m-3}\dot{x}+bx^{m-4}\dot{x}+$ &c. $+\frac{cx\dot{x}+d\dot{x}}{x^2-px+q}$, consequently (Art. 37. and 139.) $F=\frac{x^{m-1}}{m-1}+\frac{ax^{m-2}}{m-2}+\frac{bx^{m-3}}{m-3}$ + &c. + Q.

If $m=2$, then $F=x+Q$.
If $m=3$, then $F=\frac{1}{2}x^2+ax+Q$.
If $m=4$, then $F=\frac{1}{3}x^3+\frac{1}{2}ax^2+bx+Q$.

Prop. LXXIV.

Let $\dot{F}=\frac{z^{-m}\dot{z}}{z^2-pz+q}$, *to find* F.

141. Put $x=\frac{1}{z}=z^{-1}$, then $x^{m-1}=z^{-m+1}$, and $x^{m-2}\dot{x}=-z^{-m}\dot{z}$; hence, $\dot{F}=\frac{-x^{m-2}\dot{x}}{\frac{1}{x^2}-\frac{p}{x}+q}=-\frac{x^m\dot{x}}{1-px+qx^2}$

$$=-\frac{1}{q}\times\frac{x^m\dot{x}}{\frac{1}{q}-\frac{px}{q}+x^2}=\left(\text{if } \frac{1}{q}=q', \frac{p}{q}=p'\right)-\frac{1}{q}\times\frac{x^m\dot{x}}{x^2-p'x+q'},$$

which is the same as the last form.

Prop. LXXV.

Let $\dot{F}=\frac{\dot{z}}{z\sqrt{a+cz^n}}$, *to find* F.

142. First, $\dot{F} = \frac{1}{\sqrt{c}} \times \frac{\dot{z}}{z\sqrt{d^2+z^n}}$ (putting $d^2 = \frac{a}{c}$); put $z^{\frac{1}{2}n} = x$, and then $z^n = x^2$; also, $\frac{\dot{x}}{x} = \frac{\frac{1}{2}nz^{\frac{1}{2}n-1}\dot{z}}{z^{\frac{1}{2}n}} = \frac{1}{2}n \times \frac{\dot{z}}{z}$, $\therefore \frac{2}{n} \times \frac{\dot{x}}{x} = \frac{\dot{z}}{z}$; hence, $\dot{F} = \frac{2}{n\sqrt{c}} \times \frac{\dot{x}}{x\sqrt{d^2+x^2}} = \frac{1}{nd\sqrt{c}} \times \frac{2d\dot{x}}{x\sqrt{d^2+x^2}}$; and (Art. 45.) $F = \frac{1}{nd\sqrt{c}} \times \text{h. l.} \frac{\sqrt{d^2+x^2}-d}{\sqrt{d^2+x^2}+d}$. If d^2 be negative, $\dot{F} = \frac{2}{n\sqrt{c}} \times \frac{\dot{x}}{x\sqrt{x^2-d^2}} = \frac{2}{nd^2\sqrt{c}} \times \frac{d^2\dot{x}}{x\sqrt{x^2-d^2}}$, and (Art. 46.) $F = \frac{2}{nd^2\sqrt{c}} \times$ cir. arc, rad. $= d$, secant $= x$.

PROP. LXXVI.

Let $\dot{F} = \frac{\dot{z}}{z^2\sqrt{a^2+z^2}}$, *to find* F.

143. Put $x = \frac{a^2}{z}$, then $\dot{x} = -\frac{a^2\dot{z}}{z^2}$; hence, $-\frac{1}{a^2} \times \dot{x} = \frac{\dot{z}}{z^2}$; therefore $\dot{F} = -\frac{1}{a^2} \times \frac{\dot{x}}{\sqrt{a^2+\frac{a^4}{x^2}}} = -\frac{1}{a^3} \times \frac{x\dot{x}}{\sqrt{x^2+a^2}}$; hence (Art. 39.), $F = -\frac{1}{a^3} \times \sqrt{x^2+a^2}$.

PROP. LXXVII.

Let $\dot{F} = \frac{z\dot{z}\sqrt{b^2+z^2}}{\sqrt{c^2-z^2}}$, *to find* F.

144. Put $x = \sqrt{c^2-z^2}$, then $z^2 = c^2 - x^2$, therefore $z\dot{z} = -x\dot{x}$, and $\sqrt{b^2+z^2} = \sqrt{b^2+c^2-x^2} =$ (if $a^2 = b^2 + c^2$) $\sqrt{a^2-x^2}$; hence, $\dot{F} = -\dot{x}\sqrt{a^2-x^2}$. Now let

AN be a circular arc whose centre is O, (See Fig. p. 162.) and PM be perpendicular to AO, and put $a=$ OA, $x=$OP, then PM$=\sqrt{a^2-x^2}$; hence, $\dot{F}=-$the fluxion of the area OPMN (Art. 49.), consequently F $=-$ area OPMN.

PROP. LXXVIII.

$$Let\ \dot{F}=\frac{z^{n-1}\dot{z}}{(g+hz^n)\sqrt{e+fz^n}},\ to\ find\ F.$$

145. Put $\sqrt{e+fz^n}=x$, then $z^n=\frac{x^2-e}{f}$, and $g+hz^n=g+\frac{h}{f}\times\overline{x^2-e}=\frac{fg-eh}{f}+\frac{h}{f}x^2=\left(\text{if }\frac{fg-eh}{f}=a,\frac{h}{f}=b\right)a+bx^2$; also, $nz^{n-1}\dot{z}=2b\times x\dot{x}$, and $z^{n-1}\dot{z}=\frac{2b}{n}\times x\dot{x}$; hence, $\dot{F}=\frac{2b\dot{x}}{n\times\overline{a+bx^2}}$, whose fluent is found by Art. 45 or 46, according as a and b have different or the same signs.

PROP. LXXIX.

$$Let\ \dot{F}=\sqrt{\frac{e+fz^n}{g+hz^n}}\times z^{n-1}\dot{z},\ to\ find\ F.$$

146. Put $\sqrt{g+hz^n}=x$, then $z^n=\frac{x^2-g}{h}$, and $e+fz^n=e+\frac{f}{h}\times\overline{x^2-g}=\frac{he-fg}{h}+\frac{f}{h}\times x^2=\left(\text{if }\frac{he-fg}{h}=a,\frac{f}{h}=b\right)a+bx^2$; also, $z^{n-1}\dot{z}=\frac{2b}{n}\times x\dot{x}$; hence, $\dot{F}=\frac{2b}{n}\times\sqrt{a+bx^2}\times\dot{x}$, whose fluent is found by Art: 46. when b is negative and a positive; but by Art. 45. when b is positive and a either positive or negative.

Prop. LXXX.

Let $\dot{F} = \frac{z^{rm-1}\dot{z}}{(e+fz^m)\times(g+hz^m)}$, *to find* F, r *and* m *being whole positive numbers.*

147. By Art. 134. $\frac{z^{rm-1}\dot{z}}{(e+fz^m)\times(g+hz^m)} = \frac{Kz^{rm-1}\dot{z}}{e+fz^m} + \frac{Lz^{rm-1}\dot{z}}{g+hz^m}$, where K and L are known; and the fluents are found by Prop. 70.

Prop. LXXXI.

Let $\dot{F} = \overline{e+fz^n}|^m \times \overline{g+hz^n}|^r \times z^{sn-1}\dot{z}$, *to find* F, *where* s *is a whole positive number, and* r *half any whole positive number.*

148. Put $v=e+fz^n$, then $z^n=\frac{v-e}{f}$; $hz^n=\frac{h}{f}\times\overline{v-e}$; $g+hz^n=g+\frac{h}{f}\times\overline{v-e}=g-\frac{he}{f}+\frac{h}{f}\times v=$ (if $d=g-\frac{he}{f}$) $d+\frac{h}{f}\times v$; $z^{sn}=\frac{1}{f^s}\times\overline{v-e}|^s$; $snz^{sn-1}\dot{z}=\frac{s}{f^s}\times\overline{v-e}|^{s-1}\dot{v}$; $z^{sn-1}\dot{z}=\frac{1}{nf^s}\times\overline{v-e}|^{s-1}\dot{v}$; hence, by substitution we get $\dot{F}=v^m\times\overline{d+\frac{h}{f}v}|^r\times\frac{1}{nf^s}\times\overline{v-e}|^{s-1}\dot{v}$; and by expanding $\overline{d+\frac{h}{f}v}|^r$ and $\overline{v-e}|^{s-1}$, and actually multiplying each term into $v^m\dot{v}$, then when r is the half of an *odd* number (as $t+\frac{1}{2}$), $\overline{d+\frac{h}{f}v}|^r = \overline{d+\frac{h}{f}v}|^t\times\sqrt{d+\frac{h}{f}v}$, expand $\overline{d+\frac{h}{f}v}|^t$, and the fluent

can be found by Art. 39 or 41. But when r is the half of an *even* number expand $\overline{d+\frac{h}{f}v}\Big|^{r}$, and then the fluent of each term may be found by Art. 37. except m be negative, such that one of the terms be of the form $\frac{\dot{v}}{v}$, in which case the fluent of that term is found by Art. 45.

If $r=-\frac{1}{2}$, and m a positive whole number, the fluent may be found by Art. 41. And if $m=-1$, then the fluent may be found by Art. 41. except for one term in the series thence arising, whose fluent is found by Prop. 75. it being of the form $\frac{\dot{v}}{v\sqrt{d+\frac{h}{f}v}}$.

PROP. LXXXII.

Let $\dot{F}=\frac{\sqrt{a+bx^2}}{c+dx^2}\times\dot{x}$, *to find* F.

149. Multiply the num. and den. by $\sqrt{a+bx^2}$, and we get $\dot{F}=\frac{\overline{a+bx^2}\times\dot{x}}{\overline{c+dx^2}\times\sqrt{a+bx^2}}=\frac{a\dot{x}}{\overline{c+dx^2}\times\sqrt{a+bx^2}}+\frac{bx^2\dot{x}}{\overline{c+dx^2}\times\sqrt{a+bx^2}}$. But the *first* of these terms $=\frac{ax^{-3}\dot{x}}{\overline{d^2+cx^{-2}}\times\sqrt{b+ax^{-2}}}$; and in the second term, by division $\frac{bx^2\dot{x}}{c+dx^2}=\frac{b}{d}\times\dot{x}-\frac{bc}{d}\times\frac{\dot{x}}{dx^2+c}$; hence, the *second* term $=\frac{b}{d}\times\frac{\dot{x}}{\sqrt{a+bx^2}}-\frac{bc}{d}\times\frac{\dot{x}}{\overline{c+dx^2}\times\sqrt{a+bx^2}}$, and the last term of this $=-\frac{bc}{d}\times\frac{x^{-3}\dot{x}}{\overline{d^2+cx^{-2}}\times\sqrt{b+ax^{-2}}}$;

hence, $\dot{F}=\frac{b}{d}\times\frac{\dot{x}}{\sqrt{a+bx^2}}+\left(a-\frac{bc}{d}\right)\times$ $\frac{x^{-3}\dot{x}}{\overline{d^2+cx^{-2}}\times\sqrt{b+ax^{-2}}}$; and the fluent of the *first* of these terms is found by Art. 45. or 46, according to the signs of a and b, and of the *second* by Prop. 81.

LEMMA.

To resolve $\frac{1}{\overline{x+a}|^m\times\overline{x+b}|^n}$ into $\frac{H}{\overline{x+a}|^m}+\frac{K}{\overline{x+a}|^{m-1}}$ $+\frac{L}{\overline{x+a}|^{m-2}}+\&c.+\frac{P}{\overline{x+b}|^n}+\frac{Q}{\overline{x+b}|^{n-1}}+\frac{R}{\overline{x+b}|^{n-2}}+$ &c. continued to m and n quantities respectively.

150. Reduce the fractions to a common denominator, and make the numerators on each side equal, and (A) $H\times\overline{x+b}^n+K\times\overline{x+b}^n\times\overline{x+a}+L\times\overline{x+b}^n\times\overline{x+a}^2+\&c.+P\times\overline{x+a}^m+Q\times\overline{x+a}^m\times\overline{x+b}+R\times\overline{x+a}^m\times\overline{x+b}^2+\&c.=1$. Make $x+a=0$, or $x=-a$, and every term where $x+a$ enters, becomes$=0$; hence, $H\times\overline{x+b}^n=1$, or $H\times\overline{b-a}^n=1$, $\therefore H=\frac{1}{\overline{b-a}^n}$. Take the fluxion of the equation (A), and omitting $\dot{x}$, we have (B) $nH\times\overline{x+b}^{n-1}+nK\times\overline{x+b}^{n-1}\times\overline{x+a}+K\times\overline{x+b}^n+\&c.=0$; make $x=-a$, and we have $nH\times\overline{b-a}^{n-1}+K\times\overline{b-a}^n=0$; hence, $K=-\frac{nH}{b-a}=\frac{-n}{\overline{b-a}^{n+1}}$; thus by continuing to take the fluxion of the last equation, and then making $x=-a$, we shall get the values of L, &c. In like manner, if we make $x+b=0$, or $x=-b$, we find $P=\frac{1}{\overline{a-b}^m}$; then by taking the fluxion

of the last equation, and making $x=-b$, we get $Q=\frac{-m}{\overline{a-b}^{m+1}}$; and by proceeding as before, we get R, &c.

Prop. LXXXIII.

Let $\dot{F}=\frac{x^r\dot{x}}{\overline{x+a}^m\times\overline{x+b}^n}$, *to find* F, r *being a whole positive number.*

151. By the last article, $\dot{F}=\frac{Hx^r\dot{x}}{\overline{x+a}^m}+\frac{Kx^r\dot{x}}{\overline{x+a}^{m-1}}+$&c. $+\frac{Px^r\dot{x}}{\overline{x+b}^n}+\frac{Qx^r\dot{x}}{\overline{x+b}^{n-1}}+$&c. Put $x+a=z$, then $x=z-a$, therefore $x^{r+1}=\overline{z-a}^{r+1}$, and $x^r\dot{x}=\overline{z-a}^r\times\dot{z}$; hence, $\frac{x^r\dot{x}}{\overline{x+a}^m}=\frac{\overline{z-a}^r\times\dot{z}}{z^m}=z^{r-m}\dot{z}-raz^{r-m-1}\dot{z}+r\cdot\frac{r-1}{2}a^2z^{r-m-2}\dot{z}-$&c. where the number of the terms $=r+1$, and the fluent of every term is found by Art. 37. except that term where the index of z is -1, whose fluent is found by Art. 45. and the sum of all these multiplied by H, is the fluent of the first term. In like manner, the fluents of the other terms are found.

Prop. LXXXIV.

Given A *the fluent of* $\overline{e+fx^n}|^m\times x^p\dot{x}$, *to find* B *the fluent of* $\overline{e+fx^n}|^m\times x^{p+n}\dot{x}$, *and* C *the fluent of* $\overline{e+fx^n}|^{m+1}\times x^p\dot{x}$.

152. Assume $Q=\overline{e+fx^n}|^{m+1}\times x^{p+1}$, then $\dot{Q}=\overline{p+1}\times\overline{e+fx^n}|^{m+1}\times x^p\dot{x}+\overline{m+1}\times nf\times\overline{e+fx^n}|^m\times x^{p+n}\dot{x}=\overline{p+1}\times\dot{C}+\overline{m+1}\times nf\times\dot{B}$; hence, by taking the fluents, $Q=\overline{p+1}\times C+\overline{m+1}\times nf\times B$. Also, $\overline{e+fx^n}|^{m+1}\times x^p\dot{x}=\overline{e+fx^n}\times\overline{e+fx^n}|^m\times x^p\dot{x}=e\times\overline{e+fx^n}|^m\times x^p\dot{x}+$

$f \times \overline{e+fx^n}|^m \times x^{p+n}\dot{x}$, that is, $\dot{C}=e\dot{A}+f\dot{B}$, therefore $C=eA+fB$. Now from the *first* fluent, $B=\dfrac{Q-\overline{p+1}\times C}{\overline{m+1}\times nf}$, and from the *second*, $B=\dfrac{C-eA}{f}$; hence, $\dfrac{Q-\overline{p+1}\times C}{\overline{m+1}\times nf}=\dfrac{C-eA}{f}$; $\therefore C=\dfrac{Q+\overline{m+1}\times neA}{p+1+\overline{m+1}\times n}$; consequently $B=\dfrac{C-eA}{f}=\dfrac{C}{f}-\dfrac{eA}{f}=\dfrac{Q+\overline{m+1}\times neA}{p+1+\overline{m+1}\times n\times f}-\dfrac{eA}{f}$. Hence, we may continue the fluent as far as we please, increasing m by 1, and p by n.

Let $e=a^2, f=1, m=-\frac{1}{2}, p=0, n=2$; then $\dot{A}=\dfrac{\dot{x}}{\sqrt{a^2+x^2}}$, and $A=$ h. l. $x+\sqrt{x^2+a^2}$ (Art. 45.); hence, B the fluent of $\dfrac{x^2\dot{x}}{\sqrt{a^2+x^2}}=\frac{1}{2}x\times\overline{a^2+x^2}|^{\frac{1}{2}}-\frac{1}{2}a^2A$, as in Art. 124. also, C the fluent of $\overline{a^2+x^2}|^{\frac{1}{2}}\times\dot{x}=\frac{1}{2}x\times\overline{a^2+x^2}|^{\frac{1}{2}}+\frac{1}{2}a^2A$.

Prop. LXXXV.

Let $\dot{F}=vx^n\dot{x}$, *where* $v=$ *h. l.* $\dfrac{1}{1-x}$, *to find* F.

153. Assume $\dfrac{vx^{n+1}}{n+1}+r=F$, then $vx^n\dot{x}+\dfrac{x^{n+1}\dot{v}}{n+1}+\dot{r}=\dot{F}=vx^n\dot{x}$; hence $\dot{r}=-\dfrac{x^{n+1}\dot{v}}{n+1}=\left(\text{because } \dot{v}=\dfrac{\dot{x}}{1-x}\right)$ $=-\dfrac{x^{n+1}\dot{x}}{\overline{n+1}\times\overline{1-x}}=$ (by division) $-\dfrac{1}{n+1}\times$ $\overline{-x^n\dot{x}-x^{n-1}\dot{x}-\&c.+\dfrac{\dot{x}}{1-x}}$, therefore $r=\dfrac{1}{n+1}\times$

$\overline{\frac{x^{n+1}}{n+1}+\frac{x^n}{n}+\&c.-v}$; hence, $F=\frac{vx^{n+1}}{n+1}+\frac{1}{n+1}\times$
$\overline{\frac{x^{n+1}}{n+1}+\frac{x^n}{n}+\&c.-v}$.

Prop. LXXXVI.

Let $\dot{F}=vx^n\dot{x}$, *where* v *is a circular arc whose radius is* 1 *and tangent* x, *to find* F.

154. Assume $\frac{vx^{n+1}}{n+1}+r=F$; then $vx^n\dot{x}+\frac{x^{n+1}\dot{v}}{n+1}+\dot{r}$ $=\dot{F}=vx^n\dot{x}$. Let n be an *odd* number, and then $\dot{r}=$ $-\frac{x^{n+1}\dot{v}}{n+1}=$ (Art. 46.) $-\frac{x^{n+1}\dot{x}}{n+1\times\overline{1+x^2}}=-\frac{1}{n+1}\times$ $\overline{x^{n-1}\dot{x}-x^{n-3}\dot{x}+\&c.\pm\dot{v}}\left(\frac{\dot{x}}{1+x^2}\right)$, where the sign of $\dot{v}$ will be + or —, according as $\frac{n+1}{2}$ is even or odd; hence, $r=\frac{1}{n+1}\times\overline{-\frac{x^n}{n}+\frac{x^{n-2}}{n-2}-\&c.\mp v}$; therefore $F=\frac{vx^{n+1}}{n+1}+\frac{1}{n+1}\times\overline{-\frac{x^n}{n}+\frac{x^{n-2}}{n-2}-\&c.\mp v}$.

If n be an *even* number, the last term of the division will be $\pm\frac{x\dot{x}}{1+x^2}$, whose fluent is $\pm\frac{1}{2}$ h. l. $\overline{1+x^2}=$ $\pm$ h. l. $\sqrt{1+x^2}$; hence, $F=\frac{vx^{n+1}}{n+1}+\frac{1}{n+1}\times$ $\overline{-\frac{x^n}{n}+\frac{x^{n-2}}{n-2}-\&c.}\pm$ h. l. $\sqrt{1+x^2}$, where the sign of the last term is + or —, according as $\frac{1}{2}n$ is odd or even.

Prop. LXXXVII.

Let $\dot{F}=z^m x^{n-1}\dot{x}$, *where* $z=h.\ l.\ x$, *to find* F.

155. Assume $F=az^m+bz^{m-1}+cz^{m-2}+$ &c. a, b, c, &c. being variable coefficients in terms of x; hence, by taking the fluxion we have,

$$\left.\begin{array}{l}\dot{a}z^m + \dot{b}z^{m-1} + \dot{c}z^{m-2}+\&c.\\ ma\dot{z}z^{m-1}+\overline{m-1}.b\dot{z}z^{m-2}+\&c.\end{array}\right\}=z^m x^{n-1}\dot{x};\text{ but}$$

by Art. 45. $\dot{z}=\frac{\dot{x}}{x}$; hence, by transposition,

$$\left.\begin{array}{l}\dot{a}z^m+ \quad \dot{b}z^{m-1}+ \quad \dot{c}z^{m-2}+\&c.\\ -x^{n-1}\dot{x}z^m+\frac{ma\dot{x}}{x}z^{m-1}+\frac{\overline{m-1}.b\dot{x}}{x}z^{m-2}+\&c.\end{array}\right\}=0$$

therefore by Art. 110. $\dot{a}-x^{n-1}\dot{x}=0$, $\dot{b}+\frac{ma\dot{x}}{x}=0$, $\dot{c}+\frac{\overline{m-1}.b\dot{x}}{x}=0$, &c. hence, $\dot{a}=x^{n-1}\dot{x}$, $\therefore a=\frac{x^n}{n}$; $\dot{b}=\frac{-ma\dot{x}}{x}=\frac{-mx^{n-1}\dot{x}}{n}$, $\therefore b=\frac{-mx^n}{n^2}$; $\dot{c}=\frac{-\overline{m-1}\,.\,b\dot{x}}{x}=\frac{-\overline{m-1}\times -mx^{n-1}\dot{x}}{n^2}$, $\therefore c=\frac{m.\overline{m-1}\,x^n}{n^3}$; &c. hence,

$$F=\frac{x^n}{n}\times z^m-\frac{mx^n}{n^2}\times z^{m-1}+\frac{m.\overline{m-1}.x^n}{n^3}\times z^{m-2}-\&c.$$

where the law of continuation is manifest, and the series will terminate when m is a whole positive number.

Prop. LXXXVIII.

Let $\dot{F}=a^x x^n\dot{x}$, *to find* F.

156. Assume $F=a^x\times\overline{px^n+qx^{n-1}+rx^{n-2}+\&c.}$ and let $m=$h. l. a; then (Art. 114.) $ma^x\dot{x}$ is the fluxion of a^x; hence, by taking the fluxions,

$$\left.\begin{array}{l} ma^x\dot{x} \times \overline{px^n + qx^{n-1} + rx^{n-2} + \&c.} \\ a^x \times \overline{npx^{n-1}\dot{x} + n-1 . qx^{n-2}\dot{x} + \&c.} \end{array}\right\} = a^x x^n \dot{x};$$

divide both sides by $a^x\dot{x}$, and transpose x^n, and we have

$$\left.\begin{array}{l} mpx^n + mqx^{n-1} + mrx^{n-2} + \&c. \\ -x^n + npx^{n-1} + \overline{n-1} . qx^{n-2} + \&c. \end{array}\right\} = 0;$$

hence (Art. 110.), $mp - 1 = 0$, $mq + np = 0$, $mr + \overline{n-1} . q = 0$, &c. $\therefore p = \frac{1}{m}$; $q = \frac{-np}{m} = -\frac{n}{m^2}$; $r = -\frac{\overline{n-1} . q}{m} = -\frac{\overline{n-1} \times -n}{m^3} = \frac{n . \overline{n-1}}{m^3}$, &c. therefore $\overline{\text{F}} = a^x \times \frac{1}{m} x^n - \frac{n}{m^2} x^{n-1} + \frac{n . \overline{n-1}}{m^3} x^{n-2} - \&c.$ where the law of continuation is manifest, and the series will terminate when n is a whole number.

Prop. LXXXIX.

To find the fluent of $\frac{z^r\dot{z}}{\overline{1 \pm z^n}|^2}$, *given the fluent of* $\frac{z^r\dot{z}}{1 \pm z^n}$.

157. Assume $\frac{az^{r+1}}{1 \pm z^n} + \text{Q}$ for the fluent; then, by taking the fluxion, we have $\frac{\overline{r+1} \times az^r\dot{z} \times \overline{1 \pm z^n} \mp naz^{r+n}\dot{z}}{\overline{1 \pm z^n}|^2} + \dot{\text{Q}} = \frac{z^r\dot{z}}{\overline{1 \pm z^n}|^2}$, or $\frac{z^r\dot{z}}{\overline{1 \pm z^n}|^2} = \frac{1}{1 \pm z^n} \times \overline{\overline{r+1} \times az^r\dot{z} \mp \frac{naz^{r+n}\dot{z}}{1 \pm z^n}} + \dot{\text{Q}}$; but $\mp \frac{naz^{r+n}\dot{z}}{1 \pm z^n} = -naz^r\dot{z} + \frac{naz^r\dot{z}}{1 \pm z^n}$; hence, $\frac{z^r\dot{z}}{\overline{1 \pm z^n}|^2} = \frac{1}{1 \pm z^n} \times \overline{\overline{r+1} \times az^r\dot{z} - naz^r\dot{z} + \frac{naz^r\dot{z}}{1 \pm z^n}} + \dot{\text{Q}} = \overline{r+1} \times a - na \times$

$\frac{z^r\dot{z}}{1\pm z^n}+\frac{naz^r\dot{z}}{\overline{1\pm z^n}|^2}+\dot{Q}$; assume $na=1$, or $a=\frac{1}{n}$, so that the terms $\frac{z^r\dot{z}}{\overline{1\pm z^n}|^2}$ and $\frac{naz^r\dot{z}}{\overline{1\pm z^n}|^2}$ may destroy each other, and we have $\dot{Q}=\overline{1-\frac{r+1}{n}}\times\frac{z^r\dot{z}}{1\pm z^n}$; hence, if P be the fluent of $\frac{z^r\dot{z}}{1\pm z^n}$, we have $Q=\overline{1-\frac{r+1}{n}}\times P$; consequently the fluent required is $\frac{1}{n}\times\frac{z^{r+1}}{1\pm z^n}+\overline{1-\frac{r+1}{n}}$ $\times P$.

PROP. XC.

To find fluents where there are two variable quantities in the given fluxion.

158. It frequently happens, that a fluxional equation contains two variable quantities, in which case, they must either be separated, or reduced to the fluxion of some known fluent; but no general rules can be given for this purpose, and the reductions must be left to trial and the skill of the Analyst; the following Rules, however, may be of some use.

RULE 1.

Multiply or divide the given equation by some function of the unknown quantities, so as to bring them to a form whose fluents may be found by some of the rules already given, or to the fluxion of a known fluent.

EXAMPLES.

Ex. 1. Let $\frac{\dot{x}}{x}+\frac{\dot{y}}{y}=\frac{ax^m\dot{x}}{y^n}$. Multiply both sides by nx^ny^n, and it becomes $ny^nx^{n-1}\dot{x}+nx^ny^{n-1}\dot{y}=nax^{m+n}\dot{x}$; now the fluent of the first part is known from Prop. 7. to be x^ny^n, and the fluent of the other part is found (Art. 37.) to be $\frac{nax^{m+n+1}}{m+n+1}$; hence, the equation of the fluents is $x^n y^n = \frac{nax^{m+n+1}}{m+n+1}$.

Ex. 2. Let $\ddot{x} - x\dot{z}^2 = f\dot{z}^2$. As $\ddot{z}$ does not enter into this equation, conceiving it to be deduced from a fluent, $\dot{z}$ must have been supposed constant. Multiply by $\dot{x}$, and $\dot{x}\ddot{x} - x\dot{x}\dot{z}^2 = f\dot{x}\dot{z}^2$, and as $\dot{z}$ is constant, the fluent is $\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2\dot{z}^2 = fx\dot{z}^2$; hence, $\dot{z} = \frac{\dot{x}}{\sqrt{2fx + x^2}}$, whose fluent (Art. 45.) is $z =$ h. l. $\overline{f + x + \sqrt{2fx + x^2}}$.

RULE 2.

Sometimes the fluent may be found by the addition of a new variable quantity.

EXAMPLE.

Let $a\dot{z} = z\dot{x} - x\dot{x}$. Assume $z = a + x + v$, then $\dot{z} = \dot{x} + \dot{v}$; hence, by substitution, $a\dot{x} + a\dot{v} = a\dot{x} + x\dot{x} + v\dot{x} - x\dot{x}$, therefore $a\dot{v} = v\dot{x}$, or $\dot{x} = \frac{a\dot{v}}{v}$; hence (Art. 45.), $x = a \times$ h. l. v; consequently $z = a + v + a \times$ h. l. v, and by substituting for v its value $z - a - x$, we get $x = a \times$ h. l. $\overline{z - a - x}$.

RULE 3.

The fluent may sometimes be found by first putting the equation into fluxions, making one of the fluxions constant.

EXAMPLE.

Let $\frac{a\dot{x} + y\dot{x}}{\dot{y}} = x + y - \frac{x\dot{y}}{\dot{x}}$. Make $\dot{y}$ constant, and put the equation into fluxions, and $\frac{\overline{a+y}\times\dot{x}}{\dot{y}}+\dot{x}=$ $\dot{x}+\dot{y}+\frac{x\dot{y}\ddot{x}-\dot{x}^2\dot{y}}{\dot{x}^2}$; hence, $\frac{\overline{a+y}\times\ddot{x}}{\dot{y}}=\frac{x\dot{y}\ddot{x}}{\dot{x}^2}$, and $\overline{a+y}$ $\times\,\dot{x}^2=x\dot{y}^2$, consequently $x^{-\frac{1}{2}}\dot{x}=\overline{a+y}\rceil^{-\frac{1}{2}}\dot{y}$; hence, (Art. 37. and 39.) we have $2x^{\frac{1}{2}}=2\times\overline{a+y}\rceil^{\frac{1}{2}}$.

RULE 4.

If only one of the variable quantities (x *or* y) *enter, substitute for the fluxion of one of them, the fluxion of the other multiplied into a new variable quantity.*

EXAMPLE.

Let $y\dot{y}^3\dot{x}=a\dot{x}^4+2a\dot{x}^2\dot{y}^2+a\dot{y}^4$, where x is wanting. Assume $z\dot{y}=\dot{x}$, and we get $yz\dot{y}^4=az^4\dot{y}^4+2az^2\dot{y}^4$ $+a\dot{y}^4$, or $yz=az^4+2az^2+a$; hence, $y=az^3+2az$ $+\frac{a}{z}$, therefore $\dot{y}=3az^2\dot{z}+2a\dot{z}-\frac{a\dot{z}}{z^2}$, consequently $\dot{x}=z\dot{y}=3az^3\dot{z}+2az\dot{z}-\frac{a\dot{z}}{z}$, whose fluent is $x=\frac{3}{4}az^4$ $+az^2-a\times$ h. l. z; and if in this equation we substitute the value of z in terms of y, found from the equation $y=az^3+2az+\frac{a}{z}$, we shall get x in terms of y.

Prop. XCI.

In any fluxional equation of the second order, where the fluxion of one of the variable quantities ($\dot{x}$) *is constant, to transform it into one in which* $\dot{y}$ *is constant.*

159. Suppose the value of y to be expressed by

$a + bx + cx^2 + dx^3 +$ &c. then $\frac{\dot{y}}{\dot{x}} = b + 2cx + 3dx^2 +$ &c. Make $\dot{x}$ constant, and take the fluxion, and $\frac{\ddot{y}}{\dot{x}} = 2c\dot{x} + 6dx\dot{x} +$ &c. Now make $\dot{y}$ constant, and $\frac{-\dot{y}\ddot{x}}{\dot{x}^2} = 2c\dot{x} + 6dx\dot{x} +$ &c. when therefore $\dot{x}$ is constant, the value of $\frac{\ddot{y}}{\dot{x}}$ is the same as $\frac{-\dot{y}\ddot{x}}{\dot{x}^2}$ when $\dot{y}$ is constant.

Hence, we have the following

RULE 5.

If in any fluxional equation of the second order, in which $\dot{x}$ is constant, we substitute for $\frac{\ddot{y}}{\dot{x}}$ the quantity $\frac{-\dot{y}\ddot{x}}{\dot{x}^2}$, or for $\ddot{y}$ the quantity $\frac{-\dot{y}\ddot{x}}{\dot{x}}$, we shall transform the equation into one in which $\dot{y}$ is constant, and thus the fluent may be often found.

EXAMPLE.

Let $\dot{x}\dot{y} - x\ddot{y} - a\ddot{y} - \frac{x\dot{y}^2}{b} = 0$, which being supposed to have arisen from some fluent, $\dot{x}$ is constant, as $\ddot{x}$ does not enter. Substitute $\frac{-\dot{y}\ddot{x}}{\dot{x}}$ for $\ddot{y}$ (in which case $\dot{y}$ becomes constant), and we get $\dot{x}\dot{y} + x \times \frac{\dot{y}\ddot{x}}{\dot{x}} + a \times \frac{\dot{y}\ddot{x}}{\dot{x}} - \frac{x\dot{y}^2}{b} = 0$, or $\dot{x}^2 + x\ddot{x} + a\ddot{x} - \frac{x\dot{x}\dot{y}}{b} = 0$, whose fluent is $x\dot{x} + a\dot{x} - \frac{x^2\dot{y}}{2b}$, which, as the fluxion is $= 0$, must be equal to some constant quantity; let it be $c\dot{y}$*, and then $\dot{y}=\frac{2bx\dot{x}}{2bc+x^2}+\frac{2ab\dot{x}}{2bc+x^2}$, whose fluents (Art. 45. and 46.) are $y=b\times\text{L}+a\times\sqrt{\frac{2b}{c}}\times\text{A}$, where A is a circular arc whose radius is 1 and tangent $\frac{x}{\sqrt{2bc}}$, and $\text{L}=\text{h. l.}\ \overline{2bc+x^2}$.

RULE 6.

Sometimes the fluent may be found by assuming an equation with unknown coefficients, which put into fluxions shall give a fluxion of the same form as the given fluxion, and by equating the coefficients, the assumed coefficients may be found.

Let the fluent of $\frac{a\dot{x}+bx\dot{x}}{cx+x^2}$ be required. Assume $d\times\text{hyp. log.}\ \overline{cx^r+x^{r+1}}$ for the fluent, then the fluxion is $d\times\frac{rcx^{r-1}\dot{x}+\overline{r+1}\times x^r\dot{x}}{cx^r+x^{r+1}}=\frac{drc\dot{x}+d\times\overline{r+1}\times x\dot{x}}{cx+x^2}$ which we assume $=\frac{a\dot{x}+bx\dot{x}}{cx+x^2}$; hence, $drc=a$, $d\times\overline{r+1}=b$, therefore $r=\frac{a}{bc-a}$ and $d=\frac{bc-a}{c}$; and the required fluent is $\frac{bc-a}{c}\times\text{h. l.}\left(cx^{\frac{a}{bc-a}}+x^{\frac{bc}{bc-a}}\right)$.

If the fluent cannot be obtained by these means,

* The given fluxion being supposed to have arisen from some fluent, it is easy to conceive that this constant quantity must be such as $c\dot{y}$; because the equation, after taking the fluent the first time, arose from taking the fluxion of the fluential equation, and therefore $\dot{x}$ or $\dot{y}$ must necessarily enter into every term.

or any other artifices, it may be necessary to have recourse to infinite series (see Art. 111.) in order to express the fluent, in which case it will be very useful to attend to the following

RULE 7.

Let the quantity whose value is required be assumed equal to some unknown power, n, *of the other quantity, and let that power with its fluxion or fluxions be substituted for their supposed equals in the given equation.*

Let the least *exponents for an* ascending, *or* greatest *for a* descending *series, of the quantity thus substituted, be made equal to each other, and thence* n *will be found. Or if there happen to be only one or more terms having the least or greatest index, make the coefficient of that term or terms*=0, *and you get* n.

Substitute this value of n *for* n, *and take the difference between one of the equal exponents, and every other exponent of the same variable quantity.*

To these differences, write down all the least numbers which can be composed out of them by continual addition, either to themselves, or to one another, till you get as many terms as the required series is to be continued to.

Let each of these terms be increased *by* n *for an* ascending *series, and* decreased *by* n *for a* descending *series, and you have the required exponents.*

In equations where the higher order of fluxions are concerned, the series must be assumed in terms of that quantity which flows uniformly, and that is known by observing which quantity has no second, &c. fluxions.

Ex. 1. Let the equation be $a^2\dot{x}^2 + x^2\dot{z}^2 - a^2\dot{z}^2 = 0$, when z is a circular arc whose radius is a and sine x.

Assume z^n for x, then $nz^{n-1}\dot{z} = \dot{x}$, and by substitution, the equation becomes $a^2n^2z^{2n-2}\dot{z}^2 + z^{2n}\dot{z}^2 - a^2\dot{z}^2 = 0$, and the indices of z are $2n-2$, $2n$, and 0, for we conceive the last term $a^2\dot{z}^2$ to be $a^2z^0\dot{z}^2$; and putting the two least indices $2n-2$ and 0 equal, we get $n=1$; which substituted for n, the indices become 0, 2, 0, and the differences are 0, 2, and by adding 2 continually, we get the series 0, 2, 4, 6, &c. to which add n, or 1, and we get 1, 3, 5, 7, &c. for the indices. Assume therefore $x=pz+qz^3+rz^5+sz^7+$ &c. and putting $\dot{z}=1$ to shorten the operation, $\dot{x}=p+3qz^2+5rz^4+7sz^6$ + &c. and this squared and substituted into the given equation, we get

$$\left.\begin{array}{rrrrr} a^2p^2 & +\,6a^2pqz^2 & +\,10a^2prz^4 & +\,14a^2psz^6 & +\,\&c. \\ & & 9a^2q^2z^4 & +\,30a^2qrz^6 & +\,\&c. \\ + & p^2z^2\,+ & 2pqz^4\,+ & 2prz^6 & +\,\&c. \\ -a^2 & & + & q^2z^6 & +\,\&c. \end{array}\right\} = 0;$$

hence, (Art. 110.) $a^2p^2-a^2=0$, $6a^2pq+p^2=0$, $10a^2pr+9a^2q^2+2pq=0$, $14a^2ps+30a^2qr+2pr+q^2=0$; &c. and from the first, $p=1$; therefore $6a^2q+1=0$, and $q=-\frac{1}{6a^2}=-\frac{1}{2.3.a^2}$; hence, $10a^2r=-9a^2q^2-2q=-q\times\overline{9a^2q+2}=-q\times\overline{-\tfrac{3}{2}+2}=-\frac{q}{2}=\frac{1}{2.3.2a^2}$, therefore $r=\frac{1}{2.3.4.5a^4}$; also, $14a^2ps=-30a^2qr-2pr-q^2=-3a^2\times-\frac{1}{6a^2}\times\frac{1}{120a^4}-2\times\frac{1}{120a^4}-\frac{1}{36a^4}=\frac{1}{24a^4}-\frac{1}{60a^4}-\frac{1}{36a^4}=-\frac{1}{360a^4}$, therefore $s=-\frac{1}{14\times360a^6}=-\frac{1}{2.3.4.5.6.7a^6}$; hence, $x=z-\frac{z^3}{2.3a^2}+\frac{z^5}{2.3.4.5a^4}-\frac{z^7}{2.3.4.5.6.7a^6}+$ &c.

Ex. 2. Let $2ax\dot{y}^2-ay^2\ddot{x}+2x^2\dot{y}^2-2y^2\dot{x}^2=0$. Assume $x=y^n$, and $\dot{x}=ny^{n-1}\dot{y}$, and ($\dot{y}$ being constant) $\ddot{x}=n\,.\,\overline{n-1}\,.\,y^{n-2}\dot{y}^2$; therefore the equation becomes (omitting $\dot{y}^2$) $2ay^n - n\,.\,\overline{n-1}\,.\,ay^n + 2y^{2n} - 2n^2y^{2n} = 0$; here there is only one power of y having the least index, therefore we must assume $2a - n.\overline{n-1}.a = 0$, or $n\,.\,\overline{n-1} = 2$, and $n = 2$, and this is for an *ascending* series. Substitute this for n, and the indices become 2, 2, 4, 4; now the difference between one of the least indices 2, and the other indices is 0, 2, and by adding 2 continually, we get the series 0, 2, 4, 6, &c. and increasing these by n, or 2, we get 2, 4, 6, 8, &c. for the required coefficients, Assume, therefore, $x = py^2 + qy^4 + ry^6 + sy^8 +$ &c. then $\dot{x} = 2py + 4qy^3 + 6ry^5 + 8sy^7 +$ &c. (assuming $\dot{y} = 1$), and $\ddot{x} = 2p + 12qy^2 + 30ry^4 + 56sy^6 +$ &c. also $\dot{x}^2 = 4p^2y^2 + 16q^2y^6 + 16pqy^4 +$ &c. hence, by substitution, we get

$$\left.\begin{array}{r} 2apy^2 + 2aqy^4 + 2ary^6 + 2asy^8 + \&c. \\ -2apy^2 - 12aqy^4 - 30ary^6 - 56asy^8 + \&c. \\ +\ 2p^2y^4 + 4pqy^6 + 2q^2y^8 \\ +\ 4pry^8 + \&c. \\ -\ 8p^2y^4 - 32pqy^6 - 32q^2y^8 \\ -48pry^8 + \&c. \end{array}\right\} = 0;$$

hence, $2ap - 2ap = 0$; $2aq - 12aq + 2p^2 - 8p^2 = 0$; $2ar - 30ar + 4pq - 32pq = 0$; $2as - 56as + 2q^2 + 4pr - 32q^2 - 48pr = 0$; from the first equation it appears that p may be assumed at pleasure; from the second equation, $q = \frac{-3p^2}{5a}$; from the third, $r = \frac{3p^3}{5a^2}$; from the fourth, $s = \frac{-31p^4}{45a^3}$; &c. hence, $x = py^2 - \frac{3p^2}{5a}y^4 + \frac{3p^3}{5a^2}y^6 - \frac{31p^4}{45a^3}y^8$, &c.

For a *descending* series, we make the coefficients of the highest powers of $y = 0$, or $2 - 2n^2 = 0$, and $n = 1$; and the indices become 1, 1, 2, 2, and taking one of the greatest, 2, from all the rest, the remainders are -1 and 0, and by adding -1 continually, we get 0, -1, -2, -3, -4, &c. and these increased by n, or 1, give 1, 0, —1, —2, —3, &c.; hence, assume $x = py + q + ry^{-1} + sy^{-2} +$ &c. and we get, as before,

$$\left.\begin{array}{r} 2apy + 2aq + 2ary^{-1} + \&c. \\ -2ary^{-1} - \&c. \\ 2p^2y^2 + 4pqy + 2q^2 + 4qry^{-1} + \&c. \\ +4pr + 4psy^{-1} + \&c. \\ -2p^2y^2 \qquad\qquad +4pr + 8psy^{-1} + \&c. \end{array}\right\} = 0;$$

hence, $2p^2 - 2p^2 = 0$; $2ap + 4pq = 0$; $2aq + 2q^2 + 8pr = 0$; $4qr + 12ps = 0$; we may therefore assume p at pleasure, and then $q = -\frac{a}{2}$; $r = \frac{a^2}{16p}$; $s = \frac{a^3}{96p^2}$; &c. therefore $x = py - \frac{a}{2} + \frac{a^2}{16py} + \frac{a^3}{96p^2y^2} +$ &c.

Although this rule may become sometimes impracticable, yet when it can be applied, it never takes in any unnecessary terms.

SECTION XI.

ON THE SUMMATION OF SERIES.

PROP. XCII.

To find the sum of $1^n x + 2^n x^2 + 3^n x^3 + \&c. \ldots s^n x^s$.

160. Assume $x + x^2 + x^3 + \&c. \ldots x^s = \frac{x^{s+1} - x}{x - 1} = a$; take the fluxion of both sides, divide by $\dot{x}$, and multiply by x; repeat this operation, and you will raise the powers of the natural numbers an unit every time; hence,

$$1\,x + 2\,x^2 + 3\,x^3 + \&c. \ldots s\,x^s = \frac{x\dot{a}}{\dot{x}} = b;$$

$$1^2 x + 2^2 x^2 + 3^2 x^3 + \&c. \ldots s^2 x^s = \frac{x\dot{b}}{\dot{x}} = c;$$

$$1^3 x + 2^3 x^2 + 3^3 x^3 + \&c. \ldots s^3 x^s = \frac{x\dot{c}}{\dot{x}} = d;$$

Thus we may continue the operation to any power.

PROP. XCIII.

To find the sum of $1.2.3x + 2.3.4x^2 + 3.4.5x^3 + \&c. \ldots \overline{s-2}\,.\,\overline{s-1}\,.\,s x^{s-2}$.

161. Assume as before, take the fluxion, and divide by $\dot{x}$, repeat this operation till you have gotten the number of factors, and then multiply by x; hence,

$$1 + 2x + 3x^2 + 4x^3 + \&c. \ldots \ldots sx^{s-1} = \frac{\dot{a}}{\dot{x}} = b;$$

$$1.2+2.3x+3.4x^2+\&c.\ldots\overline{s-1}.sx^{s-2}=\frac{\dot{b}}{\dot{x}}=c;$$

$$1.2.3x+2.3.4x^2+\&c.\overline{s-2}.\overline{s-1}.sx^{s-2}=\frac{\dot{xc}}{\dot{x}}=d.$$

Prop. XCIV.

Given $ax^n+bx^{2n}+cx^{3n}+$ *&c.* $+mx^{vn}=A$; *to find* $\overline{p+n}\times\overline{q+n}\times ax^n+\overline{p+2n}\times\overline{q+2n}\times bx^{2n}+\&c.\ldots\ldots\overline{p+vn}\times\overline{q+vn}.mx^{vn}$.

162. Multiply the given equation by x^p, and $ax^{p+n}+bx^{p+2n}+\&c.=Ax^p=B$; take the fluxion and divide by $\dot{x}$, and $\overline{p+n}\times ax^{p+n-1}+\overline{p+2n}\times bx^{p+2n-1}+\&c.=\frac{\dot{B}}{\dot{x}}$; divide by x^{p-1}, and $\overline{p+n}\times ax^n+\overline{p+2n}\times bx^{2n}+\&c.=\frac{\dot{B}}{x^{p-1}\dot{x}}=C$. Now multiply this equation by x^q, take the fluxion, and divide by $x^{q-1}\dot{x}$, and we get $\overline{p+n}\times\overline{q+n}\times ax^n+\overline{p+2n}\times\overline{q+2n}\times bx^{2n}+\&c.=\frac{\dot{\overline{Cx^q}}}{x^{q-1}\dot{x}}$.

In this manner, any factors may be introduced, by multiplying by such powers of x as shall produce the factors required.

Prop. XCV.

Let the sum of $\frac{2x}{1}-\frac{4x^3}{3}+\frac{6x^5}{5}-$ *&c. ad infinitum be required.*

163. By Art. 54. Ex. 5. $\frac{x}{1}-\frac{x^3}{3}+\frac{x^5}{5}-\&c.=A$, A being an arc of a circle whose radius $=1$, tangent $=x$. Multiply by x, and $\frac{x^2}{1}-\frac{x^4}{3}+\frac{x^6}{5}-\&c.=Ax$; hence,

$\frac{2x}{1} - \frac{4x^3}{3} + \frac{6x^5}{5} -$ &c. $= \frac{A\dot{x}+x\dot{A}}{\dot{x}} =$ (because $\dot{A} = \frac{\dot{x}}{1+x^2}$ by Art. 46.) $A + \frac{x}{1+x^2}$.

If $x=1$, then $\frac{2}{1} - \frac{4}{3} + \frac{6}{5} -$ &c. $= A + \frac{1}{2}$.

Prop. XCVI.

To sum series by means of the fluent of $vx^n\dot{x}$, v being $= h.\ l.\ \frac{1}{1-x}$.

164. By Art. 153. the fluent of $vx^n\dot{x}$ is $\frac{vx^{n+1}}{n+1} + \frac{1}{n+1} \times \overline{\frac{x^{n+1}}{n+1} + \frac{x^n}{n} + \frac{x^{n-1}}{n-1} + \&c. - v} = v \times \overline{\frac{x^{n+1}}{n+1} - \frac{1}{n+1}} + \frac{x^{n+1}}{\overline{n+1} \times \overline{n+1}} + \frac{x^n}{\overline{n+1} \times n} + \frac{x^{n-1}}{\overline{n+1}.\overline{n-1}} +$ &c. But $v=$ hyp. log. $\frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 +$ &c. ad infinit. hence, $vx^n\dot{x} = x^{n+1}\dot{x} + \frac{1}{2}x^{n+2}\dot{x} + \frac{1}{3}x^{n+3}\dot{x} + \frac{1}{4}x^{n+4}\dot{x} +$ &c. whose fluent is $\frac{x^{n+2}}{n+2} + \frac{x^{n+3}}{2.\overline{n+3}} + \frac{x^{n+4}}{3.\overline{n+4}} + \frac{x^{n+5}}{4.\overline{n+5}}$ + &c. Make these two fluents equal, and we have $\frac{v}{n+1} \times \overline{x^{n+1}-1} + \frac{x^{n+1}}{\overline{n+1} \times \overline{n+1}} + \frac{x^n}{\overline{n+1} \times n} + \frac{x^{n-1}}{\overline{n+1} \times \overline{n-1}}$ +&c. to $n+1$ terms $= \frac{x^{n+2}}{n+2} + \frac{x^{n+3}}{2.\overline{n+3}} + \frac{x^{n+4}}{3.\overline{n+4}} +$ &c. ad infinitum.

165. If $n=0$, then $\frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{3.4} +$ &c. ad infinit. $= v \times \overline{x-1} + x$; hence, if $x = 1$, $\frac{1}{2} + \frac{1}{2.3} + \frac{1}{3.4} +$ &c. $= 1$.

166. Since $\frac{x^2}{1.2} + \frac{x^3}{2.3} + \frac{x^4}{3.4} + \text{\&c.} = vx - v + x$, multiply by $\dot{x}$, and $\frac{x^2\dot{x}}{1.2} + \frac{x^3\dot{x}}{2.3} + \frac{x^4\dot{x}}{3.4} + \text{\&c.} = vx\dot{x} - v\dot{x} + x\dot{x}$; now by Art. 153. the fluent of $vx\dot{x}$ is $\frac{1}{2}vx^2 - \frac{1}{2}v + \frac{1}{4}x^2 + \frac{1}{2}x$; also, the fluent of $v\dot{x}$ is $vx - v + x$; hence, the fluent of $vx\dot{x} - v\dot{x} + x\dot{x}$ is $\frac{1}{2}vx^2 - vx + \frac{1}{2}v - \frac{1}{2}x + \frac{3}{4}x^2$; consequently (B) $\frac{x^3}{1.2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{3.4.5} + \text{\&c.} = \frac{1}{2}vx^2 - vx + \frac{1}{2}v - \frac{1}{2}x + \frac{3}{4}x^2$. Assume $\frac{1}{2}vx^2 - vx + \frac{1}{2}v = 0$, or $x^2 - 2x + 1 = 0$; hence, $x = 1$; make $x = 1$, and $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \text{\&c.} = \frac{1}{4}$.

Let $x = \frac{1}{4}$, then $v = \text{h. l. } \frac{4}{3}$; hence, $\frac{1}{1.2.3} + \frac{1}{4^3} + \frac{1}{2.3.4} \times \frac{1}{4^4} + \text{\&c.} = \frac{9}{32} \times \text{h. l. } \frac{4}{3} - \frac{5}{64}$.

Let $x = \frac{1}{2}$, then $v = \text{h. l. } 2$; hence, $\frac{1}{1.2.3} \times \frac{1}{8} + \frac{1}{2.3.4} \times \frac{1}{16} + \text{\&c.} = \frac{1}{8} \times \text{h. l. } 2 - \frac{1}{16}$. Thus by assuming x and determining v from it, we may find the sum of the corresponding series.

In like manner, by multiplying B by $\dot{x}$ and taking the fluent, we shall get four factors in the denominator, 1.2.3.4, 2.3.4.5, &c. or if we multiply by $x\dot{x}$ and take the fluent, we shall get the factors 1.2.3.5, 2.3.4.6, &c. And, in like manner, we may add what factors we please, by multiplying by such a power of x as will produce that factor. If the Reader wish to see more instances, he may consult A. de MOIVRE's *Miscel. Anal.* Lib. VI.

Prop. XCVII.

To sum series from the fluent of $vx^n\dot{x}$, where v is a circular arc, whose radius is unity and tangent x.

167. By Art. 154. the fluent of $vx^n\dot{x}$ is $\frac{vx^{n+1}}{n+1} + \frac{1}{n+1} \times \overline{-\frac{x^n}{n}+\frac{x^{n-2}}{n-2}} - \&c. \mp v$, where the sign of v is $+$ or $-$, according as $\frac{n+1}{2}$ is odd or even, when n is an *odd* number. But (Art. 46.) $v=x-\frac{x^3}{3}+\frac{x^5}{5}-\&c.$ hence, $vx^n\dot{x}=x^{n+1}\dot{x}-\frac{x^{n+3}\dot{x}}{3}+\frac{x^{n+5}\dot{x}}{5}-\&c.$ whose fluent is $\frac{x^{n+2}}{n+2}-\frac{x^{n+4}}{3.\overline{n+4}}+\frac{x^{n+6}}{5.\overline{n+6}}-\&c.$ Make these fluents equal, and we have $\frac{1}{n+1} \times \overline{vx^{n+1} \mp v - \frac{x^n}{n} + \frac{x^{n-2}}{n-2} - \&c.} = \frac{x^{n+2}}{n+2} - \frac{x^{n+4}}{3.\overline{n+4}} + \frac{x^{n+6}}{5.\overline{n+6}} - \&c.$ ad infinitum.

Let $\frac{n+1}{2}$ be an *even* number, and assume $vx^{n+1} - v=0$, and then $x=1$; hence, $\frac{1}{n+1} \times \overline{-\frac{1}{n}+\frac{1}{n-2}-\&c.}$ to $\frac{n+1}{2}$ terms, is equal to $\frac{1}{n+2}-\frac{1}{3.\overline{n+4}}+\frac{1}{5.\overline{n+6}}-$ &c. ad infinitum.

If $n=3$, then $\frac{1}{1.5}-\frac{1}{3.7}+\frac{1}{5.9}-\&c.$ ad infinitum $=\frac{1}{4}\times\overline{-\frac{1}{3}+1}=\frac{1}{6}$.

Let $\frac{n+1}{2}$ be an *odd* number, and assume $n=1$,

$x = 1$; then v becomes an arc of 45°; and we get $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7}$ — &c. ad infinitum = arc 45° — $\frac{1}{2}$.

If n be an *even* number, then (Art. 154.) we get, in like manner,

$$\frac{1}{n+1} \times \overline{vx^{n+1} - \frac{x^n}{n} + \frac{x^{n-2}}{n-2} - \&c. \mp \text{h. l.} \sqrt{1+x^2}} = \frac{x^{n+2}}{n+2} - \frac{x^{n+4}}{3.\overline{n+4}} + \frac{x^{n+6}}{5.\overline{n+6}} - \&c. \text{ ad infinitum},$$

where the number of terms to be taken in the first series is $\frac{1}{2}n$, the first and last terms excepted, and the sign of the last term is + or —, according as $\frac{1}{2}n$ is odd or even.

If $n = 2$, and $x = 1$, then v becomes an arc of 45°; and we get $\frac{1}{1.4} - \frac{1}{3.6} + \frac{1}{5.8}$ — &c. ad infinitum = $\frac{1}{3} \times \overline{\text{arc } 45° - \frac{1}{2} + \text{h. l.} \sqrt{2}}$. For more upon this subject, see A. de Moivre's *Miscel. Anal.* Lib. VI.

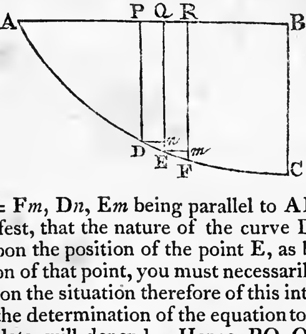
SECTION XII.

ON THE MAXIMA AND MINIMA OF CURVES.

PROP. XCVIII.

TO find the nature of curves, in which some quantities remaining invariable, others are the greatest or least possible.

168. Let ABC be any curvilinear area, PD, RF two fixed ordinates indefinitely near to each other, and the ordinate QE an arithmetic mean between them, so



that En = Fm, Dn, Em being parallel to AB. Now it is manifest, that the nature of the curve DEF must depend upon the position of the point E, as by varying the position of that point, you must necessarily vary the curve; upon the situation therefore of this intermediate ordinate, the determination of the equation to the curve, from the data, will depend. Hence, PQ, QR, are the only variable quantities.

169. Let any given quantity M be made up of A, B, C, D, E, &c. or let A + B + C + D + E + &c. = M, and at the same time let some other quantity m be

required to be a maximum or minimum, and let the corresponding parts of m be a, b, c, d, e, &c. and then will $a+b+c+d+e+$&c. $=m$, M and m being expressed in terms of the same variable quantities. Now let us suppose all the quantities in each to remain constant, except two which correspond, that is, let C and D, c and d be alone variable; then C+D is constant, and to satisfy the other condition, $c+d$ must be a maximum or minimum; hence, (Art. 21.), $\dot{C}+\dot{D}=0$, $\dot{c}+\dot{d}=0$, and from these two equations we may get the relation of the variable quantities which compose them, which will be found sufficient to determine the nature of the curve.

Prop. XCIX.

Given the points A *and* C, *to find the curve in which a body will descend from* A *to* C, *in the least time possible.*

170. Put $PD=m$, $QE=n$, $En=Fm=a$, the constant quantities, $v=PQ=Dn$, $w=QR=Em$; then $DE=\sqrt{a^2+v^2}$, and $EF=\sqrt{a^2+w^2}$. Now AB being parallel to the horizon, the velocities at D and E are as $\sqrt{m}$ and $\sqrt{n}$, by *Mechanics;* also, the times being as the spaces directly and velocities inversely, the times through DE, EF will be as $\frac{\sqrt{a^2+v^2}}{\sqrt{m}}$ and $\frac{\sqrt{a^2+w^2}}{\sqrt{n}}$; hence, as AB is given, v, w are two parts of this given quantity, whose sum $v+w$ is constant; also, $\frac{\sqrt{a^2+v^2}}{\sqrt{m}}$ and $\frac{\sqrt{a^2+w^2}}{\sqrt{n}}$ are the two corresponding parts of the minimum, whose sum $\frac{\sqrt{a^2+v^2}}{\sqrt{m}}+\frac{\sqrt{a^2+w^2}}{\sqrt{n}}=$ minimum (Art. 169.); hence, $\dot{v}+\dot{w}=0$,

and $\frac{v\dot{v}}{\sqrt{m} \times \sqrt{a^2+v^2}} + \frac{w\dot{w}}{\sqrt{n} \times \sqrt{a^2+w^2}} = 0$; $\therefore$ $\dot{w} = -\dot{v}$; consequently $\frac{v\dot{v}}{\sqrt{m} \times \sqrt{a^2+v^2}} - \frac{w\dot{v}}{\sqrt{n} \times \sqrt{a^2+w^2}} = 0$, and $\frac{v}{\sqrt{m} \times \sqrt{a^2+v^2}} = \frac{w}{\sqrt{n} \times \sqrt{a^2+w^2}}$; now these are two similar quantities, which express (in their ultimate state) the fluxion of the abscissa divided by the square root of the ordinate $\times$ fluxion of the curve; two successive values of this quantity therefore being equal to each other, shows the quantity itself to be constant; hence, put AP$=x$, PD $=y$, AD$=z$, and we have $\frac{\dot{x}}{\sqrt{y} \times \dot{z}} = \frac{1}{\sqrt{r}}$ a constant quantity, which is the property of a cycloid, the diameter of whose generating semicircle is r.

Prop. C.

To determine the nature of the curve AC, *whose length is given, when its area is a maximum.*

171. The same notation remaining, we have DE $+$ EF$= \sqrt{a^2+v^2} + \sqrt{a^2+w^2}$ a constant quantity, the sum of two parts of the given curve line AC; also, $mv + nw$ is the sum of the two corresponding parts of the maximum; hence (Art. 169.), $mv + nw =$ max. $\therefore$ $m\dot{v} + n\dot{w} = 0$, and $\frac{v\dot{v}}{\sqrt{a^2+v^2}} + \frac{w\dot{w}}{\sqrt{a^2+w^2}} = 0$; hence, $\frac{m\dot{v}}{n} = -\dot{w}$, therefore $\frac{v\dot{v}}{\sqrt{a^2+v^2}} - \frac{mw\dot{v}}{n\sqrt{a^2+w^2}} = 0$, consequently $\frac{v}{m\sqrt{a^2+v^2}} = \frac{w}{n\sqrt{a^2+w^2}}$; which being

similar quantities, we have $\frac{\dot{x}}{y\dot{z}} = \frac{1}{r}$ a constant quantity, or $r\dot{x} = y\dot{z}$ the equation of a circle by Art. 46.

Prop. CI.

Let the surface of the solid generated by the revolution of the curve AC *about* AB *be given; to find the nature of the curve, when the solid is a maximum.*

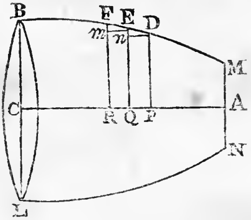
172. Put $p = 3{,}14159$, &c. then (Art. 56.) $2pm \times \sqrt{a^2+v^2} + 2pn\sqrt{a^2+w^2} =$ the sum of the two parts of the given surface generated by DE + EF, a constant quantity; also, $pm^2v + pn^2w =$ the sum of the two corresponding parts of the maximum, generated by PQED, QRFE; hence, $pm^2v + pn^2w =$ max. $\therefore$ (neglecting the constant multiplier p) $m^2\dot{v} + n^2\dot{w} = 0$, and $\frac{mv\dot{v}}{\sqrt{a^2+v^2}} + \frac{nw\dot{w}}{\sqrt{a^2+w^2}} = 0$; hence, $\dot{w} = -\frac{m^2\dot{v}}{n^2}$, which substituted for $\dot{w}$ in the second equation, we get $\frac{v}{m\sqrt{a^2+v^2}} = \frac{w}{n\sqrt{a^2+w^2}}$, which are the same quantities as in the last case; hence, the curve is a circle.

Prop. CII.

To find the nature of the curve which generates a solid of the least resistance, when moving in a fluid in the direction of its axis, its greatest diameter BL *and length* AC *being given.*

173. By the Principles of Hydrostatics, the resistance against DE is as $\frac{ma^3}{a^2+v^2}$, and against EF as $\frac{na^3}{a^2+w^2}$; hence, the sum of the two parts of the quantity which is to be a minimum $= \frac{ma^3}{a^2+v^2} + \frac{na^3}{a^2+w^2}$; also, as AC is given, $v + w$, the sum of the two corresponding parts

of the given quantity, is constant; therefore $-\frac{2ma^3v\dot{v}}{\overline{a^2+v^2}^2}$



$-\frac{2na^3w\dot{w}}{\overline{a^2+w^2}^2}=0$, and $\dot{v}+\dot{w}=0$; hence, $\dot{v}=-\dot{w}$; consequently, by substitution, $\frac{ma^3v}{\overline{a^2+v^2}^2}=\frac{na^3w}{\overline{a^2+w^2}^2}$, which being similar quantities, we have $\frac{y\dot{y}^3\dot{x}}{\dot{z}^4}=r$, a given quantity, which is the fluxional equation of the curve.

That the curve does not meet the axis at A, appears from hence; $y=r\times\frac{\dot{z}^4}{\dot{y}^3\dot{x}}=r\times\frac{ED^4}{En^3\times Dn}$, where the numerator must evidently be greater than the denominator, and therefore y must be greater than r.

174. If the greatest diameter BL, and *area* BMNL be given, then $mv+nw$ will be given, consequently $m\dot{v}+n\dot{w}=0$, which gives $\frac{\dot{y}^3\dot{x}}{\dot{z}^4}=r$, the equation of the curve.

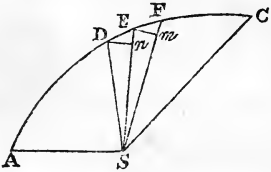
If the greatest diameter and *bulk* be given, then instead of $v+w$ being given, pm^2v+pn^2w will be given (Art. 169.); hence, $m^2\dot{v}+n^2\dot{w}=0$, which gives $\frac{\dot{y}^3\dot{x}}{y\dot{z}^4}=r$, the equation of the curve.

Although PDEQ, QEFR are here taken as increments, yet we reason upon them as fluxions, conceiving their limiting ratio to be taken, and consequently the conclusions are mathematically true.

Prop. CIII.

To find the nature of the curve AC, *so that a body may move from* A *to* C *in the least time possible, the velocity at any point* D *being as* DS^r, S *being any fixed point.*

175. Let DS, FS, be two given distances including a given angle DSF, draw SE, and Dn perpendicular



to SE, and Em to SF, and let En = mF. Put SD = m, SE = n, En = Fm = a, the constant quantities, Dn = v, Em = w, the variable quantities; then DE = $\sqrt{a^2+v^2}$ and EF = $\sqrt{a^2+w^2}$; and the time of describing DE = $\frac{\sqrt{a^2+v^2}}{m^r}$, and of EF = $\frac{\sqrt{a^2+w^2}}{n^r}$; hence, $\frac{\sqrt{a^2+v^2}}{m^r}+\frac{\sqrt{a^2+w^2}}{n^r}$ = max. and its fluxion $\frac{v\dot{v}}{m^r\sqrt{a^2+v^2}}+\frac{w\dot{w}}{n^r\sqrt{a^2+w^2}}=0$; but the $\angle$DSE is measured by $\frac{v}{m}$, and $\angle$ESF by $\frac{w}{n}$; therefore $\frac{v}{m}+\frac{w}{n}$ = $\angle$DSF, and $\frac{\dot{v}}{m}+\frac{\dot{w}}{n}=0$; hence, $\dot{w}=\frac{-n\dot{v}}{m}$, therefore $\frac{v\dot{v}}{m^r\sqrt{a^2+v^2}}-\frac{nw\dot{v}}{mn^r\sqrt{a^2+w^2}}=0$, and

$\frac{v}{m^{r-1}\sqrt{a^2+v^2}} = \frac{w}{n^{r-1}\sqrt{a^2+w^2}}$; that is, if SD $= x$, AD $= z$, D$n = \dot{y}$, then $\frac{\dot{y}}{x^{r-1}\dot{z}} = \frac{1}{c^{r-1}}$, a constant quantity.

If $r = 0$, AC is a straight line.

If $r = 1$, then $\frac{\dot{y}}{\dot{z}}$ is constant, and the curve is the log. spiral.

If $r = 2$, then $c\dot{y} = x\dot{z}$, and the curve is a circle.

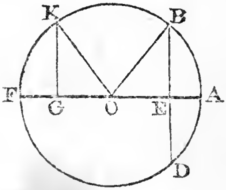
SECTION XIII.

MISCELLANEOUS PROPOSITIONS.

Prop. CIV.

GIVEN the sine EB *of an arc* AB *of a circle; to find the sine of* n *times* AB.

176. Let AB $=z$, and AK $=nz$; put OB $=1$, $y=$ OE the cosine of AB, $v=$ the sine BE $=\sqrt{1^2-y^2}$, $x=$ the cosine OG of AK, then $\sqrt{1^2-x^2}=$ the sine GK



of AK. Now (Art. 46.) $\dot{z} : -\dot{y} :: 1 : \sqrt{1^2-y^2}$, $\therefore \dot{z}= \frac{-\dot{y}}{\sqrt{1^2-y^2}}$; for the same reason, the fluxion of nz, or $n\dot{z}= \frac{-\dot{x}}{\sqrt{1^2-x^2}}$; hence, $\frac{\dot{x}}{\sqrt{1^2-x^2}} = \frac{n\dot{y}}{\sqrt{1^2-y^2}}$; multiply both denominators by $\sqrt{-1}$, and $\frac{\dot{x}}{\sqrt{x^2-1^2}} =$

$\frac{n\dot{y}}{\sqrt{y^2-1^2}}$, whose fluent (Art. 45.) is h. l. $\overline{x+\sqrt{x^2-1^2}}$

$=n\times$ h. l. $\overline{y+\sqrt{y^2-1^2}}$; hence, (Art. 109.) $x+\sqrt{x^2-1^2}=\overline{y+\sqrt{y^2-1^2}}\Big]^n=$ (Art. 34.) $y^n+ny^{n-1}\sqrt{y^2-1}+n.\frac{n-1}{2}.y^{n-2}\times\overline{y^2-1}+n.\frac{n-1}{2}.\frac{n-2}{3}y^{n-3}\times\sqrt{y^2-1}\times\overline{y^2-1}+$ &c. Now as this equation consists of quantities, partly possible and partly impossible, $\sqrt{x^2-1}$ and $\sqrt{y^2-1}$ being impossible, it is manifest, that the possible and impossible parts must be respectively equal. Hence, assuming the *impossible* parts equal, we have, $\sqrt{x^2-1}=ny^{n-1}\sqrt{y^2-1}+n.\frac{n-1}{2}.\frac{n-2}{3}y^{n-3}\times\sqrt{y^2-1}\times\overline{y^2-1}+$&c. Multiply both sides by $\sqrt{-1}$, and $\sqrt{1-x^2}=ny^{n-1}\sqrt{1-y^2}+n.\frac{n-1}{2}.\frac{n-2}{3}y^{n-3}\times\sqrt{1-y^2}\times\overline{y^2-1}+$ &c. $=$ (because $v=\sqrt{1-y^2}$, and $-v^2=y^2-1$) $ny^{n-1}v-n.\frac{n-1}{2}.\frac{n-2}{3}y^{n-3}v^3+$&c. the sine of AK.

PROP. CV.

Given as before, to find the cosine of AK.

177. Assume the *possible* parts of the above equation equal, and we have $x=y^n+n.\frac{n-1}{2}y^{n-2}\times\overline{y^2-1}+$&c. $=y^n-n.\frac{n-1}{2}y^{n-2}v^2+$&c. the cosine of AK.

Prop. CVI.

Given as before, to find the tangent of AK.

178. Put t = tangent of AB, then by Plane Trig. $t=\frac{v}{y}$, radius being unity; hence, the tangent of AK=

$$\frac{\text{sin. AK}}{\text{cos. AK}}=\frac{ny^{n-1}v-n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot y^{n-3}v^3+\&c.}{y^n-n\cdot\frac{n-1}{2}y^{n-2}v^2+\&c.}=$$ (by dividing the numerator and denominator by y^n)

$$\frac{\frac{nv}{y}-n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{v^3}{y^3}+\&c.}{1-n.\frac{n-1}{2}.\frac{v^2}{y^2}+\&c.}=\frac{nt-n.\frac{n-1}{2}.\frac{n-2}{3}.t^3+\&c.}{1-n.\frac{n-1}{2}.t^2+\&c.}$$

Prop. CVII.

To resolve $v^{2n}-2xv^n+1=0$, *into its quadratic divisors, the limits of* x *being* $+1$ *and* -1.

179. Retaining the notation in Art. 176, we have $x+\sqrt{x^2-1^2}=\overline{y+\sqrt{y^2-1^2}}\,\big|^n$. Put $v=y+\sqrt{y^2-1^2}$; transpose y and square both sides, and we get $v^2-2yv=-1^2$, $\therefore$ $v^2-2yv+1^2=0$. Also, $v^n=x+\sqrt{x^2-1^2}$; hence, by transposing x, and proceeding as before, we get $v^{2n}-2xv^n+1=0$, the given equation, of which we have one quadratic divisor $v^2-2yv+1^2=0$, v being the same in both equations. Now if to the arc AK, we add 360°, 2 × 360°, &c. we shall come again to the same point K, and consequently we shall have the same cosine, or x; hence, x is the cosine of AK, 360° + AK, 2 × 360° + AK, &c. But y is the cosine of an n^{th} part of that arc whose cosine is x; hence, y is the cosine of $\frac{\text{AK}}{n}$, $\frac{360°+\text{AK}}{n}$, $\frac{2\times360°+\text{AK}}{n}$,

&c. which cosines call a, b, c, &c. substitute therefore these values for y in the equation $v^2 - 2yv + 1^2 = 0$, and we get $v^2 - 2av + 1^2 = 0$, $v^2 - 2bv + 1^2 = 0$, $v^2 - 2cv + 1^2 = 0$, &c. for the quadratic divisors required; hence, $\overline{v^2 - 2av + 1^2} \times \overline{v^2 - 2bv + 1^2} \times$ &c. $= v^{2n} - 2xv^n + 1^{2n}$, retaining the power of the radius in the last term. Although there are an infinite number of arcs whose cosines are x, and consequently an infinite number of corresponding values of y, yet there are only n *different* values of y; because, after taking n arcs, $\frac{AK}{n}$, $\frac{360+AK}{n}$, &c. the same cosines will return again.

If $x = \pm 1$, or if AK be taken equal to the whole circumference, or half the circumference, the equation becomes $v^{2n} \mp 2v^n + 1 = 0$, whose square root is $v^n \mp 1 = 0$; now as every equation which is a square, must have to every root another equal to it, the equation $v^n \mp 1 = 0$ must contain the same roots as $v^{2n} \mp 2v^n + 1 = 0$; the roots therefore of $v^n \mp 1 = 0$ are found in like manner.

180. Hence, we may find the quadratic divisors of $v^{2n} - 2xr^n v^n + r^{2n} = 0$, which is the equation $v^{2n} - 2xv^n + 1 = 0$, having its roots multiplied by r (*Alg.* Art. 282.); multiplying the roots therefore of the above quadratics by r, we have $v^2 - 2arv + r^2 = 0$, $v_2 - 2brv + r_2 = 0$, &c. for the quadratics required. If $AK = 90°$, then $x = 0$, and the equation becomes $v^{2n} + r^{2n} = 0$.

PROP. CVIII.

To resolve $\frac{1}{1 - 2xv^n + v^{2n}}$ *into* $\frac{P - Qv}{1 - 2av + v^2} + \frac{R - Sv}{1 - 2bv + v^2} +$ &c. x *being the same as in the last proposition.*

181. Let the roots of $1-2xv^n+v^{2n}=0$, be $\frac{1}{m}, \frac{1}{p}, \frac{1}{q}$, &c. then as this is a recurring equation (*Alg.* Art. 289), the corresponding roots will be m, p, q, &c. Assume $\frac{1}{1-2xv^n+v^{2n}}=\frac{A}{1-mv}+\frac{B}{1-pv}+\frac{C}{1-qv}+$ &c. then reducing these to a common denominator, we have $A\times\overline{1-pv}\times\overline{1-qv}\times$ &c. $+B\times\overline{1-mv}\times\overline{1-qv}\times$ &c. + &c. = 1; let $1-mv=0$, then $v=\frac{1}{m}$; hence, $A\times\overline{1-\frac{p}{m}}\times\overline{1-\frac{q}{m}}\times$ &c.$=1$, or $A\times\overline{\frac{m-p}{m}}\times\overline{\frac{m-q}{m}}\times$ &c.$=1$; or if $w=\overline{m-p}\times\overline{m-q}\times$ &c. then $A\times\frac{w}{m^{2n-1}}=1$; hence, $A=\frac{m^{2n-1}}{w}$. In like manner we find B, C, &c. by making $1-pv=0$, $1-qv=0$, &c. Now as $1-2xv^n+v^{2n}=\overline{v-m}\times\overline{v-p}\times\overline{v-q}\times$ &c. take the fluxion, omitting $\dot{v}$, and $2nv^{2n-1}-2nxv^{n-1}=\overline{v-p}\times\overline{v-q}\times$ &c. $+\overline{v-m}\times\overline{v-q}\times$ &c. + &c. now let $v=m$, and it becomes $2nm^{2n-1}-2nxm^{n-1}=\overline{m-p}\times\overline{m-q}\times$ &c.$=w$; hence, $A\left(=\frac{m^{2n-1}}{w}\right)=\frac{m^{2n-1}}{2nm^{2n-1}-2nxm^{n-1}}=\frac{m^n}{2nm^n-2nx}$. For the same reason, $B=\frac{p^n}{2np^n-2nx}$, &c. Now $\frac{A}{1-mv}+\frac{B}{1-pv}=\frac{A+B-\overline{pA+mB}\times v}{1-2av+v^2}$; and as $1-2av+v^2=\overline{1-mv}\times\overline{1-pv}=1-\overline{m+p}\times v+mpv^2$, we have $m+p=2a$, and $mp=1$. Also, $A=\frac{m^n}{2nm^n-2nx}$, $B=\frac{p^n}{2np^n-2nx}$; hence, $A+B=$

$\frac{4nm^np^n - 2xn\times\overline{m^n+p^n}}{4n^2m^np^n - 4n^2x\times\overline{m^n+p^n} + 4n^2x^2}$. But $v^{2n} - 2xv^n + 1$ $=0$, therefore $v^n + \frac{1}{v^n} = 2x$; now for v substitute m, and $m^n + \frac{1}{m^n} = 2x$; but $mp=1$, and $p = \frac{1}{m}$; hence, $m^n + p^n =$ $2x$; consequently $A + B = \frac{4n - 4nx^2}{4n^2 - 8n^2x^2 + 4n^2x^2} =$ $\frac{4n\times\overline{1-x^2}}{4n^2\times\overline{1-x^2}} = \frac{1}{n}$. Also, $pA + mB = \frac{pm^n}{2nm^n - 2nx} +$ $\frac{mp^n}{2np^n - 2nx} =$

$$\frac{2nm\times m^np^n + 2np\times m^np^n - 2nxp\times m^n - 2nxm\times p^n}{4n^2\times\overline{1-x^2}}$$

(the common denominator being the same as in the value of $A + B$) $=$ (as $pm = 1$, $m + p = 2a$)

$$\frac{2n\times 2a - 2nxpm\times m^{n-1} - 2nxpm\times p^{n-1}}{4n^2\times\overline{1-x^2}} =$$

$\frac{4na - 2nx\times\overline{m^{n-1}+p^{n-1}}}{4n^2\times\overline{1-x^2}}$. Now $m^n + p^n = 2x$, where x is the cosine of an arc which is to the arc whose cosine is a, as $n : 1$; for the same reason $m^{n-1} +$ $p^{n-1} = 2e$, if e be the cosine of an arc which is to the arc whose cosine is a, as $n - 1 : 1$; therefore $pA +$ $mB = \frac{4na - 2nx\times 2e}{4n^2\times\overline{1-x^2}} = \frac{a - ex}{n - nx^2}$. Hence, $\frac{A}{1 - mv} +$ $\frac{B}{1 - pv} = \frac{\frac{1}{n} - \frac{a - ex}{n - nx^2}\times v}{1 - 2av + v^2}$. Consequently $\frac{1}{1 - 2xv^n + v^{2n}}$

$$= \frac{\frac{1}{n} + \frac{a-ex}{n-nx^2}\times v}{1-2av+v^2} + \frac{\frac{1}{n} + \frac{b-fx}{n-nx^2}\times v}{1-2bv+v^2} + \&c.$$ where f is found from b, in the same manner that e is found from a; and so on.

182. If x be negative, the given quantity becomes $\frac{1}{1+2xv^n+v^{2n}}$.

183. In like manner, $\frac{1}{1 \pm v^n}$ will be found equal to $\frac{A}{1-mv} + \frac{B}{1-pv}$ +&c. where $A=\frac{1}{n}$, $B=\frac{1}{n}$, &c. and if $\overline{1-mv} \times \overline{1-pv} = 1-2av+v^2$, then $\frac{A}{1-mv} + \frac{B}{1-pv}$

$$= \frac{\frac{2}{n} - \frac{2a}{n}\times v}{1-2av+v^2};$$ and so on; n being an *even* number.

If n be an *odd* number, then of the equation $1+v^n=0$; one root $=-1$; hence, $1+v=0$ is one of the simple equations; and as the other part is made up of quadratics, we have $\frac{1}{1+v^n} = \frac{\frac{2}{n} - \frac{2a}{n}\times v}{1-2av+v^2} + \&c. + \frac{\frac{1}{n}}{1+v}$.

If n be an *odd* number, the equation $1-v^n=0$ contains one simple equation, and $\frac{n-1}{2}$ quadratics. Now the equation $1-v^n=0$, has one root $=1$, consequently the simple equation is $1-v=0$. Hence,

$$\frac{1}{1-v^n} = \frac{\frac{2}{n} - \frac{2a}{n}\times v}{1-2av+v^2} + \&c. + \frac{\frac{1}{n}}{1-v}.$$

If n be an *even* number, $1-v^n=0$ has two roots,

$-1, +1$; therefore two of the simple equations will be $1-v=0,\ 1+v=0$; hence, $\frac{1}{1-v^n}=\frac{\frac{2}{n}-\frac{2a}{n}\times v}{1-2av+v^2}$ $+\ \&c. + \frac{\frac{1}{n}}{1-v}+\frac{\frac{1}{n}}{1+v}$.

Prop. CIX.

Let $\dot{F}=\frac{\dot{v}}{1-2xv^n+v^{2n}}$, *to find* F, *x being constant, and the same as in the last proposition.*

184. Retaining every thing as in Art. 181. we have $\dot{F}=\frac{\frac{1}{n}\dot{v}-\frac{a-ex}{n-nx^2}v\dot{v}}{1+2av+v^2}+\frac{\frac{1}{n}\dot{v}-\frac{b-fx}{n-nx^2}v\dot{v}}{1-2bv+v^2}+\&c.$ the fluent of each of which quantities is found as in Art. 139.

Prop. CX.

Let $\dot{F}=\frac{\dot{v}}{1+v^n}$, n *being an* even *number, to find* F.

185. By Art. 183. $\dot{F}=\frac{\frac{2}{n}\dot{v}-\frac{2a}{n}v\dot{v}}{1-2av+v^2}+\frac{\frac{2}{n}\dot{v}-\frac{2b}{n}v\dot{v}}{1-2bv+v^2}$ $+\ \&c.$ whose fluents are found by Art. 139.

If n be an *odd* number, then $\dot{F}=\frac{\frac{2}{n}\dot{v}-\frac{2a}{n}v\dot{v}}{1-2av+v^2}$ $+\ \&c. + \frac{\frac{1}{n}\dot{v}}{1+v}$, whose fluents are found by Art. 139, and 45.

Prop. CXI.

Let $\dot{F} = \frac{\dot{v}}{1 - v^n}$, n *being an* even *number, to find* F.

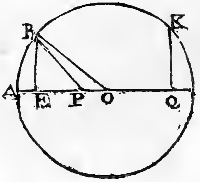
186. By Art. 183. $\dot{F} = \frac{\frac{2}{n}\dot{v} - \frac{2a}{n}v\dot{v}}{1 - 2av + v^2} + \&c. + \frac{\frac{1}{n}\dot{v}}{1 - v}$ $+ \frac{\frac{1}{n}\dot{v}}{1 + v}$, whose fluents are found by Art. 139. and 45.

If n be an *odd* number, we have $\dot{F} = \frac{\frac{2}{n}\dot{v} - \frac{2a}{n}v\dot{v}}{1 - 2av + v^2} + \&c. + \frac{\frac{1}{n}\dot{v}}{1 + v}$, whose fluents are found by Art. 139. and 45.

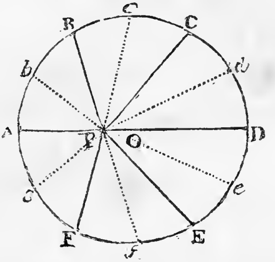
Prop. CXII.

To demonstrate Cotes's *properties of the circle.*

187. Retaining every thing as in Art. 179. we have $v^{2n} - 2xv^n + 1^{2n} = 0$, of which $v^2 - 2yv + 1^2 = 0$ is a quadratic divisor. Assume any point P, and draw PB, and put $v = PO$; then $BO^2 = BP^2 + PO^2 + 2PO \times PE$; that is, $1^2 = BP^2 + v^2 + 2v \times \overline{y - v} = BP^2 - v^2 + 2yv$; hence, $BP^2 = v^2 - 2yv + 1^2$. Also, y is the cosine of $\frac{AK}{n}, \frac{360° + AK}{n}, \frac{2 \times 360° + AK}{n}$, &c. whose cosines are a, b, c, &c. and $\overline{v^2 - 2av + 1^2} \times \overline{v^2 - 2bv + 1^2}$



$\times$&c.$=v^{2n}-2xv^{n}+1^{2n}$. Now let AK be the whole circumference C, then the above arcs are $\frac{C}{n}$, $\frac{2C}{n}$, $\frac{3C}{n}$, &c.



or the $\frac{1}{n}$, $\frac{2}{n}$, $\frac{3}{n}$, &c. parts of C; that is, if the whole circumference C be divided from A into n = parts at B, C, D, &c. then the cosines of the arcs AB, AC, AD, &c. are a, b, c, &c. and $x=1$; hence, from what we have already proved, $PB^2=v^2-2av+1^2$, $PC^2=v^2-2bv+1^2$, $PD^2=v^2-2cv+1^2$, &c. consequently $PB^2\times PC^2\times PD^2\times$&c. $=v^{2n}-2v^n+1^{2n}$; hence, by taking the square root, we get $PB\times PC\times PD\times$ &c. $=v^n-1^n$, or $1^n-v^n=PO^n-AO^n$, or AO^n-PO^n, according as PO or AO is the greater, or according as P is without or within the circle, for every thing holds the same whether P be within or without. This is one of the properties of the circle.

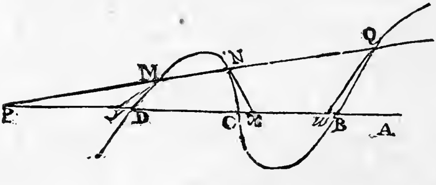
188. Let these divisions be again divided into two equal parts at b, c, d, &c. then the whole circumference will be divided into $2n$ equal parts, and therefore from what is already proved, $Pb\times PB\times Pc\times PC\times Pd\times PD\times$&c. $=AO^{2n}-PO^{2n}$, taking P within, for instance; divide this by the above equation, and we get, $\frac{Pb\times PB\times Pc\times PC\times Pd\times PD\times \&c.}{PB\times PC\times PD\times \&c.}=$

$\frac{AO^{2n} - PO^{2n}}{AO^n - PO^n}$; that is, $Pb \times Pc \times Pd \times$ &c. $= AO^n + PO^n$, which is the other property.

Prop. CXIII.

Let AP *be the abscissa of any curve*, PMNQ *an ordinate revolving about any fixed point* P, *and cutting the curve in as many points as it has dimensions; and draw the tangents* My, Nx, Qw, &c. *then will* $\frac{1}{Py} + \frac{1}{Px} + \frac{1}{Pw} +$ &c. (*the sum of the reciprocal subtangents*) *be a constant quantity.*

189. Let the equation of the curve be $y^n - \overline{a' + b'x} \times y^{n-1} +$ &c. $+ px^n - qx^{n-1} +$ &c. $= 0$; and corresponding to AP the abscissa (x), let a, b, c, &c. be the values of y; then, by the *Elements of Algebra*, Art. 267. $a \times b \times c \times$ &c. $= px^n - qx^{n-1} +$ &c. take the fluxion



of each side, and $\dot{a}bc$ &c. $+ \dot{b}ac$ &c. $+ \dot{c}ab$ &c. $+$ &c. $= npx^{n-1}\dot{x} - \overline{n-1} \times qx^{n-2}\dot{x} +$ &c. divide this latter equation by the former, and we have $\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}$ $+$ &c. $= \frac{npx^{n-1}\dot{x} - \overline{n-1} \times qx^{n-2}\dot{x} + \text{\&c.}}{px^n - qx^{n-1} + \text{\&c.}}$; hence, $\frac{\dot{a}}{a\dot{x}}$ $+ \frac{\dot{b}}{b\dot{x}} + \frac{\dot{c}}{c\dot{x}} +$ &c. $= \frac{npx^{n-1} - \overline{n-1} \times qx^{n-2} + \text{\&c.}}{px^n - qx^{n-1} + \text{\&c.}}$; but

Art. 23.) $\frac{\dot{a}}{a\dot{x}}, \frac{\dot{b}}{b\dot{x}}, \frac{\dot{c}}{c\dot{x}}$, &c. are the reciprocals of the subtangents Py, Px, Pw, &c.; hence, (dividing the numerator and denominator on the right hand side of the equation by p, which will not alter its value)

$$\frac{nx^{n-1} - \overline{n-1} \times \frac{q}{p} x^{n-2} + \&c.}{x^n - \frac{q}{p} x^{n-1} + \&c.} = \frac{1}{Py} + \frac{1}{Px} + \frac{1}{Pw} + \&c.$$

But by the *Algebra*, Art. 525, the roots of the equation $x^n - \frac{q}{p} x^{n-1} + \&c. = 0$ are AB, AC, AD, &c. whatever be the angle at P; hence, (*Algebra*, Art. 267.), the coefficients of $x^n - \frac{q}{p} x^{n-1} + \&c.$ are constant; and if P be assumed a fixed point, x is invariable; hence, $x^n - \frac{q}{p} x^{n-1} + \&c.$ is constant, and $nx^{n-1} - \overline{n-1} . \frac{q}{p} x^{n-2} + \&c.$ is constant; therefore the sum of the reciprocal subtangents is a constant quantity.

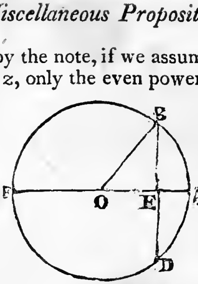
Prop. CXIV.

Given the arc of a circle; to find its sine and cosine.

190. Put the radius OA $= r$, the arc AB $= z$, its sine BE $= x$, cosine OE $= y$, and produce BE to D; then (Art. 46.) $\dot{z} : -\dot{y} :: r : x = \frac{-r\dot{y}}{\dot{z}}$. Now corresponding to the same value OE of y, z may be either AB or AD; but the arc beginning at A, if we consider AB as positive, AD will be negative, therefore every positive value of z* has a negative value equal

* If every positive value of z have a negative value equal to it, the equation whose roots are those values of z, will have only the

to it; hence, by the note, if we assume y in a series of the powers of z, only the even powers of z will enter.



Assume therefore $y = r + az^2 + bz^4 + cz^6 + \&c.$ the first term being r, because when $z = 0$, $y = r$; hence, $\dot{y} = 2az\dot{z} + 4bz^3\dot{z} + 6cz^5\dot{z} + \&c.$ therefore $x \left(= \frac{-r\dot{y}}{\dot{z}}\right) = -2raz - 4rbz^3 - 6rcz^5 - \&c.$ and $\dot{x} = -2ra\dot{z} - 3.4rbz^2\dot{z} - 5.6rcz^4\dot{z} - \&c.$ But (Art. 46.) $\dot{z} : \dot{x} :: r : y$; hence, $y\dot{z} = r\dot{x}$, and $y\dot{z} - r\dot{x} = 0$; now in this equation, instead of y and $\dot{x}$ substitute their values above found and we have

$$\left.\begin{array}{l} r\dot{z} + \quad az^2\dot{z} + \quad bz^4\dot{z} + \&c. \\ 2r^2a\dot{z} + 3.4r^2bz^2\dot{z} + 5.6r^2cz^4\dot{z} + \&c. \end{array}\right\} = 0;$$

hence, (Art. 110.) $2r^2a + r = 0$, $3.4r^2b + a = 0$, $5.6r^2c + b = 0$, &c. consequently $a = \frac{-1}{2r}$; $b = \frac{-a}{3.4r^2} = \frac{1}{2.3.4r^3}$; $c = \frac{-b}{5.6r^2} = \frac{-1}{2.3.4.5.6r^5}$; &c. hence, $y = r - \frac{z^2}{2r} + \frac{z^4}{2.3.4r^3} - \frac{z^6}{2.3.4.5.6r^5} + \&c.$ Also, $-2ra = 1$; $-4rb = \frac{-1}{2.3r^2}$; $-6rc = \frac{1}{2.3.4.5r^4}$; &c. hence, $x = z - \frac{z^3}{2.3r^2} + \frac{z^5}{2.3.4.5r^4} - \&c.$

even powers of z; for if $z=a$, $z=-a$, then $z-a=0$, $z+a=0$, and consequently the quadratic from these two will be $z^2-a^2=0$; and as every such pair of roots will form a similar quadratic, it is manifest, that the equation formed by the multiplication of these quadratics, will contain only the even power of z.

Prop. CXV.

To find the sum of the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+$ &c. *ad infinitum.*

191. Put the radius AO $=1$, EB $=x$, AB $=z$; then (Art. 190.) $x=z-\frac{z^3}{2.3}+\frac{z^5}{2.3.4.5}-$ &c. Let $x=0$, and then $z-\frac{z^3}{2.3}+\frac{z^5}{2.3.4.5}-$ &c. $=0$, or $1-\frac{z^2}{2.3}+\frac{z^4}{2.3.4.5}-$ &c. $=0$, the former equation containing one root $=0$, it being divisible by z, or $z-0$, (*Elem. Alg.* Art. 266.), which is taken away by dividing by z. But if $c=$ the semi-circumference of the circle, the other values of z, corresponding to $x=0$, will be $1c$, $2c$, $3c$, &c. ad infinitum, and by taking the arcs in a contrary direction, they will be $-1c$, $-2c$, $-3c$, &c. ad infinitum (*Elem. Alg.* 473.); hence, these values of z are the roots of the equation $1-\frac{z^2}{2.3}+\frac{z^4}{2.3.4.5}-$&c.$=0$. Put $z=\frac{1}{y}$, and the equation becomes $1-\frac{1}{2.3.y^2}+\frac{1}{2.3.4.5.y^4}-$&c. $=0$; multiply it by y^n, and it becomes $y^n-\frac{y^{n-2}}{2.3}+\frac{y^{n-4}}{2.3.4.5}-$ &c. $=0$, which equation contains n roots $=0$, the other roots remaining the same. But as $y=\frac{1}{z}$, the values of y are $\frac{1}{1c}, \frac{1}{2c}, \frac{1}{3c}$, &c. and $-\frac{1}{1c}, -\frac{1}{2c}, -\frac{1}{3c}$, &c. ad inf. Now (*Alg.* Art. 349.) the sum of the squares of the roots of the last equation is $\frac{1}{3}$; and the

squares of the positive values of y being the same as the square of the negative values, we have $\frac{2}{1^2c^2}+\frac{2}{2^2c^2}+\frac{2}{3^2c^2}+$ ad inf. $=\frac{1}{3}$, consequently $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+$ &c. ad inf. $=\frac{c^2}{6}$.

Cor. 1. In like manner we may find the sum of any of the *even* powers of the reciprocals of the natural numbers, by assuming the sum equal to its value given by the same Art. in the *Algebra*. For instance, the sum of the fourth powers of the roots of the equation is $\frac{1}{45}$; hence, $\frac{2}{1^4c^4}+\frac{2}{2^4c^4}+\frac{2}{3^4c^4}+\&c.=\frac{1}{45}$, consequently $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\&c.=\frac{c^4}{90}$.

The sum of the reciprocals of the *odd* powers cannot be found by this method, because the *odd* powers of the *negative* roots destroy those of the *positive*.

Cor. 2. By transposition, $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\&c.=\frac{c^2}{6}-\frac{1}{2^2}-\frac{1}{4^2}-\&c.=\frac{c^2}{6}-\frac{1}{2^2}\times\overline{\frac{1}{1^2}\times\frac{1}{2^2}+\&c.}=\frac{c^2}{6}-\frac{1}{2^2}\times\frac{c^2}{6}=\frac{c^2}{8}$. And in like manner, we may find the sum of the reciprocals of all the even powers of 1, 3, 5, &c.

Prop. CXVI.

Supposing the force of gravity to vary as the n^{th} power of the distance from the centre of the earth, and the compressive force of the air to vary as its density; to find the density of the air at any altitude above the surface of the earth.

192. Let the radius of the earth = 1, x = the distance of any point above the earth's surface from the centre, v = the density of the air at that point, the density at the surface being unity; h = the altitude of an homogeneous atmosphere. Now it appears by experiment, that the compressive force of the air varies as its density; consequently the fluxion of the compressive force must be to the fluxion of the density, as the compressive force is to the density, and this ratio is the same at all altitudes. Now at any distance x from the earth's centre, the fluxion of the compressive force must be in proportion to the force of gravity, the density, and the fluxion of the altitude; hence, $x^n v\dot{x}$ has a constant ratio to $-\dot{v}$, writing the latter fluxion with the sign $-$ (Art. 16.), because v decreases as x increases; and according to this representation of the compressive force, h will represent the compressive force at the surface; hence, $h : 1 :: x^n v\dot{x} : -\dot{v}$, therefore $x^n\dot{x} = -h \times \frac{\dot{v}}{v}$ and $\frac{x^{n+1}}{n+1} = -h \times \text{h. l.}\, v + C$; but when $x=1$, $v=1$, and this equation becomes $\frac{1}{n+1} = C$; hence, the correct fluent is $\frac{x^{n+1}}{n+1} = -h \times \text{h. l.}\, v + \frac{1}{n+1}$, consequently $\frac{1-x^{n+1}}{n+1} = h \times \text{h. l.}\, v$, an equation expressing the relation between the altitude and density.

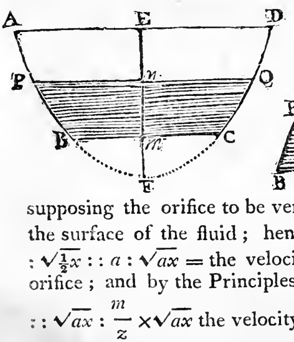
Cor. 1. If we suppose the force to vary inversely as the square of the distance, n becomes -2; hence, $\frac{1}{x} - 1 = h \times \text{h. l.}\, v$; if therefore x increase in musical progression, $\frac{1}{x}$ will decrease in arithmetic progression, and consequently the h. l. v will decrease in arithmetic progression.

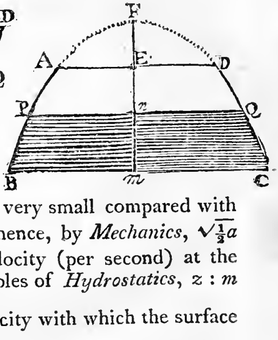
Cor. 2. If the force of gravity be supposed constant, $n = 0$; hence, $1 - x = h \times$ h. l. v; and if x increase in arithmetic progression, then $1 - x$ will decrease in arithmetic progression, consequently the h. l. v will decrease in arithmetic progression.

Prop. CXVII.

To find the time in which a vessel ABCD *filled with a fluid, will empty itself through a very small orifice* m *at the bottom.*

193. Put $a = 32\frac{1}{6}$ feet $= 386$ inches, $x = mn$ the depth of the fluid at any point of time, z = the area of the surface PQ of the fluid, m = the area of the orifice, t = the time in which the surface of the fluid descends from PQ to BC. Now it appears by experiment, that the velocity of the fluid at the orifice is that which a body acquires in falling down $\frac{1}{2}x$,





supposing the orifice to be very small compared with the surface of the fluid; hence, by *Mechanics*, $\sqrt{\frac{1}{2}a} : \sqrt{\frac{1}{2}x} :: a : \sqrt{ax}$ = the velocity (per second) at the orifice; and by the Principles of *Hydrostatics*, $z : m :: \sqrt{ax} : \frac{m}{z} \times \sqrt{ax}$ the velocity with which the surface descends; hence (Art. 81.), $\dot{t} = \dfrac{\dot{x}}{\frac{m}{z} \times \sqrt{ax}} = \dfrac{z\dot{x}}{m\sqrt{ax}}$, the fluent of which, corrected when necessary, gives t.

EXAMPLES.

Ex. 1. *Let the vessel be a cylinder or prism.*

Put $h = \mathrm{E}m$ its altitude. In this case z is constant, and $\dot{t} = \frac{z\dot{x}}{m\sqrt{ax}} = \frac{z}{m\sqrt{a}} \times x^{-\frac{1}{2}}\dot{x}$, whose fluent is $t = \frac{2zx^{\frac{1}{2}}}{m\sqrt{a}} = \frac{2z}{m} \times \sqrt{\frac{x}{a}}$, which wants no correction; and when $x = h$, $t = \frac{2z}{m} \times \sqrt{\frac{h}{a}}$, the time of emptying.

Ex. 2. *Let* ABCD *be the frustrum of a cone.*

Put $\mathrm{F}m = c$, $m\mathrm{B} = d$, $\mathrm{E}m = e$, $p = 3{,}14159$ &c. then $\mathrm{F}n = c \pm x$, the sign + or — being taken according as the less or greater end is downwards; and (FA, FD being now right lines) by similar triangles, $c : d :: c \pm x : \mathrm{P}n = \frac{d}{c} \times \overline{c \pm x}$; hence, $z = \frac{pd^2}{c^2} \times \overline{c \pm x}^2$; consequently $\dot{t} = \frac{pd^2}{mc^2\sqrt{a}} \times x^{-\frac{1}{2}} \times \overline{c \pm x}^2 \times \dot{x} = \frac{pd^2}{mc^2\sqrt{a}} \times \overline{c^2x^{-\frac{1}{2}}\dot{x} \pm 2cx^{\frac{1}{2}}\dot{x} + x^{\frac{3}{2}}\dot{x}}$, and $t = \frac{pd^2}{mc^2\sqrt{a}} \times \overline{2c^2x^{\frac{1}{2}} \pm \frac{4}{3}cx^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}}$, which requires no correction; and when $x = e$, $t = \frac{pd^2}{mc^2\sqrt{a}} \times \overline{2c^2e^{\frac{1}{2}} \pm \frac{4}{3}ce^{\frac{3}{2}} + \frac{2}{5}e^{\frac{5}{2}}}$, the whole time of emptying.

If the orifice be a circle whose radius $= r$, then $m = pr^2$; consequently $t = \frac{d^2}{r^2c^2\sqrt{a}} \times \overline{2c^2e^{\frac{1}{2}} \pm \frac{4}{3}ce^{\frac{3}{2}} + \frac{2}{5}e^{\frac{5}{2}}}$.

Cor. If the base be downwards, and we take the whole cone, then $c=e$; hence, $t=\frac{d^2}{r^2c^2\sqrt{a}}\times\frac{16}{15}c^{\frac{5}{2}}=\frac{16d^2\sqrt{c}}{15r^2\sqrt{a}}$, the whole time of emptying.

If the vertex be downwards, and the orifice be so small that we may consider Em as equal to EF, then $c=0$, $d=0$; but because c is always to d as FE : EA, $\therefore$ when c and d vanish, we may consider $\frac{d^2}{c^2}=\frac{EA^2}{FE^2}$; hence, $t=\frac{EA^2}{FE^2\times r^2\sqrt{a}}\times\frac{2}{5}e^{\frac{5}{2}}=\frac{2EA^2\times\sqrt{FE}}{5r^2\sqrt{a}}$ the whole time of emptying.

Ex. 3. *Let* BFC *be a hemisphere standing on its base.*

Put the radius $mB=mF=r$; then $Pn^2=r^2-x^2$, and $z=p\times\overline{r^2-x^2}$; hence, $\dot{t}=\frac{p\times\overline{r^2-x^2}\times\dot{x}}{m\sqrt{ax}}=\frac{p}{m\sqrt{a}}\times\overline{r^2x^{-\frac{1}{2}}\dot{x}-x^{\frac{3}{2}}\dot{x}}$, whose fluent is $t=\frac{p}{m\sqrt{a}}\times\overline{2r^2x^{\frac{1}{2}}-\frac{2}{5}x^{\frac{5}{2}}}$, which wants no correction; and when $x=r$, $t=\frac{8p}{5m\sqrt{a}}\times r^{\frac{5}{2}}$, the whole time of emptying.

If the orifice be a circle whose radius is w, then $m=pw^2$; hence, $t=\frac{8r^{\frac{5}{2}}}{5w^2\sqrt{a}}$.

If the hemisphere stand on its vertex, $Pn^2=2rx-x^2$; hence, $z=p\times\overline{2rx-x^2}$, consequently $\dot{t}=\frac{p}{m\sqrt{a}}\times\overline{2rx^{\frac{1}{2}}\dot{x}-x^{\frac{3}{2}}\dot{x}}$, whose fluent is $t=\frac{p}{m\sqrt{a}}\times\overline{\frac{4}{3}rx^{\frac{3}{2}}-\frac{2}{5}x^{\frac{5}{2}}}$, which requires no correction; and when $x = r$, $t = \frac{14pr^{\frac{5}{2}}}{15m\sqrt{a}} = \frac{14r^{\frac{5}{2}}}{15w^2\sqrt{r}}$, the whole time of emptying.

Ex. 4. *Let* BCF *be a* paraboloid *standing on its base.*

Put its parameter $= r$, its altitude $Fm = e$, then $r \times \overline{e-x} = Pn^2$, and $pr \times \overline{e-x} = z$; hence, $\dot{t} = \frac{pr}{m\sqrt{a}} \times \overline{ex^{-\frac{1}{2}}\dot{x} - x^{\frac{1}{2}}\dot{x}}$, whose fluent is $t = \frac{pr}{m\sqrt{a}} \times \overline{2ex^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}}}$, which requires no correction; and when $x = e$, $t = \frac{4pre^{\frac{3}{2}}}{3m\sqrt{a}} = \frac{4re^{\frac{3}{2}}}{3w^2\sqrt{a}}$, the whole time of emptying.

If the paraboloid stand on its vertex, $Pn^2 = rx$; hence, $z = prx$; consequently $\dot{t} = \frac{prx^{\frac{1}{2}}\dot{x}}{m\sqrt{a}}$, and $t = \frac{2prx^{\frac{3}{2}}}{3m\sqrt{a}}$, which wants no correction; and when $x = e$, $t = \frac{2pre^{\frac{3}{2}}}{3m\sqrt{a}} = \frac{2re^{\frac{3}{2}}}{3w^2\sqrt{a}}$, the whole time of emptying.

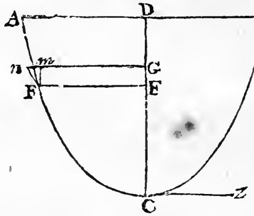
In like manner, whatever be the form of the vessel, we may find the time of emptying, substituting into the value of $\dot{t}$, the quantity z expressed in terms of x, and then taking the fluent.

Prop. CXVIII.

If a perfectly flexible chain ACB, *of uniform density and thickness, be hung upon two pins at* A *and* B; *to find the curve into which it will form itself.*

194. Let C be the lowest point, draw the axis CD perpendicular to the horizon; draw also EF, Gn perpendicular to CD; Fn a tangent at F, and Fm perpendicular to FE. Now assuming any part CF of the chain, we may consider it as if it were perfectly rigid;

for conceive CF to become perfectly rigid, and it is manifest that no alteration whatever can take place; for the gravity of the chain gives CF a certain situation; and if we make that part to become inflexible, we add no new force; we only suppose a cohesion to take place between the constituent particles whilst they are



so disposed. Considering therefore CF as a perfectly inflexible body, it is kept at rest by three forces; at C by the action of the part BC of the chain in the direction Cz of the tangent at C; at F by the action of the part FA of the chain in the direction Fn of the tangent at F; and by its gravity in a direction parallel to EC; but * Cz is parallel to mn, and CE to mF; hence, these three forces act parallel to the three sides of the triangle Fmn, and consequently will be respectively proportional to them, the body FC being at rest. Put CE$=x$, EF$=y$, CF$=z$, then (Art. 23. and 27.) F$m=\dot{x}$, $mn=\dot{y}$, F$n=\dot{z}$. Now the chain being of uniform density and thickness, the gravity of any part CF will be in proportion to its length z; also, let $a=$ the tension of the chain BC at C acting in the direction Cz, a constant quantity, it not varying by changing the point F. Hence, $a : z :: \dot{y} : \dot{x}$, $\therefore a\dot{x}=z\dot{y}$; but $\dot{z}^2=\dot{x}^2+\dot{y}^2=\dot{x}^2+\frac{a^2\dot{x}^2}{z^2}$, therefore $z^2\dot{z}^2=z^2\dot{x}^2+$

* As by *Mechanics*, these three forces must be directed to one point, if the two tangents nF, zC be produced to meet, the intersection must be in the line of direction passing through the centre of gravity of FC.

$a^2\dot{x}^2$, consequently $\dot{x} = \frac{z\dot{z}}{\sqrt{a^2+z^2}}$, whose fluent (Art. 39.) is $x = \sqrt{a^2+z^2} + C$; but when $x = 0$, then $z = 0$; hence, the equation becomes $0 = a + C$, and $C = -a$; therefore the correct fluent is $x = \sqrt{a^2+z^2} - a$, and by transposing a and squaring both sides, $x^2 + 2ax = z^2$, the equation of the curve. This curve is called the *Catenary*.

Prop. CXIX.

If the chain ACB *be of uniform thickness; to find the law of weight and density, so that it may form itself into any given curve.*

195. Let w = the weight of any part CF, d = the density at F; then by the last proposition, $a : w :: \dot{y} : \dot{x}$, therefore $w = a \times \frac{\dot{x}}{\dot{y}}$. Now $\dot{w} = d\dot{z}$; hence, $d = \frac{\dot{w}}{\dot{z}}$. But $w = a \times \frac{\dot{x}}{\dot{y}}$, and if $\dot{y}$ be made constant, $\dot{w} = a \times \frac{\ddot{x}}{\dot{y}}$; hence, $d = \frac{a\ddot{x}}{\dot{y}\dot{z}}$, which gives the law of density.

EXAMPLES.

Ex. 1. *Let the curve be a* circle *whose radius is r.*

Here, $\dot{x} : \dot{y} :: y : r - x$; therefore $w \left(= a \times \frac{\dot{x}}{\dot{y}} \right) = a \times \frac{y}{r-x} = a \times$ tan. of CF; the weight therefore of any part CF varies as the tangent of CF. Now, $y^2 = 2rx - x^2$, and $y\dot{y} = r\dot{x} - x\dot{x}$, and (making $\dot{y}$ constant) $\dot{y}^2 = r\ddot{x} - x\ddot{x} - \dot{x}^2$, therefore $\ddot{x} = \frac{\dot{y}^2 + \dot{x}^2}{r-x} = \frac{\dot{z}^2}{r-x} =$ (because $r : y :: \dot{z} : \dot{x}$) $\frac{r^2\dot{x}^2}{y^2 \times \overline{r-x}}$; also, $\dot{y} = \frac{\overline{r-x} \times \dot{x}}{y}$, and $\dot{z} = \frac{r\dot{x}}{y}$;

hence, $d\left(=\frac{a\ddot{x}}{\dot{y}\dot{z}}\right)=\frac{ar^2\dot{x}^2}{y^2\times\overline{r-x}}\times\frac{y}{\overline{r-x}\times\dot{x}}\times\frac{y}{r\dot{x}}=\frac{ar}{\overline{r-x}^2}$. The density therefore varies inversely as the square of the cosine of CF. If therefore the arc be a semicircumference, the density at the highest point is infinite.

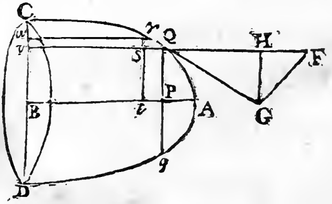
Ex. 2. *Let the curve be a* parabola.

Here, $px=y^2$; therefore, $\dot{x}=\frac{2y\dot{y}}{p}$; hence, w $\left(=a\times\frac{\dot{x}}{\dot{y}}\right)=\frac{2ay}{p}$; therefore the weight of any part CF varies as the ordinate FE. Also, (if $\dot{y}$ be constant) $\ddot{x}=\frac{2\dot{y}^2}{p}$; but (Art. 54. Ex. 3.) $\dot{z}=\frac{\overline{y^2+c^2}|^{\frac{1}{2}}\times\dot{y}}{c}$, putting $c=\frac{1}{2}p$; hence, $d\left(=\frac{a\ddot{x}}{\dot{y}\dot{z}}\right)=\frac{a}{\sqrt{y^2+c^2}}$. The density therefore varies inversely as $\sqrt{y^2+c^2}$, or inversely as the *normal* (Art. 24. Ex.).

PROP. CXX.

Let CAD *be a plane figure, or a solid generated by its revolution about its axis, moving in a fluid in the direction of its axis* BA; *to find the resistance of the curve line* CAD, *or of the surface of the solid, to the resistance on the base* CD.

196. Draw FQ*sv* and *wr* parallel to AB, *rst*,



Q*Pq* perpendicular to AB; then if $AP=x$, $PQ=y$, QA $= z$, it appears from Art. 23. and 27. that ultimately, by bringing r up to Q, Q$s = \dot{x}$, $sr = \dot{y}$, Q$r = \dot{z}$. Draw the tangent QG, and let fall the perpendicular FG upon it, and also GH upon FQ. Now let FQ represent the force of one particle of the fluid, then if that particle struck the base at v, its whole force would act to oppose the motion, because it acts perpendicularly to the base, and therefore no part of its force is lost; but striking the curve at Q obliquely, if the force FQ be resolved into GQ and FG, then GQ is here supposed to be lost by the obliquity of the stroke, and FG to be the only effective part; but this not being opposite to the motion of the body, we must resolve it into FH and HG, and then FH is that part which opposes the motion of the body, and HG is destroyed by an equal and opposite force of a particle acting at q. Hence, the force of a particle at v : force at Q :: FQ : FH :: (because FQ : FG :: FG : FH) $FQ^2 : FG^2$: : (by sim. trian.) $\dot{z}^2 : \dot{y}^2$. Now the quantity of fluid striking Qr and vw is the same, and in proportion to sr or $\dot{y}$. Hence, if we consider it as a *plane* figure, as the whole force is as the number of particles $\times$ force of each, we have the force against vw : force against Qr : : $\dot{y} : \dfrac{\dot{y}^3}{\dot{z}^2} = \dfrac{\dot{y}^3}{\dot{x}^2 + \dot{y}^2} = \dfrac{\dot{y}}{1 + \dfrac{\dot{x}^2}{\dot{y}}}$;

hence, the whole resistance on the base : that on the curve : : the fluent of $\dot{y}$, or y, : fluent (F) of $\dfrac{\dot{y}}{1 + \dfrac{\dot{x}^2}{\dot{y}^2}}$.

For a *solid*, the number of particles striking the area generated by vw will be as $vw \times$ circum. described by v, or as $vw \times y$ or as $y\dot{y}$; hence, for the same reason, the resistance on the base : that on the surface : : the flu. of $y\dot{y}$, or $\frac{1}{2}y^2$, : flu. (F) of $\dfrac{y\dot{y}}{1 + \dfrac{\dot{x}^2}{\dot{y}^2}}$.

EXAMPLES.

Ex. 1. *Let* ACD *be an* isosceles triangle.

Here the plane is a triangle, and $\dot{x} : \dot{y} :: x : y :: a$ (AB) : b (BC), $\therefore \frac{\dot{x}^2}{\dot{y}^2} = \frac{a^2}{b^2}$; hence, the resistances are as y : flu. $\frac{\dot{y}}{1+\frac{a^2}{b^2}} :: y : \frac{y}{1+\frac{a^2}{b^2}} :: b^2 + a^2 : b^2 :: AC^2 : BC^2$. The same is true for the *cone*, or for any *prismatic* solid.

Ex. 2. *Let* CAD *be a* semicircle.

Put $AB=r$, then $y^2=2rx-x^2$; hence, $\dot{x}=\frac{y\dot{y}}{r-x}=\frac{y\dot{y}}{\sqrt{r^2-y^2}}$, and $\frac{\dot{x}^2}{\dot{y}^2}=\frac{y^2}{r^2-y^2}$; $\therefore \dot{F}=\frac{\dot{y}}{1+\frac{y^2}{r^2-y^2}}=\frac{r^2\dot{y}-y^2\dot{y}}{r^2}$, and $F=y-\frac{y^3}{3r^2}$; hence, the resistances are as $y : y-\frac{y^3}{3r^2}$, which, when $y=r$, is as 3 : 2.

Ex. 3. *Let* CAD *be a* hemisphere.

Here $\dot{F}=\frac{y\dot{y}}{1+\frac{y^2}{r^2-y^2}}=\frac{r^2y\dot{y}-y^3\dot{y}}{r^2}$, and $F=\frac{1}{2}y^2-\frac{y^4}{4r^2}$; hence, the resistances are as $\frac{1}{2}y^2 : \frac{1}{2}y^2-\frac{y^4}{4r^2}$, which, when $y=r$, is as 2 : 1.

Ex. 4. *Let the solid* CAD *be generated by a* cycloid AC *revolving about* AB, BC *being the axis of the cycloid.*

If $a=BC$, then $y=z-\frac{z^2}{4a}$ by the nature of the curve;

hence, $\dot{y} = \dot{z} - \frac{z\dot{z}}{2a}$, $\therefore \dot{F} = \frac{y\dot{y} \times y^2}{z^2} = \frac{y\dot{y} \times \overline{2a - z}^2}{4a^2} = \frac{y\dot{y} \times \overline{a - y}}{a} = y\dot{y} - \frac{y^2\dot{y}}{a}$, and $F = \frac{1}{2}y^2 - \frac{y^3}{3a}$; hence, the resistances are as $\frac{1}{2}y^2 : \frac{1}{2}y^2 - \frac{y^3}{3a}$, which, when $y = a$, is as 3 : 1.

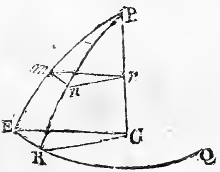
197. Considering the body as a *solid*, and the force of a particle on the base as constant, the force of a particle on the surface $\propto \frac{\dot{y}^2}{\dot{z}^2}$, and the area generated by rs being as $y\dot{y}$, the resistance against QR $\propto \frac{y\dot{y}^3}{\dot{z}^2}$.

On MERCATOR's PROJECTION.

Prop. CXXI.

If P *be the pole of the earth,* EQ *the equator,* PE, PR, *two meridians,* mn *a small circle parallel to* ER; *then the length of a degree of latitude : the length of a degree of longitude at* m :: *radius : the cosine of the latitude of* m, *supposing the earth to be a sphere.*

198. For let PC be the radius of the earth; draw



mr, nr perpendicular to it, and join EC, RC. Then mr, nr being parallel to EC, RC respectively, the angle

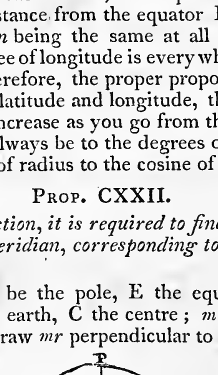
$mrn =$ ECR; hence, by similar sectors, ER : mn : : EC : mr the cosine of mE. But when the angle is given, the length of an arc of a degree is in proportion to the radius; also, the length of a degree of the great circle ER is a degree of latitude; and the length of a degree of mn is a degree of longitude at m; hence, a degree of latitude : a degree of longitude : : radius : the cosine of latitude.

In MERCATOR's Projection, the sphere is projected upon a plane, and the meridians EP, RP are straight lines parallel to each other; consequently P must be at an infinite distance from the equator EQ. In this case, the arc mn being the same at all latitudes, the length of a degree of longitude is every where the same; to preserve, therefore, the proper proportion between the degrees of latitude and longitude, the degrees of latitude must increase as you go from the equator, so that they may always be to the degrees of longitude in the proportion of radius to the cosine of latitude.

PROP. CXXII.

In this projection, it is required to find the length of an arc of the meridian, corresponding to any given latitude.

199. Let P be the pole, E the equator, PCQ a diameter of the earth, C the centre; m any place on the surface; draw mr perpendicular to PQ, and join



mC, mQ. Put Cm=r, Em=x, Cr (the sine of Em the latitude of m)=y, and the length of Em on the projection

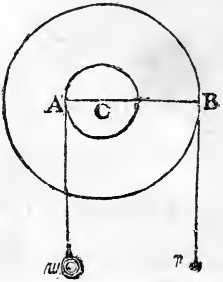
$=z$, called the *meridional parts*. Then by Prop. 121. $\sqrt{r^2-y^2}$ (cos. of lat.) : $r :: \dot{x} : \dot{z} = \frac{r\dot{x}}{\sqrt{r^2-y^2}}$; but (Art. 46.) $\dot{x} = \frac{r\dot{y}}{\sqrt{r^2-y^2}}$; hence, $\dot{z} = \frac{r^2\dot{y}}{r^2-y^2} = \frac{r}{2} \times \frac{2r\dot{y}}{r^2-y^2}$, $\therefore$ $z = \frac{r}{2} \times$ h. l. $\frac{r+y}{r-y} + C$ (Art. 45. Ex. 6.) $= r \times$ h. l. $\sqrt{\frac{r+y}{r-y}} + C$, by the nature of logarithms. But by Plane Trig. $\sqrt{r^2-y^2}$ $(mr) : r+y$ $(rQ) :: r$ (rad.) : $\frac{r \times \overline{r+y}}{\sqrt{r^2-y^2}}$ $= r\sqrt{\frac{r+y}{r-y}}$ the tangent of the angle rmQ = cotangent of rCm = cotan. of $\frac{1}{2}rQm$ = cotan. $\frac{1}{2}$ the complement of lat.; hence, $\sqrt{\frac{r+y}{r-y}} = \frac{\text{cotan. } \frac{1}{2} \text{ comp. lat.}}{r}$; consequently $z = r \times$ h. l. $\frac{\text{cotan. } \frac{1}{2} \text{ comp. lat.}}{r} + C$; but when $z=0$, cotan. $\frac{1}{2}$ comp. lat. $=r$; hence, $0 = r \times$ h. l. $\frac{r}{r} + C = r \times$ h. l. $1 + C = 0 + C$, $\therefore C=0$; consequently $z = r \times$ h. l. $\frac{\text{cotan. } \frac{1}{2} \text{ comp. lat.}}{r} = r \times$ h. l. cotan. $\frac{1}{2}$ comp. lat. $- r \times$ h. l. r, the length of the meridian Em in the projection.

Prop. CXXIII.

Given the radii BC, AC *of a wheel and axle, and the weight p which draws up w; to find w, so that the momentum communicated to it in a given time may be a maximum, the wheel and axle being supposed of no weight.*

200. Put BC$=b$, AC$=a$; then, by *Mechanics*, the forces with which w and p endeavour to descend, are aw

and bp; hence, the moving force is as $bp-aw$; also,



the inertia of each weight is (Art. 60.) as $a^2 \times w$, and $b^2 \times p$; hence, the accelerative force of the lever is as $\frac{bp-aw}{b^2p+a^2w}$; and as the acceleration of any point of a lever must (besides the accelerating force with which the lever itself is made to revolve) be in proportion to the distance of that point from the fulcrum, the accelerative force of the point A, or of w, will be as $\frac{abp-a^2w}{b^2p+a^2w}$, which is as the velocity generated in w in a given time; consequently the momentum of w will be as $\frac{abp-a^2w}{b^2p+a^2w} \times w = \frac{abpw-a^2w^2}{b^2p+a^2w}$ = a maximum, or $\frac{bpw-aw^2}{b^2p+a^2w}$ = a maximum; hence, (Art. 21.) its fluxion $\frac{\overline{bp\dot{w}-2aw\dot{w}} \times \overline{b^2p+a^2w}-a^2\dot{w} \times \overline{bpw-aw^2}}{\overline{b^2p+a^2w}|^2} = 0$, or $\overline{bp-2aw} \times \overline{b^2p+a^2w}-a^2 \times \overline{bpw-aw^2} = 0$; hence, $w = \sqrt{\frac{b^4}{a^4}+\frac{b^3}{a^3}}-\frac{b^2}{a^2} \times p$.

If $a=b$, $w=\overline{\sqrt{2}-1} \times p$.

Prop. CXXIV.

Given two weights w and p, and the radius CA of the axle, to find the radius CB of the wheel, so that p may draw up w through a given space, in the least time possible.

201. When the space is given, the time varies inversely as the square root of the accelerative force; hence (by the last Art.), the square of the time varies as $\frac{b^2p+a^2w}{abp-a^2w}$ a minimum, where b is variable; put its fluxion $=0$, and we get $b=\frac{aw}{p}+\frac{\sqrt{a^2w^2+a^2pw}}{p}$.

If $p=w$, $b=a\times\overline{1+\sqrt{2}}$.

Prop. CXXV.

If the force of gravity upon the earth's surface be represented by $32\frac{1}{6}$ feet, and r represent the radius of any circle, about the centre of which a body revolves with the velocity v, and F *represent the centripetal, and consequently the centrifugal force; then* $F=\frac{v^2}{r}$.

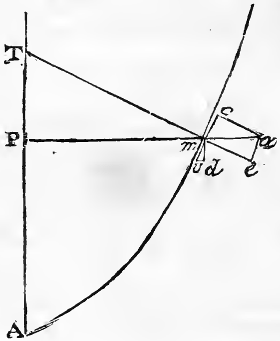
202. For let $V=$ the velocity of a body revolving in a circle at the earth's surface, about its centre, $R=$ the radius of the earth; then $\frac{V^2}{2R}=$ the sagitta of the arc described in $1''=16\frac{1}{12}$ feet; and as the forces of bodies revolving in different circles vary as the squares of the velocities directly and the radii inversely (*Newton's Prin. Lib.* 1. Prop. iv. Cor. 1.), $32\frac{1}{6} : F :: \frac{V^2}{R} : \frac{v^2}{r}$; but $32\frac{1}{6}=\frac{V^2}{R}$; hence, $F=\frac{v^2}{r}$.

Cor. If $r =$ radius of curvature of any curve; then the force being the same in the curve and the circle, the same is true for the curve, r being the radius of curvature.

Prop. CXXVI.

Let Am be a slender rod in the form of a parabola whose axis AP is perpendicular to the horizon; and let a ring which can freely move upon the rod be put upon it at any point m; then if the parabola revolve about AP with such a velocity that the ring may remain at rest, it would remain at rest at every other point of the rod.

203. Put $p = 32\frac{1}{6}$ feet, and let it represent the force



of gravity; v=the velocity of the point m, x=AP, y=Pm; then $\frac{v^2}{y}$ = the centrifugal force of the ring (Art. 202.); produce Pm to a, and let $ma = \frac{v^2}{y}$; resolve the force ma into two, one mc in the direction of the tangent to the curve, and the other me perpendicular to it, and produce it to T. Draw md perpendicular to the horizon, and let it represent p the force of gravity, and resolve it into two other forces, one mv in the direction of the tangent, and the other vd perpendicular to it. Now when the ring remains at rest, mc must be equal to mv. As the triangles acm, dvm are similar to mPT, we have

$$am\left(\frac{v^2}{y}\right) : mc, \text{ or } mv, :: m\text{T} : \text{PT}.$$

$$mv : dm\,(p) :: m\text{P} : m\text{T}.$$

$$\therefore \frac{v^2}{y} : p :: m\text{P} : \text{PT}.$$

But v varies as y; let, therefore, $v = ay$; and we have $a^2y : p :: m\text{P}\,(y) : \text{PT}$, or $a^2 : p :: 1 : \text{PT}$, which proportion, consisting only of constant quantities, must be true for every point of the curve; therefore at every point $mc = mv$, and the ring would remain at rest.

Cor. 1. If the parabola be given, PT is given, it being half the latus rectum; hence, we know $a = \sqrt{\frac{p}{\text{PT}}}$; assuming therefore any ordinate Pm (y), we know ay, or v; thus we get the velocity of the point m. Put c=6,28319 &c. then cy = the circumference described by m; hence, $cy : v$, or ay, :: 360° : the angular velocity $= 360^\circ \times \frac{a}{c}$.

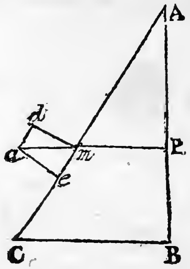
Cor. 2. Hence, if a vessel of water revolve about its axis, the water will rise up in the curve of a parabola; for the water cannot remain at rest till the two forces mc, mv destroy each other. The forces ca, vd acting perpendicularly to the surface of the fluid, cannot disturb it.

Prop. CXXVII.

Let a ring be put upon a slender rod AC, *and let the rod revolve about* AB *which is perpendicular to the horizon; it is required to find how long the ring will be in descending from* A *to* C, *the velocity of the rod, its length, and the angle* CAB *being given.*

204. Draw CB perpendicular to AB; put AB=a, BC

$=b$, AC$=c$, $d=$the velocity of the point C, $x=$Am, $v=$



the velocity of the ring at m, $m = 32\frac{1}{6}$ feet the force of gravity, and $t =$ the time of the ring's descent. Draw mP perpendicular to AB, and produce it to a, and let ma represent the centrifugal force of the point m; resolve ma into two forces, one md perpendicular to AC, and the other me in the direction AC. By similar triangles, $c : b :: x : \frac{bx}{c} =$ Pm, and $b : \frac{bx}{c} :: d : \frac{dx}{c}$ $=$ the velocity of the point m; hence (Art. 202.), the centrifugal force $ma = \frac{d^2x^2}{c^2} \times \frac{c}{bx} = \frac{d^2x}{bc}$; and by similar triangles, $c : b :: \frac{d^2x}{bc} : me = \frac{d^2x}{c^2}$; also, $c : a :: m$ (the force of gravity) : $\frac{ma}{c} =$ the accelerative force of the ring from the action of gravity; hence (Art. 81. Cor.), $\frac{d^2x\dot{x}}{c^2} + \frac{ma\dot{x}}{c} = v\dot{v}$; and $v = \sqrt{\frac{d^2x^2}{c^2} + \frac{2max}{c}} = \left(\text{if } \frac{mac}{d^2} = n\right) \frac{d}{c} \sqrt{x^2 + 2nx}$. Hence (Art. 81.), $\dot{t} = \frac{c}{d} \times \frac{\dot{x}}{\sqrt{x^2 + 2nx}}$, and (Art. 45. Ex. 5.) $t = \frac{c}{d} \times$ h. l.

$\overline{n+x+\sqrt{x^2+2nx}}+C$; but when $x=0$, $t=0$, and we have $0=\frac{c}{d}\times$h. l. $n+C$; hence, the correct fluent $t=\frac{c}{d}\times$ h. l. $\frac{n+x+\sqrt{x^2+2nx}}{n}=$ (when $x=c$) $\frac{c}{d}\times$ h. l. $\frac{n+c+\sqrt{c^2+2nc}}{n}$ the whole time of descent.

Cor. 1. The accelerative force $\frac{d^2x}{c^2}$ of the ring in the direction of the rod, arising from the centrifugal force, is always the same whatever be the inclination of the rod, the length of the rod, and the velocity of its lowest point being given.

Cor. 2. By similar triangles, $c:a::\frac{ax^2}{bc}:md=\frac{d^2ax}{c^2b}$; and by *Mechanics*, $c:b::m:\frac{bm}{c}=$ the pressure of the ring on the rod; hence, when $\frac{d^2ax}{c^2b}=\frac{bm}{c}$, the pressure of the ring on the rod $=0$, which therefore happens when $x=\frac{b^2cm}{d^2a}$.

Cor. 3. If AC become horizontal, then $a=0$, and $v\dot{v}=\frac{d^2x\dot{x}}{c^2}$. Now as in this case the ring will not begin to move from A, we must at first put it at some distance r from A. Hence, $v^2=\frac{d^2x^2}{c^2}+C$, and when $\dot{v}=0$, $x=r$; therefore the equation becomes $0=\frac{d^2r^2}{c^2}+C=0$, and $C=-\frac{d^2r^2}{c^2}$; hence, $v=\frac{d}{c}\times\sqrt{x^2-r^2}$. Also,

$\dot{t} = \frac{c}{d} \times \frac{\dot{x}}{\sqrt{x^2 - r^2}}$, whose fluent (Art. 45. Ex. 4.) is $t = \frac{c}{d}$ $\times$ h. l. $\overline{x + \sqrt{x^2 - r^2}} + C$; but when $t = 0$, $x = r$, and the equation becomes $0 = \frac{c}{d} \times$ h. l. $r + C$; therefore $C = -\frac{c}{d}$ $\times$ h. l. r; hence, $t = \frac{c}{d} \times$ h. l. $\frac{x + \sqrt{x^2 - r^2}}{r}$.

Cor. 4. If A be the lower point of the rod and C the higher; then the force $\frac{d^2 x}{c^2}$ acts upwards, and the accelerating force of the ring $= \frac{d^2 x}{c^2} \backsim \frac{ma}{c}$. Let the ring at first be at any distance from A; then if $\frac{ma}{c}$ be greater than $\frac{d^2 x}{c^2}$, the ring *descends* by the force $\frac{ma}{c} - \frac{d^2 x}{c^2}$; but if $\frac{d^2 x}{c^2}$ be greater than $\frac{ma}{c}$, the ring *ascends* by the force $\frac{d^2 x}{c^2} - \frac{ma}{c}$; and the velocity and time may be found in each case as before.

Cor. 5. Taking the position of the rod as in the last Corol., and the case when $\frac{d^2 x}{c^2}$ is greater than $\frac{ma}{c}$, let the ring at the distance r from A be projected downwards on the rod with the velocity e; then $v\dot{v} = \frac{d^2 x\dot{x}}{c^2} - \frac{ma\dot{x}}{c}$, and $\frac{v^2}{2} = \frac{d^2}{c^2} \times \frac{x^2}{2} - \frac{max}{c} + C$; but when $v = e$, $x = r$, and the equation becomes $\frac{e^2}{2} = \frac{d^2}{c^2} \times \frac{r^2}{2} - \frac{mar}{c} + C$, therefore $C = \frac{e^2}{2} - \frac{d^2}{c^2} \times \frac{r^2}{2} + \frac{mar}{c}$; hence,

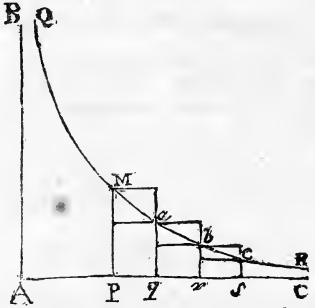
$v^2 = e^2 + \frac{d^2}{c^2} \times \overline{x^2 - r^2} + \frac{2ma}{c} \times \overline{r - x}$. Make $v=0$, and we get $x = \frac{mca}{d^2} + \sqrt{\frac{m^2c^2a^2}{d^4} + r^2 - \frac{c^2e^2}{d^2} - \frac{2macr}{d^2}}$, the distance from A, to which the ring descends when it has lost all its velocity. If the value of x be impossible, the ring will come to A without losing all its velocity. If the quantity under the radical sign $= 0$, $x = \frac{mca}{d^2}$; which is the value of x when the force $\frac{d^2x}{c^2} - \frac{ma}{c} = 0$; in this case therefore the ring will remain at rest when it has lost all its velocity. If the quantity under the radical sign be positive, then when $v = 0$, the force $\frac{d^2x}{c^2} - \frac{ma}{c}$ acting upwards, the ring will return, and continue to ascend. Put $n = \frac{mca}{d^2}$, $p = \frac{c^2e^2}{d^2} + 2rn - r^2$; and we have $\dot{t} = \frac{c}{d} \times \frac{-\dot{x}}{\sqrt{x^2 - 2nx + p}}$; let $x - n = y$, and $x^2 - 2nx = y^2 - n^2 + p = y^2 + q^2$ (putting $-n^2 + p = q^2$); also, $\dot{x} = \dot{y}$; hence, $\dot{t} = \frac{c}{d} \times \frac{-\dot{y}}{\sqrt{y^2+q^2}}$, and $t = \frac{c}{d} \times -$ h. l. $\overline{y + \sqrt{y^2+q^2}} + C$; but when $t = 0$, $x = r$, $\therefore y = r - n$; and the fluent becomes $0 = \frac{c}{d} \times -$ h. l. $\overline{r - n + \sqrt{\overline{r-n}^2 + q^2}}$ $+ C$, and $C = \frac{c}{d} \times$ h. l. $\overline{r - n + \sqrt{\overline{r-n}^2 + q^2}}$; hence, $t = \frac{c}{d}$ $\times$ h. l. $\frac{r - n + \sqrt{\overline{r-n}^2 + q^2}}{y + \sqrt{y^2+q^2}} = \frac{c}{d} \times$ h. l. $\frac{r - n + \sqrt{\overline{r-n}^2 + q^2}}{x - n + \sqrt{\overline{x-n}^2 + q^2}}$, the time of descent.

On the same principle we may find the motion of a ring on a curve line revolving in like manner.

Prop. CXXVIII.

To show when the series $\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+$ *&c. ad infinitum is finite, and when infinite.*

205. Let QR be an hyperbolic curve between the asymptotes AB, AC, which are perpendicular to each other; take AP=ordinate PM=1, and let Pq, qr, rs, &c.

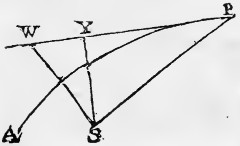


be each =1, and draw the ordinates qa, rb, sc, &c. and complete the circumscribing parallelograms, qM, ra, sb, &c. and the inscribed Pa, qb, rc, &c. and let the ordinate be equal to the inverse n^{th} power of the abscissa; then will PM$=\frac{1}{1^n}$, $qa=\frac{1}{2^n}$, $rb=\frac{1}{3^n}$, $sc=\frac{1}{4^n}$, &c. and as the bases of these parallelograms are each $=1$, the area of the parallelogram qM $=\frac{1}{1^n}$, of $ra=\frac{1}{2^n}$, of $sb=\frac{1}{3^n}$, &c. therefore the sum of all the circumscribed parallelograms $=\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+$ &c. ad infinitum; but it is manifest that the sum of all the inscribed parallelograms is less than the sum of all the circumscribed parallelograms, by the first parallelogram qM, that parallelogram being the sum of all the parallelograms, Ma, ab, bc, &c. each of which expresses the difference between its respective inscribed and circumscribed parallelogram. But the whole curvilinear area PMRC (being between the sum of the inscribed and circumscribed parallelograms) is less than the sum of all the circumscribed parallelograms, by a quantity which is less than the parallelogram qM; these two therefore differing by a finite quantity, when one is finite the other is finite, and when one is infinite the other is infinite. But by Prop. 20. Ex. 3. when n is equal to or less than unity, the area of the curve is infinite, and when n is greater than unity, the area is finite. Hence, the sum of the given series is *infinite* when n is *equal* to or *less* than unity, and *finite* when n is *greater* than unity.

Prop. CXXIX.

To determine the law of centripetal force tending to S, *so that a body may describe any given curve* AP.

206. Let SY be perpendicular to the tangent PY, and P the place of the body. Put x=SP, u=SY, F= force in the direction PS, f= that part of F which acts in the direction PY, v= the velocity at P, and z=AP. Now (Art. 81. Cor.) $v\dot{v}=f\dot{z}$; but F : f : : SP : PY : : (Art. 32.) $\dot{z} : \dot{x}$, therefore $f\dot{z}=F\dot{x}$; hence,



$v\dot{v}=F\dot{x}$, or rather $v\dot{v}=-F\dot{x}$ because (Art. 16.) when v increases x decreases; therefore $F=\frac{-v\dot{v}}{\dot{x}}$. But

(*Newton's Prin. L.* 1. Pr. 1. Cor. 1.) $v \propto \frac{1}{u}$; therefore

$v\dot{v} \propto \frac{-\dot{u}}{u^3}$; hence, $F \propto \frac{\dot{u}}{u^3\dot{x}}$.

207. Cor. Hence, whatever be the angle SPY, if v remain the same, then if $\dot{v}$ be given, $\dot{x}$ will be given; and if we suppose the angle SPY to vanish, then it follows, that if the velocity (v) of a body in the curve at P be equal to the velocity of a body in the right line SP at P, they will be equal at all other equal distances from S.

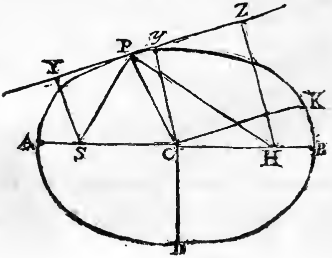
Ex. 1. Let AP be the *logarithmic* spiral, S its centre. Then $x : u :: a : b$ some constant ratio, $\therefore \dot{x} = \frac{a}{b}\dot{u}$;

hence, $F \propto \frac{b}{a} \times \frac{1}{u^3} \propto \frac{1}{x^3}$.

Ex. 2. Let AP be the *hyperbolic* spiral. Draw SW perpendicular to SP, meeting the tangent at W; then by the property of the curve, $SW = a$, a constant quantity; and $WP = \sqrt{a^2 + x^2}$; hence, by similar triangles, $\sqrt{a^2 + x^2} : x :: a : u = \frac{ax}{\sqrt{a^2+x^2}}$, and $\frac{1}{u^2} = \frac{1}{x^2} + \frac{1}{a^2}$,

therefore $\frac{\dot{u}}{u^3} = \frac{\dot{x}}{x^3}$; hence, $F \propto \frac{\dot{u}}{u^3\dot{x}} \propto \frac{1}{x^3}$.

Ex. 3. Let APB be an *ellipse* whose focus is S;



let H be the other focus, C the centre, CD the semi-axis minor, and HZ perpendicular to PY. Put $a =$ AC, $b=$CD ; then $2a-x=$PH, then by sim. tri. $x : u :: 2a-x :$ HZ $= \frac{\overline{2a-x} \times u}{x}$, and (*Con. Sect.* p. 6.) $\frac{\overline{2a-x} \times u^2}{x} = b^2$; hence, $\frac{1}{u^2} = \frac{2a}{b^2 x} - \frac{1}{b^2}$, and $\frac{\dot{u}}{u^3} = \frac{a\dot{x}}{b^2 x^2}$; therefore F $\propto \frac{\dot{u}}{u^3 \dot{x}} \propto \frac{a}{b^2 x^2} \propto \frac{1}{x^2}$.

For an *hyperbola*, $2a + x =$ PH, and the same conclusion follows.

For a *parabola*, $x \propto u^2$ (*Con. Sect.* p. 8. Cor. 2.), therefore $\frac{1}{u^2} \propto \frac{1}{x}$, and $\frac{\dot{u}}{u^3} \propto \frac{\dot{x}}{x^2}$; hence, F $\propto \frac{\dot{u}}{u^3 \dot{x}} \propto \frac{1}{x^2}$.

Hence, a force tending to the focus of any of the conic sections, varies in the inverse duplicate ratio of the distance.

Ex. 4. Let the force tend to the centre C of the ellipse. Let CK be the semi-conjugate to CP, and Cy perpendicular to Py ; CP$=x$, C$y=u$; then (*Con. Sect.* p. 13.) $a^2 + b^2 = x^2 +$ CK^2, and CK $= \sqrt{a^2 + b^2 - x^2}$; also, (*Con. Sect.* p. 11.) $ab = u \times \sqrt{a^2 + b^2 - x^2}$, and $u^2 = \frac{a^2 b^2}{a^2 + b^2 - x^2}$, therefore $\frac{1}{u^2} = \frac{1}{b^2} + \frac{1}{a^2} - \frac{x^2}{a^2 b^2}$, and $\frac{\dot{u}}{u^3} = \frac{x\dot{x}}{a^2 b^2}$; hence, F $\propto \frac{\dot{u}}{u^3 \dot{x}} \propto \frac{x}{a^2 b^2} \propto x$.

For an *hyperbola*, F $\propto - x$, which shows the force to be repulsive.

Ex. 5. Let it be the *spiral* in Article 32. Here, SY^2 $= \frac{m^2 x^{2m+2}}{t^{2m} + m^2 x^{2m}}$, and $\frac{1}{\text{SY}^2}$ or $\frac{1}{u^2} = \frac{t^{2m}}{m^2 x^{2m+2}} + \frac{1}{x^2}$, therefore

$\frac{\dot{u}}{u^3} = \frac{\overline{m+1} \times t^{2m} \dot{x}}{m^2 x^{2m+3}} + \frac{\dot{x}}{x^3}$; hence, $F \propto \frac{\dot{u}}{u^3 \dot{x}} \propto \frac{\overline{m+1} \times t^{2m}}{m^2 x^{2m+3}} + \frac{x^3}{1}$.

If $m=1$, it is the spiral of *Archimedes*, and $F \propto \frac{2t^2}{x^5} + \frac{1}{x^3}$.

If $m = -1$, it is the *reciprocal* spiral, and $F \propto \frac{1}{x^3}$.

If $m = -2$, it is the *Lituus*, and $F \propto -\frac{x}{4t^4} + \frac{1}{x^3}$.

When the negative part is *greater* than the positive, the force is repulsive, and the curve is convex to the centre; when it is *less*, the force is attractive, and the curve is concave to the centre; but at the point of contrary flexure $F=0$, or $\frac{-x}{4t^4} + \frac{1}{x^3} = 0$, and $x = t\sqrt{2}$, as found in Art. 80. And like circumstances must take place in all cases where $m+1$ is negative.

Prop. CXXX.

The velocity of a body revolving in any curve about a centre of force : velocity of a body revolving in a circle at the same distance, in the subduplicate ratio of the chord of curvature : twice the distance, or in the subduplicate ratio of $\frac{\dot{x}}{x} : \frac{\dot{u}}{u}$.

208. For (Art. 97.) let sr be a sagitta of a circle of curvature to any curve, parallel to the chord CV which passes through the centre of force; then by sim. tri. $sr : Cr :: Cr : CV$, but Cr : the arc Cr ultimately in a ratio of equality; therefore ultimately, sr : arc Cr :: arc $Cr : CV$; hence, arc $Cr = \sqrt{sr \times CV}$; but sr, dato tempore, is as the force, and Cr is as the velocity;

therefore the velocity $\propto \sqrt{\text{force} \times \text{chord curvature}}$; but at the same distance, the force is the same in the circle and in the curve, and the chord of curvature of the circle is its diameter, or twice the distance; therefore the velocity in the curve : velocity in the circle :: $\sqrt{\text{ch. curv. of the curve}} : \sqrt{\text{twice dist.}}$. But the chord of curvature (Art. 101.) is $\frac{2u\dot{x}}{\dot{u}}$; hence, the velocity in the curve : velocity in the circle :: $\sqrt{\frac{2u\dot{x}}{\dot{u}}}$: $\sqrt{2x}$ :: $\sqrt{\frac{\dot{x}}{x}} : \sqrt{\frac{\dot{u}}{u}}$.

Ex. 1. Let the curve be the *logarithmic* spiral. Here, the velocities are equal, because the chord of curvature = twice the distance; or, as $u \propto x$, therefore $\frac{\dot{x}}{x} = \frac{\dot{u}}{u}$.

Ex. 2. Let the curve be an *ellipse* with the force tending to the *focus*. Here, (Art. 207. Ex. 3.) $\frac{\dot{u}}{u^3} = \frac{a\dot{x}}{b^2x^2}$; hence, $\frac{\dot{x}}{x} : \frac{\dot{u}}{u}$ :: $\frac{1}{u^2} : \frac{a}{b^2x}$ :: $\left(\text{as } \frac{1}{u^2} = \frac{2a-x}{b^2x}\right)$ $2a - x : a$; therefore the velocity in the ellipse : velocity in the circle :: $\sqrt{2a-x} : \sqrt{a}$ :: $\sqrt{PH}$: $\sqrt{AC}$.

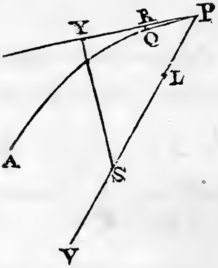
Ex. 3. Let the force tend to the *centre* of the ellipse. Here, (Art. 207. Ex. 4.) $\frac{\dot{u}}{u^3} = \frac{x\dot{x}}{a^2b^2}$; hence, $\frac{\dot{x}}{x} : \frac{\dot{u}}{u}$ :: $\frac{1}{u^2} : \frac{x^2}{a^2b^2}$ :: (as $a^2b^2 = u^2 \times CK^2$) $CK^2 : x^2$; therefore the velocity in the ellipse : velocity in the circle :: CK : x, or CP.

Ex. 4. Let the curve be the *hyperbolic* spiral. Here, (Art. 207. Ex. 2.) $\frac{\dot{x}}{x^3} = \frac{\dot{u}}{u^3}$; hence, $\frac{\dot{x}}{x} : \frac{\dot{u}}{u} :: \frac{1}{u^2} : \frac{1}{x^2} :: x^2 : u^2$; therefore the velocity in the curve : velocity in the circle :: $x : u$.

LEMMA.

If a body revolve in any curve, the velocity (V) *at any point is equal to the velocity which a body would acquire in falling down one fourth of the chord of the circle of curvature passing through the centre of force, supposing the force to remain constant.*

209. By Prop. 45. in the limiting state of the arc PQ, RQ : QP :: QP : PV $= \frac{QP^2}{RQ}$. Now whilst PQ is described by the velocity V, the body is drawn by the force through RQ, and acquires a velocity (v) which,



in the same time, would, if continued uniform, make it pass over 2RQ; and let PL be the space fallen through with the constant force at P, to acquire the velocity V. Then

$$V^2 : v^2 :: PQ^2 : 4RQ^2$$

$$v^2 : V^2 :: RQ : PL, \text{ by } \textit{Mechanics},$$

$$\therefore 1 : 1 :: PQ^2 : 4RQ \times PL;$$

hence, $PL = \frac{PQ^2}{4RQ} = \frac{1}{4} PV.$

210. Cor. Hence, if the curve be a circle, and the centre of force in the centre, a body must fall down half the radius.

Prop. CXXXI.

If a body revolve in a circle about the centre, to find its velocity.

211. Let the force of gravity on the earth's surface be denoted by unity, the radius of the earth by unity, and the velocity of a body revolving about the earth at its surface by unity; and in proportion to these, let $r =$ the radius of any circle, $v =$ the velocity of a body revolving in that circle, and the force $= x^n$; then as a body must fall down $\frac{1}{4}$ of the radius to acquire the velocity in the circle, the force remaining constant, and by *Mechanics*, the velocity varies as the square root of the force and space conjointly, we have $1 : \sqrt{1 \times \frac{1}{2}} :: v : \sqrt{x^n \times \frac{1}{2}x}$; hence, $v = x^{\frac{n+1}{2}}$.

212. Cor. As the periodic time (P) varies as the circumference of the circle directly and velocity (v) inversely, and therefore as the radius (x) directly and v inversely we have $P \propto \frac{x}{x^{\frac{n+1}{2}}} \propto x^{\frac{1-n}{2}}$.

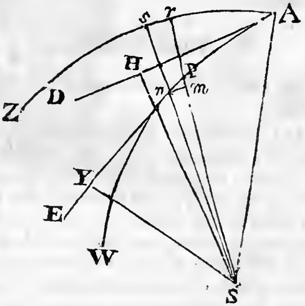
If $n = 0$, $P \propto x^{\frac{1}{2}}$. If $n = 1$, $P \propto x^0 = 1$, or P is constant. If $n = -2$, $P \propto x^{\frac{3}{2}}$.

Prop. CXXXII.

Given the law of force as any power of the distance, to find the curve which the body describes.

213. Let S be the centre of force, and let the body be projected in the direction AD, and describe the curve APW; describe the circular arc AZ with the centre S; draw the tangent PE, on which let fall the perpendicular SY, and SH on AD; also draw Sn

indefinitely near to SP, and nm perpendicular to SP, and produce SP, Sn, to r and s. Put SA $=a$, SH $=p$,



A$r=z$, SP $=x$, $b=$ the velocity at A, $v=$ the velocity at P, P$m=\dot{x}$, $rs=\dot{z}$. Now the velocity being inversely as the perpendicular, $v:b::p:\text{SY}=\frac{pb}{v}$;

therefore $\text{P}y=\sqrt{x^2-\frac{p^2b^2}{v^2}}=\frac{\sqrt{x^2v^2-p^2b^2}}{v}$; and by sim. trian. $\frac{\sqrt{x^2v^2-p^2b^2}}{v}:\frac{pb}{v}::\dot{x}:mn=\frac{pb\dot{x}}{\sqrt{x^2v^2-p^2b^2}}$;

hence, $x:a::\frac{pb\dot{x}}{\sqrt{x^2v^2-p^2b^2}}:\dot{z}=\frac{pab\dot{x}}{x\sqrt{x^2v^2-p^2b^2}}$ expressing the fluxional equation of the curve in terms of the angle described and distance. But (Art. 82. Ex. 7.) $v^2=b^2+\frac{2}{n+1}\times\overline{a^{n+1}-x^{n+1}}$; or if b (the vel. of proj.) : vel. $\left(a^{\frac{n+1}{2}}\right)$ in a circle at the same distance (Art. 211.) : : $m:1$, then $b^2=m^2a^{m+1}$; hence, $v^2=\overline{m^2+\frac{2}{n+1}\times}$ $a^{n+1}-\frac{2}{n+2}\times x^{n+1}$; therefore $\dot{z}=$

$$\frac{pab\dot{x}}{x\sqrt{m^2+\frac{2}{n+1}\times a^{n+1}\times x^2-\frac{2}{n+1}\times x^{n+3}-p^2m^2a^{m+1}}}$$

the fluent of which can only be found in particular cases.

214. At the apsides, SP = SY, or $x=\frac{pb}{v}=$

$$\frac{pb}{\sqrt{m^2+\frac{2}{n+1}\times a^{n+1}-\frac{2}{n+1}\times x^{n+1}}};$$ therefore

$$x\sqrt{m^2+\frac{2}{n+1}\times a^{n+1}-\frac{2}{n+1}\times x^{n+1}}-pb=0,$$ the equation to the apsides. Now to find the *number* of apsides, by squaring the first equation, we get $m^2+\frac{2}{n+1}\times a^{n+1}\times x^2-\frac{2}{n+1}\times x^{n+3}-p^2b^2=0$, which equation (*Algebra*, Art. 358.) may have 4 possible roots when n is an even number, and 3 when n is an odd number; but this being the square of the original equation, some of the roots are introduced by that operation, and the equation to the apsides can never have more than 2 possible roots, so that no orbit can have more than 2 apsides, that is, there are only two different distances of the apsides; but there is no limit to the number of repetitions of these, without their falling upon the same points. If n be -3, or a greater negative number, the equation can have only 1 possible root, and the orbit can have but one apside.

ANNOTATIONS.

On the LIMITING RATIO of VARIABLE QUANTITIES.

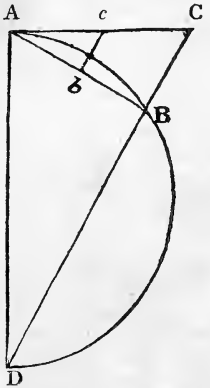
WHEN any quantity increasing or decreasing continually according to a certain law, approaches to a determinate value, and arrives nearer to it than by any assignable difference, but never absolutely equals it, that value is called its limit. Thus when a polygon is inscribed in a circle, and the number of its sides is continually increased, its area and perimeter approach to the area and circumference of the circle, as their limit (Prop. 4. and 6. book 1. Sup. to *Playfair's Geometry*). Hence, if AD be always to the given line AB either as the area of the polygon to

that of the circle, or as the perimeter of the former to the circumference of the latter, then while the polygon, by having the number of its sides increased, approaches to its limit, the point D must move toward B, or AD approach to AB as its limit. The limiting ratio of the polygon to the circle, whether the areas or perimeters be compared, is therefore said to be a ratio of equality.

And here it may be proper to observe, that as the *limiting value* of a perpetually varying quantity, is not an actual value, which it ever absolutely attains, so the *limiting ratio* of two variable quantities, is not a ratio which they bear in any actual state of those quantities. Thus, if we take AD : AB : : polygon inscribed in a circle : the similar polygon described about it, it is manifest that the point D never arrives at B, yet no point can be assigned between A and B which it does not pass.

When variable quantities become infinitely great, or indefinitely small, their *limiting ratio* may frequently be determined, though the quantities themselves, in such state, elude our comprehension.

If AC touch the circle ABD in A, and on the chord AB a right-angled triangle ABC be constructed, then



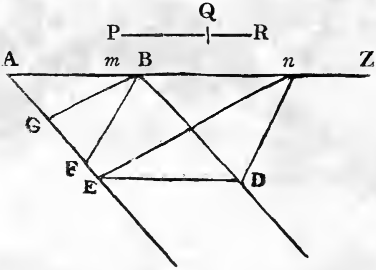
while B moves along the arc until it arrives at A, let the limiting ratio of AC to AB be required.

Produce CB till it meets the circle in D, and join AD; then since ABD is a right angle, AD is a diameter; also, the angle ADB = BAC; whence AC : AB : : AD : DB; but when B arrives at A, BD =

DA ; hence, the limiting ratio of AC : AB is a ratio of equality.

But if the point B move along the given line AB, and BC, or *bc* continue at right angles to AB, then, although the difference of AC and AB becomes less than any that can be assigned ; yet (since A*c* : A*b* : : AC : AB) their ratio is a constant ratio of inequality.

Let two points *m*, *n*, set out from A, B, and move

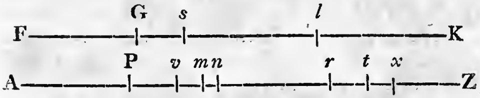


ad infinitum along the right line AZ with velocities which are always in the given ratio : PQ : PR, and let the limiting ratio of A*n* : A*m* be required.

Through A, B draw the parallels AE, BD making any angle with AZ ; make AE = BD = always to A*m*, join *n*, D ; *n*, E ; and draw BF, BG, respectively parallel to *n*D, *n*E.

Since AB : AF : : B*n* : BD (A*m*) : : PR : PQ, a constant ratio, the angle B*n*D = ABF is invariable (5. 6. El.), and AF constant ; also ED = AB, is constant ; but E*n*, D*n* increase without limit ; hence, the angle E*n*D (= GBF) is indefinitely diminished (21. 1. El.) ; consequently the difference of AF and AG becomes less than any assignable ; and since A*n* : AE (A*m*) : : AB : AG ; the limiting ratio of A*n* : A*m* is that of AB : AF : : B*n* : BD (A*m*) : : PR : PQ.

Prop. II.—Assuming the data of the proposition, let P*n* be the increment, which would be uniformly generated with the velocity at m, in the time Pm is described with the accelerated velocity; then Pn is evidently greater than Pm. Take any line GL, and make Gl : Px : : Gs : Pn, Gl : Pt : : Gs : Pm, and Gl :



Pr : : Gs : Pv ; then Gl, Pr may denote the fluxions of FK, AZ at the points G and P (Art. 3. Cor. 1.). Now Pr : Px : : velocity at P : velocity at m ; hence, Pr : rx : : vel. at P : vel. gained while Pm is described; whence, if we diminish the time of description, and consequently the acceleration, rx will decrease, while Pr remains constant: and if the increments be decreased till they vanish, the difference of velocities at P and m will vanish, consequently rx will vanish, or Px become $=$ Pr. But t lies between r and x, therefore Pt $=$ Pr ; and since, in all states of the increments, Gl : Pt : : Gs : Pm, the limiting ratio of Gs : Pm is the ratio of Gl : Pr the ratio of the fluxions.

If Pm be described with a decreasing velocity, take Pn the increment cotemporary with Gs, which would have been generated with the velocity at P ; Pv that uniformly described with the velocity at m, then Gl, Px may denote the fluxions at G and P, and it may be demonstrated as above that the limiting ratio of Gs : Pm, is the ratio of Gl : Px.

Prop. V. The binomial theorem being investigated, in Art. 34. by means of the rule derived from this and the following proposition, a solution of this problem, independent of that theorem, may be as follows.

Given ($\dot{x}$) the fluxion of x, to find the fluxion of x^2.

Let x increase uniformly by v, and become successively equal to $x+v$, $x+2v$, &c.; then x^2 will become $x^2+2xv+v^2$, $x^2+4xv+4v^2$, &c.; hence, the successive increments of x^2 will be $2xv+v^2$, $2xv+3v^2$, &c.;

consequently while x increases uniformly, x^2 does not increase uniformly; therefore to find the ratio of the fluxion of x to that of x^2, we must determine the limiting ratio of the increments. Now the increment of x : increment of x^2 :: $v : 2xv + v^2 :: 1 : 2x + v$ :: (when $v = 0$) $1 : 2x$; therefore by prop. 2. the fluxion of x : fluxion of x^2 :: $1 : 2x :: \dot{x} : 2x\dot{x} =$ fluxion of x^2.

Cor. Hence, the fluxion of $\overline{x+y}|^2 = 2.\overline{x+y} \times \overline{\dot{x}+\dot{y}}$. For put $z^2 = \overline{x+y}|^2$, then $z = x+y$ and $\dot{z} = \dot{x}+\dot{y}$, $\therefore 2z\dot{z} = 2.\overline{x+y} . \overline{\dot{x}+\dot{y}}$.

Hence, prop. 7. as solved in the text, easily follows.

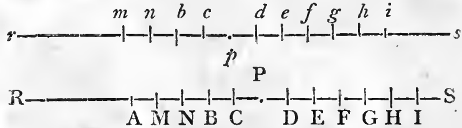
To prop. 7. we may subjoin the following

Cor. 2. The fluxion of a product, divided by that product, is equal to the sum of the fluxions of the several factors, divided by the factors themselves: $\frac{\text{flux. } xyz}{xyz} = \frac{yz\dot{x}}{xyz} + \frac{xz\dot{y}}{xyz} + \frac{xy\dot{z}}{xyz} = \frac{\dot{x}}{x} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z}$; and the same may be shown for any number of factors whatever.

From this corollary the solution of prop. 5. is thus derived.—Since $x^n = x . x . x$, &c. (x being repeated n times) $\frac{\text{flux. of } x^n}{x^n} = \frac{\dot{x}}{x} + \frac{\dot{x}}{x} + \frac{\dot{x}}{x}$, &c. to n terms $= \frac{n\dot{x}}{x}$, $\therefore$ flux. of $x^n = \frac{nx^n\dot{x}}{x} = nx^{n-1}\dot{x}$.

Art. 33. Ex. 4. "This curve is a circle." One of the points S, H will be within, and the other without the circle. (See prop. F, book 6. *Playfair's Geometry*, for a demonstration of this property.)

Art. 42. Let the points p, P, set out at the same time, from b, B, and move along the lines rs, RS, and



while p uniformly describes the equal parts bc, cd, de,

&c. let P describe the spaces BC, CD, DE, such that AB, AC, AD, AE, &c. may be in continued proportion.

Now the ratio of AB : AD is compounded of the two equal ratios AB : AC, and AC : AD; also $bd = 2bc$; the ratio of AB : AE is compounded of three ratios, each equal to that of AB : AC; also $be = 3bc$; in like manner, if any cotemporary values of AP, *bp*, be assumed as AH, *bh*, then whatever number of ratios of AB : AC is contained in the ratio of AB : AH the same multiple is *bh* of *bc*; hence, if *bc* be assumed as the measure of the ratio of AB : AC*, *bh* will measure the ratio of AB : AH.

Ratios compounded of the same number of equal ratios being equal to each other (F. 5. Elem.), we have AB : AE : : AF : AI; and *bi* (the measure of the ratio of AB : AI) $= be + bf$ (the sum of the measures of the ratios of AB : AE, and of AB : AF) In like manner it appears, that of any four proportional terms, the first of which is AB, the measure of the ratio of AB to the last is the sum of the measures of the ratios of AB to the second and third. Hence, AB being taken to represent an unit, AC, AD, &c. numbers forming with unity a series of geometrical proportionals, *bc*, *cd*, &c. any equal numbers, then *bc*, *bd*, &c. will be the logarithms of AC, AD, &c. whose property, as appears from above, is that the logarithm of the product of any two natural numbers is equal to the sum of the logarithms of the factors. For the product is a fourth proportional to the two factors and unity. And hence the principal properties of logarithms are easily inferred.

Again, since AB : AC : : AE : AF; by alternation and division AB : AE : : BC : EF, which ratio of the increments in its limiting state, when the time of description is indefinitely diminished, is the ratio of the velocity of P at B to its velocity at E. Let the velocity of P at B be to the uniform velocity of *p* as 1 : M; whence, by compounding this with the propor-

* See Art. 107.

tion above, $M \times AB : AE ::$ velocity of p ; velocity of P at B; that is, AB being $= 1$, $M : AE ::$ fluxion of be : flux. of AE, or assuming, as in the text, $y =$ any number, and $x =$ its logarithm, $M : y :: \dot{x} : \dot{y}$, $\therefore \dot{x} = M \times \frac{\dot{y}}{y}$.

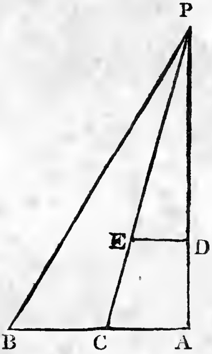
If the points p, P be supposed to move back from b, B toward r, R, still making bn, nm, &c. equal to bc, and AN, AM, &c. the cotemporary values of AP, such that AC, AB, AN, AM, &c. shall be in continued proportion; then bn, bm, &c. are the logarithms of AN, AM, &c.; but bc, bd, &c. the measures of the ratios of unity to greater numbers, being considered as positive, bn, bm, &c. the measures of the ratios of unity to less parts, must be reckoned as negative. Moreover, since the velocity of P varies as AP, or the decrements of AN, AM, &c. are as the quantities themselves, it is manifest that the number of terms in the series AB, AN, AM, &c. before P can arrive at A, must be infinite, but the velocity of p is uniform; therefore the log. of 0 is an infinite negative quantity.

From this elucidation the generation of the logarithmic curve and logarithmic spiral, are very easily shown. For if AS be placed at right angles to bs, with the point A on b, P being at B, and AS be carried along bs, so that A may describe the equal parts bc, cd, &c. while P passes over BC, CD, &c. as before; then the point P will trace the curve called the logarithmic curve. Hence, any abscissa of this curve measured from the point where the ordinate is unity, is the logarithm of its corresponding ordinate. See Art. 49. Ex. 4.

But if the point A be fixed, and AS carried uniformly round, so that a fixed point in it may describe arcs of a circle equal to bc, cd, &c. while P describes BC, CD, &c. as before, then the point P will generate the logarithmic spiral. See Art. 32. Ex. 4.

Art. 67.—" The direction in which the particle will *begin* to move." This conclusion is correct when

AB = AP. In other cases, make PD = AB; and

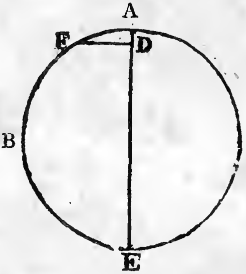


DE (parallel to AB) = PB — PA; join PE; then PD : DE : : force in the direction PA : force in the direction DE; hence, PE is the direction in which the particle will *begin* to move.

Art. 69.—The case of this problem, wherein the attraction varies inversely as the square of the distance, is article 836 of our author's *Complete System of Astronomy*, in which $2pa\dot{x}$ is made the fluxion of the force; hence, the corrected fluent is found $2pax - 2pa^2$, instead of $\frac{2px - 2pa}{x}$. The former of these expressions varies as $x - a$, the latter as $1 - \frac{a}{x}$; of which $x - a$ is increased, and $1 - \frac{a}{x}$ is diminished by increasing the value of x; the author, however, asserts, that $2pax - 2pa^2$ varies as $1 - \frac{a}{x}$; hence, one error is counterbalanced by another.

Art. 82. Ex. 6.—" By Sir I. Newton's *Principia*."

This conclusion may be thus obtained. Let ABE be a circle uniformly described by a revolving body, by



means of a force tending to the centre; AF an indefinitely small arc described in the time 1; draw FD at right angles to the diameter AE; then in the time 1, the body falls through AD, by the action of the centripetal force; but AF^2 in its nascent state $=EA \cdot AD$; now if the time be increased in the ratio of $1 : a$, the square of the arc will be increased in the ratio of $1 : a^2$; also the distance through which the body would fall by the constant central force, will be increased in the same ratio of $1 : a^2$; therefore the arc described in any time is a mean proportional between the diameter of the circle and the distance fallen through in the same time, by the constant action of the centripetal force.

Now the distance which a body falls in $1''$ by the force of gravity at the surface of the earth $= m$; therefore $1''^2 : \frac{p^2 r}{2m}(t''^2) :: m : \frac{p^2 r}{2}$, the distance fallen in t'' by the force at the earth's surface; hence, the arc described in $t'' = pr = \frac{1}{4}$ part of the circumference; whence the time of describing the whole circumference $= 4t'' = 4p \times \sqrt{\frac{r}{2m}}$.

Art. 94.—" Now it is well known," See Art. 209.

$a : a :: 6z$

$a(1+\frac{1}{x}) :: a : 6z(1+\frac{}{x})$

$a(1+\frac{1}{x}) :: a(1+\frac{1}{x}) : 6z(1+\frac{}{x})(1+\frac{}{x})$

$a+x :: a+x$

$a+x$

$ax+x^2$

a^2+ax

$a^2+2ax+x^2$

x

$x : a :: a :: x$

$x : a+x :: a+x : (1+x)(1+x)$