THE PRINCIPLES

OF

THE TRANSFORMER

BY

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This Book is Dedicated

TO

ALBERT CUSHING CREHORE

TO WHOM IT OWES ITS INCEPTION

PREFACE

It is the aim of this book to present in a connected manner the principles of the alternating current transformer. Of the important position taken by the transformer in systems of distribution for light and power, and of the need of a consecutive treatment of the laws and fundamental facts upon which the operation of the transformer depends, little need be said by way of preface. Ten years ago the transformer was born, and in one decade it has attained its maturity. During its development it has been the subject of much investigation and study, and has been carefully considered from every standpoint, so that complete novelty of treatment is now scarcely looked for, — in fact would not be desirable. There is a demand, however, for a united and logical exposition of the principles involved; to this end the writer has turned his efforts, and contributes the following pages.

In 1891, a work on transformer diagrams and the theory of the alternating current transformer was commenced by Dr. Albert C. Crehore and the writer. A systematic, geometrical treatment of the transformer was then undertaken with the idea of developing the principles of the transformer diagram so as to show the action of the transformer and the relation between the various quantities involved. This led to a series of articles entitled "The Theory of the Transformer," published under joint authorship in *The Electrical World*, in Vols. XXI. and XXII. Parallel with the graphical development of the subject, a general analytical treatment was undertaken and presented before the International Congress of Electricians at Chicago in 1893. This, together with some minor papers, likewise appeared under joint authorship, the authors being at that time colleagues in the Department of Physics at Cornell University. This literary partnership was necessarily dissolved by the appointment of Dr. Crehore at Dartmouth College in the year 1893. As it did not seem feasible to continue the work together after that time, the writer continued the work alone and undertook the preparation of this volume; unity and consecutiveness could not well be otherwise obtained. The writer regretted the loss of Professor Crehore's collaboration, as he had taken a leading part in the preparation of many of the earlier papers. Mere reference to these is but a poor indication of the credit the writer desires to ascribe to his associate.

The scope of the book has been increased since the work was undertaken, and much of it is now published for the first time. The subject, however, has been kept within well-defined limits; thus, while systems of distribution are briefly reviewed as bearing directly upon the principles of the transformer, the subjects of fuel and boilers and of central-station operation have been excluded as irrelevant. The theory of the alternator is given in brief.

The author has received valuable assistance from Mr. H. L. Duncan, in the preparation of the chapter on transformer design, and from Mr. J. E. Boyd, in the preparation of the chapter on transformer testing.

Over-abbreviation and the introduction of heterogeneous prefixes and suffixes *ad libitum* the author views as a mania of the day, which proves a source of confusion rather than an aid to the reader. Entire consistency in notation is difficult to attain; the notation in Chapter II. it is hoped will tend toward uniformity in this respect. The introduction of new terms and units has been carefully avoided. No names have been assigned to the magnetic units, for international usage has not yet been established, the recommendations of the American Institute of Electrical Engineers, 1894, and of the British Association for the Advancement of Science, 1895, being at variance. Magnetic quantities are consequently expressed throughout in C. G. S. electromagnetic units, without names. The quantities B and H are represented by italic letters, in conformity with the system of notation given in Chapter II., the small letters b and h being used for instantaneous values. The constants γ_1, γ_2, and ζ have been found convenient, and their general adoption is urged.

The use of vector diagrams in representing quantities that are not simple harmonic functions is an extension which justifies their general use. Hysteresis has been taken into account; it has been treated as producing two effects, — an increase in the primary power electromotive force and a lagging of the magnetization behind the magnetizing force. These effects are pointed out in the earlier portions of the book, but for simplicity the more complete discussion is reserved for a later chapter.

The author wishes to thank several of his students for assistance in proof reading.

F. B.

ITHACA, New York, April 12, 1896.

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THE PRINCIPLES OF THE TRANSFORMER

CHAPTER I

TRANSFORMER SYSTEMS OF DISTRIBUTION

IN the distribution of electrical power for commercial purposes, the chief questions to be considered are the economical transmission of the electrical energy from the place of production to the place of consumption, and the delivery to the consumer at a convenient electrical pressure or voltage. The transmission of power over long distances by means of low pressure circuits is commercially impossible, on account of the size and consequent cost of the conductors needed to convey the large currents. For economical transmission, high pressure is used, inasmuch as the same power may then be transmitted with a smaller current. This is evident when it is borne in mind that the power transmitted depends upon the product of the current and electromotive force. The economy may be twofold: first, in the diminution of the amount of copper required in the line; and second, in the reduction of the loss of power in transmission. For example, let us suppose that we double the pressure at which a certain power is transmitted; the current is then reduced one-half. If we maintain the same current density in the conductor, the amount of copper in the line is also reduced one-half, and its resistance is doubled. The power lost in the line being proportional to the resistance and the square of the current (namely, RI^2) is therefore reduced one-half. The percentage drop in voltage is reduced by a corresponding amount. Consequently high pressure transmission gives us better regulation, smaller outlay for copper in the construction of the line, and smaller losses of power in operation. These advantages increase as we increase the pressure at which we transmit our power, insulation being the only limitation. Power may be successfully transmitted at a pressure of 10,000 volts, and higher pressures are not uncommon.

Although the advantages of high pressure for transmission are thus obvious, we must consider the delivery to the consumer before adopting such a system. The utilization of electric power at such high potentials is impracticable, being both inconvenient and dangerous. If we retain the advantages of high potential in transmission, some means must therefore be employed for reducing the potential at the point of service (the current being at the same time correspondingly increased) to a value suitable for commercial use. The transforming of a high pressure continuous current to a current of low pressure, although possible, is not always feasible. The problem, however, finds ready solution in the **alternating current,** which may be transformed up or down to any desired pressure by means of the alternating current transformer. Furthermore, these transformations may be made with but little loss of energy, the efficiency of a good transformer at full load being 95 or 96 per cent, and the all-day efficiency 85 or 90 per cent. With small transformers the efficiency is somewhat less. A discussion of the principles of the **alternating current transformer** forms the subject matter of this book.

The alternating current transformer has the advantage that it possesses no commutator, nor, in fact, any moving parts; it consists simply of two independent circuits, a primary and a secondary, wound adjacently upon a common core of laminated soft iron. In its simplest form, it is shown in Fig. 1. The primary circuit P, consists of many turns of fine wire, and is connected with the high potential mains; the secondary circuit S, consisting of fewer turns of coarse wire, furnishes current at a low pressure for any purpose for which it is needed.

The primary electromotive force (neglecting for the moment the small drop in potential due to the ohmic resistance of the coil itself) is equal to the product of the number of primary turns, and the rate at which the magnetic flux in the magnetic circuit is changing. Similarly, the secondary electromotive force is equal to the product of the number of secondary turns and the rate of change of the magnetic flux. Now in such a transformer as shown in Fig. 1, much of the magnetic flux would leak across between *a* and *b*, and all the flux produced by the current in one circuit would not thread the other circuit. In the practical construction of transformers, however, the primary and secondary coils are arranged close together, so that nearly all the flux of one circuit passes through the other. This condition of no magnetic leakage, as it is called, is very nearly attained in commercial transformers, the magnetic flux through the primary and secondary being the same. From what has been said above concerning the values of primary and secondary electromotive forces, it is evident that their ratio is the ratio of the number of the primary and secondary turns. This ratio is termed the ratio of transformation. The ratio of the primary and secondary currents is approximately the reciprocal of this ratio of transformation; a reduction in voltage is accompanied by a corresponding increase in current. A **step-down** transformer is one which transforms from a high to a low pressure; a **step-up** transformer converts from a large current at low pressure to a small current at high pressure.

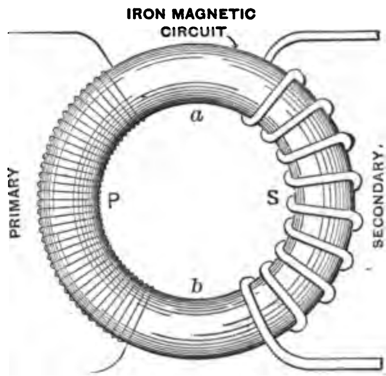
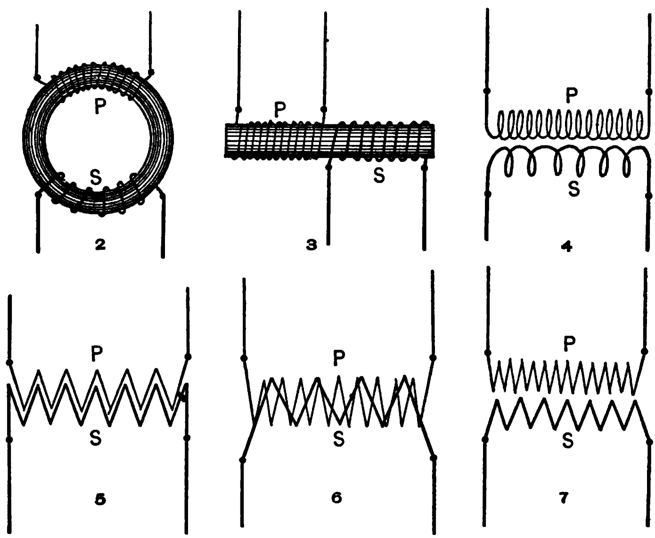


Fig. 1. The step-down transformer (typical).

The action of a transformer may be compared with that of a dynamo; the primary of the transformer corresponding to the field coil of the dynamo, and the secondary corresponding to the armature. In the dynamo an electromotive force is generated in each armature coil, due to the fact that the value of the magnetic flux threading the coil is constantly changing both in magnitude and in direction as the armature revolves.



Figs. 2–7. Diagrammatic representations of transformers.

In the transformer the electromotive force induced in the secondary coil is likewise produced on account of the changing magnetic flux through it. In this case the changes in the flux are due to the changes in the magnitude of primary current. In the dynamo, the conductors on the armature *cut* the lines of force of the field on account of the rotation of the armature; in the transformer, the convolutions of the secondary are, so to speak, *cut by* the lines set up by the primary.

Typical representations of the transformer are shown in Figs. 2–7. Where the distinction is to be made between open and closed magnetic circuit types of transformer, Fig. 2 may be taken to represent the transformer with a closed magnetic circuit, and Fig. 3 one with an open magnetic circuit. The Hedgehog transformer is of this latter type. The closed magnetic circuit transformer commonly prevails. These diagrams must not be taken as illustrations of actual transformers, for, as has been explained, the magnetic leakage would be excessive in a transformer in which the primary and secondary coils are not closely wound side by side.

Fig. 8. Westinghouse transformer, without case.

In practice a compact arrangement of parts is employed on account of the mechanical as well as electrical advantages. Some types of transformers are shown in Figs. 8, 9, and 10. An early form of Westinghouse transformer without its case is shown in Fig. 8. The Stanley transformer is shown in Fig. 9, which gives the case and arrangement of fuse blocks. Fig. 10 shows the details of the Phœnix transformer. The

Fig. 9. Stanley transformer, with case and fuse blocks.

construction of transformers is discussed later in this book. As transformers are commonly erected on poles, or placed upon the outside of houses, each transformer is usually completely encased in an iron box, which affords protection from the weather. For this reason, when we see a transformer in use, we rarely see the construction of the transformer itself, little knowledge of which can be gained from the outside box. Transformers mounted on poles are shown in Fig. 13.

In alternating current distribution in towns, or wherever great distances are not involved, it is customary to have the primary circuits supplied directly by the alternating current generator in the central station, the common electromotive force employed being

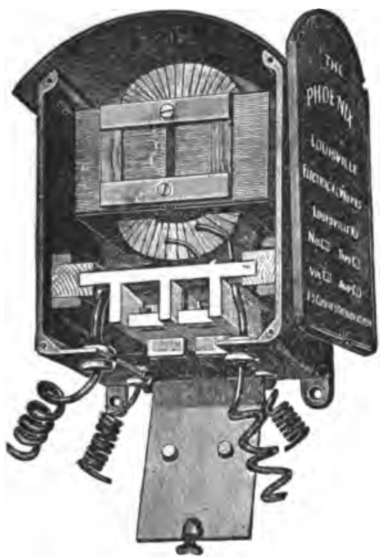


Fig. 10. Phœnix transformer.

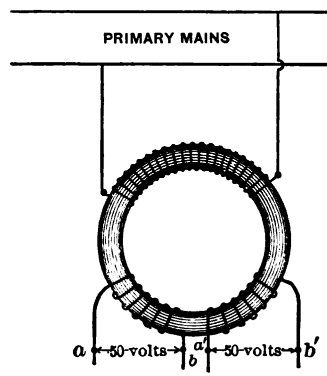


Fig. 11. Transformer, with two independent secondaries.

1000 or 2000 volts. For incandescent lighting, this is transformed down to a pressure of 50 or 100 volts in the secondary, these being the customary pressures* used on incandescent lamp circuits. Very often a transformer will have two 50-volt secondaries which may be connected in parallel to supply current at 50 volts, or may be connected in series where a difference of potential of 100 volts is desired. Such a transformer is shown in Fig. 11.

* Usage differs in this respect. A common primary potential is 2200 volts, with a secondary potential of 55 volts.

For long-distance transmission higher potentials are employed than those used in the systems just described. For example, the generator may produce an electromotive force of 2000 volts which is transformed up in the station by a step-up transformer with a ratio of 1 to 5, to 10,000 volts; at the receiving end there are corresponding step-down transformers, which transform down to the required potential.

Let us consider the alternating current system of distribution commonly employed for incandescent lighting, illustrated in Fig. 12.

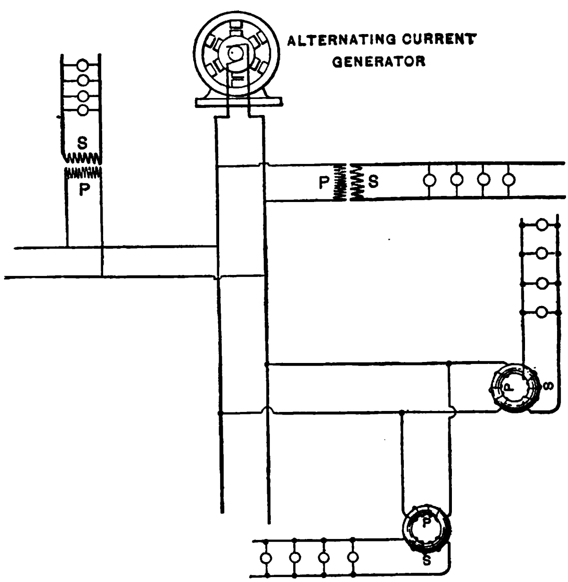


Fig. 12. Constant potential transformer system of distribution.

The alternator (situated in the central station) produces an electromotive force of 1000 or 2000 volts; the primary mains are constantly maintained at this difference of potential, supplying current therefore to the primary coils of the transformers at a constant potential, let us say of 1000 volts. Step-down transformers are employed with secondary circuits wound for 50 or 100 volts; the ratio of transformation is accordingly

20 or 10, this being the ratio of the primary to the secondary turns. The incandescent lamps arranged in parallel in the secondary circuits may be separately turned on or off; the pressure of the secondary mains remains practically constant from no load to full load, falling off but a few per cent in a transformer possessing close regulation.

The regulation of a constant potential transformer is inherent. That the secondary electromotive force remains constant, irrespective of load, provided the impressed electromotive force in the primary is constant, is evident from the explanation above, that the electromotive forces are in proportion to the number of turns in each circuit. There is inherent regulation, likewise, in the matter of the current taken by the transformer from the primary mains and the amount of power thus required for operation, both being practically proportional to the power taken from the secondary of the transformer. When the secondary circuit is open, very little current flows through the primary on account of the high self-induction of the primary circuit. In the design of transformers an effort is made to make this no-load current as small as possible, for a high efficiency is thus obtained. As lamps are turned on in the secondary, and the load is thereby increased, the increase in the value of the secondary current is accompanied by a corresponding increase in the value of the current which flows through the primary. A greater primary current is allowed to flow when the transformer is loaded on account of the apparent diminution in the self-induction of the primary, this diminution being due to the reaction of the secondary current upon the primary. We thus see that a transformer practically regulates itself, taking a primary current which is approximately proportional to the load. The actions and reactions between the primary and secondary circuits of a transformer are fully discussed in the chapters which follow.

The advantages of the alternating current over the direct current, which have been given above, become less marked as the distance of transmission is reduced. In a compactly settled district where electricity is distributed to points near at hand for purposes of light and power, the advantages of the direct current are great. The choice must be made after consideration of the conditions for the particular case at hand, the density of population, water supply, cost of coal, and other conditions affecting the economy of the system. Roughly speaking, the direct current may be used to advantage in districts, all parts of which are within about half a mile of the central station; beyond that it can rarely be operated economically. For longer distances, accordingly, the alternating current is used. Even, however, in a thickly settled district which is, in many respects, favorable for a direct current system of distribution, there are often commercial advantages in favor of the alternating current. The latter system may make it possible for the central station to be located in the outlying parts, thus occupying a cheaper site and being more advantageously placed with reference to the supply of coal and water. The many advantages which may arise from freedom in choice of location are obvious and need merely to be mentioned here.

Instead of the distribution shown in Fig. 12, the system of substations possesses many advantages. In the system just described, which is at present the usual one, each consumer has his individual transformer, the size of which depends upon the amount of power he requires. In the latter system much larger transformers are installed in a number of substations located at convenient points. The secondary circuits of these substation transformers feed into a set of secondary mains from which the consumers obtain their supply. Where the consumers are few and far between, the system employing individual transformers is the more economical. In densely settled districts it is cheaper to have larger transformers and fewer of them, located in substations, for the efficiency of a transformer increases rapidly with its size; and furthermore, so much material is not needed, since the output of a transformer per unit of material is much greater in large than in small transformers. The danger of conveying the high pressure primary currents into the premises of the consumer is also avoided, and better insulation of the high potential circuit may be obtained. A number of small transformers, on the other hand, possess the advantage of affording greater flexibility in the arrangement of the system, and smaller cost of repairs for each unit damaged.

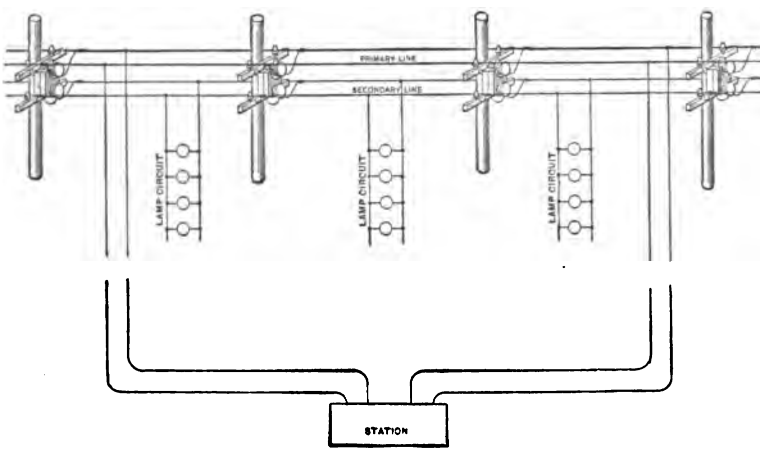


Fig. 13. System of distribution employing a set of secondary mains.

In Fig. 13, a transformer system is shown in which a complete set of primary and secondary mains is carried upon the same pole line. Such a system would not be employed where the consumers are scattered. This system possesses the advantage that when occasion demands, more lamps can be operated at any part of the system than can be supplied by the transformers located just at that part, for some current can be taken from the secondaries of transformers located at a little distance. Smaller transformers, therefore, can be employed than in the usual system, where each set of lamps is operated from its own transformer.

Where the demands are greater than can be supplied from one transformer, a number of transformers may be "banked"; that is, they may be arranged side by side, with their primaries connected in parallel to the primary mains, and their secondaries connected in parallel to the lamp circuit. Corresponding terminals of the secondary should be connected together, otherwise one transformer will short-circuit the others. This

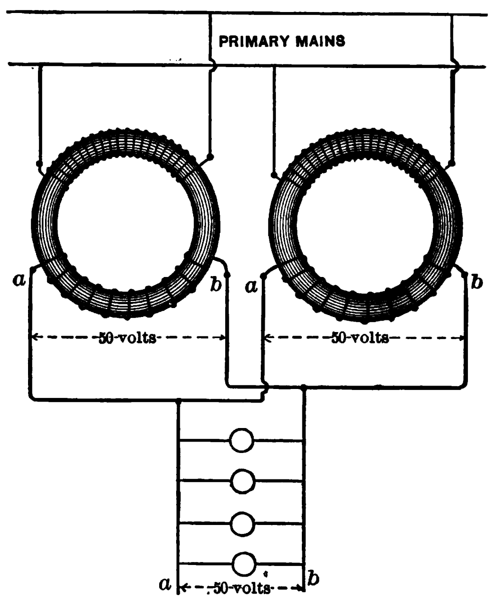


Fig. 14. "Banking" transformers; secondaries connected in parallel.

arrangement is evidently less efficient than the use of one larger transformer. Economy, however, may be gained by cutting out some of the transformers on light load. Where the lights in a building are supplied from a bank of transformers, the plan of supplying separate circuits from each transformer is sometimes adopted, so that total darkness will not ensue in the case of the blowing of a fuse. Fuses are usually located in the primary circuits (one in each leading-in main of every transformer); these are melted by the increase of primary current when the secondary is over-loaded or short-circuited, and thus prevent an excess of current, which might burn out the transformer. In the secondary, fuses are commonly used only in the separate circuits.

The correct arrangement for banking house-transformers is shown in Fig. 14, in which the corresponding terminals *a*, *a*, are connected together; any number of transformers may be

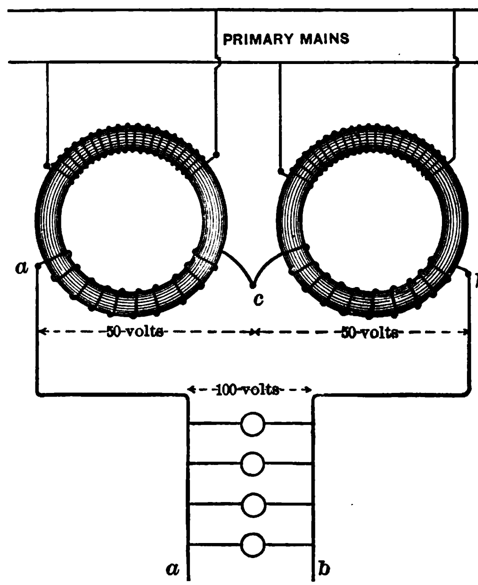


Fig. 15. Two 50-volt secondaries connected in series to supply 100-volt lamps.

so connected. The wrong connection of any one secondary, however, short-circuits all the transformers. This would also result if the primary connections of one of the transformers were reversed; that is, if it were connected up in an opposite sense from the other primaries, for this would reverse the direction of the electromotive force in the secondary.

The electromotive forces of several secondaries may be added by connecting them in series. The proper connection of secondaries in series is shown in Fig. 15. The wrong connection in this case does no damage, for the two equal and opposite electromotive forces in series just neutralize each other, and produce no difference of potential between *a* and *b*.

Where the lamps are located at a distance from the transformer, a secondary difference of potential of 100 or 110 volts is preferable on account of the economy in copper. This voltage is chosen as being the highest for which incandescent lamps

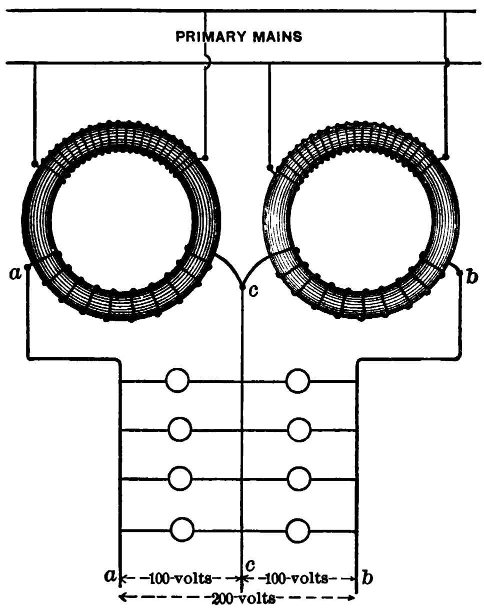


Fig. 16. Transformers connected for three-wire system.

are usually constructed and economically operated. For shorter distances, 50-volt lamps are more commonly employed.

The secondaries of two identical transformers may be connected together for supplying incandescent lamps by the three-wire system, as shown in Fig. 16. In this case each secondary is usually 100 volts; this system may well be used where some or all of the lamps are located at a distance from the transformer. For this purpose, two secondaries on one transformer may be used, instead of two transformers. The secondaries should be so connected that, when the number of lamps on each side of the system is the same, *no current will flow* in the middle wire. If one of the secondaries or primaries should be wrongly connected, the whole benefit of the system would be lost, for the middle wire would then carry the *sum* of the currents in the two side wires. An incorrect three-wire system is shown in Fig. 17.

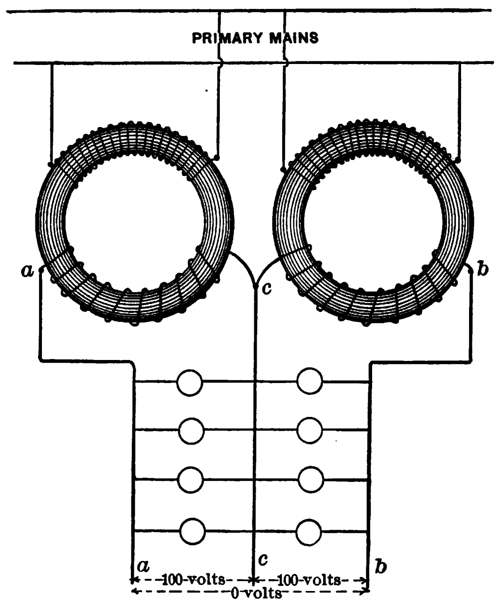


Fig. 17. Incorrect connection for three-wire system.

A convenient and common form of transformer is one with two independent primary and two independent secondary coils, each primary being designed for 1000 volts, and each secondary for 50 volts. The secondaries may be connected in series or parallel for 100- or 50-volt lamp circuits, as desired. The transformer may be operated from 1000- or 2000-volt mains by connecting the two primary coils in parallel or in series; they must be connected, however, with due regard to the *sense* of the winding.

Figure 18 shows the correct connection of primaries in parallel; the corresponding terminals *a* and *a'* are connected to one primary main, A; the terminals *b* and *b'* are connected to the other main, B. The two coils are thus arranged so that the currents in them flow around the coils in the same direction. The wrong connection of primaries in parallel is shown in Fig. 19. In this case the currents in the two coils are opposed

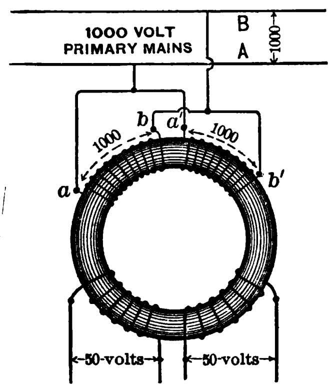


Fig. 18. Correct connection of primaries in parallel.

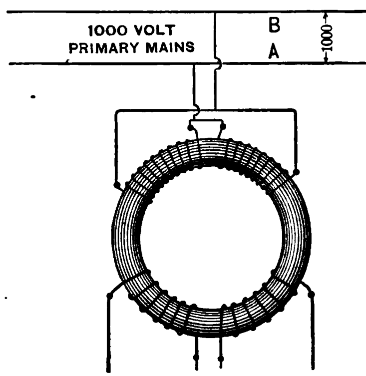


Fig. 19. Incorrect connection of primaries in parallel.

to each other and practically neutralize each other's self-induction. When the transformer is switched in, therefore, there is an excessive rush of current, and the transformer would burn out, unless protected by fuses.

The correct connection of primaries in series is shown in Fig. 20, where two coils, each designed for 1000 volts, are so connected as to practically form one continuous coil, which may be supplied from a 2000-volt main. Figs. 18 and 20 represent the same transformer; the magnetization is the same in each case, and the same current at the same pressure may be taken from the secondaries. If the primaries are connected in series in an opposite sense, as in Fig. 21, the self-induction of one just balances the other's, and the coils are practically non-inductive. The transformer would therefore be burned out when connected to the primary mains.

The usual forms of alternating current distribution have been briefly described. The constant potential system is in almost universal use. There are other systems, however, which

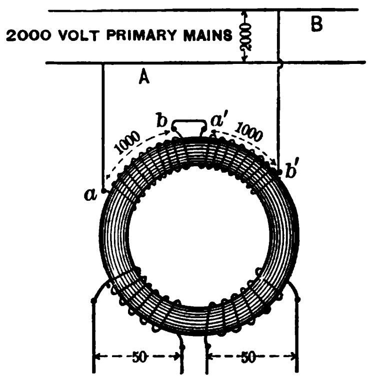


Fig. 20. Correct connection of primaries in series.

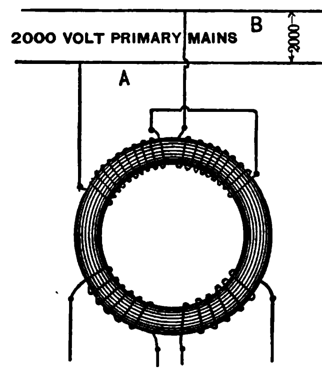


Fig. 21. Incorrect connection of primaries in series.

are not common, but are employed for special cases. The series constant-current system is shown in Fig. 22. This is a system for arc lighting. The transformers have their primaries arranged in series, and are supplied with a constant current. A constant current is produced in the secondary circuit, which

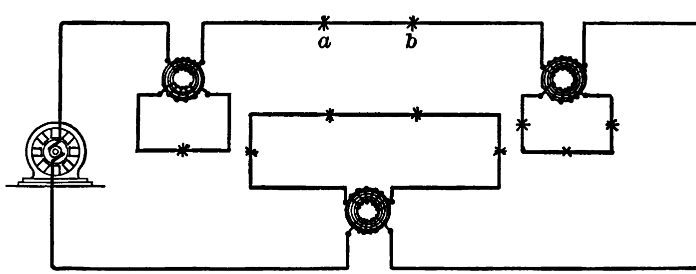


Fig. 22. Series constant-current system.

supplies arc lamps likewise arranged in series. Here the transformation is from a constant primary current to a constant secondary current, whereas in the parallel constant potential system the transformation is from a constant potential to a constant potential. The primary current in the series system may be of any desired strength; it is convenient to have it of the proper strength to operate arc lamps connected directly in the primary circuit, as *a* and *b*. The primary circuit should not be taken indoors on account of the danger from the high potential employed. Interior lighting should, therefore, be from the secondary circuits; but street lamps may be operated directly from the primary mains. In such a system, neither the primary nor the secondary circuit should be opened. Opening the primary circuit would not only cut off the current from all the other transformers, but would occasion a dangerous rise of potential at the dynamo. Opening the secondary circuit would occasion an excessive increase of primary-potential, in order to maintain the normal primary current, for the self-induction of a transformer on open circuit is greater than when the secondary is closed. On no load, either the secondary or the primary of the constant current transformer should be short-circuited. This is the proper way for cutting out the lamp circuit, and is free from danger of burning out, inasmuch as the current cannot exceed its normal value. This is quite the opposite of the requirements in the constant-potential parallel system, in which the short-circuiting of either primary or secondary would cause a burn-out. In constant potential distribution, open circuits are perfectly safe, and are commonly employed for cutting out the apparatus: short circuits must be avoided. In constant current distribution, open circuits must be avoided: short circuits must be employed to cut out apparatus.

Transformers designed with a large amount of magnetic leakage are used to operate arc lamps from constant potential mains. In this system, shown in Fig. 23, the primary circuits of the transformers are arranged in parallel. Transformers A and B are of this type; each of these transformers has in its secondary one arc light or several lamps in series. The transformation from the constant primary potential to an approximately constant current is made possible by the magnetic leakage, as explained in a later chapter. This system of distribution has not come into general use. In Fig. 23, transformers C and D are the usual constant potential transformers for the operation of incandescent lamps.

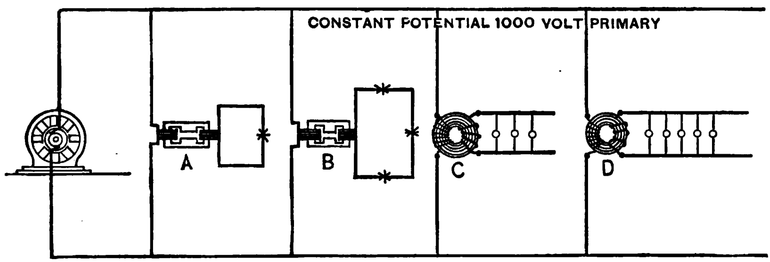


Fig. 23. Constant potential system: arc lights are supplied with an approximately constant current from transformers A and B.

For lighting side streets the system shown in Fig. 24 is used, in which the series of incandescent lamps are supplied from the ordinary constant potential mains. Each lamp has an inductive coil connected in parallel with it, so that if one lamp breaks, the others will not be thrown out.

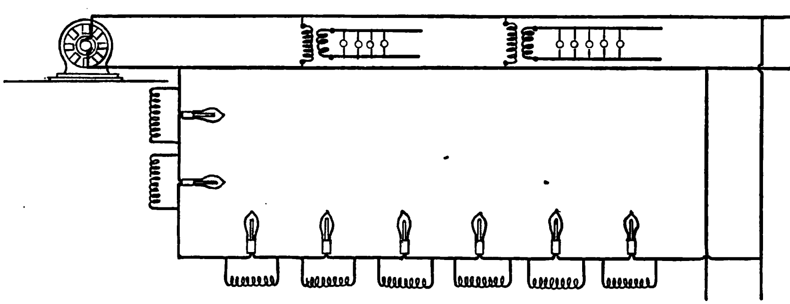


Fig. 24. Series incandescent lamps operated from constant potential system for street lighting.

These are special systems; other systems with compensators and regulators with special applications of the transformer are referred to in a later chapter. It must be borne in mind, however, that the vast majority of transformers are designed to be operated from parallel constant potential mains, and that these are usually referred to when we speak of *transformers*.

CHAPTER II

NOTATION AND CONVENTIONS

For clearness, a uniform notation is employed throughout this book. The notation is an extension of that used in *Alternating Currents.** Many of the recommendations of congresses and other assemblies have been closely followed, and radical digressions from these recommendations or from common usage have been avoided. The notation has been adopted as being most convenient for the clear exposition of the subject in hand. With uniformity of notation, ambiguity may be avoided without a detailed explanation of each symbol upon every occurrence. Where departures occur from the general system of notation given in this chapter, they will be understood from the context.

Symbols for Quantities

(Numbers in parentheses refer to pages where first used or defined.)

A	area ; sectional area of magnetic circuit.
$\mathbf{B}$	induction per square centimeter (29).
C	capacity (71).
E	electromotive force (34).
E_b	back electromotive force (104).
H	magnetizing force (28).
I	current (39).
$\mathfrak{J}$	intensity of magnetization (28).
J	impedance (66).

* Alternating Currents: An analytical and graphical treatise for students and engineers, by Bedell & Crehore, referred to as *Alternating Currents* throughout this book.

K reactance (68, 76).
L coefficient of self-induction, or inductance (34).
M coefficient of mutual induction (36).
M_0 value of M for no magnetic leakage (181).
$\mathfrak{M}$ magnetic moment (28).
N total induction (33).
O origin; center of revolution (24).
P power (45).
Q quantity of electricity (71).
R resistance (46).
$\mathcal{R}$ reluctance (40).
S number of turns (33).
T period (57); or, time.
V potential, or difference of potential (45).
W work or energy (45).

a amplitude.
d distance.
f form factor (311).
l length; length of magnetic circuit.
m strength of pole (28).
n frequency (57).
r radius.
t time.

ζ coefficient of magnetic leakage (181).
η coefficient for hysteresis (33).
$+\theta$ angle, usually of advance (24).
$-\theta$ angle, usually of lag (24).
θ_1 value of ϕ_1 on open circuit: $\tan\theta_1 = K_1 \div R_1$ (97).
θ_2 angle between any position of E_2 and of E_2 on open circuit: $\tan\theta_2 = K_2 \div R_2$ (97).
μ permeability (29).
ξ coefficient for eddy currents (314).
ϕ angle.
ϕ_1 angle between primary electromotive force and current (97).

ϕ_2 angle between primary electromotive force and secondary current (113).
$\phi_2 - \phi_1$ angle between primary and secondary currents (119).
ψ hysteretic angle (396).
ω angular velocity, $2\pi \times$ frequency (57).

Constant electric and magnetic quantities are denoted by italic capitals: thus,

$$E, I, Q, \boldsymbol{B}, \boldsymbol{H},$$

the meanings of which are given above.

For quantities varying harmonically, capitals are used to denote maximum values: thus (55),

$$E, I, Q, \boldsymbol{B}, \boldsymbol{H}.$$

When quantities are varying periodically but not harmonically, capitals denote the maximum values of the equivalent harmonic functions (59).

A bar over a capital letter denotes the virtual value of the quantity designated (square root of the mean square of the instantaneous values): thus (59),

$$\bar{E}, \bar{I}, \bar{Q}, \bar{\boldsymbol{B}}, \bar{\boldsymbol{H}}.$$

The instantaneous values of variable quantities are represented by small italic letters: thus (55),

$$e, i, q, \boldsymbol{b}, \boldsymbol{h}.$$

The equivalent or apparent value of a quantity is indicated by a prime ($'$): thus (109),

$$R', L', C', J', K'.$$

The subscript *one* designates a quantity pertaining to the primary circuit: thus,

$$R_1, L_1, C_1, K_1, J_1, E_1, I_1, e_1, i_1.$$

The primary current of a constant potential transformer with secondary circuit open is represented by I_0 (135). The primary electromotive force after deducting the drop due to ohmic resistance is denoted by E_p, this electromotive force being at right angles to the magnetization (128).

Quantities with the subscript *two* pertain to the secondary circuit: thus,

$$R_2,\ L_2,\ C_2,\ K_2,\ J_2,\ E_2,\ I_2,\ e_2,\ i_2.$$

These quantities refer to the whole secondary circuit, both within and without the transformer. Where the internal and external portions of the secondary circuit need further distinction, the subscripts *int.* and *ext.* are added: thus (129),

$$R_{2_{\text{int.}}};\ R_{2_{\text{ext.}}};\ L_{2_{\text{int.}}};\ L_{2_{\text{ext.}}}.$$

Mathematical Symbols

D symbolic operator, $\frac{d}{dt}$ (190).
c arbitrary constant of integration (199).
f arbitrary function (189).
f' first differential coefficient of the function f (190).
x independent variable.
y dependent variable.
z dependent variable.
ϵ base of Napierian logarithms, 2·718282.
π ratio of circumference to diameter of circle, 3·141593.
Σ sign of summation (41).

Constants and Abbreviations

The following letters stand for certain expressions involving the constants of a transformer; they are employed as abbreviations, but at the same time possess physical significance.

a and b are abbreviations which, in the general case of a transformer with condensers in each circuit, have the values (see 214),

$$a = -\omega^2(L_1L_2 - M^2) + \left(\frac{L_1}{C_2} + \frac{L_2}{C_1} + R_1R_2\right) - \frac{1}{C_1C_2\omega^2},$$
$$b = \omega(R_1L_2 + R_2L_1) - \left(\frac{R_1}{C_2\omega} + \frac{R_2}{C_1\omega}\right);$$
$$a = R_1R_2 - K_1K_2 + M^2\omega^2,$$
$$b = R_1K_2 + R_2K_1.$$

For a transformer without condensers, a and b become α and β with the following values (see 114):

$$\alpha = R_1R_2 - \omega^2(L_1L_2 - M^2),$$
$$\beta = \omega(R_1L_2 + R_2L_1).$$
$$\gamma_1 = \frac{M\omega}{J_1} \text{ (118)},$$
$$= \frac{M}{L_1} = \frac{S_2}{S_1}, \text{ approximately (118).}$$
$$\gamma_2 = \frac{M\omega}{J_2} = \frac{I_2}{I_1} \text{ (115).}$$

c_1, c_2, c', c'' (see context, 206).

τ reciprocal of a time constant.

τ_1, τ_2 (see 198).

τ_1', τ_2'' (see 202).

τ_1'', τ_2'' (see 202).

p and α are used as constants in the discussion of oscillatory charges and discharges (210).

Z, a constant used in the discussion of the magnetic leakage circuit (178).

Other Conventions

In vector diagrams, Roman capitals, which are always used to represent points, are placed at the ends of lines, so that lines may be referred to in the text by the letters at the extremities. They are thus distinguished from letters in Italic, which represent quantities.

In these diagrams, the origin or center about which the various vectors revolve is denoted by the letter O. The diagrams are generally lettered alphabetically, in the order of construction. Thus, in Figs. 64 and 81, the diagram would be constructed by drawing OA, OB, OC, BC, etc., in order. Where possible, corresponding lines have the same lettering in all diagrams. Accordingly, in the diagrams referred to, OH always represents the ohmic* electromotive force $R_1 I_1$; OB always represents the

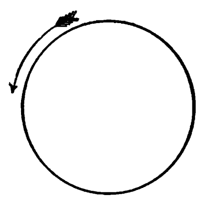


Fig. 25. Positive rotation; direction of advance; positive angle. This is the direction of rotation of all diagrams in this book.

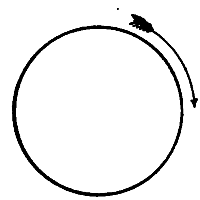


Fig. 26. Negative rotation; direction of lag; negative angle.

electromotive force $M\omega I_1$, induced in the secondary by the primary current, etc. Lines, as OH, OB, etc., frequently referred to, accordingly become known and are understood without consulting a diagram. Figures are numbered consecutively through the book. The numbering of equations begins anew with each chapter.

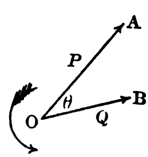


Fig. 27. P is in advance of Q by an angle $+\theta$; Q lags behind P by an angle $-\theta$.

In all polar diagrams positive rotation is counter-clockwise; direction of advance is counter-clockwise, as in Fig. 25.

Negative rotation, negative angle, and direction of lag are in the clockwise direction, as in Fig. 26.

Harmonic quantities are represented in a polar diagram by vectors equal to the maximum values of the quantities represented. Thus, harmonic quantities with

* The electromotive force to overcome ohmic resistance is termed the ohmic electromotive force.

maximum values P and Q are represented by the lines OA and OB in Fig. 27, revolving in the counter-clockwise direction about O. The instantaneous values of these quantities are shown in Fig. 28. These figures illustrate clearly the direction of lag and advance. The method of representing harmonic quantities by curves, as in Fig. 28, becomes complex

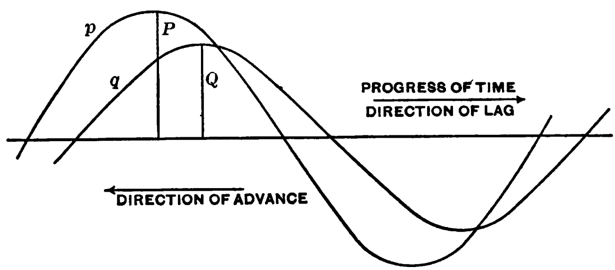


Fig. 28. P is in advance of Q; Q lags behind P, and reaches its maximum *after* P.

when many quantities are involved. The polar diagram of Fig. 27 is employed as being the simplest method of representing the magnitudes of various harmonic quantities and their relative phase positions.

The sense of the direction of a vector is indicated by an arrow, which likewise defines its magnitude. In this book,

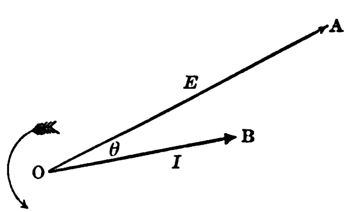


Fig. 29. Vector representation of electromotive force and current.

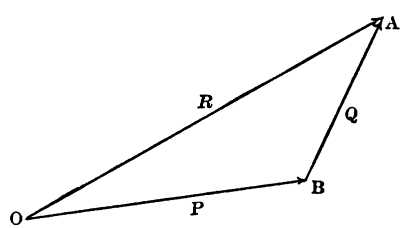


Fig. 30. Vector or geometrical addition.

currents and electromotive forces are chiefly represented by polar diagrams. For convenience, currents will be represented by closed arrowheads, and electromotive forces by open ones. In Fig. 29, the current I is represented as lagging behind the electromotive force E by an angle θ.

Vector quantities may be added by a method corresponding to the parallelogram of forces. In Fig. 30, *R* is the resultant of *P* and *Q*. Any vector quantities may be added together in this way, and upon this principle of vector addition depend the graphical constructions used in this book. Proof of this graphical method of adding currents or electromotive forces is given in Part II. of *Alternating Currents*. The method is explained in the following chapters.

In variation diagrams, as Figs. 81 and 82, an arrow on the locus of any particular quantity indicates the direction of change in that quantity as the independent variable increases.

CHAPTER III

THE MAGNETIC CIRCUIT OF THE TRANSFORMER

THE magnetic field and the magnetic circuit form elements found in all electrodynamic apparatus. The dynamo consists essentially of a magnetic circuit and an armature revolving in the magnetic field which exists in a gap in this magnetic circuit. The electric motor is the counterpart of the dynamo, the torque being obtained by the reaction between a magnetic field and a conductor carrying a current in that field. In alternating current circuits, the constant change in the intensity of the surrounding magnetic field reacts upon the circuit itself and upon all neighboring circuits, thus producing the phenomena of self and mutual induction.

The action of an alternating current transformer depends upon the mutual action between two circuits in no way electrically connected, this mutual action taking place through the medium of their common magnetic field; that is, the magnetic circuit of the transformer. Before we can study the effects of one circuit upon another, we must, therefore, investigate the nature of the magnetic circuit, the medium through which these mutual influences are felt. Let us turn our attention then to the magnetic circuit, giving particular attention to its action in connection with alternating currents.

Elementary Definitions. — Magnetism is the term applied to the phenomena observed in the region surrounding a magnet, or in the neighborhood of a conductor conveying an electric current. In these regions there exists a **magnetic force** which acts upon magnetic substances, such for instance as a compass needle or iron filings, which tend to assume definite positions according to the direction of the magnetic force. A **magnetic field** is any place where there is a magnetic force. The direction of the magnetic force throughout such a field is indicated by the so-called **lines of force,** which may be shown by the arrangement of iron filings sprinkled upon a plate in the neighborhood of the pole of a magnet or near a conductor in which a current is flowing. The intensity of the magnetic force H at any point is numerically equal, in C. G. S. units, to the force in dynes with which a unit pole would be urged when placed at that point. A **unit magnetic pole** is defined as one which exerts a force of one dyne upon another equal pole at a distance of one centimeter. The **strength of field** H at a distance of one centimeter from a unit pole is therefore unity. A pole of strength m is equal to m unit poles, and would accordingly be acted upon by a force Hm dynes when placed in a field of strength H.

The **magnetic moment** of a bar magnet is the product of the pole strength and the distance between the poles; thus a magnet with poles $+m$ and $-m$, has a magnetic moment $\mathfrak{M}=ml$, where l is the distance between the poles. The magnetization of a piece of iron is continuous throughout the whole mass; the **intensity of magnetization** $\mathfrak{J}$ is defined as the magnetic moment per unit volume. For a uniformly magnetized bar with cross-section A and length l, we have, accordingly, $\mathfrak{J}=\frac{ml}{lA}=\frac{m}{A}$; that is, the intensity of magnetization is equal to the pole strength per unit area of polar surface.

Lines of force are conventionally used to indicate the intensity of a field of force at each point, as well as its direction; we consider that the number of lines of force which pass through each square centimeter of area at right angles to the direction of the lines of force represents the intensity of the magnetic field. Thus, when we say that there are 7000 lines of force per square centimeter in any region, we mean that this is the intensity of the magnetic force in that region; or, a unit pole placed in this field would be acted upon by a force of 7000 dynes.

There are evidently 4π lines of force emanating from a unit pole, for the intensity of the field at a distance of one centimeter is unity, and a sphere described with a radius of one centimeter would accordingly have one line of force for each square centimeter of surface, or 4π lines in all. From a pole m there are accordingly $4\pi m$ lines emanating. A magnet with intensity of magnetization $\mathfrak{J}$ will have $4\pi\mathfrak{J}$ lines emanating from each square centimeter of polar surface.

If a piece of iron or other magnetic substance be placed in a magnetic field, it becomes magnetized, and we therefore speak of the magnetic force H as the **magnetizing force**, inasmuch as it is the measure of the force which tends to magnetize the iron. On account of the magnetization of the iron, there will be in addition to the H lines of force per square centimeter, $4\pi\mathfrak{J}$ lines passing from each square centimeter of polar surface. The total number of lines of force per square centimeter is accordingly $B = H + 4\pi\mathfrak{J}$. The total number of lines of force which pass through any space is called the **magnetic induction.** This magnetic induction is, as we have seen, increased by the presence of iron or other magnetic substance, and a magnetizing force of H lines per square centimeter will produce a magnetic induction of B lines per square centimeter, vastly greater than H. A magnetizing force H would produce H lines of force in air or other non-magnetic substance, but in iron the number of lines of force or induction due to this magnetizing force, H, would be B lines per square centimeter, where $B = \mu H$. The quantity μ is termed the **permeability** of the iron, and is dependent upon its quality and the degree of magnetization. The magnetic properties of iron are usually expressed in terms of B, μ, and H.

Magnetic Properties of Iron. — In a dynamo or motor a magnetic circuit of iron is used so that as strong a field as possible may be produced in which the armature may revolve; for the electromotive force generated is directly proportional to this strength of field, — the other factors determining the electromotive force being the speed and the number of conductors on the armature. Iron is used in a transformer to increase the number of lines which the primary current will set up in the magnetic circuit, so as to thread the secondary circuit of the transformer, the electromotive force induced in the secondary being directly dependent upon the amount of magnetic induction which is thus caused to pass through the convolutions of the secondary circuit. The secondary electromotive force is

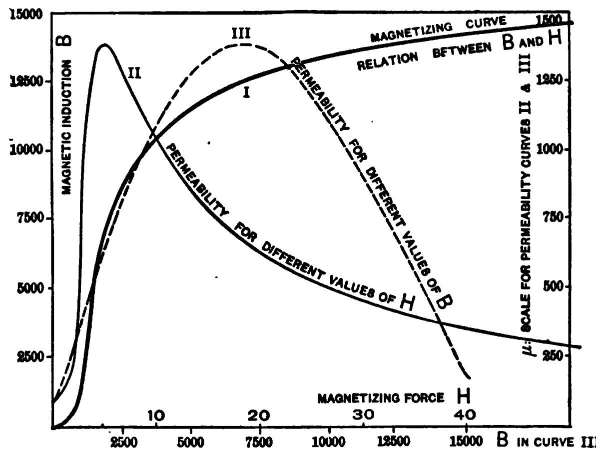


Fig. 31. Curves of magnetic induction and permeability of iron.

likewise dependent upon the number of secondary turns or conductors and the frequency or speed of alternation. There is a close analogy between the production of an electromotive force in the secondary of a transformer and the generation of an electromotive force in the armature of a dynamo. When it is considered that the permeability of iron may be many hundred or even several thousand, which means that the number of lines of force is multiplied by several hundred or thousand, on account of the presence of the iron, the utility of the magnetic circuit is at once obvious.

These magnetic properties are limited to iron (and steel), as far as practical utility is concerned. Nickel and cobalt possess similar properties to a lesser degree. In addition to these three

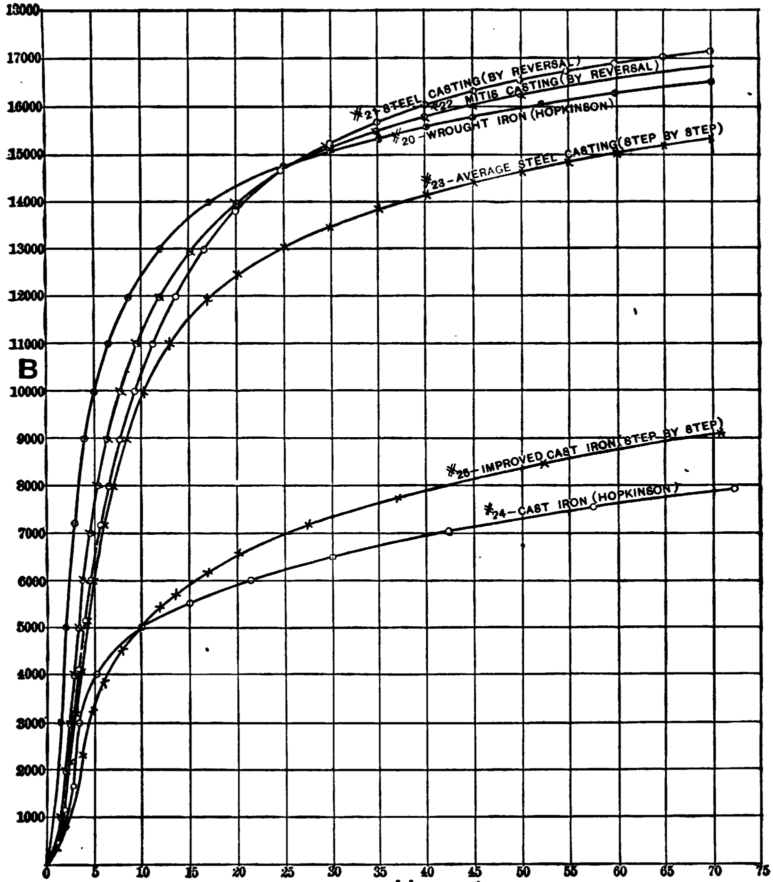


Fig. 32. Curves of magnetization.

may be named chromium, manganese, and a few other substances, which are appreciably magnetic.

In Fig. 31 is given the curve of magnetization for a specimen of sheet iron. The curve of magnetization (curve I) shows the relation between the magnetizing force ***H***, and the magnetic induction *B*, and therefore indicates as well the permeability of the specimen, the permeability being the ratio of *B* to *H*. The value of the permeability thus computed for each value of *H* is shown in curve II. The relation between the permeability and magnetic induction *B* is shown in curve III. The magnetic properties of iron depend upon the quality of the iron and upon the particular sample tested. The magnetization curves* for several samples of sheet and cast iron are given in Fig. 32.

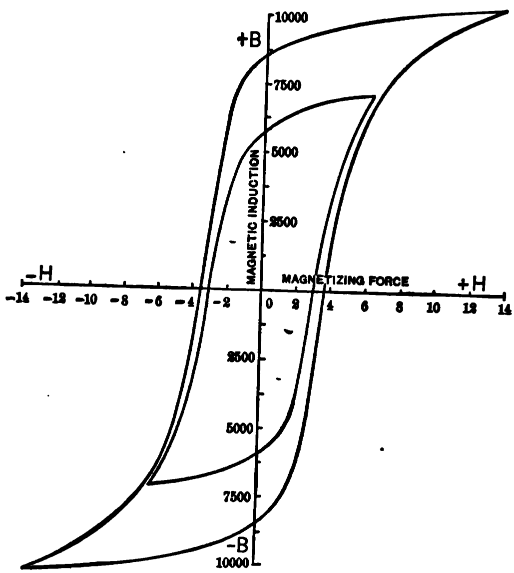


Fig. 33. Hysteresis loops.

Hysteresis. — By hysteresis in iron we mean a lagging of the magnetic induction with relation to the magnetizing force as the iron is carried through a cycle of magnetization. Curves of magnetization for a complete cycle, or "hysteresis loops," as they are called, are shown in Fig. 33. These are taken from the same sample of iron as Fig. 31. The loop depends upon the degree of magnetization to which the iron is taken. The

* The curves in Fig. 32 are from the paper of Thompson, Knight, and Bacon, *Transactions American Institute of Electrical Engineers*, Vol. IX.

name "hysteresis" is due to Professor Ewing, who has been one of the foremost investigators in this field. On account of hysteresis the reversal of the magnetism in the core of a transformer is accompanied by a dissipation of energy, this energy appearing in the form of heat and being radiated from the exposed surface. It is a definite amount for each alternation, this amount depending upon the quality of the iron. Wrought iron of good quality proves best for the construction of transformers, inasmuch as it affords the highest permeability with small hysteresis loss. This loss of energy is proportional to the area of the hysteresis loop, the loss during each cycle in each cubic centimeter of iron being (as proved later in this chapter):

$$\text{Hysteresis loss} = \frac{1}{4\pi}\int H dB. \tag{1}$$

Mr. Steinmetz has shown empirically that this loss per cycle for each cubic centimeter of iron is dependent upon the degree of magnetization; thus:

$$\text{Hysteresis loss} = \eta B^{1\cdot6}. \tag{2}$$

Here the coefficient η is a quantity dependent upon the quality of the iron, being 0·002 or 0·003, approximately, for sheet iron of good quality, such as used in the construction of transformers.

Electromagnetic Induction. — Let us consider a current flowing in a single turn of wire. A magnetic field is set up consisting of a definite amount of magnetic flux or lines of force threading the circuit and forming closed curves. When the current is increased, the amount of magnetic flux is increased. If the permeability of the medium is constant, it is found that the magnetic flux is directly proportional to the current, any change in the current being accompanied by a corresponding change in the induction. If there are S turns in the circuit, the flux N passes through each one of the S turns, and, consequently, there are SN lines threading or linked with the circuit. The quantity

SN may be termed the number of "linkages" or "flux-turns," being the product of the flux and the number of turns through which it passes. Thus, if a conductor has six convolutions or turns and a flux of 15,000 lines passes through these six turns, there are 90,000 flux-turns, or linkages, the effect being the same as if the circuit had but one turn and included 90,000 lines. In each case there are 90,000 lines linked with the circuit.

If the magnetic induction through any circuit be changed, due to any cause whatsoever, an electromotive force is developed in the circuit proportional to the rate of change of the magnetic induction, as first shown experimentally by Faraday. In C. G. S. units, this induced electromotive force is equal to the rate of change of the number of lines linked with the circuit; that is, in a single turn,

$$e = -\frac{dN}{dt},$$

the C. G. S. unit* of electromotive force being defined by this relation. The electromotive force induced in S turns is

$$e = -S\frac{dN}{dt}. \tag{3}$$

The sign indicates that an increase in the induction produces a negative electromotive force,—one which will tend to cause a current to flow in such a direction as to oppose the change in the induction.

Self-Induction.—The change in the induction to which this induced electromotive force is due may be produced by a change in the current flowing in the circuit itself, in which case the induced electromotive force is dependent upon the rate of change of current; that is,

$$e = -L\frac{di}{dt}. \tag{4}$$

Here the coefficient L is a constant of the circuit, termed the **coefficient of self-induction** or **inductance**, which depends upon the form of the circuit.

* The practical unit, the *volt*, is 10^8 times this C. G. S. unit.

From the above relation we have the definition:

The coefficient of self-induction is the ratio of the induced electromotive force in a circuit to the time rate of change of the current producing it.

The practical unit* is the henry. The self-induction of a circuit is one henry when the induced electromotive force is one volt, while the inducing current varies at the rate of one ampere per second. The henry was so defined by the Chicago Congress in 1893.

This electromotive force induced in a circuit when the current is changing is called the counter-electromotive force of self-induction.

$$\text{E. M. F. of self-induction} = -L\frac{di}{dt} = -S\frac{dN}{dt}. \qquad (5)$$

This electromotive force always opposes any change in the value of the current, in accordance with the principle of the conservation of energy. The coefficient of self-induction is constant when the permeability of the medium is constant, as is generally assumed in theoretical discussions. In a circuit with one turn, it follows that $Li = N$; that is, the induction N is proportional to the current and is equal to the current multiplied by the coefficient of self-induction. The coefficient of self-induction may then be defined as the ratio of the magnetic induction to the current producing it. When there are S turns,

$$Li = SN,$$

and

$$L = \frac{SN}{i}. \qquad (6)$$

We may then give a second definition as follows:

The coefficient of self-induction is *the ratio of the number of lines linked with a circuit (flux-turns) to the current producing them.*

* The practical unit for current, the *ampere*, is 10^{-1} times the C. G. S. unit for current; hence (since the volt is 10^8 times the C. G. S. unit), the henry is 10^9 times the C. G. S., unit of induction.

Mutual Induction. — Let us consider two coils, which we will call the primary and the secondary, with S_1 and S_2 convolutions, respectively; let there be a current, i_1, flowing in the primary coil. These coils are sufficiently near for the current in one circuit to set up lines of magnetic induction through the other. If the current in the primary circuit is changing, the induction produced by it through the secondary is changing; and an electromotive force is therefore induced in the secondary in accordance with Faraday's law. This electromotive force, like the counter electromotive force of self-induction, is dependent upon the time rate of change of the current in the primary which produces it; and we may write

$$e_2 = -M\frac{di_1}{dt}. \qquad (7)$$

Here M is the **coefficient of mutual induction,** defined from the above relation as follows:

The coefficient of mutual induction of two circuits is the ratio of the electromotive force induced in one circuit to the time rate of change of the current in the other producing it.

The same unit, the henry, is used for self and mutual induction.

This induced electromotive force may likewise be expressed in terms of the induction threading the secondary due to the current in the primary, being equal to the rate of change of the number of lines thus linked with the secondary, that is, the secondary flux-turns. Thus,

$$e_2 = -S_2\frac{dN}{dt}, \qquad (8)$$

where N is the flux through the secondary convolutions produced by the primary current. The coefficient of mutual as of self induction is constant when the permeability of the medium is constant. We may then write

$$Mi_1 = S_2N,$$

as the number of lines linked with the secondary when a current i_1 is flowing in the primary.

The coefficient of mutual induction of two circuits may then be given a second definition as follows:

The coefficient of mutual induction is the ratio of the number of lines linked with one circuit to the current in the other circuit which produces those lines.

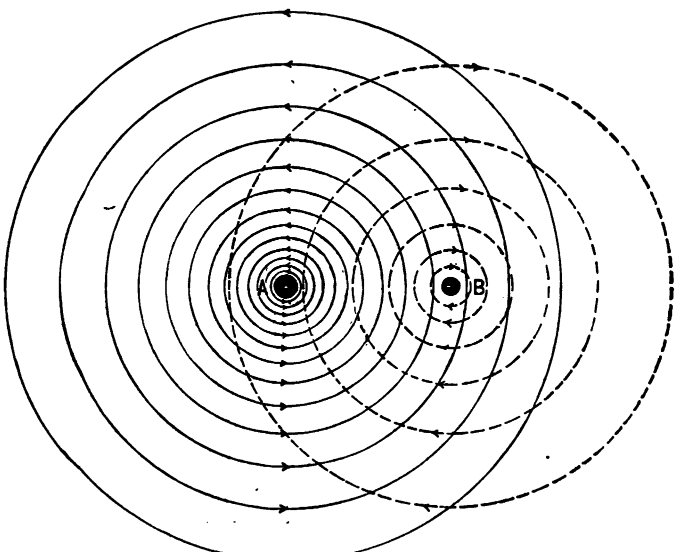


Fig. 34. Magnetic field around a conductor, A, carrying a current in a direction up from the paper; the *creation* of this field induces a current in a neighboring conductor, B, in an opposite direction.

Evidently where no induction from the primary circuit of a transformer threads the secondary circuit, there can be no electromotive force induced in the secondary due to a change in the primary current. The relation between two circuits is strictly a mutual one, the coefficient of mutual induction having the same value with either circuit as primary or secondary, as will be seen later.

The effects of mutual induction may be made more clear by a consideration of the electromotive force produced by the cutting of lines of force. In Fig. 34 a current is flowing in the conductor A in a direction up from the paper. When this current is *set up*, lines of force encircling the conductor are sent out like the ripples from a stone thrown into a pond. These rings remain stationary when the current is steady, and collapse when it decreases. The neighboring conductor, B, is cut by these lines as they go forth or collapse, and so has induced in it an electromotive force when the current in A is either increasing or decreasing. If B is a portion of a closed circuit, a current will flow in it. It is evident that an increasing current in A induces in B a current in the opposite direction, as shown in the figure. The conductor B is therefore encircled by lines of its own magnetic field (shown by the dotted lines in the figure), which in turn cut A, and thereby react on it. This illustrates the action of a transformer. The secondary B has an electromotive force induced in it by the changes of the primary current in A. The primary A has induced in it a counter-electromotive force due to its own self-induction, and a back electromotive force due to the mutual induction of B.

The Law of the Magnetic Circuit.—In order to determine the action between the primary and secondary circuits of a transformer through the medium of their common magnetic field, we must ascertain the amount of magnetic induction which will be set up by a current I flowing through a coil of S turns which embrace a magnetic circuit of known permeability.

When a current flows through a solenoid, as in Fig. 35, a magnetic field is set up, consisting of a definite number of lines of force threading the solenoid. The magnetizing force of the solenoid is proportional to the number of convolutions, S, and the current, I, flowing through them; or, more briefly, it is proportional to the number of ampere-turns, SI. We wish to ascertain the definite relation

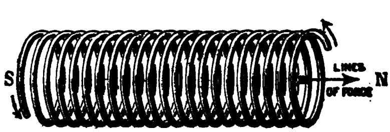


Fig. 35. Solenoid carrying a current; equivalent to a magnet with poles N and S.

between the magnetizing force in a magnetic circuit and the number of ampere-turns which produce it.

We have seen that a magnetic field exists in the neighborhood of a magnet or an electric current; the same magnetic effects are produced by either, a solenoid behaving the same as a magnet, as shown in Fig. 36.

If we have a single circuit of wire in which the current I is flowing, inclosing an area A, it may be shown by experiment that the magnetic moment of this loop of wire is IA, the product of the current and the area of the circuit.* The conductor carrying the current I may then be replaced, so far as magnetic effects are concerned, by a magnet whose magnetic moment is

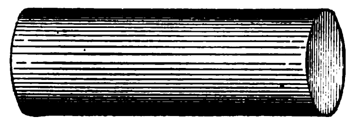


Fig. 36. Bar magnet with magnetic moment $\mathfrak{M} = ml$; equivalent to solenoid in preceding figure.

$$ml = IA.$$

If the permeability of the medium is μ, and there are S turns instead of one, the magnetic moment is correspondingly increased, and we have

$$\text{Magnetic moment} = ml = \mu ASI;$$

or,
$$m = \frac{\mu ASI}{l}. \qquad (9)$$

The conductor carrying the current is then equivalent to a magnet with uniform cross-section A, length l, and strength of pole m; but we have seen above that $4\pi m$ lines emanate from a pole of strength m.

Hence we may write for the magnetic flux,

$$N = 4\pi m = 4\pi \frac{\mu ASI}{l} = \frac{4\pi SI}{\frac{l}{A\mu}}.$$

Where we have a closed solenoid, wound uniformly about an anchor ring of uniform cross-section, as in Fig. 36 a, m is the

* Lodge, *Modern Views of Electricity*, p. 388.

pole strength of any surface formed by cutting the anchor ring.

If I is the value of the current in amperes, and other quantities are expressed in C. G. S. units as above, we have

$$N=\frac{\frac{4\pi SI}{10}}{\frac{l}{A\mu}}. \qquad (10)$$

This gives us the following law of the magnetic circuit, first formulated by Professor Rowland:

$$\text{Magnetic flux}=\frac{\text{magnetomotive force}}{\text{reluctance}}=\frac{\text{M. M. F}}{\mathcal{R}}.$$

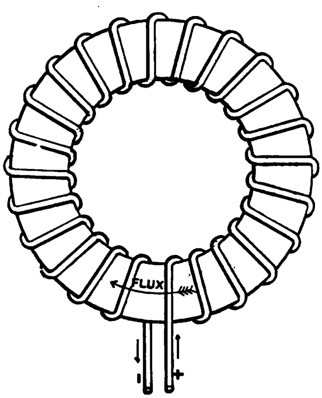


Fig. 36 a. Solenoid, wound uniformly upon a closed magnetic circuit.

This is the analogy of Ohm's law. The first member, the magnetic flux, is the analogue of current. The **magnetomotive force** (M. M. F.) is the analogue of electromotive force; it is equal to $\frac{4\pi SI}{10}$, or 1·26 times the number of ampere-turns. The **reluctance,** $\mathcal{R}$, or magnetic resistance, $\frac{l}{A\mu}$, is analogous to electrical resistance; it varies directly as the length of the magnetic circuit, and inversely as the cross-section and permeability. From this law of the magnetic circuit we can compute the magnetic flux which will be set up in any magnetic circuit by a certain number of ampere-turns.

Although the laws of the electric and the magnetic circuit are similar in many respects, it must not be supposed that these analogies hold throughout. It requires no energy to maintain the magnetic flow when once established; there is, therefore, no analogue to Joule's law for the energy continually being dissipated in heat during the current flow. There is a difference, too, between electric resistance and magnetic resistance or reluctance, inasmuch as the former is independent of current (temperature remaining constant), while the value of the reluctance of a magnetic circuit depends upon the value of the magnetic flux.

The induction $\boldsymbol{B}$, per square centimeter, may be obtained by dividing the magnetic flux N, by the cross-section A of the magnetic circuit. The magnetizing force $\boldsymbol{H}$ is equal to $\frac{\boldsymbol{B}}{\mu}$;

hence,
$$\boldsymbol{H} = \frac{N}{A\mu} = \frac{4\pi SI}{l}, \qquad (10a)$$

where the current I is in the C. G. S. units. When the current is expressed in amperes,

$$\boldsymbol{H} = \frac{4\pi SI}{10\,l} = 1{\cdot}26\frac{SI}{l}.$$

The magnetic force $\boldsymbol{H}$ is equal to the magnetomotive force per unit length; or, it is equal to $\frac{4\pi}{10}$ times the ampere-turns per unit length. These same relations may be otherwise expressed as follows: The magnetomotive force is equal to $\boldsymbol{H}l$, the product of the magnetizing force and the length of the magnetic circuit; or, it is equal to the number of ampere-turns multiplied by $\frac{4\pi}{10}$.

The relations, as described above, hold true for a uniform magnetic circuit, — one with constant cross-section and permeability. The results may readily be extended to a magnetic circuit consisting of several portions, with different cross-sections and permeabilities.

For such a circuit we have

$$\text{Reluctance} = \mathcal{R} = \frac{l_1}{A\mu_1} + \frac{l_2}{A\mu_2} + \frac{l_3}{A\mu_3} + \cdots = \Sigma\frac{l}{A\mu}.$$

The reluctance of the whole circuit is found by adding together the values of the reluctance found for each part separately.

We accordingly have

$$\text{Magnetic flux} = \frac{\frac{4\pi SI}{10}}{\Sigma \frac{l}{A\mu}}. \qquad (11)$$

The magnetomotive force for the several portions of the magnetic circuit is H_1l_1, H_2l_2, H_3l_3, etc.; and for the whole magnetic circuit is

$$\text{Total magnetomotive force} = H_1l_1 + H_2l_2 + H_3l_3 + \cdots = \Sigma Hl.$$

The more general form may be written

$$\text{Magnetomotive force} = \int Hdl,$$

which is called the *line integral of magnetizing force.*

The relation between the magnetizing force and the ampere-turns is written

$$\int Hdl = H_1l_1 + H_2l_2 + H_3l_3 + \cdots = \frac{4\pi SI}{10}. \qquad (12)$$

Here H_1, H_2, H_3, etc., are the values of the magnetizing force in each portion, l_1, l_2, l_3, etc., of the magnetic circuit.

Values of the Coefficients of Self and Mutual Inductions. — The values of the coefficients of mutual and self induction of two circuits can never be exactly obtained by computation; but in the case of two coils placed together upon the same iron core or ring, the formulæ based upon the assumption of no magnetic leakage are approximately correct; that is, each of the convolutions is supposed to be threaded by the same amount of magnetic flux.

Let us consider two circuits, which we will call the primary and the secondary, and let S_1 and S_2, respectively, represent the number of turns or convolutions in each. The coefficient of mutual induction, according to the second definition above, is the number of lines linked with the secondary circuit when there is unit current in the primary; that is, M is equal to the secondary flux-turns, S_2N, due to unit primary current. If there is no magnetic leakage, — that is, if all the lines set up by the primary current pass through the secondary circuit, — the value for M may be readily calculated from the law of the magnetic circuit. The magnetomotive force for unit primary current is $4\pi S_1$. The reluctance is $\Sigma\frac{l}{A\mu}$, where l, A, and μ may have different values in each term of the summation, according to the particular part in the circuit to which each term refers. The total magnetic flux is equal to the magnetomotive force divided by the reluctance, and in this case

$$N = \frac{4\pi S_1}{\Sigma\frac{l}{A\mu}}. \tag{13}$$

Now, every turn of the secondary is threaded by this magnetic flux N. The coefficient of mutual induction is obtained by multiplying this flux by the number of secondary turns; thus,

$$S_2N = M = \frac{4\pi S_1 S_2}{\Sigma\frac{l}{A\mu}}. \tag{14}$$

This is the general expression for the mutual induction of two circuits, irrespective of whether the magnetic circuit is continuous or not, provided there is no magnetic leakage. Of the several terms in the summation expressing the reluctance, some may refer to parts of the magnetic circuit through iron, and others to parts through air, and these terms cannot always be computed.

The value of the coefficient of self-induction for the primary is obtained in the same manner. The expressions for magnetomotive force and reluctance are the same as before. The magnetic flux N, obtained by dividing the one by the other, passes through each primary turn. Multiplying this flux by S_1, the number of primary turns, we get the total number of lines linked with the primary due to unit primary current. By definition this is the coefficient of self-induction of the primary, and so

$$S_1 N = L_1 = \frac{4\pi S_1^2}{\Sigma \frac{l}{A\mu}}. \tag{15}$$

The coefficient of self-induction of the secondary is similarly found by considering unit current flowing in the secondary, and is

$$L_2 = \frac{4\pi S_2^2}{\Sigma \frac{l}{A\mu}}. \tag{16}$$

The same expression is obtained for the coefficient of mutual induction when we consider unit current in the secondary as was obtained above by considering unit current in the primary.

The values for the coefficients of mutual and self induction given in equations (14), (15), and (16) are true irrespective of the uniformity or continuity of the magnetic circuit. The expression for the mutual induction (14) only holds true, however, when there is no magnetic leakage between the primary and secondary. The reluctance $\Sigma \frac{1}{A\mu}$ has the same value in each of the above equations, and therefore

$$M = \sqrt{L_1 L_2} = \frac{S_2}{S_1} L_1 = \frac{S_1}{S_2} L_2, \tag{17}$$

and

$$\frac{L_1}{L_2} = \frac{S_1^2}{S_2^2}. \tag{18}$$

The relation for mutual induction only holds true in case of no magnetic leakage. If there is magnetic leakage, and all the lines do not pass through the two circuits, the coefficient of mutual induction is diminished. The percentage by which M is diminished is the percentage of magnetic leakage.

In the case of a closed iron magnetic circuit of uniform cross-section, l, A, and μ have but one value for all parts of the circuit, and there is therefore only one term in the summation. The formulæ then become:

$$M = \frac{4\pi S_1 S_2 A\mu}{l}. \tag{19}$$

$$L_1 = \frac{4\pi S_1^2 A\mu}{l}. \tag{20}$$

$$L_2 = \frac{4\pi S_2^2 A\mu}{l}. \tag{21}$$

These are the expressions for the coefficients of self and mutual induction which are of practical value in the study of the closed-magnetic-circuit transformer.

Power and Energy. — The maintenance of a current in a circuit containing resistance or a counter-electromotive force requires an expenditure of energy, the rate of this expenditure of energy at any time being proportional to the current and electromotive force at that time. **Power,** sometimes termed **activity,** is defined as the rate of doing work or of consuming energy; that is,

$$P = \frac{dw}{dt},$$

where P is power, and dw is the work done or energy supplied in the time dt. Where the rate of doing work is constant, we have,

$$P = \frac{W}{T};$$

or

$$W = PT.$$

With a current I and electromotive force E, both constant in value, the power or activity is equal to their product; thus,

$$P = EI,$$

and

$$W = EIT,$$

the last expression* being the energy expended in the time T.

* This may be obtained by putting E for V, and IT for Q, in the expression $W = QV$. It may also be derived from the work done in moving a conductor of unit length in unit field, with velocity v. The conductor will be acted on by unit force, if the current it carries be unity. There will be an electromotive $E = v$, generated in it, due to the cutting of lines of force at the rate of v per second. If this produces a current I, the conductor is acted upon by a force I. The work done, being the product of force and distance, is $W = IvT = IET$. Q.E.D.

With a varying current and electromotive force, the power is

$$P = ei.$$

The general expression for the energy supplied to the circuits in the time dt is

$$dw = eidt. \qquad (22)$$

In this case the rate of expenditure of energy is changing from instant to instant.

Energy expended in Resistance.—If the energy is all expended in ohmic resistance, in which case it is dissipated in the form of heat, we may write, in accordance with Ohm's law,

$$e = Ri.$$

Hence, from equation (22), in the time dt,

$$\text{Energy expended in resistance} = Ri^2dt. \qquad (23)$$

The rate at which energy is expended in heat, in the resistance R, is Ri^2; where the current has a constant value, this may be written RI^2. That the energy dissipated in heat is proportional to the resistance and the square of the current was first shown experimentally by Joule.

Where the values of the resistance, current, and electromotive force are given in ohms, amperes, and volts, respectively, energy is expressed in terms of the practical unit, the *joule*, which is 10^7 times the C. G. S. unit of energy, the *erg*. The practical unit of power is the *watt*, which represents an expenditure of energy at the rate of one *joule per second*. The value of the power in watts is found directly by taking the product of the current in amperes and the electromotive force in volts. The English unit of power, commonly used by mechanical engineers, is the *horse-power*, one horse-power being equal to 745·9 watts.

Energy of a Magnetic Field.—Inasmuch as there is an electromotive force induced in a circuit when the current in the circuit is changing, there must be a certain amount of energy required to cause the current to flow against this counter-electromotive force. When the current is increasing, the counter-electromotive force of self-induction, due to the change in the magnetic field, is in such a direction as to oppose the flow of current. A positive electromotive force is required to overcome this electromotive force (which, by Faraday's law, is negative for increasing current), and positive work is therefore done, a certain amount of energy supplied to the circuit being stored up in the magnetic field. Conversely, when the current is decreasing, energy previously stored in the magnetic field is returned to the circuit. The energy imparted to or derived from the magnetic field in the time dt may readily be found from the general expression for energy, $eidt$, by substituting for e the value of the electromotive force necessary to overcome the counter-electromotive force of self-induction, viz.:

$$L\frac{di}{dt}.$$

Equation (22) then gives, for the time dt,

$$\text{Change in energy of magnetic field} = Li\frac{di}{dt}dt. \qquad (24)$$

The energy of the field when the current has any particular value, I, is obtained by integrating the above expression between the limits 0 and I, which gives:

$$\text{Energy of magnetic field} = \tfrac{1}{2}LI^2. \qquad (25)$$

The energy of a magnetic field may be determined in terms of $\boldsymbol{B}$ and $\boldsymbol{H}$ as follows. From above we have:

$$dw = Li\frac{di}{dt}dt.$$

With $S\frac{dN}{dt}$ substituted for $L\frac{di}{dt}$, this becomes

$$dw = Si\frac{dN}{dt}dt.$$

But

$$Si = \frac{\boldsymbol{H}l}{4\pi}; \text{ and } \frac{dN}{dt} = A\frac{d\boldsymbol{B}}{dt}.$$

Hence $$dw = \frac{1}{4\pi} HlA \frac{dB}{dt} dt;$$

or $$W = \frac{\text{vol.}}{4\pi} \int H dB. \tag{25a}$$

This value has already been discussed in connection with the hysteresis loop in Fig. 33. When the permeability is constant, the value for the coefficient of self-induction, L, may be readily obtained by writing,

$$\tfrac{1}{2}\mu H^2 \text{ for } \int H dB.$$

It then follows (from 25 a) that

$$W = \frac{1}{2}\frac{lA}{4\pi}\mu H^2.$$

But $$W = \tfrac{1}{2} LI^2;$$

hence, using (10 a), $$L = \frac{4\pi S^2 A\mu}{l}.$$

This value is identical with the one previously obtained.

Total Energy of a Single Circuit. — Expressions have been given above for the energy imparted to a circuit, the energy expended in ohmic resistance and that stored in the magnetic field. It follows from the principle of the conservation of energy that the total energy supplied to a circuit is equal to the sum of the several expenditures. In a single circuit, therefore, in which the only dispositions of energy are those just discussed, — namely, the dissipation of energy in heat and the storing of energy in the magnetic field, — we may equate the energy supplied to the energy expended in heat and stored in the field, as follows:

$$eidt = Ri^2 dt + Li\frac{di}{dt}dt. \tag{26}$$

In the second member of this equation, the first term, expressing the energy expended in ohmic resistance, is always positive for all values of the current. The last term may be positive or negative, changing periodically in the case of an alternating current. It is zero when the current is unchanging.

Energy of Two Circuits. — Let us consider two circuits in which the currents i_1 and i_2 are flowing; the coefficients of induction are L_1, L_2, and M. In addition to the electromotive forces, $R_1 i_1$, and $R_2 i_2$, necessary to overcome ohmic resistance, electromotive forces are needed to overcome the counter-electromotive force of self-induction and the back electromotive force of mutual induction in each circuit. The current i_1 requires an electromotive force $L_1 \frac{di_1}{dt}$ to overcome the counter-electromotive force induced in the circuit by the current i_1 itself, and an electromotive force $M\frac{di_2}{dt}$ to overcome the back electromotive force induced in the one circuit by the current i_2 in the other. The current i_2 requires similar electromotive forces to cause it to flow. The total electromotive forces active in the two circuits (which we will call the primary and the secondary) are,

$$\text{Total primary E. M. F.'s} = R_1 i_1 + L_1 \frac{di_1}{dt} + M\frac{di_2}{dt}; \qquad (27)$$

$$\text{Total secondary E. M. F.'s} = R_2 i_2 + L_2 \frac{di_2}{dt} + M\frac{di_1}{dt}. \qquad (28)$$

The energy required to maintain a current, being equal to the product of the electromotive force, current, and time, we have for the time dt,

$$\text{Primary energy changes} = R_1 i_1^2 dt + L_1 i_1 \frac{di_1}{dt} dt + M i_1 \frac{di_2}{dt} dt; \ (29)$$

$$\text{Secondary energy changes} = R_2 i_2^2 dt + L_2 i_2 \frac{di_2}{dt} dt + M i_2 \frac{di_1}{dt} dt. \ (30)$$

The first term in each of these equations is always positive, and denotes the energy dissipated in heat; the remaining four terms represent the changes in the energy of the magnetic field in the time dt.

The energy imparted to the magnetic field in the time dt is accordingly,

$$dw = L_1 i_1 \frac{di_1}{dt} dt + M i_1 \frac{di_2}{dt} dt + M i_2 \frac{di_1}{dt} dt + L_2 i_2 \frac{di_2}{dt} dt. \qquad (31)$$

E

There is a constant interchange of energy between the primary circuit and the magnetic field, and between the magnetic field and the secondary circuit; and hence between the primary and secondary through the common magnetic field.

When the primary and secondary currents have particular values, I_1 and I_2, the energy of the magnetic field is found by integrating the above differential equation; thus,

$$W = \tfrac{1}{2} L_1 I_1^2 + M I_1 I_2 + \tfrac{1}{2} L_2 I_2^2.$$

The energy of the magnetic field is thus seen to be composed of three parts corresponding to the three terms in the equation. The first and last terms give the value of the energy of the primary and secondary circuits, each considered separately. The term MI_1I_2 is the mutual energy due to their juxtaposition, so that the induction produced by the current in one circuit threads the other circuit. Where this is not the case, the coefficient of mutual induction is zero, and the mutual energy is likewise zero.

It is by means of the magnetic field that energy is transferred from the primary to the secondary of a transformer.

APPENDIX

Relation between Practical and C. G. S. Units

One Coulomb	$= 10^{-1}$	C. G. S. electromagnetic unit.
One Ampere	$= 10^{-1}$	C. G. S. electromagnetic unit.
One Volt	$= 10^{8}$	C. G. S. electromagnetic units.
One Ohm	$= 10^{9}$	C. G. S. electromagnetic units.
One Farad	$= 10^{-9}$	C. G. S. electromagnetic unit.
One Henry	$= 10^{9}$	C. G. S. electromagnetic units.

In this book the values of magnetic quantities are expressed in C. G. S. electromagnetic units.

CHAPTER IV

THE ALTERNATING CURRENT

The Alternating Current. — The alternating current is a current which periodically changes its direction of flow. The direction of the current is conventionally designated as *positive* or *negative*, being positive for a short interval of time (perhaps $\frac{1}{200}$ of a second) and then negative for an equal interval of time. If a piston is thrust back and forth periodically (see Fig. 37) in a closed pipe, the alternate flow of the water back

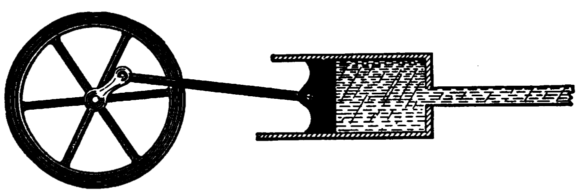


Fig. 37. Analogue of alternating electric current.

and forth corresponds to an alternating electric current. The total flow after any number of complete strokes is zero, for the flow during the second half of each stroke is equal in quantity but opposite in direction to that during the first half stroke. The value of the current at any time may be shown by a curve, as in Fig. 38, in which the abscissæ represent time and the ordinates represent instantaneous values of the current. The curve in the figure represents the current for one **period,** a period being the time required for the completion of one **cycle.** A single period is all that is needed to show the value of the current at any time, inasmuch as successive periods are repetitions of any one. This figure gives the instantaneous values of an alternating current actually found by measurements upon a constant potential 1000-volt Westinghouse generator in commercial use. In this case there were 140 periods per second; that is, the **frequency** was 140. The *period* was therefore equal to $\frac{1}{140}$ second, and a *semi-period* to $\frac{1}{280}$ second. The points indicated on the curve are from observation.

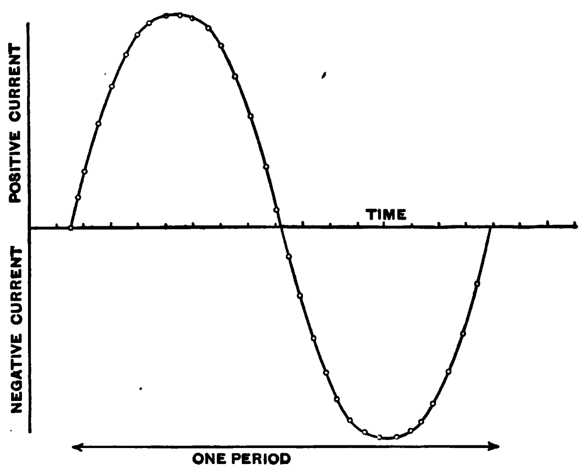


Fig. 38. Instantaneous values of an alternating current.

The Alternate Current Generator. — When the armature of a dynamo is rotated in the magnetic field between the pole pieces of the field magnets, an electromotive force is generated in each conductor on the armature, the direction of this electromotive force depending upon the direction of cutting lines of force. In a two-pole generator, which is typically represented in Fig. 39, the electromotive force in any one conductor is in one direction for half a revolution of the armature and in the opposite direction during the remaining half. In a direct current dynamo, the function of the commutator is to so connect the various armature conductors that the current in the outside circuit will always be in one direction, although being continually reversed in any one coil of the armature. The commutator thus rectifies and makes unidirectional a current which would otherwise be constantly alternating in direction. The commutator is a feature of all direct current dynamo-electric machinery.

If the commutator is omitted, and in its place are put collector rings, the current in the outside circuit will reverse with each reversal of electromotive force in the armature conductors. The simplest form of alternators is represented in Fig. 39 as consisting of a single rectangular loop revolving between two pole pieces. The electromotive force generated, and likewise the current if the circuit is closed, will reverse twice during each revolution of the armature, being positive for half a revolution and negative for the remaining half revolution. In a *two-pole*

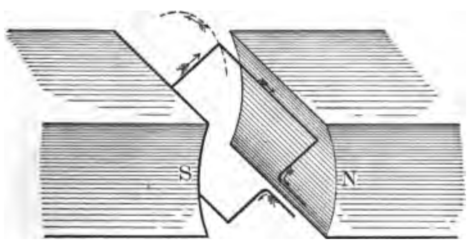


Fig. 39. Typical representation of an alternator.

generator, therefore, there is one period for each revolution of the armature. The frequency of an alternating current accordingly depends directly upon the speed of the machine, and may be increased by revolving the armature faster. There is, however, a limit to the safe speed at which a dynamo may be driven. To obtain greater frequencies with safe speeds, multipolar generators are constructed having more than two poles, commonly eight, ten, or more. In such machines *one period* is equal to the time taken by one conductor in *passing two poles;* thus, a ten-pole generator would furnish an alternating current with five periods during every revolution of the armature. A modern alternator is shown in Fig. 40, which illustrates a 450 kilowatt Westinghouse generator. The first alternator of this type, constructed in the winter of 1885–6 by William Stanley, Jr., is shown in Fig. 41.

The Generation of Alternating Currents. — To fix our ideas in regard to the generation of an alternating current, we will turn our attention to the simplest type of generator. In the simple alternator shown in Fig. 39 let us consider that the rectangular coil is revolving uniformly in a uniform field. When the coil is at right angles to the direction of lines of force, it will include the greatest number of these lines, which we may designate as $+D$, or $-D$, according to the sense in which they

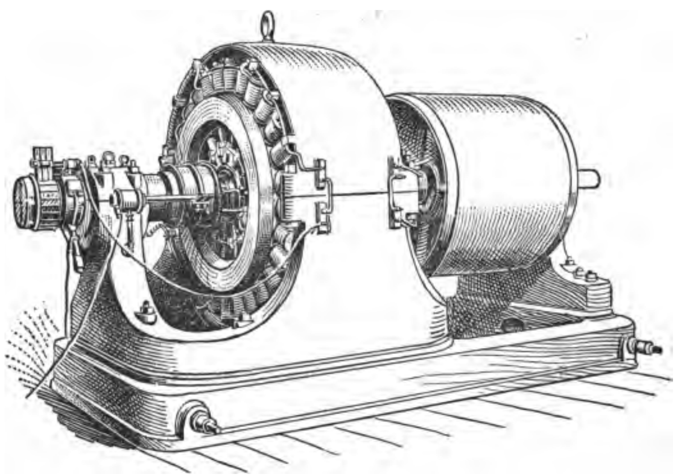


Fig. 40. A Westinghouse alternator.

thread the circuit. Let ϕ be the angle which the coil makes at any time with the position in which it includes the greatest number of lines, D. At any time, then, the number of lines included by the coil is

$$N = D \cos \phi.$$

The angle ϕ, through which the coil revolves, is directly proportional to the time, being equal to the time t multiplied by the *angular velocity* ω; that is,

$$\phi = \omega t,$$

and

$$N = D \cos \omega t.$$

By Faraday's law (Chapter III.) the electromotive force generated in the armature coil is equal at any time to the rate at which the number of lines threading the coil is changing; that is,

$$e = -\frac{dN}{dt} = D\omega \sin \omega t.$$

The electromotive force, therefore, in the case considered, varies harmonically; that is, in accordance with a simple sine law.

If we write E for the maximum value of the electromotive force, the electromotive force at any instant of time may be written

$$e = E \sin \phi;$$

$$e = E \sin \omega t.$$

Fig. 41.

When the electromotive force is harmonic, the current which flows is likewise harmonic. This is not strictly correct when hysteresis is prominent, but in this discussion it will be assumed to be true. The consideration of an equivalent sine-wave, explained further on, makes this assumption justifiable. We may write, then, for the current, giving no consideration at present to phase relations,

$$i = I \sin \omega t,$$

where I is the maximum value of the current, and i the value at any time t.

Harmonic Alternating Currents. — The nature of an harmonic current will be more clearly seen by reference to Fig. 37. If the fly-wheel there represented is at some distance from the piston and revolves uniformly, the piston will be thrust back and forth harmonically and the flow of water will be harmonic, corresponding to an harmonic alternating electric current.

A curve showing the instantaneous value of an harmonic alternating current — one following the law of sines — is shown in Fig. 42. This, as Fig. 38, is drawn to represent one *period* or *cycle*, sometimes called the time of one *complete* or *double* alternation.

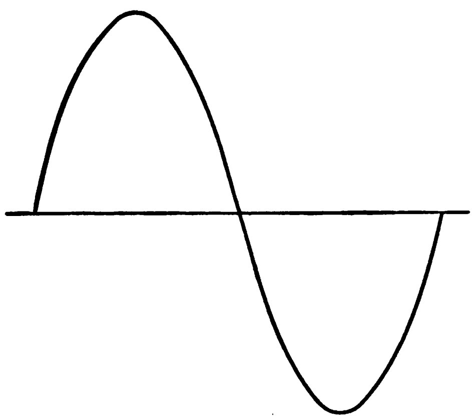


Fig. 42. Sine curve, — representing an harmonic current or electromotive force.

Although the construction of the actual alternator is quite different from the typical diagram in Fig. 39, the current which is obtained in practice is very nearly the same and is approximately harmonic. This is strikingly shown by a comparison of the sine-curve represented in Fig. 42 and the actual alternating current curve in Fig. 38. If the current is not harmonic, we may consider in its stead an equivalent harmonic current as discussed later.

Representation of an Harmonic Current or Electromotive Force. — In Fig. 43 let us consider that P and R represent the opposite conductors of a rectangle revolving between the poles N and S, the arrangement being similar to that shown in Fig. 39. The coil, and likewise the point P, are revolving in a positive direction. Throughout this book *positive direction of rotation is considered to be counter-clockwise.* The angular velocity ω being uniform, the angle ϕ described in the time t is $\phi = \omega t$. For one revolution, $\phi = 2\pi$; and $t = T$, the period or time of one revolution. Hence $\omega = \frac{2\pi}{T}$; and $\phi = \omega t = \frac{2\pi}{T} t$. The *frequency* n is the reciprocal of T; or $n = \frac{1}{T}$, where n is the number of periods

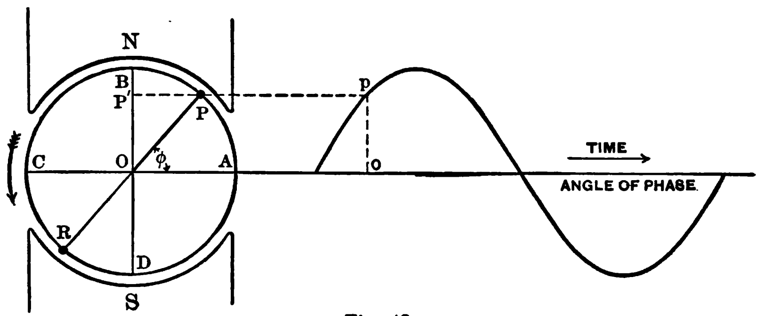


Fig. 43.

in each second, T being the length of each period in fractions of a second. As we have already seen, the electromotive force varies as $\sin \phi$; hence

$$e = E \sin \phi = E \sin \omega t = E \sin \frac{2\pi}{T} t = E \sin 2\pi n t.$$

If we take our scale so that $OP = E$, the electromotive force e, at any instant of time, is represented by OP′, the projection of OP upon the diameter BD, for

$$OP' = OP \sin \phi.$$

The sine curve on the right gives the value of OP′ for each angular position or for each instant of time, and so represents the instantaneous values of the electromotive force.

This representation is correct for any harmonic function, current, or electromotive force, irrespective of its source. The two-pole generator was introduced merely for illustration, and is not at all essential. The representation of an alternating current is the same no matter how many poles the generator may have. Any harmonic electromotive force may be represented in this same manner as shown in Fig. 44, in which the circle about O is the *circle of reference.* The maximum value of a sine function is termed its *amplitude,* and is taken as the radius of the circle of reference.

Phase is designated in angular measure *referred to the reference circle.* As the reference circle consists of 360° and corresponds to one period, each period is divided into 360°. In

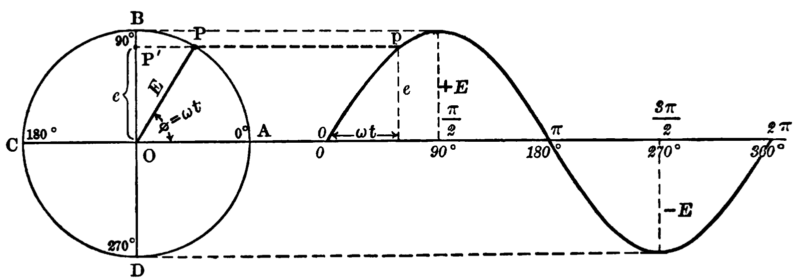


Fig. 44.

the two-pole generator, a period of 360° corresponds to one revolution of the armature. In multipolar machines, a period of 360° is equal to a portion of a revolution of the armature corresponding to two poles of the field. Thus, in an eight-pole machine, the revolution of the armature through ninety degrees gives a complete period of 360°. Phase may be measured in

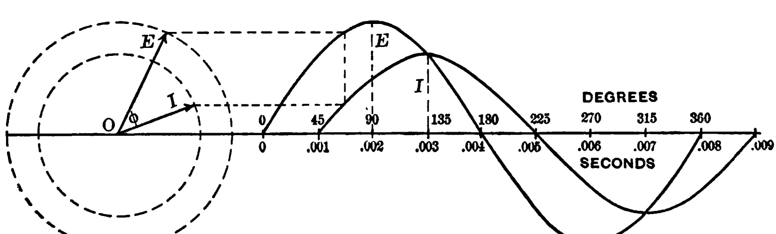


Fig. 45. Current lagging 45° behind electromotive force. Frequency = 125.

degrees or in fractions of a circumference, one circumference being equal to 2π; thus, $90° = \frac{\pi}{2}$, $180° = \pi$, etc.

The lagging of one quantity behind another is represented in Fig. 45, where the current *I* lags behind the electromotive force E by an angle θ (in this case about 45°). This means that the current comes to its maximum value $\frac{45}{360}$ of one period after its electromotive force attains its maximum value. With a frequency of 125 periods per second, each period is 0·008 second, and a difference in phase of 45° would represent a difference in time of 0·001 second. It is to be borne in mind that time advances from left to right.

Equivalent Sine Curves. — In a general way we may consider alternating currents to vary harmonically and to be faithfully represented by a curve of sines. Although actual curves may deviate from sine curves, it is true that in a vast number of cases the conclusions obtained from the sine assumption are substantially correct. When we have curves for current and electromotive force deviating considerably from the sine curve, and make the so-called sine assumption, we do not assume that the current and electromotive force *actually are* harmonic; *we assume an harmonic current and an harmonic electromotive force which are equivalent* in effect, and which may therefore represent the actual current and electromotive force. The virtual values of the equivalent harmonic functions are so taken as to be equal to the square root of the mean square values of the functions represented, as explained in the following paragraph. The phase relations are determined by a consideration of the power represented. The phase difference between the equivalent harmonic current and electromotive force is so taken that the resultant power which they represent is equal to the power determined by the actual current and electromotive force. The subject of equivalent sine curves cannot be fully discussed here. Suffice it to say that the assumption of such curves, although approximately correct, is not rigorously so. A thoroughly rigorous treatment may be employed in any case by considering a current or electromotive force curve of any form whatsoever as made up of a series of sine terms.

Virtual Values. — In referring to an alternating current, the *virtual* value is commonly employed rather than the maximum or mean value. The virtual value of an alternating current is the value of an unvarying direct current which would produce the same heating effect.* It is the virtual value which is used commercially. Thus a 50-volt lamp connected to a 50-volt (virtual value) alternating current circuit will be illuminated to the same candle power as when it is supplied from a 50-volt direct current circuit.

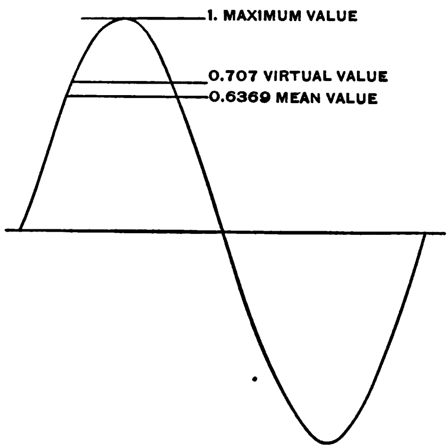


Fig. 46. Virtual and mean values of a sine function.

The heating effect of a current depends upon the square of the current, or, in the case of a varying current, upon the mean of the squares of the instantaneous values; it accordingly follows that the virtual value of an alternating current is equal to the square root of the mean square of the instantaneous value. For a sine function, the virtual value is equal to 0·707 times the maximum value; that is,

$$\text{virtual value} = \frac{\text{maximum value}}{\sqrt{2}} = 0{\cdot}707 \text{ maximum value};$$

$$\text{maximum value} = \sqrt{2} \text{ virtual value} = 1{\cdot}414 \text{ virtual value.} \qquad (1)$$

* We here neglect a slight apparent increase in the resistance of a conductor to alternating currents due to the irregular distribution of the current between the center of the conductor and the surface. The conductor may be viewed as consisting of a number of parallel filaments the currents of which act inductively upon each other. The average distance of the center filaments from all others is less than for the filaments nearer the surface; hence the current is less in the center on account of this greater impedance, and the current is crowded near the surface. On account of this crowding of the current near the surface, the heating effect is greater than it would be in case of a uniform current density; and hence the resistance is apparently increased. In ordinary conductors this effect is negligible at common frequencies. See Fleming, *The Alternating Current Transformer*, Vol. I., p. 238.

The mean or average value of an alternating current is of less importance than the virtual value. It is equal to 0·6369 times the maximum value. The mean value of a sine function is accordingly equal to 0·9 times the virtual value.*

The maximum, virtual, and mean values of a sine function are shown in Fig. 46.

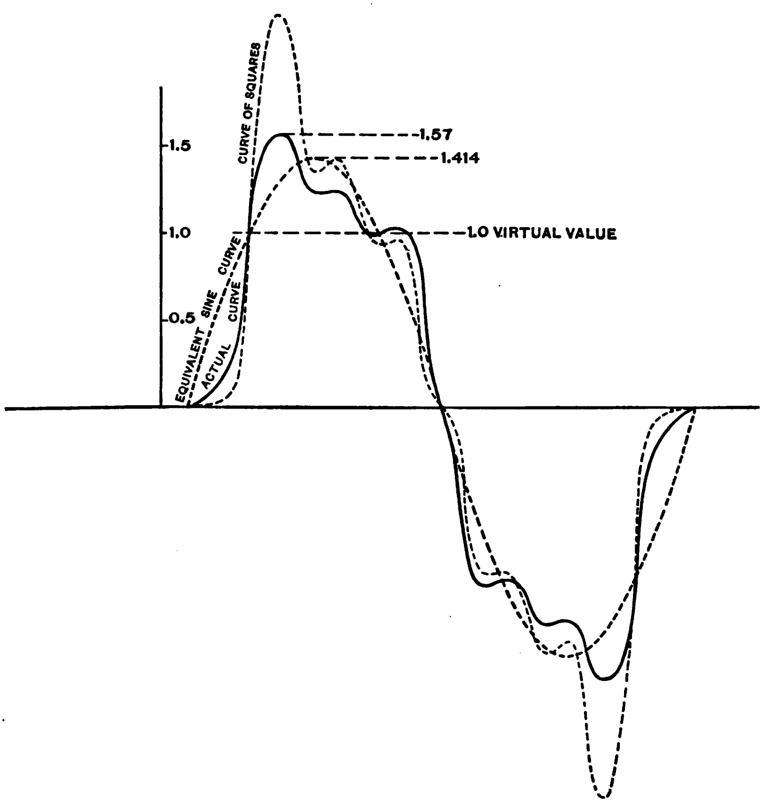


Fig. 47. Equivalent sine curve.

For irregular alternating current curves there is no definite relation between the virtual, the mean, and the maximum values. In Fig. 47 the unbroken line is an irregular alternating

* For a proof of the relation between the maximum, mean, and virtual values of a sine function, see *Alternating Currents*, p. 37.

current curve with a virtual value of 1·00. In this case, the curve being peaked, its maximum value 1·57 is greater than the maximum value 1·414 of the equivalent sine curve corresponding to the virtual value of 1·00. The equivalent sine curve (represented by the heavy broken line) is so selected that its virtual value is the same as the virtual value of the irregular curve.

In this case it so happens that the mean values of the two curves are nearly identical, being 0·90 for the sine and 0·87 for the actual curve, approximately.

The relation between the mean and virtual values of an irregular curve to a certain extent defines the form of the curve. The ratio of these values has been termed the *form factor* of the curve.* The main point to be borne in mind is that when the current or electromotive force is periodic but not harmonic, the virtual value is 0·707 times the maximum value of *the equivalent* harmonic current or electromotive force. The actual current or electromotive force may be quite irregular, and have a maximum value very different from the maximum value of the equivalent harmonic function.

The mean or average value of the ordinates of a curve may be obtained by ascertaining the area of the curve by a planimeter and dividing this area by the base line. The virtual value may be obtained by plotting a curve, each ordinate of which represents the square of the corresponding ordinate of the curve under investigation. This curve of squares is shown by the fine dotted lines in Fig. 47. From this curve of squares the mean square value may be obtained directly by means of a planimeter, and the virtual value is then found by taking the square root of the mean square value.

The equivalent sine function and the virtual value may be more simply found by plotting a curve with polar co-ordinates. This is shown in Fig. 48, in which e is a vector of varying

* Roessler, *The Electrician*, London, Nov. 24, 1895; Fleming, *Ibid*, Jan. 10, 1896. The form factor is further discussed in Chapter XV.

length revolving uniformly about a center O. The length of the vector in any position represents the value of the function at any instant of time. If a circle be drawn with an area equal to that included by the curve under consideration, this circle will represent the equivalent sine function, for which it is evident $e' = E \sin \omega t$. The virtual value of the two curves will be the same.* This may be shown as follows: Integrating the two polar curves between the limits 0 and π, we have

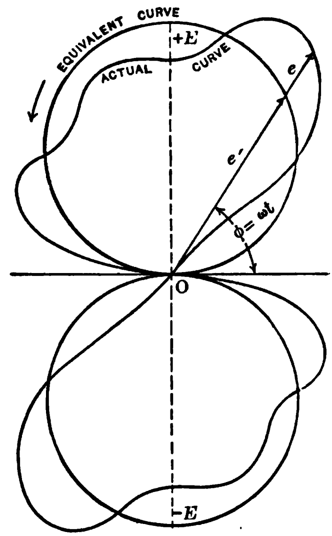


Fig. 48. Polar curve.

$$\text{Area irregular curve} = \frac{1}{2}\int_0^{\pi} e^2 d\phi.$$

$$\text{Area circle} = \frac{1}{2}\int_0^{\pi} e'^2 d\phi.$$

As the circle was drawn so that these two areas are equal, it follows that e and e' have the same mean square value, and hence the same virtual value. The virtual value is 0·707 times the diameter E of the circle; that is, 0·707 times the maximum value of the equivalent sine function.

The Flow of an Alternating Current in an Inductive Circuit.— The flow of a current in a circuit in which there are no counter-electromotive forces whatsoever is determined by the value of the impressed electromotive force and the resistance of the circuit. The electromotive force required to cause the current to flow is simply that necessary to overcome the ohmic resistance; that is, $E = RI$. This is equally true for alternating and direct currents.

If the circuit contains a counter-electromotive force of any kind, the impressed electromotive force required to cause a cur-

* This method has been shown by Mr. Steinmetz, *Electrotechnische Zeitschrift*, June 20, 1890. See also Dr. Fleming, *The Electrician*, May 10, 1895.

rent to flow will be equal to the sum of the electromotive force necessary to overcome the resistance of the circuit and an electromotive force to overcome the counter-electromotive force of the circuit, which shall be equal to the counter-electromotive force and opposite to it in sign. The counter-electromotive forces in a circuit are often the most important factors in determining the value of the current which will flow with a given electromotive force. This, likewise, is equally true for continuous and alternating currents. Thus, in a continuous current circuit supplying power to electric motors, it is the counter-electromotive force of the motors which regulates the flow of current.

In Chapter III. we have seen that any change in the flow of the current in a circuit produces an electromotive force called the counter-electromotive force of self-induction, equal to the product of the coefficient of self-induction (which depends upon the form of the circuit) and the time rate of change of the current. The equation of electromotive forces was accordingly written

$$e=Ri+L\frac{di}{dt}. \qquad (2)$$

This equation is true in all cases irrespective of the manner in which the currents and electromotive forces are varying. In all cases the impressed electromotive force at any instant is equal to the sum of the ohmic electromotive force and the electromotive force to overcome self-induction. This equation may be written in a manner analogous to Ohm's law as follows:

$$i=\frac{e-L\frac{di}{dt}}{R}, \qquad (3)$$

that is, the current at any instant of time is equal to the algebraic sum of the various electromotive forces in the circuit divided by the resistance. The general solution of this equation and an extended discussion of the current flow in inductive circuits is given in *Alternating Currents.*

For the case of an alternating current varying in regular periodic fashion, the relation between the current and the electromotive force may be readily obtained. We may assume that the electromotive force and current vary harmonically, or that they may be represented by an equivalent harmonic electromotive force and equivalent harmonic current. We have then the electromotive force

$$e = E \sin \omega t. \qquad (4)$$

Let us suppose the problem is as follows: given the electromotive force in equation (4); it is required to find the value of the current which will flow and the phase differences between it and the electromotive force. The expression * for the current under the harmonic assumption is

$$i = I \sin (\omega t + \theta). \qquad (5)$$

It is required to find the values of I and θ. By differentiation

$$\frac{di}{dt} = \omega I \cos (\omega t + \theta).$$

We have accordingly

$$\text{Ohmic electromotive force} = Ri = RI \sin (\omega t + \theta); \qquad (6)$$

$$\text{Inductive electromotive force} = L\frac{di}{dt} = L\omega I \cos (\omega t + \theta). \qquad (7)$$

The first is in phase with the current; the second at right angles to it.

Substituting these values in (2), we have

$$E \sin \omega t = RI \sin (\omega t + \theta) + L\omega I \cos (\omega t + \theta), \qquad (8)$$

which may be written,†

$$E \sin \omega t = I\sqrt{R^2 + L^2\omega^2} \sin \left(\omega t + \theta + \text{arc tan}\frac{L\omega}{R}\right). \qquad (10)$$

* The angle θ may be positive or negative, as determined later, indicating either an advance or lag of the current. The expression $+\theta$ does not therefore indicate an angle of advance, for θ may in some cases be found to have a negative value, as is seen later in the present case.

† A sine and cosine term of the same angle may always be written in terms of either by the following trigonometric formula:

$$A \sin \theta + B \cos \theta = \sqrt{A^2 + B^2} \sin \left(\theta + \text{arc tan}\frac{B}{A}\right). \qquad (9)$$

This formula is repeatedly used in the discussion of alternating currents, and is proven in *Alternating Currents*, p. 52.

The two equal sine functions in this equation must have equal maximum values * and must be together in phase, inasmuch as they are identical. Hence

$$E = I\sqrt{R^2 + L^2\omega^2},$$

which may be written,

$$I = \frac{E}{\sqrt{R^2 + L^2\omega^2}} = \frac{E}{J}, \qquad (11)$$

or

$$\text{current} = \frac{\text{electromotive force}}{\text{impedance}}.$$

The maximum value of the current is thus seen to be equal to the electromotive force divided by the impedance. The **impedance**, designated by the letter J, is defined from this relation as the ratio of the electromotive force to the current. In an inductive circuit

$$\text{impedance} = \sqrt{R^2 + L^2\omega^2}.$$

Since both members of equation (10) are always in the same phase, each being zero when t is zero, it follows that

$$\theta + \text{arc tan}\frac{L\omega}{R} = 0;$$

or

$$\theta = -\text{arc tan}\frac{L\omega}{R},$$

and

$$\tan\theta = -\frac{L\omega}{R}.$$

The magnitude and phase of the current being thus determined, we may write for the value † of the current at any time,‡

$$i = \frac{E}{\sqrt{R^2 + L^2\omega^2}}\sin\left(\omega t - \text{arc tan}\frac{L\omega}{R}\right). \qquad (12)$$

* Suppose $A \sin\alpha = B \sin\beta$.

When α is zero, $A \sin\alpha = 0$; hence, $B \sin\beta = 0$.

Therefore, $\alpha = \beta$, and the two sine functions are in phase. Their maximum values, which occur when α and β are 90°, are accordingly equal, that is, $A = B$.

† This may be obtained from the general expression, Equation (16), Chapter XII.

‡ This is the expression for the current at any time after the circuit has been made sufficiently long (a fraction of a second) for a permanent state to be reached. After the closing of the circuit, the current rises from zero and approaches its final periodic value by an exponential curve of approach, represented by an additional term $c\epsilon^{-\frac{Rt}{L}}$ in the current equation. This is fully discussed in *Alternating Currents*, p. 55, ff.

The current is thus seen to lag behind the impressed electromotive force by an angle θ, whose tangent is the ratio of the reactance of the circuit to the resistance, as explained in the next paragraph.

When the self-induction is zero, the current follows Ohm's law and is in phase with the electromotive force;

$$i = \frac{E}{R} \sin \omega t.$$

When the resistance is negligible, we have

$$i = \frac{E}{L\omega} \sin (\omega t - 90^\circ);$$

the current then lags ninety degrees behind the electromotive force.

The current in an inductive circuit always lags behind the electromotive force by an angle which cannot exceed ninety degrees.

Reactive Electromotive Force and Power Electromotive Force. — The equation (2) for the instantaneous values of the electromotive forces in an inductive circuit expresses the relation that at any instant of time the impressed electromotive force is equal to the sum of the electromotive force to overcome resistance and that to overcome the counter-electromotive force of self-induction. With alternating currents represented by harmonic functions, this leads to equation (8), in which the impressed electromotive force E is made up of two components with maximum values RI and $L\omega I$, to overcome resistance and self-induction, respectively.

The impressed electromotive force in *any* alternating current circuit may thus be divided into two components: First, the *power electromotive force* in phase with the current; and, second, the *reactive electromotive force* in quadrature* with the

* An electromotive force in phase with the current is spoken of as being in the direction of the current, as it is so represented in vector diagrams. An electromotive force differing in phase by 90° from the current is said to be in quadrature or at right angles to it.

current to overcome the reactance. The reactive electromotive force is the product of the current and the reactance. The **reactance** is, accordingly, equal * to the component of the impressed electromotive force at right angles to the current, divided by the current. Reactance is measured in ohms. The diagram of electromotive forces for an inductive circuit is shown in Fig. 49.

In general,

$$\tan\theta = -\frac{\text{reactance}}{\text{resistance}} = -\frac{K}{R}. \tag{13}$$

For a simple inductive circuit the reactance is $K = L\omega$, and the resistance R; accordingly,

$$\tan\theta = -\frac{L\omega}{R}.$$

The negative sign indicates that θ is an angle of lag, and that the current in an inductive circuit lags behind the impressed electromotive force.

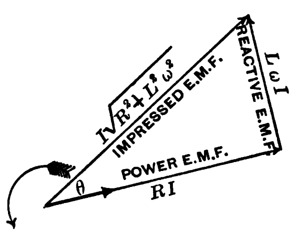


Fig. 49. Triangle of electromotive forces for an inductive circuit.

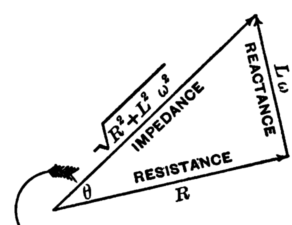


Fig. 50. Impedance diagram.

The impedance diagram in Fig. 50 is the same as the electromotive force diagram in Fig. 49 divided by the current. These constructions are justified by the foregoing equations. In a simple inductive circuit, where all the power is expended in overcoming resistance, the power electromotive force is equal to the ohmic electromotive force RI; the reactive electromotive

* As first defined by Steinmetz and Bedell, American Institute of Electrical Engineers, May, 1894. The term was first suggested by Hospitalier.

force is equal to the inductive electromotive force $L\omega I$; hence the impressed electromotive force is

$$E = \sqrt{\text{ohmic E.M.F.}^2 + \text{inductive E.M.F.}^2} \qquad (14)$$

In circuits where energy is expended in other ways than in resistance, and there are other reactive electromotive forces than that due to self-induction, the impressed electromotive may be similarly resolved into two components. In general,

$$E = \sqrt{\text{power E.M.F.}^2 + \text{reactive E.M.F.}^2} \qquad (15)$$

The terms "power electromotive force" and "active electromotive force" are used synonymously.

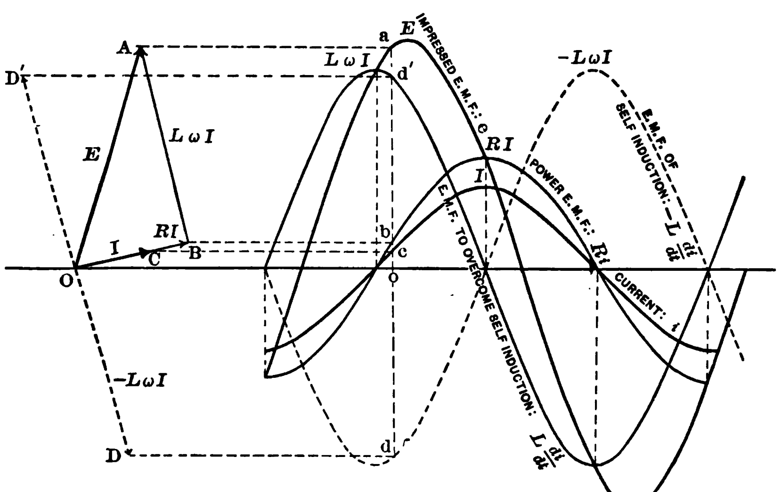


Fig. 51. Instantaneous curves of current and electromotive forces for an inductive circuit.

Instantaneous Values of Current and Electromotive Force in an Inductive Circuit. — In the triangle of electromotive forces for an inductive circuit given in Fig. 49, each line represents an harmonically varying quantity, as has been explained. In Fig. 51 curves are drawn showing the instantaneous values of these various quantities and their relations to each other. The power electromotive force is seen to be in phase with the current and a multiple of it. The dotted curve represents the counter-electromotive force *of* self-induction. The electromotive force to *overcome* self-induction is equal and opposite to this. The impressed electromotive force is equal to the sum of the two components, the power electromotive force and the electromotive force to overcome self-induction. The vector addition of the maximum values of these two components has already been explained. The instantaneous values of the two component curves are added arithmetically; thus *oa* is equal to the sum of *ob* and *od'*.

Instead of considering the impressed electromotive force as composed of two components, we may view the matter from another standpoint, and consider that there are *two* electromotive forces acting in the circuit, one an *impressed* electromotive force and the other the *counter*-electromotive force of self-induction. The power electromotive force is then the resultant of these two; thus, in the vector diagram, the power electromotive force, OB, is the sum of OA and OD. The curves representing the instantaneous values may be interpreted in the same manner. Any ordinate of the power electromotive force curves will be seen to be the sum of the corresponding ordinates of the curves for the impressed electromotive force, and the electromotive force of self-induction; thus, *ob* is the sum of *oa* and *od*.

When we have, as in Fig. 51, one curve as the resultant of two curves, it evidently follows that when one component is passing through zero the other component intersects the resultant curve. Since the power and inductive electromotive forces are in quadrature, it follows that when one is zero the other is at a maximum. The power electromotive force is proportional to the current; the inductive electromotive force is proportional to the rate of change of the current.

Circuits containing a Condenser. — The **capacity** of a condenser is defined as the quantity of electricity which it receives when subjected to a unit difference of potential; or, it is the ratio of the charge to the difference of potential between the terminals of the condenser.* A condenser of capacity C when subjected to a difference of potential V, accordingly receives a charge of electricity equal to

$$Q = CV. \tag{16}$$

The practical unit for quantity is the *coulomb*, which is 10^{-1} times the C. G. S. (electromagnetic) unit of quantity. The practical unit for capacity is the *farad;* one farad is equal to 10^{-9} C. G. S. units. The microcoulomb and microfarad are more convenient in size for ordinary use. A condenser with a capacity of one microfarad will receive a charge of one microcoulomb when subjected to a difference of potential of one volt. These units are given in the appendix to the previous chapter. The capacity of a condenser is very nearly constant,† and is usually so considered.

If a condenser is subjected to a varying difference of potential, the relation in (16) still holds; the charge varies proportionally to the electromotive force impressed upon the terminals of the condenser, and is equal to this electromotive force multiplied by the capacity C. The current which flows in and out of the condenser at any time is

$$i = \frac{dq}{dt}. \tag{17}$$

Since $q = \int idt$, we may express the value of the difference of potential at the terminals of a condenser in terms of the current flowing, it being equal to $\frac{\int idt}{C}$.

* A discussion of capacity is given in *Alternating Currents*, Chap. IV.

† Dielectric hysteresis will cause a variation in the capacity of a condenser analogous to the variation in the coefficient of self-induction due to magnetic hysteresis. The variations in capacity are, however, much less and practically negligible, inasmuch as the hysteresis loop of a condenser is small. For such a loop, see "Alternate Current Condensers and Dielectric Hysteresis," by Bedell, Ballantyne, and Williamson, *Physical Review*, Vol. I., No. 2.

The equation of electromotive forces for a circuit containing resistance and a condenser is accordingly

$$e = Ri + \frac{\int idt}{C}, \tag{18}$$

the impressed electromotive force being equal to the sum of the ohmic electromotive force and that of the condenser. The general solution of this equation and the discussion of circuits containing capacity is given in *Alternating Currents*, Chaps. V., XVIII., and XIX.

As in the case of inductive circuits, we may obtain a ready solution for the flow of an alternating current in a circuit containing a condenser by making the sine assumption. We have

$$e = E \sin \omega t;$$
$$i = I \sin (\omega t + \theta).$$

It is required to find I and θ. The ohmic or power electromotive force is

$$Ri = RI \sin (\omega t + \theta).$$

The reactive or condenser electromotive force is

$$\frac{q}{C} = \frac{\int idt}{C} = -\frac{I}{C\omega} \cos (\omega t + \theta).$$

Equating the sum of these to the impressed electromotive force, we have,*

$$E \sin \omega t = RI \sin (\omega t + \theta) - \frac{I}{C\omega} \cos (\omega t + \theta); \tag{18a}$$

$$E \sin \omega t = I\sqrt{R^2 + \frac{1}{C^2\omega^2}} \sin \left(\omega t + \theta - \text{arc tan } \frac{1}{CR\omega}\right).$$

As these sine functions are equal, their maximum values and phase relations are the same; hence,

$$E = I\sqrt{R^2 + \frac{1}{C^2\omega^2}};$$

$$\theta = + \text{arc tan } \frac{1}{CR\omega}.$$

* The sine and cosine terms are combined as before by formula (9).

For a circuit with a condenser:

$$\text{reactance} = -\frac{1}{C\omega};$$

$$\text{impedance} = \sqrt{R^2 + \frac{1}{C^2\omega^2}};$$

$$\text{tangent } \theta = -\frac{\text{reactance}}{\text{resistance}} = +\frac{1}{CR\omega}. \tag{19}$$

The instantaneous values* of the current are given by the equation

$$i = \frac{E}{\sqrt{R^2 + \frac{1}{C^2\omega^2}}} \sin\left(\omega t + \text{arc tan } \frac{1}{CR\omega}\right). \tag{20}$$

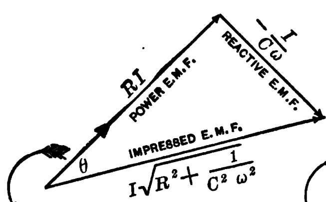


Fig. 52. Triangle of electromotive forces for a circuit containing a condenser.

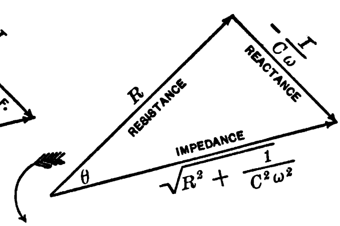


Fig. 53. Impedance diagram for a circuit containing a condenser.

The current is in advance of the electromotive force, and has a maximum value

$$I = \frac{E}{\sqrt{R^2 + \frac{1}{C^2\omega^2}}}. \tag{21}$$

These relations are shown in Figs. 52 and 53.

If the electromotive force considered is that at the terminals

* The effect immediately after the circuit is made would be expressed by an exponential term $c\epsilon^{-\frac{t}{RC}}$, which becomes negligible in a few one-hundredths of a second, as explained in the note in connection with equation (12). See, also, equations (66) and (67), Chapter XI., and the following discussion. This equation may be obtained as a particular case of the more general equation (16), Chapter XII.

of the condenser, or if the resistance in the circuit is negligible, $R = 0$; hence,

$$I = CE\omega, \qquad (22)$$
$$\theta = 90°.$$

The condenser current is then ninety degrees in advance of the electromotive force at the terminals of the condenser.

When there is no condenser * in the circuit,

$$i = \frac{E}{R} \sin \omega t.$$

The current is in accordance with Ohm's law and in phase with the electromotive force.

The Instantaneous Values of Current and Electromotive Force in a Circuit containing a Condenser.—The instantaneous values for the current and the various electromotive forces in a circuit containing resistance and capacity are shown in Fig. 54. These

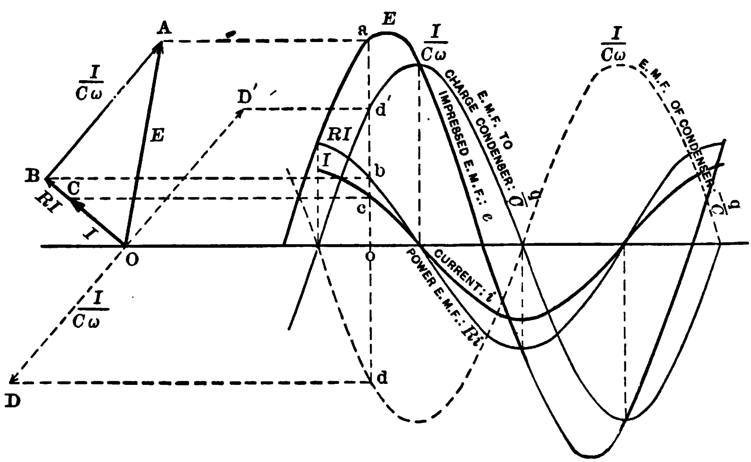


Fig. 54. Instantaneous curves of current and electromotive forces for a circuit containing a condenser.

curves are similar to the corresponding curves in Fig. 51 for an inductive circuit. The impressed electromotive force is equal to

* In this case $C = \infty$, when the circuit is complete; when the circuit is broken, $C = 0$.

the sum of the power electromotive force and electromotive force required to charge the condenser. It is to be noted that the current is ninety degrees in advance of the electromotive force at the terminals of the condenser. The equal opposing electromotive force produced by the condenser is shown by the dotted curve. Treating the condenser as an independent source of electromotive force, we may consider that the power electromotive force is the resultant of the condenser and impressed electromotive forces. The relations between the maximum and zero values should be carefully noted.

Circuits with Self-induction and Capacity.—The equation of electromotive forces for circuits containing resistance, self-induction, and capacity is

$$e = Ri + L\frac{di}{dt} + \frac{\int idt}{C}, \tag{23}$$

the several terms of which have been explained in connection with equations (2) and (18).

The corresponding equation of energy is,

$$eidt = Ri^2dt + Li\frac{di}{dt}dt + \frac{idt\int idt}{C}. \tag{23a}$$

Under the harmonic assumption

$$e = E \sin \omega t,$$
$$i = I \sin (\omega t + \theta).$$

Substituting these values for e and i in (23) [see also (10) and (18a)],

$$E \sin \omega t = RI + \left(L\omega - \frac{1}{C\omega}\right)I \cos (\omega t + \theta).$$

Treating this in a similar manner to equations (8) and (18 a), we obtain for the current

$$i = \frac{E}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}} \sin \left[\omega t + \arc\tan \frac{\frac{1}{C\omega} - L\omega}{R}\right]. \tag{24}$$

In this case

$$\text{reactance} = K = L\omega - \frac{1}{C\omega}. \tag{25}$$

$$\text{impedance} = J = \sqrt{R^2 + K^2} = \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}. \tag{26}$$

$$\text{tangent } \theta = -\frac{K}{R} = \frac{1}{CR\omega} - \frac{L\omega}{R}. \tag{27}$$

$$I = \frac{E}{\sqrt{R^2 + K^2}}. \tag{28}$$

The reactive electromotive force at right angles to the current is,

$$KI = L\omega I - \frac{I}{C\omega}. \tag{29}$$

The reactance may be either positive or negative. The reactive electromotive, being the algebraic sum of the inductive

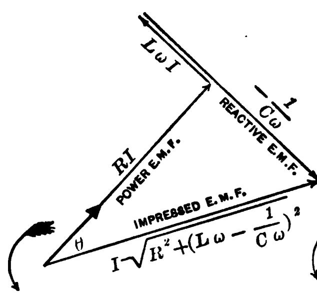


Fig. 55. Electromotive force diagram for a circuit with self-induction and capacity.

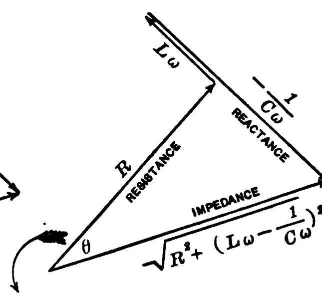


Fig. 56. Impedance diagram for a circuit with self-induction and capacity.

electromotive force and condenser electromotive force, may likewise be positive or negative. When the condenser has more effect than the self-induction, the current is in advance of the electromotive force; it lags behind the electromotive force when the self-induction is more important. In Figs. 55 and 56 the current is in advance of the electromotive force. The current and electromotive force are in phase when the self-induction and capacity just neutralize each other and the reactance is zero.

This is the condition of resonance. The current then rises to its greatest value, determined by the resistance of the circuit. When $L = 0$, this case reduces to the preceding one. By omitting the condenser terms, the results reduce to those for a simple inductive circuit.

When the capacity of the condenser bears a certain relation to the values of the resistance and inductance, rapid oscillation may occur when the current is made.*

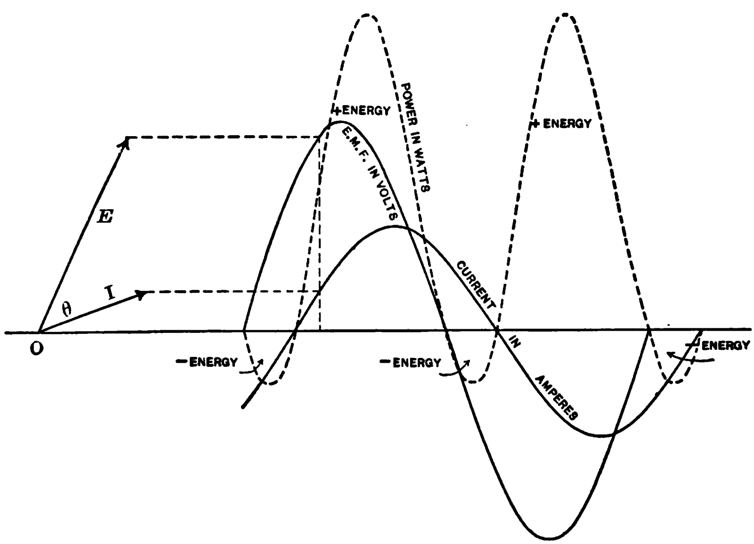


Fig. 57. Power curve for a lagging current.

Power Curves. — In a circuit in which the current and electromotive force are varying, the power or rate of expenditure of energy at any instant of time is equal to the product of the instantaneous values of the current and electromotive force at that time. The rate of expenditure of energy, therefore, varies periodically. Figure 57 represents a current lagging behind the impressed electromotive force. The dotted curve represents the power at each instant of time. The power curve passes through

* A full treatment of the effects of self-induction and capacity in a circuit is given in *Alternating Currents.*

zero whenever the current or the electromotive force is zero; that is, four times in each period. When the current and electromotive force have opposite signs, their product is negative, and consequently the power curve is negative through two portions of each cycle. The area included between the power curve and a portion of the time axis represents the energy expended or work done during that time. Positive areas indicate that energy is being transmitted from the source to the circuit; negative areas represent energy returned from the cir-

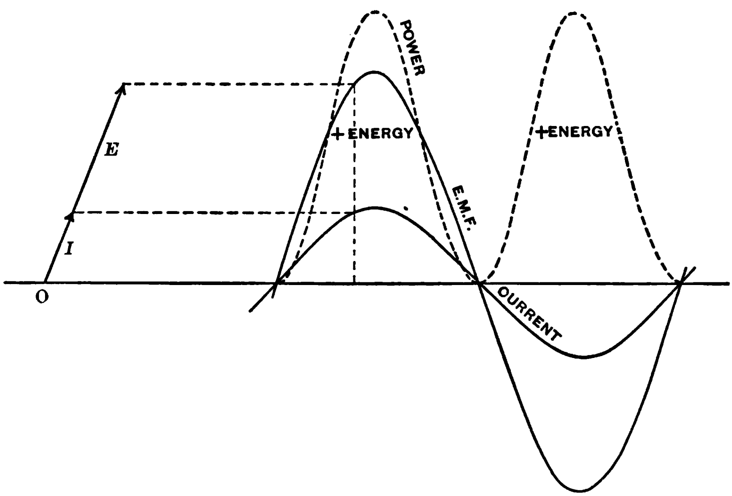


Fig. 58. Power curve for a current in phase with the electromotive force.

cuit to the source. This return of energy corresponds to that which a fly wheel may give back to its driving mechanism. The algebraic sum of the positive and negative areas is equal to the amount of energy imparted to the circuit in any one cycle.

Figure 58 represents the power curve for a circuit in which the current and electromotive force are in phase. In this case the product of the current and electromotive force is always positive, and there is no area representing negative work. Under these conditions the power transmitted by a given current and electromotive force is a maximum.

In Fig. 59 the current represented is in quadrature with the electromotive force. It will be noted that the power curve includes equal positive and negative areas, and that the total energy transmitted is zero. In general, a current and electromotive force in quadrature transmit no power. When the current differs in phase from its electromotive force by less than ninety degrees, the component of the current at right angles to the electromotive force accordingly transmits no power, the

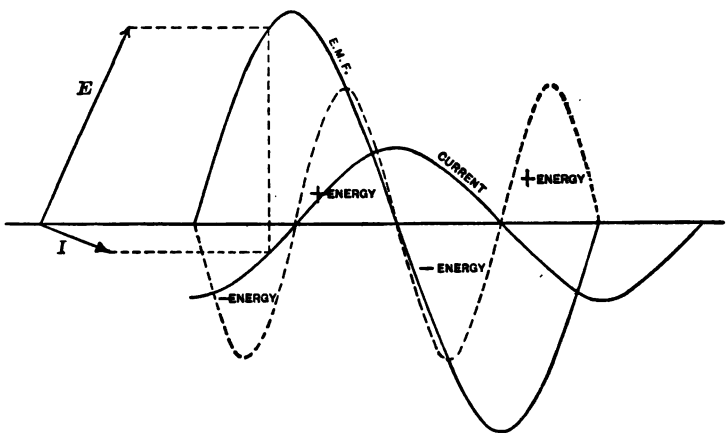


Fig. 59. Power curve for a reactive or wattless current in quadrature with the electromotive force.

energy component being the one in phase with the electromotive force.

In Fig. 60 the current which lags behind its electromotive force is resolved into the power and the reactive currents, in phase and in quadrature, respectively, with the electromotive force. The current and the electromotive force are the same as in Fig. 57. The power curve in Fig. 57 indicates, therefore, the power for the current represented in Fig. 60. We may likewise obtain the power by considering the components of the current. The power current and the electromotive force are identical with those shown in Fig. 58. The power curve in Fig. 58 therefore represents the power transmitted by the current in Fig. 60. The electromotive force and reactive current for Fig. 60 are represented in Fig. 59, which indicates zero power. The two curves in Figs. 58 and 59 are therefore components of the power curve in Fig. 57, the sum of the areas of the former being equal to the sum of the areas of the latter, and representing the amount of energy expended for one cycle.

Figure 61 represents the same electromotive force and current as Fig. 60, the electromotive force being resolved into the power

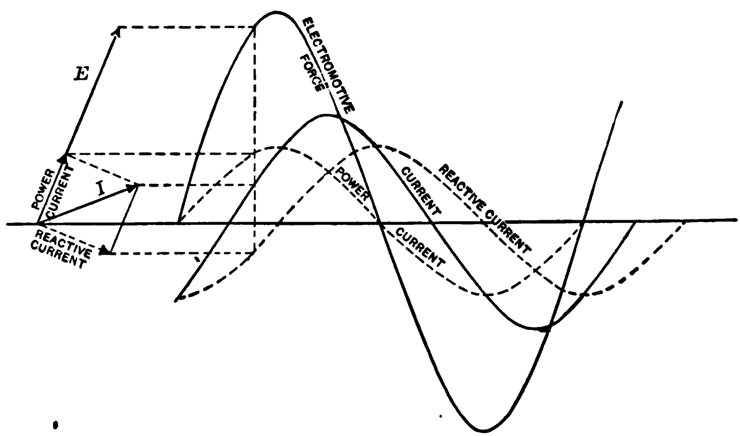


Fig. 60. Resolution of a current into a power component and a reactive (wattless) current.

and reactive components. Power curves constructed from Fig. 61 would be similar to those already discussed.

We have seen that components of currents and electromotive forces in phase represent power, and that components in quadrature represent none. These results are comprised in the following statements:

In an alternating current circuit the power is equal to the product of the current and electromotive force (virtual values) and the cosine of the angle of phase difference between them.

The power in the alternating current circuit is equal to one-half the product of the current and electromotive force (maximum values) and the cosine of the angle of phase difference between them.

The expressions * for power are accordingly

$$P = \overline{EI} \cos \theta, \quad (30)$$

$$P = \tfrac{1}{2} EI \cos \theta.$$

The factor $\frac{1}{2}$ arises from the relation that the maximum value of the current or electromotive force is $\sqrt{2}$ times the virtual values. This relation must be borne in mind in all expressions

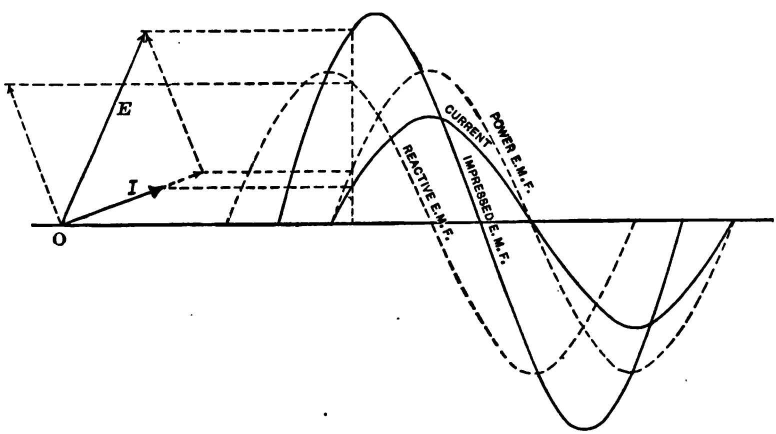


Fig. 61. Resolution of the impressed electromotive force into a power component and a reactive component.

for power. Although equations for currents or electromotive forces are generally true either for maximum or for virtual values, the same does not hold in the case of power formulæ; in these the factor 2 or $\frac{1}{2}$ must be introduced in changing between maximum and virtual values of the current and electromotive force.

Power Factor. — The product of the electromotive force in volts, and current in amperes (virtual values), is sometimes termed "apparent power" of a circuit, or the "apparent watts."

* The analytical derivation of these formulæ is given in *Alternating Currents*, p. 143.

The ratio of the real power in a circuit to the product of the current and electromotive force is termed the power factor. Thus,

$$\text{Power factor} = \frac{\text{watts}}{\text{volts} \times \text{amperes}};$$

or

$$\text{Power} = \bar{E}\bar{I} \times \text{power factor}.$$

By comparing equation (30), it is seen that the power factor is equal to the cosine of the angle of phase difference between the current and electromotive force. When the current and electromotive force are in phase, the power factor is unity; it becomes less and less as the phase difference increases, being zero when the current and electromotive force are in quadrature. The power factor of a transformer on open circuit would have a value say of 0·65; this would increase as the transformer is loaded, and become 0·99 on full load when the current and electromotive force are very nearly in phase.

Series Circuits. — When several electromotive forces are active in a single circuit, they may be added or combined as vectors; they are accordingly added together geometrically, as was done in Figs. 49, 52, and 55. Electromotive forces may be thus added when they are *in series.*

The sum of several electromotive forces in series is the **vector** *or* **geometrical** *sum of the separate electromotive forces;* this is true for maximum or for virtual values.

At any instant of time the sum of several electromotive forces in series is the **algebraic** *sum of the instantaneous values of the separate electromotive forces.*

In other words, maximum and virtual values are added geometrically; instantaneous values are added algebraically. This is true irrespective of the number of electromotive forces or their source.

In a series circuit the total resistance is the arithmetical sum of the resistances of the several parts; the total self-induction and the total reactance are likewise the arithmetical sums of the separate self-inductions and reactances respectively. Impedances are added geometrically, like electromotive forces (see Figs. 50, 53, and 56), being composed of resistances and reactances at right angles to each other. These relations do not hold true for other than series circuits.

Parallel Circuits. — If two circuits be connected in parallel as in Fig. 62, and supplied with a constant electromotive force, the current which flows in each will be independent of the current which flows in the other (provided there is no mutual induction between the circuits) and will depend upon the elec-

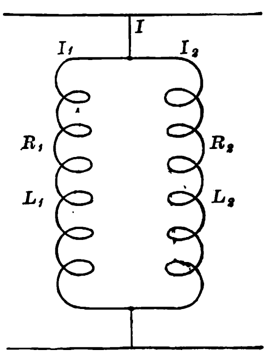


Fig. 62. Parallel circuits.

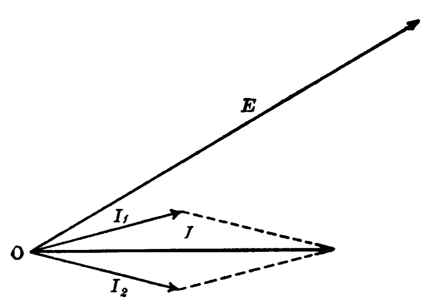


Fig. 63. Division of current in parallel circuits.

tromotive force impressed at the terminals, and the amount of resistance, self-induction, and capacity of the circuit.

At any instant of time the total or line current is equal to the algebraic sum of the instantaneous values of the branch currents. The maximum or virtual value of the main current is equal to the vector sum of the maximum or virtual values of the branch currents. This holds true for any number of circuits in parallel. In parallel circuits, currents are added in the same way that electromotive forces are added in series circuits. The diagram showing the addition of currents in two parallel inductive circuits is given in Fig. 63. For any number of parallel circuits, we may write the *vector* equation

$$I = I_1 + I_2 + I_3 + \cdots, \text{ etc.}$$

As each current is equal to the electromotive force divided by the impedance of the circuit in which it flows, denoting the joint impedance by J', we may write the vector equations,

$$\frac{E}{J'} = \frac{E}{J_1} + \frac{E}{J_2} + \frac{E}{J_3} + \cdots, \text{ etc.};$$

or

$$\frac{1}{J'} = \frac{1}{J_1} + \frac{1}{J_2} + \frac{1}{J_3} + \cdots, \text{ etc.} \tag{31}$$

The reciprocal of the joint impedance of parallel circuits is the vector sum of the reciprocals of the separate impedances.

The joint impedance is

$$J' = \frac{1}{\frac{1}{J_1} + \frac{1}{J_2} + \frac{1}{J_3} + \cdots, \text{ etc.}} \tag{32}$$

The denominator is here a vector sum.

The *admittance* of a circuit is the reciprocal of its impedance; hence, *the joint admittance of parallel circuits is the vector sum of the separate admittances.* For alternating currents, admittance and impedance correspond to conductance and resistance for direct currents; they may be similarly treated by combining them *geometrically*, instead of arithmetically, as in the case of direct currents. In parallel circuits we may add the currents or admittances of the several branches, while for series circuits, as explained above, we may add electromotive forces or impedances.

In complex circuits, each portion of the system may be separately considered.

Relation between Magnetization and Electromotive Force. — The value of an alternating electromotive force impressed at the terminals of a coil wound upon an iron core or ring may be expressed in terms of the magnetic induction, the frequency n, and the number of convolutions in the coil.

From Faraday's law (see Chapter III.), we have for the electromotive force to overcome the self-induction of a coil

$$e = S\frac{dN}{dt} = SA\frac{db}{dt}.$$

We may write the instantaneous value of the induction $\boldsymbol{b}$, in terms of the maximum value $\boldsymbol{B}$:

$$\boldsymbol{b} = \boldsymbol{B} \sin \omega t;$$

hence,
$$\frac{d\boldsymbol{b}}{dt} = \boldsymbol{B}\omega \cos \omega t.$$

Substituting in the first equation above, (33)

$$e = SA\boldsymbol{B}\omega \cos \omega t.$$

The maximum value of the electromotive force is accordingly

$$E_{max} = SA\boldsymbol{B}_{max}\omega. \qquad (34)$$

The virtual, or square root of the mean square value, is

$$\overline{E} = \frac{1}{\sqrt{2}} SA\boldsymbol{B}_{max}\omega. \qquad (35)$$

This is true where E and $\boldsymbol{B}$ are expressed in C. G. S. units. If we express E in volts and leave $\boldsymbol{B}$ expressed in C. G. S. lines of induction, writing $2\pi n$ for ω, we obtain

$$\overline{E} = 10^{-8}\sqrt{2}\,\pi n SA\boldsymbol{B}_{max} \qquad (36)$$
$$= 4{\cdot}45 \times 10^{-8} n SA\boldsymbol{B}_{max}.$$

The maximum value of the induction may likewise be written in terms of the impressed electromotive force in volts (virtual value); thus,

$$\boldsymbol{B}_{max} = \frac{\overline{E} \times 10^8}{\sqrt{2}\,\pi n SA} \qquad (37)$$
$$= \frac{\overline{E} \times 10^8}{4{\cdot}45\, nSA}.$$

These formulæ are useful in designing reactive coils and transformers. With a given electromotive force and an assumed value of the induction, the number of turns required in a reactive coil or in the primary of a transformer may be computed; or rather the product, SA, of the number of turns and the cross-section of the magnetic circuit, may be determined. The relation between S and A may be anything we please, provided the product has the proper value; the proportioning of S and A is largely a question of judgment and good practice.

Theory of the Alternator. — Consider the case of a simple alternator having but one armature coil that rotates in a magnetic field of uniform intensity about an axis at right angles to the direction of the lines of force. If successive instants of time during one revolution of the coil are counted from the instant that the coil passes a line drawn through its axis of rotation and perpendicular to both the axis of rotation and the direction of the magnetic flux, the value of the induction threading the coil at any instant during one cycle is expressed by the equation

$$N = N_{max} \cos \omega t, \qquad (38)$$

in which N_{max} equals that portion of the flux that passes through the coil at the instant the plane of the coil is at right angles to the direction of the lines of force, and ω represents its angular velocity. The instantaneous value of the electromotive force generated in the coil will be, by Faraday's law,

$$e = -\frac{dN}{dt} = \omega N_{max} \sin \omega t = E \sin \omega t, \qquad (39)$$

its maximum value being $E = \omega N_{max}$. (40)

If the coil is closed through a circuit of resistance R', inductance L', and capacity C', the resistance and inductance of the coil itself being R and L respectively, a current i will begin to circulate, and we can write the equation of electromotive forces of the circuit in the form

$$e = (R + R')i + (L + L')\frac{di}{dt} + \frac{\int i dt}{C'}.$$

From this expression we derive the equation of the current in terms of the constants of the circuit and the maximum value of the electromotive force developed in the coil, and obtain

$$i = \frac{E}{\sqrt{[R+R']^2 + \left[\frac{1}{C'\omega} - (L+L')\omega\right]^2}}$$

$$\sin\left\{\omega t + \text{arc tan}\left[\frac{1}{C'\omega(R+R')} - \frac{(L+L')\omega}{R+R'}\right]\right\} \qquad (41)$$

which expresses the instantaneous value of i as soon as a condition of cyclic stability has been attained.

Equations (38), (39), and (40) are the general equations that cover the working of alternating current dynamos; they have been subjected * to graphical analysis by W. E. Goldsborough, to whom is due the diagram shown in Fig. 63 a and its discussion.

Suppose a circuit in which the inductance is zero, the capacity † infinite, and the resistance variable, to be subjected to the influence of a simple harmonic electromotive force that is generated by an alternator having a constant armature inductance for all values of armature current, a constant field excitation, and a constant speed. Under these conditions, the virtual value of the electromotive force at the brushes of the alternator just before the circuit is closed will be

$$\overline{E} = \omega N_{\text{max}} \div \sqrt{2}, \qquad (42)$$

which is represented by the vector OA in the figure. The vector ON is laid off at right angles to OA to represent the value of the magnetomotive force producing N_{max}. It is drawn ninety degrees in advance of the electromotive force it induces, in accordance with the relation exhibited in equations (38) and (39). At the time of closing the circuit, suppose the external variable non-inductive resistance to have a value R', and that the constant armature resistance has a value R, and the con-

* "On the Alternating Current Dynamo," *The Physical Review*, 1896. Professor Goldsborough's paper is followed verbatim.

† The capacity is infinite when there is no condenser in the circuit.

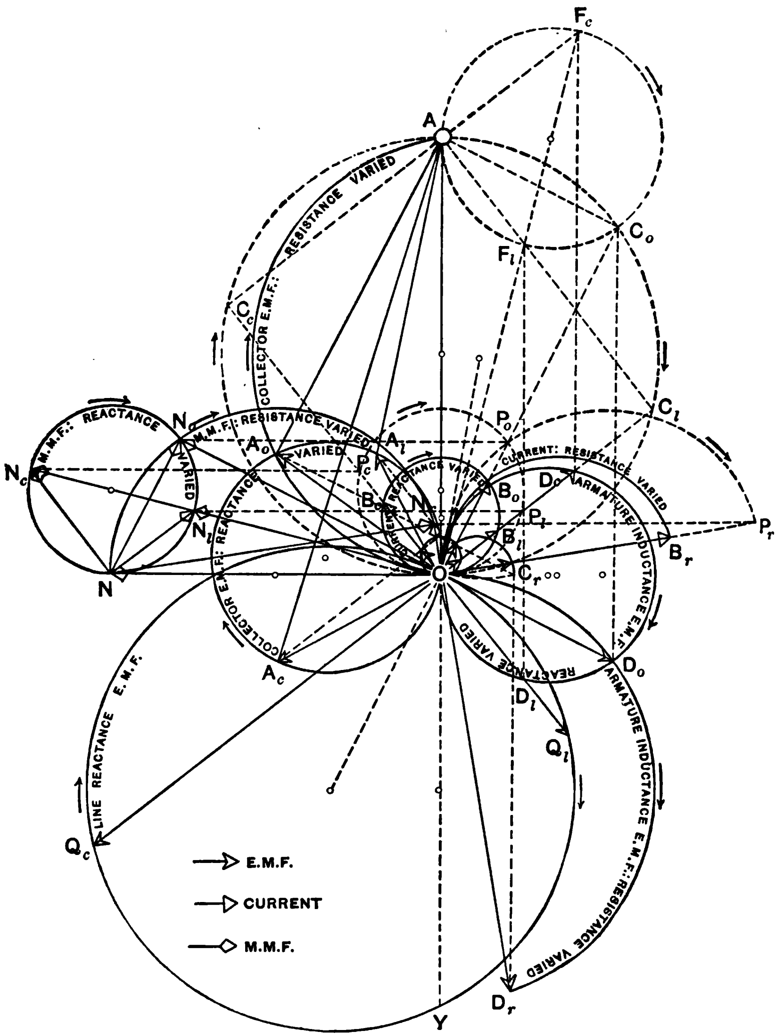


Fig. 63 a.

stant armature inductance a value L. Then the equation of the current will assume the form

$$i = \frac{E}{(\sqrt{R + R'})^2 + L^2\omega^2} \sin\left[\omega t - \operatorname{arc\,tan}\frac{L\omega}{R + R'}\right], \quad (43)$$

and its virtual value

$$\bar{I} = \frac{\bar{E}}{\sqrt{(R+R')^2 + L^2\omega^2}}, \tag{44}$$

which we can represent * by the vector OB_o lagging

$$\text{arc tan} \frac{L\omega}{R+R'} \text{ degrees}$$

behind OA. This armature current will react upon the magnetizing forces due to the constant field excitation, and by virtue of the inductance of the armature will produce an electromotive force in phase with itself which is represented by the vector NN_o, drawn parallel to the current vector from the positive extremity of ON. This armature magnetomotive force sets up a cyclic magnetization developing a counter-electromotive force OD_o lagging ninety degrees behind the current, and there is a loss of effective electromotive force due to the armature resistance that is shown by the short electromotive force vector in phase with OB_o; therefore the total loss of electromotive force in the armature will be the resultant of these two vectors or OA_o. The effective electromotive force that overcomes the resistance of the non-inductive external circuit will be the vector A_oA, since it completes the electromotive force triangle on OA and is in phase with the current OB_o. The total effective electromotive force (OC_o) that overcomes the total ohmic resistance $(R + R')$ of the circuit, is due to the cyclic magnetization set up by the magnetomotive force vector ON_o. ON_o is the resultant of ON and NN_o, and as shown by the geometry of the figure, it is ninety degrees in advance of the current, and therefore of A_oA, as it should be. The projection of NN_o on ON is the component of the armature magnetomotive force that acts against the field magnetization; *i.e.* it is *a measure of*

* The subscript (*o*) refers to the initial condition.
The subscript (*r*) refers to changes in the line resistance.
The subscript (*l*) refers to changes in the inductance of the line.
The subscript (*c*) refers to changes in the capacity of the line.

the armature reaction. The projection of NN_0 on OA is likewise a measure of the cross-magnetizing action of the armature.

Having constructed the initial diagram, we can now follow out what takes place when the resistance of the external circuit is varied. Suppose R' is reduced to a value R_r. The current vector head B_0 will move out along the semicircle OB_0B_r until equilibrium is again established in the circuit by the current reaching its maximum possible value under the new conditions.* The vectors OA and ON retaining their positions, all the other vectors involved will reach their final values corresponding to the new current by following the arcs of the circles passing through their positive extremities to the positions designated by the common subscript letter (r). The correctness of the variations indicated can be readily verified by an inspection of the geometry of the figure in connection with equation (7).

In the present case, R' has been reduced to zero; in other words, the subscripts (r) indicate what takes place when a machine whose armature inductance is large, as well as constant, is short-circuited. A_0 moves up to A, and the electromotive force at the brushes is zero. The current assumes an angle of lag of almost ninety degrees behind the total internal armature electromotive force. OA, the *armature reaction*, almost counterbalances the magnetomotive force of the fields, and the resultant magnetomotive force ON_r is just sufficient to develop the electromotive force OC_r that overcomes the resistance of the armature.

Returning to the initial conditions, suppose we increase the value of L' from zero to some value L_l; *i.e.* suppose we introduce inductance into the external circuit. The virtual value of the current will then be expressed by the equation

$$\bar{I} = \frac{\bar{E}}{\sqrt{(R+R')^2+(L+L_l)^2\omega^2}}, \qquad (45)$$

* See *Alternating Currents*, p. 223.

and it will lag behind the internal electromotive force $\bar{E}$, or OA, by an angle

$$\phi = \text{arc tan}\left(\frac{L+L_l}{R+R'}\omega\right). \tag{46}$$

Referring to the figure, the new positions assumed by the variable vectors, owing to the introduction of L_l, are designated by the subscript (l). The current will decrease and its vector head move along the circle $OB_cB_oB_lO$ until a state of equilibrium exists between the forces involved. The electromotive force that overcomes the resistance and inductance of the armature will decrease also and move to the position OA_l, its vector head following the circle $OA_cA_oA_lO$, and the electromotive force at the collector rings will first decrease and then increase to a final value A_lA. The introduction of inductance into the external circuit brings the electromotive force at the collector rings and the total internal electromotive force (OA) more nearly into phase; it, however, causes a lag angle F_lOB_l to be introduced between the collector electromotive force and the current. The inductance electromotive force of the armature decreases along the circle $OD_cD_oD_lO$ to a value OD_l, and the inductance electromotive force of the external circuit increases from zero along the circle YQ_cOQ_lY to a value OQ_l. The resultant magnetomotive force will be ON_l, and it is seen that while the armature reaction has remained very nearly constant, the cross-magnetizing effect has been reduced about fifty per cent.

From our initial conditions as indicated by the subscript (o), we can also study the effects produced by the introduction of capacity into the external circuit. If the value of C' is reduced from infinity to some value C_c, the virtual value of the current will change to

$$\bar{I} = \frac{\bar{E}}{\sqrt{(R+R')^2 + \left(\frac{1}{C_c\omega} - L\omega\right)^2}}, \tag{47}$$

and the angle between OA and the current will have a value

$$\phi = \text{arc tan}\left[\frac{1}{(R + R')C_c\omega} - \frac{L\omega}{(R + R')}\right]. \qquad (48)$$

In consequence of this change the current vector will assume the position OB_c, and the other variable vectors will move to their corresponding positions shown by the subscript letter (c). The current in its new position is not only in advance of the electromotive force (A_cO) at the brushes, but is also in advance of the electromotive force OA, since it has moved from B_o to a maximum value when passing OA, and then decreased in value.*

The collector electromotive force, on the other hand, steadily increases as the capacity decreases, till it reaches a value A_cA much greater than the open circuit electromotive force of the machine. A resonant effect comes into play here after the capacity of the line neutralizes the inductance of the armature that is very well illustrated by the figure; the line A_cA will be a maximum when it passes from A through the center of the circle $OA_cA_oA_lO$, and will represent the greatest difference of potential that can possibly exist between the brushes so long as R and R' remain unchanged in value. This rise in potential is due to the current being in *advance* of the vector OA, for the position of the armature magnetomotive force vector is also advanced, and NN_c increases the total flux in the air gap instead of diminishing it. The cross-magnetizing action of the armature, however, remains approximately the same.

The introduction of capacity into the line causes the inductance electromotive force of the armature to move to the position D_c, and the reactance electromotive force of the external circuit to decrease through zero and then increasing, assume a position Q_cO, considerably in advance of the electromotive force, and ninety degrees in advance of the current OB_c.

The arrows indicate the relative direction of motion of the

* See *Alternating Currents*, p. 297.

vectors as the resistance is varied from infinity to zero, or as the reactance is carried from zero capacity to an infinite inductance.

By following out a similar line of constructions the effects produced by variations of the armature inductance can be studied, and by successively varying the resistance, inductance, capacity, and frequency constants, and constructing corresponding diagrams, a large variety of problems involving the simultaneous variation of several terms can be successfully treated.

The action of a condenser in neutralizing the effects of magnetic leakage and secondary self-induction in a transformer is analogous to the action just discussed.

CHAPTER V

THE SIMPLE TRANSFORMER DIAGRAM

The Production of an Electromotive Force in the Secondary. — When a current is flowing in the primary circuit of a transformer, a magnetic induction N is set up in the magnetic circuit so as to pass through the secondary circuit of the transformer. This magnetic flux is embraced by the S_2 turns of the secondary circuit, and there are accordingly S_2N lines "linked" with the secondary circuit. These secondary "linkages" being equal to the product of the magnetic flux N, and the number of turns S_2 of the secondary, are likewise known as the secondary "flux-turns," sometimes referred to as the number of lines "threading" the secondary circuit. The subject of magnetic induction has been discussed in Chapter III. The number of lines thus linked with the secondary circuit when a primary current i_1 is flowing has been shown to depend upon the values of the primary current and the coefficient of mutual induction M, being equal to the primary current multiplied by M. We have, accordingly,

$$S_2N = Mi_1.$$

When the magnetic flux is unvarying, which is the case when the primary current has a steady value, there is no electromotive force induced in the secondary circuit.

If the primary current is changing in value, and the magnetic flux is therefore varying as well, there is an electromotive force induced in the secondary circuit, which may be expressed in terms of either the primary current or the magnetic flux.

By Faraday's law, the electromotive force induced in a circuit is equal to the rate of change in the number of lines linked with the circuit, a decrease in the number of these lines making a positive electromotive force. Now, the number of lines linked with the secondary is S_2N or Mi_1. Hence, e_2, the electromotive force induced in the secondary, will be determined as follows:

$$e_2 = -\frac{d(S_2N)}{dt} = -S_2\frac{dN}{dt}; \quad (1)$$

or,

$$= -\frac{d(Mi_1)}{dt} = -M\frac{di_1}{dt}.$$

According to the last relation, the secondary electromotive force induced by the primary current is proportional to the rate at which the primary current is changing, and is equal to this time rate of change multiplied by the coefficient of mutual induction.

This relation might have been stated directly as the result of experiment, in which case M would be defined as the ratio of the induced electromotive force in one circuit to the time rate of change of the inducing current in the other circuit.

The minus sign indicates that the electromotive force is induced in a direction opposite to the primary current when increasing.

We accordingly have this definite relation between the primary current and the electromotive force induced by it in the secondary.

If the electromotive force impressed upon the primary is harmonic, it may be proved * that, with constant coefficients of induction, the primary current is also harmonic and may be written, †

$$i_1 = I_1 \sin \omega t, \quad (2)$$

* From physical reasoning we would suppose an harmonic electromotive force would produce an harmonic current. This is rigorously proved later in the discussion of the general theory of the transformer (see equation (6), Chapter XII.). For simplicity in treatment, we here assume at once an harmonic current. If the coefficients of induction are not constant, the form of current will be modified. When the current or electromotive force are not harmonic, they may be represented by an equivalent harmonic current or electromotive force, as explained in the previous chapter.

† The subscripts one and two, thus e_1, e_2, refer throughout to the primary and secondary circuits respectively. The small letters e and i denote instantaneous

where ω is equal to 2π times the frequency. The time is here counted from the moment when the current is zero. By differentiation we obtain

$$\frac{di_1}{dt} = I_1 \omega \cos \omega t. \qquad (3)$$

Upon substituting this in equation (1) we have an expression for the electromotive force impressed upon the secondary; thus,

$$e_2 = -M\omega I_1 \cos \omega t = M\omega I_1 \sin(\omega t - 90^\circ). \qquad (4)$$

This shows that the electromotive force induced in the secondary by the primary current, lags ninety degrees behind the primary current. When I_1 represents the maximum value of the primary current, $M\omega I_1$ is the maximum value of the electromotive force induced in the secondary; if $\bar{I}_1$ represents the virtual or the square root of the mean square value, then $M\omega\bar{I}_1$ is the virtual value of this induced electromotive force. We thus see that the secondary impressed electromotive force is proportional to the primary current, the coefficient of mutual induction and the frequency. The coefficient of mutual induction, M, is proportional* to the number of primary and secondary turns, the permeability of the iron and the cross-section of the magnetic circuit; and inversely proportional to the length of the magnetic circuit. The coefficient M is usually quite constant and is here so considered, although it varies with the saturation of the iron and the amount of magnetic leakage. Hysteresis will cause an induced electromotive force to lag more than ninety degrees behind the current inducing it.

We may now proceed to construct the simple transformer diagram. The lines in this diagram represent the maximum values of the various quantities. All diagrams are supposed to be revolving at a uniform rate in a counter-clockwise direction; the instantaneous value of a quantity at any time is represented

values, and the corresponding capital letters, E and I, maximum values of harmonic electromotive force and current, respectively, according to the conventions given in Chapter II.

* See equation (19), Chapter III.

by the projection, upon any fixed line of reference, of the line representing the maximum value of the quantity. Referring to Fig. 64, we may draw the line OA (a vector) to represent the maximum value of the harmonic primary current given by equation (2). Ninety degrees behind OA, in the negative or lag direction, draw the vector OB having a magnitude $M\omega I_1$. This

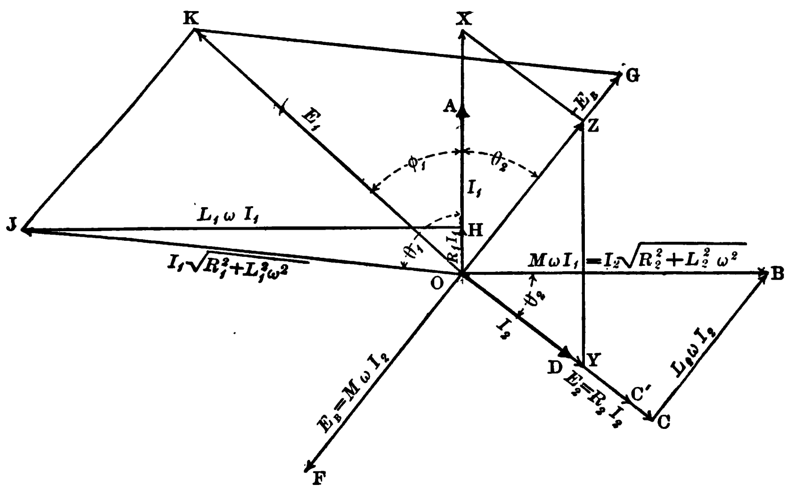


Fig. 64. Transformer diagram (Bedell and Crehore).

line represents the electromotive force induced in the secondary circuit, by the primary current, according to equation (4). It is the electromotive force observed at the terminals of the secondary, when the secondary is on open circuit. When the secondary circuit is closed, a current flows which acts inductively both upon the primary and secondary circuits. These effects will be considered after we have determined the secondary current.

The Secondary Current. — The action in the secondary of a transformer may be looked upon in three ways. In the *first* method, we consider that in the secondary circuit there is the impressed electromotive force $M\omega I_1$, induced by the primary current; we may then treat the secondary as a simple circuit in which this impressed electromotive force is resolved into components, the power electromotive force and reactive electromotive force, to overcome the ohmic resistance and the inductance of the entire secondary circuit, within and without the transformer. By the *second* method, we take into consideration two electromotive forces induced in the secondary, one $M\omega I_1$, due to primary current, and the other $L_2\omega I_2$, due to the secondary current, these two electromotive forces being at right angles to the primary and secondary currents, respectively. The power electromotive force is the resultant of these two induced electromotive forces. In the *third* method, the magnetization produced by the resultant of the primary and secondary ampere turns is determined; the secondary (power) electromotive force is ninety degrees behind the resultant magnetization, being dependent upon the rate at which the latter is changing; that is, $e_2 = - S_2 \frac{dN}{dt}$.

First Method. — Let us first regard the electromotive force $M\omega I_1$, induced in the secondary by the primary current, as acting in a single circuit, whose total resistance is R_2, and inductance L_2. For the present we will consider that the circuit connected to the secondary of the transformer consists of a non-inductive resistance, as is the case in systems of incandescent lighting. The secondary inductance L_2 is, therefore, entirely within the transformer, and there is no self-induction in the secondary external circuit. The current which will flow in the secondary under these circumstances may be readily determined by considering it as a simple circuit, containing an impressed electromotive force $M\omega I_1$.

If the electromotive force acting in a simple circuit in which there is resistance and self-induction is

$$e = E \sin \omega t, \tag{5}$$

we know * that the current will be

$$i = \frac{E}{\sqrt{R^2 + L^2\omega^2}} \sin\left(\omega t - \text{arc} \tan \frac{L\omega}{R}\right). \tag{6}$$

* See equation 12, Chapter IV.

This means that the current is harmonic and lags behind the harmonic impressed electromotive force by an angle whose tangent is the ratio of the reactance of the circuit to its ohmic resistance; the maximum value of the current is equal to the impressed electromotive force divided by the impedance.

If the electromotive force is that induced in the secondary by the primary current, as given in equation (4,) the secondary current is

$$i_2 = \frac{M\omega I_1}{\sqrt{R_2^2 + L_2^2\omega^2}} \sin\left(\omega t - 90° - \text{arc}\tan\frac{L_2\omega}{R_2}\right). \qquad (7)$$

This may be written

$$i_2 = I_2 \sin(\omega t - 90° - \theta_2).$$

The secondary current is, therefore, harmonic, lagging behind the electromotive force $M\omega I_1$ by an angle θ_2. Its maximum value is

$$I_2 = \frac{M\omega I_1}{\sqrt{R_2^2 + L_2^2\omega^2}}; \qquad (8)$$

and

$$\tan\theta_2 = \frac{L_2\omega}{R_2}.$$

We may write J_2 for the secondary impedance and make use of the following abbreviations:

$$J_2 = \sqrt{R_2^2 + L_2^2\omega^2};$$

$$\frac{I_2}{I_1} = \frac{M\omega}{J_2} = \gamma_2. \qquad (9)$$

The ratio γ_2 of the secondary to the primary current will enter into many of our later results.

Referring to Fig. 64, we may now represent the secondary current by a vector OD, whose magnitude is equal to the maximum value I_2 lagging behind OB by an angle θ_2. By the graphical construction for simple circuits we know that the right triangle OCB upon OB as an hypotenuse, represents the secondary electromotive forces; OC representing that necessary to overcome the resistance; CB, that necessary to overcome the self-induction. The hypotenuse OB represents the impressed electromotive force upon the secondary and equals the geometrical sum of the components OC and CB. The *power* electromotive force is the *ohmic* electromotive force $OC = R_2I_2$; it is in the direction of the current and is equal to the current multiplied by the resistance. The *reactive* electromotive force CB is at right angles to the power electromotive force and so represents no power or expenditure of energy.

In speaking of the secondary resistance R_2, in the construction of these diagrams and in the analytical study of the transformer which follows, the resistance of the entire secondary circuit is included; that is, R_2 is the sum of the resistance of the external secondary circuit and of the secondary coil within the transformer itself. It accordingly follows that the line OC, which is the power electromotive force R_2I_2, represents the electromotive force used in overcoming the ohmic resistance in the entire secondary circuit, both within and without the transformer. The fall in potential due to the resistance of the secondary coil is small as compared with the fall in potential in the external circuit. In a well-designed transformer it may, at full load, amount to one per cent of the secondary electromotive force, or thereabouts. If OC is the total ohmic electromotive force, the electromotive force obtained at the secondary terminals would accordingly have a smaller value, OC′, which is less than OC by the amount CC′ due to the fall in potential caused by the resistance of the secondary within the transformer.

Mention may be made here of the effects due to magnetic leakage, hysteresis, and other factors which tend to complicate the simple transformer problem. Magnetic leakage causes a further fall in potential, in addition to that produced by ohmic resistance. The drop in the secondary electromotive force, as obtained at the secondary terminals, may be analyzed and found to consist of a drop due to the ohmic resistance of the primary, a drop due to the loss of energy in hysteresis and eddy currents, a drop due to the ohmic resistance of the secondary, and a drop due to magnetic leakage. All but the last of these involve a loss of energy and, therefore, affect the efficiency of the transformer as well as its regulation. In an ideal transformer, with no magnetic leakage and with no iron or copper losses, the ratio of transformation would be constant, and under all circumstances the ratio of the primary and secondary electromotive forces (at the terminals) would be equal to the ratio of the primary and secondary turns. The departure from this ideal, in a well-designed transformer, should not exceed three, or three and one-half, per cent.

In the absence of hysteresis the induced electromotive force, OB, is exactly ninety degrees behind the primary current. Where hysteresis is present, OB would be more than ninety degrees behind OA on account of the lagging of the magnetism of the iron.* For the present we may leave this additional lag out of consideration and construct our diagrams with OB ninety degrees behind the primary current.

The observed difference of potential at the terminals of the secondary is generally spoken of as the "secondary electromotive force," which, as explained above, is represented by OC′ in the diagram. More strictly the line OC represents the secondary electromotive force — being the total power electromotive force — which would be the difference of potential at the terminals of the secondary if there were no drop due to the resistance of the secondary coils.

Second Method. — In the first method, just discussed, OB, the electromotive force induced in the secondary by the primary current, has been considered as an electromotive force impressed upon the secondary as a simple circuit containing resistance and inductance. This impressed electromotive force has been resolved into the power electromotive force OC, and the reactive electromotive force CB. Note the direction of these various electromotive forces as shown in Fig. 65.

* See Chapter XIX.

In the second method, the power electromotive force OC is obtained as the *resultant* of OB and BC; see Fig. 66. In the secondary circuit there are considered to be *two* electromotive forces induced: OB induced by the primary current and equal

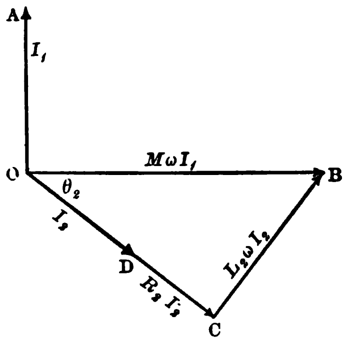


Fig. 65.

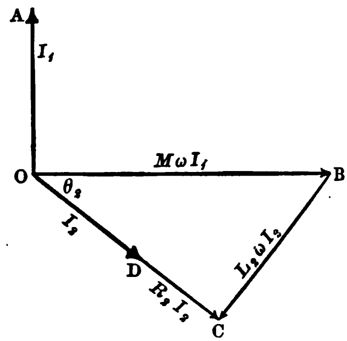


Fig. 66.

to $M\omega I_1$; and BC induced by the secondary current and equal to $L_2\omega I_2$. Each of these electromotive forces is ninety degrees *behind* the current inducing it. In the second method we consider BC, the electromotive force induced by the secondary current; in the first method we consider CB to overcome BC, and therefore equal and opposite to it. The two methods are otherwise identical, as seen by a comparison of Figs. 65 and 66. All that has been said concerning the transformer diagram in Fig. 64 is true, whichever method we employ in our construction.

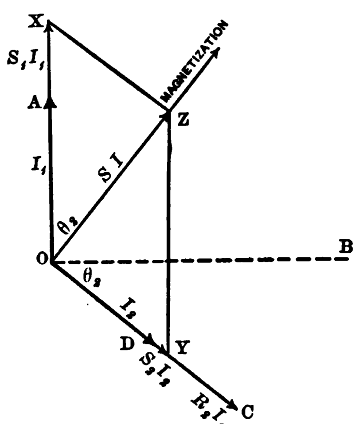


Fig. 67.

Third Method.—In the third method the primary and secondary currents are considered, as in Fig. 67. The primary ampere turns S_1I_1, and secondary ampere turns S_2I_2,—in the direction of the primary and secondary currents, respectively—are represented by OX and OY. The resultant ampere turns SI are accordingly represented by OZ, the vector sum of OX and OY. The magnetization set up is in the direction of the resultant ampere turns. (It would lag somewhat in the case of hysteresis.) The secondary electromotive force OC, being proportional to the rate of change of the magnetization, is ninety degrees behind OZ. This is more fully discussed in Chapter VII.

The three methods here given lead to identical results; for clearness it is well to consider all three. The first two methods are practically the same, but differ in the point of view. It is merely a question of the direction of the electromotive force BC, whether this shall be the self-induced secondary electromotive force, or the equal and opposite electromotive force necessary to overcome it.

In the first two methods, the various component electromotive forces are considered, from which the total electromotive force is built up synthetically. Each of these component electromotive forces, represented by a line in the transformer diagram, has a corresponding term in the equation of electromotive forces in the analytical discussion in the following chapter. Hence the graphical or synthetic method of the transformer diagram and the analytical method of the theoretical treatment are exactly parallel, and may be compared throughout, each line in the one having a corresponding term in the equations of the other.

This is not so with the third method. This method, however, possesses the advantage that it more closely represents the physical facts. There is one electromotive force induced in the secondary, due to the changing magnetization; the magnetization considered is that produced by the combined effects of the primary and secondary currents. Evidently this is more real than considering two electromotive forces produced independently by the primary and secondary currents.

The action in the secondary circuit, due to both primary and secondary current, having thus been shown, it remains to study the reactive effect of the secondary current upon the primary circuit.

The Influence of the Secondary upon the Primary. — The relation between the primary and secondary circuits is entirely a mutual one. A current flowing in the secondary must cause an electromotive force in the primary just as a current in the primary causes an electromotive force in the secondary. This electromotive force set up by the secondary current in the primary is usually called the "back" electromotive force. It may be found in exactly the same manner as is found the electromotive force impressed upon the secondary, due to the primary current. (See equation (1) above.) This back electromotive force is proportional to the time rate of change of the secondary current, and is equal to it multiplied by the coefficient of mutual induction M. Thus, writing e_B for back electromotive force, we have

$$e_B = -M\frac{di_2}{dt}. \qquad (10)$$

Differentiating the expression for secondary current, equation (7), we have

$$\frac{di_2}{dt} = I_2\omega\cos\left(\omega t - 90^\circ - \text{arc tan}\frac{L_2\omega}{R_2}\right). \qquad (11)$$

By substituting this value in (10) above, we obtain the back electromotive force:

$$e_B = M\omega I_2 \sin\left(\omega t - 180^\circ - \text{arc tan}\frac{L_2\omega}{R_2}\right). \qquad (12)$$

The back electromotive force accordingly is ninety degrees behind the secondary current, and has a maximum value, $M\omega I_2$. This bears the same relation to the secondary current both in magnitude and direction as the electromotive force $M\omega I_1$, induced in the secondary, bears to the primary current.

In Fig. 64 we may now represent the back electromotive force by a vector OF, ninety degrees behind the secondary current, OD, and equal in magnitude to $M\omega I_2$.

Having assumed the primary current to be as given in equation (2) and represented by OA in our transformer diagram, we have determined the secondary electromotive force, the secondary current, shown by the line OD and equation (7), and the back electromotive force, shown by the line OF and equation (12). It now remains to find what impressed primary electromotive force will be required to cause the primary current to flow.

The Primary Electromotive Forces. — With the secondary on open circuit the action of the transformer would be the same as though the secondary circuit were entirely removed; the primary would then behave as a simple circuit, and the impressed electromotive force would be used in overcoming the primary resistance and self-induction. Let us now consider the effect produced upon the primary by a secondary circuit in which a current is flowing.

Instead of having but two electromotive forces to overcome, as it would have in the case of a simple circuit, the primary electromotive force must now not only overcome the electromotive force of resistance and of self-induction in the primary, but also the back electromotive force induced in the primary by the current in the secondary.

The electromotive force to overcome the primary resistance always has the same direction as the primary current, and is equal to the primary current multiplied by its resistance; that is, denoting the instantaneous value of this component by e_{R_1}, we have for the primary ohmic electromotive force

$$e_{R_1} = R_1 i_1 = R_1 I_1 \sin \omega t. \qquad (13)$$

The maximum value of this ohmic electromotive force is $R_1 I_1$, and it is denoted in Fig. 64 by the vector OH in the direction of the primary current, equal to the primary current multiplied by R_1.

The component of the primary electromotive force necessary to overcome the primary self-induction is equal and opposite to the electromotive force of self-induction, and is therefore ninety degrees ahead of the current. It is represented by the equation

$$e_{L_1} = L_1 \frac{di_1}{dt} = L_1 \omega I_1 \sin(\omega t + 90°). \qquad (14)$$

The maximum value of this reactive component is $L_1 \omega I_1$, and is represented in Fig. 64 by the vector HJ, ninety degrees ahead of the primary current OA. Indeed, the triangle OHJ is the triangle of electromotive forces for the primary when no current is allowed to flow in the secondary, and the primary is considered as a simple circuit having resistance and self-induction. In this case the necessary impressed electromotive force upon the primary is OJ.

But when the secondary current is allowed to flow, the primary electromotive force must also overcome the back electromotive force due to this secondary current. This electromotive force, to overcome the back electromotive force, must be equal and opposite to it, and is represented, therefore, by equation (12) with the sign changed; that is,

$$-e_B = M\omega I_2 \sin\left(\omega t - \arctan \frac{L_2 \omega}{R_2}\right). \qquad (15)$$

The maximum value of this component is $M\omega I_2$. In Fig. 64, the line OG, being equal and opposite to OF, represents that component of the primary electromotive force necessary to overcome the back electromotive force.

Now, the primary electromotive force is easily found in the diagram, since it is the geometrical sum of the three components OH, HJ, and OG. The resultant of OH and HJ gives OJ, and the resultant of OJ and OG gives OK, the required primary impressed electromotive force.

We may find this electromotive force by adding together the three components expressed in equations (13), (14), and (15). The sum of (13) and (14) gives

$$e_{R_1 L_1} = I_1 \sqrt{R_1^2 + L_1^2 \omega^2} \sin\left(\omega t + \arctan \frac{L_1 \omega}{R_1}\right) \qquad (16)$$

$$= I_1 J_1 \sin(\omega t + \theta_1).$$

This is the electromotive force to overcome the primary impedance, and is represented in the diagram by the line OJ, which would be the total impressed electromotive force in the absence of the back electromotive force due to the current flowing in the secondary. The angle HOJ, or the difference in phase between the current and electromotive force under these conditions, is represented by θ_1.

The value of OJ is thus obtained from (16), the sum of (13) and (14), or directly from the geometry of the figure, being the geometrical sum of OH and HJ; thus,

$$\text{OJ} = I_1\sqrt{R_1^2 + L_1^2\omega^2}.$$

The total impressed primary electromotive force may be found from the sum of (15) and (16); or from the graphical construction. The value of the primary electromotive force thus obtained is

$$e_1 = E_1 \sin(\omega t + \phi_1), \tag{17}$$

where E_1 is its maximum value, and ϕ_1 is the angle of phase difference between it and the primary current.

The value of E_1 may be found as follows: From the trigonometrical relation between the three sides of a triangle, we have

$$E_1^2 = \overline{\text{OK}}^2 = \overline{\text{OJ}}^2 + \overline{\text{OG}}^2 + 2\,\text{OJ}\cdot\text{OG}\cdot\cos\text{JOG}.$$

$$\cos\text{JOG} = \cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2.$$

$$\cos\theta_1 = \frac{R_1}{J_1}; \qquad \cos\theta_2 = \frac{R_2}{J_2};$$

$$\sin\theta_1 = \frac{L_1\omega}{J_1}; \qquad \sin\theta_2 = \frac{L_2\omega}{J_2}.$$

Hence,
$$\cos\text{JOG} = \frac{R_1R_2 - L_1L_2\omega^2}{J_1J_2};$$

and
$$E_1^2 = I_1^2J_1^2 + M^2\omega^2I_2^2 + \frac{2\,M\omega I_2I_1(R_1R_2 - L_1L_2\omega^2)}{J_2}.$$

Inserting the values of the primary and the secondary impedances for J_1 and J_2, and remembering the relation (9),

$$\gamma_2 = \frac{M\omega}{J_2} = \frac{I_2}{I_1},$$

we obtain

$$E_1^2 = I_1^2(R_1^2 + L_1^2\omega^2) + I_1^2\gamma_2^4(R_2^2 + L_2^2\omega^2) + 2\gamma_2^2 I_1^2(R_1R_2 - L_1L_2\omega^2).$$

$$E_1 = I_1\sqrt{(R_1 + \gamma_2^2R_2)^2 + (L_1\omega - \gamma_2^2L_2\omega)^2}.$$

This gives us the relation between the primary electromotive force and the current which will flow under any given conditions. We may write

$$I_1 = \frac{E_1}{\sqrt{(R_1 + \gamma_2^2R_2)^2 + (L_1\omega - \gamma_2^2L_2\omega)^2}} \tag{18}$$

$$= \frac{E_1}{\text{apparent impedance}}.$$

The primary current is equal to the impressed electromotive force divided by the apparent impedance, given in the denominator above. The apparent impedance of the primary (J'_1) when the secondary circuit is closed, differs from the impedance (J_1) of the primary when the secondary circuit is open, on account of the back electromotive force induced in the primary circuit by the secondary current. The resistance of the primary is apparently increased and its self-induction diminished, which may be seen as follows:

We have the relations

$$\text{impedance} = \sqrt{\text{resistance}^2 + \text{reactance}^2},$$

or,

$$J_1 = \sqrt{R_1^2 + L_1^2\,\omega^2};$$

$$\text{appar. impedance} = \sqrt{\text{appar. resistance}^2 + \text{appar. reactance}^2},$$

or,

$$J'_1 = \sqrt{R'^2_1 + L'^2_1\,\omega^2}.$$

From an examination of (18) above, it is evident that

$$\text{apparent resistance} = R'_1 = R_1 + \gamma_2{}^2 R_2;$$
$$\text{apparent reactance} = L'_1\omega = L_1\omega - \gamma_2{}^2 L_2\omega;$$
$$\text{apparent inductance} = L'_1 = L_1 - \gamma_2{}^2 L_2.$$

The effect of closing the secondary circuit is to apparently increase the primary resistance by an amount equal to $\gamma_2{}^2$ times the secondary resistance. The primary reactance and coefficient of self-induction are apparently decreased by an amount equal to $\gamma_2{}^2$ times the secondary reactance and coefficient of self-induction, respectively. The coefficient γ_2 is equal to the ratio of the secondary and primary currents, as given in equation (9). When the secondary circuit is open, $\gamma_2 = 0$, and these apparent values reduce to the real values.

Inasmuch as the angle of lag in a circuit is determined by the ratio of the apparent reactance of the circuit to its apparent resistance, we have for ϕ_1, the angle by which the primary current lags behind the electromotive force, when the secondary is closed,

$$\tan\phi_1 = \frac{\text{apparent reactance}}{\text{apparent resistance}} = \frac{L'_1\omega}{R'_1} = \frac{L_1\omega - \gamma_2 L_2{}^2\omega}{R_1 + \gamma_2{}^2 R_2}.$$

If we let θ_1 be the value of this angle when the secondary is on open circuit, we have

$$\tan\theta_1 = \frac{\text{reactance}}{\text{resistance}} = \frac{L_1\omega}{R_1}.$$

In the figure, θ_1 is the angle HOJ; and ϕ_1 is the angle AOK. On open circuit the primary current lags almost ninety degrees behind the primary electromotive force. This angle is diminished when the secondary is closed.

From Fig. 64 we have thus been able to deduce all the essential relations between the various quantities involved in the simple transformer, obtaining these from the geometrical relations of the transformer diagram. In the following chapter these same relations will be analytically deduced, after which they will be discussed further.

CHAPTER VI

SIMPLE ANALYTICAL THEORY OF THE TRANSFORMER

Fundamental Equations. — The analytical theory of the transformer is based upon two differential equations, one showing the relation between the electromotive forces in the primary circuit at any instant of time, and the other showing the corresponding relations for the secondary circuit. These equations are written below in (1) and (2), and may be interpreted as follows. The electromotive force impressed upon the primary circuit is equal to the sum of the component electromotive forces necessary to overcome ohmic resistance, the counter-electromotive force of self-induction, and the back electromotive force induced in the primary by the secondary current. This relation is mathematically expressed in equation (1), the impressed primary electromotive force, e_1, at any instant, being equal to the sum of the ohmic electromotive force, $R_1 i_1$, and the electromotive forces $L_1 \frac{di_1}{dt}$ and $M \frac{di_2}{dt}$, to overcome self and mutual induction, respectively. In the secondary circuit, the sum of these three component electromotive forces, at any instant of time, is evidently equal to zero, inasmuch as there is no external source of electromotive force in the circuit and the internal and external electromotive forces for any circuit are always equal. It is the solution of these two equations that is necessary in order to obtain the values of the primary and secondary currents.

Our fundamental electromotive force equations are as follows:

For the primary,

$$R_1 i_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = e_1 ; \qquad (1)$$

For the secondary,

$$R_2 i_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} = 0. \tag{2}$$

These equations* may also be established from a consideration of the fact that the current which flows in any circuit at any instant is equal to the sum of the electromotive forces active in the circuit, divided by the resistance of the circuit. We have accordingly for the primary and secondary currents —

For the primary,

$$i_1 = \frac{e_1 - L_1 \frac{di_1}{dt} - M \frac{di_2}{dt}}{R_1}; \tag{1a}$$

For the secondary,

$$i_2 = \frac{-L_2 \frac{di_2}{dt} - M \frac{di_1}{dt}}{R_2}. \tag{2a}$$

Equations (1) and (2) are then obtained by transposition. The electromotive forces *of* self and mutual induction in (1 a) and (2 a) are negative in accordance with Faraday's law (Chapter III.), the corresponding electromotive forces *to overcome* self and mutual induction being positive, as in (1) and (2). From equations (1) and (2) we desire first to eliminate i_2 and to solve for the primary current i_1; and then to eliminate i_1 from the same equations and to solve for the secondary current i_2.

The method of solution will be to rearrange equations (1) and (2) so as to obtain two differential equations (6) and (8), for the primary and secondary currents respectively. These equations will then be solved by assuming that the currents and electromotive forces are harmonic, or may be represented by equivalent harmonic functions.

* The first solution for the flow of current in a primary and secondary circuit was given by Maxwell and was based upon these equations. See "A Dynamical Theory of the Electromagnetic Field," *Philosophical Transactions of the Royal Society*, p. 473, 1865. See also, E. C. Rimington, *Philosophical Magazine*, Vol. 37, p. 394.

Differentiating (1) and (2) with respect to t, we have

$$R_1\frac{di_1}{dt}+L_1\frac{d^2i_1}{dt^2}+M\frac{d^2i_2}{dt^2}=\frac{de_1}{dt}; \qquad (3)$$

$$R_2\frac{di_2}{dt}+L_2\frac{d^2i_2}{dt^2}+M\frac{d^2i_1}{dt^2}=0. \qquad (4)$$

Multiplying (3) by L_2 and (4) by M, and subtracting, we have

$$(L_1L_2-M^2)\frac{d^2i_1}{dt^2}+L_2R_1\frac{di_1}{dt}-MR_2\frac{di_2}{dt}=L_2\frac{de_1}{dt}. \qquad (5)$$

Multiplying (1) by R_2, and adding to (5), gives

$$(L_1L_2-M^2)\frac{d^2i_1}{dt^2}+(L_1R_2+L_2R_1)\frac{di_1}{dt}$$
$$+R_1R_2i_1=R_2e_1+L_2\frac{de_1}{dt}. \qquad (6)$$

Multiplying (3) by M and (4) by L_1, and subtracting, we have

$$(L_1L_2-M^2)\frac{d^2i_2}{dt^2}-MR_1\frac{di_1}{dt}+L_1R_2\frac{di_2}{dt}=-M\frac{de_1}{dt}. \qquad (7)$$

Multiplying (2) by R_1 and adding to (7), gives

$$(L_1L_2-M^2)\frac{d^2i_2}{dt^2}+(L_1R_2+L_2R_1)\frac{di_2}{dt}+R_1R_2i_2=-M\frac{de_1}{dt}. \qquad (8)$$

Equations (6) and (8) are the differential equations for the primary and secondary currents respectively. [Compare equations (26) and (29), Chapter XI.] For the simplest treatment, we may at once assume that the impressed electromotive force is harmonic, and that harmonic electromotive forces produce harmonic currents. Physical reasons would lead us to suppose that an harmonic electromotive force would give rise to harmonic currents in both primary and secondary, — the coefficients of induction being considered constant; and this is rigorously shown to be so by the more general analytical treatment given in Chapter XII.

For an harmonic electromotive force we have, then,

$$e_1 = E_1 \sin \omega t;$$

$$\frac{de_1}{dt} = E_1 \omega \cos \omega t;$$

$$i_1 = I_1 \sin(\omega t + \phi_1); \qquad i_2 = I_2 \sin(\omega t + \phi_2); \tag{9}$$

$$\frac{di_1}{dt} = I_1 \omega \cos(\omega t + \phi_1); \qquad \frac{di_2}{dt} = I_2 \omega \cos(\omega t + \phi_2);$$

$$\frac{d^2 i_1}{dt^2} = -I_1 \omega^2 (\sin \omega t + \phi_1); \qquad \frac{d^2 i_2}{dt^2} = -I_2 \omega^2 \sin(\omega t + \phi_2).$$

In these equations, the angle of phase difference between the primary current and the primary electromotive force is represented by ϕ_1; the angle of phase difference between the secondary current and the primary electromotive force is represented by ϕ_2. Evidently the secondary current differs in phase from the primary current by an angle equal to $\phi_2 - \phi_1$. All of these angles will be found to be *negative;* that is, the primary current *lags* behind the primary electromotive force and the secondary current lags behind the current in the primary.

The values for the primary and secondary currents may be obtained by substituting these expressions (9), in equations (6) and (8).

The Primary Current. — Substituting in (6), in order to obtain the solution for the primary current, we have

$$\begin{aligned} &\{R_1 R_2 - (L_1 L_2 - M^2)\omega^2\} I_1 \sin(\omega t + \phi_1) \\ &\quad + (L_1 R_2 + L_2 R_1) I_1 \omega (\cos \omega t + \phi_1) \\ &\quad = E_1 R_2 \sin \omega t + E_1 L_2 \omega \cos \omega t. \end{aligned}$$

Combined by means of the trigonometric formula,

$$A \sin \theta + B \cos \theta = \sqrt{A^2 + B^2} \sin\left(\theta + \arctan \frac{B}{A}\right), \tag{10}$$

with the abbreviations given below this becomes

$$I_1 \sqrt{\alpha^2 + \beta^2} \sin\left(\omega t + \phi_1 + \arctan \frac{\beta}{\alpha}\right) = E_1 J_2 \sin\left(\omega t + \arctan \frac{L_2 \omega}{R_2}\right). \tag{11}$$

Here the following abbreviations are employed:

$$J_2 = \sqrt{R_2^2 + L_2^2\omega^2} = \text{secondary impedance};$$

$$\alpha = R_1R_2 - \omega^2(L_1L_2 - M^2); \qquad (12)$$

$$\beta = \omega(R_1L_2 + R_2L_1). \qquad (13)$$

The relation expressed in equation (11) is true for all values of the time; hence the two sine functions must be in phase. The maximum values are accordingly identical and may be equated thus:

$$I_1\sqrt{\alpha^2+\beta^2} = E_1J_2;$$

and

$$I_1 = \frac{E_1}{\frac{\sqrt{\alpha^2+\beta^2}}{J_2}}. \qquad (14)$$

We have thus obtained the primary current in terms of the impressed electromotive force and the constants of the primary and secondary circuits. This leads to a useful expression for the value of the primary current, when the values of α and β are inserted, as in (18) below.

The apparent impedance of the primary is equal to the primary electromotive force divided by the current; that is,

$$I_1 = \frac{E_1}{\text{primary apparent impedance}} = \frac{E_1}{J'_1}; \qquad (15)$$

hence, from (14) and (15),

$$J'_1 = \frac{\sqrt{\alpha^2+\beta^2}}{J_2}. \qquad (16)$$

By inserting the values for α and β, we obtain * for the ap-

* From the values of α and β, we have

$$\alpha^2+\beta^2 = L_1^2L_2^2\omega^4 + M^4\omega^4 + R_1^2R_2^2 - 2\,L_1L_2M^2\omega^4 + 2\,R_1R_2M^2\omega^2 + R_1^2L_2^2\omega^2 + R_2^2L_1^2\omega^2$$

$$= J_2^2\left\{R_1^2 + 2\left(\frac{M\omega}{J_2}\right)^2R_1R_2 + \left(\frac{M\omega}{J_2}\right)^4R_2^2 + L_1^2\omega^2 - 2\left(\frac{M\omega}{J_2}\right)^2L_1L_2\omega^2 + \left(\frac{M\omega}{J_2}\right)^4L_2^2\omega^2\right\}. \qquad (16a)$$

With the use of the abbreviation γ_2, this leads to the desired simplified expression for the apparent impedance.

parent impedance of the primary

$$J'_1=\sqrt{(R_1+\gamma_2{}^2R_2)^2+(L_1\omega-\gamma_2{}^2L_2\omega)^2}. \qquad (17)$$

The value of the primary current is accordingly

$$I_1=\frac{E_1}{\sqrt{(R_1+\gamma_2{}^2R_2)^2+(L_1\omega-\gamma_2{}^2L_2\omega)^2}}. \qquad (18)$$

The abbreviation γ_2 is the ratio of the secondary and primary currents; thus,

$$\gamma_2=\frac{M\omega}{J_2}=\frac{I_2}{I_1}. \qquad (19)$$

[See equation (26), below; also equation (9), Chapter V.]

As seen from (17), the effect of the secondary is apparently to increase the primary resistance by an amount $\gamma_2{}^2R_2$ and to decrease the reactance by an amount $\gamma_2{}^2L_2\omega$; that is, the primary self-induction is apparently decreased by $\gamma_2{}^2$ times the value of the secondary self-induction. When the secondary is removed or is on open circuit, $\gamma_2=0$, and the primary behaves as a simple circuit. These identical results were graphically obtained in the previous chapter.

When the transformer is fully loaded, we may make an approximation by assuming that the primary and secondary ampere turns are equal; that is, $S_1I_1=S_2I_2$. We then have from (19),

$$\gamma_2=\frac{S_1}{S_2}, \text{ approximately,} \qquad (19\text{a})$$

and* $L_1\omega-\gamma_2{}^2L_2\omega=0$, approximately. The assumption can be made only when magnetic leakage is negligible. The apparent reactance of the primary is reduced to zero, and (18) becomes

$$I_1=\frac{E_1}{R_1+\frac{S_1{}^2}{S_2{}^2}R_2}. \qquad (20)$$

The primary is here represented as acting like a simple non-

* This follows from the relation $L_1 : L_2 :: S_1{}^2 : S_2{}^2$, given in Chapter III.

inductive circuit with its resistance, R_1, increased by $\frac{S_1^2}{S_2^2}$ times the resistance of the entire secondary circuit. The effect of the secondary may be thus considered as transferred to the primary. This is discussed in the following chapter. [See equation (20), Chapter VII.] Where hysteresis is present, its effect should be included in the value of R_1, as explained in Chapter XIX.

Primary Angle of Lag. — Inasmuch as in equation (11) we have two sine functions equal to each other, these sine functions being identical in phase relations as well as in amplitude, it follows that

$$\phi_1 + \text{arc tan}\,\frac{\beta}{\alpha} = \text{arc tan}\,\frac{L_2\omega}{R_2};$$

or

$$\phi_1 = -\text{arc tan}\,\frac{\beta}{\alpha} + \text{arc tan}\,\frac{L_2\omega}{R_2}. \qquad (21)$$

By a trigonometric transposition,* this becomes

$$\tan\phi_1 = -\frac{\frac{\beta}{\alpha} - \frac{L_2\omega}{R_2}}{1 + \frac{\beta}{\alpha}\cdot\frac{L_2\omega}{R_2}} = -\frac{\beta R_2 - \alpha L_2\omega}{\alpha R_2 + \beta L_2\omega}. \qquad (21a)$$

This value for the primary angle of lag becomes more significant when the values for α and β are inserted,† when it takes the following form:

$$\tan\phi_1 = -\frac{L_1\omega - \gamma_2^2 L_2\omega}{R_1 + \gamma_2^2 R_2} = -\frac{L'_1\omega}{R'_1} = -\frac{\text{apparent reactance}}{\text{apparent resistance}}. \qquad (23)$$

* The formula here employed is

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}. \qquad (22)$$

† The intermediate steps employed in the transformation are

$$-\frac{\beta R_2 - \alpha L_2\omega}{\alpha R_2 + \beta L_2\omega} = -\frac{R_2^2 L_1\omega + L_1 L_2^2\omega^3 - M^2 L_2\omega^3}{R_1 R_2^2 + R_1 L_2^2\omega^2 + M^2 R_2\omega^2}$$

$$= -\frac{L_1\omega(R_2^2 + L_2^2\omega^2) - M^2\omega^2 L_2\omega}{R_1(R_2^2 + L_2^2\omega^2) + M^2\omega^2 R_2} = -\frac{L_1\omega - \left(\frac{M\omega}{J_2}\right)^2 L_2\omega}{R_1 + \left(\frac{M\omega}{J_2}\right)^2 R_2}. \qquad (22a)$$

In the assumption of an harmonic electromotive force,

$$e_1 = E_1 \sin \omega t,$$

time is counted from the time at which the electromotive force is passing through zero. All phase relations are accordingly reckoned with reference to the primary electromotive force. The angle ϕ_1 consequently indicates the phase of the primary current with reference to the primary electromotive force. The negative sign in (23) indicates that the current *lags behind* the electromotive force by the angle ϕ_1. The tangent of this angle is equal to the ratio of the apparent reactance of the primary circuit to its apparent resistance.

The Secondary Current. — The secondary current is obtained by substituting in equation (8) the values given in equation (9) under the harmonic assumption. We thus obtain:

$$\{R_1R_2 - (L_1L_2 - M^2)\omega^2\}\, I_2 \sin(\omega t + \phi_2) +$$
$$(L_1R_2 + L_2R_1)I_2\omega \cos(\omega t + \phi_2) = -ME_1\omega \cos \omega t;$$

which reduces to

$$I_2\sqrt{\alpha^2+\beta^2}\,\sin\left(\omega t + \phi_2 + \text{arc tan}\,\frac{\beta}{\alpha}\right) = ME_1\omega \sin(\omega t - 90^\circ). \qquad (24)$$

These two sine functions being equal, their maximum values must be equal. Hence

$$I_2 = \frac{ME_1\omega}{\sqrt{\alpha^2+\beta^2}} = \frac{M\omega I_1}{J_2}. \qquad (25)$$

The last term in this equation, which is found by referring to (14), expresses the relation that the secondary current is equal to the electromotive force $M\omega I_1$ induced in the secondary by the primary current, divided by the secondary impedance. The ratio of the secondary to the primary current is thus seen to be

$$\frac{I_2}{I_1} = \frac{M\omega}{J_2} = \gamma_2, \qquad (26)$$

a relation we have already noted. Here J_2 is the impedance of the secondary both within and without the transformer.

From equation (25), the relation between the secondary current and the primary electromotive force may be obtained * by substituting the values for α and β; thus

$$I_2=\frac{ME_1\omega}{\sqrt{\alpha^2+\beta^2}}=\frac{\gamma_1 E_1}{\sqrt{(R_2+\gamma_1{}^2R_1)^2+(L_2\omega-\gamma_1{}^2L_1\omega)^2}}; \qquad (27)$$

where γ_1 is a constant with the value

$$\gamma_1=\frac{M\omega}{J_1}=\frac{M\omega}{\sqrt{R_1{}^2+L_1{}^2\omega^2}}. \qquad (28)$$

Inasmuch as R_1 is usually small as compared with $L_1\omega$, it follows that

$$\gamma_1=\frac{M}{L_1}, \text{ approximately}; \qquad (28a)$$

and, in the case of small magnetic leakage,

$$\gamma_1=\frac{S_2}{S_1}, \text{ approximately}, \qquad (29)$$

which is the ratio of transformation of the transformer.

It is thus seen from (27) that the current which flows in the secondary circuit of a transformer is the same as if the secondary circuit were isolated from the primary and contained an electromotive force equal to the primary electromotive force multiplied by the ratio of transformation; the resistance of the secondary being increased, meanwhile, by an amount $\gamma_1{}^2R_1$ and the self-induction of the secondary being diminished by an amount $\gamma_1{}^2L_1$.

This leads to a further simplified relation. The coefficient of self-induction is proportional to the square of the number of turns. Hence it is evident that

$$L_2\omega-\left(\frac{S_2}{S_1}\right)^2L_1\omega=0;$$

* This may be shown by the student by means of a transformation similar to the one employed in obtaining equation (17), given in a previous foot note.

that is, the secondary self-induction is apparently *nil*, and we have

$$I_2 = \frac{\frac{S_2}{S_1} E_1}{R_2 + \left(\frac{S_2}{S_1}\right)^2 R_1}. \qquad (30)$$

Where equal quantities of copper are used in the primary and secondary circuits, $\left(\frac{S_2}{S_1}\right)^2 R_1$ is equal to the resistance of the secondary coil itself (that part of R_2 within the transformer). Hence we have the following approximate simple relation :

The secondary current is the same as would flow if the secondary circuit were isolated, and contained an electromotive force equal to that of the primary multiplied by the ratio of transformation, the secondary inductance being nil and the resistance of the secondary within the transformer being doubled.

The Secondary Angle of Lag. — The phase relations of the secondary current may be obtained by referring to equation (24). The two sine functions in this equation being the same in all respects, the phase of the two must be the same, and we may write

$$\phi_2 + \text{arc tan}\,\frac{\beta}{\alpha} = -90^\circ ;$$

or,

$$\phi_2 = -\text{arc tan}\,\frac{\beta}{\alpha} - 90^\circ. \qquad (31)$$

This is the lag of the secondary current behind the primary electromotive force, and is usually found to be about 180°.

The angle of lag of the secondary current behind the primary current is evidently, from (21) and (31),

$$\phi_2 - \phi_1 = -90^\circ - \text{arc tan}\,\frac{L_2\omega}{R_2}. \qquad (32)$$

The secondary current, therefore, lags behind the primary current by ninety degrees plus the angle θ_2, where θ_2 is an angle whose tangent is the ratio of the secondary reactance to the secondary resistance. This angle is increased by hysteresis. The primary current itself lags behind the primary impressed electromotive force by an angle ϕ_1, whose tangent, we have already seen, is the ratio of the apparent reactance and resistance of the primary. A careful comparison of these results with the graphical construction given in the previous chapter will prove instructive.

CHAPTER VII

FURTHER DISCUSSION OF THE SIMPLE TRANSFORMER

In the two preceding chapters the simple transformer diagram and the simple theory of the transformer have been established. Some of the results there shown will now be more fully discussed, and further relations deduced.

Apparent Resistance and Reactance of the Primary of a Loaded Transformer. — In each of the two preceding chapters it has been shown that the action of a current in the secondary of a

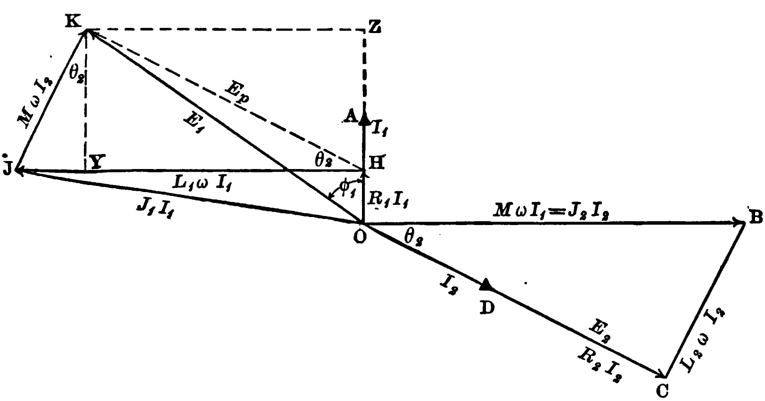


Fig. 68.

transformer is to apparently increase the resistance of the primary circuit and to decrease its self-induction, the primary current being thereby increased and more power thus obtained. This effect will be better understood from Fig. 68, which is a portion of the transformer diagram, Fig. 64. As explained in connection with Fig. 64, the primary current induces two electromotive forces, one in each circuit of the transformer, — $M\omega I_1$ in the secondary circuit and $L_1\omega I_1$ in the primary, — these electromotive forces being proportional to the number of secondary and primary turns, respectively, when there is no magnetic leakage. These are at right angles to the primary current OA, and are in the direction OB and JH; the line HJ represents the electromotive force to overcome JH. In a like manner the secondary current induces, at right angles to itself, the electromotive forces $L_2\omega I_2$ in the secondary and $M\omega I_2$ in the primary.

In Fig. 68, JK is accordingly at right angles to OD. Let KY be drawn perpendicular to HJ and OB. The angle JKY equals the angle COB, for the sides of one of these angles are perpendicular to the sides of the other. The angle JKY, therefore, equals θ_2.

When the secondary circuit is open, the primary electromotive forces are represented by the triangle OHJ. The primary electromotive force diagram, for open secondary circuits, is given separately in Fig. 69.

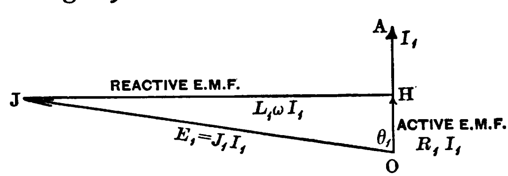


Fig. 69. Primary electromotive forces when the secondary circuit is open.

This electromotive force diagram, divided by the current, gives the impedance diagram for open circuit, in Fig. 70.

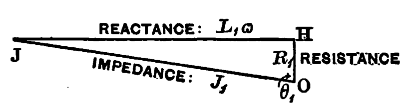


Fig. 70. Primary impedance diagram when the secondary is open.

When the secondary is closed, the primary electromotive force diagram is as shown in Fig. 68 or 71 (as explained in connection with Fig. 64). It is here seen that the active or power electromotive force is increased from OH to OZ, the amount of this increase being YK. The reactive electromotive force is decreased by an amount JY.

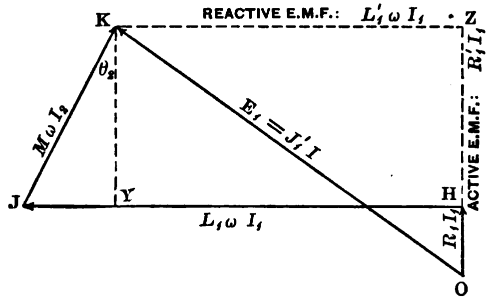


Fig. 71. Primary electromotive force diagram when the secondary circuit is closed.

The values for these quantities may be obtained thus:

$$\text{YK} = \text{JK} \cos \theta_2;$$

$$\text{JY} = \text{JK} \sin \theta_2.$$

From the triangle OBC, Fig. 68, we have

$$\cos \theta_2 = \frac{R_2 I_2}{M\omega I_1}; \qquad \sin \theta_2 = \frac{L_2 \omega I_2}{M\omega I_1}. \tag{1}$$

Hence, the increase in the power electromotive force is

$$\text{YK} = M\omega I_2 \frac{R_2 I_2}{M\omega I_1} = (\gamma_2{}^2 R_2) I_1. \tag{2}$$

The decrease in the reactive electromotive force is

$$\text{JY} = M\omega I_2 \frac{L_2 \omega I_2}{M\omega I_1} = (\gamma_2{}^2 L_2 \omega) I_1, \tag{3}$$

where

$$\gamma_2 = \frac{I_2}{I_1} = \frac{M\omega}{J_2}.$$

By dividing the primary electromotive force diagram, Fig. 71, by the primary current, the primary impedance diagram,

Fig. 72, is obtained, from which we find the values

$$\text{apparent reactance} = KZ = L'_1\omega = L_1\omega - \gamma_2^2 L_2\omega;$$
$$\text{apparent resistance} = OZ = R'_1 = R_1 + \gamma_2^2 R_2;$$
$$\text{apparent impedance} = J'_1 = \sqrt{R'^2_1 + L'^2_1\omega^2};$$
$$\tan\phi_1 = \frac{L'_1\omega}{R'_1}.$$

The line JK represents an impedance in the primary due to the action of the secondary. This is resolved into the reactance JY, which consumes no power, and the apparent resistance or power component YK. These are equal, respectively, to the reactance and resistance of the secondary multiplied by γ_2^2. The apparent resistance R'_1 is more strictly a *power coefficient*, consisting of R_1, the power coefficient for no load, and $\gamma_2^2 R_2$, a coefficient depending upon the power transferred from the primary circuit to the secondary when the transformer is loaded. As elsewhere explained (Chapter XIX.), R_1 or OH includes the effects of hysteresis when these are present.

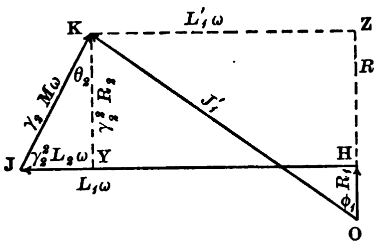


Fig. 72. Primary impedance diagram when the secondary circuit is closed.

On open secondary circuit, the impedance is OJ. Combining with this the impedance JK due to the secondary, the apparent impedance OK is found when the transformer is loaded. The direction of JK is such that the impedance of the transformer is apparently diminished by closing the secondary.

The Resultant Magnetization.—The currents and electromotive forces in the two circuits of a transformer, and the magnetic induction threading these circuits, are mutually dependent. The magnetic flux is caused by the combined magnetizing force of the primary and secondary currents; but the secondary current, to which the magnetization is in part due, depends upon the electromotive force induced in the secondary.

This secondary electromotive force is itself dependent upon the magnetic flux, and is proportional to the rate of change of the latter. No one quantity, accordingly, is independent of the others.

The magnetic flux, or magnetization, is the connecting link between the two circuits, and its relation to the currents and electromotive forces is as follows. The magnetization is due to the combined effects of the currents in the two circuits. The change in magnetization produces an electromotive force in each circuit proportional to the rate of change, being $-S_1\frac{dN}{dt}$ and $-S_2\frac{dN}{dt}$, in the primary and secondary circuits, respectively. A primary impressed electromotive force $S_1\frac{dN}{dt}$, equal and opposite to $-S_1\frac{dN}{dt}$, is required to overcome the counter-electromotive force induced in the primary by the changing magnetization.

These electromotive forces are at right angles to the magnetization, and we accordingly see that the primary and secondary electromotive forces are opposite to one another in phase, and proportional to the number of primary and secondary turns. In this statement the small effect due to the resistance of the primary is neglected. It is also assumed that the same flux threads the primary and secondary.

For simplicity we will consider that hysteresis is negligible and that the permeability μ is constant. In Chapter III. we have seen that a current of I amperes in a coil of S turns produces a magnetomotive force $\frac{4\pi SI}{10}$, and a magnetizing force (the magnetomotive force per unit length of the magnetic circuit) equal to

$$\boldsymbol{H}=\frac{4\pi SI}{10\,l}.$$

This magnetizing force $\boldsymbol{H}$ produces a magnetic induction in iron of $\boldsymbol{B}$ lines per square centimeter, where $\boldsymbol{B}=\mu\boldsymbol{H}$.

With a varying current i, the magnetizing force h varies always in proportion to it; and where the permeability is constant, the induction varies likewise in unison, being $b=\mu h$. [When the permeability is not constant and hysteresis is present, b would lag behind h; see Chapter XIX.] We therefore have

$$b=\mu h=\frac{4\pi\mu Si}{10l}. \tag{4}$$

The ampere-turns to which this induction is due, are the resultant ampere-turns due to the primary and secondary currents, as shown in Fig. 73. The resultant ampere-turns OZ may be shown to lag behind the primary current by an angle θ_2, as follows. Inasmuch as the secondary electromotive force E_2 is ninety degrees behind the magnetization, ZOY is a right angle. Since XOB is a right angle, it follows that

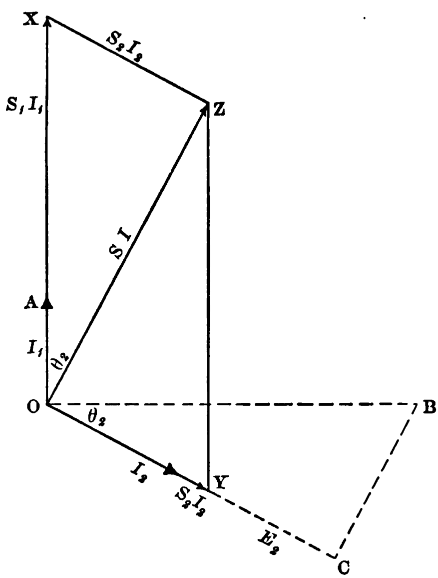


Fig. 73. Primary, secondary, and resultant ampere-turns.

$$XOZ = BOC = \theta_2.$$

Or further, in the triangle XOZ

$$\sin XOZ=\frac{S_2I_2}{S_1I_1}.$$

From the triangle OBC we have

$$\sin\theta_2=\frac{L_2\omega I_2}{M\omega I_1}=\frac{S_2I_2}{S_1I_1}, \tag{5}$$

the last term being true in case of no magnetic leakage.

The magnetization accordingly lags behind the primary current by an angle θ_2; the secondary current and electromotive force lag behind the primary current by an angle θ_2 plus ninety degrees. These results accord with those already obtained for the transformer diagram, which may be thus established.

The direction and magnitude of the resultant magnetization may likewise be determined analytically. The induction, in terms of primary and secondary currents, is

$$b = \frac{4\pi\mu}{10\,l}\left(S_1 i_1 + S_2 i_2\right). \qquad (6)$$

We may substitute the following values for the currents:

$$i_1 = I_1 \sin \omega t;$$
$$i_2 = I_2 \sin (\omega t - 90^\circ - \theta_2).$$

Hence

$$S_1 i_1 + S_2 i_2 = S_1 I_1 \sin \omega t + S_2 I_2 \sin (\omega t - 90^\circ - \theta_2).$$

From (5) we have $S_1 I_1 \sin \theta_2 = S_2 I_2$; hence

$$S_1 i_1 + S_2 i_2 = S_1 I_1 \{\sin \omega t - \sin \theta_2 \cos (\omega t - \theta_2)\}, \qquad (6\,a)$$

which may be reduced * to

$$S_1 i_1 + S_2 i_2 = S_1 I_1 \cos \theta_2 \sin (\omega t - \theta_2). \qquad (7)$$

It is thus shown that the resultant magnetizing force lags behind the primary current by an angle θ_2, and has a maximum value (in ampere-turns) of $S_1 I_1 \cos \theta_2$. The maximum value of the induction is

$$B = \frac{4\pi\mu}{10\,l} S_1 I_1 \cos \theta_2. \qquad (8)$$

Relation between Primary and Secondary Electromotive Forces. — It has already been stated that in an ideal transformer the primary and secondary electromotive forces would be opposite in phase and proportional to the number of turns in each coil.

* The steps are as follows: Omitting $S_1 I_1$, equation (6 a) expanded gives

$$\sin \omega t - \sin \theta_2 \cos \theta_2 \cos \omega t - \sin^2 \theta_2 \sin \omega t,$$

or

$$\cos^2 \theta_2 \sin \omega t - \sin \theta_2 \cos \theta_2 \cos \omega t,$$

which is (7) in an expanded form.

In Fig. 74, E_2 is ninety degrees behind the magnetization. The electromotive force E_1, at the terminals of the primary, is composed of the components OH and HK. The part HK, denoted by E_p, is the sum of the electromotive forces due to self- and mutual induction, HJ and JK respectively; it is at right angles to the magnetization, and therefore (in the absence of magnetic leakage) is opposite to E_2. If OH were zero, E_1 would equal E_p, and the electromotive force at the terminals of the primary would, accordingly, be opposite to secondary electromotive force.

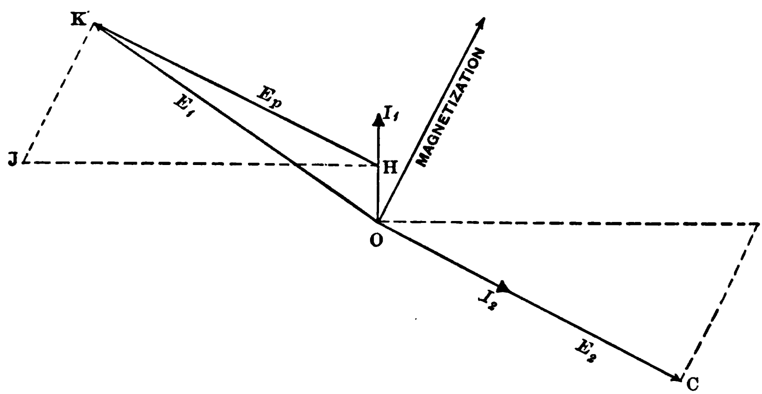


Fig. 74. Relation between primary and secondary electromotive forces.

The component OH is the power electromotive force on open circuit. In a transformer with no iron, it is the electromotive force to overcome the ohmic resistance of the primary; in general, however, it includes all expenditure of energy when the transformer is on open circuit. This component OH is always small, being only a few per cent of the total electromotive force, and is sometimes neglected in making approximate deductions; that is, E_p is considered as approximately equal to the electromotive force at the primary terminals. The component OH causes E_2 to be less than 180° behind E_1. Magnetic leakage has here been neglected. It will be seen in a later chapter that its effect is to increase this lag, so in practice E_2 is usually very nearly 180° behind E_1.

Fall in Potential for a Constant Potential Transformer. — At the close of Chapter VI. (see equations 27 and 30) an approximate expression for the secondary current was obtained, which may be written as follows: *

$$I_2 = \frac{\frac{S_2}{S_1} E_1}{\left(\frac{S_2}{S_1}\right)^2 R_1 + R_{2_{\text{int.}}} + R_{2_{\text{ext.}}}}; \qquad (9)$$

where $R_{2_{\text{int.}}}$ denotes the resistance of the secondary circuit within the transformer, and $R_{2_{\text{ext.}}}$ denotes the resistance of the secondary external circuit.

This relation may be shown for a transformer with constant primary potential by the diagram in Fig. 75, in which the

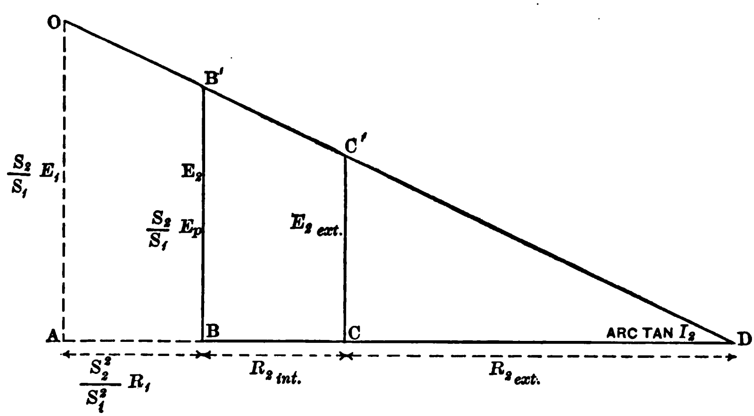


Fig. 75. Diagram showing fall of potential (referred to the secondary circuit) in a constant potential transformer.

ordinates AO, BB′, CC′ represent electromotive forces in the secondary circuit, the decrease in their values showing the fall in potential due to various causes. According to the above equation, the current in the secondary of a transformer is the same as would flow in a simple non-inductive circuit with an

* See, also, equation 10, which follows.

impressed electromotive force $\frac{S_2}{S_1}E_1$, and resistances $\left(\frac{S_2}{S_1}\right)^2 R_1$, $R_{2_{\text{int.}}}$, and $R_{2_{\text{ext.}}}$. The electromotive force $\frac{S_2}{S_1}E_1$, which has no real existence except on open circuit, is represented by AO; the several resistances are represented by AB, BC, and CD. The real electromotive force induced in the secondary is E_2, represented by BB′, and is less than the fictitious electromotive force $\frac{S_2}{S_1}E_1$, on account of the fall in potential due to the resistance $\left(\frac{S_2}{S_1}\right)^2 R_1$, which represents the effect of the primary resistance transferred to the secondary. The electromotive force at the secondary terminals is CC′, which shows a still further fall in potential due to the internal secondary resistance. As the secondary external resistance is changed, the point D moves from C, on short circuit, to an infinite distance on open circuit. The electromotive force AO remains unchanged, and the line OD swings about O as a center. The secondary electromotive force, represented by BB′ or CC′, changes with the load. In open circuit CC′ = BB′ = AO. The secondary current, from the equation above, is equal to $\frac{\text{AO}}{\text{AD}}$, the tangent of the angle at D; also,

$$I_2 = \frac{\text{BB}'}{\text{BD}} = \frac{E_2}{R_2},$$

$$= \frac{\text{CC}'}{\text{CD}} = \frac{E_{2_{\text{ext.}}}}{R_{2_{\text{ext.}}}}.$$

Where the copper losses in the primary and secondary are equal, as is often the case, AB is equal to BC. In this case

$$I_2 = \frac{\frac{S_2}{S_1}E_1}{2\,R_{2_{\text{int.}}} + R_{2_{\text{ext.}}}}. \qquad (10)$$

This diagram represents only the falls in potential due to the resistance of the primary and secondary circuits. There would likewise be a further fall in potential due to magnetic leakage and hysteresis which would further diminish the secondary electromotive force E_2.

Considering the primary of a fully loaded transformer as a single non-inductive circuit, with the effect of the secondary transferred to the primary as an added resistance, from equation (20) of the previous chapter, we may write

$$I_1 = \frac{E_1}{R_1 + \frac{S_1^2}{S_2^2}\left(R_{2\text{int.}} + R_{2\text{ext.}}\right)}. \tag{11}$$

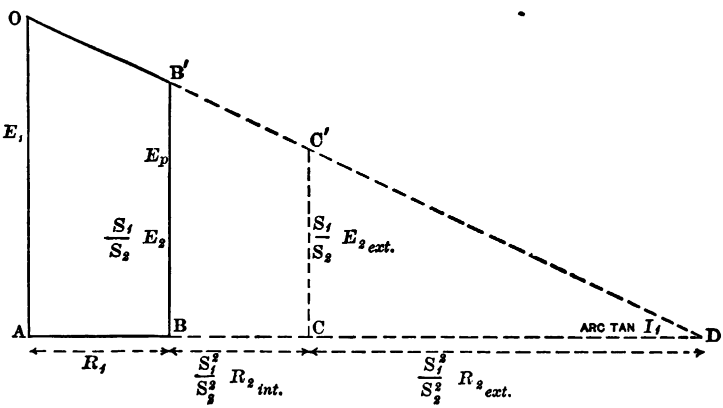


Fig. 76. Fall of potential (referred to the primary circuit) for a constant potential transformer.

The fall of potential is accordingly represented by Fig. 76. This figure is the same as Fig. 75 with the electromotive forces multiplied by $\frac{S_1}{S_2}$ and the resistances multiplied by $\frac{S_1^2}{S_2^2}$, to change from secondary to primary. The tangent of the angle at D, Fig. 75, is accordingly multiplied by $\frac{S_2}{S_1}$, and therefore represents primary current in Fig. 76. Equation (11) may similarly be obtained by multiplying (9) by $\frac{S_2}{S_1}$.

Hopkinson's Method. — The relation (equation 9) which forms the basis for the construction just discussed (Fig. 75) may be obtained from the following equations, first employed by Dr. John Hopkinson.*

$$4\pi(S_1 i_1 + S_2 i_2) = hl; \tag{12}$$

$$e_1 = R_1 i_1 + S_1 \frac{dN}{dt}; \tag{13}$$

$$o = R_2 i_2 + S_2 \frac{dN}{dt}. \tag{14}$$

Multiplying (13) and (14) by S_2 and S_1, respectively, and subtracting, we have

$$S_2 e_1 = S_2 R_1 i_1 - S_1 R_2 i_2. \tag{15}$$

From these equations may be found the values of the primary and secondary currents and the magnetization.

The Secondary Current. — By eliminating i_1 from (12) and (15), we have

$$i_2 = -\frac{\frac{S_2}{S_1}e_1 - \frac{S_2^2}{S_1^2}R_1\frac{hl}{4\pi S_2}}{\frac{S_2^2}{S_1^2}R_1 + R_2}. \tag{16}$$

When the transformer is fully loaded, the secondary and primary ampere-turns are very nearly equal and opposite; hence hl, which is the magnetomotive force to maintain the magnetization, is small as compared with either $S_1 I_1$ or $S_2 I_2$. See (12). The second term in (16) may be neglected in a first approximation, this term representing the electromotive force required to produce the magnetization; therefore,

$$i_2 = -\frac{\frac{S_2}{S_1}e_1}{\frac{S_2^2}{S_1^2}R_1 + R_2}. \tag{17}$$

The secondary current is here represented as just opposite in

* Proceedings Royal Society, March 10, 1887; see also S. P. Thompson's *Dynamo-Electric Machinery.*

phase to the primary electromotive force. Its maximum value [see also equation (9)] is

$$I_2 = -\frac{\frac{S_2}{S_1}E_1}{\frac{S_2^2}{S_1^2}R_1 + R_2}. \tag{18}$$

The Primary Current. — By eliminating i_2 from (12) and (15), we have

$$i_1 = \frac{e_1 + \frac{S_1}{S_2}R_2\frac{hl}{4\pi S_2}}{R_1 + \frac{S_1^2}{S_2^2}R_2}. \tag{19}$$

By the same approximation as above

$$I_1 = \frac{E_1}{R_1 + \frac{S_1^2}{S_2^2}R_2}. \tag{20}$$

The approximation here made introduces the least error when the transformer is fully loaded. The primary current is then nearly in phase with the electromotive force and opposite to the secondary current and electromotive force; also $\frac{I_1}{I_2} = \frac{S_2}{S_1}$. These results may be compared with equations (19 a) and (20) in the previous chapter. See also equation (25) of this chapter.

The Magnetization. — The rate of change of the magnetization, from (14), is

$$\frac{dN}{dt} = -\frac{R_2}{S_2}i_2. \tag{21}$$

Substituting in this equation the value for i_2 in (17), we obtain

$$A\frac{db}{dt} = \frac{R_2S_1e_1}{S_2^2R_1 + S_1^2R_2} = \frac{R_2S_1E_1 \sin \omega t}{S_2^2R_1 + S_1^2R_2};$$

$$Ab = \frac{R_2S_1E_1 \sin(\omega t - 90^\circ)}{\omega(S_2^2R_1 + S_1^2R_2)}. \tag{22}$$

As $S_1^2R_2$ is large* compared with $S_2^2R_1$, we may write

$$AB = \frac{E_1}{\omega S_1}. \tag{22a}$$

By substituting e_2 for R_2i_2 in (21), we have

$$A\frac{db}{dt} = -\frac{e_2}{S_2} = -\frac{E_2}{S_2}\sin \omega t,$$

$$Ab = \frac{E_2}{\omega S_2}\sin(\omega t + 90°), \tag{23}$$

$$AB = \frac{E_2}{\omega S_2}.$$

This gives us the relation

$$\omega AB = \omega N = \frac{E_1}{S_1} = \frac{E_2}{S_2}; \tag{24}$$

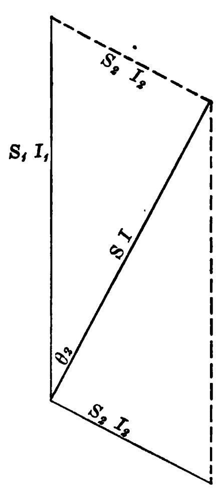


Fig. 77. Ampere-turn diagram for any transformer.

or, expressed in words, the maximum value of the electromotive force generated in a single turn of the primary or secondary is equal to ω times the maximum value of the magnetic flux.

This has already been shown for a single coil (see equation 34, Chapter IV.), it being then written in the form

$$E = ASB\omega,$$

which was derived from the reactive electromotive force $L\omega I$. It is to be borne in mind that in (24), E_1 and E_2 are the primary and secondary electromotive forces, neglecting the fall of potential in the internal resistances.

* It is to be borne in mind that R_2 is the resistance of the entire secondary circuit.

Relation between Primary and Secondary Currents in a Constant Potential Transformer. — From Fig. 73 we have the relation between the primary and secondary ampere-turns and the resultant ampere-turns, as shown in Fig. 77. This is true under any conditions, whatever be the values of the electromotive forces or currents; the primary current or the electromotive force may either one be maintained constant at all loads, or they may vary irregularly with the load. For a constant potential transformer, it leads to a simple and useful relation between the primary and secondary currents at all loads.

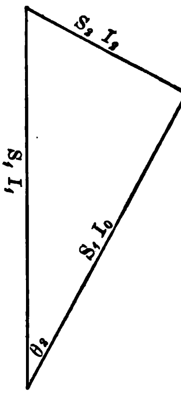


Fig. 78. Ampere-turn diagram for a constant potential transformer.

In a constant potential transformer, the magnetization is practically constant at all loads for $E_1 = S_1 \frac{dN}{dt}$. Hence the resultant ampere-turns SI, under any load, must be constant and equal to the ampere-turns at no load or open circuit. On open circuit, the ampere-turns are $S_1 I_0$, where I_0 is the primary current on open circuit. The resultant ampere-turns therefore have the constant value $S_1 I_0$ at all loads. For SI in Fig. 77, we may substitute $S_1 I_0$ and obtain Fig. 78, from which Fig. 79 is obtained by dividing by S_1. Hence:

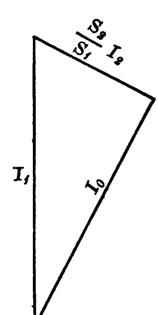


Fig. 79. Relation between currents in a constant potential transformer.

In a constant potential transformer, the primary current at any load is equal to the hypotenuse of a right triangle, in which one side is the primary current on open circuit, and the other side is equal to the secondary current multiplied by the ratio of secondary to primary turns. That is,

$$I_1 = \sqrt{I_0^2 + \left(\frac{S_2}{S_1}\right)^2 I_2^2}. \qquad (25)$$

The current I_1, which flows in the primary of a transformer, is the same as the sum of the currents flowing in two parallel circuits, Fig. 80, the values of the two currents being I_0 and $\frac{S_2}{S_1} I_2$, and their phase difference ninety degrees.

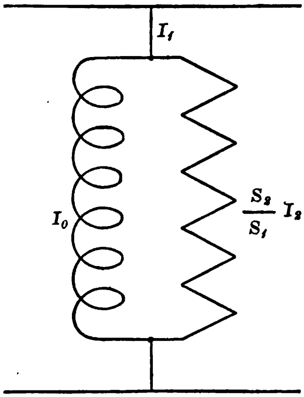


Fig. 80. Two components of the primary current in a constant potential transformer.

The primary current may be considered as made up of these two components. The magnetizing current I_0 is constant, maintaining the constant magnetization and furnishing the power for the no-load losses; $\frac{S_2}{S_1} I_2$ furnishes the power which is transferred to the secondary. Multiplying this second component by S_1, we see that it produces a magnetizing force just equal and opposite to that of the secondary current. It is an object to make I_0 as small as possible. It is usually so small that for a transformer under load, I_1 is approximately equal to $\frac{S_2}{S_1} I_2$.

This may be seen from the actual data which follow.

$\bar{I}_2$.	$\frac{S_2}{S_1} \bar{I}_2$.	$\sqrt{\bar{I}_0^2 + \left(\frac{S_2}{S_1}\right)^2 \bar{I}_2^2}$.	$\sqrt{\bar{I}_0^2 + \left(\frac{S_2}{S_1}\right)^2 \bar{I}_2^2 + 2\frac{S_2}{S_1}\bar{I}_2\bar{I}_0 \sin 45^\circ}$.	Observed $\bar{I}_1$.
0.00	0.00	0.090	0.09	0.09
6.57	0.657	0.663	0.72	0.76
7.63	0.763	0.769	0.83	0.84
8.66	0.866	0.871	0.93	0.93
9.64	0.964	0.968	1.03	0.99
11.22	1.122	1.125	1.19	1.17
12.53	1.253	1.255	1.32	1.29
14.4	1.44	1.442	1.50	1.51

The last column gives the observed values for the primary current. Columns 2, 3, and 4 give computed values. For an open magnetic-circuit transformer, the formulæ at the heads of columns 2 and 3 give values nearly correct. For a closed magnetic-circuit transformer (from which these data are taken), the formula at the head of column 4 is a closer approximation, for it assumes a lag between the magnetization and the magnetizing force. (See Chapter XIX.)

CHAPTER VIII

CONSTANT CURRENT TRANSFORMER

Constant Current Transformer. — In the preceding chapters, the transformer diagram has been established upon an analytical basis, and it has been shown how to find the electromotive force that must be impressed upon the primary in order to cause an assumed primary current to flow. We desire to know the behavior of a transformer under any given circumstances. The transformer is a sympathetic instrument, the variable quantities being so connected in mutual relationship that a slight change in any one quantity immediately affects the others. It is desirable to be able to predict what will happen in a given transformer when any one of its conditions is changed: to find, for example, how the primary current or electromotive force changes when the secondary resistance is varied, or how the power required changes with such a variation. When these questions are answered, it will be easy to see, by means of the diagrams, what conditions are necessary for good regulation; why a natural transformation is from constant primary potential to constant secondary potential, or from constant primary current to constant secondary current. The transformer diagram enables us to answer most of these questions.

In operation the transformer is connected with some constant source of supply, and the load is the independent variable. The source of supply may be one giving a constant primary current, or one maintaining a constant difference of potential at the terminals of the transformer primary, the former being the series constant current system, the latter, the parallel constant potential system in general use. The discussion of the constant current transformer, although such a transformer is less common than one designed for constant potential, will be given first, inasmuch as upon it depends the discussion of the constant potential transformer given in the following chapter. The variation diagram for a transformer supplied with constant primary current enables us also to determine the changes in the apparent impedance of the primary of a transformer as the load is varied.

Statement of the Problem. — The problem may be stated as follows:

The primary current is maintained constant, while the secondary resistance alone is varied. It is required to find how all the other quantities are affected.

The Secondary Electromotive Forces. — Referring to Fig. 81, we find reproduced the transformer diagram given in Fig. 64, the same letters designating corresponding points. OA is the

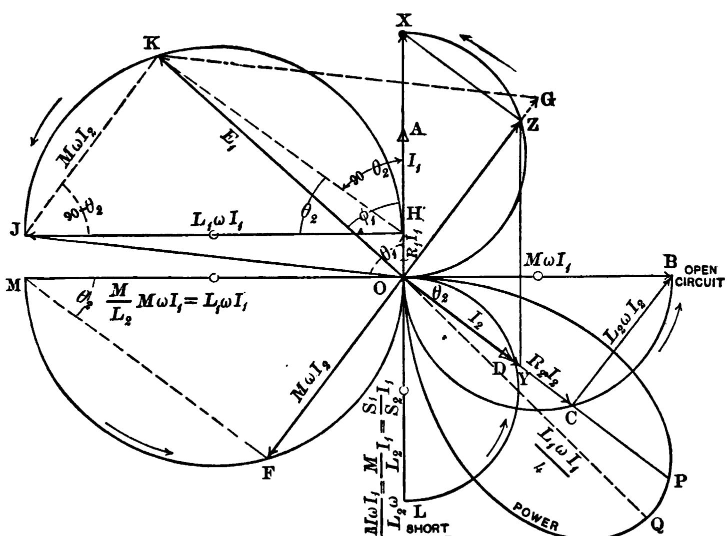


Fig. 81. Constant current polar diagram showing the variation of the quantities affected as the secondary resistance changes. Typical case.

constant primary current which is given; the electromotive force impressed upon the secondary by the primary current is $OB = M\omega I_1$, which is constant independent of any variation of secondary resistance.

The secondary electromotive force triangle OBC is a right triangle, always having OB for its hypotenuse; hence the locus of the point C is the semicircle OCB, for any variation in the secondary resistance. OC is always a chord of this circle equal to $R_2 I_2$, the secondary electromotive force. The point C moves in the direction indicated by the arrow as the secondary resistance increases. The arrows on the various loci in all variation diagrams, as Fig. 81, indicate the direction of a change due to an *increase* in the quantity varied.

The Secondary Current. — The secondary current OD is in the direction of the secondary electromotive force OC and equal to OC divided by R_2. The locus of D, as the resistance varies, is the semicircle ODL upon OL as a diameter. The magnitude of OL is $\frac{M\omega I_1}{L_2\omega} = \frac{S_1}{S_2} I_1$, and is independent of the secondary resistance. This may be shown as follows: If we draw from D a line perpendicular to OD and let it intersect OA produced at L, it will always intersect at L, a constant distance from O; for in the right triangle ODL which may be drawn, the angle

$$DLO = \theta_2, \quad OL \sin\theta_2 = OD.$$

That is,
$$OL = \frac{OD}{\sin\theta_2} = \frac{I_2}{\sin\theta_2}.$$

But
$$\sin\theta_2 = \frac{L_2\omega I_2}{M\omega I_1}.$$

Therefore
$$OL = \frac{M\omega I_1}{L_2\omega} = \frac{S_1}{S_2} I_1. \qquad (1)$$

Hence, as OL is independent of R_2, the locus of the secondary current is a semicircle ODL upon OL as a diameter. Even if the ohmic resistance were nil, the secondary current would have a finite value equal to OL, in a direction opposite to that of the primary current OA. This is the magnitude and direction of the secondary current when the secondary is short-circuited. As the secondary resistance is increased, the current OD moves in the semicircle as indicated by the arrow, becoming zero on open circuit. The secondary electromotive force then has the direction OB. It is not possible for R_2 to be reduced absolutely to zero, for it includes the internal resistance of the secondary. OL is the limit which the secondary current approaches, and is equal to the primary current multiplied by the ratio of transformation. In the limiting case, the primary and secondary ampere-turns would be equal and opposite, magnetic leakage being here neglected.

The Back Electromotive Force. — Since the back electromotive force OF is always equal to $M\omega I_2$ and is ninety degrees behind the secondary current, its locus must be similar to that of the secondary current, and ninety degrees behind it; that is, it must be a semicircle having its diameter ninety degrees behind OL, and consequently in the opposite direction to OB. The diameter of the back electromotive force circle must be equal to Mω times OL; that is, equal to $\frac{M}{L_2}M\omega I_1$. If we assume that there is no magnetic leakage, and that there is no external self-induction in the secondary, we have the relation $M^2 = L_1L_2$ (see equation 17, Chapter III.), and the diameter OM of the semicircle of the back electromotive force may be expressed,

$$\text{OM} = \frac{M}{L_2}M\omega I_1 = L_1\omega I_1. \qquad (2)$$

Hence it appears that the back electromotive force approaches in magnitude the primary electromotive force of self-induction $L_1\omega I_1$ (and would equal it in the absence of magnetic leakage), but is in the opposite direction so as to neutralize it when the secondary is short-circuited. This will be more evident after we have considered the primary electromotive forces.

The Primary Electromotive Forces.—The component of the primary electromotive force needed to overcome the primary resistance, viz., OH $= R_1 I_1$, remains constant independent of the secondary resistance; the component to overcome the primary self-induction $L_1 \omega I_1$ is also independent of R_2.

The only component of the primary electromotive force that changes with the secondary resistance is, therefore, that to overcome the back electromotive force. When this component, OG, equal and opposite to OF, is added to the constant line OJ, which is the resultant of OH and HJ, we find that the primary electromotive force OK lies upon a semicircle HKJ, similar but opposite to the back electromotive force circle, and with HJ as its diameter. When the secondary is short-circuited and R_2 is very small, the point F approaches M, and K approaches H, so that the only primary electromotive force needed is merely enough to overcome the ohmic resistance. The back electromotive force here counteracts that of the primary self-induction as before explained. In an actual transformer OH would be increased by an amount sufficient to supply all losses due to hysteresis and eddy currents. It will be shown later (see Chapter X.) that OK could never quite reach the position OH on account of the effects of magnetic leakage.

The Resultant Magnetization.—The variation in the resultant magnetization OZ is easily found, for the point Z must always be the right angle of the triangle, having OX as its hypotenuse. (See Fig. 73 and equation (5) in Chapter VII.) Since OX is a constant, depending only upon the primary ampere-turns, and being independent of any variation of the secondary resistance, it is evident that the magnetization OZ always lies upon a semicircle OZX for any variation in the secondary resistance. It is always ninety degrees in advance of the secondary current OD. The magnetization is here considered to be proportional to the magnetizing force and without lag. Hysteresis would cause the magnetization to lag; an effect discussed in Chapter XIX.

The Secondary Total Output. — The energy represented by the secondary current is expended partly within the transformer itself, being transformed into heat, and partly in the external circuit, in the form of useful work. We will include in the term "total output" P_2 both the internal power $P_{2_{int.}}$ and the external power $P_{2_{ext.}}$ of the secondary.

The total output is *

$$P_2 = \frac{1}{2}\, \mathrm{OD} \times \mathrm{OC} = \frac{1}{2} R_2 I_2^2. \tag{3}$$

Evidently this product will be zero when either OD or OC is zero. It will be a maximum when midway between these limits.

To find the curve OPQ, which expresses the total output for any value of the secondary resistance, we may more conveniently express the output (3) in terms of the angle of lag θ_2, which depends directly upon the resistance, since $\tan\theta_2 = \frac{L_2\omega}{R_2}$. Thus, by Fig. 81, we have,

$$\mathrm{OD} = \mathrm{OL} \sin\theta_2; \text{ or } I_2 = \frac{M}{L_2} I_1 \sin\theta_2;$$

and

$$\mathrm{OC} = \mathrm{OB} \cos\theta_2; \text{ or } R_2 I_2 = M\omega I_1 \cos\theta_2.$$

Combining these expressions according to (3), we find

$$P_2 = \frac{M^2\omega I_1^2}{4 L_2} \sin 2\theta_2 = \frac{L_1\omega I_1^2}{4} \sin 2\theta_2. \tag{4}$$

This would be modified in case of magnetic leakage.

By this equation it appears that the curve OPQ for the total output has the form of a loop symmetrical with respect to a line OQ bisecting the angle BOL. The maximum value of P_2 occurs when $\sin 2\theta_2 = 1$ or when $\theta_2 = 45°$; and OQ is therefore equal to $\frac{L_1\omega I_1^2}{4}$. At this point the $\tan\theta_2 = \frac{L_2\omega}{R_2} = 1$. Hence, in a constant current transformer, the value of the secondary re-

* See equation (30) Chapter IV. If the diagram is drawn to represent virtual values, instead of maximum, this would be $R_2\bar{I}_2^2$, and the following expressions would be changed to correspond.

sistance to make the total output a maximum is $R_2 = L_2\omega$. Note the experimental diagrams in Chapter XVI.

The Secondary External Output. — If we desire to ascertain the external output of the transformer alone, we may subtract from the total output that part which is lost in the transformer itself. Fig. 82 illustrates this point. It is merely a repetition of a part of Fig. 81 showing the total watt curve as before; but now there are two additional curves, the internal and external watt curves respectively, the sum of which equals the total watt

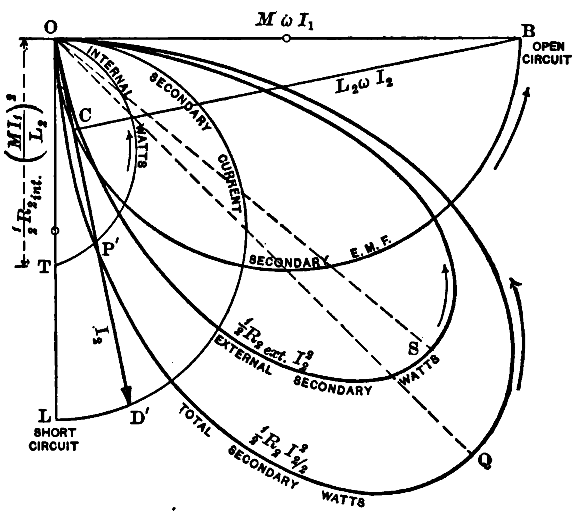


Fig. 82. Constant current polar diagram showing the secondary total, external and internal watt curves as the secondary resistance varies.

curve. Calling that part of the secondary resistance which is within the transformer $R_{2_{int}}$ and that which is external to it $R_{2_{ext}}$, we have the total output made up of two parts, viz.:

$$\tfrac{1}{2} R_2 I_2{}^2 = \tfrac{1}{2} R_{2_{int}} I_2{}^2 + \tfrac{1}{2} R_{2_{ext}} I_2{}^2; \qquad (5)$$

or,

$$P_2 = P_{2_{int}} + P_{2_{ext}}.$$

When we vary the resistance R_2, we will suppose that the external resistance alone changes, while the transformer resistance remains constant. As it was found more convenient in the case of the total output to express it in terms of θ_2, so now we write $P_{2_{int}}$ in terms of θ_2.

$$P_{2_{int}} = \tfrac{1}{2} R_{2_{int}} I_2^2; \qquad (6)$$

and since $I_2 = \frac{M}{L_2} I_1 \sin \theta_2$ (see Fig. 81), we have

$$P_{2_{int}} = \tfrac{1}{2} R_{2_{int}} \left(\frac{MI_1}{L_2}\right)^2 \sin^2\theta_2. \qquad (7)$$

By this equation it appears that the power expended within the transformer is directly proportional to the square of the sine of the angle of lag θ_2, since the coefficient

$$\tfrac{1}{2} R_{2_{int}} \left(\frac{MI_1}{L_2}\right)^2$$

is not directly dependent upon the external resistance. When $\theta_2 = 90°$, the power expended internally is a maximum equal to the coefficient above, and is represented by OT in Fig. 82. At 60 degrees it is $\frac{3}{4}$, at 45 degrees it is $\frac{1}{2}$, and at 30 degrees it is $\frac{1}{4}$ of the maximum value.

The external watt curve OS may be found by subtracting at every point from the radius of the total watt curve OP′Q the corresponding radius of the internal watt curve OP′T. This curve naturally is the one which it is desirable to ascertain, since it expresses the useful energy which can be taken from the transformer. Its equation may be found by the same principle by which we are enabled to draw the curve.

The maximum value θ_2 can have is BOD′ (a little less than ninety degrees), when the external resistance is reduced to zero and the secondary current is OD′. The maximum external output is OS, when θ_2 is less than 45° by an angle equal to $\frac{1}{2}$ LOD′.

Derivation of the Equation for $P_{2_{ext}}$. Subtracting (7) from (4),

$$P_{2_{ext}} = P_2 - P_{2_{int}} = \frac{L_1 \omega I_1^2}{2}\left(\frac{\sin 2\theta_2}{2} - \frac{R_{2_{int}}}{L_2 \omega} \sin^2 \theta_2\right).$$

Replacing the square of the sine by its equivalent, $\frac{1 - \cos 2\theta_2}{2}$, we have

$$P_{2_{ext}} = \tfrac{1}{4} L_1 \omega I_1^2 \left(\sin 2\theta_2 + \frac{R_{2_{int}}}{L_2 \omega} \cos 2\theta_2\right) - \frac{L_1 R_{2_{int}} I_1^2}{4 L_2}.$$

This may be simplified by the general formula

$$A \sin\theta + B\cos\theta = \sqrt{A^2 + B^2}\sin\left(\theta + \text{arc tan}\frac{B}{A}\right),$$

and written

$$P_{2_{\text{ext}}} = \frac{L_1 I_1^2}{4 L_2}\left[\sqrt{R_{2_{\text{int}}}^2 + L_2^2\omega^2}\sin\left(2\,\theta_2 + \text{arc tan}\frac{R_{2_{\text{int}}}}{L_2\omega}\right) - R_{2_{\text{int}}}\right]. \qquad (8)$$

This equation enables us to see the nature of the external watt curve. When $\theta_2 = 0$, that is, when $R_{2_{\text{ext}}} = \infty$ and the transformer is open-circuited, there is evidently no secondary current, and it is seen that $P_{2_{\text{ext}}}$ reduces to zero; for

$$\sin\left(2\,\theta_2 + \text{arc tan}\frac{R_{2_{\text{int}}}}{L_2\omega}\right) \text{ becomes } \sin \text{arc tan}\frac{R_{2_{\text{int}}}}{L_2\omega} = \frac{R_{2_{\text{int}}}}{\sqrt{R_{2_{\text{int}}}^2 + L_2^2\omega^2}},$$

and this multiplied by the coefficient $\sqrt{R_{2_{\text{int}}}^2 + L_2^2\omega^2}$ reduces the first term between the brackets of (8) to $R_{2_{\text{int}}}$, and the bracket itself consequently to zero.

Maximum Value of θ_2.—Again, when the transformer is short-circuited, the external resistance is practically reduced to zero, and θ_2 must have a maximum value attainable. This value of θ_2 should reduce the external work again to zero, since the external resistance is zero. Since $\tan\theta_2$ is equal to $\frac{L_2\omega}{R_{2_{\text{ext}}} + R_{2_{\text{int}}}}$, the largest value which it can have is evidently $\frac{L_2\omega}{R_{2_{\text{int}}}}$, found by making $R_{2_{\text{ext}}}$ zero. This value, substituted in (8), reduces the sine to

$$\sin\left(2\,\text{arc tan}\frac{L_2\omega}{R_{2_{\text{int}}}} + \text{arc tan}\frac{R_{2_{\text{int}}}}{L_2\omega}\right),$$

which reduces to *

$$\frac{R_{2_{\text{int}}}}{\sqrt{R_{2_{\text{int}}}^2 + L_2^2\omega^2}}.$$

Substituting in (8), we see that $P_{2_{\text{ext}}}$ becomes zero for this greatest value of θ_2. This angle is determined in Fig. 82 by the point P′, where the internal and the total watt curves intersect, since here the external watt curve is zero, being found by subtracting the internal from the total watts. The angle BOD′ is, therefore, equal to arc tan $\frac{L_2\omega}{R_{2_{\text{int}}}}$, and LOD′ is arc tan $\frac{R_{2_{\text{int}}}}{L_2\omega}$. OD′ is the largest secondary current attainable, and OC′ is the electromotive force at this point.

Maximum External Output.—To find when $P_{2_{\text{ext}}}$ is a maximum, make the sine in (8) unity; that is, put

$$2\,\theta_2 + \text{arc tan}\frac{R_{2_{\text{int}}}}{L_2\omega} = 90^\circ,$$

---

* This may be shown by a geometrical construction consisting of a right triangle with one side equal to $R_{2_{\text{int}}}$ and the other to $L_2\omega$.

Let $\qquad \alpha = \text{arc tan}\frac{L_2\omega}{R_{2_{\text{int}}}};\ 90^\circ - \alpha = \text{arc tan}\frac{R_{2_{\text{int}}}}{L_2\omega}.$

Then $\qquad \sin(2\,\alpha + 90^\circ - \alpha) = \cos\alpha = \frac{R_{2_{\text{int}}}}{\sqrt{R_{2_{\text{int}}}^2 + L_2^2\omega^2}}.$

and consequently find

$$\theta_2 = 45^\circ - \tfrac{1}{2} \text{ arc tan } \frac{R_{2_{\text{int}}}}{L_2\omega} = 45^\circ - \tfrac{1}{2} \text{ LOD}'. \qquad (9)$$

This shows that the value of θ_2, viz., BOS, Fig. 82, which makes $P_{2_{\text{ext}}}$ a maximum, brings OS in advance, by an angle equal to $\frac{1}{2}$ LOD′, of the maximum value OQ of the total watt curve. Equation (8) shows that the external watt curve is of the same nature as the total watt curve, each being in the form of a loop. The total curve has its two tangents at the origin at right angles to each other, and is symmetrical with respect to a line OQ which bisects this angle. The external curve has its tangents at the origin, making an angle BOD′ with each other, and is symmetrical with respect to a line OS which bisects this angle. The points of difference are that the external curve has its line of symmetry thrown in advance of the total curve, and at the same time the tangents at the origin make an angle with each other less than ninety degrees.

The output not being an harmonic function, the instantaneous value cannot be found from the diagrams here discussed. The curves here given represent the output as averaged over a complete cycle under a given set of conditions, the values being laid off in the direction of the secondary current. To find the output corresponding to any value of the secondary resistance, the current vector is extended until it intersects the proper output curve.

Diagrams with Rectangular Co-ordinates. — Each of the quantities whose variation we have already followed by means of the polar diagram may be represented by means of the rectangular diagram if we choose to sacrifice the advantage gained in the polar diagram of seeing the angles between the various quantities, and to know more clearly the relation between their magnitudes.

It is an advantage to see the variation of the quantities concerned, both in the polar diagram and in the Cartesian diagram; the latter shows the variation in magnitude only of the quantities already considered in Fig. 81. In Fig. 83 the abscissæ represent the angles of lag, θ_2, in the secondary as the resistance is varied. The cotangent of the angle of lag varies directly as the resistance, so that if the resistance is known for one angle, it may be calculated for any other. In Fig. 84 the abscissæ are the secondary resistances. The ordinates in every case are indicated by the name written on the curve in question. The construction of the curves in Figs. 83 and 84 is made directly from the polar diagram Fig. 81.

When the quantities are plotted as a function of R_2, the complete change between short circuit and open circuit cannot be shown, for R_2 then varies between the limits zero and infinity. They may, however, be plotted for the entire range from short circuit to open circuit, with θ_2 as abscissæ, for the corresponding change in θ_2 is from 90°, when $R_2 = 0$, to zero, when $R_2 = \infty$.

Quantities shown as a Function of θ_2 as R_2 is varied. — To construct the curves in Fig. 83 proceed as follows: Lay off equal spaces on a horizontal axis to represent angles of second-

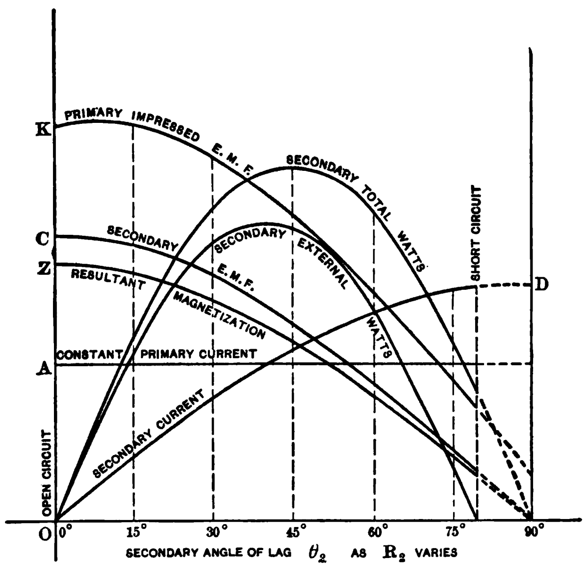


Fig. 83. Constant current rectangular diagram showing the relation between the magnitudes of the various quantities affected and the angle of lag θ_2 as the secondary resistance varies. Typical case.

ary lag θ_2. Let us suppose that it is desired, for instance, to represent the secondary electromotive force in this diagram. Referring to the polar diagram, Fig. 81, we see that the secondary electromotive force OC always lies upon the semicircle OCB. From the origin O draw radii vectores at equal intervals, say every fifteen degrees. The ordinates of the curve in Fig. 83

are the lengths of the various radii intercepted between the circumference of the semicircle OCB and the origin O. Similarly the rectangular curves may be constructed by points for all the curves in the polar diagram. A slight initial increase in the curve for the primary electromotive force may be noted.

It is noticeable that for a constant current transformer all these curves are sine curves in the rectangular diagram. This may be shown as follows: Consider the curve for secondary electromotive force. In the polar diagram we have

$$OC = OB \cos \theta_2 = OB \sin (90^\circ - \theta_2).$$

This is the polar equation of the circle OCB. If we now plot this equation with rectangular co-ordinates, it evidently is a sine curve, as the essential characteristic of the equation of a sine curve is that the dependent variable shall equal a constant times the sine or cosine of the independent variable. By the last equation, when θ_2 is zero, the secondary electromotive force has its maximum value OB, and it diminishes to zero at ninety degrees.

The resultant magnetization has a curve similar to the secondary electromotive force, but differing, of course, in magnitude. In the polar diagram the magnetization is ninety degrees in advance of the secondary electromotive force; this phase difference cannot be indicated in the curves with rectangular co-ordinates.

The primary current which is kept constant is represented by a horizontal line.

The secondary current OD has for its equation

$$OD = OL \sin \theta_2,$$

and is consequently a maximum when θ_2 is ninety degrees.

There is a limit indicated by the vertical line labeled "short circuit" in the figure, beyond which these curves could not be used in practice, inasmuch as the transformer is short-circuited before θ_2 reaches ninety degrees; the total secondary resistance is never reduced absolutely to zero.

Equation (4) gives the total output of the secondary. It starts at zero, reaching a maximum when $\theta_2 = 45^\circ$, and returning to zero again when $\theta_2 = 90^\circ$.

Quantities shown as Functions of R_2. — In order to draw the curves of Fig. 84, in which the abscissæ represent the values of the secondary resistance, instead of the angle of lag, as in Fig. 83, suppose that, in the particular position of the lines as represented in the polar diagram (OC, OD, OZ, and OK, for example, Fig. 81), the corresponding secondary resistance is R'_2. In the rectangular diagram (Fig. 84) erect ordinates equal to the lengths of the radii OC, OD, etc., corresponding to this value of the secondary resistance R'_2.

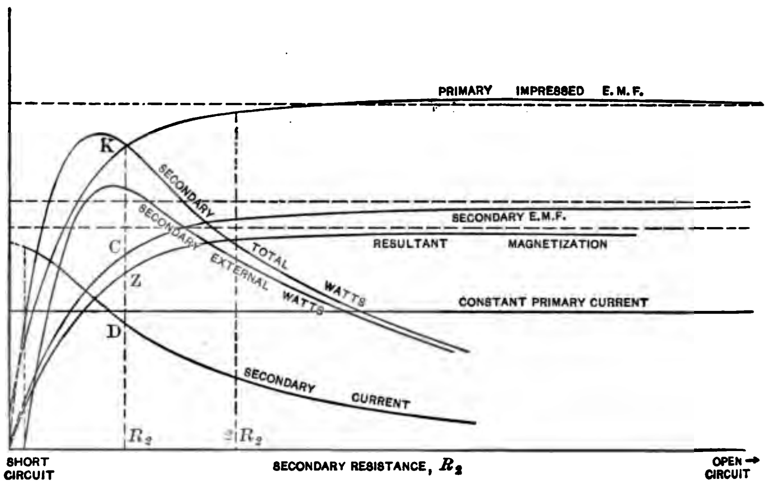


Fig. 84. Constant current rectangular diagram showing the variation in the magnitude of the quantities affected as the secondary resistance varies. Typical case.

One set of points being found, others are readily determined; for if we double the resistance, for example, the tangent of the angle of lag becomes half as great, and the values of the dependent variable for the angle which has its tangent half as great may be measured off from either the polar diagram, Fig. 81, or the rectangular one, Fig. 83. Thus all of the curves in Fig. 84 may be drawn by points. The vertical line at the left of Fig. 84, beyond which the curves are dotted, corresponds to that at the right in Fig. 83, and represents the point where the secondary is short-circuited.

Numerical Illustration. — Figs. 85 and 86 illustrate the way in which the typical diagrams Figs. 81 and 84 appear in an actual case. The data for the transformer, which are sufficient to enable one to draw the figures, are given below each figure. The very long and thin primary electromotive force triangle OHJ, in Fig. 85, is to be noticed. The points 1, 2, 3, 4, 5, 10, 15 denote the positions of the various vectors when these numbers are the values of the secondary resistance in ohms. It is because this primary electromotive force triangle is so long and thin that the secondary current OD is almost 180 degrees behind the primary electromotive force OK.

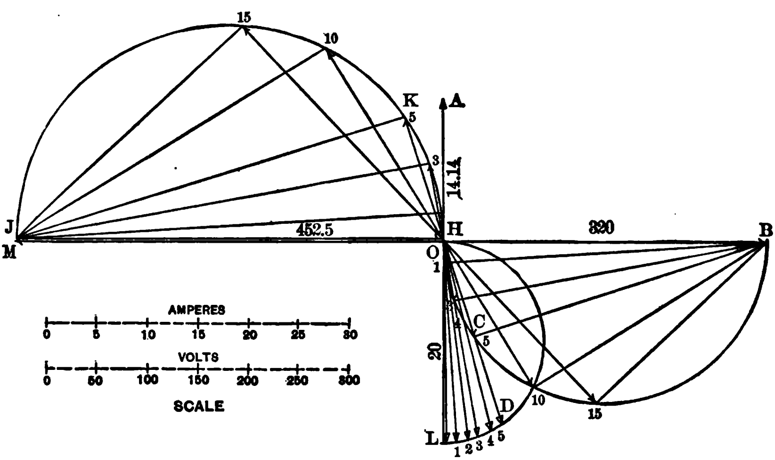


Fig. 85. Numerical example for constant current diagram. Data: $L_1 = 0{\cdot}032$ henry; $L_2 = 0{\cdot}016$ henry; $R_1 = 0{\cdot}227$ ohm; $R_{2_{int}} = 0{\cdot}16$ ohm; $\omega = 1000$. Points 1, 2, 3, etc., designate the positions of the vectors when the secondary resistance is 1, 2, or 3, etc., ohms.

This fact also makes the ratio of the primary electromotive force OK to the secondary electromotive force OC almost constant. The ratio would be constant if the points H and O coincided; that is, if $R_1 = 0$, and there were no losses due to hysteresis or eddy currents. In all cases considered in this chapter there is assumed to be no magnetic leakage.

Figure 86 is drawn by means of the diagram, Fig. 85, in the same manner as Fig. 84 was drawn from Fig. 81, and it shows how nearly constant the secondary current remains for all loads for which the transformer is designed. When the secondary resistance is increased from 0 to 5 ohms, the secondary current falls from 20 to 19 amperes, and the output becomes $\frac{R_2 I_2^2}{2} = \frac{5 \times 19^2}{2} = 902 \cdot 5$ watts. The transformer was designed to give an output of only 550 watts, so that within the range for which it is designed it transforms from a constant current to an approximately constant current.

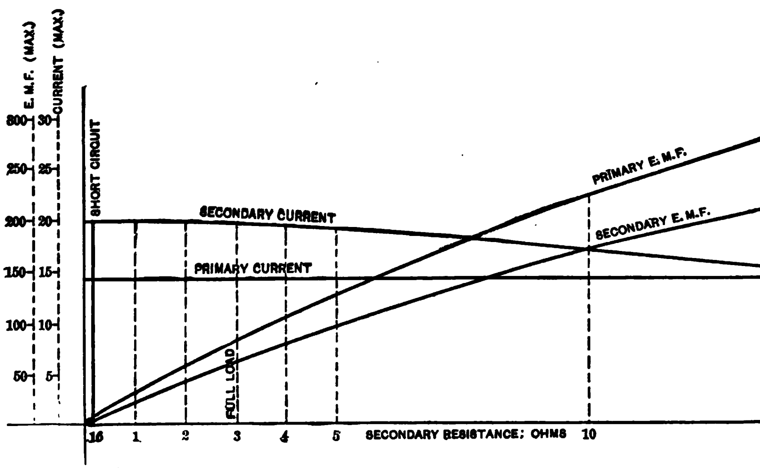


Fig. 86. Transformer rectangular diagram, numerical example. Data: $L_1 = 0{\cdot}032$ henry; $L_2 = 0{\cdot}016$ henry; $R_1 = 0{\cdot}227$ ohm; $R_{2_{int}} = 0{\cdot}16$ ohm; $\omega = 1000$.

Primary Impedance. — The construction given in Fig. 81 enables us to determine at once the changes in the value of the apparent impedance of the primary of a transformer as the secondary resistance is varied; independent of any condition regarding the value of the primary current or electromotive force.

In the preceding chapter it was shown how the impedance diagram, Fig. 72, could be obtained from the electromotive force diagram, Fig. 71. This was done by dividing every magnitude in the electromotive force diagram by the primary current. An electromotive force divided by a current gives an impedance;

hence every line which before represented an electromotive force will now represent an impedance.

By this method we obtain the impedance diagram Fig. 87 from the electromotive force diagram Fig. 81. The primary impedance J_1 on open circuit is represented by OJ, which is the geometrical sum of the primary resistance R_1 and the primary reactance $L_1\omega$. When the secondary is closed through a resistance R_2, an impedance represented by the line JK is added, as a vector to the line OJ. The locus of the point K is a semicircle.

$$\text{JK} = \frac{I_2}{I_1} M\omega = \gamma_2 M\omega.$$

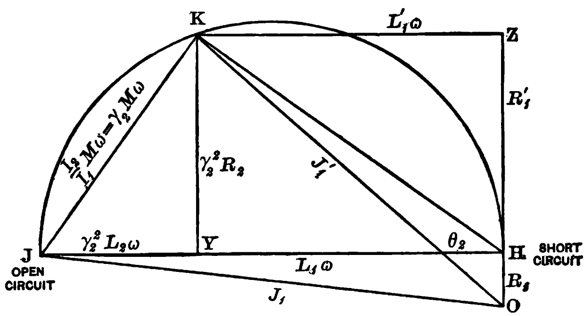


Fig. 87. Primary impedance diagram.

The reactance of the primary is apparently diminished by the closing of the secondary, and from Fig. 87 it will be seen that it is diminished continuously from open circuit, when its value is HJ, to short circuit, when it is reduced to zero in the absence of magnetic leakage.

The apparent primary resistance is equal to the numerical sum of the resistance OH, and KY due to the secondary. It will be noted that as the secondary resistance changes, KY, which is zero on open circuit, increases to a maximum value when θ_2 is 45° and diminishes again to zero at short circuit. The apparent resistance of the primary is therefore equal to the real resistance when the secondary circuit is open and when it is short-circuited, so that R_2 is zero. (This latter limit could never be quite reached, inasmuch as the internal resistance of the secondary would make it impossible to reduce R_2 to zero.)

The greatest value for the apparent resistance of the primary occurs when θ_2 is 45°; in this case the secondary resistance has such a value that the secondary power is a maximum, as shown in Fig. 82.

The impedance of the primary is diminished from open circuit to short circuit. If the point O coincided with H, this change would be a continuous one; that is, it would be a continual decrease from open circuit to short circuit. Since O does not coincide with H, it is possible for the impedance of the primary to be slightly *increased* at first when the secondary is closed. This increase will always be small and practically negligible. It is possible, therefore, on this account, for the primary current of a constant potential transformer to be slightly diminished as the secondary load is increased. (See paragraph at the close of following chapter; also experimental curve in Chapter XVI.)

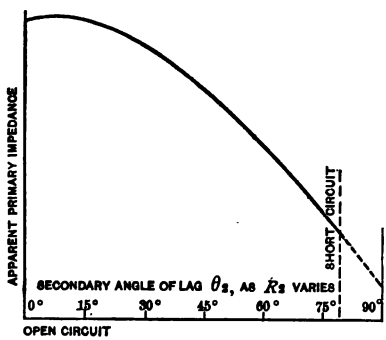


Fig. 88. Primary impedance curve, from Fig. 83.

Fig. 89. Primary impedance curve, from Fig. 84.

From the curves for the primary electromotive force given in Figs. 83 and 84 the impedance curves, Figs. 88 and 89, may be drawn. The slight increase of the value of the primary impedance as the secondary resistance is changed from open circuit to short circuit will be noted in these curves.

CHAPTER IX

CONSTANT POTENTIAL TRANSFORMER

Constant Potential Transformer. — The variation of the different quantities in a constant potential transformer when the load is changed may be determined directly from the diagram given in the previous chapter for the constant current transformer. From the variation diagram for the constant potential transformer, an admittance diagram may be constructed corresponding to the impedance diagrams given at the close of the last chapter.

Statement of the Problem. — The problem may be stated as follows:

The primary impressed electromotive force is maintained constant while the secondary resistance alone is varied. It is required to find how all the other quantities are affected.

Methods of Treatment. — This problem is in some ways of more interest than the former, because the common system of transformation is the constant potential system. In order to follow the plan here used, it is first necessary to understand the constant current diagrams, since we depend upon these for the solution in the constant potential system.

In Fig. 90 is reproduced the transformer diagram, Fig. 81, the same letters being used to designate corresponding points; thus, OA denotes the constant primary current, OCB the secondary electromotive force triangle, etc.

It has been shown that the primary impressed electromotive force varies upon the semicircle HKJ when the primary current is constant and secondary resistance varied. If we do not consider that the primary current is constant, let us suppose we change it to some other value, and draw the diagram again for this new value, this diagram being drawn for a particular value of the secondary resistance. We should find that every line of the new diagram had changed in magnitude in proportion to the change in the primary current, and all the angles had remained unchanged. This process is equivalent to either magnifying or reducing the whole diagram proportionally as the primary current changes. This will be evident when we consider that no angle depends upon the primary current, and that the primary current is a factor in the magnitude of every line, so that to double the primary current is to double every line in the diagram without changing the relative positions of the various lines.

We may magnify the diagram profitably in two ways. *First*, we may magnify the primary current as described and keep its direction always the same. The variation diagram will then be for a constant primary electromotive force varying in direction as the load changes. *Secondly*, we may make these same changes in scale and keep the primary electromotive force always in the same direction, as well as of constant magnitude. The primary current will then vary both in magnitude and direction as the load changes.

First Method: Primary Electromotive Force Constant in Magnitude; Primary Current Constant in Direction.— Let us suppose that the primary electromotive force OK is changed to OK′, which is the constant electromotive force we intend to furnish the transformer. Each radius vector in the diagram must be changed proportionally; thus, OC becomes OC′, OD becomes OD′, etc. If the diagram is so changed for every value of the secondary resistance, *i.e.* for every position of the various vectors, and OK is always changed to a constant value OK′ in the same direction as each particular value of OK, we have the diagram represented in Fig. 90. Here the primary electromotive force lies upon the arc of a circle J′K′H′, having

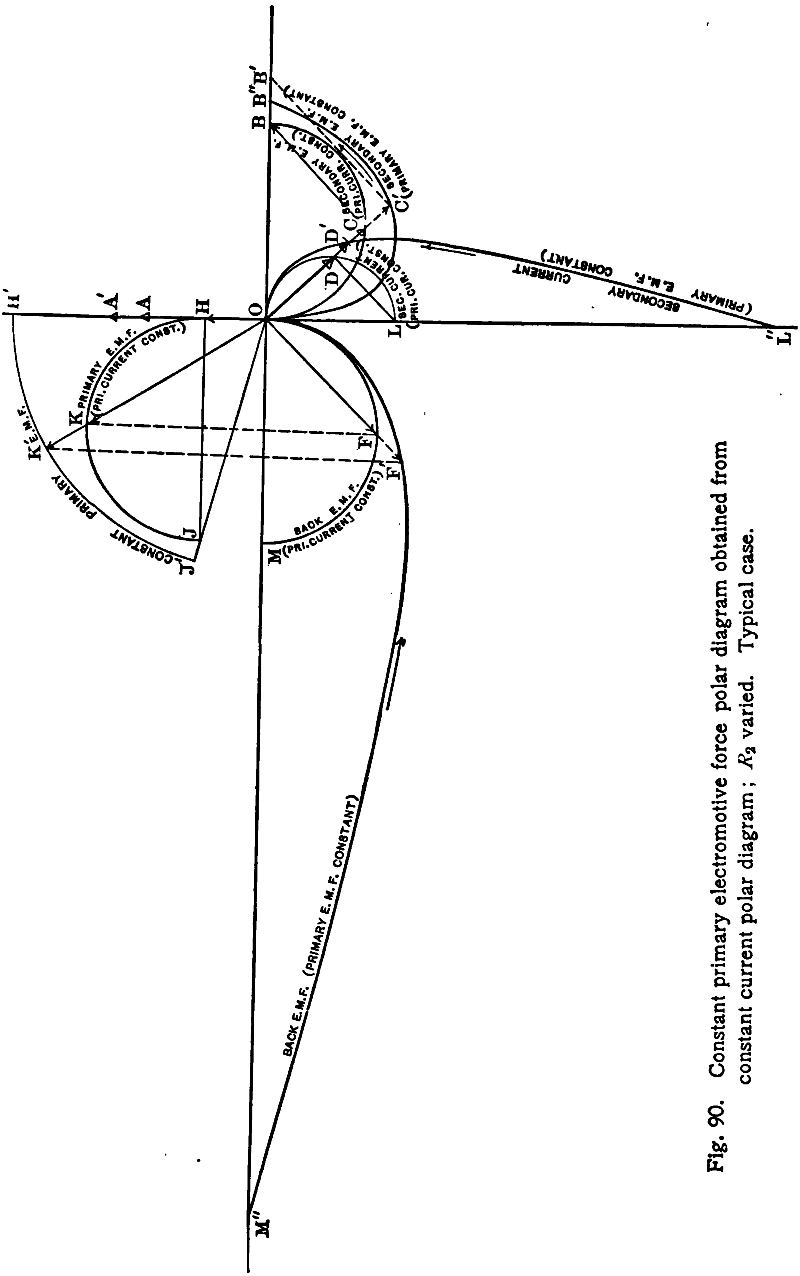


Fig. 90. Constant primary electromotive force polar diagram obtained from constant current polar diagram; R_2 varied. Typical case.

O as its center, and is therefore constant in magnitude. The primary current OA′ has a constant direction, but varies in magnitude along the line OH′.

The secondary electromotive force lies upon the curve OC′B″, which rises more abruptly than the semicircle near the

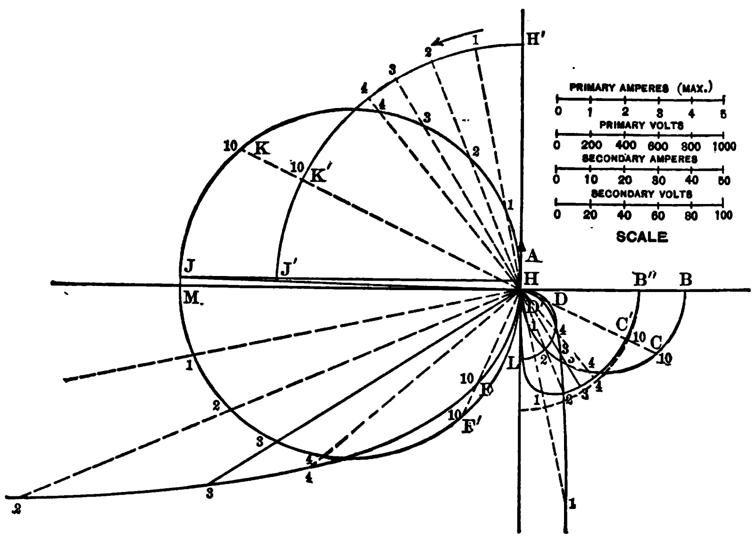


Fig. 91. Constant primary electromotive force polar diagram ($E_1 = 1000$ volts) obtained from constant current diagram ($I_1 = 1$ ampere). Actual case. Ten-light transformer, $R_1 = 50$ ohms; $R_2 = 0{\cdot}1$ ohm; $L_1 = 1{\cdot}96$ henrys; $L_2 = 0{\cdot}0047$ henry; $\omega = 1000$.

origin, and descends more gradually to B″. Upon that portion of the curve near B″ it is evident that the secondary electromotive force vector OC′ keeps nearer to a constant value for a considerable distance than does the electromotive force OC in the semicircle.

When we consider the diagram Fig. 91 for an actual transformer, this curve OC′B″ coincides approximately throughout a long range with a circle having O as a center, and the secondary electromotive force is therefore nearly constant. The smaller the line OH in comparison with HJ, the more nearly will the locus of secondary electromotive force be similar to the primary electromotive force. Usually the primary resistance is very small compared with the self-induction, so that OH is small compared with HJ, and consequently the curve OC′B″ is nearly a circle similar to J′K′H′, and the secondary electromotive force is approximately constant. That this is so nearly constant in this case is partially due to the fact that these diagrams are based upon the consideration of a constant coefficient of self-induction; that is, the magnetic medium is perfect. Hysteresis and eddy currents will increase the line OH, and consequently these, together with magnetic leakage, will increase the drop from a constant secondary electromotive force.

The secondary current, instead of lying on the semicircle ODL, in Fig. 90, now lies upon the curve OD′L″, which starts out from the origin in the direction of OB, but turns rapidly away toward L″.

In the actual transformer diagram, Fig. 91, the point L″ is very far removed in the direction of OL, and not shown in the figure, so that the locus of the secondary current is here nearly parallel with the line OL.

The back electromotive force represented in Figs. 90 and 91 by the curve OF′M″ is similar in shape but ninety degrees behind the secondary current curves. (The points M″ and O are not shown in Fig. 91.)

The transformer for which the curves in Figs. 91, 93, 95, and 97 are drawn is a ten-light transformer, in which $L_1 = 1{\cdot}96$ henrys; $L_2 = 0{\cdot}0047$ henry; $M = 0{\cdot}096$; $R_1 = 50$ ohms; $R_2 = 0{\cdot}1$ ohm. The curves for constant primary current are drawn for a current of one ampere, virtual or square root of mean square value, or 1·414 maximum. The curves for constant primary electromotive force are for 1000 volts virtual, or 1414 maximum. In Fig. 91 the numbers 1, 2, 3, 4, etc., indicate the value of the secondary resistance in ohms for that point on the curve.

Second Method: Primary Electromotive Force Constant in Magnitude and Direction. — The second method of magnifying the diagram, which possesses some advantages, inasmuch as it shows the variations in the primary current with the load, is shown in Fig. 92. Here the primary electromotive force OK′

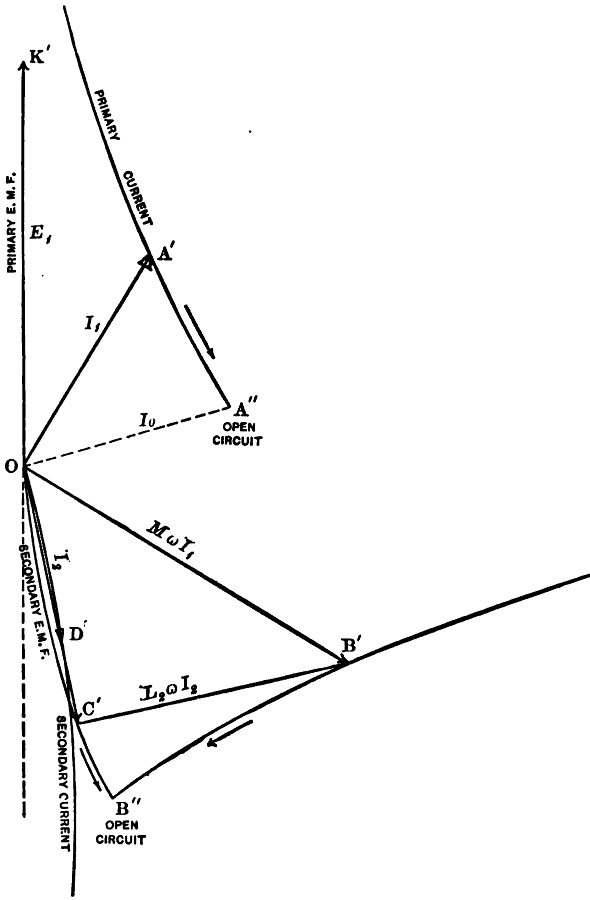


Fig. 92. Constant primary electromotive force polar diagram. R_2 varied. Typical case. Second method. Primary electromotive force constant in magnitude and direction.

is not only made constant in magnitude, as in Fig. 90, but also constant in direction, analogous to the constant primary current in the constant current diagrams. This figure is constructed from Fig. 90, by revolving each line in Fig. 90 through that particular angle which will bring the primary electromotive force OK always into a constant position OK′ arbitrarily selected.

The primary current, Fig. 92, evidently lies upon the curve A′A″, changing not only its angle with the electromotive force, but also its magnitude as the load increases. The point A′

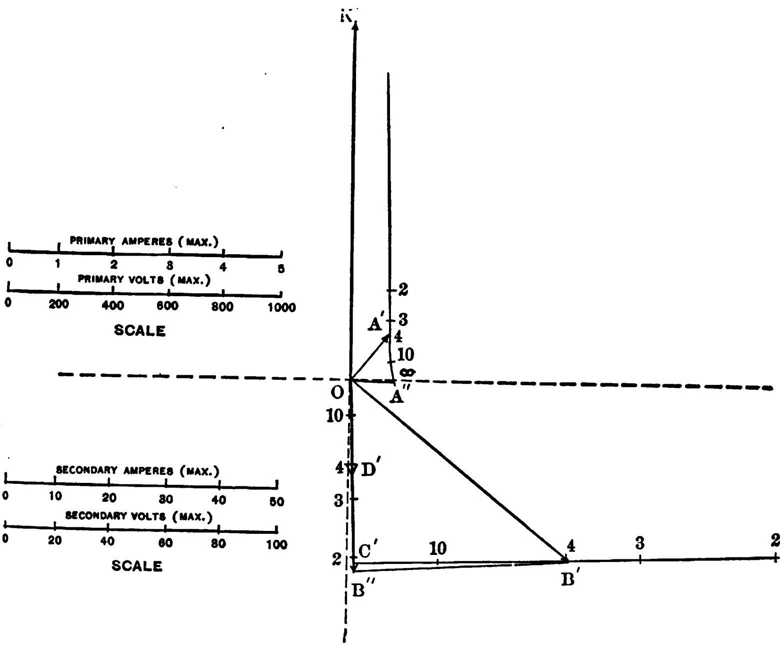


Fig. 93. Primary electromotive force constant in magnitude and direction. Actual case. Ten-light transformer. Data given under Fig. 91.

coincides with A″ when the load is nothing and the secondary is open-circuited, and the primary current is I_0. It is to be borne in mind that the arrows indicate the direction of change, due, not to an increase of load, but to an increase in the secondary resistance. The load varies inversely as the secondary resistance.

The electromotive force OB′ induced in the secondary by the primary current, being always ninety degrees behind the primary current and proportional to it, necessarily describes a curve B′B″ similar to the primary current curve.

The secondary electromotive force OC′ varies upon the curve OC′B″, in the direction of the arrow, as the resistance increases. When the resistance is infinite, and the secondary is open-circuited, the points B′ and C′ move to B″, and OB″ is the largest value of the secondary electromotive force.

The secondary current lies upon the curve OD′, which bends

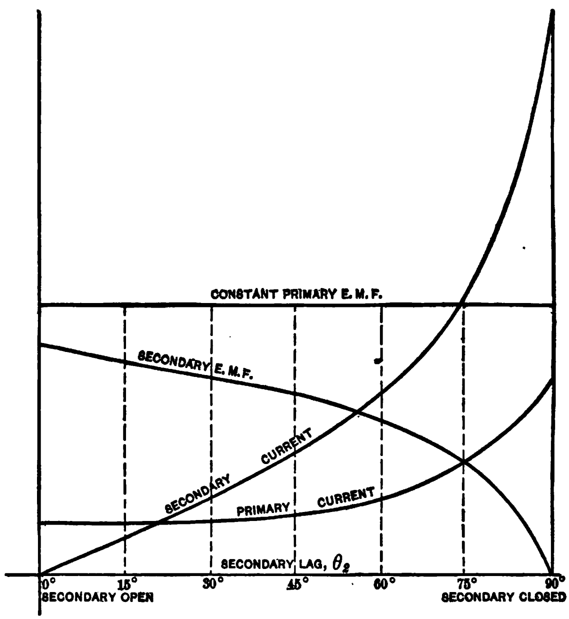


Fig. 94. Electromotive forces and currents for different values of secondary angle of lag θ_2 as R_2 is varied. Typical case.

slightly in the opposite direction to that of the electromotive force curve OC′B″; but the secondary current and electromotive force always remain nearly 180 degrees behind the primary electromotive force OK′.

Fig. 93 shows the transformer diagram for an actual case, the curves being drawn for the same ten-light transformer as in Fig. 91, the numbers 1, 2, 3, 4, etc., indicating, as before, the secondary resistance. The secondary current is almost 180 degrees behind the primary electromotive force. It is to be noticed that the locus AA′ of the primary current is nearly a straight line parallel with the line OK′.

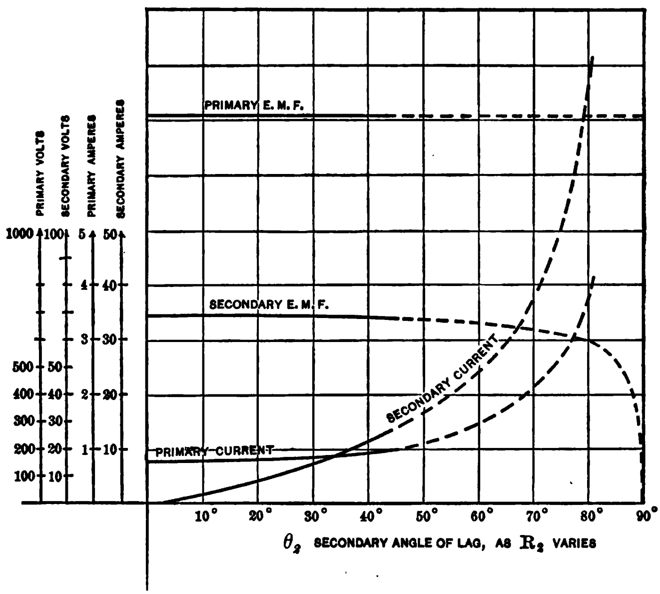


Fig. 95. Rectangular diagrams showing electromotive forces and currents for different values of secondary lag as R_2 is varied. Actual case. Ten-light transformer. Data given under Fig. 91.

Curves with Rectangular Co-ordinates. — From the polar curves already described we can construct curves plotted with rectangular co-ordinates, showing the relations between the magnitudes of the various quantities, neglecting their direction. Figure 94 is thus constructed directly from Fig. 90, and shows the primary and secondary electromotive forces and currents for various values of the secondary angle of lag θ_2 for this typical case. Figure 95 is similarly constructed, for an actual case, from Fig. 91. The heavy portions of the curves represent the variations within the working range of the transformer, the dotted lines showing these curves extended beyond the limits of practice.

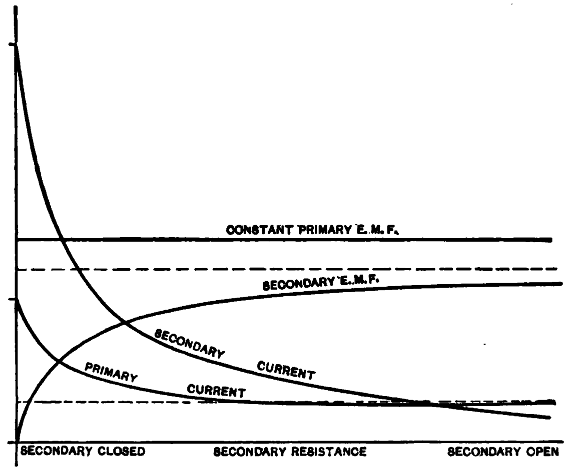


Fig. 96. Electromotive forces and currents for different values of R_2. Typical case.

Figure 96 shows, in rectangular co-ordinates, the values of the various quantities as a function of the secondary resistance, and is a typical case constructed from the polar diagram in Fig. 90.

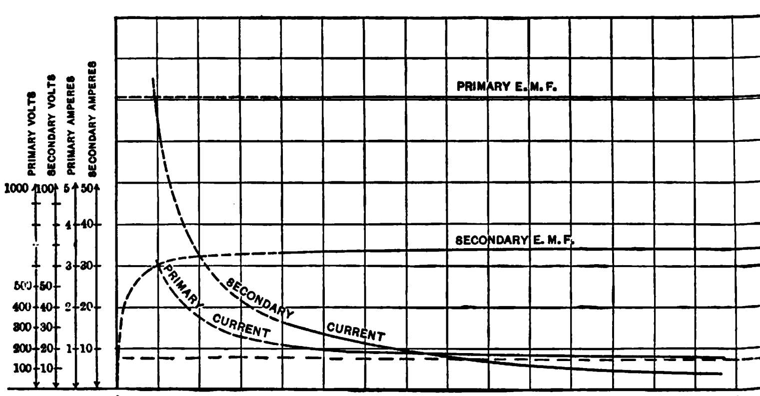


Fig. 97. Rectangular diagram showing the electromotive forces and currents for different values of R_2. Actual case. Ten-light transformer.

Figure 97 is the same for a ten-light transformer, and is constructed from Fig. 91. The heavy portions of the curves indicate, as before, the actual range of the transformer, and the dotted curves are extended beyond the working limits. It is seen how constant the secondary electromotive force is within these limits.

The curve for the primary current in Fig. 96 shows that it is possible for the primary current in a constant potential transformer to be *decreased* by closing the secondary, a point shown independently by Mr. E. C. Rimington,* and Bedell and Crehore.

Primary Admittance.—Inasmuch as a current is equal to the product of an electromotive force and an admittance, any diagram or locus for a current constructed for a constant impressed electromotive force will give an admittance diagram or locus if all values are divided by the electromotive force. From the locus for the primary current of a transformer, when the primary electromotive force is maintained constant, we may, therefore, obtain the locus for the primary admittance as the load is changed. By dividing the primary current by the electromotive force, we accordingly obtain from Fig. 92 the admittance locus, Fig. 98. The admittance being the reciprocal of the impedance, the locus of the admittance in Fig. 98 is the reciprocal of the impedance locus in Fig. 87. As the impedance locus is an arc of a circle, the admittance locus is likewise an arc of a circle. (For proof see Fig. 123.) Admittance and impedance diagrams will be discussed more generally in Chapter XII.

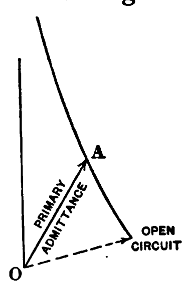


Fig. 98. Primary admittance diagram, from Fig. 92.

From the curves for primary current in Figs. 94 and 96,

* An elaborate discussion of the conditions necessary for this decrease in the primary current is given by Mr. Rimington in his paper "On the Behaviour of an Air-core Transformer when the Frequency is below a certain Critical Value"; *Philosophical Magazine*, Vol. 37, p. 394.

the admittance curves in Figs. 99 and 100 are obtained by dividing the value of the primary current by the constant primary impressed electromotive force. The slight decrease in the primary current in Figs. 94 and 96 and in the admittance in Figs. 99 and 100, in passing from open circuit to short

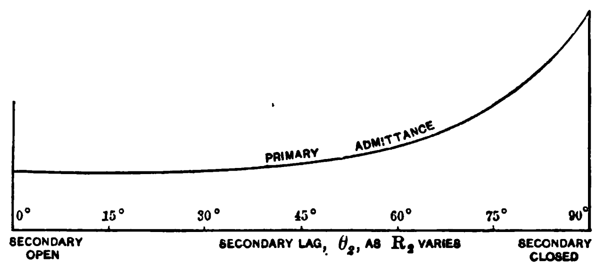


Fig. 99. Curve showing primary admittance for different values of secondary resistance, drawn from Fig. 94.

circuit, corresponds to the slight increase in the electromotive force in Figs. 83 and 84, for the constant current transformer, and in the impedance curves in Figs. 88 and 89 already discussed.

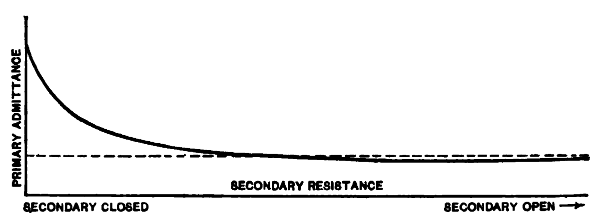


Fig. 100. Primary admittance, from Fig. 96.

CHAPTER X

EFFECTS OF EXTERNAL SELF-INDUCTION AND OF MAGNETIC LEAKAGE

External Self-induction and Magnetic Leakage. — In the two preceding chapters the effects of the variation of the secondary resistance in a transformer have been considered for the case in which the primary current is constant, and for the case in which the primary electromotive force is constant. The discussion was based upon the supposition that there was no appreciable self-induction in the secondary circuit external to the transformer itself, and that there was no magnetic leakage; in other words, it was assumed that $M^2 = L_1 L_2$. External self-induction and magnetic leakage will be found to have similar effects. Let us investigate the modifications necessary when these effects are taken into consideration.

Constant Current Transformer with External Secondary Self-induction. — The problem may be thus stated:

The primary current is maintained constant in a transformer in which there is self-induction in the secondary circuit external to the transformer itself; it is required to find how the other quantities are affected when the secondary resistance alone is varied.

If we let $L_{2_{int}}$ and $L_{2_{ext}}$ represent the self-inductions of the secondary circuit of the transformer, meaning by $L_{2_{int}}$ the self-induction within the transformer itself, and by $L_{2_{ext}}$ that external to the transformer, we have the relation

$$L_2 = L_{2_{int}} + L_{2_{ext}},$$

where L_2 denotes the total self-induction of the secondary circuit. Instead of the relation $L_1L_2 = M^2$, we now have

$$L_1 L_{2_{\text{int}}} = M^2.$$

By referring to the constant current diagram, Fig. 81, it will be found that, by introducing self-induction into the external secondary circuit, we do not change the primary current OA, which is constant by supposition, and consequently do not change the electromotive force OB, equal to $M\omega I_1$, which it

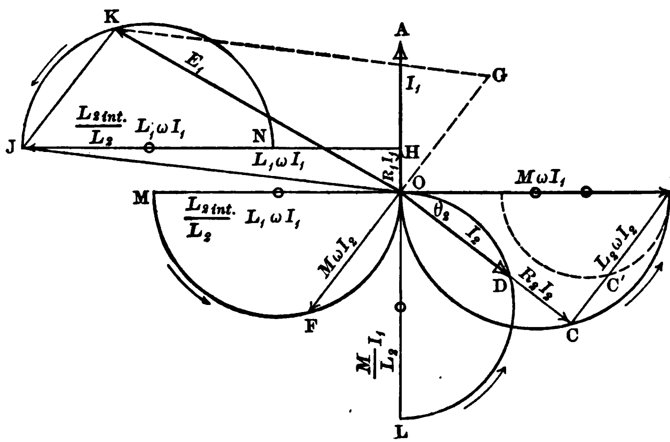


Fig. 101. Typical diagram showing effects due to variation of R_2 in a transformer with constant primary current and external secondary self-induction. Compare with Fig. 81.

induces in the secondary. The locus OCB for the secondary ohmic electromotive force is therefore not changed. In Fig. 101, the electromotive force at the terminals of the secondary of the transformer is OC′. BC′ is the inductive electromotive force due to the self-induction within the transformer. CC′ is the reactive electromotive force in the external secondary circuit. The locus of the point C′ would be a semicircle passing through B and C′, with its diameter in the direction OB. The diameter OL of the secondary current circle ODL, being equal to $\frac{M}{L_2}I_1$, in which L_2 has been increased by the addition of external self-induction, is diminished, and the current is accordingly decreased by the external self-induction. This is to be expected when we consider that the effect of additional self-induction is always to choke or lessen the current, except in those cases where there is capacity present in the circuit to neutralize the effects of self-induction.

The secondary current being diminished induces a smaller back electromotive force OF in the primary than before. The diameter OM of the back electromotive force circle has been shown to be $\frac{M}{L_2} M\omega I_1$. In the case of no external self-induction, this is equivalent to $L_1\omega I_1$, but it now becomes $\frac{L_{2\text{int}}}{L_2} L\omega_1 I_1$, which is less than the former, as the ratio $\frac{L_{2\text{int}}}{L_2}$ is always less than unity. The diagram in Fig. 101 illustrates the changes which occur when there is external self-induction. The primary electromotive force triangle OHJ (for open circuit) is not changed. The back electromotive force circle OFM has a diameter OM less than HJ, the latter being equal to $L_1\omega I_1$. Since the primary impressed electromotive force has a locus JKN, symmetrical with OFM and determined by it, we see that the primary electromotive force can never come into phase with the primary current as before, and a greater or less angle must always exist between the impressed electromotive force and primary current.

The more important features of this diagram, Fig. 101, which includes the effects of external self-induction in the secondary, will be readily seen by comparison with the constant current diagram, Fig. 81. The locus JKN of the primary impressed electromotive force is shifted to the right by the secondary self-induction. In all constant current diagrams the loci of the variable quantities are in general semicircles.

Numerical Example.—The typical diagrams do not indicate the extent of the modifications caused by the additional secondary self-induction, and it is well, therefore, to illustrate by a numerical example. In the constant current transformer already considered, Fig. 85 represents the polar diagram, and with it the data for the transformer are given. Figure 86 is the corresponding rectangular diagram showing the relations between the currents and electromotive forces. Figure 102 represents the change in Fig. 85 when self-induction is introduced into the external secondary circuit, this additional self-induction being

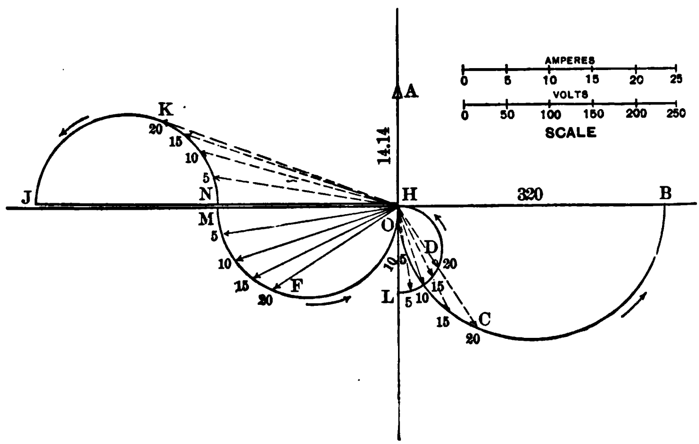


Fig. 102. Numerical example: Transformer polar diagram, constant primary current, external self-induction, 0·016 henry. Secondary resistance varied. Data: $R_1 = 0·227$ ohm; $R_{2_{int}} = 0·16$ ohm; $L_1 = 0·032$ henry; $L_{2_{int}} = 0·016$ henry; $\omega = 1000$. Points 5, 10, etc., indicate positions for $R_2 = 5, 10$, etc., ohms. Compare with Fig. 85.

equal in amount to the secondary self-induction of the transformer itself; that is, the total self-induction of the secondary is doubled, and, where the old value of L_2 was 0·016 henry, now $L_2 = 0·032$ henry, $L_{2_{int}} = 0·016$, and $L_{2_{ext}} = 0·016$ henry. The external self-induction does not change the ohmic secondary electromotive force locus OCB, but it reduces the diameter OL of the secondary current semicircle inversely as L_2 increases, so that the secondary current approaches 10 amperes instead of 20 as the secondary resistance approaches zero. At the same time the tangent of the secondary angle of lag θ_2, being equal to $\frac{L_2\omega}{R_2}$, is increased proportionately to L_2 for any given resistance. This fact allows a proportionately greater increase of the secondary resistance to reduce the secondary current to the same percentage as before. Figure 103 is the rectangular diagram for the same transformer with secondary external self-induction, and may be compared with Fig. 86. The secondary resistance may be increased to 20 ohms and cause the same percentage decrease in the current as was caused by an increase to 10 ohms when there was no external self-induction present.

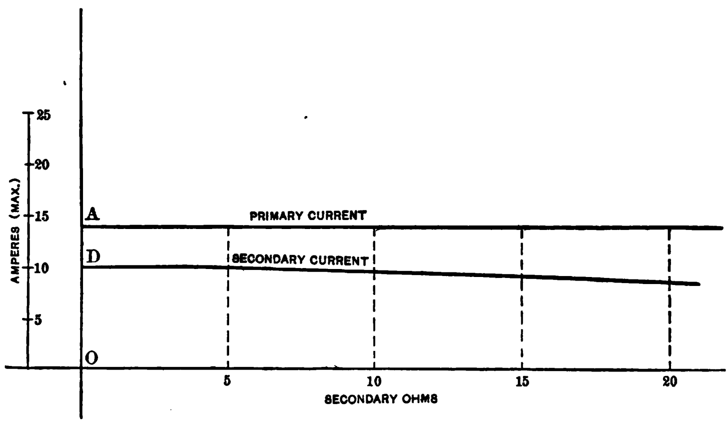


Fig. 103. Numerical example: Transformer rectangular diagram corresponding to polar diagram in Fig. 102. Compare with Fig. 86.

It may be noted that by introducing external self-induction into the secondary of a constant current transformer we may obtain a current constant throughout a greater range of ohmic resistance, although the magnitude of the current will be reduced.

A noticeable effect of the external self-induction is that it causes the secondary current OD to lag considerably more than 180 degrees behind the impressed primary electromotive force OK. According to all previous diagrams this has never been the case. The increased external self-induction makes $L_1 L_2 > M^2$. Similarly, if M alone should be diminished, L_1 and L_2 remaining the same (and this might be due to an increase of magnetic leakage in the transformer itself), we should have the same result that $L_1 L_2 > M^2$, and it follows that magnetic leakage in the transformer causes the effect just described. These results may be compared with the experimental curves in Chapter XVI.

The next problem to be considered is the investigation of the effect that the external self-induction produces when the primary electromotive force is kept constant.

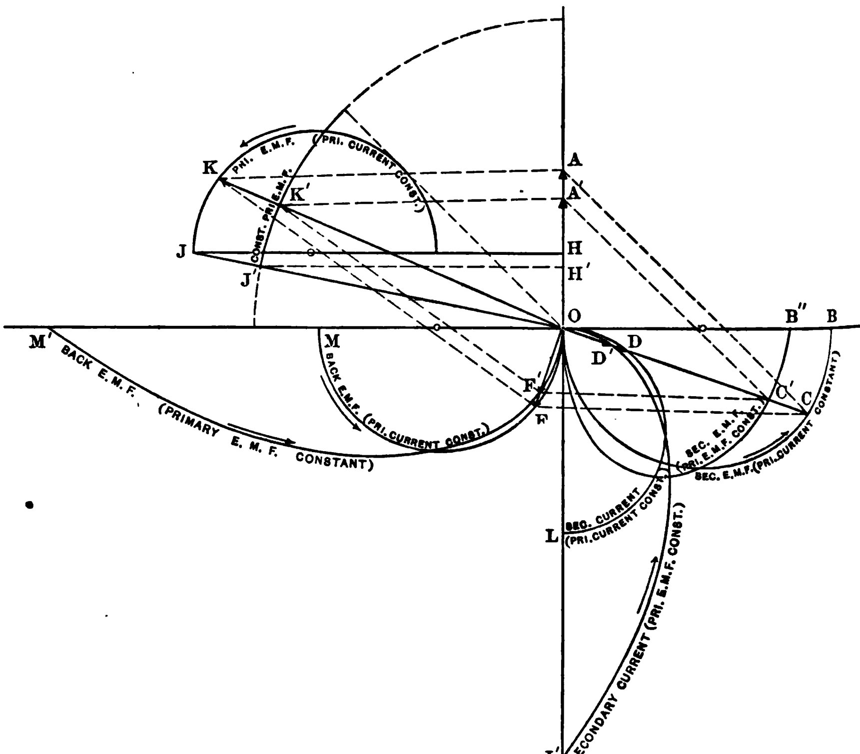


Fig. 104. Typical diagram showing effects due to a variation of R_2 in a transformer with constant primary potential and external secondary self-induction. Compare with Fig. 90.

The statement of the problem is as follows:

Constant Potential Transformer with External Secondary Self-induction. — The primary electromotive force is maintained constant in a transformer in which there is self-induction in the secondary external to the transformer; it is required to find how the various quantities are affected when the secondary resistance only is variable.

Figure 104 is a typical diagram, illustrating this case. The constant current diagram similar to Fig. 101 is given here and transformed as before in the case of a constant potential transformer, by magnifying or diminishing the whole diagram for each point until the primary electromotive force locus becomes a circle with O as a center, so as to represent the constant potential case. This diagram should be compared with Fig. 90, the constant potential diagram for the case where there is no external secondary self-induction.

Perhaps the most noticeable effect of the external self-induction, aside from diminishing the current considerably, is the change in the ohmic secondary electromotive force locus OC′B″. This does not rise so abruptly from the origin and turn off toward B″ as suddenly as it did in Fig. 90, where there was no external self-induction, but it turns more gradually and does not keep constant throughout so great a range as formerly.

Numerical Example. — These features are more easily seen by reference to the diagram in Fig. 105, which represents a numerical case, where the transformer of Fig. 91 has self-induction inserted in the external secondary circuit equal to the self-induction of the secondary within the transformer. Here the secondary current curve, instead of extending indefinitely downward, stops abruptly at L′. The secondary current is never larger than OL′ even on short circuit. When the resistance has only been decreased to 10 ohms, it is seen that the ohmic secondary electromotive force has fallen away considerably. The change is so great and the regulation for constant secondary potential so affected that the transformer would be rendered useless for the purpose for which it was designed by the presence of so great a quantity of external self-induction. The extra self-induction is accordingly detrimental for constant potential regulation, although it is an assistance towards regulation for constant current. The effect of secondary self-induction can be balanced by a condenser placed in the circuit in series.

Magnetic Leakage. — The action of a transformer, so constructed that the magnetic leakage may be neglected as being very small, has been discussed in the foregoing chapters; and

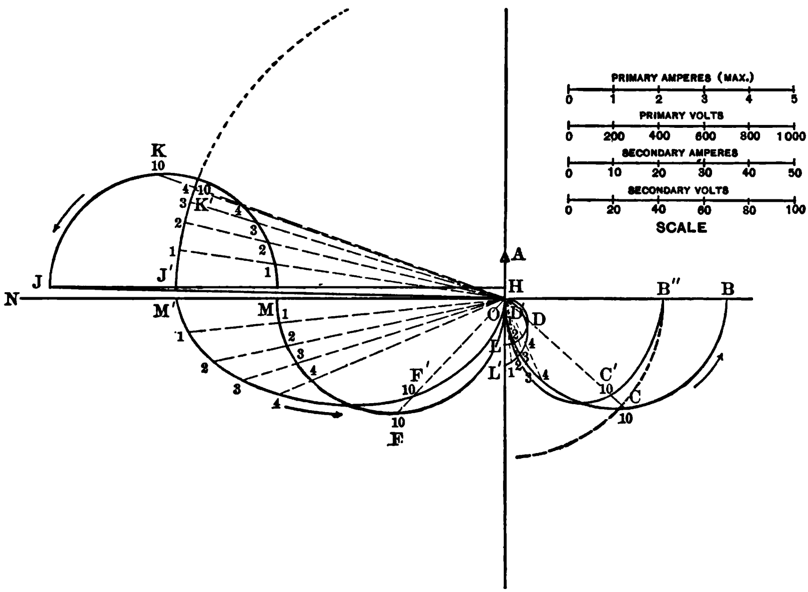


Fig. 105. Numerical example transformer diagram for constant primary potential. External secondary self-induction = 0·0047 henry. Resistance varied. Data: $R_1 = 50$ ohms; $R_{2_{int}} = 0·1$ ohm; $L_1 = 1·96$ henrys; $L_{2_{int}} = 0·0047$ henry; $\omega = 1000$. Points 5, 10, etc., indicate positions for $R_2 = 5$, 10, etc., ohms. Compare with Fig. 91.

diagrams have been given showing the manner in which the electromotive forces and currents in a transformer change when the secondary resistance is made the independent variable. For such a transformer the product of the coefficients of primary and secondary self-induction is equal to the square of the coefficient of mutual induction; that is, $L_1 L_2 = M^2$. An example (see Figs. 91, 93, and 95) was given of an actual transformer in which the measured coefficients were so nearly in accordance with this relation that the approximation in considering that there was no magnetic leakage was very close indeed.

More or less magnetic leakage may be introduced intentionally into the design of a transformer for a specific purpose; in any case a small amount will be found to exist even in the best designed transformer in which particular effort is made to avoid magnetic leakage. The effects of magnetic leakage are clearly exhibited by noting the modifications introduced by it into the transformer diagram, as the secondary resistance is changed.

Let us again consider the magnetic circuit already discussed in Chapter III.

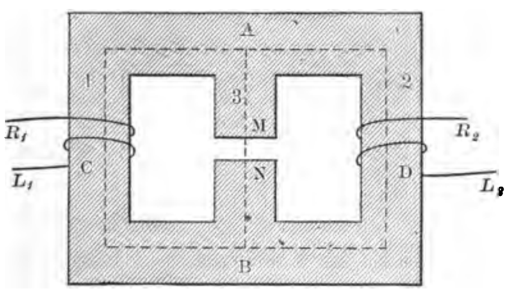


Fig. 106. A divided magnetic circuit.

Magnetic Leakage Circuit. — In saying that there exists magnetic leakage in a transformer, we mean that all the induction set up by either primary or secondary circuit does not thread the other, so that an appreciable quantity of the flux due to either coil avoids passing through the other, going around it by another path, namely, the leakage magnetic circuit. We are thus led to the idea of a divided magnetic circuit, and we might expect that the magnetic flux in such circuits would be analogous to the current flow in electric circuits, since the law of the magnetic circuit conforms so closely with Ohm's law for electric circuits.

Take a particular case, represented in Fig. 106, of a divided magnetic circuit composed of three separate branches, ACB, ADB, and AMNB. The section is taken parallel to the lines of induction. Let us suppose that the points A and B have different magnetic potentials, and that A has the higher potential. There will then be a continuous fall of magnetic potential from A to B, and there will be magnetic flux or induction from A to B, the greater part of which takes that path that has the least reluctance. If there were no magnetic material in the region between the points A and B, which for the moment we are supposing to be points of maximum and minimum potential respectively, the flux would spread out from A and converge to B in such a manner that the density of the flow at any point in the surrounding medium would change gradually from point to point, it being the most dense on the straight line AB. But by introducing a path of iron between the points we should find that nearly all the flux passed through the iron in preference to the air, since it has a much less reluctance; by introducing more than one iron path the flux would divide among them inversely as their reluctances.

Let $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ denote the reluctances of the three branches ACB, ADB, and AMNB, respectively. The values of $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ depend upon the shape of the iron circuit as well as the kind of iron used, and in some cases it might be difficult to approximate to their values; but when the cross-section of the circuit is approximately uniform throughout, as in the case before us, these values may be expressed in terms of the dimensions of the circuits. In case of the circuit ACB, we might select a line ACB (shown by the dotted line) as the mean path of induction, and denote by l the average length of the lines of induction through ACB. If A_1 denotes the average cross-section, we have

$$\mathcal{R}_1 = \frac{l_1}{\mu_1 A_1},$$

where μ_1 is the permeability of the iron used. Similarly for the branch ADB we should have

$$\mathcal{R}_2 = \frac{l_2}{\mu_2 A_2}.$$

The reluctance of the branch AMNB, which is supposed to contain an air gap MN with permeability equal to unity, would be found by adding together two portions, namely, the part through the iron and that through the air. Thus:

$$\mathcal{R}_2 = \frac{l_i}{\mu_3 A_3} + \frac{l_a}{A_3},$$

where l_i is the mean length of the iron and l_a of the air circuit.

Let us suppose that the primary of the transformer is wound on C and the secondary on D; the middle branch, MN, constitutes the leakage path. In any case, whether we can easily compute them or not, there will in general in any transformer exist these three magnetic circuits which we may say have the reluctances $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$. If the transformer has little or no leak, we would say that $\mathcal{R}_3$, the reluctance of the branch not containing the primary or secondary coil, is very great compared with $\mathcal{R}_1$ or $\mathcal{R}_2$.

The law of the magnetic circuit, developed in Chapter III., is

$$\text{magnetic flux} = \frac{\text{magnetomotive force}}{\text{reluctance}}.$$

In symbols this is expressed

$$N = \frac{M.M.F.}{\mathcal{R}},$$

where N is the flux through that path whose reluctance is $\mathcal{R}$, and $M.M.F.$ is the magnetomotive force between A and B, measured by the line integral $\int \mathbf{H} \cdot dl$ of the magnetizing force taken from A to B. This integral expresses the work done in moving a unit positive magnetic pole from B to A and is accordingly the difference of magnetic potential between the points.

The flux by each of the three paths from A to B would be respectively

$$N_1 = \frac{M.M.F.}{\mathcal{R}_1}; \quad N_2 = \frac{M.M.F.}{\mathcal{R}_2}; \quad N_3 = \frac{M.M.F.}{\mathcal{R}_3},$$

where $M.M.F.$ is the common difference of magnetic potential between the points A and B. The total flux is thus:

$$N = M.M.F. \left(\frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2} + \frac{1}{\mathcal{R}_3}\right).$$

If $\mathcal{R}$ denotes the reluctance of a single circuit, which would just replace the three above, and allow the same magnetomotive force to cause the same flux, we should have

$$N = \frac{M.M.F.}{\mathcal{R}};$$

hence,

$$\frac{1}{\mathcal{R}} = \frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2} + \frac{1}{\mathcal{R}_3},$$

the analogy of the equivalent resistance of a number of electric circuits in parallel. It was to be expected that this result must follow from the law of the magnetic circuit.

To apply this principle to the transformer, let us suppose that we have a coil of wire composed of S_1 turns upon the branch C, Fig. 106, and another of S_2 turns upon the branch D, these being the primary and secondary coils respectively.

When a current is flowing in the coil S_1, a magnetomotive force is set up that causes an induction, all of which passes through C and divides at A, part going via D and part via MN to B, and finally back through C, thus completing the circuit. The reluctance of this path is the sum of two parts, viz., that due to ACB and that due to the combination of ADB and AMNB, in parallel. Let $\mathcal{R}'$ denote this total reluctance; then

$$\mathcal{R}' = \mathcal{R}_1 + \frac{1}{\frac{1}{\mathcal{R}_2} + \frac{1}{\mathcal{R}_3}} = \mathcal{R}_1 + \frac{\mathcal{R}_2\mathcal{R}_3}{\mathcal{R}_2 + \mathcal{R}_3} = \frac{\mathcal{R}_1\mathcal{R}_2 + \mathcal{R}_1\mathcal{R}_3 + \mathcal{R}_2\mathcal{R}_3}{\mathcal{R}_2 + \mathcal{R}_3}.$$

Let Z denote the numerator of this expression; then

$$\mathcal{R}' = \frac{Z}{\mathcal{R}_2 + \mathcal{R}_3}. \qquad (1)$$

If $\mathcal{R}''$ denotes the reluctance of the circuits ADB, plus the combination ACB and AMNB in parallel, we have in a similar manner

$$\mathcal{R}'' = \mathcal{R}_2 + \frac{\mathcal{R}_1\mathcal{R}_3}{\mathcal{R}_1 + \mathcal{R}_3} = \frac{Z}{\mathcal{R}_1 + \mathcal{R}_3}. \qquad (2)$$

Values of the Magnetic Flux. — In considering magnetic flux, we need to distinguish the flux through any particular branch caused by a current in either primary or secondary coil. Let the subscripts denote the branch through which the flux passes, and the accent denote the current to which it is due; thus, N_3'' means the flux through the branch AMNB which is due to the secondary current alone.

When a current I_1 passes through the primary coil, the flux in the primary coil circuit due to it is found by the law of the magnetic circuit.

$$N_1' = \frac{4\pi S_1 I_1}{\mathcal{R}_1}. \qquad (3)$$

This flux divides at A into two parts, ADB and AMNB, which are inversely as the reluctances of the paths; that is, we have

$$\frac{N_2'}{N_3'} = \frac{\mathcal{R}_3}{\mathcal{R}_2}.$$

Hence it follows that

$$\frac{N_2' + N_3'}{N_3'} = \frac{\mathcal{R}_2 + \mathcal{R}_3}{\mathcal{R}_2},$$

and

$$\frac{N_2'}{N_2' + N_3'} = \frac{\mathcal{R}_3}{\mathcal{R}_2 + \mathcal{R}_3}.$$

But

$$N_2' + N_3' = N_1'.$$

Hence,

$$N_2' = \frac{\mathcal{R}_3}{\mathcal{R}_2 + \mathcal{R}_3} N_1';$$

and

$$N_3' = \frac{\mathcal{R}_2}{\mathcal{R}_2 + \mathcal{R}_3} N_1'. \qquad (4)$$

Substituting the value of $\mathcal{R}'$ given in (1) in the denominator of (3), we have, for the flux through the primary due to the primary current, I_1,

$$N_1' = \frac{4\pi S_1 I_1(\mathcal{R}_2 + \mathcal{R}_3)}{Z}. \qquad (5)$$

This flux, we have seen, divides at A and B into two portions, which may be found by referring to equations (4) and (5). The flux through the secondary due to the primary is

$$N_2' = \frac{4\pi S_1 I_1 \mathcal{R}_3}{Z}. \tag{6}$$

The flux through the leakage branch due to the primary alone is

$$N_3' = \frac{4\pi S_1 I_1 \mathcal{R}_2}{Z}. \tag{7}$$

In a similar manner, if we suppose a current I_2 to flow in the secondary coil, we should find the flux through the secondary, due to I_2, to be

$$N_2'' = \frac{4\pi S_2 I_2 (\mathcal{R}_1 + \mathcal{R}_3)}{Z}. \tag{8}$$

The flux through the primary due to the secondary current is

$$N_1'' = \frac{4\pi S_2 I_2 \mathcal{R}_3}{Z}. \tag{9}$$

The flux through the leakage branch due to the secondary is

$$N_3'' = \frac{4\pi S_1 I_1 \mathcal{R}_1}{Z}. \tag{10}$$

By means of these expressions we are enabled to express the coefficients of induction of the transformer.

Values for the Coefficients of Induction in a Transformer with Magnetic Leakage. — The coefficient of self-induction of the primary is the number of primary flux-turns due to a unit current flowing in the primary; hence,

$$L_1 = \frac{S_1 N_1'}{I_1} = \frac{4\pi S_1^2 (\mathcal{R}_2 + \mathcal{R}_3)}{Z}. \tag{11}$$

The secondary coefficient of self-induction is found in a similar manner to be

$$L_2 = \frac{S_2 N_2''}{I_2} = \frac{4\pi S_2^2 (\mathcal{R}_1 + \mathcal{R}_3)}{Z}. \tag{12}$$

The coefficient of mutual induction, interpreted to mean the secondary flux-turns when a unit current flows in the primary, is

$$M' = \frac{S_2 N_2'}{I_1} = \frac{4\pi S_1 S_2 \mathcal{R}_3}{Z}. \tag{13}$$

The mutual induction, interpreted to mean the primary flux-turns when unit current flows in the secondary, is

$$M'' = \frac{S_1 N_1''}{I_2} = \frac{4\pi S_1 S_2 \mathcal{R}_3}{Z}. \tag{14}$$

Hence, whether there is magnetic leakage or not, the coefficient M is the same whether we consider that it represents the induction through the secondary due to unit current in the primary, or whether it represents the induction in the primary due to unit current in the secondary, and we have

$$M' = M'' = M.$$

We see by the expression for M that it must decrease as the reluctance of the branch AMNB decreases; that is, if there is a perfectly easy path for the induction across from A to B through MN, none will go through the path ADB, and the coefficient of mutual induction will be zero. As the reluctance of the leakage path increases, the coefficient of mutual induction becomes greater, and approaches the value $M = \sqrt{L_1 L_2}$, for no magnetic leakage.

Coefficient of Magnetic Leakage. — Let us denote by M_0 the value of the coefficient of mutual induction in the absence of magnetic leakage; that is, M_0 equals $\sqrt{L_1 L_2}$. In the case of magnetic leakage, the coefficient of the mutual induction M is always less than M_0, this decrease in the mutual induction depending directly upon the amount of magnetic leakage. We may denote the amount of magnetic leakage (expressed as a decimal) by the **coefficient of magnetic leakage** ζ, which is equal to the ratio of the decrease in the mutual induction to the value of the mutual induction in the absence of magnetic leakage; that is,

$$\zeta = \frac{M_0 - M}{M_0} = 1 - \frac{M}{\sqrt{L_1 L_2}}; \tag{15}$$

$$M = (1 - \zeta)\sqrt{L_1 L_2}. \tag{16}$$

If we substitute for the coefficients of self and mutual induction the values given in equations (11), (12), and (14), where a distinct magnetic leakage circuit with reluctance $\mathcal{R}_3$ was assumed, equation (15) becomes

$$\zeta = 1 - \frac{\mathcal{R}_3}{\sqrt{(\mathcal{R}_1 + \mathcal{R}_3)(\mathcal{R}_2 + \mathcal{R}_3)}}. \tag{17}$$

A definite magnetic leakage path does not exist in the transformer as usually designed, but the effects of magnetic leakage, as they actually occur, are identical with those which would be obtained with such a magnetic leakage path.

The ratio of the actual value of the mutual induction to the value it would have in the case of no magnetic leakage is

$$\frac{M}{M_0} = \frac{M}{\sqrt{L_1 L_2}} = 1 - \zeta = \frac{\mathcal{R}_3}{\sqrt{(\mathcal{R}_1 + \mathcal{R}_3)(\mathcal{R}_2 + \mathcal{R}_3)}}. \tag{18}$$

The square of this ratio, which enters into some of the following computations, is

$$\frac{M^2}{L_1 L_2} = (1 - \zeta)^2 = \frac{1}{1 + \frac{Z}{\mathcal{R}_3^2}}. \tag{19}$$

This quantity is always less than unity, approaching unity as the reluctance, $\mathcal{R}_3$ of the leakage circuit becomes greater.

Diagram for Transformer with Magnetic Leakage. — A diagram for a transformer with magnetic leakage may be constructed by the principles already developed. Such a diagram is given in Fig. 107, and its variations are shown for the case of a constant primary current. The primary current is represented by OA, ninety degrees behind which is OB, the electromotive force induced in the secondary by the primary current. This is equal to $M\omega I_1$, and is accordingly a constant. The secondary electromotive force locus is formed by the variation of the right triangle OBC upon the constant hypotenuse OB, and is therefore a semicircle OCB. The secondary current locus will then be the locus ODL, a semicircle upon OL as

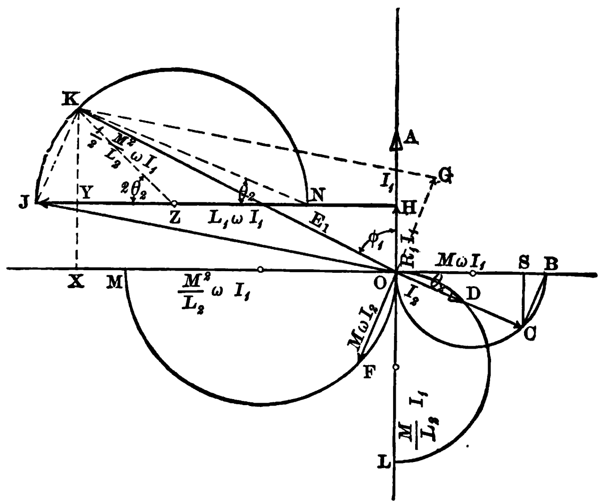


Fig. 107. Typical diagram for transformer with magnetic leakage.

diameter. The value of OL has been shown to be equal to $\frac{M}{L_2}I_1$ (see Fig. 81), and is thus a constant quantity. With no magnetic leakage or external self-induction, $\frac{M}{L_2}I_1$ reduces to $\frac{S_1}{S_2}I$.

In the case of magnetic leakage, by referring to equations (12) and (13), we have

$$\text{OL} = \frac{M}{L_2}I_1 = \frac{\mathcal{R}_3}{\mathcal{R}_1 + \mathcal{R}_3} \cdot \frac{S_1}{S_2}I_1. \qquad (20)$$

When the magnetic leakage occurs equally through the convolutions of the primary and secondary, or when $\mathcal{R}_1$ and $\mathcal{R}_2$ are approximately equal as compared with $\mathcal{R}_3$, as would usually be the case, from the above equation and equation (17) we obtain

$$OL = (1 - \zeta)\frac{S_1}{S_2} I_1. \tag{21}$$

The line OL is accordingly diminished directly in proportion to the magnetic leakage. As $\mathcal{R}_3$, the reluctance of the leakage path, is increased, $\frac{\mathcal{R}_3}{\mathcal{R}_1 + \mathcal{R}_3}$ and $(1 - \zeta)$ approach unity, and OL approaches the value $\frac{S_1}{S_2} I_1$ which it has when there is no leakage.

The back electromotive force in the primary is always ninety degrees behind the secondary current and equal to it multiplied by $M\omega$. Hence its locus is a semicircle OFM upon a diameter OM, ninety degrees behind OL and equal to OL multiplied by $M\omega$. Therefore we have

$$OM = OL \cdot M\omega = \frac{M}{L_2} M\omega I_1.$$

In case of no leakage, $M^2 = L_1 L_2$, so that OM was equal to $L_1 \omega I_1$, the primary electromotive force of self-induction represented by HJ.

We have the ratio

$$\frac{OM}{HJ} = \frac{\frac{M}{L_2} M\omega I_1}{L_1 \omega_1} = \left(\frac{M^2}{L_1 L_2}\right) = (1 - \zeta)^2. \tag{22}$$

The back electromotive force locus is accordingly a semicircle OFM, whose diameter OM is less than HJ.

The primary electromotive force locus is found directly, when the back electromotive force is known, by adding together the electromotive forces: HJ to overcome the primary self-induction, OH to overcome the primary resistance, and OG to overcome the back electromotive force. Thus we find the primary electromotive force locus to be a semicircle JKN, whose diameter JN is equal to OM. The circle lies to the left of the origin O, as it did in the case already discussed, where the secondary circuit contained external self-induction. In fact, the typical figures for the two cases are the same. External self-induction in the secondary circuit can be considered as producing the same effect as magnetic leakage. In either case all the induction generated by the primary does not pass through all the turns of the secondary; and, *vice versa*, all the induction caused by the secondary current does not thread all the turns of the primary; thus there is practically magnetic leakage when there is external self-induction. The magnitudes of the lines in each diagram may be differently expressed in the two cases.

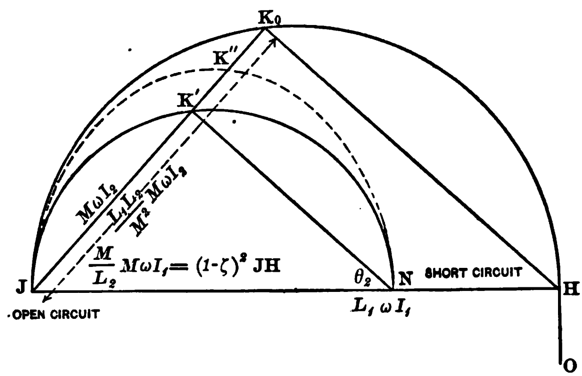


Fig. 108. Effect of magnetic leakage.

In Fig. 108, the semicircle JK_0H is the locus for the primary electromotive force in the absence of magnetic leakage. The semicircle $JK'N$ is the locus for the primary electromotive force when the coefficient of magnetic leakage ζ is constant from open circuit to short circuit. In this case we have the relations [compare equation (18)]

$$\frac{JK'}{JK_0} = \frac{JN}{JH} = \frac{M^2}{L_1L_2} = (1-\zeta)^2; \tag{23}$$

$$1-\zeta = \sqrt{\frac{JK'}{JK_0}} = \sqrt{\frac{JN}{JH}}; \tag{24}$$

$$\zeta = 1 - \sqrt{\frac{JK'}{JK_0}} = 1 - \sqrt{\frac{JN}{JH}}. \tag{25}$$

In an actual transformer the magnetic leakage is not constant, but varies with the load. The locus represented by the dotted curve $JK''N$ is for such a case; here the magnetic leakage is zero on open circuit and increases to the maximum at short circuit. Where the magnetic leakage is variable, it is determined for any point as K'' by the ratio of JK'' to JK_0.

$$\zeta = 1 - \sqrt{\frac{JK''}{JK_0}}. \qquad (26)$$

Thus let us suppose that the back electromotive force JK'', actually induced in the primary by the secondary current I_2, is eighty-one volts; and that JK_0, which would be the back electromotive force in the absence of magnetic leakage, is 100 volts. We then have the equation

$$1 - \zeta = \sqrt{\frac{81}{100}} = \frac{9}{10} = 0{\cdot}9,$$

which indicates that the mutual induction is nine-tenths of the value it would have in the case of no magnetic leakage. The coefficient of magnetic leakage is accordingly found to be ten per cent, thus:

$$\zeta = 1 - 0{\cdot}9 = 0{\cdot}10.$$

Figures 107 and 108 are drawn to represent the values of the various electromotive forces in the primary circuit of a transformer, for a given value of the primary current and for different values of the secondary resistance. If the magnitude of each line is divided by the primary current I_1, Figs. 107 and 108 represent the values for the primary impedance (without any assumption as to constant current or electromotive force) for different values of the secondary resistance. The effect of magnetic leakage upon the primary impedance of any transformer is thus shown for different values of the secondary resistance.

CHAPTER XI

GENERAL EQUATIONS FOR THE CURRENT FLOW IN A PRIMARY AND A SECONDARY CIRCUIT

THE analytical theory for the simple transformer has been discussed in Chapter VI.; the treatment there given was limited to the case of an harmonic impressed electromotive force (or an electromotive force represented by an equivalent harmonic function), and applied to a transformer with no capacity in the primary or secondary circuits. In this chapter the more general case will be discussed of two independent circuits, each of which contains resistance, self-induction, and capacity, which are connected only by means of their common magnetic field. The arrangement of circuits is shown in Fig. 109. The subject will be treated analytically for any electromotive force whatsoever. In a subsequent chapter a discussion is given of the general transformer with a condenser in each circuit, when the primary is supplied with an harmonic electromotive force, or the equivalent of an harmonic electromotive force.

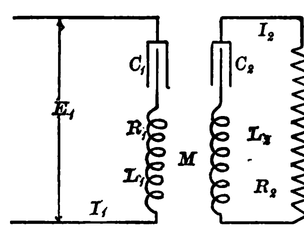


Fig. 109. Arrangement of circuits.

The method * followed will be to start with the differential equations for energy and electromotive forces in the two cir-

* See "General Discussion of the Current Flow in Two Mutually Related Circuits containing Capacity," Bedell and Crehore: Proceedings of the International Electrical Congress, Chicago, 1893, p. 100; also the *Physical Review*, Sept.–Oct., 1893, p. 117.

cuits of the transformer, and from the solution of these equations to obtain values for the primary and the secondary current.

Primary and Secondary Differential Equations of Energy and Electromotive Forces. — The differential equation of energy for a single circuit (see equation (23 a), Chapter IV.) is

$$eidt = \frac{idt\int idt}{C} + Ri^2dt + Li\frac{di}{dt}dt, \tag{1}$$

in which the first member represents the total energy imparted to the circuit by the impressed electromotive force. The second member consists of three terms, which represent respectively the three ways in which the energy is used in the circuit. The first term is the energy required to charge the condenser; the second is that expended in heat; the third is that required to produce the magnetic field.

If a second circuit is now placed in mutual relation to the first, so that they possess a common magnetic field, a fourth term must be added to equation (1), depending upon the coefficient of mutual induction of the two circuits. The equation of energy for the primary coil may be written,

$$ei_1dt = \frac{i_1dt\int i_1dt}{C_1} + R_1i_1^2dt + L_1i_1\frac{di_1}{dt}dt + Mi_1\frac{di_2}{dt}dt. \tag{2}$$

The equation of energy for the second coil, which contains no generator or source of electromotive force, is

$$0 = \frac{i_2dt\int i_2dt}{C_2} + R_2i_2^2dt + L_2i_2\frac{di_2}{dt}dt + Mi_2\frac{di_1}{dt}dt. \tag{3}$$

The second members of equations (2) and (3) are interpreted as follows: the first term in each represents the change in the time *dt* of the amount of energy stored in the primary and secondary condensers, respectively. The second terms represent the energy expended in ohmic resistance in the respective circuits. The last two terms in equations (2) and (3) together represent the changes in the energy of the magnetic field. Thus, in the time dt, the energy stored in the field undergoes a change

$$dW = L_1 i_1 \frac{di_1}{dt} dt + L_2 i_2 \frac{di_2}{dt} dt + M i_1 \frac{di_2}{dt} dt + M i_2 \frac{di_1}{dt} dt. \quad (3\text{ a})$$

The energy of the magnetic field for any particular values of primary and secondary currents may be obtained by integrating equation (3 a) from zero to I_1 and I_2. Thus, the energy of the magnetic field is

$$W = \tfrac{1}{2} L_1 I_1^2 + \tfrac{1}{2} L_2 I_2^2 + M I_1 I_2. \quad (3\text{ b})$$

Here the first and second terms represent the amounts of energy of the magnetic field due to the primary and secondary currents, individually. The last term represents the mutual energy of the two circuits due to their common magnetic field.

If we divide equation (2) by $i_1 dt$ and equation (3) by $i_2 dt$, we obtain equations of electromotive forces

$$e = f(t) = \frac{\int i_1 dt}{C_1} + R_1 i_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}. \quad (4)$$

$$0 = \frac{\int i_2 dt}{C_2} + R_2 i_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}. \quad (5)$$

These equations of electromotive forces, here derived from the corresponding equations of energy, may themselves be considered as fundamental. Equation (4) is the mathemathical statement of the relation between the various electromotive forces in the primary circuit at any time. The electromotive force e, impressed upon the primary circuit, may vary in any way whatsoever with the time, and may therefore be expressed as a function of the time; that is, $e = f(t)$. The primary electromotive force, at any time, is equal to the sum of the several components: $\int i_1 dt \div C_1$, the electromotive force at the terminals of the primary condenser; $R_1 i_1$, the electromotive force needed to overcome the primary resistance; $L_1 \frac{di_1}{dt}$, the electromotive force necessary to overcome the counter-electromotive force of self-induction; and $M\frac{di_2}{dt}$, the electromotive force to overcome the back electromotive force induced in the primary by the secondary current. Equation (5) is the corresponding equation for the secondary electromotive forces.

The solution of these differential equations will be the complete solution of the problem of circuits with constant coefficients of mutual and self induction, having resistance and capacity, and it will give us the current that flows in either the primary or secondary at any time due to any impressed electromotive force. In order to solve the equations we may eliminate from the pair either i_1 or i_2 and obtain a single equation containing but one dependent variable. We then obtain an expression for the primary or secondary current as a function of the time. To perform this elimination we may differentiate (4) and (5) enough times to obtain one more equation than we have variables to eliminate. We may then form a determinant of the coefficients of the variables to be eliminated and proceed in the usual way in elimination by determinants.

If we use the symbols $D = \frac{d}{dt}$, and $D^2 = \frac{d^2}{dt^2}$, etc., equations (4) and (5) may be written

$$f(t) = \frac{\int i_1 dt}{C_1} + R_1 i_1 + L_1 D i_1 + M D i_2; \qquad (6)$$

$$0 = \frac{\int i_2 dt}{C_2} + R_2 i_2 + L_2 D i_2 + M D i_1. \qquad (7)$$

By differentiation we obtain the six equations

$$f'(t) = \frac{1}{C_1} i_1 + R_1 D i_1 + L_1 D^2 i_1 + M D^2 i_2. \qquad (8)$$

$$f''(t) = \frac{1}{C_1} Di_1 + R_1 D^2 i_1 + L_1 D^3 i_1 + M D^3 i_2. \quad (9)$$

$$f'''(t) = \frac{1}{C_1} D^2 i_1 + R_1 D^3 i_1 + L_1 D^4 i_1 + M D^4 i_2. \quad (10)$$

$$0 = \frac{1}{C_2} i_2 + R_2 Di_2 + L_2 D^2 i_2 + M D^2 i_1. \quad (11)$$

$$0 = \frac{1}{C_2} Di_2 + R_2 D^2 i_2 + L_2 D^3 i_2 + M D^3 i_1. \quad (12)$$

$$0 = \frac{1}{C_2} D^2 i_2 + R_2 D^3 i_2 + L_2 D^4 i_2 + M D^4 i_1. \quad (13)$$

In these six equations there are five quantities which it is required to eliminate, viz., i_1, Di_1, D^2i_1, D^3i_1, and D^4i_1, in order to obtain an equation containing only i_2 and the derivatives of i_2. To obtain a single equation containing i_1 alone, it is necessary to eliminate i_2, Di_2, D^2i_2, D^3i_2, and D^4i_2.

Differential Equation for Primary Current.—To obtain the equation for primary current, we form a determinant consisting of six columns, five of which contain the coefficients of i_2, Di_2, D^2i_2, etc., in the above equations, while the sixth column contains all quantities in the equations independent of i_2, Di_2, or D^2i_2, etc. Thus, to eliminate i_2, we may write the determinant,

$$\begin{array}{cccccc} i_2 & Di_2 & D^2i_2 & D^3i_2 & D^4i_2 & \text{Remaining Terms.} \end{array}$$

$$\begin{vmatrix} 0 & 0 & M & 0 & 0 & \frac{1}{C_1} i_1 + R_1 Di_1 + L_1 D^2 i_1 - f'(t) \\ 0 & 0 & 0 & M & 0 & \frac{1}{C_1} Di_1 + R_1 D^2 i_1 + L_1 D^3 i_1 - f''(t) \\ 0 & 0 & 0 & 0 & M & \frac{1}{C_1} D^2 i_1 + R_1 D^3 i_1 + L_1 D^4 i_1 - f'''(t) \\ \frac{1}{C_2} & R_2 & L_2 & 0 & 0 & M D^2 i_1 \\ 0 & \frac{1}{C_2} & R_2 & L_2 & 0 & M D^3 i_1 \\ 0 & 0 & \frac{1}{C_2} & R_2 & L_2 & M D^4 i_1 \end{vmatrix} = 0. \quad (14)$$

Since the first column is composed entirely of zeros except the fourth row, the determinant reduces to $\frac{1}{C_2}$ times its minor determinant. One column of this minor determinant is all zeros except one row, so that the determinant reduces to one of the fourth order. By erasing the first and second columns and the fourth and fifth rows of the determinant of the sixth order, it becomes

$$\begin{vmatrix} M & 0 & 0 & \frac{1}{C_1}i_1 + R_1Di_1 + L_1D^2i_1 - f'(t) \\ 0 & M & 0 & \frac{1}{C_1}Di_1 + R_1D^2i_1 + L_1D^3i_1 - f''(t) \\ 0 & 0 & M & \frac{1}{C_1}D^2i_1 + R_1D^3i_1 + L_1D^4i_1 - f'''(t) \\ \frac{1}{C_2} & R_2 & L_2 & MD^4i_1 \end{vmatrix} = 0. \quad (15)$$

Multiplied out and expanded, this determinant gives

$$\left[(L_1L_2 - M^2)D^4 + (R_1L_2 + R_2L_1)D^3 + \left(\frac{L_1}{C_2} + \frac{L_2}{C_1} + R_1R_2\right)D^2 + \left(\frac{R_1}{C_2} + \frac{R_2}{C_1}\right)D + \frac{1}{C_1C_2}\right]i_1 = \frac{1}{C_2}f'(t) + R_2f''(t) + L_2f'''(t). \quad (16)$$

Here we have a differential equation of the fourth order containing but two variables i_1 and t, the solution of which gives the desired primary current in terms of the impressed electromotive force.

Differential Equation for Secondary Current. — The differential equation for the secondary current may be obtained in a similar manner by eliminating i_1 and its derivatives from equations (8) to (13) inclusive. In order to perform the elimination, we may write, as before, the determinant (17) which follows. Since the first column of this determinant is composed entirely of zeros except the first row, it reduces to the minor of the fifth order found by erasing the first row and first column.

$$\begin{array}{cccccc} i_1 & Di_1 & D^2i_1 & D^3i_1 & D^4i_1 & \text{Remaining Terms.} \end{array}$$

$$\begin{vmatrix} \frac{1}{C_1} & R_1 & L_1 & 0 & 0 & MD^2i_2-f'(t) \\ 0 & \frac{1}{C_1} & R_1 & L_1 & 0 & MD^3i_2-f''(t) \\ 0 & 0 & \frac{1}{C_1} & R_1 & L_1 & MD^4i_2-f'''(t) \\ 0 & 0 & M & 0 & 0 & \frac{1}{C_2}i_2 + R_2Di_2 + L_2D^2i_2 \\ 0 & 0 & 0 & M & 0 & \frac{1}{C_2}Di_2 + R_2D^2i_2 + L_2D^3i_2 \\ 0 & 0 & 0 & 0 & M & \frac{1}{C_2}D^2i_2 + R_2D^3i_2 + L_2D^4i_2 \end{vmatrix} = 0. \quad (17)$$

Since the first column of the minor determinant formed from (17) is composed entirely of zeros except its first row, the whole reduces to the determinant of the fourth order found by striking out both the first and second rows and columns. Thus,

$$\begin{vmatrix} \frac{1}{C_1} & R_1 & L_1 & MD^4i_2-f'''(t) \\ M & 0 & 0 & \frac{1}{C_2}i_2 + R_2Di_2 + L_2D^2i_2 \\ 0 & M & 0 & \frac{1}{C_2}Di_2 + R_2D^2i_2 + L_2D^3i_2 \\ 0 & 0 & M & \frac{1}{C_2}D^2i_2 + R_2D^3i_2 + L_2D^4i_2 \end{vmatrix} = 0. \quad (18)$$

Multiplying out and expanding, we obtain the desired differential equation for the secondary current.

$$\left[(L_1L_2-M^2)D^4+(R_1L_2+R_2L_1)D^3+\left(\frac{L_1}{C_2}+\frac{L_2}{C_1}+R_1R_2\right)D^2 + \left(\frac{R_1}{C_2}+\frac{R_2}{C_1}\right)D+\frac{1}{C_1C_2}\right]i_2 = -Mf'''(t). \quad (19)$$

It is noticeable that these differential equations (16) and (19) for the primary and secondary currents are very similar. The first members are alike if we write i_2 for i_1. The second members show a marked difference. The equation for primary current contains R_2, L_2, and C_2, with the three derivatives of $f(t)$, while the equation for secondary current contains only M, with the third derivative of $f(t)$.

Modifications Necessary for Ready Solution. — The general solutions of these equations of the fourth order, (16) and (19), would give the value of the primary and secondary currents at any time, where the impressed electromotive force is any function whatsoever of the time, and they would involve the literal solution of the general bi-quadratic equation, that we might find four factors of the bi-quadratic expression in order to resolve the inverse operator into four partial fractions. In their full generality, the equations are too cumbersome for practical purposes, but readily admit of solution, as they now stand, if we assume the impressed electromotive force to be harmonic, as will be shown hereafter. However, the solutions may be obtained for any impressed electromotive force whatsoever, by the introduction of certain modifications which reduce the equations to an order lower than the fourth. If the equation can be reduced to the second order, it is comparatively easy to find its general solution; but we will not attempt to write the solution of equations of the third and fourth orders, except in the case of an harmonic impressed electromotive force, which is to be discussed later.

Omission of Both Condensers. — If we omit one of the condensers from the circuit, it will reduce the equations (16) and (19) to the third order, and if we omit both condensers, the equations reduce to the second order. The resulting equations in these particular cases may be written from (16) and (19) by omitting terms and reducing the accents and the order by the required amount. To see how the order is reduced, we will form the equation in the case where both condensers are omitted. To do this let us return to (6) and (7). Omitting the condenser terms, these equations of electromotive forces become

$$f(t)=R_1i_1+L_1Di_1+MDi_2. \qquad (20)$$

$$0=R_2i_2+L_2Di_2+MDi_1. \qquad (21)$$

We need differentiate each equation only once to obtain enough equations to eliminate either of the variables i_1 or i_2.

By differentiation we have

$$f'(t)=R_1Di_1+L_1D^2i_1+MD^2i_2. \qquad (22)$$

$$0=R_2Di_2+L_2D^2i_2+MD^2i_1. \qquad (23)$$

Forming a determinant of these four equations to eliminate i_2, we have

$$\begin{array}{cccl} i_2 & Di_2 & D^2i_2 & \text{Remaining Terms.} \\ \end{array}$$

$$\begin{vmatrix} 0 & M & 0 & R_1i_1 \ +L_1Di_1 -f(t) \\ 0 & 0 & M & R_1Di_1+L_1D^2i_1-f'(t) \\ R_2 & L_2 & 0 & MDi_1 \\ 0 & R_2 & L_2 & MD^2i_1 \end{vmatrix}=0. \qquad (24)$$

Since the first column is all zeros except the third row, this determinant reduces to

$$\begin{vmatrix} M & 0 & R_1i_1 \ +L_1Di_1 -f(t) \\ 0 & M & R_1Di_1+L_1D^2i_1-f'(t) \\ R_2 & L_2 & MD^2i_1 \end{vmatrix}=0; \qquad (25)$$

and this expanded gives

$$[(L_1L_2-M^2)D^2+(R_1L_2+R_2L_1)D+R_1R_2]i_1 \\ =R_2f(t)+L_2f'(t). \qquad (26)$$

Forming the determinant to eliminate i_2, we have

$$\begin{array}{cccl} i_1 & Di_1 & D^2i_1 & \text{Remaining Terms.} \\ \end{array}$$

$$\begin{vmatrix} R_1 & L_1 & 0 & MDi_2-f(t) \\ 0 & R_1 & L_1 & MD^2i_2-f'(t) \\ 0 & M & 0 & R_2i_2 \ +L_2Di_2 \\ 0 & 0 & M & R_2Di_2 +L_2D^2i_2 \end{vmatrix}=0. \qquad (27)$$

This reduces to

$$\begin{vmatrix} R_1 & L_1 & MD^2i_2 - f'(t) \\ M & 0 & R_2i_2 \ + L_2Di_2 \\ 0 & M & R_2Di_2 + L_2D^2i_2 \end{vmatrix} = 0; \qquad (28)$$

and, when expanded, gives

$$[(L_1L_2 - M^2)D^2 + (R_1L_2 + R_2L_1)D + R_1R_2]i_2 = -Mf'(t). \qquad (29)$$

Upon comparing (26) and (29) with (16) and (19), we see that we might have written both (26) and (29) immediately from the more general forms, by reducing each exponent and accent by two, and omitting all terms containing C_1 or C_2.

The equations (26) and (29), here obtained as particular cases of the more general equations, are identical with (6) and (8), independently obtained in Chapter VI.

Omission of One Condenser. — In a similar manner we may find the equations of the third order from the general by omitting one condenser only. If we omit the primary condenser only, we may write from the general the equation in this particular case by reducing all exponents and accents by unity and omitting terms containing C_1. Thus, omitting C_1 from the primary, we have for the primary current

$$\left[(L_1L_2 - M^2)D^3 + (R_1L_2 + R_2L_1)D^2 + \left(\frac{L_1}{C_2} + R_1R_2\right)D + \frac{R_1}{C_2}\right]i_1$$
$$= \frac{1}{C_2}f(t) + R_2f'(t) + L_2f''(t). \qquad (30)$$

The secondary current equation becomes

$$\left[(L_1L_2 - M^2)D^3 + (R_1L_2 + R_2L_1)D^2 + \left(\frac{L_1}{C_2} + R_1R_2\right)D + \frac{R_1}{C_2}\right]i_2$$
$$= -Mf''(t). \qquad (31)$$

If the secondary condenser is omitted, the equations (16) and (19) become

$$\left[(L_1L_2 - M^2)D^3 + (R_1L_2 + R_2L_1)D^2 + \left(\frac{L_2}{C_1} + R_1R_2\right)D + \frac{R_2}{C_1}\right]i_1$$
$$= R_2f'(t) + L_2f''(t); \qquad (32)$$

and

$$\left[(L_1L_2-M^2)D^3+(R_1L_2+R_2L_1)D^2+\left(\frac{L_2}{C_1}+R_1R_2\right)D+\frac{R_2}{C_1}\right]i_2$$
$$= - Mf''(t). \quad (33)$$

Condition of No Magnetic Leakage — Another consideration which will enable us to reduce the order of our equations is that of no magnetic leakage. If we consider that all the lines of induction generated by the primary circuit thread the secondary, it has been shown that the product of the coefficients of self-induction equals the square of the coefficient of mutual induction of the two circuits; that is, $L_1L_2=M^2$, or $L_1L_2-M^2=0$. This approximation is so nearly realized in most of the closed magnetic circuit transformers that it is worth while to consider how it simplifies the equations. The coefficient of the terms of the highest order in any of the differential equations yet written is this quantity $L_1L_2-M^2$. If we consider, therefore, that there is no magnetic leakage in the transformer, we may reduce the general equations (16) and (19) to the third order. It will not be attempted here to write down the general solution of the equation of the third order. However, if we wish to consider the case where only one condenser is in circuit and there is no magnetic leakage, equations (30) to (33) inclusive reduce to the second order and can be readily solved. When there is no condenser in either circuit, equations (26) and (29) reduce to very simple forms, being of the first order when the consideration of no magnetic leakage is introduced.

The various differential equations have now been formed.

It is now proposed to obtain the general solutions of equations (26) and (29), which give the complete solution of the problem of two mutually related circuits without considering condensers, but with no limiting assumptions in regard to the magnetic leakage. Then will be given the solution of the same case simplified by the assumption that there is no magnetic leakage. Lastly, the cases will be considered where there is no magnetic leakage, and one of the circuits contains a condenser.

General Solution for the Currents in Two Mutually Related Circuits with No Condenser. — In symbolic notation (see Johnson's *Differential Equations*, Chap. V.), equations (26) and (29) may be written

$$i_1=\frac{1}{D^2+\frac{R_1L_2+R_2L_1}{L_1L_2-M^2}D+\frac{R_1R_2}{L_1L_2-M^2}}\frac{[R_2f(t)+L_2f'(t)]}{L_1L_2-M^2}, \quad (34)$$

and

$$i_2=\frac{-1}{D^2+\frac{R_1L_2+R_2L_1}{L_1L_2-M^2}D+\frac{R_1R_2}{L_1L_2-M^2}}\frac{Mf'(t)}{L_1L_2-M^2}. \quad (35)$$

Resolving the inverse operator into partial fractions, we have the identical equation

$$\frac{1}{D^2+\frac{R_1L_2+R_2L_1}{L_1L_2-M^2}D+\frac{R_1R_2}{L_1L_2-M^2}} \quad (36)$$

$$=\frac{L_1L_2-M^2}{\sqrt{(R_1L_2+R_2L_1)^2-4R_1R_2(L_1L_2-M^2)}}\left(\frac{1}{D+\tau_1}-\frac{1}{D+\tau_2}\right).$$

The radical expression which occurs in equation (36) may be written

$$\sqrt{(R_1L_2+R_2L_1)^2-4R_1R_2(L_1L_2-M^2)}$$
$$=\sqrt{(R_1L_2-R_2L_1)^2+4R_1R_2M^2}.$$

For simplification, the abbreviations have been used:

$$\tau_1=\frac{R_1L_2+R_2L_1-\sqrt{(R_1L_2-R_2L_1)^2+4R_1R_2M^2}}{2(L_1L_2-M^2)}, \quad (37)$$

and

$$\tau_2=\frac{R_1L_2+R_2L_1+\sqrt{(R_1L_2-R_2L_1)^2+4R_1R_2M^2}}{2(L_1L_2-M^2)}. \quad (38)$$

Placing (36) in (34), we obtain

$$i_1=\frac{1}{\sqrt{(R_1L_2-R_2L_1)^2+4R_1R_2M^2}}$$
$$\left(\frac{R_2f(t)+L_2f'(t)}{D+\tau_1}-\frac{R_2f(t)+L_2f'(t)}{D+\tau_2}\right). \quad (40)$$

Similarly, (35) becomes

$$i_2=\frac{-M}{\sqrt{(R_1L_2-R_2L_1)^2+4R_1R_2M_2}}\left(\frac{f'(t)}{D+\tau_1}-\frac{f'(t)}{D+\tau_2}\right). \qquad (41)$$

Now the linear equation of the first order may be written

$$y=\frac{1}{D+a}f(x),$$

and its solution is known* to be

$$y=\epsilon^{-ax}\int\epsilon^{ax}f(x)dx+c\epsilon^{-ax}.$$

Hence we have

$$\frac{f(x)}{D+a}=\epsilon^{-ax}\int\epsilon^{ax}f(x)dx+c\epsilon^{-ax}. \qquad (42)$$

Replacing $f(x)$ by $R_2f(t)$, and a by τ_1 in this general formula, we have

$$\frac{R_2f(t)}{D+\tau_1}=R_2\epsilon^{-\tau_1t}\int\epsilon^{\tau_1t}f'(t)dt+c\epsilon^{-\tau_1t}. \qquad (43)$$

Similarly, we should find

$$\frac{L_2f'(t)}{D+\tau_1}=L_2\epsilon^{-\tau_1t}\int\epsilon^{\tau_1t}f(t)dt+c\epsilon^{-\tau_1t}, \qquad (44)$$

and also

$$\frac{f'(t)}{D+\tau_1}=\epsilon^{-\tau_1t}\int\epsilon^{\tau_1t}f'(t)dt+c\epsilon^{-\tau_1t}. \qquad (45)$$

Substituting these values in (40) and (41), we obtain for the primary current

$$i_1=\frac{1}{\sqrt{(R_1L_2-R_2L_1)^2+4R_1R_2M^2}}\left(\epsilon^{-\tau_1t}\int\epsilon^{\tau_1t}[R_2f(t)+L_2f'(t)]dt\right.$$
$$\left.-\epsilon^{-\tau_2t}\int\epsilon^{\tau_2t}[R_2f(t)+L_2f'(t)]dt\right)+c_1\epsilon^{-\tau_1t}+c_2\epsilon^{-\tau_2t}; \qquad (46)$$

and for the secondary current

$$i_2=\frac{-M}{\sqrt{(R_1L_2-R_2L_1)^2+4R_1R_2M^2}}$$
$$\left(\epsilon^{-\tau_1t}\int\epsilon^{\tau_1t}f'(t)dt-\epsilon^{-\tau_2t}\int\epsilon^{\tau_2t}f'(t)dt\right)+c_3\epsilon^{-\tau_1t}+c_4\epsilon^{-\tau_2t}. \qquad (47)$$

* See Johnson's *Differential Equations*, p. 31.

These equations, (46) and (47), are the complete solutions for the current flowing in two mutually related circuits, containing no condenser, with no limiting assumption in regard to magnetic leakage. The electromotive force may be anything whatsoever. It is noticed, (37) and (38), that there are two time constants, as in the case of capacity and self-induction in a single circuit; but no oscillatory effect can be obtained in this case, because the expression under the radical is always real for all values of the constants R, M, or L. Further discussion will be deferred until the equations for the other cases are found.

General Solution for Two Mutually Related Circuits in which there is No Condenser, assuming No Magnetic Leakage. — Upon the introduction of the condition of no leak in equations (26) and (29), *i.e.* equating $L_1L_2 - M^2$ to zero, they become

$$\left[D + \frac{R_1R_2}{R_1L_2 + R_2L_1}\right]i_1 = \frac{R_2f(t) + L_2f'(t)}{R_1L_2 + R_2L_1}, \tag{48}$$

and
$$\left[D + \frac{R_1R_2}{R_1L_2 + R_2L_1}\right]i_2 = \frac{-Mf'(t)}{R_1L_2 + R_2L_1}. \tag{49}$$

The solution of these linear equations of the first order may be written by means of the general formula (42), and we have

$$i_1 = \frac{\epsilon^{-\tau t}}{R_1L_2 + R_2L_1}\int \epsilon^{\tau t}[R_2f(t) + L_2f'(t)]dt + c_1\epsilon^{-\tau t}, \tag{50}$$

and
$$i_2 = -\frac{M}{R_1L_2 + R_2L_1}\epsilon^{-\tau t}\int \epsilon^{\tau t}f'(t)dt + c_2\epsilon^{-\tau t}, \tag{51}$$

where
$$\tau = \frac{R_1R_2}{L_1R_2 + L_2R_1}. \tag{52}$$

These equations might have been written from the general solutions (46) and (47), but it is easier to derive them independently from the differential equations. It is remarkable how much the consideration of no magnetic leakage simplifies the mathematical expression of the results.

Before entering upon the discussion of these results, it is thought best to obtain the equations for the case in which there is a condenser in one of the circuits, when there is no magnetic leakage.

General Solution for Two Mutually Related Circuits with a Condenser in One Circuit, assuming No Magnetic Leakage. — The solutions may be obtained from the differential equations (30) to (33), after equating $L_1L_2 - M^2$ to zero. With this supposition, when we have the secondary condenser only, the equations become

$$i_1 = \frac{\frac{1}{C_2}f(t) + R_2 f'(t) + L_2 f''(t)}{\left\{D^2 + \frac{\left(\frac{L_1}{C_2} + R_1R_2\right)}{R_1L_2 + R_2L_1}D + \frac{R_1}{C_2(R_1L_2 + R_2L_1)}\right\} R_1L_2 + R_2L_1}; \tag{53}$$

$$i_2 = \frac{-1}{D^2 + \frac{\frac{L_1}{C_2} + R_1R_2}{R_1L_2 + R_2L_1}D + \frac{R_1}{C_2(R_1L_2 + R_2L_1)}} \cdot \frac{Mf''(t)}{R_1L_2 + R_2L_1}. \tag{54}$$

With the primary condenser only, they become

$$i_1 = \frac{1}{D^2 + \frac{\frac{L_2}{C_1} + R_1R_2}{R_1L_2 + R_2L_1}D + \frac{R_2}{C_1(R_1L_2 + R_2L_1)}} \cdot \frac{R_2 f'(t) + L_2 f''(t)}{R_1L_2 + R_2L_1}, \tag{55}$$

and

$$i_2 = \frac{-1}{D^2 + \frac{\frac{L_2}{C_1} + R_1R_2}{R_1L_2 + R_2L_1}D + \frac{R_2}{C_1(R_1L_2 + R_2L_1)}} \cdot \frac{Mf''(t)}{R_1L_2 + R_2L_1}. \tag{56}$$

Resolving the inverse operator in (53) and (54) into partial fractions, we have the identity

$$\frac{1}{D^2 + \frac{\left(\frac{L_1}{C_2} + R_1R_2\right)}{R_1L_2 + R_2L_1}D + \frac{R_1}{R_1L_2 + R_2L_1}} \tag{57}$$

$$= \frac{R_1L_2 + R_2L_1}{\sqrt{\left(\frac{L_1}{C_2} + R_1R_2\right)^2 - 4R_1(R_1L_2 + R_2L_1)}} \left(\frac{1}{D + \tau_1''} - \frac{1}{D + \tau_2''}\right),$$

where, for abbreviation,

$$\tau_1'' = \frac{\frac{L_1}{C_2} + R_1R_2 + \sqrt{\left(\frac{L_1}{C_2} + R_1R_2\right)^2 - 4R_1(R_1L_2 + R_2L_1)}}{2(R_1L_2 + R_2L_1)}, \quad (58)$$

$$\tau_2'' = \frac{\frac{L_1}{C_2} + R_1R_2 - \sqrt{\left(\frac{L_1}{C_2} + R_1R_2\right)^2 - 4R_1(R_1L_2 + R_2L_1)}}{2(R_1L_2 + R_2L_1)}. \quad (59)$$

Equations (55) and (56), for the case of the primary condenser only, may be similarly treated by interchanging R_1, L_1, C_1, τ_1', τ_2', with R_2, L_2, C_2, τ_1'', τ_2'', where, for the case with the primary condenser, we have the abbreviations

$$\tau_1' = \frac{\frac{L_2}{C_1} + R_1R_2 + \sqrt{\left(\frac{L_2}{C_1} + R_1R_2\right)^2 - 4R_2(R_1L_2 + R_2L_1)}}{2(R_1L_2 + R_2L_1)}. \quad (60)$$

$$\tau_2' = \frac{\frac{L_2}{C_1} + R_1R_2 - \sqrt{\left(\frac{L_2}{C_1} + R_1R_2\right)^2 - 4R_2(R_1L_2 + R_2L_1)}}{2(R_1L_2 + R_2L_1)}. \quad (61)$$

This method of obtaining the solution which has previously been given, (34) to (47), finally gives the integral equations in the case of one condenser with the consideration of no leak.

For the case in which there is a condenser in the secondary and no leak, the solutions are

$$i_1 = \frac{1}{\sqrt{\left(\frac{L_1}{C_2} + R_1R_2\right)^2 - 4R_1(R_1L_2 + R_2L_1)}} \quad (62)$$

$$\left(\epsilon^{-\tau_2''t}\int\epsilon^{\tau_2''t}\left[\frac{1}{C_2}f(t) + R_2f'(t) + L_2f''(t)\right]dt + \epsilon^{-\tau_1''t}\int\epsilon^{\tau_1''t}\left[\frac{1}{C_2}f(t) + R_2f'(t) + L_2f''(t)\right]dt\right) + c_1\epsilon^{-\tau_1''t} + c_2\epsilon^{-\tau_2''t}.$$

$$i_2 = \frac{-M}{\sqrt{\left(\frac{L_1}{C_2} + R_1R_2\right)^2 - 4R_1(R_1L_2 + R_2L_1)}} \quad (63)$$

$$\left(\epsilon^{-\tau_1''t}\int\epsilon^{\tau_1''t}f''(t)dt - \epsilon^{-\tau_2''t}\int\epsilon^{\tau_2''t}f''(t)dt\right) + c_3\epsilon^{-\tau_1''t} + c_4\epsilon^{-\tau_2''t}.$$

When there is a condenser in the primary alone, and no leak, the solutions are

$$i_1 = \frac{1}{\sqrt{\left(\frac{L_2}{C_1}+R_1R_2\right)^2 - 4R_2(R_1L_2+R_2L_1)}} \tag{64}$$

$$\left(\epsilon^{-r_2t}\int\epsilon^{r_2t}[R_2f'(t)+L_2f''(t)]dt - \epsilon^{-r_1t}\int\epsilon^{r_1t}[R_2f'(t)\right.$$

$$\left.+L_2f''(t)]\,dt\right) + c_5\epsilon^{-r_1t} + c_6\epsilon^{-r_2t}.$$

$$i_2 = \frac{M}{\sqrt{\left(\frac{L_2}{C_1}+R_1R_2\right)^2 - 4R_2(R_1L_2+R_2L_1)}} \tag{65}$$

$$\left(\epsilon^{-r_1t}\int\epsilon^{r_1t}f''(t)dt - \epsilon^{-r_2t}\int\epsilon^{r_2t}f''(t)dt\right) + c_7\epsilon^{-r_1t} + c_8\epsilon^{-r_2t}.$$

We have now obtained the equations for current flow under various conditions in regard to magnetic leakage and the location of condensers. The solutions have thus far been general; that is, there have been no limitations in regard to the nature of the impressed electromotive force, which may be any function whatsoever of the time. In (46) and (47) we have the expressions for current flow when there are no condensers, without limitation in regard to there being no magnetic leakage. In (50) and (51) we have the same simplified by the assumption of no leakage. Equations (62) and (63) are the solutions in case of no leak and a secondary condenser; and (64) and (65) are the same for a primary condenser. These general solutions will now be interpreted in turn for certain particular impressed electromotive forces, after which the solution in case of two condensers and no assumption as to absence of magnetic leak will be taken up for an harmonic impressed electromotive force, this solution giving us the general theory of the transformer.

Discussion of the General Solution for the Currents in Two Mutually Related Circuits with No Condenser. — Equations (46) and (47) are the general expressions for the current flowing in two mutually related circuits due to any electromotive force whatsoever impressed upon one of them. Each equation consists of a particular integral and a complementary function containing two arbitrary constants of integration to be determined according to the imposed conditions. In order to perform the operations indicated in the particular integrals, in which the electromotive force is expressed as $f(t)$, it is necessary to assume the electromotive force to be some particular function of the time.

General Case of "Make" or "Break."— Ordinarily this would be the case in which initially the two currents are zero and a certain electromotive force is suddenly impressed; or, the case in which initially the two currents have certain assigned values, and the electromotive force is suddenly reduced to zero. All the cases of make or break, that is, the introduction or the removal of the electromotive force, may be generally stated thus: The initial conditions are a primary current I' and the secondary current I'', due to some primary impressed electromotive force, the exact nature of which is immaterial; this electromotive force is suddenly changed to a certain known electromotive force. The initial or final values of any of these electromotive forces or currents may be zero.

Let us suppose that the final value of the impressed electromotive force is a constant, $e = f(t) = E'$. Substituting $f(t) = E'$, and $f'(t) = 0$ in the general equations (46) and (47), and performing the indicated integrations, we obtain for the primary and secondary currents,

$$i_1 = \frac{E'}{R_1} + c_1\epsilon^{-\tau_1 t} + c_2\epsilon^{-\tau_2 t}, \tag{66}$$

$$i_2 = c_3\epsilon^{-\tau_1 t} + c_4\epsilon^{-\tau_2 t}. \tag{67}$$

The arbitrary constants of integration in these complementary functions are to be obtained according to the conditions of the problem. Let I stand for the final steady value $\dfrac{E'}{R_1}$ of the primary current. Counting time from the time of alteration of the primary electromotive force, we have,

when $t = 0$, $\qquad i_1 = I' = I + c_1 + c_2,$ (68)

and $\qquad i_2 = I'' = c_3 + c_4.$

This gives two equations in which there are four unknown arbitrary constants to be determined, and evidently two more equations must be obtained before they can be found. These equations may be formed from the consideration of the quantities of electricity which will flow in the two circuits while the magnetic field is changing, on account of the change in the impressed electromotive force. The number of lines initially threading the primary circuit is L_1I' due to the primary current, plus MI'' lines due to the secondary; finally, when the primary current has the steady value I, the number of lines will be L_1I. The quantity of electricity (in C. G. S. units) which will flow in the primary, *due to the change in the magnetic field*, will be equal to the change in lines divided by the resistance, or

$$Q_1 = \frac{L_1I' + MI'' - L_1I}{R_1}.$$

Similarly, the initial number of lines threading the secondary will be $L_2I'' + MI'$, which will change to the final value MI; whence

$$Q_2 = \frac{L_2I'' + MI - MI}{R_2}.$$

Now these values of the quantities of electricity, in primary and secondary, may be obtained from the integrals of the current equations. The integrals of (66) and (67) between the limits zero and infinity will give the quantities of electricity which will flow during an infinite time from the time of changing the impressed electromotive force; thus, from (66),

$$\int_{t=0}^{t=\infty} i_1 dt = It_\infty + \frac{c_1}{\tau_1} + \frac{c_2}{\tau_2}.$$

Now the first term in the second member is the quantity which will flow, due to the final steady current $I=\frac{E'}{R_1}$, and the remaining two terms represent the quantity which will flow, due to the change in the magnetic field, or

$$Q_1=\frac{c_1}{\tau_1}+\frac{c_2}{\tau_2}.$$

The secondary flow, from (67), is

$$Q_2=\int_{t=0}^{t=\infty} i_2 dt=\frac{c_3}{\tau_1}+\frac{c_4}{\tau_2}.$$

This gives us the two additional equations containing the unknown arbitrary constants. From these two equations and the two before obtained (68), the four constants are found to be

$$c_1=\frac{\tau_1(\tau_2 Q_1+I-I')}{\tau_2-\tau_1},$$

$$c_2=\frac{\tau_2(I'-I-\tau_1 Q_1)}{\tau_2-\tau_1},$$

$$c_3=\frac{\tau_1(\tau_2 Q_2-I'')}{\tau_2-\tau_1},$$

$$c_4=\frac{\tau_2(I''-\tau_1 Q_2)}{\tau_2-\tau_1}.$$

Equations (66) and (67), with these values substituted for the arbitrary constants of integration, give the values of primary and secondary currents at any time after the alteration of the primary impressed electromotive force. The values of τ_1 and τ_2 so depend upon the constants of the circuits that they are always real, and so the change of the currents from their initial to final values is gradual and non-oscillatory.

The nature of this change in the currents is shown by the typical curves in Fig. 110, representing a case in which the primary current is changed from a value OA to a final value OB, the secondary current rising from zero to a maximum, and gradually dying away to zero again. In all cases where the primary is either increased or decreased from one steady value to another by make or by break, the secondary current curve would have a shape similar to that shown. The primary current curve as shown is typical for any case of decrease, either to a finite steady value or to zero; if inverted, it would show the change for a corresponding increase.

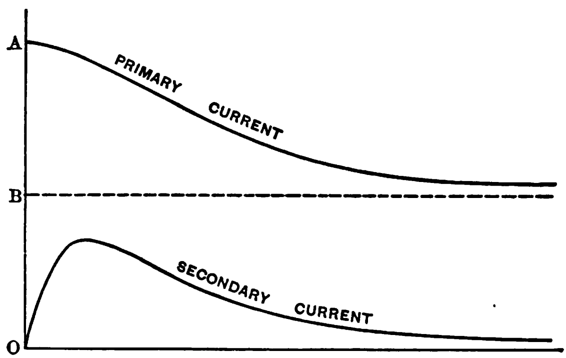


Fig. 110.

Case of No Magnetic Leakage.—In the hypothetical case of no magnetic leakage the equations take the simpler forms of (50) and (51). According to these equations, the change of the primary and secondary currents from initial to final values, due to a change of the primary impressed electromotive force from its initial to a final steady value, would be represented by exponential curves. Figure 111 typically represents such a change. The primary current is changed from a value OA to a final steady value OB in the opposite direction, the total change being represented by AB. The secondary current is represented by an exponential curve with initial value OC, and final value zero. Now in the case supposed, the secondary current will be initially zero, and to follow this exponential law it would have to immediately assume the value OC. Evidently this is impossible, and it shows that the case of absolutely no magnetic leakage is hypothetical. The dotted line shows the nature of the rise from zero, giving a curve as that shown in Fig. 110.

Discussion of Case of Mutually Related Circuits containing a Condenser. — Let us first consider the case in which there is a condenser in the secondary circuit. The general case of make or break will be treated as before by assuming the primary and secondary currents to be initially I' and I'' respectively, and the impressed electromotive force to be altered to a constant value E'. The expressions for the currents in the primary and secondary at any time t after the change, may be

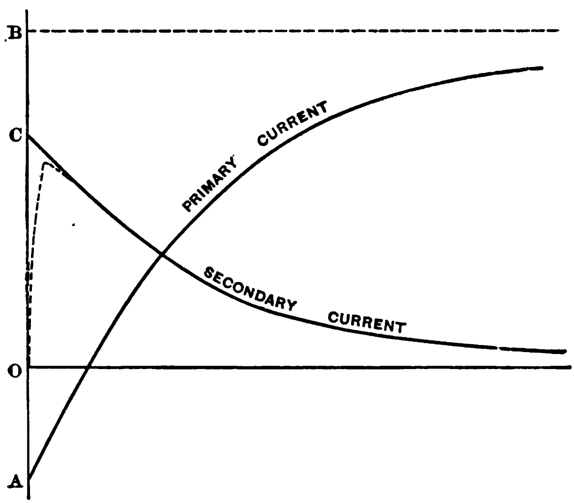


Fig. 111.

found directly from equations (62) and (63) by substituting $f(t)=E'$, $f'(t)=0$, and $f''(t)=0$. Making these substitutions and performing the integrations indicated, we obtain

$$i_1=\frac{E'}{R_1}+c_1\epsilon^{-\tau_1''t}+c_2\epsilon^{-\tau_2''t}, \qquad (69)$$

$$i_2=c_3\epsilon^{-\tau_1''t}+c_4\epsilon^{-\tau_2''t}. \qquad (70)$$

These equations are similar to (66) and (67) already discussed, and when the constants τ_1'' and τ_2'' are real, the phenomena attending the make or break do not differ from those already referred to, illustrated in Fig. 110, and need no further explanation.

Oscillatory Case. — The constants of the two circuits may have such values, however, that the constants τ_1'' and τ_2'' are imaginary, in which case the equations (69) and (70) may be transformed, by means of the exponential values of the sine and cosine, into a real form. By referring to the values of the constants τ_1'' and τ_2'' given in (58) and (59), we can note whether

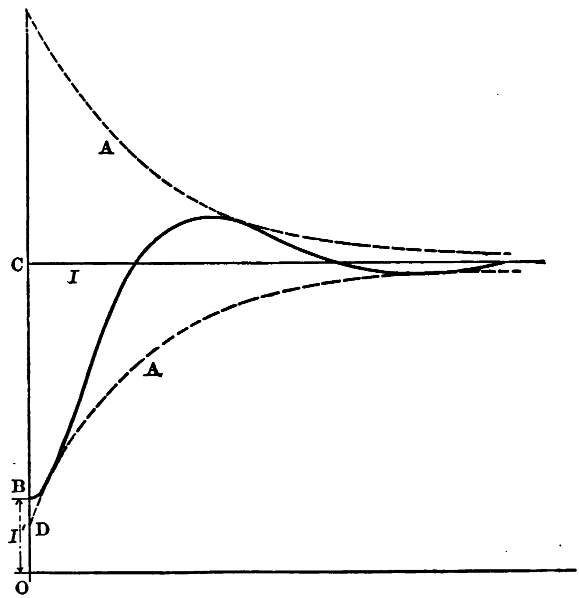


Fig. 112.

these values are real or imaginary, and whether or not a transformation is necessary. When $\left(\frac{L_1}{C_2}+R_1R_2\right)^2$ is greater than $4R_1(R_1L_2+R_2L_1)$, the values of τ_1'' and τ_2'' are real, and the equations (69) and (70) may be interpreted as (66) and (67); that is, there is no oscillation. When $\left(\frac{L_1}{C_2}+R_1R_2\right)^2$ is less than $4R_1(R_1L_2+R_2L_1)$, the values of τ_1'' and τ_2'' become imaginary. Equations (69) and (70) can be transformed, however, into the real forms,

$$i_1=\frac{E'}{R_1}+A_1\epsilon^{-pt}\sin(\alpha t+\Phi_1), \tag{71}$$

$$i_2=A_2\epsilon^{-pt}\sin(\alpha t+\Phi_2). \tag{72}$$

In the equations A_1, A_2, Φ_1, Φ_2, are constants of integration, which may be determined from the supposed conditions; p and α are constants depending upon the constants of the circuit, thus:

$$p = \frac{\frac{L_1}{C_2} + R_1 R_2}{2(R_1 L_2 + R_2 L_1)}, \tag{73}$$

$$\alpha = \frac{\sqrt{\left(\frac{L_1}{C_2} + R_1 R_2\right)^2 - 4 R_1 (R_1 L_2 + R_2 L_1)}}{2(R_1 L_2 + R_2 L_1)}. \tag{74}$$

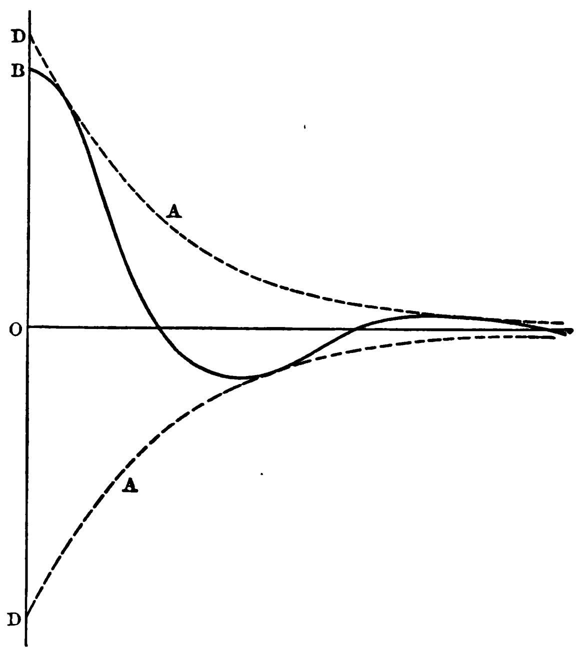


Fig. 113.

Equations (71) and (72) show that both primary and secondary currents will oscillate harmonically about their final values with a period equal to $\frac{2\pi}{\alpha}$, the maximum values of the oscillations decreasing rapidly with a logarithmic decrement depending upon the value of p. The final steady value of the primary current will be $\frac{E'}{R_1}$, and the secondary will become zero after a short interval of time. The amplitude of the oscillations depends upon the values of A_1 and A_2; their relative phase, upon the values of Φ_1 and Φ_2.

The nature of these oscillations will be more clearly understood by inspection of the typical curves in Figs. 112, 113, and 114. Figure 112 represents the value of the primary current at each point of time as it changes from an initial value I' to a

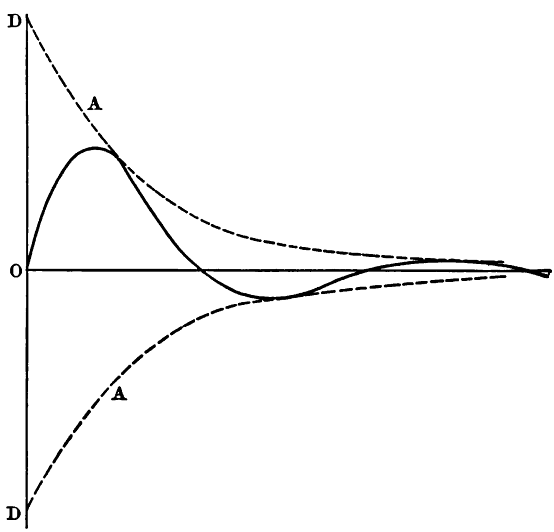


Fig. 114.

final value I. The distance BC represents the difference between the initial and final currents. The curve lies between two logarithmic envelopes AA, the initial value of which is equal to A_1 in equation (71), the rate of decay depending upon p. The relation of I' and I to the origin is immaterial; either may be the greater or may be zero. If the initial value I' is zero, the origin should be moved from O to B. If the initial value is not zero, and the final value is zero, the oscillations are as shown in Fig. 113. Where both initial and final values are zero, the curve in Fig. 114 shows the instantaneous values of the primary current.

The oscillations in the secondary circuit are similar in general character to those in the primary. The nature of the oscillations in the secondary current, as it changes from an initial value OB to a final steady value, is shown by Fig. 113. Figure 114 shows the same with the secondary current initially zero.

The foregoing explanation of the oscillations caused by make or break, when the secondary circuit contains a condenser, is simply the interpretation of equations (62) and (63). If the condenser be placed in the primary circuit instead of the secondary, the phenomena will in many respects be the same as those just described, inasmuch as the equations (64) and (65) for the currents in this case are similar to (62) and (63) for the secondary condenser, and a detailed discussion is accordingly unnecessary.

CHAPTER XII

THE GENERAL ALTERNATING CURRENT TRANSFORMER

The General Transformer. — For the discussion of the general alternating current transformer, we consider the case of a transformer with a condenser in the primary, and a condenser in the secondary circuit, as in Fig. 109. The electromotive force impressed upon the primary may be represented by a sine function of the time. The values of the primary and secondary currents may be readily obtained from the differential equations (16) and (19) in the preceding chapter, by assuming the electromotive force to have a value $e_1 = E_1 \sin \omega t$. From the solution thus obtained, the magnitudes and phase relations of the various quantities may be obtained for the general alternating current transformer. The action of such a transformer may be determined graphically, and these same relations obtained from the general transformer diagrams which are given later in the chapter.

The Primary Current. — To obtain the primary current, substitute the following values in equation (16) of the previous chapter:

$$f(t) = E_1 \sin \omega t; \qquad f''(t) = -E_1\omega^2 \sin \omega t;$$
$$f'(t) = E_1\omega \cos \omega t; \qquad f'''(t) = -E_1\omega^3 \cos \omega t.$$

The expression for primary current is accordingly found to be

$$i_1 = \frac{\frac{E_1\omega}{C_2}\cos\omega t - R_2E_1\omega^2 \sin\omega t - L_2E_1\omega^3\cos\omega t}{(L_1L_2 - M^2)D^4 + (R_1L_2 + R_2L_1)D^3 + \left(\frac{L_1}{C_2} + \frac{L_2}{C_1} + R_1R_2\right)D^2 + \left(\frac{R_1}{C_2} + \frac{R_2}{C_1}\right)D + \frac{1}{C_1C_2}}. \qquad (1)$$

Now $D \sin \omega t = \omega \cos \omega t$, and $D^2 \sin \omega t = -\omega^2 \sin \omega t$; whence $D^2 = -\omega^2$, and $D^4 = \omega^4$.

The numerator in (1) may be written as one term by the trigonometric formula,

$$A \sin \theta + B \cos \theta = \sqrt{A^2 + B^2} \sin\left(\theta + \text{arc} \tan \frac{B}{A}\right). \quad (2)$$

Combining the numerator in this manner, substituting $-\omega^2$ and ω^4 for D^2 and D^4, and multiplying by D, we may write (1) thus:

$$i_1 = \frac{DE_1 \sqrt{R_2^2 + \left(\frac{1}{C_2\omega} - L_2\omega\right)^2} \sin\left\{\omega t - \text{arc} \tan\left(\frac{1}{C_2 R_2 \omega} - \frac{L_2\omega}{R_2}\right)\right\}}{\omega b - aD}. \quad (3)$$

Here a and b are abbreviations which, in this general case, have the values

$$a = -\omega^2(L_1 L_2 - M^2) + \left(\frac{L_1}{C_2} + \frac{L_2}{C_1} + R_1 R_2\right) - \frac{1}{C_1 C_2 \omega^2}, \quad (4)$$

$$b = \omega(R_1 L_2 + R_2 L_1) - \left(\frac{R_1}{C_2\omega} + \frac{R_2}{C_1\omega}\right); \quad (5)$$

or

$$a = R_1 R_2 - K_1 K_2 + M^2\omega^2, \quad (4a)$$

$$b = R_1 K_2 + R_2 K_1. \quad (5a)$$

In the simple transformer, with condensers omitted, a and b reduce to the simpler forms α and β, given in equations (12) and (13), Chapter VI.

To free (3) from the operator D, operate upon the numerator as indicated, and multiply numerator and denominator by $\omega b + aD$. Substitute $-\omega^2$ for D^2, and again perform the operation D in the numerator. Having now become rid of the operator D, we have the required integral, which expresses the value of the primary current at any time. Thus:

$$i_1 = I_1 \sin(\omega t + \phi_1), \quad (6)$$

where

$$I_1 = \frac{E_1 J_2}{\sqrt{a^2 + b^2}}, \quad (7)$$

and $$\phi_1 = -\text{arc}\tan\frac{b}{a} + \text{arc}\tan\frac{K_2}{R_2}, \quad (8)$$

$$\tan\phi_1 = \frac{-\frac{b}{a}+\frac{K_2}{R_2}}{1+\frac{b}{a}\cdot\frac{K_2}{R_2}} = -\frac{bR_2 - aK_2}{aR_2 + bK_2}. \quad (9)$$

Here ϕ_1' is the angle between the primary current and the impressed electromotive force. For the method of reduction, see equation (22), Chapter VI.

It is to be noted that

$$\text{reactance} \equiv K = L\omega - \frac{1}{C\omega};$$

$$\text{impedance} \equiv J = \sqrt{R^2 + K^2};$$

$$\text{apparent impedance} \equiv J_1' = \sqrt{R_1'^2 + K_1'^2}.$$

The Secondary Current. — The secondary current is similarly obtained by referring to the general equation (19) of the preceding chapter, and substituting $f'''(t) = -E_1\omega^3\cos\omega t$; $D^2 = -\omega^2$; $D^4 = \omega^4$. Making these substitutions, and multiplying numerator and denominator by D, we obtain

$$i_2 = \frac{DME_1\cos\omega t}{\omega(\omega b - aD)}. \quad (10)$$

Operating upon the numerator by D, then multiplying both numerator and denominator by $\omega b + aD$, and again operating as indicated, we free the equation from the operator D, and obtain the final integral

$$i_2 = I_2\sin\left(\omega t - 90° - \text{arc}\tan\frac{b}{a}\right), \quad (11)$$

where a and b stand for the expressions given in (4) and (5), and

$$I_2 = \frac{ME\omega}{\sqrt{a^2+b^2}} = \frac{M\omega I_1}{J_2}. \quad (12)$$

These equations just obtained for the primary and secondary currents are the general equations for a transformer, subjected to an harmonic impressed electromotive force, when we have a condenser in each circuit, and make no assumption as to the absence of magnetic leakage.

Discussion of Results. — The value of ϕ_1, the angle between the primary current and electromotive force, is found by substituting in equation (9) the values for *a* and *b* given in (4 a) and (5 a). We thus have

$$\tan \phi_1 = -\frac{R_2^2 K_1 + K_1 K_2^2 - M^2\omega^2 K_2}{R_1 R_2^2 + R_1 K_2^2 + M^2\omega^2 R_2}. \tag{13}$$

By rearrangement * this may be written in the following useful form,

$$\tan \phi_1 = -\frac{K_1 - \gamma_2^2 K_2}{R_1 + \gamma_2^2 R_2} = -\frac{K_1'}{R_1'}. \tag{14}$$

The primary current may be written †

$$I_1 = \frac{E_1}{\sqrt{(R_1 + \gamma_2^2 R_2)^2 + (K_1 - \gamma_2^2 K_2)^2}} = \frac{E_1}{J_1'}, \tag{15}$$

where
$$\gamma_2 = \frac{M\omega}{J_2} = \frac{I_2}{I_1}.$$

We thus have for the primary circuit:

$$\text{apparent resistance} = R_1' = R_1 + \gamma_2^2 R_2;$$
$$\text{apparent reactance} = K_1' = K_1 - \gamma_2^2 K_2;$$
$$\text{apparent impedance} = J_1' = \sqrt{R_1'^2 + K_1'^2}.$$

The apparent primary reactance equals the real reactance diminished by a multiple of the secondary reactance. The apparent primary resistance equals the real resistance increased by the same multiple of the secondary resistance. This multiple, γ_2^2, is the square of the ratio of secondary and primary currents.

The angular relation between the primary and secondary currents is seen by a comparison of (6) and (11). The second-

* The intermediate steps are similar to those in equation (22 a), Chapter VI.

† See equation (16 a), Chapter VI.

ary current lags behind the primary by an angle of 90° plus an angle θ_2, whose tangent is $\frac{1}{C_2R_2\omega}-\frac{L_2\omega}{R_2}$. If there is no condenser in the secondary, this lag is $-90°-\arctan\frac{L_2\omega}{R_2}$, which is in accordance with the simple transformer diagram, Chapter V. See also equation (31), Chapter VI. A condenser in the secondary might reduce this angle to 90° or make it even less, as is seen from the equations, and will be shown later graphically.

Make and Break with Harmonic Electromotive Force. — In the discussion of the current flow in the primary and secondary of a transformer subjected to an harmonic electromotive force, the exponential terms which constitute the complementary function have been omitted from the equations, and the currents are simple harmonic functions of the time. These exponential terms modify the current for a short time after the make, but their effects rapidly diminish, and become negligible after a fraction of a second. The exponential terms have been discussed in the preceding chapter, and the effects there described are to be superimposed upon the simple harmonic flow of current which would take place if they were not present. Whether these terms are oscillatory or not depends, as before, upon the relation between the various constants of the circuits. When the complementary function is oscillatory, the resultant current for a short time oscillates about its final sinusoidal form, its form depending upon the relation between the period of the impressed electromotive force and the natural period of the circuit, and upon the time of introduction of the electromotive force. The periods may be such that distinct beats are obtained. These oscillations are of the same nature as those which occur after the closing of a single circuit * containing resistance, self-induction, and capacity.

* For an experimental study of these phenomena in a single circuit, see Hotchkiss and Millis, *The Physical Review*, Vol. III., p. 49; also F. E. Millis, *ibid.*, Vol. III., p. 351.

Current Flow in a Single Circuit. — The equations for a single circuit containing resistance, self-induction, and capacity are directly derivable from those for a transformer by assuming that the secondary is removed. To find the values for a and b for this case, take out from (4) and (5) the factor R_1R_2, and let $\frac{L_2\omega}{R_2}=0$; $\frac{M\omega}{R_2}=0$; $\frac{1}{C_2R_2\omega}=0$. For the primary circuit alone, we then obtain the values

$$a=R_1R_2; \qquad b=\left(\frac{L_1\omega}{R_1}-\frac{1}{C_1R_1\omega}\right)R_1R_2.$$

Substituting these values in (6), the expression for primary current becomes

$$i_1=\frac{E_1}{\sqrt{R_1^2+\left(\frac{1}{C_1\omega}-L_1\omega\right)^2}}\sin\left\{\omega t+\text{arc}\tan\left(\frac{1}{C_1R_1\omega}-\frac{L_1\omega}{R_1}\right)\right\}.$$

This expression may likewise be obtained by letting $\gamma_2=0$ in equations (6) and (15). This gives the value for the current at any time in a simple circuit containing resistance, self-induction, and capacity when subjected to an harmonic electromotive force, derived in Chapter IV., equation (24).

The General Transformer Diagram. — The general transformer diagram may be constructed in the same way as the simple transformer diagram, discussed in Chapter V., by introducing the action of a condenser as shown in Figs. 55 and 56 in Chapter IV.

In Fig. 115, let OA$=I_1$ represent the primary current. OB$=M\omega I_1$, drawn 90° behind OA, represents the electromotive force induced in the secondary by the primary current, the instantaneous value of this electromotive force being $M\frac{di_1}{dt}$.

The secondary current OC lags behind OB by an angle θ_2, whose tangent is $K_2\div R_2$. The primary impressed electromotive force E_1 is the sum of three components: OH$=R_1I_1$, to overcome primary resistance; HT$=K_1I_1$, to overcome the reactance of the primary circuit; and TF $= M\omega I_2$, to overcome the back electromotive force induced in the primary by the secondary current. These components are in the direction of the primary current, and at right angles to the primary and secondary currents respectively. When iron is present, R_1 is strictly a power coefficient, hysteretic and ohmic; OH is the

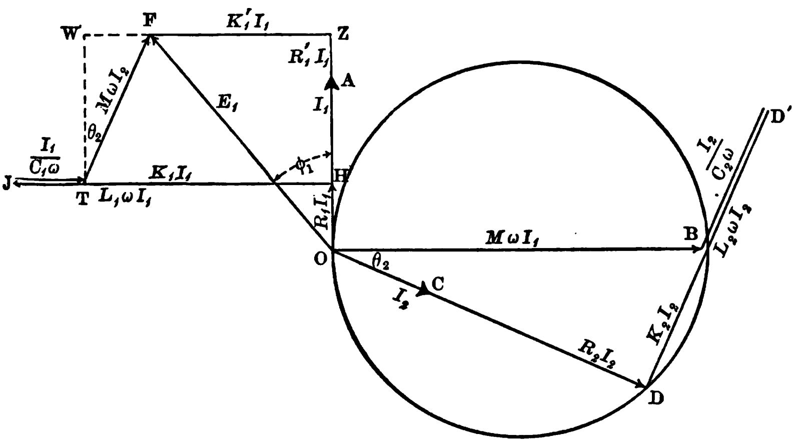


Fig. 115. General transformer diagram: θ_2, lag; ϕ_1, lag.

power electromotive force for all expenditure of power excepting in the secondary. Eddy currents and hysteresis are thus taken into consideration in so far as they occasion an expenditure of energy. Hysteresis would likewise cause a lagging of magnetization behind the magnetizing current, on account of which OB would be more than 90° behind OA.

The vector sum of the active electromotive force OH and the reactive electromotive force HT determines OT, the electromotive force to overcome the impedance proper of the primary. The impressed electromotive force OF is the sum of OT and the back electromotive force TF.

It is to be noted that the secondary current may lag behind OB (as in the case of the simple transformer), or it may be brought in advance of OB by the action of the secondary condenser; that is, θ_2 may be either an angle of lag or of advance.

The primary condenser introduces an electromotive force opposite to that of the primary self-induction, that is, HJ is diminished by the line JT. As the reactance of the primary condenser is increased, the point T moves towards H; in fact, it may be moved to the right of H. The primary current may

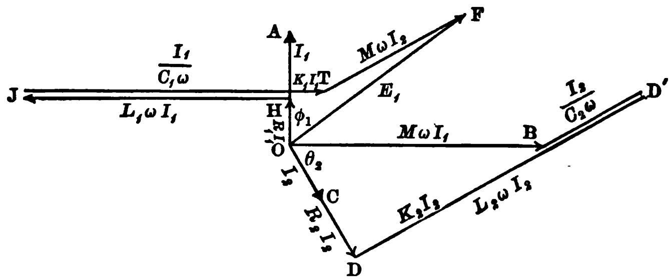


Fig. 116. General transformer diagram: θ_2, lag; ϕ_1, advance.

therefore either lag behind the primary electromotive force (as in the simple transformer) or it may be in advance of it; that is, ϕ_1 may be an angle of lag or advance. Diagrams for the general transformer may therefore be constructed for four cases, accord-

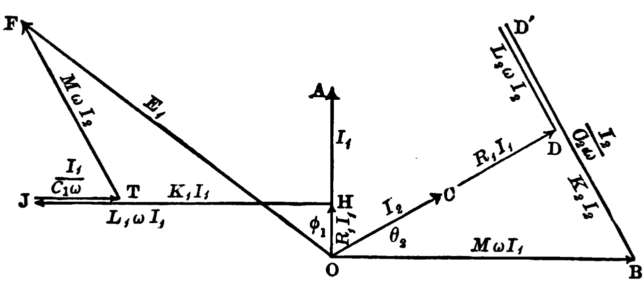


Fig. 117. General transformer diagram: θ_2, advance; ϕ_1, lag.

ing as θ_2 and ϕ_1 are angles of lag or advance. Diagrams for these four cases are given in Figs. 115, 116, 117, and 118.

The primary current may lag behind, or be in advance of the primary electromotive force by an angle which has 90° for its limit. There is consequently almost 180° variation possible in the position of the primary current with relation to the electromotive force. The angle θ_2 may approach 90° of advance or 90° of lag for its limits, and the secondary current may therefore lag between zero and 180° behind the primary current. If

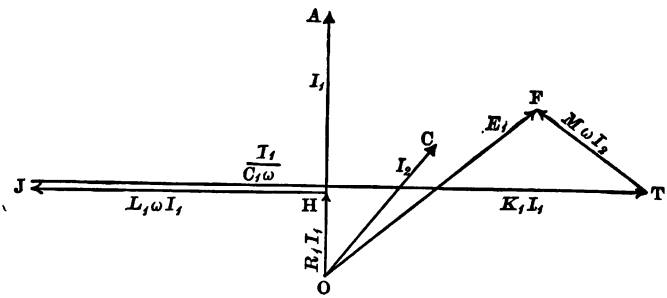


Fig. 118. General transformer diagram: θ_2, advance; ϕ_1, advance.

the limiting cases could be reached in practice, the secondary current could then have any phase position with relation to the primary electromotive force.

The Primary Electromotive Force. — The primary electromotive forces may be resolved into active and reactive components in the direction of the primary current and at right angles to it respectively.

The line TF, showing the back electromotive force due to the secondary current, may be resolved into

the active electromotive force $\mathrm{TW}=\mathrm{TF}\cos\theta_2=\left(\frac{M\omega}{J_2}\right)^2 R_2 I_1$;

the reactive electromotive force $\mathrm{WF}=\mathrm{TF}\sin\theta_2=\left(\frac{M\omega}{J_2}\right)^2 K_2 I_1$.

These are the components of the back electromotive force. Adding these to OH and to HT respectively, we obtain the total values of the active and reactive electromotive force in the primary; viz.:

$$\mathrm{OZ}=R_1 I_1+\gamma_2^2 R_2 I_1=R_1' I_1;$$

$$\mathrm{ZF}=K_1 I_1-\gamma_2^2 K_2 I_1=K_1' I_1.$$

Dividing Fig. 115 by I_1, an impedance diagram would be obtained. From the above expressions for the induced and reactive electromotive forces, the following values for the apparent resistance and reactance of the primary are thus found.

$$OZ = R_1' = R_1 + \gamma_2^2 R_2;$$

$$ZF = K_1' = K_1 - \gamma_2^2 K_2;$$

$$OF = J_1' = \sqrt{R_1'^2 + K_1'^2}.$$

Also $$\tan \phi_1 = \frac{K_1'}{R_1'};$$

$$E_1 = I_1\sqrt{R_1'^2 + K_1'^2} = I_1\sqrt{(R_1 + \gamma_2^2 R_2)^2 + (K_1 - \gamma_2^2 K_2)^2}.$$

These graphical results agree with the analytical.

Variation Diagrams. — Variation diagrams may be constructed for the general transformer similar to those constructed for the

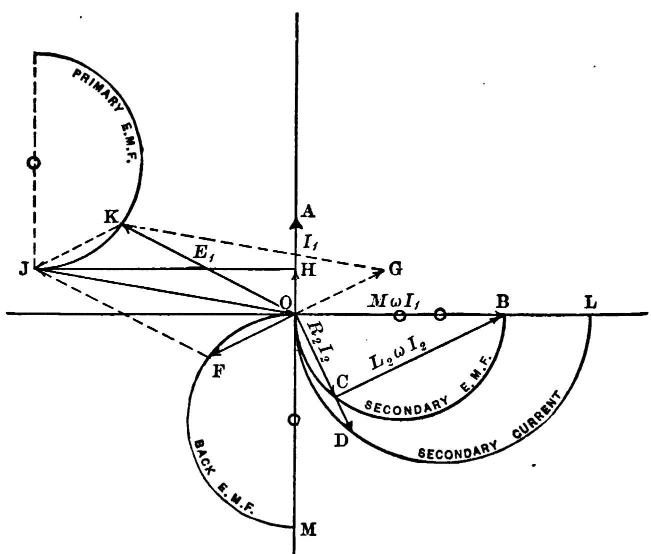


Fig. 119. Variation of the external secondary self-induction in a constant current transformer.

simple transformer in the preceding chapters. A variation not previously considered is shown in Fig. 119; which shows the effect of a change in the self-induction in the secondary circuit.

For changing from a constant current variation diagram to a constant potential variation diagram, the method of reciprocal vectors may be employed.

Reciprocal Vectors. — If any vector has an arc of a circle for its locus, a vector proportional to its reciprocal will have an arc of a circle for a locus. The method of reciprocal vectors,* explained below, is useful in constructing variation transformer diagrams, in which the loci are semicircles or arcs of circles.

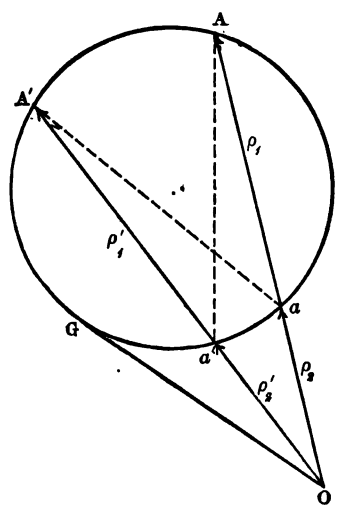


Fig. 120. Reciprocal vectors, ρ_1 and ρ_2.

In Fig. 120, let ρ_1 be any vector from the origin O, having its locus as shown upon the arc of a circle. The vector ρ_2, drawn in the direction of ρ_1, and proportional to its reciprocal, will have its locus upon an arc of a circle, which may be shown as follows. Let ρ_1 and ρ_1' represent the vector in any two positions, OA and OA′. The intercepts O*a* and O*a*′ will represent the reciprocal vectors ρ_2 and ρ_2'; for, in the similar triangles OA′*a* and OA*a*′,

$$\rho_1 : \rho_1' :: \rho_2' : \rho_2.$$

Hence $$\rho_1'\rho_2' = \rho_1\rho_2 = \text{a constant.}$$

The value of this constant product of ρ_1 and ρ_2 is $\overline{OG}^2$.

* Admittance and Impedance Loci; Bedell, The Physical Society of London, June 26, 1896.

By a suitable selection of scale, the loci of ρ_1 and its reciprocal ρ_2 may be represented as arcs of the same circle, as in Fig. 120; or they may be represented by different circles, as in Fig. 121. In the latter case, $\rho_1 \rho_2 = OG_1 \cdot OG_2 =$ a constant.

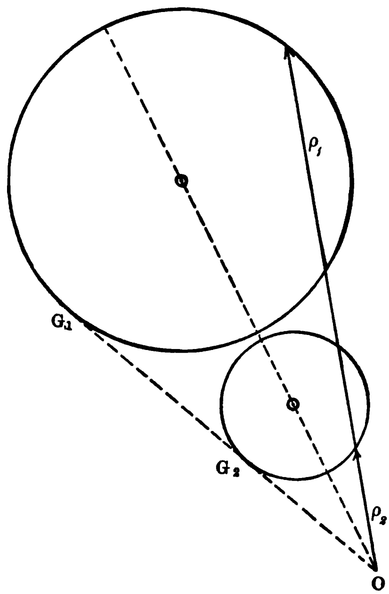


Fig. 121. Reciprocal vectors.

As the origin O approaches the circle which represents the locus of ρ_1, the center of the reciprocal circle becomes more distant, and its radius becomes greater. When the origin O is a point in the circumference of the first circle, the center of the reciprocal circle is at an infinite distance; that is, the reciprocal locus is a straight line.

These principles may be usefully applied to transformer diagrams. In many cases the locus of the primary impedance, as some particular quantity is varied, is a portion of a circle. This has been shown to be the case (see Fig 81) when the secondary resistance is varied. Since the admittance of the primary is the reciprocal of its impedance, the admittance may be represented by the vector ρ_2 in the above construction, if the impedance is represented by ρ_1. These loci may be drawn to scale for actual values.

In a constant current transformer, the primary electromotive force varies directly as the primary impedance. In a constant potential transformer the primary current varies directly as the primary admittance. But the admittance is the reciprocal of the impedance; hence, if we have an arc of a circle for the locus of the primary electromotive force when the primary current is maintained constant, we may employ the above method to obtain the arc of a circle which will be the locus for the primary current when the transformer is supplied with a constant electromotive force. The converse operation may likewise be performed.

In Fig. 122, let the circle C_1 represent the locus of the primary electromotive force E_1 during some particular change of

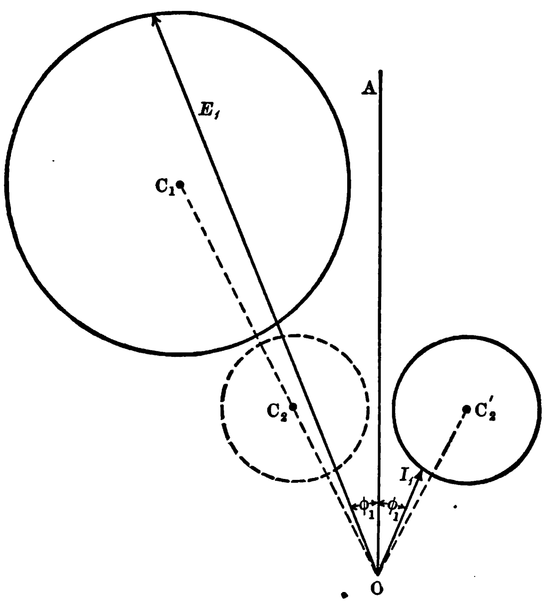


Fig. 122.

condition, the primary current meanwhile being maintained constant and in the direction OA. The difference in phase between the current and electromotive force is the angle ϕ_1. The locus of the primary current under the same change of conditions, if the primary electromotive force is maintained constant, is the dotted circle C_2, which is reciprocal to C_1. If the constant electromotive force is drawn in the direction OA, the locus of the primary current is the circle C_2', drawn so that the angles AOC_1 and AOC_2' are equal.

An application of the method of reciprocal vectors is shown in Fig. 123. The semicircle JKN represents the locus of the primary electromotive force of a transformer, when the primary current is constant and is in the direction of OA, and the secondary resistance is varied. [Compare Fig. 81.]

The electromotive force has the position OJ on open circuit, and ON on short circuit. It is to be noted that the line NH represents the effects due to magnetic leakage. It is desired to

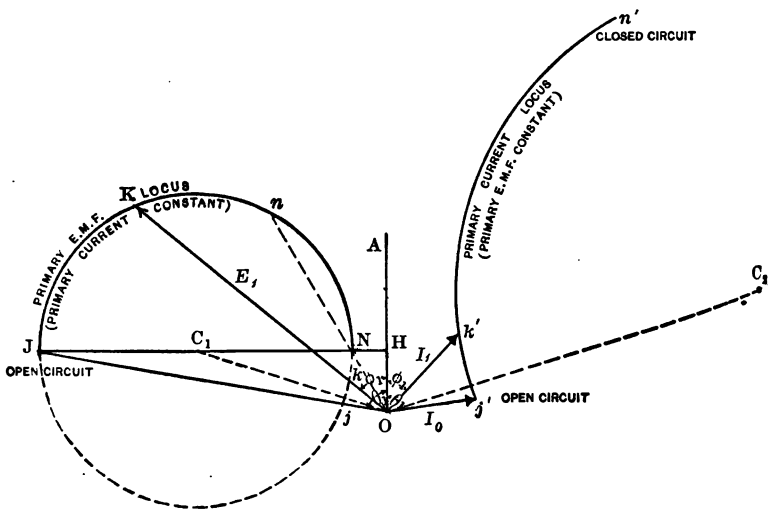


Fig. 123. Method of obtaining primary current locus by the principle of reciprocal vectors.

find the locus for the primary current when the primary electromotive force has a constant value, and is drawn in the direction OA. The angle of lag θ_1, between the primary electromotive force and current on open circuit, is JOH. Accordingly, with a constant electromotive force in the direction OA, the open circuit current I_0, is laid off lagging behind the electromotive force at an angle of $AOj' = \theta_1 = JOH$. The open circuit current I_0 may be laid off to any convenient scale. To construct the locus for the primary current, proceed as follows: Lay off the line OC_2 so that the angles AOC_1 and AOC_2 are equal. The point

C_2 is located so that $OC_2 : OC_1 :: Oj' : Oj$. The primary current locus is then drawn as the arc of a circle with C_2 as a center, passing through j'.

The limits of the primary electromotive force locus are the points J and N. The corresponding limits of the primary current locus are the points j' and n'. It will be noted that these points correspond to the points j and n on the circle C_1, which are reciprocal to the points J and N.

In the absence of magnetic leakage, the points N and H coincide. The point n' would then lie in the line OA. The deviation of the primary current locus from the line OA is produced by magnetic leakage. Compare Fig. 92, constructed for no magnetic leakage.

An experimental curve showing the primary current locus for a constant potential transformer as affected by magnetic leakage is shown in Fig. 124.

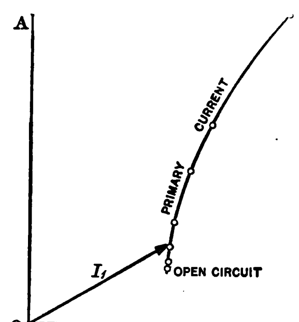


Fig. 124. Primary current locus for constant potential transformer; determined experimentally.

The method of reciprocal points is well illustrated by the following discussion of resonance in transformer circuits.

Resonance in Transformer Circuits. — The reactive action of a condenser in an alternating current circuit produces effects classed under the head of electrical resonance. The resultant phenomena under different conditions of operations are many and various; but under whatever particular sets of conditions the condenser is used, the effects produced are closely correlated and may be grouped together in this one class. In a single circuit resonance is obtained when, inductance and capacity being present, the resultant reactance is zero; that is, the reactance of the condenser is exactly equal and opposite to the reactance due to self-induction. The natural period of the circuit then equals the period of the impressed electromotive force.

Complete resonance is thus obtained only when the impressed electromotive force varies harmonically.

Effects corresponding to resonance in a single circuit may be produced by the reactive effect of one circuit upon another, occasioned by the mutual induction of the two circuits. Such effects arise in the case of a transformer containing a condenser in the secondary circuit.

Resonance, in this case, occurs when the relation between the primary and secondary is such that the apparent reactance of primary circuit is zero. The elastic influence of the condenser is transferred from one circuit to the other, on account of their mutual relationship, and the natural period of the primary circuit depends not only upon the value of its own constants, but upon those of the secondary as well. There is a surging of energy back and forth between the primary circuit and the secondary condenser, by intervention of their common magnetic field: the period of these surgings determines the period of the system. A condenser in either primary or secondary circuit may give rise to such oscillations.

In the general sense of resonance, the case here described would be a particular case, but for distinction we may apply the term "simple resonance" to the phenomena observed in a single circuit, the term "consonance," suggested* by Dr. Pupin, being used when the resonant action is transferred from one circuit to another by the mutually inductive influence between them.

The conditions necessary for consonance, although not so simple as those for simple resonance, are definite, and may be readily determined.

Problems of this nature are usually capable of two modes of treatment, leading to identical results: first, the general analytical treatment; second, the synthetical graphical treatment. Which to use is a matter of preference. These methods are

* "Electrical Consonance," M. I. Pupin, *Electrical World*, Feb. 9, 1895; see also paper by C. P. Steinmetz, *ibid.*, March 2, 1895.

well illustrated in the present instance. In the study of resonance and consonance the graphical method is the more direct. It is replete with obvious conclusions, and at once brings out relations not simply shown by analytics.

The analytical treatment is at once obtained by direct application of the formulæ derived above for the general transformer.

The conditions for consonance are readily obtained by noting the conditions under which the apparent reactance of the primary is zero; these will be discussed after the graphical treatment of the problem.

Graphical Treatment. — In the graphical treatment of resonant phenomena,* as of many alternating current problems, there are four correlated methods of representation. We may plot *electromotive forces*, as in Fig. 125; *impedances*, as in Fig. 126, obtained by dividing our electromotive force diagram by current; *admittances*, as in Fig. 127, the reciprocal of our impedance diagram; and *currents*, as in Fig. 128, obtained by multiplying the admittance diagram by an electromotive force. Figures 127 and 128 are reciprocal to Figs. 125 and 126. There is a close relation between all of these diagrams. Figure 125 is the general transformer diagram, constructed as Figs. 115, 116, 117, and 118. Our particular study is to determine the action of the condenser in the secondary circuit.

Under the assumption that I_1 is constant, evidently OB is constant. As the secondary capacity or reactance is varied, the point D moves on the circle ODB, and the point F moves similarly on the circle TQFPG. This determines the phase of the primary electromotive force OF, which may be either in advance of the current or behind it, being in phase with it at two points P and Q. In the case represented in the figure, the electromotive force is behind the current.

The diameter of this primary circle is the value of TF, when in the position TG, which occurs when $K_2 = 0$. The secondary

* "Resonance in Transformer Circuits," Bedell and Crehore, *The Physical Review*, Vol. II., p. 442.

current is then in the direction of OB, and we have $I_2 = M\omega I_1 \div R_2$. The diameter of the primary circle is then

$$M\omega I_2 = M\omega \frac{M\omega I_1}{R_2}.$$

With the secondary condenser removed, the point F would move to some point, as Y, and the point D to a corresponding

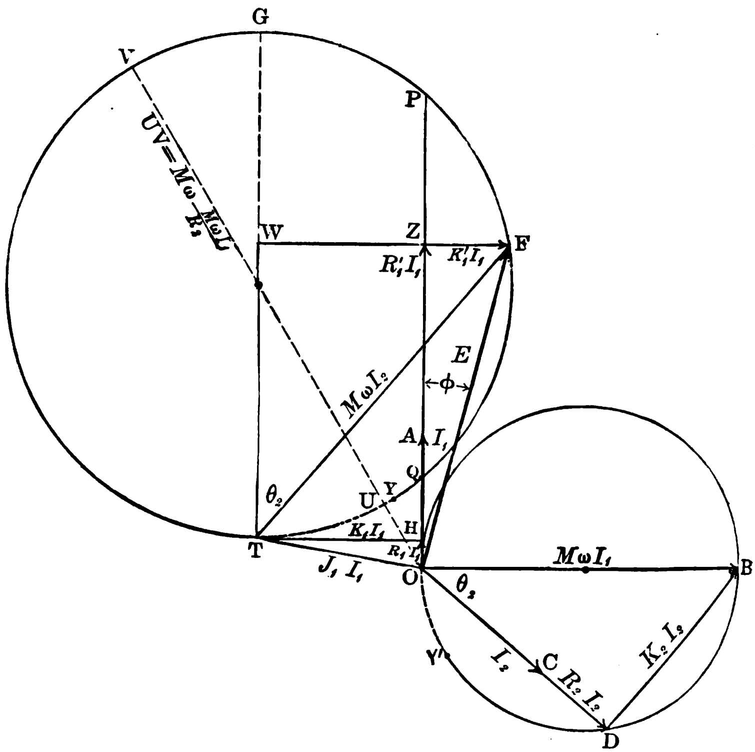


Fig. 125. Electromotive force diagram : capacity of secondary condenser varied.

point Y′, depending upon the resistance in the secondary circuit, and upon the amount of magnetic leakage, a small amount of which is here assumed present. The limiting positions, therefore, of the point F are Y and T. As TY would be the position of TF in the absence of the secondary condenser, it follows that a line drawn from Y to F in any position shows the component of the primary electromotive force due to the secondary condenser. The point F could not lie on the dotted portion of the curve.

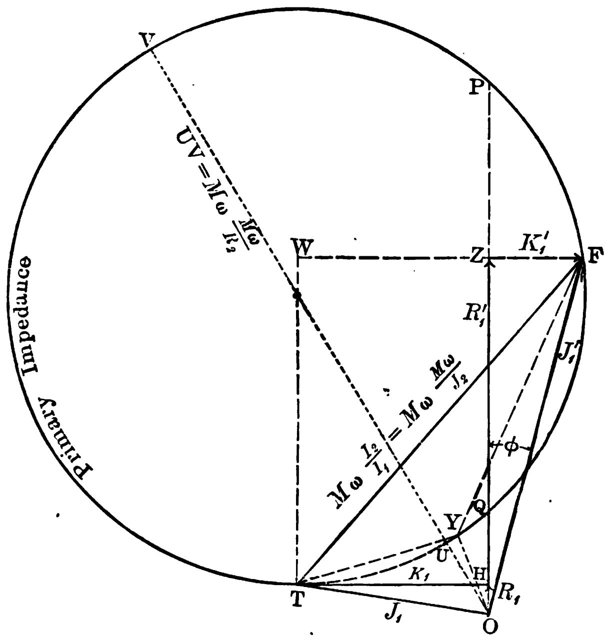


Fig. 126. Impedance diagram (obtained by dividing electromotive force diagram, Fig. 125, by I_1).

If we divide the electromotive force diagram (Fig. 125) by the value of the primary current for which it was constructed, we obtain (Fig. 126) an impedance diagram, independent of current or electromotive force. Here OH represents ohmic resistance [or a power coefficient where other losses are involved]; HT is the primary reactance; OT is the primary impedance. The apparent primary impedance OF is the sum of the impedance OT and of TF, an impedance component due to the reaction of the secondary; of this, YF is due to the presence of the secondary condenser.

From Fig. 126 we have

$$OZ = R_1' = R_1 + \gamma_2^2 R_2;$$
$$ZF = K_1' = K_1 - \gamma_2^2 K_2;$$
$$OF = J_1' = \sqrt{R_1'^2 + K_1'^2}.$$

From Fig. 125 we also have

$$E = I_1\sqrt{R_1'^2 + K_1'^2} = I_1\sqrt{(R_1 + \gamma_2^2 R_2)^2 + (K_1 - \gamma_2^2 K_2)^2}.$$

In Fig. 126, for the maximum and minimum values of the apparent impedance, we have OV and OU; for consonance we have the values OQ and OP. We may note that, while for consonance the secondary condenser decreases the apparent reactance of the primary to zero, it gives rise to a corresponding increase in the apparent resistance.

If, in Fig. 126, OF is a vector representing the apparent impedance of the primary, we may obtain the value of the admittance by taking the reciprocal of OF. Now OF is a vector drawn from the origin O, and lies on the circle TQFPV as the secondary reactance is changed. By the principle of reciprocal vectors it follows that, if the locus of the impedance vector is a circle, the locus of the admittance vector (its reciprocal) will likewise be a circle. In Fig. 127, *of* is a vector representing admittance, being always equal to the reciprocal of OF, Fig. 126. The circle *vpfqu*, in Fig. 127, is reciprocal to the circle VPFQU in Fig. 126. We

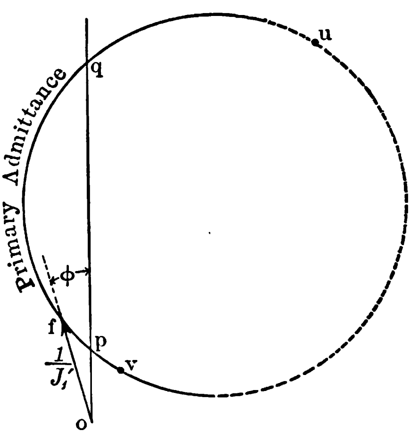


Fig. 127. Admittance diagram (the reciprocal of the impedance diagram).

may note the reciprocal points *u*, *v*, *p*, *q*, and U, V, P, Q. The points of minimum and maximum impedance U and V, correspond to the points of maximum and minimum admittance *u* and v. The admittance diagram is evidently independent of current or electromotive force.

Let us now assume an alternating electromotive force E_1, constant in value, supplied to the primary circuit. Multiplying our admittance diagram [which for simplicity is not complete] by E_1, we obtain the current diagram, Fig. 128. Here *oh* is the power current in the direction of the electromotive force; *ht* is the reactive current, and represents no work. The resultant is *ot*, the open-circuit current, which is denoted by I_0. It is to be observed that *oh* is the current representing the power expended in all ways other than in the secondary circuit. The actual current I_1, which flows in the primary when the secondary is closed, is the resultant of the open-circuit current *ot* and the component *tf*, due to the reaction of the secondary, which is equal to the secondary current multiplied by the ratio of secondary to primary turns ($S_2 \div S_1$).

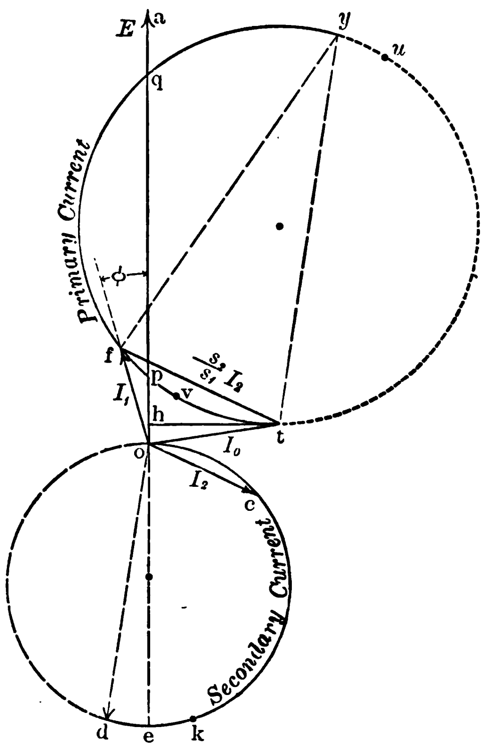


Fig. 128. Current diagram (admittance diagram multiplied by primary electromotive force E).

As the secondary reactance is varied, the primary current changes in magnitude and direction according to the reaction of the secondary, and lies upon the circle, as shown, as a locus.

The dotted portion of the circle corresponds to the dotted portion in the previous figures.

The points of consonance, *p* and *q*, correspond to P and Q above. If consonance can be obtained at all, there are two values of the primary current for which it may be obtained.

The secondary current I_2 varies proportionally to *tf*, and is drawn as *oc*, with its locus on the circle *c*, *k*, *e*, *d*. In the present case (which assumes a certain amount of magnetic leakage), the secondary current would take the position *od*, if the secondary condenser were removed from the circuit. It could only move on to the dotted part of the curve in case self-induction were added to the secondary circuit, — or, what is the same thing, the magnetic leak increased. We thus see the action of the condenser and of magnetic leak are opposed, and that magnetic leak may be thus compensated for. In fact, the secondary circuit may be given a capacity which will overcompensate for magnetic leakage; as a constant potential transformer is loaded the secondary electromotive may rise, despite the magnetic-leakage drop and the fall in potential due to ohmic resistance. In the absence of condenser, magnetic leak, and open-circuit losses (that is, if *oh* were zero), the secondary current would be in the direction *oe*. Losses on open-circuit, eddy currents, hysteresis, etc., give to the open-circuit current the power component *oh*, and in the absence of magnetic leak the secondary current would assume the position *ok*.

In the present instance the points of minimum impedance U and maximum current *u* are on the dotted portion of their respective circles, and so could not be actually obtained. This, however, is not necessarily so. By the principle of reciprocal points, it follows that the product of the maximum and minimum currents is equal to the product of the two consonant values of current; that is,

$$ou \times ov = op \times oq.$$

We may best examine the conditions of consonance by returning to Fig. 125. When the primary circle does not intersect

OP, there can be no consonance; analytically we would obtain imaginary values. If there is one condition for consonance, there must be two, corresponding to the points P and Q, except in the limiting case when the circle is tangent to OP. Then HT would equal the radius of the circle, and

$$K_1 I_1 = \frac{M\omega}{2} \frac{M\omega I_1}{R_2}.$$

For consonance to be possible, we must therefore have the relation

$$M^2\omega^2 > 2 K_1 R_2.$$

The points of consonance are determined by equating the value of primary apparent reactance to zero; thus

$$K_1' = K_1 - \gamma_2^2 K_2 = 0;$$

or

$$\frac{K_1}{K_2} = \frac{M^2\omega^2}{R_2^2 + K_2^2}.$$

Two values of K_2 will satisfy this condition. Frequency is an important factor in this criterion. The conditions for consonance are obtained when OD is perpendicular to TQ or to TP.

A particular case, which may be termed "pure consonance," is obtained when the natural period of each circuit independently is equal to the period of the impressed electromotive force. The period of the system in this case is evidently the same as the period of either circuit. We have K_1 and K_2 each equal to zero. The point T coincides with H, and OD takes the position OB.

Although it is readily shown that the loci of primary impedance and admittance are circles, as the secondary reactance is changed, the conditions are not easily obtainable for a wide range of variation. The diagrams here drawn are for an assumed case, and the secondary capacity is varied from zero to infinity. Ordinarily the range of variation would be small.

In addition to the effect of consonance, the action of a condenser to counterbalance leakage has been pointed out above. Attention may be called to another effect which may be produced by means of a properly proportioned condenser in the secondary circuit. If the secondary reactance be made zero (inside and out of the transformer), simple secondary resonance is obtained, and a constant primary current will transform to a constant secondary electromotive force.

Experimental Observations of Transformer Resonance. — The following experiments illustrate the effect of capacity in the secondary of a transformer. In these experiments resonance

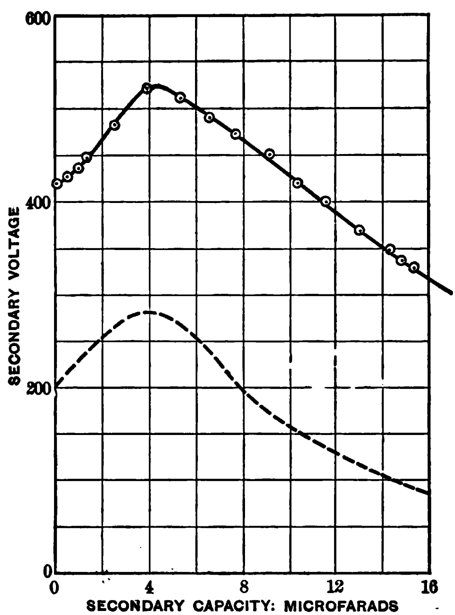


Fig. 129. Resonance in secondary circuit.

was produced by a condenser in the secondary, and this resonant effect was transferred to the primary in accordance with the foregoing theory. There was no condenser in the primary circuit. The transformer used was one of the old style Westinghouse transformers with an output of 1500 watts. The resistance of the secondary (high potential coil) was 29 ohms, and its self-induction 13·87 henrys. The transformer was used to transform up, the transformation ratio being approximately

1 : 20. It was operated from an alternator built by students at Cornell University, which was driven so as to give a frequency of 48 periods per second, in order to bring the reactance of the capacity used within the proper range. The speed of the alternator and the exciting current were kept constant; there was no resistance in the secondary circuit. Measurements were taken for different values of the capacity of the secondary condenser. Curves for the primary and secondary voltage are

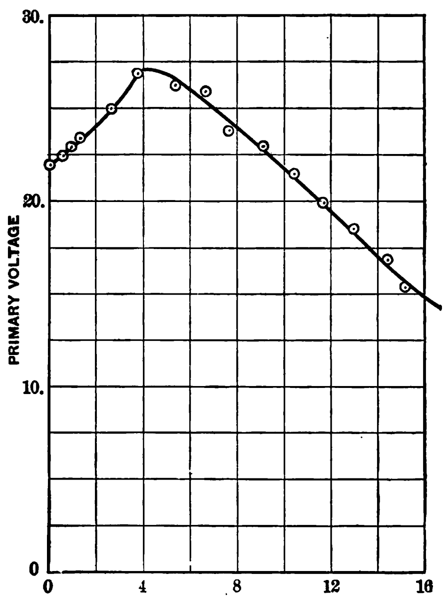


Fig. 130. Primary resonance, produced by a condenser in the secondary.

given in Figs. 129 and 130. It is seen that when the capacity in the secondary is increased, the voltage at the terminals of the secondary rises quite rapidly, reaching a maximum for a capacity of about four microfarads, and then gradually drops until at ten microfarads it is at the original value. If the capacity is still further increased, the voltage drops more and more rapidly, and would finally become zero if the capacity were made infinitely large.

It is also seen that the curve for the primary electromotive force is almost identical in form with the secondary electromotive force curve; but from a comparison of the two curves it is seen that there is a slight increase in the ratio of transforma-

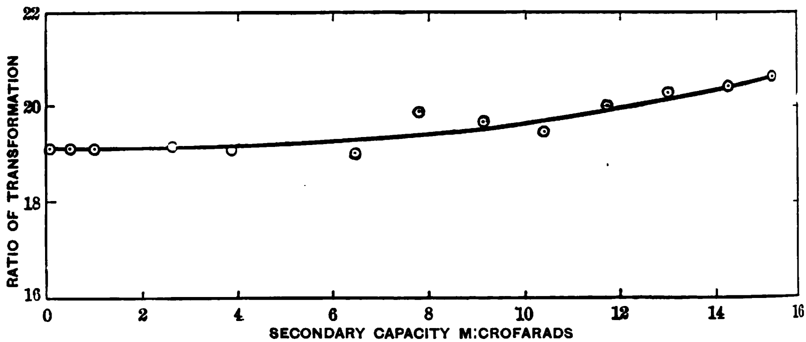


Fig. 131. Increase in the ratio of transformation due to resonance.

tion as the capacity increases. The curve for the ratio of transformation in Fig. 131 shows an increase of about 5 per cent in the range of the experiment. The curve for the primary current is shown in Fig. 132. The dotted curve in Fig. 129 is a theoretical curve, drawn to a different scale.

The increase in the ratio of transformation is due to the fact that the secondary condenser neutralizes the effect of the self-induction in the secondary, and even compensates for the drop due to magnetic leakage. When a transformer has full load, the primary and secondary currents are nearly opposite in phase, and their magnetomotive forces are consequently opposed and thus give rise to magnetic leakage. The secondary condenser advances the secondary current; accordingly its magnetizing force, instead of opposing the magnetizing force of the primary current, is in a direction to increase its effect.

The condenser always acts back through the transformer, and has the same effect upon the generator as though it were located in the primary circuit. If located in the primary (low potential) circuit, however, the capacity to produce the same effect would have to be much larger.

The resonant rise of potential in the primary is due to the influence of the condenser upon the armature reactions of the alternator, this rise being analogous to the increased ratio of transformation, caused by the influence of the secondary condenser upon the magnetic leakage of the transformer.

Concentric Cables. — A notable instance of the resonant rise of potential discussed above occurred in the case of the concentric Ferranti cables.* The capacity of the Ferranti mains is about one-third of a microfarad per mile. These mains were 15½ miles long, and took a current (condenser current so called) of 12 or 14 amperes when on open circuit, and subjected to a difference of potential of 10,000 volts, $2\pi \times$ frequency being 400. When supplied from step-up transformers, the condenser effect of these cables apparently increased the ratio of transformation of the transformers by about one-sixth, and likewise increased the generator electromotive force. On open circuit the difference of potential between the inner and outer conductors of the cable was everywhere the same; on closed circuit there was a fall of

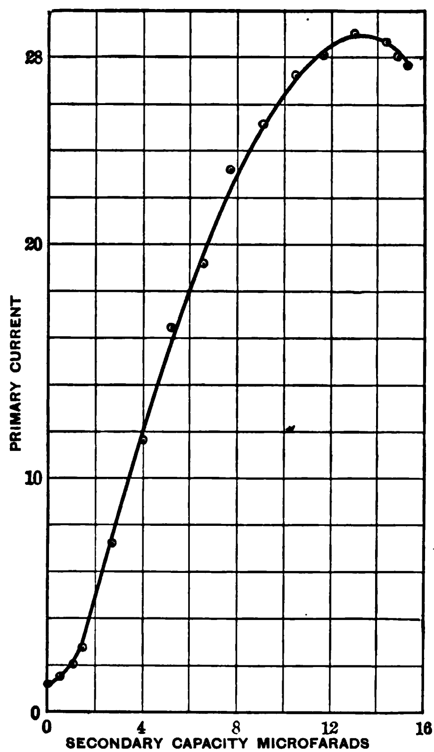


Fig. 132.

* A full discussion of these resonant phenomena is given by Dr. Fleming in the second volume of his book, "The Alternating Current Transformer," to which the reader is referred.

potential along the cable, more current entering the cable than flowed from it.

In this case, in addition to the ordinary resonant effect due to the neutralization of self-induction by capacity, two closely associated phenomena were involved; namely, the increased ratio of transformation of the transformers, and the rise of potential at the generator, as were found in the experiment described above.

These experiments are similar to some carried out at the Siemens works (see the *Alternating Current Transformer*, Vol. II., p. 395), in which the secondary condenser was connected to a cable immersed in a tank of water. The secondary capacity was changed by varying the length of the cable; simultaneous measurements were made of the primary and secondary electromotive forces. The results obtained were similar to those shown in Figs. 129, 130, and 131, but were even more marked. The secondary electromotive force varied from 2500 volts to 8500 volts.

Effect of Armature Reactions and Magnetic Leakage in the Foregoing Experiments. — The effect of armature reactions may be explained as follows. When the current and electromotive force are in phase in an alternating current generator or motor, the effects due to armature reactions are small, for the relation of the armature coils to the pole pieces is such that the armature current increases the magnetic flux in one portion of the circuit about as much as it decreases it in another. A lagging current, such as occurs in an inductive circuit, causes the current to be a maximum at a time when the armature coils are not symmetrically situated with reference to the pole pieces. In a generator, the magnetic flux is accordingly diminished, and the generator electromotive force falls off. The armature self-induction will suffice to make the current lag, and to produce this effect even when the external circuit is non-inductive.

The reverse effect is caused by a current in advance of the electromotive force. In this case the reactive effect of the armature current increases the magnetic flux; that is, the exciting current in the generator field is aided, and the electromotive force of the generator is consequently increased as the generator becomes loaded. The necessary condition for this increase of potential is an armature current in advance of the generated electromotive force, a condition which may be obtained by a condenser or by a synchronous motor * operated by the generator. A condenser or synchronous motor † may thus overcompensate for inductive drop, and for drop due to armature reactions. An increase of 10 per cent in the potential at the brushes of an alternator thus operated has been observed by the writer. A condenser or an over-excited synchronous motor may thus assist in the regulation of a constant potential system.

Divided Circuit with Mutual Induction.‡ — A divided circuit with mutual induction between the two branches is shown in Fig. 133, being the same as a transformer with the primary and secondary circuits connected in parallel. The electromotive force equations for the two circuits are similar, the internal electromotive forces in each being equal to the same impressed electromotive force. The electromotive force of mutual induction will be positive or negative according to the sense or direction in which the coils are connected. If the coils are connected as in Fig. 134, so that the ampere-turns

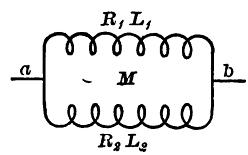


Fig. 133. Divided circuit with mutual induction.

* See "Action of a Single-Phase Synchronous Motor," Bedell and Ryan, *Journal of the Franklin Institute*, March, 1895.

† A synchronous motor does not necessarily behave as a condenser, this behavior depending upon its field excitation; in fact, below a certain value of field excitation, the effect of the motor corresponds to that of self-induction, the current lagging behind the electromotive force. As the excitation is increased, the current and electromotive force are brought nearer and nearer into phase. When in phase, the current becomes a minimum. Further increase of the excitation brings the current in advance, and the behavior of an over-excited synchronous motor is the same as that of a condenser.

‡ The division of an alternating current in parallel circuits with mutual induction; Bedell, Liverpool meeting of the British Association for the Advancement of Science, 1896.

of the two coils assist each other, the electromotive forces of self and mutual induction will be of the same sign, and the coefficient of mutual induction will be positive. If the coils are connected so as to oppose each other, as in Fig. 135, the electromotive force of mutual induction will be opposite in sign to

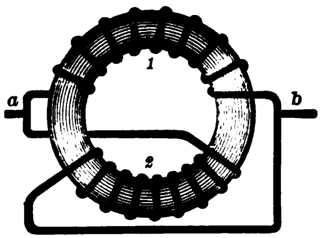


Fig. 134. Coils additive.

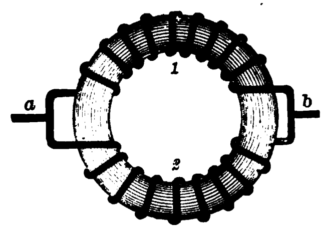


Fig. 135. Coils opposed.

that of self-induction. The coefficient of mutual induction may accordingly be $+M$ or $-M$. Writing the electromotive force as a function of the time, we have the following equations:

$$e = f(t) = R_1 i_1 + L_1 D i_1 \pm M D i_2; \qquad (16)$$

$$e = f(t) = R_2 i_2 + L_2 D i_2 \pm M D i_1. \qquad (17)$$

The elimination of i_1 or i_2 might be performed as in Chapter VI.; the elimination by determinants, as in Chapter XI., is more direct.

By differentiation, the above equations become

$$f'(t) = R_1 D i_1 + L_1 D^2 i_1 \pm M D^2 i_2;$$

$$f'(t) = R_2 D i_2 + L_2 D^2 i_2 \pm M D^2 i_1.$$

Eliminating i_2 by determinants, we have

i_2	Di_2	$D^2 i_2$	OTHER TERMS.
0	$\pm M$	0	$R_1 i_1 + L_1 D i_1 - f(t)$
0	0	$\pm M$	$R_1 D i_1 + L_1 D^2 i_1 - f'(t)$
R_2	L_2	0	$\pm M D i_1 - f(t)$
0	R_2	L_2	$\pm M D^2 i_1 - f'(t)$

$= 0.$

Canceling out the first column and third row, as indicated by fine type, we obtain

$$M^2[\pm MD^2 i_1 - f'(t)] \mp MR_2[R_1 i_1 + L_1 D i_1 - f(t)] \mp ML_2[R_1 D i_1 + L_1 D^2 i_1 - f'(t)] = 0. \quad (18)$$

From this equation, the value for the current i_1 is obtained:

$$i_1 = \frac{R_2 f(t) + (L_2 \mp M) f'(t)}{(L_1 L_2 - M^2) D^2 + (R_1 L_2 + R_2 L_1) D + R_1 R_2}. \quad (19)$$

A similar expression is found for i_2.

For an harmonic electromotive force

$$e = f(t) = E \sin \omega t;$$
$$f'(t) = E\omega \cos \omega t.$$

Since $D^2 \sin \omega t = -\omega^2 \sin \omega t$, we may write $-\omega^2$ for D^2.

In equation (19) we may make these substitutions for $f(t)$, $f'(t)$, and D^2. By combining the sine and cosine terms into one term, and multiplying numerator and denominator by D, we have

$$i_1 = \frac{DE\sqrt{R_2^2 + (L_2 \mp M)^2 \omega^2} \sin\left(\omega t + \arctan \frac{(L_2 \mp M)\omega}{R_2}\right)}{\alpha D - \omega \beta}; \quad (20)$$

where
$$\alpha = R_1 R_2 - \omega^2 (L_1 L_2 - M^2),$$
$$\beta = \omega (R_1 L_2 + R_2 L_1).$$

To free equation (20) from the operator D, perform the operation D as indicated in the numerator (by differentiating the numerator), and multiply numerator and denominator by $\alpha D + \omega\beta$. Substitute $-\omega^2$ for D^2 in the denominator and again perform the operation D in the numerator. We thus obtain an expression for i_1 which is free from D; this may be written in the following form:

$$i_1 = I_1 \sin(\omega t + \phi_1); \quad (21)$$

where
$$I_1 = \frac{E}{\dfrac{\sqrt{\alpha^2 + \beta^2}}{\sqrt{R_2^2 + (L_2 \mp M)^2 \omega^2}}}. \quad (22)$$

$$\phi_1 = -\arctan \frac{\beta}{\alpha} + \arctan \frac{(L_2 \mp M)\omega}{R_2}. \quad (23)$$

Similarly, for the current in the second branch,

$$i_2 = I_2 \sin(\omega t + \phi_2); \tag{24}$$

where

$$I_2 = \frac{E}{\dfrac{\sqrt{\alpha^2+\beta^2}}{\sqrt{R_1^2+(L_1 \mp M)^2\omega^2}}}. \tag{25}$$

$$\phi_2 = -\text{arc tan}\,\frac{\beta}{\alpha} + \text{arc tan}\,\frac{(L_1 \mp M)\omega}{R_1}. \tag{26}$$

The ratio of the two currents is

$$\frac{I_1}{I_2} = \frac{\sqrt{R_2^2+(L_2 \mp M)^2\omega^2}}{\sqrt{R_1^2+(L_1 \mp M)^2\omega^2}}. \tag{27}$$

Their phase difference is

$$\phi_1 - \phi_2 = \text{arc tan}\,\frac{(L_2 \mp M)\omega}{R_2} - \text{arc tan}\,\frac{(L_1 \mp M)\omega}{R_1}. \tag{28}$$

Where the coils are opposed and nearly similar, the angle of phase difference between the currents depends largely upon the amount of magnetic leakage.

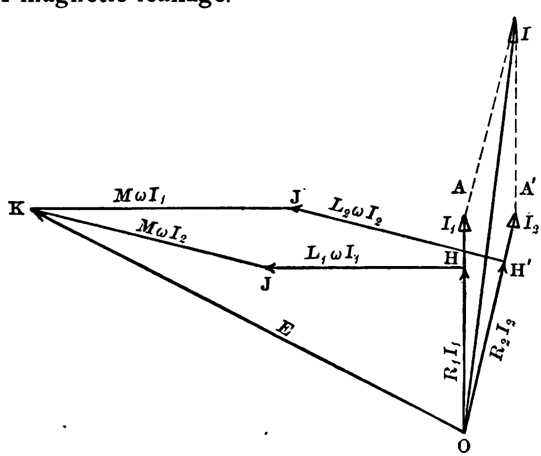


Fig. 136. Coils additive; M positive.

The diagram for the electromotive forces in a divided circuit in which the effects of self and mutual induction are additive is shown in Fig. 136. The two currents are I_1 and I_2, respectively, the main current being their vector sum I.

In branch one, OH is the power electromotive force (in the absence of branch two); HJ and JK are the electromotive forces of self and mutual induction, being at right angles to the currents I_1 and I_2, respectively. The corresponding electromotive forces for the second circuit are shown in the same diagram. The sum of the several electromotive forces in either circuit add up to E, the impressed electromotive force.

Ordinarily the branch current and the main current would lag behind the impressed electromotive force by a large angle approaching ninety degrees.

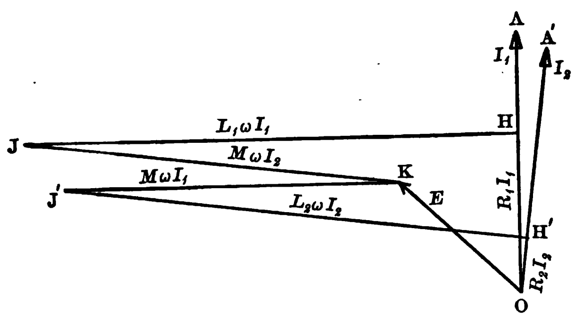


Fig. 137. Coils opposed; M negative.

When the two circuits are identical, the branch currents I_1 and I_2 would have the same magnitude and direction; the points H′ and J′ would coincide with the points H and J, respectively. An inspection of the figure will show that for this case, in the absence of magnetic leakage, the effect of the mutual induction would be to double the tangent of the angle of lag.

Figure 137 shows the corresponding diagram for the case in which the electromotive forces of self and mutual induction are opposed. The current I_1 produces an electromotive force HJ in the first branch, and J′K in the second branch; these electromotive forces being at right angles to the current I_1, and opposite to each other in direction, on account of the sense in which the two coils are connected. Similarly, H′J′ is opposite in direction to JK. The effects of self and mutual induction consequently tend to neutralize each other. The electromotive force OK, required to cause the currents I_1 and I_2 to flow, is accordingly made less on account of the mutual induction, and would be reduced to the value of the ohmic electromotive force in the case of two similar coils wound oppositely. In this case the angle of lag would be reduced to zero, and the coil would be non-inductive.

The equivalent resistance and self-induction for the two coils together, whether they are additive or opposed, may be found by resolving E into two components, one in the direction of the main current, and the other at right angles to it. These components will be equal to $R'I$ and $L'\omega I$, respectively. Their magnitudes may be obtained graphically, and from them the values of the equivalent resistance R' and the equivalent self-induction L'. The equivalent resistance and self-induction of either branch may be obtained in the same manner.

The conclusions relating to the special case of two identical coils wound without magnetic leakage, stated above in connection with the graphical construction, may be reached by substituting in the formulæ the values $R_1 = R_2 = R$; and $L_1 = L_2 = \pm M = L$. Also, $\beta \div \alpha = 2 L\omega \div R$. The two cases are as follows:

When M is positive.—When M is positive, the coils being wound in the same direction,

$$\phi = -\text{arc tan}\frac{2 L\omega}{R} + \text{arc tan } 0 = -\text{arc tan}\frac{2 L\omega}{R}. \qquad (29)$$

The current in each branch is

$$I_1 = I_2 = \frac{E}{\sqrt{R^2 + (2 L\omega)^2}}. \qquad (30)$$

For this case, therefore, the effect produced is the same as though the mutual induction were zero and the self-induction of each circuit doubled, the tangent of the angle of lag being doubled and each current being diminished accordingly.

The main current is

$$I = I_1 + I_2 = \frac{2E}{\sqrt{R^2 + (2L\omega)^2}} = \frac{E}{\sqrt{\left(\frac{R}{2}\right)^2 + L^2\omega^2}}, \qquad (31)$$

which is the same that would flow in a single coil, with the same self-induction and half the resistance as either branch.

When M is negative. — When the coils are identical and opposed,

$$\phi = -\text{arc tan}\frac{2L\omega}{R} + \text{arc tan}\frac{2L\omega}{R} = 0; \qquad (32)$$

$$I_1 = I_2 = \frac{E}{R}. \qquad (33)$$

Each coil acts in this case as though non-inductive, the magnitude of the current being in accordance with Ohm's law, and the angle of lag being zero.

Cases where the two coils are not identical may be treated by substituting particular values for the resistance and coefficients of induction in the general equations.

CHAPTER XIII

SPECIAL PROBLEM: EFFECTS PRODUCED BY CHANGING THE RELUCTANCE OF THE MAGNETIC CIRCUIT AND HENCE THE COEFFICIENTS OF INDUCTION OF A TRANSFORMER

In order to show the function of the magnetic circuit, the medium through which all the action between the primary and secondary circuits takes place, let us ascertain the effects of a variation in its reluctance or magnetic resistance. Evidently this problem is quite determinate, inasmuch as it is by means of the magnetic medium alone that the primary has any influence whatsoever upon the secondary. The problem is complicated by the presence of magnetic leakage, or external secondary self-induction, and for simplicity we will, therefore, take it up first, neglecting the effects of these quantities; their effects will be discussed in the latter part of the chapter.

Effects Produced by a Variation in the Coefficients of Induction Caused by Changing the Reluctance of the Magnetic Circuit in a Transformer with No Magnetic Leakage or External Self-Induction

We will first consider the case in which the primary is supplied with a constant harmonic current. It is required to find the variations in the secondary electromotive force and current, and in the primary impressed electromotive force as the magnetic reluctance, and consequently the coefficients of induction, are varied.

The values of the coefficients of induction are:

$$L_1 = \frac{4\pi S_1^2}{\text{reluctance}};$$

$$M = \frac{4\pi S_1 S_2}{\text{reluctance}};$$

$$L_2 = \frac{4\pi S_2^2}{\text{reluctance}}.$$

The ratios of the values of these coefficients are constant; namely,

$$\frac{L_1}{M} = \frac{S_1}{S_2}; \quad \frac{M}{L_2} = \frac{S_1}{S_2}; \quad \frac{L_1}{L_2} = \frac{S_1^2}{S_2^2}.$$

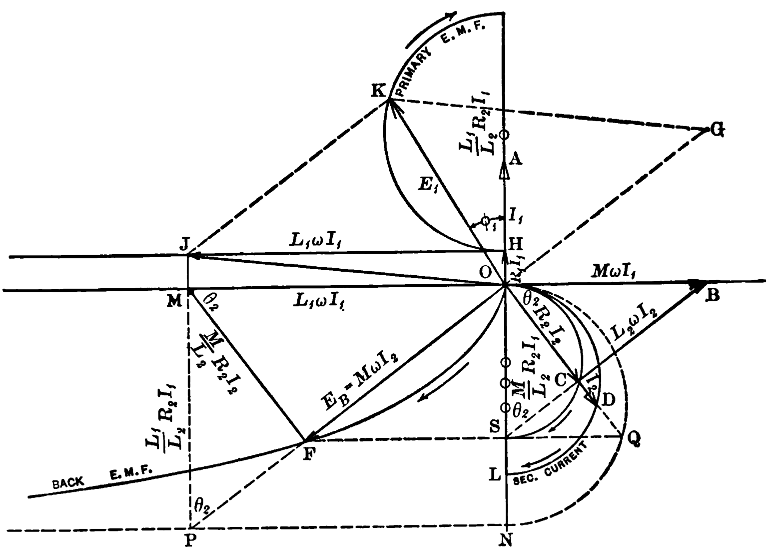


Fig. 138. Constant current transformer with no magnetic leakage or external self-induction. Typical diagram for a variation in the coefficients of induction. Arrows indicate direction of change as the coefficients of induction increase.

The coefficients of induction vary inversely as the reluctance; their ratios are independent of the reluctance.

In Fig. 138, draw the line OA to represent the constant primary current in magnitude and direction. Ninety degrees behind OA draw OB equal to $M\omega I_1$, the electromotive force induced in the secondary by the primary current. Construct the secondary electromotive force triangle OBC, and the back electromotive force OF, ninety degrees behind the secondary current. These are all drawn for a certain fixed value of the secondary resistance, and any selected value for M. The primary electromotive force triangle is drawn in the usual way, and also the impressed primary electromotive force E_1.

The various electromotive forces and currents are thus obtained for one value of M. Suppose that the value of M is increased by diminishing the reluctance. The various quantities will be affected as follows.

The Secondary Current. — The electromotive force OB is increased in proportion to M. The secondary angle of lag θ_2 increases so that the point C moves upon the semicircle OCS in the direction of S. That the curve OCS is a semicircle may be proved as follows: Produce the line BC to S, so that OS is perpendicular to OB.

In the triangle OBS, we have $\text{OS} = \dfrac{M\omega I_1}{\tan\theta_2} = \dfrac{M}{L_2} R_2 I_1$. Now $\dfrac{M}{L_2}$ has been shown to be a constant, independent of any change in M or L_2, due to a change in the reluctance of the magnetic circuit. OS is therefore constant. Since OCS is a right triangle, it follows that the point C always lies upon a semicircle.

The secondary current OD, being equal to OC divided by a constant R_2, also lies upon a semicircle the diameter of which is equal to OS divided by R_2; that is,

$$\text{OL} = \frac{M}{L_2} I_1.$$

The Back Electromotive Force. — The back electromotive force may be found for any value of M as follows: From the point F, already found for one value of M, draw FM perpendicular to OF until it intersects OB produced at M. We have the angle $\text{OMF} = \theta_2$, and $\text{OF} = M\omega I_2$; hence,

$$\text{FM} = \frac{M\omega I_2}{\tan\theta_2} = \frac{M}{L_2} R_2 I_2.$$

It appears then that FM varies directly with the current I_2, and is always equal to I_2 multiplied by a constant. Let us take OQ (= FM) equal to this constant multiple of the secondary current. Since the locus of the secondary current OD is a semicircle, the locus of OQ will be a semicircle with a diameter ON equal to OL multiplied by $\frac{M}{L_2}R_2$; that is,

$$ON = \frac{M^2}{L_2{}^2}R_2I_1 = \frac{L_1}{L_2}R_2I_1.$$

We may now find the locus of F, which will give us the value of the back electromotive force for any value of M. To find any point F, on the back electromotive force locus, draw OQ to some point on the semicircle OQN; from Q draw QF parallel to OB, and find its intersection at F with the line OF, drawn perpendicular to OQ. The point F thus determined by the intersection of QF and OF will always lie upon the curve marked "back E. M. F." in the figure. This curve approaches the asymptote NP, but practically the point F cannot go beyond a certain limit, since it is not possible to increase the value of M beyond a certain amount by changing the reluctance.

The locus of the back electromotive force may be shown to be a curve known as the cissoid. This may be shown as follows: Let us first find the polar equation of the curve, referred to O as an origin. If we denote the angle of MOF by ϕ, and the line of OF by ρ, we have

$$\rho = \text{FM} \cot\phi; \text{ and FM} = \text{MP} \cos\phi.$$

Hence,
$$\rho = a\cos\phi\cot\phi,$$

where a is a constant equal to $\text{MP} = \text{ON} = \frac{L_1}{L_2}R_2I_1$. This polar equation may be transformed into one with rectangular co-ordinates, where OM is the axis of x, and ON the axis of y, by substituting the values

$$\cos\phi = \frac{x}{\rho};\ \cot\phi = \frac{x}{y};\ \rho = \sqrt{x^2+y^2}.$$

The equation then becomes

$$\sqrt{x^2+y^2} = a \cdot \frac{x}{\sqrt{x^2+y^2}} \cdot \frac{x}{y};$$

or,
$$y^3 + x^2y = ax^2.$$

The curve represented by this equation is known as the cissoid.

The Primary Electromotive Force. — The locus of the primary electromotive force is a semicircle HKV with HV, its

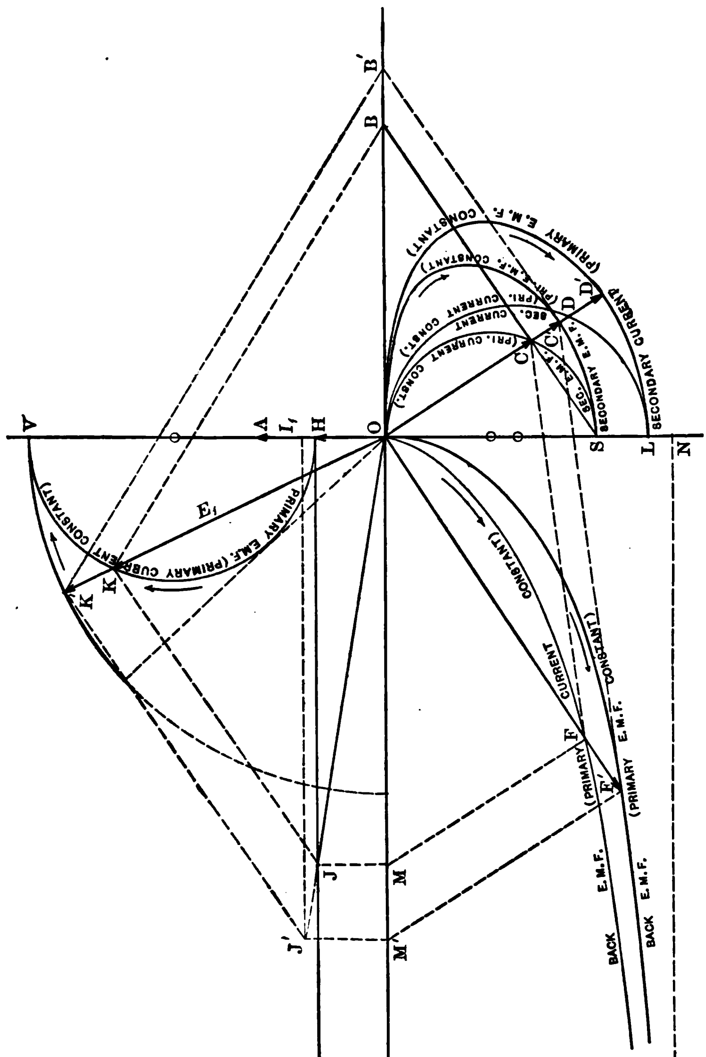


Fig. 139. Constant potential transformer with no magnetic leakage or external self-induction. Typical diagram for a variation in the coefficients of induction.

diameter, equal to $ON = \frac{L_1}{L_2} R_2 I_1$. This may be shown as follows: The primary electromotive force E_1 is the vector sum of OH, HJ, and OG. The back electromotive force FO may be resolved into FQ and QO. Now FQ is equal and opposite to HJ. The primary electromotive force is therefore the

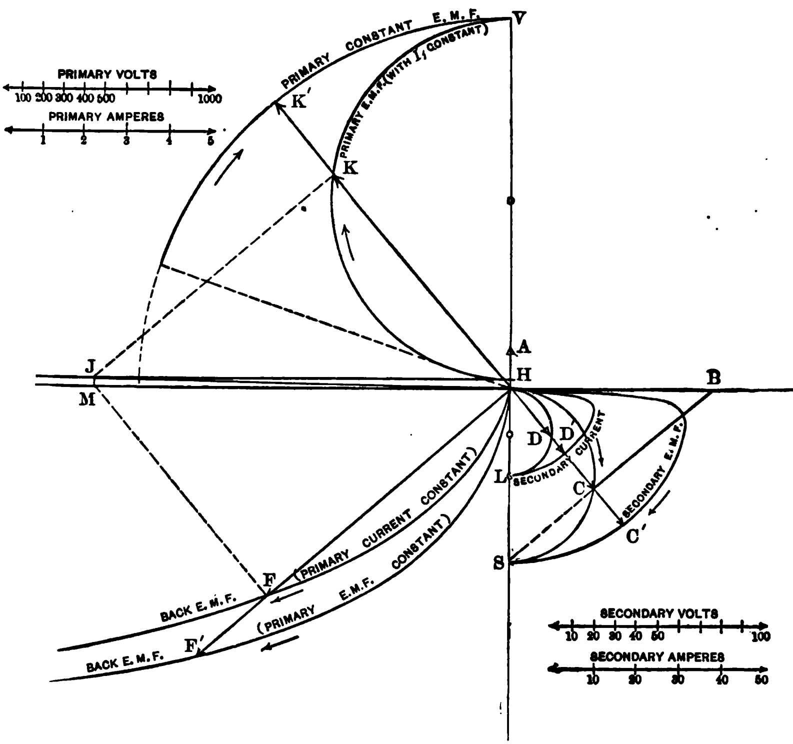


Fig. 140. Effects of the variation in the coefficients of induction of a ten-light Westinghouse transformer when $R_2 = 4$ ohms.

resultant of OH and QO. But OH is constant, and QO has a semicircle for a locus. The primary electromotive force, therefore, has for a locus the semicircle HKV, which is equal to the semicircle OQN. E_1 increases as M increases. It is to be noticed that in either limit ϕ_1, the angle between primary current and electromotive force is zero, and that it has a maximum value for a certain value of M.

The Secondary Output. — The secondary output (including the power expended in the secondary itself) is $\frac{1}{2} R_2 I_2^2 = R_2 \bar{I}_2^2$. It is, therefore, proportional to the square of the secondary current, and is a maximum when the current is a maximum and approaches the limit of OL.

Constant Potential Transformer. — If the primary is supplied with a *constant electromotive force*, the changes in the various

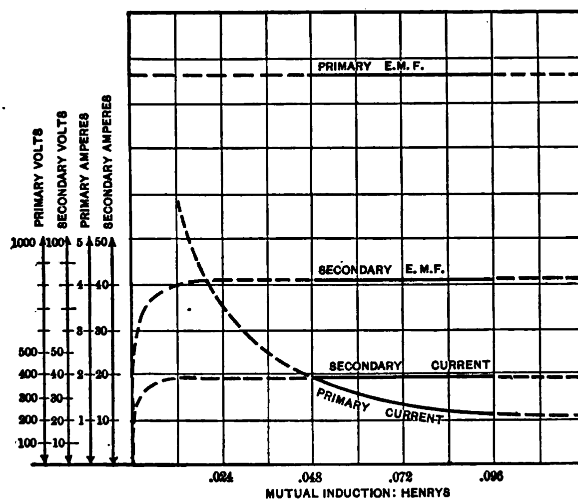


Fig. 141. Effects of the variation in the coefficients of induction of a ten-light Westinghouse transformer when $R_2 = 4$ ohms.

quantities due to a variation in M may be shown by a diagram obtained from the constant current diagram, by a proper magnifying or diminution, by the method explained in Chapter IX. Such a diagram is shown in Fig. 139. The figure is first drawn for a constant primary current OA; HKV is then the locus for the primary electromotive force. For a *constant* primary electromotive force, the locus K′V is drawn as an arc of a circle with E_1 for a radius and O as a center. It is evident that for the particular position of E_1, represented in Fig. 139 by OK′, the primary electromotive force has been increased from the value OK, which it had when the primary current was constant, in the ratio of OK′ to OK. All other quantities, therefore, must be increased in this same proportion: OB to OB′; OC to OC′; OD to OD′; OF to OF′; OM to OM′; and OJ to OJ′. In this way points are obtained on the various curves corresponding to the constant primary electromotive force E_1 in the particular position indicated in the figure by OK′. By locating other points in the same way, the locus OC′S is found for the secondary electromotive force, OD′L for the secondary current, and OF′ for the back electromotive force.

In Fig. 140 a similar diagram is shown for a ten-light Westinghouse transformer, when $R_2 = 4$ ohms. In Figs. 139 and 140 it is seen that the magnitude of the secondary current changes but little, as M is varied, except when M is very small.

The changes for the same transformer are plotted with rectangular co-ordinates in Fig. 141. The dotted portions of the curves are beyond the possible limits for operating the transformer.

Experimental Curves.—The effects produced by variation in the reluctance of the magnetic circuit of a transformer may be experimentally investigated. The range of variation is limited in experimental work, and the results are not as marked as those obtained by a variation of the secondary resistance. Figure 142 shows the changes in the primary electromotive force, and in the secondary electromotive force, current and power for a constant current transformer, with 128 primary turns and 90 secondary turns. The secondary resistance is 20 ohms. This transformer was used in the experiments described in Chapter XVI.

Figure 143 shows the results of experiments upon the same transformer, used as a step-up transformer instead of step-down as before (that is, primary turns = 90, secondary turns = 128), when supplied with a constant potential. The secondary resistance in this case was 25 ohms. The variation in the angles of lag were small in both experiments.

The amount of effect produced by a change in mutual induction depends largely upon the value of the secondary resistance, and this fact must be borne in mind in connection with all the curves given in this article, the typical ones as well as those for actual transformers. With larger values of the secondary resistance the effects produced by a change in M increase, and

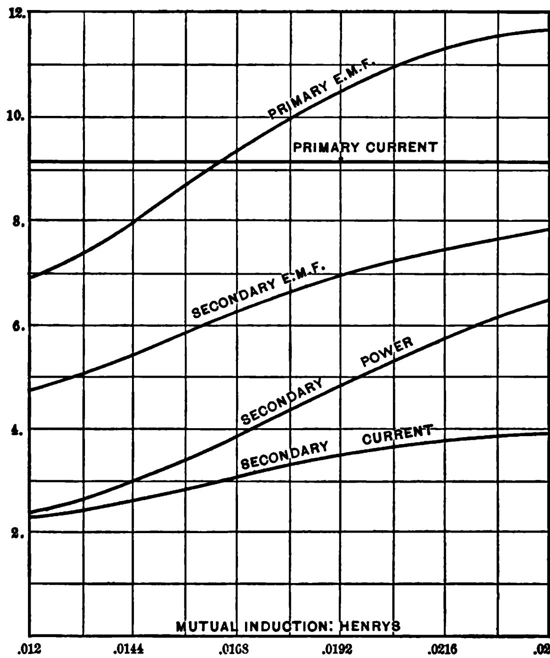


Fig. 142. Experimental curve for constant current transformer. Primary turns = 128. Secondary turns = 90. $R_2 = 20$ ohms. Scale of ordinates: One division represents 10 volts, 1 ampere, or 50 watts.

are greatest when the secondary is an open circuit. Let us consider the case of a constant current transformer, in which magnetic leakage is neglected. On open circuit the secondary electromotive force is $M\omega I_1$, and accordingly varies directly as M. For large values of the secondary resistance the same holds approximately true; and so, in a rectangular diagram, the secondary electromotive force and current would be, approximately, oblique straight lines. On short circuit the secondary current

$$\frac{M\omega I_1}{\sqrt{R_2^2 + L_2^2\omega^2}}$$

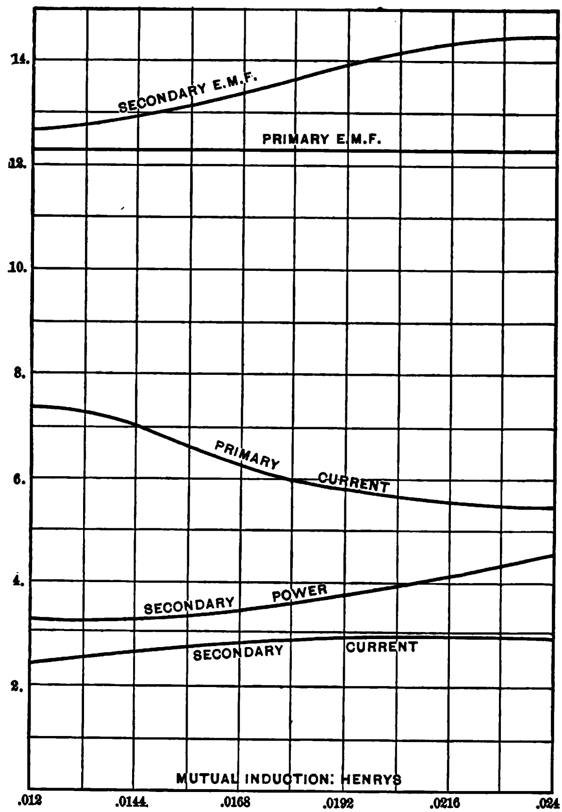


Fig. 143. Experimental curves for constant potential transformer. Primary turns = 90. Secondary turns = 128. $R_2 = 25$ ohms. Scale of ordinates: One division represents 5 volts, 1 ampere, or 50 watts.

becomes approximately $\frac{M}{L_2}I_1$, which is a constant, independent of any variation in M. For this case then—that of a constant primary current and small secondary resistance—the secondary electromotive force and current would be constant for different values of M, and would be represented by horizontal straight lines in a diagram drawn with rectangular co-ordinates. The effects of a change in M for other values of the secondary resistance are intermediate to those just described. Experimental results accord with these conclusions.

The treatment thus far has been confined to transformers in which there is no magnetic leakage or external self-induction in the secondary. The modifications introduced by the consideration of external self-induction may now be considered.

Effects Produced by a Variation in the Coefficients of Induction Caused by Changing the Reluctance of the Magnetic Circuit in a Transformer with External Secondary Self-Induction

We will first consider the case in which the primary current is maintained constant.

External to the transformer is a constant self-induction $L_{2_{ext}}$ in the secondary circuit. The coefficient of mutual induction M, and the self-inductions of the transformer, primary and secondary, L_1 and $L_{2_{int}}$ respectively, vary inversely with the reluctance of the magnetic circuit, while all other quantities remain constant. The secondary self-induction, being the sum of the internal, or that in the secondary coil itself, and the external, is

$$L_2 = L_{2_{int}} + L_{2_{ext}}.$$

We also have the relation $M^2 = L_1 L_{2_{int}}$. This means that there is no leakage within the transformer, the effects of magnetic leakage being included in the external self-induction.

Secondary Electromotive Force. — In Fig. 144, OA represents the constant primary current; OB is the electromotive force induced by the primary current for a particular value of M, and OC the corresponding secondary electromotive force which overcomes ohmic resistance. As M varies, the position of the point C will change. It is now required to find the locus upon which it will always lie.

From the triangle OBC, we have

$$OC = M\omega I_1 \cos \theta_2;$$

and

$$\tan \theta_2 = \frac{L_2\omega}{R_2} = \frac{L_{2_{\text{int}}}\omega}{R_2} + \frac{L_{2_{\text{ext}}}\omega}{R_2} = \frac{M^2\omega}{L_1 R_2} + \frac{L_{2_{\text{ext}}}\omega}{R_2}.$$

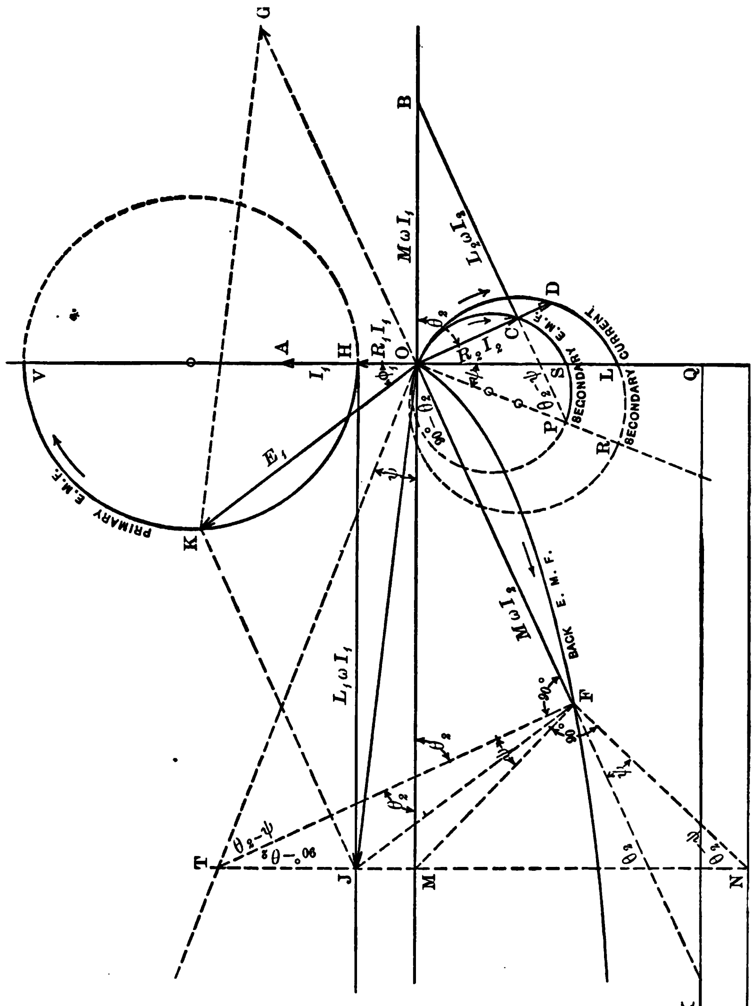


Fig. 144. Constant current transformer with external secondary self-induction. Typical diagram for a variation in the coefficients of induction.

Hence, $$M\omega = \frac{L_1}{M}R_2 \tan\theta_2 - \frac{L_1}{M}L_{2_{ext}}\omega,$$

and $$OC = \frac{L_1 I_1 R_2}{M}\sin\theta_2 - \frac{L_1 I_1 L_{2_{ext}}\omega}{M}\cos\theta_2; \qquad (1)$$

or, $$OC = \frac{L_1 I_1}{M}(R_2 \sin\theta_2 - L_{2_{ext}}\omega\cos\theta_2).$$

Transforming into terms of the sine alone, we have

$$OC = \frac{L_1 I_1}{M}\sqrt{R_2^2 + L_{2_{ext}}^2\omega^2}\sin(\theta_2 - \psi), \qquad (2)$$

where $$\tan\psi = \frac{L_{2_{ext}}\omega}{R_2} = \text{a constant.}$$

In equation (2) θ_2 is the only variable. The locus of C is therefore a circle upon a diameter

$$\frac{L_1 I_1}{M}\sqrt{R_2^2 + L_{2_{ext}}^2\omega^2}.$$

To locate this diameter in Fig. 144, make the angle SOP equal to ψ and produce BC to P. The angle OPC is equal to $(\theta_2 - \psi)$. Hence $OC = OP \sin(\theta_2 - \psi)$. Evidently the locus of C is a semicircle with a diameter

$$OP = \frac{L_1 I_1}{M}\sqrt{R_2^2 + L_{2_{ext}}^2\omega^2}.$$

This is geometrically shown from equation (1) in Fig. 145. Lay off OS and OX at right angles to each other, and equal, respectively, to the coefficients of the sine and cosine in equation (1). Erect the semicircles OQX and OTS. We have from the figure:

$$OT = \frac{L_1 I_1 R_2}{M}\sin\theta_2; \quad OQ = \frac{L_1 L_{2_{ext}}\omega I_1}{M}\cos\theta_2.$$

Hence from equation (1)

$$OC = OT - OQ.$$

Since the projection of OP on QT is always equal to the difference of the projections of OS and OX upon it; the locus of C is accordingly a semicircle whose diameter OP is the diagonal of the rectangle on the component diameters OS and OX. When there is no external self-induction, OX disappears. OX increases with $L_{2_{ext}}$ and has the ratio $\frac{L_{2_{ext}}\omega}{R_2}$ to OS.

Secondary Current. — The secondary current OD (in Fig. 144) is equal to OC (which is equal to $R_2 I_2$), divided by R_2, and

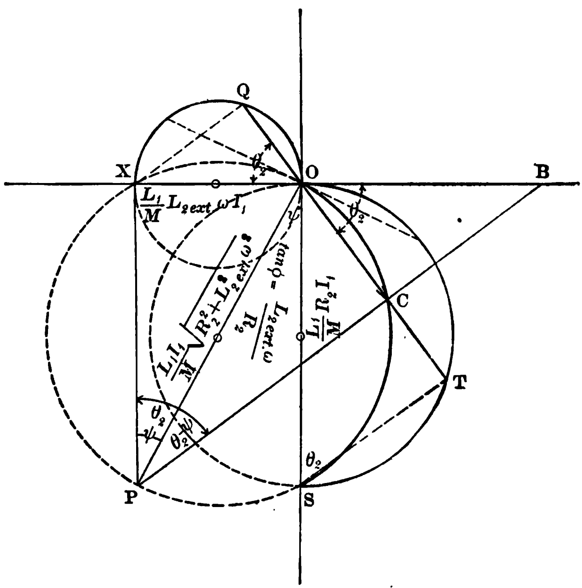


Fig. 145. Detail portion of Fig. 144.

must, therefore, have a similar locus. Referring to the expressions for OC in (1) and (2), we have

$$I_2 = \text{OD} = \frac{L_1 I_1}{M}\left(\sin\theta_2 - \frac{L_{2_{ext}}\omega}{R_2}\cos\theta_2\right) = \frac{L_1 I_1}{M}(\sin\theta_2 - \tan\psi\cos\theta_2)$$

$$= \frac{L_1 I_1}{M\cos\psi}\sin(\theta_2 - \psi) = \text{OR}\cdot\sin(\theta_2 - \psi). \qquad (3)$$

The locus of D is a semicircle ODLR with a diameter

$$\text{OR} = \frac{L_1 I_1}{M R_2}\sqrt{R_2^{\,2} + L_{2_{ext}}^{\;2}\omega^2}. \qquad (4)$$

It is to be noticed that the lines OS and OL in Figs. 144 and 145 have the same values as in Figs. 138 and 139.

Back Electromotive Force. — The back electromotive force E_B is always equal to $M\omega I_2$, and is ninety degrees behind the secondary current. Referring to equation (3), we have

$$E_B = \text{OF} = M\omega I_2 = \frac{L_1\omega I_1}{\cos\psi}\sin(\theta_2 - \psi). \tag{5}$$

If a line OT be drawn in Fig. 144, making an angle ψ with OM, and if JT be drawn from J perpendicular to HJ, we have $\text{OT} = \frac{L_1\omega I_1}{\cos\psi}$. The line OT is a variable, depending upon the value of L_1. Substituting in (5):

$$E_B = \text{OF} = \text{OT}\sin(\theta_2 - \psi).$$

Hence the point F is upon a semicircle with OT for a diameter, and FT is perpendicular to OF.

The locus for the back electromotive force may be constructed by points located graphically in accordance with the above relations. The line OF is drawn at right angles to OD, the secondary current for any particular values of the coefficients of induction. On the line OT the point T is located, corresponding to these same values of the coefficients of induction. From T a line TF is drawn perpendicular to OF. The intersection F is a point on the back electromotive force locus. Other points are obtained in the same way, and the whole curve thus determined. The curve is tangent to OP, and approaches QX as an asymptote.

Primary Impressed Electromotive Force. — The primary electromotive force is the vector sum of FO, OH, and HJ, and is therefore equal to FJ. Drawn from O as an origin, the primary electromotive force is $E_1 = \text{OK} = \text{FJ}$. The locus of K is a semicircle upon the diameter HV, as may be shown as follows:

In the triangle MFT, the angle MFT is equal to ψ, since the points M and F lie upon a semicircle which may be described upon OT as a diameter, and the angle MFT = MOT = ψ, each subtending the same arc MT. In the triangle MFT we accordingly have

$$\text{MF} : \text{MT} :: \cos\theta_2 : \sin\psi.$$

But MT $= L_1\omega I_1 \tan\psi$. Hence

$$\text{MF} = \frac{L_1\omega I_1}{\cos\psi}\cos\theta_2 = \frac{L_1 M\omega I_1}{M\cos\psi}\cos\theta_2 = \frac{L_1\cdot\text{OC}}{M\cos\psi}.$$

But OC = OP $\sin(\theta_2 - \psi)$. Hence

$$\text{MF} = \frac{L_1\text{OP}}{M\cos\psi}\sin(\theta_2 - \psi). \qquad (6)$$

If FN is drawn perpendicular to FM, we have in the triangle FMN

$$\text{MF} = \text{MN}\sin(\theta_2 - \psi). \qquad (7)$$

It follows directly from (6) and (7) that

$$\text{MN} = \frac{L_1\cdot\text{OP}}{M\cos\psi}. \qquad (8)$$

Since, from Fig. 145, OP $= \frac{L_1 I_1}{M}\sqrt{R_2{}^2 + L_{2_{\text{ext}}}{}^2\omega^2}$ and $\cos\psi = \frac{R_2}{\sqrt{R_2{}^2 + L_{2_{\text{ext}}}{}^2\omega^2}}$, we have

$$\text{MN} = \frac{L_1{}^2 I_1}{M^2 R_2}(R_2{}^2 + L_{2_{\text{ext}}}{}^2\omega^2) = \text{a constant}. \qquad (9)$$

Since MFN is a right angle, the point F always lies upon a semicircle with a diameter MN constant in value, but shifting in position as HJ increases so as to remain directly below J. The back electromotive force locus might have been constructed in this way.

To find the locus of the primary electromotive force, draw HV equal and parallel to MN. Then HK will be equal and parallel to FM, and OK equal and parallel to FJ. Evidently the locus of K is the semicircle HKV with a diameter HV = MN as given in (9).

As the external secondary self-induction is made smaller, the angle ψ is diminished. When there is no external self-induction, $\psi = 0$. The value of HV then takes the simpler form given in Fig. 138.

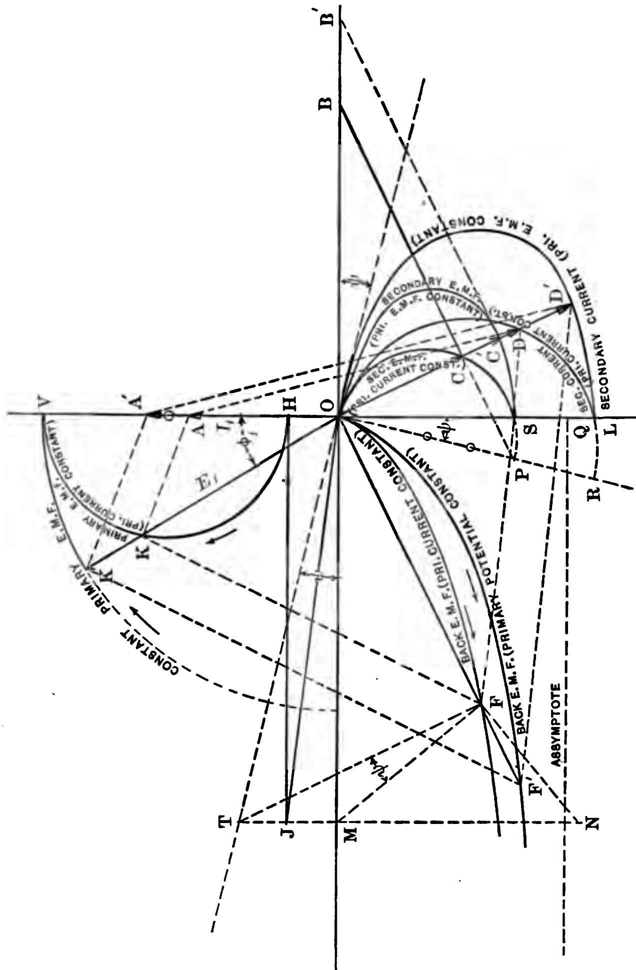


Fig. 146.

Constant Primary Potential. — The figure showing the effects produced by varying the coefficients of induction in a constant potential transformer is given in Fig. 146. The construction is first made for a constant primary current OA. Any particular value OK of the primary electromotive force must be changed from OK to OK′ for constant potential, and all other magnitudes changed in the same proportion; that is, OC becomes OC′, OD becomes OD′, OF becomes OF′, etc., as already explained in the construction of other constant potential diagrams.

CHAPTER XIV

SPECIAL PROBLEM: ACTION OF A TRANSFORMER WITH A CONDENSER IN PARALLEL WITH THE SECONDARY *

Statement of the Problem. — This problem treats of the action of a transformer, across the secondary terminals of which is shunted a condenser. The condenser is, therefore, in parallel with the secondary load, which is supposed to be non-inductive, as would be the case with incandescent lamps. In the previous chapters the action of a transformer has been investigated under various conditions, when supplied with a constant current and when supplied with a constant potential, the results for the latter case being derived from the results of the former, inasmuch as the graphical construction is simpler for the constant current case than for the case with constant primary electromotive force. The method for converting constant current diagrams into those for constant potential has been given for so many of the previous problems that in this discussion we will confine ourselves to transformers with constant primary current.

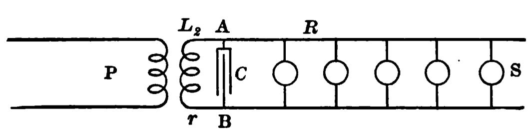


Fig. 147. Arrangement of circuits.

Hitherto the secondary has been considered as a single circuit, all the current from the secondary of the transformer passing through the secondary load. It is now proposed to consider the simplest case of a divided circuit, where the transformer secondary current divides, part passing through the condenser and part through the external resistance.

* This problem is given merely as an exercise for the student in applying the principles of the foregoing chapters to a special case.

This problem is rendered easy by the use of graphical methods; for, although in the transformer we have mutual induction, yet, to construct the diagram for a transformer with a constant primary current, we know that the impressed electromotive force upon the transformer secondary is equal to $M\omega I_1$ and is therefore a constant. This consideration reduces the matter to the simple case of the transformer circuit in series with a divided circuit; there is a constant impressed electromotive force, independent of the mutual induction of the transformer.

The diagrammatical representation of the combination of transformer and condenser, to be discussed in this problem, is shown in Fig. 147. The primary circuit is denoted by P, and secondary by S. The condenser, with capacity C, is shunted between the terminals, A and B, of the secondary. The line resistance is R; the resistance and reactance of the secondary of the transformer are r and $L_2\omega$ respectively.

Since the impressed electromotive force upon the transformer secondary is $M\omega I_1$, we may, if we choose, represent the secondary circuit apart from the primary, as in Fig. 148. Here $M\omega I_1$ is a generator supposed to give an impressed electromotive force $M\omega I_1$ exactly like the transformer, but supposed to offer no impedance to the current.

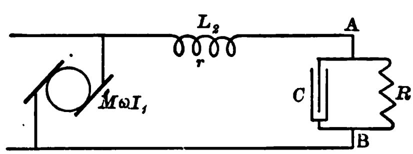


Fig. 148.

There are two methods which present themselves by which the solution of this problem may be obtained. One is partly analytical and partly graphical, and the other entirely graphical. The former method will first be briefly indicated. The latter will be given in greater detail.

Diagram established Analytically. — The electromotive force $M\omega I_1$ impressed upon the circuit by the primary current is the geometrical sum of two components; one to overcome the transformer impedance, equal to $I_t(r^2 + L_2^2\omega^2)^{\frac{1}{2}}$; the other to overcome the combination of condenser and resistance in parallel between the points A and B (Fig. 147). The equivalent resistance and capacity of circuits in parallel for the general case,* where there are any number of branches to the divided

* See *Alternating Currents*, pp. 303–308.

circuit, may be thus expressed: The equivalent resistance is

$$R' = \frac{A}{A^2 + B^2\omega^2}.$$

The equivalent capacity C' may be found from the expression

$$\frac{1}{C'\omega} = \frac{B\omega}{A^2 + B^2\omega^2},$$

where

$$A = \Sigma \frac{R}{R^2 + \frac{1}{C^2\omega^2}}$$

and

$$B\omega = \Sigma \frac{C\omega}{C^2R^2\omega^2 + 1}.$$

The summation gives one term for each of the branches. In the present case, where there are but two branches, and one branch contains no capacity, the other no resistance, the values of A and $B\omega$ reduce to

$$A = \frac{1}{R}\text{; and } B\omega = C\omega.$$

This gives the values of R' and $\frac{1}{C'\omega}$:

$$R' = \frac{R}{1 + R^2C^2\omega^2}\text{; and } \frac{1}{C'\omega} = \frac{C\omega R^2}{1 R + {}^2C^2\omega^2}.$$

The impedance of the combination of the divided circuits is

$$\left(R'^2 + \frac{1}{C'^2\omega^2}\right)^{\frac{1}{2}} = \left(\frac{R^2 + C^2\omega^2R^4}{(1 + C^2\omega^2R^2)^2}\right)^{\frac{1}{2}} = \frac{R}{(1 + C^2\omega^2R^2)^{\frac{1}{2}}}.$$

The current through the transformer may now be found as follows: Replace the condenser and resistance branches by a single circuit, having resistance R' and capacity C', and we have now a simple circuit whose total impedance is

$$\left\{(r + R')^2 + \left(\frac{1}{C'\omega} - L_2\omega\right)^2\right\}^{\frac{1}{2}}.$$

The angle of phase difference between the electromotive force $M\omega I_1$ and the transformer current is

$$\text{arc tan}\frac{\left(\frac{1}{C'\omega} - L_2\omega\right)}{r + R'}.$$

Referring to Fig. 149, lay off OB equal to $M\omega I_1$ and make the angle BOV equal to the angle of phase difference obtained in the last expression above. Draw BV perpendicular to OV. Let I_t denote the total current through the transformer secondary. Then OV equals $(r+R')I_t$, and

$$BV=\left(\frac{1}{C'\omega}-L_2\omega\right)I_t.$$

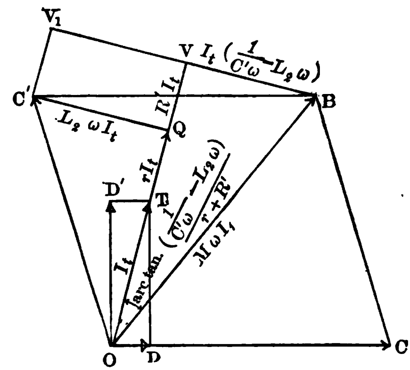


Fig. 149.

Divide OV by $(r+R')$ and obtain the current I_t represented by OT. Also divide OV into two parts, OQ and QV, proportional to r and R' respectively. Then OQ is that portion of the impressed electromotive force used for the transformer resistance. Draw QC′ perpendicular to OQ, in advance of it, and make it equal to $L_2\omega I_t$. Then OC′ is that portion of the impressed electromotive force OB which goes to overcome the transformer secondary impedance. Since OB is the sum of *two* components, OC′ for the transformer impedance, and a second component for the combination circuit, it follows that C′B, equal to OC, is the electromotive force at the terminals of the divided circuit. The line current OD is necessarily in phase with the electromotive force OC at its terminals, and the condenser current OD′ is a right angle ahead of OC. From T draw a perpendicular upon OC and upon OD′, and we have the transformer current OT, the sum of the currents OD and OD′ in the line and condenser.

Thus, by the consideration of the equivalent resistance and capacity of divided circuits, we may easily find the relation between the electromotive forces and currents in the various branches. This, however, is not the way of looking at the problem that will most obviously give us the means to draw diagrams illustrating the manner in which the various quantities concerned change as we alter some one quantity. The method that lends itself more readily to this is the entirely graphical method which follows:

Diagram established Graphically. — It is our aim to construct the diagrams representing the changes which occur when the primary current is constant, while any one quantity, as, for instance, the line resistance or condenser capacity, is varied. As already explained, the constant current diagrams naturally precede the constant electromotive force diagrams, and are usually much simpler, inasmuch as the variations usually take place upon the arc of a circle. The constant electromotive force diagrams may be directly derived from the constant current figures.

A constant primary current induces a constant electromotive force $M\omega I_1$ upon the transformer secondary. Any variation in one of the quantities of the secondary circuit will naturally cause the electromotive force at the condenser terminals to change. If this condenser potential were to remain constant, it would be a very simple matter to draw the diagram of variations. It is proposed, first, to construct diagrams representing constant condenser electromotive force, and then to show how these may be changed to represent constant transformer induced electromotive force $M\omega I_1$.

Let OC, Fig. 150, represent the constant electromotive force at the condenser terminals A and B (Fig. 147). Then the condenser current OD′ is a right angle in advance of OC, and equal to $C\omega$ times OC. The line current OD has the direction of the electromotive force OC, and is equal to OC divided by R. The transformer current OT is the geometrical sum of OD and OD′. The electromotive force for the transformer resistance is OQ, equal to OT times r. The electromotive force for transformer reactance is QC′, equal to $L_2\omega I_t$, a right angle in advance of OT. The electromotive force for transformer impedance is OC′, the sum of OQ and QC′. The total impressed electromotive force in the transformer is the geometrical sum of the components OC and OC′, which is OB. This completes the diagram, and gives a figure like Fig. 149, obtained from the idea of equivalent resistance and capacity.

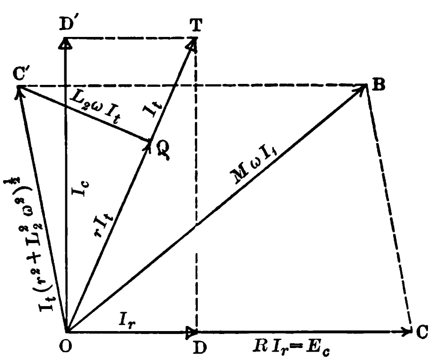


Fig. 150.

Effects of a Variation in the Capacity of the Condenser when the Electromotive Force at the Condenser Terminals is Constant. — In Fig. 151 is reproduced Fig. 150, with lines added which show the variation in the quantities when the capacity of the condenser is changed. The same letters designate corresponding points in the different diagrams. As the capacity is varied, the condenser current changes its magnitude

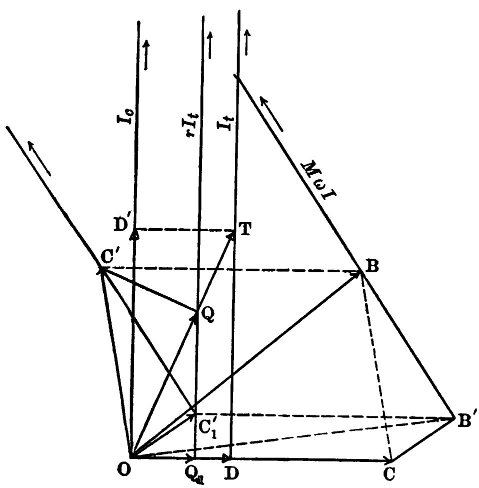


Fig. 151. Constant condenser potential diagram illustrating change of capacity.

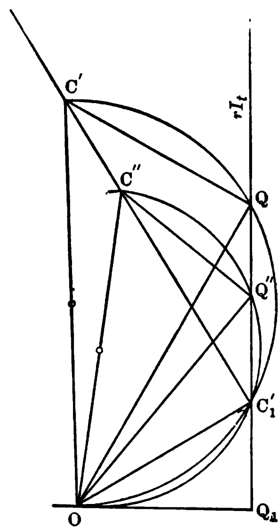


Fig. 152.

only. Its direction is always at right angles with OC, and its locus therefore upon the straight line labeled I_c. The line current OD does not change with the capacity, since the electromotive force at the terminals of the line is supposed constant. Hence the transformer current OT has for its locus the line labeled I_t parallel with I_c.

The electromotive force OQ for the transformer resistance is proportional to OT, and hence moves upon the line labeled rI_t. The angle C′OQ is always equal to arc tan $\frac{L_2\omega}{r}$, and is therefore constant for all positions of the triangle C′OQ. The angle at Q is always a right angle. Hence C′OQ is always similar to itself in any position. Under these circumstances, it follows that the locus of the point C′

is the straight line $C'C_1'$; and the point C_1' is a point upon Q_1Q, such that tan Q_1OC_1' equals $\frac{L_2\omega}{r}$.

This may be proved as follows: Fig. 152 represents that portion of Fig. 151 under consideration. The angle Q_1OC_1' equals arc tan $\frac{L_2\omega}{r}$. Draw $C_1'C'$ at right angles to OC_1', from C_1'. It will appear that this line is the locus of the vertex C' of the triangle $C'OQ$ as it moves with its right angle upon QQ_1, always being similar to itself. Take any point Q, and from it draw QC' perpendicular to OQ, until it meets $C_1'C'$ at C'. Likewise, take another point Q'', and, in a similar manner, draw $Q''C''$. The triangles QOC' and $Q''OC''$ are now similar; for, describe a semicircle $C'QC_1'O$ about OC' as diameter, and another, $C''Q''C_1'O$, about OC'' as diameter. Each of these arcs passes through the point C_1', because $OC_1'C'$ is a right angle. Now, the arcs $C'Q$ and $C''Q''$ measure equal angles at the centers of their respective circles, since each subtends the inscribed angle $C'C_1'Q$. Hence the inscribed angle $C'OQ$ is equal to the inscribed angle $C''OQ''$, as each is subtended in the respective circles by the arcs $C'Q$ and $C''Q''$. Hence the triangles $C'OQ$ and $C''OQ''$ are similar. But as these represent any two positions of the triangles, the proposition is proved.

Returning to the diagram (Fig. 151), it has been proved that OC' varies on the locus $C_1'C'$. This is that part of the impressed electromotive force which goes to overcome the transformer impedance. OC is that part which overcomes the impedance of the divided circuit. The resultant of OC and OC' is OB, the transformer impressed electromotive force. Since OC is constant, OB varies along the line BB', parallel with $C_1'C'$, as the condenser capacity changes. When the capacity is zero, D' coincides with O, T with D, Q with Q_1, C' with C_1', and B with B'.

This completes the diagram showing the effect of any change in the capacity, and, before altering the diagram to represent the case where the electromotive force OB is kept constant, we will draw the diagram showing the effect of a change in the resistance R.

Effects of a Variation of Secondary Load when the Electromotive Force of the Condenser is Constant. — We have reproduced in Fig. 153 the lines of Fig. 150, and added more to show the variation of the different quantities as the secondary resistance is changed. As the capacity is constant, OD′ remains fixed, but OD, the line current, varies inversely as R, and consequently D moves upon the line OC. The resultant of OD and OD′, viz., OT, therefore varies upon the line labeled

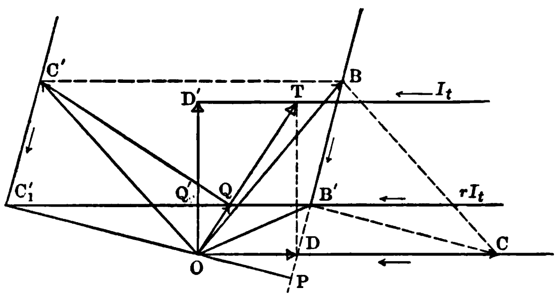


Fig. 153. Constant condenser potential diagram illustrating change of line resistance.

I_t, which is parallel with OC. The electromotive force OQ, to overcome the transformer resistance, being proportional to OT, lies upon the line rI_t. It may be proved, in a similar manner to the proof already given, that the locus of C′ is upon the straight line $C_1'C'$, which makes an angle arc tan $\frac{L_2\omega}{r}$ with the line rI_t. The locus of $M\omega I_1$, or of OB, is consequently a line BB′, parallel with $C'C_1'$, so that when the resistance is infinite, B coincides with B′, Q with Q′, T with D′, and C′ with C_1'.

This completes the diagram, as far as a variation of the resistance alone affects it, from the standpoint of constant condenser potential.

Having obtained the diagrams for each case, viz., when the condenser capacity is varied, and when the secondary resistance is altered, from the consideration of constant potential at the condenser terminals, it is desirable to so alter these diagrams that the electromotive force induced in the transformer secondary will be constant, while the other quantities change. This occurs when the transformer primary is supplied with a constant current. The diagrams will then conform to the usual standard constant current form, which may easily be compared with others already obtained. They may also be changed so as to represent a constant primary impressed electromotive force.

In order to change the previous diagrams into those which are desired, the following general remarks in regard to the relation between directed or vector quantities may prove of assistance, if they are not indeed essential to the method of solving the problem in hand. In the question before us, referring to Fig. 151, we have seen that, if OC represents the constant electromotive force at the condenser terminals, then OB, which represents the impressed electromotive force in the transformer, varies upon a straight line BB′ as the condenser capacity is changed.

The question now is, how would OC, the condenser electromotive force, vary, if we select some value of the secondary electromotive force, as OB, to be kept constant? We know that for any given angle

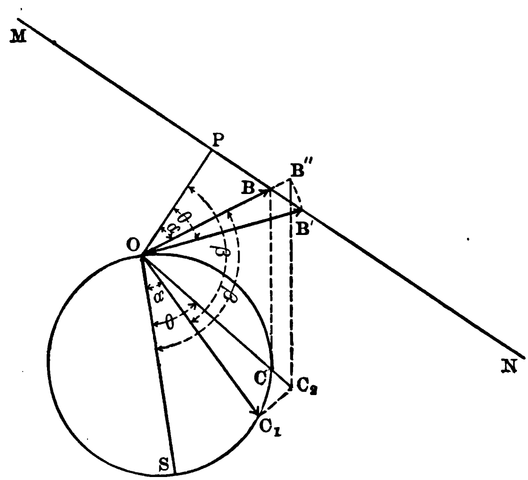


Fig. 154. Illustrating general theorem.

of phase difference between OC and OB, as COB, the relative magnitudes, that is the ratio, of the lines, must always remain the same, independent of absolute magnitude. This enables us to determine the character of the locus of OC when OB is taken as constant. To make the proposition general, refer to Fig. 154. Here we have any line MN along which a variable vector OB′ moves; and let OC represent some vector constant both in magnitude and in its direction with relation to OB′. Let OB be a certain position of the variable vector, which it is desired to maintain constant. To find the locus of the varying positions of OC, which, when variable, we call OC_1, so moving that the ratio of OC to OB is the same as that of OC_1 to OB′ for any constant angle or phase difference, we may construct the locus by points as follows. OC_1 is a particular value of the variable which makes the angle $B'OC_1$ with the constant OB'. Revolve OB' to OB'' and OC_1 to OC_2 so that the angle $B'OC_1 = B''OC_2$. Then change the magnitudes of both OB'' and OC_2 proportionally, making OB'' take the value OB. Then OC_2 becomes OC, which gives the required point on the locus. The change from OC_2 to OC may be effected by drawing BC parallel with $B''C_2$. It will be found that the locus is a circle whose diameter OS makes the same angle with OB as OC_1 makes with the perpendicular OP upon MN.

Proof of Theorem.—These principles may be shown as follows: An equation of a straight line MN may be written,

$$OB' \cos \theta = OP = \text{a constant.}$$

Now OC_1 is a constant vector making an angle β with OP. Hence,

$$OC_1 \cos \beta = mOP, \tag{1}$$

where $OC_1 \cos \beta$ is the resolved part of OC_1 along OP and is consequently some multiple m of OP. By the preceding equations

$$OB' \cos \theta = \frac{OC_1 \cos \beta}{m} = OP, \text{ a constant.}$$

In this equation OB' and θ are *variable* co-ordinates. But if it is desired to keep some value of OB' a constant, say OB, and to make OC_1 variable, say OC, we still have $OB \cos \theta = \frac{OC \cos \beta}{m}$, or, solving for OC, $OC = \frac{mOB \cos \theta}{\cos \beta}$. Here $\frac{mOB}{\cos \beta}$ is a constant, equal to OS, we will say. Then $OC = OS \cos \theta$.

This is the equation of a circle, and shows that OC is a variable in a circle with diameter OS. The position of the diameter OS is determined by the relation above

$$mOB = OS \cos \beta. \tag{2}$$

Dividing (1) by (2),

$$\frac{OC_1}{OS} = \frac{OP}{OB} = \cos \alpha.$$

Thus OS makes the same angle α with OC_1 as OB makes with the perpendicular OP; this defines both the magnitude and direction of OS, the diameter of the circle. This general theorem may be stated as follows:

If a variable vector OB' has its origin at a point O, and terminus upon a straight line MN, while OC_1 represents some constant vector; then another variable vector, moving upon a circle OCS (whose diameter OS makes the same angle β with a constant vector OB as the constant OC_1 made with the perpendicular OP upon the straight line MN), preserves the same relative magnitude or ratio to its constant OB as the first variable OB' bears to its constant OC for the same angle of phase difference.

Effects of a Variation of the Condenser Capacity when the Primary Current is Constant. — We are now prepared to construct the diagram to represent a transformer supplied with a constant primary current, the circuit being that of Fig. 147, when the condenser capacity is varied.

Referring to Fig. 155, let OA represent the constant primary current; OB, equal to $M\omega I_1$, and 90° behind the primary current, is the constant impressed electromotive force in the transformer secondary. Take OB′, Fig. 151, as the selected value of this secondary electromotive force,

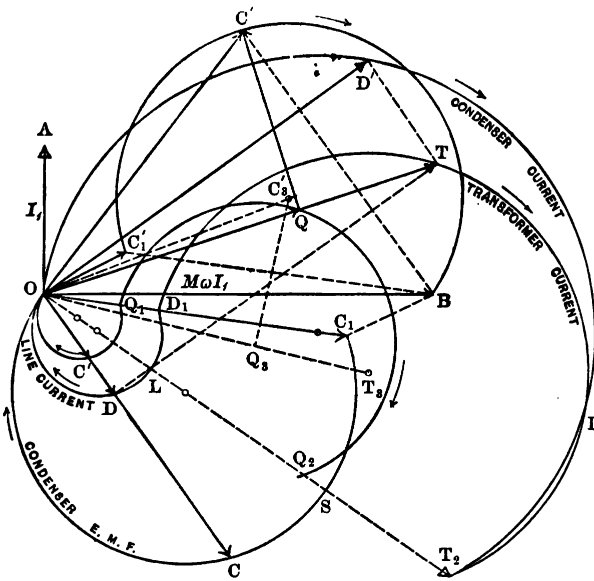


Fig. 155. Effects of a variation of the condenser capacity when the primary current is constant.

which is to be kept constant. This is made equal to OB, Fig. 155. When the condenser capacity is zero, we have the parallelogram OC_1BC_1', Fig. 155, equal to $OCB'C_1'$, Fig. 151; and so OC_1, OD_1, and OQ_1 are lines in Fig. 155 that correspond with OC, OD, and OQ_1 in Fig. 151, and these are values of corresponding quantities when the condenser capacity is zero, or when the condenser is absent. Lay off OS, Fig. 155, so that the angle BOS equals arc tan $\frac{L_2\omega}{r}$ equal to the angle COC_1', Fig. 151, which OC makes with the perpendicular upon the line BB′. Then will OS be the *direction* of the diameter of the circle upon which OC varies, according to the previous theorem, and, since one point C_1 on the circle is determined, this determines the circle C_1SCO, which may now be drawn. OC is any position of the variable. The resultant of OC, the condenser electromotive force, and OC′ the electromotive force for the impedance of the transformer, gives the total impressed electromotive force OB. Hence the locus of C′ is a circle $C_1'C'B$ similar to the locus of C. The electromotive force for the transformer impedance is now resolved into its two components OQ and QC′, at right angles to one another, making the angle QOC′ equal to arc tan $\frac{L_2\omega}{r}$ equal to BOS. OQ is then that component of the total electromotive force for the transformer resistance alone. QC′ is that for the transformer self-induction. OQ has the direction of the transformer current OT, and is equal to OT multiplied by the transformer resistance r. Hence we have one position of the transformer current OT, which corresponds with the position OC of condenser electromotive force. Since the current OD through the line is proportional to the electromotive force at the condenser terminals OC, and equal to it divided by the resistance R, the locus of OD is a circle similar to OCS upon OL as diameter. And since OT, the transformer current, is the geometrical sum of OD, the line current, and OD′, the condenser current, we have OD′ equal and parallel with DT.

It is evident that the locus of OQ is the circle Q_1QQ_2 having a center Q_3 situated so as to make $C_3'OQ_3$ a triangle similar to C′OQ; as C′ moves around its circle the point Q always lies behind it by an angle arc tan $\frac{L_2\omega}{r}$, and when C′ arrives at B, Q arrives at Q_2 upon the line OS. The center Q_3 of this circle is similarly situated behind C_3', the center of the other circle, as the points in the circumference are so situated. Likewise the locus of OT is a circle D_1TT_2 similar to Q_1QQ_2, having its center at T_3, so that OT_3 equals QQ_3 divided by r. Having the loci of T and of D, that of D′ is found since OT is always the resultant of OD and OD′. The locus of OD′, the condenser current, is the circle $OD'D_1'T_2$ upon OD_1' as diameter, which is OC_1 prolonged. That the diameter has this direction appears from the following: OD′ is always at right angles to OC, but when OC coincides with OC_1, the condenser current is zero, and OD′ starts out at the origin perpendicularly to OC_1. Hence the diameter has the direction of OC_1. The circle passes through T_2 because the transformer and condenser current coincide when the capacity is infinite, inasmuch as either current is very large compared with the line current OD at this point. When the capacity is infinite, it is equivalent to short-circuiting the transformer by a stout wire, so that no current will pass through the line. In this case the diagram becomes simple and the triangle of electromotive forces is as usual OBQ_2, so that BOQ_2 equals arc tan $\frac{L_2\omega}{r}$, and OT_2 equals OQ_2 divided by r.

Having obtained the current OT through the secondary of the transformer, the back electromotive force and primary electromotive force are readily obtained by methods discussed in previous papers. The construction for this is omitted from the diagram to avoid complexity. The reaction of the secondary upon the primary will be taken up later.

Diagram for Actual Transformer when the Condenser Terminals have a Constant Potential Difference, and the Capacity is Varied. — In practice there is a modification which amounts to a simplification that enters into the construction of this diagram. The transformer inductance is usually very large as compared with the internal resistance, so that the angle arc tan $\frac{L_2\omega}{r}$ is almost ninety degrees.

Take the particular example of a constant current transformer. The data for the transformer are as follows: $L_1 = 0{\cdot}032$ henry, $L_2 = 0{\cdot}016$ henry, $R_1 = 0{\cdot}227$, $r_2 = 0{\cdot}16$ ohm, $\omega = 1000$. A calculation gives,

$$\frac{L_2\omega}{r_2} = \frac{0{\cdot}016 \times 1000}{0{\cdot}16} = 100.$$

Hence, arc tan $\frac{L_2\omega}{r_2} = 90°$, approximately.

To draw the actual diagram, Fig. 156, corresponding to the typical diagram, Fig. 151, take OC arbitrarily equal to 150 volts. Let the line resistance R be 15 ohms, a rather high value. Then OD equals $150 \div 15 = 10$ amperes. The condenser current OD′ varies upon the line I_c, and the transformer current upon I_t. When the condenser capacity is ten microfarads, the condenser current, $C\omega$ times OC, is equal to $1000 \times 150 \times 10^{-5} = 1{\cdot}5$ amperes, and is represented by OD′. This gives the transformer resultant current OT a trifle more than ten amperes.

When the condenser capacity is zero, T coincides with D, and the impressed electromotive force to drive this current through the transformer is $L_2\omega$ times the current OD, and is 90° in advance of it. $L_2\omega \times OD = 0{\cdot}016 \times 1000 \times 10 = 160$ volts. Hence OC$_1'$, ninety degrees in advance of OD, and equal to 160 volts, is the electromotive force for the transformer impedance. Since the electromotive force OC′ is always 90° in advance of OT, and equal to $L_2\omega$ times OT, the locus of C′ is the line C$_1'$C′ at right angles to OC$_1'$. The transformer impressed electromotive force OB has for its locus BB′ in the same straight line with C′C$_1'$. This evidently corresponds completely with the diagram Fig. 151, when the angle COC$_1'$ becomes 90°. The lines BB′ and C$_1'$C′ become one and the same straight line parallel with OC.

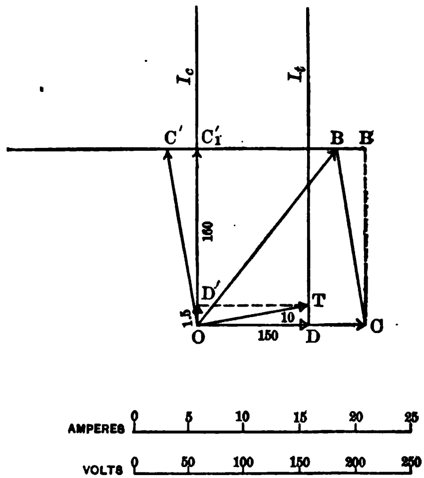


Fig. 156.

The Actual Transformer Diagram for Constant Primary Current, when the Capacity is Varied. — The typical diagram, Fig. 155, representing a general case, is likewise modified in the actual case supposed, since the angle C′OT becomes approximately a right angle. Let OA, Fig. 157, represent the primary current of 14·14 amperes supplied to the transformer, the data of which appear above. Then the impressed electromotive force OB upon the secondary is 320 volts. When the condenser capacity is zero, we have a rectangle OC$_1$BC′ upon OB as diagonal similar to that in Fig. 156, viz.: OCB′C$_1'$. The angle BOC$_1$ is arc tan $\frac{L_2\omega}{r_2+R}$, and in the case supposed is

$$\text{arc tan}\,\frac{0{\cdot}016 \times 1000}{0{\cdot}16 + 15} = 46{\cdot}5^\circ \text{ approximately.}$$

The circle OSC$_1$ here has its diameter OS at right angles with OB; and since C$_1$ is a point on the circle, the length of the diameter OS is found by drawing a line from C$_1$ perpendicular to OC$_1$, until it intersects the perpendicular to OB in S. This line C$_1$S is then the line BC$_1$, prolonged to S. The locus of OC′, the electromotive force for the transformer impedance, is now determined by a circle upon BC_2' as diameter parallel with OS and symmetrically situated with respect to OB. The locus of OD, the current through the line, is a circle OLD upon a diameter OL, having the direction of OS, and equal to OS divided by R. The condenser current, being always at right angles in advance of the electromotive force OC, has for a locus a circle $OD'D_1'P$, whose diam-

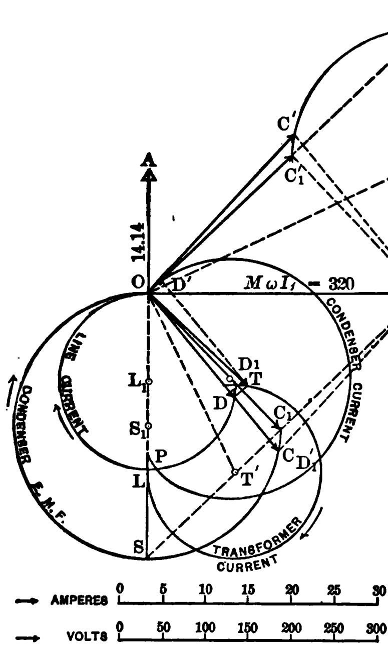


Fig. 157. Constant primary current diagram, showing effects of varying capacity.

eter OD_1' has the direction of OC_1, as explained in Fig. 155. The resultant or transformer current is determined by the two component currents already drawn; but it is a circle whose center T′ is on a line at right angles in this case behind the center C_3', as explained in Fig. 155.

Effects of the Variation of Secondary Load when the Primary Current is Constant. — By means of Fig. 153, which has been explained, we may construct the constant current diagram for a variation in the line resistance R. This diagram is represented in Fig. 158, where OB is the constant electromotive force impressed upon the transformer secondary.

Construct the triangle BOC_1, Fig. 158, similar to $B'OC$, Fig. 153. The line OC_1 then represents the position of the condenser electromotive force upon open circuit. The diameter of the circle upon which OC_1 varies is found by drawing a line OS, so that the angle BOS, Fig. 158, is equal to the angle COP, Fig. 153, between the fixed line OC and the perpendicular OP upon the line BB′. The magnitude of the diameter may be found by drawing from C_1 a perpendicular to OC_1, and producing until it meets OS. The locus of OC, the condenser electromotive force, is then represented by the arc OCC_1. The locus of the electromotive force for the transformer impedance is now determined, because the sum of OC and OC′ gives OB. Hence OC′ lies upon a circle $C_1'C'B$, symmetrically situated with respect to OB to the circle OCC_1.

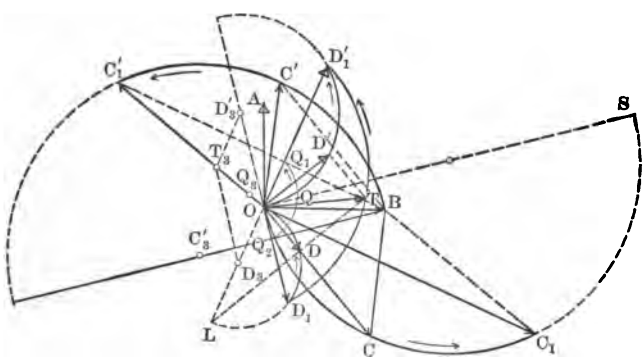


Fig. 158. Constant primary current diagram, showing effects of varying the line resistance.

The condenser current is now determined; for it is represented in Fig. 153 by the constant line OD′, a right angle in advance of OC, and equal to $C\omega$ times OC. In Fig. 158 this is represented by the arc $OD'D_1'$, the diameter of which is a right angle in advance of OS, and equal to $C\omega$ times OS.

The locus of OQ, the electromotive force for the transformer resistance, is found by drawing a circle with center Q_3, a point which lags behind C_3' by the constant angle $\arctan \frac{L_2\omega}{r}$. This gives for the locus of QO the arc Q_1QQ_2. From this the transformer current OT is determined, for it is always a constant multiple of OQ, $OQ = r.OT$. One point on the locus is known to be D_1', for here the line current is zero.

Another point is known to be D_1; for here the condenser current is zero. The center of the circle T_3 lies upon OQ_3 produced, so that $OQ_3 = r.OT_3$.

The locus of the line current is upon the arc ODD_1L, the diameter of which, OL, is a right angle behind OC_1, or two right angles behind OD_1'. This will appear from the consideration that OT is the geometrical sum of OD and OD′, or that OT_3 is the sum of OD_3' and OD_3 the lines drawn from O to the centers of the respective circles. The heavy part of the circles indicates the range within which a variation may take place, and the arrow shows the direction of such variation with an increase of line resistance.

Actual Diagram for Constant Condenser Potential when the Line Resistance is Varied. — Figure 153 represents the case where the line resisttance is varied and the condenser potential constant; but it is the typical case, and the lines are not drawn to scale. The self-induction, too, is not very large compared with the transformer resistance. On the other hand, Fig. 159 represents the diagram drawn to scale for the particular case of the transformer, the data of which have already been given. The condenser and line potential is arbitrarily chosen as 150 volts, and the capacity of the condenser in parallel with the transformer is 10 microfarads. The particular

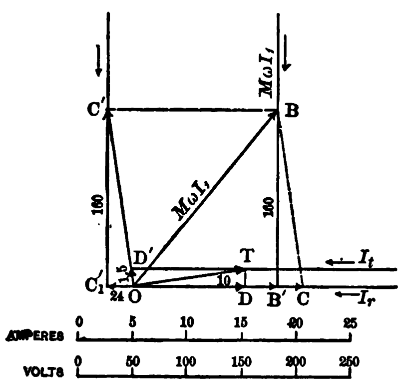


Fig. 159. Actual constant condenser potential diagram when $\frac{L_2\omega}{r}$ is very large.

value of the line resistance represented is 15 ohms, a rather high value. This makes the constant condenser current OD′ equal to 1·5 amperes, and the line current 10 amperes, for this particular resistance, if $\omega = 1000$. The transformer current OT will then be only a little more than 10 amperes. The electromotive force OC_1' for the transformer impedance is equal to $L_2\omega$ times OD′, upon open line circuit. This gives OC_1' equal to 24 volts. Hence OB′ is 150 — 24, or 126 volts. The reactance $L_2\omega$, being equal to 0·016 × 1000, or 16, is large compared with the internal resistance 0·16 ohm, and therefore the angle C′OQ of Fig. 153, the typical diagram, becomes here a right angle, C'OT. Thus the locus of the impressed electromotive force OB becomes a line perpendicular to OC.

This diagram may now be easily transformed into the constant current diagram.

Actual Constant Current Diagram when the Line Resistance is Varied.— Figure 160 is the constant current diagram for the case under consider-

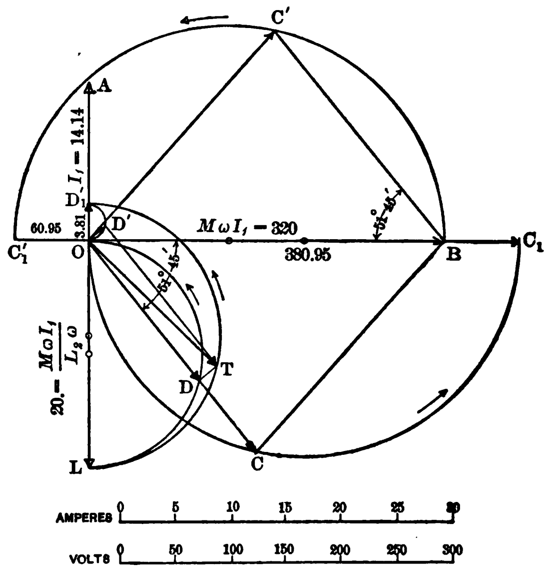


Fig. 160. Actual constant current diagram when the line resistance is varied and $\frac{L_2\omega}{r}$ is large.

ation. It may be derived directly either from Fig. 159 or from Fig. 158. If we take it from Fig. 158, it is simply necessary to make the angle $C_1'OQ_1$ a right angle, and all other angles which immediately result from it. Thus the angle BOS of Fig. 158 vanishes in Fig. 160 so that OB coincides with the axis of the semicircle OCC_1. The diameters of the current circles OL and OD_1' become perpendicular to the impressed electromotive force OB; for (Fig. 158) the radius OD_3' is always at right angles with OS, but in Fig. 160, OS coincides with OB. So also OL is always at right angles to OC_1, but in Fig. 160 OC_1 and OB coincide in direction. The resultant or transformer current circle has its diameter coincident with and equal to the sum of OL and OD_1'.

To consider the figure more in detail, the primary current of 14·14 amperes maximum value OA, causes an impressed electromotive force OB, upon the transformer secondary which is equal to $M\omega I_1$, or to $\sqrt{L_1 L_2}\,\omega I_1$; and numerically gives $\sqrt{0{\cdot}032 \times 0{\cdot}016} \times 1000 \times 14{\cdot}14$, or approximately 320 volts.

Upon open line circuit we have a single series circuit through transformer and condenser, the impedance of which is

$$\sqrt{r^2 + \left(\frac{1}{C\omega} - L_2\omega\right)^2}.$$

Numerically this is

$$\sqrt{0{\cdot}16^2 + \left(\frac{1}{10^{-5} \times 10^3} - 0{\cdot}016 \times 1000\right)^2},$$

or approximately 84 ohms. Therefore the condenser current which flows upon open circuit is equal to the impressed electromotive force 320 divided by the impedance 84; that is, 3·8095 amperes. This is the value of the diameter OD_1' of the condenser current locus. The electromotive force OB, or 320 volts, is the algebraic sum of two components, one, OC_1, to drive this current through the condenser, and the other for the transformer impedance. The condenser electromotive force OC_1 is equal to the current OD_1' divided by $C\omega$; that is, is equal to $\frac{3{\cdot}8095}{10^{-5} \times 10^3} = 380{\cdot}95$. So also the transformer part is equal to $L_2\omega$ times this current, or $0{\cdot}016 \times 1000 \times 3{\cdot}8095 = 60{\cdot}95$ volts. The difference between these electromotive forces, viz., $380{\cdot}95 - 60{\cdot}95 = 320$, the impressed electromotive force. Upon short circuit the condenser terminals are joined by a wire of zero resistance, and there is then no potential difference at the condenser terminals, and consequently no condenser current. All that we have then is a simple circuit of impedance

$$\sqrt{r^2 + L_2^2\omega^2} \text{ or } \sqrt{(0{\cdot}16)^2 + 16^2} = 16 \text{ approx.},$$

and impressed electromotive force 320 volts. This makes the transformer and line current equal to $\frac{320}{16} = 20$ amperes. This value of the line current upon short circuit is the same as without a condenser. The circle ODL representing the line current locus is the same as if the condenser were not present at all. But this difference is to be noted, *that while the locus is the same it does not follow that for any given line resistance the line current will be the same in the two cases either in direction or in magnitude.* This may be best illustrated by a numerical example. We will suppose that the line resistance is 15 ohms, and that we first have no condenser present, and then secondly connect in the condenser.

In the first case, with no condenser, we have OD lagging behind OB by an angle

$$\text{arc tan}\,\frac{L_2\omega}{r+R},\ \text{or arc tan}\,\frac{16}{15\cdot16}=\text{arc tan } 1\cdot056.$$

The angle of lag is then 46° 30′. The current is then equal to

$$\frac{320}{15\cdot16}\cos 46^\circ\ 30',\ \text{or to } 14\cdot5 \text{ amperes.}$$

In the second case, when the condenser is present, we may perhaps most easily find the angle of lag by reference to Fig. 159, from which Fig. 160 is derived. The angle BOC is the same in either figure. It may be obtained as follows from Fig. 159. The tangent of BOC is equal to BB′ divided by OB′, and OB′ is equal to

$$OC - OC_1' = 150 - 24 = 126.$$

But BB′ = $C'C_1'$. We have the proportion from the similar triangles $OC'C_1'$ and ODT that

$$\frac{OD}{TD}=\frac{C_1'C'}{OC_1'}.$$

Hence we have

$$C'_1C' = \frac{OD \times OC_1'}{TD} = \frac{10 \times 24}{1\cdot5} = 160 \text{ volts.}$$

This makes BOC = arc tan $\frac{160}{126}$ = arc tan 1·27 ; and $\theta = 51^\circ\ 45'$, which is a larger angle of lag than when the condenser was absent.

Referring again to Fig. 160, we may find the magnitude of the current by taking the value of OL sin 51° 45′, or 20 sin 51° 45′, which is 15·7 amperes approximately.

The example shows that shunting the condenser around the secondary actually increases the line current which existed before, and of course also the effective electromotive force at the line and condenser terminals, but at the expense of increasing the phase difference between the current and impressed electromotive force.

Up to this time no mention has been made of the reaction of the secondary current upon the primary electromotive force and current. It is thought that a single typical example illustrating the case just considered, where the secondary self-induction is large as compared with the internal resistance, will suffice.

Constant Current Diagram showing Secondary Reaction upon the Primary as the Line Resistance is Varied. — Figure 161 illustrates the

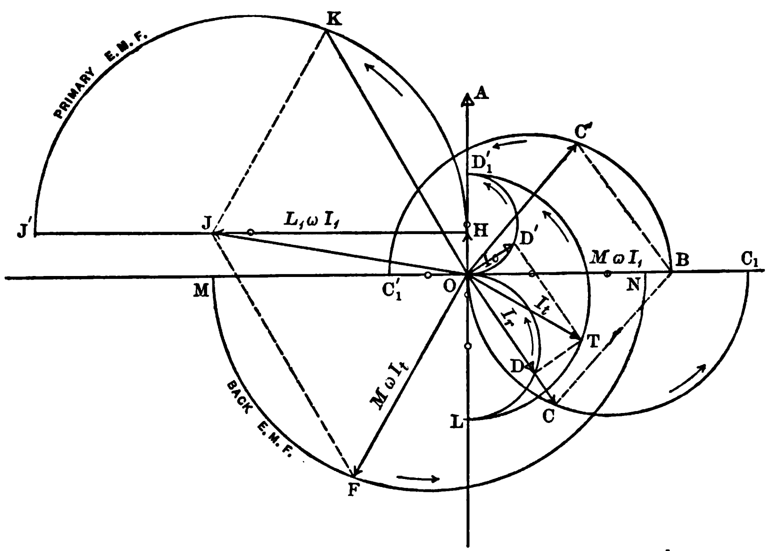


Fig. 161. Typical constant primary current diagram when the $\tan \frac{L_2\omega}{r}$ is large, showing reaction upon the primary.

$$OC_1 = \frac{M\omega I_1}{1 - L_2C\omega^2}. \quad OC_1' = \frac{M\omega I_1}{\frac{1}{L_2C\omega^2} - 1}. \quad OL = \frac{M}{L_2} I_1. \quad OD_1' = \frac{M\omega I_1}{\frac{1}{C\omega} - L_2\omega}. \quad HJ' = MN.$$

complete constant current diagram for a variation of the line resistance showing the back and primary electromotive forces.

The secondary currents and electromotive forces have been explained. The back electromotive force OF lies upon a circle MF, so that OF is always ninety degrees behind the transformer current OT and equal to $M\omega$ times OT. This makes the distance OM equal to $L_1\omega I_1$, the primary electromotive force of self-induction. The diameter of the circle has the direction of OM and terminates at a point N, so that the distance ON $= M\omega$ times OD_1'. The primary electromotive force OK has its locus similar to the back electromotive force circle, with diameter HJ′ equal to MN. Thus it appears that upon open circuit the primary electromotive force must be considerably larger than it is when the condenser is absent. It is this extra primary electromotive force which is used for driving the secondary condenser current upon open line circuit.

Below the typical diagram, Fig. 161, there are to be found the expressions for the principal lines and angles which may be derived from the figures from which this one is constructed.

The more common case where the primary is supplied with a constant potential still remains to be considered. Such diagrams may be constructed directly from the constant current diagrams just discussed. It is considered unnecessary to draw them here, inasmuch as the methods for constructing constant potential from constant current diagrams has been fully given in previous chapters.

CHAPTER XV

DESIGN AND CONSTRUCTION

The Evolution of the Transformer. — The alternating current transformer of to-day is essentially a modern contrivance. Its period of evolution has been short. Successive forms have been tried, each containing features more or less distinctive,

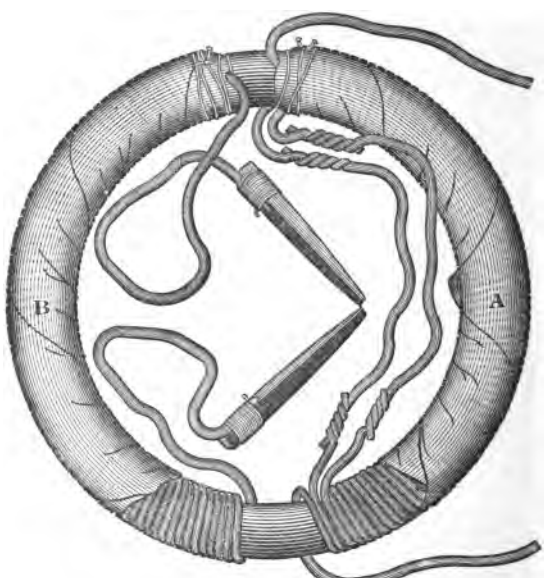


Fig. 162. Faraday's type of transformer.

leading to the adoption of the present commercial type, for reasons of economy in construction and efficiency in operation. The transformer is commonly considered to be a development of what is generally known as Ruhmkorff's induction coil, — more properly, the induction coil of Page. The construction of the induction coil of Faraday was, however, nearer that of the present type. Faraday announced his discovery of electromagnetic induction in 1831 in a paper read before the Philosophical Society.* He established the laws which are fundamental in the subject of electromagnetism and form the basis for the study of the principles of the transformer. The first transformer of Faraday consisted of two solenoids, one placed within the other. A galvanometer was connected to the secondary coil and a battery to the primary. The making and breaking of the primary circuit was observed to cause a throw of the galvanometer in the secondary. Faraday afterwards adopted a ring type of transformer, in which the primary and secondary coils were wound upon a continuous ring of iron, shown in Fig. 162 and Fig. 163.† The closed magnetic circuit was thus early introduced.

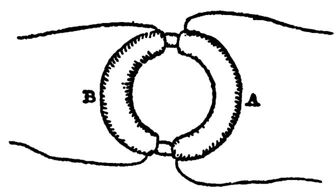


Fig. 163.

Faraday's transformer contained the essential elements of the transformer of to-day. We cannot, however, look upon Faraday as the inventor of the transformer, even although his apparatus was similar to that of the present day. The core of Faraday's induction coil consisted of a solid iron ring, lamination being a later invention. The primary and secondary coils were not wound continuously, so that there was a great deal of magnetic leakage. Furthermore, Faraday's induction coil, although close to the modern type, was not, strictly speaking, used as a transformer, for it was operated by interrupted or intermittent currents, rather than by an alternating current; moreover,

* *Transactions*, 1832; see also Faraday's *Experimental Researches*.

† Figure 162 is taken from the *History of the Transformer*, by F. Uppenborn, from which some of the following data have been obtained. Figure 163 is from the first volume of Faraday's *Experimental Researches*, it being the first illustration in this work.

the ratio of the primary and secondary turns was about unity. Although Faraday discovered the laws of electromagnetic induction, and constructed the first induction coil consisting of a primary and secondary circuit wound about a closed ring of iron, the modern transformer cannot be ascribed directly to him. The laws of magnetic induction, like many great discoveries, were not discovered by one man alone. During the years that Faraday was conducting his investigations, Joseph Henry of Albany was applying himself to practically the same line of study; for some time each was unaware of the other's work. As we now go back and look at their earlier writings, it is interesting to note the development of the same ideas in the minds of each, and how one was, perhaps, a little in advance in certain discoveries, only to be anticipated by the other in other discoveries equally important. Henry* independently discovered the phenomena of electromagnetic induction within less than a year of the time that they were first announced by Faraday. Henry at the same time noted the effects of *self-induction*, being the first to observe this phenomenon and to give it a name. The experiments of Henry were soon followed by similar ones performed by Dr. Page of Washington. The first experiments in self-induction of Henry and Page were made upon coils of copper ribbon. An interesting sketch of the historical development of the induction coil and transformer is given by Dr. Fleming,† who points out that the modern transformer is descended not so much from Faraday's ring as from Henry's flat spirals. Callan, Sturgeon, and Barker were among other experimenters with induction apparatus.

The induction coil, commonly known as Ruhmkorff's coil, was first constructed by Professor Page with essentially its

* *Silliman's Journal*, 1832. This volume and the following volumes contain many papers by Henry and Page relative to the early development of the induction coil. See likewise Sturgeon's *Annals of Electricity*, Vol. I.

† The reader is referred to his work, *The Alternating Current Transformer*.

present form. The induction coil of Professor Page,* constructed in 1838, consisted of a bundle of soft iron wire around which were wound (cylindrically) the primary and secondary coils. This is shown in Fig. 164. It was operated by an automatic make-and-break device, consisting of a mercury contact, the electrode being drawn out of the mercury by the motion of an armature drawn toward the core with every magnetization. The apparatus of Page made it possible to obtain a spark eight

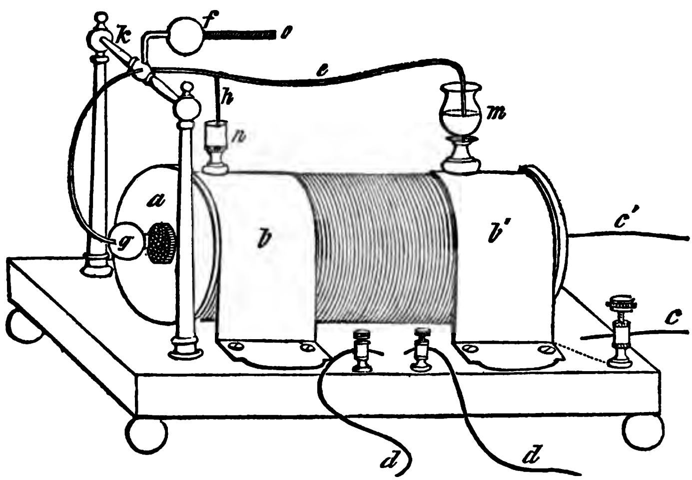


Fig. 164. Induction coil of Page.

inches long; the coils were sufficiently powerful to hold suspended a core weighing over 500 kilograms placed in the solenoid.

The induction coil of Page was little known in Europe at the time, over ten years later, that Ruhmkorff first made his spark inducer or induction coil, now known by his name.† The effects obtained from Ruhmkorff's first coil were on a much smaller scale than those obtained from the earlier coil of Page.

The Ruhmkorff induction coil is a transformer of the "open

* *Silliman's Journal*, 1839.

† A few years earlier Masson and Breguet had employed a toothed wheel as an interrupter.

magnetic circuit" type; its form is therefore not so closely allied to the common transformer of to-day as is the apparatus of Faraday shown in Figs. 162 and 163. The Hedgehog transformer of James Swinburne is of the open magnetic circuit type.

Poggendorff and Foucault both experimented with a Ruhmkorff coil. Their papers were published in 1855 and 1856. The induction coil had now been known in one form or another for almost twenty years and had been used by various experimenters; the open magnetic circuit was commonly employed. In 1856 Varley * described an induction coil with a closed magnetic circuit consisting of a bundle of iron wires with the ends bent back on the outside of the copper coils, a construction similar to that employed later by Ferranti, Fig. 165. In the modern transformer the attempt is made to reduce the reluctance of the magnetic circuit so as to obtain as large a mutual induction as possible; hence the closed magnetic circuit prevails. The use of induction coils in systems of distribution had not at this time been developed. Bright, Jablochkoff, and Harrison devised systems of distribution by induction coils for purposes of lighting.

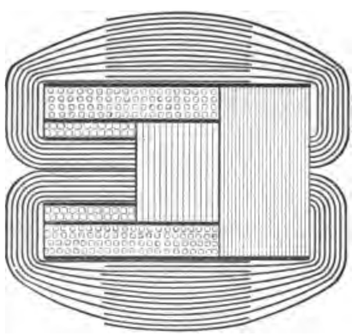


Fig. 165. Ferranti transformer.

At the Paris Exhibition in 1878 Jablochkoff's system of distribution by induction coils was in operation; he used an alternating as well as an interrupted direct current; this was the last attempt in the direction of the use of interrupted currents for distribution with induction coils. In the same year systems employing alternating currents and induction coils, with various modifications in arrangement, were employed by Sir C. T. and

* British Patent Specification, 3059; for further references, chiefly to British Patent Specifications, the reader is referred to S. P. Thompson's *Dynamo Electric Machinery*.

E. B. Bright, E. Edwards, and A. Normandy (who used ribbon conductors), Strumbo (who placed primary and secondary coils side by side), Fuller, Harrison, and Meritens (the last two distinctly describing the arrangement of transformer primaries in series). Fuller employed a method of adjusting the magnetic leakage in order to obtain regulation. His work was conducted in Brooklyn. His activity of a few years gave much promise, but an early death prevented the completion of his work.

The practical commercial use of transformers was not yet attained. Moreover, the object was not to develop a system of transformers in the present sense of the term. The attempt was not made to employ transformers as *transformers:* that is, for the conversion from a high to a low pressure or *vice versa.* The object of the modern transformer system of distribution is to transmit electric energy at a high pressure and to deliver it at a suitable low pressure, — the advantage arising from the economy in conductors where high pressure is employed, as explained in Chapter I. At this time this advantage arising from the employment of transformers was not appreciated, and the chief end sought in the employment of induction coils was the subdivision of the current. Regulation was not sought for, in fact was not understood. With the invention of the glow lamp by Edison and the introduction of the parallel system for incandescent lighting, attention was turned toward the requirements of regulation, so that lamps could be turned on or off without disturbing the supply of current furnished to other lamps in the system. To obtain the advantage of the high pressure transmission and small current, the employment of the series system was deemed necessary, inasmuch as this was found to work successfully for series arc lighting. In 1882 Gaulard and Gibbs patented a system for the commercial operation of incandescent lamps from transformers, and exhibited it at the exhibition at the Royal Aquarium in London in 1883. They still made use of the series distribution, and were evidently not familiar with the principles of transformer regulation. The transformer of Gaulard and Gibbs was of the Ruhmkorff coil type, with an equal number of primary and secondary turns. Their system was again tried in some of the stations of the Metropolitan Railway in London, and the series system found to be unsatisfactory. Kennedy (June 9, 1883, *Electrical Review*) pointed out that the parallel arrangement of transformers would make self-regulation possible. He did not have in mind the advantages of high potential transmission, for he considered at the time that the size of the conductors would be an objection to the parallel system. With the primary potential the same as the secondary, this system would be practically the same as one furnishing current directly to incandescent lamps in parallel, and there would be no gain from the use of transformers. The opinions expressed as to the advantages or disadvantages of the Gaulard and Gibbs series system generally displayed a lack of understanding of the principles involved. Those advocating the system usually failed to recognize the influence of the secondary current in inducing a back electromotive force in the primary, and considered that the primary current could be caused to flow through any desired number of transformers in series, and that it would be unaffected by changes of load in the secondary. This was a serious error. Some of those who recognized the disadvantages in the system arising from the influence of the secondary were equally at fault. In a constant current system, the primary and secondary electromotive forces vary with the amount of resistance placed in the secondary circuit. When no secondary current is flowing the electromotive force in the primary is a maximum. This has led to the extraordinary statement, that with series constant current transformers, the smaller the load the greater the energy consumed, and that with the secondary circuit open the energy used would be many times as great as under full load. This conclusion is evidently reached by neglecting phase relations. The angle of lag of the primary current behind the primary electromotive force changes with the load in such a way that the exact opposite of the above statement is true and in fact the energy consumed varies directly with the output of the apparatus.

The problem of an efficient self-regulating transformer system of distribution had not yet been properly attacked. The conditions were now such that the necessity of this kind of a system was recognized. The time had come for its development, and a number of inventors came forth at about the same time with valuable improvements. The parallel system of constant potential high-pressure distribution was brought out in the year 1885 by Zipernowsky, Deri, and Blathy, of Budapest, and for its development we are largely indebted to them and to William Stanley of this country.

The objects to be attained were economy of transmission and constancy of regulation; these were the real requirements rather than the so-called "division" of the current. The largest commercial applications of the transformer have been made by the Westinghouse Company, who were likewise among the earliest to adopt this system. The early use of the transformer by this company may therefore be considered as one of the important factors in the general introduction of the transformer. Early in 1885 William Stanley became persuaded that the transformer or converter system afforded the proper solution of the problem of how to distribute electric energy with least cost for conductors. He was at that time associated with the Westinghouse Company, who were considering the Gaulard and Gibbs system of series distribution. Mr. Stanley attempted to show the futility of this system, and offered in its place a parallel system employing transformers with closed magnetic circuits. As the Westinghouse Company were not at that time convinced of the value of the parallel system as proposed by Mr. Stanley, the latter went to Great Barrington, Mass., where he constructed with his own hands his first transformer, and operated it from November, 1885, to May, 1886. He supplied 150 sixteen-candle-power lamps at a distance of half a mile, the primary electromotive force being 500 volts. At this same time

Mr. Stanley constructed the first alternator of the radial pole type, now in general use, which is shown in Fig. 41. The work, which had thus far been conducted by Mr. Stanley at his own expense, was now taken up by the Westinghouse Company, by whom it was perfected and developed into the present system. Although Mr. Stanley took such an important part in the introduction of the modern transformer system, he was anticipated by the similar inventions of Zipernowsky, Deri, and Blathy, which were developed a little earlier in the year 1885. In that year three patents were awarded for their invention. Their system was displayed at the public exhibitions in Budapest, London, and Antwerp. Zipernowsky, Deri, and Blathy developed a complete system of constant potential distribution with high pressure primary mains. At the Inventions Exhibition in London in the summer of 1885, they displayed their systems in operation, employing a primary pressure of 1000 volts. It is they who must be looked upon as the first to solve the problem of transformer distribution. They employed two types of transformers; their first type consisted of a coil of iron wire upon which were wound the primary and secondary coils. This type is shown in Fig. 166. They likewise described a transformer in which the primary and secondary circuits formed a circular coil which is wound on the outside with soft iron wire; this form is shown in Fig. 167. The latter type was first used, but was replaced later by the first form. They later employed a laminated core more nearly like the modern type. Their first patents were soon followed by others. Although the transformer problem was solved in 1885, it was not immediately recognized that this was the solution. In 1886 Professor Forbes maintained that the parallel operation of transformers was impracticable. He considered that if the parallel system were used it would be necessary to have separate leads for each transformer, an arrangement which would be too complicated for commercial use. He maintained that the series system was the proper one.

At about the time of the inventions of Zipernowsky, Deri, and Blathy, and immediately after, improvements in transformers were patented by John and Edward Hopkinson, Kennedy, Ferranti, and others.

Mention has been made of the earlier types of transformers employed by Zipernowsky, Deri, and Blathy, consisting of a coil of iron wire, wound with copper wire for the primary or secondary, and the other form constructed in a similar way with

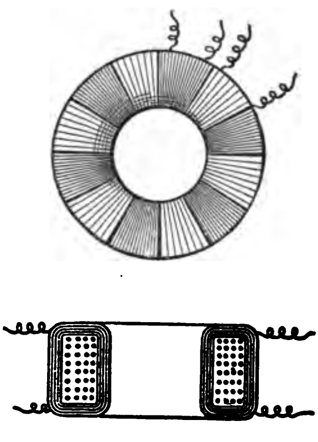


Fig. 166. Transformer of Zipernowsky, Deri, and Blathy; coils wound upon iron core.

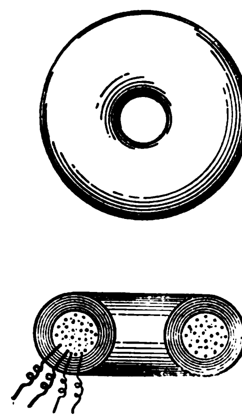


Fig. 167. Transformer of Zipernowsky, Deri, and Blathy; coils wound with iron wire outside.

the primary and secondary coils wound with iron wire, completely enclosing them (Fig. 167). The construction of these forms was laborious and expensive. Furthermore, they did not afford facilities for ready repair. The form later adopted and manufactured by the Ganz Company was constructed as follows: The magnetic circuit was built up of annular disks, stamped from thin sheet iron, these disks being separated by paper to avoid eddy currents. The primary and then the secondary coils were wound continuously over this core. This form offered ample surface for the radiation of heat.

The development of the transformer into its present type was now nearly complete. Various types and forms have been tried

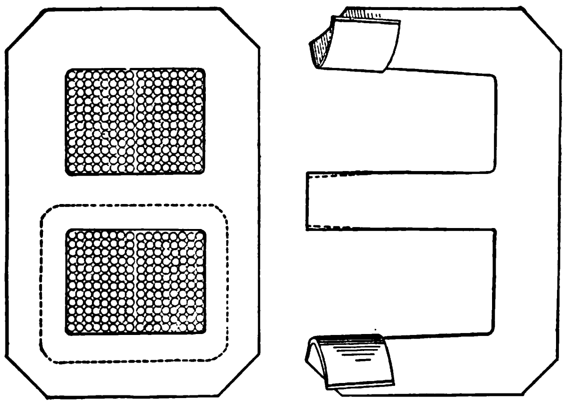


Fig. 168.

and adopted. With the notable exception of the Hedgehog transformer, the closed magnetic circuit is practically universal.

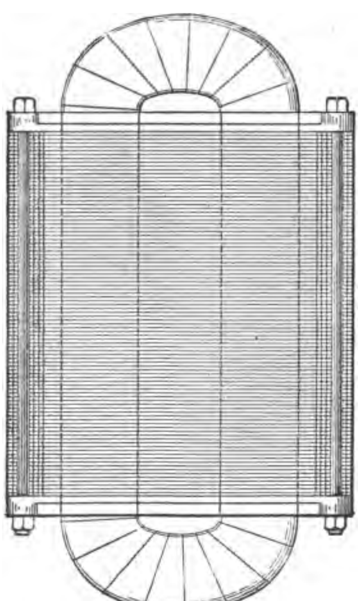


Fig. 169.

The modern transformer employs a closed magnetic circuit built up of plates of soft sheet iron (see Fig. 168), which are usually annealed after stamping. The magnetic circuit is then laminated in a direction at right angles to that in which the primary and secondary circuits flow. Eddy currents in the iron are thus reduced. The arrangement of coils and plates is shown in Fig. 169. The lamination does not increase the reluctance of the magnetic circuit, inasmuch as the magnetic lines of force pass around in a plane at right angles to the primary and secondary currents, and therefore do not have to pass from one plate to the next. The employment of the transformer is almost universal, and much attention has been given to its design and construction, to which we may now proceed.

Principles of Transformer Design. — In the design of any apparatus, the aim of the designer should be to produce the desired result with the smallest yearly expense. The yearly expense includes not only maintenance, interest, and attendance, but also cost of power wasted in the apparatus, a just allowance for repairs, and also for depreciation and final replacement. The ideal design must be based upon a consideration of each of these items of yearly expense as affected by the particular conditions under which the apparatus is to be operated; the best design for one set of conditions may not and probably will not be the best for other conditions.

Let us confine our attention to the design of a constant potential transformer having a closed magnetic circuit. The function of such a transformer is to transform an alternate current of a known periodicity and potential, and in any quantity up to a fixed maximum, into an alternate current of the same periodicity, but of another potential. This is to be accomplished with a prescribed excellence of regulation, as the load is changed between the limits of no load and full load. This is the object to be attained at a minimum yearly expense. The expense in the case of a transformer includes interest on first cost and a small percentage allowed for depreciation and repairs, and the cost of power wasted in hysteresis, eddy currents, and copper resistance, in addition to the cost of power delivered. These are the given conditions in the problem of transformer design; let us look at some methods of undertaking the problem.

Methods of Design. — The complete analytical treatment of transformer design is extremely difficult, if not impossible. The only other methods are the tentative, or trial and error method, and the empirical method. The first consists of designing, with variations of all the important dimensions, a large number of transformers, each of which has the required output. If the ranges of variation cover the most desirable dimensions, the best transformer for the work can be selected from the number designed. This method, while perfectly correct in theory, involves an enormous amount of labor; it may be advantageously employed in the design of new types of apparatus.

In the empirical method a study is made of the best existing transformers; curves and equations are thus obtained, showing the relations between the quantities most important in design. From these equations the main dimensions of the new transformer are determined. This method gives transformers just as good as those on which the equations are based. It is only an improved method of interpolating between the standard sizes, except so far as the equations are modified to suit new conditions. Only the most important of the designing quantities are determined in this way: such as $B_{max.}$, the maximum magnetic density used in the iron; A, the cross-section of the tongue of the iron core, shown in Fig. 176; and S_1 and S_2, the numbers of turns in the primary and secondary coils. After these are ascertained a large part of the work is still to be performed. All the proportions, arrangements, and details of construction must be determined from the designer's experience and his knowledge of the best practice and the properties of the materials used. Some of the considerations which determine these minor dimensions will now be discussed.

Proportions. — After the cross-section of the tongue is known, we wish to ascertain a proper relation between its length and breadth. In Fig. 176, $A = 2\,bc$; it remains to determine c. Let r denote the ratio $\frac{c}{2\,b}$. In the practice of a few years ago, r had a value from 6 to 8. In some recent large transformers, designed for low frequency, one company makes $r = 1$, so that the cross-section of the tongue is square. The effects of these changes may be easily followed by reference to Fig. 170. Suppose A and the cross-section of the primary and secondary coils to remain constant while r is varied. If r is small b is large, and the weight of iron and the iron loss are large; while the amount of copper and copper loss are small. If r is large the opposite conditions hold true. Analytical investigation shows that r should never be made as low as 1, even under extreme conditions, while for ordinary practice it should be greater than 2, from 3 to 7 being good values. Then

$$\frac{c}{2b} = r = 3 \text{ to } 7\text{; and } A = 4rb^2. \qquad (1)$$

Both primary and secondary coils lie side by side, in the same holes in the iron core; S_1 is the primary and S_2 the secondary coil, in Fig. 170. Suppose that at full load the primary ampere-turns at some instant are such as to produce the magnetism indicated by the heavy arrows; the induced secondary current is in such a direction that it tends to oppose this magnetism and tends to set up magnetic induction as indicated by the light arrows. Accordingly there is a large magnetomotive force established between X and X_1 along the line of separation between the two coils. Lines of force will leak across here; and since all the magnetism actually circulating around the iron circuit is caused by the primary coil, these leakage lines pass through the primary but do not pass through the secondary coil. There are magnetomotive forces, although less in amount, over the whole surfaces X, X_1, producing magnetic

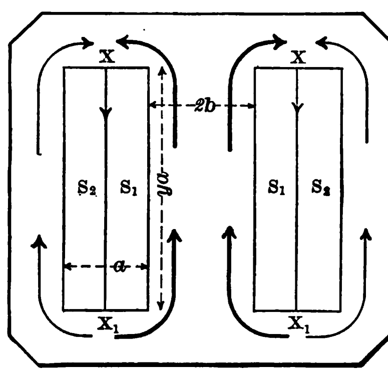


Fig. 170. Magnetic flux in transformer.

leakage. The primary flux-turns are constant, since the primary voltage is maintained constant. Magnetic leakage makes the induction through the secondary less at full load than that at no load; consequently the secondary potential at full load is less than at no load. The magnetomotive forces X, X_1 are constant in amount for the same number of ampere-turns, so the leakage flux varies inversely as the reluctance of the leakage path.

To obtain good regulation with two coil transformers, the coils should be arranged as shown in Fig. 170, which possesses a leakage path of large reluctance. The cross-section of a coil should be a thin rectangle whose length is the length of the hole in the iron core. The ratio y of the length to breadth of the slot for the coils, should be about 3 for ordinary lighting transformers; that is,

$$y = 2{\cdot}5 \text{ to } 3{\cdot}5. \qquad (2)$$

An increase of y increases the amount of iron required for the core. A value of y larger than necessary for good regulation means a useless addition to the cost of the transformer.

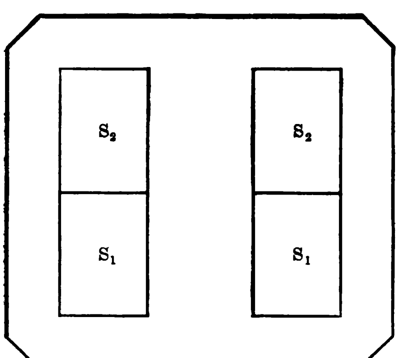


Fig. 171. Arrangement of coils in old-style Westinghouse transformer.

The old Westinghouse type of transformer, Fig. 171, gives the greatest opportunity for magnetic leakage, and should be avoided for the reasons given above; the drop due to magnetic leakage was found* to be excessive.

A recent novelty in design is shown in Fig. 172; although awkward in appearance, it possesses many excellent features.

* *Ryan on Transformer*, American Institute of Electrical Engineers, 1889, Vol. VII., p. 1.

The leakage drop is the same as in the ordinary form for equal values of y. The amount of iron is likewise the same, but there is slightly more copper in the coils because of the broader and

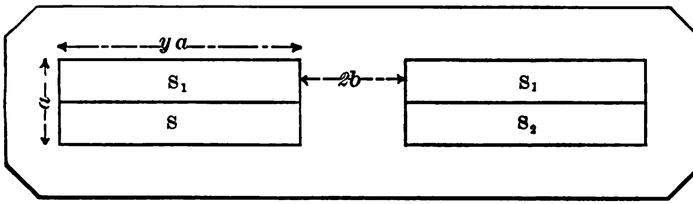


Fig. 172. Special arrangement of coils.

longer end connections. In many cases this would be more than compensated for by the greater ease of construction, for the coils are much easier to wind.

Multicoil transformers contain more than one primary and more than one secondary coil. If there are the same number of primary and secondary coils arranged alternately in the same hole, the leakage is reduced. The leakage in two transformers otherwise alike is inversely as the square of number of pairs of coils arranged in this way.

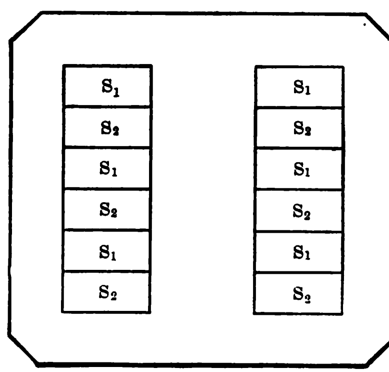


Fig. 173. Arrangement of coils in multicoil transformer.

In Fig. 173 the magnetic leakage is one-ninth that of Fig. 171. The multicoil construction is especially useful in large units and may be found in recent Westinghouse transformers; the arrangement of coils in one of their transformers is shown in Fig. 174.

Iron Core. — The iron core is an important feature of the transformer, being the link that connects the two circuits. The whole of the core is subjected to alternate magnetization of the same frequency as the alternate current. The hysteresis caused by this reversal of magnetism represents by far the greater part of the loss in ordinary transformer distribution. The iron loss takes place all the time that the transformer is in circuit, usually 24 hours a day; while the copper losses occur only while the current is used, perhaps 4 or 5 hours a day. The important efficiency of the transformer, the ratio of total output to total input or "all day" efficiency, is influenced much more by the iron than by the copper loss, as shown later.

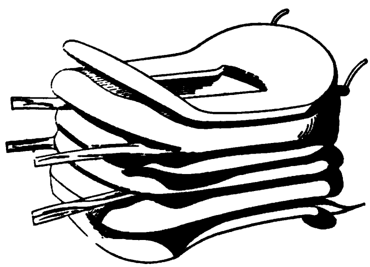


Fig. 174.

The hysteresis loss in the iron is made small by using the best quality of iron, by careful annealing just before assembly, and by employing a low magnetic density. Excessive eddy current loss is prevented by building up the core of thin iron plates 0·015″ thick. The oxid is relied on to insulate the plates from one another sufficiently for this purpose, although shellac, paper, or other insulation is sometimes employed.

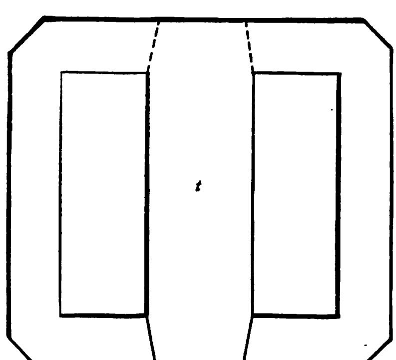


Fig. 175. Westinghouse transformer-plate.

Transformer iron should be of the best obtainable quality, as free from hysteresis as possible, and of high permeability. The magnetic circuit should be designed to offer as little reluctance as possible; there should accordingly be few joints. The form of stamping almost universally used in this country is the Westinghouse pattern, shown in Fig. 175. It is made from one piece of metal, and the tongue (t) is separated at one end by the two cuts shown, so that the plate may be sprung over the coils. These cuts form one joint in each branch of the magnetic circuit. The reluctance introduced is made very small by the method of "stacking" the plates, alternate plates having the tongue pointing in opposite directions, as indicated. The joints thus occur first on one side of the core and then on the other. When the plates are clamped firmly together, the reluctance of the joint is almost zero.

There are other methods of designing the magnetic circuit; a marked exception to the closed magnetic circuit type being the Swinburne Hedgehog transformer. The iron circuit is open and is composed of fine iron wire, as in an ordinary induction coil, the ends of the wire being spread out, as the name of the transformer implies. While the iron loss is much less in this transformer than in the standard type, it has a large no-load current which increases the copper losses in the line and generator, although it does not materially affect the efficiency of the transformer.

Joints are sometimes made in the iron circuit for convenience of construction and repair. Fig. 176 shows such a construction. The cover-plate is built up of laminations fastened rigidly together, and fits closely upon the mass of E-shaped plates. This style of practice is generally condemned.

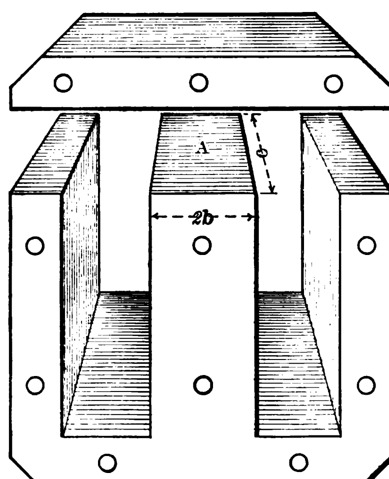


Fig. 176.

The Primary and Secondary Coils. — The copper conductors in the primary and secondary coils should be made large enough to carry the normal current without undue heating, and to be safe in case of a momentary rush of current somewhat in excess of the normal. An allowance of 1000 to 1500 circular mils* per ampere is sufficient and in accord with average practice. Conductors with only 800 circular mils per ampere are sometimes used. Conductors are designed for a current that will give the rated power at a power factor of unity. This method of rating is universal; the transformer is only supposed to work up to its rating when it has a non-inductive load, such as incandescent lamps.

As a large solid conductor, especially if of circular cross-section, has the alternate current crowded out toward the surface, and its apparent resistance thereby increased,† no wire larger than No. 8 B. & S. should be used. Large conductors are also excluded on account of difficulty in winding. Beyond this size and up to 50,000 circular mils, conductors may be made up of a number of wires in parallel, although ribbon conductors are preferable. For conductors larger than 50,000 circular mils, flat copper ribbons are always used. These ribbon conductors are not subject to the same objection as large wires; they fit together better and make a firmer coil. The copper strip is rounded on the edges so that any burr is removed, and is made up into a conductor of the desired cross-section. The strips are laid together without insulation between them; the ribbon is then insulated as a whole. To give the desired flexibility, the copper strip is not made thicker than $\frac{1}{8}''$ for every foot in the shortest radius over which the conductor is to be bent.

Insulation. — The wire used in transformer construction is usually double cotton covered. For moderate sizes this insulation adds 0·02″ to the diameter. Ribbon conductors are insulated from each other in two ways. The conductor may be

* A mil = 0·001 inch. The area of a circular wire in *circular mils* is found by squaring its diameter given in mils.

† See paragraph "Virtual Values," Chapter IV. See also a paper "On the Transmission of High Frequency Currents by Wires and Cables," by E. Merritt; Proceedings of the Cornell Electrical Society, p. 1, Vol. III.

taped before winding with an insulating tape that will add about 0·08″ to its thickness; or a strip of insulating cloth or tape, one tenth of an inch wider than the ribbon, may be wound into the coil under the conductor, as in Fig. 177. Such an insulating strip would be perhaps 0·02″ thick. All the turns of the section will thus be insulated from one another by one thickness of this cloth, which will project at the sides of the section. The section is then lightly taped as a whole to insulate and to preserve the form.

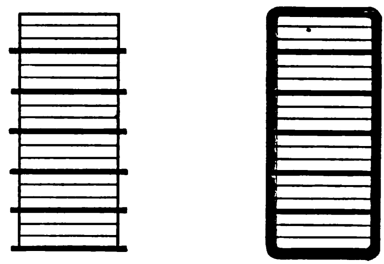


Fig. 177. Arrangement of ribbon conductors.

In winding wire into coils, care must also be taken to insulate between the layers, in addition to the insulation of the separate conductors, especially if each layer contains many turns. The cotton insulation of two neighboring conductors should not be relied upon to withstand more than 50 volts. The conductors of two layers have sometimes a potential difference of 500 volts; they must be insulated by several layers of prepared cloth or paper.

Fig. 178 shows the potential difference between conductors of two layers, when each layer generates 200 volts; it also

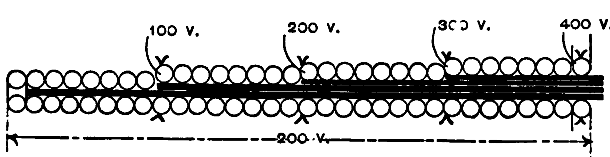


Fig. 178. Potential difference between different parts of coil.

shows what would be a proper mode of insulation, if we suppose that each thickness of insulating material can safely stand 100 volts.

Some of the prepared cloths and papers give remarkable insulation, and may be safely employed at a pressure of 500 volts (factor of safety of 5) with a thickness less than 0·01″. For high pressures some form of mica insulation should be used.

Insulation between layers is objectionable, for this insulation often occupies as much space as the wire itself. To avoid the use of insulating material between the layers of a coil, which is necessary when there are many turns in a layer, coils are often made with many layers and few turns per layer. Such a coil is very easy to construct and proves satisfactory. A contrast of the two extreme methods is shown in Figs. 179

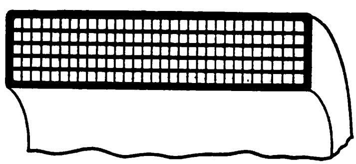


Fig. 179. 30 turns per layer, 5 layers.

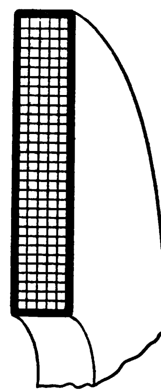


Fig. 180. 5 turns per layer, 30 layers.

and 180. The second would probably require no layer insulation at all.

All transformer coils are taped externally, and are usually painted with some insulating compound for insulation and protection and to preserve their shape. High potential coils must be taped so as to insure good insulation from the low potential coils and from the core and case. Immersion in oil, mentioned later, greatly increases this coil insulation.

Ventilation. — The energy wasted in a transformer appears as heat, and this heat, often large in amount, must be radiated without too great a rise of temperature. Transformers are generally enclosed in cast iron cases when intended for ordinary outdoor service. Formerly an air space was left between the case and the transformer. Radiation was thereby greatly retarded, and the rise of temperature was often excessive. For this old practice, an allowance of six square inches of radiating surface on the transformer itself per watt loss is considered sufficient. The radiating surface means the total exposed surface of the coils and core, independent of the case.

Transformers when intended for indoor use are not enclosed in a water-tight box. They are merely enclosed sufficiently to protect them from injury, and the case is arranged so that air can circulate freely about the transformer. A good example of this is seen in the Stanley transformer, Plate I., appended.

During the last few years a new feature of practice has appeared. The transformer is placed in the case and then the case is filled with a mineral oil of about 300° flash test. This oil improves coil insulation greatly and also allows much freer radiation. The limit of heat radiated is now determined by the radiating surface of the case, an allowance of three square inches per watt loss being sufficient. With large transformers (over 50 k.w.) ventilation becomes a serious problem, and special expedients are often employed. Forced circulation of oil* or a circulation of cooling water in connection with oil immersion† are used with encased transformers. Air circulating through special passages in the core‡ is used with open transformers.

Transformer Losses. — The losses in a transformer are those due to the heating of the primary and secondary conductors, equal to $R_1\bar{I}_1^2$ and $R_2\bar{I}_2^2$, respectively, and the losses due to eddy currents and hysteresis in the iron core. The copper losses vary as the square of the load, approximately. At full load they usually amount to from 2 to 4 per cent of the output;

* Carborundum works transformer, General Electric Co.

† Niagara type, Westinghouse transformer.

‡ General Electric construction at Pittsburgh Reduction Co., Niagara.

in large transformers of good design the copper losses may be $1\frac{1}{2}$ per cent, or less. The iron loss, including losses due to hysteresis and Foucault currents, is the same at all loads in a constant potential transformer, and the effort is made to reduce this as much as possible. In a constant current transformer the conditions are different; the copper losses are constant, and should be small. In this case the iron losses vary with the load, and may be larger than in a constant potential transformer; the magnetization may accordingly be higher.

For a small open circuit current, and for small losses at light loads in a constant potential transformer, a magnetic circuit with low reluctance is required. A low resistance for both the primary and the secondary is desired for good regulation, and for high efficiency at full load. Good design consists in the proper proportioning between the amount of copper and iron employed, which necessitates a compromise between the above conflicting requirements. The conditions for a small magnetizing current are determined as follows, from the formula for magnetizing force:

$$H = \frac{4\pi}{10} \frac{S_1 I_0}{l};$$

from which we obtain

$$I_0 = \frac{10}{4\pi} \frac{Hl}{S_1} = \frac{10}{4\pi} \frac{Bl}{S_1 \mu}. \qquad (3)$$

To have I_0 small, the magnetic induction should be low, the magnetic circuit short, the permeability large, and the primary should have many turns.

The power lost in hysteresis in the iron core of a transformer depends upon the quality of the iron, its volume, the degree of magnetization, and the frequency of alternation. The empirical formula of Steinmetz gives the hysteresis loss in ergs per cubic centimeter for one cycle as $\eta B^{1.6}_{max.}$, where η is a coefficient depending upon the particular iron used, and $B_{max.}$ is the maximum value of the number of C. G. S. lines per square centimeter. If n is the number of cycles per second, the loss is Vol. $\times\, n\eta B^{1\cdot6}_{max.}$ ergs per second, or

$$\text{Power lost in hysteresis} = \frac{\text{Vol.}}{10^7} n\eta B^{1\cdot6}_{max.} \text{ watts.} \qquad (4)$$

The volume is here given in cubic centimeters. The value of the coefficient η varies from 0·002 to 0·02. In transformer iron the limits would be about 0·002 and 0·0045.

Kind of Iron.	Coefficient η.	Kind of Iron.	Coefficient η.
Very soft iron wire	0·0020	Soft annealed cast steel	0·0080
Very thin soft sheet iron	0·0024	Soft machine steel	0·0094
Thin good sheet iron	0·0030	Cast steel	0·0120
Thick sheet iron	0·0033	Cast iron	0·0162
Mitis metal	0·0043	Hardened cast steel	0·0250
Sheet iron	0·0045	Magnetite	0·0204

Hysteresis is increased by hardening and decreased by annealing. The more pure the iron, the less is the value of η. For practical purposes these values are independent of frequency. Transformer plates should be annealed *after* being stamped; otherwise, although well annealed sheet iron is used, it will become hardened by stamping, and the hysteresis loss will thus be increased.

As the hysteresis loss varies directly with the frequency, the magnetic density at which a transformer should be worked depends upon the frequency, if the hysteresis loss is to be the same. A much lower induction is used in this country, where the frequency is commonly over 100 periods per second, than is employed in Europe, where the frequency as low as 50 is often employed.

Influence of the Form Factor upon Transformer Losses. — We have thus far generally considered the action of a transformer subjected to an alternating potential difference, the instantaneous values of which form a sine curve. Where this is not the case, the equivalent sine wave offers a convenient means of treating the problem analytically and graphically; but we must not lose sight of the fact that the sine wave is not the exact equivalent of the curve represented. In some of its effects it may be very different; thus, there is a difference in the transformer losses caused by different wave forms of electromotive force. An elaborate treatment of the subject is given by Dr. G. Roessler,* which is here followed.

If E_{av} is the average or mean value of the electromotive force for half a period, the ratio of E_{av} to $\bar{E}$, the virtual or square root of mean square value, is the form factor f, which is a constant for the particular wave form.

$$\text{Form factor} = f = \frac{E_{av}}{\bar{E}} \tag{5}$$

The form factor varies for different electromotive force waves, being unity for a rectangular wave and decreasing in value as the wave form becomes more and more pointed. For the sine curve, $f = 0{\cdot}9$, as given in Chapter IV.

The induction $B_{max.}$ in terms of E_{av} is †

$$B_{max.} = \frac{E_{av} \times 10^8}{4AnS}. \tag{6}$$

* "Behavior of Transformers under the Influence of Alternating Currents of Different Wave Forms," *Electrician*, Nov. 24, 1895, and following. Form factor, as used by Dr. Fleming, is the reciprocal of that used by Dr. Roessler, which is here adopted.

† This may be proved as follows:

From Faraday's law, $e = 10^{-8}\,SA\,\dfrac{db}{dt},$

e being in volts. Hence, $db = \dfrac{10^8}{SA}\,edt.$

During half a period, b changes from $-B_{max.}$ to $+B_{max.}$, while e rises from zero to a maximum and falls to zero again.

$$\int_{-B}^{+B} db = \frac{10^8}{SA}\int_0^{\frac{T}{2}} edt.$$

But $\dfrac{2}{T}\displaystyle\int_0^{\frac{T}{2}} edt = E_{av}.$

Writing $\dfrac{T}{2}E_{av}$, or $\dfrac{1}{2n}E_{av}$, for the above integral,

$$2B_{max.} = \frac{10^8}{2nSA}E_{av}.$$

Substituting $f\bar{E}$ for E_{av}, from (5),

$$\boldsymbol{B}_{max.} = \frac{f\bar{E} \times 10^8}{4AnS}. \qquad (7)$$

The maximum value of the magnetic induction varies, therefore, directly with the form factor f for the same virtual value of the electromotive force $\bar{E}$. The flatter the electromotive force wave, the higher is the magnetic density. The hysteresis losses are therefore dependent upon the form factor. By Steinmetz' formula the hysteresis loss per cycle in each cubic centimeter is

$$\text{Hysteresis loss} = \eta \boldsymbol{B}^{1\cdot6}_{max.}.$$

By substituting the value for $\boldsymbol{B}_{max.}$ given in (7);

$$\text{Hysteresis loss} = \eta\left(\frac{f\bar{E} \times 10^8}{4AnS}\right)^{1\cdot6}. \qquad (8)$$

The hysteresis loss in a transformer, when supplied with an electromotive force of a certain virtual value $\bar{E}_1$, varies as the 1·6 power of the form factor.

The more sharp the electromotive force wave, the less will be the hysteresis losses when the transformer is used at the same primary potential. This is important in the case of transformers worked on open circuit, where nine-tenths of the loss of the transformer system is due directly or indirectly to the iron loss. The no-load current, that is responsible for much line loss and a low plant efficiency, is cut down by the sharp electromotive force curve, because less magnetizing current is required for the lower magnetic density. These considerations have helped to make the T-toothed iron-clad alternator, whose electromotive force curve is sharper than a sine curve, particularly adapted to lighting purposes.

The influence of the form factor upon eddy current losses is as follows:

Suppose that the small eddy circuit in the iron is non-inductive; then the power wasted in the differential eddy circuit equals the square of its virtual voltage divided by its resistance:

$$dP_e = \frac{(d\bar{E}_e)^2}{dR_e};$$

$$P_e = \int \frac{(d\bar{E}_e)^2}{dR_e}$$

The electromotive forces of the eddy circuits are produced by the same magnetic flux that produces the counter electromotive force equal to $\bar{E}$; hence,

$$P_e \propto \bar{E}^2. \tag{9}$$

But from (7)

$$\bar{E} = \frac{4\,AnS\boldsymbol{B}_{\text{max.}}}{f \times 10^8};$$

hence, for each cubic centimeter,

$$\text{Eddy current loss} = P_e \propto \left(\frac{4\,AnS\boldsymbol{B}_{\text{max.}}}{f \times 10^8}\right)^2. \tag{10}$$

This may be written

$$P_e = \frac{\xi n^2 \boldsymbol{B}^2_{\text{max.}}}{f^2}, \tag{11}$$

where ξ is a coefficient dependent upon the quality of the iron and upon the degree of lamination.

In any transformer working at a constant virtual primary electromotive force, the eddy loss is independent of the wave form (9), but is inversely as the square of the form factor if $\boldsymbol{B}_{\text{max.}}$ is constant (11).

In Mr. Steinmetz' empirical formula for eddy current losses, the form factor f has been neglected; that is, it is considered to be a constant (0·9).

Method of Professor Ryan. — The following method and empirical equations are due to Professor Harris J. Ryan, and are used by him in his course on transformer design. The forms of the equations were deduced from some examples of 1891 practice; the constants employed have, however, been changed to bring the method into accordance with current practice. The empirical equations given below are limited to high frequency transformers of 100 to 135 periods per second, and are not suited to any other frequency. They give good commercial transformers up to an output of about 25 kilowatts. When applied to the design of transformers intended to be operated under one set of conditions, the equations may be used with one set of constants throughout the whole range for which they are intended.

The empirical formulæ for the maximum value of the magnetic induction per square centimeter, in terms of the secondary power, P_2, expressed in watts, are:

For transformers in which the output P_2 is less than 1000 watts,

$$\boldsymbol{B}_{\text{max.}} = \frac{200{,}000}{P_2} + 2800; \tag{12}$$

For transformers in which P_2 is more than 1000,

$$\boldsymbol{B}_{\text{max.}} = 3050 - \frac{P_2}{20}. \tag{13}$$

The number of primary turns is determined by an empirical formula, the constant of which depends upon whether the transformer is to be used for lighting or power, the difference arising from the fact that a power transformer is in circuit only when loaded, while a lighting transformer is usually in circuit 24 hours a day, although only fully loaded a few hours each day, as already explained. The best transformer for lighting service is one with a smaller iron loss and a greater copper loss than desirable in a transformer for power service.

For power transformers:

$$S_1 = \frac{\overline{E}_1 \times 10^4}{\sqrt{1{\cdot}11\, n P_2 \boldsymbol{B}_{\text{max.}}}}. \tag{14}$$

For lighting transformers:

$$S_1 = \frac{1{\cdot}9\,\overline{E}_1 \times 10^4}{\sqrt{n P_2 \boldsymbol{B}_{\text{max.}}}}. \tag{15}$$

Virtual values for $\overline{E}_1$ and $\overline{E}_2$ are used. The secondary turns are

$$S_2 = S_1 \frac{\overline{E}_2}{\overline{E}_1}. \quad (16)$$

The cross-section A_0, in square centimeters,* of the solid iron in the transformer tongue, is determined by equation (37) Chapter IV.:

$$A_0 = \frac{\overline{E}_1 \times 10^8}{4{\cdot}45\, n S_1 B_{max.}}. \quad (17)$$

Numerical Example. — We will apply these equations to the design of a 5 kilowatt transformer to be used for power purposes to transform from 1000 volts to either 50 or 100 volts, using a frequency of 125.

Using formula (13)

$$B_{max.} = 3050 - \tfrac{5000}{20} = 2800.$$

Formula (14) for power transformers gives

$$S_1 = \frac{10{,}000{,}000}{\sqrt{1{\cdot}11 \times 125 \times 5000 \times 2800}} = 227.$$

The secondary is to be made in two sections, each of which will give 50 volts. We will keep this fact in mind, but otherwise design the transformer as though the secondary was to furnish 100 volts.

$$S_2 = \frac{100 \times 227{\cdot}5}{1000} = 22{\cdot}7.$$

* Expressing the gross area of tongue A'' in inches, and keeping $B_{max.}$ as the induction per square centimeter of solid iron, (17) becomes

$$A'' = \frac{3{\cdot}875\, \overline{E}_1 \times 10^6}{S_1\, n B_{max.}}. \quad (18)$$

The watts lost in hysteresis per cubic inch (gross) of iron is

$$\frac{n B^{1{\cdot}6}_{max.}}{2{\cdot}83 \times 10^8}.$$

To obtain an even number of turns in each section of the secondary, we will adopt

$$S_1 = 220; \quad S_2 = 22.$$

From formula (17) the net area of the iron core equals

$$A_0 = \frac{100,000,000,000}{4{\cdot}45 \times 125 \times 220 \times 2800} = 292 \text{ square centimeters};*$$

$$A_0'' = \frac{292}{2{\cdot}54^2} = 45{\cdot}2 \text{ square inches.}$$

We will rely on oxidation to prevent eddy currents. Since a laminated core is only 90% solid iron,

$$A'' = \frac{A_0''}{0{\cdot}9} = \frac{45{\cdot}2}{0{\cdot}9} = 50{\cdot}2 \text{ square inches.}$$

This gives us A'', the area of the laminated tongue in square inches.

From equation (1)

$$A = 2\,bc = 4\,rb^2, \text{ where } r = \frac{c}{2\,b}.$$

We will assume $r = 4$, and take the area A'' in square inches.

$$b^2 = \frac{50{\cdot}2}{16} = 3{\cdot}14 \text{ square inches.}$$

$$b = 1{\cdot}77; \quad 2\,b = 3{\cdot}54 \text{ inches.}$$

We will take $2\,b = 3{\cdot}5$ inches;

$$c = \frac{50{\cdot}2}{3{\cdot}5} = 14{\cdot}375 \text{ inches.}$$

If our stampings are 0·015 inch thick, the number required will be

$$\frac{14{\cdot}375 \times 0{\cdot}90}{0{\cdot}015} = 862 \text{ stampings.}$$

* A_0 is the net and A the gross area in centimeters. The mark $''$ indicates area in inches.

Design of the Coils.—The secondary current will be

$$\bar{I}_2 = \frac{P_2}{\bar{E}_2} = \frac{5000}{100} = 50 \text{ amperes.}$$

The primary current is

$$\bar{I}_1 = \frac{S_2}{S_1}\bar{I}_2 = \frac{50}{10} = 5 \text{ amperes.}$$

Allowing 1200 circular mils per ampere, the conductors will have the following cross-section:

60,000 circular mils, for the secondary.

6000 circular mils, for the primary.

For the primary conductor we will adopt No. 12 B. & S. wire, with bare diameter 0·0808 inch, and with an area of 6530 circular mils.

We now make a preliminary estimate of the cross-section of the coils.

Primary copper space $220 \times \overline{0{\cdot}0808}^2 = 1{\cdot}43$ square inches.

Assuming that the secondary will occupy the same amount of space as the primary, and, for our preliminary estimate, that the total space required is twice the cross-section of the bare copper,

$$\text{Total coil space} = 4 \times 1{\cdot}43 = 5{\cdot}72 \text{ square inches.}$$

ya^2 = the cross-section of the coils (see Fig. 183), and we will assume $y = 3$.

$$a = \sqrt{\frac{5{\cdot}72}{3}} = 1{\cdot}38 \text{ inches;}$$

$$ya = 4{\cdot}14 \text{ inches.}$$

We will assume that the coils are arranged as in Fig. 183 and adopt $ya = 4$ inches.

The Primary Coil.—We will allow $\frac{1}{8}$ inch in the breadth of the coil for clearance and $\frac{1}{8}$ inch for taping. The breadth of the coil or actual winding is accordingly 3·75 inches.

Adding 0·02 inch for double cotton insulation to the bare diameter (0·0808 inch) of the primary conductor, we have 0·10 inch for its insulated diameter. The number of turns per layer and the number of layers will accordingly be

$$\text{Primary turns per layer} = \frac{\text{breadth of winding}}{\text{diam. of wire}} = \frac{3 \cdot 75}{0 \cdot 10} = 37.$$

$$\text{Number of layers} = \frac{S_1}{\text{turns per layer}} = \frac{220}{37} = 5\tfrac{35}{37};$$

this gives 6 layers, the last layer having only 35 turns.

The thickness of the section of the primary is

6 layers of wire,	0·6	inch.
10 thicknesses linseed cloth,	0·2	inch.
Outside taping,	0·125	inch.
Thickness of section, taped,	0·925	inch.

This will permit the section to be located in the 4-inch space, with proper allowance for clearance (see Figs. 181 and 183).

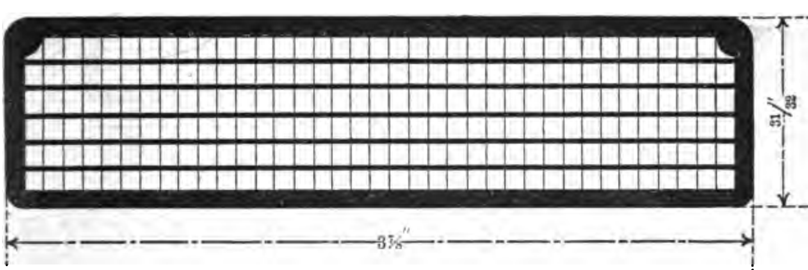


Fig. 181. Section of primary coil.

Secondary Coil.—The probable least radius over which the secondary will be bent is $\frac{1}{4}$ foot. Allowing $\frac{1}{8}$ inch thickness per foot radius, gives an allowable thickness of copper strip of $\frac{1}{32}$ inch. The thickness of the ribbon conductor should be so chosen that too much space will not be wasted in insulating, and also that an even number of sections may be placed side by side, since the transformer is to have two 50-volt secondaries. We will allow $\frac{1}{16}$ inch increase in width of each section for light taping. The conductors will be insulated from one another by being interwound with linseed cloth as explained in connection with Fig. 177. Assuming six sections side by side in the breadth of the coil,

$$\frac{3{\cdot}75 - 6 \times \frac{1}{16}}{6} = 0{\cdot}56 \text{ inch, the width of copper strip.}$$

The area of the copper ribbon is

$$60{,}000 \text{ circular mils} = \frac{60{,}000 \times \frac{\pi}{4}}{1{,}000{,}000} = 0{\cdot}047 \text{ square inch.}$$

$$\text{Thickness ribbon conductor} = \frac{0{\cdot}047}{0{\cdot}56} = 0{\cdot}084 \text{ inch.}$$

Copper strips come in thicknesses corresponding to the Birmingham wire gauge (B. W. G.), and are cut in any width. This conductor may be made of two No. 19 B. W. G. strips, 0·56 inch wide, giving the exact area, the thickness of each strip 0·042 inch being nearly within the desired limit, $\frac{1}{32}$ inch.

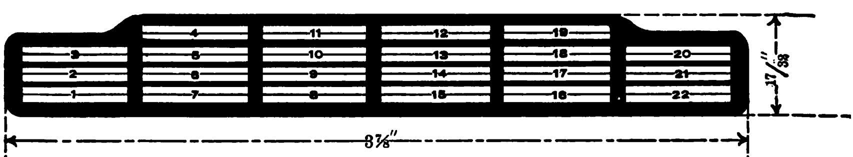


Fig. 182. Section of secondary coil.

The secondary coil will be arranged as in Fig. 182; have 22 turns as determined above and connected in the order indicated.

The thickness of coil is determined as follows:

Four ribbons 4 × 0·084 =	0·336	inch.
Three thicknesses linseed cloth =	·06	inch.
Double outside taping =	·125	inch.
	·521	inch.
Clearance,	·125	inch.
Space for secondary with clearance,	·646	inch.
Space for primary with clearance,	1·05	inches.
Space for both coils = a =	1·696	inches.

If we assume $a = 1\cdot75$ inches, we may give the primary coil a heavier taping.

Iron Losses.—The volume of the core (see Fig. 183) is

$$\text{Vol.} = 4\,bc(a + ya + 2\,b).$$

The volume of the iron is 90 per cent of the above:

$$\text{Vol. of iron} = 2\,A_0(a + ya + 2\,b).$$

Using the final dimensions:

$$\text{Vol.} = 2 \times 45\cdot2\,(3\cdot5 + 4 + 1\cdot75) = 836 \text{ cubic inches};$$

$$\text{Vol.} = 836 \times 16\cdot4 = 13{,}700 \text{ cubic centimeters.}$$

The hysteresis loss is found from equation (4). We have

$$B_{\text{max.}} = 2800; \text{ and } 2800^{1\cdot6} = 332{,}000.$$

Taking the hysteresis coefficient $\eta = 0\cdot0024$, we have

$$\left.\begin{array}{r}\text{Hysteresis loss}\\ \text{per cubic centimeter}\end{array}\right\} = \frac{0\cdot0024 \times 125 \times 332{,}000}{10^7} = 0\cdot01 \text{ watt.}$$

Total hysteresis loss = 137 watts; this is constant for all loads.

Copper Losses.—Suppose the ends of the coils are circular, as in Fig. 183; the average length of a turn in each coil is found by adding the length of the straight parts and the circular ends.

Average length, primary turn

$$= 2\,(14\cdot375) + 2\,\pi\left(\frac{3\cdot5 + 1\cdot05}{2}\right)$$

$$= 43\cdot05 \text{ inches.}$$

Average length, secondary turn

$$= 2(14\cdot375) + 2\pi\left(\frac{3\cdot5 + \cdot65}{2} + 1\cdot05\right)$$

$$= 48\cdot35 \text{ inches.}$$

Length primary coil $= 220 \times 43\cdot05 = 9480$ inches.

Length secondary coil $= 22 \times 48\cdot35 = 1063$ inches.

The resistance of a conductor varies directly as its length and specific resistance and inversely as its cross-section. If

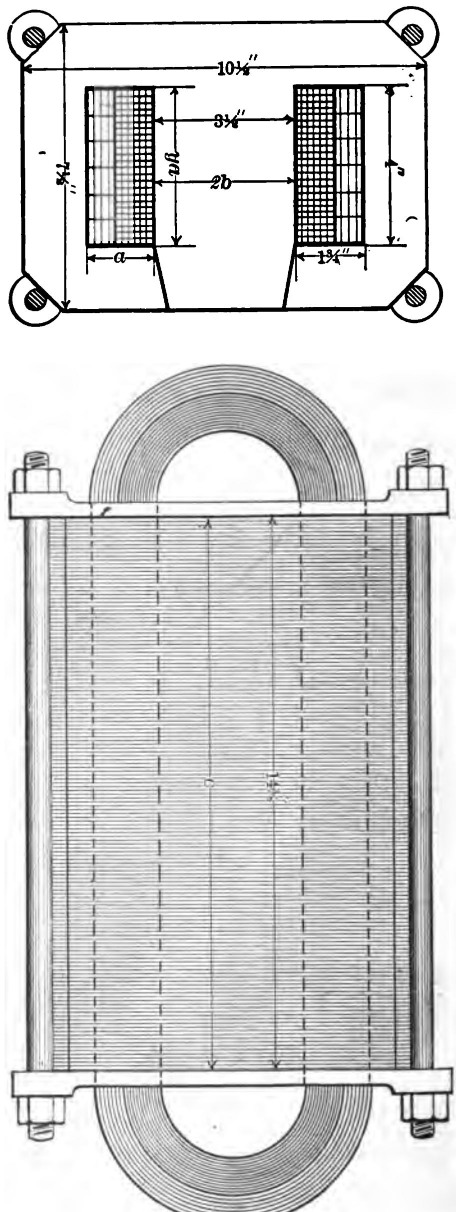


Fig. 183. Design for 5-kilowatt power transformer: one-fifth size.

we know the resistance of a mil-foot* of the material, the resistance of a conductor may be computed in ohms as follows:

$$R = \frac{\text{resistance per mil-foot} \times \text{length in feet}}{\text{area in circular mils}}.$$

If we take 12 ohms as the resistance of a mil-foot of copper used in transformers, the resistance in ohms is

$$R = \frac{\text{length in inches}}{\text{area in circular mils}}.$$

The resistances of the primary and secondary are accordingly

$$R_1 = \frac{9480}{6530} = 1{\cdot}453 \text{ ohms.}$$

$$R_2 = \frac{1063}{60{,}000} = 0{\cdot}0177 \text{ ohm.}$$

The copper losses at full load are:

Primary copper losses $= R_1 \bar{I}^2_1 = 36{\cdot}3$ watts.
Secondary copper losses $= R_2 \bar{I}^2_2 = 44{\cdot}3$ watts.
Total copper losses $=$ $80{\cdot}6$ watts.

Since $\bar{I}_1$ and $\bar{I}_2$ vary almost in proportion to the output, the copper loss varies approximately as the square of the output.

Calculating the losses for various outputs, we obtain the following efficiencies:

Load.		Copper Loss.	Total Loss.	Efficiency.
%	Watts.			
10	500	0·8	137·8	78·4
20	1000	3·2	140·2	87·6
30	1500	7·3	144·3	91·2
40	2000	12·9	149·9	93·0
50	2500	20·2	157·2	94·1
60	3000	29·0	166·0	94·7
70	3500	39·4	176·4	95·3
80	4000	51·5	188·5	95·5
90	4500	65·4	202·4	95·7
100	5000	80·6	217·6	95·84

* A conductor one foot in length and with a cross-section of one circular mil.

The no-load current of a transformer is calculated from the formula

$$Hl = \frac{4\pi}{10} S_1 I_0.$$

The value of H that corresponds to B is taken from a curve of magnetization for iron such as that used. In the present example, H is about 1·5, if the iron is good. The length l of the magnetic circuit is 47 cm.; therefore, the no-load current

$$I_{0_{max.}} = \frac{Hl}{1 \cdot 26\, S_1} = \frac{1 \cdot 5 \times 47}{1 \cdot 26 \times 220} = 0 \cdot 254 \text{ ampere.}$$

If the current is a sine curve, the virtual value is

$$\bar{I}_0 = 0 \cdot 254 \times 0 \cdot 707 = 0 \cdot 18 \text{ ampere.}$$

Special Uses of Transformers. — Aside from the systems in which transformers are commonly used, there are a large number of special arrangements employed for particular purposes. The principles of operation are the same as in a simple transformer and need no separate discussion. The following are a few of these special devices:

Reactive Coil. — For purposes of regulation in systems of distribution and adjusting the illumination of incandescent lamps, the reactive coil, sometimes called a "dimmer," has been used, which is essentially a transformer with an adjustable mutual induction. Such a coil is shown in Fig. 184. It consists of a laminated iron core, on one side of which the primary is wound. The secondary is short circuited, and consists of several turns of heavy copper, which may be revolved by a handle so as to slide completely over the primary. In the previous chapters it has been explained that the action of the secondary circuit is to diminish the self-induction of the primary. This reactive coil (that is, its primary circuit) is placed in series with the lamps or other devices in which the current is to be regulated. The apparent self-induction of the reactive coil may be increased or diminished by moving the short-circuited secondary circuit to and fro. When the secondary is over the primary circuit, the self-induction is apparently *nil*, and little impedance is given to the current. When the secondary is removed to the opposite part of the iron core, the self-induction of the primary apparently increases and impedes the current. In this case the lines of force from the primary pass through

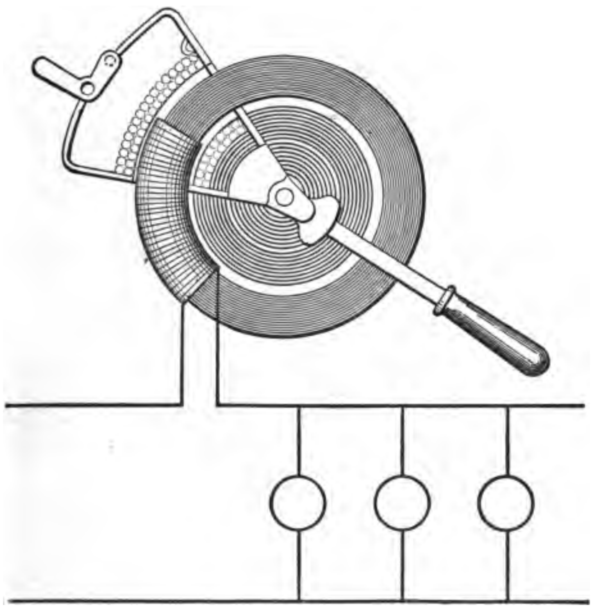


Fig. 184. Thomson's reactive coil or dimmer.

the movable iron disc which is concentric with the iron ring, and the apparatus acts like a simple impedance coil.

Compensated Voltmeter. — For the determination of the difference of potential between mains at a distance from the station, an indicating device consisting of an arrangement of transformers, as shown in Fig. 185, has been used by Mr. Kapp. The transformer A is connected in series with one of the mains. The fall in potential in the main is proportional to the current, and may be represented by the difference of potential between the secondary terminals of the transformer A, inasmuch as the secondary electromotive force on open circuit is proportional to the primary current; that is, $E_2 = M\omega I_1$. The secondary voltage of A therefore indicates the drop in the line RI. The difference of potential at the further end of the line is therefore equal to the potential at the dynamo, less this drop. The

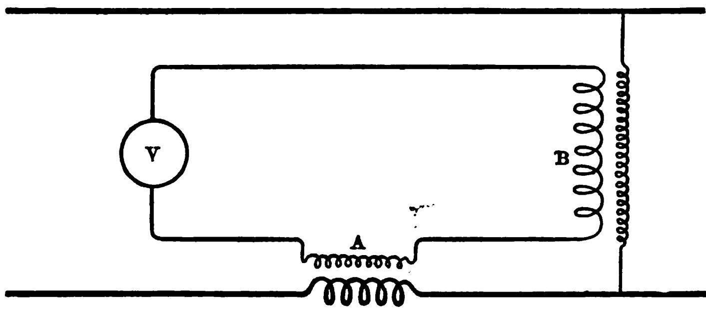


Fig. 185. Compensated voltmeter.

arrangement shown in Fig. 185 makes it possible to determine this directly. Transformer B is arranged in shunt so as to measure the station electromotive force E. The secondaries of A and B are so connected that the voltmeter indicates the

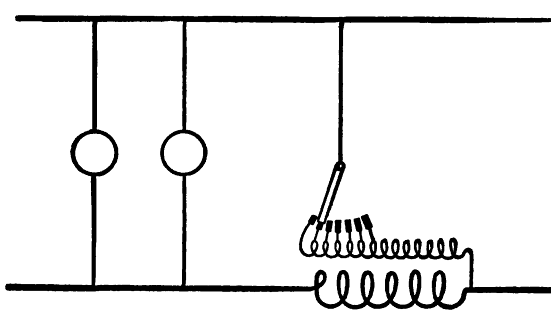


Fig. 186. Potential regulator.

difference in their electromotive forces; that is, it gives the reading proportional to $E - RI$, which is the difference of potential at the further end of the line. For accuracy, the secondary terminals of transformer A should be connected through a non-inductive resistance; otherwise there will be a phase difference between the secondary voltages, and their difference will not be a true indication of difference of potential on the line. This resistance, however, is not commonly used.

Hopkinson had previously described a similar method of indicating pressure at the distant end of a continuous current

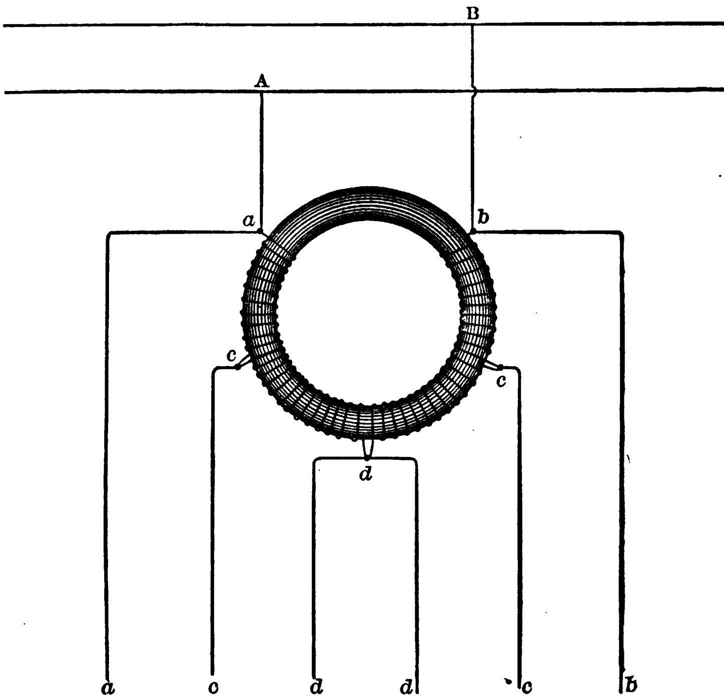


Fig. 187. Single coil transformer. Thomson's compensator system.

circuit by means of a differential galvanometer with one coil in shunt and the other in series with a line circuit.*

A similar device was employed by Zipernowsky, Deri, and Blathy for maintaining a constant difference of potential at a distant point in a system of transformer distribution. Two

* See British Patent Specification, No. 3576; also Fleming's *Alternate Current Transformer*, Vol. II., p. 92.

transformers were employed, as in Fig. 185, one in series and one in shunt with the primary mains. These secondaries were connected differentially to a regulating device, which raised or lowered the electromotive force obtained from the generator, so as to keep the potential constant at the far end of the line. This end has been accomplished by Mr. Kapp by an arrangement shown in Fig. 186, consisting of a transformer with one coil in parallel and the other in series with a feeder circuit. By adjusting the number of turns and the direction of current in the shunt coil, the potential may be raised or lowered as desired.

Compensator System of Elihu Thomson. — This system was devised by Professor Thomson in order to obtain the advantage of high pressure transmission in supplying incandescent lamps. The compensator consists essentially of a single coil transformer, as shown in Fig. 187. This system was brought out before the modern transformer system was extensively employed. The primary mains were connected directly to the terminals of the single coil. Conductors were connected to the coil at various points, dividing it into equal parts, and these conductors furnished current for the multiple wire system for operating incandescent lamps.

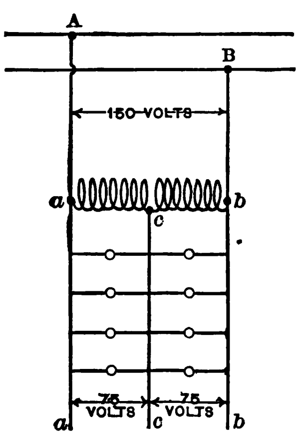


Fig. 188. Single coil transformer.

In Fig. 188 is shown the representation of the simplest form of this system. The terminals of the single coil transformer are supplied, let us suppose, with 150 volts from the line. The conductor *c*, connected with the middle wire of the coil, forms the middle wire of the three-wire system *a*, *b*, *c*.

Considering the compensator coil as a transformer, the primary circuit is *ab*, and the secondaries *ac* and *cb* connected in the three-wire system, as shown. This system is similar, therefore, to that shown in Fig. 16, Chapter I. The sum of the pressures of the so-called secondary circuits *ac* and *cb* must be equal to the pressure of the mains AB. For a line potential of 1000 or 2000 volts, therefore, a multiplicity of secondary circuits would be necessary. In the ordinary transformer, Fig. 11, in which the primary and secondary coils are independent, there is no such restriction, and the primary and secondary turns may be so taken as to give any desired ratio of transformation.

In Fig. 189 a five-wire compensator system is shown. In this arrangement of a single coil transformer, when the secondary circuits are loaded equally they are supplied directly with current from the mains A, B, and the current in the coil is not increased by the load. When the secondary circuits are not equally loaded, there is sufficient current flowing in different portions of the coil to equalize the potential. This system not only has the disadvantage mentioned above of necessitating a large number of secondary circuits to obtain the advantage of high pressure primary mains, but it lacks likewise the advantage of the usual transformer system in that the high pressure is brought to the lamp circuit in the house.

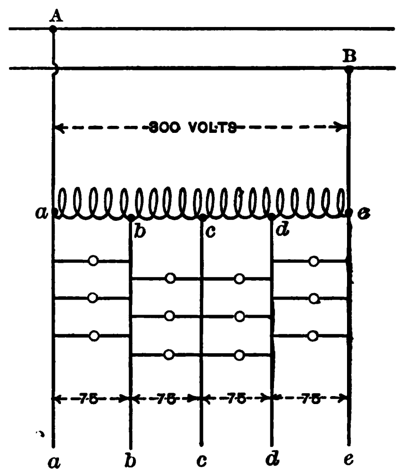


Fig. 189. Single coil transformer.

Stillwell Regulator. — The regulator known as the Stillwell Regulator, manufactured by the Westinghouse Company, is, in fact, a single coil transformer. It is shown in Fig. 190, which is a transformer in which A is the primary and B the secondary circuit, the number of turns in the latter being adjustable. The pressure in the lamp circuit may be raised or lowered (by a few volts) by adjusting the number of secondary turns, which is done by a switch conveniently arranged for the purpose. When the secondary turns are fewer than the primary, the current in the mains is less than that in the lamp circuit; this is a case of a divided circuit with one branch current greater than the main current.

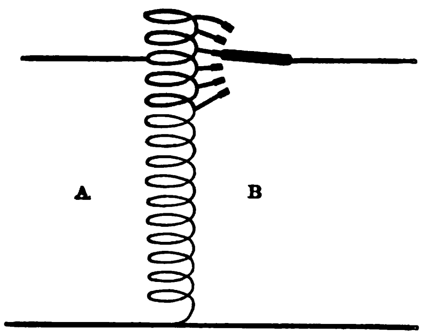


Fig. 190. Stillwell regulator.

Westinghouse Stage Regulator. — This device, like the preceding, is essentially a single coil transformer, with a movable core by which the pressure in the lamp circuit may be continuously altered. This is shown in Fig. 191, and diagrammatically in Fig. 192. The lamps are brightest when the core is down, for then the induction in the lower part of the coil is the greatest, and the self-induction of the upper part of the coil becomes small. When the core is raised, the inductance of the upper or series part of the coil becomes considerable and impedes the current.

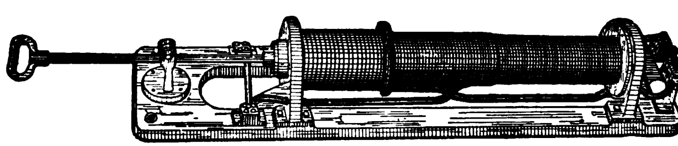


Fig. 191. Westinghouse stage regulator.

Furthermore, the lower part of the coil has now little impedance and acts as a shunt to the lamps in parallel with it.

Electric Welding. — For the purposes of electric welding a very large current with small electromotive force is required, and this is obtained by employing a transformer in which the secondary consists of one massive turn of copper.

All-Day Efficiency. — In considering the relative economy of various types of transformers, we must take into consideration the specific duty to be performed. If the transformer is to be operated only at its full capacity, high efficiency at full load is the desideratum, and it is immaterial whether the efficiency at any other load is high or low; similarly, if the transformer is to be operated a large portion of the time at some particular load, less than the full capacity of the transformer, let us say at one-fourth of its full load, high efficiency upon light load is most important. When the load of the transformer is different at different hours of the day, it is the "all-day" efficiency which is to be considered. The data from

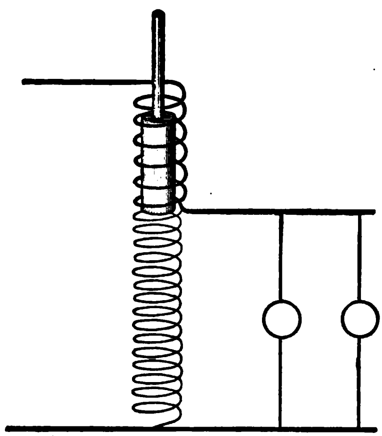


Fig. 192. Westinghouse stage regulator.

which to obtain such an efficiency are not often readily obtainable. If we know the load upon the transformer for each hour in the day, we may construct a load diagram in which P_2 represents the secondary output at each time, and P_1 the corresponding power supplied to the primary. The average efficiency is the ratio of the areas included by the first and second curves. Where the load varies from day to day and from month to month, the average efficiency for the year may be obtained in the same manner.

The features which determine the relation between the iron and copper losses in a transformer for the greatest ultimate economy are the cost of coal (or power), the cost of the materials used in the transformer, the nature of the system in which the transformer is to be operated, and the load diagram of the transformer. Inasmuch as copper losses are important only when the transformer is loaded, and iron losses are the same even at no load, so long as the primary circuit remains connected to the main, it follows that the reduction of iron losses is of considerable importance for high all-day efficiency. In the absence of data from which to construct a load diagram, we may assume the following basis for computation of all-day efficiency. Let us suppose that the transformer is connected in circuit continuously, and that it is operated at full load for 5 hours, and that there is no load upon the transformer for the remaining 19 hours. Let us see how the all-day efficiency is affected by the relation between the iron and copper losses. Assume that the efficiency at full load is 95 per cent; that is, that the total loss amounts to 5 per cent. Table I. gives the

EFFICIENCY OF TRANSFORMERS UNDER DIFFERENT CONDITIONS.

TABLE I.

Copper Losses. %	Iron Losses. %	Total Losses. %	Efficiency, Full-Load.	Efficiency, All-Day.
0	5	5	95	79·8
1	4	5	95	82·5
2	3	5	95	85·3
3	2	5	95	88·3
4	1	5	95	91·5
5	0	5	95	95·0

all-day efficiency of such a transformer, in which the total loss of 5 per cent is differently divided between the copper and iron losses. Although the efficiency at full load is the same in all cases, we note a marked rise in the all-day efficiency as the iron losses are diminished.

Let us take a case in which a decrease of 1 per cent in the iron losses is paid for by an increase of 3 per cent in the copper losses. Computations on this basis are given in Table II. As the iron losses are diminished, the total losses are increased on account of the great increase in copper losses, and the efficiency at full load is thereby diminished. We note, however, an increase in the all-day efficiency, notwithstanding the great increase in the total losses in the transformer at full load, and in this case the highest all-day efficiency is obtained from the transformer which has the lowest efficiency at full load.

EFFICIENCY OF TRANSFORMERS UNDER DIFFERENT CONDITIONS.

TABLE II.

Copper Losses. %	Iron Losses. %	Total Losses. %	Efficiency, Full-Load.	Efficiency, All-Day.
0	5	5	95	79·8
3	4	7	93	80·7
6	3	9	91	81·7
9	2	11	89	82·7
12	1	13	87	83·8
15	0	15	85	85·

In the design of the transformer for a specific purpose, or in the selection of one, we must, then, consider not only the efficiency at full load, but the efficiency with the conditions under which it is to operate. The computation of all-day efficiency by assuming 5 hours of full load and 19 hours of no load is crude, and by no means covers all cases. It is correct for the particular set of conditions; for other conditions corresponding assumptions must be made.

Relation between Size, Output, and Efficiency.—It is interesting to note the relation between the size of a transformer and its output and efficiency. These relations have been pointed out by Mr. Kapp in a paper before the Royal Engineers. Let us suppose that all linear dimensions are made twice as large; that is, the weight of the transformer is increased eightfold.

Suppose that the induction per square centimeter is kept the same, and likewise the number of primary and secondary turns. The total induction is then increased fourfold, and so, too, the electromotive force. Each coil has now twice the length of wire with four times the cross-section, hence the resistance is reduced one-half. Let us assume the same per cent copper losses and voltage drop as before. As we have the electromotive force increased fourfold, and the resistance reduced one-half, the current will be eight times its former value under the assumption of the same per cent voltage drop. The output, being the product of the electromotive force and the current, is now thirty-two times as much as before. We have, then, the following relations: the output of a transformer varies as the fifth power of its linear dimensions; the weight, and hence the cost of material, varies as the volume, that is, as the cube of the linear dimensions; the weight and cost per unit of output accordingly vary inversely as the square of the linear dimensions. In the case assumed above the copper losses will be the same per cent as before. The iron losses vary as the third power of linear dimensions; the percentage iron loss will be reduced to one-fourth of its previous value. The all-day efficiency is thus much increased. If we wish to increase the output of the transformer x-fold, the weight and cost per kilowatt and percentage loss in the iron will be multiplied by the factor $x^{-\frac{2}{5}}$, and so reduced.

The above computation, based upon the assumption of the same per cent copper loss, doubles the current density, since all areas are increased fourfold, and the current is eight times its former value. Let us make the computations anew, assuming the current density to remain unchanged, and that the magnetic induction per square centimeter is kept the same, as in the previous case. Doubling the linear dimensions will then increase the current and electromotive force each fourfold, and the output sixteen fold, while the iron and copper losses are only increased by the factor eight. The losses increase half as rapidly as the output. The increase in efficiency is thus greater than in the previous case, but there is not so much decrease in cost of material or the weight per kilowatt, for the large transformer has one-half instead of one-fourth the weight per kilowatt of the small transformer. Gain in efficiency is obtained by a greater cost of construction.

Even should we employ the same amount of material per kilowatt, the efficiency of the larger transformer will be greater, although there is little gain in first cost; if the efficiency were 92 per cent for the smaller, it would be 96 per cent for the larger transformer.

There is one modification which must be taken into consideration. The energy lost in the copper and in the iron is dissipated in the form of heat radiated from the surface of the transformer. The rise in temperature should not exceed 45° C., although a greater rise, 60° C., would be safe. The rise of temperature increases the resistance of the coils and so affects the efficiency and regulation. In many good transformers the rise of temperature is kept below 35° C. A radiating surface of from 3 to 10 square inches for each watt lost in the transformer is found necessary. If this were disregarded, and large transformers were made with the same proportions as smaller ones in accordance with the relations given above, the heating would be excessive. The size of large transformers must therefore be determined by the necessity of providing a proper means of cooling. There is then a limit to the increased economy of large transformers. The rate at which heat is dissipated must be increased by some special device in large transformers, and this adds materially to the cost of large sizes, — over 50 kilowatts. The advantages of large transformers are in part counterbalanced by this fact.

CHAPTER XVI

EXPERIMENTAL DETERMINATION OF TRANSFORMER DIAGRAMS

THE experiments described in the following chapter were undertaken with the object of obtaining data from which to construct polar transformer diagrams, for comparison with the theoretical diagrams discussed in the foregoing chapters, with particular reference to the effects caused by the variation of the resistance in the secondary circuit. Experimental transformer diagrams are not adapted to show the results of commercial tests; they are, however, well adapted to show the phase relations of the different currents and electromotive forces and to verify, accordingly, the theoretical work of the preceding pages.

The conditions under which the experiments were performed were those best suited to give results directly comparable with the results of theory. The ratio of transformation was small, so that the primary and secondary measurements may be represented conveniently to the same scale. On account of an air gap in the magnetic circuit, the hysteresis was small. Furthermore, the magnetic leakage was large, and shows pronounced effects in the diagrams obtained. These are not commercial conditions, but are suited for the present purpose.

The transformer * employed was particularly adapted for the

* This transformer was constructed by Dr. Crehore and the writer for experimental work. The experiments which follow are described in a paper "Transformer Diagrams Experimentally Determined," by F. Bedell, *Proceedings of the International Electrical Congress*, Chicago, 1893, p. 234, and were performed with the assistance of Messrs. A. W. Berresford, W. M. Craft, and B. Gherardi, Jr.

present investigation, since it was provided with means for varying the reluctance of the magnetic circuit and thereby the coefficients of self and mutual induction. This variation was produced by the moving of an iron tongue in and out of a gap

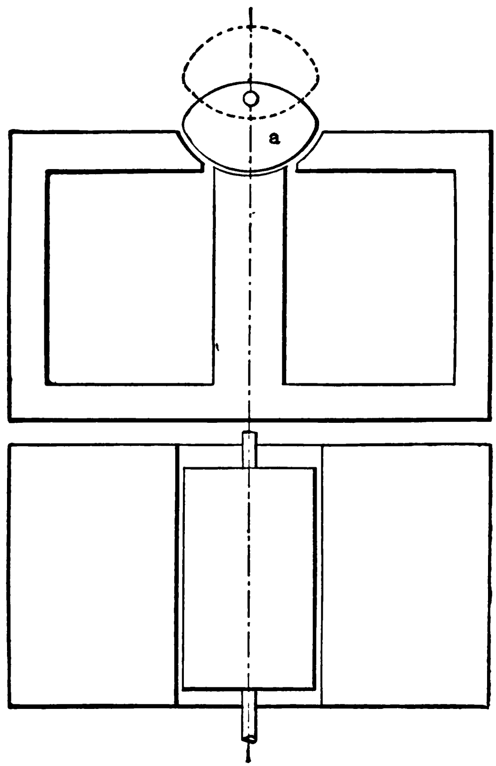


Fig. 193. Arrangement of transformer (half size)

in the magnetic circuit. Figure 193 shows in diagram the magnetic circuit of the transformer and the arrangement of the movable tongue. A laminated iron tongue, *a*, is arranged so that it may be rotated about an axis occupying the central line of the circular gap which it fills. In the position in which the tongue is represented in the figure, the magnetic circuit is as nearly complete as it can be made. When the tongue is moved through 180° to the dotted position in the figure, the magnetic circuit is most imperfect and the reluctance is highest. The tongue may be moved by an arm and secured in any desired position, and the position is indicated by a pointer moving over a graduated scale.

The essential dimensions of the transformer are as follows:

Length of core (perpendicular to lamination)	$2\frac{3}{4}$ inches.
Width of each plate	5 inches.
Length of each plate	3 inches.
No. turns large coil	128
No. turns small coil	90
Size of wire in both coils	No. 10 B. & S.
Resistance of large coil	0·2272 ohm.
Resistance of small coil	0·160 ohm.
Cross-section magnetic circuit	12 sq. cm.

Either coil may be used as primary or secondary.

Changing the position of the tongue alters the reluctance of the magnetic circuit, and alters the coefficients of self and mutual induction accordingly. These coefficients increase as the reluctance decreases, and have the highest values when the tongue is all in.

Determination of the Coefficients of Induction.—Preliminary to the experiments proper upon the transformer, the values of the coefficients of self and mutual induction of the coils were determined for the different positions of the tongue.

TABLE OF CONSTANTS.

Position of Tongue.	L_S.	L_L.	M Observed.	M Computed.	$\omega = 863$.			$\omega = 566$.			$\frac{L_L}{L_S}$
					$L_S\omega$.	$L_L\omega$.	$M\omega$.	$L_S\omega$.	$L_L\omega$.	$M\omega$.	
0°	·00908	·01743	·0122	·0126	7·83	15·04	10·53	5·15	9·87	6·91	1·92
70°	·00988	·01924	·01258	·0138	8·52	16·6	11·08	5·58	10·78	7·26	1·95
90°	·01111	·02199	·01501	·0156	9·60	18·95	12·95	6·30	12·45	8·50	1·98
120°	·01247	·02525	·01774	·0179	10·73	21·8	15·30	7·05	14·30	10·03	2·02
150°	·01427	·02918	·0209	·0204	12·31	25·32	18·05	8·09	16·65	11·83	2·05
180°	·01578	·03310	·02338	·0229	13·6	28·6	20·17	8·93	18·73	13·24	2·10

The values of these coefficients for six different positions of the tongue are given in the accompanying table.

The position of the tongue is indicated in degrees by the reading of the pointer over the graduated scale, the tongue being all out at 0° and all in at 180°. The two coils of the transformer, either of which may be primary or secondary, are distinguished as the "large" and the "small" coil, and by the subscript letters $_L$ and $_S$, respectively.

In obtaining the coefficients of self and mutual induction for different positions of the tongue, observations were made for different values of the current. The values obtained for the coefficients decreased as larger currents were used, this varia-

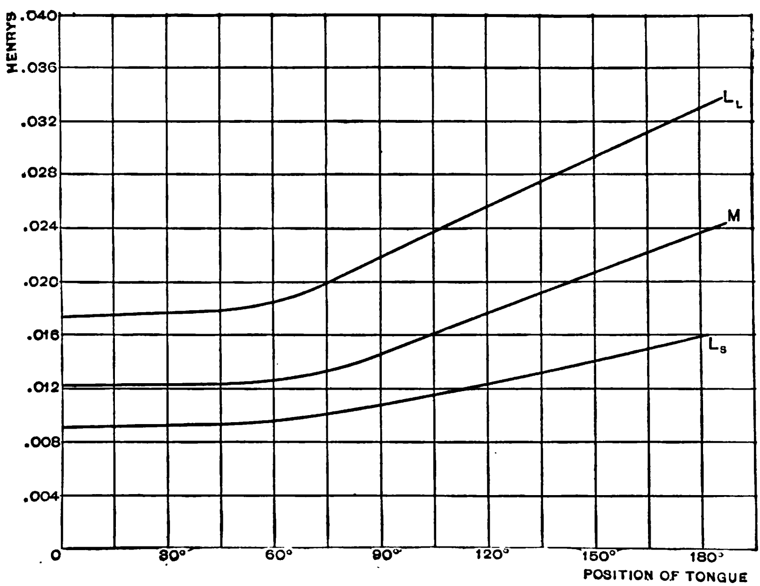


Fig. 194. Variation of constants.

tion extending through a range of five or six per cent. Mean values were taken corresponding to the currents used during the experiments and the coefficients were considered constant.

The coefficients increase steadily as the arm is moved in from 0° to 180°, but not by equal increments. The positions 0°, 70°, 90°, 120°, 150°, and 180° were afterwards used as giving approximately equal increments to L and M. The values of

L and M for different positions of the tongue are plotted in Fig. 194.

The column headed M (computed) in the table gives the values of the coefficient of mutual induction computed from the values of the coefficients of self-induction of the two coils, according to the formula based on the assumption of no magnetic leakage,

$$M = \sqrt{L_1 L_2}.$$

The values of M thus computed agree very closely with the observed values.

The column headed $\frac{L_1}{L_s}$ gives the ratio of the coefficients of self-induction for the two coils, which we would expect to find equal to the square of the ratio of the number of turns of the respective coils, or

$$(\tfrac{128}{90})^2 = 2.$$

The observed ratio approximates quite closely to this.

Experiments were made with the transformer on constant potential and also on constant current. The constant potential circuit was supplied with a current of such a frequency that $\omega = 2\pi n = 863$; while for the constant current, $\omega = 2\pi n = 566$. In the table are given the values of the reactance obtained by multiplying the coefficients of induction by ω.

Method of Measurement. — In the experiments proper with the transformer the connections were as shown in Fig. 195. In series with the primary was placed a non-inductive resistance consisting of incandescent lamps. The three-voltmeter method was used to determine the primary power and the angle of lag ϕ_1 between primary current I_1 and electromotive force E_1. This necessitated

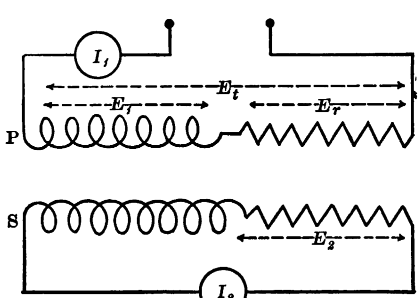


Fig. 195. Diagram of connections.

a measurement of the primary current, obtained by a Thomson balance, and the differences of potential between the terminals of the transformer, around the non-inductive resistance, and around resistance and transformer together. The secondary load was non-inductive, and the secondary current and electromotive force measured directly.

The quantities observed in making the runs were:

E_1. The electromotive force around the primary.

E_r. The electromotive force around the external resistance.

E_t. The total electromotive force around the primary and resistance.

I_1. The current in the primary.

E_2. The electromotive force in the secondary.

I_2. The current in the secondary.

From this observed data the following results were computed:

ϕ_1. The angle of lag between the primary electromotive force and current.

P_1. The power supplied to the primary.

P_2. The power in the secondary.

R_2. The secondary resistance.

θ_2. The angle of lag of secondary current behind the electromotive force in the secondary induced by the primary current.

The primary angle of lag is calculated by the formula for the three-voltmeter method

$$\cos\phi_1 = \frac{E_t^2 - E_1^2 - E_r^2}{2\,E_1 E}.$$

The primary power is $P_1 = \overline{E}_1 \overline{I}_1 \cos\phi_1$.

The secondary power is $P_2 = \overline{E}_2 \overline{I}_2$.

The secondary resistance, being non-inductive, is calculated by the fall of potential, thus,

$$R_2 = \frac{E_2}{I_2}.$$

The tangent of the secondary angle of lag is

$$\tan \theta_2 = \frac{L_2 \omega}{R_2},$$

from which θ_2 can be readily obtained.

Constant Current Experiments. — The transformer was supplied in these experiments with a constant primary current, the square root of mean square value of which was $9\frac{1}{3}$ amperes, and the maximum value about 13·2 amperes. The frequency

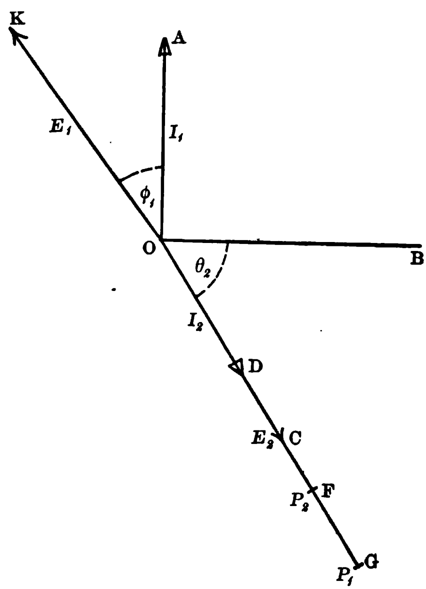


Fig. 196.

was such that $\omega = 2\pi n = 566$. The transformer was used both as a "step-up" and as a "step-down" transformer. The step-up and step-down experiments were substantially the same; inasmuch as the ratio of primary and secondary turns was not far from unity, the results of the constant current experiments will be given for the step-down transformation only. A run was first made with the movable tongue secured by an arm in the zero position, that is, in the position in which the reluctance of the magnetic circuit is the highest and the coefficients of induction the smallest. The secondary non-inductive load of incandescent lamps was then varied from short circuit to open circuit, and the primary and secondary measurements taken as described for successive values of the secondary resistance. The tongue was then moved in and secured by means of an arm in the 70° position, and a complete set of readings again taken of primary and secondary for different loads. Runs were then successively made with the tongue secured at 90°, 120°, 150°, and all in at 180°, and the same measurements taken.

In this way data was obtained for all values of the coefficients of induction of the transformer, within the range of change produced by the moving in and out of the tongue, for all values of secondary resistance. For each observation, the primary and secondary currents and electromotive forces were known both in magnitude and direction, — sufficient data for the construction of a polar transformer diagram. Such a diagram is represented in Fig. 196.

Polar Diagram. — In Fig. 196, the lines OA, OB, OC, etc., represent the maximum values of the several electromotive forces and currents in their relative phase positions. OA is

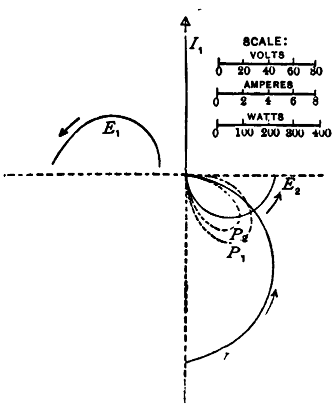


Fig. 197. Constant current. Step-down. Arm at 0°.

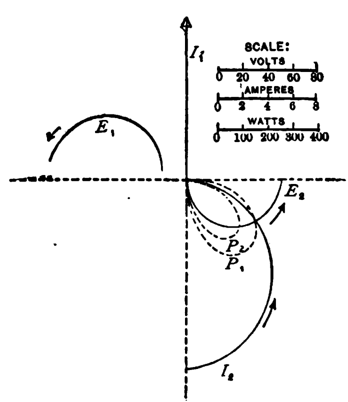


Fig. 198. Constant current. Step-down. Arm at 70°.

drawn equal to the primary current, I_1. The primary impressed electromotive force is in advance of the primary current by an angle ϕ_1, and is represented by the line OK drawn to any convenient scale. The electromotive force induced in the secondary due to the primary current is OB, 90° behind the primary current. In this transformer, the effect of hysteresis was small on account of the air gap in the magnetic circuit; hence, little error was introduced by assuming OB to be 90° behind OA. It was so drawn in order that the result might be better compared with the theoretical diagrams in the preceding chapters. The secondary external resistance is non-inductive, and the current is therefore in phase with the electromotive force measured at the terminals of the transformer secondary, but the current and measured secondary electromotive force lag behind the secondary impressed electromotive force OB by an angle θ_2 due to the self-induction in the secondary coil. The angle θ_2 is drawn so that $\tan\theta_2 = \frac{L_2\omega}{R_2}$. The line OD is drawn equal to the secondary current I_2, and OC equal to the electromotive force E_2 at the terminals of the secondary.

The primary and secondary power, P_1 and P_2, although not vector quantities, may for convenience be represented by the lines OG and OF drawn in the direction of the secondary current.

Diagrams similar to Fig. 196 may be drawn for each load of the transformer, and the variations in the different quantities observed by a study of the locus formed for each of the several quantities, that change as the secondary resistance is altered. It is in this way that the following diagrams are plotted. Fig. 197 shows the variation in the primary electromotive force, secondary current and electromotive force, and the variation in the primary and secondary power for all loads of the transformer, from $R_2 = 0$ to $R_2 = \infty$, for a constant primary current when the tongue is in the 0° position, that is, when it is all out. The arrows on E_1, E_2, and I_2 indicate the direction of the

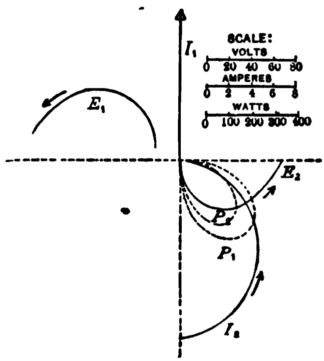


Fig. 199. Constant current. Step-down. Arm at 90°.

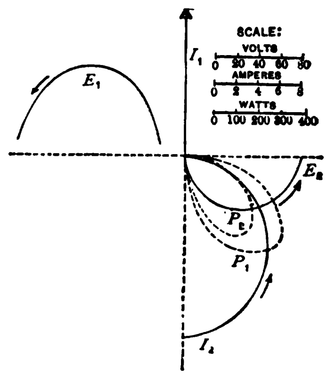


Fig. 200. Constant current. Step-down. Arm at 120°.

change as the secondary resistance increases. Figs. 198–202 are drawn in the same manner for other positions of the tongue. In all cases the primary current is constant.

The agreement of these curves with the theoretical ones given in Chapters VIII. and X. is marked. The primary

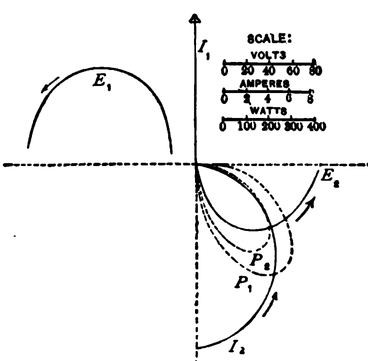


Fig. 201. Constant current. Step-down. Arm at 150°.

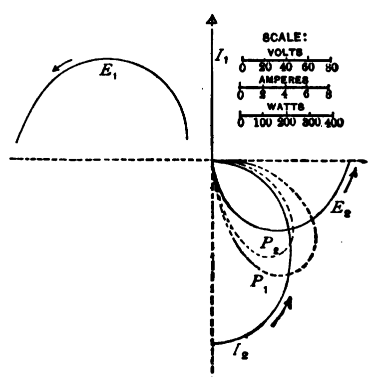


Fig. 202. Constant current. Step-down. Arm at 180°.

electromotive force curve, and the curves for the secondary electromotive force and current, are approximately semicircles, as theory would indicate. The power curves, P_1 and P_2, are symmetrical lobes at about 45°. The shifting of the primary electromotive force curve to the left is caused by magnetic leakage, and is also in perfect accordance with theory. The changes in the successive diagrams, from Fig. 197 with the tongue all out, to Fig. 202 with the tongue all in, show the effects of diminished reluctance and increased coefficients of induction. The diameter of the semicircle for the secondary electromotive force increases in direct proportion to the increase in the coefficient of mutual induction. The primary electromotive circle increases in proportion to the coefficient of self-induction of the primary. The diameter of the secondary current circle is nearly constant. The comparison of these curves is better seen in Fig. 203, in which the preceding diagrams are combined so as to show the successive changes in the currents and electromotive forces for different tongue positions. A composite figure is shown in the same way for the power curves, P_1 and P_2, in Fig. 204.

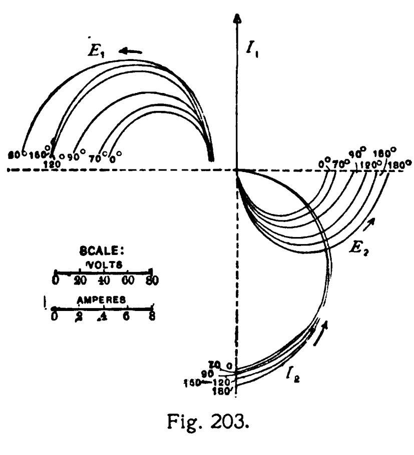


Fig. 203.

Rectangular Diagrams. — Rectangular diagrams do not indicate the direction of the quantities represented, but they show quantities in such a way that they are often superior to polar diagrams for indicating relative magnitude. For this purpose rectangular diagrams were drawn from the polar diagrams, Figs. 197–202, and two of these are shown in Figs. 205 and

206, for the tongue all out and all in, respectively. These figures give the currents, electromotive forces, angles of lag, the power for the primary and for the secondary, and the

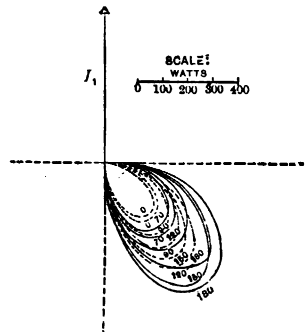


Fig. 204.

efficiency of the transformer for different values of the secondary resistance. The curves explain themselves. It is interesting to note that the primary angle of lag on short circuit has

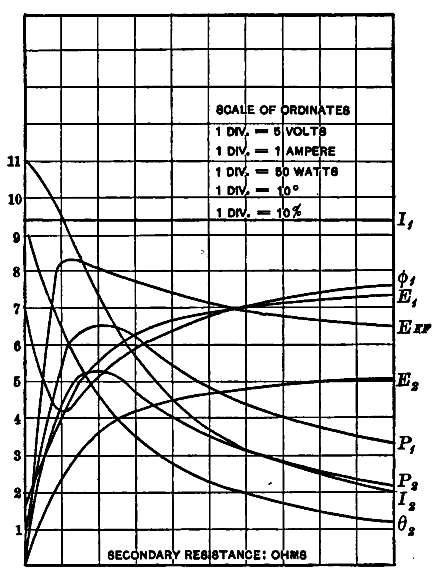


Fig. 205. Constant current. Step-down. Arm at 0°.

about the same value, 70°, for all positions of the tongue. As the secondary resistance is increased, the angle decreases at first and then increases again. A study of Fig. 203 will show that this is explained by the shifting of the primary electromotive force semicircle in Fig. 203, to the left, due to magnetic leakage. If the leakage were diminished, the primary semicircle would be moved toward the primary current OA.

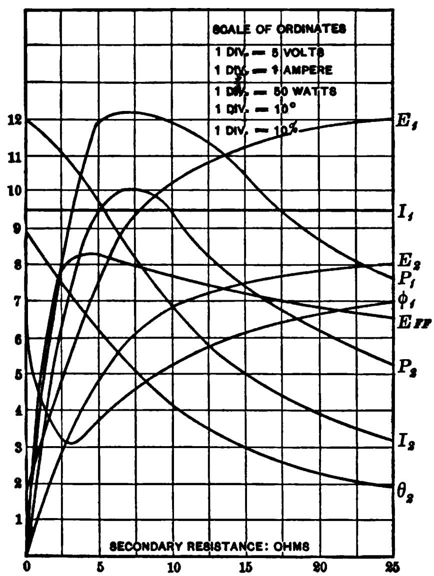


Fig. 206. Constant current. Step-down. Arm at 180°.

In the rectangular diagrams, a diminishing of the magnetic leakage would cause the minimum point in the curve for ϕ_1 to shrink and disappear in the case of no leakage.

Constant Potential Experiments. — Runs were made with the transformer supplied with a constant potential of about 62 volts (square root of mean square value), with the movable tongue secured in the different positions 0°, 70°, 90°, 120°, 150°, and 180°, as before. The electromotive force of the generator was about 110 volts, but varied a little, so that it was necessary to maintain the electromotive force supplied to the transformer constant by adjusting the non-inductive resistances. Measurements were made as in the constant current experiments, and

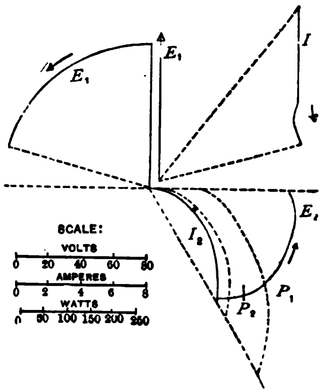


Fig. 207. Constant potential. Step-up. Arm at 0°.

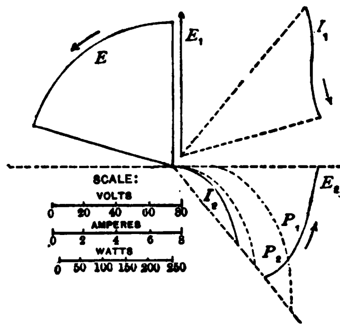


Fig. 208. Constant potential. Step-up. Arm at 70°.

the values of the various quantities obtained as before. From these data, the polar diagrams, Figs. 207–212, were constructed. In these the direction of the primary current is taken as vertical, fixed in direction, but varying in magnitude. The primary

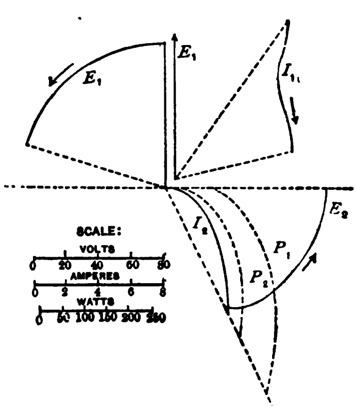


Fig. 209. Constant potential. Step-up. Arm at 90°.

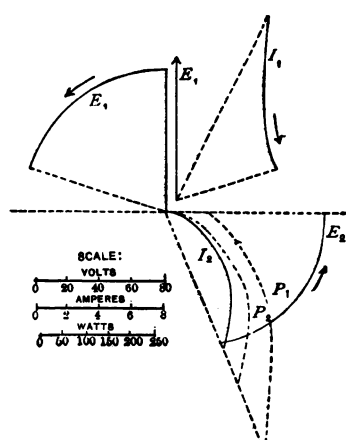


Fig. 210. Constant potential. Step-up. Arm at 120°.

electromotive force is constant in magnitude, but varies in direction according to the value of the angle ϕ_1 between it and the fixed primary current. A diagram thus drawn does not show the changes in the magnitude of the primary current, and therefore an additional diagram is given in the upper right-hand corner of each figure in which the primary electromotive force is constant both in magnitude and direction. The locus of the primary current in this supplementary diagram shows clearly the changes in the magnitude of the primary current as well as its direction. As in all the polar diagrams in this chapter, the loci are drawn to show the variation in the differ-

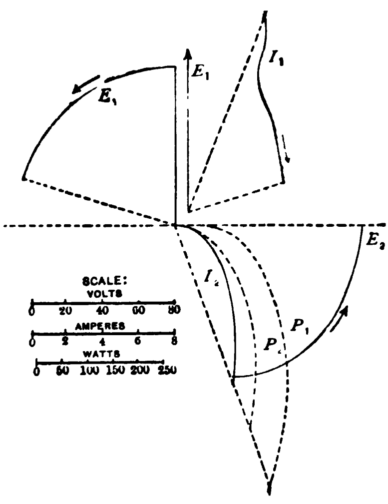


Fig. 211. Constant potential. Step-up. Arm at 150°.

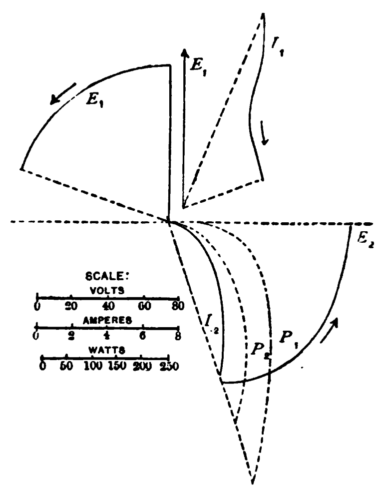


Fig. 212. Constant potential. Step-up. Arm at 180°.

ent quantities represented, caused by a change in the secondary resistance, the direction of the variation with increased secondary resistance being indicated by an arrow. These diagrams correspond with those derived theoretically in Chapter IX.

Figs. 213 and 214 are rectangular diagrams, drawn from the corresponding polar diagrams, Figs. 207 and 212, representing the relative magnitudes only of the various quantities.

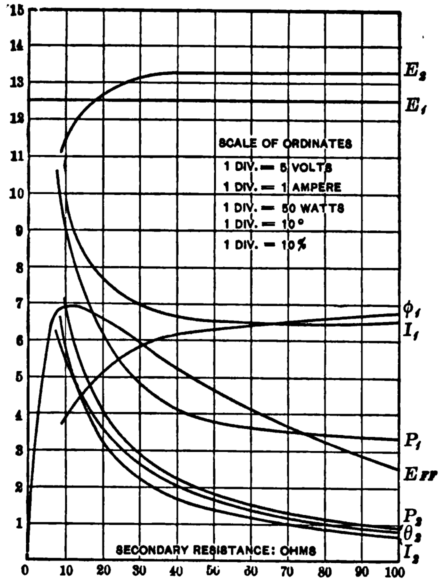


Fig. 213. Constant potential. Step-up. Arm at 0°.

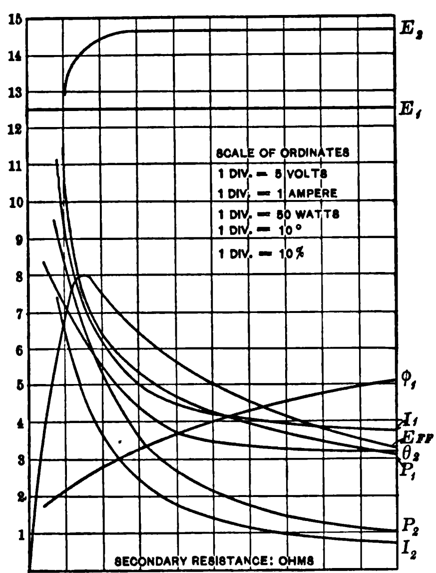


Fig. 214. Constant potential. Step-up. Arm at 180°.

When this investigation was undertaken with a view of experimentally developing the method of polar diagrams just described, it was with a little uncertainty as to whether the method would admit of ready practical application, and would give a clear exposition of the results obtained. It not only proves to be a convenient method for showing results, but tells the whole story in a comprehensive way, bringing out many points not otherwise shown.

It proves likewise that polar diagrams based upon the sine assumption or upon the assumption of equivalent sine waves are actually correct for cases in practice in which the currents and electromotive forces never are exact sine functions.

CHAPTER XVII

INSTANTANEOUS CURVES OF TRANSFORMER CURRENTS AND ELECTROMOTIVE FORCES

Method of Instantaneous Contact. — The complete action of a transformer is shown by curves which represent the values of the various currents and electromotive forces at each instant of time throughout a complete cycle. Curves giving the instantaneous values of the various quantities may be obtained by the so-called method of instantaneous contact. This method employs a revolving contact device which rotates synchronously with the generator, and completes the circuit to the measuring instrument once in each revolution. The time when this contact is made may be adjusted so that the measuring instrument will be in circuit at any particular phase of the alternation. The method was employed by Joubert in the year 1880 in measurements upon the Jablochkoff candle. It was independently devised by B. F. Thomas in this country in the same year. Eight years later it was used by Dr. Duncan and others in the study of the transformer, and in the following year by Merritt and Ryan. Since then it has been subjected to many modifications, and has come to be a well-recognized method for laboratory investigation.*

Figure 215 shows one arrangement † of the apparatus employed in obtaining curves by the method of instantaneous

* The history of the development of this method, and a description of its details, is given in a paper "Hedgehog Transformer and Condensers," by Bedell, Miller, and Wagner, *Transactions*, American Inst. of Electrical Engineers, Vol. X., p. 497. From this paper are taken the experiments described in the following pages.

† Employed by Professor Ryan and the writer.

contact. The contact-maker is used with an electrostatic voltmeter. The difference of potential between *a* and *b* is to be measured. The condenser C_1 is kept charged to this potential to be measured, being connected through the contact-maker once in each revolution. The condenser C_2 in series with the voltmeter is kept charged to a known constant difference of potential in order to displace the zero of the voltmeter, so that readings may be obtained upon the most favorable part of the scale. If the difference of potential between two points, as A and B, beyond the range of the instrument, is to be determined, these points are connected by a non-inductive resistance, and the difference of potential measured around a known portion *ab* of this resistance, and the whole difference of potential between A and B calculated. Measurement of the current is made by determining the fall in potential around a known non-inductive resistance in the circuit.

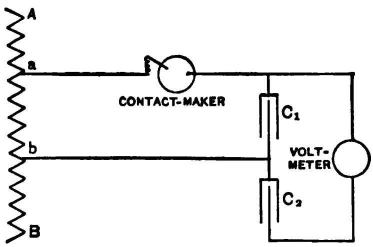


Fig. 215. Method of instantaneous contact.

The curves which follow were obtained from a Hedgehog transformer. The Hedgehog transformer, having an open magnetic circuit, has only a small loss due to hysteresis; the primary angle of lag is large (that is, the power factor is small), and the primary current is large, as compared with a transformer of the same output having a closed magnetic circuit. This is particularly noticeable at no load. This is to be borne in mind in studying the results which follow. This point will be made more clear from the no load curves for a closed magnetic circuit transformer, given later in Fig. 223 for comparison with the corresponding curves in Fig. 216 for the Hedgehog transformer.

Test of Hedgehog Transformer alone. — The transformer upon which these experiments were made is a 60-light, Hedgehog transformer, built on the general lines of an old-fashioned induction coil. A thin casting of gun-metal, cross-shaped in section, runs through its center and supports the flanges of the spool. The four angles of this casting are filled with soft iron wires, running lengthwise, thus forming a cylindrical core. These wires are considerably longer than the castings, and are spread out at their free ends, thus making a partial return

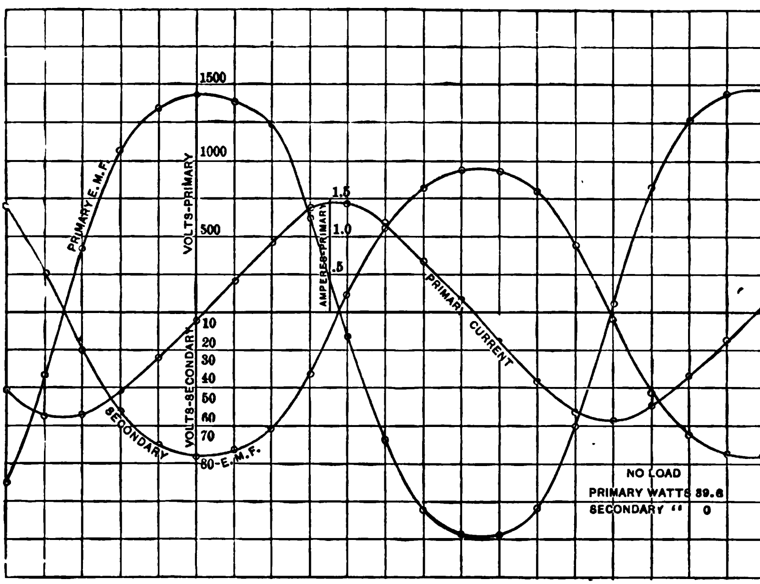


Fig. 216.

for the magnetic lines. Half the secondary turns are first wound around this core, then all the primary, and finally the other half of the secondary. The primary consists of 1426 turns of wire 0·072 inch in diameter, arranged in 12 layers. It has a resistance of 2·748 ohms and weighs 29 pounds. The secondary consists of 73 turns of 19/0·058 cable, in two layers. Its resistance is 0·0149 ohm, and weight 12·5 pounds. The transformer is designed for a primary potential of 1000 volts, with a frequency of 130.

This transformer was supplied with current from a Westinghouse alternator with a frequency of 133. The potential was kept constant at 1154 volts by varying the exciting current. The primary electromotive force was measured by measuring the fall of potential around a portion of a non-inductive resistance, which consisted of a series of 50-volt incandescent lamps arranged between the mains. The secondary elec-

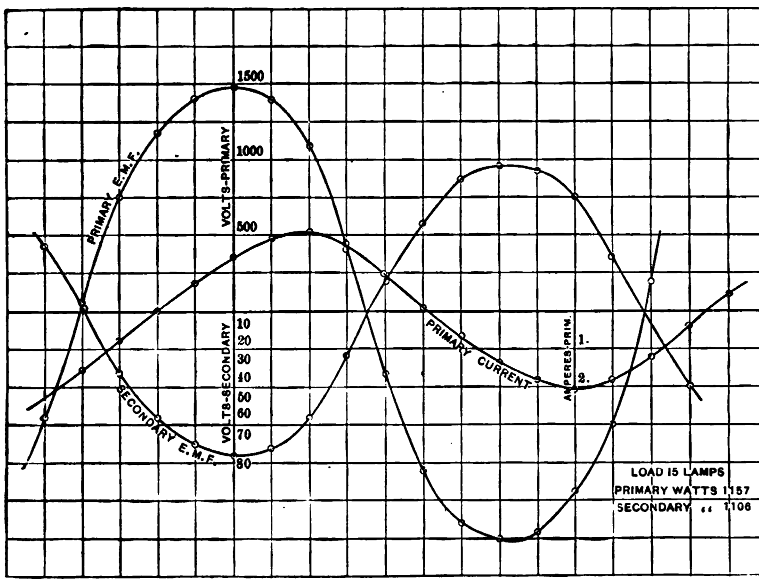


Fig. 217.

tromotive force was similarly found, by measuring the fall of potential around one of three lamps placed in series across the secondary terminals. The primary current was found from the fall of potential around incandescent lamps placed in the primary circuit. The number of these lamps was so adjusted that the fall in potential would be in the range of the voltmeter. The lamps were carefully calibrated for current at different pressures by means of a direct current, and curves drawn, from which the current could be determined for any pressure.

It is evident that the drop through these lamps would diminish the total impressed electromotive force of the primary, and a correction was accordingly made for this, by deducting from the instantaneous values of the total line electromotive force the instantaneous values of the fall in potential around these lamps in obtaining the actual impressed electromotive force of the primary.

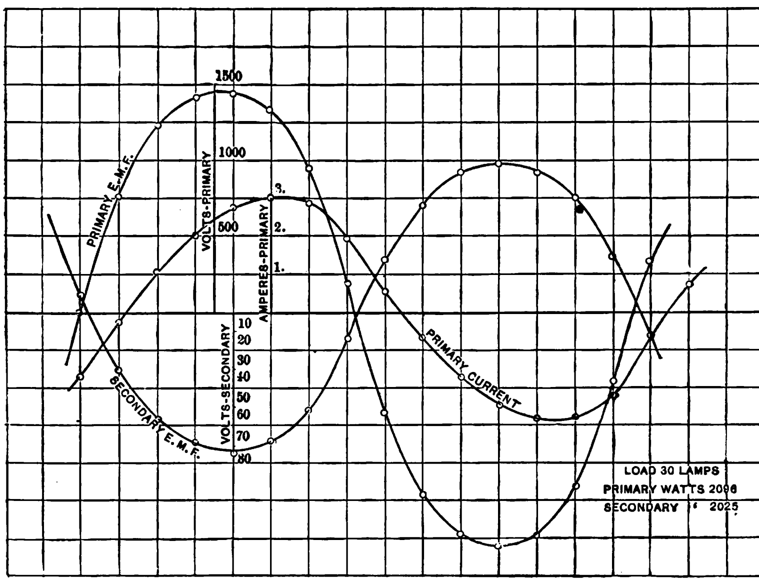


Fig. 218.

Wires were led from the terminals of the various resistances, around which measurements were taken, to a small hard-rubber switchboard above the voltmeter, by means of which connections were readily made for measuring the several electromotive forces. Instead of obtaining each complete curve independently, a reading corresponding to a certain position of the armature was taken for each curve, without changing the position of the contact-maker. The contact-maker was then moved to a new position, and readings again taken, one for each curve. This method prevents the relative displacement of the different curves.

The secondary load consisted of a non-inductive resistance, composed of 60 fifty-volt incandescent lamps, mounted upon a frame in such a manner that any number of lamps could be turned on or off. It was planned to measure the secondary current by means of a calibration of these lamps, but it was found on recalibrating them, at the end of the work, that their resistance had increased considerably, so this method had to be

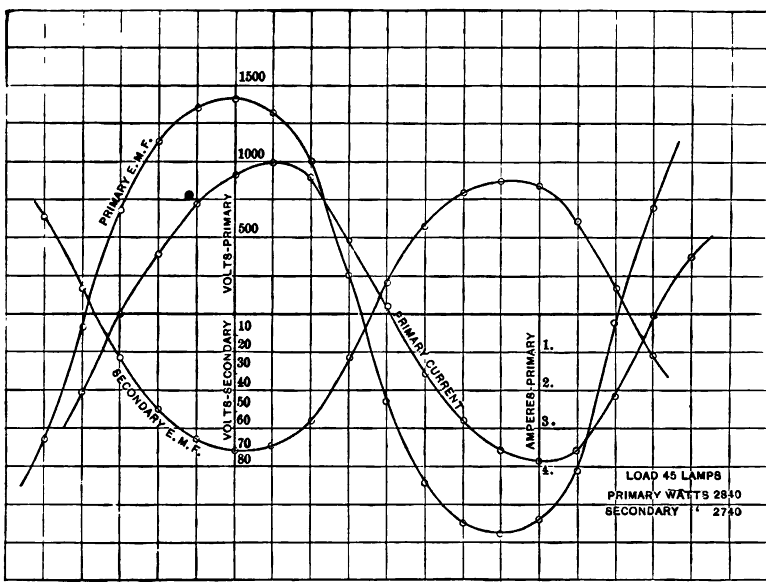


Fig. 219.

abandoned. The secondary current was finally determined by constructing a right triangle, the hypothenuse of which was equal to the primary current at the given load, and one side of which was equal to the primary current at no load. The third side, multiplied by the ratio of transformation, gave the secondary current. In the case of a transformer with an open magnetic circuit, there is little error in computing current in this way.

The square root of the mean square value of the secondary electromotive force was measured directly by the multicellular voltmeter.

Five separate runs were made, first for no load, with the secondary on open circuit, and then with loads of 15, 30, 45, and 60 fifty-volt incandescent lamps. With lamps of this resistance, the last run was quite beyond full load, inasmuch as the primary electromotive force was 15 per cent above that for which the

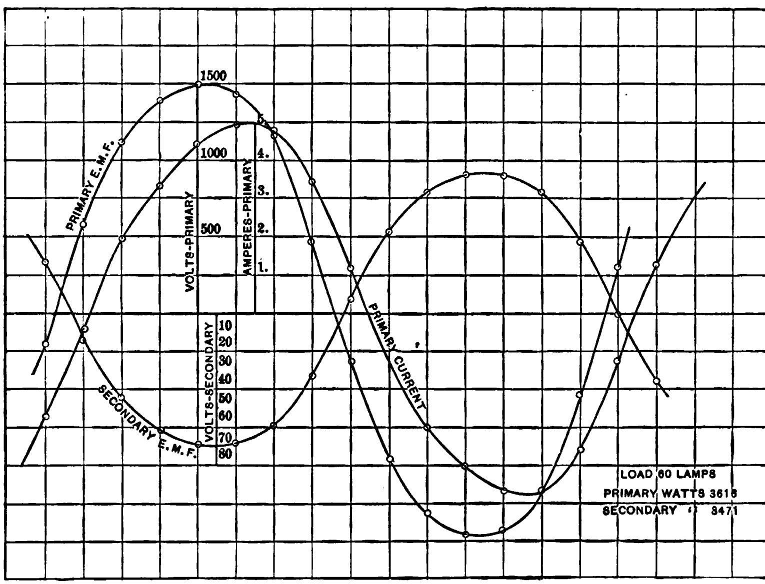


Fig. 220.

transformer was intended. The results of these five runs are shown in Figs. 216–220.

At no load, the lag of the primary current behind the primary electromotive force is almost 90°, which is greatly in excess of the lag shown by similar experiments upon a transformer with a closed magnetic circuit. The lag diminishes rapidly as the load increases, as seen in the successive curves, but it is always greater than for the corresponding load of a closed magnetic circuit transformer. This is on account of the large magnetizing current in the open magnetic circuit transformer, the effect of this component of the total primary current being quite marked, even at full load.

The secondary electromotive force at no load is almost opposite in phase to the primary electromotive force; that is, it is almost 180° behind it. As the load increases, a small but distinct increase in this lag is noticed, due largely to increased leakage.

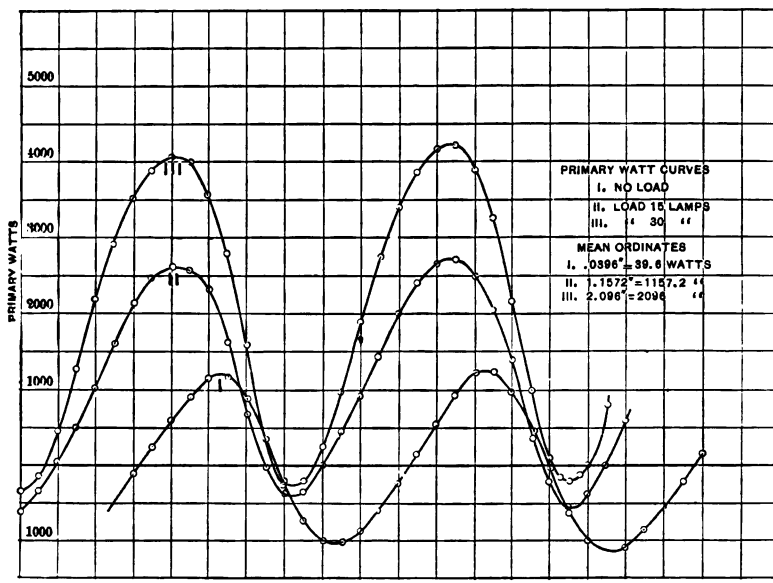


Fig. 221.

The power supplied to the transformer was found by finding the products of corresponding instantaneous values of the primary electromotive force and current. These products were positive or negative, according to whether the two factors were of the same or different sign. The products thus found were plotted as curves, showing the primary power, Figs. 221 and 222, positive values being plotted above the axis, and negative below. These areas were found by a planimeter, their algebraic sum taken, and the mean ordinate thus determined.

Although at no load the primary current is large, the fact that it lags so nearly 90° behind the primary electromotive force causes the energy expended to be small. This was found to be a total of 39·6 watts. Of this the primary copper losses, $R_1\bar{I}_1^2$, amounted to 2·58 watts, leaving 37·02 for the core losses, — hysteresis and eddy currents.

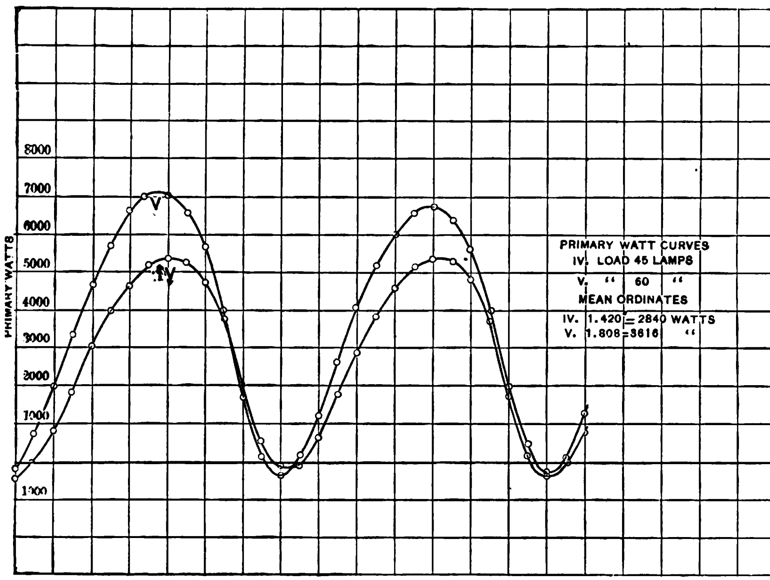


Fig. 222.

In any transformer, if the primary electromotive force is a sine curve, the current curve would also be one, in case there were no hysteresis or eddy currents. In the no-load diagram of this transformer, the primary current curve is not materially different from a sine curve 90° behind the electromotive force; while in the no-load diagrams of closed magnetic circuit transformers supplied from this same alternator, the curves for primary current depart widely from sine curves, and are much nearer the primary electromotive force in phase. Thus the form and position of the current curve in the open magnetic circuit transformer show the hysteresis and eddy currents to be small, as compared with the transformer with closed magnetic circuit. No-load curves for a Stanley (closed magnetic circuit) transformer taken by Professor Ryan are shown in Fig. 223.

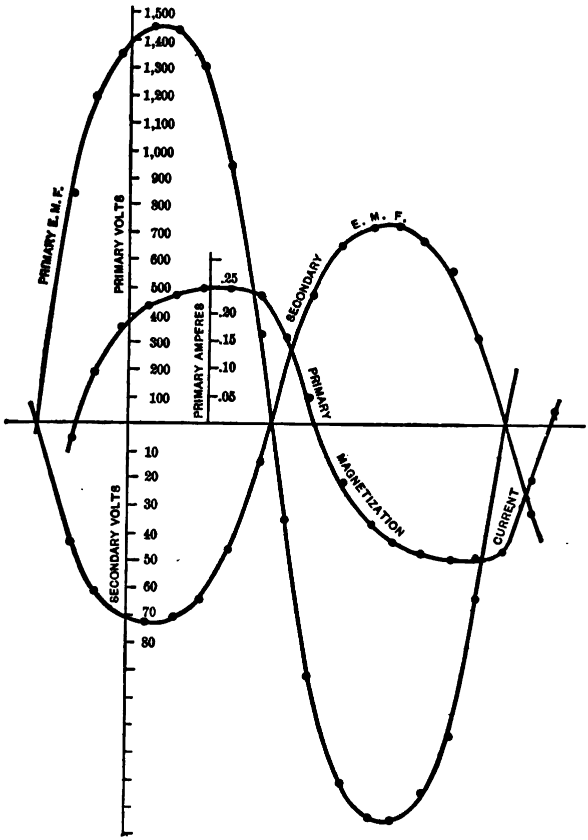


Fig. 223. Curves for a 17,500-watt transformer with closed magnetic circuit. $\overline{E}_1 = 1000$ volts. $\overline{E}_2 = 50$ volts. Frequency = 133.

The efficiency of the transformer was found by taking the ratio of the secondary to primary power. The primary power was found from the watt curves in Figs. 221 and 222. The secondary power was found by deducting the primary and secondary copper losses, $R_1\bar{I}_1^2$ and $R_2\bar{I}_2^2$, and the core losses, from the primary power thus found.

The regulation is seen by the curve in Fig. 224, showing the secondary electromotive force for different loads. There is a fall of about 2·5 volts in the secondary, between no load and full load. This curve is drawn for a primary electromotive force of 1000 volts.

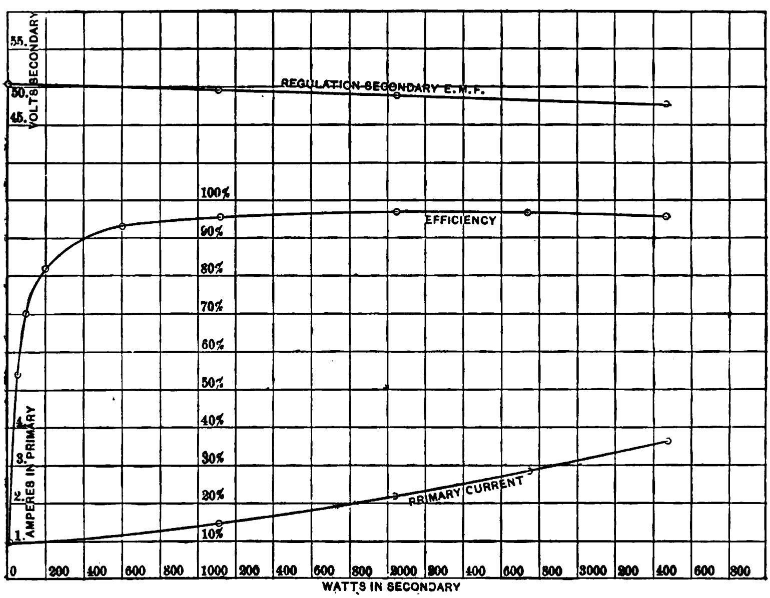


Fig. 224.

The primary current is shown by a curve in the same figure. Even at no load it is quite large, although representing very little energy expended. Clearly the disadvantage of such a current is that it increases the losses in the line and in the primary conductor. If this line current can be successfully reduced, one of the chief objections to the open magnetic circuit transformer will be removed. This can be done by the use of condensers.

Test of Hedgehog Transformer with Condensers. — This series of experiments was the same as those just described, with the addition of a set of condensers of proper capacity placed between the terminals of the primary, in order to reduce the line current. The electromotive force of the supply was in this case 1089 volts. The instantaneous values of the line current, that is, the current supplied to the transformer and the condensers, was determined by the fall of potential around a non-inductive resistance in the line.

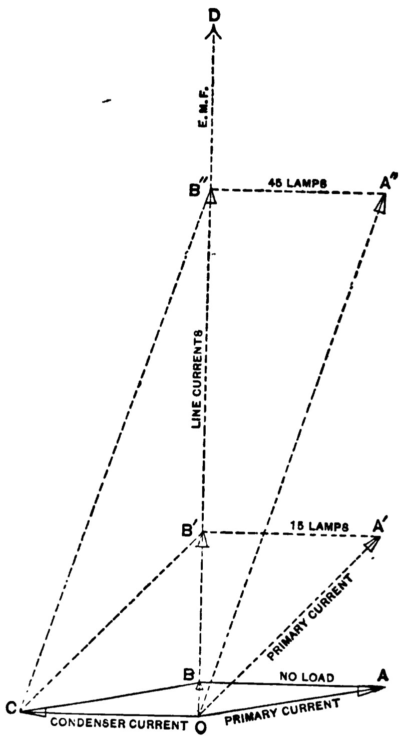


Fig. 225. Addition of condenser and line currents.

The condensers used were six in number, from the Stanley Laboratory, and were intended to be used commercially on a 500-volt circuit. The plates are of tinfoil, the useful part of which is of the following dimensions: length, $10\frac{1}{2}$ inches; width, 8 inches; thickness, 0·0007 inch. The dielectric is of waxed paper, 0·0043 inch thick. There are sixty-five sheets of tinfoil (total) in each slab, and two of these slabs are placed together in one tin case. The capacity of each condenser was about 1·5 microfarads. When one condenser was supplied with an electromotive force of 574 volts, the loss was found to be 4·4 watts, representing an efficiency of 96·9 per cent.

By arranging the condensers in various combinations, it was possible to obtain a number of different capacities, ranging from 0·25 to 2·25 microfarads, without subjecting any one condenser to more than 500 volts difference of potential.

The proper capacity to be used in parallel with the primary is that which will bring the line current in phase with the electromotive force. When this result is brought about, the line current is a minimum. The amount of capacity to be used may be predetermined theoretically, or ascertained experimentally. Predetermined graphically, the proper capacity was found to be very close to one microfarad. Fortunately, the

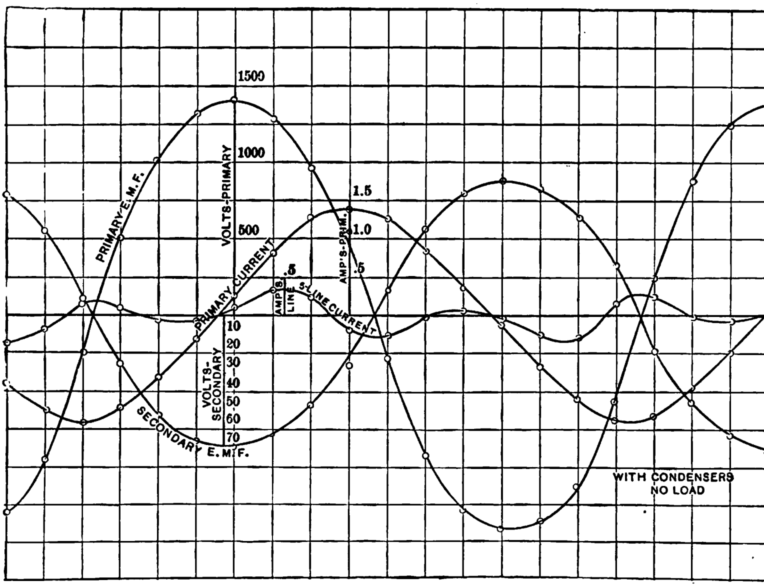


Fig. 226.

capacity required is almost constant for all loads of the transformer. The proper capacity for the condensers was ascertained by trial as follows: the transformer was run on open circuit with the condensers arranged in various combinations, so as to obtain different capacities, and the fall of potential around the non-inductive resistance in the line was measured by the multicellular voltmeter. Square roots of mean square values were obtained by direct reading, no instantaneous values being taken. The minimum drop occurs when the current is a minimum and in phase with the electromotive force. This was obtained by a capacity of almost one microfarad, the same capacity as calculated. The six condensers were therefore used, three in series and two in parallel, giving a capacity of 1·02 microfarads.

Three runs were now made, one with the secondary circuit open, and the other two with loads of 15 and 45 lamps each.

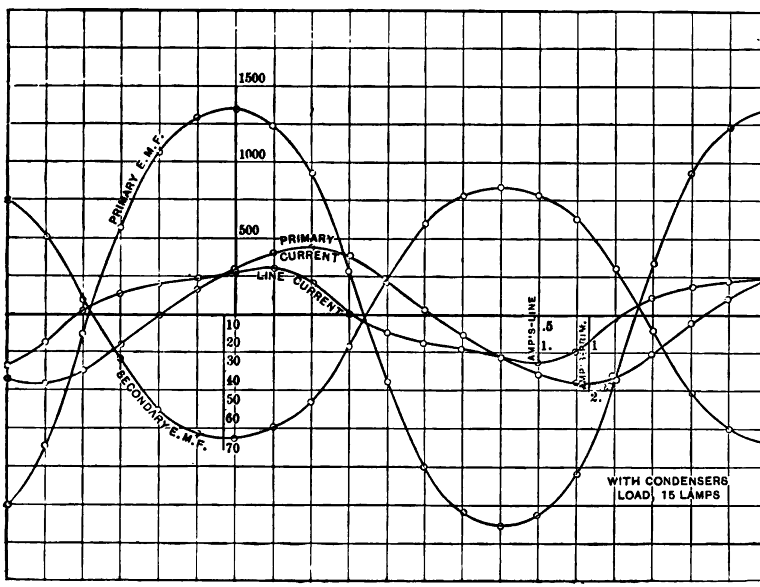


Fig. 227.

Readings were taken for determining the primary electromotive force, secondary electromotive force, primary current, and line current. By primary current we mean the current in the primary of the transformer, and by line current, the current supplied to the condensers and transformer together.

The results of these three runs are given in the following table, showing clearly the effect of the condenser in diminishing the line current, especially at no load.

LAMPS IN SECONDARY.	PRIMARY CURRENT.	LINE CURRENT.
0	0·95	0·187
15	1·24	0·867
45	3·07	2·73

At no load, the line current is seen to be less than one-fifth of the primary current, or the value it would have if the condensers were absent; even under load the reduction is considerable. These results are better shown by the polar diagram in Fig. 225, drawn on the supposition that the currents are harmonic. On no load OC is the condenser current, OA the primary current, and OB the resultant line current, much smaller than either component. For 15 lamps, the condenser current is, as before, OC, the primary current is OA′, and the resultant current OB′. Similarly for 45 lamps, the resultant current is OB″. In all cases the condenser current is 90° ahead of the impressed electromotive force OD, and has a value of about 0·93 ampere. The fact that the currents are not strictly sinusoidal causes the representation to be an approximation, which although not exact is, however, useful for practical calculations and for illustration.

The results of the three runs are shown by the curves in Figs. 226, 227, 228. The line current is the algebraic sum of the instantaneous values of the primary and condenser currents; and if these currents were harmonic, it would be in phase with the electromotive force, if the condenser were properly proportioned. In Figs. 227 and 228 this is seen to be practically the case. The irregular character of the line current curve in Fig. 226 is explained thus. It is the algebraic sum (the arithmetical difference) of two almost equal and opposite currents, and is very small compared with either. If these two currents were exactly sinusoidal, the resultant curve would be sinusoidal. As a matter of fact, although very nearly sinusoidal, they are not exactly so, and these variations have a marked effect upon their resultant, which is their arithmetical difference. The line current curve has a marked depression, which brings the portion which would naturally be the crest over the axis; that is, for a brief interval in each half period it has the opposite sign, and the current flows in a direction opposite to its flow during the other part of the same half period. This shows the presence

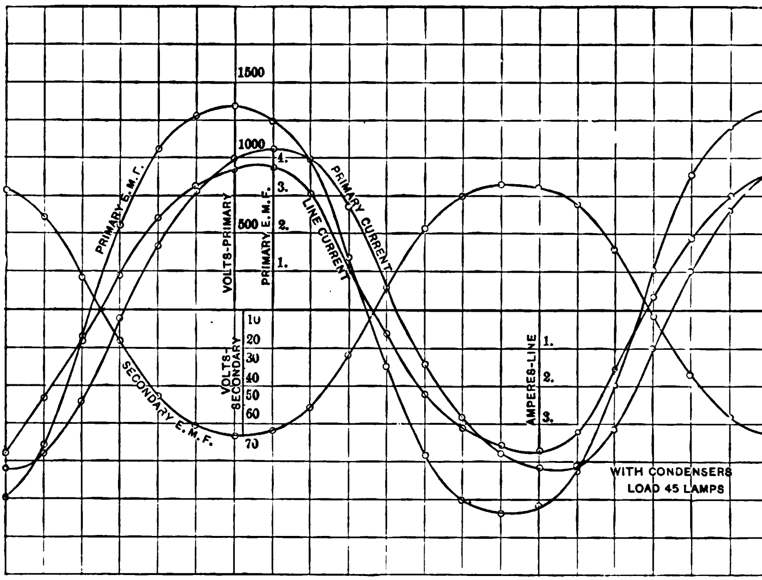


Fig. 228.

of harmonics with three and five times the fundamental frequency. With a load of 15 lamps this depression has a slight effect, but is scarcely noticeable for a load of 45 lamps, the line current curve in Fig. 228 differing but little from the primary current curve. It is ahead in phase, and with harmonic currents and the exact capacity required, it would be in phase with the electromotive force as shown in Fig. 225.

CHAPTER XVIII

TRANSFORMER TESTING

Efficiency. — The efficiency of a transformer is the ratio of the power taken from its secondary to the power supplied to its primary. Designating these by P_2 and P_1 respectively, we have

$$\text{efficiency} = \frac{P_2}{P_1}.$$

Denoting the losses in the transformer by p, we have

$$\text{efficiency} = \frac{P_1 - p}{P_1} = \frac{P_2}{P_2 + p}. \qquad (1)$$

With direct currents the power in watts equals the current in amperes multiplied by the electromotive force in volts, but with alternating currents this is true only when there is no phase difference between the current and the voltage. In general [see equation (30), Chapter IV.], the power is equal to the power component of the electromotive force multiplied by the current, so that

$$\text{power} = \bar{I}\bar{E} \cos \phi.$$

When the load is a bank of incandescent lamps, ϕ is so small that its cosine differs from unity by an inappreciable amount; and when such a load is used on the secondary circuit of a transformer, the power P_2 may be obtained from readings of a voltmeter and an ammeter, and we may write

$$P_2 = \bar{E}_2 \bar{I}_2.$$

To measure the power supplied to the primary we must have recourse to other methods. Of these, the most convenient, as well as the most accurate, are the wattmeter methods.

Wattmeter Methods.—In a Siemens dynamometer, the current to be measured is sent through two coils in series, one of which is fixed and the other suspended with its plane at right angles to the plane of the first. The torque varies as the product of the field of the fixed coil by that of the suspended one, and the angular readings are accordingly proportional to the mean square values of the current.

If now, instead of connecting the two coils in series, we put one in the main circuit as we would an ammeter, and connect the other, in series with a suitable resistance, across the mains as we would a voltmeter, a wattmeter is the result. This is shown diagrammatically in Fig. 229. Here *ab* is the circuit the power in which is to be measured; *cd* is a coil in series with *ab*, which we will call the current coil; and *ef* is a coil which, in series with the resistance *R*, is connected across the mains *mm* from *c* to *b*.

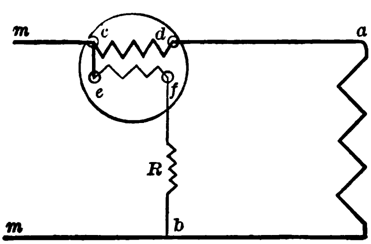


Fig. 229. Method for using wattmeter.

Now the torque at any instant is proportional to the product of the currents in the coils at that instant. But the current in the voltage coil is proportional to the voltage at the terminals; consequently the torque is proportional to *ei*, the product of the instantaneous values of the current and voltage, and the deflection multiplied by a constant is the power required.

With the voltage terminals connected as shown in Fig. 229, the drop in the current coil is added to that in the load, and the reading includes the watts expended in heating this coil. If the connection be changed from *c* to *d*, the current of the voltage coil is added to that of the load and the reading is again too large, this time the excess being the watts expended in the voltage coil. Some wattmeters are provided with a compensating coil, that is, a coil of fine wire having the same number of turns as the current coil is wound with it and connected in series with the voltage coil. In Fig. 230 the current flowing from *g* to *e* neutralizes the field of an equal current in *cd*, and the reading of the instrument gives the power expended in the circuit *ab* without any correction.

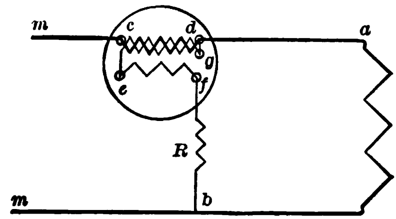


Fig. 230. Compensated wattmeter.

In order that the indications of a wattmeter may be practically correct with loads of any character whatever, it is necessary that the self-induction of the voltage circuit be small as compared with its resistance. To this end, the coil should be wound with as few turns as possible and put in series with a large non-inductive resistance. Since such an arrangement gives a weak field to the coil, in order to render the instrument sufficiently sensitive, the current coil must have a strong field. To diminish the disturbing effects of circuits outside of the instrument, the coil producing the weak field should be the suspended one.

The effect of self-induction in the voltage coil is twofold. If the wattmeter has been calibrated with direct currents, the readings will be too small with alternating currents, since with the same terminal voltages the currents in the coil in the two cases will be in the ratio $1 : \cos\theta$, where θ is the angle of lag in the coil in question. Moreover, the current in the voltage coil will not be in phase with the voltage, but will lag behind it. With harmonic curves the expression for the power is

$$\text{true watts} = \bar{E}\bar{I}\cos\phi.$$

When there is inductance in the voltage circuit, the current lags behind the voltage by an angle θ and leads * the current in the main circuit by an angle $(\phi - \theta)$. The reading of the wattmeter is

* In this discussion, ϕ is the angle between the current and electromotive force in the circuit *ab*, the power of which is being measured ; θ is the angle of lag in the shunt or voltage circuit.

then proportional to $\bar{E}\bar{I}\cos(\phi - \theta)$. Introducing both factors, we have

$$\text{wattmeter reading} = \bar{E}\bar{I}\cos\theta\cos(\phi - \theta). \qquad (2)$$

$$\text{true watts} = \frac{\text{wattmeter reading} \times \cos\phi}{\cos\theta\cos(\phi - \theta)}. \qquad (3)$$

The instrument gives theoretically correct readings only when $\phi = \theta$. When ϕ is less than θ, the readings are too small, and when ϕ is greater than θ, they are too large. The correction is of greater importance in cases where ϕ is large, as in the determination of the losses in a transformer having an open magnetic circuit. This is evident from the formula, since the cosine changes rapidly as the angle approaches 90°. (For further discussion, see Fleming, *Alternate Current Transformer*, Vol. I., p. 139; Kapp, *Alternating Currents of Electricity*, p. 47.)

Example.—The self-induction of the voltage coil of a wattmeter as measured by means of the bridge, telephone and Ayrton and Perry standard of self-induction, was found to be 0·02383 henry. The resistance was 747·7 ohms, and the frequency, 111 periods per second.

$$\text{Tan}\,\theta = \frac{2\pi \times 111 \times 0{\cdot}02383}{747{\cdot}7} = 0{\cdot}0223;$$

$$\theta = 1^\circ\ 16'.$$

Taking, as an extreme case, that of a coil without a core, the wattmeter reading was 102·4 watts when the current was 16·3 amperes and the potential 66 volts. Cos θ being practically unity, we have, from equation (2),

$$\cos(\phi - \theta) = \frac{102{\cdot}4}{\bar{E}\bar{I}} = \frac{102{\cdot}4}{1076} = 0{\cdot}0952,$$

$$(\phi - \theta) = 84^\circ\ 32',$$

$$\phi = 85^\circ\ 48',$$

$$\cos\phi = 0{\cdot}0732,$$

$$\text{true watts} = 102{\cdot}4\frac{0{\cdot}0732}{0{\cdot}0952} = 78{\cdot}7.$$

Wattmeters are now constructed with the self-induction of the voltage circuit less than 0·004 henry, and with resistances of 1250 ohms for 75-volt circuits, and 25,000 ohms for 1500-volt circuits. With a frequency of 100 this gives 7 minutes and less than ½ minute, respectively, as the values of θ corresponding to the two resistances above. With a seven-minute lag, the correction is eight-tenths of one per cent when ϕ is 75° and two tenths of one per cent when ϕ is 45°. In case the high resistance is used, the correction is much less. For all measurements on transformers with closed magnetic circuits, and for most measurements on transformers with open magnetic circuits, such an instrument may be used without any correction. It would not be quite accurate, if used to measure the no-load losses of a transformer with open magnetic circuit with the low pressure coil as primary.

Shunting a Wattmeter. — When there is self-induction in the voltage circuit, the current in it lags behind the potential by an angle $\theta = \text{arc tan } \frac{L\omega}{R}$. Now if we can produce in the current coil an equal lag behind the current in the load circuit, the factor depending upon the phase difference becomes equal to one. In order to effect this, it is necessary to shunt the current coil. In the diagram, Fig. 231, AC represents the current in the coil, AB that in the shunt, and AD the resultant current which flows in the main circuit. The angle CAD, which measures the lag of the current in the coil of the instrument behind the resultant current, should be equal to θ, the lag of the current in the voltage circuit behind the electromotive force. This gives a method of correcting an instrument which has an excessive lag in the voltage coil, as well as a means of shunting a wattmeter when it is desired to use it for larger cur-

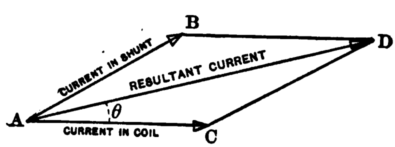


Fig. 231. Diagram of wattmeter currents.

rents than its current coil will carry.* In the latter case it is sometimes a positive advantage to have self-induction in the shunt.

In using a wattmeter, care must be taken that no disturbance is produced by currents in circuits near it. To be sure that there is no error from this source, pass a current through the movable coil alone, and if there is no deflection when the circuit is closed through the apparatus in question, there is no disturbance from this cause.

While efficiency tests may be made by means of a wattmeter on the primary, and a voltmeter and an ammeter on the secondary, it is more convenient to use a wattmeter on each circuit. These should be compared, preferably with alternating currents of the same frequency as those of the circuits to be tested. Since the ratio of the two readings is of the principal importance in determining the efficiency, neither instrument may give correct readings in watts, and yet the result may be sufficiently accurate.

Kennelly's Differential Wattmeter. — In order to be able to measure directly the difference between the primary and secondary watts, Mr. A. E. Kennelly has devised a differential wattmeter.† The instrument consists essentially of two wattmeters, one of which is placed vertically above the other, with their movable coils united rigidly by a rod. One wattmeter is connected to the primary and the other to the secondary in such a way that their torques are opposite. If they are adjusted so as to have the same constant, the resulting torque will be proportional to the difference between P_1 and P_2.

The diagram, Fig. 232, shows the essential features of the apparatus. The movable coil of the upper wattmeter C_1, in series with the resistance R_1, forms the voltage circuit for the primary. C_2 and R_2 constitute the voltage circuit of the

* See J. E. Boyd, *Proceedings Cornell Electrical Society*, Vol. III.

† *Electrical Engineer*, New York; *Electrician*, Vol. 30, Jan. 13, 1893.

secondary. The fixed coils B_1 and B_2 carry the primary and secondary currents respectively.

In order that the suspended coil of one may not be influenced by the current in the fixed coil of the other, the two wattmeters are placed at right angles; that is, the suspended coil of one and the fixed coil of the other lie in the same plane. Mr. Kennelly found it necessary to add auxiliary coils of a single turn, the positions of which could be adjusted to compensate

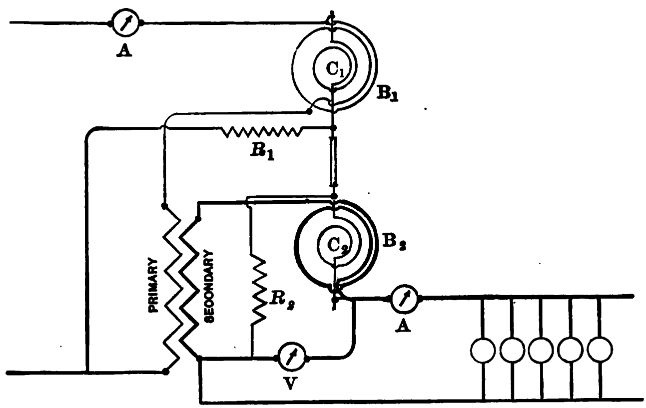


Fig. 232. Kennelly's differential wattmeter.

for the induction between the fixed coil of one and the movable coil of the other.

To get the constants of the two instruments the same, connect the current coils in series, and the voltage coils in parallel, and adjust the resistances R_1 and R_2 until the deflection becomes zero. The value of this constant may be determined in the ordinary way with either instrument. If the load be non-inductive, the secondary watts may be computed from voltmeter and ammeter readings. We will then have the losses in the transformer from the reading of the differential wattmeter; knowing the power in the secondary, we have

$$\text{efficiency} = \frac{P_2}{P_2 + p} = 1 - \frac{p}{P_2 + p}.$$

If the range of the instrument is sufficiently great, the power in one circuit may be measured directly by one of the wattmeters.

In the instrument constructed by Mr. Kennelly, R_1 was about 20,000 ohms and R_2 about 4000 ohms. The constant was one watt per division. In a transformer test, the following results were obtained:

Wattmeter Reading.	Primary Current.	Secondary Current.	Secondary Terminal Potential.	Secondary Watts, $\bar{E}_2\bar{I}_2$.
91	0·1	0	52·4	0
91	0·25	2·046	52·4	107
92	0·3	4·092	52·3	214
93	0·4	7·818	51·6	404
95	0·6	11·46	51·5	590
99	0·79	15·28	51·0	779
110	1·13	21·32	50·3	1072
127·5	1·55	38·44	50·0	1422
91	0·1	0	52·4	0

Efficiency.	$R_1\bar{I}_1^2$.	$R_2\bar{I}_2^2$.	$R_1\bar{I}_1^2+R_2\bar{I}_2^2$.	Hysteresis and Eddy Current Losses.
0	0·1	0	0·1	90·9
0·54	0·6	0·1	0·7	90·3
0·699	0·9	0·2	1·1	90·9
0·813	1·8	0·7	2·5	90·5
0·861	3·1	1·4	4·5	90·5
0·887	6·8	2·6	9·4	89·6
0·907	14·5	5·0	19·5	90·5
0·918	27·2	8·9	36·1	91·4
0	0·1	0	0·1	90·9

R_1 and R_2 above are the resistances of the primary and secondary coils of the transformer, and not the resistances of the wattmeter.

With the connections as shown in the diagram, if there are no compensating coils, the upper wattmeter measures the watts put into the transformer plus those expended in the current coil of the instrument itself. The lower wattmeter measures the output minus the losses in its current coil. In the wattmeter reading, then, we have the transformer losses together with the losses in these two coils, and unless the resistance of the coils is small, a considerable error may be introduced.

A compensating coil may be used on the upper wattmeter, as shown in the diagram Fig. 230. In the lower wattmeter the compensating coil is reversed, and the current in it flows in the same direction as that in the main fixed coil. If in Fig. 230 *ab* is regarded as a source of electrical energy, then the wattmeter connected as shown will give the true output. The deflection, however, will be in the opposite sense, and it will be necessary to reverse the terminals of the movable coil.

A slightly different method of using this differential wattmeter may sometimes be convenient, in which one of the resistances R_2 or R_1 is adjusted until there is no deflection. Suppose that the constants of the two instruments are the same, and that the resistance of R_1 is 20,000 ohms. If when used in testing a transformer it is found that 1000 ohms must be added to bring the deflection to zero, the efficiency is known at once to be $20 \div 21$, or 0·952. This method, of course, requires special resistances suitable for high potentials.

Sumpner's Differential Method.* — When two similar transformers are to be tested, the differential method of Sumpner may be employed to advantage. This method, besides being one of great accuracy, has the additional advantage that the output of one transformer goes into the other, so that it is only necessary to draw on the source for power equivalent to the losses in the two.

Figure 233 shows the connections; T_1 and T_2 are the transformers to be tested, the coils a_1, a_2, being preferably those of high potential. T_3 is a small auxiliary transformer whose primary c is connected to the source through the resistance R, and whose secondary is in series with the low potential coil of T_2. A wattmeter W_1 carries the current from the line to the transformers to be tested, but not to the auxiliary transformer.

* *Electrician*, Vol. 29, July 1, 1892, p. 223, and Oct. 7, 1892, p. 665. *Electrical World*, Oct. 8, 1892.

W_2 is a second wattmeter arranged to measure the output of T_3. The resistance R may be adjusted to vary the secondary voltage of T_3.

With switches S, S, and S_3 closed and S_2 open, close S_1; a current equal to the magnetizing current of the two transformers T_1 and T_2 will flow from the line through the watt-

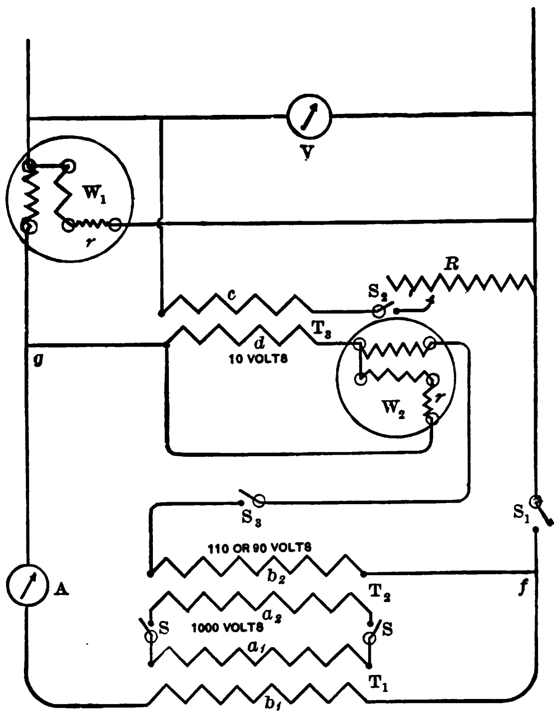


Fig. 233. Sumpner's differential method.

meter W_1, the ammeter A, and the coil b_1. The transformer T_1 is now a step-up transformer, producing current sufficient to magnetize T_2. If T_1 and T_2 are exactly equal, there will be no current in b_2 unless d is short-circuited. In the latter case both primaries will get the same current, and it will make no difference whether S and S are open or closed. The wattmeter W_1 in either case will measure the no-load losses of the two.

If, however, the transformers T_1 and T_2 be not properly connected, the result will be different. If the electromotive force of T_2 as a step-down transformer be in the same direction as that of the line, we have the case of two transformers in parallel with their secondaries short-circuited in series, which will result in blowing the transformer fuses and burning out the wattmeter W_1. To be safe, connect transformer T_1 and the right side of transformer T_2 to the line, leaving the left side of T_2 open. With S, S, and S_1 closed, measure the potential between the left terminals of b_1 and b_2, at the switch S_3 for example, by means of a voltmeter or some incandescent lamps. If no potential difference is found, the final connection may be made. If the connections are wrong, the potential will be twice that of b_1 or b_2, and one pair of terminals must be reversed.

With S, S, S_1, and S_3 closed, connect S_2. There will now be an electromotive force in *d*. Suppose that this is 10 volts, and that the potential of the source is 100. We will now have a potential of 90 or 110 volts at the terminals of b_2. If the electromotive force of *d* is not in phase with that of the source, this potential may be anything from 90 volts to 110 volts. If it is more than 100, T_2 becomes a step-up transformer, taking energy from the line, from the auxiliary transformer, and from T_1, which converts down. The wattmeter W_1 measures the power coming direct from the line, and W_2 that from the secondary of T_3. The sum of these two readings is the loss in T_1 and T_2 and in the instruments.

The output of T_1 is determined from the readings of the ammeter A and the voltmeter V. Denoting the reading of W_1 by p_1, that of W_2 by p_2, and the losses in the instruments and line l, we have as the loss of the double transformation p_1+p_2-l. If P be the output of T_1, the total power put into T_2 is $P+p_1+p_2-l$, and the efficiency of the double transformation is

$$1-\frac{p_1+p_2-l}{P+p_1+p_2-l}.$$

If the switch S_1 be opened, while the others are closed, there is the electromotive force of d in the circuit $gdS_3b_2fb_1g$, and since the current flows from left to right in b_2 while flowing from right to left in b_1, the high potential coils are in series. The magnetization of the iron is low and the iron losses relatively small, and the wattmeter W_2 measures the copper losses in transformers and instruments together with this small iron loss.

When, with S_1 open, the coils b_2 and b_1 are short-circuited, we get from W_2 the losses in the instruments and connections alone. This does not include the loss in W_1, which may best be eliminated by means of a compensating coil.

The efficiency of a single transformation is the square root of the efficiency of the double transformation if the efficiency of the two transformers is the same.

This method was used to test two 1500-watt transformers in the Physical Laboratory of Cornell University. The losses in the instruments were measured, and it was found that when the current was 17·6 amperes, the reading of W_2 was 41·6. The resistance of the instruments and connections to alternating currents was accordingly $41·6 \div (17·6)^2$ or 0·134 ohm.

The following table shows the results of a set of measurements at a frequency of 107:

Current in W_1.	Watts in W_1.	Current in A.	Watts in W_2.	Volts in V.	Loss in Instruments.	p_1+p_2-l.	$\bar{E}_2\bar{I}_2$.	Efficiency. Single Transformer.
1·84	122	—	0	107·4	0	—	—	0
1·70	116·2	4·2	16	106·8	4·7	127·5	448	0·882
1·69	114·5	5·2	24·8	106·7	6·4	132·9	552	0·897
1·67	113	7·6	44·1	106·6	11·5	145·6	814	0·921
1·65	113	10·4	69·9	107·6	19·5	163·4	1119	0·934
1·64	113	12·6	92·8	107·4	28·2	177·6	1353	0·940
1·61	111	14·4	111	106·6	34	188	1535	0·944

In computing the loss in the instruments and line, the current in b_2 was used. The current in A and a part of the connections was smaller than this, but the error introduced is inconsiderable.

Computation of Efficiency from No-Load Losses and Resistance. — The efficiency of a transformer may be obtained readily from wattmeter measurements of the no-load losses, and the resistance of the primary and secondary coils. At a given frequency the hysteresis and eddy current losses are practically constant. (See results of Kennelly above.) When the transformer is loaded, the only additional losses are the copper losses in the conductors, which may be computed if the resistance and current are known.

In the transformer tested by the method of Sumpner, the mean primary resistance was 6·7 ohms, and that of the secondary was 0·065 ohm. The no-load loss, as determined by a wattmeter, was 60 watts, at 106 volts. The transformation ratio being 10 to 1, the copper loss for a given secondary current is equal to $\bar{I}_2^2(0.065 + 0.067)$.

For 4 amperes in the secondary, the copper losses are 2·1 watts, and the total losses 62·1 watts. The output at 106 volts is 424 watts, and the efficiency is $\frac{424}{486 \cdot 1} = 0 \cdot 871$ per cent.

In a like manner for 6, 8, 10, 12, 14, and 15 amperes we get efficiencies of 0·908, 0·925, 0·936, 0·942, 0·945, and 0·947.

The copper losses may be measured for any current. Short-circuit the secondary and adjust a resistance in the primary until the current is that for which the copper losses are required, and measure the input by means of a wattmeter.* With this arrangement the magnetization is extremely low, so that the iron losses are practically nothing, and the wattmeter reading is that of the copper losses in the two coils. It is generally best to use the low potential coil as primary.

Use of Wattmeter with Coils on Separate Circuits.— In the ordinary way of using a wattmeter we have the current to be measured in one coil, while the other coil bears a current

* See D. C. Jackson, *The Electrical Engineer*, New York, Aug. 27, 1895.

in phase with the electromotive force. Now the electromotive force of the secondary is 180° behind that of the primary. To measure the power put into a transformer, we may connect the voltage coil of the wattmeter across the secondary terminals instead of the primary terminals, and the reading thus obtained, multiplied by the transformation ratio $\frac{S_1}{S_2}$, is the power required; for, the potential of the secondary being that of the primary multiplied by $\frac{S_2}{S_1}$, and being 180° behind, its power component, with respect to the primary current, is $\frac{S_2}{S_1}$ times the power component of the primary potential.

If there is considerable magnetic leakage, the transformation ratio will vary, and the lag of the secondary potential will change so that the method is no longer correct. Also with a loaded transformer, there is a fall of potential due to the resistance, and the reading will be too small by the amount of the copper losses.

Split Dynamometer Methods. — The split dynamometer method* is practically a wattmeter method, differing from the one last given in that all the energy of the secondary is consumed in the wattmeter circuit. The connections are shown in Fig. 234, in which A_1, A_2, are alternating current ammeters in the primary and secondary circuits, respectively; and D is a dynamometer, one coil of which carries the primary current, and the other the secondary current. The readings of this dynamometer are proportional to the mean value of the products of these currents.

If K be the constant of the dynamometer when the coils are in series (that is, the square of the current which gives a deflection of one division), and d the deflection when used as a split dynamometer, then the input is $\frac{S_1}{S_2} KdR_2 + R_1\bar{I}_1^2$, where

* Blakesley, *Philosophical Magazine*, April, 1891, p. 346; Ayrton and Taylor, *ibid.*, 1891, p. 354.

R_2 is the total resistance of the secondary. This is evident when we remember that Kd is the mean value of $i_1 i_2$, and that

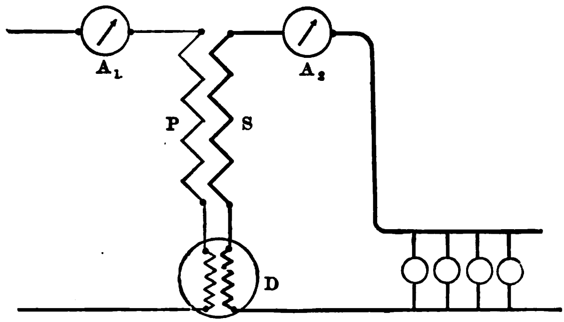


Fig. 234. Blakesley's split dynamometer method.

$\frac{S_1}{S_2} R_2 i_2 = e_1$. The output is $R_{2_{ext}} \bar{I}_2^2$, where $R_{2_{ext}}$ is the external secondary resistance; hence we have

$$\text{efficiency} = \frac{R_{2_{ext}} \bar{I}_2^2}{R_1 \bar{I}_1^2 + \frac{S_1}{S_2} R_2 Kd}.$$

The external secondary resistance may be measured by placing a voltmeter across the terminals. This resistance must be non-inductive, or the same errors will be introduced as with a wattmeter having a lag in the voltage coil. Since the mean instantaneous products of the two currents are measured, the method is independent of the form of the current waves. It is not accurate, however, when there is magnetic leakage; for in this case $\frac{S_1}{S_2} R_2 i_2$ no longer equals the primary counter-electromotive force.

Another method, also due to Mr. Blakesley, is shown in Fig. 235. D is a split dynamometer. At the point a between the coils is connected a non-inductive resistance R. Let $\bar{I}_1$ and $\bar{I}_r$ be the currents in the primary and the resistance respectively, i_1 and i_r being their values at any instant. The current in the coil *ca* of the dynamometer at any instant is $i_1 + i_r$; and in *ab* it is i_1. The torque, then, is measured by

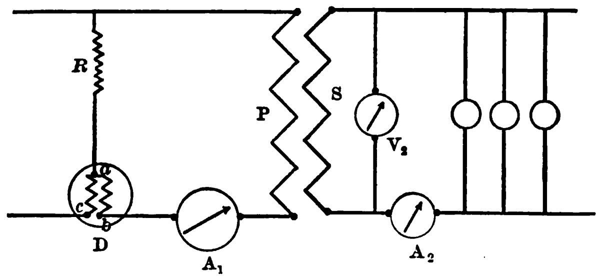


Fig. 235. Split dynamometer.

the mean product $i_1(i_1 + i_r)$, which for convenience we may represent as

$$\frac{1}{T}\int_0^T i_1(i_1 + i_r)\,dt.$$

Hence, $Kd = \frac{1}{T}\int_0^T i_1(i_1 + i_r)dt = \frac{1}{T}\int_0^T i_1^2 dt + \frac{1}{T}\int_0^T i_1 i\,dt.$

The first term of the second member is the mean square of the primary current $\bar{I}_1$, which may be obtained from the readings of the ammeter A_1. Also, $i_r = \frac{e_1}{R}$; hence the second term above is

$$\frac{1}{R} \times \frac{1}{T}\int_0^T i_1 e_1 dt = \frac{P_1}{R}.$$

Accordingly, $$Kd = \bar{I}_1^2 + \frac{P_1}{R};$$

and the primary power is

$$P_1 = R(Kd - \bar{I}_1^2).$$

The secondary power is $P_2 = \bar{E}_2\bar{I}_2$; hence,

$$\text{efficiency} = \frac{\bar{E}_2\bar{I}_2}{R(Kd - \bar{I}_1^2)}.$$

This arrangement differs from the wattmeter only in that the coil *ac* carries the primary current $\bar{I}_1$ in addition to the voltage current $\bar{I}_r$, while as a wattmeter it would carry the voltage current alone. To change to a wattmeter, connect R to c and a to the line. Then we have

$$Kd = \frac{1}{T}\int_0^T i_1 i_r \, dt = \frac{1}{T}\int_0^T \frac{i_1 e_1}{R} dt,$$

$$P_1 = \frac{1}{T}\int_0^T i_1 e_1 \, dt = RKd.$$

Using the instrument in this way, we secure greater accuracy and no longer need the ammeter A_1, a suggestion made by Professor B. F. Thomas.

Other similar methods which depend upon the difference of two comparatively large quantities, and which usually require a considerable waste of power, have been published from time to time in the technical journals. To quote from Feldmann (*Wechselstrom Transformatoren*, p. 378): A good wattmeter at a single reading will give results for a transformer with either open or closed magnetic circuit, which are as accurate as can be obtained by the other methods requiring the use of several instruments under the most favorable conditions.

Three-Voltmeter Method. — Ayrton and Sumpner's three-voltmeter method* of measuring power is shown in Fig. 236. A non-inductive resistance R is connected in series with an induc-

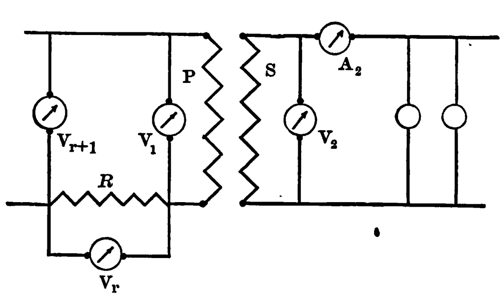


Fig. 236. Three-voltmeter method.

* *Electrician*, Vol. 26, p. 735. See also Dr. Roessler, *Electrotechnische Zeitschrift*, Oct. 24, 1895; Nichols' *Laboratory Manual of Physics and Applied Electricity*, Vol. II., p. 124; *Alternating Currents*, p. 229.

tive resistance which in this case is the primary of the transformer. Of the three voltmeters, V_{r+1} measures the total potential, V_1 the terminal voltage of the primary, and V_r the drop in the resistance. Calling the instantaneous values of these potentials, e_{r+1}, e_1, and e_r, respectively, we have

$$e_{r+1} = e_1 + e_r;$$
$$e_{r+1}^2 = e_1^2 + e_r^2 + 2\, e_1 e_r;$$
$$2\, e_r e_1 = e_{r+1}^2 - e_r^2 - e_1^2;$$
$$\frac{2}{T}\int_0^T e_r e_1\, dt = \frac{1}{T}\int_0^T e_{r+1}^2\, dt - \frac{1}{T}\int_0^T e_r^2\, dt - \frac{1}{T}\int_0^T e_1^2\, dt;$$
$$2\, R \frac{1}{T}\int_0^T e_1 i_1\, dt = \bar{E}_{r+1}^2 - \bar{E}_r^2 - \bar{E}_1^2.$$

But $\frac{1}{T}\int_0^T e_1 i_1\, dt$ is the power put into the primary; hence

$$P_1 = \frac{1}{2\, R}(\bar{E}_{r+1}^2 - \bar{E}_r^2 - \bar{E}_1^2).$$

For R we may substitute $\bar{E}_r \div \bar{I}$.

We may construct a triangle ABC (Fig. 237) with the virtual

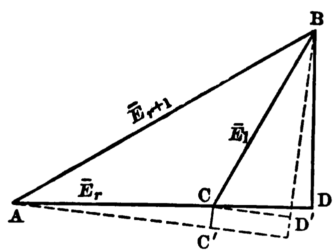


Fig. 237. Three-voltmeter method.

voltages $\bar{E}_{r+1}$, $\bar{E}_r$, and $\bar{E}_1$ as the sides. The line AC $= \bar{E}_r = R\bar{I}_1$, and

$$\mathrm{CD} = \bar{E}_1 \cos \mathrm{BCD} = \frac{\bar{E}_{r+1}^2 - \bar{E}_r^2 - \bar{E}_1^2}{2\, \bar{E}_r} = \frac{\bar{E}_{r+1}^2 - \bar{E}_r^2 - \bar{E}_1^2}{2\, R\bar{I}_1} = \frac{P_1}{\bar{I}_1}.$$

CD is then the power component of the terminal voltage of the primary coil.

If now there be some lag in the resistance R, the potential at its terminals will be ahead of the current, which in this case may be represented in direction by the line AC′, and the power component of $\overline{E}_1$, instead of being equal to CD, will be equal to CD′, the projection of BC upon a line parallel to AC′. If the lag in the primary be considerable, as is the case with transformers with open magnetic circuits, a very small lag in the resistance, which is assumed non-inductive, will make a large error in the result. A bank of incandescent lamps forms a satisfactory resistance for this work.

The method gives the best results when the resistance is such that $\overline{E}_r$ and $\overline{E}_1$ are approximately equal. It has the disadvantage that it requires an impressed electromotive force considerably higher than the potential of the transformer to be tested, also three readings of potential are necessary to determine the primary power. These three readings may all be taken with one instrument, provided a proper switch is used to change it from one position to the other, a method which will give satisfactory results only when the voltage is steady. The method is best employed only as a check on other methods.

Three-Ammeter Method. — The three-ammeter method has the advantage of not requiring a source of higher potential than is to be used in the transformer; but on the other hand

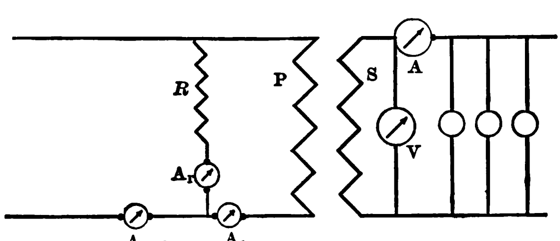


Fig. 238. Three-ammeter method.

it takes more current, a requirement, however, which is easily met. Figure 238 shows the connections. A non-inductive resistance R is placed across the mains in series with an ammeter A_r. A second ammeter A_1 measures the current in the primary, and a third A_{r+1} measures the total current.

Calling the currents in the three ammeters at any instant i_r, i_1, and i_{r+1} respectively, we have

$$i_{r+1} = i_r + i_1\,;$$
$$i_{r+1}^2 = i_r^2 + i_1^2 + 2\,i_r i_1\,;$$
$$i_{r+1}^2 - i_r^2 - i_1^2 = 2\,i_r i_1 \doteq \frac{2\,e_1}{R}\,i_1\,;$$
$$\frac{R}{2}(\bar{I}_{r+1}^2 - \bar{I}_r^2 - \bar{I}_1^2) = \text{mean value of } e_1 i_1 = P_1.$$

As before, P_2 may be found by means of an ammeter and voltmeter, or by a wattmeter in the secondary circuit.

As in the case of the three voltmeter method, the resistance must be non-inductive. It is not advisable to use a single ammeter for all readings. The instruments A_r and A_1 should be of low resistance and small self-induction.

A modified method is to use a voltmeter in parallel with the resistance, in place of the ammeter A_r. Then $\bar{I}_r$ is equal to the voltmeter reading divided by R. Unless the voltmeter is an electrostatic one, it will take some current which must be added to $\frac{\bar{E}_1}{R}$, in case the ammeters are read while the voltmeter circuit is closed. (See Nichols' *Laboratory Manual*, Vol. II., p. 126.)

Regulation. — The regulation is an important point to be considered in a transformer. The drop in voltage from no load to full load must be small, in order that the lamps supplied by it may be kept at as near the same voltage as possible.

To test the regulation, it is only necessary to take readings of the secondary potential while that of the primary is kept constant. The primary voltage may be measured by an electrostatic voltmeter if one is at hand, or a second transformer which is not loaded may be used. In the latter case one voltmeter will do for both, readings being taken alternately on the secondary of each transformer. It is better to use a separate voltmeter on each, and to take simultaneous readings when the reading of the voltmeter on the unloaded transformer is at its normal value. The voltmeters may be interchanged and errors due to difference in the instruments thus eliminated.

A 1500-watt transformer, tested with one voltmeter at no load, half load, and full load, gave the following regulation:

Auxiliary Transformer.	Transformer Tested.		Potential Difference.	Percent Difference.	Percent Difference Less Difference at No Load.
$\overline{E}_2$	$\overline{E}_2$	$\overline{I}_2$			
101·3	101·09	0	0·21	0·21	
103·2	101·51	7·7	1·69	1·64	1·43
102·7	99·66	14·8	3·04	2·96	2·75

Insulation. — On account of the danger to life due to defective insulation, that point must be carefully considered. The insulation resistance between primary and case and between primary and secondary should be measured. A convenient practical method is by means of a voltmeter. Suppose one terminal of a high potential generator be connected to the primary and the other connected through the voltmeter to the secondary. The potential of the source minus the voltmeter reading is to the voltmeter reading, as the insulation resistance between primary and secondary is to the resistance of the instrument.

If the potential of the source is 500 volts, the reading when in series with the insulation 2 volts and the resistance of the instrument 40,000 ohms, the insulation resistance is then

$$\frac{40{,}000 \times 498}{2} = 9{,}960{,}000 \text{ ohms},$$

or practically 10 megohms.

The insulation resistance may be more accurately determined* by noting the rate of discharge through it of a condenser of known capacity.

* Nichols' *Laboratory Manual of Physics and Applied Electricity*, Vol. II., p. 159.

CHAPTER XIX

EFFECTS OF HYSTERESIS AND FOUCAULT CURRENTS

THE treatment of problems involving simple harmonic currents and electromotive forces by means of vector diagrams is complete and rigorous. Simple harmonic currents and electromotive forces, however, only exist in circuits without iron. In the presence of iron, the magnetization produced lags behind the current or magnetizing force which produces it, on account of hysteresis, and the coefficients of induction are not constant. Under such circumstances it may be proved* that the current and electromotive force cannot both be harmonic.† If the impressed electromotive force is a simple sine function, the current will be composed of sine terms ‡ of 1, 3, 5, etc., times the fundamental frequency; or, conversely, if the current is a simple sine function, these odd harmonics will exist in the electromotive force which causes it to flow.

It is true that non-harmonic currents and electromotive forces cannot be accurately replaced by equivalent sine functions in *all* their effects.§ The use of polar diagrams therefore, based upon equivalent sine functions, is not entirely correct in general; it may be justified, however, in many instances. We will note

* See Rowland, *Philosophical Magazine*, Vol. XXXIV., p. 54; also, *Electrical World*, July 9, 1892.

† Harmonic is here used as meaning a simple sine function.

‡ Only odd harmonics can exist if each successive semi-period is a repetition of the preceding one.

§ *E.g.* capacity and self-induction cannot completely neutralize each other, unless the current is an exact sine function, as proved rigorously, *Transaction Amer. Inst. of Electrical Engineers*, p. 364, Vol. IX.

some of the effects produced by hysteresis and harmonics upon polar diagrams.*

It may be shown that the phase position of odd harmonics do not influence the virtual or square root of the mean square value. Furthermore, the virtual values of periodic quantities that are not harmonic may be added and subtracted as vectors.

Influence of the Odd Harmonics upon the Virtual Values.—Let $y = a\sin(x+\alpha) + b\sin 3(x+\beta) + \cdots h\sin p(x+\gamma) + k\sin q(x+\delta)$ represent the instantaneous values of current or electromotive force.

The mean square value of y is $\frac{1}{\pi}\int_0^\pi y^2dx$.

$$y^2 = a^2\sin^2(x+\alpha) + b^2\sin^2 3(x+\beta) + \cdots + h^2\sin^2 p(x+\gamma)$$
$$+ k^2\sin^2 q(x+\delta) + 2\,ab\sin(x+\alpha)\sin 3(x+\beta)$$
$$+ \cdots 2\,hk\sin p(x+\gamma)\sin q(x+\delta).$$

We may write the following integrals:

$$\int_0^\pi a^2\sin^2(x+\alpha) = a^2\left[\frac{x}{2} - \frac{\sin(x+\alpha)\cos(x+\alpha)}{2}\right]_0^\pi = \frac{a^2\pi}{2};$$

$$\int_0^\pi k^2\sin^2 q(x+\delta) = k^2\left[\frac{x}{2} - \frac{\sin q(x+\delta)\cos q(x+\delta)}{2q}\right]_0^\pi = \frac{k^2\pi}{2};$$

$$\int_0^\pi \sin p(x+\gamma)\sin q(x+\delta)$$
$$= \frac{1}{p^2-q^2}\Big[q\sin p(x+\gamma)\cos q(x+\delta) - p\cos p(x+\gamma)\sin q(x+\delta)\Big]_0^\pi.$$

When both p and q are odd, the last integral is zero.

Hence $$\frac{1}{\pi}\int_0^\pi y^2dx = \frac{a^2+b^2+\cdots+h^2+k^2}{2}.$$

The magnitude of the virtual value of a function made up of odd harmonics accordingly depends only upon the amplitude of these harmonics, and is independent of their phase positions.

* Discussed in the *Electrical World*, 1896, by F. Bedell and J. E. Boyd.

Vector Representation of Non-Harmonic Functions. — The vector representation of non-harmonic currents or electromotive forces may be illustrated and justified in the following manner.

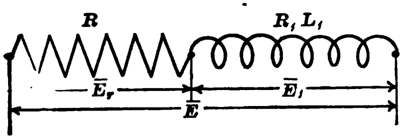


Fig. 239.

Figure 239 represents an inductive and a non-inductive circuit in series. Let $\overline{E}_1$ and $\overline{E}_r$ represent the virtual values of the electromotive forces at the terminals of the two portions of the circuit, and let $\overline{E}$ be the total electromotive force. Without any assumption as to the nature of these electromotive forces, the power expended in the inductive portion of the circuit is in general in accordance with the principles of the three-voltmeter method,*

$$P_1 = \overline{E}_1 \overline{I} \frac{\overline{E}^2 - \overline{E}_r^2 - \overline{E}_1^2}{2\,\overline{E}_r \overline{E}_1}.$$

If we represent the electromotive forces, $\overline{E}$, $\overline{E}_r$, and $\overline{E}_1$ by vectors as in Fig. 240, and the current $\overline{I}$ as a vector in the direction of $\overline{E}_r$, it follows from the geometry of the figure that

$$\frac{\overline{E}^2 - \overline{E}_r^2 - \overline{E}_1^2}{2\,\overline{E}_r \overline{E}_1} = \cos\phi.$$

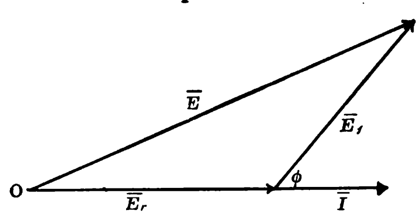


Fig. 240. Vector representation of non-harmonic currents and electromotive forces.

Here ϕ is the angle between the vectors representing the current and the electromotive force $\overline{E}_1$ at the terminals of the inductive circuit. From the above equations it follows that

$$P_1 = \overline{E}_1 \overline{I} \cos\phi.$$

* See "Three-Voltmeter Method," Chapter XVIII.

But this is the same expression for the power that would be obtained in case $\bar{E}_1$ and $\bar{I}$ were harmonic functions, and the angle ϕ was the true angle of lag between them. Accordingly Fig. 240 is correct for currents and electromotive forces that are not harmonic, and it has been seen that ϕ represents the equivalent angle of lag; that is, the angle of lag of the equivalent harmonic current behind the equivalent harmonic electromotive force.

A similar vector diagram may be constructed for the currents in an inductive and non-inductive circuit in parallel, in accordance with the principles of the three-ammeter method.

Vector Addition of Non-Harmonic Currents and Electromotive Forces. — A further justification of the use of vectors in representing functions that are not harmonic follows from the fact

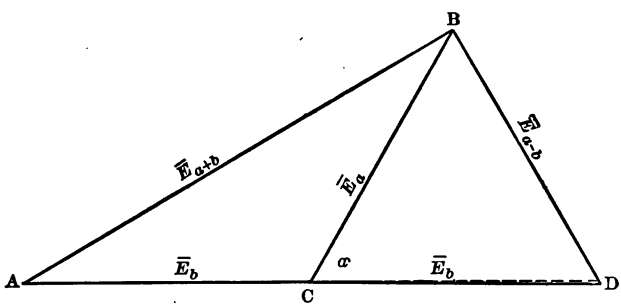


Fig. 241. Addition and subtraction of vectors representing quantities that are not harmonic.

that such vectors may be added or subtracted geometrically in the same manner as vectors representing true harmonic functions. Given two electromotive forces $\bar{E}_a$ and $\bar{E}_b$ of any form whatsoever: let their sum be $\bar{E}_{a+b}$ and their difference be $\bar{E}_{a-b}$. The vector addition is shown in the triangle ACB in Fig. 241; the subtraction is shown in the triangle BCD. If this vector addition and subtraction is correct, the angle between $\bar{E}_a$ and $\bar{E}_b$ in the two cases should be the same; that is, if the two triangles are placed together, as in the figure, with the side $\bar{E}_a$ of the one coinciding with the side $\bar{E}_a$ of the other, the side

$\overline{E}_b$ of each should be in the same direction, and ACD should be a straight line. That this is so may be proved as follows:

If ACD is a straight line, we have the relations

$$\overline{E}^2_{a-b} = \overline{E}_a^2 + \overline{E}_b^2 - 2\,\overline{E}_a\overline{E}_b \cos \alpha;$$
$$\overline{E}^2_{a+b} = \overline{E}_a^2 + \overline{E}_b^2 + 2\,\overline{E}_a\overline{E}_b \cos \alpha.$$

By addition

$$\overline{E}^2_{a+b} + \overline{E}^2_{a-b} = 2\,\overline{E}_a^2 + 2\,\overline{E}_b^2.$$

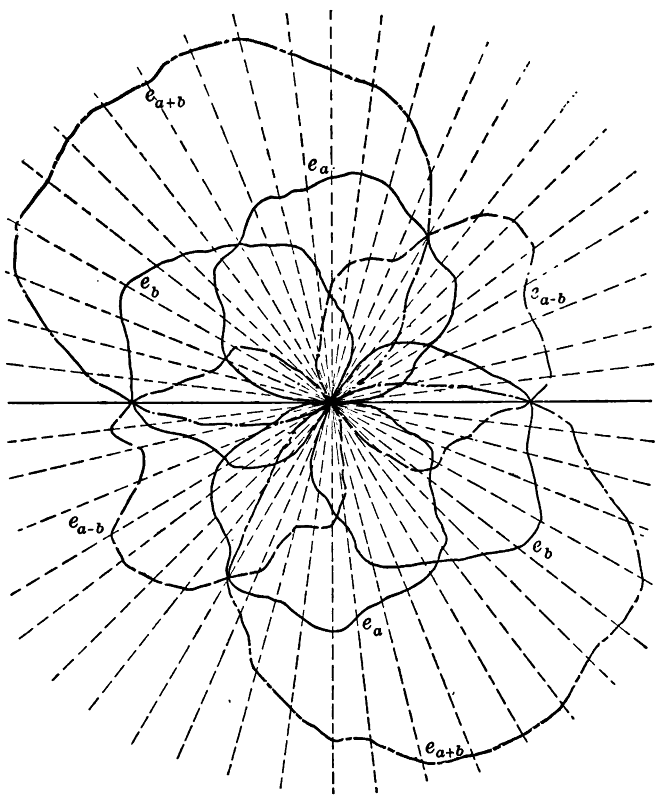


Fig. 242. Addition and subtraction of periodic quantities.

Now $\frac{1}{2\pi}\int_0^{2\pi}(e_a - e_b)^2 dt = \overline{E}^2_{a-b} = \frac{1}{2\pi}\int_0^{2\pi}(e_a^2 - 2\,e_a e_b + e_b^2)\,dt;$

and $\frac{1}{2\pi}\int_0^{2\pi}(e_a + e_b)^2 dt = \overline{E}^2_{a+b} = \frac{1}{2\pi}\int_0^{2\pi}(e_a^2 + 2\,e_a e_b + e_b^2)dt.$

Hence by addition

$$\overline{E}^2_{a+b} + \overline{E}^2_{a-b} = \frac{1}{2\pi}\int_0^{2\pi} 2\, e_a^2 dt + \frac{1}{2\pi}\int_0^{2\pi} 2\, e_b^2 dt = 2\,\overline{E}_a^2 + 2\,\overline{E}_b^2.$$

Hence ACD is a straight line, and the vector addition of periodic quantities which is not harmonic is proved to be correct.

The graphical test of this principle is shown in Figs. 241 and 242. In Fig. 242, e_a and e_b are two irregular electromotive forces represented by polar curves; their sum and difference are shown in curves e_{a+b} and e_{a-b}. The virtual values of these four curves, found by means of a planimeter, are plotted to scale in Fig. 241. The results are in accordance with the above proof, the line ACD proving to be practically a straight line, differing from one by an almost inappreciable amount, indicated by the dotted line. This discrepancy is within the error of the work.

Effect of Hysteresis upon Vector Diagrams.—Figure 243 is the diagram of electromotive forces for a simple circuit with resistance and self-induction without hysteresis. The current $\overline{I}$ lags behind the electromotive force $\overline{E}_1$ by an angle ϕ. The power electromotive force OH, being the component of $\overline{E}_1$ in the direction of $\overline{I}$, is in this case equal to $R\overline{I}$, since power is expended in ohmic resistance alone. The reactive electromotive force HJ, at right angles to $\overline{I}$, represents no power. The magnetizing force ***H*** is in the direction of the current, and equal to $\frac{4\pi SI}{l}$. In the absence of hysteresis, the magnetization ***B*** is in the direction of the magnetizing force ***H***.

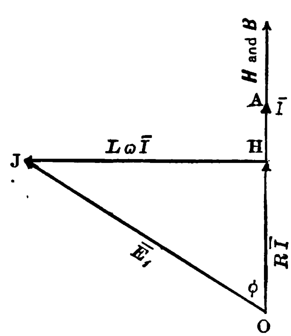


Fig. 243. Diagram for inductive circuit without iron.

Figure 244 is the diagram for a similar circuit when hysteresis is present. The power electromotive force OH comprises the electromotive force $R\bar{I}$ expended in ohmic resistance and $R_h\bar{I}$ expended in hysteresis. Together these equal $R'\bar{I}$, where R' is the apparent resistance, or more properly, a *power coefficient*. The counter electromotive force of the coil, that is, the whole electromotive force after the ohmic electromotive force has been deduced, is $\bar{E}'$. The magnetizing force ***H*** is in the direction of

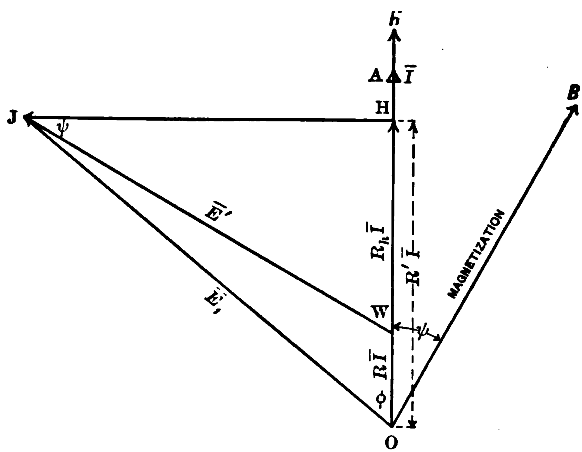


Fig. 244. Diagram for inductive circuit with iron.

the current, but the induction ***B*** now lags behind ***H*** by an angle ψ on account of hysteresis. The counter electromotive force $\bar{E}'$, it is to be observed, is at right angles to the magnetization ***B*** in accordance with the relation

$$e' = -S\frac{dN}{dt} = -SA\frac{db}{dt}.$$

The angle ψ is, therefore, equal to the angle between $\bar{E}'$ and JH, and is consequently dependent upon the value of $R_h\bar{I}$, the electromotive force used in hysteresis.

If the single coil, for which the diagrams in Figs. 243 and 244 are drawn, is the primary coil of a transformer with a secondary on open circuit, the electromotive force $\bar{E}_2$ induced in the secondary would be 90° behind the magnetization $\boldsymbol{B}$ in accordance with the relation

$$e_2 = S_2 A \frac{db}{dt}.$$

On open circuit, therefore, the secondary electromotive force is $90° + \psi$ behind the primary current, where ψ is the hysteresis angle, which is zero in Fig. 243.

Eddy Currents. — Eddy or Foucault currents introduce effects similar to those of hysteresis. Eddy currents react as secondary currents upon the primary, introducing a power component in the electromotive force and causing the magnetization to lag due to the magnetizing force of the eddy current ampere-turns. The effects of eddy currents and hysteresis may accordingly be considered together.

Effect of Hysteresis on Current Diagrams. — In Figs. 243 and 244 the several components of the electromotive force in a single circuit have been considered. The current may be similarly resolved into a reactive or wattless current, at right angles to the electromotive force, and a power current in the direction of the electromotive force. The power current represents expenditures of energy both ohmic and hysteretic. This is seen in Fig. 245.

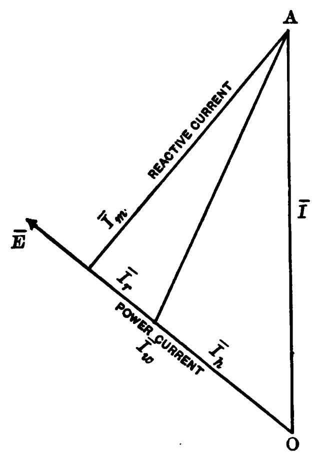


Fig. 245. Reactive and power currents.

The reactive or wattless current $\bar{I}_m$ is in quadrature with the impressed electromotive force $\bar{E}$. The power current $\bar{I}_w$, in the direction of the impressed electromotive force, comprises two parts, $\bar{I}_h$ and $\bar{I}_r$, in proportion to the iron and copper losses, in accordance with the relations

$$\bar{I}_w = \frac{\text{power}}{\bar{E}}; \qquad \bar{I}_h = \frac{\text{iron loss}}{\bar{E}}; \qquad \bar{I}_r = \frac{\text{copper loss}}{\bar{E}}.$$

We have, further,

$$\text{Power factor} = \frac{\bar{I}_w}{\bar{I}} = \frac{\bar{E}_w}{\bar{E}}.$$

Influence of Hysteresis upon the Transformer Diagram. — In the transformer diagram the effect of hysteresis is the same as that described above: it introduces an hysteretic lag ψ of the

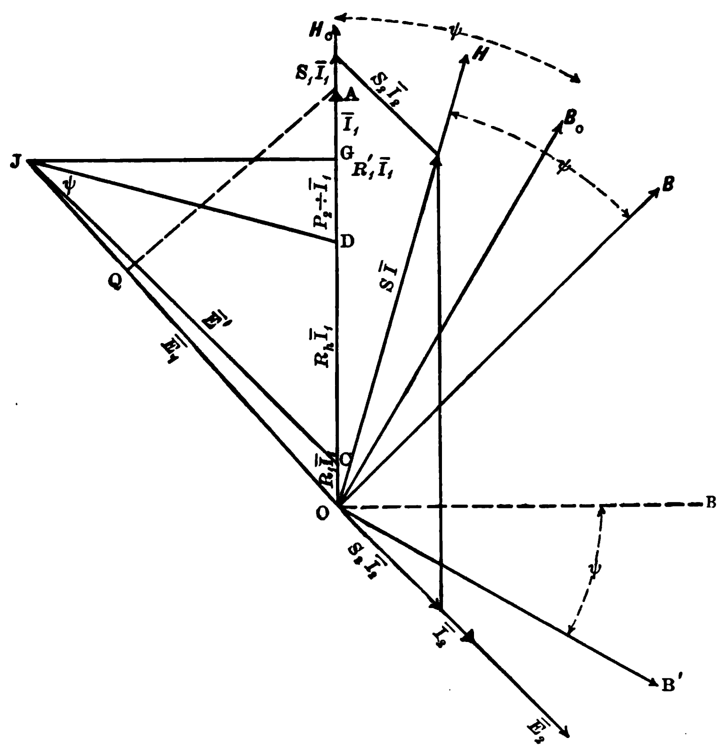


Fig. 246. Diagram for transformer on light load

magnetization *B* behind the magnetizing force *H*, which is in the direction of the resultant ampere-turns; and introduces an hysteretic resistance R_h in the primary. This is shown in Fig. 246, the diagram for a transformer under light load.

The power electromotive force OG, obtained by projecting the impressed electromotive force upon the current, is

$$R'_1\bar{I}_1 = R_1\bar{I}_1 + R_h\bar{I}_1 + \frac{P_2}{\bar{I}_1}.$$

The last term is the power transferred by the transformer from the primary to the secondary. For the case shown in the figure, this is much less than the losses, inasmuch as the diagram is drawn for a transformer on light load. On open circuit the magnetization $\boldsymbol{B}_0$ lags by an angle ψ behind $\boldsymbol{H}_0$, the magnetizing force in the direction of the primary ampere-turns. The secondary electromotive force on open circuit would be OB′, ninety degrees behind $\boldsymbol{B}_0$. When the secondary is closed, the magnetizing force $\boldsymbol{H}$ is in the direction of the ampere-turns $S\bar{I}$, the resultant of $S_1\bar{I}_1$ and $S_2\bar{I}_2$. The magnetization lags behind $\boldsymbol{H}$ by the hysteretic angle ψ. The secondary electromotive force $\bar{E}_2$, and the electromotive $\bar{E}'$, the primary electromotive neglecting ohmic resistance, are at right angles to the magnetization $\boldsymbol{B}$, and opposite to each other in direction. The angle ψ may have a value from a few degrees, in the case of a transformer with an open magnetic circuit or without any iron, to forty or fifty degrees in the case of a closed magnetic circuit transformer.

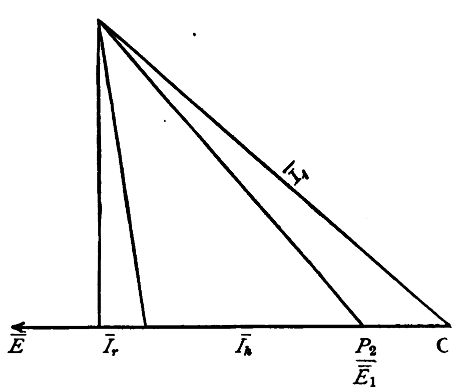


Fig. 247. Primary current resolved into components.

In Fig. 247, the primary current $\bar{I}_1$ is resolved into the reactive current and the active or power current, the latter being composed of the parts $\bar{I}_r$ and $\bar{I}_h$, in proportion to the primary copper and the iron losses, and $\frac{P_2}{E_1}$, which is proportional to the power transferred from the primary to the secondary of the transformer.

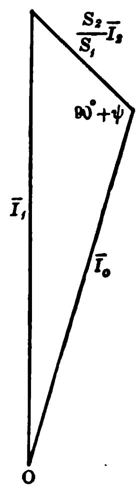


Fig. 248. Relation between primary and secondary currents.

In Fig. 246 the primary and secondary ampere-turns and the resultant ampere-turns are seen to form a triangle. In a constant potential transformer, the magnetization, and hence the resultant ampere-turns, are constant. The resultant ampere-turns $S\bar{I}$ are accordingly always equal to $S_1\bar{I}_0$, their value at no load. Making this substitution in the ampere-turn triangle in Fig. 246, and dividing each side by S_1, the triangle in Fig. 248 is obtained which shows the relation between the primary current of a constant potential transformer under load, and the secondary and magnetizing currents. From this triangle the following relation is obtained:

$$\bar{I}_1 = \sqrt{\bar{I}^2_0 + \left(\frac{S_2}{S_1}\bar{I}_2\right)^2 + 2\frac{S_1}{S_2}\bar{I}_2\bar{I}_0 \sin\psi}.$$

When there is no hysteresis, $\psi = 0$ and the expression under the radical reduces to the first two terms. For a closed magnetic circuit transformer, a close approximation is obtained by putting $\psi = 45°$, as will be seen from the table at the end of Chapter VII.

For a transformer with a closed magnetic circuit, the value ψ may be computed approximately by equating the power factor on open circuit to cos $(90° - \psi)$.

Experimental Determination of Hysteretic Lag. — To determine the angle ψ experimentally, connections were made as shown in Fig. 249. A non-inductive resistance 1, 2, was connected in series with the primary. Voltmeter readings were taken of $\bar{E}_a$, the fall in potential in this resistance, of the secondary potential $\bar{E}_b$, and of the difference of potential around the resistance and secondary in series. The last were taken in two ways. First: points 2 and 4 were connected and the voltmeter terminals placed at 1 and 5; then 2 and 5 were connected, and the poten-

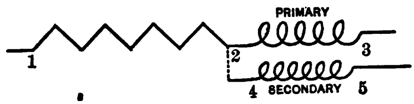


Fig. 249. Diagram of connections.

tial read between 1 and 4. Denoting these two readings by $\overline{E}_{a+b}$ and $\overline{E}_{a-b}$, we have the triangles of Fig. 241. The angle ψ is equal to $90° - \alpha$, and is determined from the relation

$$\sin\psi = \cos\alpha = \frac{\overline{E}^2_{a+b} - \overline{E}^2_{a-b}}{4\,\overline{E}_a\overline{E}_b}$$

In the case of a coil of 237 turns of No. 7 copper wire wound in six layers, some of the results were as follows:

FREQUENCY.	$\overline{E}_a$.	$\overline{E}_b$.	$\overline{E}_{a+b}$.	$\overline{E}_{a-b}$.	$\sin\psi$.	ψ.
107	48·30	14·15	50·02	49·68	0·0124	0° 43′
107	46·83	14·15	49·11	48·87	0·0084	0° 31′
107	43·48	15·75	46·44	46·16	0·0087	0° 30′
107	41·50	16·20	44·78	44·45	0·0106	0° 37′

We find then that the effect of eddy currents in the coil is to introduce a lag of 35 minutes in the secondary potential.

Placing a bundle of iron wires in this coil as a core, the results obtained were:

FREQUENCY.	$\overline{E}_a$.	$\overline{E}_b$.	$\overline{E}_{a+b}$.	$\overline{E}_{a-b}$.	$\sin\psi$.	ψ.
107	33·72	19·10	39·70	38·25	0·044	2° 32′
107	31·55	20·47	38·45	37·00	0·042	2° 25′
107	27·73	22·00	36·00	34·50	0·041	2° 21′

The same experiment was tried with a 3000-watt hedgehog transformer, and the value of ψ was found to be 4° 20′ when the secondary potential was 52 volts. In all open magnetic circuit transformers, the lag due to eddy currents and hysteresis is so small as to be practically negligible, as far as transformer diagrams are concerned.

On a small transformer having a closed magnetic circuit, the readings were

$\bar{E}_a$	$\bar{E}_b$	$\bar{E}_{a+b}$	$\bar{E}_{a-b}$	ψ	$\bar{I}_1$
57·9	57·6	108·3	41	49° 30′	0·82

It was desired to study the effect of load upon the phase relations of the magnetization and secondary electromotive force. To take the secondary current from the coil 4, 5 (Fig. 249) to which the voltmeter terminals were attached, would introduce complications; for if there is any self-induction in the load, or in the secondary coil, the potential at the terminals 4, 5 will not be in phase with the secondary electromotive force, except in the one case where the time constants of coil and load are the same. [The self-induction of the secondary which we have here to consider is that due to the lines of force which do not thread through the primary; that is, it depends upon the leakage.] To simplify the work, two secondary coils were used, the voltmeter terminals being attached to one and the current taken from the other. The first of these had the same number of turns as the primary, while the second had one-third that number.

The results for loads of various types are given below:

$\bar{E}_a$	$\bar{E}_b$	$\bar{E}_{a+b}$	$\bar{E}_{a-b}$	χ	$\bar{I}_1$	$\bar{I}_2$
			Non-inductive load.			
73·6	36·8	109·1	41·3	71° 40′	1·0	1·4
			Inductive load without iron; lag angle about 85°.			
92·5	31·4	109·5	85·4	23° 50′	1·28	2·45

Here χ is the amount by which the magnetization lags behind the primary current; it is equal to ψ when there is no current in the secondary.

With the non-inductive load we find χ greater than ψ, shown in the diagram, Fig. 246. The secondary ampere-turns being more than 90° and less than 180° behind the primary ampere-turns, the resultant magnetizing force is shifted backwards.

The result is different when we use a load producing a large lag in the secondary current. Here we have χ less than ψ, and we see that the magnetization is shifted forwards with respect to the primary current. Figure 250 is a graphical representation of what occurs. The magnetization lies in the direction ***B***, making an angle 23° 50′ with the primary current. Assuming that the phase difference between the magnetization and the magnetizing force is the same as in the case of no load, namely, 49° 30′, the resultant ampere-turns must lie in the direction ***H***, 25° 40′ in advance of $\bar{I}_1$.

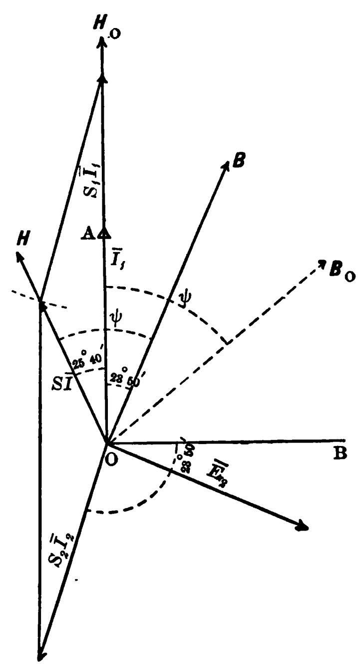
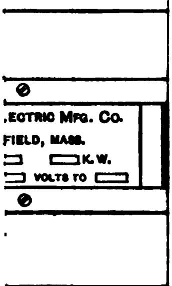


Fig. 250.

We now have the direction and magnitude of the primary ampere-turns, the magnitude of the secondary ampere-turns, and the direction of the resultant. These determine our parallelogram of magnetizing forces. The secondary ampere-turns $S_2\bar{I}_2$, we find to lie very nearly 85° behind the secondary electromotive force, giving a check on the results, as the lag in this coil at the given frequency had been previously found to be about that amount.

The above represents an extreme case. Generally the secondary current will be less than 180° behind the primary current, and the resultant magnetization will lag behind the primary ampere-turns. Since $S_1\bar{I}_1$ is greater than $S_2\bar{I}_2$, the resultant ampere-turns always lie in the two upper quadrants, and with a given secondary current an increase of lag occasioned by replacing resistance by inductance shifts the magnetization forward.

Where the lag due to hysteresis and eddy currents is not known, it may be safely neglected, and assumed zero in the case of a transformer with an open magnetic circuit, and assumed to be 45° as an approximation in a closed magnetic circuit transformer.



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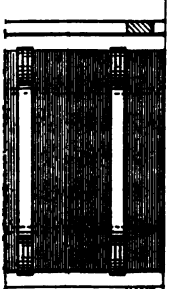

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