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ROYAL SOCIETY OF EDINBURGH.

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NOVEMBER 1908 TO JULY 1909.

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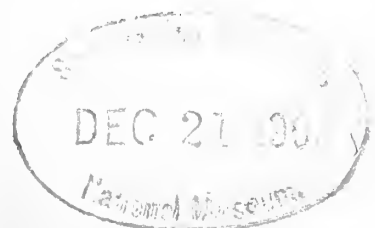
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PROCEEDINGS

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1908-9.

I.—The Shift of the Neutral Points due to Variation of the Intensity of Mechanical Vibrations or Electric Oscillations superposed upon Cyclic Magnetisation in Iron. By James Russell.

(Read November 18, 1907. MS. received July 18, 1908.)

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WHEN mechanical vibrations or electric oscillations are superposed at all points of a normal hysteresis loop, the induction at the cyclic extremes is increased, the residual magnetisation decreased, and those differences of magnetic condition to which hysteresis gives rise lessened. The curves delineating these changes form a continuous loop, and neutral points occur where vibrations superposed upon decreasing cyclic field produce no induction change whatever.

In a former communication* it was found that the position of the neutral points in the first and third quadrants depends not only upon the cyclic amplitude, but upon the intensity of the vibrations. To quote: "The smaller the cyclic field" (amplitude) "and the greater the vibrational intensity, the closer is the neutral point thrust towards the vertical axis; the higher the cyclic field and the less the vibrational intensity, the closer is the neutral point thrust towards the cyclic extreme." For any given loop, therefore, the position of the neutral points depends wholly on the intensity of the vibrations.

In a recent paper, Maurain,† experimenting with electric oscillations, obtained a different result. He found that the position of the neutral points in the first and third quadrants is where "*la courbe d'aimantation stable*," as defined, cuts the particular hysteresis loop upon which the oscillations are superposed. Their position, thus definitely fixed, is, according to these experiments, independent of the intensity of the oscillations. On the other hand, Dr Eccles,‡ also experimenting with electric oscillations produced by single minute sparks, obtained yet another result. Although the "spark effect" increases the induction at both cyclic extremes, the first recorded readings, as these positions are departed from, show in all cases decrease of induction. There is here no evidence of neutral points, and the curves plotted from the observations are discontinuous in the same way that dB/dH plotted against cyclic field produces curves discontinuous at the cyclic extremes.

Further, in an earlier series of experiments, Garibaldi found that Hertzian waves *increased* or *decreased* the permanent magnetism of steel according as it *had* or had not been previously subjected to a demagnetising field immediately withdrawn. On the other hand, Maurain, in the paper above referred to, obtained a result which he describes as "*presque nulle*" when oscillations are superposed upon residual magnetisation which had been reached by the same process of withdrawing a demagnetising field. The author of a paper § which I have been unable to refer to wrote me as follows:—"I found that if a piece of iron or steel is magnetised by a direct current, and then demagnetised by reversing the current, the effect

* "The Superposition of Mechanical Vibrations (Electric Oscillations) upon Magnetism, and conversely, in Iron, Steel, and Nickel," *Trans. R.S.E.*, vol. xlv., part ii., p. 510. *Phil. Mag.*, Oct. 1907. *Electrician*, July 5, 1907.

† "Les Détecteurs magnétiques et l'action des Oscillations électriques sur l'aimantation," *Journal de Physique*, Janvier 1907.

‡ *Proceedings of the Physical Society of London*, vol. xx. *Phil. Mag.*, August 1906.

§ Note presented at the Meeting of the American Association for the Advancement of Science, by G. F. Stradling. (Reports have not been received by the R.S.E. for several years.)

of a series of taps is first to cause the development of magnetic poles in the sense of those produced by the magnetising current. The strength of the poles reaches a maximum and diminishes. I was not able to get this effect by tapping the iron after the magnetising current was broken, but only when the iron was at least approximately demagnetised. Fromme seems to have found this effect, though he says very little about it."

Eccles connects the *magnitude* of the spark effect "at any point" with the slope of the hysteresis loop; Maurain, with the whole magnetic history of the iron. According to the latter, the *direction* of the induction change depends upon the position of the point where oscillations are superposed relative to "*la courbe d'aimantation stable*"; according to Garibaldi and Stradling, it must depend also on how that point had been reached.

None of these apparently discordant results are, however, complete statements of the whole facts as observed by the author. All alike have failed to recognise the shift of the neutral points with the intensity either of the mechanical vibrations or electric oscillations, which implies that within the limits of the range of shift their intensity determines whether the induction will be increased or decreased.

I shall endeavour to show that the introduction of this factor, which the present investigation fully confirms and extends, co-ordinates the apparently discordant results; is not only in harmony with the molecular theory of magnetism, but a necessary deduction from it.

I. MECHANICAL VIBRATIONS.

Apparatus.

The mild steel wire (100 cms. long and .092 cms. diameter), one of the wires with which the former experiments had been made, was, after demagnetisation by decreasing reversals, reannealed by passing a bunsen flame along its entire length. It was wound with a few turns of soft woollen yarn and inserted in the axis of the horizontal magnetising coil, 41 cms. long, placed at right angles to the earth's magnetic field. One of its ends was soldered to the edge of the gong of the electric bell, the whole apparatus, as previously described, being suspended from the roof by means of india-rubber tubing to prevent as far as possible any external disturbances reaching the wire. The damped trains of mechanical vibrations were produced in the wire by a steel ball so arranged as to strike the gong once after rolling down the angle between two inclined planes. An exploring coil and ballistic galvanometer (11 seconds complete period), with com-

pensating coils, measured as before the change of magnetic intensity at approximately the central portion of the wire.

Experimental Methods.

Cyclic Field.—After demagnetisation by decreasing reversals (the revolving commutator being used), the field, put on by steps of increasing reversals, is reversed about forty times at some pre-arranged extreme value. One-half the average of two consecutive readings determines the induction at this cyclic extreme; from which steps (the first being zero) are then taken to a sufficient number of points in the first and third quadrants from positive and negative amplitudes respectively, the induction change being noted. At each point the galvanometer readings are again noted, on the superposition of a series of nine damped trains of vibrations of increasing intensity, produced by increasing the heights through which the steel ball falls in arithmetical progression. On the completion of each series, the field is put off in steps of decreasing reversals. In no case are further vibrations superposed at the same or any other point until the demagnetising process, followed by reversals at the cyclic amplitude, has intervened.

Cyclic Residual Magnetisation.—After about forty reversals at a high value of field amplitude, the induction is first determined as above described. Noting the galvanometer reading on withdrawing the field determines the residual magnetisation, which is now subjected to various values of a reversed field, so selected that when withdrawn the steel wire is left residually magnetised in zero field at a sufficient number of points of what may very well be called *cyclic residual magnetisation* between positive and negative maxima of residual magnetisation in the usual sense. A number of these points are shown in the reference diagram accompanying figs. III. and IV., and their positions on the vertical axis are determined by noting the two galvanometer readings when the reversed field is put “on” and “off,” taken in conjunction with the value of the residual magnetisation previously determined. At each point the galvanometer readings are again noted on the superposition of the nine damped trains of mechanical vibrations of increasing intensity produced as above described. On the completion of each series the cyclic extreme is again reverted to, and in no case are further vibrations superposed at the same or any other point until after forty reversals of the cyclic field, followed by the cyclic field process as above described. Owing to the high field amplitude, demagnetisation by decreasing reversals was considered unnecessary.

Intensity of Vibrations.

The intensity of the mechanical vibrations may be increased either by increasing the weight (W) of the steel ball or the vertical distance (D) through which it falls before striking the gong to which the experimental wire is soldered. When W and D are varied simultaneously so that the product of $W \times D^{0.7}$ is not changed, it is found that, under the same conditions, the induction change in the wire as measured by the ballistic galvanometer also remains unchanged within the limits of accuracy attainable in these experiments. The *intensity* of the vibrations is used in the sense that it varies with the function $W \times D^{0.7}$. In any series of experiments the same ball is used, the distance through which it falls alone being varied.

Diagrams.

Field (H), induction (B), and induction change (dB) are in C.G.S. units in all the diagrams. The intensity of the vibrations is proportional to $W \times D^{0.7}$, the weight (W) of the steel ball being in grams, the vertical distance (D) through which it falls in cms. All the points plotted are obtained by averaging the readings taken at symmetrical points in reference to the positive and negative cyclic extremes. They are, however, represented in all the diagrams as if taken in reference to the positive extremes. This applies both to cyclic fields and cyclic residual magnetisation.

Cyclic Field.

Experimental Data.—In fig. I. a series of nine trains of damped mechanical vibrations increasing in intensity are superposed in succession upon a sufficient number of points of that part of the hysteresis loop decreasing for $H=1.9$, $B=3170$ (cyclic extreme) to $H=0$, $B=2000$. The abscissæ represent the intensity of the vibrations, increasing from $W \times D^{0.7} = 2.2$ to 10.4 , superposed at eight values of field, viz. $H=1.9$, 1.72 , 1.6 , 1.34 , 1.1 , 0.95 , 0.52 , and 0 ; the ordinates, the summation of the induction changes following thereon. Fig. I. taken in conjunction with Table I. which supplies the actual data from which two of the curves, viz. $H=1.34$ and 1.1 , have been plotted, is, without further explanation, self-explanatory.

Fig. I.a shows the same curves drawn to a slightly contracted horizontal scale, to a largely expanded vertical scale, in so far as they can be shown within the limits of about $B = \pm 60$. The rectangles in dash lines show the contraction and expansion of the horizontal and vertical ordinates

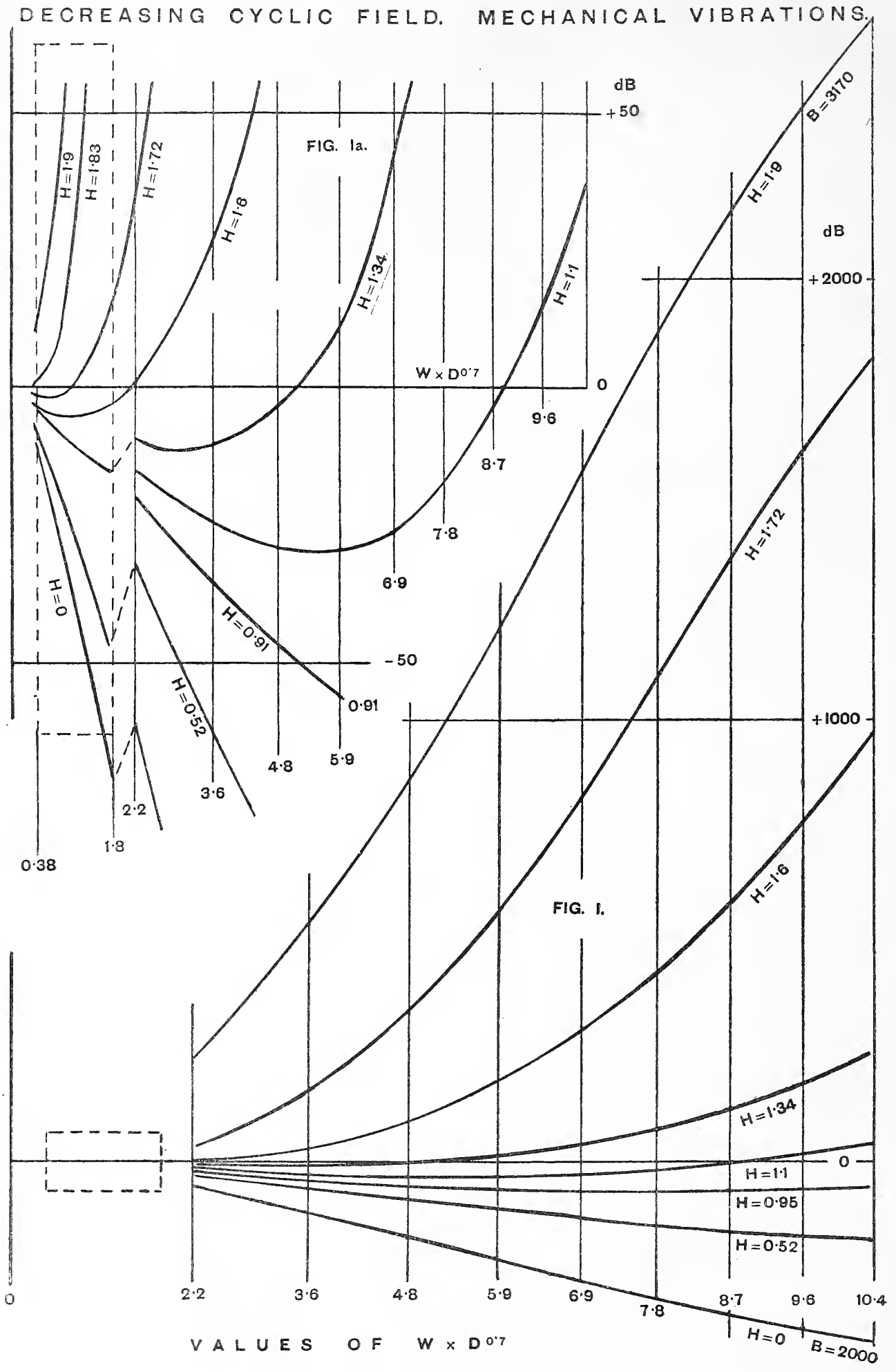


TABLE I.—DECREASING CYCLIC FIELD. MECHANICAL VIBRATIONS.

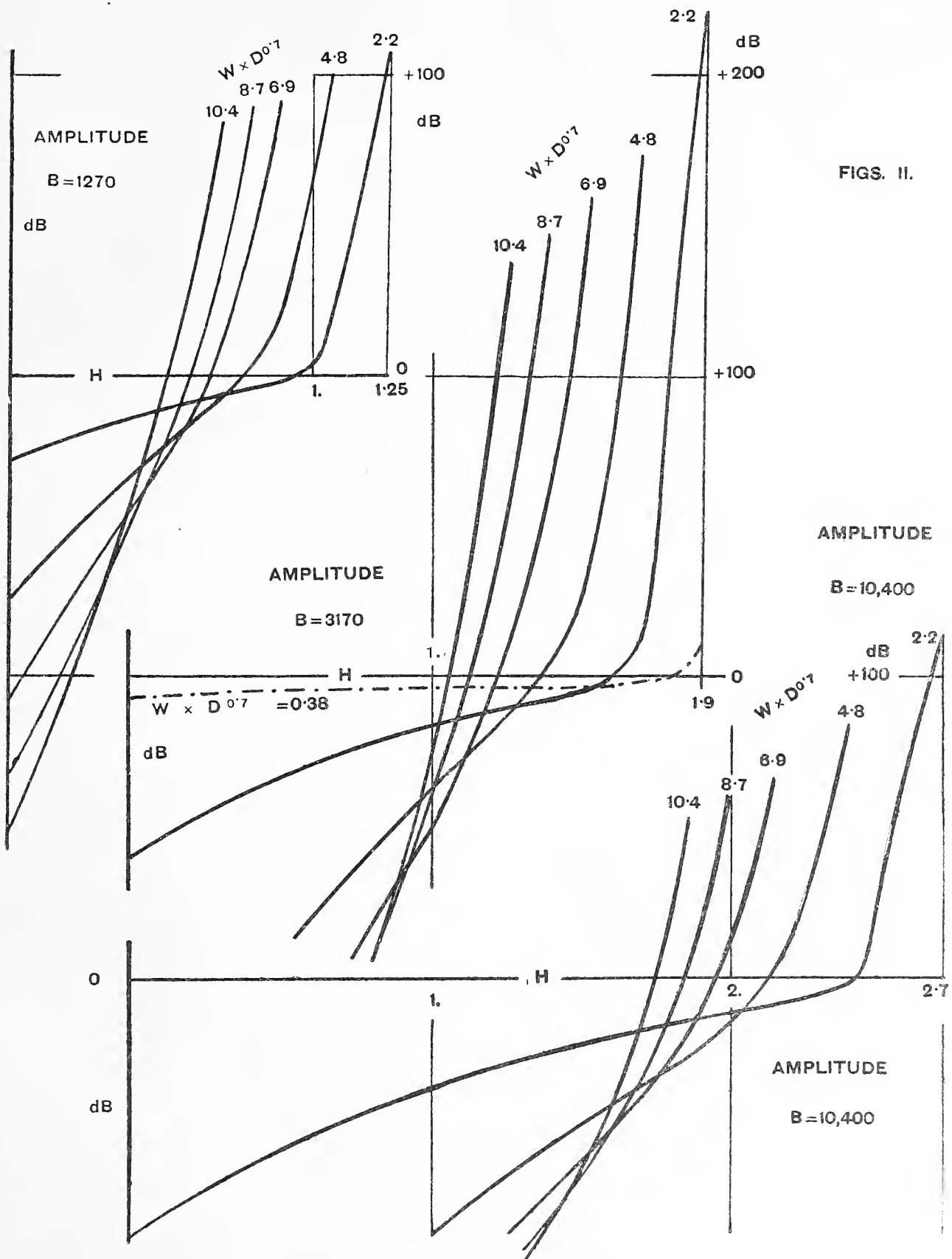
$W \times D^{0.7}$.	Vibrations superposed upon		Vibrations regarded as superposed upon $H = +1.34$.			Curve $H = 1.34$ of fig. I.
	$H = +1.34$ reduced from $H = +1.9$.	$H = -1.34$ reduced from $H = -1.9$.	Average readings.	Summation.	Summation $\times 2.95$ = Total change of B.	
2.2	-3	3	-3	-3	-9	
3.6	-0.5	0.5	-0.5	-3.5	-10.5	
4.8	2	-3	2.5	-1	-3	
5.9	4	-6	5	4	12	
6.9	11	-9	10	14	41	
7.8	14	-10	12	26	77	
8.7	12	-18	15	41	120	
9.6	22	-24	23	64	189	
10.4	26	-18	22	86	254	

$W \times D^{0.7}$.	Vibrations superposed upon		Vibrations regarded as superposed upon $H = +1.1$.			Curve $H = 1.1$ of fig. I.
	$H = +1.1$ reduced from $H = +1.9$.	$H = -1.1$ reduced from $H = -1.9$.	Average readings.	Summation.	Summation $\times 2.75$ = Total change of B.	
2.2	-5	5	-5	-5	-15	
3.6	-3	3	-3	-8	-24	
4.8	-2	1.5	-1.75	-9.75	-29	
5.9	-1	-0.5	-0.25	-10	-30	
6.9	1	-1	1	-9	-27	
7.8	4.5	-3.5	4	-5	-15	
8.7	2	-4	3	-2	-6	
9.6	6	-8	7	5	15	
10.4	9	-7	8	13	38	

respectively in fig. I.a as compared to fig. I. Fig. I.a also shows a second series of experiments, conducted in exactly the same way, in which, by means of a smaller steel ball (0.252 grams instead of 1.475 grams), nine trains of vibrations of increasing intensity are superposed in succession between the values of $W \times D^{0.7} = 0.38$ and 1.8 for the various values of H shown. Had these curves been drawn to the same scale as that adopted in fig. I., they would have fallen within the area of the small rectangle shown with that figure. Fig. I.a therefore enables the curves to be more readily traced closer to the vertical axis, the lowest value of $W \times D^{0.7}$ (0.38) being twenty-seven times less in intensity than the highest value (10.4). Note that the curves for the same values of H in the two series of experiments are not in all cases continuous. The conditions are different. Thus it is obvious from the discontinuity of the curves when $H = 1.34$, 0.52, and 0, that the demagnetising effect of *nine* vibrational trains of increasing intensity superposed in succession between the limits of $W \times D^{0.7} = 0.38$ and 1.8 is greater than that of the initial single vibrational train $W \times D^{0.7} = 2.3$ of the first series of experiments. On the other hand, when $H = 1.72$ and 1.6 the curves appear to be quite continuous, notwithstanding the dissimilarity of the conditions.

The experiments illustrated in fig. I. were repeated for a lower and a higher value of cyclic amplitude, and all the results are shown in the three diagrams of fig. II., the field and induction at cyclic amplitudes being $H = 1.25, 1.9, 2.7$, and $B = 1270, 3170, 10,400$ respectively. The induction changes are in this instance not plotted against $W \times D^{0.7}$ for various values of field as in fig. I., but against field (decreasing from the positive cyclic extreme) as abscissæ for various values of $W \times D^{0.7}$. The curves of the middle diagram are thus derived from the same experimental data as those of fig. I. In the latter case each curve represents one series of readings obtained consecutively; in the former case each curve is obtained by selecting the same particular values of $W \times D^{0.7}$ from eight distinct series of readings. The curves for the vibrational intensities $W \times D^{0.7} = 3.6, 5.9, 7.8$, and 9.6 have been omitted in these diagrams to avoid unnecessary crowding. The openness of the vertical scale shows clearly the regular sequence in which the curves cross each other, and also the shift of the neutral points where each curve crosses the horizontal axis; but it precludes the delineation of the complete curves, with the exception of those for the lowest vibrational intensity in each case, viz. $W \times D^{0.7} = 2.2$. It may, however, be stated that the curves do not appear to cross each other at any other points than those shown in these diagrams, nor do they cross each other at any point whatever when induction change is plotted against

DECREASING CYCLIC FIELD. MECHANICAL VIBRATIONS.



FIGS. II.

$W \times D^{0.7}$ for the various values of field decreasing from the cyclic extremes (fig. I.).

The broken line curve of the middle diagram of fig. II. has not yet been referred to. It is derived from the same experimental data from which the second series of curves (within the dash line rectangle) of fig. I.a were plotted, by selecting from eight distinct series of experiments for various values of H the readings obtained upon the superposition of the initial single vibrational train only in each case. These readings are given fully in Table II. The intensity of the vibrations is represented by $W \times D^{0.7} = 0.38$, and the necessity of taking readings at values of decreasing field close to the cyclic extreme, for this the lowest vibrational intensity used in these experiments, may be noted.

TABLE II.—DECREASING CYCLIC FIELD. MECHANICAL VIBRATIONS.

Values of H + or -.	Vibrations superposed upon		Vibrations regarded as superposed upon + H .		Broken line curve fig. II. $W \times D^{0.7} = 0.38$.
	$H +$.	$H -$.	Average readings.	$\times 2.95 =$ change of B .	
1.9	3.0	-3.5	3.25	9.5	
1.86	1.25	-0.75	1.0	3	
1.83	0.25	-0.25	0.25	0.75	
1.72	-0.5	0.25	-0.375	-1	
1.6	-1.0	0.75	-0.875	-2.5	
1.34	-1.0	1.0	-1.0	-3	
0.52	-1.75	1.75	-1.75	-5	
0	-2.75	1.75	-2.25	-6.5	

Results.—When damped trains of mechanical vibrations of increasing intensity are superposed upon that part of a normal hysteresis loop when the field is decreasing, the induction is for various values of field neither at nor too far removed from the cyclic extremes, first decreased, afterwards increased. The curves representing these changes (figs. I. and I.a) first fall somewhat below, afterwards may rise largely above, the horizontal axis. Any one of these curves is sufficient to show that the *direction* of the induction change is not independent of the vibrational intensity, while a comparison of the curves confirms the conclusion previously arrived at

(by a different experimental method) that for any given normal loop the neutral points are thrust from or towards the cyclic extremes according as the vibrational intensities are greater or less. Thus for a cyclic amplitude of $H=1.9$, $B=3170$ (fig. I.a), the neutral points for the following ascending values of the decreased cyclic field, viz. $H=1.1$, 1.34 , 1.6 , and 1.72 , occur with vibrations of decreasing intensity of approximately $W \times D^{0.7} = 9, 5, 2$, and 1 respectively.

It is also further evident that the neutral points are not only thrust towards the cyclic extremes the weaker the vibrations, but that they may occur quite close to it, provided the vibrational intensity is sufficiently reduced. For instance, with a field value of $H=1.83$ removed from the cyclic extreme by so small an amount as 0.07 , although the curves fail to fall below the horizontal axis, it has every appearance of doing so had the intensity of the initial vibrational train been only a little less than $W \times D^{0.7} = 0.38$ (fig. I.a). In any case, a *higher limiting value* of the neutral points occurs well within $H=0.1$ of the cyclic extremes, for this particular value of the amplitude, if the mechanical vibrations are weak enough (see also broken line curve, fig. II.). On the other hand, the neutral points are thrust from the cyclic extremes the greater the vibrational intensity, until a lower limiting value of the decreased cyclic field is reached, beyond which vibrations of all intensities produce only induction decrease (fig. I., $H=0.95, 0.52$, also the middle diagram of fig. II.).

The above results apply to a cyclic amplitude of $H=1.9$, $B=3170$. At lower and higher amplitudes, viz. $H=1.25$, $B=1270$, and $H=2.7$, $B=10,400$, the various neutral points with different vibrational intensities occur at lower and higher values of field respectively, and each corresponding neutral point for any given vibrational intensity is thus thrust from the vertical axis the higher the amplitude (fig. II.). At these three increasing values of cyclic amplitudes, the range of field through which the neutral points may shift with intensity, viz. $H=0.43, 0.54$, and 0.67 respectively, is also increased as the amplitudes are increased (fig. II.), but this must again contract as saturation values are reached.

To sum up, the conclusions may now be stated more definitely and fully than in the previous communication. The neutral points which occur when vibrations are superposed upon the decreasing cyclic field symmetrical about the origin are thrust towards the vertical axis the less the cyclic amplitude and the greater the vibrational intensity, towards the cyclic extreme the greater the cyclic amplitude and the less the vibrational intensity. There are thus lower and higher limiting values of the decreasing field through which the neutral points may shift, and both must vanish in

the origin or the cyclic extremes according as the amplitude is sufficiently decreased (zero amplitude at the origin) or sufficiently increased (saturation values of induction at the extremes). The maximum range of possible shift, which may be extensive, occurs between these two extremes. Between the lower and higher limiting values of the range of shift, vibrations increase or decrease the induction according as the vibrations are sufficiently increased or sufficiently decreased. Below the lower limiting value vibrations of all intensities, however great, decrease the induction. Above the higher limiting value, vibrations of all intensities, however weak, increase the induction.

In Section III. these results will be co-ordinated with the immediately preceding field change.

Cyclic Residual Magnetisation.

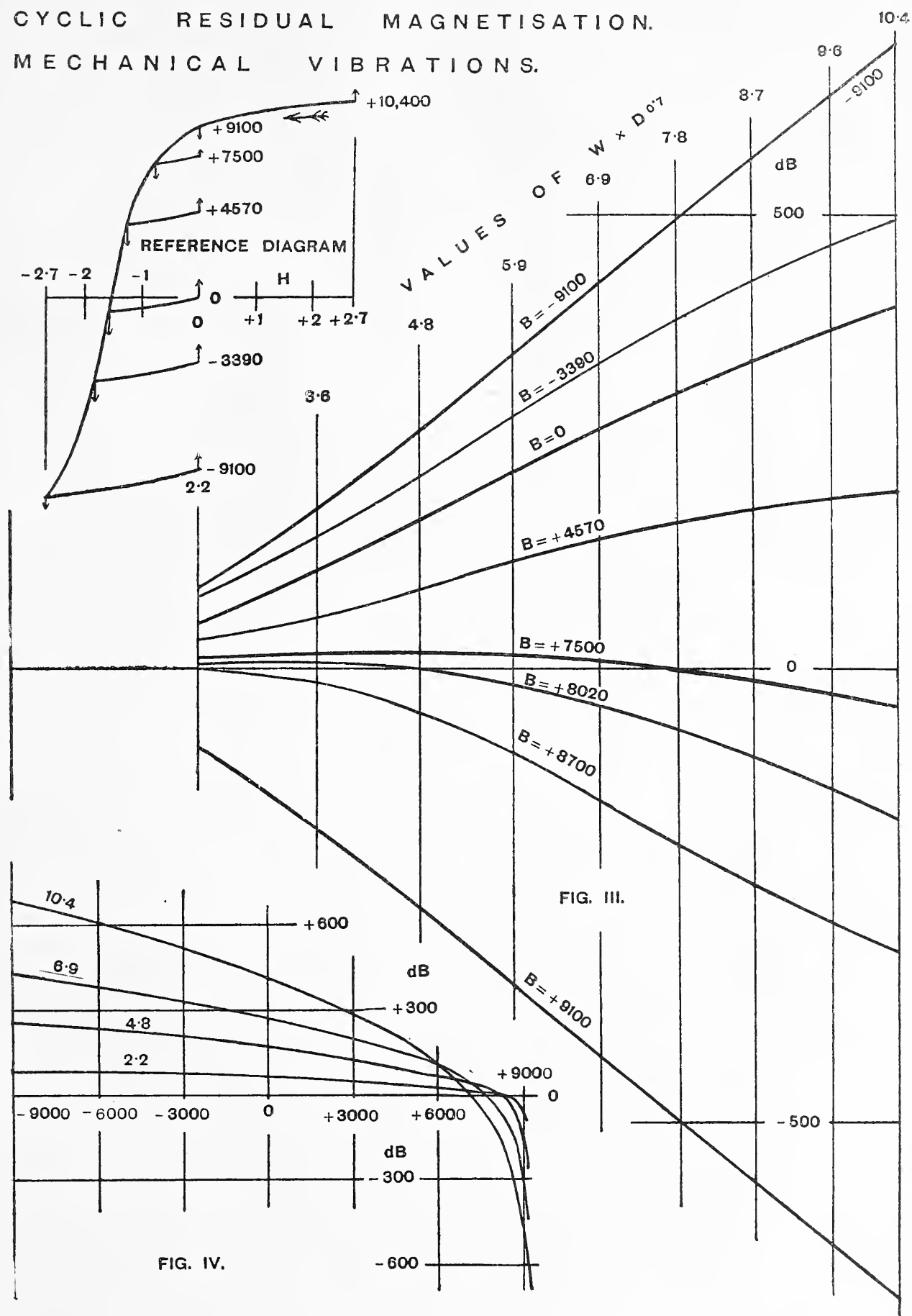
Experimental Data.—In fig. III. the series of nine trains of damped vibrations, increasing in intensity, are superposed in succession upon a sufficient number of values of cyclic residual magnetisation as already defined, between the limits of $B = +9100$ and -9100 , the field being zero. The reference diagram shows these values as reached from the *positive* cyclic extreme, viz. $H = 2.7$, $B = 10,400$, just as the results for cyclic field have been represented as occurring with the decreasing positive field. The abscissæ (fig. III.) represent (as in fig. I.) the intensity of the vibrations increasing from $W \times D^{0.7} = 2.2$ to 10.4 superposed upon eight values of cyclic residual magnetisation, viz. $B = +9100$, $+8700$, $+8020$, $+7500$, $+4570$, 0 , -3390 and -9100 ; the ordinates, the summation of the induction changes following thereon.

In fig. IV. the induction changes from the same experimental data are plotted, not against $W \times D^{0.7}$ for various values of cyclic residual magnetisation, as in fig. III., but against cyclic residual magnetisation as abscissæ for the following values of vibrational intensity, viz. $W \times D^{0.7} = 2.2$, 4.8 , 6.9 , and 10.4 . The curves for the other five values of vibrational intensities have been omitted, to avoid confusion.

Obviously figs. III. and IV. for cyclic residual magnetisation correspond to figs. I. and II. respectively for decreasing cyclic field.

Results.—When damped trains of mechanical vibrations are superposed at the extreme values of cyclic residual magnetisation, the resulting induction changes are, for reasons of symmetry, the same quantitatively, but of reversed sign—negative in reference to positive magnetisation, positive in reference to negative magnetisation. In other words, the residual magnetisation is for all values of vibrational intensity reduced. In fact, the extreme values of

CYCLIC RESIDUAL MAGNETISATION.
MECHANICAL VIBRATIONS.



cyclic residual magnetisation are reached in the ordinary way by withdrawing the field at the cyclic amplitudes. But as these extreme and symmetrical positions are departed from, the curves for the same intermediate values of positive and negative cyclic residual magnetisation plotted against increasing values of $W \times D^{0.7}$ become wholly unsymmetrical. To so great an extent is this the case that the decrease of the cyclic residual magnetisation has, for the lowest value of vibrational intensity, viz. $W \times D^{0.7} = 2.2$, wholly vanished when $B = +8700$ is reached, to be immediately replaced by increase at only slightly lower values, viz. $B = +8020$ and $+7500$ (fig. III.). These curves further show that the direction of the induction change is not independent of the vibrational intensity. They first rise somewhat above, afterwards may fall largely below the horizontal axis. The neutral points which therefore occur are thrust from or towards the positive extreme of residual magnetisation according as the vibrational intensities are greater or less. Thus, for values of cyclic residual magnetisation between the limits of $B = \pm 9100$, the neutral points for the following ascending values of $B = +7200, +7700, +8020,$ and $+8700$ occur with vibrations of decreasing intensity, viz. $W \times D^{0.7} = 10.4, 6.9, 4.8,$ and 2.2 (fig. IV.). Thereafter, vibrations of all intensities between the above experimental limits increase the cyclic residual magnetisation to a greater extent the less its value. When the zero of residual magnetisation is passed, the positive induction change, now decrease in reference to the negative magnetisation, continues at somewhat increased values until the negative extreme is reached.

These results do not differ essentially from those obtained with cyclic field. The shift of the neutral points is subject to essentially the same laws, although the direction of the induction changes is opposite in the two cases. At a later stage (Section III.) it will be shown that even these apparently opposite effects disappear when the results for cyclic field and cyclic residual magnetisation are expressed, not in terms either of magnetisation or of field, but of *field change*.

II. ELECTRIC OSCILLATIONS.

Apparatus.

Experiments were in the first place tried with the mild steel wire .092 cms. diameter, already used, and the results obtained for transverse oscillations will be given later. Owing to the effect of the oscillations being mainly confined to the surface of the wire, the galvanometer readings were,

as anticipated, much less than with mechanical vibrations. Lengths of iron wires 100 cms. long and 0·028 cms. diameter, annealed and varnished, were therefore made up into bundles of five wires each. Four of these bundles could be inserted into either of two narrow glass tubes 41 cms. long. Round the first, five silk-covered copper wires 0·056 cms. diameter were closely wound in a single layer, through which oscillations could be passed in parallel. This formed *oscillation solenoid No. 1*, with three (15/5) effective turns of wire per cm. of length. The second glass tube was wound with a single silk-covered copper wire of the same diameter, also in a single layer throughout its entire length, through which oscillations could be passed in series. This second glass tube formed *oscillation solenoid No. 2* with fifteen (effective) turns per cm. of length.

Either of these solenoids, after insertion in the magnetising coil at right angles to the earth's field and as previously described, could be connected with the secondary of an air oscillation transformer consisting of eight turns of copper wire 0·264 cms. diameter, insulated from its supports by means of india-rubber tubing. The radius of the secondary was 8·7 cms., and it was earthed at a point approximately equidistant from its terminals.* The primary, consisting of one turn of copper wire insulated in the same way, 0·264 cms. diameter and 15 cms. radius, was connected with the terminals of a small Wimshurst machine (8-inch plates) with one of its small Leyden jars in series. A row of spark gaps, each 1 mm. in length, formed a shunt across the terminals of the Wimshurst, and any number from one to ten could be introduced into this parallel circuit by means of a sliding wire connection, pointed at one of its ends so as to rest securely in small holes drilled in the spark balls. These were of brass, 1·2 cms. diameter, and only approximately spherical. They were cemented to vertical glass supports. The above description is illustrated in fig. V., which may be regarded as a plan of the apparatus drawn to scale. In the preliminary experiments with the single steel wire (·092 cms. diameter) a single micrometer spark gap was used. It could be increased in length from 1 to 12 mm.

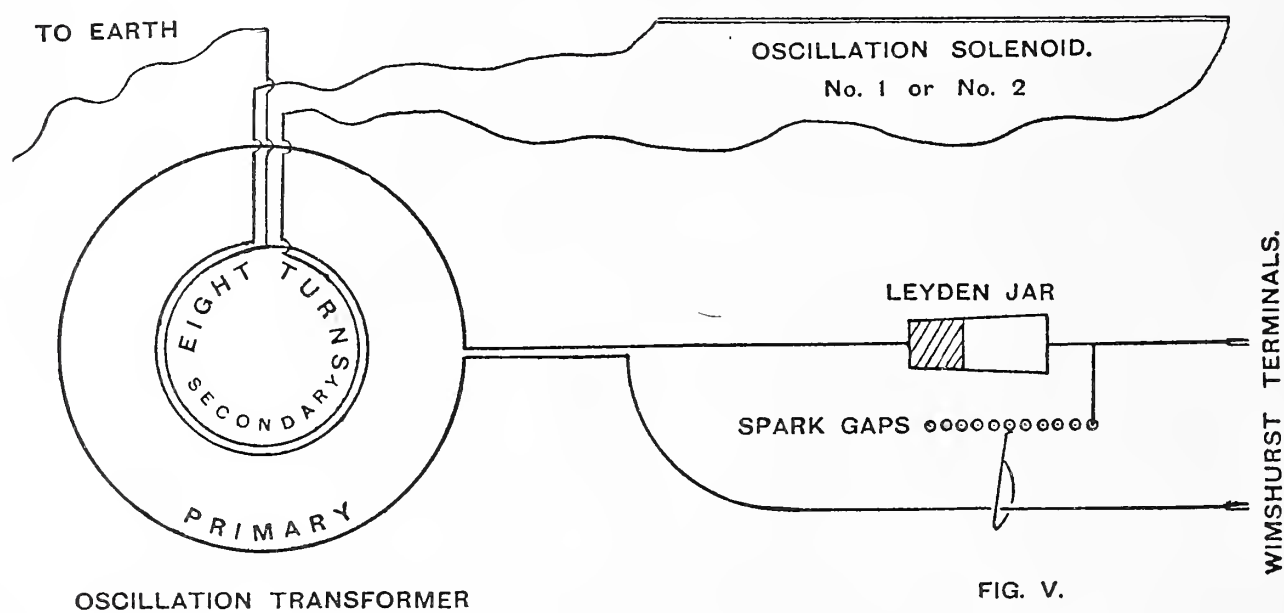
Experimental Methods.

Cyclic Fields.—The sequence of operations observed and the galvanometer readings taken were entirely similar to those which have been fully described for mechanical vibrations, with the following exception. It was

* This was found to be absolutely necessary. Without the earth connection, the spot of light moved over a large part of the scale, more especially just before the passage of the spark; with the earth connection, a sharp deflection was obtained on the passage of the spark, in the vast majority of cases from a steady zero position.

not considered necessary, with the high induction amplitude alone used, viz. $B=8000$, that demagnetisation by decreasing reversals should follow upon the completion of each series of readings. In no case, however, were further trains of oscillations of increasing intensity superposed at the same or any other point until about forty reversals of field at the cyclic amplitude had intervened.

The superposed trains of damped secondary electric oscillations were increased in intensity by progressively increasing either the number of 1 mm. spark gaps in series or the length of the single micrometer spark gap. In the former case, oscillation solenoids No. 1 and No. 2 were used, which subjected the bundles of fine iron wires to oscillations co-directional



with the magnetising cyclic field; in the latter case the steel wire $\cdot 092$ cms. diameter was subjected to transverse oscillations, its ends being directly connected with the secondary of the air transformer.

With co-directional oscillations it was found necessary to reverse the connections between the secondary of the oscillation transformer and either of the oscillation solenoids. Under these conditions the Wimshurst machine did not reverse its polarity during any series of continued experiments, although it might do so from day to day. The results obtained by reversing the connections between the oscillation primary and the Wimshurst could only be explained by the polarity of the latter reversing in an erratic manner. This method was therefore discarded from the first.

Intensity of Electric Oscillations.

In the first part of this paper a very definite meaning could be attached to the *intensity* of the mechanical vibrations. It was found that their intensity varied with a function of the weight (W) of the ball and the distance (D) through which it fell, the function being approximately $W \times D^{0.7}$. With electric oscillations the same definite signification does not apply. It is assumed that the *intensity* of the resultant trains of damped oscillations in the secondary of the air transformer is increased or decreased as the number of 1 mm. sparks in series or the length of the single spark is increased or decreased.

Diagrams.

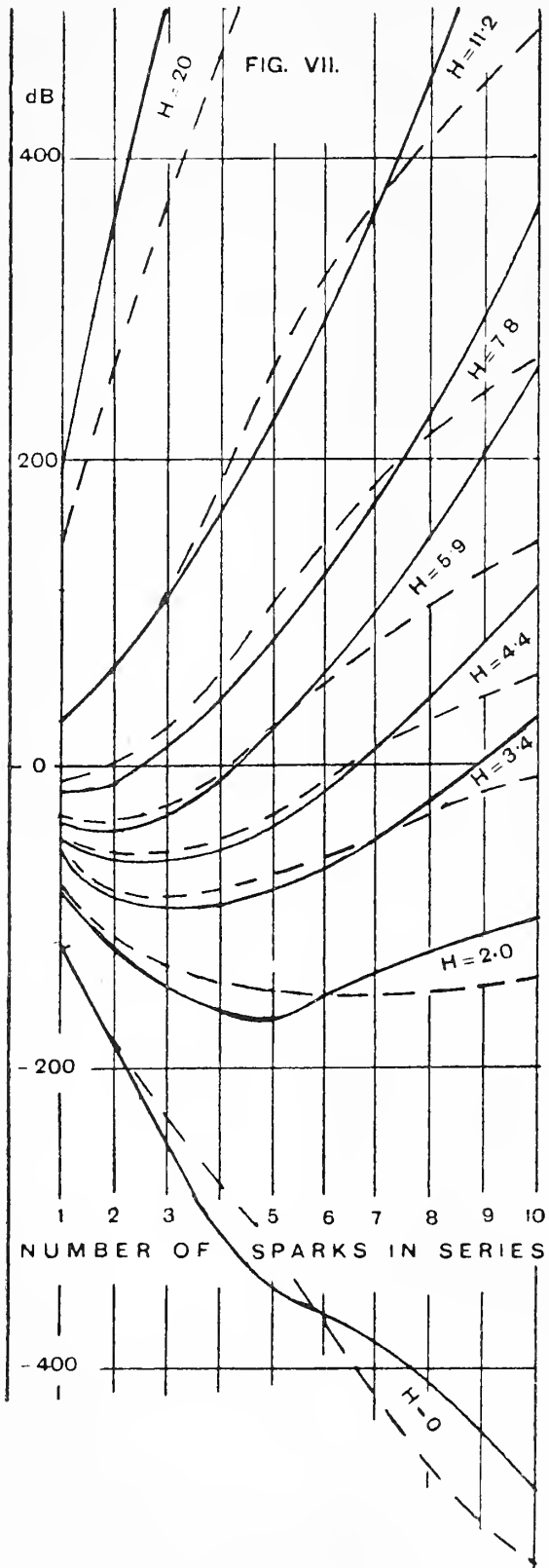
These are plotted from the experimental data in the way already described for mechanical vibrations, the intensity of the electric oscillations being represented by the number of 1 mm. sparks in series or the length of the single spark, for co-directional and transverse oscillations respectively. Field (H), induction (B), and induction change (dB) are likewise in C.G.S. units.

Cyclic Field.

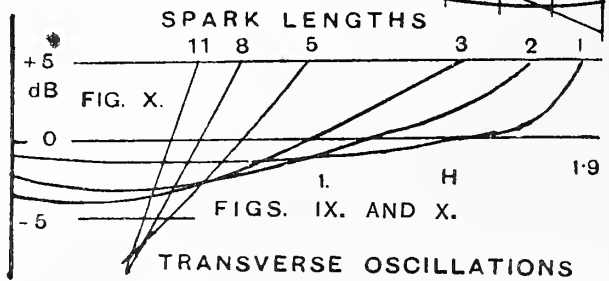
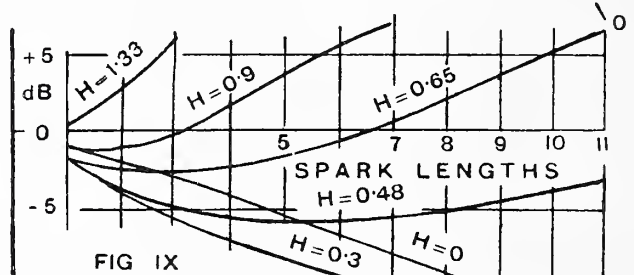
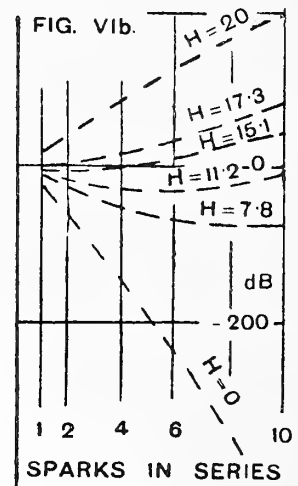
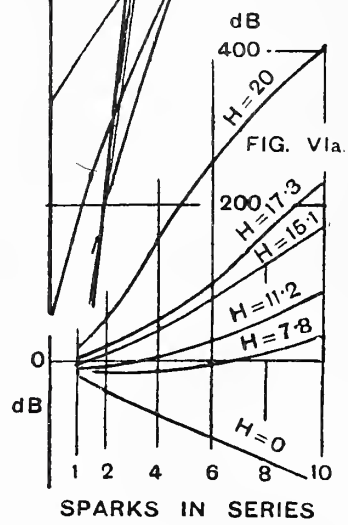
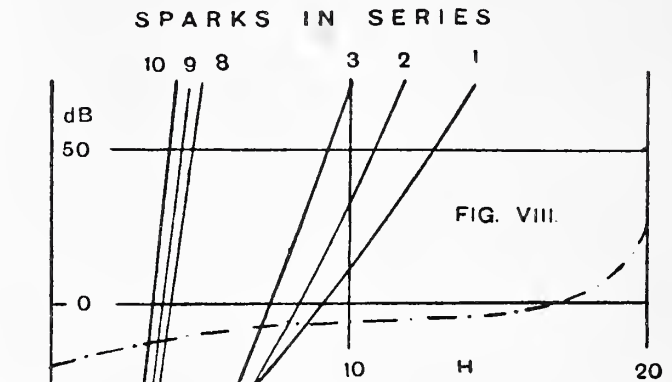
Experimental Data.—In fig. VI. a b and fig. VII., with the twenty iron wires, 0.028 cms. diameter, inserted in oscillation solenoids Nos. 1 and 2 respectively, a series of ten trains of damped electric oscillations, increasing in intensity and co-directional with the magnetising field, are superposed in succession upon a sufficient number of points of that part of the normal hysteresis loop decreasing from $H = 20$, $B = 8000$ (cyclic extreme), to $H = 0$, $B = 6900$. The abscissæ represent the number of 1 mm. sparks in series, increasing from one to ten, superposed in the case of figs. VI. a b at six values of field, viz. $H = 20$, 17.3, 15.1, 11.2, 7.8, and 0, and in the case of fig. VII. at eight values of field, viz. $H = 20$, 11.2, 7.8, 5.9, 4.4, 3.4, 2.0, and 0; the ordinates, the induction changes following thereon.

Observe that for each value of field there are pairs of curves, represented in the case of each pair by the continuous and dash line curves respectively. Tables III. and IV. show how one of the pairs of curves in figs. VI. a b (curves $H = 15.1$) and in fig. VII. (curves $H = 4.4$) respectively have been obtained from the experimental data. If, when the field is positive, the connections between the secondary of the air transformer and the oscillation solenoid are not crossed, it is found necessary to reverse or cross them to obtain approximately the same readings when the experiments are repeated with negative field. On the other hand, if the oscillation connections are crossed

DECREASING CYCLIC FIELD. ELECTRIC OSCILLATIONS.



FIGS. VIa., VIb., VII., AND VIIi.
CO-DIRECTIONAL OSCILLATIONS



FIGS. IX. AND X.
TRANSVERSE OSCILLATIONS

TABLE III.—DECREASING CYCLIC FIELD. CO-DIRECTIONAL OSCILLATIONS.
OSCILLATION SOLENOID No. 1.

No. of mm. sparks in series.	Oscillations superposed upon		Oscillations regarded as superposed upon $H = +15.1$.					
	$H = +15.1$ reduced from $H = +20$.	$H = -15.1$ reduced from $H = -20$.	Average readings.	Summation.	Summations $\times 1.54$ = Total change of B.			
1	Secondary oscillation connections uncrossed.	- 1	Secondary oscillation connections crossed.	0	- 0.5	- 0.5	- 1	Curve $H = 15.1$ of fig. VI.a.
2		1		0	0.5	0	0	
3		11		- 15	13	13	20	
4		13		- 17	15	28	43	
5		11		- 15	13	41	63	
6		13		- 13	13	54	83	
7		13		- 17	15	69	107	
8		14		- 12	13	82	127	
9		15		- 16	15.5	97.5	151	
10		13		- 13	13	110.5	171	
1	Secondary oscillation connections crossed.	- 2.5	Secondary oscillation connections uncrossed.	2.5	- 2.5	- 2.5	- 4	Curve $H = 15.1$ of fig. VI.b.
2		- 1.5		0.5	- 1	- 3.5	- 5.5	
3		- 0.5		- 2	1	- 2.5	- 4	
4		2		- 5	3.5	1	1.5	
5		5		- 3	4	5	8	
6		2		- 5	3.5	8.5	13	
7		4		- 7	5.5	14	21	
8		3		- 7	5	19	29	
9		4		- 5	4.5	23.5	36	
10		3		- 3	3	26.5	41	

TABLE IV.—DECREASING CYCLIC FIELD. CO-DIRECTIONAL OSCILLATIONS.
OSCILLATION SOLENOID No. 2.

No. of mm. sparks in series.	Oscillations superposed upon		Oscillations regarded as superposed upon $H = +4.4$.					
	$H = +4.4$ reduced from $H = +20$.	$H = -4.4$ reduced from $H = -20$.	Average readings.	Summation.	Summations $\times 1.54$ = Total change of B.			
1	Secondary oscillation connections uncrossed.	- 34	Secondary oscillation connections crossed.	34	- 34	- 34	- 52	Full line curve $H = 4.4$ of fig. VII.
2		- 7		8	- 7.5	- 41.5	- 64	
3		0		- 1	0.5	- 41	- 63	
4		4		- 4	4	- 37	- 57	
5		14		- 8	11	- 26	- 40	
6		18		- 18	18	- 8	- 12	
7		19		- 22	20.5	11.5	14	
8		23		- 21	22	33.5	52	
9		24		- 22	23	56.5	87	
10		27		- 25	26	82.5	127	
1	Secondary oscillation connections crossed.	- 32	Secondary oscillation connections uncrossed.	34	- 33	- 33	- 51	Dash line curve $H = 4.4$ of fig. VII.
2		- 6		8	- 7	- 40	- 62	
3		0		0	0	- 40	- 62	
4		6		- 8	7	- 33	- 51	
5		14		- 17	15.5	- 17.5	- 27	
6		12		- 12	12	- 5.5	- 8	
7		13		- 11	12	6.5	10	
8		12		- 11	11.5	18	28	
9		13		- 9	11	29	45	
10		13		- 6	9.5	38.5	59	

when the field is positive, the differences in the series of readings obtained are immediately apparent, and to repeat these different readings for negative field the connections must be uncrossed. Compare the upper and under halves of the third and fifth columns of both tables. On the right-hand side of the tables the summations of the average readings under these two distinct conditions, relative to the reversal of the secondary oscillation connections, multiplied by the factor 1.54 gives the induction (B) in C.G.S. units plotted in fig. VI.a b, $H=15.1$, and in fig. VII., $H=4.4$, as if all the observations had been made with positive field. These curves are typical of all the others. The continuous-line and dash-line curves represent for positive values of field uncrossed and crossed oscillation connections respectively. With oscillation solenoid No. 1, figs. VI.a b, although the curves of each pair do not cross, they could not have been represented on the same diagram without unnecessary confusion. With oscillation solenoid No. 2, on the other hand, they can readily be shown on the same diagram, fig. VII. In this case the crossings of the curves of each pair, with the exception of those for $H=20$ (the cyclic extreme), may be in the meantime noted.

The full-line curves of fig. VIII., obtained from the same experimental data as those of fig. VII. (oscillation solenoid No. 2, with 15 turns per cm. of length), show the induction changes plotted, not against oscillations of increasing intensity for various values of field (as in fig. VII.), but against field as abscissæ for various values of the oscillation intensity, *i.e.* against the number of 1 mm. sparks in series. Six curves only have been selected. Each of those might have been shown in pairs corresponding to crossed and uncrossed oscillation connections in reference to positive field. This, however, has not been done, the *average* induction changes under these two distinct conditions having been plotted. The number of 1 mm. sparks in series are 1, 2, 3, 8, 9, and 10, represented by the corresponding vertical ordinates of fig. VII., the four intermediate values 4, 5, 6, and 7 having been omitted.

On the other hand, the broken line curve of fig. VIII. is plotted from the same experimental data as that from which the curves of fig. V.a b were obtained (oscillation solenoid No. 1, with three effective turns per cm. of length). It shows the *average* induction changes produced on the superposition of the initial 1 mm. spark obtained from distinct series of experiments for the six values of H referred to above, represented by the corresponding vertical ordinates (1) of figs. VI.a and VI.b with crossed and uncrossed oscillation connections (with reference to positive field) respectively. These results are also tabulated in Table V.

Fig. VIII. thus shows the induction changes (ordinates) due to co-direc-

tional oscillations superposed on decreasing cyclic field (abscissæ) between the wide limits of oscillation intensity represented by 1 mm. spark with oscillation solenoid No. 1 (three effective turns per cm. of length), and 10 mm. sparks in series with oscillation solenoid No. 2 (fifteen turns per cm. of length).

The preliminary experiments with the single steel wire 0.092 cms. diameter already mentioned may now be referred to. The oscillations were passed directly through the steel wire (transverse oscillations) and also through oscillation solenoid No. 2 (co-directional oscillations). In the latter case the general results, in so far as carried out, were entirely similar to those already described for the bundles of iron wires 0.028 cms. diameter. The

TABLE V.—DECREASING CYCLIC FIELD. CO-DIRECTIONAL OSCILLATIONS.
1 MM. SPARK. OSCILLATION SOLENOID NO. 1.

Oscillations regarded as superposed upon +H.				
With secondary oscillation connections uncrossed.		Average.	With secondary oscillation connections crossed.	
H.	B →	B.	← B	H.
20	32	25	18	20
17.3	5	2	- 1	17.3
15.1	- 1	- 2.5	- 4	15.1
11.2	- 3	- 4.5	- 6	11.2
7.8	- 5	- 8	- 11	7.8
0	- 14	- 20	- 26	0
B ÷ 1.54 = average readings with + H uncrossed connections, and - H crossed connections.			B ÷ 1.54 = average readings with + H crossed connections, and - H uncrossed connections.	
See broken line curve of fig. VIII.				

magnetic qualities of the wires were also very different in the two cases. It may be mentioned that the position of the neutral points indicated strong oscillation intensities, while the small galvanometer readings obtained showed that the effects of the oscillations were largely confined to the surface of the wire.

TABLE VI.—DECREASING CYCLIC FIELD. TRANSVERSE OSCILLATIONS.

Length of single spark in mms.	Oscillations superposed upon		Oscillations regarded as superposed upon $H = +0.9$.		
	$H = +0.9$ reduced from $H = +1.9$.	$H = -0.9$ reduced from $H = -1.9$.	Average readings.	Summation.	Summations $\times 2.34$ = Total change of B.
1	-0.25	0.5	-0.375	-0.375	-0.9
2	0	0	0	-0.375	-0.9
3	0.5	-0.25	0.375	0	0
4	0.75	-0.75	0.75	0.75	1.7
5	0.75	-0.75	0.75	1.5	3.5
6	0.75	-0.75	0.75	2.25	5.3
7	0.75	-0.75	0.75	3.0	7.0
8	1.0	-1.0	1.0	4.0	9.3
9	1.0	-1.0	1.0	5.0	11.7
10	1.0	-1.25	1.125	6.125	14.3
11	1.25	-1.25	1.25	7.375	17.2

Curve $H = 0.9$ of fig. IX.

Length of single spark in mms.	Oscillations superposed upon		Oscillations regarded as superposed upon $H = +0.65$.		
	$H = +0.65$ reduced from $H = +1.9$.	$H = -0.65$ reduced from $H = -1.9$.	Average readings.	Summation.	Summations $+ 2.34$ = Total change of B.
1	-0.75	0.75	-0.75	-0.75	-1.7
2	-0.25	0.25	-0.25	-1.0	-2.3
3	0	0.25	-0.125	-1.125	-2.6
4	0.25	-0.25	0.25	-0.875	-2.0
5	0.5	0	0.25	-0.625	-1.4
6	0.5	-0.5	0.5	-0.125	-0.3
7	0.75	-0.25	0.5	0.375	0.9
8	0.5	-0.5	0.5	0.875	2.0
9	0.5	-0.75	0.625	1.5	3.5
10	0.75	-0.75	0.75	2.25	5.3
11	0.75	-0.75	0.75	3.0	7.0
12	0.75	-0.5	0.625	3.625	8.5

Curve $H = 0.65$ of fig. IX.

Figs. IX. and X. show the results obtained when the secondary terminals of the oscillation transformer were connected directly with the ends of the single steel wire (transverse oscillations). In these experiments the length of a single spark is increased from 1 mm. to 11 or 12 mms. by increments of 1 mm. Fig. IX. shows for the following values of decreasing positive cyclic field, viz. $H=1.33, 0.9, 0.65, 0.48, 0.3$ and 0, the results when the induction change due to superposed transverse oscillations increasing in intensity by the means above indicated are plotted against the length of the spark as abscissæ. Table VI. shows the experimental data from which two of the curves, viz. $H=0.9$ and 0.65 , have been plotted. Observe that the galvanometer readings are very small, and have been read to 0.25 of a mm. division of the scale (*i.e.* to $B=0.6$). Notwithstanding this, the readings obtained with positive and negative fields are nearly the same throughout (as a rule identical), so that with transverse oscillations it is unnecessary, for obvious reasons, to reverse the oscillation connections to obtain similar curves differing only in sign, as was found absolutely necessary with co-directional oscillations.

Fig. X. shows from the same experimental data the induction changes plotted against the decreasing cyclic field as abscissæ for various oscillation intensities corresponding to spark lengths of 1, 2, 3, 5, 8 and 11 mms. In these experiments the cyclic amplitude is $H=1.9$, $B=3170$ and the residual magnetisation $B=2000$ when $H=0$.

Figs. IX. and X. for transverse oscillations exhibit differences when zero field is approached, not observed either with co-directional oscillations or mechanical vibrations, but not such as to affect the general results. Compare these two figures with figs. VII. and VIII. for co-directional oscillations, and with figs. I. and II. for mechanical vibrations.

Results.—When damped trains of electric oscillations are superposed upon magnetisation, it is necessary to distinguish between co-directional and transverse oscillations. In the latter case the oscillation phase neither directly assists nor directly opposes the magnetisation change, and the same results are obtained if the secondary oscillation connections are reversed relative to either positive or negative magnetisation. In the former case the oscillation phase must either directly assist or directly oppose the magnetisation change. If it could be assumed that the initial amplitude of each damped train of secondary oscillations was invariably greater than that of succeeding amplitudes, it might be anticipated that the curves representing the induction changes due to oscillations superposed upon that arm of the hysteresis loop decreasing from the *positive* extreme would fall *above* or *below* each other according as the phase of the first amplitude

opposed or assisted respectively the magnetisation change. But no such assumption can be made. The oscillation constants of the primary and secondary of the air transformer vary with the number of 1 mm. sparks in series, with oscillation solenoids No. 1 and No. 2, and with the permeability of the iron wires. Hence there may be an alternate rising and falling resultant of the damped oscillations in the secondary circuit. The amplitude of the first secondary oscillation may therefore be less than a succeeding amplitude of opposite phase.

When, however, the number of 1 mm. sparks in series are sufficiently increased, the large amount of damping thus introduced into the primary circuit may become the dominating factor, independent to a large extent of any other. A comparison of fig. VI.a b (oscillation solenoid No. 1) and fig. VII. (oscillation solenoid No. 2) supports this assumption. For strong oscillations, the curves with uncrossed oscillation connections relative to the decreasing positive field are in both cases invariably higher than those obtained with crossed connections. It is therefore highly probable that the amplitude of the first oscillation is greater than that of those which follow, and that its phase opposes the magnetisation change. On the other hand, the relative position of these curves is reversed for weak oscillations when solenoid No. 2 is used (fig. VI.). This result is at least not unintelligible in view of a rising and falling resultant of the damped secondary oscillations under the varied resonance conditions under which the experiments were made.

In any case, it is obvious that the induction change in the iron wires is sensibly affected in a very definite way by the conditions under which the damped secondary oscillations are produced.

Notwithstanding these differences, the conclusions arrived at relative to the shift of the neutral points when the cyclic field is decreasing, and the consequent dependence, within the limits of the range of shift of the *direction* of the induction change upon intensity (see p. 11), are fully applicable to electric oscillations, whether co-directional or transverse. This is sufficiently evident from a cursory comparison of the diagrams in each case, without again entering into details. Figs. VI.a b and fig. VII. (co-directional oscillations) and fig. IX. (transverse oscillations), showing the induction changes due to superposed oscillations of increasing intensity for various values of decreasing cyclic field plotted against the number of 1 mm. sparks in series as abscissæ, are entirely similar in their main features to figs. I. and I.a showing the induction changes due to superposed mechanical vibrations of increasing intensity for various values of the decreasing cyclic field plotted against $W \times D^{0.7}$ as abscissæ. So also are fig. VIII. (co-directional oscillations), fig. X. (transverse oscillations), and fig. II. (mechanical vibrations) entirely

similar in their main features, in which the induction changes are plotted against decreasing values of field as abscissæ for various intensities represented as the case may be by the number of 1 mm. sparks in series for co-directional oscillations, the length of the single spark for transverse oscillations, and the function $W \times D^{0.7}$ for mechanical vibrations.

It may be noted, however, that the range of decreasing cyclic field over which the neutral points may shift with variation in the intensity of the oscillations (fig. VIII. co-directional oscillations, fig. X. transverse oscillations) is much greater than that obtained with mechanical vibrations, fig. II.

Cyclic Residual Magnetisation.—Garibaldi obtained increase, Maurain a very small decrease,* of cyclic residual magnetisation upon the superposition of electric oscillations. These isolated results are in harmony with those obtained with mechanical vibrations. Any elaborate experiments were therefore considered superfluous with electric oscillations.

III. RESULTS WITH CYCLIC FIELD AND CYCLIC RESIDUAL MAGNETISATION IN TERMS OF FIELD CHANGE.

The experimental results for cyclic field and cyclic residual magnetisation now fall to be compared, the one with the other. The comparison will be facilitated if the starting-point be made from one or other of the cyclic extremes in both cases, say from the positive extreme, as represented in all the diagrams. In the former case, when vibrations or oscillations are superposed at all points of the descending arm of the normal hysteresis loop, the induction change at the cyclic extreme is positive, as the cyclic extreme is just departed from also positive, afterwards negative or positive as the intensity of the vibrations or oscillations is weak or strong, beyond the lower limiting value of the neutral points and until zero field is reached negative, between zero field and the negative cyclic extreme also negative. In the latter case, where vibrations or oscillations are superposed at all, values of cyclic residual magnetisation reached by the cyclic field process from the positive amplitude, the induction change at the residual extreme is negative, as this position is just departed from also negative, afterwards positive or negative as the intensity of the vibrations or oscillations is weak or strong; beyond the lower limiting value of the neutral points (for the strongest intensity used) and until zero magnetisation is reached positive, and finally between zero magnetisation and the negative extreme of cyclic residual magnetisation also positive.

With cyclic field and cyclic residual magnetisation alike, the shift of the

* “*presque nulle*”; see p. 2.

neutral points, therefore, is from or towards the positive extremes according as the vibration or oscillation intensity is strong or weak. In both cases the neutral points may occur close to the positive extremes if the intensity is sufficiently reduced, towards the lower limiting value of the neutral points if the intensity is sufficiently increased.

On the other hand, the *direction* of the induction change due to superposed vibrations or oscillations is at all corresponding points *reversed* for cyclic residual magnetisation as compared with cyclic field. This reversal can neither be co-ordinated with the induction nor with the presence or absence of the field. It can, however, be directly co-ordinated with the field change immediately preceding the superposition of vibrations or oscillations, which is of opposite sign in the two cases.

The results in both cases may now be stated as follows:—When vibrations or oscillations are superposed at the cyclic field amplitudes or at the extreme values of cyclic residual magnetisation, the direction of the induction change due to superposed vibrations or oscillations corresponds with that of the immediately preceding field change. As these extreme positions are departed from, the direction of the induction change is first opposed to, afterwards corresponds with, that of the immediately preceding field change, and continues to do so until the opposite extremes of cyclic field amplitude or cyclic residual magnetisation are reached. The position of the neutral points depends upon the intensity of the vibrations or oscillations, and it is thrust from the extremes the greater their intensity, but reaches a lower limiting value before zero field in the one case or zero magnetisation in the other case is reached. On the other hand, the neutral points are thrust towards an upper limiting value the weaker the intensity of the vibrations or oscillations, which may occur close to but not at the cyclic field amplitude in the one case, or close to but not at the extreme value of cyclic residual magnetisation in the other case, provided always the vibrations or oscillations are sufficiently weak. Within the upper and lower limiting values of the neutral points, the directional of the induction change may either correspond with or be opposed to the immediately preceding field change according as the intensity of the vibrations or oscillations are weak or strong. In both cases, also, the direction of the induction change therefore will correspond with that of the immediately preceding field change throughout a wider and wider range the weaker the vibrational or oscillational intensity, but never quite throughout the whole range.

IV. DR ECCLES AND PROFESSOR MAURAIN'S EXPERIMENTAL RESULTS.

Dr Eccles plots the induction changes due to superposed oscillations produced by single minute sparks against cyclic field as abscissæ. The curves are discontinuous in the same way that the curves obtained by plotting dI/dH against cyclic field are discontinuous at the cyclic extremes. My previous paper showed that the curves representing the effect of vibrations or oscillations superposed at all points of a normal loop form a continuous loop. In this respect Professor Maurain's experiments and those of others agree with my own. It was also pointed out that the neutral points are thrust towards the cyclic extremes the less the intensity of the vibrations or oscillations. The present paper shows how very close the neutral points may approximate the cyclic extremes, but they can only vanish there when the effect of vibrations or oscillations superposed *at the cyclic extremes* is zero.

My own experiments, however, differ from those of Eccles to a less extent than those of Maurain, who varies the conditions of the former's experiments. Eccles superposes oscillations of the same intensity throughout upon hysteresis loops of various amplitudes. Maurain superposes oscillations of four different intensities upon the same loop throughout. One of the results is that "*plus les oscillations agissantes sont faibles, plus le maximum de leur action s'éloigne du maximum de la courbe des distances D et se rapproche du maximum de dI/dH .*" This is clearly shown in fig. 3 of the paper, as also that the maximum of each curve is thrust from the maximum of dI/dH in the direction of the vertical axis the stronger the intensity of the oscillations. Maurain continues:—" *En somme, la considération de la distance D à la courbe d'aimantation stable, qui permet déjà de prévoir le sens de l'action des oscillations, est importante également pour prévoir l'intensité de cette action, et cela d'autant plus que les oscillations sont plus intenses.*"

In fact, Maurain's curves (fig. 3) show (1) that dI/dH becomes a more and more important factor than the distance D in determining the maximum of the curves due to oscillations the weaker their intensity; and, conversely, (2) that the distance D becomes the more important factor the stronger their intensity. Thus it is that Eccles' results obtained with minute sparks approximate more nearly to Maurain's result (1) with "*très faibles*" oscillations than to result (2) when the intensities are taken stronger and stronger. They approximate still more closely to my own results with the weakest vibrations or oscillations used, because under these conditions, with the neutral points thrust in my case close

to the cyclic extremes, the *direction* of the induction change coincides with that of the immediately preceding field change throughout a very much wider range of cyclic field. Consequently a greater opportunity is afforded for at least a general correspondence between the magnitude of the "spark effect" and the slope of the hysteresis loop which Eccles sought to establish. Between the upper limiting value of the neutral point and the cyclic extreme this correspondence is impossible, because the direction of the "spark effect" according to my results is opposed to that of the field change.

The *direction* of the induction changes due to superposed oscillations is, however, according to Maurain, wholly independent of their intensity, and is determined by "*la considération de la distance D à la courbe d'aimantation stable.*" These differences therefore appear to be more fundamental than those between Eccles' results and my own. In the series of experiments above referred to, Maurain starts with oscillations sufficiently strong to completely annul the hysteresis. Curve 1 is thus obtained. Curve 2 results from oscillations rather less strong. Curve 3 is obtained with oscillations described as "*plus faibles,*" and curve 4 with oscillations "*très faibles.*"

On the other hand, my experiments begin with the weakest intensity, producing in the case of mechanical vibrations and electric oscillations an induction change of $B = 10$ and 25 respectively at the cyclic extremes. In the latter case the maximum induction change obtained where approximately the slope of the loop is steepest (and consequently not given in the tables) was $B = 100$. Maurain's maximum induction change under the same conditions appears to have been of the order of $B =$ several thousands for the weakest intensity of oscillations (curve iv. fig. 2, Maurain). My experiments end with the strongest vibrational or oscillational intensity used, which appears to have reduced the residual magnetisation to a rather less extent than the induction obtained with the intensity of oscillations described as "*plus faibles*" (curve iii. fig. 2, Maurain). Curve iv., obtained with the weakest oscillation with which Maurain experimented, is therefore the only curve which may probably fall within my range of intensities, either for mechanical vibrations or electric oscillations.

I found it impossible, however, to show with accuracy, or even to show at all, the gradual shift of the neutral points towards the cyclic extremes in a diagram which also exhibits, as in Maurain's case, the normal hysteresis loop. The angles which the curves make with the horizontal axis become much too small as the intensities of the oscillations are reduced. In fig. VIII. (present paper), for instance, the ordinate scale has been

expanded 100 times in comparison with that usually adopted for hysteresis loops.

Maurain, commenting on Eccles' results, says: "*Les courbe qui, dans le mémoire de M. Eccles représentent l'action des oscillations aux différents points d'une des branches d'un cycle d'aimantation, paraissent comporter une discontinuité dans les valeur de cette action autour du sommet d'un cycle; je n'ai pu me rendre compte de la cause de cette différence entre ses résultats et les miens.*"

Between the upper and lower limiting values of the neutral points, which, as has been shown in this paper, may cover a very wide range of decreasing cyclic field, Eccles obtains only induction decrease when oscillations of very weak intensity are superposed; Maurain, experimenting in all cases with oscillations of much greater intensity, only induction increase. In the former case the neutral points are thrust towards their upper limiting values, which may occur quite close to the cyclic extremes, and the curves are discontinuous, because readings do not appear to have been taken between these two positions. In the latter case the neutral points are thrust from the cyclic extremes towards their lower limiting value, where the curves crowd together as the intensity of the oscillations are more and more increased. The lower limiting value of the neutral point may correspond to "*la courbe d'aimantation stable.*"

In the view of the author, the molecular theory of magnetism lends no support to the possibility that the positions of the neutral points (where vibrations or oscillations superposed upon cyclic fields produce no induction change) are independent of their intensity, nor that within definite limits vibrations or oscillations may not, when superposed at the same point, produce induction change in opposite directions, according as their intensities are sufficiently weak or sufficiently strong.

V. MOLECULAR THEORY.

It now remains to be shown whether the whole of the experimental results obtained with mechanical vibrations (Section I.) and with electric oscillations (Section II.), and restated in terms of field change irrespective of the presence (cyclic field) or absence (cyclic residual magnetisation) of the field (Section III.), are not only in harmony with, but a necessary deduction from, the molecular theory of magnetisation.

It will be convenient that the word *vibrations* when used alone signifies in the remainder of this paper either mechanical vibrations or electric oscillations.

With field increasing towards a cyclic extreme, the molecular groups will tend to place themselves more and more in alignment with the direction of the field, so that when (say) the positive cyclic extreme is reached, the superposition of vibrations will assist the rotation of the molecules of very different degrees of stability, and the induction will be increased. This induction change will necessarily coincide in direction with that of the immediately preceding field change, and it will be independent of the intensity of the vibrations. When the field is decreased, the decrement may obviously be made so small that even the least stable molecular groups will remain unaffected, and the rotation of the molecules will, on the superposition of vibrations, remain unchanged in direction. But the induction change will now be against the immediately preceding field change, and this result also will be independent of the vibrational intensity. If, however, the field decrease had been a little greater than above indicated, by an amount sufficient to rotate negatively the more favourably placed molecules of the least stable groups, their rotation with the field change will also be assisted, and the induction change due to superposed vibrations will now be definitely reduced in amount. On a further field decrease the reduced positive induction change will more readily pass into negative induction change if the intensity of the vibrations is weak enough to leave uninfluenced the more stable molecular groups, the less stable molecules rotating negatively in response to the negative field change alone being assisted. The induction change will now be with the field change, and once this has been established, will continue to be so for all greater decrements of field, provided the vibrational intensity be not unduly increased. But the decrease of field remaining as before, if the vibrations had been of a somewhat greater intensity to influence also a few of the more stable molecular groups not yet sensibly effected by the comparatively small field decrease, and whose last rotation was in the direction of the field (*i.e.* of the field change before the positive extreme was departed from), their tendency to rotate positively would also be assisted. Obviously the intensity of the vibrations could be adjusted so that their superposition at this particular value of decreasing field would produce no induction change whatever, the negative rotations of the less stable balancing the positive rotations of the more stable molecular groups. Further, if the vibrations had been of yet greater intensity, groups of yet greater stability, whose last rotation was positive, would have been brought into action, the molecular balancing would be upset, and the induction change would again be against the field change, a reversion to what took place at a smaller field reduction with weaker vibrational intensity. To re-establish a balance for

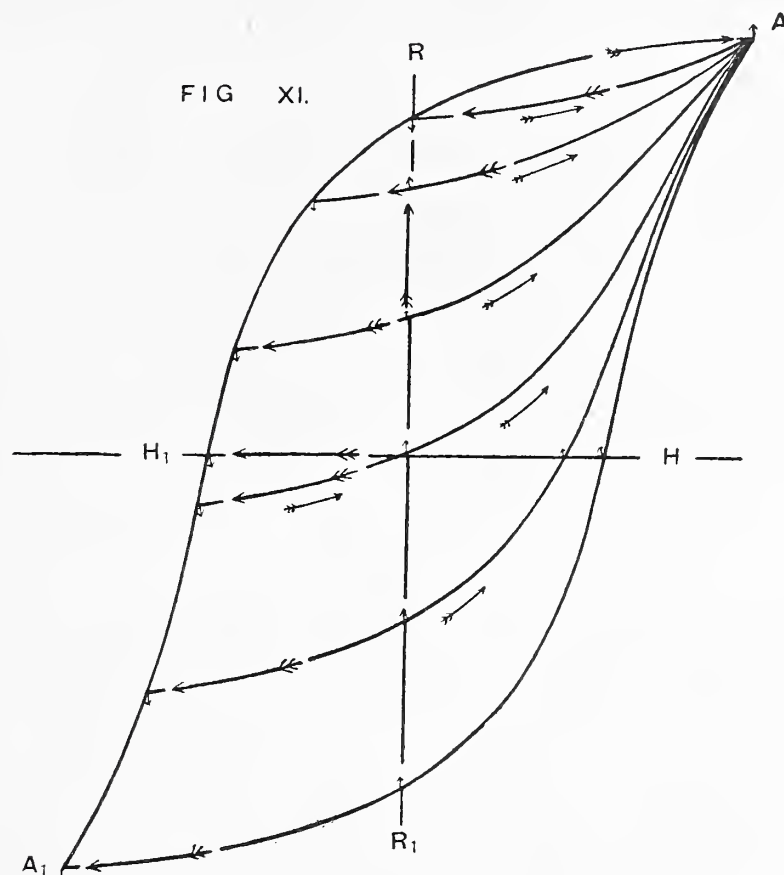
this greater vibrational intensity, a larger proportion of the molecules must tend to rotate negatively, and this could be accomplished by a further increase of the field decrement. The position where the opposing molecular rotations balance (induction change neutral point) is therefore thrust from the cyclic extreme the greater the vibrational intensity. If the intensity of the vibrations be further increased, the molecular balancing would be again upset by bringing into play yet more stable groups tending to rotate positively, and the induction change would for the third time oppose the field change.

Ultimately, however, this alternating process must cease when, under the influence of a sufficiently large decrement of field (or even increment of negative field, which this argument does not exclude), the increasing number of molecular groups tending to rotate negatively could just be balanced, and no more, by all the available remaining molecules still tending positively. Thereafter, negative rotation of the molecules would predominate, and the direction of the induction change would, on the superposition of vibrations of all intensities, however great, coincide with the direction of the field change. This process would continue until, with a sufficiently large increment of negative field, all the molecular groups would tend to place themselves more and more in alignment with the direction of the field as the negative cyclic extreme is approached. The induction change due to superposed vibrations would coincide with that of the immediately preceding field change, the condition of things from which this hypothesis started at the positive cyclic extreme.

The above deductions from the molecular theory of magnetisation are obviously in harmony with the experimental results obtained for cyclic field symmetrical about the origin. The shift of the neutral points within definite limits, and the correspondence between the direction of the induction change and that of the immediately preceding field change throughout a wider and wider range of cyclic field the weaker the intensity of the superposed vibrations or oscillations, are rendered immediately intelligible.

It will be readily perceived that these deductions are not confined to cyclic fields symmetrical about the origin. It need only be supposed that a preponderance of the molecules are rotating in the direction of the field change as the cyclic amplitudes (not even necessarily of opposite sign) are reached for the above deductions to become fully applicable. The area of a large normal hysteresis loop may, for instance, be mapped out by a series of loops unsymmetrical about the origin between the cyclic limits of, say, the positive extreme and increasing values of the negative field, as shown in

diagram XI. In the descending curve A R the long arrow represents the theoretical shift of the neutral points towards A the weaker the vibrational intensity. According to the above deductions, the shift of the neutral points is towards the cyclic extreme where the last reversal of field change took place, and consequently in the ascending curve R A it will be towards R the weaker the vibrational intensity. In the two branches of this little loop, therefore, between which the induction does not differ greatly, the shift of the neutral points will be in opposite directions, as indicated by the arrows,



and is therefore independent of the values either of the field or of the induction.

Similarly, in all the other ascending curves the shift of the neutral points will be towards the negative field extremes of their respective loops the weaker the intensity of the vibrations, until finally the last of the series A₁ R₁ A becomes symmetrical with A R A₁, the curve descending from the positive extreme.

Further, the ascending branches of these and all other intermediate loops cut either the vertical or horizontal axis, or both, at all points in a *positive* direction; while the descending branches, which coincide with one another, cut the vertical and horizontal axes only at their positive and negative extremes respectively. Hence the direction of the induction change due to vibrations superposed at all points of the axes included within the large

symmetrical loop will be *positive* throughout a wider range of either the less the vibrational intensity, but never quite throughout the whole range. The direction of the shift of the neutral points represented by the large arrows will be towards the negative extreme H_1 of the horizontal axis and towards the positive extreme R of the vertical axis the less the vibrational intensity, because a nearer approach is in both cases made to the negative cyclic extremes of the unsymmetrical loops at which the last reversals of field change took place. These deductions relative to *cyclic residual magnetisation* (vertical axis-zero field) and what may similarly be called *cyclic zero induction* (horizontal axis) are inseparably connected with each other. In the former case residual magnetisation is not necessarily decreased by vibrations. In neither case can the sign of the induction change due to superposed vibrations be determined apart from (1) the direction of the path by which the points on either axis have been reached, nor apart from (2) the vibrational intensity.

The experimental results with cyclic residual magnetisation show the wide range throughout which these conclusions deduced from the molecular theory of magnetisation are applicable. Vibrations may increase the residual magnetisation within the limits of $B=0$, and a point, $B=8700$, not far removed from the extreme possible limit under the conditions, viz. $B=9100$.

CONCLUSION.

The above deductions from the molecular theory of magnetisation depend primarily, therefore, upon the molecular condition in which the iron is left when the cyclic field process is arrested at any point, and not upon the actual values of the induction or field, either or both of which may be zero. If the cyclic field change be large enough a preponderance of the molecules must be rotating in the direction of the field change as the cyclic amplitudes, not necessarily symmetrical about the origin, are approached. On a reversal of field change the most favourably placed molecules of the least stable molecular groups must be the first to experience a reversal of their rotation, followed by those of more and more stable groups as the field change is taken greater and greater. Finally, the opposite cyclic extremes will be reached.

The intensity of vibrations superposed at any point will necessarily differentiate between molecular groups of various degrees of stability whose molecules have been arrested in their passage from positions of less to positions of greater stability under the conditions. The rotation of the molecules of a greater number of molecular groups of all degrees of stability

will be assisted in the direction of their last rotations, which may or may not coincide with that of the immediately preceding field change, the stronger the vibrational intensity. On the other hand, the rotation of the molecules of the least stable groups, to the exclusion of those of greater stability, will be assisted in the direction of their last rotations the weaker the vibrational intensity. But as these are the first to rotate in the direction of the cyclic field change after reversal, it follows that the induction change due to superposed vibrations will coincide in direction with that of the immediately preceding field change throughout a wider range of cyclic field the less their intensity. According to this hypothesis, neutral points occur where the rotations of the molecules in opposite directions balance, and these positions must be thrust towards the cyclic extreme at which the last reversal of field change took place the weaker the vibrational intensity.

The above deductions due to a complex magnetic æolotropy produced by reversals of field change are thus in harmony with the experimental results obtained for cyclic field and cyclic residual magnetisation with mechanical vibrations and electric oscillations, and summarised in terms of the immediately preceding field change in Section III. of this paper.

In cyclic fields symmetrical about the origin there are higher and lower limiting values of the decreasing cyclic field between which the neutral points may shift. These vanish in the origin or the cyclic extremes according as the amplitude is sufficiently decreased (zero amplitude at the origin) or sufficiently increased (saturation values of induction at the extremes).

The locus of the lower limiting value of the neutral points may approximate to the stable curve of Duhem (theoretical) or Maurain (experimental). The locus of the higher limiting value of the neutral points approximates to the cyclic extremes, but can only vanish there when the effect of the vibrations superposed *at the extremes* is zero. Between these loci, which may thus cover a very wide range of the decreasing cyclic field, neither unduly increased nor decreased, superposed vibrations may either increase or decrease the induction according as the vibrational intensity is sufficiently strong or sufficiently weak.

With cyclic residual magnetisation (zero field) there are also lower and higher limiting values between which the neutral points may shift. The direction of shift is towards the higher limiting values, which may occur close to but not at one or other of the extremes of residual magnetisation, the less the vibrational intensity. It will be towards the positive extreme if the slope of the unsymmetrical loops by which points on the vertical axis were reached be positive, and *vice versa*, because a nearer approach is

thereby made to those points of the loops where the last reversal of field change took place. It may therefore be repeated, that residual magnetisation is not necessarily decreased by vibrations. The sign of the induction change cannot be determined apart from the direction of the path by which points on the vertical axis (cyclic residual magnetisation) have been reached, nor apart from the vibrational intensity.

In this short summary of deductions from the molecular theory of magnetisation and of the experimental results the word "vibrations" refers both to mechanical vibrations and electric oscillations (co-directional and transverse). In the former case the *intensity* of the mechanical vibrations was found to be proportional to a definite function ($W \times D^{0.7}$) of the weight (W) of the steel ball and the distance (D) through which it fell. The accuracy of this experimental method is obvious, and no difficulty was experienced in obtaining as many as twenty distinct curves between the limits of $W \times D^{0.7} = 0.38$ and 10.4 (measured in gram-centimetres), from which the conclusions are drawn.

Although, in the latter case, a similar definite signification would be less readily obtained, the same general conclusions are equally applicable to both. With damped electric oscillations the experimental results are necessarily more complex. The induction changes due to superposed oscillations are the resultant of the irreversible* (vibrational) effects, and the reversible effects of the oscillatory current, just as the induction changes due to loading and unloading (or *vice versa*), are the resultant of the irreversible (vibrational) effects (due to a complex molecular æolotropy in cyclic field) and the reversible effects of the "ons" and "offs" (or *vice versa*) of the load.† The irreversible effects are subject to the same laws which govern the induction changes due to superposed mechanical vibrations in cyclic fields. In fact, these results may be taken to be general, not only applying to electric oscillations, weak alternating currents, and the

* This compounding of the irreversible with the reversible effects may very easily be lost sight of. The following appears to be worth further consideration. Professor Rutherford,¹ in his determination of the damping of oscillations by the magnetic method, assumes that if the initial phase of the oscillation train be such as to increase the magnetisation of the bundle of steel needles, it will produce no magnetic effect, because these have been magnetised to saturation. That is to say, as the initial phase increases to a maximum the magnetisation will increase, and as the initial phase falls to zero the decrease of magnetisation will not exceed the previous increment. But the irreversible (vibrational) effect opposes the increase and assists the decrease, hence the residual magnetisation may fall below its original value. The proof as stated would therefore appear to be incomplete.

† "The Effect of Load and Vibrations upon Magnetism in Nickel," pp. 38-56. Read in conjunction with this paper.

¹ "On a Magnetic Detector of Electric Waves, and some of its Applications," *Phil. Trans. Roy. Soc.*, London, A., 1897, vol. clxxxix., p. 1.

irreversible effects of loading and unloading, but to the irreversible effects of all other disturbances, whether produced by electric, mechanical,* or thermal means (temperature change). Wherever these are associated with a complex magnetic æolotropy such as may be impressed upon the magnetic material by cyclic field change, there the shift of the (vibrational) neutral points, and the consequent dependence of the *direction* of the irreversible induction change upon the intensity of the disturbance, independent of the values of the magnetisation or the field, either or both of which may be zero, will be in evidence.

As on former occasions, I desire to thank the Royal Society of London for placing at my disposal Government grants for furthering these researches.

* In a later communication than that already referred to (p. 2), Professor Maurain states that "la courbe normale dépend un peu de l'amplitude des cycles de torsion." See "Sur l'action de la torsion sur l'aimantation," *Journal de Physique*, May 1907. In other words, the vibrational neutral points shift with the intensity of the torsional disturbance.

(Issued separately November 16, 1908.)

II.—The Effect of Load and Vibrations upon Magnetism in Nickel. By James Russell.

(MS. received April 13, 1908. Read November 18, 1907, and March 2, 1908.¹)

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THE initial irreversible effects of applying or of removing stress resemble those produced by vibrations, although these processes “may be conducted in such a way that no actual vibration takes place.”* Ewing has investigated the initial effects of tension in iron,† Cree the initial effects of pressure in cobalt.‡ In iron and cobalt, tension and pressure respectively increase the induction before the Villari reversal in both metals is reached. Vibrations also in non-cyclic fields increase induction. Consequently in low fields the initial irreversible and final reversible effects of tension and pressure in iron and cobalt respectively augment each other. In nickel, on the other hand, the irreversible and reversible effects of tension must oppose each other; the latter effect lowering the induction at all stages of field increasing from zero.§ In Ewing and Cowan’s paper, however, no mention is made of any initial effects tending towards induction increase.

Nickel has therefore been selected as the subject of these experiments, and annealed nickel, because in this condition the effects of mechanical vibrations are large, and in comparison with quenched nickel entirely normal in character.|| The irreversible and reversible effects of load with and without permanently acting mechanical vibrations are investigated (1) in fields increasing from zero; (2) in cyclic fields; and (3) in zero field with what may be called cyclic residual magnetisation, which will be defined later. To differentiate these effects the loads used must be small.

* *Magnetic Induction in Iron*, Ewing, 3rd edition, p. 216.

† *Phil. Trans.*, 1885, p. 603.

‡ *Phil. Trans.*, 1890, p. 329.

§ Lord Kelvin, “Electro-Dynamic Qualities of Metals,” part vii., *Phil. Trans.*, 1879; Reprint of Papers, vol. ii., pp. 332–407. “Magnetic Qualities of Nickel,” Ewing and Cowan, *Phil. Trans.*, 1888, pp. 325–333.

|| “The Superposition of Mechanical Vibrations upon Magnetisation,” etc., *Trans. Roy. Soc. Edin.*, xlv. p. 491.

APPARATUS.

One end of the annealed nickel wire, 100 cms. long and .092 cm. diameter, formerly used, was soldered to the gong of the electric bell, while its other end was linked by means of a short length of thread (to prevent torsion) to the vertical arm of an L-shaped lever. Its horizontal arm carried a light scale pan, counterpoised so that when empty the wire was entirely free from load. To prevent vibrations in the ordinary sense when load was put "on" or "off," the scale pan was cushioned with india-rubber, and a soft woollen thread was, for the same purpose, wound round the nickel wire in a loose spiral. The whole apparatus, which has previously been more fully described, || was suspended from the roof by means of india-rubber tubing, to prevent external vibrations reaching the nickel wire. The wire was in a horizontal position, at right angles to the earth's magnetic field, and coincided with the axis of the magnetising solenoid 41 cms. long. An exploring coil and ballistic galvanometer (complete period eleven seconds) measured the magnetic intensity at approximately the central position of the wire.

SUPERPOSITION OF FIELD, LOAD, AND VIBRATIONS.

The above apparatus enabled the relative superposition of load, field, or cyclic residual magnetisation, as distinguished below, to be performed either with or without permanently acting vibrations. In the former case the nickel wire was kept in a state of continuous vibration by ringing the electric bell.

A *conditions*.—Loading, followed by repeated "offs" and "ons" of load, may be superposed at a sufficient number of points on the normal BH curve obtained at each point by field reversals increasing from zero (A1), or on the normal hysteresis loop (A2). This latter case may, when so stated, include the superposition of "offs" and "ons" of load on a hysteresis loop performed with permanent load. Loading and unloading may also be superposed, field being zero at all values of *cyclic residual magnetisation* between positive and negative maxima (A3)—the cyclic residual magnetisation being produced, as shown in fig. V., by *withdrawing* the cyclic field increasing negatively (positively) always after the positive (negative) cyclic extreme has been departed from. Loading and unloading may be performed in steps. In all these cases each set of observations performed at any given point is independent of those performed at any other point.

|| Reference previous page.

Either demagnetisation by decreasing reversals intervenes, or under conditions (A2) and (A3) and only where expressly stated many reversals of high amplitude alone may intervene.

B conditions.—After demagnetisation by decreasing reversals, the nickel wire being loaded, field reversals increasing from zero may be superposed (B1). When the loading is zero the normal BH curve is obtained.

In this paper the A and B conditions refer exclusively to the relative superposition of load and field. Mechanical vibrations are never superposed; they either are or are not permanently acting.

DIAGRAMS.

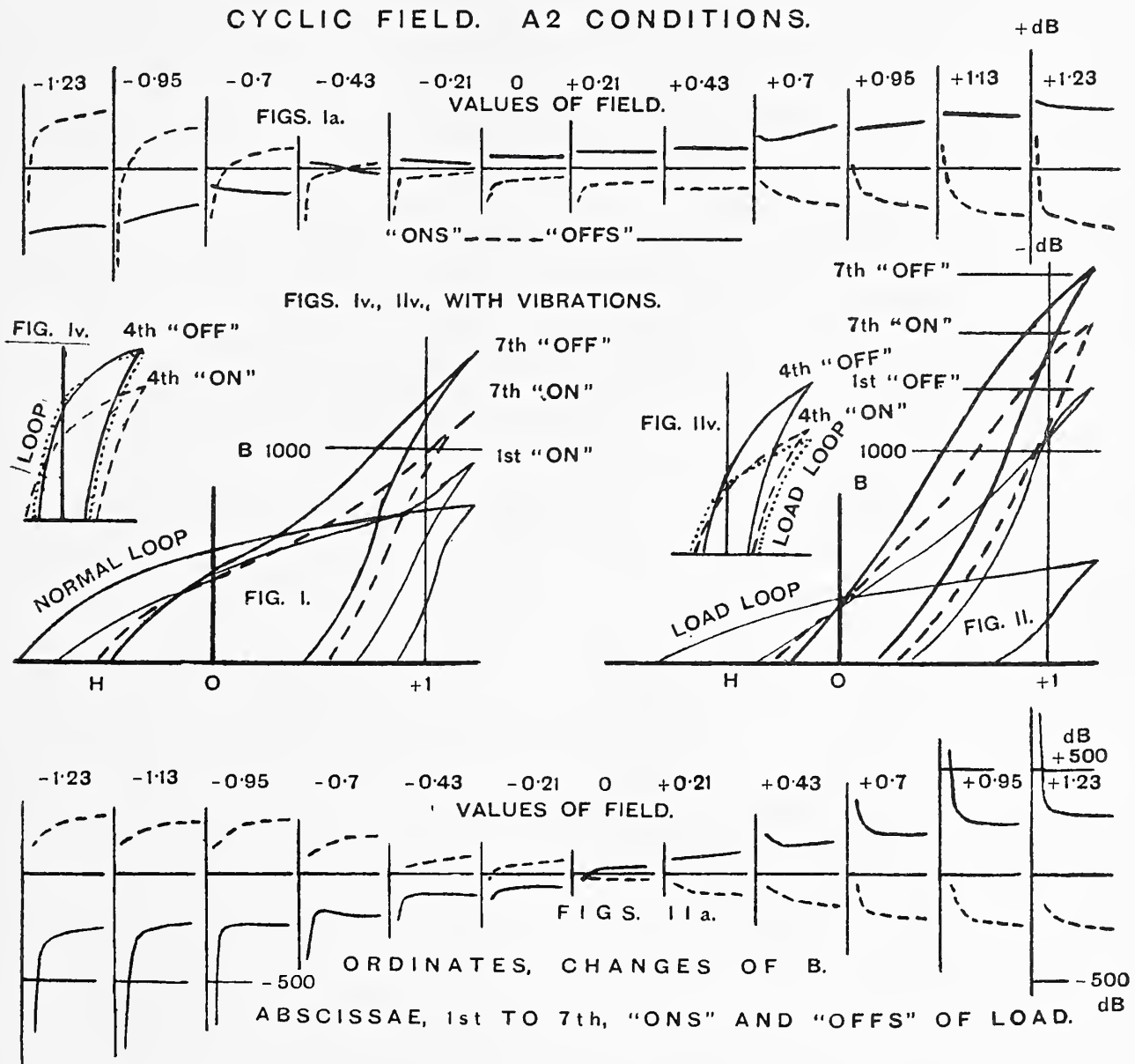
Field (H), induction (B), and induction change (dB) are in C.G.S. units in all the diagrams. The curves are either obtained from the average readings taken with both positive and negative fields increasing from zero; or in the case of cyclic fields, the results plotted for each arm of a hysteresis loop are obtained from the average results taken with both arms; the same method being adopted for cyclic residual magnetisation.

The occurrence of Villari reversals, found to exist with cyclic fields and with cyclic residual magnetisation, appears to offer the explanation that results not so easy of satisfactory repetition obtained in very low fields increasing from zero may be fully accounted for by the difficulty of reaching in every case an absolutely neutral position by the method of demagnetisation by decreasing reversals in nickel. For this reason the data and conclusions relating to cyclic fields and cyclic residual magnetisation will be taken first; fields increasing from a neutral magnetic zero last.

CYCLIC FIELDS.

Experimental Data.—Fig. I. shows the effects of the first “on” and of the seventh “on” and “off” of a load of 11 ozs. (0.5 kilos per square mm. of sectional area), superposed under the A2 conditions, at a sufficient number of points, upon the *normal hysteresis loop without load*. Fig. II. shows the effects of the first “off” and of the seventh “off” and “on” of the same load, superposed under the same conditions, upon the *hysteresis loop performed with load “on.”* In both cases the value of the field $H = 1.23$ at cyclic extremes is the same. In both cases nearly final results are reached with the seventh “on” and “off” of load. The series of small diagrams show the inductive changes (dB) which occur between the first

and seventh "ons" and "offs" (fig. I.a) and between the first and seventh "offs" and "ons" (fig. II.a) for a sufficient number of field values between $H = +1.23$ and -1.23 , *i.e.* as if the observations had been taken on the descending arms of the loops. The induction changes (dB) are plotted as ordinates; the number of times the load has been put "on" and "off" as abscissæ.



Figs. I. and I.a must thus be taken together, as also figs. II. and II.a. In fig. I.a when $H = +1.23$, for instance, the positive reading obtained with the first "on" (dash line curve), corresponds with the increase of induction at the cyclic extreme of fig. I. for the first "on." The ordinate difference in the latter fig. between the seventh "on" and the seventh "off" corresponds to the readings obtained at the extreme right of fig. I.a ($H = +1.23$), as measured by the distance between the horizontal axis and *either* the "on" curve (dash line) or the "off" curve (full line). Again, in

fig. I.a when $H = +0.43$ the "ons" and the "offs" decrease and increase the induction respectively by nearly equal amounts from the first, and either of these amounts corresponds to the difference between the "on" and "off" curves of fig. I. at this particular value of field, irrespective of whether the first or the seventh "on" curves be taken.

When these experiments are repeated with permanently acting vibrations, fig. I.v shows the curves representing the fourth "on" and "off" of load superposed on a loop without load, fig. II.v the curves representing the fourth "off" and "on" superposed upon a loop performed with permanent load. Under these conditions the first "on" differs very little from the fourth "on" (fig. I.v), and the first "off" very little from the fourth "off" (fig. II.v). These figures are the experimental results obtained for the same amplitude of cyclic field as in figs. I. and II., but the scale is three times closer. For approximately the same induction amplitude, and consequently lower field amplitude, the crossings of the curves remain essentially the same.

In the above diagrams the curves representing the (practically) final "ons" or "offs" of load are obtained by the summation of all the readings. In those now to be described, where "ons" and "offs" of load are superposed forty times in succession, this method becomes wholly inapplicable. If the cyclic amplitude be sufficiently increased, the induction at cyclic extremes may be assumed to be independent of the "ons" and "offs" of load superposed during the cyclic. That this assumption is sufficiently correct is supported by the fact that there is no "sagging" in either direction when the field is reversed several times on the conclusion of any series of readings. *The actual position of each final "off" of load may therefore readily be obtained by noting the induction change when a single step is taken to the opposite cyclic extreme to that from which the observations started.* This also fixes the position of the immediately preceding "on" of load. Fig. III. shows the curves, obtained in this manner, representing the fortieth "on" and the fortieth "off" of the 11 ozs. load (0.5 kilos per mm.²) superposed under the A conditions upon the normal hysteresis loop, the values of field and induction at cyclic extremes being respectively $H = 11.1$ and $B = 4200$. The series of smaller diagrams (fig. III.a) show the instantaneous induction changes which occur between the first and fortieth "ons" and "offs" of load for various values of field between $H = +11.1$ and $H = -11.1$.

In figs. IV. and IV.a the corresponding results are recorded when mechanical vibrations are permanently acting.

Figs. III. and IV. eliminate not only possible errors very liable to occur

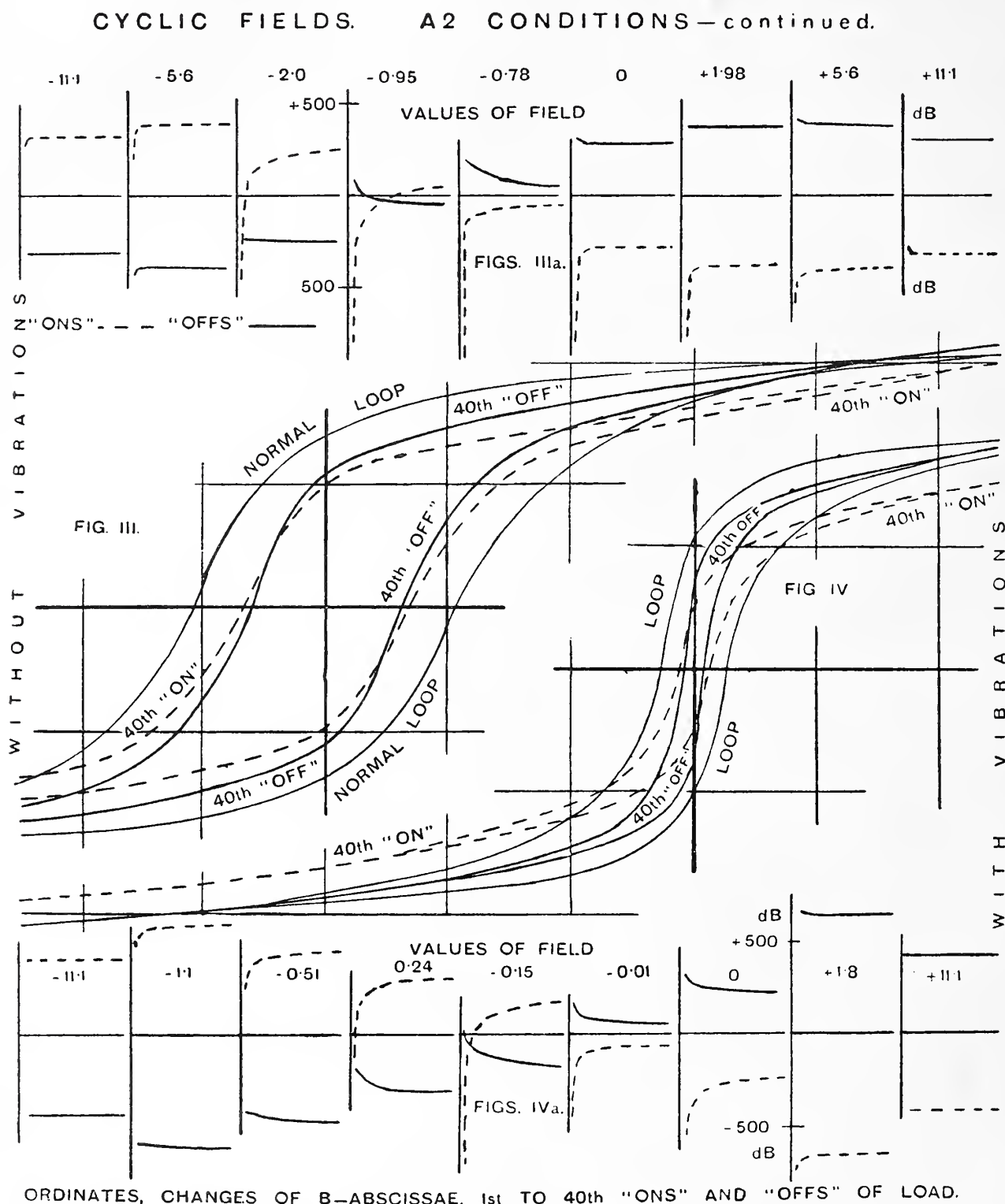
where the curves are obtained by the summation of a series of readings as in figs. I. and II., but also the effects of magnetic viscosity. The same results, however, are obtained in both cases.

In figs. III.a and IV.a the induction changes (dB) are plotted as ordinates; the *number* of times the load has been put "on" and "off" as abscissæ. The curves of these figs., and those of figs. I.a and II.a, are merely introduced to show the *general nature of the sequence of changes* which occur when "ons" and "offs" of load (in the case of II.a "offs" and "ons" of load) many times repeated are superposed at all stages of that arm of the loops descending from the positive to the negative cyclic extreme.

Results.—The effects of the first "on" and the first "off" of load superposed at all points of hysteresis loops not unduly increased, without (fig. I.) and with (fig. II.) load respectively, are essentially vibrational in character. In other words, induction at cyclic extremes is increased, residual magnetisation decreased, neutral points occur in the first and third quadrants, and, generally speaking, those differences to which magnetic hysteresis gives rise are reduced, irrespective of whether "on" is superposed upon a normal loop without load, or load "off" superposed upon a loop performed with load. The *differences* between the curves representing the first "on" (fig. I.) and the first "off" (fig. II.) might have been anticipated from the known effects of load in nickel. Load "on" (fig. I.) increases the induction to a less extent, decreases the residual magnetisation to a greater extent, than load "off" (fig. II.). Load "on" therefore thrusts the neutral point towards, load "off" from, the cyclic extreme. These neutral points are not Villari critical points due to load, but irreversible vibrational effects inseparable from the putting "on" or "off" of load.

Although it is the *first* "on" (figs. I.a, III.a) or the first "off" (fig. II.a) superposed upon hysteresis loops without and with load respectively which produces the most marked irreversible induction changes, repeated loading and unloading, as these figures show, also produce irreversible effects by no means negligible. These, however, gradually disappear, leaving finally equal and opposite induction changes due to "ons" and "offs" when the process is sufficiently often repeated. A study of these diagrams shows that these combined changes occur at all stages of the loop in orderly sequence, and that both effects consequently may readily be distinguished from each other. Repeated loading and unloading therefore accentuate all the vibrational effects already mentioned due to the superposition of the first "on" (fig. I.) or the first "off" (fig. II.). The induction at cyclic extremes is further increased, and the differences due to magnetic hysteresis

further reduced. The former effect tends to disappear as the cyclic amplitudes are sufficiently increased, and, as in fig. III, the final "off" curve alone exceeds the normal induction at the cyclic extreme, the final "on" not being increased sufficiently to rise above the normal cyclic



amplitude as in figs. I. and II. at lower amplitudes (see "Fields increasing from Zero," p. 48).

But there are points in the first and third quadrants, when the cyclic extremes are being departed from, where the irreversible effects vanish, and loading and unloading decrease and increase the induction by practi-

cally equal amounts from the first to the final "on" and "off" of load. These points occur with small induction amplitudes, as in fig. I., when $H = 0.43$ (fig. I.a); at higher amplitudes, as in fig. III., not quite at but near the cyclic extreme (fig. III.a). They are thrust towards or from the cyclic extremes as the amplitude is greater or less, and in this respect are subject to the same laws which govern the superposition of mechanical vibrations and other forms of disturbances superposed under the A conditions. || The positions where the initial irreversible effects of repeated loading and unloading vanish are therefore vibrational (but not load) neutral points. It may be observed that in figs. II. and II.a the vibrational neutral points, as above defined, are not quite so perfectly marked, and occur much closer to the vertical axis, just as is the case with the actual neutral point already referred to when "off" is first superposed upon the loop performed with permanent load, the reason in each case being the same.

Passing now from these irreversible changes essentially vibrational *in relation to the loops* upon which loading and unloading are superposed, the curves representing the final effects *in relation to each other* may now be considered. Load "on" invariably decreases, load "off" invariably increases, not only the induction at cyclic extremes, but the residual magnetisation. At points in the *second and fourth quadrants*, where induction and field oppose each other, loading and unloading produce (after the initial vibrational effects, which may be considerable, are over) no induction change whatever. Thereafter, until zero induction is reached, loads "on" and "off" *increase and decrease* induction respectively. There thus exist in cyclic fields well-marked Villari reversals in nickel.

Fig. III. may more especially be referred to. Fig. III.a shows how the Villari critical point is reached. It occurs at some definite field value between $H = 0.78$ and $H = 0.95$. The curves, after forty "ons" and "offs" of load, have practically reached asymptotic values. With a field of $H = -0.78$ (the induction being positive) it is evident that the Villari critical point will not be reached however often the loading and unloading be repeated. With a larger negative field of $H = -0.95$ the critical point has been passed, but it may be observed that the crossing of these curves does not, strictly speaking, constitute a Villari neutral point, as, in the immediate neighbourhood of this point, "ons" and "offs" are both producing negative induction change. It must therefore occur at some intermediate value of field for which, after the irreversible vibrational effects are over, repeated "ons" and "offs" would give rise to no induction change whatever. With lower cyclic amplitudes the Villari reversal occurs closer to the vertical

axis (figs. I. and I.a), and when the hysteresis loop is performed with load "on" apparently closer still (figs. II. and II.a).

Results with permanent vibrations.—The great differences which occur between the curves representing the effects of "ons" and "offs" of load superposed upon a normal loop (fig. I.), and "offs" and "ons" superposed upon a loop performed with permanent load (fig. II.), to a large extent disappear when the nickel wire is kept in a state of continuous vibration (irrespective of whether the experiments are repeated for the same field amplitude, or approximately for the same induction amplitude). This is immediately apparent on comparing figs. I. and II. without vibrations with figs. I.v and II.v with vibrations. The latter figures, taken in conjunction with the text, show that loading and unloading decrease and increase respectively both the induction at cyclic extremes and the residual magnetisation from the first. Nevertheless, the irreversible vibrational effects have not entirely been eliminated. The relation of the fourth "off" curve (fig. I.v) and of the fourth "on" curve (fig. II.v) to the normal loop and to the loop performed with load upon which they are respectively superposed is obviously vibrational. The induction at cyclic extremes is in both cases slightly increased, the residual magnetisation decreased. At higher induction amplitudes, however (figs. III. and IV.), the irreversible vibrational effect of repeated loading and unloading entirely vanishes at the cyclic extremes when mechanical vibrations are acting—see fig. IV.a, $H = 11.1$. This is not quite the case without vibrations, fig. III.a, $H = 11.1$. As the cyclic extremes are departed from the irreversible effects in both cases reassert themselves, reaching maximum values, which may be opposite in sign to the final effect of load, where the slope of the cyclic curves is great. At all stages the initial irreversible effects with vibrations (fig. IV.a) are much less than without vibrations (fig. III.a), and in the former case are confined to a much narrower range of low field.

While this is so in reference to the initial effects, the final reversible effects of load, *i.e.* the ordinate difference between the "on" and "off" curves, are much greater with (figs. IV. and IV.a) than without (figs. III. and III.a) permanently acting vibrations. Also the Villari reversals in the second and fourth quadrants remain practically unchanged although they occur at much lower values of the cyclic field.

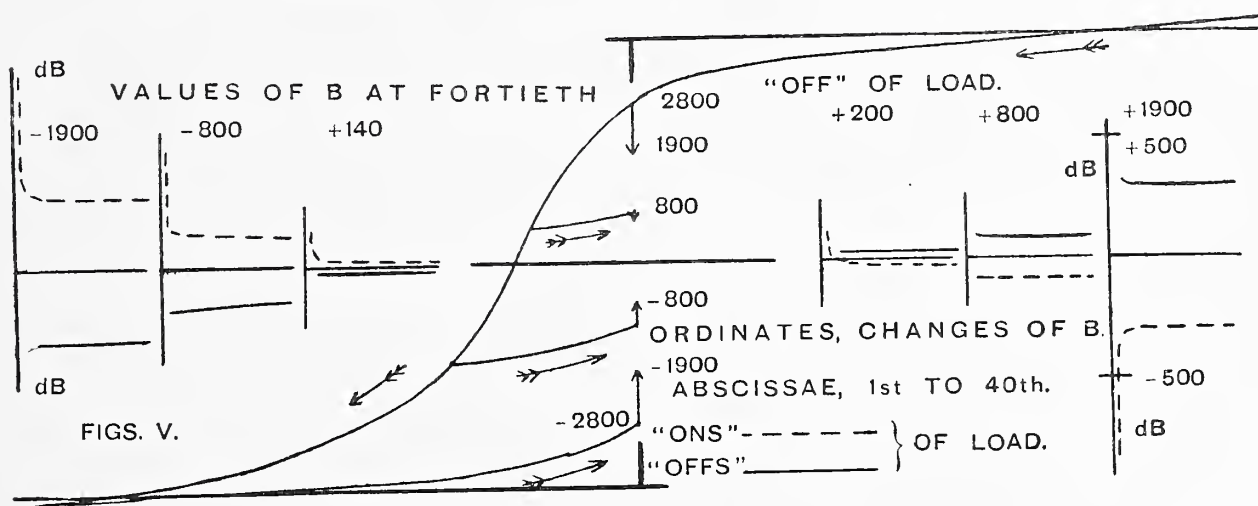
CYCLIC RESIDUAL MAGNETISATION.

Experimental Data.—Fig. V. shows the results obtained with what may be called cyclic residual magnetisation between the limits of $B = \pm 2800$

obtained by simply withdrawing the field $H = \pm 11.1$. Intermediate values of residual magnetisation are, however, obtained by withdrawing the negative (or positive) cyclic field at a sufficient number of points of the loop descending from the same positive (or negative) induction amplitude. The six small diagrams show the changes which take place between the first and fortieth "on" and "off" of the same load (11 ozs. = 0.5 kilos per mm.²), as if the cyclic field had always started from the positive extreme, as illustrated in the large central reference diagram. The value of the induction given with each diagram is obtained by measuring the induction change when, after the fortieth "off" of load, a single step is taken to the opposite cyclic extreme to that from which the cyclic process started. Between each set of observations, thirty or forty reversals of the highest field used intervene.

Results.—The initial irreversible and final reversible effects of loading

CYCLIC RESIDUAL MAGNETISATION. A3 CONDITIONS.



and unloading at the extreme values ± 2800 of residual magnetisation, shown in fig. V., must, for reasons of symmetry, be each the reflection of the other in the horizontal axis. In each case the irreversible vibrational effect lowers the induction from 2800 to 1900, while the reversible effects of loads "on" and "off" many times repeated decrease and increase respectively the residual magnetisation. But when these positions are departed from, the symmetry of the curves between the first and fortieth "ons" and "offs" of load disappears. This is well illustrated in the next pair of curves, where, quite accidentally, the residual magnetisation is $B = +800$ and $B = -800$ respectively. At the former positive value of induction there is practically no irreversible vibrational effect, loading and unloading taking their final values almost at once. At the latter negative value of induction, however, the initial lowering of the residual magnetisation which marks more especially the first "on" of load is still largely in

evidence, while the final reversible effect is distinctly greater than at the same positive value of magnetisation. If this want of symmetry be continued at lower values, a point must be reached where at some positive value of residual magnetisation the final reversible effect of loading and unloading will be nil. The next pair of curves shows that such a point is reached between the values $B = +200$ and $B = +140$, which constitutes a Villari critical point for cyclic residual magnetisation. Thereafter, until zero magnetisation is reached, loading and unloading produce *increase* and *decrease* of residual magnetisation respectively, as shown in fig. V., when $B = +140$.

These results therefore in zero field, with cyclic residual magnetisation, do not differ essentially from those obtained in cyclic fields. The absence of the *initial irreversible* effects at a comparatively large value of residual magnetisation constitutes a vibrational neutral point, and corresponds to that which occurs in the first (and third) quadrants with cyclic fields where the effects of loading and unloading assume their final values at once. The absence of the *final reversible* effects (after the irreversible effects are over) at a smaller value of residual magnetisation constitutes a Villari critical point, and corresponds to that which occurs in the second (and fourth) quadrants with cyclic fields where the final reversible effects of loading and unloading are nil. The asymptotic nature of the curves are equally well marked in both cases (figs. V. and III.a).

FIELDS INCREASING FROM ZERO.

Experimental Data.—Figs. VI. and VII. show the induction changes (ordinates) due to the first superposition of loads of 3, 11, and 20 ozs., corresponding to 0.14, 0.5, and 0.9 kilos per sq. mm. of sectional area, for values of induction (abscissæ) increasing from zero without and with permanently acting vibrations respectively. Figs. VIII. and IX. show the effects of cyclic load (abscissæ) without and with vibrations respectively. In both cases the same value of field supports an induction of $B = 650$ before the superposition of load. A total load of 14 ozs. (0.64 kilos per sq. mm.) is put “on” in steps of 3, 3, 2, 2, and 2 ozs., and “off” in inverse order. The first two complete load cycles, and a practically closed loop obtained after ten such cycles have been performed, are shown without (fig. VIII.) and with (fig. IX.) permanently acting vibrations.

When load is superposed under the A conditions many distinct curves may be obtained, showing the irreversible effects of the first, second, etc. “ons” and “offs.” The effects of the first “on” have been shown in figs.

VI. and VII., plotted against induction. When "ons" and "offs" are repeated a sufficient number of times, equal magnetic changes occur in either case. The full and faint dash line curves of fig. X. show respectively for all values of field the induction reached when load is put "off" and "on" for the fortieth time, without vibrations. These curves represent the final reversible results with a load of 11 ozs. (0.5 kilos per sq. mm.) under the A conditions. The other two pairs of curves in this diagram are obtained under the B conditions, the load used being the same. The full and faint continuous lines are respectively the normal BH curve without load and the BH curve with load. The full and faint dotted lines show the corresponding curves without and with load respectively, but with permanently acting vibrations.

The ordinate differences between each of the above three pairs of curves with and without load are in fig. XI. plotted against induction. The dash line curves show this *load effect* under the A conditions without mechanical vibration, while the continuous line curve represents within the limits of experimental error the load effects under the B conditions irrespective of whether vibrations are or are not permanently acting.

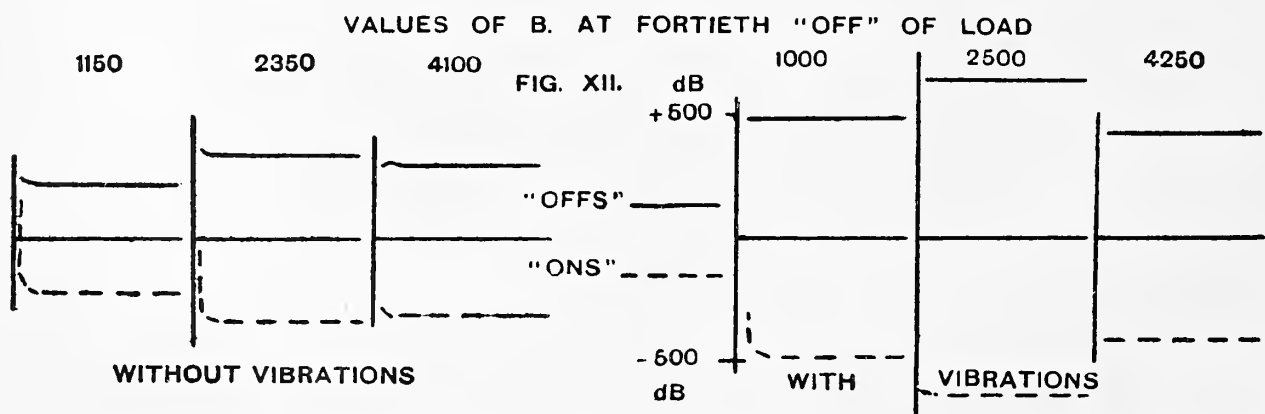
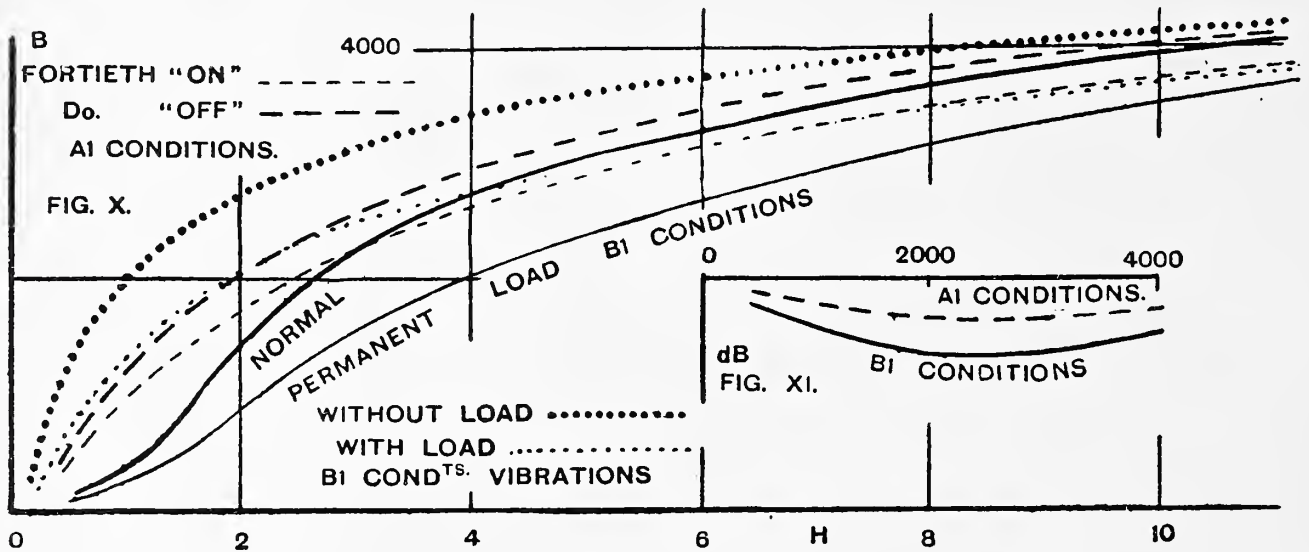
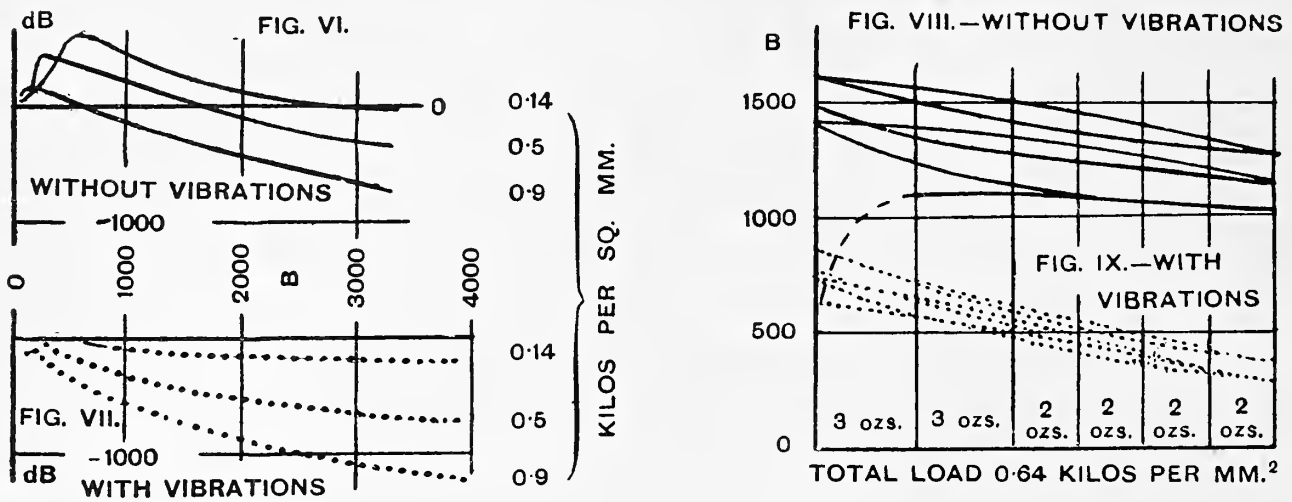
Fig. XII., without vibrations, shows for three different values of induction the changes due to loading and unloading which take place between the first and fortieth "ons" and "offs" of load, plotted from the experimental data from which the dash line curves of fig. X. were obtained. The remaining diagrams of the fig. show for three very similar values of induction the corresponding changes under the same A conditions, but with permanently acting vibrations. The induction values given with each figure are obtained after the fortieth "off" of load by taking a single step to a sufficiently high value of field, as already explained.

Results.—In fields increasing from zero, the effect of the *first* "on" of load superposed under the A conditions is to increase or decrease induction as field (induction) is low or high; within the experimental limits, the less the load the greater is the increase in low fields; the greater the load, the greater is the decrease in high fields. The neutral points are thrust towards the origin the greater the load, from the origin the less the load (fig. VI.). Although in this respect the *initial* irreversible effect of load in nickel is similar to the *final* reversible effect of load in iron, the resemblance is in appearance only. With permanently acting vibrations the initial effects nearly vanish, the first superposition of load decreasing the induction (almost) from the origin (fig. VII.). It thus becomes apparent that these initial effects of load are at all stages the result of the vibrational effect due to the first superposition of load which is always

positive, and the reversible effect which is always negative. At low values of field and induction the former predominates, at high values the latter.

Successive positive induction changes may be obtained with successive

FIELDS INCREASING FROM ZERO.



ORDINATES, CHANGES OF B.—ABSCISSAE, 1st TO 40th "ONS" AND "OFFS" OF LOAD.

increments of load, but this possibility is limited with fields increasing from zero to a very narrow range (fig. VI.). In general, the second and further increments of load will either produce no effect or lower the induction (fig. VIII.). Successive "ons" and "offs," however, augment the

irreversible vibrational effect observed upon the first superposition of load. The increase of induction due to this cause is not so completely eliminated with permanently acting vibrations as the increase due to the first superposition (figs. IX. and VII.). Permanently acting vibrations, while increasing the induction change due to load, decrease the area of the loops formed during a cyclic load process, and observed in these figures when loading and unloading is performed in steps. A comparison of fig. VIII. with, say, fig. 108 given on p. 218 of Ewing's *Magnetic Induction*, 3rd ed., shows that the irreversible vibrational effect of cyclic loading does not differ from the same effect in iron, except that it is the first increment of load that in general gives rise to induction increase, and not successive increments as in iron. This is due to the fact that in nickel the irreversible vibrational and the reversible effects of load oppose each other, while in iron they assist each other. Nevertheless, the vibrational effect of repeated loading and unloading tends towards induction increase in nickel as in iron.

This latter effect is especially marked in low fields (fig. X., dash line curves). The final "off" curve is for all values of field the higher of the two, but lower than that obtained under the B conditions with permanently acting vibrations of sufficient intensity. The final "on" curve (A conditions) and the permanent load curve (0.5 kilos per mm.²—B conditions) with vibrations fall above or below the normal BH curve as the fields are low or high respectively. The permanent load curve without vibrations (B conditions) is the lowest throughout of the six curves compared, that with permanently acting vibrations of sufficient intensity without load (B conditions) the highest; the load curves therefore fall below the corresponding curves without load. This almost universally recognised result is irrespective of whether mechanical vibrations are or are not acting. Permanently acting vibrations, however, eliminate in fields increasing from zero the irreversible first effects of loading and unloading to a greater extent that at some stages of cyclic field (fig. XII. and figs. III.a, IV.a respectively).

But the load effect, *i.e.* the ordinate differences between each pair of curves, is at all values of *induction* decidedly less under the A conditions without vibrations than under the B conditions irrespective of whether vibrations are or are not acting. It may be noted that the approximation of the load effect under the A conditions with vibrations to the load effect under the B conditions will depend upon their intensity.

DISCUSSION AND SUMMARY.

Loading and unloading may be superposed upon magnetisation (A conditions) with field increasing from zero (A1), with cyclic field (A2), or with zero field, cyclic residual magnetisation (A3) alone being dealt with. Reversals of field increasing from zero may also be superposed (B1 conditions), the nickel wire being either loaded or unloaded.

Under the A conditions, the magnetic changes following upon load changes are compounded of (*a*) the irreversible vibrational effects due to molecular instability and (*b*) the reversible effects of loading and unloading. In general, the initial effects (*a*) are most marked on the first superposition of load (or on the first superposition of load "off" if the nickel wire be already loaded). They decrease and finally vanish when "ons" and "offs" of load are sufficiently often repeated, leaving only the final reversible effects (*b*). Both effects are clearly distinguishable from each other and traceable in regular sequence at all stages of increasing field (A1), cyclic field (A2), or cyclic residual magnetisation (A3). Permanently acting vibrations lessen or eliminate the irreversible effects (*a*), but increase the reversible effects (*b*).

The curves representing the *first* "on" of various loads under conditions A1 are of interest. The irreversible vibrational effect (*a*) increases the induction, while the reversible effect (*b*) decreases the induction to a greater extent, the greater the load. At low inductions the former predominates. Hence the first "on" curve falls above or below the normal BH curve as the field is low or high, and the neutral points are thrust towards the origin the greater the load, from the origin the less the load, just as the Villari critical points are in iron. Under conditions A2 (cyclic field), the first "on" or the first "off" of load increases the induction at cyclic extremes (not unduly increased), decreases the residual magnetisation, and lessens those differences to which hysteresis has given rise, irrespective of whether "on" is superposed upon a normal loop without load, or load "off" is superposed upon a loop performed with load.

The irreversible changes (*a*) due to *repeated* loading and unloading under conditions A1 (increasing field) accentuate the induction increase due to the first superposition of load. This increase is especially well marked in low fields, but tends to vanish as field is sufficiently increased. Under conditions A2 (cyclic fields) also, the results mentioned immediately above as due to the first superposition of load "on" or "off" are further increased. The vibrational neutral points in the first and third quadrants are where the changes due to loading and unloading (induction decrease and increase

respectively) assume their final reversible values from the first—the (*a*) effect being zero. *These neutral points are thrust towards or from the cyclic extremes as the amplitudes are greater or less.* Under condition A3, a neutral point as above defined (where the changes due to loading and unloading assume their final reversible values from the first) occurs at a comparatively large value of cyclic residual magnetisation, which will be positive or negative as the cyclic field process has been conducted from positive or negative extremes respectively. At all other values the residual magnetisation will be either increased or decreased according as these values lie between or beyond the limits of the neutral point and the zero of cyclic residual magnetisation. Further, the irreversible *increase* (*a*) due to loading and unloading *between* these limits will be positive or negative according as the immediately preceding field change had been positive or negative. But it has been shown in the preceding paper,* read in conjunction with this communication, that a phenomenon of this kind can be co-ordinated with *the shift of the neutral points in a cyclic field from or towards the extreme at which the last reversal took place according as the vibrational intensity is strong or weak.*

Permanently acting vibrations eliminate the irreversible effects (*a*) to a greater extent under conditions A1 than under conditions A2 where the cyclic fields are steepest.

Under all the above conditions, therefore, the irreversible effects (*a*) are subject to the same laws which govern the superposition of mechanical vibrations. They depend in cases A2 (cyclic field) and A3 (cyclic residual magnetisation) upon the tendency of different molecular groups of unequal stability to rotate in opposed directions, consequent upon the nickel wire having been previously subjected to at least one reversal of field change, as fully discussed in the paper referred to above.* Under the A1 conditions, the magnetic æolotropy is less complex, no such opposing tendencies having been impressed upon different molecular groups, the field change having been in one direction only.

On the other hand, the reversible effects (*b*) are such that, under the A1 conditions (increasing field), loading and unloading sufficiently often repeated decrease and increase the induction by equal amounts respectively. This cyclic load process encloses an area, the curve for loading, as pointed out by Ewing, everywhere lying above that for unloading. This statement, however, only holds true after the reversible effects (*a*)

* "The Shift of the Neutral Points due to Variation of the Intensity of Mechanical Vibrations or Electric Oscillations superposed upon Cyclic Magnetisation in Iron," pp. 1-37. Read in conjunction with this paper.

producing induction increase are over. Permanently acting vibrations increase the range of magnetic change due to load change, but decrease the area of the loops formed during the cyclic load process. The load effect is thus negative if the induction be regarded as positive, and when plotted against the latter as abscissæ the resultant curve is concave upwards. Consequently Villari critical points have been looked for at very small and very large values of induction.* Hydweiller,† contrary to the almost universally accepted view, claimed to have found such a result at low values of field with various small loads, essentially similar to the Villari reversal in iron. Messrs Honda and Shimizu‡ have, however, recently repeated these experiments, and they find that no such reversal exists. They maintain that Hydweiller never reached a magnetic zero, the necessary initial starting-point from which such experiments must be conducted. In any case, the experiments described under A2 and A3 conditions show that the latter's method of demagnetisation by a combined process of field reduction and tapping might itself create a molecular condition favourable to the occurrence of a Villari reversal in nickel, even although a magnetic zero, as measured by magnetometer methods, had been reached.

The results of the comparisons instituted between the curves representing the final reversible effects of "ons" and "offs" of load and those obtained under the B1 conditions with and without permanently acting vibrations need not be repeated in detail. They may, however, be shortly summarised as follows. The curves obtained under the B1 conditions without vibrations exhibit magnetic but not load hysteresis; those under the A1 conditions exhibit load hysteresis, but eliminate to a large extent magnetic hysteresis; while the curves with permanently acting vibrations eliminate both magnetic and load hysteresis. The load curve always falls below the corresponding curve without load.

The reversible effects (*b*) under conditions A2 and A3 are more complex. At points in the second and third quadrants where induction and field oppose each other, loading and unloading produce, after the irreversible changes due to molecular instability are over, no induction change whatever. Thereafter, until zero induction is reached, loads "on" and "off" *increase* and *decrease* induction respectively. There is thus in cyclic fields well-marked Villari reversals in nickel. With cyclic residual magnetisation (A3 conditions, zero field) essentially similar phenomena are

* Kelvin (*loc. cit.*).

† *Wied. Ann.*, lii. 1894, p. 462. *Phil. Mag.*, (5) xxxv., 1893, p. 469.

‡ *Physikalische Zeitschrift*, 5, 1904, p. 254 and p. 631.

observed. The reversals occur at somewhat lower values than the vibrational neutral points where "ons" and "offs" of load assume their final values from the first, just as the Villari critical points occur at lower values of induction (second and fourth quadrants) than the vibrational neutral points (first and third quadrants) with cyclic fields. Permanently acting vibrations increase the irreversible effects (*b*) under the A2 as under the A1 conditions. The Villari critical points remain.

These Villari reversals, occurring as they do under both the A2 and A3 conditions, depend therefore upon the molecular condition impressed upon the nickel by the cyclic field process, independent of the presence of the field itself. To distinguish from the Villari critical points which occur in iron and cobalt in opposite senses, apart altogether from cyclic conditions, these phenomena may more appropriately be called the *cyclic Villari reversals* in nickel.

Obviously, the effects of loading and unloading under cyclic conditions must be more complicated in iron and cobalt than in nickel.

Just as the initial irreversible effects (*a*) can be co-ordinated under the A2 and A3 conditions with molecular groups of unequal stability tending to rotate in opposite directions, see p. 53, so may the reversible effects (*b*) of loading and unloading be co-ordinated with the molecular groupings which have, in continuation of this process, actually assumed opposite polarities, as Professor Hughes* found to be the case when zero magnetisation was reached or approximated to. In comparison with so essentially an isotropic process (isotropic co-directionally, not transversely, in reference to the field) as that of demagnetisation by decreasing reversals, the methods of reaching a magnetic zero by a reduced field reversal under the A2 conditions, or by subsequently withdrawing a slightly greater value of reduced field reversal under the A3 conditions, do not differ essentially from those adopted by Hughes, who reached a position of neutrality or approximate neutrality by tapping or by heating to redness an iron wire left residually magnetised. He showed that as the surface was dissolved away by means of nitric acid the successive longitudinal layers were of opposite polarities, and that it is the summation of these opposing layers which under such conditions constitute magnetic neutrality. Under both the A2 and A3 conditions the demagnetising process precludes perfect symmetry of the molecular groupings of opposite polarities. It therefore follows that the conditions constituting magnetic neutrality will not be such that the final effects (*b*) of loading and unloading can possibly be zero. Such a condition—where the summation of the positive and negative magnetic changes due to

* "Magnetic Neutrality and Polarity," *Proc. Roy. Soc.*, London, xxxvi., 1883-4, p. 405.

loading, and of the negative and positive changes due to unloading, on the molecular groups of opposite polarities respectively, is zero—must occur at some other position than that of magnetic neutrality. If this hypothesis be correct, then, after the irreversible vibrational effects (*a*) have secured permanent stability under the conditions of the various molecular groups, these positions, where this nice balancing of the opposing magnetic changes due to loading and unloading occurs, constitute *cyclic Villari critical points* in nickel.

I desire to express my indebtedness to the Royal Society of London for placing at my disposal Government grants to prosecute these researches.

(*Issued separately December 5, 1908.*)

III.—On the Recalescence Temperatures of Nickel. By T. A. Lindsay, M.A., B.Sc., Carnegie Scholar, Edinburgh University.
Communicated by Professor J. G. MACGREGOR.

(Read July 20, 1908. MS. received September 25, 1908.)

THE object of the experiments described in this paper was to study the cooling of nickel, and to endeavour to get recorded any recalescence which might occur. A specimen of pure nickel could not be obtained, but the nickel worked with, supplied by Messrs Johnston & Sons, contained less than 2 per cent. of impurities.

The critical ranges of nickel, or the temperature ranges over which changes in certain of the physical properties of nickel occur, have for long been a subject of investigation, and various workers have attempted to get evidence of evolution of heat or "recalescence" at these temperatures, but no successful attempt seems to be on record. Take,* reviewing the work done in this direction, says: "It may be stated that neither anomalous changes of length nor recalescence phenomena have been observed in the case of nickel."

In the most recent work on the recalescence of metals the "differential" method introduced by Roberts-Austen has been adopted, and Rosenhain,† in a discussion of the various methods of taking cooling curves, indicates that this "differential" method is the best. In my preliminary experiments I used the Roberts-Austen arrangement, but abandoned it for the simpler modification used by Carpenter and Keeling ‡ in their work on iron-carbon alloys, as I found the latter gave more readily interpretable results. In the method as used by Carpenter and Keeling, blocks of the metal under observation and of a metal which cools regularly without recalescence are placed close together in a furnace; thermo-electric junctions of the same kind of wires are inserted in the metal blocks, and so connected to a galvanometer that when heated they send current through the galvanometer in opposite directions, and in this way the galvanometer indicates any difference of temperature between the two metals; another thermo-couple inserted in the metal under observation is connected to another

* *Magnetische Untersuchungen, Diss. Marburg, 1904, p. 88.*

† *Physical Society of London, Jan. 24, 1908.*

‡ *Journal Iron and Steel Institute, 1904, vol. i. p. 224.*

galvanometer and thus determines the temperature of this metal. Carpenter and Keeling used platinum as the "blank" metal or metal without recalcination; but as platinum was not available, and I did not wish to work above 1000° , I used pure copper as being suitable for this purpose.

A diagram of my arrangement is shown in fig. 1. FF represents the lagging of the furnace, and AB its porcelain tube; C is a copper block, drilled with a hole for the insertion of a thermo-couple, and N the nickel block with two holes for thermo-couples; D is the galvanometer for the double thermo-couple or "differential" circuit; and T the galvanometer indicating temperature of N.

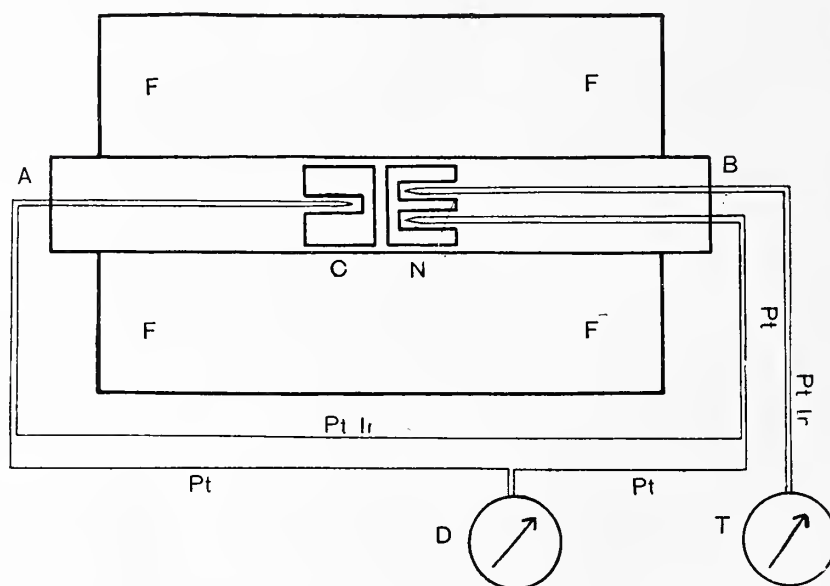


FIG. 1.

An electrical resistance furnace made by the Cambridge Scientific Company was used; it consists of a tube of English porcelain 62 cm. long, of diameter 4 cm., and lagged with asbestos to a diameter of 19 cm., this tube being heated electrically by a coil of platinum foil.

The metal blocks were 2 cm. long and $3\frac{1}{2}$ cm. in diameter, drilled with holes 1.3 cm. deep and .6 cm. in diameter; these were placed about .6 cm. apart in the centre of the furnace tube, the rest of the tube being loosely packed with asbestos, and under these conditions a cooling from 900° to 180° occupies about $3\frac{1}{4}$ hours, the heating current being completely shut off.

The thermo-junctions were of platinum and platinum-iridium wires welded together in the blowpipe; the wires of the "differential" circuit were .005 cm. in diameter, and those of the "temperature" circuit .02 cm. in diameter; in each case the wires were insulated by means of thin porcelain tubes, and were soldered to copper leads to their respective

galvanometers, these copper junctions being kept in test-tubes of alcohol surrounded by melting ice.

The galvanometers were both of the D'Arsonval type. The one used in the differential circuit was of 1098 ohms resistance, and of such sensitiveness that it gave a deflection of $\frac{1}{40}$ inch with 1 volt on 3950 megohms at 40 inches distance from the scale, and although not aperiodic it was not far from being so. The galvanometer of the temperature circuit was more sensitive than I required, and was therefore used with an external resistance, as its scale had to register from 100° to 1000° .

Preliminary experiments showed that the nickel did not oxidise in the furnace to an extent likely to affect any exhibition of recalescence, so I did not think it necessary to keep the metal in a non-oxidising gas.

The scale of the temperature galvanometer was calibrated by making use of the known boiling-points of water, naphthaline, and sulphur, and determining the position of these known temperatures on the scale, the error limit amounting to ± 1 mm., or less than $.5^\circ$. The boiling-point of naphthaline was taken as 218° , and that of sulphur as 444.5° . An open V. Meyer tube 48 cm. long, with a bulb 8 cm. in diameter, and jacketed outside with asbestos, was used to boil the naphthaline and sulphur in, and the thermo-junction, protected by a thin-walled Bohemian glass tube, was lowered to within 4 cm. of the boiling liquid. The vessel used for the water was a wide-necked flask 17 cm. in diameter, fitted with an escape tube and lined outside with asbestos; precautions to prevent bumping were observed, and the thermo-junction inserted as before. A curve drawn through these three determinations, as plotted on a temperature deflection diagram, then furnished the means of getting the temperature corresponding to any deflection on the galvanometer scale.

The general procedure was to heat up the furnace and contents slowly in about $1\frac{3}{4}$ hours to about 1000° , then shut off heating current and connect up galvanometers. During the cooling, readings of the two galvanometers were taken simultaneously at equal falls of temperature—about $4\frac{1}{2}^\circ$ —as registered by the temperature galvanometer, photographic or other self-recording appliances not being available.

In the cooling curves drawn to detect recalescences, the deflections of the differential-circuit galvanometer were plotted as abscissæ, and as ordinates the actual temperatures of the nickel determined by the deflections of the temperature galvanometer. These curves are rather complex in form, and as they might be expected to be simpler the more uniform the temperature of the furnace at the beginning of the cooling, I at first kept the furnace at its maximum temperature for about half an hour, in order to

get the thermal conditions as steady as possible; but this was not found to simplify the form of the curves, and subsequently, and in the case of the curves illustrated, the cooling was started immediately after the maximum temperature was reached.

The general form of the cooling curves may now be discussed. Under ideal conditions, with two similar pieces of the same metal, similarly situated in a furnace cooling uniformly, and initially at a uniform temperature throughout, the metals will always be at the same temperature during cooling, and the galvanometer of the differential circuit will be continuously at zero, so that the curve will coincide with the temperature axis. If we consider, however, the actual conditions under which I worked, we see that with two pieces of different metals, though neither of them metals which exhibit recalescence, in general there will be a difference of temperature between them throughout the cooling, as the furnace is not initially uniform in temperature throughout. Also this difference will be of different magnitude and sign according to the actual distribution of temperature in the furnace, while it will also be affected by the relative sizes and natures of the bodies, position in the furnace, and surroundings generally, so that the exact form of the curve cannot be foreseen. We can, however, say that (1) the initial abscissa will in general differ from zero, since there will in general be an initial difference of temperature between the metals; (2) the curve must finally arrive at the temperature axis, since the two metals must finally both come to room temperature; (3) if one of the metals cools more rapidly than the other, and is initially at the higher temperature, the curve may run down more or less steadily to the temperature axis, or it may cut the axis before the final temperature, and attain a maximum on the other side, before returning to the temperature axis again at the room temperature; (4) if the metal which cools more slowly is initially at the higher temperature, this difference of temperature may increase at first, though finally it must become zero at room temperature as before; (5) the maximum points indicated above will not occur at any definite temperature, but may correspond to a high or low temperature; in fact, the maximum difference of temperature between the metals will depend on their initial difference of temperature and on the conditions under which the cooling occurs, which will vary according to the distribution of temperature in the furnace.

The result of the occurrence of recalescence in one of the metals may be a rise in temperature of that metal, or at least a slower rate of fall than there would otherwise have been, and the effect in the curve will be a more or less sudden change of slope. If the recalescence be of small range, the curve will immediately return to approximately its former slope, and the result

will be a more or less pronounced but small hump on the curve; but if it be of large range, the result will be either a large hump or a long-drawn-out and flat hump, according to the rate of evolution of heat. Under actual conditions these humps may not be readily recognised, being concealed by the sinuosities of the curve due to errors of observation; but as these sinuosities differ from experiment to experiment, while the recalescence humps preserve their position, except in so far as it may be altered by repeated heatings, it is quite possible to recognise a recalescence hump even

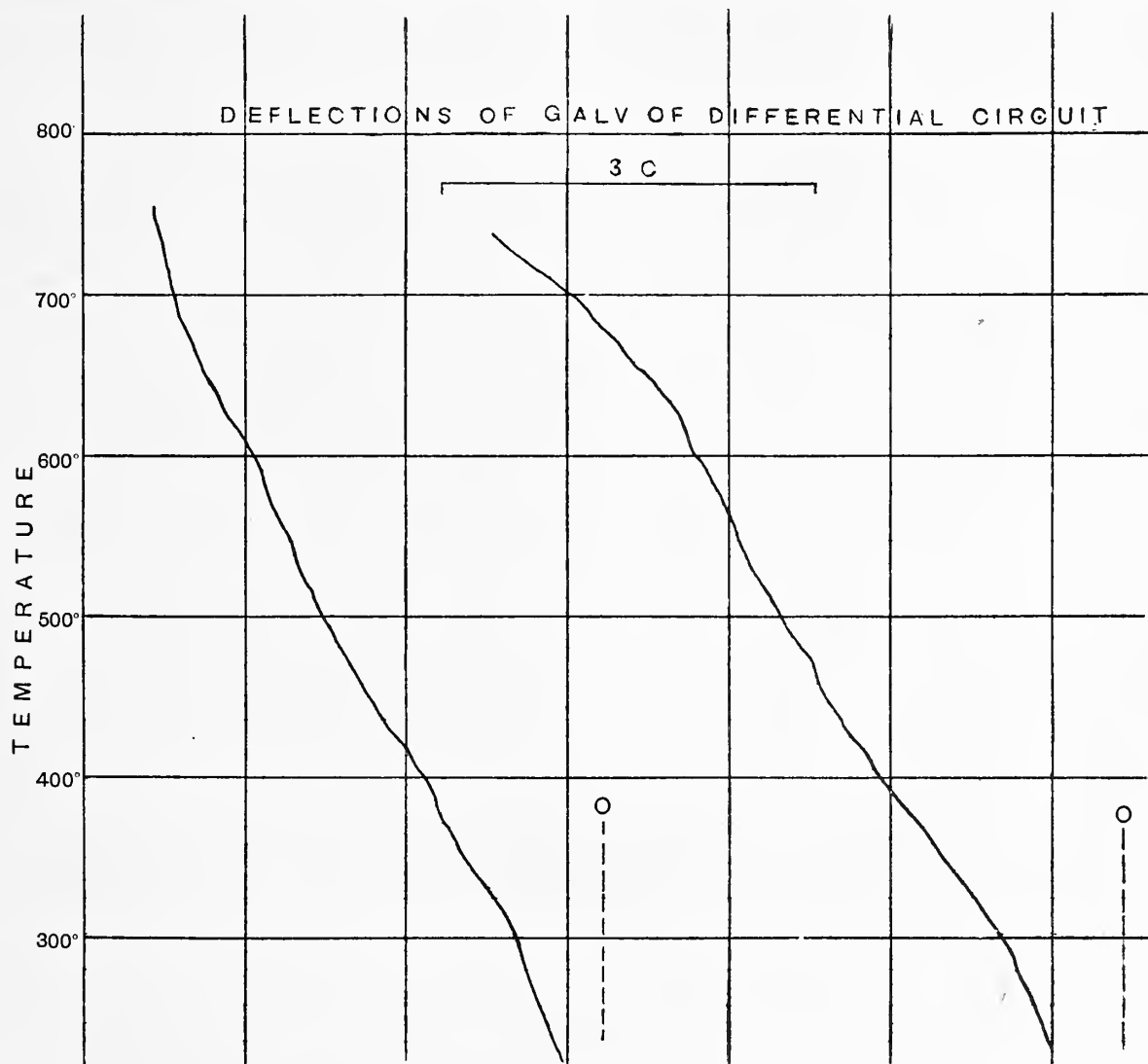


FIG. 2.

of comparatively small size, by comparing with one another a series of cooling curves.

Fig. 2 shows typical specimens of curves with no recalescence humps; they are taken with two pieces of copper of different size, and are approximately straight lines, except for the slight sinuosities, which give an idea of the limits of size within which humps could not be interpreted as recalescences. In these cases the galvanometer indicated a difference of temperature between the two pieces of metal initially, and the curve is constantly running down towards the temperature axis so far as it is recorded.

Fig. 3A shows curves obtained with nickel and copper in successive heatings of a new specimen of nickel. Here the galvanometer indicated that the metals had started at about the same temperature, but the copper cooling more quickly than the nickel, probably on account of its smaller specific heat, we get a difference of temperature established, which grows to a maximum just above A (most of the curves just begin to be recorded here, but Nos. 1 and 3 show part of this maximum), and thereafter the curves tend to get back to the temperature axis, except where they are interfered with by recalescences, the general trend of the curves being from right to left. These curves show two small humps, R_1 and R_2 , occurring at the same temperatures, about 660° and 525° respectively, in all cases; also a large hump occurring within the same temperature range in all cases, viz. 440° – 285° , being very evident as a change of slope at R, the original slope being regained at B. The large recalescence consists of two distinct parts, as at R^1 the slope again abruptly changes till the curve is nearly horizontal, indicating that heat is being given out at the greatest rate there; the first part, from R to R^1 , extends from 440° – 370° , and the latter part, R^1 to B, from 370° – 285° . The general slope of the curves is seen above R_1 and again from B to C, while between R_1 and R_2 and between R_2 and R the curve has not regained this slope, showing a small and gradual evolution of heat here also.

Fig. 3B shows the next two coolings taken of the same nickel and copper; here the galvanometer indicated that the copper had started at a higher temperature than the nickel, and the general trend of the curve is therefore from left to right towards the temperature axis, which was crossed during the large recalescence, and when this is finished at B the curve turns back again towards the temperature axis. These curves again show the recalescences in same places, R_1 , R_2 , and R to R^1 and B, though they have a less marked effect in changing the slope of the curve, owing to its general trend being from left to right.

It may be noted here that the recalescence temperatures mentioned above cannot claim to be closer than to 5° , as I was only able to take readings of galvanometer deflection at intervals of $4\frac{1}{2}^\circ$ or 5° .

A large number of coolings were taken of the specimens, partly to gain experience in interpreting and partly to test the uniform occurrence of the recalescences, and the curves shown are a few typical specimens.

Fig. 4 shows the effect of repeated heatings on the nickel; these curves were taken after the specimen had been subjected to from fifty to sixty coolings. The effect is apparently that the temperatures at which the recalescences occur are unaltered, but R_1 at 660° is disappearing—it is just

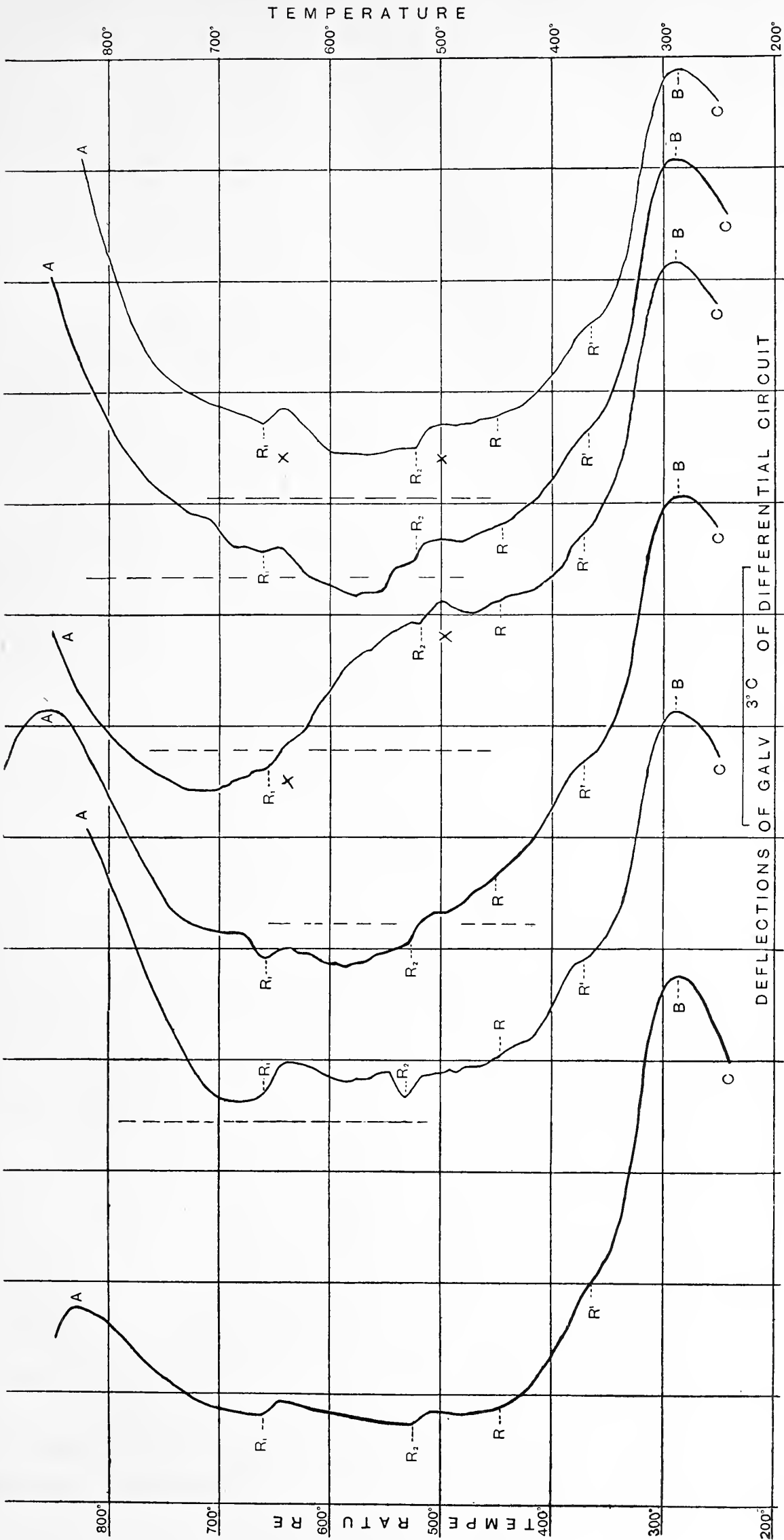


FIG. 3A.

evident as a slight bulge in the curve—and that at 525° is getting smaller in size also, while the gradual evolution of heat from R_1 to R_2 seems also to have disappeared, though this is not quite certain. These results are corroborated in fig. 5, these curves being also taken at about the same time. In the case of fig. 4 curves, the metals had started about the same temperature and attained maximum difference of temperature just above A, and thereafter tended towards the temperature axis just as the curves in fig. 3A did.

Rosenhain* has found that recalescence occurs in silica porcelain at

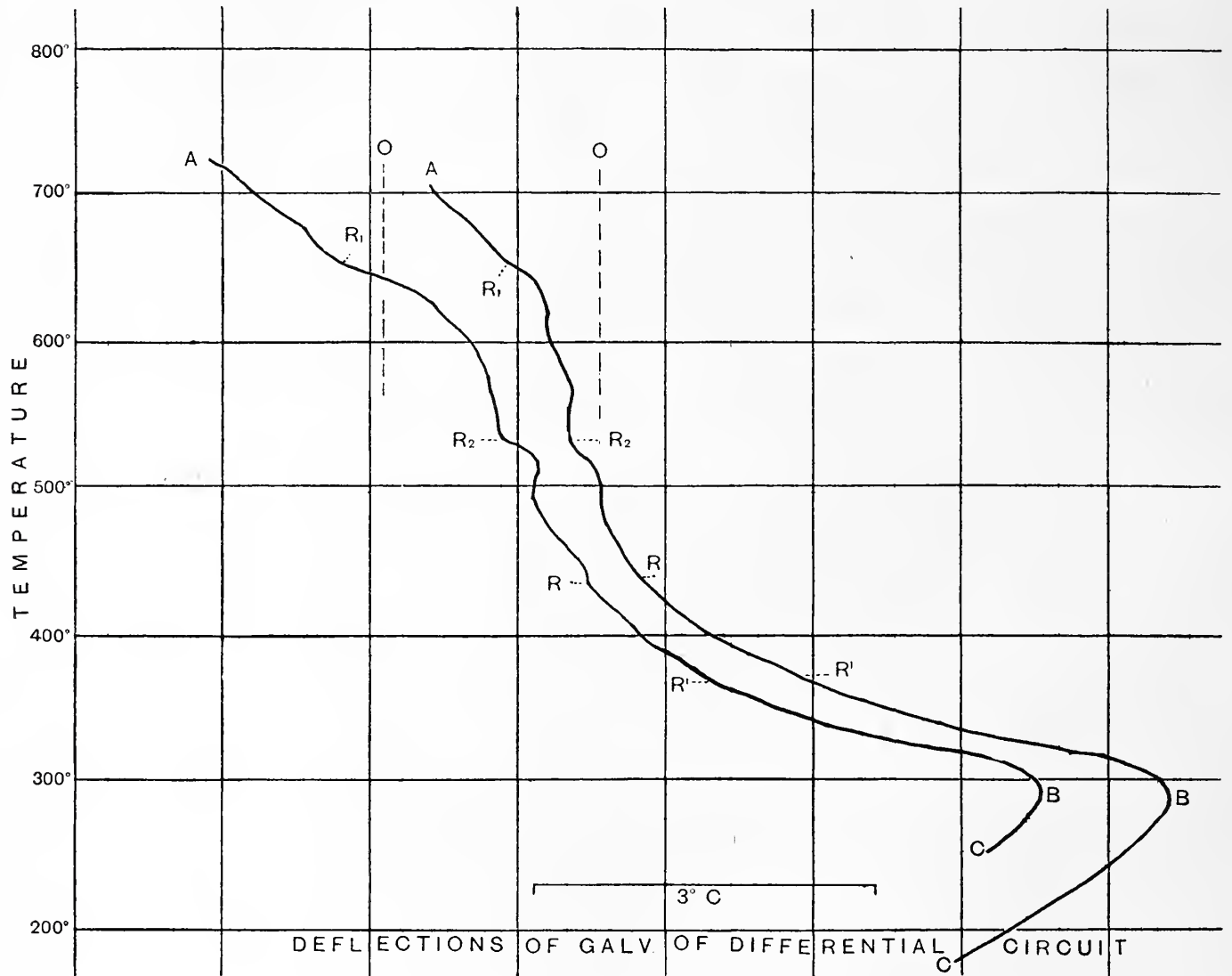


FIG. 3B.

about 550° , and as this is present in some furnaces, it gives rise to misleading results. No such indication of recalescence is given in my curves of copper and copper in fig. 2; but to find whether or not my results were due to recalescence of the materials of the furnace used, I took coolings in a fireclay gas muffle, packing the specimens in loosely with asbestos. The curves got are shown in fig. 5, and are in agreement with the other curves in their recalescence indications; it will be noticed that curves 1 and 2 have the humps smaller, but this is due to the galvanometer being shunted

* Physical Society of London, January 24, 1908.

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Part II.]

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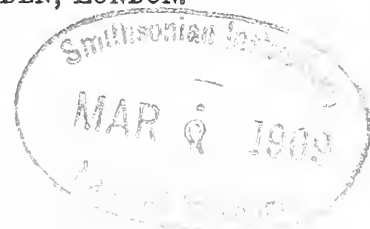
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[Continued on page iii of Cover.]

in these curves. The explanation of the form of these curves is the same as in the case of fig. 3B.

The insulation of the thermo-couple wires was also changed to asbestos, but this also had no effect on the exhibition of recalescence.

The results recorded by different workers on the transition temperatures of nickel differ somewhat, probably due to the fact that the specimens

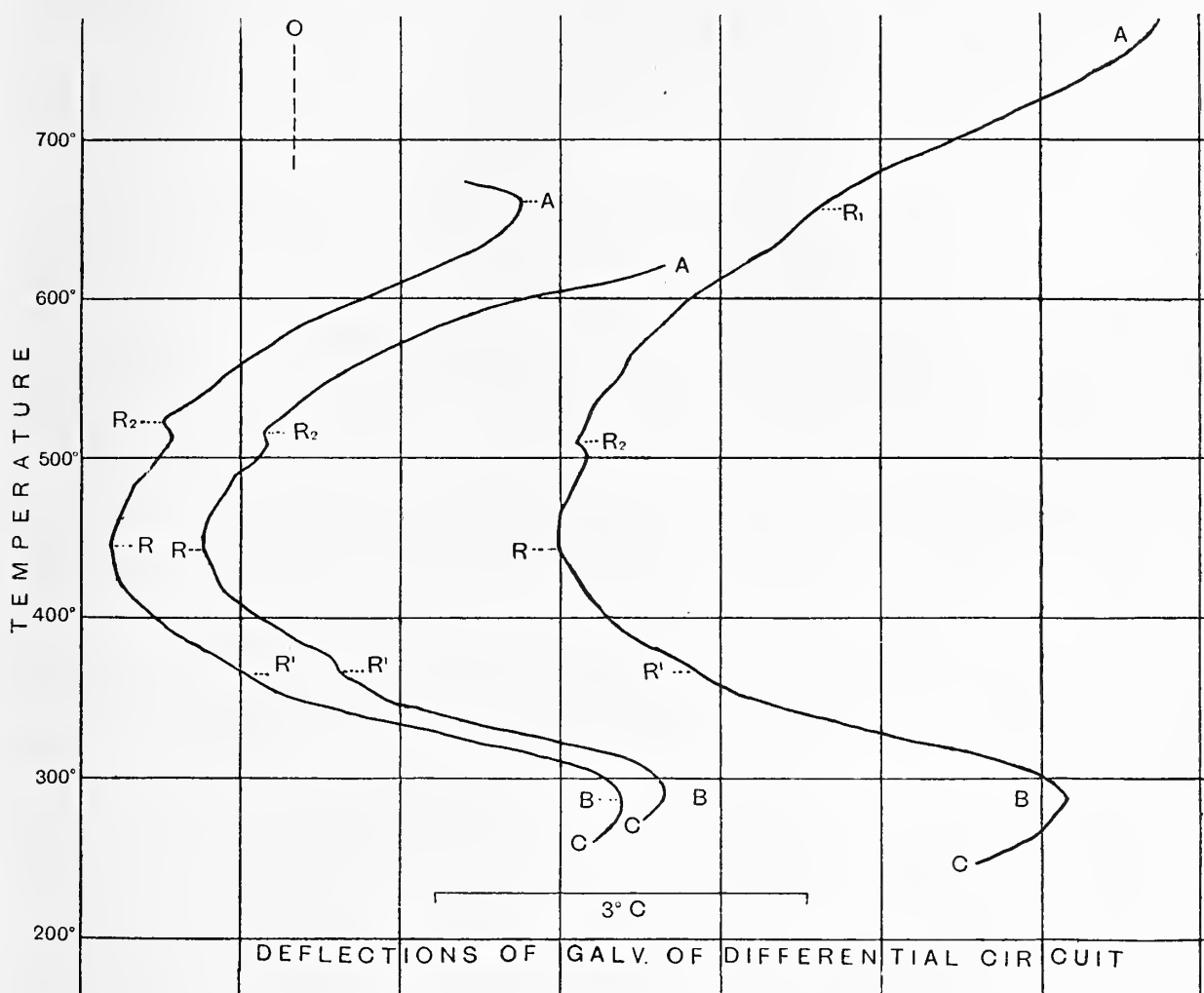


FIG. 4.

used were different in composition. Harrison* using the same specimen of nickel throughout, correlates the changes which occur in the magnetic, thermo-electric, and electrical resistance properties, the results being:—

Range of temperature extending from sudden drop in magnetic permeability to its vanishing-point:

$$\left. \begin{array}{l} 320^{\circ}-374^{\circ} \\ 300^{\circ}-373^{\circ} \\ 310^{\circ}-374^{\circ} \end{array} \right\} \begin{array}{l} \text{according to previous} \\ \text{thermal treatment.} \end{array}$$

Range in which Peltier coefficient undergoes change, $290^{\circ}-375^{\circ}$.

Range in which change in slope of the resistance-temperature curve occurs, $320^{\circ}-375^{\circ}$.

* *Phil. Mag.* (6) 8, 179, 1904.

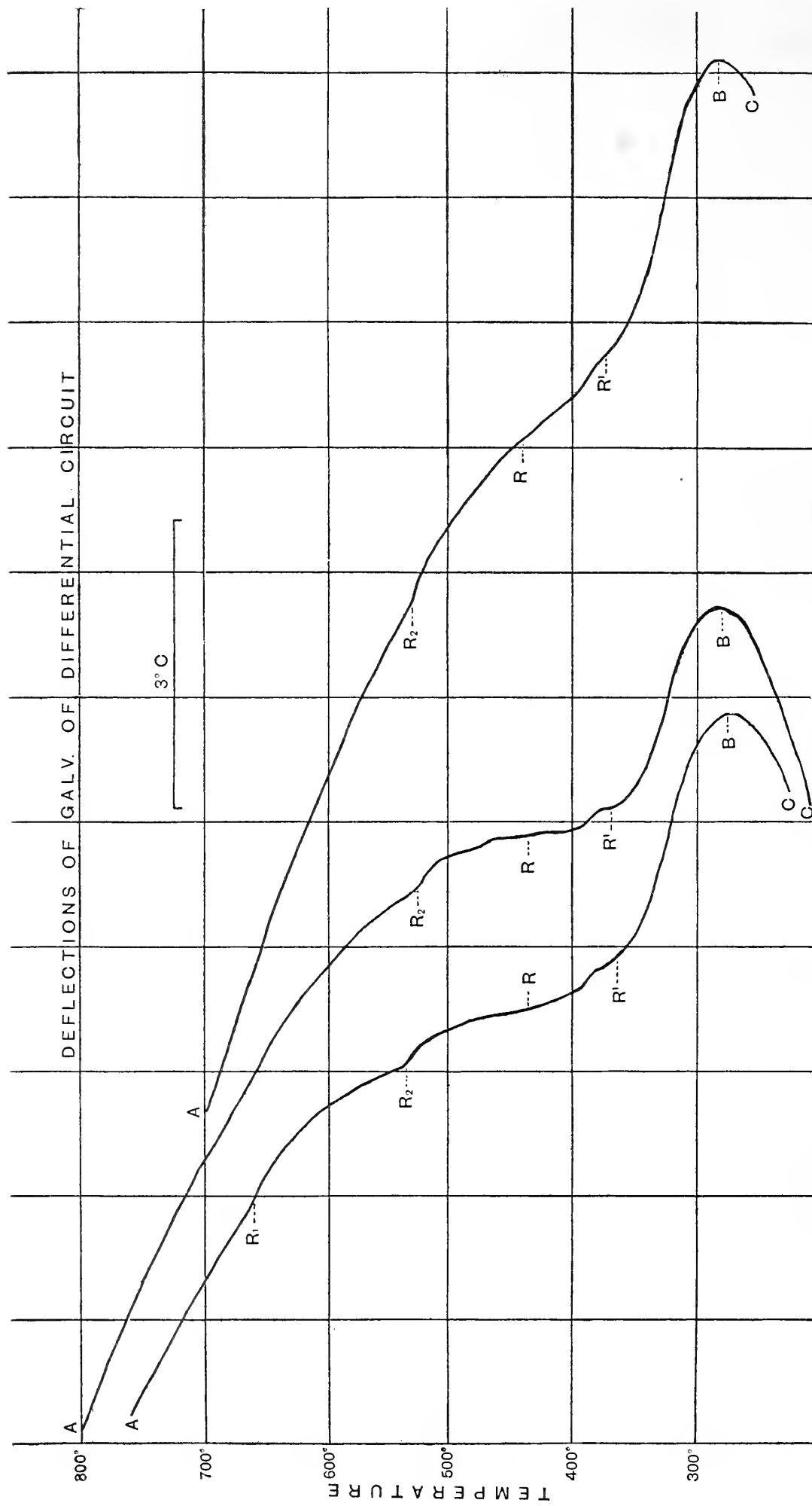


FIG. 5.

Now, the range of temperature over which I have found heat is given out at the greatest rate, viz. the latter part of the large recalescence, is from 370° - 285° , and this corresponds very well with the ranges given for the changes noted above, by Harrison.

My conclusion, then, is that nickel (98 per cent.) in cooling from 900° to 180° gives out heat gradually over a long range (roughly speaking, from about 700° to about 285°), but in addition there are three places where heat is given out at a greater rate. We have recalescences of small extent at about 660° and at about 525° , and to a much larger extent from 440° to 370° , and again to a larger extent from 370° to 285° . Also the range over which we have greatest evolution of heat corresponds to that range over which we have the well-known changes in the magnetic and thermo-electric properties of nickel occurring.

I hope to repeat these experiments with a purer specimen of nickel, to determine whether or not the recalescences observed occur in the nickel itself or are due to impurities.

The expenses of this research were defrayed in part by a grant from the Tait Memorial Fund.

(Issued separately December 22, 1908.)

IV.—On a Question in Absorption Spectroscopy. By Robert A. Houstoun, M.A., Ph.D., D.Sc., assisted by Alexander S. Russell, M.A. *Communicated by Professor A. GRAY, F.R.S.*

(MS. received October 12, 1908. Read November 2, 1908.)

IN the third volume of his *Spectroscopie*, p. 91, Kayser has raised the question whether on mixing two coloured solutions which do not act on one another chemically the absorption spectrum of each of the components remains unchanged. Melde thought he had discovered such an effect; he stated that when a solution of carmine in ammonia which has two sharp bands in the green, was added to a solution of potassium dichromate which absorbs the violet end of the spectrum, or to an ammoniacal solution of copper sulphate which absorbs the red end, that the carmine bands were in each case displaced towards the end absorption in question. He ascribed this to a physical action between the molecules. It was, however, pointed out by Schuster that a shift of this nature would be seen if, instead of mixing, the one solution was merely placed behind the other. Bostwick* and Krüss† repeated Melde's work, and came to the conclusion that there was a real shift in addition to the apparent shift pointed out by Schuster. Since then additional evidence has been adduced by Formánek‡ and has been quoted by Kayser in the section cited above. The object of the research recorded in this paper was to investigate those cases with the most accurate means possible, and, if a shift was established, to decide if it was physical.

We have theoretical grounds for expecting such a shift. Suppose that we have an electric doublet vibrating according to the equation

$$m \frac{d^2x}{dt^2} + h \frac{dx}{dt} + kx = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where x specifies the position of the negative electron relatively to the

* A. E. Bostwick, "Preliminary Note on the Absorption Spectra of Mixed Liquids," *Amer. J.* (3), xxxvii., pp. 471–473 (1889).

† G. Krüss, "Ueber die Constitution von Lösungen," *Ber. Chem. Ges.*, xv., pp. 1243–1249 (1882).

‡ J. Formánek, "Untersuchung und Nachweis Organischer Farbstoffe auf Spectroscopischem Wege," *Zs. f. Farben- u. Textil-Chemie*, i. Heft 11 (1902).

positive one, which is regarded as fixed. Let a second doublet of co-ordinate x_1 and different free period commence vibrating near the first. It will act on the first electron, and this action may be represented by introducing a new term lx_1 in equation (1), l being small in comparison with k . On account of their rapidity we may regard l as a constant as far as the light vibrations are concerned. Equation (1) now runs,

$$m\frac{d^2x}{dt^2} + h\frac{dx}{dt} + kx + lx_1 = 0. \quad (2)$$

We have a similar equation for the second doublet. On solving, we find that the mutual action has slightly altered the periods of both vibrations. As an absorption band corresponds to a free period, absorption bands might thus affect one another. The effect of the mutual actions of the electrons on the theory of dispersion and absorption has been fully worked out by one of the authors.*

On the other hand, we should expect this shift to be too small to observe. The absorption bands of organic colouring matters usually appear at different positions in the spectrum when different solvents are used. Thus the fuchsine band in water is at 540 $\mu\mu$, while in alcohol it is at 555 $\mu\mu$. Kundt stated the law that the absorption band is farther to the red the greater the dispersion of the solvent; but investigation has shown that this law is as often as not disobeyed. It is nevertheless still given in the text-books. This dependence of the position of the absorption band on the solvent—the “Kundt effect”—is supposed to be due to a physical action between the molecules of the solvent and the molecules of the colouring matter. If changing the whole medium in which the doublet is placed displaces the band only by something of the order of 15 $\mu\mu$, then adding a small quantity of another colouring matter in the proportion of at the most 5 gms. per litre should have no appreciable effect.

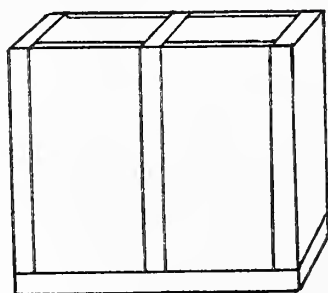
The apparatus was a spectrophotometer which has been described in an article in the *Phil. Mag.*† This instrument has two slits, one about 7 mm. vertically above the other. Both slits are illuminated by the same source, and the liquids, the absorption of which is to be compared, are placed one in front of each slit. The nicol eyepiece in the telescope may be replaced

* R. A. Houstoun, “Untersuchungen über den Einfluss der Temperatur auf die Absorption des Lichtes in isotropen Körpern,” Diss. Göttingen, 1906, *Ann. d. Phys.* (4), xxi. (1906), p. 535.

† *Ibid.*, “A New Spectrophotometer of the Hüfner Type,” *Phil. Mag.*, February 1908.

by an ordinary eyepiece with cross-wires and the instrument used as a spectroscope.

First of all the effect was tried of mixing uranyl nitrate in succession with cobalt chloride, copper sulphate, potassium dichromate, and nickel sulphate, and of mixing cobalt chloride in succession with copper sulphate, potassium dichromate, and nickel sulphate. In order to obtain the greatest accuracy, a differential method was used. Double glass cells of the form shown in the diagram were made for us by Leybold's Nachfolger, Cologne, each compartment measuring $2 \times 2 \times 1$ cm. inside. Solutions of the two salts to be mixed were prepared of a suitable strength. One compartment of each cell was filled with the one solution and the other compartment with the other solution. The cells were then placed above one another, one before each slit, so that the light passed through both solutions in each cell, and the position of the nicol read carefully for different wave-lengths. It was, of course, practically the same for all colours. Then the top cell was emptied,



a mixture prepared of equal parts of the two solutions, and both compartments of the cell filled with this mixture. The cell was then replaced and the nicol again read. Any difference in the nicol readings could thus be due only to the mixing.

The effect of mixing was also examined spectroscopically. This method is most suitable when dealing with sharp bands, the centres of which can be measured with the cross-wires. In this case light which passed through the mixed solutions was compared straight away with light which had passed through the solutions in succession.

If c denote the concentration of the solution in gram-molecules per litre and d the thickness of solution passed through, then A the "molecular extinction coefficient" is defined by

$$I = I_0 10^{-Acd},$$

where I and I_0 are respectively the intensities of the light before and after passing through.

The molecular extinction coefficients of the five salts were found to be as follows:—

CuSO ₄ ·5H ₂ O.		CoCl ₂ ·6H ₂ O.		NiSO ₄ ·7H ₂ O.		K ₂ Cr ₂ O ₇ .		UO ₂ (NO ₃) ₂ ·6H ₂ O.	
λ	A	λ	A	λ	A	λ	A	λ	A
598	·883	633	·514	693	2·112	697	·14	630	·035
590	·672	620	·518	660	1·951	633	·46	567	·039
572	·399	598	·527	630	1·493	588	·68	521	·044
537	·122	591	·552	617	1·092	568	1·10	498	·0605
450	·026	584	·620	596	·672	552	2·11	491	·181
		576	·738	586	·445	544	3·69	488	·404
		568	·970	567	·259	537	7·06	485	·413
		552	1·817	550	·185	529	12·75	483	·355
		537	3·120	535	·158	521	23·47	480	·361
		522	4·561	521	·114	510	65·69	475	·421
		515	4·827	509	·097	499	119·3	470	·902
		510	5·238	498	·085	489	179·1	466	·941
		504	5·509	488	·104	480	251·9	455	1·445
		499	5·172	479	·186	471	322·6		
				470	·293	463	394·4		
				462	·381				
				448	·562				

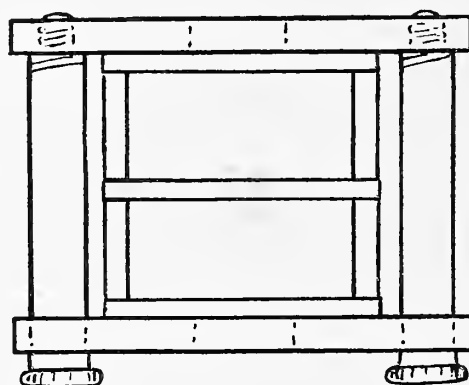
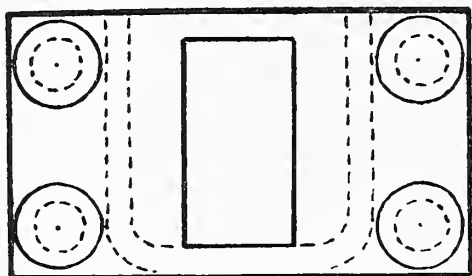
In the case of uranyl nitrate with cobalt chloride and potassium dichromate and in the case of cobalt chloride with copper sulphate, potassium dichromate, and nickel sulphate, no change could be detected as a result of mixing. Different strengths were used, and about seven points in the spectrum were examined. An alteration of $\frac{1}{20}$ of the intensity transmitted could have been detected in the most favourable cases.

It was different in the case of uranyl nitrate with copper sulphate and nickel sulphate. The spectrum of uranyl nitrate consists of bands in the violet, the first two of which at 486 and 473 are very easily seen with the eye although the table of molecular extinction coefficients does not show the second well. On mixing, in each of the above cases, both these bands were displaced towards the red. The shift varied with the strengths of the solutions used; it seemed to be proportional to the quantity of sulphate present divided by the quantity of nitrate. As the first two bands of uranyl sulphate are at 491 and 477, as no shift was produced by mixing copper nitrate and uranyl nitrate and as the same shift was produced by adding sulphuric acid to uranyl nitrate, there seems no doubt that the shift is chemical.

We next repeated Melde's work with carmine. This was done only spectroscopically. A solution of carmine in ammonia which has bands at

570 and 528 and then slowly increasing absorption in the blue and violet, was mixed with an ammoniacal solution of copper sulphate. The strength of the carmine solution was 0.15 gms. per litre. The strongest copper sulphate solutions had about 10 gms. copper sulphate per litre. Strengths down to $\frac{1}{32}$ nd of this were used. A definite shift of both bands to the red was obtained, amounting as a rule to 20 Ångström units which seemed to remain pretty much the same as the concentration of the copper sulphate was diminished. The measurements were always made on the centre of the bands. The bands were so ill-defined that 10 Ångström units was the smallest shift one could be sure about. In this case there is nothing to show whether the change is physical or chemical.

The effect was tried next of mixing the ammoniacal solution of carmine with potassium dichromate. The strength of the latter solution was 4 gms. per litre and less. We could not say whether there is a shift to the violet



or not. If there is one it cannot be more than 5 Ångström units. But there is no doubt whatever that the ammonia acts on the dichromate, changing it into the chromate, and pushing back the beginning of the continuous absorption much further into the violet. Bostwick, it must be noted, obtained a definite shift in both cases of the order of 40 Ångström units. It is difficult to say what the discrepancy is due to. We varied our conditions repeatedly but cannot explain it.

We finally repeated some of Formánek's experiments. Here we had trouble with the cells. The cement absorbed the aniline dyes used, sometimes rapidly, sometimes just appreciably. Cells made by Leybold, cells of unknown manufacture, and home-made cells were alike unsatisfactory; and cells made without cement, of plate glass held together with rubber bands, leaked. It was therefore necessary, to do the work thoroughly, to have cells made without cement by R. & J. Beck, Limited, with first quality glass surfaces, the different pieces being kept together by a brass frame. The figures show how this was done. The cells were double and were used

in the same way as the other double cells, and the compartments had the same internal dimensions. Each cell consisted, in addition to the frame, of five pieces of glass, three plates and two pieces of square tubing with the top side ground off. They were satisfactory, but had to be put together with very great care. The effect of the cement on the dyes does not seem to have been mentioned by other workers, and may explain some effects observed in the photometry of the aniline colouring matters.

In the article cited, Formánek gives five cases where on mixing dyes the bands affect one another. In his book he gives the sources from which his dyes were obtained, but we were able only to obtain those necessary for four of these cases. These cases are as follows:—

(1) The bands of malachite green and brilliant green in water (6183, 6255) run together on mixing. A new band is formed which lies between 6183 and 6255.

(2) If a little malachite green (band at 6183) be added to a suitably diluted solution of methylene blue (chief band at 6678, subsidiary band at 6081) in water, the subsidiary band of the methylene blue at 6081 runs together with the band of malachite green, and in the spectrum we see only the chief band of methylene blue in its original position and the band of malachite green somewhat displaced. If malachite green be added until the proportions of both colouring matters are about the same, the absorption band of malachite green returns to its original place.

(3) If Nile blue A (band at 6448) and methylene blue are mixed in water, we obtain only one band which lies near the position of the Nile blue band.

(4) On mixing methylene blue and methyl violet 6 B, which has a band at 5935, the methyl violet band is displaced towards the red.

We first of all examined these cases spectroscopically taking different strengths. *In no case did we see the slightest displacement as a result of mixing nor could any alteration of intensity be detected.* The spectrophotometric method was then employed, one critical point in the spectrum being taken for each case, *e.g.*, a point was taken between the bands if they were expected to move together on mixing. In the second case only could a small change of intensity be detected, which was not very consistent; as it was about $\frac{1}{40}$ th of the intensity it was too small to be investigated. It could not in any case be detected by the spectroscopic method, which was the method employed by Formánek.

Formánek seems to have used only test-tubes for the liquids under examination, and we are forced to the conclusion that the shift he observed is merely an apparent one caused by the one maximum being on the slope of

the other. It is in fact the effect pointed out by Schuster. There is no action between the molecules.

We have thus met with no evidence in favour of the existence of an effect of the nature mentioned by Kayser.

The research was carried out in the Physical Laboratory of the University of Glasgow with the aid of a grant from the Carnegie Trust for the Universities of Scotland.

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V.—Dissymmetrical Separations in the Zeeman Effect in Tungsten and Molybdenum. By Robert Jack, M.A., B.Sc., Ph.D., 1851 Exhibition Research Scholar. *Communicated by* Professor A. GRAY, LL.D., F.R.S.

(MS. received October 12, 1908. Read November 2, 1908.)

It has been mentioned by Professor Voigt of Göttingen in his newly published book * and by Professor Zeeman of Amsterdam in the *Physikalische Zeitschrift*,† that I have found examples of strongly marked dissymmetry in studying the Zeeman Effect in tungsten and molybdenum. Professor Zeeman has also discovered and published such cases of dissymmetry in other elements. Not only have many examples of normal dissymmetry been found, but almost as many cases of abnormal dissymmetry. To explain those terms, *normal* and *abnormal*, let us consider that the single spectrum line is broken up, when the light is in the magnetic field, into the three components, 1, 2, 3, where the numbers begin from the component which has the shortest wave-length. In the normal dissymmetrical triplet the middle component is nearer the component on the red side than that on the violet one, *i.e.* for the normal type the interval 1-2 is greater than the interval 2-3, but in the abnormal dissymmetrical triplet 2 is nearer to 1 than to 3. These observations of Professor Zeeman and myself, which were made at the same time in the Universities of Amsterdam and Göttingen, having been communicated to Professor Voigt, he wrote and published in the above-mentioned book an extension to his and Professor H. A. Lorentz's theories of the Zeeman Effect. In his original theory Professor Voigt had shown that, considering the electrons as uncoupled, cases of normal dissymmetry might arise among the Zeeman triplets, this dissymmetry being accompanied by a greater intensity of the red component than the violet one.‡ He pointed out also that the "absolute" dissymmetry or the difference between the absolute displacements of the red and violet components should be independent of the magnetic field strength used to produce the Zeeman Effect. To explain the large numbers of complicated types of Zeeman Effect which have been found—in the study of the Zeeman Effect in tungsten I discovered lines with no

* W. Voigt, *Magneto- und Elektrooptik*.

† P. Zeeman, *Phys. Zeit.*, x. 340, 1908 ; W. Voigt, *Phys. Zeit.*, xi. 353, 1908.

‡ Thus normal dissymmetry is the more simply explained and first discovered type, whereas abnormal dissymmetry requires a more complicated theory, and when first observed was contradictory to any existing theory.

fewer than 17 to 19 components, the largest numbers hitherto found—Professors Voigt and Lorentz made use in their theories of couplings between electrons of the *same* vibration frequencies. The latest step taken by Professor Voigt as a result of these newest observations on dissymmetry is to introduce couplings between two electrons of *different* vibration frequencies. The theory shows that cases of normal and abnormal dissymmetrical triplets arise, that the intensities of the two outer components are the same, and that the displacement of the lines which produces the dissymmetry is proportional to the square of the magnetic field strength used. All these conclusions may be compared with the results arrived at for the uncoupled electrons. Thus the subject of the dissymmetry of the components is important, as it may lead to a more accurate knowledge of the grouping of the electrons and the constitution of the atom. While Professor Voigt's book was being printed, observations appeared by P. Gmelin * showing that in the case of the dissymmetrical triplet, $\lambda = 5790$, † of the mercury spectrum, the displacement of the middle line was proportional to the square of the field strength.

In the following my observations are given, and an attempt is made to show that in addition to the possibility of the presence of such couplings in some cases, as Professor Voigt has assumed, there is evidently some connection between the rotation of the plane of polarisation produced by the concave grating apparatus used and the dissymmetry observed in many of the cases. There are, however, also examples of dissymmetry which are exceptions to such a rule; and further, there are symmetrical separations which seem to be wholly uninfluenced by the rotation of the plane of polarisation.

With the concave grating apparatus of the Physical Institute, University of Göttingen, photographs of the transversal Zeeman Effect were in the first place taken, *i.e.* the light at right angles to the magnetic lines of force was used, and hence in the case of triplets all three components were present. From these photographs both the type and the amount of the dissymmetry were observed, but one could decide nothing about the displacement of the components which had caused the dissymmetry. The variation in the separations of the components is very striking, the difference between the separations towards red and violet amounting in some cases to over 25 per cent.

The grating apparatus in that laboratory has already been described. ‡

* P. Gmelin, *Phys. Zeit.*, ix. 212, 1908.

† All wave-lengths will be given in ångströms.

‡ *Phys. Zeit.*, vi. 25, 1905.

To obtain the light, a thin layer of the salt to be used is put on strips of carbon, 1 mm. thick and 5 mm. long; these are placed against the poles of the magnet, and a spark is passed between these electrodes. To focus the light of the spark on the slit a quartz lens is used, as glass absorbs the violet rays. The maximum thickness of the lens used is 6.6 mm. The light passing through the slit and falling on the grating comes from a portion of the lens 4 cms. in diameter, which, if spark, lens and slit are in a line, and the lens at the proper height, is at the middle of the lens, and then the difference between the maximum and minimum thicknesses used is about 2 mm. Zeeman has shown * that the intensities, when a diffraction grating is used, depend on the angle which the plane of the vibrations makes with the lines of the grating. By introducing quartz plates of thicknesses to give different rotations, he was able to vary the intensities in the case of a triplet, from having a strong middle component with weak side components, to the reverse distribution of intensity. When the plane of vibration is parallel to the lines of the grating the intensity is a maximum, and when at right angles to this the intensity is a minimum. The influence of the magnetic field gives vibrations in two planes at right angles to each other, and by rotation first one plane of vibration and then the other can be brought parallel to the ruled lines, and so the maximum and minimum for each kind of vibration got. When the planes of vibration make 45° with the lines, then the vibrations are in the same relative positions to the rulings, and then it was found that in the case of triplets the intensities were as Lorentz had predicted, namely, the middle component twice as strong as the side components. In the apparatus used the quartz lens produces rotation. When the lens is placed as mentioned above it has the same effect as a quartz plate of about 1 mm. thickness. Thus, for approximately $\lambda = 3450$ the middle component was twice as strong as the side components, that is, the planes of vibration made 45° with the rulings. As the amount of the rotation varies with the wave-length for the region $\lambda = 2800$ to $\lambda = 2900$, the planes had been rotated through 45° , and now the middle component had a minimum intensity. The cycle of variations is passed through more quickly as one proceeds further towards the violet, on account of the greater variation in rotation there. Thus the relative intensities of the components in the Zeeman triplets give an indication of the amount of the rotation of the plane of polarisation produced by the quartz lens for the different parts of the spectrum. In the case of overlapping spectra of different orders, it was quite easy, by means of this property, to distinguish the lines belonging to the same order, for the

* P. Zeeman, *K. Akad. v. Wet.*, Amsterdam, Oct. 1907.

different wave-lengths overlapping had different rotations, and hence different relative intensities for their components.

In the following table the examples of symmetry found in tungsten are

TABLE I.—TUNGSTEN.
Field strength = 24,650 gauss.

Wave-length.	Dissym.	Distance from mid. Compt. (- towards Violet, + towards Red).	Intensity.	Remarks.	Wave-length.	Dissym.	Distance from mid. Compt. (- towards Violet, + towards Red).	Intensity.	Remarks.
2488·89	<i>n</i>	$\begin{cases} -0\cdot147 \\ 0 \\ +0\cdot116 \end{cases}$	$\begin{cases} 4 \\ 3 \\ 4 \end{cases}$		2856·20	<i>n</i>	$\begin{cases} -0\cdot151 \\ 0 \\ +0\cdot137 \end{cases}$	$\begin{cases} 5 \\ 1 \\ 5 \end{cases}$	
2522·14	<i>n</i>	$\begin{cases} -0\cdot146 \\ 0 \\ +0\cdot117 \end{cases}$	$\begin{cases} 3 \\ 2 \\ 3 \end{cases}$		3049·80	<i>n</i>	$\begin{cases} -0\cdot289 \\ 0 \\ +0\cdot252 \end{cases}$	$\begin{cases} 5 \\ 2 \\ 5 \end{cases}$	
2555·23	<i>n</i>	$\begin{cases} -0\cdot152 \\ 0 \\ +0\cdot114 \end{cases}$	$\begin{cases} 4 \\ 3 \\ 4 \end{cases}$		3311·53	<i>a</i>	$\begin{cases} -0\cdot159 \\ 0 \\ +0\cdot181 \end{cases}$	$\begin{cases} 8 \\ 8 \\ 8 \end{cases}$	
2580·63	<i>n</i>	$\begin{cases} -0\cdot128 \\ 0 \\ +0\cdot101 \end{cases}$	$\begin{cases} 3 \\ 2 \\ 3 \end{cases}$		3361·25	<i>a</i>	$\begin{cases} -0\cdot124 \\ 0 \\ +0\cdot139 \end{cases}$	$\begin{cases} 3 \\ 4 \\ 3 \end{cases}$	
2606·50	<i>a</i>	$\begin{cases} -0\cdot149 \\ 0 \\ +0\cdot155 \end{cases}$	$\begin{cases} 3 \\ 2 \\ 3 \end{cases}$		3373·88	<i>a</i>	$\begin{cases} -0\cdot078 \\ 0 \\ +0\cdot087 \end{cases}$	$\begin{cases} 6^* \\ 7 \\ 4 \end{cases}$	
2633·24	<i>n</i>	$\begin{cases} -0\cdot135 \\ 0 \\ +0\cdot101 \end{cases}$	$\begin{cases} 3 \\ 1 \\ 3 \end{cases}$	[] Less accurate.	3413·09	<i>a</i>	$\begin{cases} -0\cdot084 \\ 0 \\ +0\cdot108 \end{cases}$	$\begin{cases} 6^* \\ 6 \\ 6 \end{cases}$	* Shaded towards violet.
2697·81	<i>n</i>	$\begin{cases} -0\cdot169 \\ 0 \\ +0\cdot150 \end{cases}$	$\begin{cases} 5 \\ 3 \\ 5 \end{cases}$		3429·75	<i>a</i>	$\begin{cases} -0\cdot069 \\ 0 \\ +0\cdot084 \end{cases}$	$\begin{cases} 6^* \\ 7 \\ 4 \end{cases}$	
2774·12	<i>n</i>	$\begin{cases} -0\cdot177 \\ 0 \\ +0\cdot133 \end{cases}$	$\begin{cases} 4 \\ 2 \\ 4 \end{cases}$		3448·96	<i>a</i>	$\begin{cases} -0\cdot077 \\ 0 \\ +0\cdot088 \end{cases}$	$\begin{cases} 4 \\ 5 \\ 2 \end{cases}$	Mid. compt. twice as strong $\lambda=3450$ approx.
2774·60	<i>n</i>	$\begin{cases} -0\cdot153 \\ 0 \\ +0\cdot136 \end{cases}$	$\begin{cases} 5 \\ 2 \\ 5 \end{cases}$		4298·55	<i>n</i>	$\begin{cases} -0\cdot535 \\ -0\cdot128 \\ 0 \\ +0\cdot126 \\ +0\cdot454 \end{cases}$	$\begin{cases} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{cases}$	
2792·85	<i>n</i>	$\begin{cases} -0\cdot172 \\ -0\cdot083 \\ 0 \\ +0\cdot083 \\ +0\cdot159 \end{cases}$	$\begin{cases} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{cases}$	Min. intensity of mid. compt. $\lambda=2850$ approx.					

given. The measurements are stated in millimetres, it being unnecessary for the purpose here to change these to variations in the vibration frequencies. With the apparatus used, the error in measurement will very

rarely exceed 0.005 mm. The accuracy depends, of course, greatly on the character of the lines, their sharpness, intensity, separation, etc. The measurements have been made sometimes in the first, sometimes in the second, and at other times in the third order, according as the conditions were most favourable to accurate measuring.

a and n denote respectively abnormal and normal dissymmetry. One can see that in most cases the intensities of the two outer components of a triplet are the same. In the above table are the most striking examples of dissymmetry in the spectrum of this element, but certainly there must be many more which are not so apparent to the eye. Only a few examples with more than three components are given, as it is desirable to limit the consideration to the simplest cases.

It will be at once apparent from the table that concurrently with the change in the intensities there is a change in the type of dissymmetry found. As the middle component passes through its minimum value there is a change from the normal dissymmetry to the abnormal, and this change persists till the intensities pass through another critical point where the intensity of the centre component is twice as strong as that of the outer components. This would point to the rotation of the plane of polarisation due to the quartz being accompanied by a change in the type of dissymmetry. Against this it may be argued that there are lines between those cited whose planes of vibration would be so situated as to give dissymmetry, but that these lines have their components symmetrically placed. Again, there are exceptions to the rule that dissymmetry and plane of vibration of the components change correspondingly. For instance, $\lambda = 2606.50$ shows abnormal dissymmetry; and as the others in the neighbourhood show normal dissymmetry, this should also have been a normal type. These are points which at present I cannot explain, but the evidence for the connection between the dissymmetry found with the apparatus used and the rotation of the plane of polarisation is too strong to be overthrown by these exceptions. There are also deviations from the rule established by the theory that the outer components have the same intensity. The lines $\lambda = 3373.88$ and 3429.75 show this, but the shadows on the violet side indicate that there is some other external cause to bring about this change.

To show that dissymmetry could really be due to the rotation of the plane of polarisation, another test was made. The very bright line in molybdenum, $\lambda = 5533.26$, was observed. By moving the quartz lens the beam of light was made to pass through different thicknesses of the quartz. This had the effect of altering the rotation, and hence also the intensities of the components. At the same time as the intensities altered from the outer

components being the stronger to their being the weaker, the positions of the components changed from an abnormal dissymmetrical arrangement to a normal dissymmetry. On sending the current in the opposite direction, and thus reversing the magnet, there was no alteration in the dissymmetry. The same thing was done with a glass lens instead of a quartz one. The glass lens was of almost the same dimensions as the quartz one, but on moving it, neither the intensities nor the positions of the components changed.

The following table gives the details relating to examples of dissymmetry found in the spectrum of molybdenum.

TABLE II.—MOLYBDENUM.

Field strength = 24,650 gauss.

Wave-length.	Dissym.	Distance from mid. Compt. (- towards Violet, + towards Red).	Intensity.	Remarks.	Wave-length.	Dissym.	Distance from mid. Compt. (- towards Violet, + towards Red).	Intensity.	Remarks.
2672·91	<i>n</i>	$\begin{cases} -0\cdot142 \\ 0 \\ +0\cdot123 \end{cases}$	$\begin{cases} 7 \\ 3 \\ 7 \end{cases}$	Min. intensity of mid. compt.	3087·73	<i>a</i>	$\begin{cases} -0\cdot178 \\ 0 \\ +0\cdot217 \end{cases}$	$\begin{cases} 5 \\ 3 \\ 5 \end{cases}$	Edge of plate.
2924·45	<i>a</i>	$\begin{cases} -0\cdot149 \\ 0 \\ +0\cdot162 \end{cases}$	$\begin{cases} 4 \\ 2 \\ 4 \end{cases}$		Middle of plate.				
2941·37	<i>a</i>	$\begin{cases} -0\cdot100 \\ 0 \\ +0\cdot122 \end{cases}$	$\begin{cases} 6 \\ 2 \\ 6 \end{cases}$		Mid. compt. twice as strong.				
2944·97	<i>n</i>	$\begin{cases} -0\cdot160 \\ 0 \\ +0\cdot150 \end{cases}$	$\begin{cases} 6 \\ 1 \\ 6 \end{cases}$						
2946·10	<i>a</i>	$\begin{cases} -0\cdot110 \\ 0 \\ +0\cdot131 \end{cases}$	$\begin{cases} 4 \\ 2 \\ 4 \end{cases}$						
2993·00	<i>a</i>	$\begin{cases} -0\cdot160 \\ 0 \\ +0\cdot188 \end{cases}$	$\begin{cases} 5 \\ 3 \\ 5 \end{cases}$						
3014·29	<i>a</i>	$\begin{cases} -0\cdot141 \\ 0 \\ +0\cdot144 \end{cases}$	$\begin{cases} 4 \\ 3 \\ 4 \end{cases}$						
						4200·72	<i>n</i>	$\begin{cases} -0\cdot230 \\ 0 \\ +0\cdot205 \end{cases}$	$\begin{cases} 5 \\ 5 \\ 5 \end{cases}$

Again the concurrent changing of the dissymmetry with that of the intensities is striking. There are further exceptions also. The line $\lambda = 3087\cdot73$ is interesting. At the centre of the plate the light had passed through the centre of the lens, but at the edge of the plate the light was

mainly from a part nearer the edge of the lens. Due to the difference in rotation thus produced, a difference in the relative intensities of the components was observed, and at the same time a difference in the dissymmetry. The measurements show that these changes are in no way due to a difference of field strength acting on the light which falls on these two parts of the plate. Such a difference between the middle and the edge often occurs. These results tend to confirm the connection between the rotation and the dissymmetry produced by the apparatus used. The exceptions show that this does not explain everything.

The problem was now to discover what component or components moved to produce the dissymmetry. Photographs of the same lines in tungsten were made in such a way that on the upper half of the plate came the spectral lines without separation, and under these were the separated components produced under the influence of the magnetic field. By means of a sheet, first the one half and then the other was covered. The same was also done with the lines in the iron spectrum. All the observations show that the centre component may sometimes remain stationary, and at other times be moved towards the red, or again towards the violet. The same is also true of the mean of the outer components.

In the following table the results for tungsten are given:—

TABLE III.—TUNGSTEN.
Field strength = 22,000 gauss.

Wave-length.	Dissym.	Distance from Spectral line (- towards Violet, + towards Red).		Intensity.	Wave-length.	Dissym.	Distance from Spectral line (- towards Violet, + towards Red).		Intensity.
		(mm.)	(Ångström.)				(mm.)	(Ångström.)	
2488·89	<i>n</i>	-0·117	-0·114	5	2774·12	<i>a</i>	-0·138	-0·135	4
		+0·004	+0·003	2			-0·005	-0·005	2
		+0·118	+0·115	5			+0·143	+0·140	4
2522·14	<i>n</i>	-0·116	-0·112	4	2774·60	<i>a</i>	-0·126	-0·124	4
		+0·003	+0·003	2			-0·006	-0·006	1
		+0·117	+0·114	4			+0·130	+0·128	4
2555·23	<i>n</i>	-0·122	-0·119	5	2856·20	<i>a</i>	-0·131	-0·129	4
		+0·006	+0·006	3			-0·010	-0·010	1
		+0·127	+0·123	5			+0·133	+0·130	4
2697·81	<i>a</i>	-0·141	-0·137	6	3049·80	<i>a</i>	-0·246	-0·242	2
		-0·005	-0·005	4			-0·011	-0·011	1
		+0·155	+0·151	6			+0·255	+0·251	2

In this experiment the lens was moved a little distance from its former position (Table I.), which gave a difference in rotation and an accompanying

change in dissymmetry. The distances are here given from the unseparated spectral line, and these measurements have been transformed to ångströms to show the alteration in wave-length produced. The field strength was slightly smaller than before, being 22,000 gauss. It will be seen on comparing this with Table I., that the amounts of dissymmetry have changed, and that some cases of normal dissymmetry have even become abnormal, owing to the new arrangement of the apparatus. The motions of the centre components and of the mean of the outer components from the original position of the spectral line are in some cases quite appreciable, and beyond the range of the errors of measurement.

A further experiment was made, in which the light, in addition to passing through the quartz lens, was made to penetrate a quartz plate, 5 mm. thick. The distances of the outer components from the middle component are given. In this case the field strength was a little higher, being now equal to 25,700 gauss.

TABLE IV.—LIGHT THROUGH QUARTZ PLATE, 5 MM. THICK.

Field strength = 25,700 gauss.

Wave-length.	Dissym.	Distance from mid. Compt. (- towards Violet, + towards Red).	Intensity.	Remarks.	Wave-length.	Dissym.	Distance from mid. Compt. (- towards Violet, + towards Red).	Intensity.	Remarks.
2606·50	<i>n</i>	$\begin{cases} -0\cdot161 & 3 \\ 0 & 2 \\ +0\cdot154 & 3 \end{cases}$		Min. intensity of mid. compt. $\lambda=2630$ approx.	3025·01	<i>n</i>	$\begin{cases} -0\cdot157 & 4 \\ 0 & 3 \\ +0\cdot146 & 4 \end{cases}$		Components broadened and difficult to measure.
2697·81	<i>a</i>	$\begin{cases} -0\cdot151 & 3 \\ 0 & 6 \\ +0\cdot175 & 3 \end{cases}$			3041·96	<i>a</i>	$\begin{cases} -0\cdot138 & 3 \\ 0 & 4 \\ +0\cdot168 & 3 \end{cases}$		
2856·20	<i>n</i>	$\begin{cases} -0\cdot159 & 4 \\ 0 & 5 \\ +0\cdot141 & 4 \end{cases}$		3049·80	<i>n</i>	$\begin{cases} -0\cdot296 & 4 \\ 0 & 4 \\ +0\cdot278 & 4 \end{cases}$			
2935·10	<i>n</i>	$\begin{cases} -0\cdot158 & 5 \\ 0 & 3 \\ +0\cdot142 & 5 \end{cases}$		3311·53	<i>n</i>	$\begin{cases} -0\cdot184 & 4 \\ 0 & 8 \\ +0\cdot169 & 4 \end{cases}$			
2997·94	<i>a</i>	$\begin{cases} -0\cdot147 & 2 \\ 0 & 1 \\ +0\cdot171 & 2 \end{cases}$		3361·25	<i>n</i>	$\begin{cases} -0\cdot141 & 2 \\ 0 & 3 \\ +0\cdot130 & 2 \end{cases}$			
				Min. intensity of mid. compt. $\lambda=2960$ approx.					

As there is now a much greater thickness of quartz, the amount of rotation is much increased, and hence the cycle of changes in the intensities is passed through very much more quickly. From observing all the lines

on the plates, the turning points in the intensity of the middle component are approximately as stated in the table. It will be seen that corresponding to this more rapid variation in intensities, there is a similar variation in the character of the dissymmetry found in the lines. Again, there are exceptions to the rule that the dissymmetry depends on the amount of the rotation of the plane of polarisation.

These experiments show that in using a quartz lens and a concave grating to obtain the Zeeman Effect cases of dissymmetry arise which are due to the angle between the planes of vibration of the components and the lines of the grating, for this dissymmetry varies with the intensities, which, in their turn, vary with that angle. That all cases of dissymmetry cannot be so accounted for is shown by the exceptions to this rule where abnormal dissymmetry is found instead of the normal which might be expected, and *vice versa*. These point to the possibility of the presence of the couplings between electrons of different vibration frequencies assumed by Professor Voigt, in consequence of which the two triplets containing the coupled electrons become respectively an abnormal and a normal dissymmetrical triplet. The intensities of the outside components are in most cases the same, and this also is in accordance with the theory. In the photographs of these substances, tungsten and molybdenum, are also to be found many other examples of dissymmetry among lines which break up into many components.

This research was carried out in the University of Göttingen, and I take this opportunity of expressing my gratitude to Professor Voigt, and the pleasure I had in having the apparatus there, which is so splendidly adapted for such research, set at my disposal.

(*Issued separately December 30, 1908.*)

VI.—On the Reducing Action of Electrolytic Hydrogen on Arsenious and Arsenic Acids when liberated from the Surface of Different Elements. By William Thomson, F.I.C.

(Read June 15, 1908. MS. received July 12, 1908.)

THIS research was commenced with a view to find the velocity at which arsenic was liberated as arseniuretted hydrogen from cathodes of different elements :

(a) From arsenic in the form of arsenious acid.

(b) „ „ „ „ arsenic acid.

The experiments were carried out in the apparatus already described by me,* which consists of a porous pot containing 30 c.c. of dilute sulphuric acid (one of strong acid to six of water by volume), with the cathode, when that was possible, in the form of a cylinder 60 mm. long and 10 mm. diameter, immersed to a depth of 27 mm., giving a surface exposed to the electrolyte of 8.48 square centimetres, the end passing through an india-rubber cork fitted into an opening in the glass stopper, which was accurately ground into the mouth of the porous pot and made air-tight with vaseline; the liberated hydrogen was passed from a glass tube fused into the same glass stopper, which ended in a two-way tap and T-piece. Another similar apparatus was placed in series with the first-mentioned, so as to liberate hydrogen from pure dilute sulphuric acid. This apparatus was also provided with a two-way tap and T-piece, so that a stream of pure hydrogen, dried through a calcium chloride tube and passed over a small roll of dry basic lead acetate paper to remove any trace of H_2S , could be passed through one or other of two drawn-out hard glass tubes which were heated to redness near to the drawn-out ends to receive the arsenic mirrors. These were deposited on the drawn-out part, the diameter of which was graduated by inserting a wire, the one end being 1.6 mm., the other 1.1 mm. diameter, the length between these two points being 2 mm., the tube being cooled from the larger diameter. This apparatus was capable of easily detecting 0.000,000,541,3 gramme (roughly about $\frac{1}{2000000}$ of a gramme) of elemental arsenic when contained in the 30 c.c. of acid in the porous pot. This

* *Memoirs and Proceedings of the Manchester Literary and Philosophical Society*, vol. xlviii. part iii., No. 17.

quantity was taken as a unit, and for each experiment 5 c.c. of a solution containing 0.000,005,413 grammes per c.c. was taken in the form of :

(a) Arsenious acid.

(b) Arsenic „

This makes 50 units, each of which can be easily detected. This amount was deduced from another standard in which 50 c.c. of a liquid containing $\frac{1}{1000}$ th of a grain per gallon produces a well-defined mirror of arsenic on the cooled drawn-out part of the glass tube.

Having filled the apparatus with hydrogen by passing a current of 3 ampères through 10 c.c. of dilute sulphuric acid in the porous pot, 5 c.c. of the standard solution of arsenic above mentioned is made up to 15 c.c. with dilute sulphuric acid and poured into the apparatus by a funnel fused into the glass stopper of the porous pot, stoppered by a glass rod at the bottom, the funnel being rinsed into the porous pot with 5 c.c. of the same dilute acid. After the apparatus has worked for $2\frac{1}{2}$ minutes and a mirror has been deposited, the two-way tap is turned so that the hydrogen from the apparatus in series generating pure hydrogen sweeps the tube for a few moments. The tube is then removed and another inserted to receive a fresh mirror, the second mirror being produced by the two-way tap diverting the flow of the hydrogen containing arseniuretted hydrogen for $2\frac{1}{2}$ minutes to the second tube, through which pure hydrogen had previously been passing, and which was heated to redness for the purpose of receiving the second mirror. In this way mirrors were received without loss of arsenic on the drawn-out portion of hard glass tubes, cooled at the point at which the mirror was to be deposited by means of cold water flowing continually over them. The mirrors thus received were sealed off in an atmosphere of dry hydrogen, and the quantity on each estimated by comparison with a series of standard mirrors examined and compared by a magnifying glass.

Plotting the amounts of arsenic deposited in successive intervals of $2\frac{1}{2}$ minutes as the ordinates, and the time as abscissæ, curves were obtained for all these experiments. It was observed that, however carefully each experiment was repeated, no two results agreed absolutely, and this disagreement was greater for arsenic acid than for arsenious acid.

Thirteen different elements were tested—viz., lead, zinc, cadmium, tin, silver, graphite, iron, platinum, aluminium, gold, cobalt, nickel, and palladium*

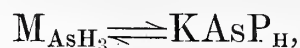
* The following metals were used in the form of cylinders:—Lead, zinc, cadmium, and tin.

Iron was used in a cylindrical form made by wrapping pure iron wire on a cylinder of glass, giving about the same surface exposed to the electrolyte as the metallic cylinders.

The graphite electrode was cut in the form of a rectangular block giving about the same

—and the order given above is the order in which they are capable of removing arsenic from solution as arseniuretted hydrogen when it exists in solution in the form of arsenious acid. This order, however, does not hold for the relative power of the different elements for removing arsenic from solution as arseniuretted hydrogen when it exists as arsenic acid in the electrolyte.

Chapman and Law* suggest that the reducing efficiency of hydrogen with respect to arsenious and arsenic oxides depends largely upon the supertension† of the cathode at which the hydrogen is liberated, and they put forward a formula to explain this, as follows:—



“where As represents the amount of unreduced arsenious oxide and K a constant factor. If the equation be disturbed by the addition of As, *i.e.* if the factor As be increased, reduction takes place or the reaction proceeds from right to left, and the same result is reached if P_H is made larger. On the other hand, if P_H is made smaller, As must be made larger to preserve the equilibrium. In other words, there is always left in solution after each experiment a certain residuum of unreduced arsenious oxide.”

M_{AsH_3} is a minute quantity of arseniuretted hydrogen always left in solution, and P_H the potential of the hydrogen effecting the reduction. The results we have obtained contradict this hypothesis.

The following table shows the relative velocities of thirteen different elements in decomposing arsenious and arsenic acids respectively, and the supertension of each element is also given when obtainable.

The figures in Table I. represent units of arsenic as above described liberated in 25 minutes from 50 units (taken as 100 half-units) of elemental arsenic contained in the electrolyte (*a*) as arsenious acid, and (*b*) as arsenic acid.

It was found that the current density had little influence on the result. The metals which could be easily melted were formed into cylinders 1 cm. surface as the cylinder, and all the others were used in the sheet form; but it was found that the area of the cathode did not materially affect the result.

* Chapman and Law on “The Reducing Action of Hydrogen,” part ii., “The Estimation of Traces of Arsenic by the Marsh-Berzelius Method and the Insensitiveness of Zinc,” *Analyst*, 1906, vol. xxxi., p. 3.

† “Supertension” of an electrode, according to Caspari (“Ueberspannung,” *Zeit. physikal. Chem.*, 1899, xxx., 89) is the excess of electromotive force necessary for the liberation of hydrogen at that electrode over the electromotive force required for the reversible production of hydrogen on a cathode of platinised platinum.

The values of the “excess-voltage” or supertension for different metals are given by Caspari in his paper, and also in Leffeldt’s *Electrochemistry*, part i., p. 176.

diameter and immersed to the depth of 2.7 cm., giving a current density of $C_{100} = 35$ (*i.e.* the current density was at the rate of 35 amperes distributed over 100 square centimetres of surface). When the cathode had to be used as a sheet, the current density was $C_{100} = 3.5$, the surface being 10 times greater.

TABLE I.—SHOWING THE RELATIVE AMOUNTS OF ARSENIC REMOVED AS ARSENIURETTED HYDROGEN FROM (a) ARSENIUS ACID AND (b) ARSENIC ACID FROM 100 PARTS INTRODUCED, IN 25 MINUTES, WHEN WORKING WITH 3 AMPERES.

Metal.	(a) Arsenious Acid.	(b) Arsenic Acid.	Supertension (Volts).
Lead	100	98.5	0.64
Zinc	100	41	0.70
Cadmium	100	27	0.48
Tin	100	18.5	0.53
Silver	100	0	0.15
Graphite	100	4	...
Iron	93	40.5	0.08
Platinum	88.5	0	0.09
Aluminium	82	9	...
Gold	74	4	0.02
Cobalt	43.5	0	...
Nickel	42.5	0	0.21
Palladium	38	0	0.46

It will be seen that lead is the most efficient of the elements examined, and it is remarkable that it liberates arsenic as arseniuretted hydrogen (AsH_3) when it exists in solution as arsenic acid almost with the same velocity as when it exists as arsenious acid. Lead has a high supertension, but not so high as zinc; and whilst both of these metals liberate arsenic at about the same velocity from arsenious acid, lead liberates as AsH_3 $\frac{98}{100}$ ths of the arsenic present as arsenic acid in 25 minutes, whilst zinc only liberates $\frac{41}{100}$ ths of the amount present in the same time.

The next remarkable metal in the series is silver, with a very low supertension of 0.15, as compared with lead 0.64 and zinc 0.70. It liberates arsenic as AsH_3 from arsenious acid with about the same velocity as lead or zinc, whilst it is powerless to liberate any arsenic when it exists in the form of arsenic acid under the conditions of the experiment.

Again, palladium, which has a comparatively high supertension, is at the bottom of the series as regards efficiency in decomposing either arsenious or arsenic acid. Gold, which is at the bottom of the series as regards supertension, has the power of decomposing arsenious acid to the extent of three-quarters of the most efficient elements, and it

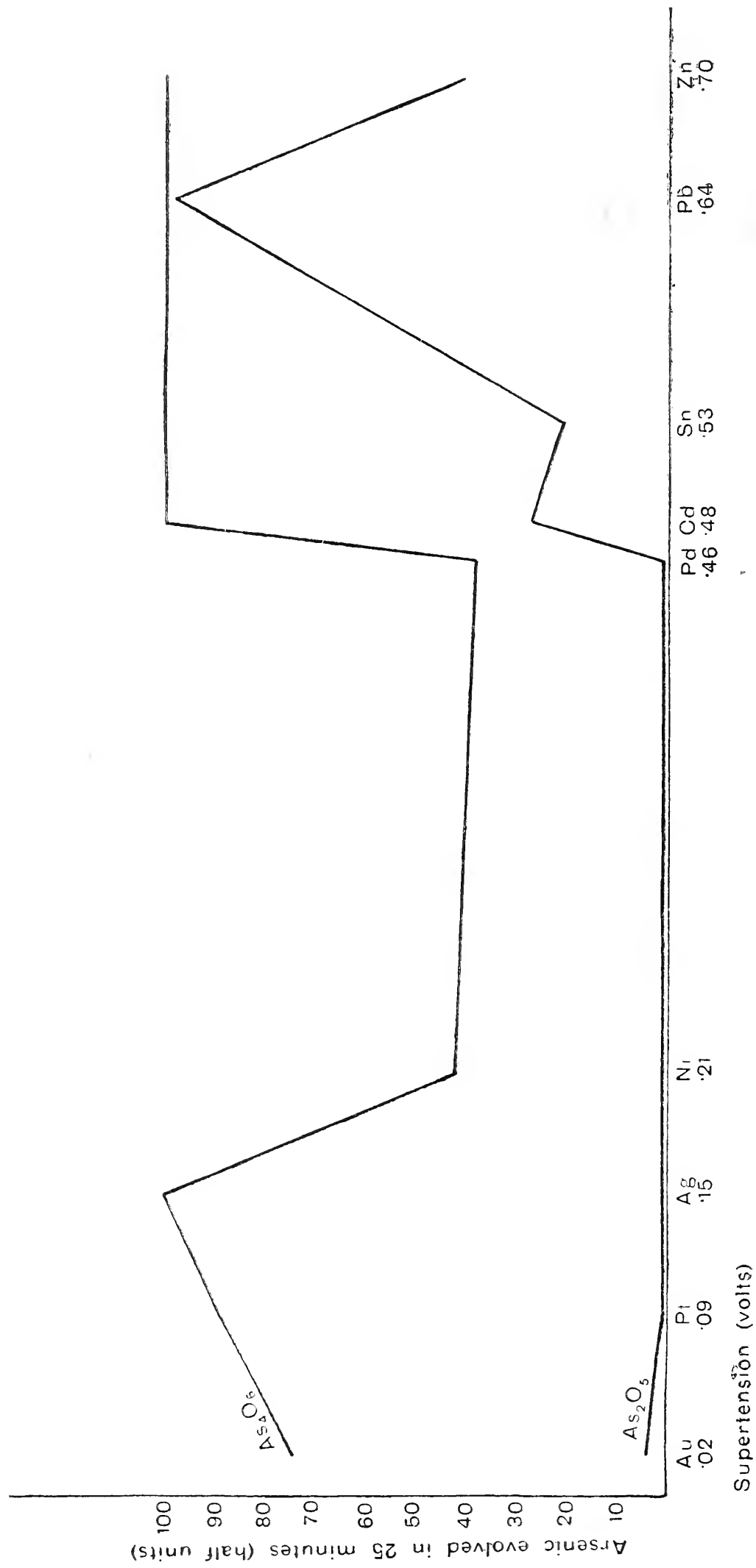


DIAGRAM I.

does decompose some arsenic acid, whilst palladium is powerless to decompose any.

Iron, in this respect, is a remarkable metal, with the lowest but one supertension; it decomposes arsenious acid nearly as rapidly as lead, and it stands second (next to lead) in its power of decomposing arsenic acid.

These experiments show, I think, that the power of any element to decompose arsenious and arsenic acid is a peculiarity of the individual metal, and appears to be independent of its supertension.

The diagram No. 1 shows the various elements arranged as abscissæ in their relative supertension values, whilst the ordinates show the amount of arsenic liberated from each in 25 minutes.

Table II. gives the results of the experiments made with the different elements as electrodes to discover the velocities at which the cathodes liberate the arsenic in periods of $2\frac{1}{2}$ minutes.

If we take the first five elements, it will be seen that the average velocities with which they decomposed arsenious acid into arseniuretted hydrogen are:—

TABLE II.—“UNITS” OF ARSENIC EVOLVED AS AsH_3 .

Intervals in Minutes .	$2\frac{1}{2}$	5	$7\frac{1}{2}$	10	$12\frac{1}{2}$	15	$17\frac{1}{2}$	20	$22\frac{1}{2}$	25	Totals.
Lead	9	13	8.5	7	5.5	4	2.25	1.5	1	0.5	52.25
Zinc	8.5	14.5	7.5	6	4	3	2.5	1.75	2	1	50.75
Cadmium	8.5	12.5	8.5	6.5	5	3.5	2.5	2.25	1.5	1.25	52.0
Tin	9	12.5	8	6.5	4.25	3.5	2.75	2	1.75	1.25	51.5
Silver	6	12	10.7	7.3	5	3.8	2.7	1.3	.8	.3	49.9

It will be seen that the first figure is always lower than the others; this was due to residual hydrogen in the apparatus, although it was made as small as possible, the total volume being 20 c.c. of air-space in tubes and apparatus with 30 c.c. of electrolyte (3 amperes liberate 21 c.c. per minute).

These results are averages of two or more experiments. Allowing for experimental error, a constant is obtained for these first five metals for a unimolecular velocity reaction, and this is one of the few electro-chemical reactions in which such velocities can be measured.

If a is the original concentration of the arsenious acid, and x the quantity transformed at the time t , the rate of transformation at that time will be, according to the unimolecular formula,

$$\frac{dx}{dt} = k(a - x),$$

where dx represents the very small quantity transformed in the very small

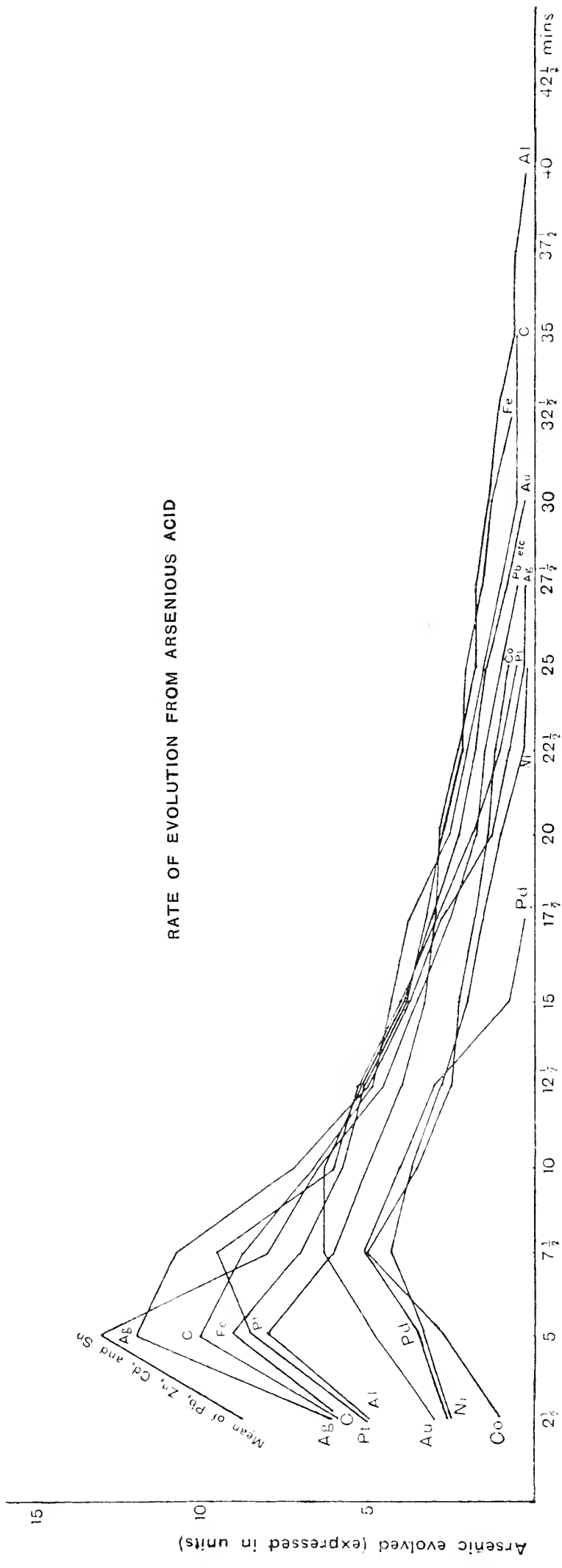


DIAGRAM 2.

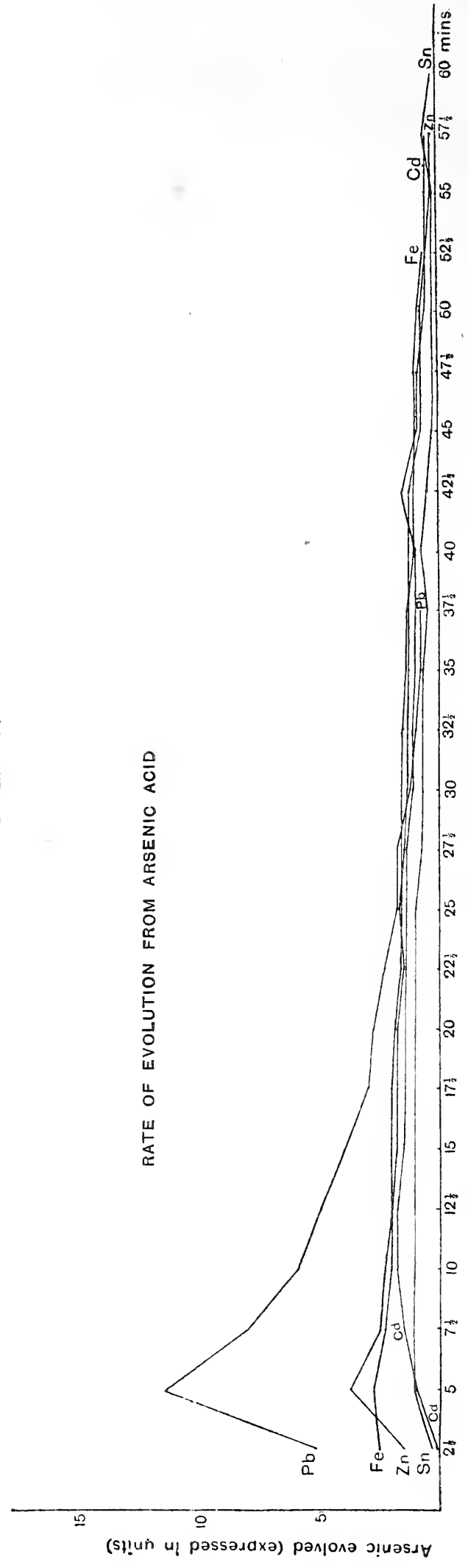


DIAGRAM 3.

interval dt starting at the beginning of the time t , and k is the coefficient of the velocity of the action. From this equation the integral calculus enables us to find a relation between x and t , the corresponding values for any stage of the reaction, in terms of the original concentration and the velocity constant.

This relation has the form,

$$\frac{1}{t} \log \frac{a}{a-x} = k.$$

It was observed that, if the amount of arsenic in the form of arsenic acid were increased in the cathode chamber, one arrived at a point when the cathode elements, which liberated no arseniuretted hydrogen under the ordinary conditions of the experiment, were capable of giving a small mirror on working the apparatus continually for 45 minutes, and the following Table shows the results:—

TABLE III.—ELEMENTAL ARSENIC IN THE FORM OF ARSENIC ACID IN 30 C.C. EXPRESSED IN—

	Units as above described.	Millionths of a gramme.
Nickel	20	10·8
Platinum	90	48·7
Silver	140	75·8
Palladium	191	103·5

An attempt was made to measure the velocity of the reduction when using a copper cathode. No arseniuretted hydrogen was given off until the copper had become covered with a brownish-black deposit which gave by two analyses of small quantities the formula Cu_3As .

Chapman and Law found the same deposit, which they regarded as one of elemental arsenic. On making other experiments we obtained no deposit on the copper and no arseniuretted hydrogen evolved.

With an electrolyte composed of 5 per cent. sodium hydrate solution we did get arseniuretted hydrogen evolved as follows:—

Minutes	2½	5	7½	10	12½	15	17½
Units obtained	0	2	3·5	2·5	2·5	1·5	1 = 13

We tried the element magnesium as a cathode in a similar way to the other metals, but found it dissolved rapidly in the electrolyte whilst the current was passing.

Table IV. shows the results of the different experiments, with the averages for each, also additions of the quantities evolved in each experi-

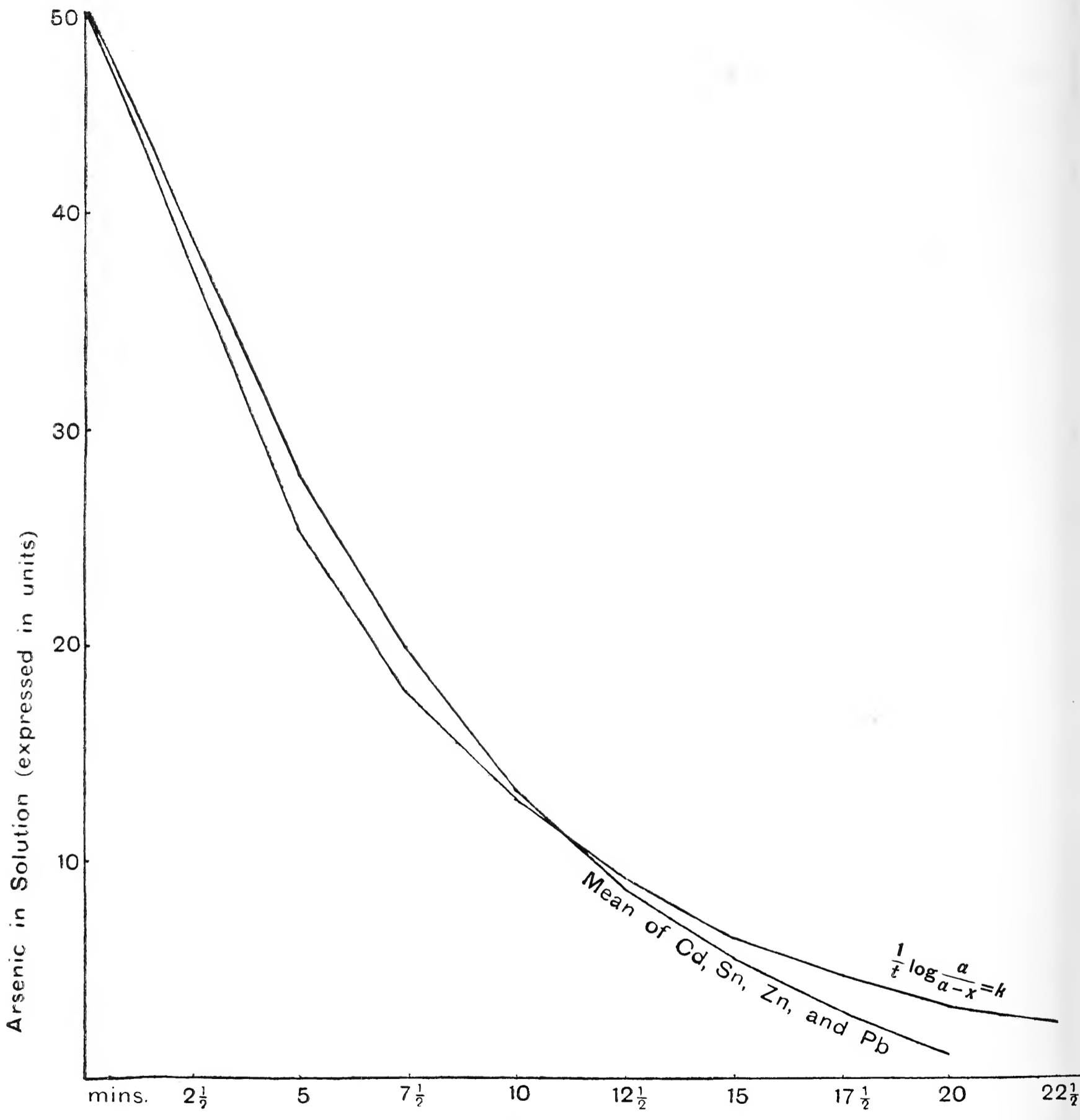


DIAGRAM 4.

ment to show the total amounts obtained as adjudged from the estimations of the quantities represented by each mirror, when compared with standard mirrors produced by eliminating arsenic from arsenious acid by lead and by zinc cathodes each advancing by one unit of arsenic, as above described.

Diagram 1 shows the elements whose supertensions were known to me arranged graphically as abscissæ, the ordinates showing the amounts of arsenic liberated (*a*) from arsenious acid and (*b*) from arsenic acid in 25 minutes from the different elements.

Diagram 2 shows the average results obtained from the different elements examined, when acting on arsenious acid; the ordinates representing units of elemental arsenic, and the abscissæ the time.

Diagram 3 is similar to diagram 2, with the exception that arsenic acid was used instead of arsenious acid, and it is drawn on a smaller scale than diagram 2.

Diagram 4 shows in one line the curve obtained by taking the average of the four elements, lead, zinc, cadmium, and tin, adding together the mirrors obtained in the first two periods of $2\frac{1}{2}$ minutes, and taking them as a 5-minutes interval, the others being taken at $2\frac{1}{2}$ -minutes intervals. The other line shows the curve obtained by calculating x for the corresponding periods when using the mean constant, k showing that for these four metals the reaction is practically a unimolecular one.

I have to acknowledge the services of my assistant, Mr L. L. Bircumshaw, for the perseverance and industry which he has exercised continuously during the last fourteen months in carrying out this investigation with me.

(Issued separately January 21, 1909.)

VII.—Preliminary Note on the Action of Nitric Anhydride on Mucic Acid. By Professor A. Crum Brown, F.R.S. and G. E. Gibson, B.Sc.

(MS. received October 12, 1908. Read July 21, 1908.)

IN the course of his researches on mucic acid, Professor Crum Brown found that this acid could not readily be nitrated in the ordinary way.

Mucic acid dissolves, to a certain extent, in a mixture of concentrated sulphuric and nitric acids, but on standing separates out again apparently unchanged. An exceedingly small quantity of a crystalline nitrate can, however, be obtained by pouring this mixture into a large volume of acid water and immediately shaking up with ether. On evaporating the ethereal layer the nitrate is obtained, but the yield is so small that this method of preparation is not practicable.

With nitric anhydride, however, a good yield of a crystalline nitrate is obtained.

The method is the same as that described in a previous paper for the preparation of tartaric acid dinitrate. Not more than 20 grams of mucic acid should be treated in one operation, and both the mucic acid and the nitric anhydride should be cooled in ice before mixing.

If these precautions are not taken, considerable decomposition of the nitrate may take place.

In one experiment, 16·3 grams of Merck's pure mucic acid were mixed with about 50 grams of nitric anhydride. The mixture was left overnight in an evacuated desiccator containing solid caustic soda. It was then treated with ether in a Soxhlet tube, the filter of which had previously been weighed. The ethereal extract was evaporated at the ordinary temperature with the help of a steam-jacketed "Geryk" vacuum pump.

Twenty-five grams of the nitrate were obtained, and 0·5 gram of unchanged mucic acid remained on the filter. If no mucic acid had been lost in the process of preparation, these figures would correspond to the formation of the trinitrate.

There is, however, always some loss caused by decomposition of the nitrate, so that there can be little doubt that the substance which is obtained is the tetranitrate of mucic acid.

Many different preparations were analysed, both by Walter Crum's mercury method and by combustion in vacuo, and the results of these

analyses confirm this conclusion. The figures vary with each new preparation, but correspond more closely with the tetranitrate than with the trinitrate.

Pure tetranitrate, free from ether, has not yet been obtained. On evaporating the ethereal solution, the substance crystallises in well-defined colourless needles, but on standing in air or in vacuo over concentrated sulphuric acid, these soon fall to a white powder. This change is apparently due to partial decomposition. The substance behaves in a similar manner when crystallised from alcohol.

The white powder thus obtained does not decompose further if kept perfectly dry, but in the air it soon begins to decompose, with evolution of nitric acid and oxides of nitrogen. Mucic acid is one of the products of this decomposition. Tartaric acid dinitrate behaves in a similar manner when exposed to moist air, tartaric acid being the chief product of decomposition. Mucic acid tetranitrate is, in many other respects, very similar to tartaric acid dinitrate. It is readily soluble in water, alcohol, and ether, but not in chloroform or benzene.

Treated with ammonium sulphhydrate, crystals of the sparingly soluble ammonium mucate are obtained.

On heating the nitrate it decomposes violently.

When an ethereal solution of the nitrate is allowed to stand in contact with a little water, drops of a syrupy substance, which is soluble in water, separate from the ethereal layer. In the course of a few days crystals of oxalic acid separate from the aqueous layer.

On the analogy of tartaric acid dinitrate, tartaric acid and tetraketo adipic acid should be formed as intermediate products of decomposition.

As yet, neither of these acids has been detected.

(Issued separately February 19, 1909.)

VIII.—Temperature Observations in Loch Garry (Inverness-shire).
With Notes on Currents and Seiches. By E. M. Wedderburn,
LL.B., W.S.

(MS. received November 2, 1908. Read November 2, 1908.)

DURING the first seven months of the year 1908 I had the good fortune to be living at Invergarry in Inverness-shire, with sufficient time at my disposal to make temperature observations in Loch Garry, and, through the generosity of Sir John Murray and Mr Laurence Pullar, the funds of the Lake Survey (Pullar Trust) were put at my disposal to defray the expenses of observation. I was further fortunate in securing the services of Mr Wm. Macdonald, Mount Pleasant, Fort Augustus, who was previously boatman to the Lake Survey on Loch Ness, and who proved himself a most painstaking and eager observer, and by whom by far the greater number of the actual observations were made.

One reason for my anxiety to obtain observations in Loch Garry was that it is a lake of the temperate class, of considerable size, and fairly uniform basin. A description of the lake will be found in the Lake Survey Reports, *Geographical Journal*, vol. xxx. p. 401, October 1907. The lake is divided into two parts. The main part of the loch is about 4 miles long, and is a simple basin with a maximum depth of about 220 feet. At the eastern end there is a shallow basin cut off from the main loch by a large promontory; the channel between the two basins is very narrow, and is so shallow that when the level of the loch is low it is difficult to get a rowing boat from one basin to the other. This smaller basin, which is about a mile in length, is little over 40 feet deep, and in summer large portions of it are covered with weeds. Nearly all the observations were made in the main basin, but occasional observations were made in the small loch for the sake of comparison, and no noteworthy differences in temperature between the basins were observed.

The chief point of interest in the observations was to discover whether temperature changes similar to those found in Loch Ness occurred in such a body of water as Loch Garry. The presence of the temperature seiche in Loch Ness has been fully established, but it was considered doubtful whether a temperature seiche could occur in a lake of moderate length and depth such as Loch Garry.

It was hoped that further use could be made of the electrical installation

which was used in Loch Ness, so that a continuous record of the temperature in the deepest part of the lake could be obtained. The Callendar Recorder was set up on the southern shore of the lake opposite the deepest part, but great difficulties were experienced with the platinum thermometer and the cable. The thermometer had been designed to stand a pressure of 300 feet of water, but it was found that the pressure at a depth of 200 feet was too much for it. A new covering was procured for the thermometer, but after that, it was found that the insulation of the cable, which was the one used on Loch Ness, had become very faulty, and the use of the Callendar Recorder had finally to be given up without any trustworthy records being obtained, and mercury thermometers had to be relied on for all observations.

Observations by means of mercury thermometers were carried on every day so far as weather and circumstances permitted. Readings of the air



FIG. 1.—Macdonald's Messenger.

temperature were made by a maximum and minimum thermometer (un-screened) placed on the shore of the loch. A buoy was moored over the deepest part of the loch, and most of the observations were made from a boat attached to this buoy. A light anchor was also carried in the boat, and by means of it the boat could be readily anchored at any part of the loch. In this way one person was able to make the observations, save in very stormy weather.

The ordinary reversing mercury thermometer was used with a messenger designed by Macdonald, which proved admirable. The usual dividing messengers are troublesome to use in cold weather, as one or both of the parts is very liable to get lost owing to the numbness of the observer's hands while working in cold water. Macdonald's messenger was formed by twisting a bar of metal into a spiral which could then be screwed on to the sounding line without much fear of losing it. The construction of the messenger sufficiently appears from the photograph which forms fig. 1.*

* For a description of methods of thermometry, see Dr H. R. Mill's "Clyde Sea Area," part iii. and plate i., *Trans. R.S.E.*, vol. xxxviii. p 3.

It is a great improvement on any of the messengers which I have seen, both in point of convenience and of cost.

The drainage area of Loch Garry is large, being 137·33 square miles—about 78·5 times the area of the loch, whereas in Loch Ness the drainage area is only 31·6 times the area of the loch. Assuming that the average rainfall over this area is 72 inches (a moderate estimate), the volume of water received by this drainage area in the course of a year is, roughly, 23,000 million cubic feet, and, assuming a 20 per cent. loss* by evaporation, absorption, etc., the amount of water flowing into Loch Garry in the course of a year is 18,400 million cubic feet, or nearly five times the normal capacity of the loch. These figures are given as showing the importance of rainfall in considering the temperature of lakes having a large drainage area, and in this connection it may be mentioned that the level of the loch rises as much as 25 feet above its summer level during floods. A rise of 10 feet in twenty-four hours is not uncommon. As a rule, however, the water which enters a lake is lighter than the bulk of water in the lake,† and spreads itself along the surface, and does not very greatly affect the abysmal temperature.

Periodical observations were made of the temperature of the upper River Garry, which is by far the largest stream entering the lake. For reference these observations are given in the following table:—

TABLE I.
Temperature of Upper River Garry in 1908.

Date.	Temperature.	Date.	Temperature.	Date.	Temperature.
Jan. 31	35·8	Mar. 16	38·0	May 27	49·5
Feb. 3	36·4	„ 19	39·2	June 2	63·7
„ 7	39·0	„ 23	40·2	„ 4	62·6
„ 19	38·8	Apr. 2	40·2	„ 11	53·4
„ 29	34·0	„ 6	41·0	„ 15	52·6
Mar. 3	35·9	„ 17	47·0	„ 20	51·2
„ 6	35·7	„ 28	43·3	„ 26	63·2
„ 9	35·0	May 6	49·2	July 3	72·0
„ 13	38·0	„ 13	48·5	„ 8	65·1
„ 14	38·5	„ 15	48·7	„ 17	57·8

The temperature of water in a shallow river responds rapidly to changes in the temperature of the atmosphere and to sunshine, and is therefore a fair guide to the quantity of heat which the lake is receiving—much more so than observations of extremes of heat and cold made with a maximum and minimum thermometer.

* See Lake Survey Report, *Geogr. Journal*, vol. xv. p. 342, April 1900.

† But see Forel's *Le Leman*, vol. ii. p. 358.

Approximate estimates were made of the quantity of heat stored in the lake for successive weeks during the period of observation. The method employed was to calculate roughly from the contour of the lake basin the quantity of water between the surface and a depth of 10 feet, between depths of 10 and 25 feet, 25 and 50 feet, and so on. The mean was then taken of all observations at the centre of the lake during each week, at the surface, 10 feet, 25 feet, 50 feet, etc.; and, on the assumption that these mean values were the means for all parts of the lake, a rough calculation was made of the quantity of heat passing into or out of the lake from week to week. As the observations from which the means were taken were all made about the centre of the lake, it is thought that the assumption that they are typical of the whole lake is sufficiently accurate for the present purpose.

Table II. gives the results of these calculations. The positive sign denotes that heat entered the lake, the negative sign that there was a loss of heat. A separate column gives the mean surface temperature for the successive weeks.

TABLE II.

Week ending	Gram calories $\div 10^{10}$	Mean Surface Temperature.	Week ending	Gram calories $\div 10^{10}$	Mean Surface Temperature.
Feb. 3		38.3	Apr. 27		41.3
„ 10	- 3200	37.8	May 4	+ 1200	41.5
„ 17	+ 4800	38.6	„ 11	+ 2800	43.7
„ 24	+ 1500	38.3	„ 18	+ 8000	46.0
Mar. 2	- 4200	38.2	„ 25	+ 11700	46.3
„ 9	- 2900	37.2	June 1	+ 13700	55.0
„ 16	- 1800	37.0	„ 8	+ 12800	56.0
„ 23	+ 2100	37.7	„ 15	+ 4700	52.3
„ 30	+ 2600	38.1	„ 22	- 800	53.1
Apr. 6	+ 4600	39.2	„ 29	+ 6600	58.5
„ 13	+ 4500	39.8	July 6	+ 8800	66.8
„ 20	+ 4100	40.9	„ 13	+ 200	61.4
„ 27	+ 3800	41.3	„ 20	- 1800	59.4

Comparing this table with the temperatures given in Table I., which are, as already explained, a rough measure of the heat supplied to the lake, it is seen that when the river temperature is below the surface temperature the lake is losing heat, and when it is above the lake is gaining in heat. Thus in the middle of February there was a very mild spell of weather during which there was an accession of heat, followed by very cold weather during which there was a considerable loss of heat, and the second week of March is the time at which the lake is at its coldest. Thereafter there is a steady accession of heat until the middle of June, the most rapid gain being during the warm and sunny weather which occurred about the end of May. About the middle of June there was actually a loss of heat. The weather during a large part of June was sunless, and during the week ending 22nd June the mean maximum air temperature was under 50° F. In the end of June and beginning of July there was again a spell of warm weather followed by an accession of heat, but during the sunless weather of the third week of July there was again a loss of heat. As the observations were discontinued shortly thereafter, I cannot say whether this loss was arrested, or whether the lake continued to lose heat during the autumn.

During the period of observation the difference between the greatest and least quantity of heat in the lake is about 9×10^{14} gram calories, which may be considered as the quantity of heat stored up in the lake during spring and summer. For Loch Ness I calculated that the quantity of heat stored up was 1.9×10^{16} gram calories, which is more than twenty times as much as in the case of Loch Garry, while the area of Loch Ness (through which the heat enters the lake) is only twelve times that of Loch Garry. According to Knott the solar energy supplied is equivalent to 60.73×10^{14} gram calories.

The disparity between the ratios of surface area to quantity of heat in Loch Ness and in Loch Garry may appear at first surprising, but when the quantity of water passing through Loch Garry, and the fact that it is the warmest surface water which flows out of the loch, is considered, it is apparent that more heat is lost by this means in the case of Loch Garry than in the case of Loch Ness, where the outflow is comparatively very small. Another reason for the smallness of the ratio in the case of Loch Garry is that owing to the less bulk of water the surface water rises in temperature much more rapidly than in Loch Ness, and accordingly the date at which the lake ceases to gain in temperature by conduction is much earlier. Moreover, though no sunshine observations are available, I think it is safe to state that the average amount of sunshine over Loch Garry is

less than over Loch Ness. The period of observation in Loch Garry was particularly sunless.

Some interesting observations on the freezing of the lake were made during the spring of the year. It does not follow that because a lake is of the temperate class that it will become frozen over every year. The classification depends solely upon the temperature of the water reaching the maximum density point. When that point is reached the lake is in a suitable condition for freezing, for water which is cooled at the surface by conduction from the atmosphere or by radiation remains at the surface and does not sink. In point of fact, however, the water in a lake is usually cooled considerably below the maximum density point before freezing occurs. There is usually an interval of time between the date at which the temperature of the water in a lake reaches the maximum density point and the date at which the temperature of the atmosphere falls sufficiently to cause freezing; and during this time the water all through the lake is falling in temperature, owing to the circulation of the water produced by winds; and though, as will be seen from Table II., the surface temperature fell as low as 37° F., freezing all over the lake did not take place. On several occasions, however, notably in the month of March, during frosty, clear nights large irregular patches of ice were formed in a single night, with a thickness in one case of as much as half an inch. Isolated patches of ice were formed in mid lake, and when observed in the early morning these appeared not to be connected with the shallow shore waters, where one would naturally expect freezing to take place most readily. These patches rapidly disappeared whenever a breeze sprang up to mix the thin surface layer, which had been rapidly cooled during the night, with the warmer waters below. It is thought that the irregularity of the patches of ice formed may be due to differences of surface tension in the lake, which would have an effect on the ripples produced by light winds, and thus on the mixing of the surface layers of the lake. Local differences at the surface may also cause differences in the rate of radiation from the lake and evaporation at the surface, and further contribute to the irregular formation of ice. In lakes with shallow shores and bays, freezing would probably occur first by the formation of a shore fringe of ice, gradually extending out into the lake; but the shores of Loch Garry are for the most part steep, and it is probable that when the lake becomes covered with ice (as it does in any ordinary winter) it will be covered all over by a thin sheet of ice in a single night in calm weather, and that if the weather continues calm this sheet will gradually grow thicker by conduction. For further discussion of the observations on formation of ice in Loch Garry, reference is made to

a paper on the "Freezing of Lakes" published in the Scottish Meteorological Society's *Journal* for 1908.*

The chief point of interest of the observations in the early months of the year, apart from the freezing of the lake, was the great uniformity in the temperature of the lake from top to bottom, even although considerable changes of temperature of the water took place from day to day.

TABLE III.

Date.	Temperature, ° F.		Difference.
	Surface.	200 Feet.	
1908.			
Jan. 30	38·3	38·2	- ·1
Feb. 5	37·8	37·7	- ·1
„ 12	38·5	38·7	+ ·2
„ 19	38·7	38·8	+ ·1
„ 28	38·0	38·2	+ ·2
Mar. 3	37·8	37·7	- ·1
„ 10	37·0	37·5	- ·5
„ 11	36·5	37·5	- 1·0
„ 12	37·2	37·5	+ ·3
„ 13	37·5	37·5	·0
„ 23	38·0	38·0	·0
Apr. 2 {	38·8	38·8	·0
(10 a.m.) }			
Apr. 2 {	38·9	38·9	·0
(3 p.m.) }			
Apr. 3	39·0	39·0	·0

Table III. gives a few observations which illustrate this point, and the explanation is to be found in the mixing of the water due to winds. Thus on 10th and 11th March there was a calm, with the result that we do not find this uniformity of temperature from top to bottom. The surface water has been cooled down, and has not mixed with the rest of the water in the lake. On the other hand, on 2nd and 3rd April there were strong winds, and the observations show how rapidly the heat which enters at the surface is distributed over the whole lake. This is, I think, very good evidence that the currents produced by wind are appreciable to the bottom of the lake in the spring of the year.

On to the end of April there was never any great variation in temperature from the surface to the bottom. On 29th April, observations at the surface gave a temperature of 41·1° and at the bottom 41·0°. There had, prior to that, been variations of as much as 1°, but these differences always disappeared before a strong wind. A reference to Table II. will show that

* Vol. xiv. p. 219.

in May the rate at which heat enters the lake begins rapidly to increase, and at the same time the temperature gradient in the lake increases, indicating that when the rate of heat-supply is sufficiently great, wind currents are not sufficiently strong to produce a thorough mixing of the water. Increasing differences of density also tend to make wind-produced currents less far-reaching, and the temperature gradient in the lake rapidly increases.

On 6th May the temperature at the surface and 200 feet was respectively 44.5° and 41.2° , there having been little variation in the bottom temperature during the previous ten days. Owing to stormy weather no observations were made on the 7th, but on the 8th the bottom temperature had risen to 42° —a rise of $.8^{\circ}$ —showing the influence of currents produced by winds. A more marked case occurred about a week later. On 15th May the surface temperature was 46.3° F., and at the bottom 42.3° ; again there had been very little variation in the bottom temperature. On the 16th and 17th the wind was very strong and no observations could be made, but on the 19th it was found that the bottom temperature had risen 1.7° to 44.0° . There was a continuance of moderately strong winds, with the result that the bottom temperature had risen to 44.5° by 22nd May, and to 45.0° by the 27th. Variable winds were experienced till the 5th of June, when the bottom temperature was only 45.2° , showing little variation for the previous nine days, but the wind increased on the 5th, and observations on 6th June showed a bottom temperature of 45.5° , and on the 10th 46.0° .

As will be seen in Table II., about the middle of June the lake, instead of gaining in heat, began to lose it. The result is the formation of a slight temperature discontinuity.

TABLE IV.

Observations at Centre of Loch Garry, June 1908.

	6th.	8th.	9th.	10th.	11th.	12th.	13th.	15th.	16th.	17th.	18th.	19th.	20th.
Surface	56.8	52.2	53.1	52.8	52.8	52.0	51.9	51.8	51.4	51.9	52.0	52.2	52.0
25	50.5	51.5	50.0	52.0	51.9	52.0	51.8	51.5	51.2	51.3	51.1	51.4	51.5
50	48.0	50.9	48.9	48.7	49.0	50.4	51.0	51.3	51.1	51.0	50.5	50.5	50.9
75	47.0	49.0	47.0	47.0	46.7	47.0	46.8	46.7	47.6	47.0	47.6	47.5	47.3
100	46.0	45.9	46.4	46.2	46.2	46.4	46.3	46.3	46.0	46.3	46.2	46.4	46.2
150	45.8	45.7	45.9	46.0	46.0	46.0	46.0	46.0	46.0	46.0	46.0	46.0	46.0
200	45.5	45.5	45.8	46.0	45.8	45.9	45.9	46.0	46.0	46.0	46.0	46.0	46.0

Table IV. gives some of the observations made at this time, and they show the gradual appearance of a discontinuity between 50 and 75 feet,

and confirm the opinion I have previously expressed,* that the discontinuity makes its appearance whenever there is no further gain of heat. During the time covered by the observations given in Table IV. the winds were light and variable, and this may have favoured the formation of the discontinuity. The bottom temperature remained about $46\cdot0^{\circ}$ until the end of June, and during the month of July there was a gradual and fairly continuous rise to $46\cdot5^{\circ}$, with moderate and variable winds. On 6th September, when an isolated observation was made, the temperature at the bottom was $47\cdot0^{\circ}$, showing that the gradual rise in bottom temperature was continued. The rise in bottom temperature after the formation of the discontinuity is very gradual, whereas before its formation changes in bottom temperature took place by small leaps following high winds. This gradual rise is probably due to the very slow currents which I believe to exist at the bottom of stratified lakes, and also to convection currents started by the temperature seiche. It also indicates that after the formation of the discontinuity layer in a lake the direct return currents take place above the discontinuity. Otherwise there would still be spasmodic increases of temperature following high winds.

It is unfortunate that I was unable to continue my observations on into the autumn, for though the discontinuity made its appearance during the cold weather of June, it did not become very distinct, and was masked somewhat by the warmer weather of July which followed. That the discontinuity did become more marked in the autumn is shown by an observation which I caused to be taken on 6th September, which shows a steep temperature gradient between 75 and 85 feet (see Table V.).

TABLE V.

Temperature on Loch Garry, 6th September 1908.

Surface	54·0° F.
25 feet	53·8
60 "	53·7
70 "	52·4
75 "	51·2
80 "	50·0
85 "	48·0
100 "	47·3
150 "	47·0
200 "	47·0

Observations made in August and September would probably have given good examples of the temperature seiche. As it was, hourly or two-hourly

* *Proc. R.S.E.*, vol. xxviii. p. 7.

observations were made at one end of the lake for consecutive days and nights on 10th and 11th July, and again from 23rd to 25th July. These observations are shown in figs. 2 and 3, and that there is an oscillation in progress with a period of about twelve hours is apparent. The

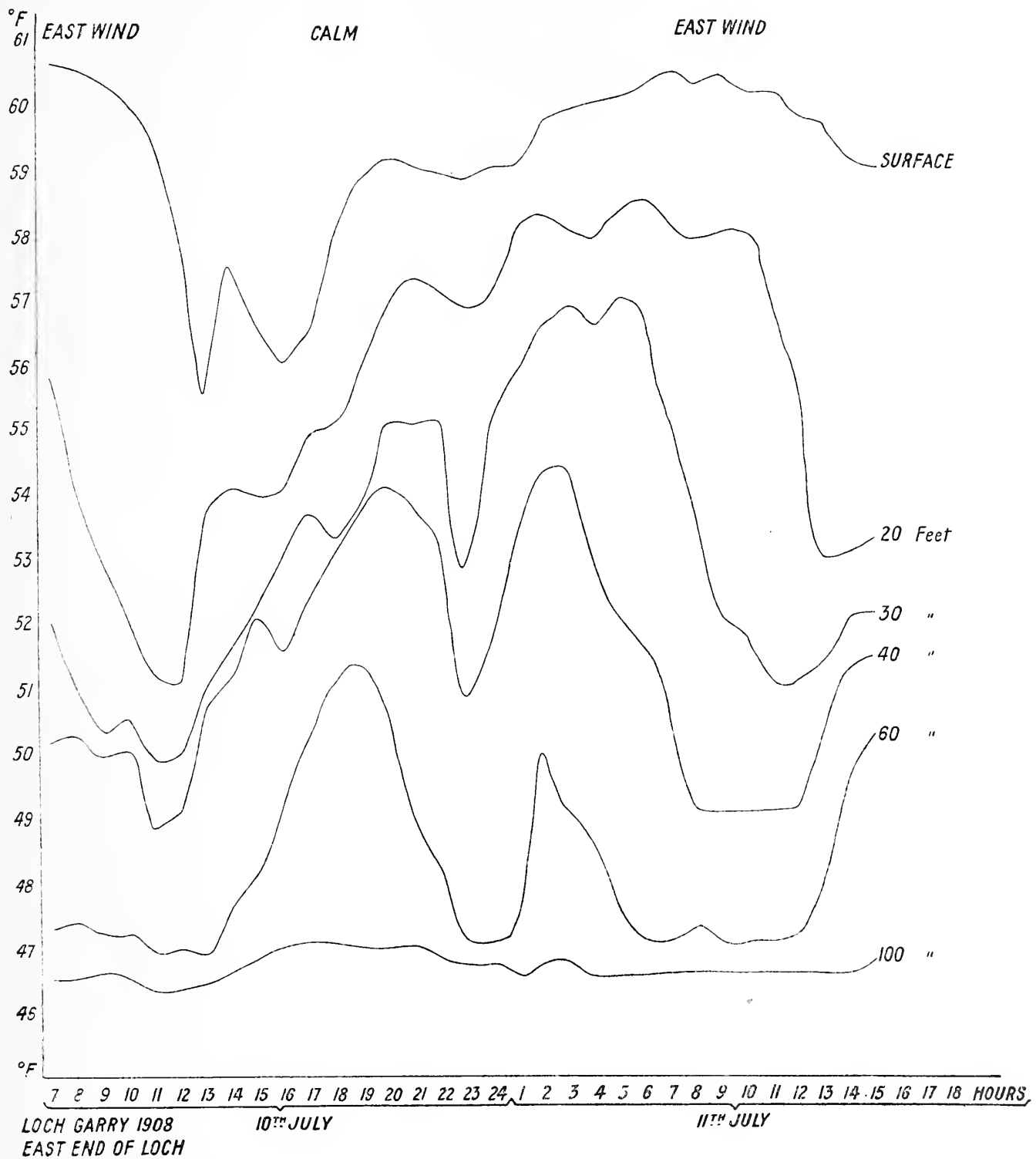


FIG. 2.

oscillations may appear to be very irregular, but the wonder is that they should exist at all, when the indefiniteness of the discontinuity is taken into consideration.

The majority of the observations were made at the centre of the lake, and thus at the node of the temperature seiche. Temperature oscillations

were, in fact, not observed there, which is additional evidence of the nature of the temperature changes. It was unfortunate that more continuous observations were not made at the ends of the lake, but the difficulty of observing day and night, with only two observers, in an open boat, and

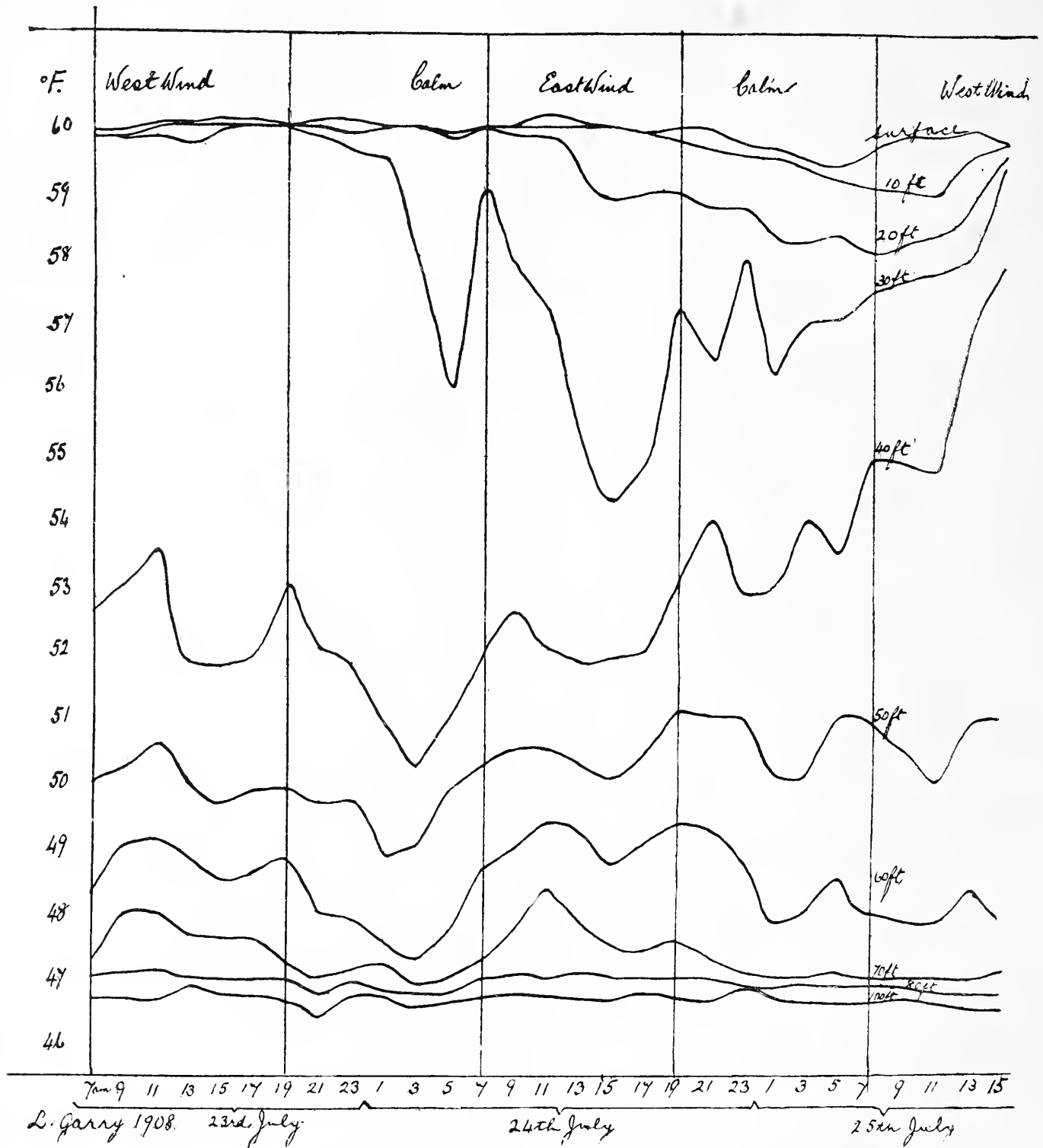


FIG. 3.

usually in pouring rain, was great. From the point of view of the temperature seiche it was unfortunate that most of the observations were made at the centre of the lake, but I was anxious to observe in as deep water as possible so as to follow the bottom temperature changes. It was intended by observing at both ends of the lake to show opposition in phase at the two ends, but the loss of a thermometer at the beginning of this series of

observations upset the experiment. By bicycling from one end of the lake to the other, however, and making observations alternately at the two ends of the lake, rough opposition in phase was shown. A change of wind during this series of observations further spoilt the experiment. A rough calculation from the approximate formula of the period of the temperature seiche gives a value of the same order as the observed value.

The important points in the Loch Garry observations may be summarised as follows:—

1. They give a complete series of observations in a temperate lake for the portion of the year during which the lake is gaining heat, and are comparable to the observations in Loch Ness, which is a tropical lake.

2. They show the apparently fortuitous manner in which freezing may take place in the larger temperate lakes.

3. They show how strong winds have the effect of producing currents at considerable depths.

4. They indicate that the formation of the discontinuity layer in a lake takes place whenever the surface layer begins to cool.

5. They indicate that after the formation of the discontinuity the return current is not directly appreciable at the bottom of a lake.

6. They show that the temperature seiche is not a phenomenon peculiar to Loch Ness or to very deep lakes; but that it is possible to have temperature oscillations even in small temperate lakes, and even when the discontinuity is not pronounced.

The observations suggest lines on which observations in other lakes of moderate size may be made. As a rule, observers have contented themselves with observations at wide intervals of time. It is difficult to interpret properly such isolated observations, and they may be very misleading; but when consecutive observations are made at near intervals of time the temperature changes can be closely followed, and a number of such series of observations in different lakes would be invaluable for the purpose of comparison. The observations in Loch Garry are given in the Appendix to this paper, so that they may be available for reference to other observers.

OBSERVATIONS WITH CURRENT METRE.

With a view to measure directly the currents occurring in a lake, I procured one of Ekman's propeller current metres.* The instrument

* The cost of this apparatus and the necessary gear was partially defrayed by a grant from the Moray Bequest of the University of Edinburgh.

and the method of working will be described in a subsequent communication to the Society dealing with observations on Loch Ness. One or two preliminary observations were made in Loch Garry to test the working of the metre, and as they are not without interest they may be summarised here.

I. 26TH MARCH 1908. LIGHT N.E. WIND.

Depth.	Velocity of Current. Cm. sec.	Direction (Mean).
1 foot	1·8	...
1 ft. 6 in.	2·3	S. 60° E.
200 ft.	1·5	N. 50° W.

Showing current at bottom in direction opposite to current at surface.

II. 28TH MARCH 1908. WEST WIND OF MODERATE STRENGTH.

Depth.	Rate of Current. Cm. sec.	Direction.
2½ feet	4·8	...
25 "	2·1	N. 30° E.
100 "	4·8	S. 65° E.
200 "	3·3	E.

Showing distinct current at bottom in opposite direction to the wind.

III. 31ST MARCH 1908. STRONG WEST WIND.

Depth.	Rate of Current. Cm. sec.	Direction.
Surface.	28·0	W.
1 foot	25·3	W. 22° N.
25 feet	3·9	W. 50° N.
100 "	3·7	E. 50° N.
200 "	3·1	E. 10° N.

This series of observations (fig. 4) shows very clearly a current to a depth of over 25 feet in the same direction as the wind, and below that a return current in the opposite direction.

SEICHE OBSERVATIONS.

A few seiche observations were made, as occasion permitted, by means of an index limnograph. Six of the best of these observations are shown in fig. 5, and it will be seen that the curves are most disappointingly irregular, and on several other occasions when observations were made there were no oscillations to measure. The period of the uninodal

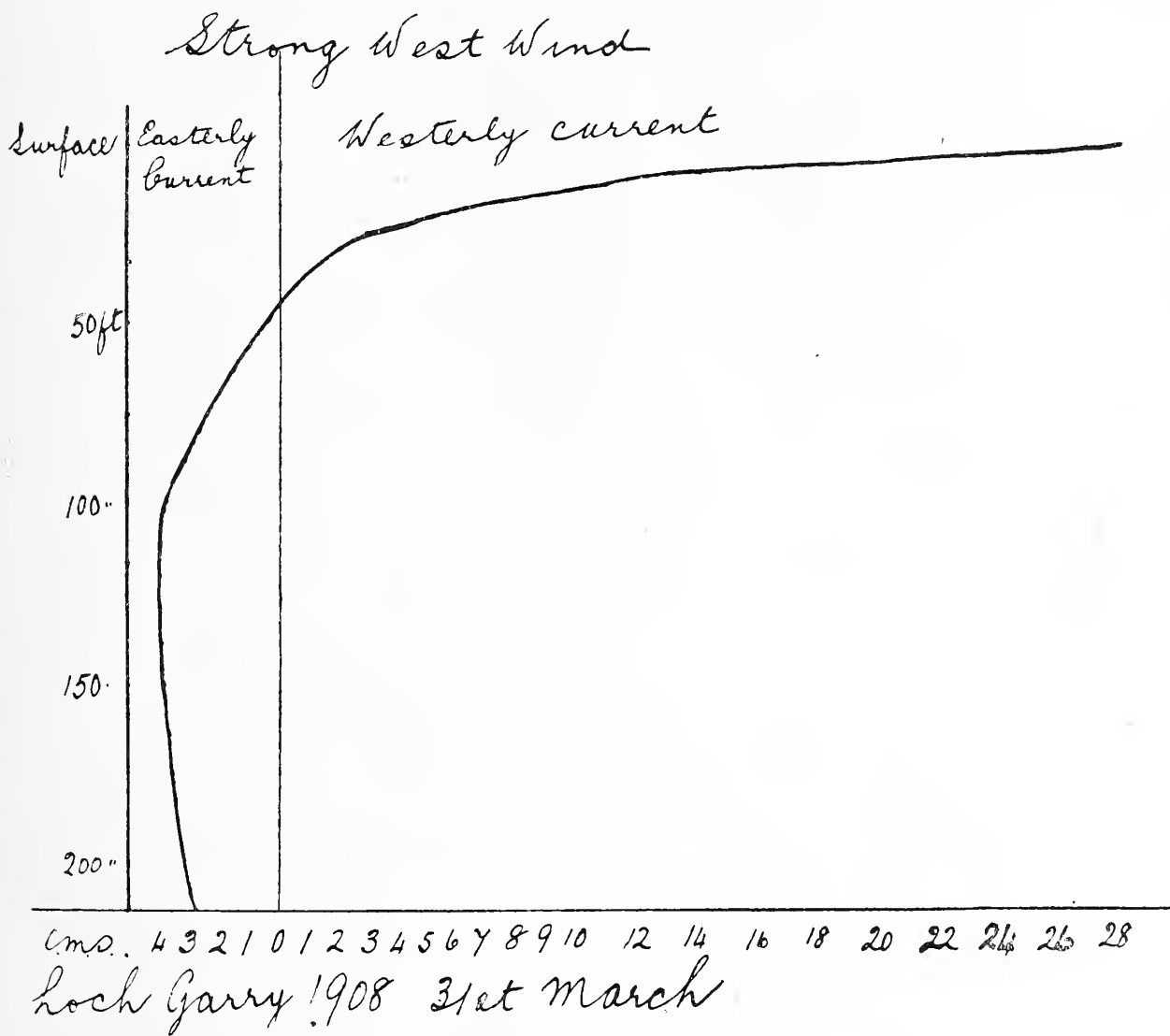


FIG. 4.

seiche is from 10.5 to 11.1 minutes, and of the binodal about 5.5 minutes. The greatest amplitude observed was 3 inches. The basin of Loch Garry is pretty regular, and its axis is practically straight; so that it is difficult to assign any cause for the behaviour of the lake. It may have been my misfortune always to observe at a time when seiches were irregular or absent, and of course with the index limnograph it is impossible to observe in stormy weather. But if the observations had been made in Loch Earn, it would hardly have been possible to observe on six occasions with such barren results as we had in Loch Garry. It may be that the small basin

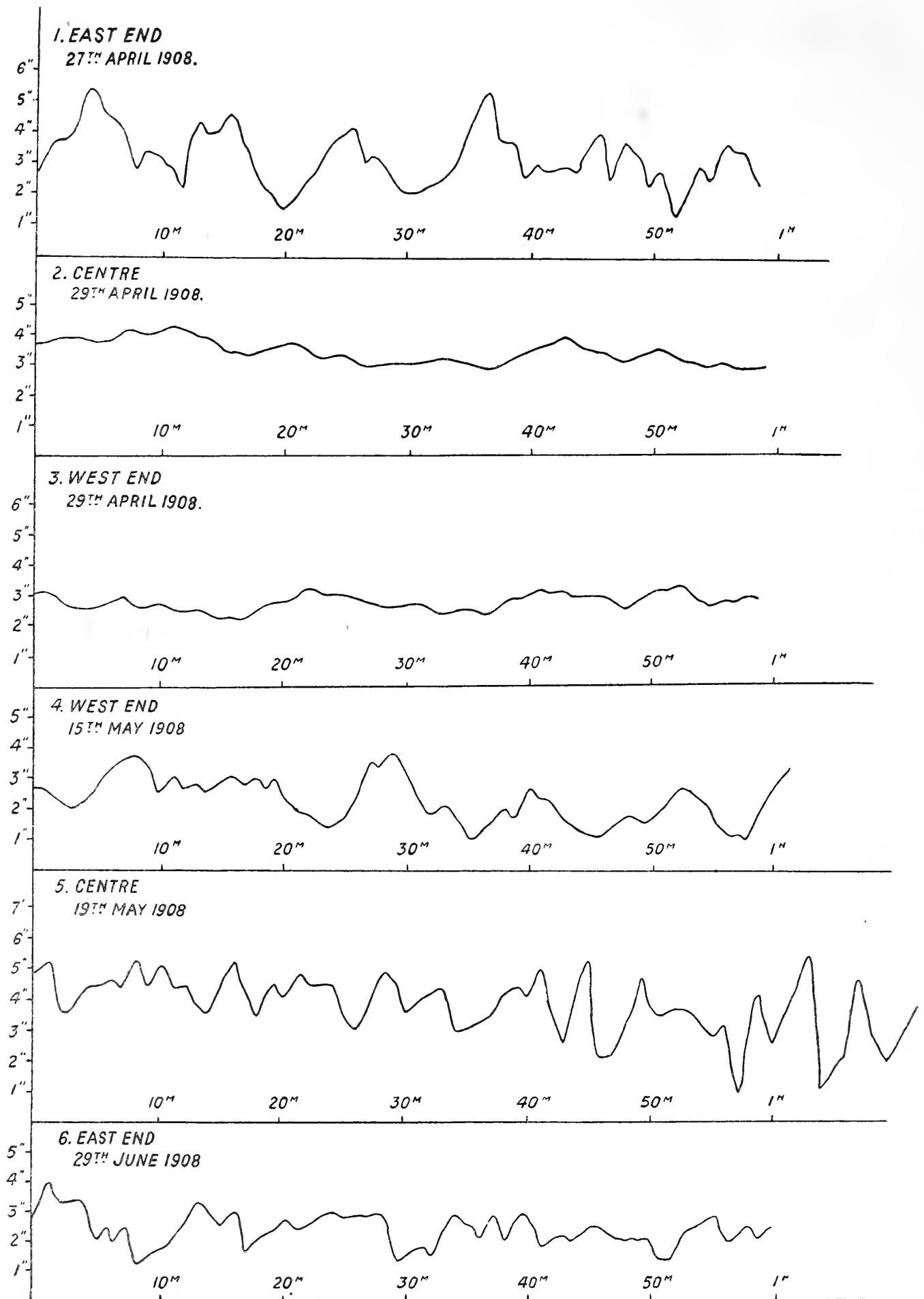


FIG. 5.

at the east end of Loch Garry has a disturbing and damping effect, but the channel between the two basins is so narrow and so shallow that I do not think this likely.

The lake runs almost due east and west, and the east end of the lake is little more than a mile from the line of the Great Glen, and the interference of disturbances passing along Glen Garry and the Great Glen may explain to some extent the irregular behaviour of the lake.*

* Since the above communication was made to the Society, I have received a copy of Dr F. M. Exner's paper, "Über eigentümliche Temperaturschwankungen von eintägiger Periode im Wolfgangsee" (*Sitz. Akad. der Wiss. Wien. math.-nat. Kl.*, Bd. cxvii., Jan. 1908), in which the author discusses observations made by electrical means in St Wolfgangsee which show a temperature seiche having a period of one day. In the case of this lake there appear to be three fairly distinct layers of water, with the result that the oscillations are rather complicated. There appears to be an oscillation of the uppermost layer of opposite phase to the oscillation in the lowest layer, while the middle layer acts as a sort of buffer between the top and bottom layers.

APPENDIX.

NOTE.—A map of the lake is published in the *Geographical Journal* for October 1907. In the following Tables the points of observation are denoted by letters, the positions of which are shown in fig. 6, representing a cross-section of the lake. The temperatures are measured in degrees Fahrenheit,

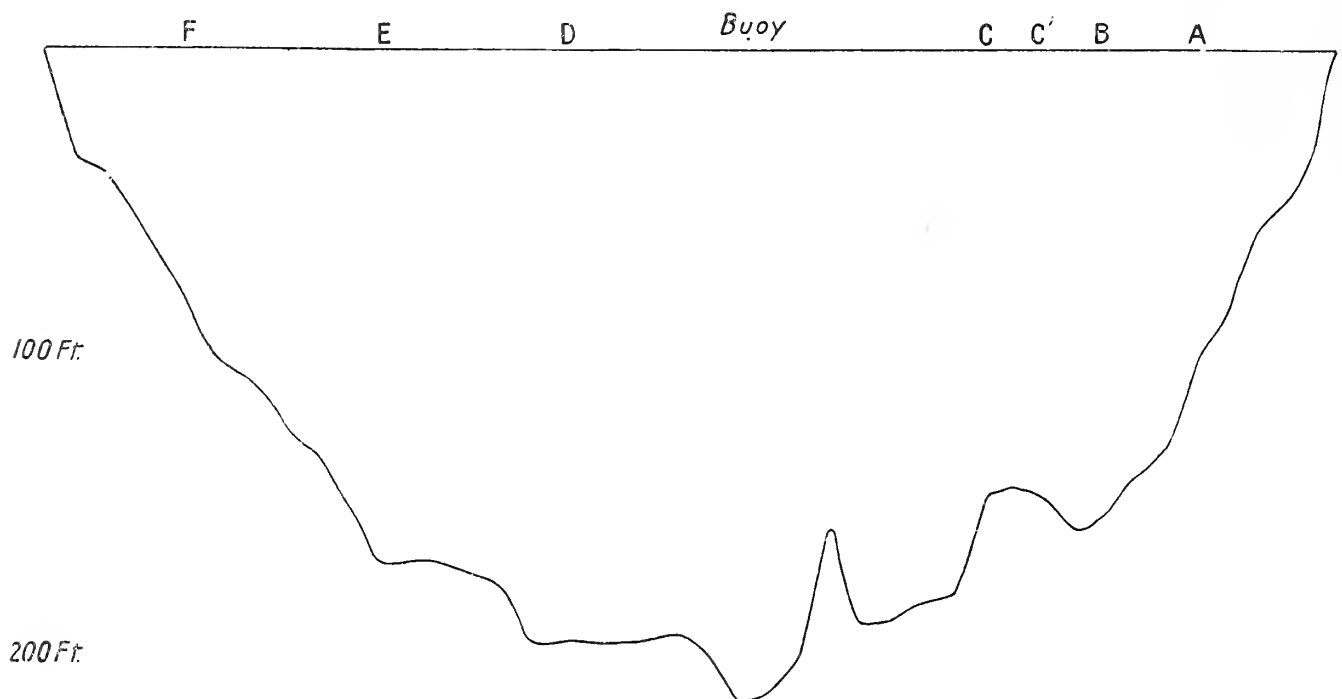


FIG. 6.

but for convenience of those accustomed to deal with the Centigrade scale, a conversion table is prefixed. A number appearing in brackets below an observation indicates the depth at which the observation was made, being different from the depth given at the head of the column. The wind force was estimated according to the Beaufort scale.

TABLE FOR CONVERSION FROM FAHRENHEIT TO CENTIGRADE SCALE.

° F.	0.	·1.	·2.	·3.	·4.	·5.	·6.	·7.	·8.	·9.
	°C.	°C.	°C.	°C.	°C.	°C.	°C.	°C.	°C.	°C.
35	1·7	1·7	1·8	1·8	1·9	1·9	2·0	2·1	2·1	2·2
36	2·2	2·3	2·3	2·4	2·4	2·5	2·6	2·6	2·7	2·7
37	2·8	2·8	2·9	2·9	3·0	3·1	3·1	3·2	3·2	3·3
38	3·3	3·4	3·4	3·5	3·6	3·6	3·7	3·7	3·8	3·8
39	3·9	3·9	4·0	4·1	4·1	4·2	4·2	4·3	4·3	4·4
40	4·4	4·5	4·6	4·6	4·7	4·7	4·8	4·8	4·9	4·9
41	5·0	5·1	5·1	5·2	5·2	5·3	5·3	5·4	5·4	5·5
42	5·6	5·6	5·7	5·7	5·8	5·8	5·9	5·9	6·0	6·1
43	6·1	6·2	6·2	6·3	6·3	6·4	6·4	6·5	6·6	6·6
44	6·7	6·7	6·8	6·8	6·9	6·9	7·0	7·1	7·1	7·2
45	7·2	7·3	7·3	7·4	7·4	7·5	7·6	7·6	7·7	7·7
46	7·8	7·8	7·9	7·9	8·0	8·1	8·1	8·2	8·2	8·3
47	8·3	8·4	8·4	8·5	8·6	8·6	8·7	8·7	8·8	8·8
48	8·9	8·9	9·0	9·1	9·1	9·2	9·2	9·3	9·3	9·4
49	9·4	9·5	9·6	9·6	9·7	9·7	9·8	9·8	9·9	9·9
50	10·0	10·1	10·1	10·2	10·2	10·3	10·3	10·4	10·4	10·5
51	10·6	10·6	10·7	10·7	10·8	10·8	10·9	10·9	11·0	11·1
52	11·1	11·2	11·2	11·3	11·3	11·4	11·4	11·5	11·6	11·6
53	11·7	11·7	11·8	11·8	11·9	11·9	12·0	12·1	12·1	12·2
54	12·2	12·3	12·3	12·4	12·4	12·5	12·6	12·6	12·7	12·7
55	12·8	12·8	12·9	12·9	13·0	13·1	13·1	13·2	13·2	13·3
56	13·3	13·4	13·4	13·5	13·6	13·6	13·7	13·7	13·8	13·8
57	13·9	13·9	14·0	14·1	14·1	14·2	14·2	14·3	14·3	14·4
58	14·4	14·5	14·6	14·6	14·7	14·7	14·8	14·8	14·9	14·9
59	15·0	15·1	15·1	15·2	15·2	15·3	15·3	15·4	15·4	15·5
60	15·6	15·6	15·7	15·7	15·8	15·8	15·9	15·9	16·0	16·1
61	16·1	16·2	16·2	16·3	16·3	16·4	16·4	16·5	16·6	16·6
62	16·7	16·7	16·8	16·8	16·9	16·9	17·0	17·1	17·1	17·2
63	17·2	17·3	17·3	17·4	17·4	17·5	17·6	17·6	17·7	17·7
64	17·8	17·8	17·9	17·9	18·0	18·1	18·1	18·2	18·2	18·3
65	18·3	18·4	18·4	18·5	18·6	18·6	18·7	18·7	18·8	18·8
66	18·9	18·9	19·0	19·1	19·1	19·2	19·2	19·3	19·3	19·4
67	19·4	19·5	19·6	19·6	19·7	19·7	19·8	19·8	19·9	19·9
68	20·0	20·1	20·1	20·2	20·2	20·3	20·3	20·4	20·4	20·5
69	20·6	20·6	20·7	20·7	20·8	20·8	20·9	20·9	21·0	21·1

LOCH GARRY.

Date.	Hour.	Position.	Wind.	Surface.	5	10	15	20	25	30	35
1908.											
Jan. 28	1.40 p.m.	Buoy.	...	38.7
" 29	10.30 a.m.	"	...	38.3
" 30	2.45 p.m.	"	W.N.W.	38.3
			Calm.								
" 31	12.40 "	F.	W. 4.	36.5
" 31	2.30 "	East end.	W. 4.	37.0
Feb. 3	12.45 "	F.	W. 4.	38.0
" 3	2.15 "	Buoy.	...	38.0
" 3	4.45 "	East end.	...	36.8
" 4	10.0 a.m.	Buoy.	Calm.	37.5
" 4	1.0 p.m.	East end.	...	37.4
" 5	2.0 "	Buoy.	W. 4.	37.8	37.8
" 5	3.45 "	Small loch.	...	38.0	37.8
" 6	11.40 a.m.	Buoy.	W. 4.	37.8
" 6	4.30 p.m.	"	...	37.5
" 6	5.15 "	East end.	Airs.
" 7	9.40 a.m.	"	...	37.8
" 7	1.40 p.m.	F.	W. 3.	38.0	38.0
" 7	3.0 "	Buoy.	...	38.0
" 7	3.50 "	East end.
" 8	2.10 "	Buoy.	W. 3.	38.0
" 11	5.0 "	"	W. 2.	38.3
" 12	10.30 a.m.	"	Calm.	38.3
" 12	4.0 p.m.	"	"	38.5
" 13	2.30 p.m.	"	Variable.	38.3
" 14	11.0 a.m.	F.	W. 2.
" 14	11.45 p.m.	Buoy.	W. 5-6.
" 14	2.30 "	East end.	...	38.2
" 15	10.20 a.m.	"	W. 2.	38.2
" 15	Noon	Buoy.	W. 4.	38.8
" 15	2.0 p.m.	F.	W. 3.	39.0
" 17	Noon	Buoy.	W. 1.	38.8	38.9
" 17	2.0 p.m.	C.	W. 2.	38.5	38.5
" 17	2.45 "	B.	W. 3.	38.5	38.5
" 17	3.15 "	A.	W. 3.	38.5	38.5
" 18	12.30 "	Buoy.	W. 3-4.	39.0	39.0
" 19	Noon	F.	W. 2.	38.8	38.8
" 19	12.45 p.m.	E.	...	38.8	38.8
" 19	1.0 "	D.	W. 1.	38.1
" 19	2.0 "	Buoy.	W. 1.	38.7
" 19	3.15 "	C.	...	38.7
" 19	3.45 "	A.	...	38.7
" 19	4.0 "	East Bay.	...	38.7	38.7
" 20	10.30 a.m.	A.	Calm.	38.8	38.8
" 20	11.30 "	Buoy.	W. 1.	38.8	38.8
" 20*	12.30 p.m.	D.	W. 3.	38.8
" 24	1.45 "	F.	W. 1.	38.6
" 24	2.20 "	E.	W. 1.	38.6

* On 21st and 22nd February

LOCH GARRY.

40	45	50	55	60	65	70	75	80	85	90	100	125	140	150	175	200
...	38.5
...	...	38.3	38.5
...	38.2
...
...	...	38.0
...
...	...	37.3	37.5	37.5	...	37.6
...
...	...	37.8	37.8	37.7
...	...	37.8
...	...	37.6	37.7
...	37.8	37.5
...	...	38.0
...	...	37.7
...
...	37.7	37.7
...	...	38.0
...	...	38.0	38.0	...	38.0
...	...	38.0	38.3	38.3
...	...	38.3	38.8	38.8	...	38.6
...	...	38.5	38.8	39.0	...	38.7
...	...	38.5	39.0	39.0	...	39.0
...	38.0
...	39.0
...	38.2
...	...	38.2
...	...	38.8	38.7	39.0	...	39.0
...	...	38.8	39.0
...	...	39.0	38.8	38.8	...	38.8
...	...	38.5	38.8	...	38.8
...	...	38.4	38.8	...	38.5
...	...	38.6	38.8
...	...	39.0	38.8	38.8	...	38.8
...	...	38.8	39.0
...	(96)
...	...	38.8	38.8	39.0
...	(168)
...	...	38.8	38.8	38.8	...
...	(180)	...
...	...	38.7	38.8	38.8	...	38.8
...	...	38.7	38.8	38.8
...	(143)
...	...	38.7	38.7
...
...	...	38.8	38.8
...	(86)
...	...	38.8	38.8	38.8	38.8	...	38.8
...	...	38.8	38.8	38.8
...	(170)
...	...	38.6	38.6
...	...	38.6	38.6
...	(160)

there was a west gale, force 6-9.

—continued.

40	45	50	55	60	65	70	75	80	85	90	100	125	140	150	175	200
...	...	38.6	38.7	...	38.7 (180)	...
...	...	38.7	38.7	38.7	...	38.7
...	...	38.6	38.6	38.6
...	...	38.6	38.6	38.6
...
...	...	38.6	38.6	38.6	38.6	...
...	...	38.0	38.0	38.0	...	38.2
...	...	38.0	38.0
...	...	38.0
...	...	38.0	38.0	38.0
...	...	38.0	38.0	38.0
...	...	37.8	37.8	37.8	...	37.8
...	...	38.0	38.0	38.0
...	...	37.4	37.4	37.5
...	...	37.0	37.4
...
...	...	37.9	38.0
...	...	37.8	37.8	37.9
...	...	37.7	37.7	37.8	...	37.7
...	...	37.8	37.8	37.8	...	37.8
...	...	37.9	38.0
...	38.0
...	...	37.5	37.6
...	...	37.4	37.8	...	37.8
...	...	37.5	37.7	37.8	38.0 (180)	...
...	...	37.9	38.0	37.9	...	38.0
...	...	38.0	38.0
...	...	37.8	37.9	37.9	...	37.9
...	...	37.5	37.5	37.5	...	37.7
...	37.5
...	...	37.7	37.5	37.8
...	...	37.2	37.4	37.5	...	37.5
...	...	37.0	37.2	37.4	...	37.5
...	...	37.3	37.5	37.5	37.5 (190)	...
...	...	37.0	37.4	37.5	...	37.4
...	...	37.3	37.5	37.5 (130)
...	...	37.5	37.5	37.7
...	...	37.3	37.5
...	...	37.0	37.4
...	...	37.0	37.5
...	...	37.2	37.4	37.4	...	37.5
...	...	37.2	37.4	37.4	...	37.7
...	...	37.5	37.5	37.5	37.8 (195)	...
...	37.5	37.5
...	...	37.9	37.8
...	...	37.3	37.4	37.5	...	37.5
...	...	37.3	37.5	37.5	...	37.8
...	38.0
...	...	37.7	37.5	37.6
...	...	37.4	37.4	37.8	...	37.5

—continued.

40	45	50	55	60	65	70	75	80	85	90	100	125	140	150	175	200
...	...	37.4	37.5	37.6	...	37.7
...	...	37.8	38.0
...	...	37.7	37.7	...	38.0
...	...	37.4	37.5	37.5
...	...	37.3	37.5	37.5	...	37.6
...	...	37.4	37.5	...	37.5
...	...	37.4	37.4	...	37.5
...	...	37.5	37.5
...	...	37.6	37.7	...	37.8
...	...	37.5	37.5	37.7	...	38.0
...	...	37.5	37.5	37.7	...	37.8
...	...	37.5	37.7
...	...	37.4	37.4	37.6
...	...	37.4	37.5	(120)
...	...	37.5	37.5	37.6
...	...	37.5	37.5	37.7
...	...	37.5	37.8	(145)
...	...	37.8	37.8	38.0	...	38.0
...	...	37.8	37.8	37.8
...	...	37.8	37.9	...	37.9
...	...	38.0	38.5
...	...	37.7	37.8	37.8	...	37.9
...	...	37.8	37.8	37.8	37.9	...
...	...	38.0	38.0	...	38.4
...	...	38.0	38.0
...	...	37.8	37.8	37.9	...	38.0
...	...	37.0	37.8	38.0	...	37.8
...	...	37.8	37.8	...	37.8
...	...	37.8	37.8	...	(130)
...	...	37.8	38.0	37.8	...	38.0
...	...	37.8	37.8	37.9	...	37.8
...	...	38.1	38.0
...	...	37.9	37.9	...	37.8
...	...	38.0	38.0	...	(130)
...	...	38.0	38.0	38.0
...	...	38.0	38.0	...	38.0	38.0
...	...	38.0	38.0
...	...	38.0	38.0	38.0	...	38.0
...	...	38.0	38.0	38.0	...	38.0
...	...	38.0	38.1	38.2	...	38.2
...	...	38.2	38.2	38.2	...	38.2
...	...	38.2	38.2	38.2	...	38.2
...	...	38.3	38.2	38.2	...	38.2
...	...	38.8	38.8	38.7	...	38.7
...	...	38.8	38.8	38.8	...	38.8
...	...	39.0
...	...	38.9	38.8	38.8	...	38.9
...	...	39.0	39.0	39.0	...	39.0
...	...	39.0	39.0
...	...	39.0	39.0	...	39.0
...	...	39.0	39.0	...	39.0
...	...	39.0	39.0	...	(130)

LOCH GARRY

Date.	Hour.	Position.	Wind.	Surface.	5	10	15	20	25	30	35
1908. May 8	11.15 a.m.	D.	Calm.	43.0	42.3
" 8	Noon	E.	"	42.5	42.0
" 8	12.30 p.m.	F.	...	43.3	42.0
" 8	1.30 "	Buoy.	Calm.	44.0	42.7
" 8	4.30 "	C.	...	44.0	43.2
" 8	5.0 "	A.	E. 1.	44.4	43.0
" 9	9.0 a.m.	Buoy.	W. 3-4.	43.3	43.2
" 12	10.0 "	"	Calm.	45.0	44.4
" 12	12.30 p.m.	"	"	46.0	44.4
" 12	2.15 "	"	E. airs.	45.7	44.3
" 12	3.45 "	B.	...	45.2	44.5
" 12	4.15 "	A.	...	45.3	44.4
" 13	11.30 a.m.	"	W. 1.	45.6	44.3
" 13	Noon	B.	...	45.7	44.4
" 13	12.30 p.m.	C.	...	46.0	44.2
" 13	1.45 "	Buoy.	N. 1.	47.0	44.5
" 13	3.15 "	D.	...	47.1
" 13	4.0 "	E.	...	49.0	44.2
" 13	4.30 "	F.	...	48.7	44.2
" 14	10.30 a.m.	Buoy.	E. 1.	46.3	44.6
" 14	3.0 p.m.	"	...	45.9	44.1
" 15	10.30 a.m.	"	E. 1.	46.3	44.5
" 15	Noon	D.	E. 2.	47.0	46.1
" 15	12.45 p.m.	E.	E. 1-2.	47.8	46.0
" 15	1.30 "	F.	E. 1-2.	48.8	47.8
" 19	10.15 a.m.	Buoy.	W. 2-3.	46.3	46.1
" 19	4.45 p.m.	"	W. 3-4.	45.8	45.8
" 21	1.30 "	"	W. 2-3.	46.2	46.0
" 21	6.15 "	"	W. 1-2.	46.2	46.1
" 22	8.30 a.m.	"	W. 2-3.	46.0	46.0
" 22	5.0 p.m.	"	W. 2-3.	46.7	46.6
" 23	10.45 a.m.	"	W. 1-2.	46.5	46.4
" 23	11.30 "	C.	...	47.0	46.8
" 23	12.15 p.m.	A.	...	47.1	46.5
" 25	12.15 "	Buoy.	W. 3-4.	46.8	46.7
" 27	11.30 a.m.	F.	W. 1.	47.7	46.0
" 27	12.15 p.m.	E.	...	47.0	46.4
" 27	1.0 "	D.	...	47.0	46.8
" 27	2.15 "	Buoy.	W. 1-2.	47.5	47.3
" 27	3.0 "	C.	...	47.8	47.4
" 27	3.30 "	B.	...	48.2	47.5
" 27	4.10 "	A.	...	48.4	48.0
" 28	12.15 "	Buoy.	W. 1.	49.7	47.4
" 28	5.45 "	"	E. 1-2.	52.3	47.5
" 29	Noon	"	E. airs.	54.4	47.7

—continued.

40	45	50	55	60	65	70	75	80	85	90	100	125	140	150	175	200
...	...	42.3	42.3	42.0	42.0	...
...	...	42.0	42.0	41.9	(170)	...
...	...	42.0	41.9
...	...	42.5	42.4	42.3	...	42.0
...	...	43.0	42.7	42.2
...	42.8	(130)
...	...	43.0	42.2	42.0	...	41.9
...	...	43.8	42.4	42.3	...	42.3	...	42.1
...	...	44.0	42.4	42.4	...	42.4	...	42.1
...	...	44.0	42.5	42.3	...	42.1
...	...	43.5	42.4	...	42.1
...	...	43.0	42.3	...	(145)
...	...	44.0	42.4
...	...	43.2	42.4	42.2
...	...	43.3	42.3	42.1
...	...	43.7	42.8	42.5	...	42.4
...	...	43.7	42.4	42.4	42.4	...
...	...	43.5	42.5	...	42.4	...	(170)	...
...	...	43.9	42.4
...	...	43.9	43.0	42.5	42.5	...	42.4	42.4	42.3
...	...	44.0	42.8	42.5	42.4	...	42.4	42.3	42.3
...	...	43.9	43.0	42.6	42.5	...	42.4	...	42.3
...	...	44.1	42.5	42.3	42.3	...
...	...	44.4	42.4	42.3
...	...	44.9	43.4	42.4
...	...	45.5	45.0	44.6	...	44.0
...	...	45.8	45.0	44.5	...	44.0
...	...	45.9	45.4	44.5	...	44.4
...	...	46.0	45.4	44.5	...	44.2
...	...	46.0	45.3	44.9	...	44.3
...	...	46.2	45.6	44.8	...	44.5
...	...	46.2	46.0	45.0	44.8	...	44.5
...	...	46.2	46.0	44.7	44.7
...	...	46.3	45.1	44.8
...	...	46.7	45.9	45.0	...	44.3
...	...	45.4	45.1	45.0
...	...	46.2	46.0	45.5	45.0	...	45.0
...	...	46.5	46.0	45.3	...	45.1	45.0	...
...	...	46.8	46.6	45.6	45.0	...	45.0
...	...	46.8	46.7	45.4	45.0
...	...	47.0	46.7	45.5	45.0
...	...	47.0	46.7	46.0
...	...	46.8	45.8	45.6	45.3	...	45.0
...	...	46.9	46.5	45.9	45.5	...	45.2
...	...	46.9	45.9	45.2	...	45.2

LOCH GARRY

Date	Hour.	Position.	Wind.	Surface.	5	10	15	20	25	30	35
1908.											
May 29	3.0 p.m.	Buoy.	...	61.0	...	49.4	47.5
" 30	9.45 a.m.	"	Calm.	57.4	...	50.0	47.4
" 30	1.15 "	"	E. 1.	59.4	...	49.5	47.3
" 30	6.15 p.m.	"	Calm.	56.1	...	48.0	47.3
June 1	10.0 a.m.	"	E. 1-2.	55.3	...	53.4	48.4
" 1	12.15 p.m.	D.	E. 2-3.	56.0	...	55.4	48.6
" 1	6.30 "	Buoy.	E. 1-2.	56.6	...	52.2	49.0
" 2	9.0 a.m.	"	E. 1.	57.3	...	53.8	50.1
" 3	10.0 "	"	W. 1-2.	57.4	...	56.2	49.4
" 3	3.30 p.m.	D-E.	W. 0-1.	60.1	...	55.0	...	50.1	49.0
" 3	4.45 "	Buoy.	W. airs.	60.3	...	57.9	49.5
" 4	3.30 "	"	W. 3-4.	56.5	...	55.3	54.0
" 5	1.0 "	"	W. 3.	53.0	...	53.0	51.4
" 5	3.0 "	E.	W. 3.	49.0	...	48.4	48.0
" 5	4.0 "	D.	W. 4.	52.2	...	49.6	49.2
" 6	10.0 a.m.	D.	E. 1	56.0	...	53.4	51.0
" 6	11.0 "	E.	...	56.0	...	54.0	52.7
" 6	Noon	F.	W. 1.	55.3	...	54.1	53.7
" 6	1.30 p.m.	Buoy.	W. 1.	56.8	...	53.6	50.5
" 6	2.30 "	C.	...	56.0	...	52.8	49.6
" 6	3.30 "	B.	...	57.5	...	53.0	50.2
" 6	4.0 "	A.	...	57.6	...	53.0	50.5
" 6	4.15 "	A-E. end	...	57.3	...	55.2	50.7	...	49.5
" 8	8.0 a.m.	Buoy.	W. 3.	52.9	...	52.9	52.8
" 8	7.30 p.m.	"	W. 2.	52.2	...	52.2	51.5
" 9	11.30 a.m.	A.	W. 1-2.	54.0	...	53.5	54.5
" 9	Noon.	B.	W. 2-3.	53.7	...	53.3	49.8
" 9	4.0 p.m.	Buoy.	W. 2-3	53.1	...	52.8	50.0
" 10	10.0 a.m.	"	W. 2-3.	52.4	52.3	52.2	52.2
" 10	1.0 p.m.	"	W. 3.	52.8	52.3	52.2	52.0
" 10	2.30 "	D.	W. 3-4.	51.4	51.2	51.2	51.1
" 10	5.0 "	Buoy.	W. 3-4.	52.5	52.4	52.3	51.8
" 11	8.30 a.m.	"	W. 1.	52.8	52.7	52.5	51.9
" 11	11.0 "	F.	W. 1.	51.2	51.0	51.0	50.9
" 11	Noon.	E.	W. 1.	51.8	51.7	51.5	51.0
" 11	1.0 p.m.	D.	W. 2.	52.0	51.8	51.8	51.4
" 11	2.0 "	Buoy.	W. 2.	52.0	52.0	51.8	51.9
" 11	3.0 "	C.	W. 2.	52.8	52.9	52.9	52.4
" 11	4.0 "	B.	W. 2-3.	53.4	53.3	53.2	53.0
" 11	5.0 "	A.	W. 1.	54.0	53.8	53.6	53.0
" 12	11.0 a.m.	Buoy.	W. 2-3.	52.0	52.0	52.0	52.0
" 12	5.30 p.m.	"	W. 3-4.	52.0	52.0
" 13	8.0 a.m.	"	W. 3-4.	51.9	51.9	51.9	51.8
" 13	Noon	"	W. 3-4.	51.9	51.9	51.9	51.8
" 15	10.0 a.m.	"	W. 1.	51.8	51.8	51.7	51.5
" 15	11.0 "	D.	...	51.8	51.7	51.5	51.4
" 15	Noon	E.	...	51.8	51.6	51.5	51.0
" 15	1.0 p.m.	F.	E. 1.	52.0	51.9	51.5	51.1
" 15	5.0 "	Buoy.	E. 2-3.	52.0	51.8	51.8	51.5

--continued.

40	45	50	55	60	65	70	75	80	85	90	100	125	140	150	175	200
...	...	46.9	45.9	45.2	...	45.1
...	...	46.9	46.1	45.4	45.1	...	45.1
...	...	46.6	46.0	45.4	45.1	...	45.0
...	...	46.7	45.8	45.2	...	45.1
...	...	47.1	46.2	45.4	45.2	...	45.2	...	45.2
...	...	46.9	45.9	45.4	45.0
...	...	47.0	46.3	45.8	45.2	...	45.1
...	...	47.0	46.3	45.6	45.3	...	45.1
...	...	47.0	46.3	45.6	45.4	...	45.2	45.2	45.1
...	...	47.0	45.9	45.3	45.1	...
...	...	47.0	46.3	45.5	45.3	...	45.2	...	45.2
...	...	48.4	46.5	45.7	45.3	...	45.1
...	...	47.4	46.2	45.5	45.3	...	45.2
...	...	46.4	45.7	45.5	45.4
...	...	47.8	47.1	46.0	45.4	...	45.2
...	...	48.5	47.2	46.2	45.9	...	45.8	45.7	(195)
...	...	47.8	46.2	46.0	46.0	...	45.4	45.4	...
...	...	48.0	46.2	45.9
...	...	48.0	47.0	46.0	45.8	...	45.8	45.5	45.5
...	...	48.0	47.0	46.3	45.3	45.1
...	...	49.0	47.3	46.9	46.0	45.5
...	...	48.8	47.3	46.8
...
...	...	47.2	45.7	45.6	...	45.5
...	...	50.9	49.0	45.9	45.7	...	45.5
...	...	49.5	47.6	46.7
...	...	48.1	46.8	46.0	45.5	...	45.5
...	...	48.9	47.0	46.4	45.9	45.8
...	...	48.3	46.0	46.0	...	45.9	45.9	45.9
...	...	48.7	47.0	46.2	46.0	...	46.0	46.0	46.0
...	...	48.7	46.4	46.2	46.0	...	45.9	45.9	...
...	...	51.2	48.0	46.4	46.0	...	45.9	45.9	45.9
...	...	49.0	46.7	46.2	46.0	...	46.0	45.9	45.8
...	...	48.0	46.8	46.7
...	...	48.7	46.6	46.0	45.9	...	45.9
...	...	49.0	46.9	46.0	45.9	...	45.7	45.7	...
...	...	50.0	46.4	46.0	...	45.9
...	...	50.6	47.3	46.1	...	45.8
...	...	51.0	47.7	46.4	46.0	45.8
...	...	51.0	47.7	46.4
...	...	50.4	47.0	46.4	46.0	...	46.0	45.9	45.9
...	...	51.8	47.2	46.5	46.0	...	46.0	46.0	45.9
...	...	51.0	...	47.8	46.8	46.3	46.0	...	46.0	45.9	45.9
...	...	51.5	...	49.5	47.1	46.4	46.0	...	46.0	45.9	45.9
...	...	51.3	47.0	46.7	46.3	46.1	...	46.0	...	46.0
...	...	51.2	...	49.4	46.7	46.2	46.0	...	46.0	...	46.0
...	...	50.0	...	48.0	46.7	46.4	46.0	...	46.0
...	...	49.4	...	47.8	47.0	46.2
...	...	51.0	...	47.9	46.5	46.1	46.0	...	46.0	46.0	46.0

LOCH GARRY

Date.	Hour.	Position.	Wind.	Surface.	5	10	15	20	25	30	35
1908.											
June 16	9.30 a.m.	Buoy.	W. 2-3.	51.4	51.4	51.2	51.2
" 16	Noon	C.	W. 3-4.	51.5	51.4	51.4	51.4
" 16	1.0 p.m.	B.	...	52.2	52.0	52.0	51.9
" 16	1.30 "	A.	...	52.3	52.2	52.1	52.0
" 16	4.30 "	Buoy.	W. 1-2.	52.0	51.9	51.8	51.2
" 17	9.0 a.m.	"	E. 1.	51.9	51.9	51.8	51.3
" 17	5.0 p.m.	"	E. 1.	52.0	52.0	52.0	51.9
" 18	9.0 a.m.	"	W. airs	52.0	52.0	51.7	51.1
" 18	1.30 p.m.	"	W. 1.	51.8	51.0
" 18	4.0 "	D.	"	51.5	...	51.5	51.3
" 18	5.0 "	Buoy.	...	51.5	51.5
" 19	10.0 a.m.	D.	E. airs	51.7	51.5
" 19	2.0 p.m.	Buoy.	W. 1.	52.2	...	51.5	51.4
" 19	5.0 "	D.	E. 1.	52.0	...	51.9	51.9
" 19	6.0 "	Buoy.	E. airs	52.0	...	52.0	51.4
" 20	7.30 a.m.	"	Calm.	52.0	51.9	51.9	51.5
" 20	10.30 "	F.	"	53.3	51.8	51.5	51.1
" 20	11.0 "	E.	E. airs	53.8	52.0	51.8	51.1
" 20	Noon	D.	...	54.8	52.0	51.8	51.4
" 20	1.0 p.m.	Buoy.	Calm.	56.0	52.0	51.8	51.4
" 20	2.0 "	C.	...	59.4	53.7	52.2	51.8
" 20	3.0 "	B.	Variable.	56.0	52.8	52.0	51.2
" 20	3.30 p.m.	A.	...	56.8	52.6	52.0	51.5
" 20	5.0 "	Buoy.	Calm.	58.0	52.4	52.0	51.4
" 22	10.0 a.m.	"	W. 1.	55.1	54.8	54.2	52.8	51.2	...
" 22	1.0 p.m.	"	W. 2-3.	54.7	...	54.4	53.7	...	51.1
" 22	5.30 "	"	W. 2-3.	54.0	53.7	53.4	53.3	...	51.8
" 23	9.30 a.m.	"	W. 1.	55.6	55.2	54.2	53.8	...	53.3	...	52.0
" 23	1.30 p.m.	"	W. 1.	56.0	55.9	54.8	53.4	...	53.0	...	51.9
" 23	3.30 "	E.	W. 1-2.	54.0	53.8	53.3	53.0	...	52.4	...	51.8
" 23	4.30 "	D.	W. 1-2.	54.8	54.5	54.2	53.5	...	52.8	...	52.0
" 23	5.30 "	Buoy.	W. 2-3.	55.1	55.0	55.0	54.6	...	52.8
" 23	8.15 "	"	...	54.5	...	54.4	53.9
" 24	1.0 "	"	W. 1.	56.2	56.0	55.5	54.2	...	53.8	53.2	...
" 24	3.30 "	C.	W. 2-3.	56.6	56.4	56.2	56.0	...	54.1	54.0	...
" 24	6.0 p.m.	Buoy.	W. 3.	55.2	55.2	55.0	54.2	54.0	...
" 25	12.30 "	"	W. 1-2.	55.0	...	55.0	54.9	...	54.4	...	54.0
" 25	4.30 "	D.	W. 1-2.	54.6	...	54.5	...	54.3	53.5	53.3	...
" 25	6.0 "	Buoy.	W. 1.	55.3	...	55.2	55.2	55.0	...
" 26	8.0 a.m.	"	Calm.	58.0	56.5	55.4	55.0	54.8	54.1	53.4	52.0
" 26	Noon	F.	"	60.0	55.6	55.1	54.5
" 26	1.0 p.m.	E.	"	64.0	56.7	55.8	...	54.8	...	53.7	...
" 26	2.0 "	D.	...	62.0	57.8	56.2	54.9	54.7	...	53.5	...
" 26	4.0 "	Buoy.	Calm.	64.3	56.3	55.3	...	54.5	...	53.4	...
" 26	5.0 "	C'.	...	63.0	56.4	55.1	...	54.4	...	52.9	...
" 26	5.30 "	B.	E. airs.	66.5	56.3	55.3	...	54.9	...	52.5	...

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— continued.

40	45	50	55	60	65	70	75	80	85	90	100	125	140	150	175	200
...	...	51.1	...	51.0	...	48.2	47.6	46.0	46.0	...	46.0	46.0	46.0
...	...	51.2	...	51.2	...	50.0	47.0	46.2	46.1	45.4
...	...	51.6	49.1	46.3	46.0
...	...	51.8	50.0	46.5
...	...	51.0	...	50.8	47.1	46.1	46.0	...	45.9	...	45.9
...	...	51.0	...	49.0	...	47.5	47.0	46.3	46.0	...	46.0	46.0	46.0
...	...	51.0	...	49.0	47.8	46.4	46.0	...	46.0	...	46.0
...	...	50.5	...	49.1	...	48.5	47.6	46.2	46.0	...	46.0	46.0	46.0
...	...	50.5	...	49.0	...	48.5	47.4	46.2	46.0	46.0
...	...	51.0	...	48.9	...	47.2	46.5	46.2	46.1	...	46.0	...	46.0
...	...	49.8	...	49.5	...	49.0	47.2	46.4	46.0	...	46.0	...	46.0
...	...	50.2	47.1	46.3	46.0	46.0
...	...	50.5	...	49.4	48.5	47.5	47.5	46.4	46.0	46.0
...	...	49.9	...	49.1	...	47.8	47.1	46.0	46.0	...
...	...	50.6	49.4	48.0	47.4	46.4	46.0	46.0
...	...	50.9	...	49.1	...	48.5	47.3	46.2	46.0	46.0
...	...	50.6	...	47.4	46.2	46.2
...	...	50.5	...	47.7	46.5	46.2	46.0	...	46.0
...	...	50.8	...	49.3	46.5	46.1	46.0	46.0	...
...	...	50.8	...	49.1	47.5	46.3	46.0	46.0
...	...	50.3	...	49.5	47.8	46.6	46.0	46.0	...
...	...	50.4	...	49.2	47.4	46.2	46.0	46.0
...	49.1	48.8	47.0
...	...	50.0	...	49.5	47.2	46.2	46.0	46.0
...	...	49.8	48.1	...	47.3	46.2	46.2	...	46.2	46.0	46.0
...	...	50.6	49.0	...	48.1	46.4	46.2	...	46.0	46.0	46.0
...	...	50.6	49.8	...	48.3	46.4	46.2	...	46.0	46.0	46.0
...	...	50.4	48.5	...	47.6	46.3	46.2	...	46.0	46.0	46.0
...	...	50.3	48.5	...	47.7	46.3	46.1	...	46.0	46.0	46.0
...	...	49.3	47.0	...	46.9	46.3	46.0
52.0	...	50.0	...	48.9	46.8	46.2	46.1	...	46.0	46.0	46.0
51.4	...	50.0	47.6	46.4	46.2	...	46.0	46.0	46.0
...	...	50.6	47.1	46.3	46.1
51.1	...	50.4	...	49.8	47.6	46.3	46.1	...	46.1	46.0	46.0
51.7	...	50.9	...	49.9	48.2	46.3	46.2	...	46.0	46.0	...
51.0	...	50.8	...	49.2	47.5	46.3	46.2	...	46.0	46.0	46.0
52.5	...	50.4	48.3	...	47.0	46.1	46.1	...	46.0	...	46.0
51.8	...	49.5	...	48.0	47.0	46.3	46.1	...	46.0	46.0	...
53.5	...	50.5	...	48.7	47.0	46.2	46.1	...	46.0	...	46.0
51.0	50.3	50.0	49.5	48.5	48.1	47.4	47.0	46.3	46.1	...	46.0	46.0	46.0
53.0	50.1	48.4	47.5	46.7	46.4	...	46.2
52.2	...	50.0	49.0	...	47.0	...	46.8	46.3	...	46.0
52.1	...	50.1	49.2	...	47.8	...	46.8	46.1	46.0	46.0	...
51.0	...	50.2	48.0	...	47.2	46.1	46.0	46.0
51.0	...	50.0	...	48.9	47.7	46.3	46.1	...	46.0
51.0	...	50.0	...	48.8	47.0	46.1	46.0

LOCH GARRY

Date.	Hour.	Position.	Wind.	Surface.	5	10	15	20	25	30	35
1908.											
June 26	6.30 p.m.	A.	...	65.8	56.0	55.0	...	54.0	...	52.3	...
" 27	10.0 a.m.	Buoy.	Calm.	65.6	57.9	55.4	55.0	54.8	54.0	53.3	52.7
" 27	4.0 p.m.	"	W. 1.	64.0	59.8	...	54.9	54.4	54.0	53.4	52.8
" 28	2.0 "	Buoy.	E. 1-2.	59.5	59.5	57.5	55.7	54.8	53.9	53.3	52.0
" 29	10.0 a.m.	"	E. 1.	62.2	60.5	59.1	58.0	55.4	54.8	53.0	51.8
" 29	4.0 p.m.	"	E. 3.	60.8	60.4	59.0	57.9	56.0	55.0	53.9	...
" 30	11.30 a.m.	"	E. airs.	63.4	62.0	61.7	58.5	55.9	55.0	53.2	52.8
" 30	3.30 p.m.	"	E. 1.	64.0	63.0	60.7	58.0	...	54.9	...	52.5
July 1	9.30 a.m.	"	E. 0- $\frac{1}{2}$.	66.6	63.3	62.8	59.0	56.0	54.2	53.0	51.2
" 1	3.30 p.m.	D.	Calm.	72.0	64.0	59.2	57.4	56.0	55.0	54.0	53.0
" 1	5.30 "	Buoy.	W. 1-2.	70.4	63.8	60.4	57.9	56.2	55.0	54.4	52.7
" 2	8.30 a.m.	"	Calm.	67.4	65.8	63.0	57.8	55.4	54.7	54.0	52.3
" 2	11.15 "	"	"	70.2	65.6	62.0	57.6	56.0	55.0	54.0	53.0
" 3	10.30 "	F.	E. 1.	69.7	69.4	69.0	58.8	56.9	55.3	54.1	53.0
" 3	Noon.	E.	E. 2.	69.8	69.5	62.6	58.0	56.5	55.4	54.3	53.8
" 3	1.45 p.m.	D.	...	69.0	68.8	63.0	58.1	57.0	55.3	54.5	53.3
" 3	3.45 "	Buoy.	E. 3.	66.5	65.4	59.3	57.2	56.8	55.0	52.6	51.5
" 3	6.0 "	C.	E. 2-3.	64.6	46.2	62.0	57.1	55.3	53.7	52.3	51.8
" 4	10.15 a.m.	A.	E. 1.	66.4	66.0	64.0	60.5	...	53.9
" 4	11.0 "	B.	E. 2.	66.4	...	63.5	...	56.9	53.7
" 4	Noon	C.	E. 2-3.	67.1	67.0	62.7	58.7	56.0	53.6	52.0	51.0
" 4	3.0 p.m.	Buoy.	E. 3.	65.6	65.4	62.6	57.8	56.1	54.8	53.7	53.0
" 4	4.30 "	D.	E. 3-4.	65.8	65.8	65.5	62.8	58.8	55.9	54.5	52.8
" 6	9.45 a.m.	Buoy.	Calm.	66.6	65.3	64.5	60.9	59.0	55.1	52.6	52.3
" 6	1.15 p.m.	"	"	68.3	65.8	63.0	61.3	58.4	55.5	53.5	52.8
" 6	4.0 "	C.	E. 1.	67.7	65.5	63.5	62.4	59.0	55.0	53.2	52.0
" 7	10.0 a.m.	Buoy.	"	64.5	64.1	63.5	60.7	59.9	55.2	53.8	52.0
" 7	Noon	D.	Calm.	65.1	64.6	63.1	61.3	60.4	55.4	54.6	53.2
" 7	5.0 p.m.	C.	W. airs.	64.6	64.5	64.0	60.8	60.0	57.3	55.0	53.9
" 8	7.0 a.m.	D.	E. airs.	63.8	63.8	63.7	61.3	58.4	55.0	54.0	53.0
" 8	8.30 "	E.	E. 1.	64.4	64.3	64.3	61.1	60.0	55.0	53.5	52.2
" 8	10.0 "	F.	E. airs.	64.3	64.3	63.6	62.3	59.7	55.0	53.7	52.4
" 8	2.30 p.m.	Buoy.	E. 1.	63.7	63.7	63.4	...	59.4	55.1	53.1	52.4
" 8	3.45 "	C.	E. 2.	63.0	63.0	62.3	58.0	56.9	54.5	53.0	52.1
" 8	5.30 "	B.	E. 2.	62.4	62.0	59.0	58.7	55.3	53.3	52.2	51.5
" 8	6.45 "	A.	E. 1.	62.0	62.0	58.0	56.0	55.3	53.4	52.6	51.5
" 9	9.30 a.m.	B.	E. 1.	61.0	61.0	61.0	60.9	58.5	55.9	54.0	52.1
" 9	11.30 "	"	E. airs.	61.0	61.0	61.0	60.0	57.8	55.7	54.0	53.3
" 9	1.30 "	"	E. airs.	61.9	61.3	61.0	60.5	58.9	56.7	54.7	53.4
" 9	4.0 p.m.	Buoy.	E. 1.	62.7	61.9	56.4	54.6	...	52.2
" 9	5.30 "	B.	E. 1.	61.5	60.0	57.1	54.8	...	52.0
" 10	7.0 a.m.	"	E. 1.	60.6	60.6	60.5	59.2	55.8	52.6	52.0	50.7
" 10	8.0 "	"	E. 2-3.	60.5	60.5	60.4	58.7	54.0	52.0	50.9	50.5
" 10	9.0 "	"	E. 2.	60.3	60.2	60.1	55.0	53.0	52.0	50.3	50.0
" 10	10.0 "	"	E. 3.	60.0	60.0	59.8	53.4	52.0	51.0	50.5	49.0
" 10	11.0 "	"	E. 2-3.	59.5	59.4	53.8	52.0	51.2	50.5	49.8	50.2
" 10	Noon.	"	E. 2.	58.3	58.0	58.0	52.0	51.0	50.8	49.9	49.0
" 10	1.0 p.m.	"	E. 1.	55.5	55.2	54.9	54.0	53.6	51.2	51.0	50.8
" 10	2.0 "	"	E. 1-2.	57.5	55.9	54.3	54.1	54.0	53.0	51.4	51.2
" 10	3.0 "	"	E. 2.	56.6	55.5	54.3	54.0	53.9	53.7	52.0	52.0

—continued.

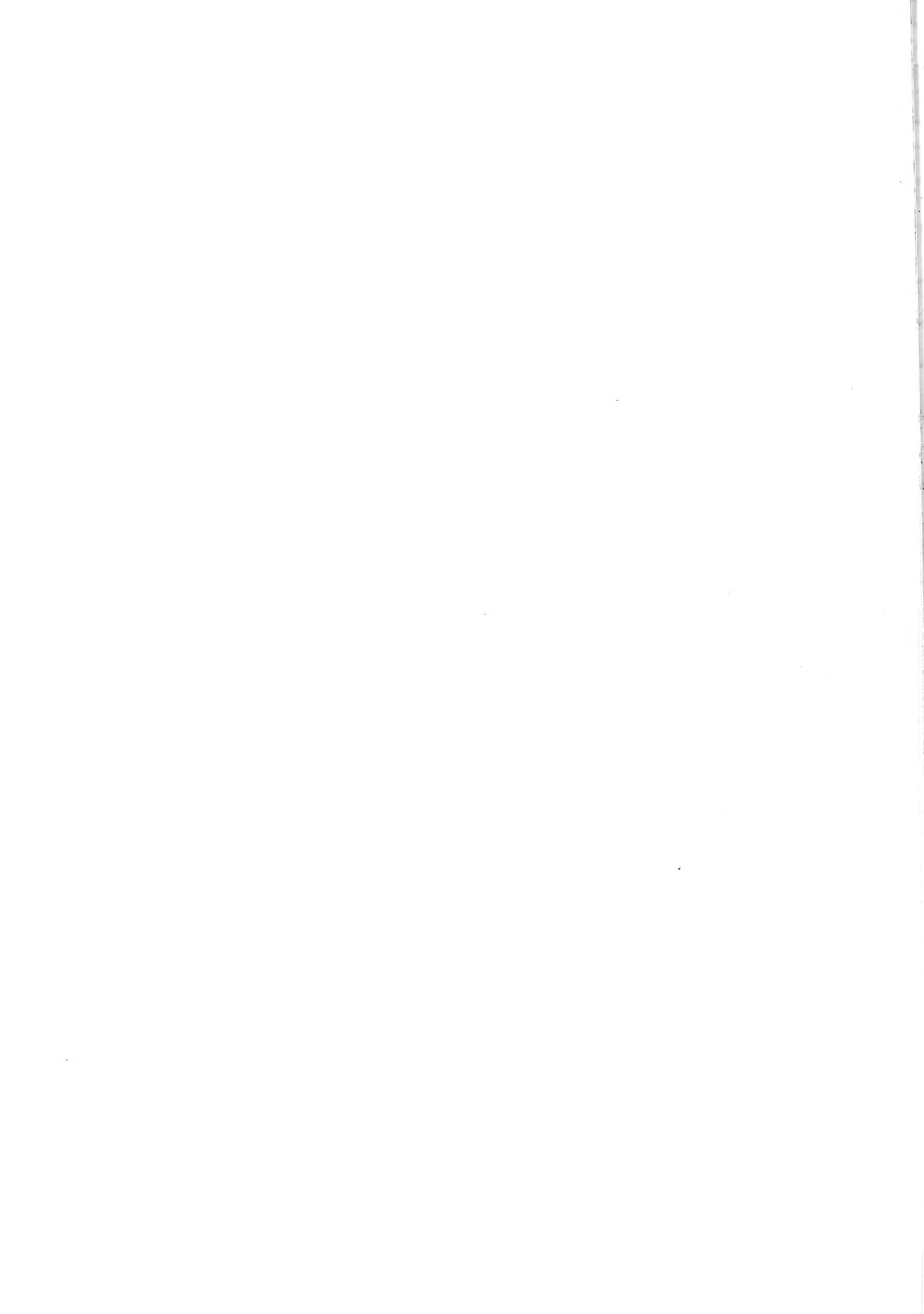
40	45	50	55	60	65	70	75	80	85	90	100	125	140	150	175	200
51.1	...	50.0	...	49.0	47.0
51.5	...	50.4	49.0	48.0	46.9	46.4	46.1	...	46.0	46.0	46.0
51.1	...	49.9	49.2	47.9	46.9	46.3	46.2	46.0	46.0
50.9	...	50.0	49.4	...	48.2	...	47.4	46.5	46.2	...	46.0	46.0	46.0
50.7	50.3	49.4	48.0	...	47.4	47.0	46.7	46.4	46.2	...	46.1	46.0	46.0
51.0	50.8	50.0	48.0	...	47.6	...	46.8	46.3	46.2	...	46.1	...	46.0
51.5	50.1	50.0	49.0	48.0	47.9	47.1	47.1	46.5	46.3	...	46.0	46.0	46.0
...	...	49.8	48.8	48.0	47.0	46.4	46.4	...	46.2	...	46.1
51.0	50.1	49.4	48.8	48.0	47.7	47.5	47.0	47.0	46.2	...	46.2	46.1	46.1
51.1	50.8	50.0	49.3	48.0	47.9	47.2	46.8	46.4	46.2	...	46.1	46.0	...
51.1	50.5	50.0	48.8	48.3	47.4	47.1	46.7	46.4	46.2	...	46.2	46.1	46.1
51.0	50.5	50.2	49.5	48.8	47.8	47.3	46.3	46.2	46.1	...	46.1	46.1	46.1
51.7	51.0	50.4	49.8	49.4	47.1	46.5	46.1
52.8	52.0	51.0	50.6	50.4	49.3	48.0	46.9	46.7
53.0	52.3	51.0	50.5	49.7	49.0	48.1	47.3	46.4	46.3
53.0	51.0	50.1	49.4	48.9	47.5	...	47.0	46.6	46.3
51.2	50.4	50.0	49.0	48.4	47.7	47.5	47.0	46.5	46.2	...	46.2	46.1	46.1
51.0	50.8	50.0	49.9	48.8	48.5	47.8	47.0	47.0	46.5	46.4
49.0	...	47.6	...	46.9	...	46.7
49.8	...	47.4	...	46.8	46.5	46.5	46.3
50.7	50.0	49.0	48.0	47.1	47.0	46.9	46.6	46.5	46.3	...	46.2	46.2	...
															(165)	
52.0	50.7	50.0	48.9	48.2	47.7	47.5	47.2	46.7	46.5	...	46.3	46.2	46.2
51.7	50.3	49.0	48.2	47.0	47.0	46.9	46.9	46.7	46.5	...	46.3	46.2	...
51.4	50.6	49.5	48.8	47.4	47.4	47.0	46.9	46.4	46.3	...	46.3	46.2	46.2
51.6	50.5	49.4	48.0	47.9	47.8	47.3	47.1	46.9	46.6	...	46.3	46.2	46.2
51.6	50.6	49.3	49.2	48.2	47.8	47.6	47.0	46.5	46.4	...	46.2	46.2	...
															(170)	
52.0	51.0	49.7	49.0	47.8	47.3	47.0	47.0	46.5	46.3	...	46.3	46.2	46.2
52.8	51.7	51.0	50.0	48.9	48.2	47.9	47.6	46.7	46.5	...	46.5	46.3	...
53.0	52.0	50.4	49.7	49.0	48.0	47.9	47.1	46.5	46.2
51.5	51.0	50.2	49.0	48.0	47.7	47.4	47.0	46.8	46.5	...	46.4	46.3	...
															(180)	
52.0	51.0	50.0	48.8	48.2	48.0	...	46.9	46.8	46.5	46.4
52.0	50.8	49.0	48.5	48.0	47.8	47.0	47.0	46.5
											(95)					
51.4	51.0	50.1	49.5	48.7	48.0	47.5	47.3	46.5	46.4	...	46.4	46.3	46.3
51.9	51.0	50.0	49.1	48.4	48.0	47.3	46.9	46.4	46.3	...	46.3
51.0	50.6	50.0	49.0	48.1	47.9	47.4	47.0	46.5	46.3
															(135)	
51.0	50.2	50.0	49.2	48.0	48.0
51.9	50.7	50.0	48.7	47.4	47.2	47.0	47.0
53.1	52.0	50.3	49.5	48.5	48.1	47.1	46.9
53.0	52.0	50.8	49.5	49.0	48.0	47.4	47.0
...	50.8	50.2	49.8	...	48.3	...	47.4	46.9	46.1
...	50.8	49.2	49.0	...	47.1	46.2
50.1	49.8	48.0	47.8	47.3	47.0	46.9	46.9	46.6
50.2	49.4	48.0	47.8	...	47.2	...	46.7
49.9	48.5	47.9	47.3	...	47.0	...	47.0
50.0	48.6	48.0	47.4	...	47.0	...	46.7
48.8	48.0	47.4	47.0	...	46.8	...	46.6	46.3
49.0	48.8	48.2	47.9	...	47.0	...	46.9	46.4
50.6	50.0	48.9	47.9	...	47.1	...	46.8	46.5
51.1	50.9	50.3	48.9	...	48.0	...	47.4	46.7
52.0	51.4	50.8	50.3	49.8	49.0	...	47.6	47.0

LOCH GARRY

Date.	Hour.	Position.	Wind.	Surface.	5	10	15	20	25	30	35
1908.											
July 10	4.0 p.m.	B.	E. 2.	56.0	55.7	55.1	54.2	54.0	53.9	53.0	52.0
" 10	5.0 "	"	E. airs.	56.5	56.1	55.9	55.0	54.9	54.8	53.6	52.3
" 10	6.0 "	"	E. airs.	58.0	57.0	56.3	56.0	55.0	53.8	53.1	53.0
" 10	7.0 "	"	Calm.	58.8	58.3	57.5	57.0	55.9	54.8	53.8	53.8
" 10	8.0 "	"	Calm.	59.1	58.8	57.5	57.2	56.8	55.8	55.0	54.8
" 10	9.0 "	"	E. 0-1.	59.0	58.8	58.3	58.0	57.3	55.7	55.0	54.6
" 10	10.0 "	"	E. 0-1.	58.9	58.8	58.2	58.0	57.1	56.0	55.1	54.7
" 11	2.0 a.m.	"	Calm.	58.8	58.6	58.0	57.8	56.8	55.0	52.8	51.4
" 11	3.0 "	"	Calm.	59.0	58.8	58.3	58.0	57.0	56.0	54.9	52.7
" 11	4.0 "	"	Calm.	59.0	59.0	58.8	58.6	58.0	57.3	55.8	54.8
" 11	5.0 "	"	E. 0-1.	59.7	59.3	59.0	58.5	58.2	57.3	56.5	56.0
" 11	6.0 "	"	"	59.9	59.8	59.1	...	58.0	57.6	...	56.2
" 11	7.0 "	"	E. airs.	60.0	59.4	59.0	58.6	57.9	57.3	56.5	54.7
" 11	8.0 "	"	E. 1.	60.1	59.2	58.7	58.6	58.3	57.7	57.0	54.4
" 11	9.0 "	"	"	60.3	60.1	59.5	59.0	58.5	58.0	56.6	53.0
" 11	10.0 "	"	E. airs.	60.5	60.5	59.5	58.9	58.1	57.5	55.1	51.8
" 11	11.0 "	"	E. 1.	60.3	60.3	59.4	58.3	57.9	57.0	53.5	51.2
" 11	Noon.	"	"	60.5	60.0	59.0	58.4	58.0	54.7	52.0	51.9
" 11	1.0 p.m.	"	E. 2.	60.2	60.0	59.8	58.7	58.0	55.0	51.7	50.3
" 11	2.0 "	"	E. 2-3.	60.2	60.0	59.6	58.3	56.8	53.0	51.0	49.5
" 11	3.0 "	"	E. 2.	59.8	59.7	59.7	58.8	55.5	52.9	51.0	49.9
" 11	4.0 "	"	"	59.7	59.5	59.0	55.1	53.0	52.1	51.3	50.5
" 11	5.0 "	"	E. 1-2.	59.2	59.2	58.5	55.5	53.0	52.3	52.0	51.4
" 11	6.0 "	"	E. 2.	59.0	58.9	58.7	55.5	53.3	53.2	52.0	51.8
" 12	10.0 a.m.	Buoy.	E. 1.	61.3	61.3	60.8	60.6	59.7	58.5	55.3	53.3
" 13	3.0 p.m.	"	E. 2-3.	58.9	58.9	58.8	58.8	58.8	58.7	57.0	52.5
" 13	4.30 "	"	E. 2-3.	58.8	58.8	58.8	58.8	58.8	58.7	57.4	52.9
" 14	11.0 a.m.	"	Variable.	59.4	59.4	59.3	58.7	58.4	57.8	55.7	53.6
" 14	Noon.	"	E. 1.	59.4	59.3	59.3	59.0	58.4	57.9	54.8	53.9
" 14	1.30 p.m.	"	...	60.0	59.6	59.0	58.5	58.3	58.0	55.4	54.0
" 14	3.0 "	"	E. 1.	59.9	59.7	59.4	59.0	58.3	58.0	55.7	53.7
" 14	5.0 "	"	Variable.	59.5	59.5	59.5	59.3	58.2	57.8	55.3	54.0
" 14	9.0 "	"	...	59.4	...	59.4	...	58.3	...	55.4	...
" 15	9.30 a.m.	"	Calm.	59.6	59.2	59.0	59.0	58.5	57.6	55.5	52.9
" 15	3.0 p.m.	"	W. 1-2.	59.4	59.4	59.2	59.0	58.3	57.7	55.8	53.3
" 16	7.0 a.m.	B.	Calm.	59.0	...	59.0	...	58.9	...	54.9	...
" 16	8.0 "	"	W. 2.	59.1	...	59.0	...	59.0	...	54.0	...
" 16	10.45 "	"	Calm.	59.3	...	59.0	...	59.0	...	55.3	...
" 16	3.15 p.m.	"	...	59.7	59.0	...	54.3	...
" 16	6.0 "	"	...	59.3	59.0	...	54.7	...
" 16	7.0 a.m.	F.	W. 1.	59.2	...	58.8	...	57.3	...	56.4	...
" 16	8.0 "	"	W. 1.	59.2	...	59.0	...	57.7	...	56.3	...
" 16	9.0 "	"	W. 1.	59.3	...	59.1	...	59.0	...	56.4	...
" 16	1.0 p.m.	"	Variable.	59.4	...	59.1	...	58.1	...	56.5	...
" 16	1.45 "	"	E. airs.	59.1	...	59.1	...	57.3	...	56.7	...
" 16	4.30 "	"	E. 1.	59.3	...	59.3	...	57.9	...	57.0	...
" 17	10.0 a.m.	Small loch	Variable.	58.9	58.9	58.7	58.2	57.0	56.5	55.3	51.0
" 17	1.0 p.m.	F.	N.E.	59.1	59.0	59.0	59.0	58.9	58.0	52.7	51.5
" 17	5.0 "	Buoy	2-3. N.E.	58.9	58.9	58.9	58.8	58.7	57.3	55.0	53.9
" 18	10.0 a.m.	"	2-3. W. 1.	58.5	58.5	58.4	58.4	58.2	57.9	56.0	53.9
" 20	10.0 "	F.	W. 1-2.	58.8	58.7	58.7	58.5	57.9	57.2	55.0	53.8
" 20	Noon.	E.	W. 1.	58.6	58.6	58.5	58.4	56.3	54.8	52.8	51.2
" 20	2.0 p.m.	D.	W. 2-3.	58.8	58.8	58.7	58.5	58.5	58.5	54.6	53.2

—continued.

40	45	50	55	60	65	70	75	80	85	90	100	125	140	150	175	200
51.5	51.3	51.0	50.8	50.1	49.8	49.3	48.8	...	47.0
52.3	51.6	51.4	51.3	50.9	50.5	50.2	50.0
53.0	52.3	52.0	51.9	51.1	51.0	51.0	50.3
53.7	53.7	53.0	52.7	52.2	52.0	51.3	50.0	...	47.2	...	46.6
54.0	53.9	53.1	52.6	52.0	51.2	50.6	49.0	...	47.2
53.7	53.7	53.0	52.3	51.3	51.2	48.9	47.8	47.2	47.1	...	47.0
53.1	53.0	51.2	...	49.0	48.2	48.2	47.7	47.0	46.8	...	46.8
50.8	50.5	49.7	48.3	48.0	47.1	47.1	47.1	47.1	46.8	...	46.7
51.7	51.3	50.7	49.0	48.2	47.3	47.0	47.0	46.8	46.8	...	46.7
53.3	53.2	51.7	50.4	49.0	47.7	47.3	47.1	47.0	46.9	...	46.5
54.2	53.8	52.0	51.0	49.9	48.2	47.4	47.2	47.0	46.9	...	46.7
54.2	53.0	52.3	51.1	49.0	48.2	47.4	47.1	...	47.0	...	46.7
52.9	51.8	51.0	49.9	48.5	47.7	47.1	47.0	46.9	46.8	...	46.5
52.0	51.9	50.3	49.0	47.5	47.2	...	47.0	46.9	46.8	...	46.5	46.5
51.5	50.0	48.9	47.7	47.0	47.0	47.0	47.0	47.0	46.7	...	46.5
50.2	48.8	48.3	47.2	47.0	47.0	47.0	47.0	47.0	46.8	...	46.5
49.0	48.1	47.9	47.8	47.2	47.0	46.9	46.9	46.9	46.9	...	46.5
49.0	47.5	47.4	47.1	47.0	47.0	46.9	46.9	46.8	46.8	...	46.5
49.0	48.9	47.4	47.2	47.0	47.0	47.0	47.0	47.0	46.7	...	46.5	46.4
49.0	48.5	47.6	47.3	47.0	46.9	46.9	46.8	46.7	46.7	...	46.5
49.0	48.9	48.0	47.4	47.1	47.0	46.9	46.9	46.9	46.8	...	46.5
50.2	50.0	49.6	49.0	48.0	47.3	47.1	47.0	47.0	46.7	...	46.5
51.2	51.2	50.5	50.0	49.5	48.0	47.4	47.0	46.9	46.9	...	46.5
51.4	51.3	51.0	50.8	50.2	49.8	49.2	48.3	47.2	47.0	...	46.8
51.9	50.8	50.2	49.4	48.8	47.7	47.3	47.0	47.0	47.0	...	46.6	46.6	...	46.4	46.4	46.4
51.2	50.7	50.2	49.3	48.2	47.8	47.3	47.0	47.0	47.0	...	46.7	46.4	...	46.3	46.3	46.3
51.8	50.7	50.2	49.4	48.2	47.9	47.3	47.2	47.0	46.9	...	46.7
53.4	51.7	50.5	49.7	48.8	47.5	47.2	47.2	47.0	47.0	...	46.8	46.8	...	46.4	...	46.3
52.7	51.1	50.4	49.8	48.7	47.4	47.1	47.1	47.0	47.0	...	46.7
52.5	51.2	50.8	49.4	48.2	47.5	47.2	47.0	47.0	47.0	...	46.7
52.8	50.9	50.6	49.5	48.0	47.8	47.6	47.2	47.0	47.0	...	46.7
51.4	50.3	50.2	...	48.9	48.4	48.0	47.7	47.3	47.0	...	46.7
52.4	...	50.7	...	49.3	...	47.9	...	47.2	...	47.0	47.0
51.7	51.0	50.9	50.1	49.4	48.7	47.9	47.7	47.5	47.0	...	46.7	46.5	...	46.5	46.4	46.4
51.8	50.7	50.7	49.9	48.6	48.4	47.8	47.3	47.2	47.0	...	46.5
52.2	...	51.2	...	50.7	...	50.0	47.4	...	47.0
52.5	...	51.6
52.7	...	52.4	...	51.0	...	47.3	46.7	...	46.7
52.0	...	49.6	...	47.3	...	47.0	46.6	...	46.6
51.3	...	49.3	...	47.1	...	47.0	46.8	...	46.8
51.0	...	47.4	...	47.0	...	47.0	46.9	...	46.6
50.7	...	48.7	...	47.8	...	47.0	46.8	...	46.8
50.9	...	49.3	...	47.8	...	47.0	46.7	...	46.7
51.5	...	49.8	...	47.8	...	47.0	46.8	...	46.6
51.9	...	49.8	...	48.2	...	47.2	46.9	...	46.7
54.0	...	51.0	...	49.2	...	47.7	46.6	...	46.6
50.3
51.5	51.0	50.0	49.4	48.4	47.9	47.3	47.0	47.0	46.9	...	46.4
52.0	51.2	50.2	49.2	48.8	48.0	47.3	47.0	47.0	46.9	...	46.6	46.4
51.2	50.4	49.6	48.1	47.7	47.4	47.0	47.0	47.0	46.9	...	46.6	46.5	...	46.5	46.4	46.4
51.6	51.4	49.0	48.0	47.5	47.2	47.0	46.9	46.8	46.7	...	46.5
50.2	49.6	49.4	48.4	47.3	47.0	46.8	46.7	46.7	46.6	...	46.6	46.5
52.3	51.7	50.1	49.0	47.9	47.5	47.3	47.0	46.9	46.8	...	46.7	46.5	...	46.5	46.4	...



—continued.

40	45	50	55	60	65	70	75	80	85	90	100	125	140	150	175	200
53·6	51·6	50·0	49·0	48·4	47·9	47·8	47·3	47·0	46·8	...	46·7	46·6	...	46·6	...	46·5
51·8	50·6	50·0	49·0	48·2	47·7	47·0	47·0	47·0	46·9	...	46·6	46·5	...
51·0	50·0	49·0	48·0	47·6	47·1	...	47·0	...	46·7	...	46·6	46·5
52·4	51·4	50·1	49·9	48·2	48·1	48·0	47·5	47·1	47·0	...	46·8	46·6
52·3	51·1	49·6	48·5	47·6	47·4	47·2	47·0	47·0	46·8	...	46·7	46·6	...	46·5	46·5	46·5
52·6	...	50·0	...	48·3	...	47·2	...	47·0	46·7
53·0	...	50·2	...	49·0	...	48·0	...	47·1	46·7
53·5	...	50·6	...	49·1	...	48·0	...	47·1	46·7
51·8	...	50·0	...	48·9	...	47·6	...	47·0	46·9
51·8	...	49·7	...	48·5	...	47·6	...	47·0	46·7
51·9	...	49·9	...	48·6	...	47·6	...	47·0	46·7
53·0	...	49·9	...	48·8	...	47·2	...	47·0	46·6
52·0	...	49·7	...	48·0	...	47·0	...	46·7	46·4
51·8	...	49·7	...	47·8	...	47·1	...	46·9	46·7
50·9	...	48·9	...	47·6	...	47·2	...	46·8	46·7
50·3	...	49·0	...	47·3	...	46·9	...	46·8	46·5
51·0	...	49·8	...	47·7	...	47·0	...	46·8	46·6
52·0	...	50·3	...	48·7	...	47·3	...	47·0	46·7
52·6	...	50·5	...	49·0	...	47·7	...	47·1	46·7
52·0	...	50·5	...	49·4	...	48·4	...	47·0	46·7
51·8	...	50·3	...	49·3	...	47·9	...	47·1	46·7
51·9	...	50·1	...	48·8	...	47·5	...	47·0	46·6
52·0	...	50·5	...	49·1	...	47·4	...	47·0	46·7
53·0	...	51·1	...	49·4	...	47·6	...	47·0	46·7
54·0	...	51·0	...	49·3	...	47·3	...	47·0	46·6
52·9	...	51·0	...	48·8	...	47·1	...	46·9	46·9
53·0	...	50·1	...	47·8	...	47·0	...	46·8	46·7
54·0	...	50·0	...	48·0	...	47·0	...	46·9	46·6
53·5	...	51·0	...	48·5	...	47·1	...	46·9	46·6
54·9	...	50·9	...	48·0	...	47·0	...	46·9	46·6
54·9	...	50·5	...	47·9	...	47·0	...	46·9	46·7
54·8	...	50·0	...	47·9	...	47·0	...	46·8	46·6
56·8	...	50·8	...	48·4	...	47·0	...	46·8	46·6
58·0	...	51·0	...	47·9	...	47·1	...	46·8	46·5
...	...	53·7	...	53·7	...	52·4	51·2	50·0	48·0	...	47·3	47·0	...	47·0
...	54·0	47·6	47·3	47·0
...	(90)	(95)
...	54·0	52·4	51·1	47·1
...	(95)	(100)

(Issued separately March 1, 1909.)

IX.—On the Conditions for the Reversibility of the Order of Partial Differentiation. By W. H. Young, Sc.D., F.R.S. (Communicated by J. H. MACLAGAN WEDDERBURN, D.Sc.)

(MS. received November 4, 1908. Read February 1, 1909.)

§ 1. IN the case of a function of two variables $f(x, y)$ there is, in general, no connection between the results of proceeding to the various limits, upper, lower, and intermediate, first with respect to x and then with respect to y , and first with respect to y and then with respect to x . We cannot even assert that the repeated upper limit obtained in one way is not less than the repeated lower limit obtained in the other way.

The so-called necessary and sufficient conditions for the equality of the two repeated limits do little more than express in ϵ -language the fact of the equality. There is, however, one simple case in which we can assert with confidence the existence of the repeated limit, this is, when a unique double limit exists.

It is no accident, therefore, that the simplest and best-known set of conditions for the reversibility of the order of partial differentiation—that associated with the names of Dini and Schwarz—is based on this property.

The main object of the first of the three parts of the present paper is to obtain these Schwarz-Dini conditions in a more precise form than has yet been given to them. It is found that the axial cross through the point (a, b) considered may be omitted in obtaining the limit of the values of $\frac{d}{dy} \cdot \frac{df}{dx}$ in the neighbourhood of the point (a, b) , and indeed that the existence of $\frac{d}{dy} \cdot \frac{df}{dx}$ on the axial cross is nowhere required for the purposes of the proof, except at the point (a, b) itself.

It is hoped that the mode of proof adopted, in which the ϵ -method is avoided, will be found somewhat simpler and more concise than the usual mode of presentation.

In the second part of the paper I obtain certain properties of the repeated partial differential coefficients, or, more generally, of the derivates with respect to one variable of the differential coefficient, or of the derivates, with respect to the other variable. I am thus led to state

and prove certain extensions of the Schwarz-Dini conditions considerably less narrow than those previously given.

In the third part of the paper I obtain another set of sufficient conditions which I believe to be new both in the form given to them and in the mode of proof. These are as follows:—

If both $\frac{d^2f}{dx^2}$ and $\frac{d^2f}{dy^2}$ are continuous functions of the ensemble (x, y) at the point (a, b) and both $\frac{d}{dy} \cdot \frac{df}{dx}$ and $\frac{d}{dx} \cdot \frac{df}{dy}$ exist at the point (a, b) , then these two latter repeated partial differential coefficients are equal at (a, b) .

Stolz in his *Grundzüge der Differenzial- und Integralrechnung*, vol. i. pp. 141-147, has given a set of conditions, also due to Schwarz, included in mine as a very special case. The additional demands made by Schwarz on the function consist of (1) the existence of $\frac{d}{dy} \cdot \frac{df}{dx}$ and $\frac{d}{dx} \cdot \frac{df}{dy}$ in a closed neighbourhood of the point (a, b) , and (2) the continuity of $\frac{d}{dy} \cdot \frac{df}{dx}$ with respect to y and of $\frac{d}{dx} \cdot \frac{df}{dy}$ with respect to x at the point (a, b) .

The interesting question then arises: What is the relation of these conditions to those for the equality of two repeated limits? The method of proof adopted shows that in this case also the equality virtually depends on the existence of a unique double limit.

It may be noted also that in their relation to the repeated partial differential coefficients $\frac{d}{dy} \cdot \frac{df}{dx}$ and $\frac{d}{dx} \cdot \frac{df}{dy}$, or to the repeated derivatives, the new conditions are less narrow than the Schwarz-Dini conditions. They do not demand even the continuity of the derivatives of derivatives at the point, still less the existence and continuity of $\frac{d}{dy} \cdot \frac{df}{dx}$ in the neighbourhood of the point considered.

PART I.

§ 2. With the object of rendering what follows as complete in itself as possible, we begin by formulating a few properties in the theory of limits, on the proper grasp of which the understanding of the subject hinges. We have first to explain exactly what we mean by a double and a repeated limit.

If $x_1, x_2, \dots, x_n, \dots$ be a series of values of a variable x , forming

a sequence with a as limiting value, the value a being expressly excluded from being a possible value for any x^n , and further, the series

$$f(x_1), f(x_2), \dots f(x_n), \dots$$

has only one limiting value u (which may be $+\infty$ or $-\infty$), then we say that u is one of the single limits of $f(x)$ at a , and write

$$u = \text{Lt}_{n=\infty} f(x_n) = \text{one of the } \text{Llt}_{x=a} f(x).$$

The function $f(x)$ may or may not be defined for the value $x=a$; but, if defined, the value $f(a)$ must be disregarded in considering the limits at a .

If $f(x, y)$ is a function of two independent variables x and y , it becomes a function of x alone when we keep y constant, and has a corresponding set of simple limits,

$$\text{Llt}_{x=a} f(x, y).$$

If there is only one such limit for each value of y ,* this limit defines a function of y , and has, as such, a set of limits for $y=b$; these are called *the repeated limits of $f(x, y)$ first with respect to x and then with respect to y* , and written

$$\text{Llt}_{y=b} \text{Lt}_{x=a} f(x, y).$$

Similarly, if there is only one limit when y is kept constant,

$$\text{Llt}_{x=a} \text{Lt}_{y=b} f(x, y)$$

denotes *the repeated limits of $f(x, y)$ first with respect to y and then with respect to x* .

The idea of *double limits* of $f(x, y)$ is different.

Let x_1, x_2, \dots be a sequence of values of x having a as limit, but not including a , and y_1, y_2, \dots a sequence of values of y having b as limit, but not including b , then $f(x, y)$ has at the points $(x_1, y_1), (x_2, y_2), \dots$

* The same is true if a law is given by means of which one of the limits is defined for each value of y ; e.g. the maximum limit. In this case a number of quantities can be identified as limits of limits of $f(x, y)$, or repeated limits; e.g. the upper upper limit, the upper lower limit, etc. The whole set of such limits, which may theoretically be denoted by

$$\text{Llt}_{y=b} \text{Llt}_{x=a} f(x, y),$$

can only be regarded as *perfectly* defined when all possible laws which can be used are in some manner specified.

a series of values and at (a, b) a corresponding set of limits.* If one of these is A (which may be $+\infty$ or $-\infty$), then A is said to be a *double limit* of $f(x, y)$ for the ensemble of values (a, b) .

If the variables x and y are continuous, we may regard the ensemble (x, y) as a point of the plane. The points $(x_1, y_1), (x_2, y_2), \dots$ then form a sequence with the point (a, b) as limiting point. They do not, however, form the most general such sequence, since the individual points are explicitly excluded from lying on the axial cross

$$\begin{aligned} x &= a, \\ y &= b, \end{aligned}$$

through the limiting point.†

Lemma 1.—*Any repeated limit is a double limit.*

For convenience we shall use $F(m, n)$ for $f(x_m, y_n)$.

By the definition of a repeated limit there is then a sequence of values of $F(m, n)$, keeping n constant, having a unique limit $v(n)$, when m is indefinitely increased; and the quantities $v(n)$, as n is indefinitely increased, have the repeated limit, say u , as one of their limits; that is

$$\begin{aligned} v(n) &= \text{Lt}_{m=\infty} F(m, n) \\ u &= \text{one of the } \text{Llt}_{n=\infty} v(n). \end{aligned}$$

First consider the quantities $v(n)$ to be all finite.

Represent the values of the various functions on a straight line,

$$\begin{aligned} f(m, n) &\text{ by the point } P_{m,n}, \\ v(n) &\text{ by the point } Q_n, \\ u &\text{ by the point } Q. \end{aligned}$$

Then, under the given conditions, Q is the limiting point of the sequence Q_1, Q_2, \dots and Q_n is, for each value of n , the limiting point of the sequence $P_{1,n}, P_{2,n}, \dots$

Since all these points, except possibly Q , are finite points, any interval d containing Q as internal or end-point determines a point Q_n where n is the first integer greater than some chosen number such that Q_n lies inside d . The interval d then determines a point $P_{m,n}$ such that n is the first integer greater than a chosen number, and such that $P_{m,n}$ lies inside d .

Taking a sequence of such intervals d_1, d_2 having Q as sole common

* These may be regarded as falling under the heading of simple limits, since they are $\text{Llt}_{n=\infty} F(n)$, where $F(n) = f(x_n, y_n)$.

† It is unnecessary here to enter into the modifications necessary when the point (a, b) is at infinity.

internal or end-point, we get in this manner a constantly increasing set of integers m_1, m_2, \dots and another n_1, n_2, \dots such that Q is the sole limiting point of the points P_{m_i, n_i} . The corresponding series $F(m_i, n_i)$ has then u for limit, which shows that u is a double limit of $F(m, n)$.
Q.E.D.

If a finite number of the quantities $v(n)$ are infinite, we can omit them from consideration. If, however, this is not the case, u is $+\infty$ or $-\infty$ and is equal to all but a finite number of the $v(n)$'s. In this trivial case it is clear that, corresponding to any $v(n)=u$, we can find an integer m such that $F(m, n)$ is numerically greater than any chosen quantity, and in this way we can construct a series $F(m_i, n_i)$ having u as limit.

Thus in any case the theorem is demonstrated.

Lemma 2.—If $f(x, y)$ has only one double limit at a point (a, b) , and the simple limit

$$\text{Lt}_{h=0} f(a+h, b+k)$$

is unique for all values of k in a certain neighbourhood of the zero point, then the repeated limit

$$\text{Lt}_{k=0} \text{Lt}_{h=0} f(a+h, b+k)$$

exists.

This is an immediate consequence of Lemma 1.

N.B.—If the simple limit is not unique, it follows from Lemma 1 that, however we define the limiting function as a function of y , all its limits are double limits of $f(x, y)$. Hence if, as in Lemma 2, $f(a+h, b+k)$ has only one double limit, all the limits of all possible limiting functions coincide. In particular the upper upper and lower lower limits are equal.

NOTE 1.—It may evidently happen that even when $\text{Lt}_{x=a} f(x, y)$ does not exist as a unique limit, the upper and lower limits, and therefore all intermediate limits, have one and the same definite limit as y approaches b . If we agree with some writers to call this the repeated limit, it is evident that Lemma 2 still holds, omitting the assumption as to the uniqueness of the simple limit.

NOTE 2.—The existence of a unique limit of $f(x, y)$ for each fixed value of x does not, of course, involve the existence of a unique limit when y varies with x , which would be concomitant to the existence of a unique double limit.

§ 3. A rectangle whose axes are parallel to the axes of coordinates, and whose lower left-hand and upper right-hand corners are respec-

tively the points (a, b) , $(a+h, b+k)$, will be referred to as the rectangle $(a, b; a+h, b+k)$, and its double incrementary ratio defined as follows:—

$$m(a, b; a+h, b+k) = \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{hk}. \quad (1)$$

Here h and k may have any positive or negative values.

Using $m_x(b, b+k)$ for the single incrementary ratio of $f(x, y)$ when x is constant,

$$m_x(b, b+k) = \frac{f(x, b+k) - f(x, b)}{k}. \quad (2)$$

we clearly have the following identity:—

$$m(a, b; a+h, b+k) = \frac{m_{a+h}(b, b+k) - m_a(b, b+k)}{h}. \quad (3)$$

so that $m(a, b; a+h, b+k)$ is the incrementary ratio with respect to x of the incrementary ratio with respect to y of $f(x, y)$, and similarly it is the incrementary ratio with respect to y of the incrementary ratio with respect to x .

If in (2), keeping x constant, we let k approach zero in any manner, and there is only one limit, this is, by definition, the partial differential coefficient with respect to y , $f_y(x, b)$, or $\frac{df}{dy}$. Forming the single incrementary ratio of $f_y(x, b)$ for the pair of values $x=a$ and $x=a+h$, if this has a unique limit as h approaches zero, this is, by definition, the repeated differential coefficient with respect first to y and then to x , and will be denoted by f_{xy} or $\frac{d}{dx} \cdot \frac{df}{dy}$ or $\frac{d^2f}{dxdy}$.

If f_y exists, but its single incrementary ratio has more than one limit, the various limits are the various derivates at the point (a, b) of f_y , and may be called the repeated derivates of $f(x, y)$ first with respect to y and then to x .

If f_y does not exist, the upper and lower left- and right-hand derivates which appear instead form functions whose derivates with respect to x may be treated as the repeated derivates of $f(x, y)$ first with respect to y and then to x .

The following theorem will for convenience be referred to as the Repeated Theorem of the Mean.

The Repeated Theorem of the Mean.

If $f(x, y)$ is a finite continuous function of x at every point of the closed rectangle $(a, b; a+h, b+k)$, and has, except possibly on the bounding ordinates,

$$x = a \text{ and } x = a + h,$$

a differential coefficient f_x with respect to x , which is itself a finite continuous function of y with, at every internal point of the rectangle, a differential coefficient

$$f_{yx}(x, y) = \frac{d}{dy} \cdot \frac{df}{dx} = \frac{d^2f}{dydx},$$

then there is an internal point of the rectangle at which the repeated differential coefficient $f_{yx}(x', y')$ is equal to the double incrementary ratio of the rectangle, viz.

$$m(a, b; a+h, b+k) = f_{yx}(x', y').$$

For, applying the Theorem of the Mean to the identity (3), we get

$$m(a, b; a+h, b+k) = \frac{d}{dx} m_x(b, b+k) \text{ at some point } x' \text{ between } a \text{ and } (a+h) \text{ not}$$

inclusive

$$= \frac{f_x(x', b+k) - f_x(x', b)}{k},$$

whence the result follows by a second application of the Theorem of the Mean, since f_x is continuous with respect to x on the sides $y=b$ and $y=b+k$ of the rectangle.

NOTE.—Assuming the simple result that, when $f(x, y)$ is a continuous function of the ensemble (x, y) the double incrementary ratio assumes at points internal to a rectangle $(a, b; a+h, b+k)$ every value between its upper and lower bounds,* this shows that if f_{yx} then exists and is finite at every internal point of this rectangle it also assumes every value between its upper and lower bounds.

§ 4. Theorem 1 (on the existence of f_{yx} at the point (a, b)).—If f_x exists in a closed neighbourhood of a point (a, b) , while in the completely open neighbourhood, excluding the axial cross,† it has a differential coefficient f_{yx} with respect to y , then, if f_{yx} has only one double limit as we approach the point (a, b) in any manner by means of points not on the axial cross, f_{yx} exists also at the point (a, b) itself.

For, by the Repeated Theorem of the Mean,

$$m(a, b; a+h, b+k) = f_{yx}(x', y'),$$

where the point (x', y') does not lie on the axial cross, and has (a, b) as limiting point when h and k each approach zero in any manner without assuming the value zero.

Since $f_{yx}(x, y)$, and therefore $f_{yx}(x', y')$, has only one double limit at the point (a, b) , the same is true of $m(a, b; a+h, b+k)$ when h and k have

* A proof of this is given in § 7 below.

† That is, not on $x=a$ nor on $y=b$.

zero as limit. Hence, a repeated limit being a double limit (Lemma 1), all the repeated limits of $m(a, b; a+h, b+k)$ are equal.

But, by hypothesis, f_x is defined on the ordinate $x=a$, so that

$$\frac{f_x(a, b+k) - f_x(a, b)}{k} = \text{Lt}_{k=0} m(a, b; a+h, b+k),$$

and has, therefore, by what has been shown, only one limit when k has zero as limit; that is to say, f_x has a differential coefficient f_{yx} with respect to y at the point (a, b) .

NOTE.—It should be noticed that it does not follow that f_{yx} is continuous at (a, b) with respect to either variable, still less with respect to the ensemble (x, y) . In fact, f_{yx} need not exist on the axial cross, except at the point (a, b) itself. It will be proved below (§ 8, Theorem 7, Cor. 2), however, that when it does exist on the axial cross, it is continuous at (a, b) .

Theorem 2 (the Dini-Schwarz Theorem).—*If in addition to the requirements of the preceding theorem, f_y exists along the line $y=b$ at and in the neighbourhood of the point (a, b) , then f_{xy} also exists at the point (a, b) and has the same value as f_{yx} .*

For, in this case,

$$\frac{f_y(a+h, b) - f_y(a, b)}{h} = \text{Lt}_{h=0} m(a, b; a+h, b+k),$$

and has, therefore, as was shown in the preceding proof, only one limit when h has zero as limit; that is to say, f_y has a differential coefficient f_{xy} with respect to x at the point (a, b) .

Since the value of $f_{xy}(a, b)$, like that of $f_{yx}(a, b)$, is thus the unique double limit of $m(a, b; a+h, b+k)$,

$$f_{xy}(a, b) = f_{yx}(a, b). \quad \text{Q.E.D.}$$

NOTE 1.—It has nowhere been assumed that the unique limit postulated is finite; it may be $+\infty$ or $-\infty$.

NOTE 2.—The arguments used in proving the Repeated Theorem of the Mean, as well as Theorems 1 and 2, which depend on it, being based on the Theorem of the Mean in one dimension, do not require the full assumption that f_{yx} exists at every point internal to the rectangle. *It suffices, in fact, if there is no distinction of right and left* with respect to the derivatives of f_x regarded as a function of y , as follows*

* Thinking of the representation in two dimensions, right and left with respect to y is, of course, "up and down."

from the more general statement of the Theorem of the Mean given in the paper "On Derivates and the Theorem of the Mean," by W. H. and G. Chisholm Young, *Quart. Journal of Math.*, Oct. 1908.

NOTE 3.—It should be noticed that, without making any properly two-dimensional hypothesis, *we can prove that f_{yx} exists at the point (a, b) if we postulate that f_{yx} exists and is finite along the ordinate $x=a$ in some open neighbourhood of the point (a, b) but not at the point itself, and that it has a unique limit as we approach the point (a, b) along that ordinate.*

This is an immediate consequence of the Theorem of the Mean for a single variable applied to f_x regarded as a function of y at points of the ordinate $x=a$.

For this reason, as well as from the fact that the axial cross can, as we have seen, be omitted, the usual statements of the Schwarz-Dini conditions seem to leave something to be desired.*

It may further be remarked that in Note 2 it is sufficient if the derivates of f_x with respect to y on the ordinate $x=a$ present no distinction of right and left.† All these sets of conditions are, of course, sufficient but not necessary. It is obvious that sufficient conditions of a less restricted character can be formulated; some of these will be found below.

NOTE 4.—In the proof of the existence of a unique double limit for $m(a, b; a+h, b+k)$, the assumption (1) that f_x exists in the open neighbourhood of the point (a, b) , excluding the ordinate $x=a$, was rendered necessary in order to apply the Repeated Theorem of the Mean.

This being postulated, the further assumption (2) that f_x also exists on the ordinate $x=a$, at and in the neighbourhood of the point (a, b) , is needed in order to ascribe a meaning to the expression

$$f_x(a, b+k) - f_x(a, b)$$

and so to prove the existence of f_{yx} at the point (a, b) .

Similarly, without postulating (2), the existence of f_{xy} requires the assumption (3) that f_y exists on the line $y=b$, at and in the neighbourhood of the point (a, b) .

* Stolz, *Grundzüge der Differenzial- und Integralrechnung*, 1893, p. 147. Hobson, "Partial Differential Coefficients and Repeated Limits," *Proc. L.M.S.*, 1906, series ii., vol. v. p. 234. See also *Functions of a Real Variable*, p. 318 and the errata. It should be noticed that the account in the book is really earlier than that in the paper.

† See footnote on preceding page.

The assumptions (2) and (3) are independent, and precisely of the same importance in the Schwarz-Dini conditions. Both could be omitted if we suitably enlarged our definition of a repeated differential coefficient, for without them we could still prove that the double incrementary ratio had a unique limit.*

PART II.

§ 5. We now proceed to entirely drop the assumption as to the existence of a differential coefficient with respect to y of $\frac{df}{dx}$, and begin by proving certain properties of the derivatives of $\frac{df}{dx}$ with respect to y .

The following lemma is an immediate consequence of the definitions and Lemma 1.

Lemma 3.—If f_x exists at a point (a, b) and on its ordinate in the neighbourhood, any derivative of f_x with respect to y is a double limit of $m(a, b; a+h, b+k)$.

For, putting

$$m_x(b, b+k) = \frac{f(x, b+k) - f(x, b)}{k}$$

$$m(a, b; a+h, b+k) = \frac{m_{a+h}(b, b+k) - m_a(b, b+k)}{h}.$$

Therefore, by the definition of a differential coefficient,

$$\text{Lt}_{h=0} m(a, b; a+h, b+k) = \frac{d}{dx} m_x(b, b+k) \text{ at } x=a,$$

$$= \frac{f_x(a, b+k) - f_x(a, b)}{k}.$$

Hence, by the definition of the derivatives,

$\text{Lt}_{k=0} \text{Lt}_{h=0} m(a, b; a+h, b+k) =$ the derivatives of f_x with respect to y at the point (a, b) .

Thus these derivatives are repeated limits, and therefore, by Lemma 1, double limits of $m(a, b; a+h, b+k)$. Q.E.D.

Lemma 4. — If f_x exists at every point of an open rectangle $(a, b; a+h, b+k)$, excluding the bounding ordinates, and is a finite continuous function of y at every internal point, then, if $f(x, y)$ still exists and is a continuous function of y on the boundary,

$$L \leq m(a, b; a+h, b+k) \leq U,$$

* It should be noticed that the difference between Hobson's conditions, *loc. cit.*, and those given by Schwarz consists in the omission of assumption (3), and that this is only possible in the light of the extended definitions suggested by Hobson. If such a definition be adopted, there is, it would appear, no reason for making the assumption (2) either.

where L and U are the lower and upper bounds of any derivate of f_x with respect to y at points internal to the rectangle.

For, since

$$m(a, b; a+h, b+k) = \frac{m_{a+h}(b, b+k) - m_a(b, b+k)}{h},$$

and the conditions of the Theorem of the Mean are satisfied,

$$\begin{aligned} m(a, b; a+h, b+k) &= \frac{f_x(x', b+k) - f_x(x', b)}{k} \quad \text{where } a < x' < a+h, \\ &= \text{incrementary ratio of function } f_x(x', y) \text{ at the pair of} \\ &\quad \text{values } y = b, y = b+k. \end{aligned}$$

Hence

$$L_{x'} \leq m(a, b; a+h, b+k) \leq U_{x'}$$

where $L_{x'}$ and $U_{x'}$ are the bounds of this incrementary ratio at pairs of values of y internal to the interval $(b, b+k)$. Since these bounds are the same as those of any derivate of $f_x(x', y)$ with respect to y at points internal to the same interval (that is, at certain points internal to the rectangle), we have, *a fortiori*,

$$L \leq m(a, b; a+h, b+k) \leq U. \quad \text{Q.E.D.}$$

Theorem 3.—If f_x exist in a closed neighbourhood of the point (a, b) , and any derivate of f_x with respect to y has only one double limit as we approach the point (a, b) by means of points not on the axial cross, then f_x has a differential coefficient f_{yx} with respect to y at the point (a, b) .

For, by Lemma 4,

$$L \leq m(a, b; a+h, b+k) \leq U,$$

where L and U are the lower and upper bounds of the derivate in question in the completely open rectangle $(a, b; a+h, b+k)$.

But, under the given condition, L and U evidently have the same limit when h and k each approach zero in any manner whatever. Hence $m(a, b; a+h, b+k)$ has only one double limit, viz. the common limit of L and U .

Further, f_x being defined on the ordinate $x=a$, the derivatives of f_x with respect to y are, by Lemma 3, double limits of $m(a, b; a+h, b+k)$. They are therefore all equal to the unique double limit of $m(a, b; a+h, b+k)$, so that f_x has a differential coefficient at (a, b) with respect to y .

NOTE.—If f_x exist on the ordinate $x=a$, and any derivate with respect to y of f_x is continuous with respect to y at the point (a, b) , then f_{yx} exists at the point (a, b) .

This follows, of course, from the well-known theorem in one dimension.

§ 6. I now proceed to give the property which may be regarded as constituting the *raison d'être* of the sets of sufficient conditions for the reversibility of the order of partial differentiation with which we are at present occupied, in the case when f_x and f_y exist throughout an area. This property is that *all the derivates of f_x with respect to y , and all the derivates of f_y with respect to x , have the same upper and lower bounds in every completely open area, and these bounds are the same as those of the double incrementary ratio in the same area.* This follows at once from the following theorem, which results immediately from Lemmas 3 and 4.

Theorem 4.—*The upper and lower bounds of all the derivates of f_x at all the internal points of a closed rectangle with sides parallel to the axes, throughout which f_x exists, and at every internal point of which f_x is a continuous function of y , are the same and are equal to those of the double incrementary ratio $m(x, y; x', y')$ at pairs of points internal to that rectangle.*

For, if (a, b) is internal to the rectangle, so is a portion of its ordinate. Hence any derivate of f_x with respect to y at (a, b) , being a repeated limit of $m(a, b; a+h, b+k)$, \geq lower bound of $m(x, y; x', y')$, since we can restrict h and k so that the point $(a+h, b+k)$ is also internal to the rectangle considered. Therefore

$$\text{lower bound of derivate} \geq \text{lower bound of } m(x, y; x', y'). \quad (1).$$

But, by Lemma 4,

$$\text{lower bound of derivate} \leq m(x, y; x', y')$$

when the points (x, y) , and (x', y') and \therefore the rectangle $(x, y; x', y')$, are internal to the region. Therefore

$$\text{lower bound of derivate} \leq \text{lower bound of } m(x, y; x', y'). \quad (2).$$

From (1) and (2) the equality of the lower bounds of the derivate and the double incrementary ratio follows. Similarly, the equality of the upper bounds may be proved.

Cor. 1.—*The bounds of the double incrementary ratio in any rectangle $(a, b; a+h, b+k)$ are unaltered if we include in the rectangle any or all of its boundary points.*

For the argument used in proving (2) is equally valid if the points (x, y) and (x', y') are on the boundary. Hence

$$\text{lower bound of derivate in the open rectangle} \leq \text{lower bound of } m(x, y; x', y') \\ \text{in the closed rectangle.}$$

But we must take the sign of equality, since the lower bound of

$m(x, y; x', y')$ in the closed rectangle is \geq that in the open rectangle, and therefore, by (1), \geq the lower bound of derivate in the open rectangle.

Cor. 2.—*The bounds of the derivates of f_x with respect to y in any rectangle $(a, b; a+h, b+k)$ are unaltered if we include the left-hand and the right-hand bounding ordinates of the rectangle, and the bounds of the right-hand (left-hand) derivates are unaltered if we include the lower (upper) bounding lines.*

For in any of these cases the argument used in proving (1) holds. Thus the result (1) still holds. The lower bound of the derivate having certainly not been increased by the introduction of the new points, (2) also holds, whence it follows that the lower bound of the derivate is still equal to that of the double incrementary ratio.

The results of this theorem and its corollaries may be shortly summed up in the following general statement:—

The bounds of any derivate of f_x in any rectangle $(a, b; a+h, b+k)$, open or closed, are the same as those of the double incrementary ratio in the same rectangle, including or not some or all of its boundary points.

Here the expression “derivate of f_x in a rectangle” is to be so understood that only such derivates are included at boundary points as result from operations in the rectangle; e.g. at the left-hand bottom corner, only the right-hand derivates with respect to y of f_x .

Cor. 3.—*The bounds of any derivate of f_x and of the double incrementary ratio in any neighbourhood of a point P are unaltered if we omit the axial cross through P.**

Cor. 4.—*All the derivates of f_x have the same associated plane limiting functions ϕ and ψ , and these are the same whether the values at the point itself be included or not, and are still the same if in calculating ϕ and ψ we omit the axial cross through P.*

In particular, therefore, all the derivates of f_x lie between their ϕ and ψ , and if one of these derivates is continuous with respect to the ensemble (x, y) , so are they all, and they are all equal. Moreover, the same is true if it is only known that one of these derivates has a unique limit as we approach the point by points not lying on the axial cross.

§ 7. The fact that the bounds of the double incrementary ratio in a rectangle $(a, b; a+h, b+k)$ are unaltered, when we include some or all of its boundary points, is at first sight remarkable, considering that we have not made the assumption that $f(x, y)$ is continuous with respect to the

* Or any isolated set of axial crosses; that is, any set of axial crosses such that, taking any point P internal to the region considered, we can find a region containing P as internal point and intersected by at most one of the axial crosses.

ensemble (x, y) . If we make this assumption, the invariability of the bounds results at once from the simple fact that the double incrementary ratio assumes every value between its upper and lower bounds in the closed rectangle at pairs of points internal to the rectangle; a proof of this is appended, similar to that given for one dimension in the paper on derivatives quoted, and easily generalisable for n -dimensions.

The assumptions actually made so far, however, are only that $f(x, y)$ should be continuous with respect to x , and f'_x with respect to y . From this it follows that $f(x, y)$ is a continuous function of x and of y (though not necessarily of the ensemble (x, y)), save for an arbitrary function of y alone, which disappears, being purely additive, from the double incrementary ratio.

It will be found later that even the existence of f'_x is not essential; it is sufficient if any one of the derivatives of $f(x, y)$ with respect to one variable is a continuous function of the other variable (§ 14).

Theorem 5.—*The double incrementary ratio*

$$m(x, y; x', y') = \frac{f(x, y) - f(x, y') - f(x', y) + f(x', y')}{(x - x')(y - y')}$$

where $f(x, y)$ is a continuous and finite function of the ensemble (x, y) , assumes every value between its upper and lower bounds in a closed rectangle $(a, b; c, d)$ at points at which

$$a < x' < x < b, \text{ and } c < y' < y < d \quad . \quad . \quad . \quad . \quad (1)$$

Consider the four-dimensional function $m(x, y; z, w)$. This is definite and continuous at every point of the closed four-dimensional parallelepiped $(a, a, b, b; c, c, d, d)$, bounded by the eight hyperplanes

$$x = a, x = b, y = a, y = b, z = c, z = d, w = c, w = d \quad . \quad . \quad (2)$$

excepting along the diagonal hyperplanes

$$x = y, z = w \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Moreover, identically

$$m(x, y; z, w) = m(y, x; z, w) = m(x, y; w, z) \quad . \quad . \quad (4)$$

so that all the values which are assumed by $m(x, y; z, w)$ in the parallelepiped, and on its boundary are the same as those assumed at points of the four-dimensional wedge

$$a \leq x' < x \leq b, c \leq y' < y \leq d \quad . \quad . \quad . \quad . \quad (5)$$

bounded by the six hyperplanes

$$x = a, y = b, z = c, w = d, \text{ and } x = y, z = w \quad . \quad . \quad . \quad (6).$$

The points of the wedge in the last two hyperplanes, which may be called its basal hyperplanes, do not come into consideration, since $m(x, y; z, w)$ is undefined there.

Let k be any value between the upper and lower bounds (not included) of $m(x, y; x', y')$ in the primary closed rectangle. Then we can find two finite values k_1 and k_2 , both of which are *assumed* by the double incremental ratio, where

$$k_1 < k < k_2.$$

By what precedes there is a point P_1 of the four-dimensional wedge (5) at which

$$m(x, y; z, w) = k_1,$$

and a point at which

$$m(x, y; z, w) = k_2.$$

If this second point is not on one and the same hyperplane boundary as P_1 , let it be P_2 . In the contrary case take as P_2 a neighbouring point inside the wedge at which

$$m(x, y; z, w) = k'_2,$$

where

$$k < k'_2;$$

this is possible by reason of the continuity of $m(x, y; z, w)$.

The stretch P_1P_2 will then be such that every one of its internal points is internal to the wedge, so that $m(x, y; z, w)$ is definite at every internal point of P_1P_2 , and has at the end-points the definite values k_1 and k_2 (or k'_2).

At every point of the closed stretch P_1P_2 , therefore, $m(x, y; z, w)$ is a continuous function of the ensemble (x, y, z, w) , and therefore assumes at some point internal to P_1P_2 , and therefore to the wedge, the value k intermediate between its values at the end-points P_1 and P_2 . This proves the theorem.*

Cor.—*The upper and lower bounds of the double incremental ratio in the rectangle are unaltered if we omit any or all the boundary points from the possible positions of the points (x, y) and (x', y') .*

§ 8. Returning to § 6, the result of Cor. 4 is sufficiently interesting to merit separate proof.

Theorem 6.—*If f_x exists throughout a closed area and is a continuous function of y , the value of any derivate of f_x with respect to y at any internal point P of the area lies between the upper and lower bounds of the derivate in a non-axial † neighbourhood of P .*

* The extension of this result to the case when $f(x, y)$ is a quasi-finite function, which, in the case of one variable, was given in the paper quoted on p. 144, line 2, is not elaborated here, the attention being concentrated in this paper, in the first instance, on finite functions.

† That is, omitting the axial cross $x=a$ and $y=b$.

Further, the upper and lower bounds in the non-axial neighbourhood are the same as those in the closed neighbourhood of P.

For

$$m(a, b; a+h, b+k) = \frac{m_{a+h}(b, b+k) - m_a(b, b+k)}{h} = \frac{d}{dx} m_{a+\theta h}(b, b+k) \\ = \frac{f_x(a+\theta h, b+k) - f_x(a+\theta h, b)}{k}$$

where $0 < \theta < 1$.

Now this latter single incrementary ratio lies between the upper and lower bounds of the derivatives of $f_x(a+\theta h, b+\theta k)$, where $0 < \theta < 1$, θh being constant; *a fortiori*, between the bounds of the same when θh is variable; and again, *a fortiori*, between the upper and lower bounds of any derivate of f_x at all points (x', y') of the neighbourhood not on the axial cross.

This being so, it is true of the unique limit of $m(a, b; a+h, b+k)$ with respect to h , that is, of $\frac{f_x(a, b+k) - f_x(a, b)}{k}$, and therefore of all the limits of this last incrementary ratio, that is, the derivatives of f_x at the point (a, b) .

To prove the second part of the theorem, we only have to notice that the value of any such derivate of f_x on the axial cross being a limit of values in the non-axial neighbourhood, lies between the bounds in that neighbourhood.

Cor. 1.—If ϕ and ψ are the associated plane limiting functions of any derivate of f_x , that derivate lies between ϕ and ψ at every point, and ϕ and ψ are the same whether defined in Baire's* manner or that usually adopted by myself† or calculated always with reference to a non-axial neighbourhood.

It should be noticed that the corresponding results for the first derivatives, obtained from one-dimensional theory, are less extended. Any derivate with respect to x of $f(x, y)$, when $f(x, y)$ is continuous with respect to x throughout a closed area, lies between its upper and lower bounds in any neighbourhood omitting the ordinate of the point. Hence the omission of the ordinate does not affect the bounds, and the plane associated functions are the same whether defined as by Baire, or in my usual manner, or calculated from neighbourhoods omitting the ordinate.

Cor. 2.—If any derivate of f_x has only one limit in a non-axial neighbourhood of a point P, it is continuous at P with respect to the ensemble (x, y) , and therefore in particular is continuous at P with respect to y .

* Closed neighbourhood.

† Open neighbourhood, omitting values at the point itself.

This corollary, together with the one-dimensional theorem given in the note on Theorem 3, constitutes a proof of Theorem 3. (Cp. also the note at the end of Theorem 1. It is unnecessary to do more than call attention to the special case in which f_x has a differential coefficient with respect to y throughout the non-axial neighbourhood.)

§ 9. Theorem 7.—If $f(x, y)$ is a finite continuous function of x at every point of a closed rectangle $(a, b; a+h, b+k)$, and has, except possibly on the bounding ordinates $x=a$ and $x=a+h$, a differential coefficient f_x with respect to x , which is a finite continuous function of y : (1) there is a point internal to the rectangle at which one lower derivate of f_x with respect to $y \geq m(a, b; a+h, b+k)$, and the upper derivate on the other side $\leq m(a, b; a+h, b+k)$; (2) each of the derivates of f_x with respect to y assumes values $\geq m(a, b; a+h, b+k)$ and values $\leq m(a, b; a+h, b+k)$.

For

$$\begin{aligned} m(a, b; a+h, b+k) &= \frac{m_{a+h}(b, b+k) - m_a(b, b+k)}{h} = \frac{d}{dx} m_{a+\theta h}(b, b+k) \\ &= \frac{f_x(a+\theta h, b+k) - f_x(a+\theta h, b)}{k} \end{aligned}$$

= incrementary ratio of $f_x(a+\theta h, y)$ regarded as a function of y .

Hence, by Theorem 4, p. 10, of the paper "On Derivates and the Theorem of the Mean," already quoted, the first result follows, and by Theorem 5 of the same paper we get the result (2).

§ 10. We now return to the question touched on as to the definition of the second partial differential coefficients. So far we have followed the usual definition adopted by Schwarz and others. This postulates that $\frac{d}{dx} \frac{df}{dy}$ should only be considered to exist at a point (a, b) if $\frac{df}{dy}$ exists on the abscissa $y=b$ at and in the neighbourhood of the point (a, b) , and similarly that $\frac{d}{dy} \frac{df}{dx}$ should only be considered to exist at the point (a, b) if $\frac{df}{dx}$ exists on the ordinate $x=a$ at and in the neighbourhood of the point (a, b) . This definition was historically inevitable. To speak of a second differential coefficient implies that we are dealing with a differential coefficient of a differential coefficient. Extension of knowledge has, however, always led to an extension of meaning, and there is no *a priori* reason why we should not, if we find it convenient, extend the definition as follows:—

If $\frac{df}{dx}$ exists at the point (a, b) but not in the neighbourhood of (a, b) on the ordinate $x=a$, and yet it happens that the derivates with respect to x of $f(x, y)$ (upper and lower, left and right) should all four have a

differential coefficient with respect to y , and, further, all these differential coefficients are equal, this common value shall be called the partial differential coefficient $\frac{d^2f}{dydx}$.

Hobson, who has recently adopted this definition,* is thus enabled to omit from the Schwarz-Dini conditions the condition that $\frac{df}{dy}$ should exist on the abscissa $y=b$, except at the point (a, b) itself.

It is evident from § 4, Note 4, of the present paper that with this definition we must, if we are to be consistent, omit the condition included in that specifically retained by Hobson in a gloss on his condition (1),† that $\frac{df}{dx}$ exists in the neighbourhood of the point (a, b) on the ordinate $x=a$.

In fact, the assumption that $\frac{df}{dx}$ exists at the point (a, b) is equivalent to the statement that, whatever sequence of values with zero as limit we ascribe to h ,

$$\frac{f(a+h, b) - f(a, b)}{h}$$

has the same limit.

At any point $(a, b+k)$ let us take a sequence of values of h , giving the upper right-hand derivate of $f(x, y)$ with respect to x as the unique limit of

$$\frac{f(a+h_{k,n}, b+k) - f(a+h_{k,n}, b)}{h_{k,n}}.$$

At the point (a, b) , however, any sequence of values of h will do, by the assumption made. Hence the incrementary ratio of this derivate may be written

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{f(a+h_{k,n}, b+k) - f(a+h_{k,n}, b)}{h_{k,n}} &= \lim_{n \rightarrow \infty} \frac{f(a+h_{k,n}, b) - f(a, b)}{h_{k,n}} \\ &= \lim_{n \rightarrow \infty} m(a, b; a+h_{k,n}, b+k). \end{aligned}$$

Thus the derivates with respect to y of the upper right-hand derivate of $f(x, y)$ with respect to x are repeated limits, and therefore double limits, of the double incrementary ratio, so that they all coincide at (a, b) if a unique double limit exists.

As pointed out in Note 4, § 4, the existence of $\frac{df}{dx}$ on the ordinate $x=a$ is immaterial in the proof of the existence of a unique limit for the double incrementary ratio. Thus the assumption in question is superfluous in

* *Loc. cit.*

† Line 13, *loc. cit.*

proving that all the derivates with respect to y of the upper right-hand derivate with respect to x are equal at the point (a, b) . For definiteness the upper right-hand derivate was taken, but it is clear that the same proof applies for any derivate. Thus, without assuming the existence of f_x on the ordinate $x=a$ except at the point (a, b) , all the derivates with respect to y of all the derivates with respect to x of $f(x, y)$ are equal at the point (a, b) , so that, in Hobson's sense, the partial differential coefficient $\frac{d^2f}{dydx}$ exists at the point (a, b) .

Adopting Hobson's definition, the set of sufficient conditions for the reversibility of the order of partial differentiation discussed by Schwarz, Dini, and Hobson, reduce, in the light of what proceeds, to the following simpler form:—

If (1) $\frac{d^2f}{dydx}$ exists at all points in a non-axial neighbourhood of the point (a, b) ;

(2) The values of $\frac{d^2f}{dydx}$ in a non-axial neighbourhood of the point (a, b) have a unique limit at (a, b) , finite or infinite;

(3) $\frac{df}{dx}$ not only, as is implied in (1), exists, and is a finite and continuous function of y in a non-axial neighbourhood of (a, b) , but also exists on the abscisse $y=b$, at and in the neighbourhood of (a, b) ;

(4) $\frac{df}{dy}$ exists at the point (a, b) ;

then, in Hobson's sense, $\frac{d^2f}{dydx}$ and $\frac{d^2f}{dxdy}$ both exist and are equal to the unique limit specified in (2).

§ 11. It should, however, be noticed that the retention of the existence of the first differential coefficient in Dr Hobson's definition of the second differential coefficient must be regarded, from the point of view adopted by him, as somewhat arbitrary. It seems not less unreasonable to assert the existence of $\frac{d^2f}{dydx}$ when all the derivates with respect to y of all the derivates with respect to x coincide at the point (a, b) , which may, of course, be the case without $\frac{df}{dx}$ existing at (a, b) .

Adopting this definition, if we know that the double incrementary ratio has a unique limit at the point (a, b) , we can easily show that all the derivates of derivates of $f(x, y)$ are equal at (a, b) . For, *although the derivates of derivates are not necessarily themselves repeated limits of*

$m(a, b; a+h, b+k)$, they lie between the upper and lower double limits of $m(a, b; a+h, b+k)$ when h and k are indefinitely diminished.

To prove this let $v(x, y)$ denote any derivate of $f(x, y)$ with respect to x , e.g. the right-hand upper derivate; then, for each value of y there is a sequence of values of h which determines the value of $v(x, y)$ as a limit, viz.

$$v(x, y) = \text{Lt}_{n=\infty} \frac{f(x+h_{y,n}, y) - f(xy)}{h_{y,n}}.$$

Therefore

$$\text{the derivates of } v(xy) \text{ at } (a, b) = \text{Lt}_{k=0} \text{ Lt}_{n=\infty} \frac{1}{k} \left\{ \frac{f(x+h_{k,n}, b+k) - f(x, b+k)}{h_{k,n}} - \frac{f(a+h_{0,n}, b+k) - f(a, b)}{h_{0,n}} \right\}.$$

Changing $h_{k,n}$ in the first fraction into $h_{0,n}$, we get $m(a, b; a+h_{0,n}, b+k)$.

Remembering that the sequence $h_{k,n}$ gave us the highest possible limit for the fraction, this change may diminish and cannot increase the value of the right-hand side of the preceding identity. Hence

$$\begin{aligned} \text{the derivates of } v(x, y) &\leq \text{Lt}_{k=0} \text{ Lt}_{n=\infty} m(a, b; a+h_{0,n}, b+k) \\ &\leq \text{a double limit of } m(a, b; a+h, b+k) \\ &\leq \text{lowest double limit of } m(a, b; a+h, b+k). \end{aligned} \quad (1)$$

Similarly, changing $h_{0,n}$ to $h_{\bar{k},n}$, we get

$$\text{the derivates of } v(x, y) \geq \text{greatest double limit of } m(a, b; a+h, b+k), \quad (2)$$

which proves the statement made above.

Hence it follows that, *if the limits of* $m(a, b; a+h, b+k)$ *are finite, the first derivates with respect to one variable are continuous functions of the other variable.*

In particular, this will be the case at every internal point of a rectangle throughout which $m(x, y; x', y')$ *is bounded.*

It must here be pointed out that, in speaking of the derivates of derivates, and in deducing the above results, we have tacitly assumed that the first derivates are finite, or quasi-finite. Unless this is the case, their derivates are not properly defined, and it would only be by introducing fresh conventions that we could apply the above results to such cases. Thus, for instance, if $f(x, y)$ is the sum of a function of x and a function of y , $m(x, y; x', y')$ is zero always, the same is therefore true of the repeated derivates, where they are defined. If the function of x has an infinite derivate, or differential coefficient anywhere, in particular if it is a non-differentiable function of x , in which case such points are dense

everywhere* on the x -axis, for each such value of x this derivate, regarded as a function of y , has its derivates undefined. We might, of course, agree still to define its derivates as having the value zero, since a change in the value of y makes no change in that of the function. In this case the above result would hold in a more extended sense.

§ 12. From the inequality of the preceding article it follows that if $m(a, b; a+h, b+k)$ has a unique double limit, all the derivates of its derivates coincide at the point (a, b) ; so that, in the sense indicated at the beginning of this article, we might choose to say that $\frac{d^2f}{dydx}$ and $\frac{d^2f}{dxdy}$ both exist at (a, b) and are equal.

In this sense the conditions for this to be the case, obtained by omitting superfluous conditions from the Schwarz-Dini conditions, are as follows:—

(1) $\frac{d^2f}{dydx}$ exists at all points in a non-axial neighbourhood of the point (a, b) ;

(2) The values of $\frac{d^2f}{dydx}$ in a non-axial neighbourhood of the point (a, b) have a unique limit at (a, b) , finite or infinite;

(3) $\frac{df}{dx}$ exists and is a finite and continuous function of y in a neighbourhood of the point (a, b) not necessarily including any point on the ordinate $x = a$.

§ 13. Hitherto we have always had to make some assumption as to the existence of $\frac{df}{dx}$, although it has been found unnecessary to assume its existence at the point itself or on its ordinate. If we make no assumption as to the existence of $\frac{df}{dx}$, it is still possible to enunciate a set of sufficient conditions for the equality of all the derivates of derivates at the point (a, b) , which is a direct generalisation of the Schwarz-Dini conditions.

To do this we require a theorem of the bounds analogous to Theorem 4, and to prove this we require a theorem analogous to Theorem 8. We begin with some preliminary remarks and inequalities.

We had the identities

$$m_x(b, b+k) = \frac{f(x, b+k) - f(x, b)}{k} \quad (1)$$

$$m(a, b; a+h, b+k) = \frac{m_{a+h}(b, b+k) - m_a(b, b+k)}{h} \quad (2)$$

* See a paper by the author "On Non-differentiable Functions," *Mess. of Math.*, September 1908.

Hence we see that the various derivatives of the function $m_x(b, b+k)$ at the point $x=a$ are the various limits of $m(a, b; a+h, b+k)$ with respect to h, k being constant.

There are two special sequences of values of h on the right which need consideration. The first is that which yields the upper derivative with respect to x of $f(a, b+k)$, say $f^{+x}(a, b+k)$, as the limit of

$$\frac{f(a+h, b+k) - f(a, b+k)}{h}.$$

This will yield a limit, or limits, of

$$\frac{f(a+h, b) - f(a, b)}{h}$$

not greater than the upper derivative $f^{+x}(a, b)$, and therefore for $m(a, b; a+h, b+k)$ a limit, or limits, not less than

$$\frac{f^{+x}(a, b+k) - f^{+x}(a, b)}{k} \quad \dots \quad (3)$$

Hence, by what was said above, this last expression is \leq the upper right-hand derivative of $m_x(b, b+k)$.

The second sequence is that which yields $f^{+x}(a, b)$. The consideration of this sequence shows in like manner that the expression (3) is \geq the lower right-hand derivative of $m_x(b, b+k)$.

Thus the expression (3) lies between the upper and lower right-hand derivatives of $m_x(b, b+k)$.

Similarly, considering the sequences which yield respectively the lower right-hand derivatives, we find that the incrementary ratio of the lower right-hand derivative, that is,

$$\frac{f_{+x}(a, b+k) - f_{+x}(a, b)}{k},$$

lies between the upper and lower right-hand derivatives of $m_x(b, b+k)$.

Similarly, the incrementary ratios of the left-hand derivatives of $f(x, y)$ lie between those of $m_x(b, b+k)$.

§ 14. Theorem 8.—*If $f(x, y)$ is a finite continuous function of x at every point of a closed rectangle $(a, b; a+h, b+k)$, while any one of the four derivatives (upper, lower, left, and right) of $f(x, y)$ with respect to x is a finite continuous function of y , except possibly on the bounding ordinates, then any derivative with respect to y of this derivative with respect to x assumes values both \leq and also \geq the double incrementary ratio $m(a, b; a+h, b+k)$ of the rectangle at points internal to the rectangle.*

For $m_x(b, b+k)$ is then also finite and continuous with respect to x in

the closed x -interval $(a, a+h)$. We may therefore apply Theorem 5 of the paper "On Derivates," already quoted. Hence any derivate of $m_x(b, b+k)$ assumes at points internal to this x -interval values both \leq and also $\geq m(a, b; a+h, b+k)$, since this is the incrementary ratio of $m_x(b, b+k)$ for the pair of x -values a and $a+h$. But, by the preceding article, the incrementary ratio of the upper right-hand derivate

$$\frac{f^{+x}(x, b+k) - f^{+x}(x, b)}{k}$$

lies between the upper and lower right-hand derivates of $m_x(b, b+k)$ at every point, and therefore also assumes values both \leq and also $\geq m(a, b; a+h, b+k)$.

Since, by the preceding article, a similar argument applies to any other derivate of $f(x, y)$ with respect to x , this proves the theorem.

Theorem 9.—*Under the same assumptions as in the preceding theorem, the upper and lower bounds of the derivates with respect to y of the chosen derivate with respect to x in the completely open rectangle $(a, b; a+h, b+k)$ are the same as those of the double incrementary ratio $m(x, y; x', y')$.*

For, if L' and U' denote the lower and upper bounds of $m(x, y; x', y')$, it follows from the preceding theorem that the repeated derivate considered assumes values \geq any quantity $< U'$ and values \leq any quantity $< L'$.

Hence

$$\text{the lower bound of the repeated derivate} \leq L' \leq U' \leq \text{upper bound of repeated derivate} \quad (1)$$

But, by § 10, the repeated derivate in question lies at any point (x, y) internal to the rectangle between the bounds of $m(x, y; x', y')$, when the point (x', y') approaches (x, y) as limit. Therefore the repeated derivate at the point (x, y) lies between quantities which themselves lie between L' and U' , so that $L' \leq$ chosen derivate at $(x, y) \leq U'$, and therefore

$$L' \leq \text{lower bound of repeated derivate} \leq \text{upper bound of same} \leq U' \quad (2)$$

from (1) and (2) L' is the lower and U' the upper bound of the repeated derivate. Q.E.D.

Cor. 1.—*The bounds of the double incrementary ratio in any rectangle $(a, b; a+h, b+k)$ are unaltered if we include in the rectangle any or all of its boundary points.*

For the argument used in proving (1) is equally valid if L' and U' denote the bounds of $m(x, y; x', y')$ in the closed rectangle. Also, since this change does not increase L' nor decrease U' , (2) still holds; hence the

bounds of $m(x, y; x', y')$ in the closed rectangle are the same as those of the repeated derivate in the open one, and therefore are the same as the bounds of $m(x, y; x', y')$ in the open rectangle.

Cor. 2.—*The bounds of a repeated derivate in a rectangle $(a, b; a+h, b+k)$ are unaltered by the inclusion of boundary points, in as far as at each such boundary point the repeated derivate in question is defined by means of sequences of points lying in the rectangle.*

Thus, for instance, in dealing with the right-hand derivatives with respect to y of the right-hand derivatives with respect to x , we may include the left-hand bottom corner; and in dealing with the right-hand derivatives with respect to y of the left-hand derivatives with respect to x , we may include the right-hand bottom corner.

For in this case the values of h and k used in the reasoning of § 10 do not take us out of the rectangle, so that (2) still holds, the bounds both of the derivate and the double incrementary ratio being taken with respect to the rectangle including the boundary points in question. Since, by the preceding corollary, the bounds of the double incrementary ratio are the same as in the open rectangle, (1) also still holds, since the introduction of new points does not increase the lower bound or decrease the upper bound of the repeated derivate.

Hence the bounds of the repeated derivate are the same as those of the double incrementary ratio, and are therefore unaltered.

Cor. 3.—*The bounds of any repeated derivate of $f(x, y)$ and of the double incrementary ratio are unaltered, if we omit the axial cross through P.*

Cor. 4.—*All the derivatives with respect to y of any derivate of $f(x, y)$ with respect to x , which is a continuous function of y in a neighbourhood of a point P, have at P the same associated plane-limiting functions ϕ and ψ , and these are the same whether the values at the point itself be included or not, and are still the same if in calculating ϕ and ψ we omit the axial cross through P. In particular, therefore, all the derivatives of such derivatives lie between their ϕ and ψ , and if one of these repeated derivatives is continuous with respect to the ensemble (x, y) , so are they all, and they are equal. Moreover, the same is true if it is only known that one of these repeated derivatives has a unique limit as we approach the point by points not lying on the axial cross.*

§ 15. It should be noticed that the restriction in the preceding theorem and corollaries, implied in the fact that we only consider derivatives with respect to one variable of a function or derivate *which is itself a continuous function of that variable*, is an essential one. In fact, the

theorems of the paper "On Derivates and the Theorem of the Mean," used in proving those of the present paper, do not in general hold if we omit the condition of continuity.

Thus, for instance, denoting by $w(x)$ the function graphically represented by the bisector of the angle between the axes of x and y from the origin to $x = \frac{1}{3}$ both inclusive, and by the line perpendicular to this through the point $(1, 0)$ from $x = \frac{1}{3}$ not inclusive to $x = 1$ inclusive (fig. 1), the function $w(x)$ does not obey the theorems of the paper quoted, although it is continuous except only at $x = \frac{1}{3}$. In fact, the left-hand derivate is $+1$ from $x = 0$ to $x = \frac{1}{3}$, and is -1 afterwards. But the right-hand derivate,

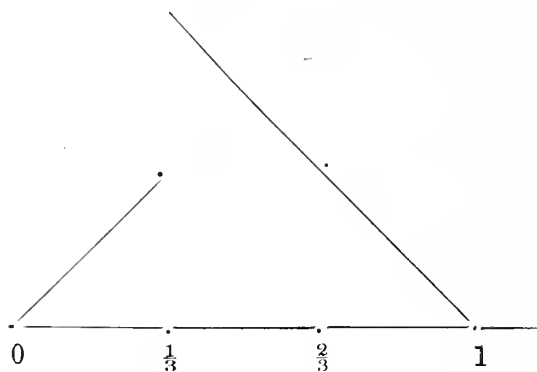


FIG. 1.

though it agrees with the left-hand one, except at the discontinuity $x = \frac{1}{3}$, has there the value $+\infty$.

From this function it is easy to construct a function of x and y which violates the result of the preceding theorem, owing to the fact that it is not a continuous function of x , or that its derivates with respect to x are discontinuous functions of y .

Example 1.—Let

$$f(x, y) = yw(x).$$

Here the left-hand derivate with respect to x is always either y or $-y$, so that the left-hand derivates with respect to y of the left-hand derivate with respect to x are always finite, being either 1 or -1 . But the right-hand derivates of the same derivate, though agreeing with its left-hand derivates except on the ordinate $x = \frac{1}{3}$, have there the value $+\infty$. Thus the upper bound is in the case of one repeated derivate finite and in another infinite.

Example 2.—In the preceding example $f(x, y)$ was itself discontinuous with respect to x . If we write

$$f(x, y) = xw(y),$$

we get a function which is continuous with respect to x , and which yet

violates the result of Cor. 4 owing to the fact that the first derivatives with respect to x are not continuous with respect to y .

In both these examples the appearance of infinite values for certain of the first derivatives disturbs the existence of certain of the second derivatives. The following example is, for this reason, perhaps more striking.

Example 3.—

$$f(x, y) = y - x, \text{ if } x \leq y, \\ = x^2 - y^2, \text{ if } x \geq y.$$

The region considered is the square $(0, 0; 1, 1)$.

Here $f(x, y)$ is a continuous function of the ensemble (x, y) ; its derivatives are, however, discontinuous.

$$f_{-x} = f^{-x} = -1, \text{ if } x \leq y, \\ = 2x \text{ if } x > y.$$

Therefore the right-hand derivatives with respect to y of the left-hand derivative with respect to x is zero everywhere, but the left-hand derivative with respect to y , though zero except on the line $x = y$, has here the value $+\infty$, since it is the limit of

$$\frac{-1 - 2x}{h}.$$

Again,

$$f_{+x} = f^{+x} = -1, \text{ if } x < y, \\ = 2x \text{ if } x \geq y.$$

Therefore the left-hand derivative with respect to y of the right-hand derivative with respect to x is zero everywhere, but the right-hand derivative with respect to y , though zero except on the line $x = y$, has here the value $+\infty$. Hence

$$L = 0,$$

while, with an obvious notation for the upper bounds,

$$U_{+y, -x} = U_{-y, +x} = 0, \quad U_{-y, -x} = U_{+y, +x} = +\infty.$$

§ 16. Of course, if all the derivatives with respect to y of some derivative of $f(x, y)$ with respect to x are finite at any point, that first derivative is bound to be continuous at the point with respect to y . Hence, as a special case of Cor. 4, we have the following:—

If $f(x, y)$ is a continuous function of x and of y , and if the derivatives with respect to one variable of one or more derivatives of $f(x, y)$ with respect to the other variable are all finite in a neighbourhood of a point P , they all have the same plane associated limiting functions ϕ and ψ at P , and lie between them.

Hence if one of them is a continuous function of the ensemble (x, y) , so are they all, and they are equal. The same is true if one of them has a unique limit for all modes of approach to the point by means of points not on the axial cross through the point.

PART III.

§ 17. Lemma.—If f_x is a continuous function of the ensemble (x, y) at the point (a, b) , and f_y exists at the point (a, b) , then

$$\frac{f(a+h, b+k) - f(a+h, b)}{k}$$

has a unique double limit f_y for all modes of approach of the point $(a+h, b+k)$ to the point (a, b) such that $\left| \frac{k}{h} \right|$ has not zero for one of its limits.

For

$$\begin{aligned} \frac{f(a+h, b+k) - f(a, b+k)}{h}h + \frac{f(a, b+k) - f(a, b)}{k}k &= f(a+h, b+k) - f(a, b) \\ &= \frac{f(a+h, b+k) - f(a+h, b)}{k}k + \frac{f(a+h, b) - f(a, b)}{h}h. \end{aligned}$$

Hence, using the Theorem of the Mean,

$$f_x(a+\theta h, b+k)h + (f_y(a, b) + \eta_k)k = \frac{f(a+h, b+k) - f(a+h, b)}{k}k + (f_x(a, b) + \eta_h')h,$$

where

$$0 < \theta < 1,$$

θ being an otherwise unknown function of h and k ; η_k is a function of k , but not of h , which vanishes with k ; η_h' is a function of h , but not of k , which vanishes with h .

Therefore

$$\left| \frac{f(a+h, b+k) - f(a+h, b)}{k} - f_y(a, b) \right| \leq \left| f_x(a+\theta h, b+k) - f_x(a, b) \right| \left| \frac{h}{k} \right| + \left| \eta_k \right| + \left| \eta_h' \right| \left| \frac{h}{k} \right|.$$

Let us approach in such a way that

$$\left| \frac{k}{h} \right| \geq m \quad \dots \quad (1)$$

where m is any positive quantity, finite or infinite. Then the right-hand side of the last inequality has the limit zero, therefore the same is true of the left-hand side.

Since this is true for all positive values of m , it is true for all modes of approach such that $\frac{k}{h}$ has not zero for one of its limits, which is equivalent to the statement to be proved.

Theorem 10.—*If f_{xx} is a continuous function of the ensemble (x, y) at the point (a, b) , and f_{yx} exists at the point (a, b) , then the double incremental ratio has a unique double limit whose value is $f_{yx}(a, b)$ for all modes of approach other than those for which $\left|\frac{k}{h}\right|$ has zero for one of its limits.*

For

$$m(a, b; a+h, b+k) = \frac{m_{a+h}(b, b+k) - m_a(b, b+k)}{h},$$

where

$$m_x(b, b+k) = \frac{f(x, b+k) - f(x, b)}{k}.$$

Therefore

$$\begin{aligned} m(a, b; a+h, b+k) &= \frac{d}{d_x} m_x(b, b+k) \text{ for some value of } x > a \text{ and } < a+h, \\ &= \frac{f_x(a+\theta h, b+k) - f_x(a+\theta h, b)}{k} \end{aligned} \quad (1)$$

where

$$0 < \theta < 1,$$

and θ is an otherwise unknown function of h and k .

But since, by hypothesis, $\frac{d}{d_x} f_x$ is a continuous function of the ensemble (x, y) at the point (a, b) , and $\frac{d}{d_y} f_x$ exists there, we may apply the lemma.

But if $\left|\frac{k}{h}\right|$ has not zero for one of its limits, $\left|\frac{k}{\theta h}\right|$ certainly has not zero for one of its limits, since, whatever function θ is of h and k ,

$$\left|\frac{k}{\theta h}\right| > \left|\frac{k}{h}\right|.$$

Hence the lemma holding for the function $f_x(x, y)$

$$\frac{f_x(a+\theta h, b+k) - f_x(a+\theta h, b)}{k}$$

has a unique limit, and this is $f_{yx}(a, b)$. Hence, by (1) the same is true of $m(a, b; a+h, b+k)$ for all the specified modes of approach of h and k to zero.

Theorem 12.—*If f_{xx} and f_{yy} are both continuous functions of the*

ensemble (x, y) at the point (a, b) , and f_{xy} and f_{yx} both exist at the point (a, b) , then

$$f_{xy}(a, b) = f_{yx}(a, b),$$

and $m(a, b; a+h, b+k)$ has a unique double limit equal to both these repeated partial differential coefficients.

For, since f_{xx} is a continuous function of (x, y) , and f_{yx} exists at the point (a, b) , it follows by the preceding theorem that $m(a, b; a+h, b+k)$ has a unique limit for all modes of approach for which $\left|\frac{k}{h}\right|$ has not the limit zero, and this limit is $f_{yx}(a, b)$.

Again, since f_{yy} is a continuous function of (x, y) , and f_{xy} exists at the point (a, b) , it follows that $m(a, b; a+h, b+k)$ has a unique double limit for all modes of approach for which $\left|\frac{h}{k}\right|$ has not the limit zero, and that this limit is $f_{xy}(a, b)$.

Hence $f_{yx}(a, b) = f_{xy}(a, b) =$ unique limit of $m(a, b; a+h, b+k)$ for all modes of approach which do not make either $\left|\frac{k}{h}\right|$ or $\left|\frac{h}{k}\right|$ have the limit zero.

But for modes of approach which make $\left|\frac{k}{h}\right|$ have zero for a limit, $m(a, b; a+h, b+k)$ has, by the above, the limit $f_{yx}(a, b)$, and for modes of approach which make $\frac{h}{k}$ have the limit zero, it has $f_{xy}(a, b)$ for limit. Hence for all modes of approach $m(a, b; a+h, b+k)$ has the unique limit $f_{yx}(a, b) = f_{xy}(a, b)$.

(Issued separately March 2, 1909.)

X.—Laboratory Note on a Study of Polarisation by means of the Dolezalek Electrometer. By A. F. Ewan, Physical Laboratory, Edinburgh University. (*Communicated by Professor J. G. MACGREGOR.*)

(Read July 20, 1908. MS. received September 15, 1908 ; revised January 23, 1909.)

THE following experiments originated in a series of laboratory exercises which I performed on the use of the Dolezalek electrometer. The instrument which I used was made by Bartels, and had its needle suspended by a phosphor-bronze strip. This needle was raised to a potential of about 160 volts by being connected to one terminal of a battery of eighty small secondary cells, the other terminal of which was connected to earth. When at this potential the instrument gave, for a potential difference of 1 volt between its two pairs of quadrants, a deflection of 200 divisions on a scale 200 centimetres distant from it. The maximum deflection in the experiments described was 166 divisions, *i.e.* 33·2 cms., giving $\tan^{-1} \cdot 166$ as maximum angle of deflection; and therefore, by taking the deflection as proportional to the potential difference, a maximum error of 0·7 per cent. is introduced.

Although these experiments are merely preliminary ones, I desire to give an account of the results obtained, as I am unable to go on with them at present. They refer to two subjects—(1) the variation of the electromotive force of polarisation of a cell with the difference of potential between the electrodes, and (2) the applicability of Wiedeburg's formula for the variation, with time, of the electromotive force of polarisation during the flow of the current.

I. VARIATION OF THE ELECTROMOTIVE FORCE OF POLARISATION WITH THE DIFFERENCE OF POTENTIAL BETWEEN THE ELECTRODES.

It is usually held that there is no such variation, but that, for a given electrolyte, given electrodes, etc., it depends only on the current density. This assumption seems never to have been tested; and as it underlies certain methods of using the Wheatstone bridge in the measurement of the conductivity of electrolytes, a test is desirable. For this purpose I passed a current through two cells of very different resistances, but otherwise

exactly similar, arranged in series. Their electrodes consequently differed considerably in potential. The differences of potential between the electrodes and the liquid in the cells immediately behind the electrodes were then measured, at observed times, by means of the electrometer and according to Fuchs'* method; and the corresponding curves were plotted for all four electrodes. The time was observed by means of the second-hand of an ordinary watch.

Each cell consisted of two small rectangular troughs connected by an upturned U-tube, each U-tube being fitted, for convenience in filling, with a branch tube closed by a clip. The electrodes consisted of rectangular pieces of platinum of approximately the same area, viz. 1.3 square inches, with platinum wires welded to them at their upper edges. The electrolyte was connected with the electrometer by means of calomel electrodes constructed according to the method described in Findlay's *Practical Physical Chemistry* (1906), page 202.

To enable me to connect the electrometer quickly to the various electrodes and the liquid in which they were placed, the opposite pairs of quadrants of the electrometer were connected by copper wires to a mercury-pool commutator, with which were also connected by copper wires the four platinum electrodes and the corresponding calomel electrodes.

To make sure that the nozzles of the calomel electrodes were out of the lines of flow of the current, the platinum electrodes were so placed, and the depth of liquid in the troughs so arranged, that the electrodes almost completely filled the cross-section of the electrolyte in the troughs; and the calomel electrodes were then placed so as to have their nozzles immediately behind the platinum plates, thus being in that part of the trough through which the current does not pass directly.

The troughs were provided with glass covers to diminish evaporation and to keep out dust; and before performing each experiment the mercury in the commutator pools was cleaned, and the electrodes were white-heated.

The U-tubes of the two cells differed in length and cross-section to such an extent that the resistances of the liquid they contained were about 90,000 and 2000 ohms respectively. Thus the difference of potential of the electrodes of the one cell would be about forty-five times as great as that of the other.

The electrolyte used in the two cells was the same as in the calomel electrodes, viz. potassium chloride, and the concentration was the same also. Thus there would be no electromotive force at the nozzles of the calomel electrodes.

* *Pogg. Ann.*, clvi. 158, 1875.

Two of the more successful series of observations of the cotemporaneous variation of the potential difference between the electrodes and the electrolyte directly behind them are shown graphically in fig. 1. They are marked (*a*) and (*b*) respectively. The graphs were obtained by plotting the polarisation against the time for each of the four electrodes—the polarisation being the excess of the potential difference between the electrode and the electrolyte behind it over the potential difference just before starting the polarising current.

The set of curves marked (*a*) was the first of the two obtained. It showed that the anode curves and cathode curves of the two cells were neither coincident nor of exactly the same form, although in the case of the cathode curves the difference in form was slight. Differences in form were to be expected, because of differences in the temperatures of the cells and in the convection currents in the neighbourhood of the electrodes, no precautions having been taken in these preliminary experiments to exclude such differences. These differences in form must prevent exact coincidence; but there was obviously not even general coincidence, both the anode and cathode of the high-resistance cell being throughout more highly polarised than those of the low-resistance cell. This might be due either to the dependence of the polarisation on the potential difference of the electrodes or to differences in the current density at the electrodes caused by inequality of their area. If it was due to differences in the current density, interchanging the electrodes might be expected to change the relative positions of the curves. Accordingly, the anodes and cathodes respectively of the two cells were interchanged, and a set of observations represented by the curves marked (*b*) (fig. 1) obtained. As the polarising current in the two series of observations (*a*) and (*b*) would not be exactly the same, the two sets of curves might be expected on this account to differ slightly in form. It will be seen that this difference is very small in the case of the anode curves, while all the four cathode curves are of approximately the same form. From the general agreement in form of the anode curves of the same cell, it would seem that the form of the curves was determined by the structure of the cell rather than by the electrode. Notwithstanding the differences in form, it is clearly seen from set (*b*) that after a short time (in which the disturbing influences might be expected to be most marked) the anode and cathode of the low-resistance cell are more highly polarised than those of the high-resistance cell, and approximately by the same amount as, in the case of set (*a*), the anode and cathode of the high-resistance cell were the more highly polarised.

This shows us that the non-coincidence of the curves for the high-

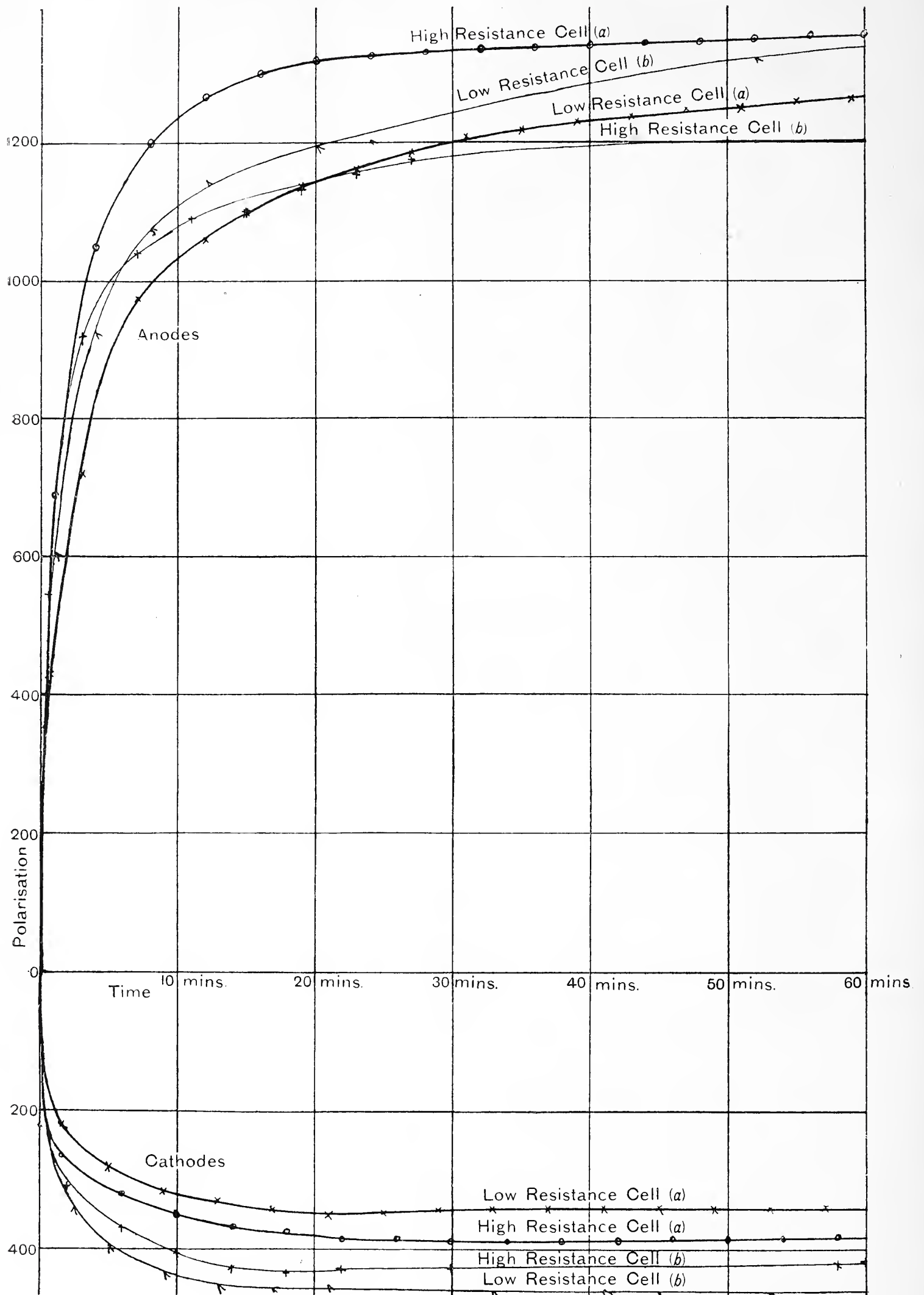


FIG. 1.

resistance and low-resistance cells, which at first sight suggested variation of polarisation with potential difference of electrodes, may have been largely, and possibly wholly, due to differences in the area of the electrodes. Additional observations would need to be made, with exactly cut electrodes and more complete precautions against the sources of error mentioned above, before a test of this kind would be completely satisfactory. But it may be concluded at least that this experiment showed no evidence of variation of the electromotive force of polarisation with the difference of potential of the electrodes.

2. WIEDEBURG'S FORMULA FOR THE VARIATION OF POLARISATION WITH TIME DURING THE FLOW OF THE CURRENT.

Wiedeburg,* making simple assumptions as to the way in which the polarisation itself occurs, and as to the occlusion and diffusion which go on at the same time, has developed the formula :

$$p = \frac{1 - e^{-yt}}{a - be^{-yt}}$$

for the polarisation, as a function of the time, during the passage of the polarising current, the circuit being of constant resistance and containing a cell of constant electromotive force, and the electrodes being of equal area. The a , b , and y in his formula are functions of one or more of the electromotive force of the galvanic cell, the resistance of the circuit, etc., and of the initial conditions.

To test this formula, I passed a current from a secondary cell through the same circuit as in the former case ; but instead of observing all four electrodes I watched only one, thus being able to keep the electrometer permanently attached to one platinum electrode and the corresponding calomel electrode. Consequently the deflections were much more accurately determined.

Three of the series of observations thus made are given graphically in fig. 2. It will be seen that, even with an ordinary watch, the curves can be followed quite a considerable distance from the bend towards the origin by means of the Dolezalek electrometer.

Wiedeburg tested his formula by the aid of the law, obtained empirically by Bouty,† that the polarisation within very short intervals of time from the starting of the polarising current is given by the formula :

$$p = \frac{at}{1 + \beta t},$$

where a and β are constants.

* *Wied. Ann.*, li. 302, 1894.

† *C.R.*, cxvi. 628, 1893.

He did so by showing that Bouty's formula is a particular case of his own. For since

$$e^{-yt} = 1 - yt + \frac{y^2}{1.2}t^2 - \frac{y^3t^3}{1.2.3} + \text{etc.},$$

we have, for sufficiently small values of t ,

$$e^{-yt} = 1 - yt.$$

Substituting this expression for e^{-yt} into Wiedeburg's formula, we have

$$p = \frac{yt}{a - b(1 - yt)}$$

which is of the same form as Bouty's. We may write Bouty's formula as

$$\frac{p}{t} + p\beta = a,$$

and in this form it indicates that if $\frac{p}{t}$ be plotted against p , the graph obtained should be rectilinear within the time to which this law is applicable.

Fig. 3 shows the curves of $\frac{p}{t}$ against p , as given by my observations for the two anodes No. 1 and No. 2 of fig. 2. It will be seen that between the points corresponding to 0.1 and 0.18 min. on the one curve, and 0.1 and 0.2 min. on the other, the curvature is not appreciable. Above this the curves begin to bend as we might expect. Difficulty of observation both of time and polarisation makes it impossible for me to take points on my curves corresponding to smaller values of the time, but from these curves we can conclude that Bouty's form of Wiedeburg's formula holds in the cases investigated within a time range of about a fifth of a minute.

For larger values of the time which necessitate the inclusion of the term $\frac{y^2t^2}{2}$, but allow of the term $\frac{y^3t^3}{2 \times 3}$ being excluded from the expansion of e^{-yt} , we have

$$e^{-yt} = 1 - yt + \frac{y^2t^2}{2}.$$

Wiedeburg's formula becomes, in that case,

$$p = \frac{t - At^2}{B + Ct - Dt^2},$$

where $A = \frac{y}{2}$; $B = \frac{a-b}{y}$; $C = b$; $D = \frac{by}{2}$.

My observations enable me to make at least a partial test of the applicability of this formula within time ranges which are beyond the

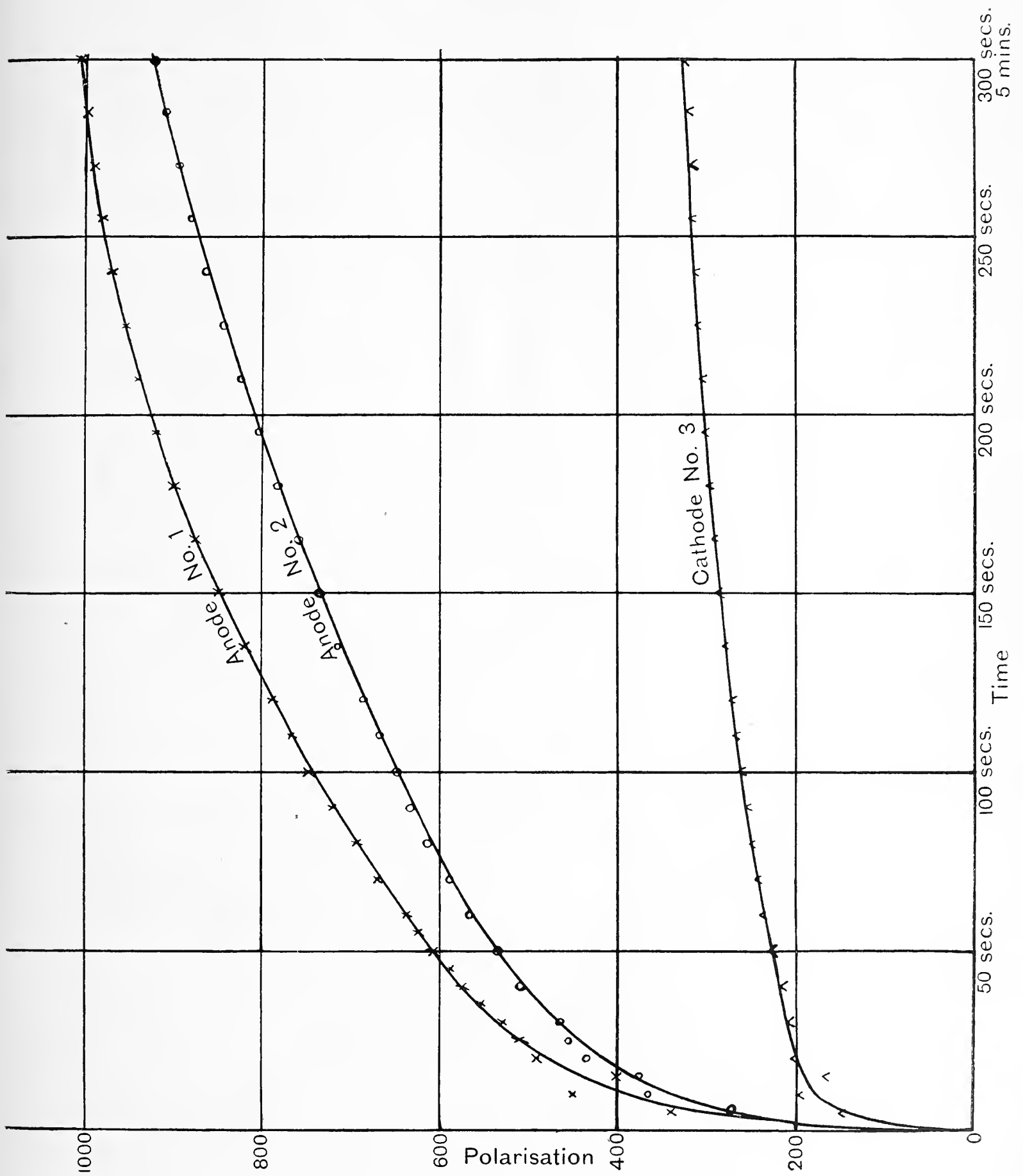


Fig. 2.

limit of Bouty's law. For this purpose I have, in the case of each of the curves in fig. 2, obtained values of the constants A , B , C , and D by selecting four points on the curves, reading off the corresponding values of p and t , substituting into the above expression, and solving the four equations thus obtained. The four points selected had to be on a part of the curve for which the values of t were, at the one end, large enough to give trustworthy observations of p , and, at the other end, small enough to be within the assumed

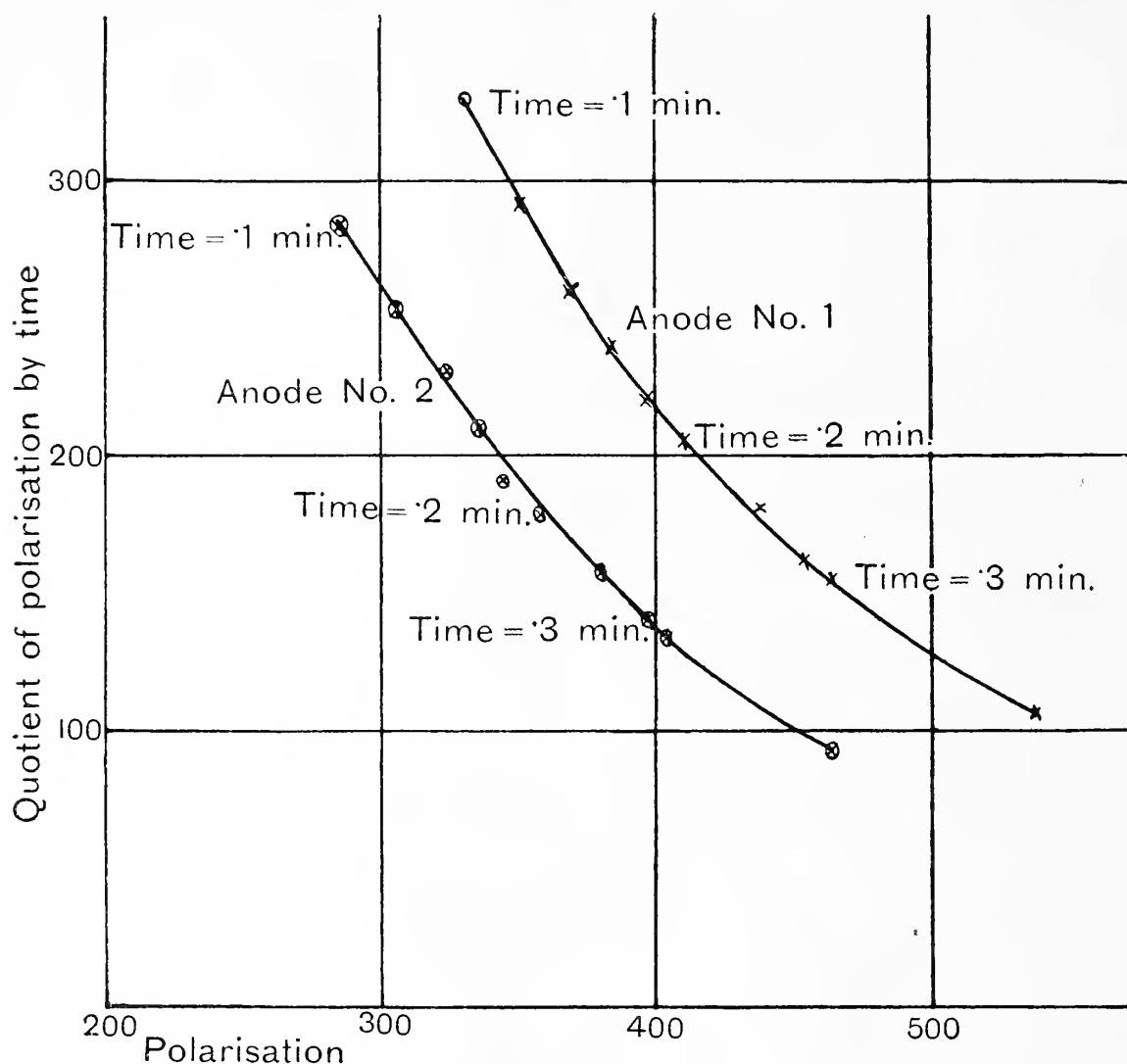


FIG. 3.

time limit of the applicability of the formula. They had, consequently, to be taken from comparatively small portions of the curves. It is clear, however, that according to the expression under consideration we must have $p=0$ for $t=0$, and that the origin is thus necessarily a point on the curve.

Tables I., II., and III. give the values of the polarisation calculated by the above formula, the values observed, and the differences between them.

It will be seen that in all three tables (especially in the first and third) the observed and calculated values of the polarisation show considerable

TABLE I.

CURVE FOR ANODE NO. 1.

$$A = -\cdot803, B = \cdot000090, C = \cdot002202, D = -\cdot000526.$$

The constants were calculated from the values of the polarisation, as obtained from the curve, corresponding to the times 0·5, 1, 1·5, and 2 mins.

Time (in mins.).	Polarisation.		
	Observed.	Calculated.	Difference.
1	450	400	50
1	490	479	11
1	530	530	0
1	587	589	- 2
1	638	640	- 2
2	790	790	0
3	900	895	5
4	969	974	- 5
5	1006	1034	- 28

TABLE II.

CURVE FOR ANODE NO. 2.

$$A = -\cdot645, B = \cdot000075, C = \cdot00244, D = -\cdot000428.$$

The constants were calculated as above from the values of the polarisation at the times 0·5, 1, 2, and 3 mins.

Time (in mins.).	Polarisation.		
	Observed.	Calculated.	Difference.
1	365	374	- 9
1	435	433	2
1	465	469	- 4
1	565	558	7
2	686	687	- 1
3	783	783	0
4	864	858	6
5	923	920	3
6	959	970	- 11

TABLE III.

CURVE FOR CATHODE NO. 3.

$$A = -8.461, B = .00005, C = .0100, D = -.02962.$$

The constants were calculated as above from the values of the polarisation at the times 0.3, 0.4, 0.5, and 0.6 mins.

Time (in mins.).	Polarisation.		
	Observed.	Calculated.	Difference.
$\frac{1}{12}$	155	131	24
$\frac{1}{6}$	200	158	42
$\frac{1}{3}$	205	191	14
$\frac{2}{3}$	210	210	0
$\frac{5}{6}$	220	223	-3
$\frac{5}{6}$	230	232	-2
1	240	239	1
$1\frac{1}{2}$	258	251	7

disagreement for large and for small values of the time. For the larger values of the time the disagreement is of no moment, for they are beyond the limit within which the formula was assumed to hold.

In the case of the small values of the time large differences were to be expected, because time was observed by means of an ordinary watch, and also because the curves are exceedingly steep for these values of the time. Comparatively small errors in determining the time of starting the current would produce large differences of uniform sign (see Tables I. and III.), while small errors in determining the times of the observation of the earlier stages of polarisation would give rise to differences of variable sign (see Table II.). Throughout the range which can fairly be tested by these experiments the agreement is fairly good.

In the original form of Wiedeburg's formula there are only three constants, whereas I have used four. These must therefore be connected by some relation, which can readily be shown to be $AC=D$, by using the values of A, B, C, and D obtained on page 170. In none of my three sets of constants does this relation hold; the product of AC being in all cases just about three times as great as D. This seems to indicate that the points used for the determination of these constants have been taken too far along the curve, *i.e.* at places where Wiedeburg's formula may no longer be applicable. It would be possible by repeated trials to find a range of points which would furnish such values of the constants as to satisfy the condition $AC=D$, but as these observations are merely

preliminary ones I have not thought it advisable to do so. I hope before long to repeat the experiments with more satisfactory precautions against the sources of error referred to above.

The expenses of the above experiments were defrayed in part from the Tait Memorial Fund.

(Issued separately April 12, 1909.)

XI.—A Special Form of Photographic Camera for Recording the Readings of the Scales of Scientific Instruments. By James Robert Milne, D.Sc.

(MS. received February 23, 1909. Read December 21, 1908.)

SOME years ago, when employed in work that involved the use of a polarimeter, there was brought forcibly to my notice the great loss of time which the frequent reading of the Nicol dial necessitates. To avoid this, I had a special form of camera made for the purpose of recording the readings automatically. This method proved successful from the first, but it is only lately that the mechanism of the camera attained its final form; for it has required several years of experiment and alteration to evolve a form of camera which should be wholly automatic and yet reasonably simple. It will be readily understood that the function of the camera is to make a series of photographs of the scale-and-vernier of the polarimeter or other instrument, which, on subsequent examination, will show the relative position of the scale-and-vernier at the time of each exposure.

When in use, the camera is so placed that its lens is focussed on the scale-and-vernier, and the latter is lit up by an incandescent gas burner with a piece of ground-glass interposed to diffuse the light evenly, or by an incandescent electric lamp having a "frosted" bulb. The scale-and-vernier must be so arranged that it is the scale, and not the vernier, which moves. The camera can be placed at any reasonable distance from the polarimeter according to the focal length of the lens employed, and of course it is not attached to the polarimeter in any way. Nor need the camera be placed directly at right angles with the scale-and-vernier to be photographed, as I find that a mirror can be used to deflect the rays of light, which allows of the camera's being placed on one side. The mirror need not be silvered on the front surface, as the reflection from the glass is not sufficiently bright to give rise to a double image on the negative, hence a piece of ordinary plate mirror-glass answers very well. If it be desired to keep the room dark, it is not a difficult matter to arrange some form of covering for the path of the light on its way to the camera.

The exposure required with the ordinary silvered brass scale is only a fraction of a second.

A single plate is sufficient for many records, as only the vernier and the

part of the scale adjacent to it at the time require to be shown in each of the records. In the camera about to be described, 60 photographs in 6 rows of 10 each are obtained on a 5 × 4-inch plate.

A general idea of the appearance of the camera may be gained from fig. 1, which for the sake of greater clearness is diagrammatic only, as are all the subsequent figures. A is a wooden upright, in the front of which is fixed the tube B carrying the lens. Behind A comes the "carriage" C, which rolls backwards and forwards by means of two wheels running on the top of A. To the back of C is attached a light metal frame F, which moves up and down C on guides, and in which is inserted the dark slide. It will be noticed that the upright A is slightly tilted; this causes C, when free, to roll down DE to the lower end E. Similarly, when free, F falls down the guides on C.

A horizontal slot is cut in C, so that the light from the lens tube B may

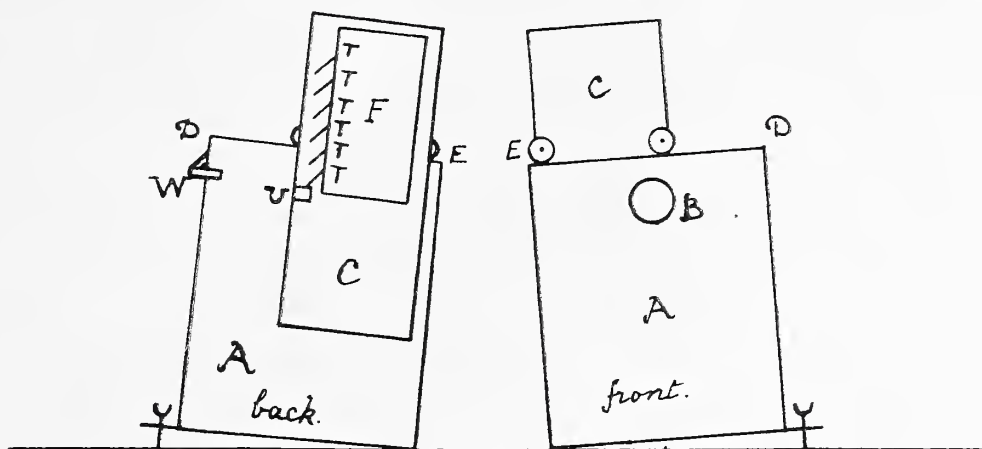


FIG. 1.

pass to the plate behind. In order to exclude the entrance of extraneous light between A and C, a short camera bellows is attached to the front of C, and its plush-coated front edge rubs lightly against the back of A. A similar bellows is attached to F, and rubs against C.

The mechanism of the camera is shown in fig. 2. The part of it which actuates the shutter is like an "Atwood's machine"; A with its weighted arms being the wheel, and B and C respectively the weights. C is a long strip of metal which moves up and down through the lens tube. A long-shaped opening is cut in the middle of it, and during each movement there is a brief interval when light can pass along the lens tube and reach the plate behind. The arms of the axle A increase its inertia, so that at the completion of each revolution it operates the lever D without fail. Below the weight B there is hung a weighted lever R, pivoted at P, which is shown drawn up by the cord attached to the pin of the disc F. As B is somewhat lighter than C, no movement will take place until the experimenter gives

the knob G at the end of the axle H a half-turn, thus releasing R, and adding its weight to B. B and R together then overcome C and fall; C rises, and the axle A with its arms revolves. After one revolution the motion is stopped because the pin projecting from the ball K (the ball Q has no pin) now comes in contact with the lower side of the lever D. When the experimenter gives another half-turn to the knob G, R is again raised, B is then overweighted by C, and the ball K revolves back to its original position, where it is arrested by the pin's coming in contact with the upper side of D. Thus every time the experimenter gives the knob G a half-turn an exposure is made.

It remains to be explained how the plate is moved after each exposure.

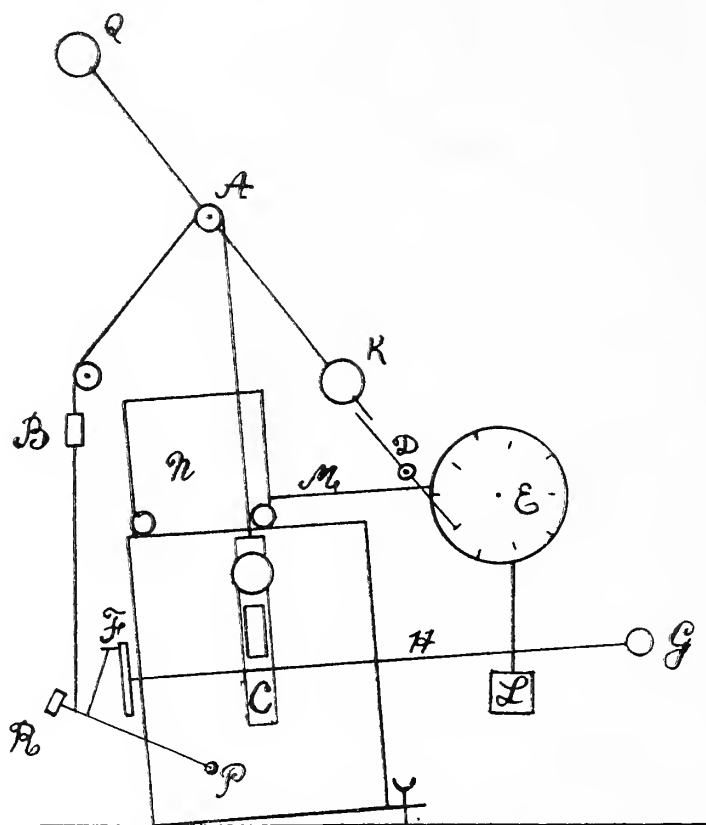


FIG. 2.

The parts D and E may be compared to the "escapement" of a clock. Each time D is pushed from one side to the other by the pin of the ball K, a tooth on the wheel E is freed, because the teeth are spaced alternately nearer to and farther from the centre of the wheel, and E revolves under the action of the weight L* until it is stopped by the engagement of the next tooth with D. In this way the cord M wound round a small wheel on the axle of the wheel E is made to draw the carriage N one step along, at the *conclusion* of each exposure. This small wheel (S, fig. 3) is not fixed to the axle of the wheel E, and is only carried round with it because of its ratchet P, which is drawn forward by the arm Q attached to

* The weight L requires to be removed each time a fresh plate is inserted.

the axle *O*. On completing a revolution the projecting end of *P* comes in contact with a fixed stop *R*, and is depressed, which raises the other end free from *Q*, thus releasing the small wheel from the axle, so that the carriage can run back to its starting-point, the small wheel turning backwards freely on the axle until its ratchet once more on the completion of a revolution re-engages with *Q*.

The return of the carriage *C* (fig. 1) brings about also the vertical movement of the plate. It will be noticed that the frame *F* containing the dark slide is furnished with a row of 6 catches *T*, *T*, . . . , each of which in turn engages with a stop *V* fixed to the carriage *C*. When *C* runs back along *DE*, the stop *W* fixed to *A* strikes against the catch which is in action for the time being, and pushes it clear of *V*, thus permitting of the descent of the frame *F* until the catch next above in its turn engages with *V*.

Fig. 4 shows a full-size reproduction of a negative, from which it will be seen that the record made by the camera is a very legible one.

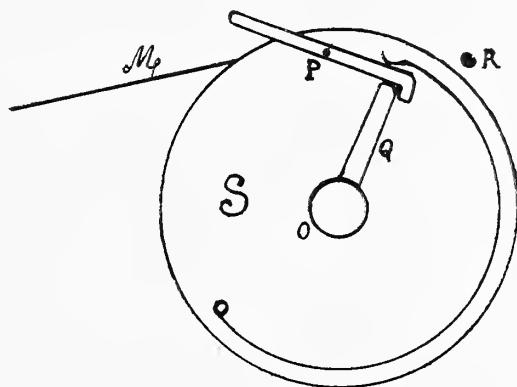


FIG. 3.—*M* is the cord to the carriage, which is wound round the grooved circumference of the wheel *S*.

In order that the observer may be able to identify individual photographs, or to indicate the commencement of a new series of records, a means is provided whereby he can produce at will on the photographs either a single or a double wedge-shaped mark. Two small levers are pivoted so that either may be swung in front of the scale into the field of view of the camera by a touch of the observer's finger, their ends being respectively in the shape of a single and of a double wedge. In my own work I do not, as a rule, require to use these marks, as I always take the average of ten readings, and the record of each experiment occupies just one line on the plate. In order to relieve the observer from having to count the exposures, I have arranged an electric bell which rings at each return of the carriage when the ten exposures have been made. The observer is thus able to give his undivided attention to the work of adjusting the polarimeter, and the speed at which the readings can be made is greatly

increased, for the eye of the observer need never leave the telescope of the polarimeter, nor is he required to remove his hand from the screw actuating the Nicol dial.

I am accustomed to use a "backed ordinary" plate, and a hydroquinone developer, but any of the usual plates and developers may be employed.



FIG. 4.

No special care need be taken with the development of the plate, because all that it is necessary to obtain is a negative which is legible, and this can be secured without any difficulty. "Prints" of the negatives need not be made, as it is so easy to store the latter. The strip of blank space which appears at the foot of each negative is convenient for the purpose of writing particulars upon relating to the experiment whose readings it records.

The advantages of this method of making photographic records of the readings of an instrument are—

(1) The saving of labour and strain to the eye of the observer, which is left in a better condition for its principal work.

(2) The elimination of the possibility of errors of reading on the part of the observer.

(3) The great saving of time which is effected.

(4) The elimination of all personal bias due to a knowledge of the results that are being obtained.

(5) The securing of a permanent record of the readings.

(6) The entire separation of the recording apparatus from the polarimeter or other instrument, whose mechanism is not added to or interfered with in any way.

I have to express my great obligation to the Carnegie Trustees for granting me the funds necessary to defray the expenses of the construction of the apparatus.

DEPARTMENT OF PHYSICS,
EDINBURGH UNIVERSITY.

(Issued separately April 17, 1909.)

XII.—On an Improved Form of Magnetometer and Accessories for the Testing of Magnetic Materials at Different Temperatures. By James G. Gray, B.Sc., Lecturer on Physics in the University of Glasgow, and Alexander D. Ross, M.A., B.Sc., Assistant to the Professor of Natural Philosophy in the University of Glasgow. *Communicated by Professor A. Gray, F.R.S.*

(MS. received January 29, 1909. Read February 1, 1909.)

IN the usual form of magnetometer the magnetising solenoid is placed with its axis in the magnetic east and west line passing through the magnetometer needle. The effect of the current is balanced at the needle by means of a compensating coil connected up in the circuit. This latter coil has its axis coincident, or nearly so, with that of the solenoid. When a feebly magnetic specimen is under examination the solenoid, and consequently the compensating coil, must of necessity be brought up close to the needle. If large magnetising currents are employed, any small shift of the coils from their correct positions may be sufficient to seriously impair the balance. In consequence of this the operation of adjusting the position of the compensating coil (the solenoid is usually clamped once for all in a convenient position) is a difficult one, especially as the slight inevitable movement of the coil which results from clamping it in position generally results in the balance being interfered with.

Even if this adjustment be accomplished with the requisite accuracy for the undisturbed position of the magnetometer needle, it does not necessarily follow that the compensation is complete for the needle in its deflected position. In practice the axes of the solenoid and compensating coils are in general slightly inclined to one another and to the east and west line passing through the needle. The effect of this is to increase the directive force on the needle for one direction of the current and to diminish it for the other. That this is the case will be seen from fig. 1, in which the want of alignment of the coil and solenoid has been greatly exaggerated. The magnetometer needle is situated at the point P, and it has been assumed that the solenoid and coil are so placed that they produce fields at P in the directions PS and PC respectively. If the intensity of the field due to the solenoid be denoted by F_s , and that due to the coil by F_c , then since the coils balance for the undisturbed position of the needle it follows that $F_s \cos \theta_1 = F_c \cos \theta_2$. There are left, however, the components of the intensities

in the north and south direction, and it is evident from the figure that if H is the horizontal component of the earth's magnetic field at P , the total directive force at the needle is $H + (F_s \sin \theta_1 + F_c \sin \theta_2)$. If the current is reversed in the circuit the directions of F_s and F_c change, and the directive force at the needle becomes $H - (F_s \sin \theta_1 + F_c \sin \theta_2)$.

The presence of the effect referred to may be made apparent by placing a permanent magnet close to the magnetometer, and thus deflecting the needle. On reversing a current in the circuit a change in the deflection will in general be observed. The magnitude of the errors introduced may be determined in this way for various parts of the scale and allowed for in the results, or the coils may be rotated until the effect disappears. If the former method is adopted, the labour of computing the results is much increased, and, further, it is difficult to make a proper correction, since the

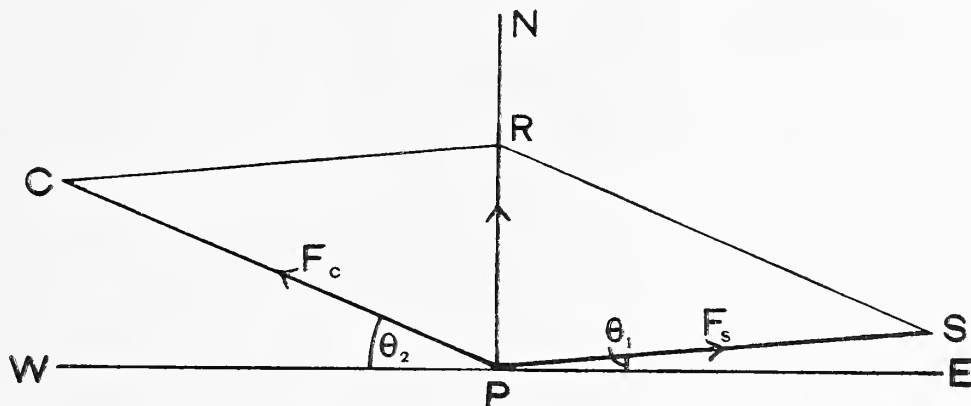


FIG. 1.

allowance to be made is a function both of the angle of deflection and the strength of the current. The second method can only be used if the coils are capable of being rotated on their stands, and the adjustments would be difficult and troublesome to carry out.

The necessity for attending to this source of inaccuracy was first pointed out by Erhard,* who investigated the magnitude of the errors which were involved by neglecting it. In the case of a magnetometer of the usual type examined by him, it was found that with a magnetising field of 128.3 C.G.S. units in the solenoid there was a change of 6.8 per cent. in the directive force on the needle on reversal of the current. Erhard advised that the magnitudes of the errors introduced should be determined for various parts of the scale and allowed for in the results.

While carrying out a research on certain feebly magnetic alloys the authors found that the elimination of the aforementioned sources of error caused very considerable delay in the progress of the work. An attempt

* "Eine Fehlerquelle bei magnetometrischen Messungen," *Ann. der Phys.*, 1902, p. 724.

was therefore made to design a form of magnetometer which would overcome these disadvantages which are common to instruments of the usual type. In planning the apparatus the following requirements were kept constantly in view: (1) the magnetometer must be capable of accurate and rapid adjustment, (2) there must be no resultant Erhard effect, (3) the instrument must be suited for testing specimens at all temperatures from that of liquid hydrogen to the critical temperature, (4) it must be alike efficient for testing strongly magnetic and feebly magnetic specimens, (5) the magnetising solenoid must be capable of furnishing fields up to at least 400 C.G.S. units, (6) the instrument must be rigid, all parts being fitted on one bed-plate, and the coils must be capable of being clamped without danger of destroying the compensation in so doing.

The general principle of the instrument which has been evolved will be seen from fig. 2. ns represents the magnetometer needle provided with a

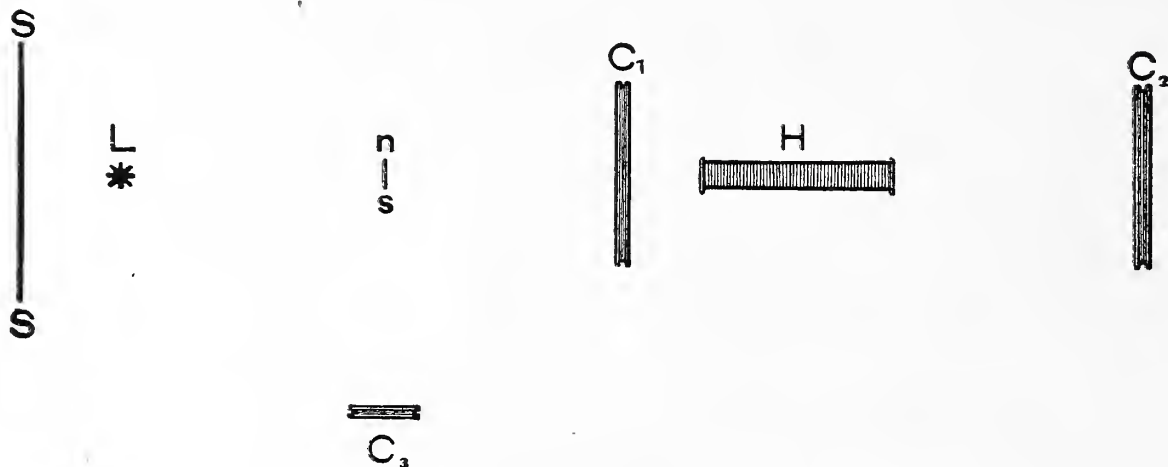


FIG. 2.

concave mirror, by means of which and a source of light L , its movements are observed on the scale SS . H is the magnetising solenoid placed due east or west of the magnetometer needle and clamped in a convenient position. C_1 and C_2 are compensating coils placed with their axes approximately in coincidence with that of the solenoid. In adjusting the apparatus the effect of the current in H on the needle ns is first approximately annulled by means of C_1 , which is then clamped in position. The final adjustment of the compensation, so far as the undisturbed position of the needle is concerned, is carried out by means of C_2 , which on account of its great distance from the needle contributes only a small fraction of the balancing field, and thus provides an adjustment of great refinement. The position of C_2 necessary for balance having been obtained, it is clamped in position; obviously, since the distance of C_2 from the needle is great, any slight movement caused by doing so produces no perceptible effect on the compensation.

If the axes of C_1 , H , and C_2 were coincident and passed through the magnetometer needle, the adjustment would now be complete. If, however, the needle ns is deflected by means of a permanent magnet, and a large current is reversed in the circuit, in general an alteration in the scale reading on SS will be observed. A coil C_3 , placed with its axis in the magnetic north and south line passing through the needle, is now included in the circuit. By properly adjusting the direction of the current in C_3 , and altering the distance of C_3 from the needle, the compensation can be made perfect for all positions of ns .* In a magnetometer where ns , C_1 , H , and C_2 are all carried on stands moving in one channel in the bed-plate there should be little departure of the axes of the coils from coincidence. Accordingly the resultant magnetic field, due to the coils and solenoid, in the north and south direction will be small. The coil C_3 is therefore made of little power, and a small change in its position brings about only a very slight alteration in its effect upon the needle. It can therefore be clamped without any risk of upsetting the balance. The manner of making the adjustments will be fully explained later.

The instrument with its compensating coils and other fittings is shown in plan in fig. 3. The bed-plate is in the form of a cross, and is built of well-seasoned mahogany planks 22 cm. broad and 2.5 cm. thick. The length over all is 350 cm., and the breadth from end to end of the arms 135 cm. The cross-piece is at a distance of 100 cm. from one end of the main length. Like the main portion of the bed-plate, it is formed from one piece of wood, the two lengths being set accurately at right angles and half checked into one another. The junction is made rigid by means of glue and brass screws. A channel 11.5 cm. broad is formed over the entire length of the cross-piece by means of two mahogany strips which are square in section and fixed parallel to the edges of the arms. A similar channel runs down the main length of the bed-plate, being discontinued where it is crossed by the channel already mentioned. The wooden strips forming these channels are permanently fixed by glueing and by brass screws driven in from the under side of the base-board. After they have been constructed they are made of perfectly uniform width by sand-papering, the width being tested from time to time during the process by means of a wooden gauge.

A is a mahogany box consisting of bottom, sides, and top, the ends which face east and west being left open. In the bottom is a slot running parallel to the cross-piece of the bed-plate. A brass screw projecting upwards from the base-board of the magnetometer passes through this slot and is provided

* A side coil has been used by Dr G. E. Allan in his magnetometric work for giving compensation throughout the scale, but it does not permit of the adjustment here described.

with a brass washer and locking-nut. By this means the box can be moved through a small distance in the north and south direction, and securely clamped in position. On the upper surface of the box is fastened a plate of glass on which stands the magnetometer proper. This part of the instrument is also constructed of mahogany. A wooden pillar 20 cm. in height has a narrow hole drilled longitudinally down through it. This hole terminates in a small cell with a glass window in front. The cell is just large enough to contain the mirror of the magnetometer—a concave mirror, 1 cm. in diameter, having a focal length of 50 cm. The mirror has attached to its back a small piece of magnetised watch-spring about 8 mm. in length. The needle and mirror are suspended by a fine quartz fibre from a screw at

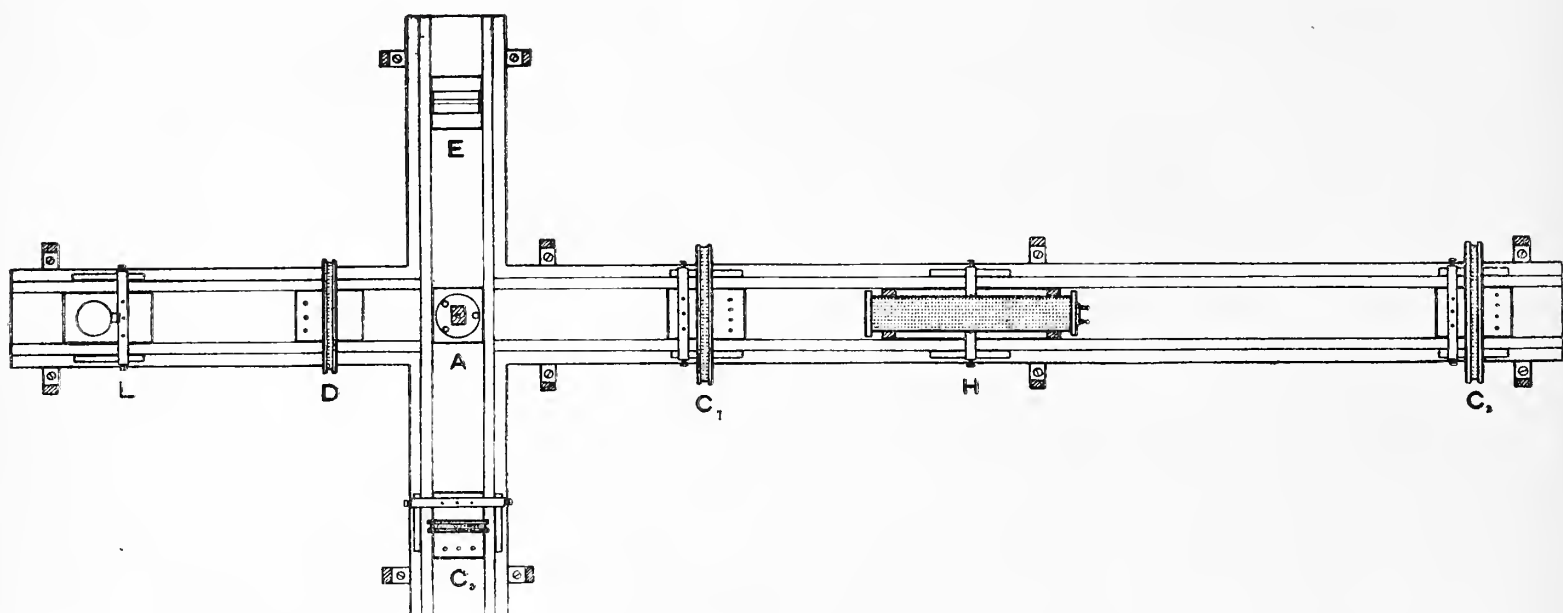


FIG. 3.—Plan of the Magnetometer.

the top of the upright pillar of the magnetometer. By means of this screw, the axis of which is vertical, all torsion can be removed from the fibre when the needle is hanging in its equilibrium position; and by giving the screw an observed number of complete turns a determination of the torsional rigidity of the fibre can be made. The pillar of the magnetometer is attached to a circular base provided with three small brass levelling-screws. The position of these feet on the glass top of the box-stand is defined by the hole, slot, and plane method.

A A (fig. 6) is the magnetising solenoid. Two brass tubes 45 cm. in length are connected at their ends by brass rings so as to form a water-jacket B B measuring 4 cm. in internal and 6 cm. in external diameter. On the outside of this is wound 868 turns of No. 15 s.w.g. copper wire in four layers (only one layer is shown in the figure). The wire is double silk-covered, and each layer is varnished over after winding. The terminals

of the coil are mounted on an ebonite block at one end of the solenoid. D and C are the inlet and outlet tubes of the water-jacket. Although the water-jacket is somewhat narrow it is found to be effective in keeping the helix of wire cool, even though the interior is raised to a temperature of over 1000° C. by means of an electric furnace. The water-jacket is made small in capacity in order to keep down the mean radius of the solenoid, and hence maintain the end effect of the solenoid small. The field at the centre of a coil of length $2l$ and radius a is less than that given by $0.4\pi nC$ in the ratio $(l^2 - 2a^2)/l^2$, where n is the number of turns in the coil per unit length and C is the magnetising current in amperes. In the case of the solenoid now described the reduction in the field from the value $0.4\pi nC$ due to the finite length of the coil is only 1.14 per cent. The solenoid is carried on a mahogany base-board provided with two vertical supports terminating in V-shaped grooves to receive the coil. The position of the solenoid carrier in the channel of the magnetometer board may be fixed by means of a brass clamp (shown in fig. 3). This friction clamp is furnished with two screws which press mahogany blocks against the outer surfaces of the wooden strips forming the channel of the magnetometer bed-plate.

C_1 and C_2 (figs. 2 and 3) are circular coils of 15 cm. radius erected on wooden stands provided with brass clamps as in the case of the solenoid. Each coil is wound in three sections, the terminals of which are screwed into the base of the stand. The sections in the case of C_1 contain 5, 7, and 9 turns of wire respectively, and in the case of C_2 6, 8, and 10 turns. These sections may be used singly or in combination, and accordingly there is a wide range of variability in the powers of the coils. C_3 is a coil of similar construction, but has a radius of only 6 cm., and is built in two sections of 1 and 3 turns of wire respectively.

D is a coil having a radius of 12 cm., and its function is to prevent loss of time due to the needle vibrating about its position of equilibrium. It is connected up in series with a single cell and a reversing key; and by properly tapping the key a series of impulses is communicated to the needle, which is thus quickly brought to rest.

L is a further sliding stand carrying the object screen. This consists of a vertical wire placed in front of a window of obscured glass fitted in a metal box containing an incandescent lamp. By altering the position of this stand, the image of the cross-wire formed by the mirror of the magnetometer can be produced at any distance from 110 cm. upwards. From 150 cm. to 200 cm. is in most cases a suitable value. At this distance it is received on an engine-divided glass scale of the usual type.

E is a deflector stand on which a small permanent magnet may be

mounted in the "B" position of Gauss. The construction of the stand is similar to that of the stand which carries the magnetometer proper. On the top of it is fixed a rectangular block of wood provided with a groove for receiving the magnet.

The bed-plate of the magnetometer is mounted on six pairs of mahogany feet, which are fastened to a rigid table by means of brass screws.

The process of setting up the apparatus is as follows. The centre of the magnetometer needle has first to be placed on the axis of the solenoid. To accomplish this, coil C_1 (fig. 3) is removed, and the solenoid H is moved along the bed-plate towards A until its inner end is almost in contact with the back of the magnetometer casing. The stand A is then moved in its channel until the needle is brought exactly on to the axis of the helix, and is then permanently fastened in this position by means of the clamping screw already mentioned. The table carrying the magnetometer is now placed so that the long channel of the bed-plate lies due east and west, the adjustments being carried out and tested by means of the well-known method described in Gray's *Absolute Measurements in Electricity and Magnetism*. The method is as follows:—A wire is stretched out vertically beneath the needle, and accurately parallel to the short channel of the bench. On passing a current through this wire a deflection of the needle is produced. If the current is reversed in direction the deflection will have the same numerical value as before, provided that the wire lies exactly magnetic north and south. The table is so placed that this condition is fulfilled, and its feet are then clamped to the floor by means of L-shaped brass brackets. The scale is erected on a separate table in order that the movements of the observer may not set up oscillations of the needle. The coils C_1 , H, and C_2 are now connected up in series with the storage battery, ammeter, and variable resistances, etc., care being taken that the direction of the current in C_1 and C_2 is opposite to that in H. The permanent adjustments of the instrument are now complete.

When a specimen has to be tested the solenoid H is moved to a convenient distance from the magnetometer needle and firmly clamped. The coil C_2 is placed at the far end of the magnetometer table, and a current two or three times greater than the maximum to be used in the subsequent test is sent through the complete circuit. Coil C_1 is then moved until it just falls short of balancing the effect of the solenoid on the needle. It is then securely clamped. Coil C_2 is next slowly moved up towards the magnetometer needle until the deflection of the latter is brought exactly to zero; C_2 is now clamped, and the accuracy of the compensation verified by suddenly reversing the current in the coils. No measurable change in the scale reading should

result. The current having been interrupted, a small permanent magnet is next placed east and west on the stand E, and the stand moved along the cross channel in the magnetometer bed-plate until the spot rests near one extremity of the scale. The current is again made and reversed, and if any appreciable deflection of the spot on the scale is observed coil C_3 is included in the circuit, the current being so directed through it that the deviations of the needle from its equilibrium position are diminished. The coil is gradually moved closer to the magnetometer until the Erhard effect is completely wiped out, and is then clamped in position. The compensation now holds for all parts of the scale, and the apparatus is ready for carrying out magnetic tests.

The several sections in which the three compensating coils are built allow the adjustment to be completely made with the coils in several different positions. This is a great advantage, as it always affords a means of escape from any arrangements of the coils which might prove awkward when specimens are in the solenoid.

With reference to the adjustment of the coils described above, it should be noted (1) that the method is systematic, and that there is no possibility of failure to secure a balance—all the adjustments are carried out in a perfectly definite manner; (2) that the method is delicate; for, owing to the great distance of coil C_2 from the magnetometer needle, it may generally be moved several millimetres without causing any appreciable error in the compensation; (3) that the method is capable of furnishing a high degree of accuracy; with the near end of the solenoid at a distance of only 12 cm. from the magnetometer needle it is quite possible to arrange that the change in the scale reading brought about by reversing a current of 15 amperes in the circuit is only a fraction of 1 mm. with the scale at a distance of 175 cm. from the needle; (4) that the operations can be carried out with great rapidity; unless the solenoid is very close up to the magnetometer, the changing over of the apparatus from one degree of sensibility to another can be carried out within the space of two minutes.

The magnitude of the directive force at the needle is easily determined by passing a measured current through one of the balancing coils and noting the deflection of the magnetometer needle produced. The value of the directive force is then easily calculated.

Fig. 4 is a photograph of the apparatus when adjusted for the examination of a strongly magnetic specimen; fig. 5 shows the arrangement when a feebly magnetic specimen is being dealt with. When the solenoid has to be placed very close to the magnetometer needle to allow of a very feebly magnetic specimen being examined, the coil C_1 is placed on the opposite side

of the needle to the solenoid. For general use, however, it is convenient to have the solenoid and coil on the same side. It is worthy of remark, in passing, that even if C_1 is placed as close up as possible to the end of the

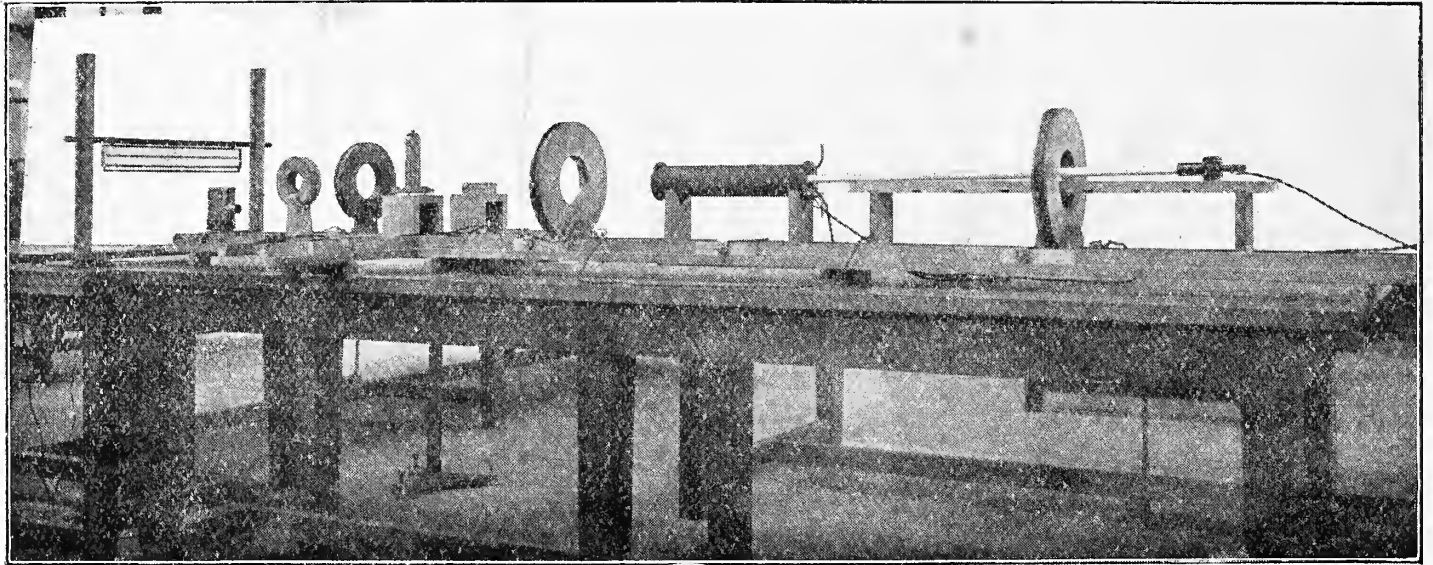


FIG. 4.

solenoid, it can alter the field at the centre of the specimen by not so much as $\frac{1}{4}$ per cent.

When used for testing specimens at temperatures higher than that of the room, an electric furnace of a type similar to that devised by Dr G. E. Allan,* is placed within the helix. In fig. 6 it is shown in position. A

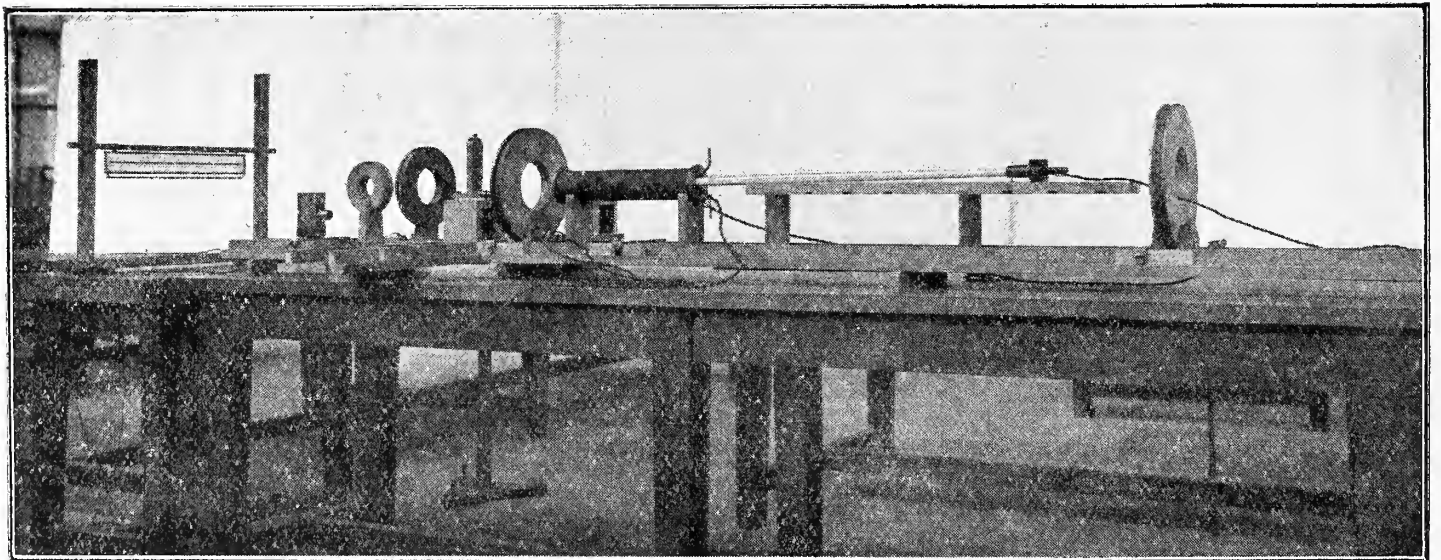


FIG. 5.

tube E of unglazed porcelain of about the same length as the solenoid, having an internal diameter of 23.5 mm. and a thickness of about 2 mm., is wound non-inductively with fine platinum wire; the ends of this wire are brought out to two terminals mounted on a slate frame at F. The tube

* *Phil. Mag.*, 1904, p. 46.

is enclosed in a tube G of Jena glass, which fits as a cartridge within the magnetising solenoid. The space H H between the glass and porcelain tubes is packed with dry kaolin clay, which performs the double duty of supporting the furnace and preventing the coils of the platinum wire from changing their positions when expanded by heat. A cylinder of electrolytic

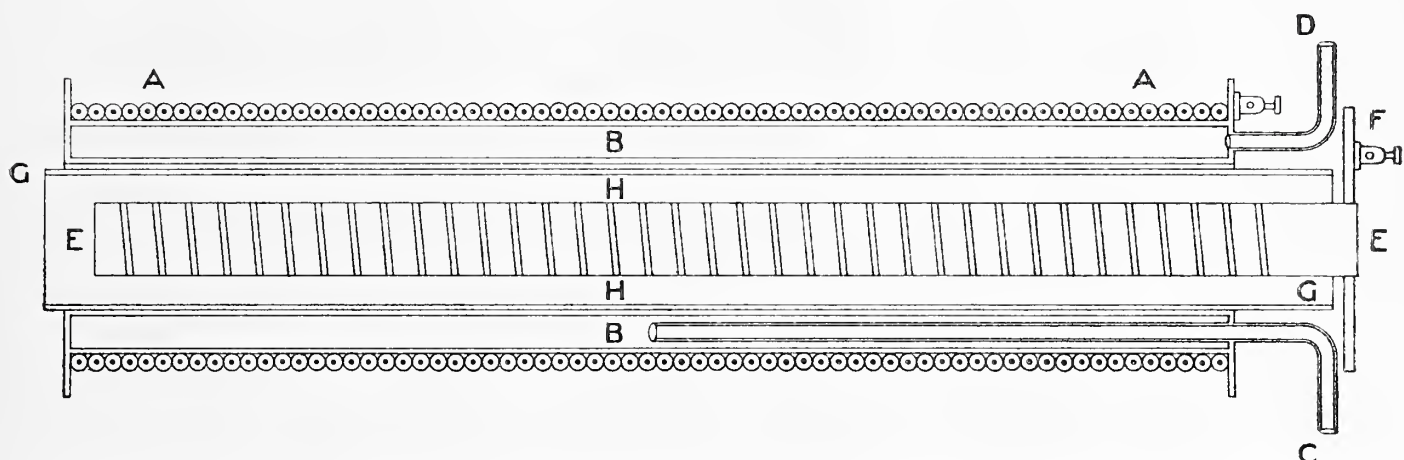


FIG. 6.

sheet-copper is placed within the tube E, and assists in maintaining a very uniform temperature over the space occupied by the specimens.

In the figure the platinum wire is shown equally spaced over the porcelain tube. In reality this is far from being the case. The proper winding of the tube is an exceedingly troublesome operation, and can only be accomplished by repeated trial.

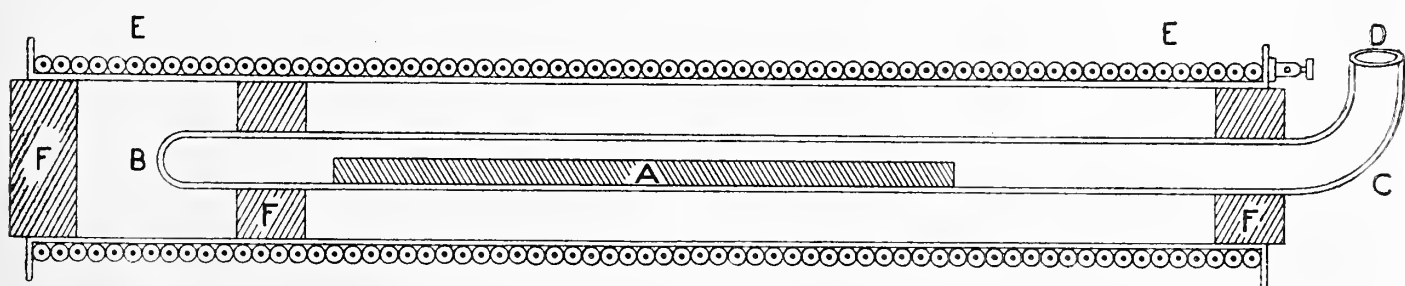


FIG. 7.

The temperature of the furnace is measured by means of the ordinary thermo-element or a platinum resistance thermometer. The two wooden stands used for the pyrometer are shown in position in figs. 4 and 5. As will be seen at once, the several slots in the horizontal carrier for fitting on the tops of the stands permit of these latter being placed clear of the sliding bases of the compensating coils.

For tests at the temperature of liquid air the arrangement shown in fig. 7 is employed. The specimen A is enclosed in a glass tube B C D, of which the end B is closed and the end D is open and bent up. Cork bungs

F, F, are fitted on the tube so as to bring the axis of the specimen into coincidence with that of the solenoid. A third bung F or a pad of cotton-wool is used to prevent access of warm air into the interior of the solenoid, and a covering of cotton-wool on the portion C D prevents it from warming up and conducting heat to the specimen. Instead of closing the glass tube at B, a cork may be used to stop up the opening. The cork, however, if dry, is liable to loosen and permit the liquid air to leak out, or if it is at all damp it expands and fractures the tube.

Where tests have to be made as the specimen slowly warms up from the temperature of liquid air a Dewar tube is used, with its mouth closed by a cork which has two bent tubes passed through it—one for pouring in the liquid air, and the other for the bringing out of the leads from one or more thermo-elements in contact with the specimen.

The dimensions given above for the internal diameter of the solenoid will be found sufficient for receiving a double vacuum Dewar tube for tests at -252° C. on specimens immersed in liquid hydrogen.

A slightly modified form of the stand supporting the solenoid permits of the latter being carried in an east and west position on one of the arms of the cross-piece of the magnetometer. The apparatus is therefore available for use with specimens in either the "A" or "B" position of Gauss; the methods described in Gray's *Absolute Measurements in Electricity and Magnetism* for the determination of the effective lengths of the specimens thus become available.

The considerable height of the magnetometer needle above the level of the magnetometer base-board (18 cm.) would also permit the apparatus to be readily adapted for testing by the "one-pole" method.*

Several instruments of the above type have been built in the Physical Institute of the University of Glasgow, and are giving every satisfaction.

* See Ewing's *Magnetic Induction in Iron and other Metals*, p. 39.

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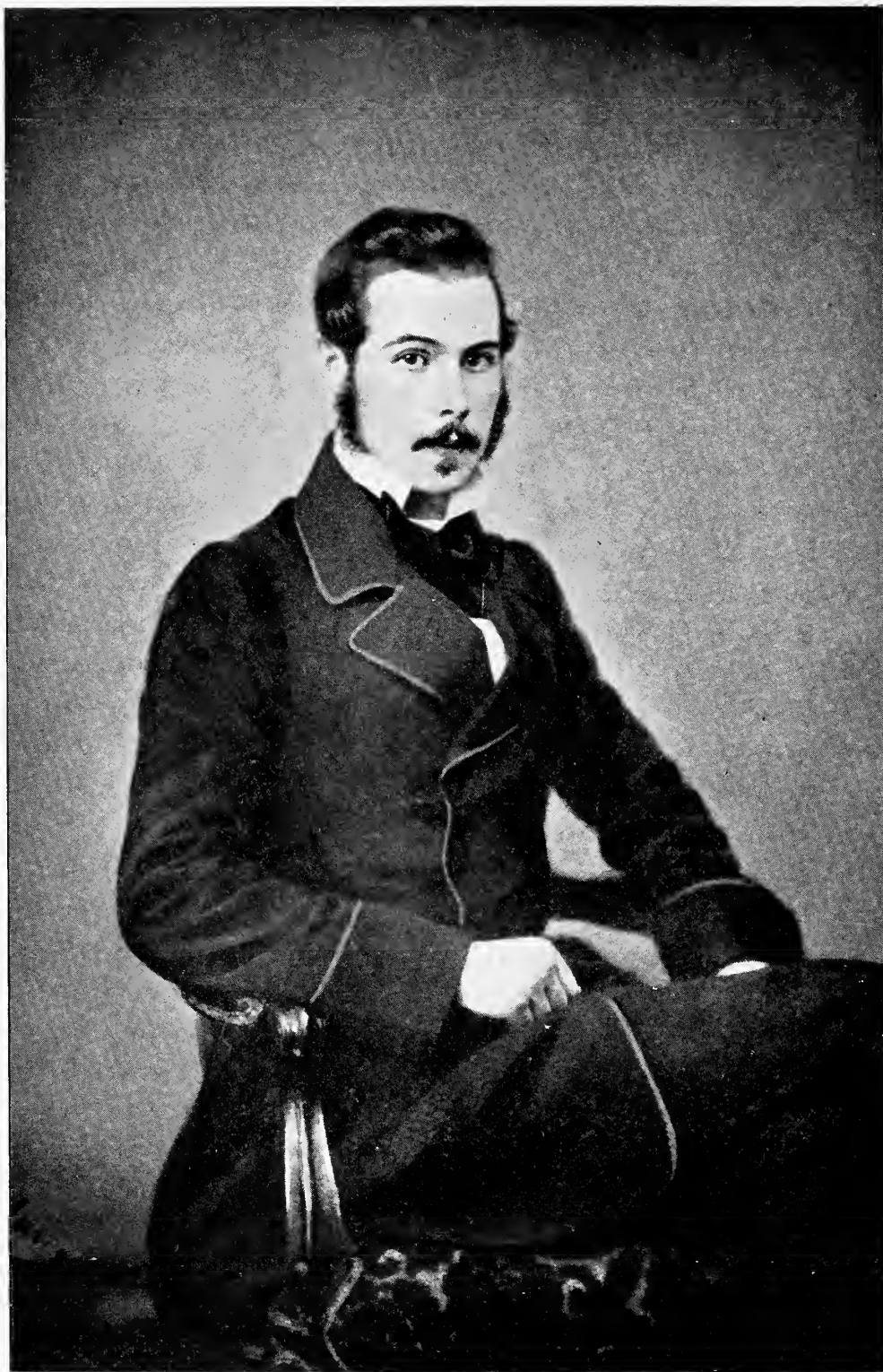
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[Continued on page iii of Cover.]



Arch^d. S. Couper

XIII.—**Life and Chemical Work of Archibald Scott Couper.** By **Richard Anschütz**, Ph.D., LL.D., Professor of Chemistry in the University of Bonn. *Translated and communicated by* Emeritus-Professor **A. CRUM BROWN**, M.D., D.Sc., LL.D.

(MS. received January 25, 1909.)

PREFACE.

ARCHIBALD SCOTT COUPER is one of the most singular appearances in the history of the development of Organic Chemistry in the nineteenth century. He comes on the scene at the time—the end of the 'fifties—when the valency theory began its victorious entrance into our science.

His two experimental investigations, "On some Derivatives of Benzene," and "On Salicylic Acid," as also his communication on "A New Chemical Theory," were published within the limits of one year, and then with startling suddenness his scientific career comes to an end.

Belonging to none of the then existing chemical schools, and furnished with an excellent philosophical training, Couper boldly attacked with sharp criticism the theories prevailing in chemistry at the time.

No claim of priority can be made for his assumption of the concatenation of carbon atoms, but, as will be shown in this paper, his statement was quite independent of that published somewhat earlier by Kekulé.

Without any doubt, Couper deserves the credit of having introduced into constitutional formulæ the lines indicating union of atoms, and of having thus produced what are now called structural formulæ. The results of his work on salicylic acid, long doubted by all the chemists who had repeated his experiments, were at last shown by the present writer to be accurate.

In my efforts to help to his historic rights a fellow-chemist as distinguished as he was unfortunate, and to obtain information as to his origin and life, I have been assisted by my honoured colleagues Heinrich Debus, Greville Williams, Adolf Lieben, Albert Ladenburg, and Alexander Crum Brown. Above all, I have to thank the sympathetic zeal with which Alexander Crum Brown, at my request, gave himself to the discovery of biographical details of his countryman's life. All of that, which I have been able to communicate, as also the translation of this paper into English, is his work.

Although in physical science there are no national frontiers, still we may expect that the life and achievements of a distinguished investigator will be of special interest to his countrymen. This consideration moved me to offer my paper on the life and chemical work of Archibald Scott Couper to the Royal Society of Edinburgh, and I had the pleasure of seeing it accepted.

Couper's three short papers from the *Comptes rendus de l'Académie des Sciences* have been reprinted here. Couper published his paper on salicylic acid also in English in the *Edinburgh New Philosophical Journal*, adding his new structural formulæ for the derivatives of salicylic acid prepared by him. He gave a full account of his new chemical theory in a paper in the *London, Edinburgh, and Dublin Philosophical Magazine*, and at the same time published a French translation of this paper, with some additions, in the *Annales de chimie et de physique*. I have thought it convenient to print the papers on salicylic acid and the two full accounts of the new chemical theory in both languages, on alternate pages, so as to facilitate reference.

I am hopeful that this unambitious paper may conduce to place the memory of Archibald Scott Couper in its right place in the history of our science, and I thank the Royal Society of Edinburgh for the way in which it has met my endeavour to do this.

RICHARD ANSCHÜTZ.

IN the history of the development which organic chemistry (or the chemistry of the compounds of carbon) underwent in consequence of the introduction of the hypothesis of the quadrivalence of carbon and the concatenation of carbon atoms, we meet alongside of the conspicuous name of Friedrich August Kekulé that of Archibald Scott Couper.

But Couper had to submit to the hard fortune that his paper "On a New Chemical Theory" was, owing to no fault of his, published too late. It is true that this paper is referred to in chemical history books, but Couper himself fell so completely out of memory that I have not been able to find his name in any of the dictionaries of scientific biography.

As Couper's three papers appeared for the first time, soon after one another, in the *Comptes rendus hebdomadaires des séances de l'Académie des Sciences*, many chemists seem to have taken him for a Frenchman, although his Christian name "Archibald" * points to Scotland, which is indeed his native land.

* In the *Comptes rendus* the name is given as A. Couper; in the *Annales de chim. et de phys.* as A. S. Couper.

Besides the theoretical paper, Couper published a communication "On some Derivatives of Benzene," and, somewhat later, a very excellent experimental work, "Researches on Salicylic Acid." But just as Couper, through no fault of his, came too late with his theoretical paper, so also with his work on salicylic acid, in which he investigated the action of phosphorus pentachloride on salicylic acid, he had the misfortune that two of the most distinguished German chemists, August Kekulé and Hermann Kolbe, as well as some others, repeated his experiments, but were unable to confirm his results.

Here Couper's appearance in chemistry comes to an end. Although his gifts seemed eminently to qualify him for a brilliant scientific career, no further communication of his is to be found in any scientific journal.

How did this come about? What became of Archibald Scott Couper?

My interest in Couper was first awakened by his work on the action of phosphorus pentachloride on salicylic acid, a subject on which I also was specially engaged. My sympathy with Couper grew when, in the course of the studies required for the preparation of a complete biography of Kekulé, I found it necessary to go very thoroughly into Couper's paper "On a New Chemical Theory." This paper of Couper's must indeed always take its place beside that of Kekulé "On the Constitution and Metamorphoses of Chemical Compounds and on the Chemical Nature of Carbon."

The first information I obtained as to Couper's life was from a letter of Greville Williams, to whom my honoured friend Heinrich Debus applied in order to help me. Some time ago, with Debus' permission, I appended this letter as a note to my paper "On the Action of the Pentachloride and of the Trichloride of Phosphorus on Substituted o-Phenolcarboxylic Acids," published in *Liebig's Annalen*,* and I repeat it here:—

"21 BOURNEVALE ROAD, STREATHAM, S.W.,
10th August 1903.

"DEAR SIR,—I grieve to say that I know nothing of the origin of poor Couper. I first became acquainted with him when I was assistant to Dr (afterwards Lord) Playfair in the University of Edinburgh, where Couper was a student in the laboratory, but he soon left. I only saw him once more, when he came up to me on the seashore at Dunoon on the Clyde, but he was then a complete wreck. I believe his trouble originated in sunstroke. I deeply regret being unable to give you more information about this great but unfortunate genius.—Yours very truly,

"GREVILLE WILLIAMS."

* cccxvi. 291 (1906).

At my request Debus now applied to Dr Alex. Crum Brown, Playfair's successor in the Edinburgh Chair of Chemistry. Although the problem seemed hopeless, he attacked it with the greatest earnestness and uncommon skill, and he succeeded.

I am thus enabled to give the following notes of Couper's life, and here thank my honoured friend and distinguished colleague for his kindness in placing in my hands the results of his successful efforts. The communications which I have used in these biographical notes were obtained by Crum Brown in the first place from Couper's cousins, Mrs Little and Miss Tait, Kirkintilloch, and Mr Dollar, the eminent veterinary surgeon, London, and from a university friend of Couper, Mr Berring, Coblenz.

Archibald Scott Couper was born March 31st, 1831, at Townhead,* Kirkintilloch, Dumbartonshire, about six miles north-east of Glasgow. He was the only surviving son of Archibald Couper, proprietor of a large cotton-weaving establishment, employing from 600 to 700 weavers. His mother's maiden name was Helen Dollar. Couper's father inherited the business from his father, who was married to Janet Scott. Couper's second name, "Scott," thus came to him from his paternal grandmother.

There is not much to say about Couper's early youth; it is obvious that he had a good and careful education at home.

In the summer or autumn of 1851 Couper went with his intimate friend, Alexander Hamilton,† to Germany. They had become acquainted in Glasgow, where a similarity of taste in literature and philosophy drew them together. The two young Scotsmen, in a family in Halle to which they had been recommended, applied themselves so diligently to the study of the German language (which is said to be easier for Scotsmen than for Englishmen), that they were soon able to use it with ease.

During the winter session 1851-52, Couper attended Latin and Greek classes in the University of Glasgow.

In the summer of 1852 we again find Couper and Hamilton in Berlin, where Berring ‡ made their acquaintance. Berring reports that at that time Couper followed no special line of study, but took a look at various fields of knowledge.

In August 1852 Couper returned to Scotland, and exchanged the University of Glasgow for that of Edinburgh. From this continued

* The house, which is in the street, is now converted into shops, and the garden partly covered with buildings.

† Alexander Hamilton, M.A. 1851, D.D. 1872; United Presbyterian minister, Kilmarnock 1855-1872, Brighton 1871-1896. Died at Brighton, 1902.

‡ Afterwards Geheimrat, and director of the engineering works in connection with the navigation of the Rhine.

university attendance we may conclude that Couper had decided not to follow his father's business, and that his father was pleased to allow his talented son the choice of a learned career.

In Edinburgh the study of philosophy took its place beside that of language. Couper attended the lectures of Sir William Hamilton on logic and on metaphysics, and of Professor MacDougal on moral philosophy. Still Couper does not seem to have taken to chemistry; at least, we find no mention of scientific studies in his notebooks.

Next summer, 1853, Couper and Hamilton returned to Germany and spent some weeks on a visit to Berring's family in Minden, Westphalia. They then made a tour through South Germany, North Italy and the Tyrol, and returned to Scotland.

In the autumn of 1854, or the spring of 1855, Couper came again to Berlin, which evidently had an attraction for him. His friendship with Berring, who was then studying engineering at the "Bauakademie" in Berlin, led the two to take rooms together in a small private hotel, 75 Dorotheenstrasse, where they lived together till Couper left Berlin for Paris in August 1856.

In the meantime Couper had made up his mind for physical science in general and for chemistry in particular. We have no means of knowing what influences or considerations led to this decision, but in any case the turn for chemistry was not present in his early youth, but must have gradually developed during the years of his university life. Berring cannot now with certainty give the name of Couper's teacher of chemistry in Berlin, but he thinks he worked under Rammelsberg. However that may be, there is no doubt that he attended Sonnenschein's lectures on analytical chemistry, and worked for two months in the summer of 1856 in his laboratory. In short, he used the three or four sessions he spent in Germany to perfect himself in analytical chemistry.

While many of his countrymen went to Heidelberg or to Munich, to work under Bunsen or to hear Liebig's lectures, Couper turned to Paris and found a place in Wurtz's laboratory.

In passing, it may be mentioned that in the same year Kekulé began to lecture in Heidelberg, his chemical student days lay behind him when Couper's had just begun.

With astonishing rapidity, after only three or four sessions of chemical study, Couper had acquired the knowledge and the skill necessary to enable him to carry out independently experimental chemical investigations.

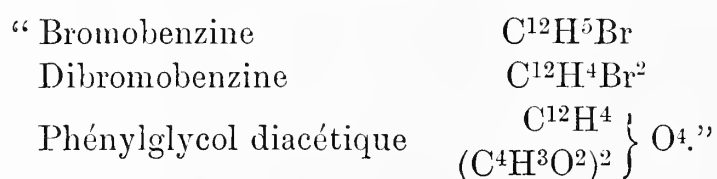
On Couper's arrival in Paris, Wurtz had just published his brilliant

discovery of glycol, by the treatment with caustic potash of the glycol diacetate obtained by the interaction of ethylene iodide and silver acetate. Couper's first work, published in August 1857, is so closely connected with this discovery that one is led to surmise that Wurtz had suggested its subject to Couper—indeed, Couper's paper, "Recherches sur la Benzine,"* immediately follows a paper by Wurtz, "Note sur la liqueur des Hollandais," † in the *Comptes rendus*.

COUPER'S INVESTIGATIONS ON BENZENE.

Couper started with the supposition that it is possible to convert benzene into the corresponding alcohol and the corresponding glycol. With this view, he treated boiling benzene with bromine vapour and thus discovered brombenzene (boiling point 150°), as well as the dibrombenzene fusing at 89° and boiling at 219° , our *p* dibrombenzene. He nitrated and sulphonised the monobrombenzene. He heated both the brombenzenes to 200° with silver acetate in sealed tubes. He ascertained that monobrombenzene scarcely acts on silver acetate. With better hope of success he tried the action of silver acetate on dibrombenzene, but unfortunately lost his material by an explosion, and had to postpone the continuation of the experiment.

In his communication Couper uses the small equivalents $C=6$, $O=8$, as Wurtz also always did at that time, and wrote, for instance:—



From this it will be seen that Couper was, at that time, by no means so much under the influence of Gerhardt's type theory as Kekulé was so early as April 1854; see his famous paper, "On a New Series of Sulphurated Acids." ‡ He had not, like Kekulé, had the opportunity of developing his scientific opinions by close scientific intercourse with Gerhardt in Paris, and then with Odling, and especially with Williamson in London. No doubt Couper uses for his expected "phénylglycol diacétique" a typical formula; but it is of a sort then generally used even in Germany, and derived from the multiple type of water H^2O^2 , and not from the type H_2O

* *Comptes rendus*, xlv. 230–232. See Appendix to this paper, p. 235.

† *Ibid.*, xlv. 228–230.

‡ *Proceedings of the Royal Society of London*, vii. 37–40.

of Gerhardt and Williamson. I specially note this point, because Couper, in the new theory which he produced in the course of the next year, freed himself, at least partially, from these erroneous assumptions.

Although Couper's paper "Sur une nouvelle théorie chimique"* appeared somewhat later than his second and last experimental work, "Recherches sur l'acide salicylique," † I think it more convenient to consider Couper's theoretical views here. For it is not until the close of the paper "On a New Chemical Theory," ‡ that Couper applies this theory to salicylic acid, whereas in the French paper on salicylic acid, as also in that on benzene, he uses almost exclusively empirical molecular formulæ.

In order to bring his work before his own countrymen, Couper published a full exposition of his new theory in the *London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*. ‡ He translated this paper into French, and, with some additions, published it in the *Annales de chimie et de physique*. § He also translated the paper on salicylic acid into English, applying his new theory to the formulation of the derivatives of salicylic acid discussed in it. This translation was printed in the *Edinburgh New Philosophical Journal*, 1858. ||

ON COUPER'S NEW CHEMICAL THEORY.

As soon as Kekulé, at that time Privatdocent in Heidelberg, saw Couper's paper in the *Comptes rendus*, he brought forward his claims to the two most important propositions contained in it. Already, in the report of the meeting of the French Academy on the 30th August 1858, we find Kekulé's communication, "Remarques de M. A. Kekulé à l'occasion d'une note de M. Couper sur une nouvelle théorie chimique." ¶ We shall therefore most conveniently follow Couper's statements along with Kekulé's remarks on them. In this connection there came to my mind a remark which Kekulé introduced into his speech at the celebration of the twenty-fifth year of the benzene theory, held in Berlin on the 11th of March 1890: ** "Some ideas at some time lie in the air: if one does not give expression to them, another will soon do so." Or, as I may add in expansion, different thinkers often come independently and about the same time to proclaim

* *Comptes rendus*, xlvi. 1157-1160. See Appendix to this paper, p. 237.

† *Ibid.*, xlvi. 1107-1110. See App., p. 266.

‡ xvi. 104-116. See App., p. 240.

|| viii. 213-217. See App., p. 267.

** *Ber. d. Deutsch. Chem. Ges.*, xxiii. 1304.

§ [3], liii. 469-489. See App., p. 241.

¶ *Comptes rendus*, xlvii. 378-380.

the same idea, when the hour of its birth has struck. For, in Couper's paper, "On a New Chemical Theory," the hypothesis of the quadrivalence of carbon and of the concatenation of carbon atoms was developed shortly after the same had been done by Kekulé, and it was—as I have already indicated—only by an accident fateful for Couper that his paper did not appear simultaneously with that of Kekulé, "Über die Constitution und die Metamorphosen der chemischen Verbindungen und die chemische Natur des Kohlenstoffs,"* which became so famous.

No doubt Kekulé had already set up the marsh-gas type in his paper on mercuric fulminate, and in a note to the paper, "Über die sogenannten gepaarten Verbindungen und die mehratomigen Radicale," sent to the editors of *Liebig's Annalen* on the 15th of August, 1857, expressed himself thus †:—"Der Kohlenstoff ist, wie sich leicht zeigen lässt und worauf ich später ausführlicher eingehen werde, vierbasisch oder vieratomig d.h. 1 Atom Kohlenstoff = C = 12 ist äquivalent 4 At. H." In this the hypothesis of the concatenation of carbon atoms was also contained implicitly.

All the same, authentic communications which I have received from highly valued sources leave no doubt that Couper's paper "On a New Chemical Theory" was in the hands of Adolphe Wurtz before the number of *Liebig's Annalen* issued on the 19th of May, and containing Kekulé's paper, dated 16th March, was published. ‡

In a letter dated 29th March 1906, Adolph Lieben writes to me:—

"Couper's Arbeit ist ganz unabhängig von der Kekulé's, was Niemand besser weiss als ich. Couper, der damals, so wie ich, in Wurtz's Laboratorium arbeitete, war gewöhnt seine Entwürfe und Ideen mit mir durchzusprechen und übergab mir auch, vor der Publication, seine später in den *Compt. rend.* 1858 erschienene Abhandlung zur Prüfung, dann übergab er sie Wurtz. Mittlerweile erschien das Ende Mai ausgegebene Heft der *Annalen* mit Kekulé's ähnlicher Arbeit und Couper war über dieses Zusammentreffen äusserst bestürzt."

In a letter dated 12th May 1906, Albert Ladenburg gives the following from his recollection:—

"Couper arbeitete bei Wurtz in Paris und bat dessen seine Abhandlung über die Vierwertigkeit des Kohlenstoffs der Akademie zu überreichen. Wurtz, der damals selbst noch nicht Mitglied war, musste die Abhandlung an einen Anderen der Mitglied war (gewöhnlich Balard) geben. Er verbummelte dies ein wenig und so erschien Kekulé's Mitteilung, ehe die von Couper der Akademie vorgelegt war. Darob grosser Zorn von Couper, der Wurtz zur Rede stellte und ausfällig wurde.

* *Liebig's Annalen*, cvi. 129–159.

† *Ibid.*, civ. 133, Anm.

‡ *Ibid.*, cvi. 129–159.

Wurtz liess sich das nicht gefallen und verwies ihn aus dem Laboratorium. Couper scheint sich dies sehr zu Herzen genommen zu haben, und in Paris glaubte man den Anfang seiner Krankheit daher zu datieren. Die Geschichte selbst ist authentisch, ich habe sie von Wurtz."

Now that it is thus settled that Couper's paper "On a New Chemical Theory," presented by Dumas to the French Academy, was independent of the corresponding exposition published somewhat earlier by Kekulé, I turn to its contents. It is the case that the line of thought, and so also the mode of expression of the two, have often a startling similarity, though there is no lack of individual peculiarities. Kekulé points out that the opening sentences of Couper's paper—"Je remonte aux éléments eux-mêmes dont j'étudie les affinités réciproques. Cette étude suffit, selon moi, à l'explication de toutes les combinaisons chimiques"—completely agree in meaning with the passage in his own German paper,* which he gives in French thus:—"Je crois nécessaire pour l'explication des propriétés des combinaisons chimiques de remonter jusqu'aux éléments eux-mêmes qui les constituent." A little further on Couper says of carbon: "La puissance de combinaison la plus élevée que l'on connaisse pour le carbone est celle du second degré, c'est-à-dire 4."

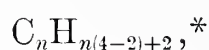
To this comes as the second property of carbon: "Il entre en combinaison avec lui-même.

"Ces deux propriétés suffisent à mon avis pour expliquer tout ce que la chimie organique offre de caractéristique. Je crois que la seconde est signalée ici pour la première fois. A mon avis, elle rend compte de ce fait important et encore inexpliqué de l'accumulation des molécules de carbone dans les combinaisons organiques. Dans les composés où 2, 3, 4, 5, 6, etc., molécules de carbone sont liées ensemble, c'est le carbone qui sert de lien au carbone."

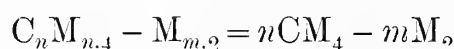
Kekulé seems to have overlooked the fact that Couper does not claim for himself the hypothesis of the quadrivalence of carbon, but only that of the concatenation of the carbon atoms, for he remarks on the passage just quoted (from which he omits the words, "Je crois que la seconde est signalée ici pour la première fois") as follows:—"Nous ne saurions lui accorder que ces propriétés soient signalées par lui pour la première fois. Déjà dans mon premier Mémoire [page 133, note] j'ai dit expressément que le carbone était de nature quatriatomique, c'est-à-dire, que 1 atome de carbone ($C=12$) est équivalent à 4 atomes d'hydrogène ($H=1$), j'ai ajouté que, par conséquent, les combinaisons les plus simples du carbone avec des éléments du premier groupe (éléments monatomiques) étaient CH_4 , CCl_4 , etc. Dans

* *Liebig's Annalen*, cvi. 136.

mon second Mémoire j'ai donné, en outre, plus de développement à cette idée [page 153] et j'en ai tiré comme corollaire [page 154] que dans les substances contenant plusieurs atomes de carbone, on ne peut expliquer cette accumulation que par l'hypothèse que les atomes du carbone lui-même soient liés entre eux, en neutralisant ainsi une partie de leur affinité générale. J'ai cru pouvoir fonder cette hypothèse sur divers exemples trop prolixes pour les rappeler ici; je me contenterai de faire remarquer que, moi aussi, j'ai donné une formule générale qui exprime, pour une certaine classe de combinaisons, le nombre d'atomes d'hydrogène combinés avec n atomes de carbone, dans les termes suivants:



tandis que M. Couper, de son côté, l'exprime de cette manière:

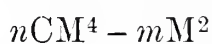


où m est $< n$."

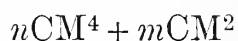
It seems to me that Kekulé has misunderstood Couper's general formulæ, for Couper expressly says:—"Toutes les combinaisons du carbone peuvent être ramenées à deux types. L'un d'eux est représenté par le symbole



l'autre par le symbole



où m est $< n$, ou bien



où n peut devenir nul. On peut citer, comme exemple du premier type, les alcools, les acides gras, les glycols, etc."

Now, if the alcohols, the fatty acids, the glycols, etc., are examples of Couper's first type, so that all saturated compounds were obviously included by him under that type, I can understand this only by assuming that M does not represent the atom of a univalent element, but the valency engaged. Only on that assumption is nCM^4 the expression for all saturated carbon compounds. And the expression $nCM^4 - mM^2$ is then the general form for the formulæ of the unsaturated carbon compounds. Couper places beside it the form $nCM^4 + mM^2$. The unsaturated carbon compounds, those which, to use a later form of speech, contain doubly and trebly united carbon, can be brought under this form. Carbonic oxide can be derived from either of the two, if we put $n = m = 1$ in the first, or $n = 0$ and $m = 1$ in the second.

* Misprinted in the *Comptes rendus* " $C_n H_{n(1-2)+2}$." See *Liebig's Annalen*, cvi. 154.

Founding on his mode of regarding combination, Couper writes formulæ in which the mutual bonds of the atoms are indicated by means of connecting lines, dotted in the papers in the *Comptes rendus* and in the *Philosophical Magazine*, continuous in that in the *Annales de chimie et de physique*—true structural formulæ in our present meaning of the term.

Kekulé, in his derivation of the constitution of the radicals, resolved the formulæ as completely as Couper; but when he had done this in words, he put the contracted radical formulæ into typical formulæ. He thus retained Gerhardt's mode of writing, and developed the idea of the typical formulæ. In the first part of his textbook, issued in the spring of 1859, he gave a graphic representation of the mode of union of the atoms, but undoubtedly our present manner of writing structural formulæ was started by Couper.

There is, however, one point on which Kekulé expressly establishes a difference between his view and that of Couper. Couper speaks, in a passage quoted above, of “la puissance de combinaison la plus élevée que l'on connaisse pour le carbone.” He distinguishes two kinds of affinity in an element:—

- “1. L'affinité de degré,
2. L'affinité élective.”

What Couper wished to be understood by “l'affinité de degré” is explained by reference to carbon:—

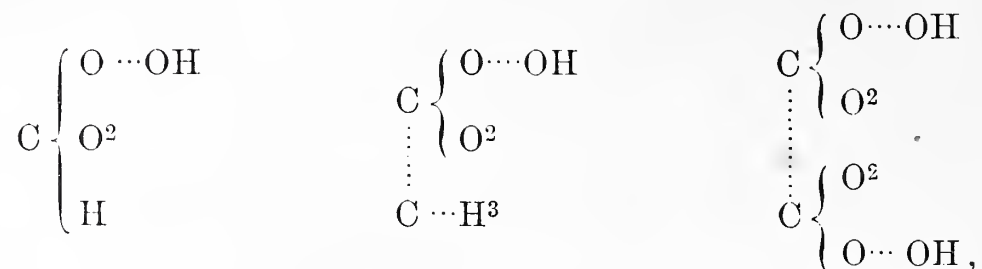
“Prenant pour exemple le carbone, je trouve qu'il exerce son pouvoir de combinaison en deux degrés. Ces degrés sont représentés par CO^2 et CO^4 , c'est-à-dire par l'oxyde de carbone et l'acide carbonique, en adoptant pour les équivalents du carbone et de l'oxygène les nombres 12 et 8.”

Without discussing the example chosen by Couper to explain the “affinité de degré,” Kekulé says:—“Si M. Couper croit avoir découvert la cause de cette différence de basicité dans l'existence d'une espèce spéciale d'affinité, l'affinité de degré, je suis le premier à reconnaître que je n'ai aucun droit à lui contester cette priorité.”

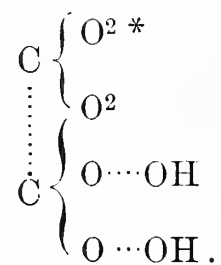
But Couper's “affinité de degré” is obviously nothing else than what was afterwards called “variable valency,” and the passage quoted above from Kekulé's claim of priority is his first clear expression of his disbelief in variable valency.

That Couper assumes $\text{O}=8$, while he puts $\text{C}=12$, is an inconsistency to which Kekulé makes no reference; I shall return to this matter later.

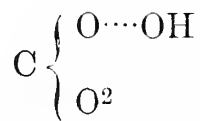
Couper writes the formulæ of formic acid, acetic acid, and oxalic acid thus:—



or the latter, “Si l'on veut réunir l'oxygène négatif à l'un des poles de la molécule, par la formule:—



Couper attaches to this a remark on the electronegative or electropositive state of the oxygen atoms that are united in the group



The oxygen atoms are in general in an electronegative state. But one oxygen atom—that one which is united to an electropositive element, such as hydrogen—thus takes on an electropositive state. The presence of the other two oxygen atoms united to carbon is necessary in order to bring the negative oxygen into the state which gives the substance the properties which are usually designated by the term acid.

He adds: “Ceci est un cas particulier d'une loi générale; car on peut voir, d'après cette théorie, comment la valeur électropositive ou électro-négative des éléments modifie et conditionne mutuellement la valeur électropositive ou électronegative des autres éléments.”

At the close of his paper Couper deduces from his new views the formulæ of salicylic acid and of the substance formed from it by the action of phosphorus pentachloride, to which he gave the name “trichloro-phosphate de salicyle.” I shall postpone the consideration of these formulæ till I have explained the reaction itself, and now turn to Couper's fuller English paper “On a New Chemical Theory.” The French version of this paper, which appeared soon after the publication in the *Philosophical Magazine*, is by Couper himself, and contains some additions made by him.

* In the *Comptes rendus* “C²” is printed in error for “O².”

Both give us a much deeper view into Couper's mode of thought. They show us how powerfully he was affected by his early philosophical training. In the introduction he develops in a general way the conditions which every sound theory must fulfil. The English paper begins with the two propositions: "The end of chemistry *is its theory*. The guide in chemical research *is a theory*." These he translates in the French version:—

"L'étude de la chimie doit avoir pour but l'établissement de la théorie de cette science; une théorie elle-même est un guide qui nous conduit dans les recherches chimiques."

Couper then criticises the theories prevalent at that time, especially Gerhardt's type theory as the most important, but also the radical theory and the theory of copulæ. He rejects them all. He objects to the type theory that the peroxides do not fit into the types. Gerhardt's types are, as he himself says, types of double decomposition, but we cannot regard as double decomposition such actions as the formation of sulphuric acid from its anhydride and water. The multiple water types existed only in Gerhardt's imagination.

Couper now examines in how far Gerhardt's type theory is in accordance with the fundamental requirements of philosophy, which demand that it should explain the greatest number of facts in the simplest possible way. In the next place, a test must be applied to see if it explains the facts at all; and thirdly, if that is so, how it explains them.

To these questions Couper answers that Gerhardt's type theory does indeed, from a certain point of view, compare every chemical compound with every other, but that it does not explain the facts at all, so that obviously the how of the third question has no application.

Couper's sarcastic vein appears in a remarkable parallel which he draws in order to show the absurdity of the type theory:—"Suppose that some one were to systematize the formation of letters into words that formed the contents of a book. Were he to begin by saying that he had discovered a *certain word which would serve as a type, and from which by substitution and double decomposition all the others are to be derived*,—that he, by this means, not only could form new words, but new books and books almost *ad infinitum*,—that this word also formed an admirable point of comparison with all the others,—that in all this there were only a few difficulties, but that these might be ingeniously overcome,—he would state certainly an empirical truth. At the same time, however, his method

would, judged by the light of common sense, be an absurdity. But a principle which common sense brands with absurdity is philosophically false and a scientific blunder.”

The following considerations may be regarded as the outcome of his philosophical studies at the university:—“The sure and invincible method of arriving at every truth which the mind is capable of discovering is always one and the same. It is that, namely, of throwing away all generalization, of going back to first principles, and of letting the mind be guided by these alone. . . . To reach the structure of words we must go back . . . to the letters. . . . In mathematics the starting-point is not generalizations, but axioms. . . . In metaphysics Descartes led the way of progress by analysing till he thought he could reach some ultimate elements beyond which it was impossible for him to go, then studying their force and power, and proceeding synthetically. . . . On the other hand, . . . Gerhardt’s generalization . . . leads him . . . to restrict chemical science to the arrangement of bodies according to their decomposition, and to deny the possibility of our comprehending their molecular constitution.”

Couper examines less fully the radical theory and the theory of copulæ. The radical theory ascribes to the radicals the character of elements, of ultimate powers, and thus stops at the very point where an explanation is required. The theory of bodies conjugated by addition divides chemical compounds, if possible, into two parts, but gives no account of the force which holds these parts together.

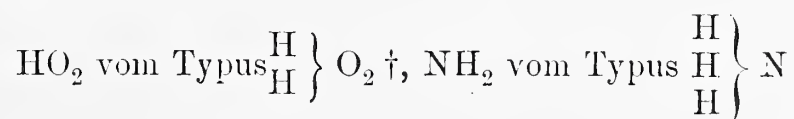
We cannot but see that it is, for Couper, a philosophical necessity, in seeking an explanation of the molecular structure of chemical compounds, to go back to the chemical elements of which they are composed, and in the very first place to find out the properties and functions of these elements. The whole of chemistry must, he says, be considered as one. The general principles common to all the elements and those peculiar to each must be determined. In each known chemical compound the properties and relations of each element contained in it must be taken into consideration.

After this philosophical introduction Couper develops his theory, establishing individual points more fully here than in the short communication in the *Comptes rendus*. Nowhere does the contrast in the mode of thought between Couper and Kekulé appear more clearly than in the philosophical foundation which Couper gives to his theory, rejecting Gerhardt’s type theory, the radical theory, and the theory

of copulæ. Kekulé, as is well known, deduced his valency theory as the idea lying at the root of Gerhardt's type theory, as he himself puts it, as a conscious, immediate development of Dalton's atomic theory, without severing the connection with the type theory and the radical theory.

Couper's complete French paper called forth the opposition of Alexander Butlerow. In a paper in *Liebig's Annalen*, entitled "Bemerkungen über A. S. Couper's neue chemische Theorie,"* he finds fault with Couper's statement, "Die Aufstellung einer Theorie" "ist der Zweck wissenschaftlicher Untersuchungen."† This translation of Butlerow's is not quite accurate; the French sentence is: "L'étude de la chimie doit avoir pour but l'établissement de la théorie de cette science." The English paper in the *Philosophical Magazine*, which Butlerow does not seem to have known, begins, as already stated, with the sentence: "The end of chemistry is its theory." "Meiner Ansicht nach," says Butlerow, "ist die Ausbildung einer Theorie die notwendige Folge vorhergegangener Untersuchungen, der Zweck aber ist eher die Kenntniss der Gesetze, nach welchen chemische Metamorphosen vor sich gehen." Couper would certainly not have disputed the statement that chemical investigations ought to teach us laws, but laws have to be explained, and to do that there is need of hypothesis or theory. Thus the ultimate aim remains still the establishment of a theory, which makes the laws intelligible, as Dalton's atomic theory did for the laws of constant composition and of multiple proportions.

I am inclined to think that the chief inducement for Butlerow to occupy himself with Couper's new theory was the opportunity of stating that nine months before, that is, early in March (his paper is dated Kazan, 1/13 December 1858), he had communicated similar theoretical views to members of the Chemical Society of Paris, although he had not then printed them. He repeats what he then said as follows:—"Wenn man unter den Radicalen nur Reste versteht, welche ihre Constitution in einer gewissen Anzahl der Reactionem beibehalten, so können mit demselben Rechte, wie die organischen Gruppen und Elemente, auch die Reste

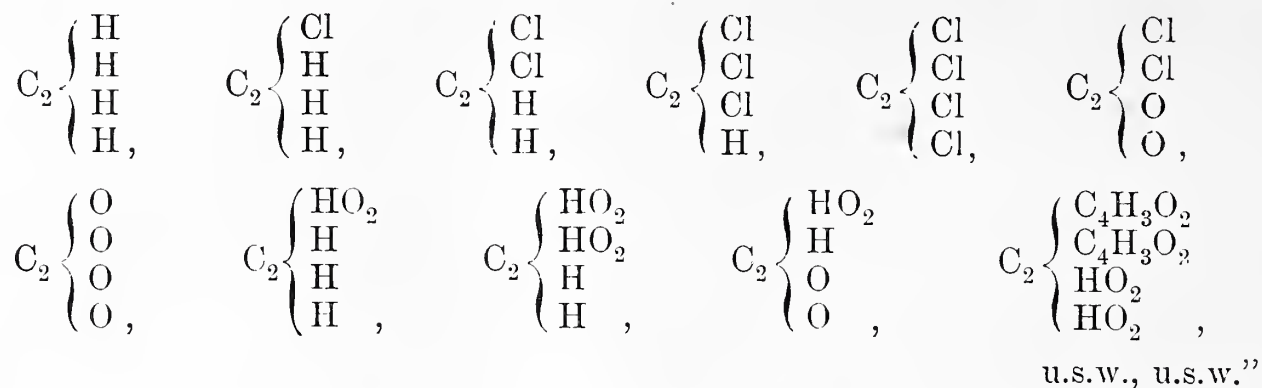


als Radicale betrachtet werden, und man könnte annehmen, dass sie ein Atom eines Elementes in verschiedenen Gruppen substituieren können. In

* ex. 51-66 (Heft 1, issued 16th April 1859).

† "Wie es Laurent annahm."

diesem Sinne würde man zum Beispiel folgende Körper als zu einem und demselben Typus der Molecularstructure gehörend ansehen:



From these formulæ we see that Butlerow was at that time still an adherent of the small atomic weights, or equivalents, for carbon $C=6$, and for oxygen $O=8$. He had thus, in this respect, remained behind Kekulé, and, so far as carbon is concerned, behind Couper also.

Couper set up, as I have already pointed out, two types for all carbon compounds, which, at the beginning of his complete paper, he formulates as follows (still with $C=6$):—

$$\begin{array}{l}
 n\text{C}^2\text{M}^4 \\
 n\text{C}^2\text{M}^4 - m\text{M}^2.*
 \end{array}$$

Further on he adopts $C=12$ for the following reasons: "There is no reaction found where it is known that C^2 is divided into two parts. It is only consequent therefore to write, with Gerhardt, C^2 simply as C , it being then understood that the equivalent of carbon is (12) twelve."

The two types then take the forms exclusively used by Couper in his short paper in the *Comptes rendus*:—

$$\begin{array}{l}
 n\text{CM}^4 \\
 n\text{CM}^4 - m\text{M}^2
 \end{array}$$

where m is less than n .

Butlerow had difficulty in understanding these two formulæ. I give here his criticism: "Setzt man in dem ersten von Couper vorgeschlagenen Ausdruck $n\text{C}_2\text{M}_4$ z.B. $n=6$, so wird man $\text{C}_{12}\text{H}_{24}$ haben; die Grenze des Verbindungsvermögens für das complexe Kohlenstoffmolecul C_{12} ist aber M_{14} (allgemeiner Ausdruck = $\text{C}_{2n}\text{M}_{2n+2}$).† Will man die typische Formel von Couper mit den Thatsachen in Übereinstimmung bringen, so muss man unter $n\text{M}_4$ Affinitätseinheiten verstehen, und zu denselben die Affinität des Kohlenstoff selbst (zwei Affinitätseinheiten für jedes Molecul C_2), ausser der, die einem Molecule C_2 gehört, mitrechnen." As already ex-

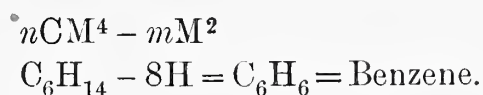
* In the *Phil. Mag.* misprinted $n\text{C}^2\text{M}^4 - m\text{M}$. See *Ann. de chim. et phys.* (3), liii. 480, line 10 from top.

† Butlerow has obviously here in mind Kekulé's general expression $\text{C}_n\text{H}_{2n+2}$.

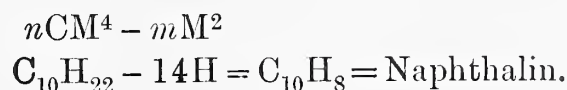
plained, I completely agree with Butlerow in this interpretation of nM_4 . But if we make this assumption for the type $nC_2M_4 (=nCM_4)$, we must do the same for the second type $nC_2M_4 - mM_2$, and that is what Butlerow does not do, for he says of this type:—"Die zweite typische Formel $nC_2M_4 - mM_2$ " ($nCM^4 - mM^2$) "wenn der Strich (-) das Zeichen der Verbindung vorstellen soll, wie in anderen Formeln von Couper, stimmt mit den That-sachen auch nicht überein; soll sie aber als nC_2M_4 minus mM_2 verstanden werden, so bekommt man für $n=7$, $m=6$ z.B. $C_{14}H_{28} - 12H = C_{14}H_{16}$, das heisst eine Verbindung mit der möglichst grossen Menge von Wasserstoff. —Die Verbindung $C_{12}H_6$ aber, ihre Homologen, Naphtalin, u.s.w. wo der Verbindungsgrad nicht der Formel $C_{2n}H_{2n+2}$ und sogar nicht der Proportion $C_{2n}H_{2n}$ entspricht, durch welche allgemeine Formeln sollen sie vorgestellt werden?"

Kekulé also, as I have shown above, misunderstood the expression.

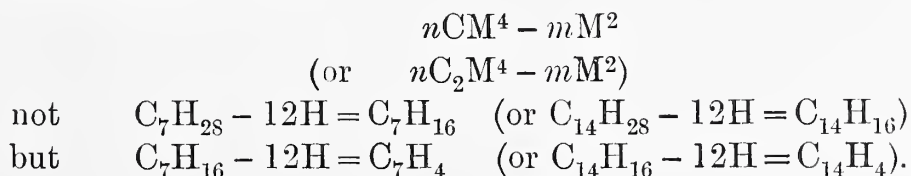
I note first that in the formula the stroke (-) is clearly a minus sign and not a mark of combination. If we keep in view that the first term of the second type is identical with the first type, and must have the same meaning, so that M stands for a unit of affinity, then we see that nCM^4 is the type for all saturated carbon compounds in the second type also. If then nM^2 units of affinity are to be subtracted from nCM^4 (because they are not saturated), the expression $nCM^4 - mM^2$ is the type of those carbon compounds which we call unsaturated. Benzene and naphthalin, the examples chosen by Butlerow, can be referred without any difficulty to this type. Let $n=6$ and $m=4$, then, if no element but hydrogen is present with the carbon, we have:



Let $n=10$ and $m=7$, then we have



In the first case supposed by Butlerow, on the assumption of $n=7$ and $m=6$, we should have for



That would be a formula in which 7C (or 7C²) are united by 12 carbon affinities, leaving 4 affinities over for union with hydrogen.

The most difficult thing to understand in Couper's view is how he

comes to the conclusion that the equivalent, atom or molecule (he does not distinguish between these terms), of oxygen must be taken as 8 and not as 16. It seems to me that Couper, still entangled in the doctrine of Berzelius, considered the hydrates of the oxygen acids and the salts of these acids as composed of two electrically opposite components. He takes a hydrated acid as a compound of the anhydride and water. In the formation of a salt from a hydrated acid and a basic hydroxide, according to him, it is not the hydrogen of the acid which changes place with the metal of the basic hydroxide, but the hydrogen takes with it an atom of oxygen ($O=8$) from the acid, and the metal carries an oxygen atom from the basic hydroxide into the salt. The oxygen atom of the acid radical, with which is united the oxygen atom combined with the metal, is negative, while the latter oxygen atom is rendered positive by its union with the metal. The reaction between hydrated acid and basic hydroxide is possible because the affinity between the positive and the negative oxygen is less than that between metal and oxygen, hydrogen and oxygen, and acid residue and oxygen. "A consequence of this truth is, that it is impossible to double the equivalent of oxygen, if the chemical equivalents are to be understood as not being in direct contradiction to any chemical truth or essential feature in the properties of an element. Carbon differs entirely in this respect from oxygen."

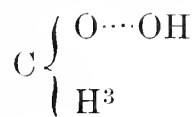
If Couper had not been so fundamentally opposed to Gerhardt's type theory he would no doubt have freed himself from these untenable views. Butlerow, too, thought Couper's adherence to the equivalent 8 for oxygen unfounded. He sets against Couper's notion of double decomposition that of Kekulé, according to which the two interacting molecules, in the first moment of the reaction, unite, and then the resulting complex splits in a new direction.

In Couper's three papers there are to be observed some variations in the mode of writing the examples of constitutional formulæ of carbon compounds. In the first short note in the *Comptes rendus*, dotted lines and brackets are used to indicate the bonds. Brackets are also used in the fuller paper in the *Annales de chimie et de physique*, with continuous instead of dotted lines. In the use of brackets we can well see a concession to the French and German chemists, who had been accustomed to their use by Gerhardt. In the English paper in the *Philosophical Magazine* the dotted lines are retained and the brackets are altogether discarded.

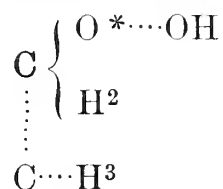
I here give Couper's constitutional formulæ for carbon compounds containing oxygen, from the *Comptes rendus* paper, in which he takes

C=12 and O=8 throughout. I here omit the formulæ of salicylic acid and the products obtained from it by means of phosphorus pentachloride, as these will be discussed under the next head.

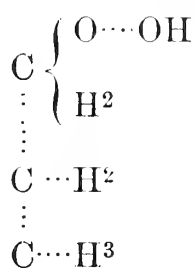
Methyl Alcohol.



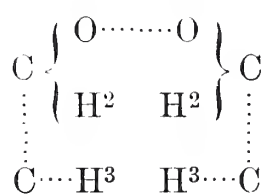
Ethyl Alcohol.



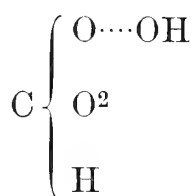
Propyl Alcohol.



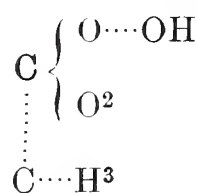
Ethyl Ether.



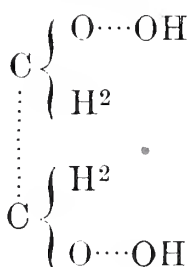
Formic Acid.



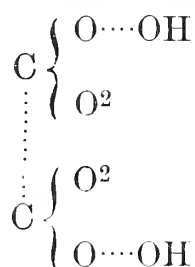
Acetic Acid.



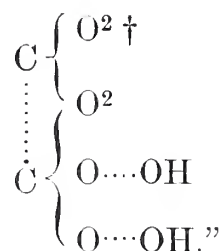
Glycol.



Oxalic Acid.



“Ou, si l'on veut réunir l'oxygène négatif à l'un des poles de la molécule, par la formule



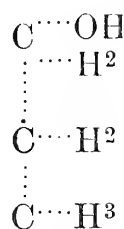
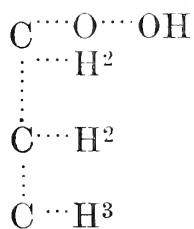
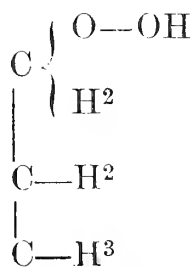
In the two full papers Couper then gives quite a number of constitutional formulæ with C=6; these I omit. Then he explains why C should

* In the paper the first “O” has been omitted in error.

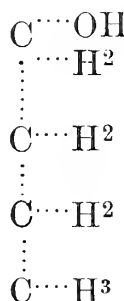
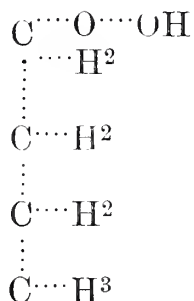
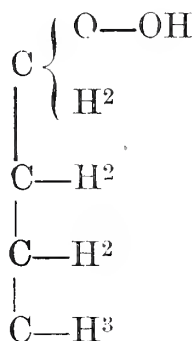
† Misprinted “C²” for “O²” in the paper.

be taken = 12, but he retains, as I have already more than once stated, O = 8. I now give, in three columns:—1st, the formulæ from the *Annales de chimie et de physique* paper; 2nd, those from the *Philosophical Magazine* paper; 3rd, the formulæ obtained from these by putting O = 16. While in the *Comptes rendus* paper Couper contented himself with quite simple examples, in the fuller papers he gives formulæ for substances with a more complicated composition.

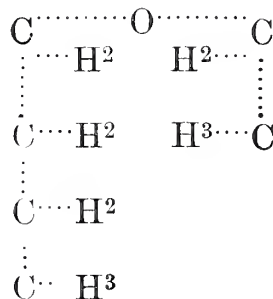
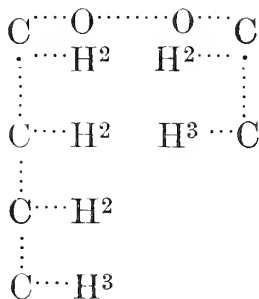
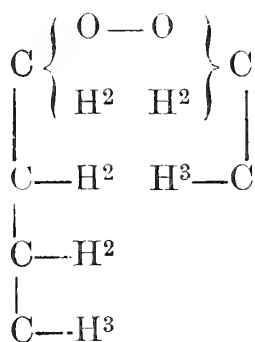
Propyle Alcohol.



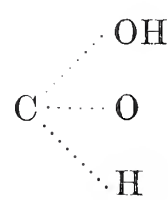
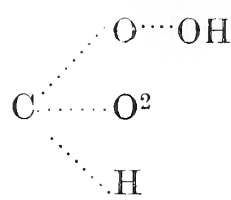
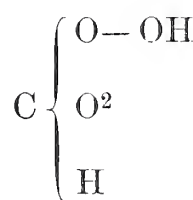
Butyle Alcohol.



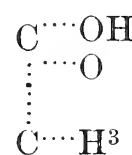
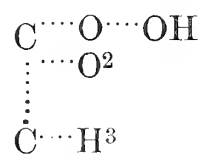
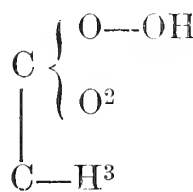
Butylic-ethylic Ether.



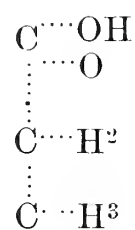
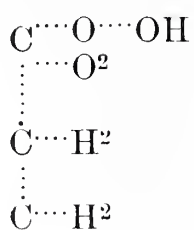
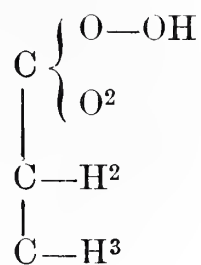
Formic Acid.



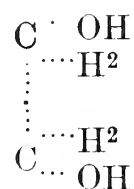
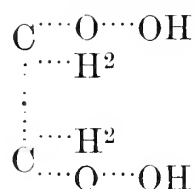
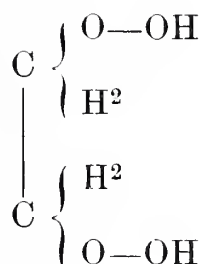
Acetic Acid.



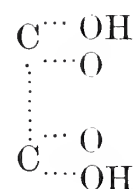
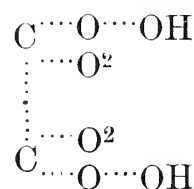
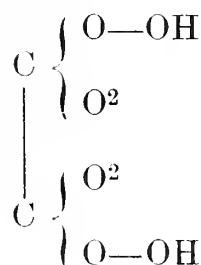
Propionic Acid.



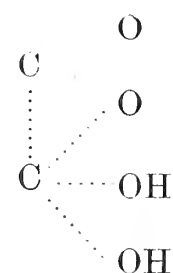
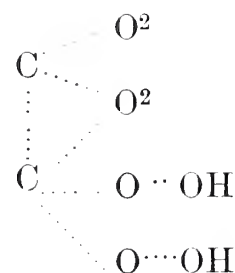
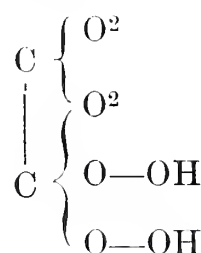
Glycol.



Oxalic Acid.



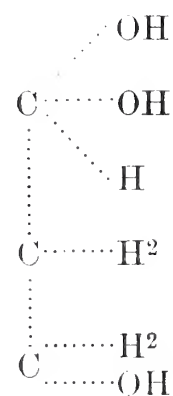
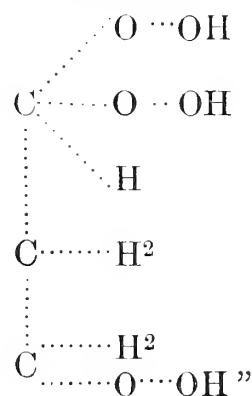
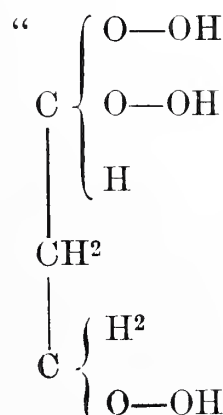
or, as before,



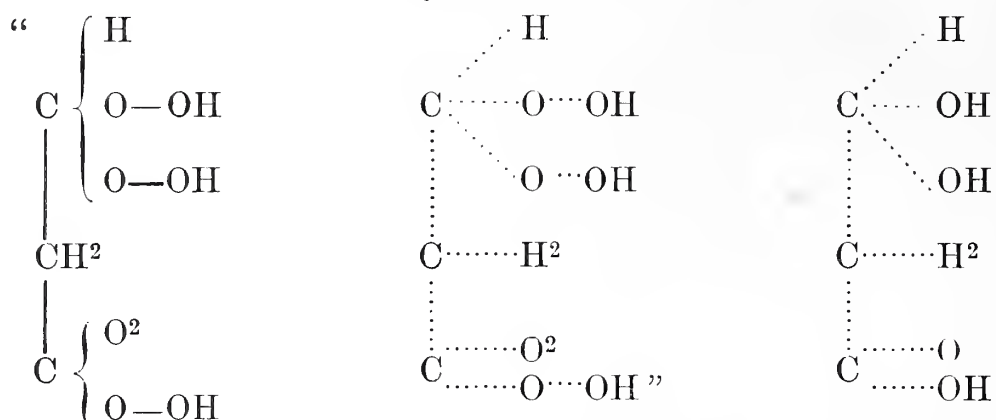
Couper thus gives an alternative formula for oxalic acid, in which the two (...O...OH) groups are united to one and the same carbon atom.

While Couper's formulæ, with the exception of this last, pass immediately into the constitutional formulæ now in use, if we take O = 16, he is less happy in formulating glycerin and glyceric acid.

Glycerin.

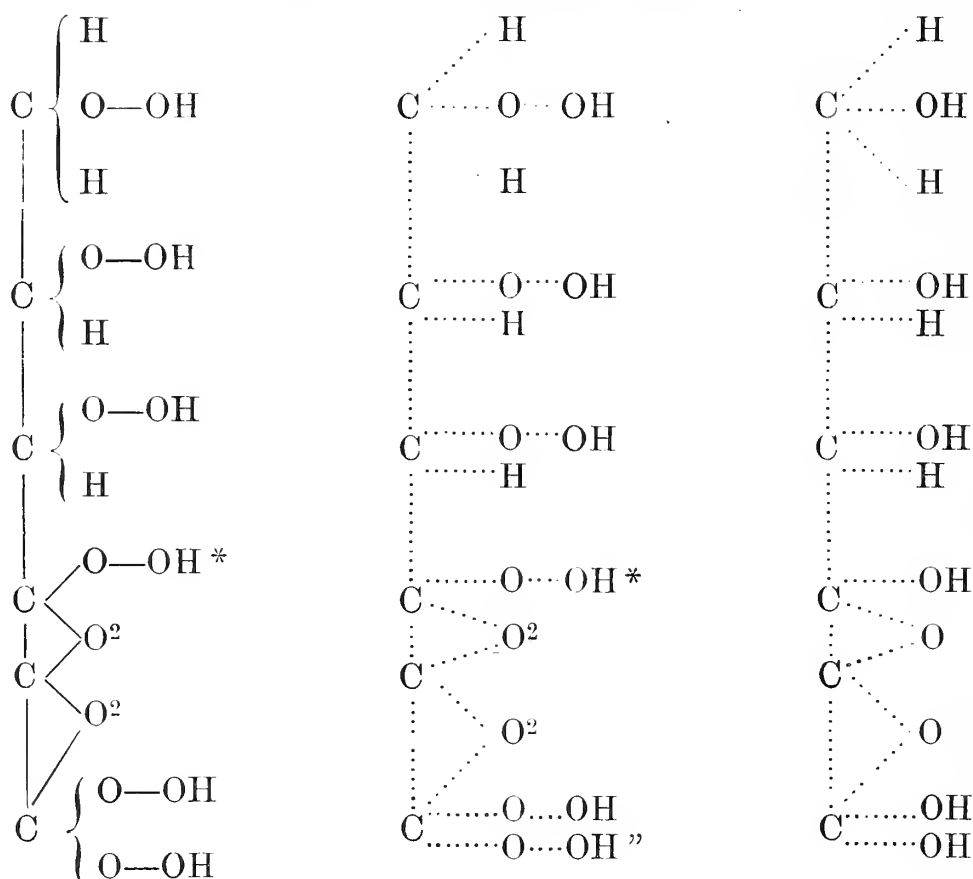


Glyceric Acid.

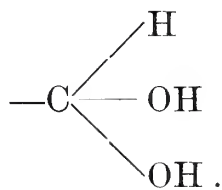


"Glucose has been perhaps too little investigated to afford data sufficient to determine definitely its formula. Taking, however, mucic and saccharic acids as starting-points, these bodies may be meanwhile represented as:—

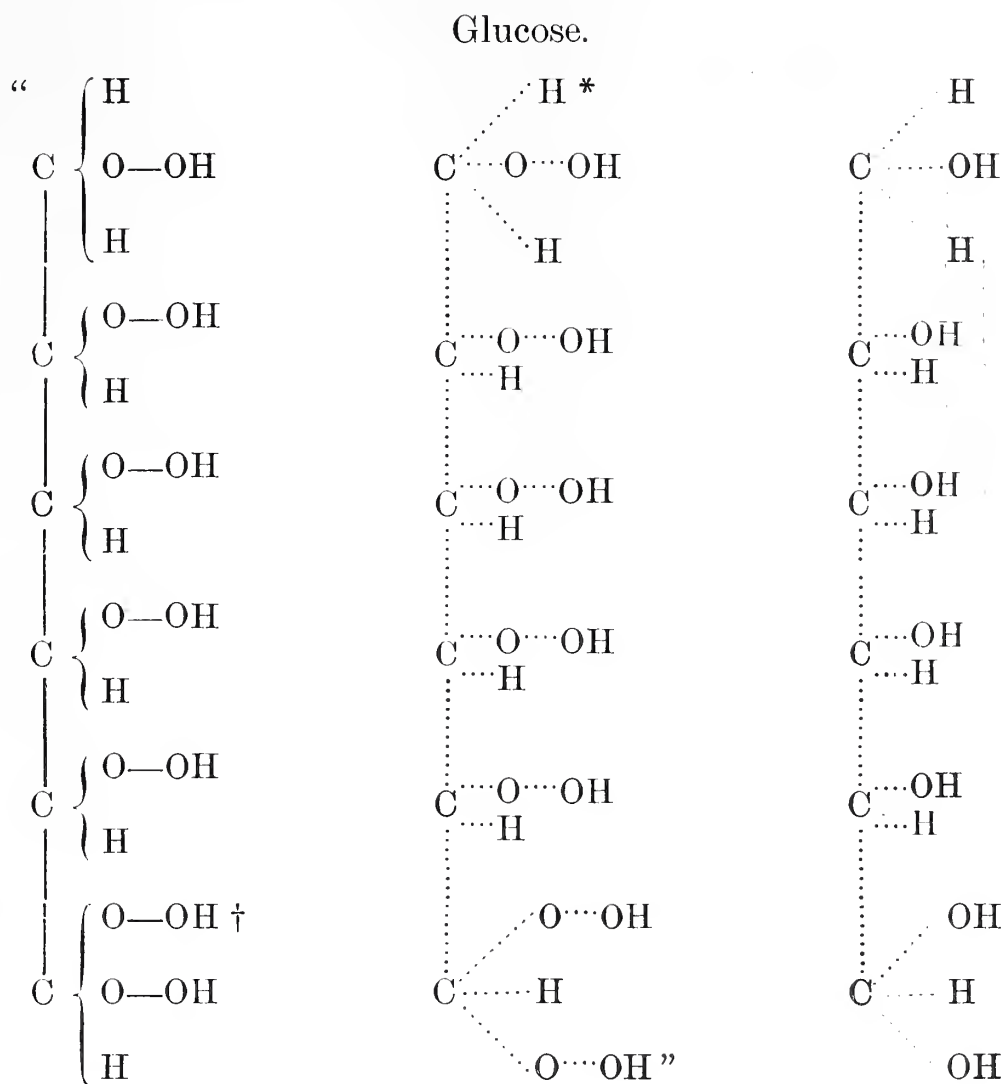
Mucic and Saccharic Acids.



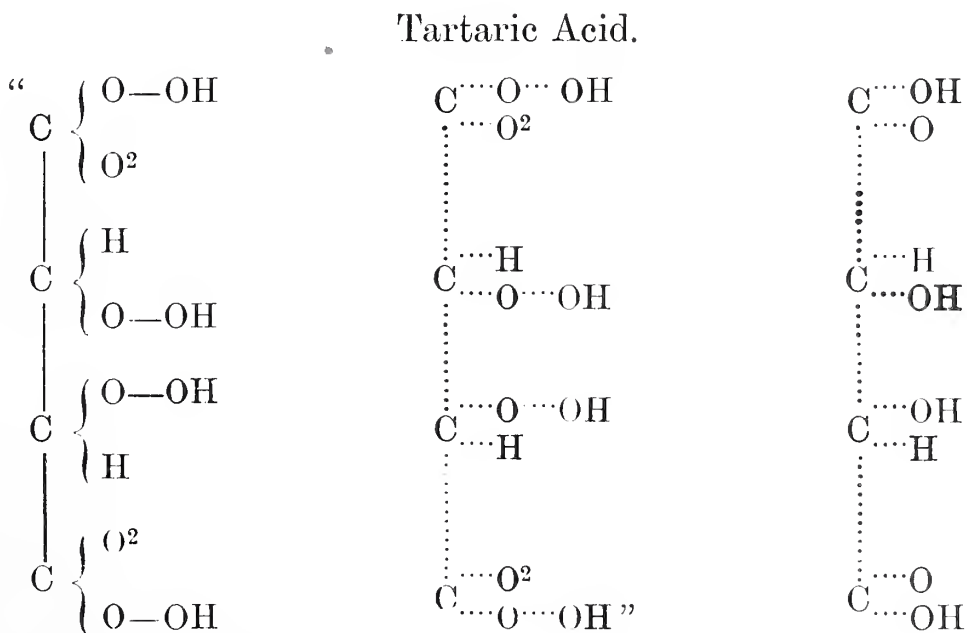
The error here is, essentially, the application to mucic and saccharic acids of the second possible mode of formulating oxalic acid, with the two acid hydroxyls on the same carbon atom. On the other hand, the formula for glucose is right, if we suppose the aldehyde group $-\text{CHO}$ transformed, by the addition of water, into the group



* In the papers this "OH" has been omitted in error.



Couper's formula for tartaric acid is, if we put $O=16$, identical with our structural formula for that acid, as here Couper distributes the two acid hydroxyls to the two end carbon atoms.

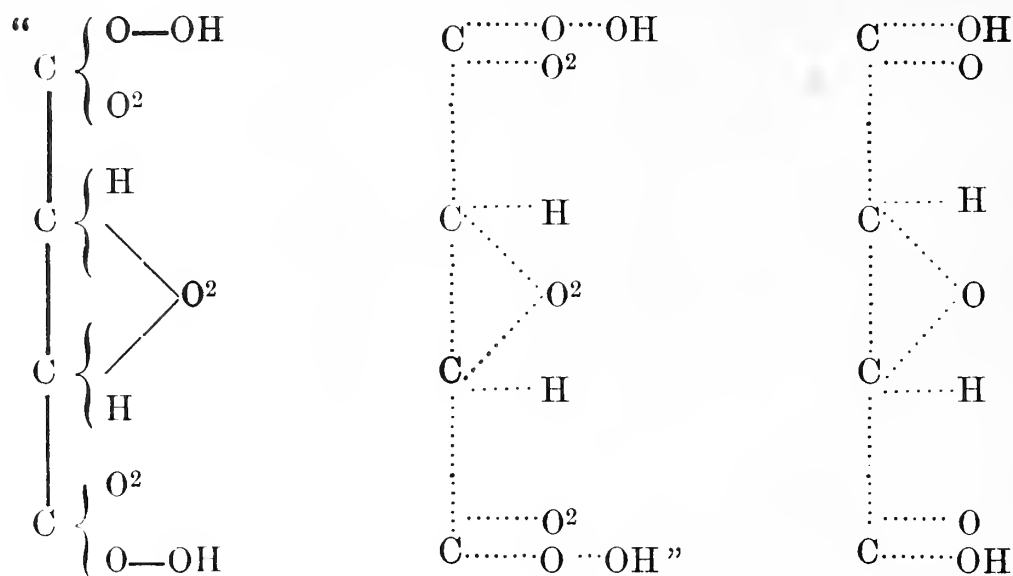


* In this formula the dotted lines joining the six carbon atoms have been omitted in the paper.

† Misprinted "H" for "O—OH."

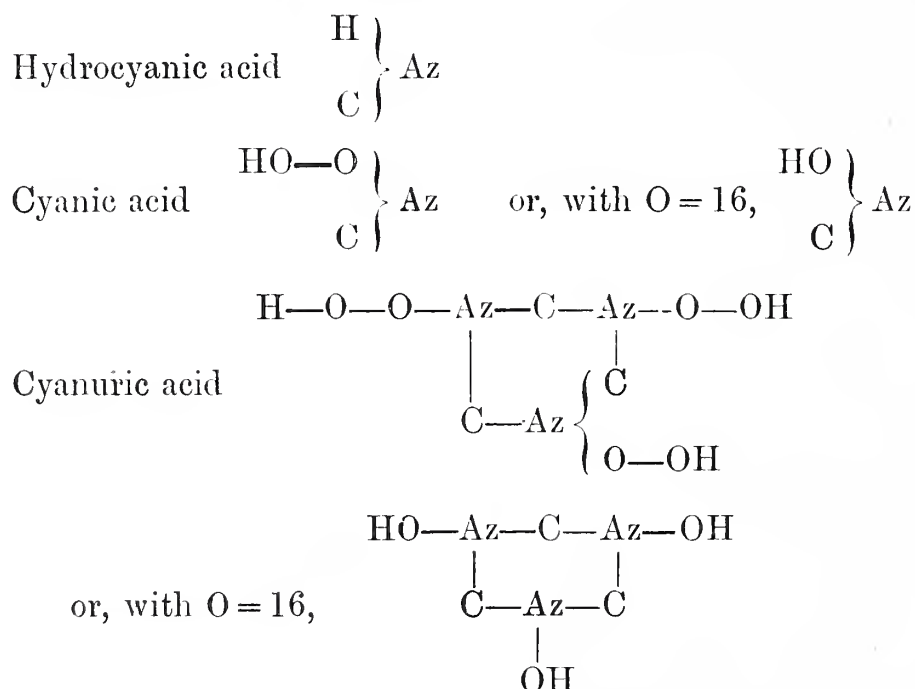
For tartrelic acid, formed by loss of water from tartaric acid, Couper gives the following formula:—

Tartrelic Acid.



All these compounds belong, according to Couper, to the type $n\text{CM}^4$.

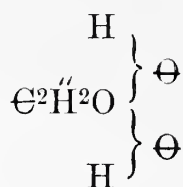
At the close of his paper in the *Philosophical Magazine*, Couper expresses his intention to discuss, in a later paper, the second type ($n\text{CM}^4 - m\text{M}^2$) and to apply his principles to the cyanogen compounds. His paper in the *Annales de chimie et de physique* closes with formulæ for hydrocyanic acid, cyanic acid, and cyanuric acid:—



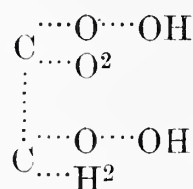
Thus in cyanuric acid Couper assumed that the three carbon atoms and the three nitrogen atoms are combined alternately in ring form.

By his method of writing formulæ Couper took an important step forward. His formulæ, in fact, enabled him, in a simple and clear way, to express his views as to the mutual combination of the atoms in the

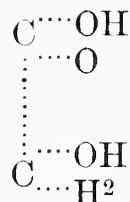
molecules of chemical compounds. Kekulé still, at that time, made shift, if we may use the expression, with typical formulæ. Although his ideas were no less clear than those of Couper, yet the formulæ he used were still quite those of the type theory. He wrote glycollic acid:—



deriving it from the double water type. Couper, as is clearly shown by his tartaric acid formula, would have written glycollic acid thus:—



and from this we at once get our present structural formula for glycollic acid:—



It was not until the spring of 1859 that Kekulé, in the first part of his *Lehrbuch*, in remarks on the mutual union of atoms, gave his well-known graphic representation in which the basicity of the atoms is indicated by a difference in the size of their symbols—a difference in size which, as Kekulé expressly says, is not intended to express any difference in the actual size of the atoms, but only the number of chemical units which an atom represents, that is, the number of hydrogen atoms to which it is equivalent.

We have already become acquainted with Butlerow's remarks on Couper's new chemical theory. But before him, immediately after the appearance of Couper's paper in the *Annales de chimie et de physique*, Wurtz expressed his views in reference both to Kekulé's paper, "Über die Constitution und die Metamorphosen der chemischen Verbindungen und über die chemische Natur des Kohlenstoffs,"* and to Couper's "Sur une nouvelle théorie chimique."† His remarks were published in the *Répertoire de chimie pure et appliquée*, a journal he had just founded for notes on current chemical literature. The distinguished position of Wurtz, and his relation to Couper, make his criticism specially worthy of our consideration.

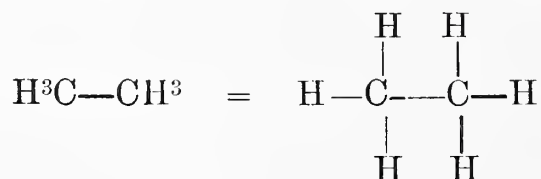
* Rép. 20-24.

† Rép. 49-52.

After a short note of the contents of the paper, he combats Couper's assumption of $O=8$, and says: "Cette théorie électrochimique d'un nouveau genre aurait besoin de s'appuyer sur quelques faits."

But there is not wanting a word of recognition by Wurtz:—"Je dois faire remarquer encore que les idées énoncées par M. Couper et qui me paraissent ingénieuses et acceptables, si on les dégage certaines hypothèses accessoires et de quelques nuages qui en enveloppent l'exposition, ne sont nullement en contradiction avec la doctrine des radicaux, et même avec cette des types moléculaires contre lesquelles l'auteur a cru devoir rompre une lance." Wurtz closes his notice with the following statement of opinion:—"En général, je trouve les formules de M. Couper trop arbitraires, trop éloignées de l'expérience. Par nos formules rationnelles nous n'avons pas la prétention de représenter la constitution intime des combinaisons. Ces formules ne représentent que les métamorphoses, c'est-à-dire des faits accessibles à l'expérience et démontrés par elle. Voilà leur avantage. Dans les formules de M. Couper, au contraire, la place de chaque atome se trouve marquée, non seulement par le pouvoir basique des éléments, mais encore par je ne sais quelle attraction électrique ou polaire. C'est trop d'hypothèses, et l'on a tort de nous présenter toutes ces choses comme la loi et les prophètes. A cet égard, M. Kekulé, qui me paraît avoir mieux compris le sens et la portée des idées, qu'il a énoncées le premier, a dit sagement à la fin de son Mémoire: 'Pour mon compte, je n'attache qu'une importance secondaire à des considérations de cet ordre là.'"

And here Wurtz should not have let the opportunity slip of putting it on record that Couper had given him the MS. of his paper for the *Comptes rendus* before Kekulé's paper appeared in *Liebig's Annalen*. Further we may note how Wurtz expressed himself some years later as to formulæ in which, exactly as had been first proposed by Couper, the mutual relations of the atoms forming a compound were represented by means of lines joining the symbols. In his work, *La théorie atomique*, Paris, 1879, livre ii., "Atomicité, ou valence des atomes dans les combinaisons," Wurtz discusses very fully the distribution of the hydrogen atoms in ethane, and says (p. 156): "Telle est la signification de la formule:



dans laquelle cet échange d'unités de saturation est marqué par les traits qui séparent les lettres." To this he appends a note:—"Cette notation généralement usitée aujourd'hui a été employée pour la première fois dans

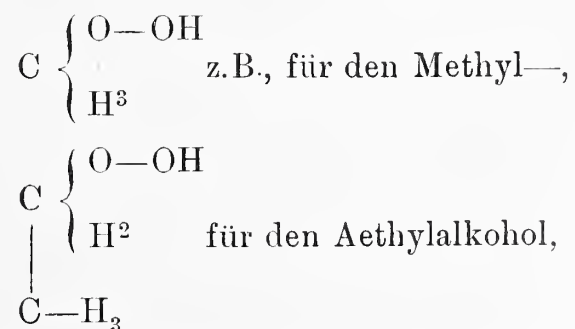
les leçons que j'ai faites au Collège de France pendant l'été de 1863, et qui ont été publiées d'abord dans le *Moniteur scientifique* du Dr Quesneville, et plus tard sous le titre *Leçons de philosophie chimique*, Hachette, 1864. Je renvoie à cet égard aux pages 140, 143, 145, 158 et 182 de cet opuscule." When Wurtz wrote these lines he had evidently forgotten the formulæ in Couper's paper which he had condemned as too fantastic.*

Butlerow, Couper's other critic, returns to the question in 1868, † about ten years later, after organic chemistry had undergone so mighty a development under the influence of the valency theory, and now fully recognises Couper's position. He says:—"Dass ferner die von Couper (leider nur kurz) ausgesprochenen Ansichten, mit den jetzt fast allgemein anerkannten identisch sind, verneine ich nicht, und sicher sind die von Couper gegebenen Formeln rationelle Formeln im gegenwärtigen Sinne des Wortes, d.h. Constitutionsformeln oder Formeln chemischer Structur."

I shall now sum up the results. It is quite certain that Kekulé proclaimed the hypothesis of the quadrivalence of carbon and the concatenation of carbon atoms before Couper. In contrast to Kekulé, Couper started the idea of affinity of degree in chemical elements, and illustrated it by the examples of carbonic oxide and carbonic acid. In the former he regarded carbon as bivalent, in the latter as quadrivalent, thus recognising a change of atomicity or valency.

Albert Ladenburg, in his lectures *On the Development of Chemistry during the last Hundred Years*, published in 1869, refers as follows to Couper's formulæ:— ‡ "Hier begegnen wir zum ersten Male Constitutionsformeln im heutigen Sinne des Wortes, Symbolen, welche aus der Erkenntniss der Atomigkeit der Elemente hervorgegangen sind"; and a little further on:—"Diese beiden Abhandlungen von Kekulé und Couper bilden die Grundlagen unserer Anschauungen über den Aufbau der Verbindungen."

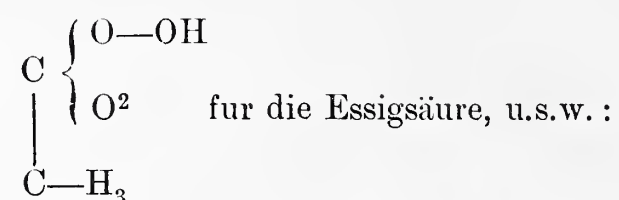
Hermann Kopp, in his work, *Die Entwicklung der Chemie in der neueren Zeit*, gives a similar decision as to Couper's formulæ:—"Die Formeln, durch welche Couper seine hierauf bezüglichen Vorstellungen ausdrückte:—



* Compare also Ernst von Meyer's *Geschichte der Chemie*, Leipzig, 1889, S. 270, Anm. 3.

† *Liebig's Annalen*, cxlvi. 260 (Heft ii., issued on 6th May 1868).

‡ S. 269.



sie waren bereits solche, wie sie als s.g. Strukturformeln weiter zu entwickeln und fester zu begründen viele Chemiker nachher als eine der ihrer Wissenschaft vorzugsweise gestellten Aufgaben betrachteten, viele jetzt es thun.”*

I close this section with an extract from Ernst von Meyer's *Geschichte der Chemie von den ältesten Zeiten bis zur Gegenwart*, first published in 1889:—“Ferner legte Couper grossen Nachdruck auf die Fähigkeit der Kohlenstoffatome sich unter einander zu vereinigen und zwar so, dass ein Theil des ihnen eigenen Bindungsvermögens ausgeglichen wird. Diese Vereinigung von Atomen versinnlichte er durch Striche, welche zwischen den mit einander verbundenen Theilchen in den chemischen Symbolen angebracht wurden, er legte so den Grund zu den sogenannten Strukturformeln.”†

COUPER'S INVESTIGATIONS ON SALICYLIC ACID.

Couper's second and last experimental investigation was laid before the French Academy of Sciences at the meeting of the 7th June 1858.‡ It was entitled “Recherches sur l'acide salicylique.” He says that he undertook this work in order to throw some light on the disputed questions as to the constitution and basicity of salicylic acid. This subject had engaged Chiozza, who worked at it in Gerhardt's laboratory, Gerhardt himself, and his pupil Drion. Unknown to Couper, Kekulé and also Kolbe, along with Lautemann, were at the same time occupied with it. Following these investigations, which had then for their aim the clearing up of the question of the constitution of salicylic acid, we see Gerhardt, Kekulé, Kolbe, and Couper writing formulæ for salicylic acid, and we shall have no difficulty in deciding which of these four investigators made the boldest and most successful advance in this region.

In 1852 Chiozza, probably on the suggestion of Gerhardt, examined the action of phosphorus pentachloride on gaultheria oil, § at that time the usual source of salicylic acid. He found that, on the distillation of the product of the reaction, a liquid was obtained which behaved like an acichloride, giving chlorobenzoic acid on treatment with water.

* Page 829.

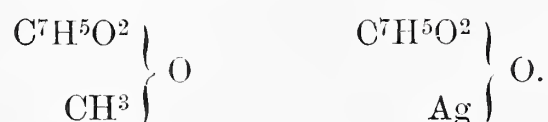
† Page 270.

‡ *Comptes rendus*, xlv. 1107–1110.

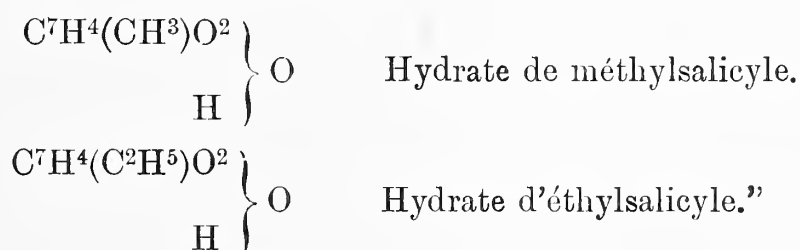
§ *Ibid.*, xxxiv. 850–851, séance du lundi, 31 May 1852.

A succeeding paper by Gerhardt* introduces us to the problem as it presented itself to chemists at that time, eleven years before the foundation of Kekulé's benzene theory. Gerhardt found two anomalies in the behaviour of the ethers of salicylic acid. First, that they give well-defined salts when treated with bases. Second, that treatment with chlorine or bromine does not lead to substitution in the alcohol radicals, but to the formation of ethers of chloro- or bromo-salicylic acid.

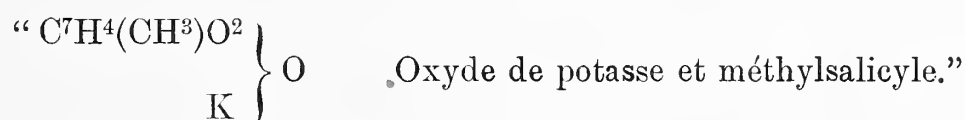
At first Gerhardt gave "salicylate de méthyle (huile de gaultheria)" a formula similar to that of silver salicylate:—



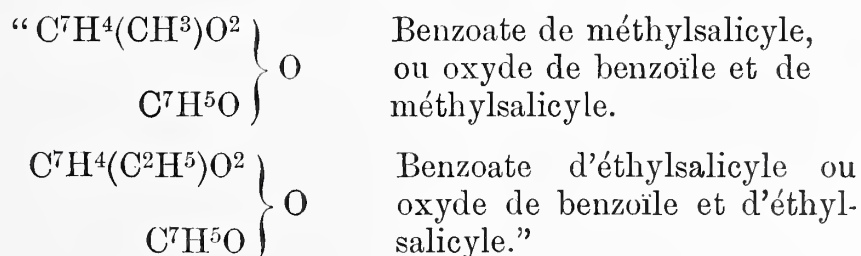
The first of the two anomalies—the only one that interests us here—led Gerhardt to regard the ethers of salicylic acid not "comme une molécule d'eau dont 2 atomes d'hydrogène étaient remplacés l'un par du salicyle et l'autre par du méthyle ou d'éthyle," but "comme une molécule d'eau dont 1 atome d'hydrogène seulement était remplacé par le groupe méthylsalicyle ou éthylsalicyle, c'est-à-dire par du salicyle contenant déjà lui-même du méthyle ou de l'éthyle en substitution à de l'hydrogène:—



Correspondingly, he gives the potash salt of gaultheria oil the formula and name:—



He adds:—"L'expérience a pleinement justifié mes présomptions. Rien n'est plus facile, en effet, que d'éthérifier les éthers salicyliques, comme on éthérifie l'alcool ou l'esprit-de-bois." For, by treatment with benzoyl chloride, the salicylic ethers give the corresponding benzoyl derivatives.



In order to obtain the "chlorure de méthylsalicyle," Gerhardt sub-

* *Comptes rendus*, xxxviii. 32-34, séance du lundi, 9 Jan. 1854.

jected gaultheria oil to the action of phosphorus pentachloride; but, to his surprise, he obtained, after a vigorous action, both from the methyl and from the ethyl ether, a new chloride, “chlorure de salicyle,” $\left. \begin{array}{l} \text{C}^7\text{H}^5\text{O}^2 \\ \text{Cl} \end{array} \right\}$. It is obvious that Gerhardt did not distil the product as Chiozza had done, but drew his conclusion as to its nature only from the action on it of methyl and ethyl alcohol, by which it is retransformed into ethers of salicylic acid. Gerhardt does not quote Chiozza’s paper, nor does he discuss its contents.

Gerhardt’s pupil, Drion, further extended Gerhardt’s experiments on salicylic ethers.* His observations are of interest to us here, as he describes the preparation of salicyl chloride somewhat more in detail than Gerhardt had done:—“Le chlorure de salicyle que j’emploie pour la préparation du salicylate d’amyle a été obtenu, pour la première fois, par M. Gerhardt, en faisant agir le perchlorure de phosphore sur l’huile de gaulthéria. Dans cette réaction remarquable, il ne se forme que des traces d’oxychlorure de phosphore; mais il se dégage beaucoup d’acide chlorhydrique, et j’ai constaté également la production abondante de chlorure de méthyle. Le chlorure de salicyle peut être chauffé jusque vers 200 degrés sans se décomposer, mais on ne peut le distiller.

“Dans le but de l’obtenir pur j’ai cherché à le distiller sous une pression moindre que celle de l’atmosphère. Bientôt d’abondantes fumées d’acide chlorhydrique sont sorties de la pompe et m’ont contraint de renoncer à l’emploi de cet appareil. J’ai continué la distillation sous la pression atmosphérique, et j’ai recueilli dans le récipient un liquide fumant, présentant tous les caractères des chlorures organiques.”

Treated with water, the distillate yielded a mixture of salicylic and chlorobenzoic acids, from which Drion concluded that the “chlorure de chlorobenzoïle, $\text{C}^7\text{H}^4\text{ClO}, \text{Cl}$,” was formed by the decomposition of the “chlorure de salicyle.” “Il avait été obtenu déjà par M. Chiozza en faisant agir le perchlorure de phosphore sur l’acide salicylique.”

I sum up the collection of observations made in Gerhardt’s laboratory by Chiozza, Gerhardt, and Drion on the action of phosphorus pentachloride on gaultheria oil. The violent reaction of the two substances takes place with the evolution of hydrochloric acid and methyl chloride: only traces of phosphorus oxychloride are produced. The undistilled product of the reaction, when treated with methyl alcohol, gives methyl salicylate, and was therefore pronounced to be salicyl chloride, the acichloride of salicylic acid. But the supposed salicyl chloride cannot be distilled unchanged, either under ordinary or under reduced pressure. The product obtained

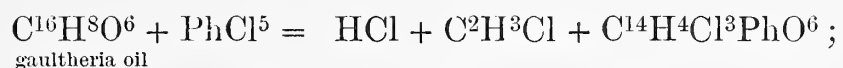
* *Comptes rendus*, xxxix. 122–125, séance du lundi, 10 juillet 1854.

by distilling under ordinary pressure gives, with water, salicylic acid and monochlorobenzoic acid, and was therefore regarded as a mixture of the acichlorides of these acids.

The puzzling thing was, what had become of the phosphorus oxychloride which must have been formed by the reaction of the hydroxyl of the salicylic acid with the phosphorus pentachloride? The production of chlorbenzoyl chloride from the supposed salicyl chloride was also unintelligible. Besides, most of the conclusions lacked the support of analysis.

The problem had, therefore, attractiveness enough to lead a sharp-sighted investigator like Couper to attack it again.

I use, for reference in the meantime, only Couper's first paper, that in the *Comptes rendus*,* in which he contented himself with the use of empirical molecular formulæ, as we should now call them, using the atomic weights C=6 and O=8. He describes his method and the course of the reaction thus:—He added gaultheria oil in small quantities at a time to the phosphorus pentachloride, in the proportion of one equivalent of the former to two of the latter. After the reaction (which ran exactly the same course when salicylic acid was used) was finished, he distilled the product under ordinary pressure. When the trace of phosphorus oxychloride and the excess of the pentachloride had been driven off, there passed over, between 285° and 295°, a colourless liquid, which he named “trichlorophosphate de salicyle, C¹⁴H⁴Cl³PhO⁶,” formed according to the equations:

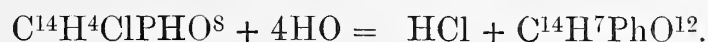
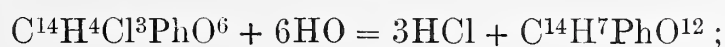


gaultheria oil
salicylic acid

The “trichlorophosphate de salicyle” is instantly decomposed by hot water, giving hydrochloric, phosphoric, and salicylic acids; according to the equation:

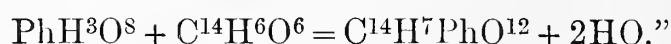


If the “trichlorophosphate de salicyle” is quickly distilled, it partially decomposes with abundant evolution of hydrochloric acid. Above 300° there passes over a liquid which, when sealed up in a glass tube, deposits large crystals of “monochlorophosphate de salicyle, C¹⁴H⁴O⁸ClPh.” Both the trichlorophosphate and the monochlorophosphate take up moisture from the air and give rise to a new acid, “acide phosphosalicylique, C¹⁴H⁷PhO¹².” This is formed in accordance with the equations:



* xlv. 1107-1110.

The "acide phosphosalicylique" is tribasic, and Couper regarded it "comme une combinaison conjugée d'acide phosphorique et d'acide salicylique,



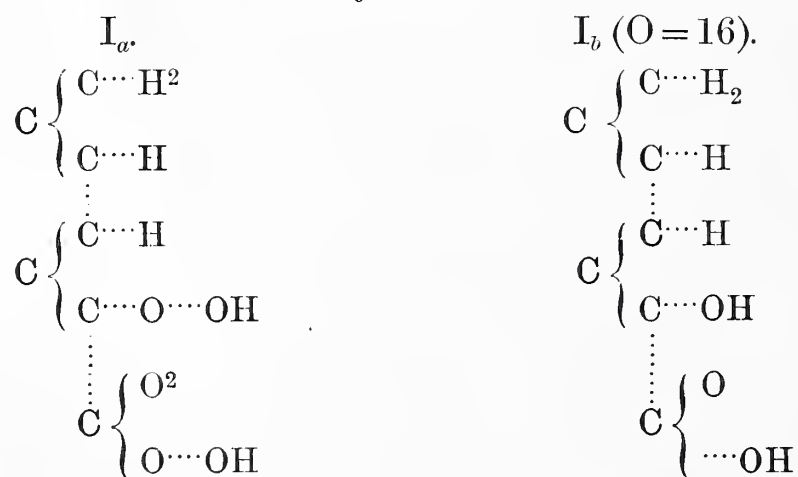
These results, says Couper, do not, in all points, agree with the observations published by Gerhardt, Chiozza, and Drion, and raise a doubt as to the existence of the salicyl chloride described by Gerhardt, a product that has never been analysed. Couper then points out that the formation of salicyl chloride from salicylic acid or from gaultheria oil could not take place without the production of phosphorus oxychloride, whereas he had observed only traces of this substance. Whether these contradictions were the result of some accident or of special conditions under which the various observers worked, could only be decided by new investigations.

The formulæ of the three new compounds containing phosphorus, which he had obtained from salicylic acid, were substantiated by Couper by means of a series of analyses. He used empirical molecular formulæ and the small equivalent weights $\text{C} = 6$ and $\text{O} = 8$.

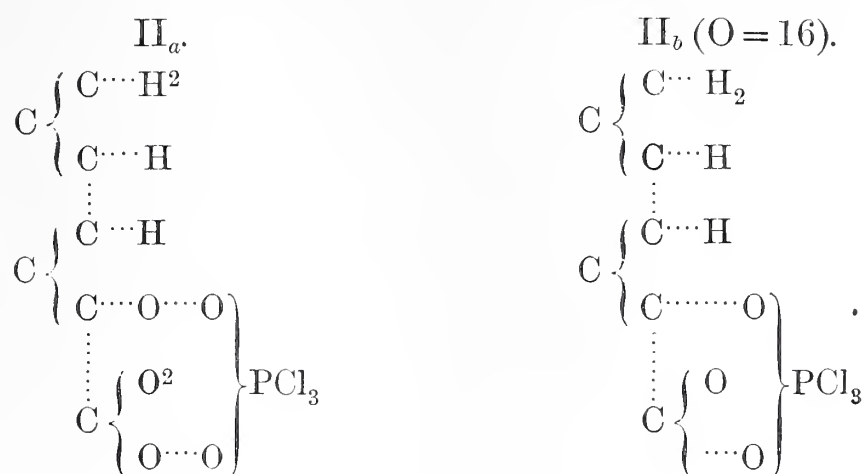
A week after the publication of this paper on salicylic acid, there appeared, in the *Comptes rendus*, Couper's first communication on a new chemical theory. He applies his new theory to the derivation of constitutional formulæ for salicylic acid and for his "trichlorophosphate de salicycle," assuming $\text{C} = 12$, but retaining $\text{O} = 8$.

In the somewhat later paper, published in the *Edinburgh New Philosophical Journal*, he gives constitutional formulæ, not only for these, but also for the "monochlorophosphate de salicycle" and the "acide phosphosalicylique." These formulæ are given below as I_a , II_a , III_a , and IV_a , and beside them I have given, as I_b , II_b , III_b , and IV_b , the formulæ obtained by doubling the atomic weight of oxygen:

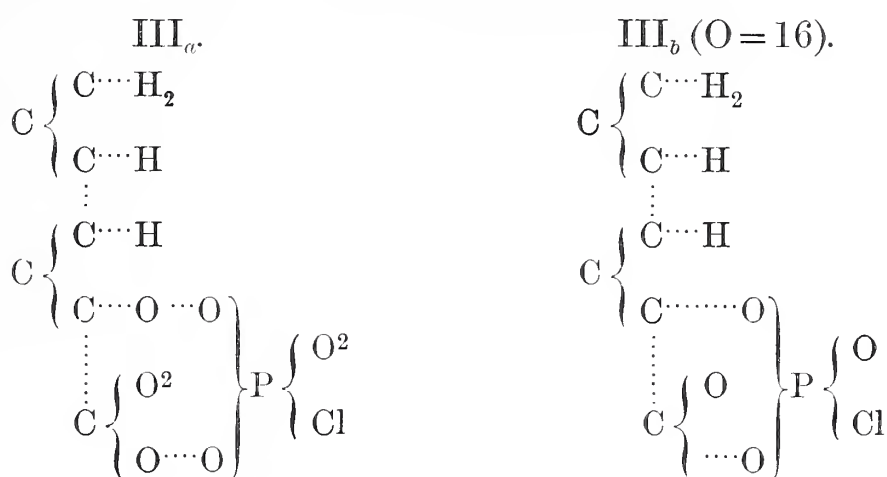
Salicylic Acid.



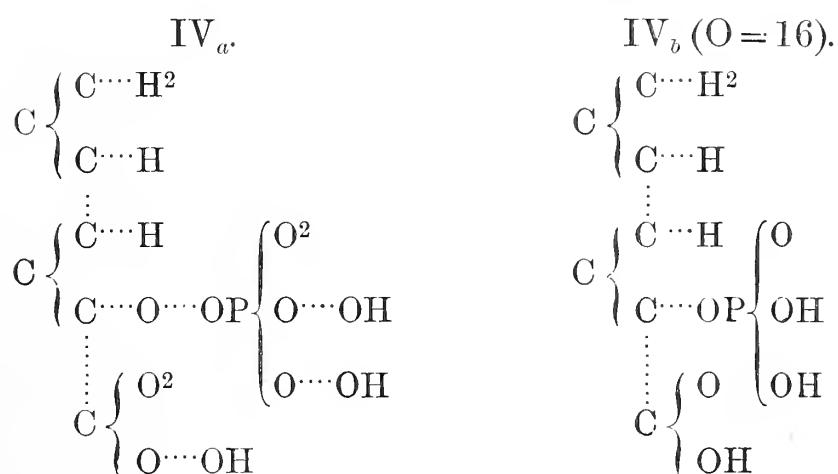
From this formula (I_a) for salicylic acid Couper derives the following formula for the "terchlorophosphate of salicycle":—



Monochlorophosphate of Salicylc.



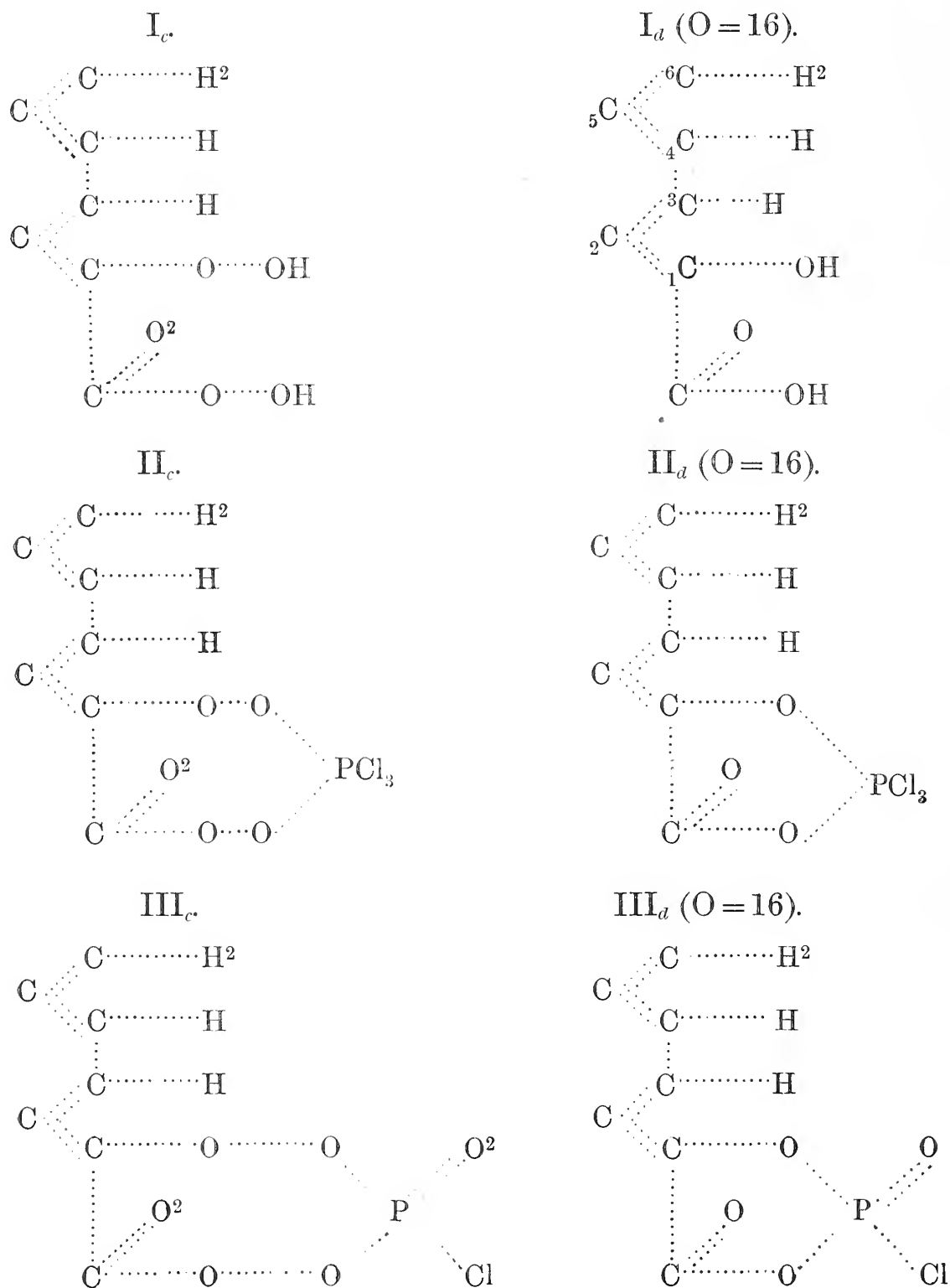
Phosphosalicylic Acid.

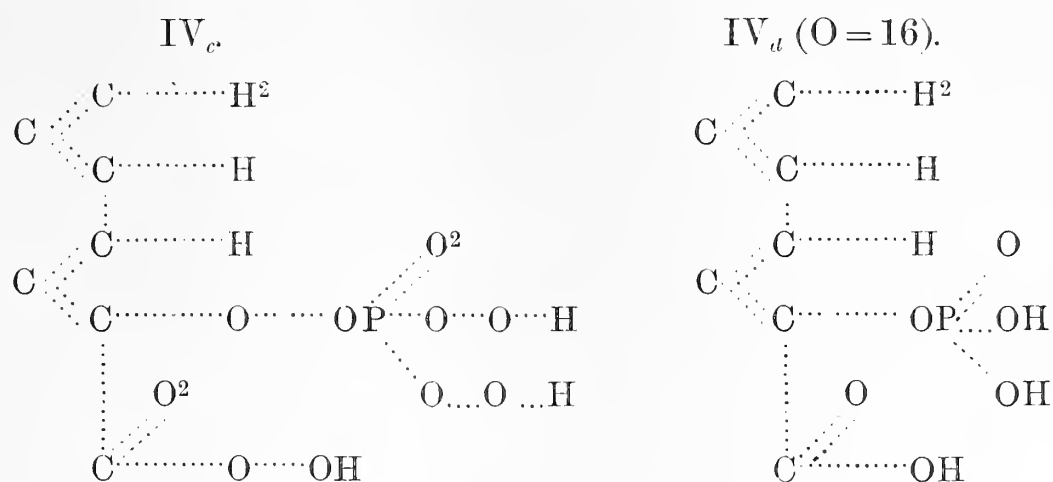


These formulæ, I_a, II_a, III_a, and IV_a, are structural formulæ, in our present meaning of the term. They strike us as still more modern when, in them, we use O=16, that is, divide the number of oxygen atoms by 2 (or write O for Couper's O² or -O-O-), as is shown in the formulæ I_b, II_b, III_b, IV_b. The formulæ II_a and II_b show us what has become of the phosphorus pentachloride, the residue of which replaces the hydrogen of the hydroxyl and of the carboxyl in the salicylic acid. We see that Couper supposed the hydroxyl and the carboxyl to be attached to the same carbon atom.

Thus, so early as seven years before the establishment of Kekulé's benzene theory, Couper endeavoured to take account of the mode of union of the seven carbon atoms of salicylic acid.

As I have pointed out in the preceding section, Couper, in his fuller paper in the *Philosophical Magazine*, "On a New Chemical Theory," everywhere replaces the brackets of his structural formulæ by lines indicating the union of the atoms. If we carry this out in the case of the eight formulæ given above, and represent, as has been done in later times, the double union by two parallel lines, we obtain the following eight forms, I_c , II_c , III_c , IV_c , and I_a , II_a , III_a , IV_a :—





As I said some time ago, in discussing these formulæ* :—“ If my reconstruction of Couper’s formulæ is justifiable, these formulæ clearly show what Couper lacked for the discovery of the benzene theory. Four carbon atoms—I have numbered them in formula I_a; they are 2, 3, 4, and 5—are represented by Couper in the same state of combination with one another as Kekulé’s formula gave them at a later date.” Couper had only to take one step more, to unite the carbon atoms 6 and 1, and close the ring: this would have involved the transference of a hydrogen atom from 6 to 5, and of the hydroxyl from 1 to 2. And Couper had already arrived at the assumption of the union of multivalent atoms in ring form, as is shown by the formula, given in an earlier part of this paper (p. 216, and Appendix, p. 265), by which Couper expressed the constitution of cyanuric acid. “ Undoubtedly Archibald Couper was, at that time, August Kekulé’s most dangerous rival.”

Of the three chemists whose work was criticised in Couper’s salicylic acid paper, Gerhardt was gone (having died August 19, 1856, in Strassburg, soon after his settlement there), Chiozza said nothing, and only Drion attempted to defend himself and Gerhardt.† He maintained that the existence of the salicyl chloride observed by Gerhardt, although not isolated in a pure state owing to its nonvolatility, was indubitably proved by the readiness with which it reproduced the ethers of salicylic acid when treated with alcohols. It will be seen that Drion makes his task too easy: he tries neither to refute Couper’s description of the course of the reaction nor to explain what became of the phosphorus oxychloride, which must have been produced in the formation of salicyl chloride, but which neither he nor Couper had observed.

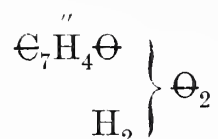
It is remarkable that Couper and Kekulé encountered one another not only in the theoretical but also in the experimental field. Kekulé, in

* *Liebig’s Annalen*, ccxlv. 290.

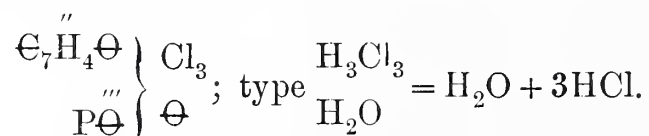
† *Comptes rendus*, xlvi. 1238, séance du lundi, 21 juin 1858.

connection with the formation of glycollic acid from monochloroacetic acid, tried to convert monochlorobenzoic acid into salicylic acid. He prepared the chlorobenzoic acid from its chloride, which he obtained, according to Chiozza's directions, by distilling the product of the reaction of phosphorus pentachloride on salicylic acid. His attempts to pass from the sodium salt of chlorobenzoic acid to salicylic acid, as he * and Reinhold Hoffmann † had obtained glycollic acid from chloroacetic acid, convinced Kekulé of the small reactive power of the chlorine in chlorobenzoic acid. He could as little replace it by hydroxyl as Couper could the bromine in brombenzene and dibrombenzene. This communication of Kekulé, "Bildung von Glykolsäure aus Essigsäure," which closes with the experiments as to salicylic acid, was sent to the editors of *Liebig's Annalen* on the 25th December 1857. Couper's paper on salicylic acid appeared in the first June number of the *Comptes rendus*, ‡ so that Kekulé could not have known anything of this work of Couper. But, after the appearance of Couper's paper, Kekulé, who had in the meantime moved from Heidelberg to Ghent, applied himself with the greatest zeal to the continuation of his work on salicylic acid. In his communication laid before the Belgian Academy on the 4th August 1860, § entitled, "Faits pour compléter l'histoire de l'acide salicylique et de l'acide benzoïque," which appeared in the number of *Liebig's Annalen* || published 4th February 1861, under the title, "Beiträge zur Kenntniss der Salicylsäure und der Benzoessäure," Kekulé minutely discusses Couper's work.

Kekulé gives salicylic acid the formula



The phosphorus compound, $\text{C}_7\text{H}_4\text{Cl}_3\text{PO}_3$, described by Couper, is regarded by Kekulé as an indirect combination of phosphorus oxychloride with the anhydride of salicylic acid, or as a compound of the radicals salicyl and phosphoryl, belonging to a mixed type, according to the formula



He cannot confirm Couper's statements as to the products of the action of phosphorus pentachloride on salicylic acid or on its methyl ether. He writes, in the *Annalen*:—"Ich habe diesen Versuch mehr als 20mal

* *Liebig's Annalen*, cv. 286-292 (Heft iii. issued 27th März 1858).

† *Ibid.*, cii. 12.

§ *Bull. Acad. Roy. Belg.* (2), x. 337-350.

‡ xlv. 1107-1110.

|| cxvii. 145-164.

wiederholt und dabei mehrmals beträchtliche Mengen der Materialien und in sehr wechselnden Mengen in Arbeit genommen. Ich habe niemals den von Couper beschriebenen Körper erhalten, der nach diesem Chemiker bei 285° bis 295° überdestilliert. Ich habe vielmehr stets beobachtet, dass, sobald die Temperatur der überdestillierenden Dämpfe auf höchstens 280° gestiegen ist, der Rückstand in der Retorte sich unter heftigem Aufblähen und mit Hinterlassung einer schwarzen blasigen Masse zersetzt. Ich habe mich ausserdem überzeugt, dass 1 Molecül Methylsalicylsäure" [that is, gaultheria oil, the methyl ether of salicylic acid] "nur 1 Molecül Phosphorsuperchlorid zu zersetzen im Stande ist, und dass alles im Überschuss zugesetzte Phosphorsuperchlorid bei der ersten Destillation unverändert überdestilliert. Ich habe ferner gefunden, im Widerspruch mit den Angaben von Couper dass eine sehr beträchtliche Menge von Phosphoroxychlorid gebildet wird." "Ich habe endlich gefunden, dass der Rückstand in der Retorte, man mag die Destillation zu Ende führen oder in irgend einer Periode unterbrechen, bei Zersetzung mit Wasser oder Kali wesentlich Salicylsäure liefert, die nur Spuren von Chlorbenzoesäure enthält."

The key to Kekulé's want of success is to be found in the fact that he had kept the product of the reaction too long at a temperature of 180°-200°, to expel the phosphorus oxychloride and the excess of phosphorus pentachloride, before proceeding to distil, so that he obtained chlorbenzoyl chloride. This is shown with a probability bordering on certainty by the fact that Kekulé found, in the undistilled chloride, only 3 per cent. of phosphorus,* while Couper's "trichlorophosphate de salicyle" contains 11.5 per cent.

As Kekulé was induced by his researches on glycollic acid to occupy himself with salicylic acid, so Hermann Kolbe was led to salicylic acid from the transformation of lactic acid into chloropropionic acid. His work, in which he was seconded by E. Lautemann, appeared under the title, "Über die Constitution und Basicität der Salicylsäure," in the number of *Liebig's Annalen*,† issued 11th August, 1860. Kekulé's communication on salicylic and benzoic acids was presented to the Belgian Academy on the 4th of August of the same year, so that it was only in the German version of his paper in the number of the *Annalen* issued 4th February 1861, that he could take notice of Kolbe and Lautemann's work; but Kekulé's remarks in opposition to Kolbe have no reference to the action of phosphorus pentachloride on salicylic acid, and therefore need not be referred to here.

Kolbe and Lautemann make no reference to Couper's salicylic acid

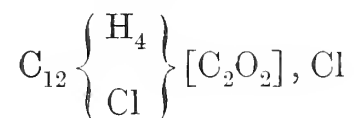
* Compare *Liebig's Annalen*, cxvii. 148, Anm.

† *Ibid.*, cxv. 157-206.

paper, which, indeed, they do not seem to have seen. They prepared the chloride of chlorobenzoic acid (called by them chlorosalicylic acid) by Chiozza's* method, as follows:—"Wir brachten in eine tubulierte Retorte 3 Theile (2 Aeq.) gepulverten Fünffach Chlorphosphor, den wir durch längeres Eintauchen des Retortenbauchs in Eiswasser stark erkälteten, und hierauf 1 Theil (1 Aeq.) trockene pulverige Salicylsäure. Beide wurden durch Umschütteln oder durch Umrühren mit einem krummen Glasstab gut gemischt.

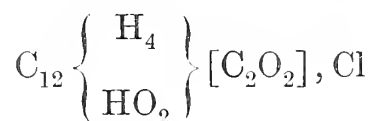
"Als bald erfolgt eine heftige Reaction, wobei der Inhalt der Retorte sich unter Aufblähen verflüssigt und Salzsäure in Strömen entweicht. Bei nachherigem Erhitzen geht der grösste Theil der flüssigen Masse als farbloses Liquidum über. Der Rest bläht sich zuletzt stark auf und es hinterbleibt schliesslich eine leichte schwammige Kohle. Bei der Rectification des Destillates geht zuerst Phosphoroxychlorid über, hernach steigt die Siedetemperatur ziemlich rasch bis 260° C. Als das Thermometer 240° C. anzeigte wurde die Vorlage gewechselt. Der grösste Theil der noch übrigen Flüssigkeit geht dann zwischen 260° und 270° C. über, erst ganz zuletzt steigt die Temperatur noch bis auf 300° C. Was über 240° C. abdestilliert, besteht hauptsächlich aus

Chlorsalicylsäurechlorid.



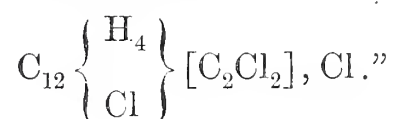
enthält aber daneben noch

Salicylsäurechlorid.



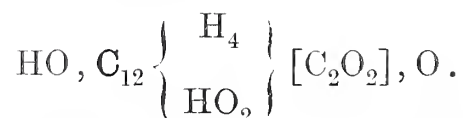
wie die Bildung von Salicylsäure beim Zusammenbringen mit Wasser beweist, und ausserdem

Chlorsalicyltrichlorid.



Kolbe wrote, for salicylic acid itself, the following formula:—

Salicylic Acid.



Obviously, Kolbe and Lautemann, in the distillation of the product of

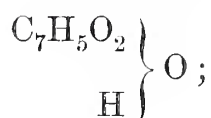
* *Annales de chim. et de phys.* (1852) [3], xxxvi. 102–107.

the reaction, had obtained in the distillate some of Couper's "trichloro-phosphate de salicyle," which had passed over undecomposed. But they regarded it as a mixture of salicyl chloride and chlorosalyl chloride, because with water they obtained salicylic and chlorosalyl (chloro-benzoic) acids. They did not test the distillate for phosphorus, and took no note of the possibility of the formation of a derivative of salicylic acid containing phosphorus.

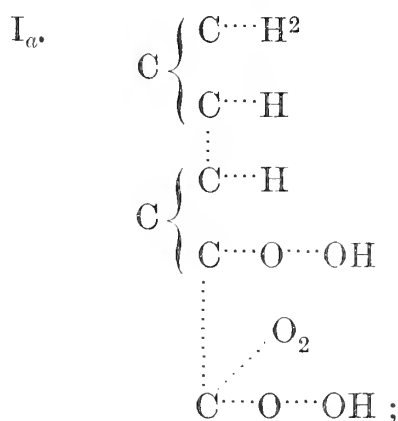
I now place together the formulæ assigned to salicylic acid by Gerhardt, by Couper, by Kekulé, and by Kolbe, appending to Couper's formula (I_a , p. 224 of this paper) the formula I have deduced from it by putting $O=16$, and replacing the brackets by lines representing bonds (I_a , p. 226).

Salicylic Acid:

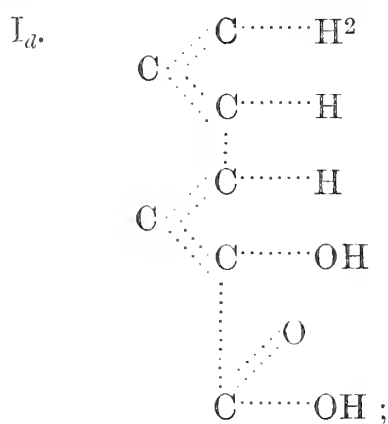
according to Gerhardt:



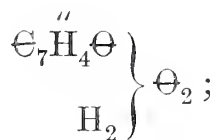
according to Couper:



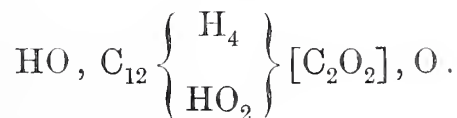
reconstructed with $O=16$, and lines in place of brackets:



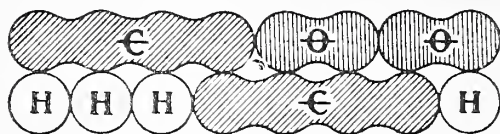
according to Kekulé:



according to Kolbe:



Kekulé himself, as will be seen, had not yet, in the year 1860, ventured to express an opinion as to the mode in which the seven atoms of carbon are united to one another, although he had already, in the first part (published in the spring of 1859) of his *Lehrbuch der organischen Chemie*, sought to represent the structure of simple inorganic and organic compounds by means of graphic formulæ, which for lucidity leave nothing to be desired, as, for instance, the formula of acetic acid: *



Involuntarily the thought arises—how would Archibald Scott Couper have supported his observations and his views in the face of the opposition of his fellow-chemists? Would not organic chemistry have undergone a more rapid development if he had succeeded in fully working out his ideas as to the mode of constructing the formulæ of carbon compounds? Couper was just the man to do that.

But what was this investigator like, who, so young and, after so short an acquaintance with chemistry and at the time of which we are speaking, ventured to pronounce on the mode of union of the atoms in such complicated compounds as salicylic acid, tartaric acid, mucic acid, grape sugar, cyanuric acid, etc.?

Mr Berring gives us from the treasury of his memory the following picture of his friend:—

“Couper was a very handsome man, tall and slender, of a distinguished, aristocratic aspect. His fine face, with its glowing colour, was animated by the almost miraculous brilliancy of his deep black eyes. He had no appearance of weakness, but yet his health was delicate, and I have heard Hamilton say that his mother was always anxious about him. The basis of his character, as is often the case with Scotsmen, was deeply religious. He was very fond of music—classical, serious, and lively—and seldom missed a good concert when he was in Berlin.”

Mr Berring wrote to Crum Brown, on receiving from him a copy of the portrait prefixed to this biography: “This is an excellent picture, and will let you see that Couper was the strikingly handsome man I described.” The original of the portrait is a hand-coloured photograph in the possession of Mr Dollar. It was taken in Paris in 1857, or in the beginning of 1858, and is therefore of the same date as Couper’s scientific work. Couper’s features betray an energetic will combined with a penetrating understanding. In his theoretic paper, his pleasure in the philosophical, critical

* Compare Bd. i., Seite 164, Anm.

treatment of the fundamental principles of chemistry is unmistakable. His mode of thought is quite his own, his statements of his observations, supported by excellent analyses, are clear and definite.

Is it not strange that such an investigator should have made no answer to the criticisms of his theoretical views by Kekulé, Wurtz, and Butlerow; that he made no attempt to refute the doubts of Drion and of Kekulé as to the accuracy of his work on salicylic acid? But no further writing came from Couper's pen. As a scientific man he had disappeared, and twenty-seven years had to elapse before the correctness of his statements as to the action of pentachloride of phosphorus on salicylic acid was demonstrated by another.

In the meantime, quite a number of chemists had busied themselves with the supposed salicyl chloride, mostly in order to study its action on other substances. Carius,* with the view of obtaining thiosalicylic acid, brought the undistilled product of the action of phosphorous pentachloride on salicylic acid into reaction with an aqueous solution of excess of potassium sulphide. In Kolbe's laboratory Glutz,† in 1867, showed that, by continued heating with a reflux condenser, for a day, of gaultheria oil with two equivalents of phosphorus pentachloride, the latter ultimately completely disappears, and a better yield of chlorsalyl chloride is obtained. He states that, on distilling, after the phosphorus oxychloride has been driven off, the temperature quickly rises to 230°-260°, and that below 240° the distillate consists half of chlorsalyl chloride and half of salicyl chloride.

Pierre Miquel‡ prepared "salicyl carbimide" by the action of undistilled "salicyl chloride" on metallic thiocyanates, and remarks: "Le chlorure de salicyle s'obtient difficilement dans un état de pureté satisfaisant, il renferme toujours des composés visqueux et retient avec beaucoup d'énergie de l'oxychlorure de phosphore."

Otto Fischer,§ in the same year, by heating "the chloride $C_6H_4\overset{OH}{COCl}$, freed as much as possible from phosphorus compounds," to 180° with dimethyl aniline and zinc dust in a current of carbonic anhydride, obtained the saliceïn of dimethyl aniline.

Schreib,|| in 1880, states that in the action of phosphorus pentachloride on salicylic acid only one molecule of the pentachloride enters into reaction; a second molecule is without effect and is recovered unchanged on heating to about 110°. He goes on to say: "Bei weiterem Erhitzen geht

* *Liebig's Annalen*, cxxix. 11 (Heft i., issued 8th Jan. 1864).

† *Ibid.*, cxliii. 194 (Heft ii., issued 20th July 1867).

‡ *Annales de chim. et de phys.* (1877) [5], xi. 304.

§ *Berl. Ber.* (1877), x. 954.

|| *Ibid.* (1880), xiii. 465.

zuerst ein grosser Theil des gebildeten Phosphoroxychlorid und später Orthochlorbenzoylchlorid über, bis sich, bei circa 260°–270° der Retorteninhalte unter starkem Aufschäumen zersetzt."

From all these statements it will be seen that it is difficult to obtain Couper's "trichlorophosphate de salicyle" from the product of the action of phosphorus pentachloride on salicylic acid by distilling under ordinary pressure. Contrary to Drion's statement, the difficulties disappear, as I showed in the year 1885,* when the distillation is conducted under diminished pressure. By working exactly according to Couper's description, with the materials he used, and distilling quickly, immediately after the reaction is finished, under ordinary pressure, "trichlorophosphate de salicyle" is also obtained, although the yield is not so good. In the course of exhaustive investigations, carried out along with George Dunning Moore, we were able also to confirm Couper's statements as to the "monochlorophosphate de salicyle" and the "acide phosphosalicylique."

And thus, certainly late enough, there came a brilliant justification of Couper's work on salicylic acid, work which chemists had gradually come to look on as so certainly inaccurate that no notice was taken of it in any of the treatises on organic chemistry.

How often since then have I longed to know what became of Couper!

No doubt, there have been cases of English chemists who, after a meteor flash of one brilliant piece of scientific work, taken up by other engagements, have for many years been absent from the purely scientific field; still, they have always attained a position which could not allow their names to be overlooked, not to say to be forgotten. It was otherwise with Archibald Scott Couper. He disappeared so suddenly and so completely from the scientific arena that there was not time for his name to gain entrance into the English, German, or French books of scientific biography.

This enigma was completely solved by the investigations of his countryman, Alexander Crum Brown. It was not his disappointment that, by no fault of his, his paper, "Sur une nouvelle théorie chimique," was not presented to the French Academy of Sciences until after the publication of Kekulé's famous paper, "Über die Constitution und die Metamorphosen der chemischen Verbindungen und die chemische Natur des Kohlenstoffs," that broke Couper down, but a severe attack of illness.

Having returned, late in the autumn of 1858, from France to Scotland, he obtained, in the end of December, the post of second laboratory assistant to the distinguished Professor Lyon Playfair, Edinburgh.

* *Liebig's Annalen*, ccxxviii. 308–321 (Heft. iii., issued 18th May 1885).



LAUREL BANK, KIRKINTILLOCH. JUNE 23, 1906.



ENTRANCE TO KIRKINTILLOCH CEMETERY. JUNE 23, 1906.

To face page 235.

His future academic position seemed thus made secure.

But soon after his entry on his new office, Couper suffered from a serious breakdown in health, and for two months was under special medical care. On his recovery he went on a fishing expedition, when over-exertion and long exposure to the sun caused a return of his illness, necessitating retirement and medical treatment for a longer time. Greville Williams was therefore quite right in his recollection of a sunstroke. He never completely recovered, and was incapable of undertaking any serious work, but lived in retirement, tenderly cared for by his mother—at first in the old home in the Townhead, and after 1880 in the comfortable house, Laurel Bank, Kirkintilloch, which she had built specially as a quiet home for him. His father died on the 30th December 1859, in his sixty-second year.

Couper's health seems to have somewhat improved, and he was able to take a long walk every day, to converse occasionally with friends, and now and then write a letter. Every morning and evening he read aloud to the household a chapter of the New Testament, and went regularly to church on Sundays.

He died, unmarried, at Laurel Bank, on the 11th March 1892, almost sixty-one years old.

His mother had the sad satisfaction of nursing him to the end. She died at Laurel Bank, 15th April 1895, at the advanced age of ninety-three.

The house, Laurel Bank, in which Couper lived for twelve years, and where he and his mother died, as also the picturesque entrance gate of the Kirkintilloch cemetery, where he and his ancestors are buried, are shown in the plates taken from photographs by Crum Brown.

In the history of organic chemistry the sorely tried Archibald Scott Couper deserves a place of honour beside his more fortunate fellow-worker, Friedrich August Kekulé.

APPENDIX I.

RECHERCHES SUR LA BENZINE; PAR M. A. COUPER.*

Supposant qu'il serait possible de transformer la benzine en alcool et en glycol phényliques, j'ai été conduit à faire avec ce carbure d'hydrogène les expériences suivantes.

Lorsqu'on fait arriver dans un appareil convenable, de la vapeur de brome dans de la benzine bouillante, il se dégage de l'acide bromhydrique, et l'on obtient successivement deux composés bromés, la bromobenzine et la dibromobenzine.

La *bromobenzine* (bromine de phényle), $C^{12}H^5Br$, passe à 150 degrés. C'est

* *Comptes rendus*, t. xlv. pp. 230-232 (10 août 1857).

un liquide parfaitement incolore, doué d'une odeur analogue à celle de la benzine elle-même. Il ne se solidifie pas à -20 degrés. Sa densité de vapeur a été trouvée égale à 5,631. La densité de vapeur théorique est de 5,4237. Elle a donné à l'analyse :

	Expérience.	Théorie.
Carbone	45,40	45,86
Hydrogène	3,27	3,18
Brome	50,84	50,94

Elle possède à un degré remarquable la stabilité bien connue de la benzine. Elle réagit à peine sur l'acétate d'argent à la température de 200 degrés. Chauffée avec de l'acide nitrique fumant, elle se transforme en un composé cristallin fusible au-dessous de 90 degrés et volatil sans décomposition.

Ce composé renferme d'après mes analyses $C^{12}H^4(AzO^4)Br$ et se transformerait probablement en bromaniline sous l'influence des agents réducteurs.

La bromobenzine se dissout dans l'acide sulfurique fumant; la solution abandonnée à l'air laisse déposer, en absorbant l'humidité, des cristaux d'acide sulfobromobenzinique. Cet acide est très-déliquescent. Lorsqu'on ajoute de l'ammoniaque à sa solution aqueuse, il se forme un sel ammoniacal qui est presque insoluble dans l'eau et qui cristallise immédiatement.

Ce sel renferme $C^{12}H^5Br$, S^2O^6 , AzH^3 , comme le prouvent les analyses suivantes :

	Expérience.	Théorie.
Carbone	27,86	28,34
Hydrogène	3,47	3,14
Brome	31,35	31,48
Azote	5,67	5,51

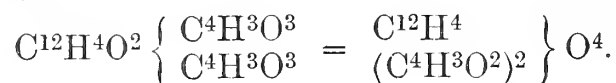
Lorsqu'on laisse pendant longtemps la monobromobenzine en contact avec un excès de brome, de l'acide bromhydrique se forme et se dégage continuellement, et un corps solide se dépose en cristaux au fond du vase. Ce corps est la *dibromobenzine* qu'on purifie facilement en le faisant cristalliser dans l'éther. La dibromobenzine renferme $C^{12}H^4Br^2$, comme le prouve l'analyse suivante :

	Expérience.	Théorie.
Carbone	30,30	30,50
Hydrogène	1,92	1,69
Brome	67,81
		100,00

Elle cristallise en magnifiques prismes obliques.

Elle fond à 89 degrés et distille sans altération à 219 degrés.

Elle réagit sur l'acétate d'argent, mais très-lentement et en formant sans doute du phénylglycol diacétique



Cette expérience ayant été perdue par suite d'une explosion, je me réserve d'en faire le sujet d'une nouvelle étude.

APPENDIX II.

SUR UNE NOUVELLE THÉORIE CHIMIQUE; PAR M. A. COUPER. (NOTE PRÉSENTÉE PAR M. DUMAS.)*

J'ai l'honneur d'exposer à l'Académie les traits principaux d'une nouvelle théorie chimique que je propose pour les combinaisons organiques.

Je remonte aux éléments eux-mêmes dont j'étudie les affinités réciproques. Cette étude suffit, selon moi, à l'explication de toutes les combinaisons chimiques, sans qu'on ait besoin de recourir à des principes inconnus et à des généralisations arbitraires.

Je distingue deux espèces d'affinité, savoir :

1°. L'affinité de degré ; 2°. l'affinité élective.

J'entends par affinité de degré, l'affinité qu'un élément exerce sur un autre avec lequel il se combine en plusieurs proportions définies. Je nomme affinité élective, celle que différents éléments exercent les uns sur les autres, avec des intensités différentes. Prenant pour exemple le carbone, je trouve qu'il exerce son pouvoir de combinaison en deux degrés. Ces degrés sont représentés par CO^2 et CO^4 , c'est-à-dire par l'oxyde de carbone et l'acide carbonique, en adoptant pour les équivalents du carbone et de l'oxygène les nombres 12 et 8.

En ce qui concerne ses affinités électives, le carbone s'éloigne des autres éléments et montre, pour ainsi dire, une physionomie particulière. Les traits qui caractérisent cette affinité élective du carbone sont les suivants :

1°. Il se combine avec des nombres d'équivalents égaux d'hydrogène, de chlore, d'oxygène, de soufre, etc., qui peuvent se remplacer mutuellement pour satisfaire son pouvoir de combinaison.

2°. Il entre en combinaison avec lui-même.

Ces deux propriétés suffisent à mon avis pour expliquer tout ce que la chimie organique offre de caractéristique. Je crois que la seconde est signalée ici pour la première fois. A mon avis, elle rend compte de ce fait important et encore inexplicé de l'accumulation des molécules de carbone dans les combinaisons organiques. Dans les composés où 2, 3, 4, 5, 6, etc., molécules de carbone sont liées ensemble, c'est le carbone qui sert de lien au carbone.

Ce n'est pas l'hydrogène qui peut lier ensemble les éléments des corps organiques. Si, comme le carbone, il avait le pouvoir de se combiner à lui-même, on devrait pouvoir former les composés H^4Cl^4 , H^6Cl^6 , H^8Cl^8 .

En ce qui concerne l'oxygène, j'admets qu'un atome de ce corps en combinaison exerce une affinité puissante sur un second atome d'oxygène qui lui-même est combiné à un autre élément. Cette affinité est modifiée par la position électrique des éléments auxquels se sont respectivement attachés les atomes d'oxygène. Les développements qui vont suivre feront comprendre cette pensée.

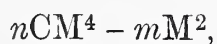
La puissance de combinaison la plus élevée que l'on connaisse pour le carbone est celle du second degré, c'est-à-dire 4.

La puissance de combinaison de l'oxygène est représentée par 2.

Toutes les combinaisons du carbone peuvent être ramenées à deux types. L'un d'eux est représentée par le symbole



l'autre par le symbole



* *Comptes rendus*, t. xlvi, pp. 1157-1160 (14 juin 1858).

où m est $< n$, ou bien $nCM^4 + mCM^2$, où n peut devenir nul. On peut citer, comme exemple du premier type, les alcools, les acides gras, les glycols, etc.

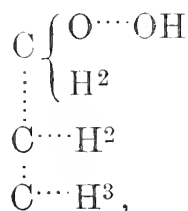
Les alcools méthylique et éthylique seront représentés par les formules



On verra facilement que pour l'alcool méthylique la limite de combinaison du carbone est égale à 4, le carbone y étant combiné à 3 d'hydrogène et à 1 d'oxygène. Cet oxygène, dont le pouvoir de combinaison est égal à 2, est à son tour combiné à un autre atome d'oxygène uni lui-même à 1 d'hydrogène.

Dans le cas de l'alcool ordinaire, chacun des deux atomes de carbone satisfait son pouvoir de combinaison d'un côté en s'unissant à 3 atomes d'hydrogène ou d'hydrogène et d'oxygène, et de l'autre côté en s'unissant à l'autre atome de carbone. L'oxygène y est combiné de la même manière que dans l'exemple précédent. Dans ces cas, on verra que le carbone appartient au premier type, chaque atome étant combiné au second degré.

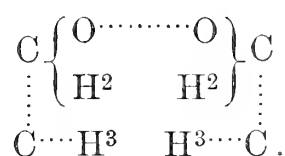
Dans l'alcool propylique,



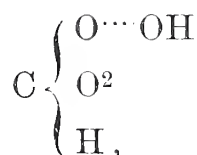
la puissance de combinaison de l'atome de carbone qui est situé au milieu est réduite à 2 pour l'hydrogène, puisqu'il est combiné chimiquement à chacun des deux autres atomes de carbone.

Des formules analogues aux précédentes expriment la constitution des autres alcools.

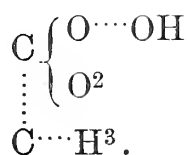
La constitution de l'éther est représentée par la formule



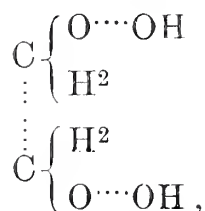
L'acide formique est



l'acide acétique

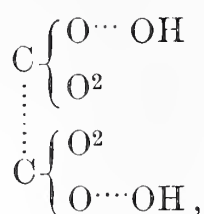


La constitution du glycol est représentée par la formule

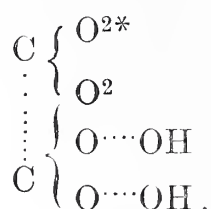


* This "O" is omitted in error in the original.

celle de l'acide oxalique par la formule



ou, si l'on veut réunir l'oxygène négatif à l'un des pôles de la molécule, par la formule



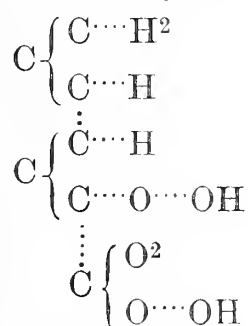
Quoi qu'il en soit cependant, on peut voir d'après cette théorie que, dans la constitution des acides organiques du premier type, la présence de 2 atomes d'oxygène combinés ensemble de manière que tous les deux sont attachés directement au carbone et situés près de l'oxygène négatif, c'est-à-dire de l'oxygène qui entraîne avec lui l'oxygène constitué dans un état électropositif par sa combinaison avec 1 atome d'un élément relativement électropositif, que la présence, dis-je, de ces atomes d'oxygène est nécessaire pour que l'oxygène négatif se trouve dans cet état électrique qui donne au corps les propriétés généralement désignées par le nom d'*acides*.

Ceci est un cas particulier d'une loi générale; car on peut voir, d'après cette théorie, comment la valeur électropositive ou électronégative des éléments modifie et conditionne mutuellement la valeur électropositive ou électronégative des autres éléments.

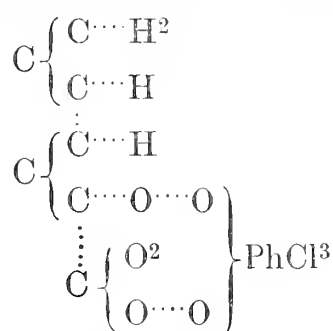
Cette loi diffère de l'hypothèse électrique que les chimistes ont défendue autrefois, mais qui n'a jamais pu recevoir une application complète à leurs vues sur la chimie organique; celle au contraire que j'énonce s'accorde parfaitement avec l'application aux faits de la théorie que je propose.

Il ne me reste qu'à ajouter la manière dont je formule l'acide salicylique et le trichlorophosphate de salicyle que j'ai fait connaître dans un travail soumis à l'Académie dans sa dernière séance.

Acide salicylique.



Trichlorophosphate de salicyle.



Ces formules suffisent, pour le moment, pour indiquer mes idées sur la constitution des corps.

* Misprinted "C²" for "O²" in the original.

APPENDIX III *a*.

ON A NEW CHEMICAL THEORY. BY ARCHIBALD S. COUPER, ESQ.
(COMMUNICATED BY THE AUTHOR.*)

The end of chemistry *is its theory*. The guide in chemical research *is a theory*. It is therefore of the greatest importance to ascertain whether the theories at present adopted by chemists are adequate to the explanation of chemical phenomena, or are, at least, based upon the true principles which ought to regulate scientific research.

Among those which have lately been developed, there is one, on account of its apparently numerous merits, which particularly claims investigation, and respecting which we deem that it would not be unprofitable were either new proofs of its scientific value furnished, or, on the contrary, should considerations be adduced establishing not only its inadequacy to the explanation, but its ultimate detriment to the progress of science. I allude to the system of types as advocated by Gerhardt.

This system, striking alike for the breadth of its conception, and the logical and consequent manner in which it has been developed, has been controverted from the point of view afforded by theories less far-reaching than the one under consideration, and even based upon a one-sided and restricted appreciation of certain chemical reactions. The consequence is that this opposition has not impaired the favour with which the unitary system has been received, but has rather tended to display it in a more advantageous light.

Imposing as this theory is, it is nevertheless all the more necessary to submit it to a strict investigation; for there is nothing so prejudicial in the search for truth as the blind spirit of conservatism. A rational belief demands the test of a preliminary doubt.

There are two conditions which every sound theory must fulfil:—

1. It must be proved to be empirically true.
2. It must no less be philosophically true.

I admit that this theory is for the most part empirically true; that is to say, it is not contradicted by many of the facts of the science. Evidence that this condition is only partially fulfilled, is to be found—

1. In the circumstance that the peroxides, for instance, do not fit very satisfactorily into the types.

2. The principle of double decomposition cannot well be applied to the conversion of the anhydrous sulphuric acid into the hydrate of that acid by the action of one equivalent of water, the formulæ of these bodies being, according to Gerhardt, in their free state $O.SO^2$ and H^2O . Combined, they become simply SH^2O^4 .

The same remark applies in like manner to carbonic acid. In these instances the wonted consequence of Gerhardt is missed. The fact of the density of the vapour of these bodies being the same in the free as in the combined states, may have prevented him from doubling the formulæ of these anhydrous acids. The types of this theory being *essentially types of double decomposition*, this instance of a simple

* *Phil. Mag.*, vol. xvi., 4th series (1858), pp. 104–116.

APPENDIX III b.

SUR UNE NOUVELLE THÉORIE CHIMIQUE; PAR M. A. S. COUPER.*

L'étude de la chimie doit avoir pour but l'établissement de la théorie de cette science; une théorie elle-même est un guide qui nous conduit dans les recherches chimiques. Il est donc de la plus grande importance de s'assurer si les théories actuellement admises par les chimistes suffisent pour l'explication des phénomènes chimiques, ou si elles sont au moins basées sur les vrais principes auxquels doivent se soumettre les recherches scientifiques.

Parmi les théories récemment développées, il en est une qui, en raison des nombreux avantages qu'elle paraît offrir, mérite une étude particulièrement approfondie; il nous a semblé d'ailleurs que la science ne pourrait qu'y gagner, soit que cet examen vienne apporter de nouvelles preuves en faveur de cette théorie, soit au contraire qu'il établisse son insuffisance et les dangers qu'elle présente pour les progrès de la science. Je veux parler de la théorie des types, telle qu'elle a été défendue par Gerhardt.

Ce système, remarquable en même temps par la largeur de sa conception et par le développement logique qu'il a reçu, a été combattu au point de vue de théories beaucoup moins satisfaisantes, basées sur une appréciation incomplète de certaines réactions chimiques. Il en est résulté que cette opposition, loin de diminuer la faveur avec laquelle le système unitaire avait été reçu, a plutôt contribué à le montrer sous un jour plus avantageux.

Quelque imposante que soit cette théorie, il n'en est que plus nécessaire de la soumettre à un sévère examen; car rien n'est nuisible, dans la recherche de la vérité, comme l'aveugle attachement aux idées reçues. Une croyance rationnelle exige l'épreuve préliminaire du doute.

Toute bonne théorie doit remplir deux conditions:

- 1°. Il faut qu'elle s'accorde avec l'expérience;
- 2°. Il n'est pas moins nécessaire qu'elle soit philosophiquement vraie.

J'admets que la théorie unitarienne s'accorde pour la plupart des cas avec l'expérience ou plutôt qu'elle n'est pas contredite par beaucoup des faits de la science.

Cependant les remarques suivantes feront voir que cette condition n'est remplie qu'en partie:

1°. Les peroxydes ne rentrent pas d'une manière bien satisfaisante dans les types;

2°. Le principe de double décomposition ne peut pas bien s'appliquer à la transformation de l'acide sulfurique anhydre en hydrate par l'action de 1 équivalent d'eau: la formule de ces deux corps étant, d'après Gerhardt, à l'état libre, OSO^2 et H^2O , combinés ils deviennent simplement, SH^2O^4 .

La même remarque s'applique d'une manière semblable à l'acide carbonique.

Dans l'explication de ces faits, on ne retrouve plus la conséquence habituelle de Gerhardt. La densité de vapeur des acides anhydres de ces corps, étant la même à l'état de liberté et à l'état de combinaison, aurait dû l'empêcher d'en doubler la formule. Les types de Gerhardt étant essentiellement des types de double décom-

* *Annales de chimie et de physique*, [3], t. liii. (1858), pp. 469-489.

combination diminishes somewhat the value of the otherwise great logical merit of this system.

Having taken notice of such exceptions, the empirical truth of the theory may be otherwise admitted.

The philosophical test demands that a theory be competent to explain the greatest number of facts in the simplest possible manner.

In applying this test, three aspects of it require to be taken into consideration :—

1. As to the extension of the theory.
2. The explanation it affords of the facts.
3. The manner of this explanation.

As to the first: this theory indeed brings every chemical combine under a certain comparative point of view with every other. Herein apparently is its merit. Nevertheless, should our test be applied to its full extent, it will be found that it is fatal to this system, in other respects so imposing. The comparative point of view which it adopts is fundamentally false.

As to the second: it does not explain the facts at all; consequently the most essential point of the test is unfulfilled.

3. This condition of the test is in like manner unfulfilled from the fact of the second not being complied with.

Why is it that Gerhardt's theory so signally fails in these two essential requisites? Because it is based upon an old but vicious principle, which has already retarded science for centuries. It begins with a generalization, and from this generalization deduces all the particular instances. But it does not come within the limits of a chemical paper to enter upon a discussion which is purely metaphysical. Nevertheless, the theory of Gerhardt can only be combated upon metaphysical grounds, because it is only in overturning a general principle of research that the theory can be proposed. Gerhardt's generalization lacks, moreover, the merit of being represented by a type having a known existence. $nO \begin{smallmatrix} H \\ H \end{smallmatrix}$, from which he derives every chemical combine,

being in itself indefinite, cannot of course be contained or be produced in any definite body. That, however, which may be demanded of the type is, that in itself it should afford at least an instance of that which it is meant to represent. Now the part "n" of the type represents the notion of indefinite multiples of $O \begin{smallmatrix} H \\ H \end{smallmatrix}$. But not

a single instance of a multiple of $O \begin{smallmatrix} H \\ H \end{smallmatrix}$ has been proved to exist; much less has it been proved that there exists, or can exist, multiples of this body in an indefinite series. The perfection or imperfection of the type meant to represent the generalized notion is, however, a matter of comparatively inferior moment. It is the principle involved in this generalization which is essentially pernicious.

Should the principle which is therein adopted be applied to the common events of life, it will be found that it is simply absurd. Suppose that some one were to

position, cet exemple de simple combinaison directe diminue en quelque degré la valeur logique de cette théorie, qui est d'ailleurs si grande.

Cette exception une fois constatée, on peut admettre pour le reste la vérité empirique du système.

Il reste à examiner si elle remplit la condition non moins importante de ne pas se trouver en désaccord avec les principes philosophiques.

Ces principes demandent que la théorie puisse expliquer le plus grand nombre possible de faits, de la manière la plus simple.

En soumettant une théorie à cette épreuve, il faut examiner :

- 1°. Son étendue ;
- 2°. L'explication qu'elle donne des faits ;
- 3°. La manière dont elle donne cette explication.

Quant au premier point, la théorie unitaire met chaque combinaison chimique dans certains rapports de comparaison avec tous les autres. C'est là que se trouve en apparence son mérite. Cependant si nous approfondissons l'examen de cette théorie, nous trouverons que ce mérite même est fatal pour elle.

Le point de vue qu'elle prend pour ses comparaisons, est un point de vue essentiellement pernicieux.

Pour le second point, elle n'explique pas les faits du tout, de sorte que la condition la plus importante n'est pas remplie.

La deuxième condition n'étant pas remplie, la troisième ne l'est pas davantage.

Comment se fait-il que la théorie de Gerhardt ne réponde pas, sur ces points essentiels, aux exigences de la philosophie ?

C'est parce qu'elle est basée sur un principe ancien mais vicieux, qui a autrefois retardé la science pendant des siècles.

Elle prend pour point de départ une généralisation dont elle déduit ensuite tous les cas particuliers. Mais ce n'est pas dans un travail chimique qu'il est possible d'entreprendre une discussion purement métaphysique, quoique la théorie de Gerhardt ne puisse être combattue efficacement que par des raisons métaphysiques, puisqu'elle ne peut être mise en avant qu'en renversant un principe général des recherches scientifiques.

La généralisation qui fait le fond du système de Gerhardt n'a pas même le mérite d'être représentée par un type ayant une existence connue. $nO \begin{smallmatrix} H \\ H \end{smallmatrix}$, dont il dérive tous les composés chimiques, étant lui-même indéfini, ne peut être contenu dans aucun corps défini. Cependant on a le droit de demander à un type de fournir en lui-même au moins un exemple de ce qu'il est censé représenter. Or la portion n du type représente l'idée de multiples indéfinis de $O \begin{smallmatrix} H \\ H \end{smallmatrix}$, et non-seulement il n'existe pas de multiples de $O \begin{smallmatrix} H \\ H \end{smallmatrix}$ en série indéfinie, mais on n'a pas même prouvé l'existence d'un seul de ces multiples.

La perfection ou l'imperfection du type qui doit représenter l'idée générale, est toutefois d'une importance relativement inférieure. C'est le principe même de la généralisation qui est essentiellement pernicieux.

Si ce principe était appliqué à la vie ordinaire, on le trouverait tout simplement absurde. Supposez, par exemple, que quelqu'un veuille systématiser la réunion des

systematize the formation of letters into words that formed the contents of a book. Were he to begin by saying that he had discovered a *certain word which would serve as a type, and from which by substitution and double decomposition all the others are to be derived*,—that he by this means not only could form new words, but new books, and books almost *ad infinitum*,—that this word also formed an admirable point of comparison with all the others,—that in all this there were only a few difficulties, but that these might be ingeniously overcome,—he would state certainly an empirical truth. At the same time, however, his method would, judged by the light of common sense, be an absurdity. But a principle which common sense brands with absurdity, is philosophically false and a scientific blunder.

Suppose the book that had formed the basis of this system were a German one, where all the words were found to be composed at least of two letters, still even in this language the viewing and systematizing of words as a series of double decompositions would be no less ridiculous.

The sure and invincible method of arriving at every truth which the mind is capable of discovering is always one and the same. It is that, namely, of throwing away all generalization, of going back to first principles, and of letting the mind be guided by these alone. It is the same in common matters. It is the same in science. To reach the structure of words we must go back, seek out the undecomposable elements, viz., the letters, and study carefully their powers and bearing. Having ascertained these, the composition and structure of every possible word is revealed. It would be well to call to recollection the parallelism of chemical research with that of every other search after truth; for it has been in overlooking this, that in chemistry false and vacillating theories have been advocated and a wrong route so often pursued. In mathematics the starting-point is not generalizations, but axioms, ultimate principles. In metaphysics, Descartes led the way of progress by analysing till he thought he could reach some ultimate elements beyond which it was impossible for him to go, then studying their force and power, and proceeding synthetically. The recognition of this method wrought the regeneration of science and philosophy.

On the other hand, look where Gerhardt's generalization of Williamson's generalization leads him, and legitimately too,—a fact which his logical spirit clearly discerned. He is led not to explain bodies according to their composition and inherent properties, but to think it necessary to restrict chemical science to the arrangement of bodies according to their decomposition, and to deny the possibility of our comprehending their molecular constitution. Can such a view tend to the advancement of science? Would it not be only rational, in accepting this veto, to renounce chemical research altogether?

These reflections naturally lead to the inquiry after another theory more adequate to satisfy the just demands which can be made upon it. There is one which, as it is still supported by many distinguished chemists, cannot be passed over altogether unnoticed. It is that of the theory of certain combinates in organic chemistry which are to be viewed as analogous to, "playing the part of," inorganic elements. These are denominated radicals, and are supposed to be contained in all organic chemical products.

In addition to this, and also in connexion with it, there is a doctrine describing many combinates to be copulated, conjugated, by addition.

lettres en mots formant un livre. S'il commençait par dire qu'il a découvert un certain mot pouvant servir de type, duquel tous les autres peuvent se dériver par substitution et par double décomposition, que par ces moyens on peut former non-seulement des mots nouveaux, mais des livres en quantité presque infinie, que ce mot forme ainsi un admirable point de comparaison pour tous les autres, que dans tout cela il n'y a que quelques difficultés peu nombreuses pouvant être ingénieusement tournées, cet homme établirait certainement une vérité expérimentale. Cependant, en même temps, sa méthode, jugée à la lumière du sens commun, serait une absurdité. Or un principe condamné par le sens commun est philosophiquement faux et ne peut être qu'une erreur scientifique.

Supposez que le livre pris pour base du système dont nous venons de parler, soit un livre allemand, où tous les mots sont composés au moins de deux lettres; même dans cette langue il serait ridicule de considérer les mots comme résultant de séries de doubles décompositions.

La méthode sûre et infaillible d'arriver à toute espèce de vérité est toujours la même. Elle consiste en particulier à mettre de côté toute généralisation, à remonter aux premiers principes, et à prendre ceux-ci pour seuls guides de l'esprit. Ceci est vrai pour les affaires ordinaires, et tout autant pour la science.

Pour étudier la structure des mots, il faut remonter aux éléments indécomposables des mots, aux lettres, et se rendre compte soigneusement de leurs propriétés.

Celles-ci une fois établies, la composition de tout mot possible est expliquée.

Il serait utile de rappeler la nécessité de suivre, en chimie, la même marche que dans tout autre genre de recherche de la vérité; car c'est en oubliant cette nécessité, qu'on a défendu, en chimie, des théories fausses et vacillantes, et qu'on a tant de fois marché dans une mauvaise voie.

En mathématiques, le point de départ ne se trouve pas dans des généralisations, mais dans des axiomes. En métaphysique, Descartes a montré le chemin du progrès en continuant son analyse jusqu'à ce qu'il crut avoir atteint des éléments derniers, au delà desquels il lui était impossible d'aller, en étudiant ensuite leurs forces et leurs propriétés, et en procédant enfin par synthèse. C'est le triomphe de cette méthode qui a régénéré la science et la philosophie.

D'un autre côté, voyez où Gerhardt est conduit par sa généralisation d'une première généralisation de Williamson: il arrive nécessairement à un fait que son esprit logique lui a clairement montré; il renonce à expliquer la constitution des corps d'après leur composition et leurs propriétés inhérentes, et croit nécessaire de restreindre la chimie à un arrangement systématique des corps d'après leurs décompositions, niant même la possibilité de comprendre leur constitution moléculaire. Une semblable manière de voir peut-elle tendre à l'avancement de la science? et ne serait-il pas raisonnable devant un pareil *veto*, de renoncer complètement aux études chimiques?

Ces réflexions conduisent naturellement à rechercher une théorie répondant mieux aux justes exigences de la raison. Il en est une, appuyée encore par beaucoup de chimistes distingués, que nous ne pouvons pas passer sous silence. C'est celle qui regarde certains composés de la chimie organique comme analogues aux corps simples de la chimie minérale, et jouant le même rôle qu'eux. Ces corps sont appelés *radicaux*, et on en admet l'existence dans tous les composés organiques.

Il se lie à cette théorie une doctrine, qui regarde beaucoup de combinaisons comme copulées ou conjuguées par addition.

It is impossible here to enter upon any extensive criticism of this theory. I can only remark that it is not merely an unprofitable figure of language, but is injurious to science, inasmuch as it tends to arrest scientific inquiry by adopting the notion that these quasi elements contain some unknown and ultimate power which it is impossible to explain. It stifles inquiry at the very point where an explanation is demanded, by putting the seal of elements, of ultimate powers, on bodies which are known to be anything but this.

Science demands the strict adherence to a principle in direct contradiction to this view. That first principle, without which research cannot advance a step, dare not be ignored; namely, that a whole is simply a derivative of its parts. As a consequence of this, it follows that it is absolutely necessary to scientific unity and research to consider these bodies as entirely derivative, and as containing no secret ultimate power whatever, and that the properties which these so-called quasi elements possess are a direct consequence of the properties of the individual elements of which they are made up.

Nor is the doctrine of bodies being "conjugated by addition" a whit in advance of that which I have just been considering. This doctrine adopts the simple expedient of dividing certain combinates, if possible, into two imaginary parts, of which one or both are bodies already known. Then it tells us that these two parts are found united in this body. But how they are united, or what force binds them together, it does not inquire. Is this explication arbitrary? Is it instructive? Is it science?

I may now be permitted to submit a few considerations relative to a more rational theory of chemical combination.

As everything depends upon the method of research employed, it will in the first place be necessary to find one that may be relied upon. If the method is good, and conscientiously carried out, stable and satisfactory results may be expected. If, on the contrary, it is vicious, we can only expect a corresponding issue. A satisfactory method is, however, not difficult to find, nor is it difficult in its application.

The principle which ought to guide all research is in every case the same. It is that of analysing till it is impossible to reach more simple elements, and of studying these elements in all their properties and powers. When all the properties and powers of the individual elements are known, then it will be possible to know the constitution of the combinates which their synthesis produces. It is necessary therefore in chemical research, in order to ascertain the various qualities and functions of the different elements,—

1. To consider the whole of chemistry as one.
2. To take into consideration every known combine, and to study the character, functions, and properties displayed by each element for itself, in each of these combinates in all their different conditions and aspects. It is by a comparison of the different bodies among themselves that we are able to trace the part that is performed by each element separately.
3. To trace the general principles common to all the elements, noting the special properties of each.

This method is essentially different from that where one class of bodies is

Il est impossible d'entrer ici dans une critique détaillée de cette théorie. Je dois me borner à faire remarquer qu'elle n'est pas seulement une manière de parler inutile, mais qu'elle nuit encore à la science en tendant à arrêter l'analyse scientifique par l'idée que ces quasi-éléments renferment quelque force dernière inconnue, qu'il est impossible d'expliquer.

En donnant le caractère d'éléments, de forces dernières à des corps qu'on sait n'être rien moins que cela, elle arrête les recherches au point même dont on demande l'explication.

La science réclame une stricte fidélité à un principe directement opposé à cette manière de voir. Ce principe, sans lequel les recherches scientifiques ne peuvent pas faire un pas, c'est qu'un tout est simplement un dérivé de ses parties.

Comme conséquence, il en résulte qu'il est absolument nécessaire pour l'unité de la science et pour le progrès des recherches de considérer ces corps qu'on a appelé *radicaux*, comme dérivés et comme ne renfermant aucune force dernière cachée, et d'admettre que leurs propriétés sont une conséquence directe des propriétés individuelles des éléments qui les composent.

La doctrine des corps conjugués par addition n'est pas un progrès sur celle que nous venons de considérer. Cette doctrine adopte le simple expédient de diviser, lorsque c'est possible, certains composés en deux portions imaginaires, dont l'une ou bien toutes deux sont des corps déjà connus. Elle déclare ensuite que ces deux corps se trouvent unis dans le composé en question. Mais elle ne s'inquiète pas de savoir comment elles sont unies ou quelle force les lie. Cette explication n'est-elle pas arbitraire? Nous apprend-elle quelque chose? Est-ce là de la science?

Il me sera permis maintenant de faire valoir quelques considérations relatives à une théorie plus rationnelle des combinaisons chimiques.

Comme tout dépend de la méthode de recherche employée, il est avant tout nécessaire d'en trouver une dans laquelle nous puissions avoir confiance. Si la méthode est bonne et si elle est appliquée consciencieusement, nous pouvons en attendre des résultats certains et satisfaisants. Si au contraire elle est vicieuse, nous ne pouvons attendre qu'un mauvais résultat. Heureusement, il n'est pas difficile de trouver une bonne méthode, qui ne présente pas de difficultés dans son application.

Le principe qui doit guider toutes les recherches est dans tous les cas le même. C'est celui d'analyser jusqu'à ce qu'il soit impossible d'atteindre des éléments plus simples, et d'étudier ces éléments dans leurs propriétés et leurs forces. Les forces et les propriétés des éléments étant toutes connues, il sera possible alors de connaître la constitution des combinaisons produites par leur synthèse.

Il est donc nécessaire, dans les recherches chimiques, pour s'assurer des propriétés et des fonctions des différents éléments :

1°. De considérer la chimie comme formant un ensemble unique.

2°. D'étudier tous les composés connus et de se rendre compte du caractère, des fonctions et des propriétés de chaque élément, dans chaque composé, sous tous les points de vue et dans toutes les conditions différentes.

C'est par la comparaison des différents corps entre eux que nous pouvons reconnaître le rôle que joue chaque élément séparément.

3°. De rechercher les principes généraux communs à tous les éléments, en prenant note des propriétés spéciales de chacun d'eux.

Cette méthode est essentiellement différente de celle où une classe de corps est

chosen as a point for the restriction of our views of the properties of the others—where only the qualities found in the first are to be measured out to the rest.

I shall now proceed to inquire how its more thorough application tends to the development of a rational chemical theory.

It has been found that there is one leading feature, one inherent property, common to all the elements. It has been denominated chemical affinity. It is discovered under two aspects:—(1) affinity of kind; (2) affinity of degree.

Affinity of kind is the special affinities manifested among the elements, the one for the other, etc., as carbon for oxygen, for chlorine, for hydrogen, etc.

Affinity of degree is the grades, or also limits of combination, which the elements display. For instance, C^2O^2 and C^2O^4 are the degrees of affinity of carbon for oxygen. C^2O^2 may be called the first degree, and C^2O^4 may be termed the second degree, and, as a higher degree than this is not known for carbon, its ultimate affinity or combining limit. Affinity of degree in an element may have only one grade. It may have, however, and generally has more than one. Here then is an inherent property common to all elements, by the removal of which the chemical character of an element will be destroyed, and by virtue of which an element finds its place marked out in a complex body.

It is such a property that is required to form the base of a system. Nor would its suitability for this purpose be affected by the discovery that the elements are themselves composite bodies, which view the chemist is perhaps not unwarranted to adopt. For, in such a case, the necessity would doubtless still be found to exist of adopting the principle of affinity, or something at least equivalent to it, as the basis of the explanation of chemical combinates. In applying this method, I propose at present to consider the single element carbon. This body is found to have two highly distinguishing characteristics:—

1. It combines with equal numbers of hydrogen, chlorine, oxygen, sulphur, etc.
2. It enters into chemical union with itself.

These two properties, in my opinion, explain all that is characteristic of organic chemistry. This will be rendered apparent as I advance.

This second property is, so far as I am aware, here signalized for the first time. Evidence as to its being a property of carbon may therefore be required.

It will be found in the following:—What is the link which binds together bodies composed of 4, 6, 8, 10, 12, etc., equivalents of carbon, and as many equivalents of hydrogen, oxygen, etc.? In these you may remove perhaps all the hydrogen or oxygen, and substitute so many equivalents of chlorine, etc. It is then the carbon that is united to carbon. Further, that it is not the hydrogen that is the binding element in these combinates is evident; thus—

choisie pour restreindre nos idées sur les propriétés des autres corps, et où celles seulement reconnues dans les premiers sont accordées aux derniers.

Je vais maintenant chercher comment l'application plus complète de cette méthode conduit au développement d'une théorie rationnelle de la chimie.

On a trouvé qu'il existe un trait saillant, une propriété commune à tous les éléments. On a appelé cette propriété *affinité chimique*. Elle se présente de deux manières différentes :

1°. Elle peut s'exercer comme affinité élective ;

2°. Elle peut s'exercer comme affinité de degré.

L'affinité élective est l'affinité que les éléments montrent les uns pour les autres : ainsi le carbone pour l'oxygène, pour le chlor, pour l'hydrogène, etc.

L'affinité de degré est l'affinité qui s'exerce entre deux éléments en proportions multiples ; ce sont des limites de combinaison. Par exemple C^2O^2 et C^2O^4 sont les degrés de l'affinité du carbone pour l'oxygène.* On peut appeler, C^2O^2 premier degré, et C^2O^4 second degré, et comme on ne connaît pas, pour le carbone, de degré plus élevé, affinité dernière ou limite de combinaison. L'affinité de degré, pour un élément, peut n'avoir qu'un degré. Cependant elle peut en avoir et en a généralement plus d'un.

C'est donc là une propriété inhérente aux éléments, qui leur est commune à tous, et dont la suppression entraînerait la destruction du caractère chimique de l'élément ; elle marque à chaque élément sa place dans un corps composé.

Il faut une propriété de ce genre pour former la base d'un système ; elle resterait encore suffisante pour cela, quand bien même on découvrirait, ce que les chimistes n'ont pas le droit de regarder comme impossible, que les éléments eux-mêmes sont des corps composés ; car dans ce cas, sans aucun doute, on se trouverait encore dans la nécessité d'adopter le principe de l'affinité ou de moins quelque chose d'équivalent, comme base de l'explication des combinaisons chimiques.

Pour le moment toutefois, il est impossible de remonter à des éléments plus simples. Il est donc nécessaire provisoirement de partir des affinités et des propriétés découvertes dans les éléments, pour arriver à la théorie de leurs combinaisons.

Comme application de cette méthode, considérons maintenant le seul élément carbone. Ce corps possède deux caractères qui le distinguent particulièrement :

1°. Il entre en combinaison avec des nombres égaux d'équivalents d'hydrogène, de chlor, d'oxygène, de soufre, etc.

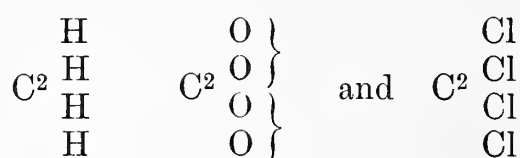
2°. Il entre en combinaison avec lui-même.

Dans mon opinion, ces deux propriétés suffisent pour expliquer tout ce que la chimie organique présente de caractéristique ; c'est que je démontrerai plus loin.

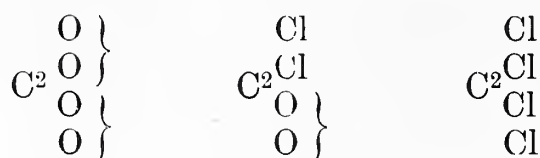
La seconde de ces propriétés est, je crois, signalée ici pour la première fois.

On peut demander de prouver que ce soit là une propriété du carbone. Ce qui suit va le démontrer. Quel est le lien qui tient ensemble les composés de 4, 6, 8, 10, 12, etc., molécules de carbone et d'un pareil nombre d'équivalents d'hydrogène, d'oxygène, etc. ? On peut enlever de ces composés tout l'hydrogène et tout l'oxygène peut-être, et le remplacer par autant d'équivalents de chlor, etc. C'est donc le carbone qui est uni au carbone. De plus, il est évident que ce n'est pas l'hydrogène qui sert de lien dans les combinaisons, car on a

* Misprinted "l'hydrogène" in the original. See the English paper.

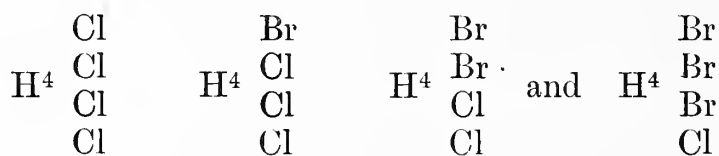


Here the whole four of hydrogen are not bound by a mutual affinity; for each element of hydrogen can be substituted for one of chlorine in regular series, beginning with the first and ending with the last. The atoms of oxygen are, on the contrary, united in pairs (which will be more fully developed hereafter), and only for two atoms of oxygen two of chlorine can be substituted; thus—



In the same manner, with bodies that contain multiples of C² united to hydrogen, etc.

Take the inverse of this. If the four atoms of hydrogen were bound together, we could evidently expect to form such bodies as



or for bodies like C⁴H⁴, C⁶H⁶, C⁸H⁸, one would naturally expect to find the carbon substituted for chlorine, and find bodies like H⁴ $\begin{array}{c} \text{Cl}^2 \\ \text{Cl}^{2*} \end{array}$, H⁶Cl⁶, H⁸Cl⁸, etc.

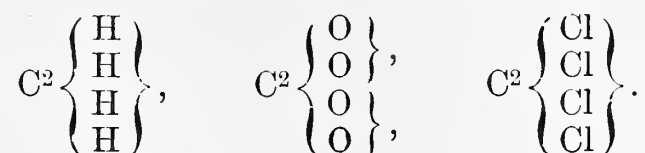
These bodies are not only unknown, but the whole history of hydrogen might be investigated and not a single instance be found to favour the opinion that it has any affinity for itself when in union with another element.

Now, on the other hand, carbon remains chemically united to carbon, while perhaps 8 equivalents of hydrogen are exchanged for 8 equivalents of chlorine, as in naphthaline. Analogous to this is the conversion of alcohol, C⁴ $\begin{array}{c} \text{O} \cdots \text{OH} \\ \text{H}^5 \end{array}$, and the hydrocarbide C⁴H⁶ into C⁴Cl⁶. All the countless instances of substitution of chlorine, etc., tend in the same direction. They prove beyond doubt that carbon enters into chemical union with carbon, and that in the most stable manner. This affinity, one of the strongest that carbon displays, is perhaps only inferior to that which it possesses for oxygen.

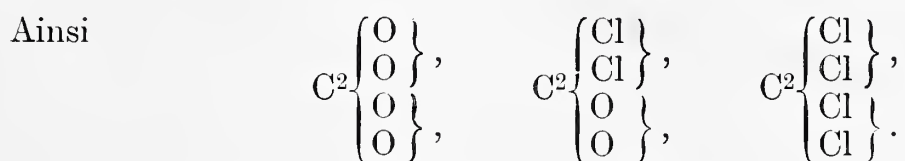
Another feature in the affinity of carbon is, that it combines by degrees of two; thus, C²O² and C²O⁴, C⁴H⁴ and C⁴H⁶, C⁶H⁶ and C⁶H⁸, C⁸H⁸ and C⁸H¹⁰, etc.: from these last it is especially evident that two is the combining grade of carbon. It becomes still more apparent when we compare the bodies C⁴H⁴ and

C⁴H⁵Cl, that is, C⁴H $\begin{array}{c} \text{Cl} \\ \text{H}^4 \end{array}$ etc. Many such proofs might be added, while, on the

* Misprinted "C²" in the original. See the French paper.

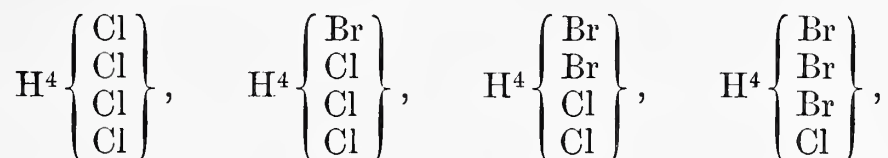


Ici les 4 molécules d'hydrogène ne sont pas liées ensemble par une affinité mutuelle, car chaque élément d'hydrogène peut être remplacé par un élément de chlore, en commençant par le premier et finissant par le dernier. Les atomes d'oxygène sont, au contraire, unis par paires (c'est ce qui sera plus complètement développé plus bas), et on ne peut remplacer que 2 molécules d'oxygène par 2 de chlore.



Il en est de même pour les corps qui renferment des multiples de C^2 unis avec de l'hydrogène, etc.

Prenons l'inverse de ce raisonnement. Si les 4 atomes d'hydrogène étaient unis ensemble, nous aurions le droit d'attendre la formation de corps tels que



ou bien pour des corps tels que C^4H^4 , C^6H^6 , C^8H^8 , on s'attendrait naturellement à voir le carbone remplacé par du chlor et à trouver des corps tels que H^4Cl^4 , H^6Cl^6 , H^8Cl^8 , etc.

Non-seulement ces corps sont inconnus, mais on pourrait encore étudier toute l'histoire de l'hydrogène sans trouver un seul exemple en faveur de l'opinion qu'il aurait quelque affinité pour lui-même, lorsqu'il est combiné avec un autre élément.

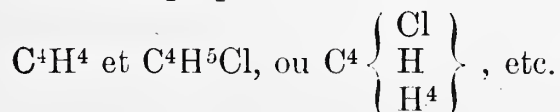
On peut remarquer aussi que le carbone reste chimiquement combiné avec lui-même, pendant que peut-être huit atomes d'hydrogène sont remplacés par huit atomes de chlore, comme dans la naphthaline.

La transformation de l'alcool $C^4 \begin{array}{c} O-OH \\ H^5 \end{array}$ et du carbure d'hydrogène C^4H^6 en chlorure de carbone C^4Cl^6 , sont des faits analogues.

Ainsi que tous les nombreux exemples de substitution de chlore, les faits précédents prouvent, sans qu'il soit possible de conserver aucun doute, que le carbone s'unit chimiquement avec le carbone, et cela de la manière la plus stable. Cette affinité, l'une des plus énergiques parmi celles que montre le carbone n'est peut-être inférieure qu'à celle qu'il a pour l'oxygène.

Un autre trait saillant de l'affinité du carbone est le suivant : il se combine par degrés pairs. Ainsi on a C^2O^2 et C^2O^4 ,— C^4H^4 et C^4H^6 ,— C^6H^6 et C^6H^8 ,— C^8H^8 et C^8H^{10} , etc. Ces derniers composés, en particulier, prouvent évidemment que le carbone se combine avec des nombres pairs d'atomes.

Cette propriété devient encore plus évidente lorsque nous comparons les corps



other hand, there are no instances contradictory of this point. Hence the circumstance that it must ever remain impossible to isolate a combine of the form C^2H^3 or C^4H^5 , etc.

Carbon having only two grades of combination of two atoms each, a fact which is easily traced throughout all organic chemistry, this inherent property of the element may legitimately furnish two grand types for all its combines.

The first type will be nC^2M^4 .

The second type will be $nC^2M^4—mM^2$.*

As examples belonging to the first type, may be mentioned the alcohols of the æthylic form; their æthers, the fatty acids, etc.

Thus methylic alcohol has the formula $C^2 \begin{array}{c} \dots O \dots OH \\ \dots H^3 \end{array}$,

and æthylic alcohol, $\begin{array}{c} C^2 \begin{array}{c} \dots O \dots OH \\ \dots H^2 \end{array} \\ \vdots \\ C^2 \dots H^3 \end{array}$.

In these instances it will be observed, that for each double atom of carbon the combining power is (4) four, which is the ultimate limit of combination for carbon in all bodies yet produced.

In the latter instance it is apparent, inasmuch as if the combining limit of two C^2 s be each reduced by 3 in hydrogen or oxygen, there still remains a combining power of *one* to each of the two C^2 s which each expends in uniting with the other;

therefore $\begin{array}{c} C^2 \dots H^3 \\ \vdots \\ C^2 \dots H^3 \end{array}$, or, what is the same thing, $\begin{array}{c} C^2 \begin{array}{c} \dots O \dots OH \\ \dots H^2 \end{array} \\ \vdots \\ C^2 \dots H^3 \end{array}$ belongs to the type nC^2M^4 .

Again, the inherent properties of the elements may be viewed as dividing bodies into primary, secondary, tertiary, and so on, combines. These may be termed so many orders of complicity. Thus C^4H^6 is a primary combine, or it belongs to the first order of complicity; but $C^4 \begin{array}{c} \dots O \dots OH \\ \dots H^5 \end{array}$ is a secondary combine, or belongs to the second order of complicity. C^2O^2 and C^2O^4 are primary, while C^2O^4 , $2OH$ and C^2O^4 , $2OKa$ are secondary.

A primary combine is then nC^2 united to nM^4 or to $nM^4—mM^2$ in such a

* Misprinted "mM" in the original.

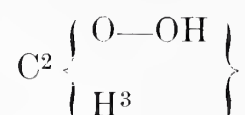
On pourrait ajouter bien d'autres preuves en sa faveur, tandis qu'il n'y a point de faits en contradiction avec elle. Il résulte de là qu'il doit être impossible d'isoler un composé de la forme C^2H^3 ou C^4H^5 , etc.

Le carbone ne se combinant qu'en deux degrés différents et chaque fois par deux atomes (fait que l'on peut vérifier aisément dans toute la chimie organique), cette propriété fondamentale de l'élément peut légitimement fournir deux grands types pour toutes les combinaisons dans lesquelles il entre.

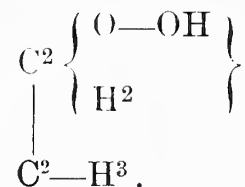
Le premier type sera nC^2M^4 , et le second $nC^2M^4 - mM^2$, où m est un nombre moindre que n .

Les alcools de la forme éthylique, leurs éthers, les acides gras, etc., rentrent dans le premier type.

Ainsi l'esprit-de-bois a pour formule

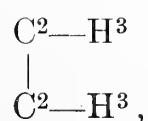


et l'alcool

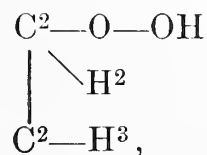


On remarquera que, dans ces exemples, la puissance de combinaison de chaque double atome de carbone est de 4; c'est là la dernière limite de combinaison du carbone dans tout les corps obtenus jusqu'à ce jour.

Le dernier exemple montre dans le carbone un même pouvoir de combinaison; car, en déduisant du pouvoir de combinaison de chacun des doubles atomes de carbone trois forces employées par l'hydrogène et l'oxygène qui leur sont combinés, il reste encore libre une force qui s'exerce dans l'union des doubles atomes l'un avec l'autre; il en résulte que



ou, ce qui revient au même,



appartient au type nC^2M^4 .

On peut considérer les propriétés inhérentes des éléments comme divisant les corps en primaires, secondaires, tertiaires, etc.; c'est ce qu'on peut appeler les divers degrés de complication. Ainsi C^4H^6 est un composé primaire, c'est-à-dire qu'il appartient au premier degré de complication. Mais $C^4 \begin{array}{l} O-OH \\ H^5 \end{array}$ est un composé secondaire ou appartient au second degré de complication. C^2O^2 et C^2O^4 sont primaires, tandis que $C^2O^4-O^2H^2$ et $C^2O^4-O^2K^2$ sont secondaires.

Une combinaison primaire est donc composée de nC^2 uni à nM^4 ou à $nM^4 - mM^2$,

manner that the combining energy of the complement (nM^4 , etc.) either *potentially* or *actually* does not extend beyond nC^2 .

A secondary combine is one in which the combining energy of the complement is not all expended upon nC^2 , but is extended further to one or more elements.

On the same principle there are tertiary combines, etc.

These orders of complicity ought in reality to be subdivided. This, however, I do not think it necessary for the present to enter upon. It will now be understood why an alcohol belongs to the type nC^2M^4 , and on the same principle why a free

æther belongs to the same type, thus
$$\begin{array}{ccc} C^2 \cdots O \cdots O \cdots & & \\ \vdots \cdots H^2 & H^2 \cdots & C^2 \\ \vdots & & \vdots \\ C^2 \cdots H^3 & H^3 \cdots & C^2 \end{array} *$$
, while they are at the same time secondary combines.

A secondary combine, that is to say, a body belonging to the second order of complicity, is, as will be understood from the principle which forms the ground of the rational theory, a direct consequence of an inherent property of one or more of the elements which form the complement to the carbon.

In the instance before us, it is a certain property of the oxygen which is the cause of the secondary combine. This property is the affinity which one atom of oxygen in combination always exerts towards another atom of oxygen likewise in combination.

This affinity is modified by the electric position of the element to which the respective atoms of oxygen are bound. From this property results the fact, that in organic combines the atoms of oxygen are always found double.

For instance, the combining limit of oxygen being two, when two molecules of
$$\begin{array}{c} C^2 \cdots O \cdots \\ \vdots \cdots H^2 \\ \vdots \\ C^2 \cdots H^3 \end{array}$$
 are set at liberty, the free affinities of the oxygen instantly produce the

union of these molecules. The cause of the union of two molecules of C^2H^3 has been already remarked. In the two cases, the causes of the union of the respective molecules are in so far different, that the one is the result of a property of the carbon, while the other is the result of a property of the oxygen.

The view here adopted of the nature of oxygen is, I am convinced, alone in conformity with the reactions where the properties of this body develop themselves.

The vapour of anhydrous sulphuric acid, for instance, is conducted into anhydrous æther. The following will then be the reaction:—
$$S^2 \begin{array}{c} \cdots O \\ \cdots O \\ \cdots O^2 \\ \cdots O^2 \end{array}$$
 entering into communi-

cation with $C^4 \cdots O \cdots O \cdots C^4$, the two atoms of the oxygen of the sulphuric acid

* The vertical dotted line between these two "C²s" is omitted in the original.

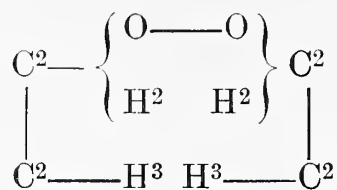
de telle sorte que le pouvoir de combinaison du complément (nM^4), soit virtuellement, soit actuellement, ne dépasse pas celui de nC^2 .

Les combinaisons secondaires sont celles où le pouvoir de combinaison du complément, au lieu de se porter entièrement sur nC^2 , s'étend encore sur un ou plusieurs autres éléments.

Il existe de même des combinaisons tertiaires, etc.

Ces degrés de complication devraient à la rigueur être subdivisés. Toutefois je ne crois pas nécessaire, pour le moment, d'entrer dans ces détails.

On comprendra maintenant comment un alcool appartient au type nC^2M^4 , ainsi qu'un éther libre ; par exemple,



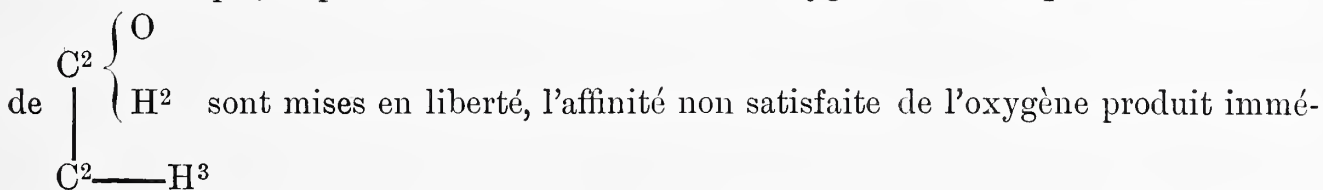
et que tous deux sont des composés secondaires.

Les combinaisons secondaires, c'est-à-dire les corps appartenant au second degré de complication, prennent naissance (on le comprendra en partant du principe qui forme la base de la théorie rationnelle), en vertu d'une propriété appartenant à un ou plusieurs des éléments combinés avec le carbone.

Dans les exemples précédents, c'est une certaine propriété de l'oxygène qui est cause de la complication du corps. Cette propriété consiste dans l'affinité qu'une molécule d'oxygène en combinaison exerce toujours sur une seconde molécule d'oxygène elle-même combinée. Cette affinité est modifiée par l'état (électrique?) des éléments auxquels les deux atomes sont liés.

Il résulte de là que, dans les composés organiques, les atomes d'oxygène vont toujours par deux.

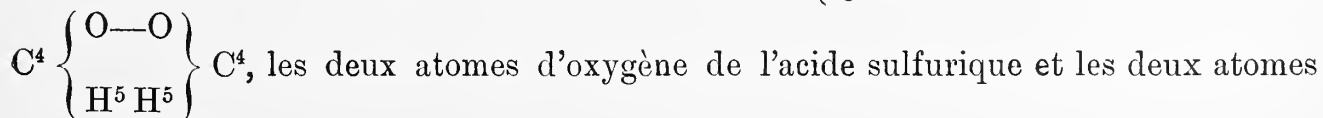
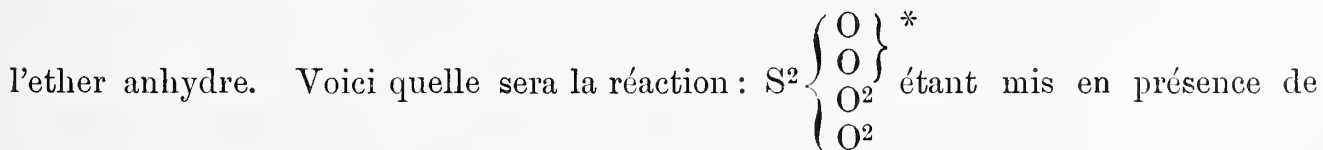
Par exemple, le pouvoir de combinaison de l'oxygène étant 2, quand 2 molécules



diatement l'union de ces molécules. On a déjà vu la cause de l'union de deux molécules de C^2H^3 . Dans ce dernier cas, l'union des molécules est due à une propriété du carbone, et dans le premier à une propriété de l'oxygène.

Les vues émises ici sur la nature et les fonctions de l'oxygène sont, j'en suis convaincu, seules conformes aux réactions dans lesquelles les propriétés de ce corps sont en jeu.

Par exemple, on fait arriver la vapeur d'acide sulfurique anhydre dans de



* The two upper "O's" have been omitted in the original. See the English paper.

and the two atoms of the oxygen of the æther (now in presence of each other) being in different (perhaps different electric) conditions, mutually loosen their former affinities and reunite themselves to the (electrically?) different atoms of oxygen of these respective combinates.

The same principle may naturally be expected to display itself with regard to acids and bases. The oxygen of an acid unites itself to the (electrically?) different oxygen of water. The oxygen of a base on the same principle has an affinity for the electrically different oxygen of water.

It will be observed—

1. That the oxygen of the water of an acid can only be expelled by that of a base and *vice versâ*.

2. It is to be remarked that *it is not* the metal of a base which exchanges places with the hydrogen of the hydrate of an acid; for if that were the case, the affinity of the oxygen of the metal, and also of the acid, *would be greater for the oxygen of the water* than the affinity of the hydrogen for that same oxygen. But this is not so. The very opposite is the truth. If one atom of hydrogen be withdrawn from the hydrate of an acid or from *the hydrate of an oxide, it is universally accompanied by an atom of oxygen*. It is evident, then, that the affinity between *the positive and negative atoms of oxygen is less than that which attaches these atoms to the element with which they form a primary combine*.

A consequence of this truth is, that it is impossible to double the equivalent of oxygen, if the chemical equivalents are to be understood as not being in direct contradiction to any chemical truth or essential feature in the properties of an element. Carbon differs entirely in this respect from oxygen.

There is no reaction found where it is known that C^2 is divided into two parts. It is only consequent therefore to write, with Gerhardt, C^2 simply as C, it being then understood that the equivalent of carbon is (12) twelve.

This value of the atom will be adopted in the following part of this paper.

Sulphur, selenium, etc., being bodies displaying properties similar, not to carbon, but to oxygen, it will be necessary to retain the equivalent value that has generally been assigned to them.

I have now shown how ordinary alcohol, $C^2H^6O^2$ common æther, and the hydro-carbide, C^2H^6 , belong to the type nCM^4 . The phenomena which necessitate this view of the constitution of these bodies have a like consequence in regard to the other alcohols, glycols, acids, and æthers of this series.

Propyle alcohol is
$$\begin{array}{c} C \cdots O \cdots OH \\ \vdots \cdots H^2 \\ \vdots \\ C \cdots H^2 \\ \vdots \\ C \cdots H^3 \end{array}$$
, where it will be seen that the atom of carbon

situated between the two others, on account of being chemically united to these, is reduced to the combining power of two for hydrogen, oxygen, etc. One combining power is given up to the carbon upon the one side, and a second to the carbon upon the other.

It will be observed also that the primary combinates ought in rigour to be themselves enumerated in an inverse order. The type nCM^4 becomes then in reality the

d'oxygène de l'éther se trouvant dans un état (peut-être électrique) différent, les affinités mutuelles des atomes d'oxygène d'un même corps s'affaiblissent, et ils s'unissent aux atomes (électriquement ?) différents d'oxygène de l'autre composé.

La même chose se passe naturellement entre les acides et les bases. L'oxygène d'un acide s'unit avec l'oxygène (électriquement ?) différent de l'eau. L'oxygène de la base, en vertu du même principe, s'unit avec l'oxygène (électriquement ?) différent de l'eau.

On remarquera : 1° que l'oxygène de l'eau contenue dans un acide ne pourra être chassé que par celui d'une base ;

2°. Que ce n'est pas le métal de la base qui prend la place de l'hydrogène de l'acide hydraté ; car si cela avait lieu, l'affinité de l'oxygène combiné au métal, ainsi que celle de l'oxygène de l'acide, serait plus grande pour l'oxygène de l'eau que l'affinité de l'hydrogène pour ce même oxygène. Mais il n'en est pas ainsi ; bien au contraire, si un atome d'hydrogène est enlevé à l'acide hydraté ou à une base hydratée, il est universellement accompagné d'un atome d'oxygène. Il est donc évident que l'affinité entre les atomes d'oxygène positif et négatif est moindre que celle qu'ils ont pour l'élément qui forme avec eux un composé primaire.

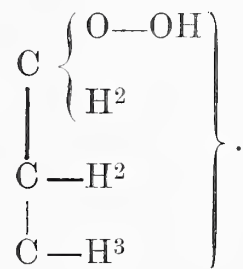
Il résulte de tout ceci, qu'il est impossible de doubler l'équivalent de l'oxygène, si l'on veut que les équivalents chimiques ne se trouvent en contradiction avec aucune vérité chimique, avec aucun trait essentiel de la physionomie propre d'un élément.

Le carbone diffère complètement sous ce rapport de l'oxygène. On ne connaît pas de réaction où C^2 soit divisé en deux parties. Il faut donc, pour être conséquent, écrire avec Gerhardt C^2 simplement C, en portant à 12 l'équivalent du carbone. C'est cette valeur du poids atomique que nous admettrons dans le reste de ce travail.

Le soufre, le sélénium, etc., étant des corps qui offrent des propriétés semblables à celles de l'oxygène et non à celles du carbone, il sera nécessaire de conserver les équivalents généralement admis.

J'ai montré maintenant comment l'alcool $C^2H^6O^2$, l'éther et le carbure d'hydrogène C^2H^6 , appartiennent au type nCM^4 . Les phénomènes qui conduisent à cette manière de voir sur la constitution de ces corps ont une conséquence semblable pour les autres alcools, les glycols, les acides, les éthers de cette série.

L'alcool propylique * est

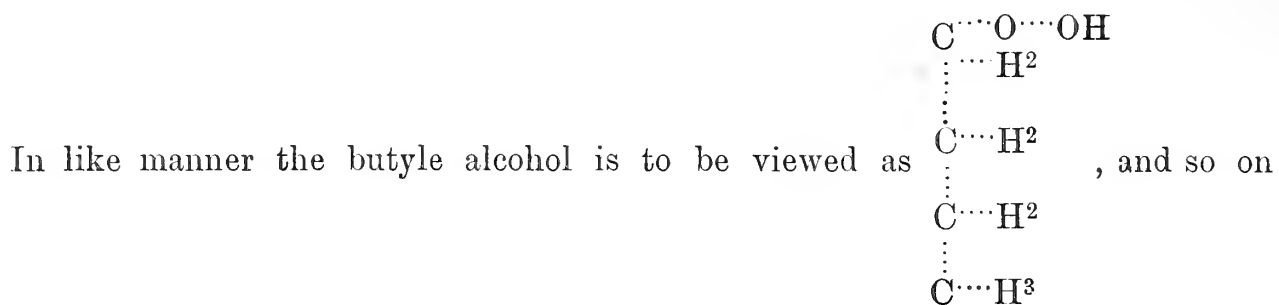


On remarquera que l'atome de carbone situé entre deux autres, étant chimiquement combiné avec chacun d'eux, son pouvoir de combinaison est réduit à 2 pour l'hydrogène, l'oxygène, etc. Une des forces de combinaison s'exerce sur un élément de carbone d'un côté, et une seconde sur un élément de carbone de l'autre.

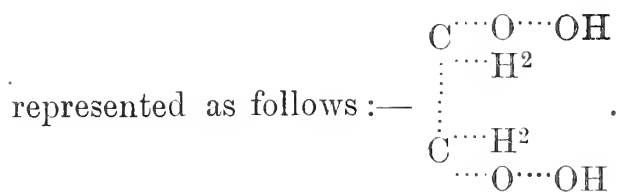
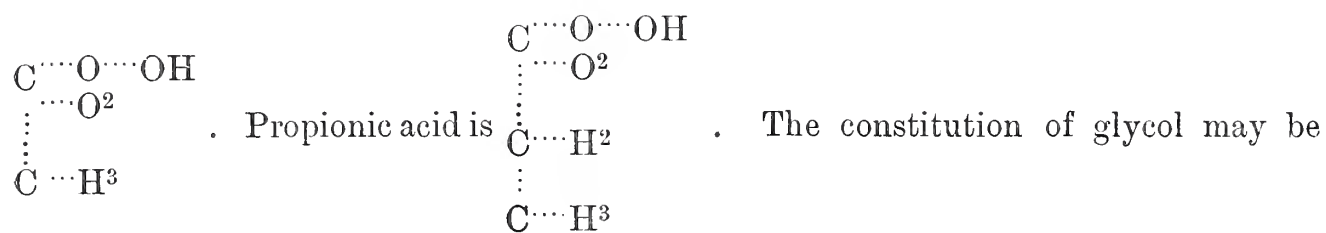
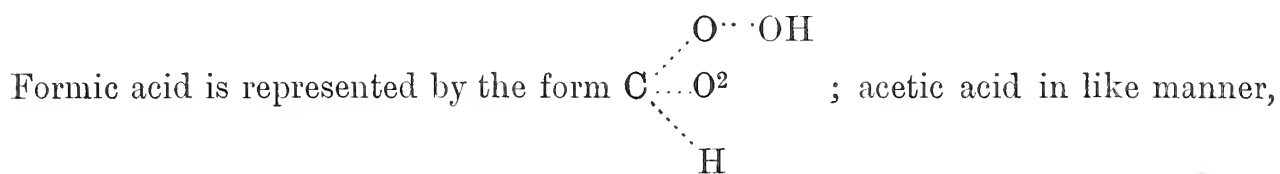
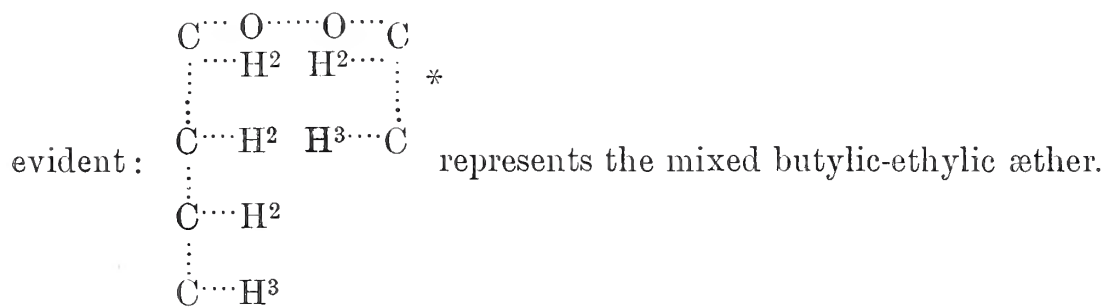
Les composés primaires devraient être à la rigueur eux-mêmes énumérés dans un ordre progressif de complication.

* Misprinted "Le propylglycol" in the original.

type CM⁴. This enumeration, however, does not appear to possess any great practical utility, and it is perhaps preferable simply to denote it in an indefinite manner by adding "n" to the true type CM⁴.



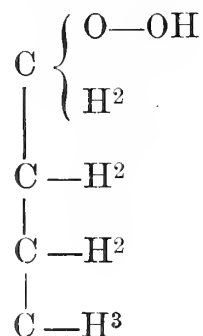
throughout all the series of these alcohols. The constitution of the æthers will be



* The vertical dotted lines between the two "C's" is omitted in the original.

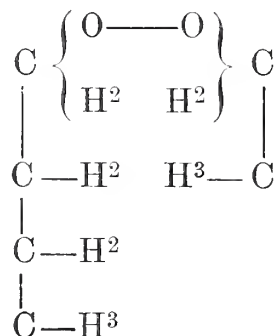
Le type $n\text{CM}^4$ devient alors en réalité le type CM^4 . Mais cette énumération ne paraît pas avoir une grande utilité pratique, et il est peut-être préférable de désigner le type d'une manière indéterminée en ajoutant n au vrai type CM^4 .

L'alcool butylique est de même représenté par

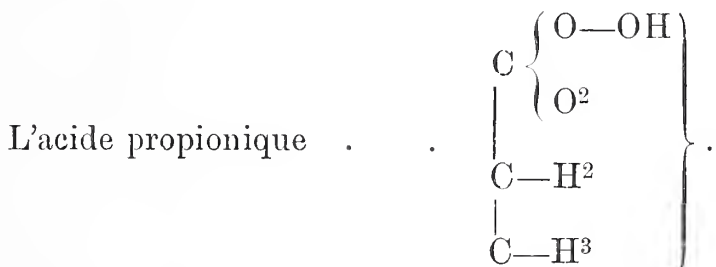
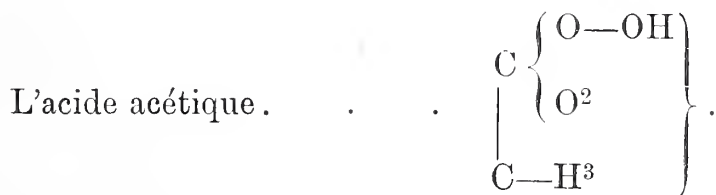
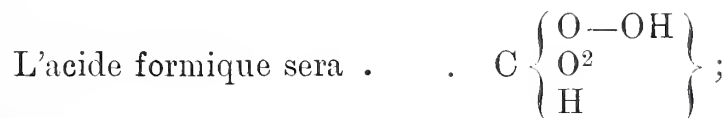


et ainsi de suite en parcourant toute la série de ces alcools.

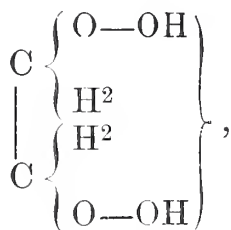
La constitution des éthers n'est pas moins évidente.



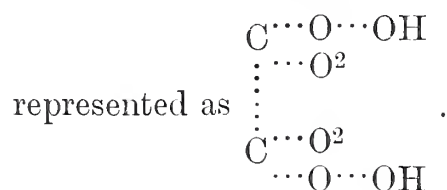
représentera l'éther mixte butyléthylique



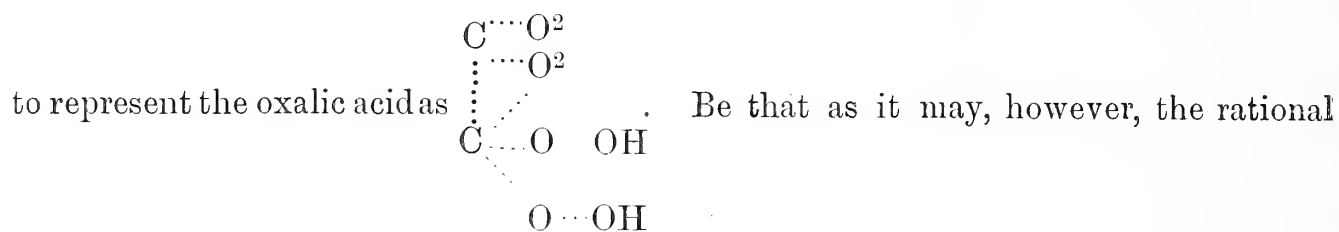
On peut écrire le glycol ainsi



In like manner as to the acids of these glycols ; oxalic acid, for instance, may be



Respecting these acids, it may perhaps be allowable to suggest the possibility of the molecule having two poles, and that especially the atom of oxygen situated at one or perhaps both, and near to two atoms of oxygen bound together, and forming no secondary combinate, may be in a state presenting great affinity for basic oxygen. Analogy with electric poles may perhaps demand the opinion that all the negative oxygen be situated upon one side of the molecule. It will in that case be preferable



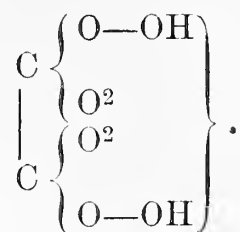
method of investigation proves it to be a law, that in acids of the type $n\text{CM}^4$ the presence of two atoms of oxygen bound together so as to form only a primary part of the same molecule, and situated close to the negative oxygen, is necessary to the calling forth or production of this negative state.

This is a particular instance, but it moreover *shows generally how the electro-positive or the electro-negative value of the elements mutually modify and condition the electro-positive or electronegative value of each other when in combination.*

This law is different from the electric hypothesis which chemists have formerly defended, but which never could be traced throughout a thoroughgoing application of their views to organic chemistry.

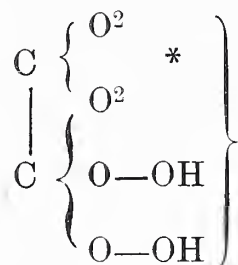
The law here distinctly enounced coincides exactly with, and is rendered apparent by the application of the theory of chemical combination which I support.

et l'acide oxalique



Relativement aux acides, il sera peut-être permis d'émettre la supposition que les molécules possèdent deux pôles, et que la molécule d'oxygène située auprès de l'un des pôles (ou peut-être des deux) et dans le voisinage des deux molécules d'oxygène liées ensemble et ne donnant pas lieu à la formation d'un composé secondaire, que cet oxygène, dis-je, se trouve dans un état tel, qu'il possède une grande affinité pour l'oxygène basique.

L'analogie avec les pôles électriques exige peut-être que tout l'oxygène soit placé d'un côté de la molécule. Il vaudrait mieux, dans ce cas, représenter l'acide oxalique par



Quoi qu'il en soit, la méthode rationnelle d'analyse prouve que c'est une loi que, dans les acides du type $n\text{CM}^4$ la présence de deux atomes d'oxygène liés ensemble de manière à former une partie primaire de la même molécule, et situés dans le voisinage de l'oxygène négatif, est nécessaire pour produire dans cet oxygène l'état négatif où il se trouve.

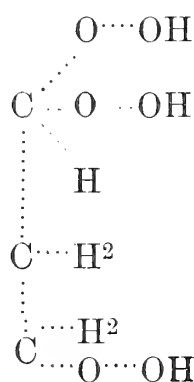
Ceci n'est qu'un cas particulier d'un fait plus général, car il résulte de ce qu'en général l'état électropositif ou électronégatif des éléments modifie ou entraîne l'état électropositif ou électronégatif des éléments combinés avec eux, et réciproquement.

Cette loi diffère de l'hypothèse électrique que les chimistes ont défendue autrefois, mais qui n'a jamais pu recevoir une application complète à leur vues sur la chimie organique.

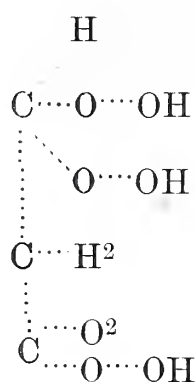
Celle au contraire que j'énonce ici s'accorde parfaitement avec la théorie que je défends, et s'en déduit d'une manière simple.

* Misprinted $\begin{array}{c} \text{C} \\ | \\ \text{C} \end{array} \left\{ \begin{array}{l} \text{O}^2 \\ \text{O}^2 \\ \text{O—OH} \\ \text{O—OH} \end{array} \right\}$ in the original. See p. 260 and also p. 239.

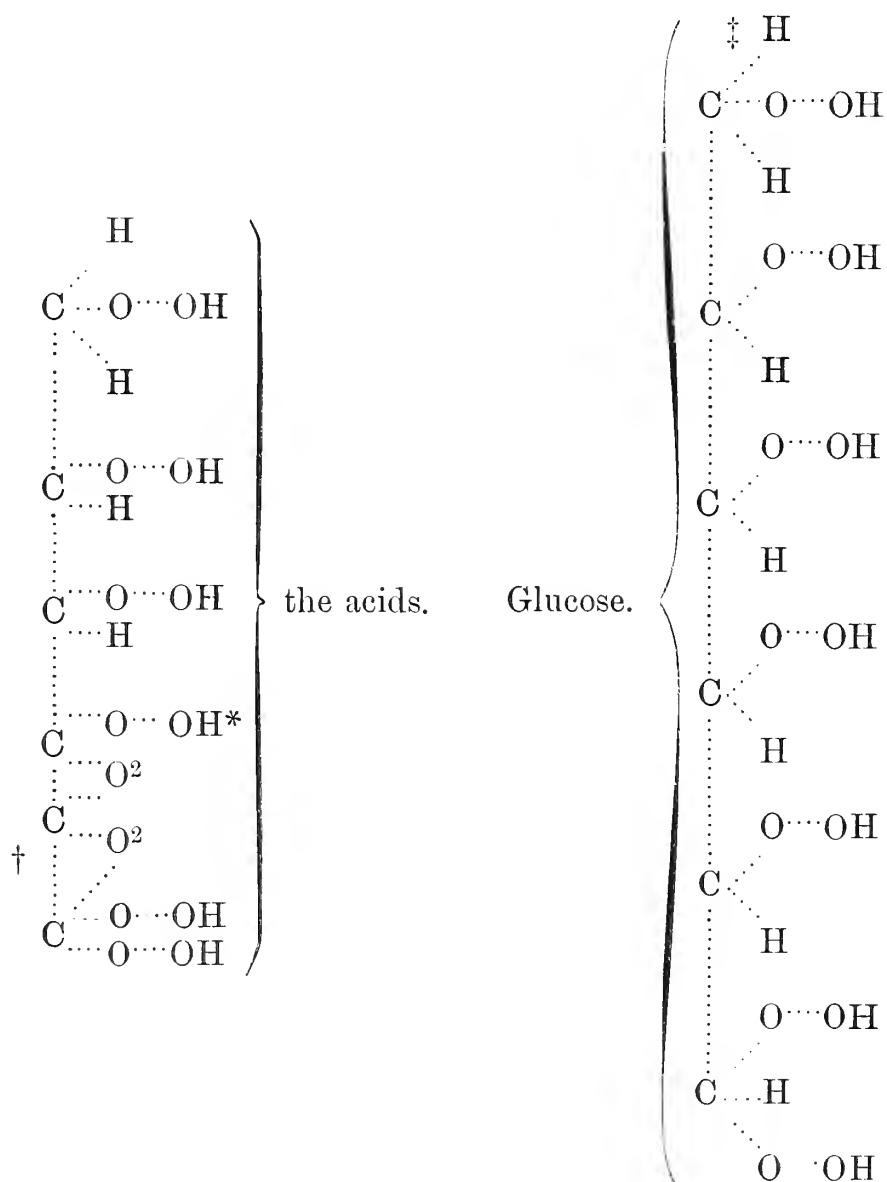
But to return. Glycerine is



, and glyceric acid



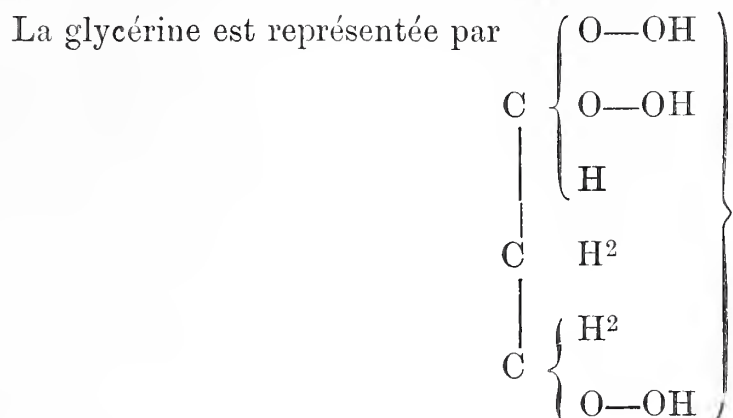
Glucose has been perhaps too little investigated to afford data sufficient to determine definitely its formula. Taking, however, mucic and saccharic acids as starting-points, these bodies may be meanwhile represented as:—



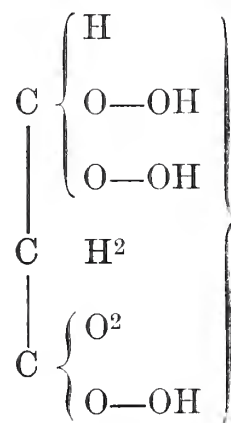
It will thus be seen that these combinates all belong to type nCM⁴.

* This "OH" is omitted in the original.
 † The oblique dotted line here is omitted in the original.
 ‡ The vertical dotted line between the "C's" is omitted in the original.

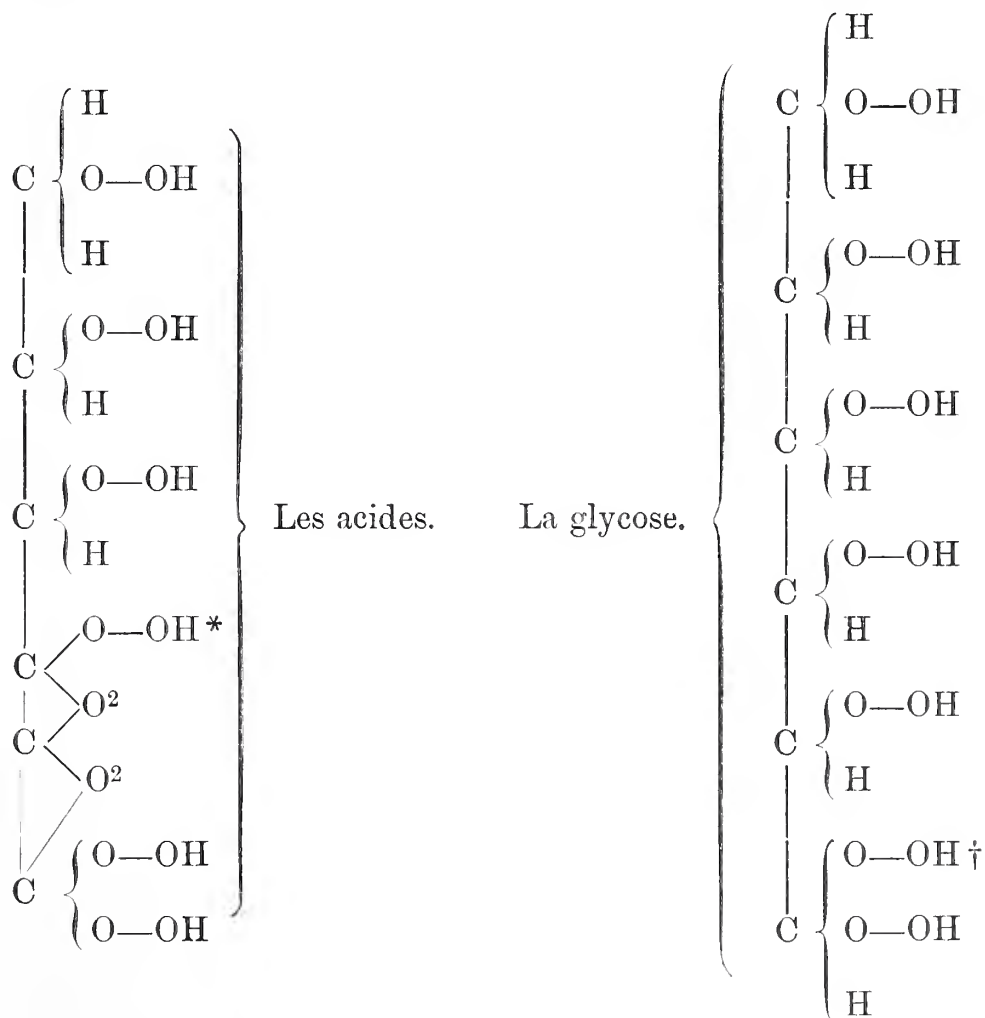
Mais revenons aux formules des combinaisons les plus importantes.



et l'acide glycérique par



La glycose est peut-être trop mal étudiée pour qu'on puisse déterminer définitivement sa formule. Mais en prenant l'acide mucique et l'acide saccharique comme points de départ, ces trois corps peuvent être provisoirement représentés par

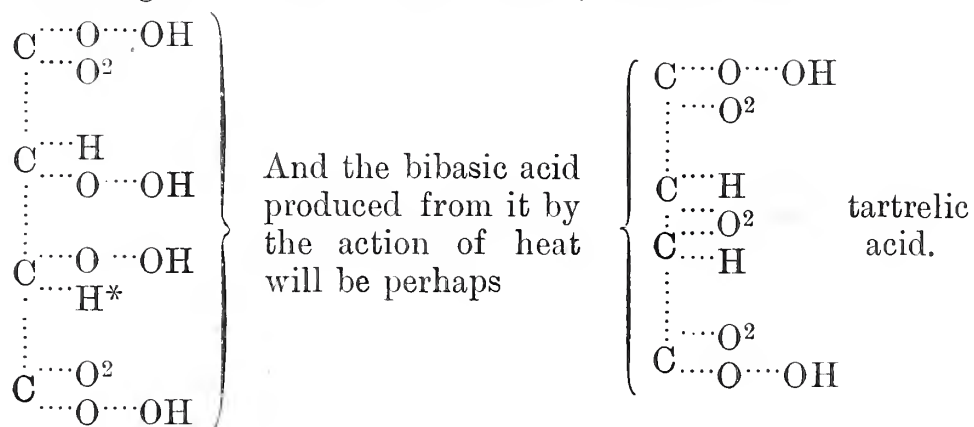


Il en résulte que ces composés appartiennent au type $n\text{CM}^4$.

* This "OH" is omitted in the original.

† Misprinted "H" for "O—OH" in the original.

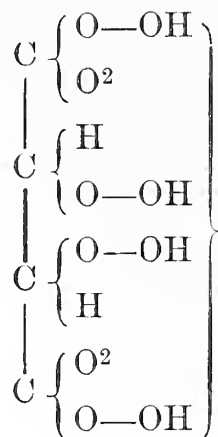
Many others might be added. For instance, tartaric acid:—



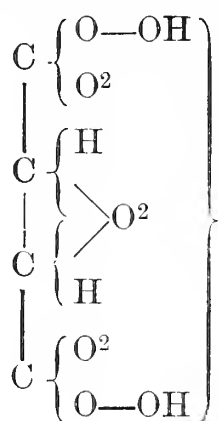
It is my intention to consider, in a future communication, the second type, and to apply my views to the cyanogen combinates, etc.

* A horizontal dotted line is erroneously printed here in the original.

On pourrait ajouter beaucoup d'autres corps, par exemple l'acide tartrique :



et l'acide bibasique dérivé de l'acide tartrique par l'action de la chaleur sera peut-être



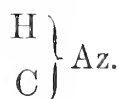
Dans un autre Mémoire je me propose de m'occuper du second type.

En attendant, j'ajouterai seulement la manière dont je considère la constitution des principaux composés cyaniques.

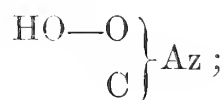
Des raisons entièrement semblables à celles qui me font admettre 4 pour limite du pouvoir de combinaison du carbone, me conduisent à assigner 5 comme limite de combinaison à l'azote. Le premier degré de combinaison de cet élément se rencontre dans l'ammoniaque et équivaut à 3. Le second, qui est égal à 5, se trouve, entre autres composés chimiques, dans le chlorure et dans l'oxyde d'ammonium ainsi que dans l'acide azotique.

Il résulte de là que le carbone et l'azote combinés, de manière à atteindre tous deux les limites de leur pouvoir de combinaison, formeront un corps dont l'affinité libre s'exercera en fixant un équivalent d'hydrogène ou d'un autre élément.

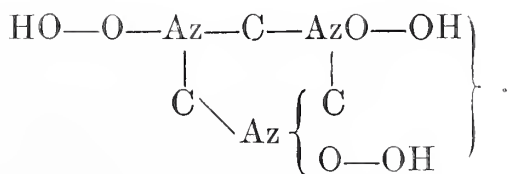
Ainsi la formule de l'acide cyanhydrique sera



L'acide cyanique sera



l'acide cyanurique



Dans cette dernière formule, les atomes de carbone et d'azote sont liés par 2 unités d'affinité et non par 4, comme dans les deux premiers exemples.

APPENDIX IV *a*.

RECHERCHES SUR L'ACIDE SALICYLIQUE ; PAR M. COUPER.*

Les recherches que j'ai l'honneur de soumettre à l'Académie ont pour objet l'action du perchlorure de phosphore sur le salicylate de méthyle. Je les ai entreprises dans le but de jeter quelque lumière sur une question controversée : la constitution et la basicité de l'acide salicylique.

Une violente réaction se manifeste au contact de l'huile de gaulthéria et du perchlorure de phosphore. Il est nécessaire, pour la maîtriser, d'ajouter par petites portions l'huile essentielle au perchlorure dans le rapport de 1 équivalent du premier corps à 2 équivalents du second. De l'acide chlorhydrique et du chlorure de méthyle se dégagent pendant tout le cours de l'opération.

Le produit obtenu est soumis à la distillation fractionnée. Une trace seulement de chloroxyde passe d'abord, un excès assez considérable de perchlorure de phosphore distille ensuite, et lorsque la température a atteint 160 degrés, le résidu constitue un liquide noir. Si l'on continue la distillation, la température s'élève rapidement à 285 degrés. La plus grande portion du produit passe entre 285 et 295 degrés, sous la forme d'un liquide incolore ou légèrement coloré en jaune. On le recueille séparément. Il reste une masse noire qui se solidifie par le refroidissement.

Le liquide recueilli vers 290 degrés a donné à l'analyse les résultats suivants :

	Expériences.			Théorie.
	30,86	29,4	29,9	
Carbone . . .	30,86	29,4	29,9	30,65
Hydrogène . . .	1,58†	1,59	1,51	1,46
Chlore . . .	41,01	41,05	...	38,86
Phosphore . . .	12,2	11,5

* *Comptes rendus*, xlvii., 1107-1110, séance du lundi, 7 juin 1858.

† Misprinted in original "1,38" for "1,58". Compare the English text.

APPENDIX IV *b*.

RESEARCHES ON SALICYLIC ACID. BY ARCHIBALD S. COUPER.*

Conflicting opinions being entertained among chemists respecting the constitution and basicity of salicylic acid, in order, if possible, to throw some additional light upon this question, I have been induced to undertake the investigation of the action of two equivalents of pentachloride of phosphorus upon the salicylate of methyl.

When these two bodies are brought into contact, the reaction which ensues is exceedingly violent, and it is necessary to add very gradually the one equivalent of oil of gaultheria to the two equivalents of perchloride of phosphorus. Vapour of hydrochloric acid, mingled with that of the chloride of methyl, is disengaged during the whole course of the operation.

The residue of this reaction is submitted to distillation. A trace of oxychloride passes over, and is followed by something approaching to an equivalent of the perchloride of phosphorus.

The temperature having now risen to about 160°, the residue has darkened in colour. Submitted to a further distillation, a colourless or slightly yellow liquid passes over, while the temperature rises rapidly to about 285° and 295° Cent. The portion passing between these temperatures is collected apart. It constitutes the larger part of the product of distillation. There remains a pitchy residue, which solidifies on cooling. The liquid collected, as described, on being submitted to analysis, furnished the following results:—

1. 0·41 grms. of substance gave 0·464 grms. of carbonic acid, and 0·0585 grms. of water.

2. 0·49 grms. of substance gave 0·528 grms. of carbonic acid, and 0·0705 grms. of water.

3. 0·515 grms. of substance gave 0·565 grms. carbonic acid, and 0·071 grms. of water.

1. 0·438 grms. of substance gave 0·7595 grms. of chloride of silver.

2. 0·34 grms. of substance gave 0·563 grms. of chloride of silver.

1. 0·651 grms. of substance gave 0·283 grms. of the pyrophosphate of magnesia.

These analyses lead to the crude formula,—



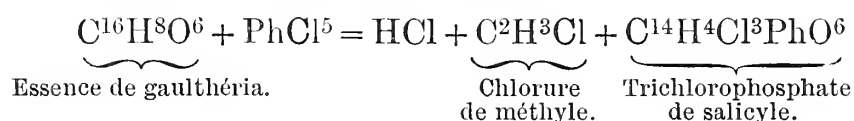
	Found.			Calculated.
	I.	II.	III.	
Carbon	30·86	29·4	29·9	30·65
Hydrogen	1·58	1·59	1·51	1·46
Chlorine	41·01	41·05	38·86
Phosphorus	12·2	11·5

* *The Edinburgh New Philosophical Journal*. New series, vol. viii., July to October 1858, 213-217.

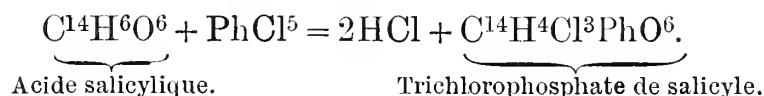
La composition de ce corps est représentée par la formule



Il se forme, en vertu de la réaction suivante :

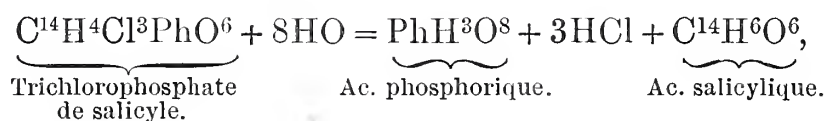


Je me suis assuré que le trichlorophosphate de salicyle prend aussi naissance par l'action du perchlorure de phosphore sur l'acide salicylique

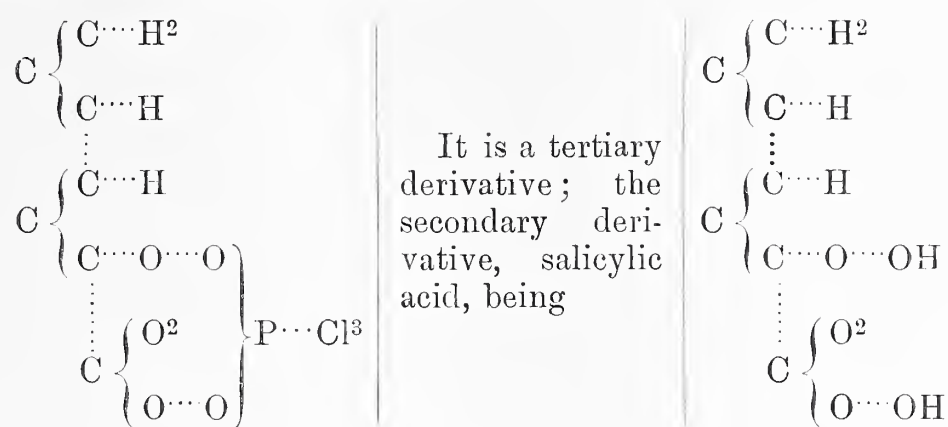


Entre ces deux corps solides, la réaction est moins violente qu'avec l'essence. Il se dégage de l'acide chlorhydrique, et lorsque l'action est terminée, le résidu est le même que celui qu'on obtient avec l'huile de gaulthéria. Les deux produits distillent exactement de la même manière et à la même température, et les liquides obtenus possèdent la même composition et les mêmes propriétés.

Le trichlorophosphate de salicyle obtenu par l'un ou l'autre de ces procédés se décompose bientôt au contact de l'eau froide et immédiatement lorsqu'on le chauffe avec ce liquide. Les produits de cette réaction sont l'acide chlorhydrique, l'acide phosphorique et l'acide salicylique.



According to the rational theory which I seek to develop in another paper,* the constitution of this body may be represented as:—



The reaction, of which this body is the result, is represented by the following equation †:—



This is the same body which, unpurified, Gerhardt calls the chloride of salicyl, ‡ and Chiozza the chloride of chlorobenzoil, § or the hydrochlorate of monochlorobenzoic acid. In order to determine rigorously whether the body produced by Chiozza from salicylic acid was really the same as that which Gerhardt and I have obtained from the oil of gaultheria, I investigated the action of the perchloride of phosphorus upon the pure acid. The reaction which takes place between the two solid bodies is less violent than when the oil was employed. Nevertheless, in this instance also the action is prompt. After the vapour of hydrochloric acid has passed off, the residue in the flask is identical with that which remains when the oil of gaultheria is employed. These two products distil in identically the same manner, with the same physical appearances, while the products of distillation produce exactly the same results upon analysis.

These products display also entirely the same results in all their reactions.

The terchlorophosphate of salicyl obtained by either process is very soon decomposed by water in the cold, and immediately, upon the application of heat, the products being hydrochloric, phosphoric, and salicylic acids.

Chiozza's monochlorobenzoic acid, which he believes to be produced along with hydrochloric acid in the decomposition of this body, is only salicylic acid || rendered impure by hydrochloric and phosphoric acids.

Chiozza did not succeed in producing the reaction for salicylic acid by the perchloride of iron. Nor is it to be observed till the liquid containing the phos-

* *Phil. Mag.*, August 1858.

† Couper gives here the equation representing the action of PCl_5 on salicylic acid (of which he speaks a few lines further on) instead of that for the action on gaultheria oil.

‡ *Comptes rendus*, xxxviii. 34.

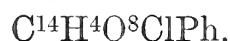
§ *Annales de chim. et de phys.* [3], xxxvi. 102.

|| Couper is mistaken in this. Chiozza certainly had in his hands chlorobenzoyl chloride and chlorobenzoic acid.

Je me suis assuré, par l'analyse, qu'il se forme véritablement dans cette réaction, de l'acide salicylique et pas de l'acide monochlorobenzoïque. Lorsque le trichlorophosphate de salicyle est rapidement distillé, il se décompose en partie en émettant d'abondantes vapeurs d'acide chlorhydrique. Au-dessus de 300 degrés il passe un corps liquide, qui, lorsqu'on le conserve pendant quelques jours dans un tube fermé, dépose de volumineux cristaux qui renferment :

					Théorie.
Carbone	40,2	39,16			38,44
Hydrogène	2,3	1,96			1,83
Chlore	17,07	...			16,25

Cette analyse, le mode de formation de ces cristaux et surtout leur dédoublement par l'eau que je vais indiquer plus loin me portent à leur attribuer la composition représentée par la formule suivante :



Comme cette substance, que je nomme monochlorophosphate de salicyle, se décompose à l'air dont il attire l'humidité et qu'elle est formée d'ailleurs par un liquide bouillant à une température très-élevée, je ne me suis pas arrêté à la pensée de l'obtenir sous une forme plus pure, et j'ai dû me contenter de l'analyse précédente.

phoric acid is separated from the crystals of salicylic acid, but as soon as this is done, the application of the test gives the intense and characteristic colour produced by the acid in question. An analysis which I have made confirms this result.

0.301 grms. of these crystals, expressed between folds of bibulous paper, re-crystallised from water, and dried, upon combustion, gave 0.668 grms. of carbonic acid, and 0.116 grms. of water.

	Found.	Calculated.
Carbon	60.52	60.86
Hydrogen	4.27	4.34

In another analysis 0.2205 grms. of substance gave 0.4885 grms. of carbonic acid, and 0.0875 grms. of water, which, calculated, gives—

60.42 % for carbon, and 4.4 % for hydrogen.

This acid agrees also in all its physical aspects with the salicylic acid.

It is evident, then, that those three bodies,—namely, the chloride of salicyl of Gerhardt, the chloride of chlorobenzoil, and the chlorobenzoilic acid of Chiozza, are not yet known.

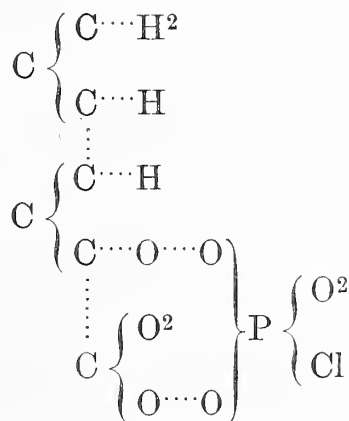
When the terchlorophosphate of salicyl is quickly distilled, hydrochloric acid is given off in considerable quantity, while the body, at the same time, acquires a dark colour. The last portion of the product of this distillation, which passes at a temperature above 300° Cent., being allowed to stand for a day or two in a closed tube, deposits large crystals, which, having been pressed between folds of bibulous paper, and submitted to analysis, gave the following results:—

- 1°. 0.3565 grms. of substance gave 0.5255 carbonic acid, and 0.074 grms. water.
- 2°. 0.57 grms. of substance gave 0.826 grms. carbonic acid, and 0.101 grms. of water.
- 1°. 0.255 grms. of substance gave 0.176 grms. of chloride of silver.

Compared with the crude formula $C^7H^4O^8ClP$, these results will be:—

	Found.		Calculated.
	I.	II.	
Carbon	40.2	39.16	38.44
Hydrogen	2.3	1.96	1.83
Chlorine	17.07	...	16.25

The rational formula is—



The monochlorophosphate of salicyl being a body which is quickly decomposed by exposure to the air, and only to be obtained from a liquid passing at an exceedingly high temperature, I did not think it easily possible to obtain a purer body, and have contented myself with the above analysis. Although these analyses might in themselves leave some doubt as to the composition of the body, still, when taken in

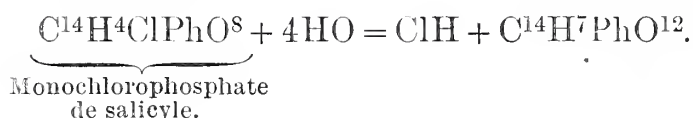
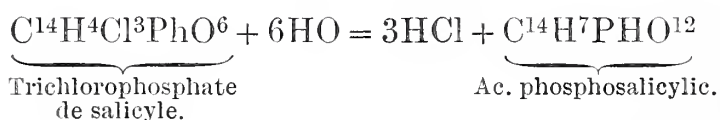
L'action lente que l'humidité exerce sur les produits à la fois chlorés et phosphorés que je viens de décrire confirme la composition que je leur attribue. Lorsqu'on les expose à l'air, ils en attirent peu à peu la vapeur d'eau ; le chlore qu'ils renferment se combine à l'hydrogène, et est remplacé par de l'oxygène. Il se forme ainsi un acide nouveau, que je nomme *acide phosphosalicylique*, et dont la composition est exprimée par la formule



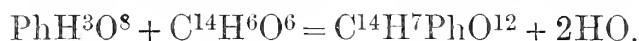
Ce produit, qui est solide, a donné à l'analyse les résultats suivants :

	Expériences.	Théorie.
Carbone	38,05	38,53
Hydrogène	3,39	3,21
Phosphore	14,48	14,22

Il prend naissance en vertu des réactions suivantes :



L'acide phosphosalicylique est un acide tribasique. On peut l'envisager comme une combinaison conjuguée d'acide phosphorique et d'acide salicylique



Les expériences que je viens de décrire sommairement ne s'accordent pas en tous points avec les observations qui ont été publiées sur le même sujet par MM. Gerhardt,* Chiozza,† et Drion.‡ Elles semblent en particulier jeter quelques doutes sur l'existence du chlorure de salicyle de M. Gerhardt, produit qui n'a jamais été analysé. Nous ferons remarquer d'ailleurs que la formation de ce produit par l'action du perchlorure de phosphore sur l'acide salicylique ou sur l'huile de gaulthéria devrait être accompagnée de celle du chloroxyde de phosphore. Or, dans les réactions dont il s'agit il ne se forme que des traces de cette substance, comme l'indiquent d'ailleurs les auteurs que nous venons de citer. Les divergences que nous signalons tiennent-elles à quelque circonstance fortuite ou à des conditions particulières dans lesquelles se sont placés les observateurs, c'est ce que de nouvelles expériences devront décider.

* *Comptes rendus*, t. xxxviii. p. 34.

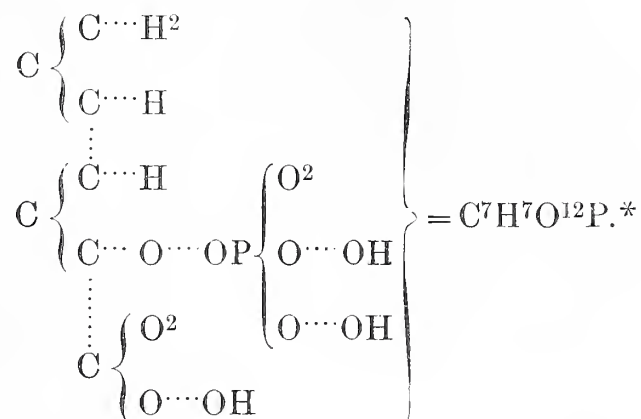
† *Annales de Chimie et de Physique* (3^e série), t. xxxvi. p. 102.

‡ *Comptes rendus*, t. xxxix

connection with a combinate which I am about to describe, and which is produced from the monochlorophosphate of salicyl, as well as from the terchlorophosphate, I think I may be warranted in ascribing definitely to it the above formula.

This compound is produced by the exposure to the atmosphere of the bodies just mentioned. The chlorine contained in them decomposes the water of the atmosphere combining with an equivalent of hydrogen, while the oxygen is taken up by the bodies in question in replacement of the chlorine. This new oxygen, however, enters into the state in which it is found in acids, and has a great affinity for the oxygen of water and bases. It therefore attracts the water of the atmosphere, and the combinate becomes an hydrated acid.

This body being submitted to analysis, gave results according with the rational formula—



1°. 0.2845 grms. of substance gave 0.397 grms. of carbonic acid, and 0.086 grms. of water.

0.1785 grms. of substance gave 0.103 grms. of the pyrophosphate of magnesia.

	Found.	Calculated.
Carbon	38.05	38.53
Hydrogen	3.39	3.21
Phosphorus	14.48	14.22

This phosphosalicylic acid is a tribasic acid, and forms insoluble salts with baryta, lead, and silver, containing three equivalents of these metals.

I propose taking into consideration more fully the constitution of salicylic acid in a subsequent communication. The formula here given of that body may be in the meanwhile suggestive.

* Couper has here for phosphorus the symbol "Ph" as throughout the French paper. Elsewhere in English he uses "P".

(Issued separately April 30, 1909.)

XIV.—On the Magnetic Properties of certain Copper Alloys. By Alexander D. Ross, M.A., B.Sc., Assistant to the Professor of Natural Philosophy in the University of Glasgow, and Robert C. Gray, Thomson Experimental Scholar in the University of Glasgow.

(MS. received February 5, 1909. Read February 15, 1909.)

IN May 1907 a paper was read before this Society by one of us on the magnetic properties of the Heusler alloy.* After that date the investigation was extended to two other copper-manganese-aluminium alloys, and latterly the scope of the research has been further extended by a series of tests carried out by the present authors conjointly on simple aluminium and manganese bronzes, and on the constituent metallic elements in the pure form, while additional tests have been made on the manganese-aluminium bronze. Some of the more interesting results obtained since the 1907 paper are given in the present communication.

Specimens.—The specimens referred to in this paper were cast by Gray and Caldwell, Paisley, and Steven and Struthers, Glasgow, and had the following compositions:—

- (1) Casting B.—62 per cent. copper, 25 per cent. manganese, 12·5 per cent. aluminium, and a trace of lead.
- (2) Casting D.—76 per cent. copper, 16 per cent. manganese, and 8 per cent. aluminium.
- (3) Casting E.—55 per cent. copper, 30 per cent. manganese, and 15 per cent. aluminium.
- (4) 5 per cent. aluminium-bronze (that is, 5 per cent. aluminium, 95 per cent copper).
- (5) 10 per cent. aluminium-bronze.
- (6) 60 per cent. aluminium-bronze.
- (7) 70 per cent. aluminium-bronze.
- (8) 30 per cent. manganese-bronze.
- (9) Pure aluminium.
- (10) Pure manganese.

It will be observed that B is the ordinary Heusler alloy, and that in D and E the same relative proportions (viz., those of the atomic weights) of manganese and aluminium have been maintained, while the amount of

* A. D. Ross, "Heusler's Magnetic Alloy," *Proc. Roy. Soc. Edin.*, xxvii. part 2, p. 88.

copper has been varied. This scheme was adopted because the effect of variation of the ratio of manganese to aluminium had already been investigated by Heusler, Starck, and Haupt,* and also because certain considerations seemed to indicate that copper played a not unimportant part in the magnetism of the Heusler alloy.†

The specimens from castings B, D, and E were of uniform size, viz., cylinders of length 20 cm. and diameter about 1 cm. They were tested in the condition as cast, except that the ends were cut off square. No further dressing was given as that process would probably have altered the magnetic properties slightly owing to the heat evolved in the operation.

The materials included under (4)-(10) in the above list were too feebly magnetic to lend themselves to testing in the usual magnetometric method. The specimens were accordingly cast in the form of small cylinders 8 cm. in length and 6 mm. in diameter, and were magnetised by means of a powerful electromagnet.

Apparatus.—For testing the castings B, D, and E the Gray-Ross magnetometer ‡ was used. Alloys D and E exhibited only feeble magnetic properties, and the obtaining of a hysteresis cycle with the material in the quenched condition was a matter of very considerable difficulty. Such tests had to be carried out in the early morning hours, when the slight disturbance of the magnetometer zero due to electric earth-currents was at a minimum.

The magnetisation of the specimens included under (4)-(10) in the above list was carried out with the large electromagnet belonging to the Physical Laboratory of Glasgow University. The magnet is of cast steel with a permeability almost coincident with that of wrought-iron. It is rectangular in shape, measuring about 100 cm. in length and 40 cm. in height. In addition to the coils on the poles, the whole length of the yoke is wound. During the experiments the pole-pieces were kept 11 cm. apart in order to freely admit the specimens when enclosed in vessels for holding liquid air. With this separation a field of 4400 C.G.S. units was obtained on exciting the magnet with the current from a 110-volt storage battery. The permanent magnetism so induced in the specimens was measured by a delicate magnetometer, the motion of whose needle and mirror was observed by means of a scale placed more than two metres distant.

Quenching and Liquid Air Tests.—In the previous paper an account has

* "Magnetisch-chemische Studien," *Ber. d. Deutsch. Phys. Gesell.*, 1903, S. 220.

† See article by J. G. Gray and A. D. Ross, *Phys. Zeit.*, x., No. 2, S. 59.

‡ J. G. Gray and A. D. Ross, "On an Improved Form of Magnetometer, etc.," *Proc. Roy. Soc. Edin.*, 1908-9.

been given of the effects of liquid air on alloy B both in the cast and in the quenched condition. Similar tests were now made with alloy D. As it had been found that quenching of such bronzes at temperatures exceeding 600°C . produced cracking of the material, the quenching was carried out at

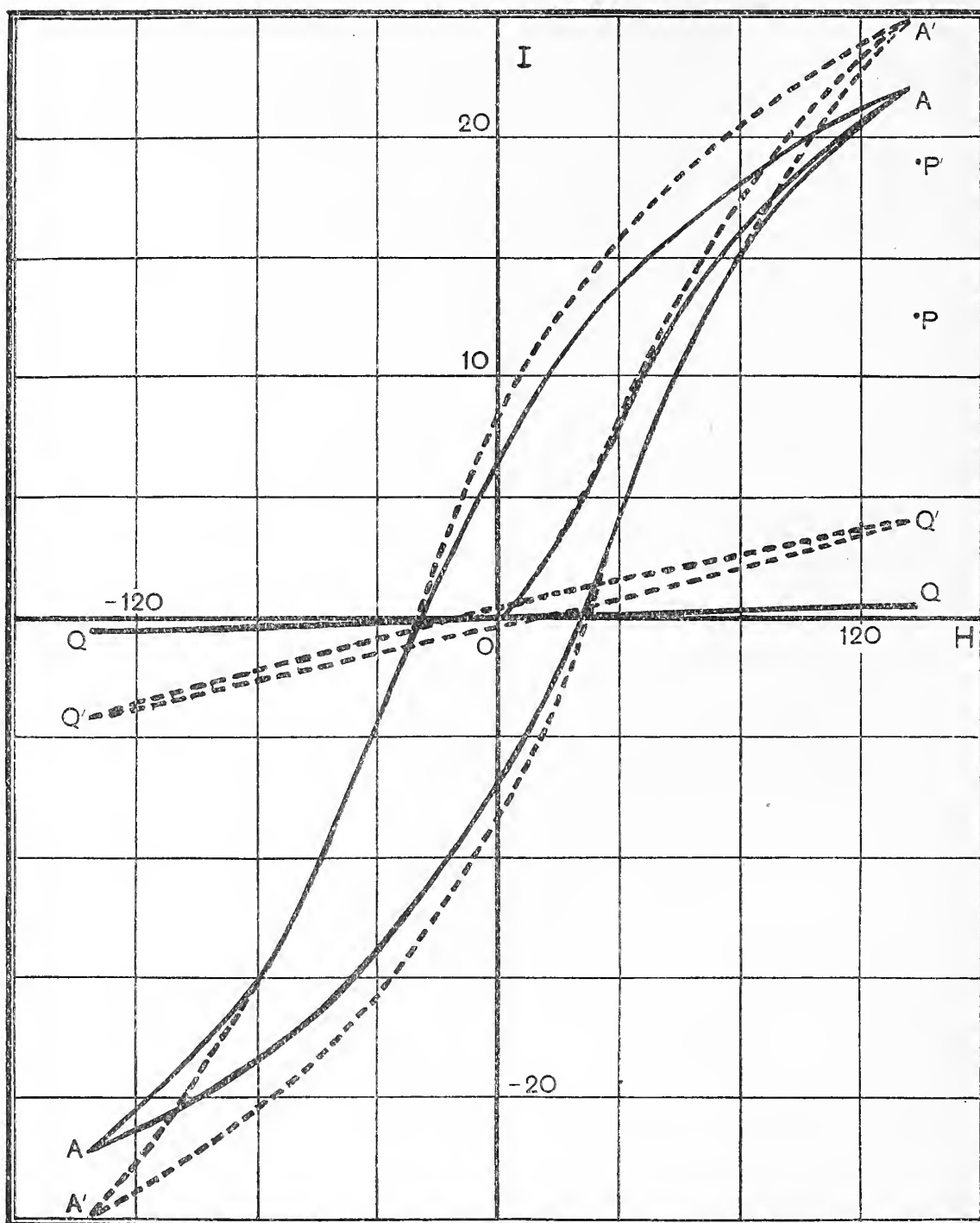


FIG. 1.

450°C . Fig. 1 exhibits the results obtained. While the most important point here is the percentage changes produced by the varying treatment, the absolute values of the intensity of magnetisation (I) and the strength of the magnetising field (H) are also given for facilitating comparison with other materials. In calculating H, allowance has been made for the end effect of the magnetising solenoid, and Du Bois's factors* are employed in

* Du Bois, *Magnetische Kreise in Theorie und Praxis*, Berlin, 1894.

deriving the true field from that due to the current in the helix. The values of I are obtained on the assumption that the polar distance of the cylindrical specimens is $\frac{5}{6}$ their total length—an assumption which has been justified by comparison of such specimens with others turned down into ellipsoids of revolution. The curve AA shows the magnetic condition of the alloy in its original state. On re-testing with the specimen at the temperature of -190°C . the dotted line curve $A'A'$ was obtained. The effect of cooling to liquid air temperature has been to increase the permeability

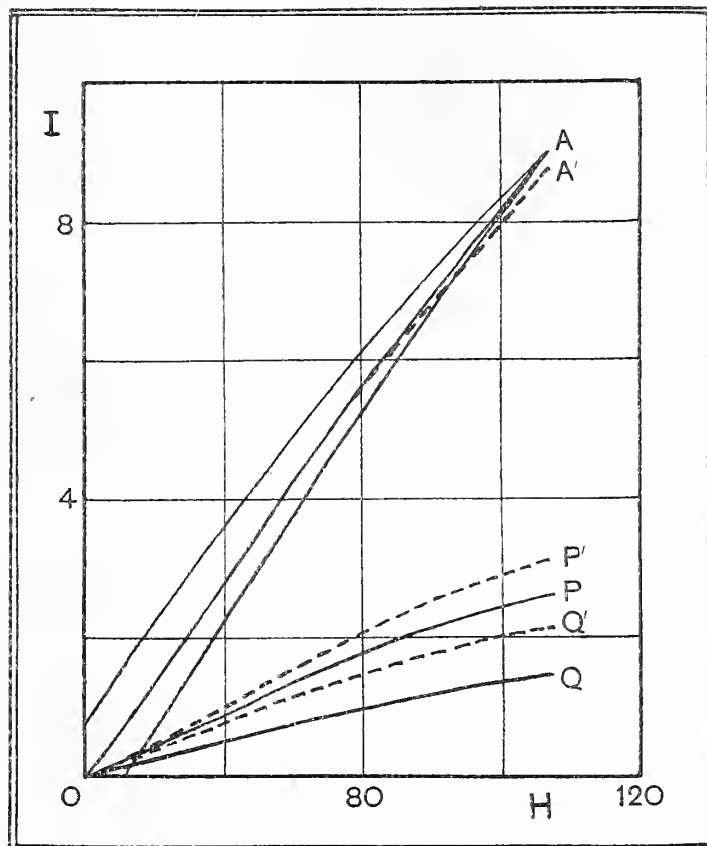


FIG. 2.

and raise the saturation-point. The coercive force has been augmented but to a very slight degree, while the residual magnetism has increased about 30 per cent. After being quenched at 450°C . the specimen was almost non-magnetic as indicated by QQ . On cooling, however, to -190°C . there was a marked increase in permeability and hysteresis as shown by the curve $Q'Q'$. Another specimen which had been very rapidly cooled in the process of casting gave P and P' respectively for the highest points on the 15°C . and the -190°C . magnetisation cycles. It exhibits a large liquid air effect, and therein resembles the quenched material.

Fig. 2 shows similar tests on alloy E—an alloy much poorer in copper. A is the magnetisation curve for the initial condition; and A' the curve obtained when the specimen has been cooled down to -190°C . The ascending and descending limbs of the hysteresis cycle have been omitted, as the

cycle does not differ much from that obtained at room temperature. Q and Q' are the magnetisation curves obtained at 15° C. and -190° C. respectively. The hysteresis is now almost inappreciable, and it will be observed that the liquid air effect has been reversed in direction. A chilled casting tested without further thermal treatment gave the curves P and P', which closely resemble those for the quenched material.

Effect of Baking the Alloys.—The effect of baking the alloys D and E at various temperatures was now investigated, it being thought possible that such treatment would result in an improvement of the magnetic quality. Various temperatures up to 500° C. were employed, but all resulted in a lowering of the magnetisation curve. Fig. 3 exhibits the nature of the results obtained. In the upper diagram is shown the effect of maintaining alloy D at a temperature of 260° C. The ordinates of the graph are the intensities of magnetisation for the fields mentioned, while the abscissæ give the duration of the baking in hours. The continuous lines show the changes in the intensity of magnetisation for fields of 125 and 30 C.G.S. units respectively, the tests being made with the specimens cooled down to room temperature after the baking process. The dotted curves are obtained with the specimen cooled to -190° C. after the baking. The result of this thermal treatment has therefore been (i) to lower the saturation intensity continuously towards a limiting value, (ii) to cause at first a slight increase in the susceptibility for low fields, (iii) to reduce the susceptibility for all fields towards a definite limiting value, (iv) to reverse the liquid air effect in specimens baked for more than 8 hours. The hysteresis was also found to increase with time towards a maximum value.

The lower part of fig. 3 similarly exhibits the effect of baking alloy E at 210° C. The results are shown only for $H=125$, as they are of the same nature for all fields. The alloy undergoes a steady and rapid deterioration in quality, and after 10 hours' baking has almost acquired an equilibrium condition. The liquid air effect in this alloy is reversed after 2 hours' baking at 210° C., but it will be noticed that its direction was originally opposite to that in alloy D.

The effect of baking alloy B has also been investigated by the authors. It has been found that the best magnetic condition of the alloy is obtained by exposing the material for a period of 6 to 8 hours at a temperature of 170° C. This results in an increase of over 10 per cent. in the saturation value of I_* while the short duration of the baking process does not produce any marked increase in the hysteresis loss in taking the material through

* Baking, however, always reduces the values of I corresponding to small fields.

a magnetic cycle. Prolonged heating at the same temperature slightly reduces the susceptibility, and greatly augments the coercive force and the hysteresis loss.

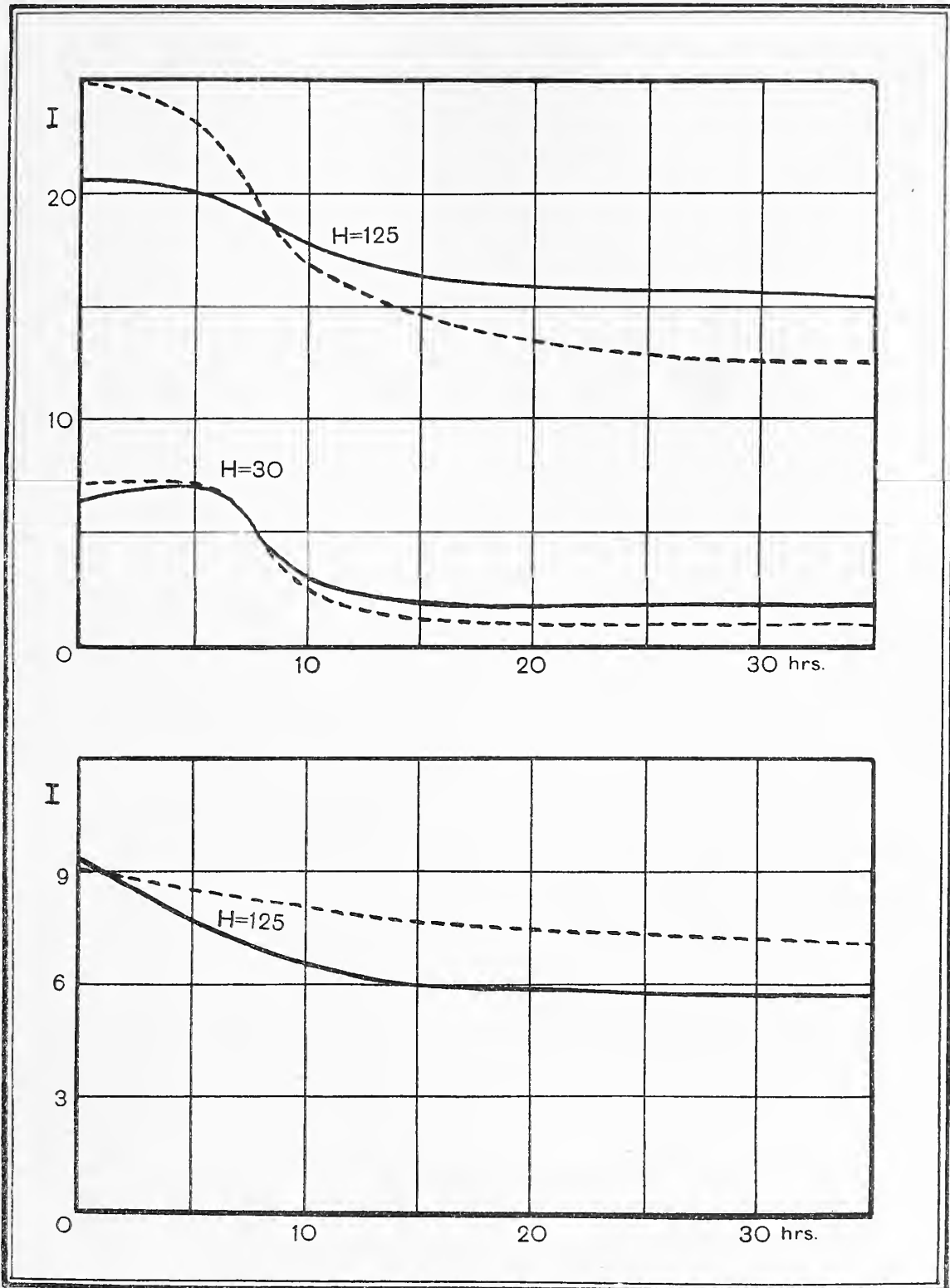


FIG. 3.

Fig. 4 shows the results of exposing the alloy B to steadily increasing temperatures. In each case the alloy was placed in a furnace at the required temperature, kept there for 3 hours, and then the furnace was shut off and the alloy allowed to cool in position and tested cold. The tests were made for a series of temperatures from 50° to 480° C., each

temperature employed exceeding the previous one by about 30°. The ordinates in the diagram are the intensities of magnetisation, while the abscissæ are the temperatures of baking. The continuous lines are obtained by testing the baked specimen at room temperature in fields of 8, 16, and 110 C.G.S. units respectively. The dotted lines give the results of

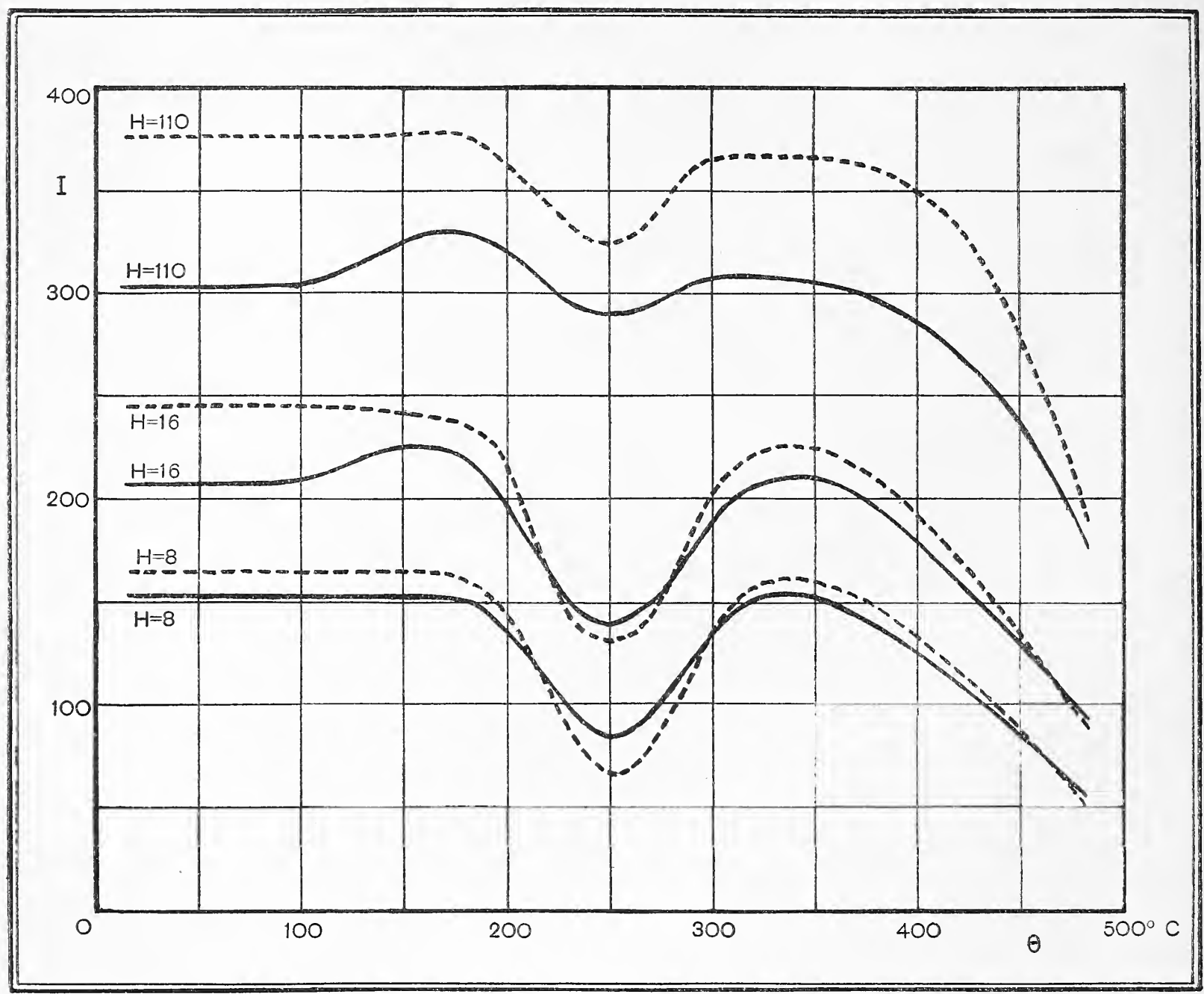


FIG. 4.

similar tests made with the baked alloy cooled to -190° C. The curves show the improvement of the material for high fields with baking at temperatures in the neighbourhood of 170° C. They also exhibit a marked deterioration in the magnetic quality after baking of the specimen about 250° C., and a subsequent recovery prior to the rapid deterioration after 330° C. This dip in the curves is very remarkable, and points to some extensive change in constitution of the material. It is noteworthy, too,

that the growth of hysteresis is specially rapid in the case of specimens exposed to temperatures between 200° and 250° C., and the maximum limit attained is greater than in specimens which have been subjected to prolonged heating at about 320° C. Indeed, a specimen which has been baked for some time at say 250° C., and thus gives a hysteresis cycle of large area, can have its hysteresis loss much reduced by a subsequent short heating at, say, 330° C. The variation in the liquid air effect shown in fig. 4 is also interesting. With the material in the condition as cast, cooling to -190° C. improves the quality for all fields. In the case of specimens which have been baked at temperatures in the neighbourhood of 250° C. the effect is reversed in sign for fields below 30 C.G.S. units, and it is much reduced in magnitude for high fields. The quality of specimens baked at 330° C. is much the same as in the unbaked condition, both as regards tests at room temperature and at that of liquid air. In specimens exposed to still higher temperatures the susceptibility once more falls off, and this is again accompanied by a lessening and final reversal of the liquid air effect.

Critical Temperatures.—In a paper* read before this Society by Mr J. G. Gray, of Glasgow University, results are given for the change in susceptibility produced by heating and cooling an alloy of similar composition to B while a constant applied field is maintained. An interesting contrast is also made with the results obtained by Hopkinson† for nickel-steels. Similar tests have been carried out with alloys D and E, and the results are set forth in fig. 5. The ordinates are the intensities of magnetisation, while the abscissæ are the temperatures of the electric furnace within the magnetising solenoid as registered by a platinum, platinum-iridium pyrometer. The continuous lines indicate the values of I corresponding to an applied field of 125 units, and the dotted lines those for $H=30$, while the arrow-heads distinguish the heating from the cooling curves. It will be observed that in the case of alloy D (shown in the upper diagram) the susceptibility falls off continuously with increasing temperature, as in the results given by Gray for alloy B. The critical temperature is, however, much lower, being approximately 280° C. instead of 500° C., and the deterioration of the specimen due to the temperature cycle is more marked. In the case of alloy E there is a pronounced increase in susceptibility with increasing temperature to nearly 200° C., and then a rapid falling off as the critical temperature (345° C.) is approached. A similar maximum of susceptibility

* "Experiments with Heusler's Magnetic Alloy," *Proc. Roy. Soc. Edin.*, vol. xxviii., part 5, p. 403.

† *Proc. Roy. Soc.*, December 12, 1889; January 23, 1890; May 1, 1890.

is attained during the process of cooling; but the resultant effect of the complete temperature cycle is to produce a great deterioration in the magnetic quality of the material.

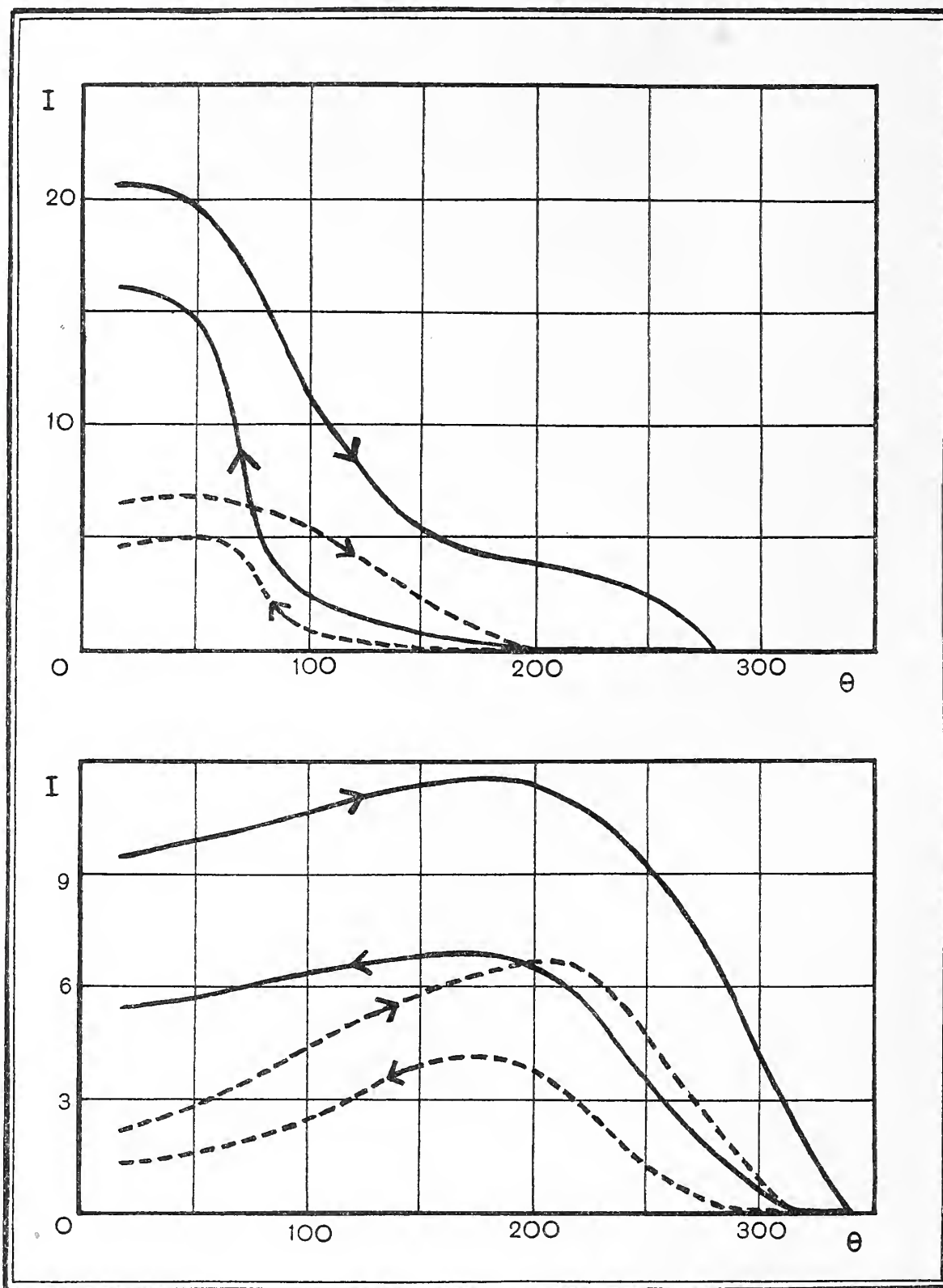


FIG. 5.

A series of such successive heatings and coolings carried out with each of the alloys D and E resulted in a gradual, but lessening, deterioration in quality. It was not found possible to restore the quality to such a marked degree as effected by Mr B. V. Hill in the case of an alloy having

the composition copper 60 per cent., manganese 25 per cent., aluminium 15 per cent.*

Simple Bronzes, Manganese, and Aluminium.—With a view to throwing some light on the magnetic properties of the Heusler alloy, tests were carried out on the alloys (4)–(8) already mentioned, and on pure manganese and aluminium. These materials were magnetised with a field of 4400 C.G.S. units, and their permanent magnetism was subsequently measured. Tests were also made with the specimens surrounded with liquid air and thus cooled to -190° C., and the whole procedure was thereafter repeated with the material quenched at 500° – 600° C. In the accompanying table the results are given and compared. The results obtained with similar specimens were

No.	Percentage Composition of Material.			Intensity of Magnetisation.			
				Cast Material.		Quenched Material.	
				Magnetised at		Magnetised at	
				15° C.	– 190° C.	15° C.	– 190° C.
4	95	...	5	·010	·091	·029	·110
5	90	...	10	·184	·228	·145	·175
6	40	...	60	?	?	?	?
7	30	...	70	?	?	·021	·027
8	70	30	...	·057	·072	·096	·118
9	100	?	?	?	?
10	...	100	...	4·76	4·68	3·74	3·70

? indicates that the value of I is less than 0·01.

in excellent agreement, and hence the percentage changes occurring between the separate tests on any specimen are thoroughly reliable. At the same time, owing to the uncertainty as to the exact dimensions of the specimens, their polar separation, and their distance from the magnetometer needle, the absolute values tabulated for I are only approximate. They are given as indicating the order of the intensity of magnetisation, and as affording a convenient method of comparing the properties of the different materials. In the "Eighth Report of the Alloys Research Committee" are given the equilibrium curves of the copper-aluminium alloys.† It will there be found that (a) the 5 per cent. aluminium-bronze is a solid solution of aluminium in copper, (b) the 10 per cent. aluminium-bronze is a mixture of Cu_4Al and

* *Physical Review*, vol. xxi. p. 335.

† *Proc. Inst. Mech. Eng.*, 1907, p. 204.

Cu_3Al , (*c*) the 60 per cent. bronze consists of CuAl_2 and a eutectic alloy, (*d*) the 70 per cent. bronze is chiefly eutectic with the addition of a solid solution of copper in aluminium. Accordingly, the two aluminium alloys which show a specially marked improvement in magnetic quality when cooled to liquid air temperature are the two bronzes which contain free copper. The same two (*viz.*, the 5 per cent. and 70 per cent. bronzes) also show an increased retentivity in the quenched state. This again agrees with the results obtained for metallic copper.* The 10 per cent. aluminium-bronze does not contain free copper, and does not improve on quenching. It shows, however, an increased retentivity at liquid air temperature. Accordingly, one or both of the compounds Cu_4Al † and Cu_3Al is magnetic. The 60 per cent. aluminium-bronze, which consists of CuAl_2 and eutectic, is practically non-magnetic. Turning now to the 30 per cent. manganese-bronze, we have another alloy containing free copper. It exhibits the same peculiar properties as the two aluminium-bronzes which had free copper, in so far as its retentivity is greater at liquid air temperature than at 15°C ., and is greater in the quenched than in the cast condition. The manganese which is a constituent of this bronze is, comparatively speaking, strongly magnetic,‡ but it will be seen that the effects of liquid air and quenching are the reverse of those found in the manganese-bronze. In a paper shortly to be laid before this Society by Mr J. G. Gray and one of us, it will be shown that manganese-steel does not give an increased residual magnetism after quenching, as does manganese-bronze, and further that when in the normal condition its quality degenerates slightly when the material is cooled to -190°C . Manganese-steel is therefore somewhat similar to pure manganese in its magnetic properties, and gives effects altogether unlike those found in the purest copper. Accordingly, as manganese-bronze resembles copper in its magnetic properties and is essentially different from manganese and manganese-steel, it seems natural

* J. G. Gray and A. D. Ross, "Über die Herstellung permanenter Magnete aus Proben nahezu reinen Kupfers," *Phys. Zeit.*, x., No. 2, S. 59.

† The existence of the compound Cu_4Al mentioned by Carpenter and Edwards in the report, *loc. cit.*, has been questioned by other experimenters. See in this connection the investigations of Gulliver, *Proc. Inst. Mech. Eng.*, 1907, p. 345; Curry, *Journ. Phys. Chem.*, 1907, p. 425; Gwyer, *Zeit. Anorg. Chem.*, 57, S. 125, 1908; Guillet, *Revue de métallurgie*, 5, p. 413. The occurrence of Cu_3Al is generally accepted, and the existence or non-existence of a compound Cu_4Al does not materially affect the present discussion.

‡ The manganese used in this test was a specially purified sample supplied by E. de Haën of Seelze. That its magnetism was not due to iron impurity was shown by chemical analysis. The amount of iron present was probably less than one part in 10,000, and certainly did not exceed one part in 5000. Taking the upper limit, it follows that the iron present was less than $\frac{1}{20}$ of that which would be required to account for the residual magnetism.

that its magnetism may be due in great measure to the free copper contained, and not to a manganese-copper constituent. Lastly, the tests given in the table indicate that pure aluminium (such as had been used in making the aluminium-bronzes and the Heusler alloys) is not susceptible of retaining any measurable trace of permanent magnetism. The result of the tests on the materials (4)-(10) is therefore to show that specimens containing free copper undergo a marked improvement in magnetic quality on cooling from 15° C. to -190° C., and that quenching is likewise accompanied by an increase of residual magnetism—except, of course, in so far as these effects may be masked to a greater or less extent by those due to the presence of other magnetic materials.

The Magnetism of the Heusler Alloy.—These results have an important bearing on the magnetism of the Heusler alloy. A characteristic feature of these bronzes is that their retentivity at the temperature of liquid air is greater than at room temperature. This holds even in the case of baked specimens of alloy B or D, where the cooling diminishes the saturation value of I. Such an effect is of the same kind as that given by free copper. It is also similar to that found in the 10 per cent. aluminium-bronze, but its great magnitude in such cases as the quenched castings D is more suggestive of the free copper than of the copper-aluminium compounds. Hitherto the magnetism of the Heusler alloy has been often ascribed to the manganese alone, whose transformation temperature it has been conjectured—though neither proof nor evidence can be adduced in support—is lowered by its solution in the other constituents. The one point always referred to in this connection is the list of several elements arranged by S. Meyer* in descending order of atomic susceptibility (that quantity being defined as the mean susceptibility of a space containing one gram atom of the substance per litre). The series is as follows:—Ho, Er, Gd, Mn, Fe, Sa, Co, Yt, Nd, Ni, Pr. This list is of course based to a very large extent on the magnetic properties exhibited by salts of the elements. Manganese therefore comes before iron. For although in general the magnetism of pure iron is incomparably greater than that of manganese, yet the susceptibility of many manganous salts exceeds that of the corresponding ferrous and ferric compounds.† The experiments described in the present paper have, however, a more immediate bearing on the subject. The authors do not claim that the investigations afford an explanation of the magnetic phenomena of the Heusler alloy, but they consider that the magnetic properties of the alloy show a suggestive similarity to those of Cu_3Al , and that the presence

* *Ann. d. Phys.*, 1889, lxi., S. 236.

† *Ber. d. Deutsch. Chem. Gesell.*, 1900, xxxiii. S. 448.

of free copper in the alloy may at least play a not unimportant part in determining the magnetic behaviour of the bronze. That manganese is also an important factor is unquestionable, as its omission destroys the remarkable properties of the bronze just as the exclusion of the copper would do. It should be noticed, in support of the view here adduced, that the liquid air effect in the Heusler alloy increases in magnitude with increasing copper content, and hence probably with increasing amount of free copper.

The investigations described in this paper have been carried out in the Physical Laboratory of the University of Glasgow, and the authors desire to express their thanks to Professor Gray for the facilities he gave for the carrying out of the work.

SUMMARY.

1. Manganese-aluminium bronzes exhibit less hysteresis, but are also less magnetic, after quenching.

2. Such alloys, if rich in copper, are more magnetic at -190° C. than at 15° C.

3. Continued baking at steady temperatures reduces the susceptibility and increases the hysteresis towards definite limits.

4. These effects are not simply related to the temperature of baking.

5. The liquid air effect in baked specimens is often (especially for low fields) of opposite sign to that in the virgin alloy.

6. In Heusler alloys containing 76, 62, and 55 per cent. copper respectively, and the remainder manganese and aluminium in atomic proportions, the critical temperatures are 280° , 500° , and 345° C.

7. The quenching and liquid air effects in the Heusler alloys are very similar to those in Cu_3Al modified by effects due to the presence of free copper and free manganese.

(Issued separately May 3, 1909.)

XV.—Low Temperature Experiments in Magnetism. By James G. Gray, B.Sc., Lecturer on Physics in the University of Glasgow, and Hugh Higgins, M.A., Thomson Experimental Scholar in the University of Glasgow. *Communicated by Professor A. GRAY, F.R.S.*

(MS. received February 15, 1908. Read same date.)

THE alterations brought about in the magnetic moments of magnets, composed of various metals and alloys, by alternate cooling and warming between ordinary room temperature and that of liquid air have been very fully investigated by Dewar and Fleming.* It was found by these experimenters that in the case of most of the metals examined the effect of the first cooling upon the magnet—which had previously been magnetised to saturation in the field of an electromagnet—was to bring about a very considerable reduction in its magnetic moment. On allowing the magnet to warm to room temperature its magnetic moment still further diminished. On cooling once more to the liquid air temperature the magnetic moment increased, and from and after this stage it was found that the magnetic moment of the magnet when cold exceeded that of the magnet when at room temperature by a definite amount. The changes brought about by the treatment were found to be much greater in the annealed than in the quenched condition of the material.

In a few cases, notably that of the chromium steels, the effect of the first cooling was to increase the magnetic moment; in other cases the treatment caused very little alteration in the magnetic moments of the specimens.

In the experiments of Dewar and Fleming the specimens were magnetised initially at room temperature. It occurred to one of the authors of the present paper to repeat the procedure, starting with a specimen magnetised to saturation when at the temperature of liquid air. On carrying out such an experiment it was found that the changes produced by the treatment differed in many respects from, and were even more remarkable than, those observed by the aforementioned investigators.

The method of experimenting was as follows:—A short bar of the metal to be examined was placed within a glass tube bent up at one end to admit liquid air. The end of the tube through which the specimen entered was closed by means of a cork, and the tube thoroughly wrapped up in cotton-

* *Proc. Roy. Soc.*, vol. 60, p. 57.

wool. The tube, containing the specimen, was then placed between the conical pole-pieces of a very powerful electromagnet. On exciting this magnet the specimen was exposed to a magnetising field of 4400 C.G.S. units. The tube was next withdrawn and mounted on a special stand so that the specimen lay at a considerable distance from a magnetometer needle, and with its axis lying along the magnetic east and west line passing through the needle. By means of the magnetometer the magnetic moment of the specimen was measured under the following conditions: (1) at room temperature immediately after being magnetised, (2) after cooling the specimen to -190° C., (3) after the specimen had regained room temperature, (4) on re-cooling to -190° C., and so on. This is the procedure of Dewar and Fleming.

The tube was now withdrawn and placed in position between the poles of the electromagnet and the specimen brought to the temperature of liquid air. The magnet was then excited and the tube withdrawn and placed once more in position on the magnetometer. The magnetic moment of the magnet was observed (1) at -190° C. following on the magnetisation of the specimen, (2) after the specimen had warmed to room temperature, (3) on cooling once more to -190° C., and so on.

The results obtained differ from those observed by Dewar and Fleming in that the reversible condition described above is arrived at after the first warming up of the specimen from -190° C. to room temperature, and the percentage change in magnetic moment then brought about by cooling from room temperature to that of liquid air is much greater than is the case for a specimen which has been magnetised initially at room temperature.

The following table gives the dimensions and chemical compositions of the specimens employed in the experiments:—

DIMENSIONS AND CHEMICAL COMPOSITION OF SPECIMENS EMPLOYED.

Description of Material.	Length of Specimen in Cms.	Diameter of Specimen in Cms.	Mass in Grms.	Percentage Composition.				
				Carbon.	Manganese.	Silicon.	Sulphur.	Phosphorus.
Steel wire . . .	7.7	0.27	3.36	0.755	0.660	0.066	0.017	0.027
Special hard steel . .	7.2	0.96	40.0	1.321	0.339	0.143	0.023	2.745
Manganese steel . .	7.0	0.96	39.0	0.71	6.00
„ „ . .	7.0	0.96	39.0	0.60	12.00

RESULTS FOR STEEL WIRE SPECIMEN.

Specimen as Received.—The results obtained on testing the specimen in this condition are exhibited in fig. 1. After being magnetised at room temperature the effect of the first cooling was to bring about a diminution in its magnetic moment of 45 per cent. On warming it to room temperature the magnetic moment still further diminished by about 11 per cent. On cooling the specimen once more the magnetic moment increased by about 9 per cent., after which stage the effect of the cooling always resulted in an increase in the magnetic moment of about this amount.

In the figures the firm lines give the magnetic moments of the speci-

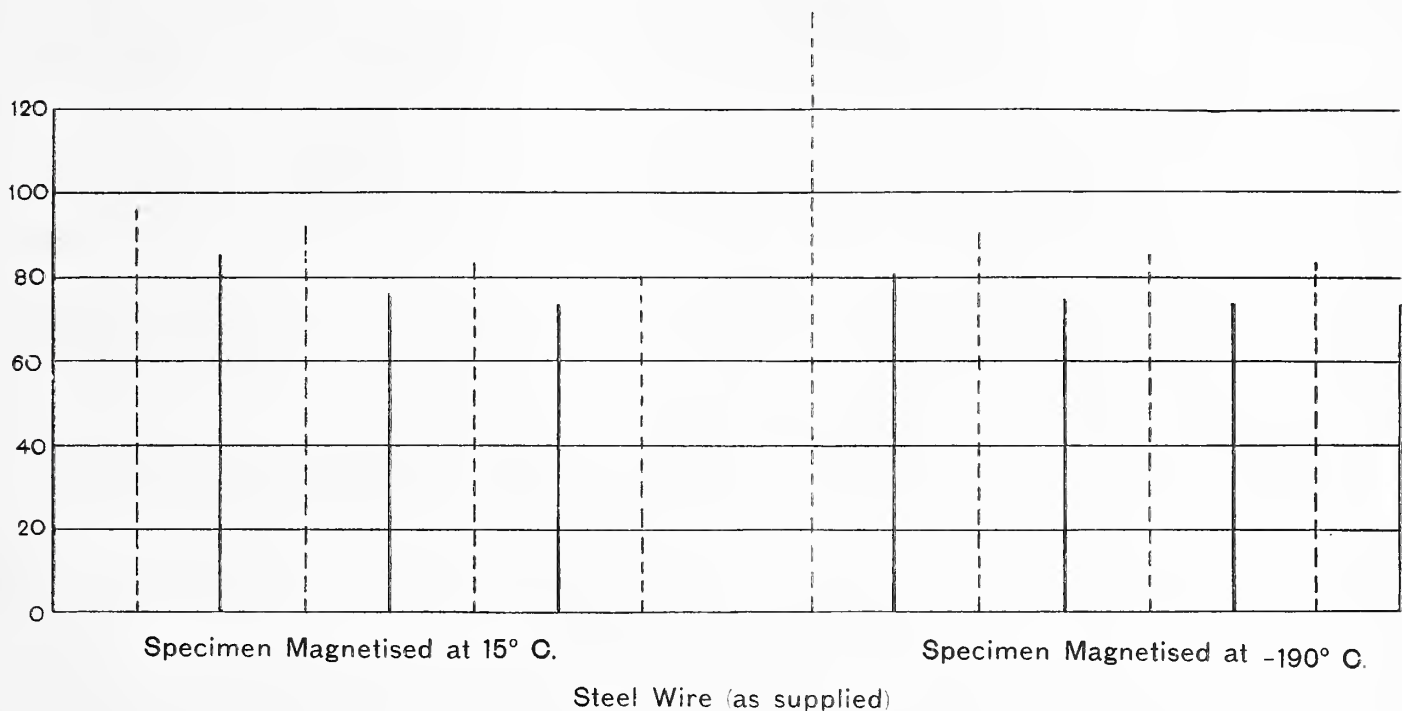


FIG. 1.

mens at room temperatures, and the dotted lines the magnetic moments at -190° C.

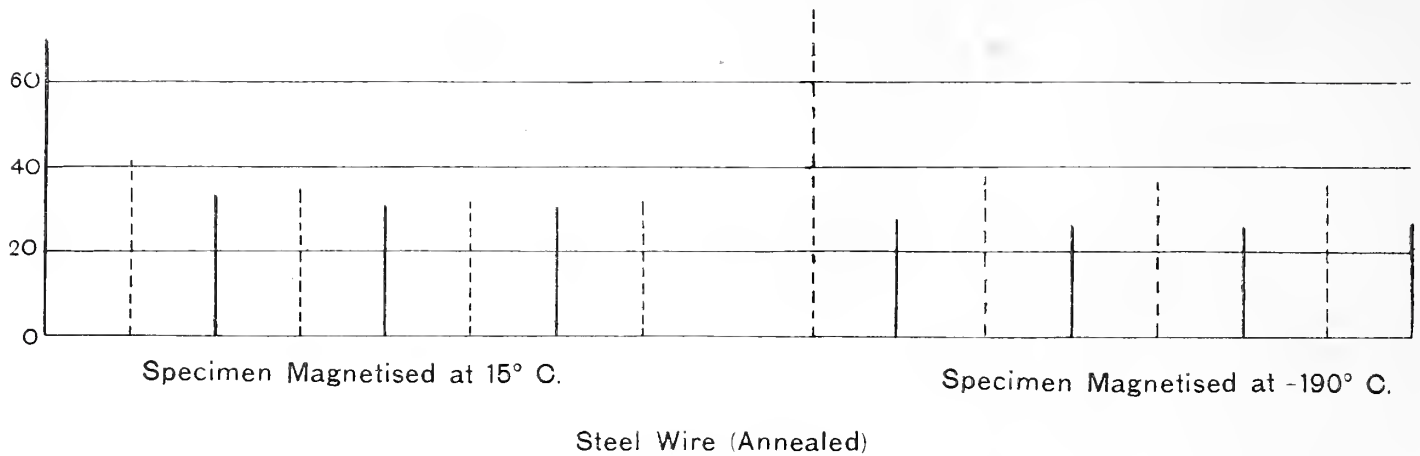
Starting with the specimen magnetised when at -190° C., the effect of the first warming was to bring about a diminution in the magnetic moment of about 44 per cent. After this stage had been reached, cooling to -190° C. caused an increase in the magnetic moment of nearly 12.5 per cent.

Specimen in Annealed Condition.—(1) *Magnetised at room temperature.*—The effect of the first cooling was to reduce the magnetic moment by about 40 per cent. On warming to room temperature the magnetic moment further diminished by 20 per cent., after which stage the effect of cooling the specimen was to increase the moment by about 5 per cent.

(2) *Specimen magnetised at -190° C.*—The effect of the first warming was to diminish the magnetic moment by about 65 per cent., after which

each cooling resulted in an increase in the magnetic moment of nearly 40 per cent.

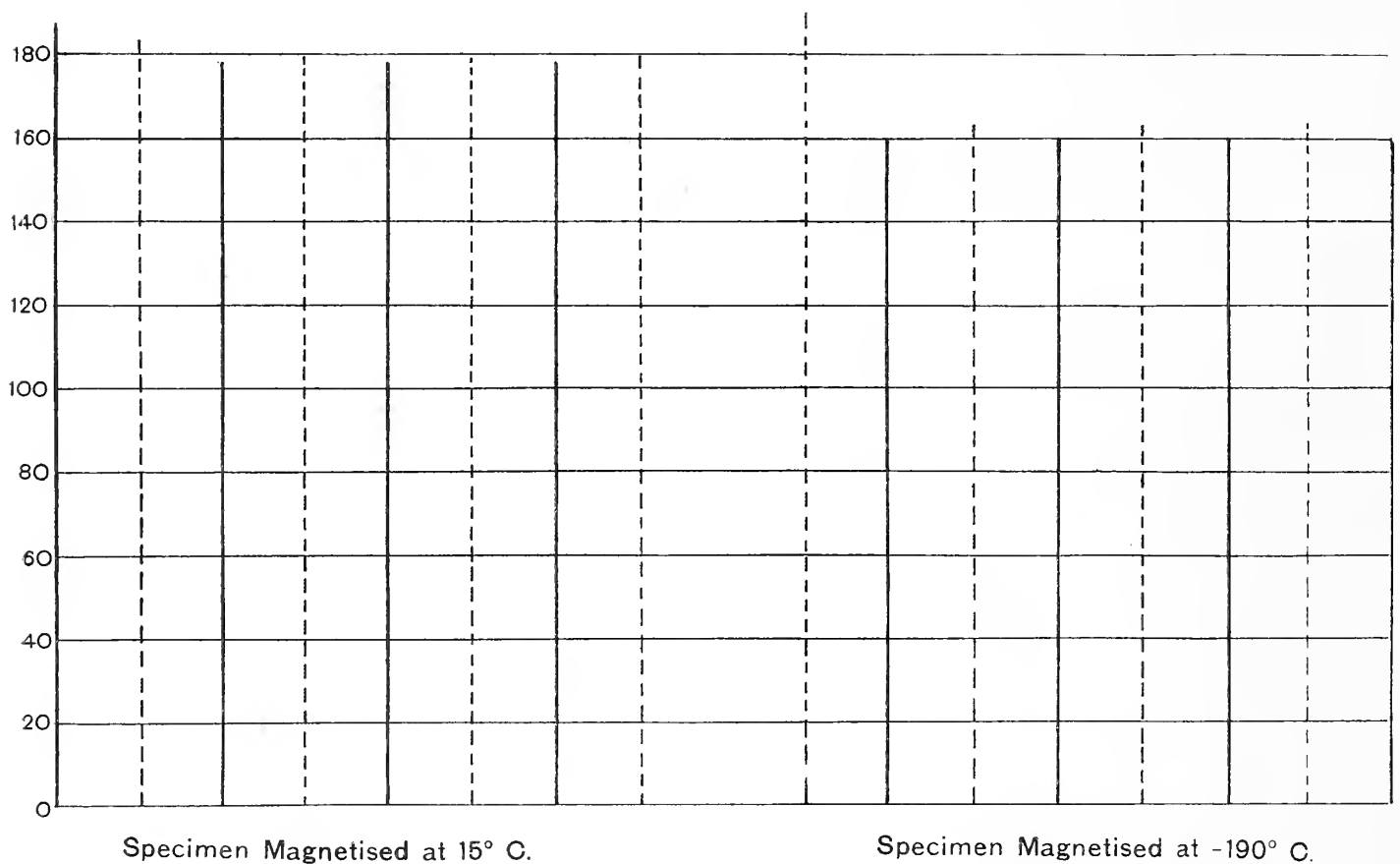
The results obtained for the material in the annealed condition are shown in fig. 2.



Steel Wire (Annealed)

FIG. 2.

Specimen in Quenched Condition.—(1) *Specimen magnetised at room temperature.*—The effect of the first cooling was to diminish the magnetic moment by about 2·5 per cent., and on warming to room temperature a



Steel Wire (Quenched)

FIG. 3.

further reduction of 3·5 per cent. took place. Further cooling always resulted in an increase in the magnetic moment of nearly 1 per cent.

(2) *Specimen magnetised at -190° C.*—The effect of allowing the speci-

men to warm to room temperature was to diminish its magnetic moment by 16 per cent. The magnet was now in a condition in which its magnetic moment at -190° C. exceeded its magnetic moment at room temperature by about 1.5 per cent.

The results for the steel wire in the quenched condition are exhibited in fig. 3.

RESULTS FOR SPECIAL HARD STEEL.

The results obtained on testing a specimen of this variety of steel are shown in figs. 4, 5, and 6. They resemble the results obtained for the steel

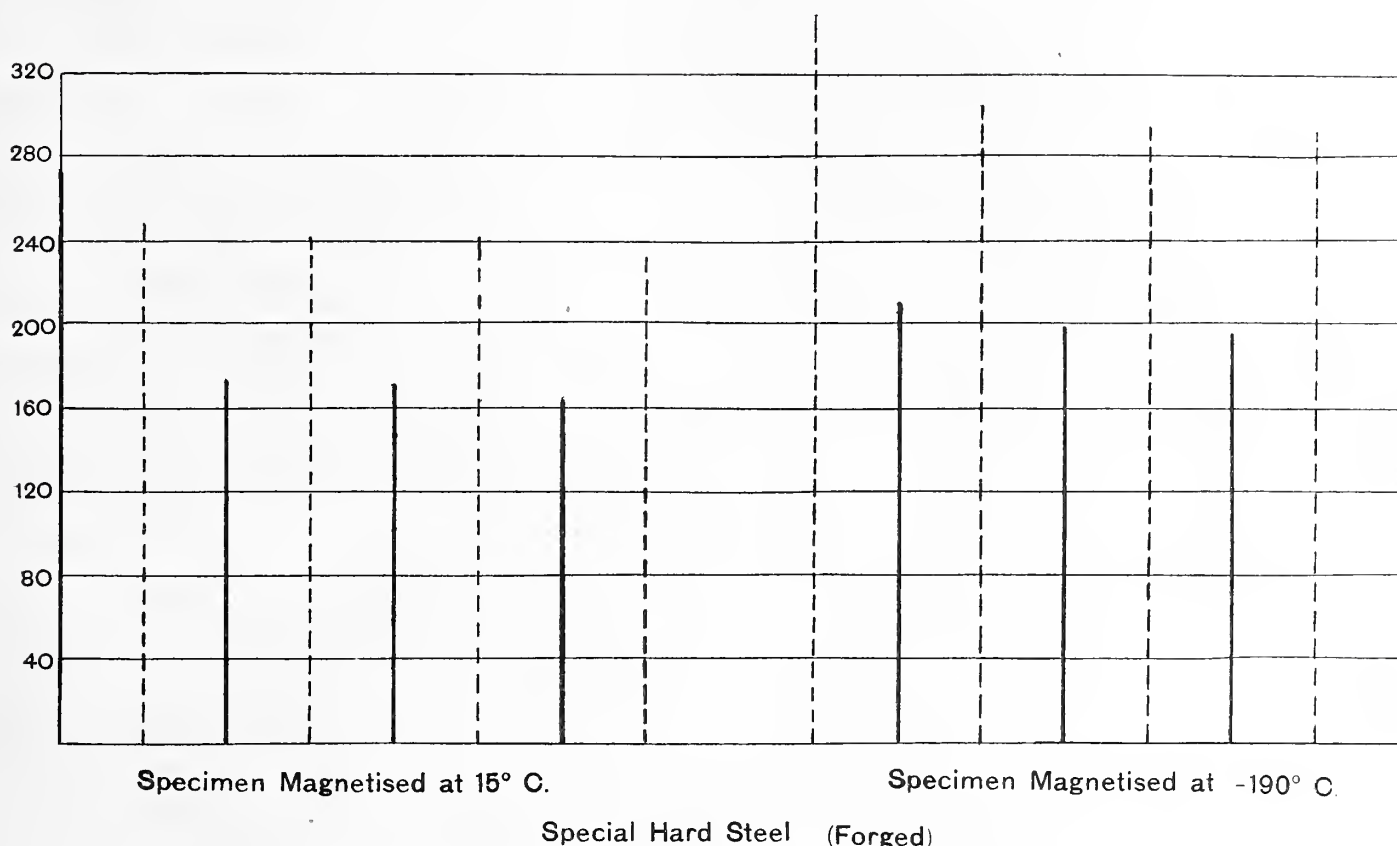


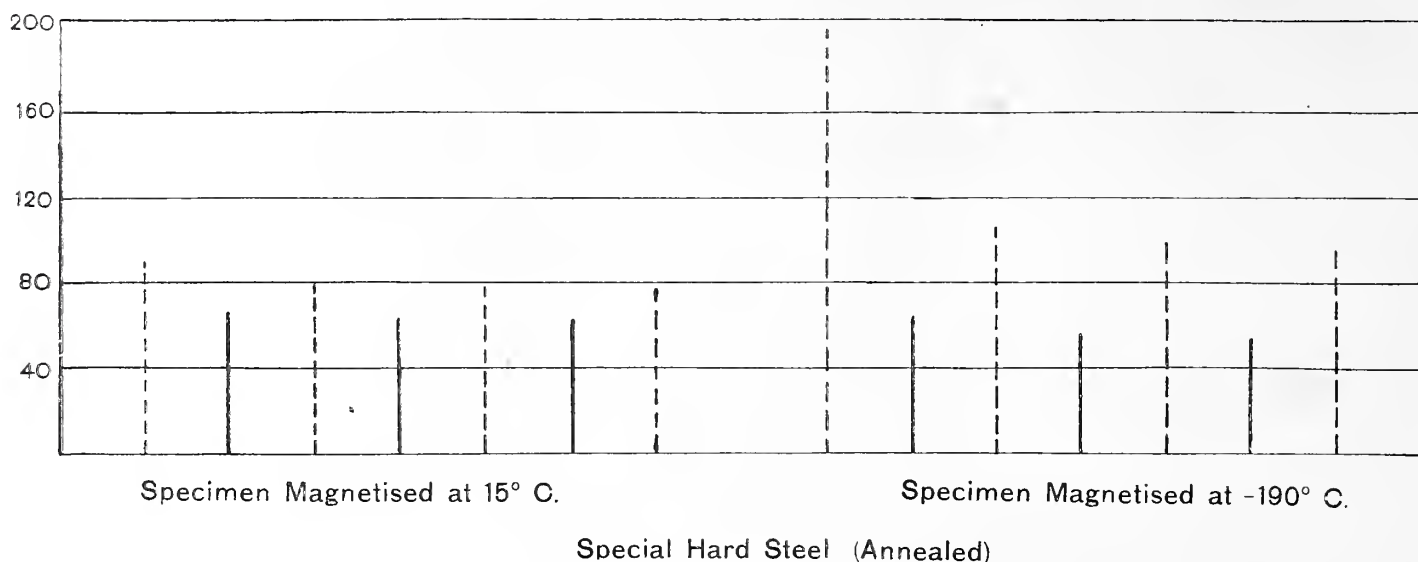
FIG. 4.

wire. It will be seen that after the reversible state has been arrived at the magnitudes of the changes brought about by the cooling are much more marked in the case of the specimen when magnetised at -190° C. than when magnetised at room temperature.

RESULTS FOR MANGANESE STEEL (6 PER CENT. MN).

The results obtained for this steel in the annealed condition are shown in fig. 7, and are very remarkable. On magnetising the specimen at the temperature of the room its magnetic moment was found to be 250 C.G.S. units. On cooling the specimen to -190° C. its magnetic moment diminished to 93 C.G.S. units; and on allowing it to warm to room temperature the magnetic moment fell to 59 C.G.S. units. From and

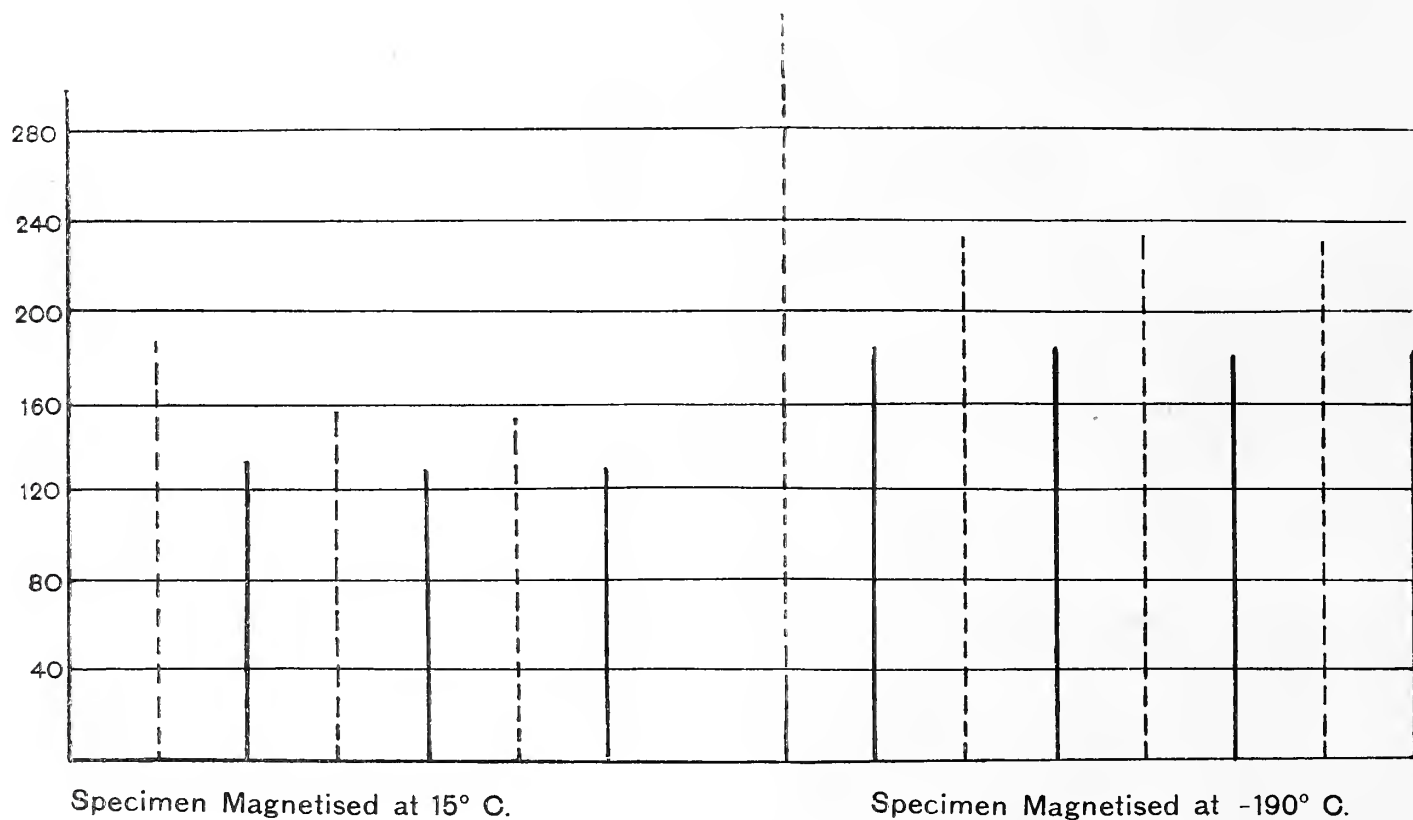
after this time the effect of cooling the specimen to -190° C. was to bring about an increase in the magnetic moment of about 9 per cent.



Special Hard Steel (Annealed)

FIG. 5.

Starting with the specimen magnetised when at the temperature of liquid air, the initial magnetic moment was 590 C.G.S. units. On warming



Special Hard Steel (Quenched)

FIG. 6.

to room temperature, the magnetic moment fell to 387 C.G.S. units; and the effect of cooling once more to -190° C. was to bring about an increase in the magnetic moment of about 13 per cent.

The specimen was now remagnetised at room temperature, and the

procedure repeated. It will be seen that the results obtained differ very greatly from those yielded by the freshly annealed specimen. The initial magnetic moment was now 557 C.G.S. units. On cooling the specimen to -190° C. its magnetic moment diminished to 416 C.G.S. units. Warming to room temperature resulted in the magnetic moment still further diminishing to 406 C.G.S. units, and after this stage any subsequent cooling resulted in a small increase taking place in the magnetic moment.

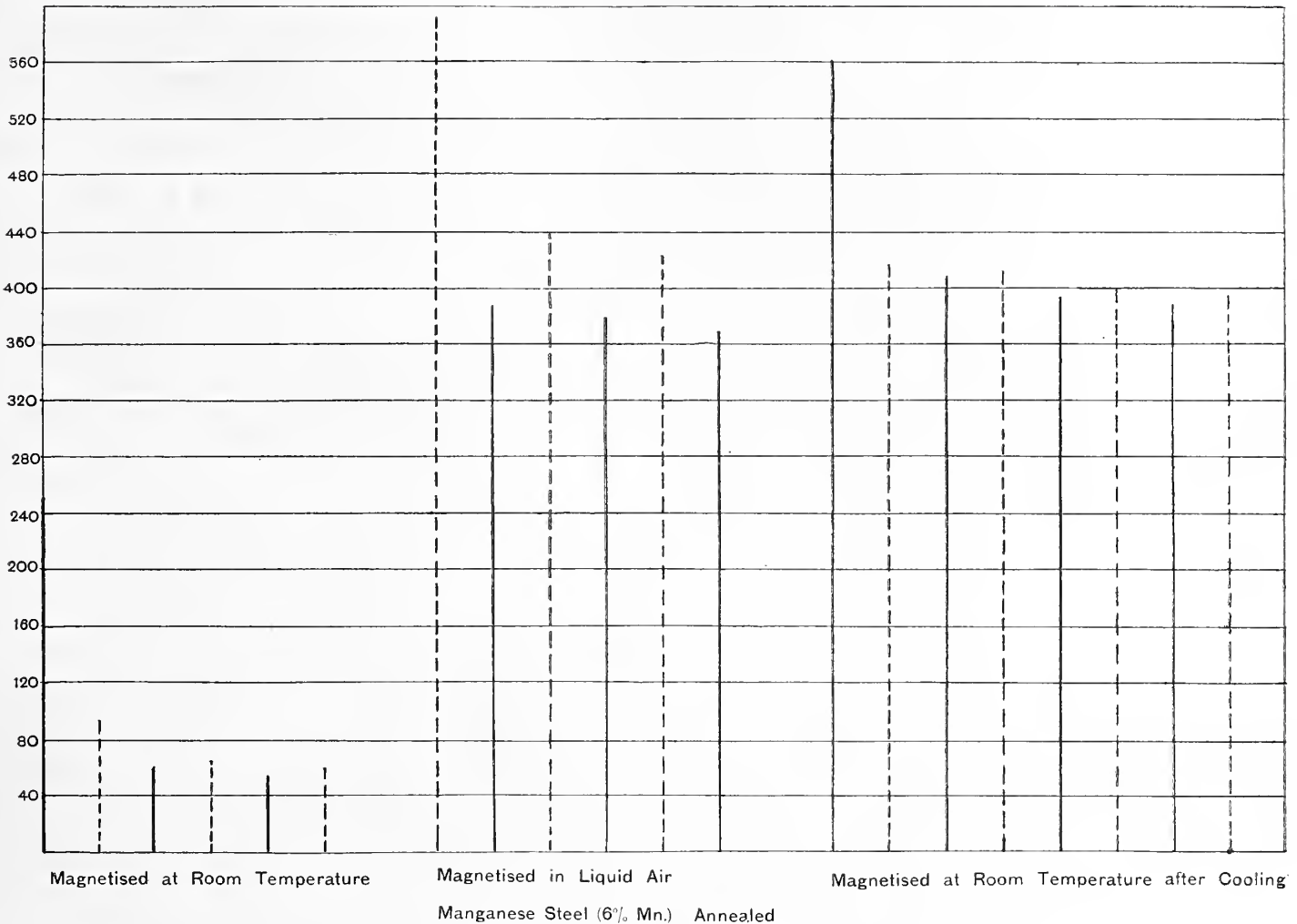


FIG. 7.

Quenching produces very little change in the properties of this steel, and for this reason the results obtained with the specimen after quenching are not included.

The properties of this steel for low and moderate fields have been investigated by A. D. Ross and one of the Authors. The results obtained will be communicated shortly to this Society. In the annealed condition the material is, comparatively speaking, feebly magnetic. Cooling to the temperature of liquid air results in the permeability being greatly increased, and on warming to room temperature a further large improvement in magnetic quality is brought about.

RESULTS FOR MANGANESE STEEL (12 PER CENT. MN).

On placing a specimen of this variety of steel in the field of the electromagnet a weak magnet was obtained. No appreciable change in its magnetic moment or in the magnetic quality of the material was brought about by the cooling in liquid air.

The experiments described in the present paper were carried out in the Natural Philosophy Institute of the University of Glasgow, and the authors desire to express their thanks to Professor Gray for the facilities granted them for the carrying out of the work.

(Issued separately May 11, 1909.)

XVI.—On the Discharge of Water from Circular Weirs and Orifices.

By G. H. Gulliver, B.Sc., A.M.I.Mech.E., Lecturer in Engineering in the University of Edinburgh.

(MS. received October 9, 1908. Read November 2, 1908.)

MANY investigators have measured the flow of water through orifices, and over weirs of various shapes and sizes. The discharge through a vertical circular orifice, situated at distances below the water surface much greater, relatively, than the diameter of the aperture, has been made the subject of study on several occasions, but the case of a circular weir, or notch—that is, a circular hole discharging across only a portion of its area—has been neglected hitherto. The author has had occasion recently to investigate the discharge of such a circular weir, more especially with regard to its suitability for measuring the flow of streams, a purpose for which weirs of rectangular shape are used almost exclusively at present. The problem forms an interesting example of the utility of graphical methods in cases for which solutions would be unobtainable by purely mathematical means.

1. Circular Weir, or orifice partly drowned.

Fig. 1 represents an orifice of radius R . Let AB be the water surface, which is situated at a height H from the lowest point of the orifice. Let CD be an elementary strip of vertical thickness dh , and at a mean depth h below the water surface.

The area of the strip, CD , is $2CF.dh$.

But

$$(CF)^2 + (OF)^2 = (OC)^2 = R^2,$$

and

$$OF = OK + KF = (R - H) + h;$$

therefore

$$\begin{aligned} (CF)^2 &= R^2 - (R - H + h)^2 \\ &= 2R(H - h) - (H - h)^2 \\ &= (H - h)\{D - (H - h)\}, \end{aligned}$$

where D , the diameter of the orifice, is written instead of $2R$.

The area of the elementary strip, CD , is

$$2 \sqrt{(H - h)\{D - (H - h)\}}.dh.$$

The discharge of water across this area, neglecting losses due to friction and contraction of the jet, is

$$2 \sqrt{(H - h)\{D - (H - h)\}}.dh. \sqrt{2gh} \text{ cubic feet per second.}$$

The total discharge of water from the orifice is

$$2 \sqrt{2g} \int_0^H \sqrt{h(H-h)\{D-(H-h)\}} \cdot dh \text{ cubic feet per second,}$$

$$= 12 \cdot 05 \int_0^H \sqrt{h(H-h)\{D-(H-h)\}} \cdot dh \text{ gallons per minute,}$$

if D , H , and h , in the last expression, are measured in inches; these are the units most commonly used in practice. In fig. 1 the water surface, AB , has

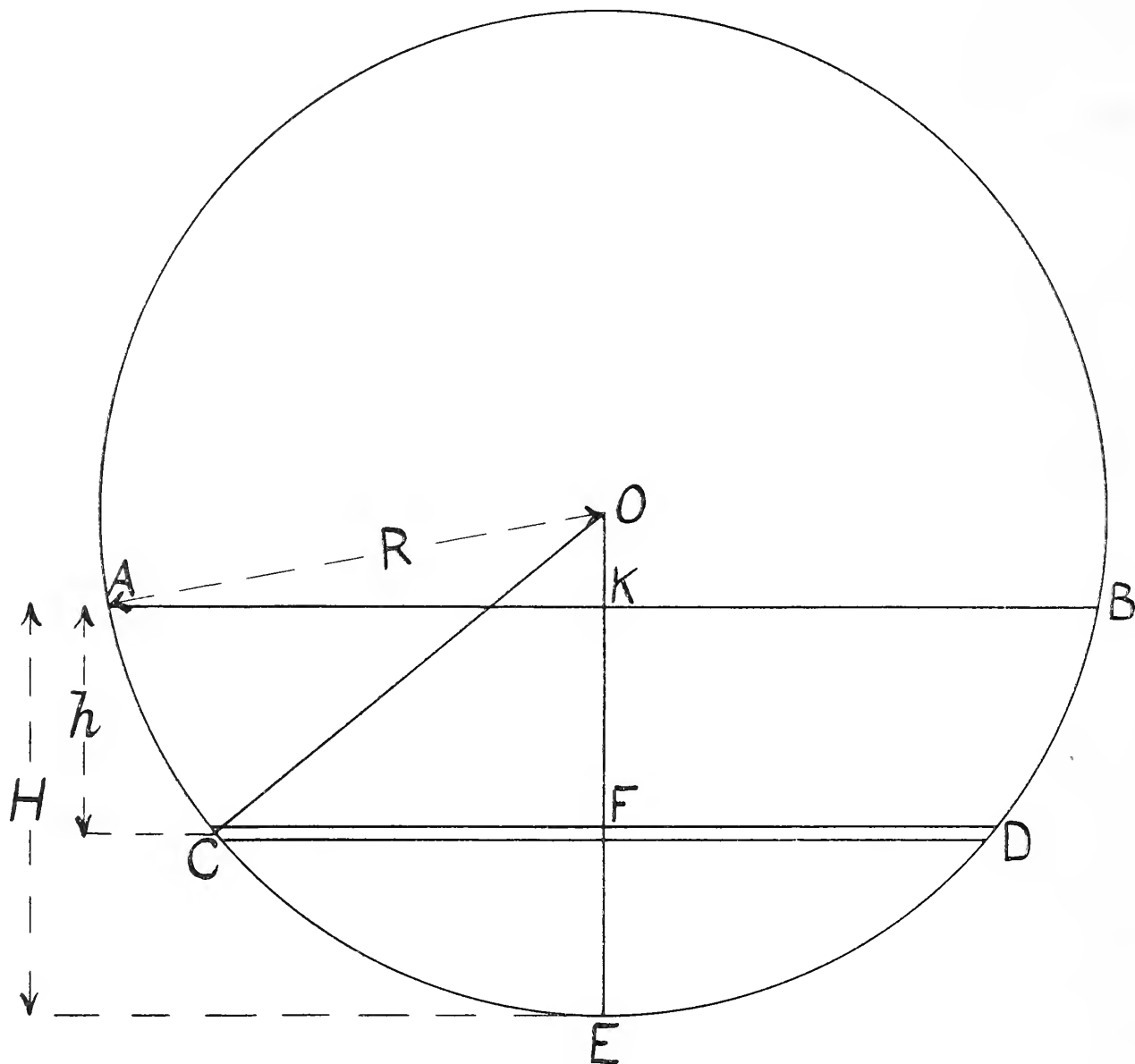


FIG. 1.

been taken between the centre and the bottom of the hole, but the form of the expression for the discharge remains the same whatever the position of this surface.

The integral is elliptic, and does not admit of a direct solution, but the result may be obtained by a graphical method, as follows. A definite value of H , in terms of D , is taken, and numerical solutions of the expression

$$\sqrt{h(H-h)\{D-(H-h)\}}$$

are found by giving to h various values between zero and H , again in terms

of D . Thus the discharge is calculated for a number of layers between the water surface and the bottom of the orifice, each result being in the form of a product of $D^{3/2}$ and a number. The values of these elementary discharges are then plotted as abscissæ, with the corresponding values of h as ordinates, and a curve is obtained which shows the variation in the discharge from the bottom of the aperture to the water surface. The area under this curve, divided by the scale of the drawing, gives the value of the integral for the particular head chosen, in the form of a product of $D^{5/2}$ and a number.

The values of H actually taken were $0.1D$, $0.2D$, and so on by tenths

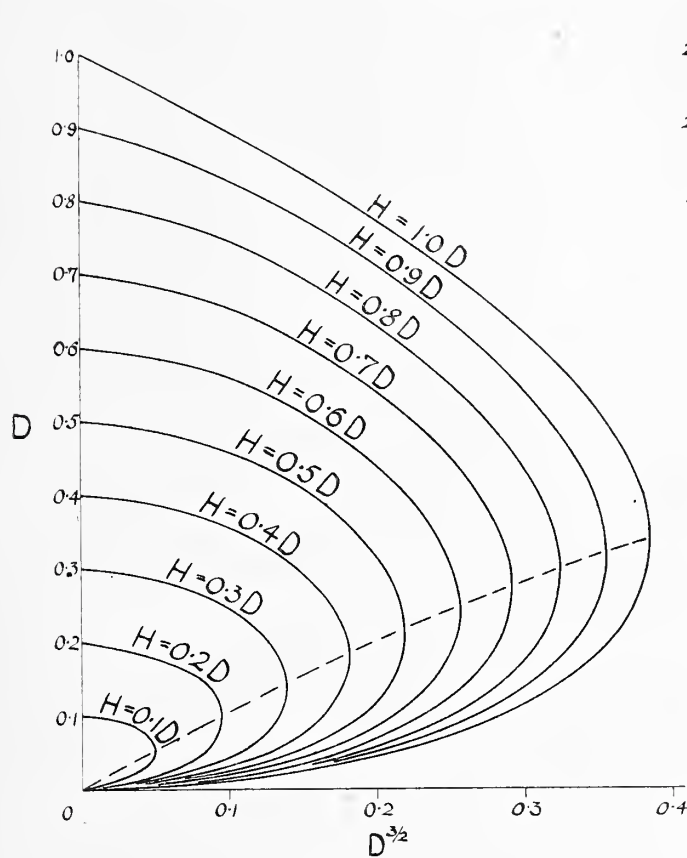


FIG. 2.

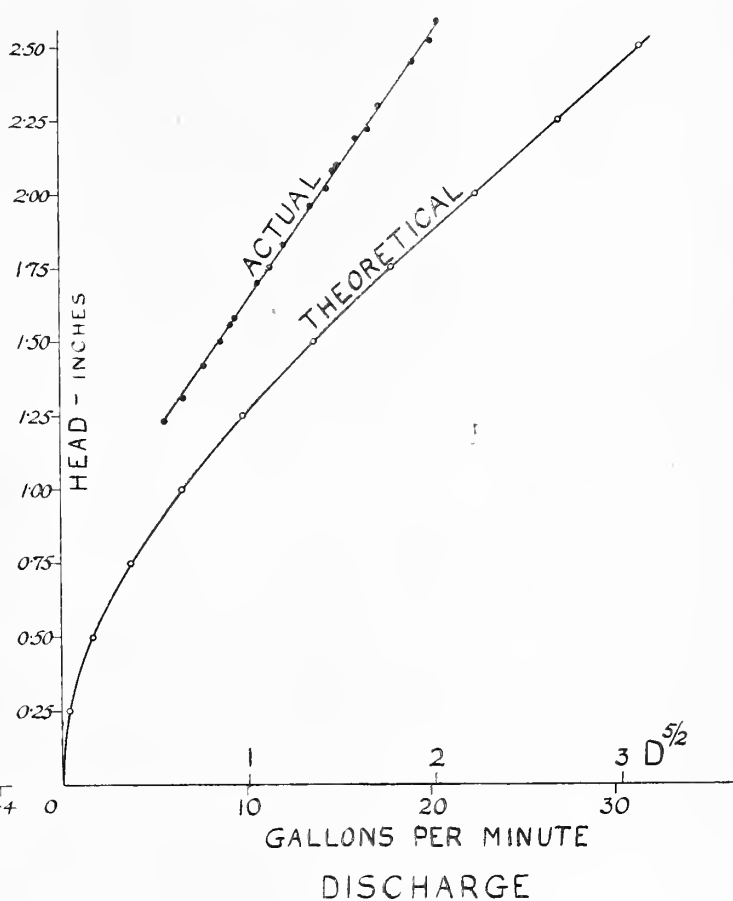


FIG. 3.

of D up to $1.0D$. To obtain the total discharge for each definite head of water, the elementary discharges were calculated for layers at depths of $0.05D$, $0.10D$, etc., from the water surface—that is, at every twentieth part of the diameter between the water surface and the bottom of the orifice. The curves obtained in this manner are shown in fig. 2. The area under each curve gives the total discharge of the weir for the corresponding head of water. For the original drawing the horizontal scale is:—1 inch = $0.05 D^{3/2}$, and the vertical scale is:—1 inch = $0.1D$, so that 1 square inch represents $0.005 D^{5/2}$. By multiplying each area by $0.005 D^{5/2}$, and then by the constant 12.05, the corresponding discharge is obtained, as given in the second column of Table I. The discharge for any size of orifice can be

calculated from these figures; if D is measured in inches the results are in gallons per minute. The figures entered in the third column of Table I are the discharges calculated for an orifice $2\frac{1}{2}$ inches in diameter.

TABLE I.

Head = H .	Theoretical Discharge.		Height of Layer of Maximum Discharge.
	Orifice of any Diameter. Gallons per minute.	Orifice of $2\frac{1}{2}$ inches Diameter. Gallons per minute.	
$0.1D$	$0.0434D^{5/2}$	0.428	$.053D$
0.2	0.1711	1.689	.102
0.3	0.3808	3.758	.145
0.4	0.6652	6.563	.183
0.5	1.004	9.904	.217
0.6	1.398	13.79	.247
0.7	1.831	18.07	.274
0.8	2.290	22.59	.298
0.9	2.747	27.11	.319
1.0	3.169	31.27	.337
1.1	3.519	34.72	.353
1.2	3.820	37.69	.366
1.3	4.109	40.55	.377
1.4	4.362	43.04	.386
1.5	4.615	45.54	.395
1.6	4.832	47.68	.403
1.7	5.061	49.94	.409
1.8	5.278	52.08	.414
1.9	5.471	53.98	.418
2.0	5.675	56.00	.422

If the areas under the curves of fig. 2 are taken as abscissæ, and the corresponding values of the head as ordinates, a curve is obtained which shows the total discharge of the weir for any head; this curve is given in fig. 3. The discharge for the orifice, under a head of water equal to some fraction of the diameter of the hole, can be obtained by multiplying the abscissa of fig. 3, measured at the corresponding head, by $D^{5/2}$, D being taken in inches. The curves, of course, are applicable to any circular orifice, since all circles are similar.

The results obtained until now refer to what is generally known as "theoretical discharge"—that is, the quantity of water which would escape from the orifice if there were no friction, and no contraction of the jet. The actual discharge is considerably less than the theoretical, chiefly on account of the fact that the free jet is smaller than the area through which it has emerged. The ratio of the actual to the theoretical discharge is usually called the "coefficient of discharge."

Some experiments were carried out in order to find the actual quantity of water passing over a circular weir under various heads, and to ascertain the value of the coefficient of discharge. The diameter of the orifice employed was $2\frac{1}{2}$ inches, and the aperture had the usual sharp edges, so as to leave the jet free to contract. The head of water was measured in the ordinary way, by means of a hook gauge. The results of the experiments

TABLE II.

Head. Inches.	Actual Discharge by Experiment. Gallons per minute.	Theoretical Discharge from Curve. Gallons per minute.	Coefficient of Discharge.
1.23	5.66	9.62	.588
1.31	6.69	10.80	.619
1.42	7.80	12.50	.624
1.50	8.73	13.79	.633
1.56	9.23	14.66	.630
1.58	9.49	15.05	.631
1.70	10.68	17.03	.627
1.75	11.40	18.07	.631
1.83	12.15	19.40	.626
1.96	13.59	21.72	.625
2.02	14.51	22.85	.635
2.08	14.82	23.92	.620
2.10	15.02	24.32	.618
2.19	16.06	25.90	.620
2.22	16.75	26.50	.632
2.30	17.32	27.90	.621
2.45	19.17	30.50	.629
2.52	20.21	31.50	.642
2.59	20.53	32.60	.630
2.69	21.39	34.00	.629
2.74	22.11	34.70	.637
2.85	22.72	36.10	.629
2.99	23.68	37.80	.626
3.13	24.78	39.25	.631
3.28	25.81	40.80	.633
3.40	26.59	42.00	.633
3.59	27.63	44.00	.628
3.76	28.66	45.65	.628
3.90	29.63	46.90	.632
4.03	30.35	48.20	.630
4.18	30.87	49.40	.625
4.35	31.91	50.80	.628
4.49	32.74	52.00	.630

are given in Table II., and are plotted in fig. 3. A scale of discharge in gallons per minute for the orifice employed is appended to the horizontal axis of the diagram, and the values of the theoretical discharge, given in the third column of Table II., were obtained by multiplying the lengths of the abscissæ of the theoretical curve by this scale. The fourth column of the table gives the coefficient of discharge—that is, the ratio of actual dis-

charge to theoretical discharge. The numerical value of this coefficient varies somewhat irregularly, but not more so than could be expected with the somewhat imperfect apparatus used in the experiments. For all heads less than D (2.5 inches), excluding the first reading, the average coefficient is 0.626. No measurements could be made for heads below 1.23 inches, as the water began to run down the face of the apparatus, and there was imperfect contraction of the jet. This head is roughly equal to one-half the diameter of the hole, but the effect occurs when the velocity of the jet is insufficient to prevent the adhesion of the water to the metal, and thus depends upon the absolute head, and not upon the ratio of head to diameter; with a larger orifice the experiments could have been carried down to a relatively lower water surface.

The most striking result, from a practical point of view, is that the curve of discharge is very nearly a straight line for values of the head between D and $D/2$. In fig. 3 the line through the points found experimentally has been drawn straight, and it represents, very closely, the results of the observations. The theoretical curve is straight between D and $0.7D$, but is curved between $0.7D$ and $0.5D$, though the departure from the straight line is not great. Thus there would be a considerable simplification in working out results if such a weir were adopted for gauging streams. The corresponding curve of discharge for a rectangular weir is of parabolic form, and only approximates to a straight line with relatively high heads of water. The circular weir has the disadvantage that it cannot be used for shallow streams, unless a deeper pool is excavated behind it. This is not always practicable, though it would be of great use in eliminating, to a considerable extent, the uncertainty in measuring the flow of the stream due to the "velocity of approach" of the water before it reaches the weir.

The equation to the straight line, which represents approximately the discharge for any head between the centre and the top of the orifice, is easily obtained as

$$Q = c(4.46 D^{3/2} \cdot H - 1.29 D^{5/2}),$$

where Q is the actual discharge, in gallons per minute; H is the head of water, measured in inches from the bottom of the orifice; D is the diameter of the orifice, in inches; and c is the coefficient of discharge, which may be taken as 0.625. For any particular size of aperture this simplifies to the form

$$Q = aH - b,$$

where a and b are numerical constants. The equation for the orifice used in the above experiments is

$$Q = 11H - 8 \text{ gallons per minute (very nearly).}$$

2. Circular Orifice, completely drowned.

It seemed profitable to continue the investigations and experiments with the orifice completely drowned. An approximation to the theoretical discharge of a circular orifice is obtained easily, if the area of the hole is multiplied by the velocity corresponding with a head measured from the water surface to the centre of the aperture; or the discharge

$$Q = \sqrt{2gH'} \cdot \pi D^2 / 4 \text{ cubic feet per second}$$

$$= 4.731 D^2 H'^{1/2} \text{ gallons per minute,}$$

if D and H' are measured in inches for the last expression, H' being the

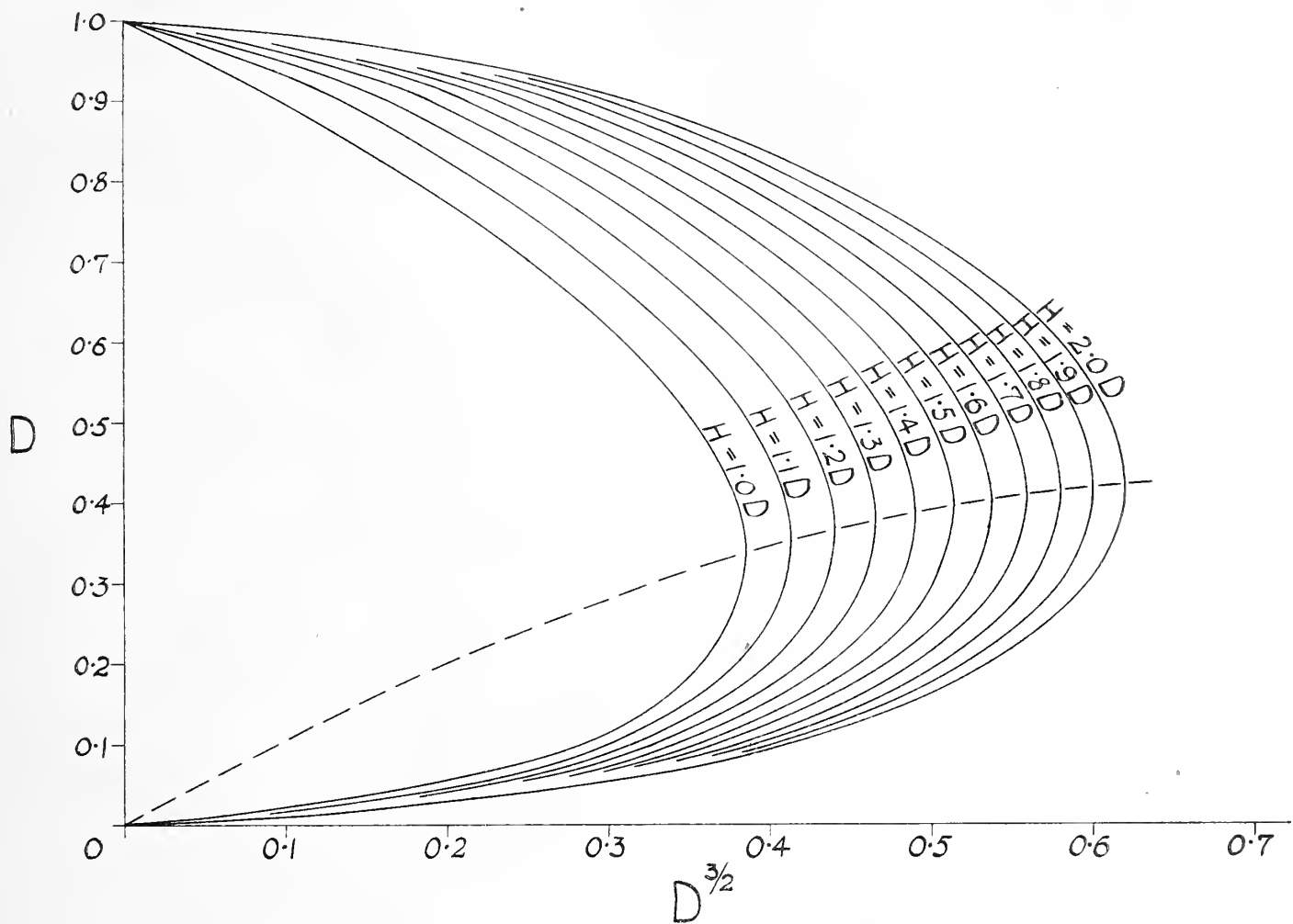


FIG. 4.

distance from the water surface to the centre of the orifice. This result is very nearly accurate when the head is large compared with D , but for relatively low values of H' it becomes less so. The true theoretical discharge was therefore investigated up to a head equal to $3D/2$ above the centre of the orifice—that is, to a height above the top of the hole equal to the diameter.

The expression for the discharge found on page 2 applies here also, and the method of solution adopted was the same. Fig. 4 shows curves similar to those of fig. 2, except that now the value of H begins at D , and is increased by steps of $0.1D$ up to $2D$. The outermost curve of fig. 2 and the

innermost one of fig. 4 are for the same head D . The areas under this second series of curves are given in the lower part of Table I., and have been plotted in fig. 5 as a curve of discharge for the orifice; this curve has been prolonged downwards by adding that of fig. 3. The change in curvature when the orifice is just completely filled is very noticeable. A curve of actual discharges for an aperture $2\frac{1}{2}$ inches in diameter, corresponding with the experimental results given in Table II., is shown also on fig. 5.

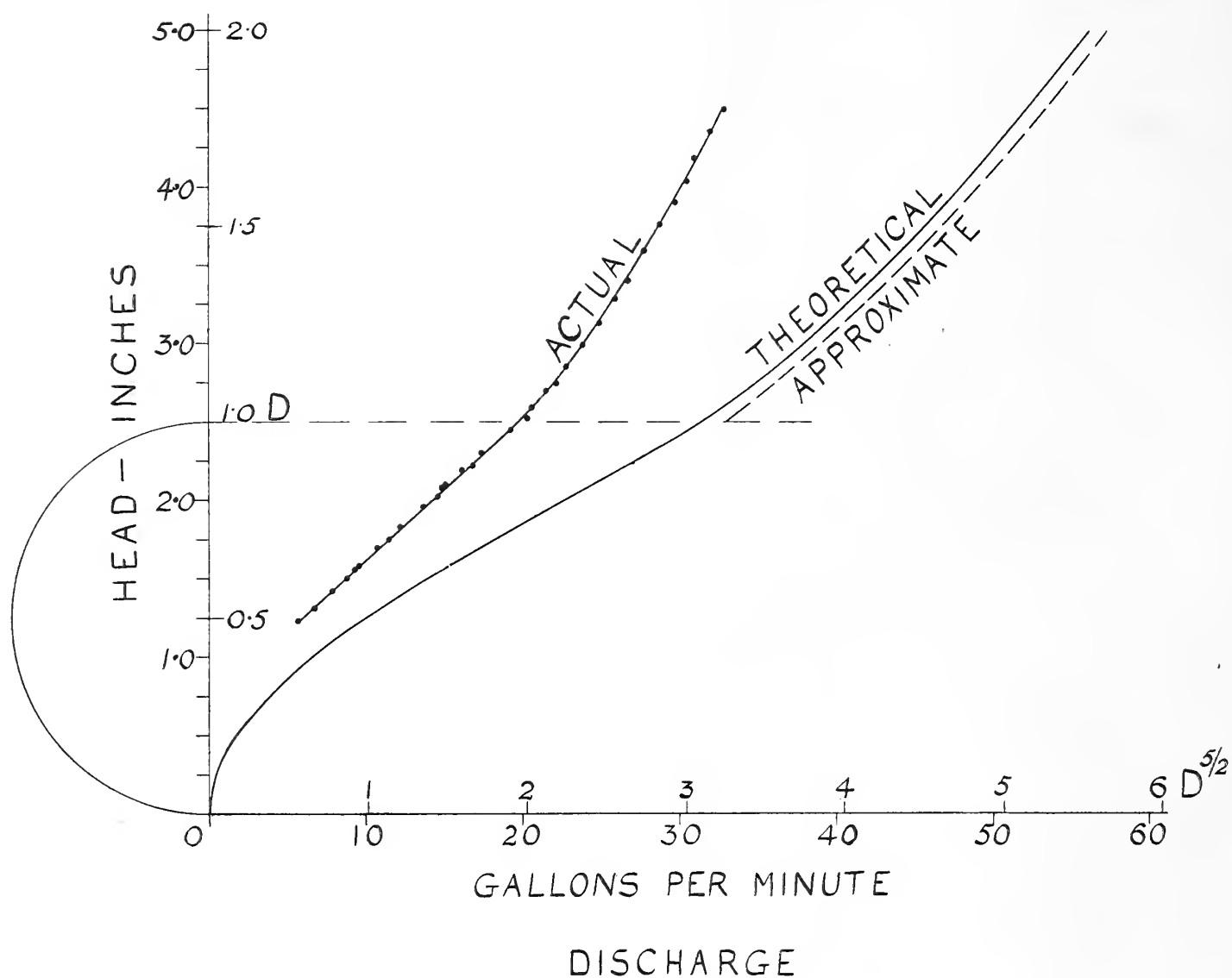


FIG. 5.

The head could not be carried above 4.49 inches because this represents the full supply of water to the apparatus.

When the water level is in the neighbourhood of the top of the orifice there is some irregularity in the flow; sometimes the hole runs full, and at other times there is a small air-space at the top, the surface conditions being unstable. It may be well to point out here, though the effect is of most importance when the orifice is not completely drowned, that the height of the water at the vertical plane of the aperture itself is always less than the "still water head" measured at some distance away from the point of dis-

charge. Thus the conditions of actual discharge are not exactly the same as those assumed in calculating the theoretical flow.

The coefficients of discharge for the drowned orifice are given also in Table II.; the average value of the coefficient for heads above the top of the orifice is 0.631. The dotted line in fig. 5 shows the theoretical discharge as calculated from the approximate formula given above. The difference between the results obtained from the accurate expression and those from the approximate one is not great; it amounts to about 6 per cent. when the water is just level with the top of the hole, and diminishes to $2\frac{1}{2}$ per cent. at the highest head taken. The results obtained graphically from the accurate expression may be in error to the extent of about 1 per cent., due partly to drawing and partly to the planimeter.

It is of interest to note the position of the layer of maximum discharge for any height of the water surface. This is marked upon each of the curves of figs. 2 and 4, and dotted lines are drawn through the points. The heights of the points were measured, and are given in the last column of Table I., in terms of D . The height of the plane of maximum discharge gradually rises and approaches $D/2$ as the head is increased, a result which was to be expected.

SUMMARY.

The discharge of water through a vertical circular orifice is represented by an expression which does not admit of a direct mathematical solution, but from which results have been obtained by graphical means. The shape of the curve of actual discharge is similar to that found for the theoretical discharge; both curves are sensibly straight lines for positions of the water surface between the centre and the top of the orifice. The value of the coefficient of discharge probably lies between 0.625 and 0.630 for low heads.

(Issued separately May 11, 1909.)

XVII.—The Electromotive Force of Iodine Concentration Cells with One Electrode saturated with Iodine. By A. P. Laurie, D.Sc., M.A. Cantab.

(MS. received January 16, 1909. Read February 15, 1909.)

THIS investigation was undertaken with the view of determining the number of free iodine ions present in saturated solutions of iodine in potassium iodide of varying strengths.

The Nernst equation for iodine concentration cells can be written as follows:—

$$E = 0.02955 \left(\log \frac{I_2}{(I)^2} - \log \frac{I'_2}{(I')^2} \right) \quad (1);$$

where I_2 and I'_2 are the concentrations of free iodine, and I and I' are the number of iodine ions present at the two electrodes.

To consider first the neighbourhood of the electrode surrounded by the dilute solution of iodine: The distribution between potassium iodide and iodine is conditioned by the mass equation

$$\frac{KI \times I_2}{KI_3} = k \quad (2);$$

or, if we call a the total number of potassium iodide molecules added, and b the total number of iodine molecules added, and x the total number of free iodine molecules present, then we have

$$\frac{(a - b + x)x}{b - x} = k \quad (3).$$

On expanding the quadratic we get for the first term

$$\frac{kb}{a - b + k} = x \quad (4),$$

which, within the limits of the solutions actually used, can be utilised for determining the value of x . The mean value of k , taken from Jakowkin's first four tables, is 0.0138.*

Applying equation (4), we can calculate the number of free iodine molecules present—that is, the value of I'_2 . Further, if the amount of iodine added is very small as compared with the total amount of potassium iodide, we may assume that the number of iodide ions present can be calculated from the known ionisation constants for potassium iodide of the strength used. We thus get the total value of the expression $\log \frac{I'_2}{(I')^2}$.

If we now consider the solution saturated with iodine surrounding the other electrode, and if we assume that the solubility of iodine in water is

* *Zeits. Phys. Chem.*, xx. 36.

not affected by the presence of KI, except in so far as they enter into combination, then we get for the value of I_2 the solubility of iodine in water at 25° C.—that is, .00134 molecules in 1000 c.c.

If we could neglect the contact electromotive force between the solutions we could therefore calculate the number of iodine ions present in the saturated solution by simply putting the known values into equation (1) and measuring the E.M.F. of the cell.

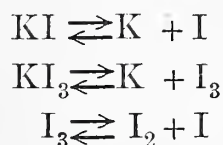
As far as potassium iodide is concerned, there is so little difference in the velocity of the potassium and iodide ions that this must be very small, but we have also the presence of I_3 ions to consider. Where the number of I_3 ions is very small compared with the number of iodide ions, this electromotive force is no doubt negligible, and in former measurements of the electromotive force of such iodine cells I showed that when the total amount of iodine present is small as compared with the potassium iodide, the experimental results showed that the contact E.M.F.'s may be neglected.

In these experiments, however, we have a very considerable concentration of iodine in proportion to the total potassium iodide round one electrode, and a very small concentration round the other electrode. We can therefore no longer neglect this possible source of error in the calculation of the number of iodine ions present.

With the view of determining the amount of this contact electromotive force I used the method devised by Cumming.* Having first measured the electromotive force of the cell, with potassium iodide solution between the respective solutions of iodine in potassium iodide, I replaced the potassium iodide by 10 normal ammonium nitrate solution, and remeasured the electromotive force.

In the subsequent tables the electromotive force in potassium iodide is given, and also the correction obtained by replacing the potassium iodide by ammonium nitrate as the connecting liquid.

Before giving the experimental results it is necessary to consider a little more closely the mass equation investigated by Jakowkin. It is evident that in the form in which it is written above no consideration is taken of the relative amount of ionised and un-ionised potassium iodide, and potassium tri-iodide present. Dawson † shows that it is necessary in this connection to consider separately the three mass equations



* A. C. Cumming, *Trans. Faraday Society*, vol. ii. part 3.

† *Chem. Soc. Trans.*, vol. lxxix. p. 239.

in order to completely investigate Jakowkin's equation. Dawson's investigation is made on the assumption that KI_3 is the only polyiodide present. The presence of higher polyiodides would evidently complicate the matter still further, but it has been proved by Jakowkin and Dawson's results that such higher polyiodides can at most be present in very small quantities in dilute solutions.

If we then consider a dilute solution of potassium iodide saturated with iodine, and if we determine the total amount of iodine present, then this amount less the solubility of iodine in water (0.00134) will give us the total amount of combined iodine, which in a dilute solution may be regarded as wholly present as ionised or un-ionised KI_3 .

The number of iodide ions present having been determined as described above, we can evidently determine the total number of uncombined KI molecules by applying the ionisation factor for that strength of potassium iodide. Further, if our methods of investigation are correct, the total amount of uncombined KI molecules (determined from the measurement of the number of iodide ions by means of the E.M.F.) added to the total number of combined KI molecules (determined by the titration of the saturated iodine solution) should equal the total number of KI molecules added to the solution, as long as we are justified in assuming that the only polyiodide present in appreciable quantity is KI_3 . If, however, an appreciable amount of polyiodides higher than KI_3 are present, this would no longer be true; for the number of combined KI molecules present, as determined by the E.M.F., would be less than the total number of combined iodine molecules determined by titration.

The actual measurements of the E.M.F. of the iodine concentration cells were made with a Dolazelek electrometer, the solutions being enclosed in the small stoppered electrodes which are figured in the paper above referred to.*

The correction for the contact E.M.F. was found by transferring the stoppered electrodes from the potassium iodide solution in which they were immersed to the ammonium nitrate solution of the same temperature, and taking readings after the electrodes had had an hour or two to settle down.

The results obtained were a little irregular, but I consider the error as probably not more than a millivolt in any single case.

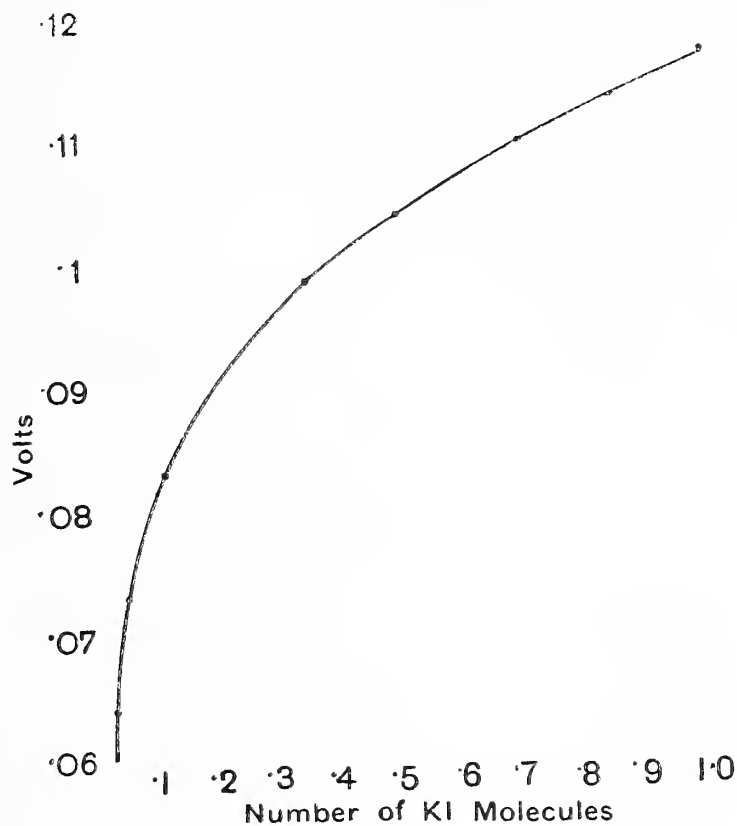
The iodine saturation-point was determined as follows. A solution of potassium iodide of the required strength, and containing the small quantity of iodine required for the one electrode, was first prepared. This electrode was then filled with this solution and stoppered, and the same solution

* Laurie, *Proc. Roy. Soc. Edin.*, vol. xxviii. (1908), pt. 5, p. 382; *Zeit. Phys. Chem.*, lxiv. 5.

poured into the bottle in which the stoppered electrodes were to be immersed.

The other electrode was then filled up with coarsely powdered iodine, and some more of the same solution was run in among the powdered iodine by means of a fine pipette till the electrode was full. It was then stoppered, and finally immersed in the bottle along with the other electrode. The whole cell was then sealed with paraffin-wax and placed in the thermostat. The electromotive force usually rose rapidly, reaching its maximum in

E.M.Fs. of saturated solution.



GRAPH 1.

about twenty-four hours and then remaining constant for some days. Readings taken after a week showed usually a slow dropping of E.M.F., probably due to the gradual diffusion of the iodine from the strong electrode, through the solution and into the weak electrode.

The iodine and potassium iodide used were Merck's pure preparations. As a check, a preparation of specially pure iodine was made by dissolving Merck's pure iodine in strong potassium iodide, precipitating by dilution with water, washing, drying over calcium nitrate, and finally subliming. This iodine introduced into the electrode instead of the iodine formerly used did not cause any appreciable change in the E.M.F.

Although all the values given are not the result of more than one measure-

ment, the more important values were determined more than once with freshly prepared solutions, and found to agree very closely (in dilute solutions, usually within $\cdot 0003$ of a volt). The intermediate values when plotted give a perfectly smooth curve, graph 1. In strong solutions the saturation-point cannot be determined quite so sharply, but an error of a millivolt is the outside limit.

The following table contains (1) the number of molecules of potassium iodide present, (2) the number of molecules of iodine present in the dilute iodine electrode, (3) the E.M.F. with potassium iodide of the same strength connecting the stoppered electrodes, (4) the correction in E.M.F. obtained by immersing the stoppered electrodes in ammonium nitrate, (5) the number of iodine ions present calculated from the corrected E.M.F., and (6) the total amount of uncombined KI present.

All the ammonium nitrate corrections given were experimentally determined except the one for $\cdot 7$ KI, which was obtained by interpolation.

TABLE I.—E.M.F. FOR SATURATED IODINE AGAINST $\cdot 0005$ IODINE MOLECULES.

No. of KI Molecules.	No. of Iodine Molecules.	E.M.F.	NH_4NO_3 Correction.	No. of Iodine Ions.	Total Free KI.
$\cdot 025$	$\cdot 0005$	$\cdot 0640$	$+\cdot 0030$	$\cdot 0117$	$\cdot 0127$
$\cdot 05$	$\cdot 0005$	$\cdot 0728$	$+\cdot 0034$	$\cdot 0226$	$\cdot 025$
$\cdot 115$	$\cdot 0005$	$\cdot 0830$	$+\cdot 0039$	$\cdot 05$	$\cdot 056$
$\cdot 344$	$\cdot 0005$	$\cdot 0987$	$+\cdot 0044$	$\cdot 135$	$\cdot 157$
$\cdot 5$	$\cdot 0005$	$\cdot 1040$	$+\cdot 0046$	$\cdot 184$	$\cdot 22$
$\cdot 7$	$\cdot 0005$	$\cdot 1099$	$+\cdot 0055$	$\cdot 23$	$\cdot 273$
$\cdot 85$	$\cdot 0005$	$\cdot 1136$	$+\cdot 0064$	$\cdot 256$	$\cdot 307$
$1\cdot 0$	$\cdot 0005$	$\cdot 1172$	$+\cdot 0008$	$\cdot 264$	$\cdot 317^*$

* This does not agree with the value calculated by Maitland (*Z. f. Electroch.*, 1906, 265), but his calculation involves Crotono's E.M.F. for a normal KI sat. iodine cell, which probably requires a considerable correction for contact E.M.F. due to I_3 ions.

In order to determine the amount of combined iodine present the amount was calculated for the dilute solutions by interpolation from the results of Noyes and Seidenstecker.* As the solutions used by me do not differ greatly from theirs in the amount of KI present, this can be done safely. At the same time independent determinations were made for these dilute solutions and found to agree so closely with the figures obtained by interpolation that they were taken.

For solutions above $\cdot 1$ normal KI, fresh determinations of the saturation-points were made in the following manner. The potassium iodide solution with excess of solid iodine was introduced into a closely stoppered bottle sealed outside with paraffin-wax and placed in a thermostat at 25°C . After being kept rotating in the thermostat for more than a week, the

* *Zeits. Phys. Chem.*, xxvii. 357.

solution was allowed to settle in the thermostat for some hours, a little quickly withdrawn in a graduated pipette through a plug of glass-wool, introduced into a flask containing some potassium iodide solution and titrated. The bottle was then closed, rotated for several more days, and a fresh sample removed and titrated. The second titration was found to agree very closely with the first, as the following figures show:—

TABLE II.—IODINE SATURATIONS.

No. of KI Molecules.	No. of Iodine Molecules.		
·025	·0125 + ·00134		
·05	·025 + ·00134		
·115	·0575 + ·00134		
	} Interpolated from Noyes's results.		
	<i>1st Determination.</i>	<i>2nd Determination.</i>	<i>Mean.</i>
·344	·189	·190	·1895
·5	·295	·297	·296
·7	·440	·442	·441
·85	·579	·578	·5785
1·0	·712	·711	·7115

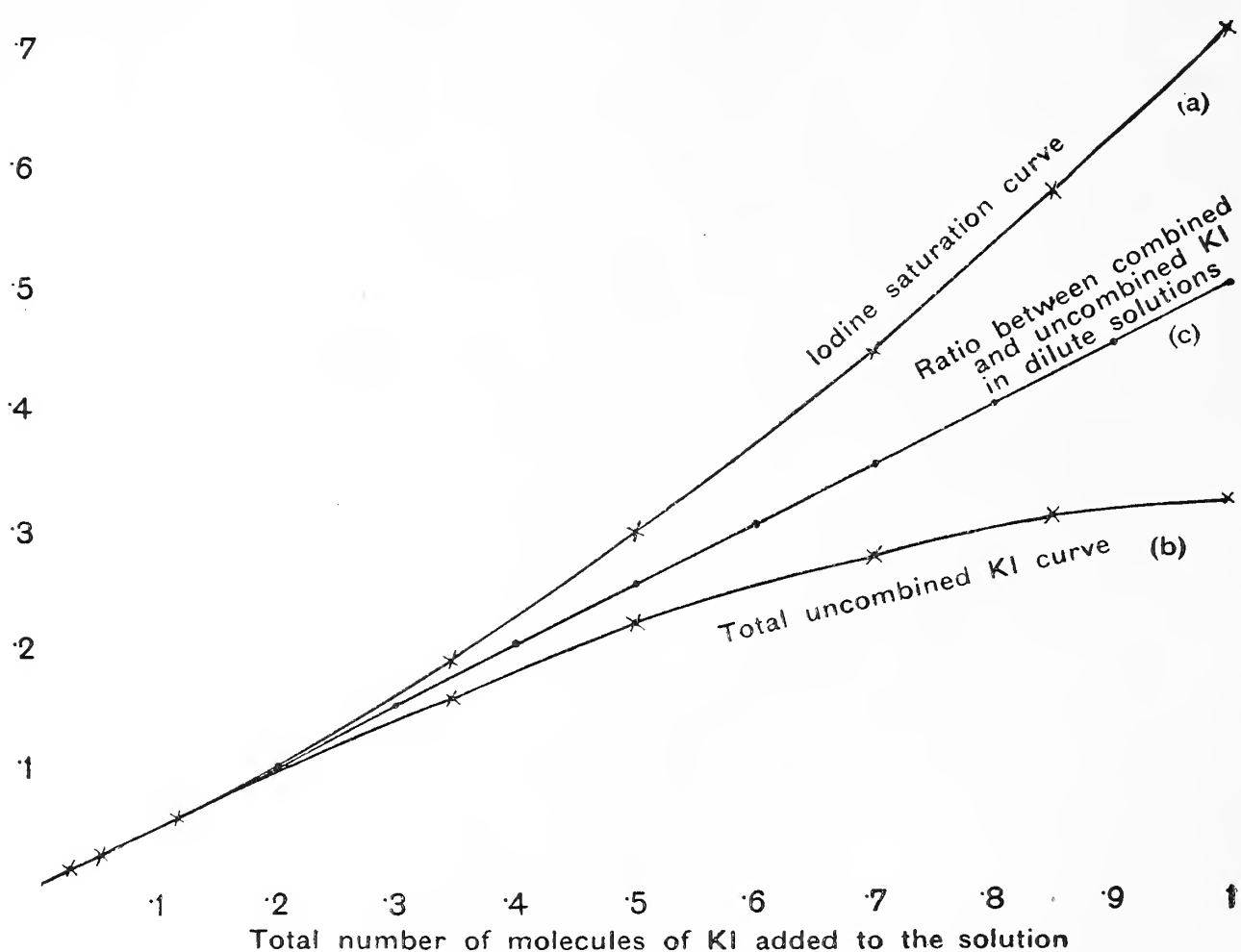
These values will be found to be a little higher than those obtained by Bruner,* but they have been taken as correct for the purposes of this paper. From these values, less the solubility of iodine in water, we obtain the total number of molecules of iodine in combination with the potassium iodide. The following table embodies the results obtained. Column I. gives the total number of potassium iodide molecules present; column II., the total amount of uncombined KI as calculated from the E.M.F.; column III., the total amount of combined iodine as obtained from the titration; and column IV., the sum of the values in columns II. and III.

TABLE III.

I. No. of KI Molecules.	II. Total Free KI.	III. Total Com- bined Iodine.	IV. Sum of II. + III.
·025	·0127	·0125	·0252
·05	·025	·025	·05
·115	·056	·0575	·1135
·344	·157	·189	·346
·5	·22	·295	·515
·7	·273	·440	·713
·85	·307	·578	·885
1·0	·317	·711	1·028

* *Zeits. Phys. Chem.* xxvi. 150.

This table shows that the total uncombined KI molecules added to the total combined iodine molecules equals the total number of KI molecules present in dilute solutions, while in the stronger solutions up to normal KI the iodine is in slight excess, indicating the presence in small quantities of higher polyiodides. For the dilute solutions, the results justify the method, as the total amount of the uncombined KI agrees very closely with the results obtained by Jakowkin and Noyes. As the potassium iodide solutions grow stronger and the solubility of the iodine increases, the



GRAPH 2.

amount of uncombined potassium iodide diminishes. The results are more clearly shown in graph 2. The total numbers of potassium iodide molecules present are taken as ordinates—for curve (a) the total number of combined iodine molecules, and for curve (b) the total number of free potassium iodide molecules, are taken respectively as abscissæ. The straight line (c) shows the relation which exists between combined iodine and uncombined KI in dilute solutions.

These results show clearly that up to normal KI there is very little polyiodide present higher than KI_3 , if we assume the method to be reliable for solutions up to this strength.

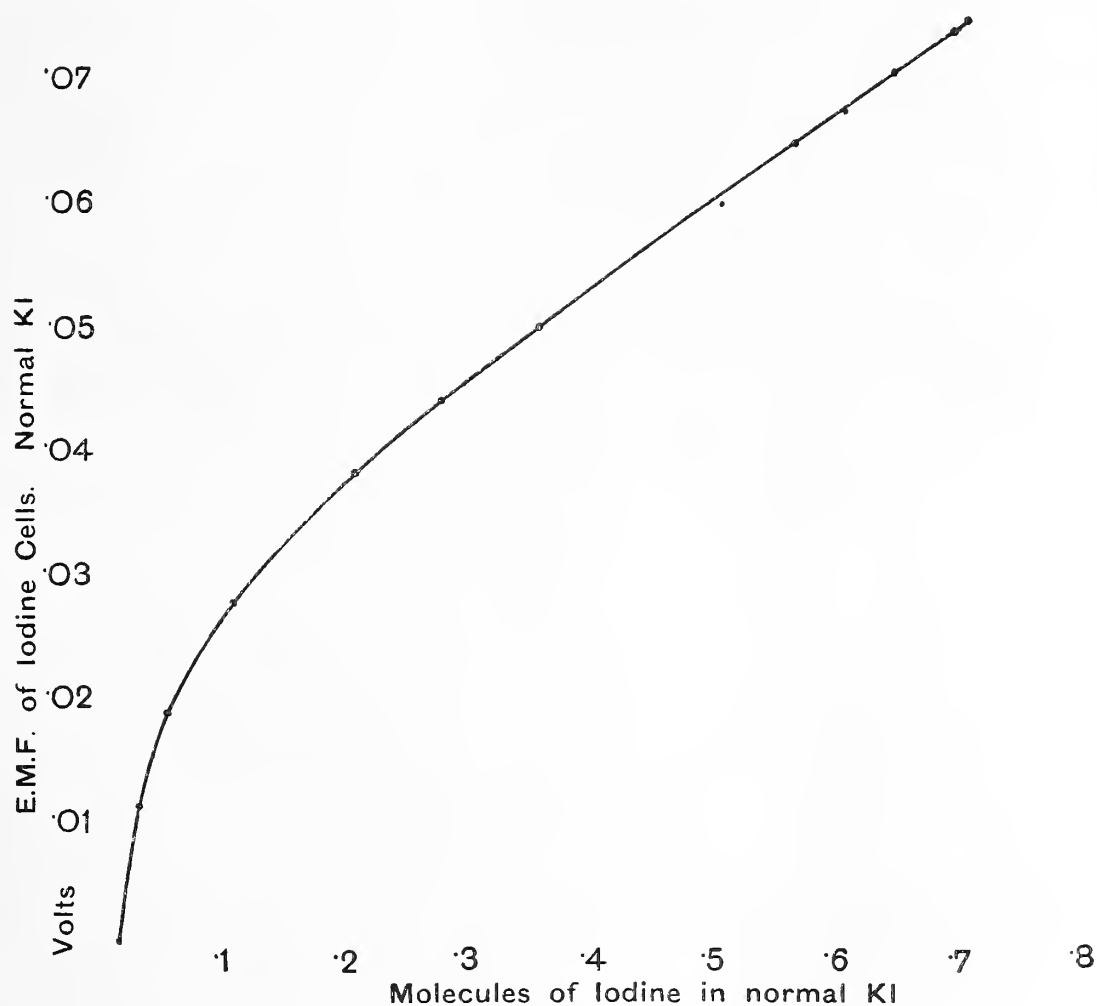
While the application of the Nernst equation is more and more likely

to yield erroneous results as the solutions get stronger, yet it seemed of some interest to measure the E.M.F. of a cell containing thrice normal KI., as this solution dissolves a number of molecules of iodine equal to the number of molecules of potassium iodide, so that it might be regarded as containing KI_3 only. The results were as follows:—

EMF = $\cdot 101$; NH_4NO_3 correction = $\cdot 02$; number of iodine ions = $\cdot 23$.

Total uncombined KI = $\cdot 275$.

It will be noted that while the total uncombined KI has not diminished

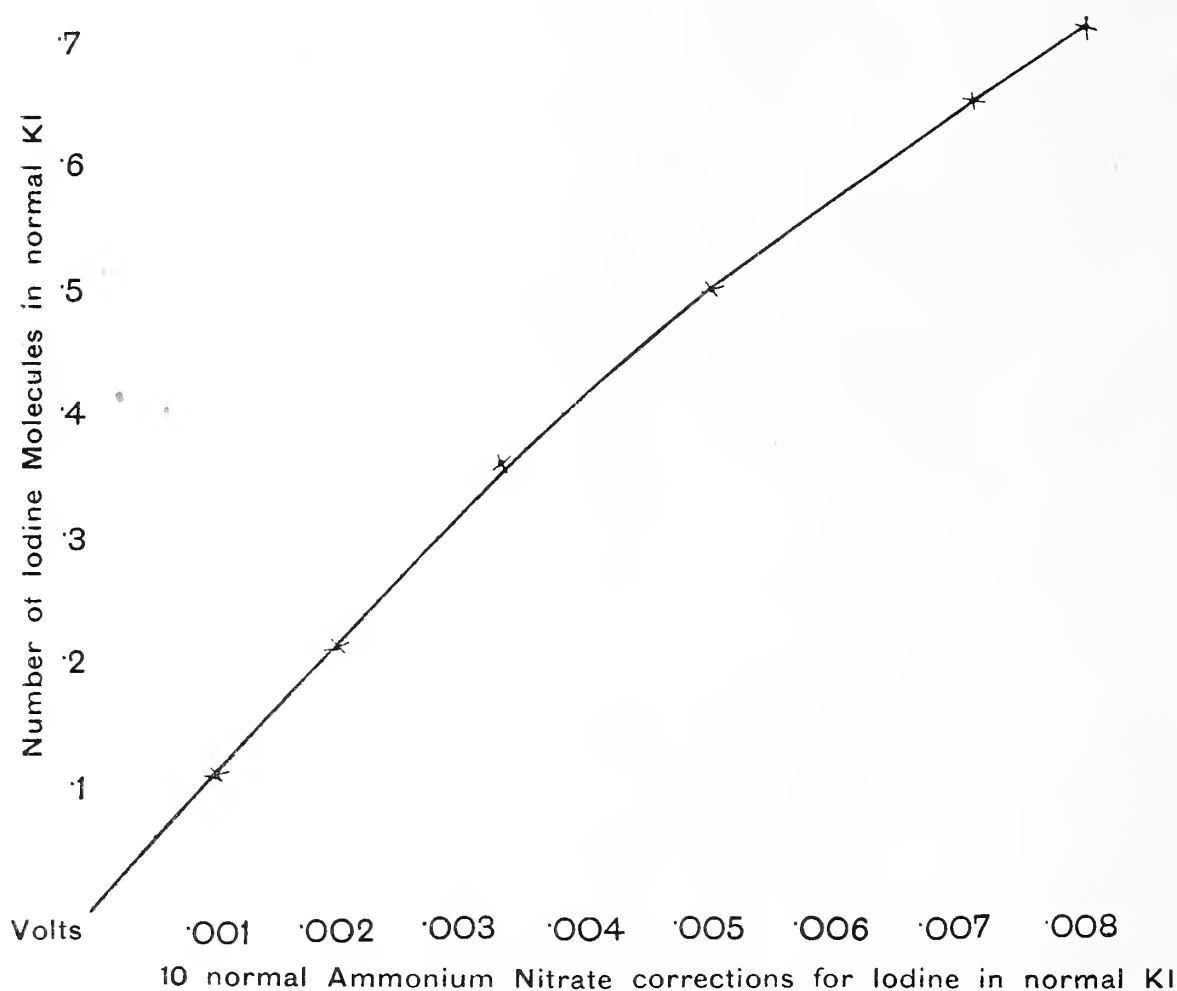


GRAPH 3.

much, it now represents a very small percentage of the total KI present, so that even this solution contains very little polyiodide higher than KI_3 . The amount, however, is now quite perceptible, and doubtless from this point onwards rapidly increases.

It seemed also of interest to investigate the number of iodine ions present in a series of solutions consisting of normal KI, with successive additions of iodine up to saturation-point. Jakowkin has shown that in such solutions the value of k is $\cdot 001365$ where little iodine is present, but progressively diminishes with addition of more iodine. In my former paper I determined the E.M.F. of a series of cells containing $\cdot 0139$ molecules of

iodine in normal KI at one electrode (the value of k being for this solution $\cdot 001365$), and the series of strengths of iodine given in Jakowkin's table at the other electrode, up to and including $\cdot 508$ of iodine. These values were measured at $20\cdot 4^\circ$ C., but have been corrected for 25° C. and included in Table IV. Additional measurements have also been made for stronger solutions up to and including the saturation-point. These E.M.F.'s are also given in Table IV., as well as the correction for contact E.M.F. as determined for certain of the solutions. Graph 3 shows the E.M.F.'s plotted against the



GRAPH 4.

iodine concentrations. Graph 4 shows the corrections plotted against the iodine concentrations.

These results can also be used to calculate the number of iodine ions in the strong solutions. For, in the dilute solution of iodine we can calculate the amount of free iodine present from the value given by Jakowkin for K for this solution, namely $\cdot 001365$, and we can without serious error assume the number of iodide ions present to be given by the ionisation ratio for normal KI. On the other hand, the amount of free iodine at the strong iodine electrode can be calculated by using the value of k as determined by Jakowkin for that particular strength of solution.

It is true that Jakowkin has not determined the value of k for strength

beyond .508 of iodine; but if the changing values of k , subtracted from .001365, be plotted against the iodine concentrations, it will be found that the graph is very nearly a straight line. If this straight line be produced until it passes through the value for saturation with iodine (.71), and if the value for k so obtained be used to calculate the amount of free iodine present in a saturated solution, then the value obtained is very nearly .00134. It is therefore justifiable to use this graph to determine the values of k between .508 and .71 of iodine. The following table contains the number of iodine ions calculated, as above described, from the E.M.F.'s as measured.

TABLE IV.—E.M.F.'s OF IODINE SOLUTIONS IN NORMAL KI WITH .00139 I₂ IN THE DILUTE ELECTRODE.

No. of Iodine Molecules.	E.M.F.	NH ₄ NO ₃ Correction.	No. of Free Iodine Ions.
.0322	.0110	.0003	.758
.056	.0187	.0005	.738
.109	.0277	.0010	.708
.209	.0382	.0020	.636
.279	.0441	.0026	.566
.362	.0498	.0033	.522
.508	.0597	.0051	.402
.57	.0644	.006	.353
.61	.0672	.0065	.325
.65	.0701	.0071	.299
.70	.0734	.0078	.274
.712	.0745	.008	.262

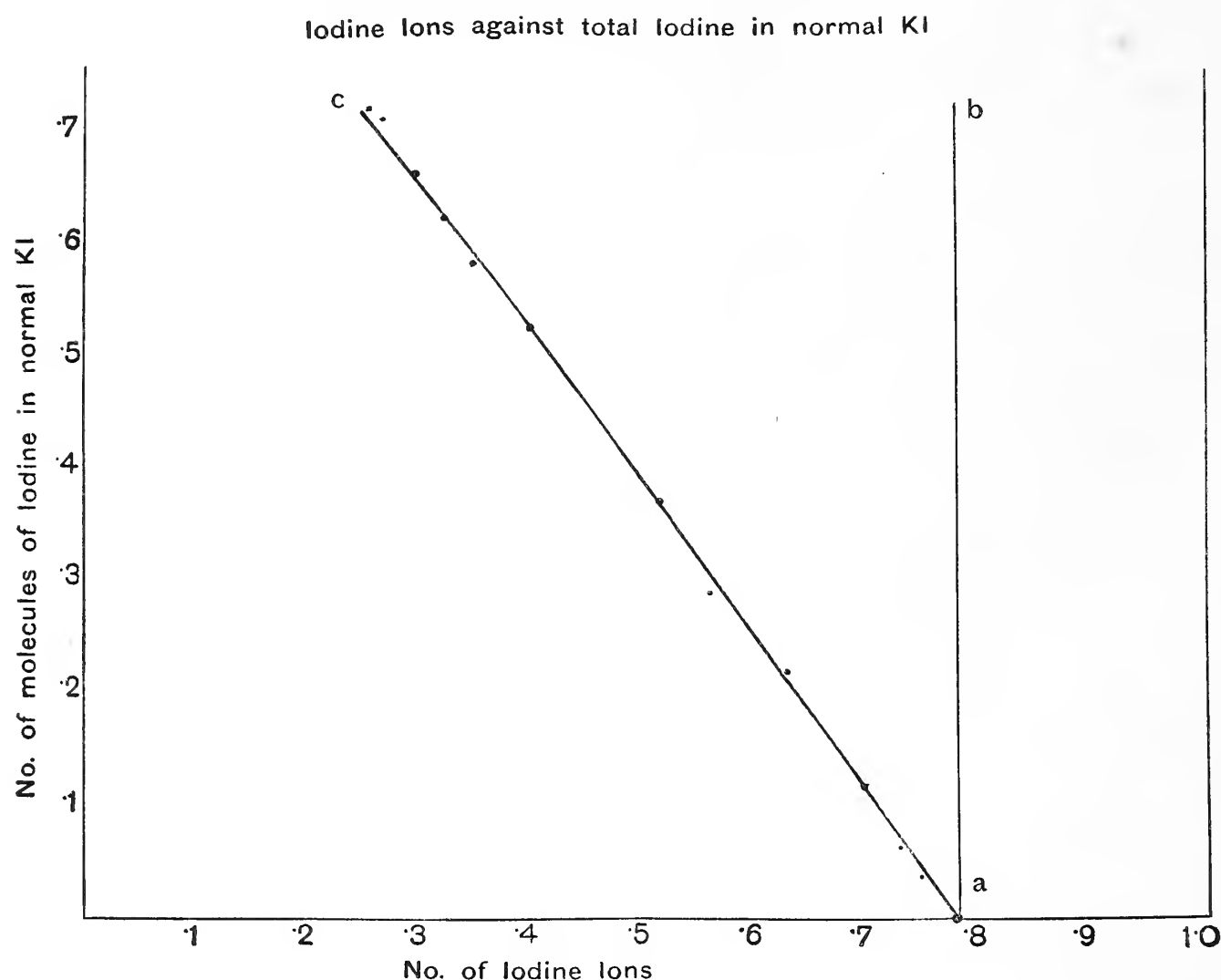
It will be noted that the number of iodide ions present in the saturated solution with .0139 molecules of iodine in the weak electrode agrees very closely with the number as determined from the solution containing .0005 of iodine. (If $k = .00138$ is used instead of $k = .00136$ in making the calculation, this number is slightly altered.) When it is remembered that in the one case the concentration of the iodine was .0005 and in the other case it was .0138, the agreement between these numbers is very satisfactory.

At the other end of the table is to be found the number of iodide ions for a solution containing .03 of iodine. If the number of I₂ ions corresponding to this amount of iodine be added to the number of iodide ions as measured from the E.M.F., then the total number of iodide ions agrees with that derived from the conductivity of a normal solution of KI.

A very interesting graph (5) can be constructed in the following manner from these results. Take the number of iodine ions as ordinates, and the respective strengths of iodine as abscissæ. Mark on the base line the number of iodide ions present in a normal KI solution, and opposite .71 of

iodine mark the number of iodide ions present in the saturated solution, and join these two points by a straight line. The intermediate values will be found to lie along this line, not departing from it more than can be accounted for by the experimental error.

Dawson has shown that the ionisation of KI_3 is the same as the ionisation of KI , so that if we consider the ratio of combined to uncombined K ions, it is evident that the un-ionised K ions will remain constant, and will be represented in the diagram by a perpendicular line ab , drawn from .79



GRAPH 5.

on the base line, while the distribution of the K ions between I and I_3 ions is given by the line ac for any given amount of iodine in solution. It appears further, from a consideration of this graph, that KI_3 is practically the only complex present. For if we consider the solution containing .5 molecules of iodine, we find that the number of free iodine ions present is .41, which means that the amount of free KI present is just about .5, thus leaving .5 of KI molecules combined with the .5 iodine molecules as KI_3 . If, on adding more iodine to this solution higher complexes were formed, the graph would not continue as a straight line.

If we consider again the table in Jakowkin's paper for normal KI , it

is evident that in a normal KI solution containing very little iodine the dissociation constant for $I_3 \rightleftharpoons I_2 + I$ is the same as in dilute solutions.

But as the amount of iodine in the solution is increased this dissociation constant alters, the I_3 molecules becoming more stable. In dilute solutions the amount of iodine can be varied without affecting this constant, but in presence of a very strong iodine solution, such as that present in normal KI, the tendency of the KI_3 to dissociate diminishes. It is for this reason that it is possible to dissolve more iodine in these solutions. For the total amount of iodine which can be dissolved is controlled by the solubility of iodine in water and by the value of k . If k is smaller, then, considering the equation $\frac{x(a-b+x)}{b-x} = k$, it is evident that the ratio of total iodine dissolved to total KI present can be pushed further, till the limiting value of x is reached, namely .00134.

While, therefore, the increased solubility of the iodine is accounted for up to a certain strength of KI by the alteration of the dissociation constant of I_3 ions, the formation of still higher polyiodides in still stronger solutions is no doubt due to the fact that it is only under the influence of very high concentrations that they are stable, and therefore begin to be present in considerable quantities.

In conclusion, I have to thank Mr King for his assistance, and the Carnegie Trust for the grant which enabled me to carry out these experiments.

(Issued separately May 11, 1909.)

XVIII.—*Cynomacrurus Piriei*, Poisson abyssal nouveau recueilli par l'Expédition Antarctique Nationale Ecossoise. Note préliminaire, par Louis Dollo, Sc.D. (Cantab.), For.Mem.G.S., C.M.Z.S., à Bruxelles (Musée). Présentée par M. R. H. TRAQUAIR, M.D., F.R.S., V.P.R.S.E.

(MS. received March 12, 1909. Read same date.)

I. INTRODUCTION.

I.—1. L'existence des *Macruridæ* à l'intérieur du Cercle Polaire Arctique est connue depuis longtemps.

C'est Sven Lovén qui en rapporta le premier exemplaire (*Macrurus berglax*), de Hammerfest (Finmark), lors de la première Expédition suédoise au Spitzberg, sur l'*Enigheten*, en 1837.*

2. Par contre, ce n'est que depuis dix ans (1899) que la présence des *Macruridæ* a été constatée à l'intérieur du Cercle Polaire Antarctique, par l'Expédition de la *Belgica* (1897–1899), avec le *Nematonurus Lecoointei*,† à une latitude presque identique :

1. *Enigheten*.—Arctique : 70° 39' 15" N.

2. *Belgica*.—Antarctique : 70° 40' 00" S.

II.—1. Jusque maintenant, *une seule espèce* de *Macruridæ* a été recueillie à l'intérieur du Cercle Polaire Arctique.‡

* C. J. Sundevall, "Om de två Nordiska arterna af fisksläktet *Macrurus* (Lepidoleprus Risso) och bådas förekommande vid Norrige," *Kongliga Vetenskaps-Academiens Handlingar*, för år 1840, p. 1, Stockholm, 1842.

R. Collett, "Meddelelser om Norges Fiske i Aarene 1884–1901 (3die Hoved-Supplement til 'Norges Fiske'), ii.," *Forhandlinger i Videnskabs-Selskabet i Christiania*, Aar 1903, No. 9, p. 76, Christiania, 1904 : "I Finmarken har den været kjendt siden 1837, da Lovén bragte det første europæiske Expl. fra Hammerfest til Riks-Museum i Stockholm."

Je remercie MM. les Professeurs R. Collett, Directeur du Musée zoologique de Christiania, et E. Lönnberg, Conservateur au Musée royal d'Histoire naturelle de Stockholm, des renseignements qu'ils ont bien voulu me fournir sur les *Macrurides* arctiques.

† L. Dollo, "Poissons de l'Expédition Antarctique Belge," *Résultats du Voyage du S.Y. "Belgica" en 1897, 1898, 1899, sous le commandement de A. de Gerlache de Gomery*, p. 44, Anvers, 1904.

‡ A. Brauer, "Die Tiefsee-Fische (I. Systematischer Teil)," *Wissenschaftliche Ergebnisse der deutschen Tiefsee-Expedition auf dem Dampfer "Valdivia" 1898–1899*, pl. xvii., Iéna, 1906.

2. Renversant l'ordre du progrès des connaissances dans ce domaine, il était réservé à l'Expédition Antarctique Nationale Ecossaise de découvrir une *deuxième espèce* de *Macruridæ* à l'intérieur du Cercle Polaire *Antarctique*, et, en même temps, le *Macruride le plus antarctique* actuellement capturé :

1. *Scotia*.—Antarctique : 71° 50' S.
2. *Belgica*.—Antarctique : 70° 40' S.

III.—L'Expédition de la *Scotia* (1902-1904) est, d'ailleurs, la *seule* Expédition Antarctique, depuis la *Belgica*, qui ait ramené des *Macruridæ* de l'intérieur du Cercle Polaire, car les autres :

- | | |
|---|------------------------------------|
| 1. <i>Southern Cross</i> * (1898-1900), | 4. <i>Antarctic</i> § (1901-1903), |
| 2. <i>Discovery</i> † (1901-1904), | 5. <i>Français</i> (1903-1905), |
| 3. <i>Gauss</i> ‡ (1901-1903), | |

n'en mentionnent pas, ou n'ont pas même pénétré à l'intérieur du Cercle en question.

IV.—Le *Macruride* pris par la *Scotia* à l'intérieur du Cercle Polaire Antarctique est *nouveau* et nettement caractérisé, notamment par sa dentition : je l'appellerai *Cynomacrurus Piriei*, en l'honneur du Docteur J. H. Harvey Pirie, médecin et géologue de l'Expédition, dont l'importante découverte de Graptolites aux Orcades du Sud est encore présente à la mémoire de tous. ¶

II. MALACOCEPHALUS ET CYNOMACRURUS.

I.—Qu'on se serve de la *Clé* donnée par M. A. Günther, Conservateur honoraire au British Museum, dans sa monographie des Poissons abyssaux

* G. A. Boulenger, "Pisces," *Report on the Collections of Natural History made in the Antarctic Regions during the Voyage of the "Southern Cross,"* p. 174, Londres, 1902.

† G. A. Boulenger, "Fishes," *National Antarctic Expedition (1901-1904), Natural History*, vol. ii. (Zoology), *Vertebrata* (iv.), Londres, 1907.

‡ E. von Drygalski, "Zum Kontinent des eisigen Südens," *Deutsche Südpolarexpedition : Fahrten und Forschungen des "Gauss" (1901-1903)*, p. 478, Berlin, 1904.

§ E. Lönnberg, "The Fishes of the Swedish South Polar Expedition," *Wissenschaftliche Ergebnisse der schwedischen Südpolar-Expedition 1901-1903*, vol. v., No. 6, pp. 9 et 51, Stockholm, 1905.

L'Expédition Antarctique Suédoise signale deux *Macrurides*, mais l'un provient du Canal du Beagle et l'autre du Détroit de Bransfield, donc pêchés en dehors du Cercle Polaire.

|| L. Vaillant, "Poissons," *Expédition Antarctique Française (1903-1905)*, p. 12, Paris, 1906.

¶ J. H. Harvey Pirie, "On the Graptolite-bearing Rocks of the South Orkneys," *Proceedings of the Royal Society of Edinburgh*, vol. xxv. p. 463, Edimbourg, 1905.

du *Challenger*,*—ou de celle que j'ai publiée dans mon mémoire sur les Poissons de la *Belgica*,†—par :

1. Ses dents prémaxillaires sur deux rangs,
2. Ses dents mandibulaires sur un rang,
3. Sa deuxième épine dorsale lisse,
4. Ses deux dorsales séparées par une distance plus grande que la base de la première,

le Macruride recueilli par la *Scotia* à l'intérieur du Cercle Polaire Antarctique semble venir se classer dans le genre *Malacocephalus*.‡

II.—Comparons donc le Poisson de l'Expédition Antarctique Nationale Ecossaise au type du genre *Malacocephalus*, qui est le *Malacocephalus levis*, Lowe, 1843:§

MALACOCEPHALUS, Günther, 1862.	CYNOMACRURUS, Dollo, 1909.
1. Pas de <i>croc prémaxillaire</i> .	1. Un <i>croc prémaxillaire</i> .
2. Pas de grands <i>crocs mandibulaires</i> .	2. Des grands <i>crocs mandibulaires</i> .
3. <i>Ecailles</i> , très petites : plus de 16, en série transversale, entre l'épine de la première dorsale et la ligne latérale.	3. <i>Ecailles</i> : 8, en série transversale, entre l'épine de la première dorsale et la ligne latérale.
4. <i>Anus</i> , entre les ventrales, et près de la base de celles-ci.	4. <i>Anus</i> , immédiatement au devant de l'anale.
5. <i>Ventouse larvaire</i> , représentée par des espaces nus au voisinage de l'anus.	5. <i>Espaces nus</i> , manquant complètement.
6. <i>Barbillon</i> , bien développé.	6. <i>Barbillon</i> , absent.
7. <i>Œil</i> , grand (orbite = $\frac{1}{3}$ tête, environ), sans membrane orbitaire.	7. <i>Œil</i> , petit (orbite = $\frac{1}{7}$ tête, environ), avec large membrane orbitaire.
8. <i>Cavité buccale</i> , blanche.	8. <i>Cavité buccale</i> , noire.

* A. Günther, "Report on the Deep-Sea Fishes," *Voyage of H.M.S. "Challenger" during the years 1873-1876, Zoology*, vol. xxii. p. 124, Edimbourg, 1887.

† L. Dollo, *Poissons de l'Expédition Antarctique Belge, etc.*, p. 38.

‡ Depuis la publication de mon mémoire sur les Poissons de la *Belgica*, M. D. S. Jordan, Président de la Leland Stanford Junior University, à Palo Alto (Californie), a créé le genre *Nezumia*, pour un Macruride du Japon, mais cet ichthyologiste semble avoir perdu de vue que *Chalinurus Murrayi* (A. Günther, *Deep-Sea Fishes, etc.*, p. 146) a douze rayons à chaque ventrale. D'autre part, *Dolloa* a les dorsales presque contigues, et non séparées par une distance plus grande que la base de la première.

Quoiqu'il en soit, *Cynomacrurus* se distingue aisément de *Nezumia*, par l'absence de dents villiformes et par sa deuxième épine dorsale lisse.

D. S. Jordan and E. C. Starks, "List of Fishes dredged by the Steamer 'Albatross' off the Coast of Japan in the Summer of 1900, with Descriptions of New Species and a Review of the Japanese Macrouridæ," *Bulletin of the United States Fish Commission*, vol. xxii. (1902), pp. 602 et 620, Washington, 1904.

§ A. Günther, *Deep-Sea Fishes, etc.*, p. 148.

|| F. A. Smitt, *A History of Scandinavian Fishes*, Part II p. 582, Stockholm, 1895.

III.—Le Macruride de la *Scotia* est, par conséquent, un tout autre Poisson que le *Malacocephalus laevis*, et il ne peut rentrer dans le même genre, au sens donné actuellement à ce mot dans la famille à laquelle il appartient : d'où le nom de *Cynomacrurus*, destiné à rappeler, en outre, les crocs caractéristiques de l'animal.

III. DIAGNOSE DU CYNOMACRURUS PIRIEI.

I. *Sous-Famille* : *Macrurinae*. — Par sa première fente branchiale réduite, notre Macruride se range dans la sous-famille des *Macrurinae*.

Voici, d'ailleurs, la description de son appareil branchial :

1. *Ouïes*, largement ouvertes.
2. *Membranes branchiostèges*, soudées seulement à la partie tout à fait antérieure de l'isthme et unies entre elles sur une faible étendue.
3. *Branchies*, au nombre de 4, avec lamelles bien développées.
4. *Fentes branchiales* :

- I. Réduite, mais beaucoup plus grande que la cinquième.
- II. Largement ouverte.
- III. Largement ouverte.
- IV. Largement ouverte.
- V. Simple boutonnière.

5. *Branchiospines* :

Arc I. En avant, réduites, non épineuses.

En arrière, claviformes, robustes, espacées, à extrémité libre épineuse, un peu plus courtes que les lamelles branchiales, alternant avec les branchiospines antérieures de l'arc suivant.

Arcs II et III. En avant et en arrière, comme en arrière de l'arc précédent.

Arc IV. En avant, comme les deux arcs précédents.

En arrière, absentes.

Arc V. Absentes.

6. *Pseudobranchies*, absentes.

II. *Genre* : *Cynomacrurus*, Dollo, 1909.—Dans ce qui va suivre, j'indiquerai en italiques les caractères *différentiels* de *Cynomacrurus* et de *Malacocephalus*, qui sont les deux genres de Macrurides les plus voisins, d'après les clés en usage aujourd'hui :

1. *Dents prémaxillaires*, sur deux rangs.

Rang interne : petites dents, assez serrées et régulièrement espacées d'un bout à l'autre de la série.

Rang externe : en arrière, comme le rang interne ; vers l'avant, plus espacées ; au tournant de la face latérale et de la face antérieure, *un croc bien marqué* ; sur le devant, petites dents, de nouveau, mais un peu plus grandes que les dents latérales, et préhensiles.

2. *Dents mandibulaires*, sur un rang.
Grands crocs irréguliers, espacés, dont les plus grands, énormes, dépassent en longueur le croc prémaxillaire.
3. *Deuxième épine dorsale*, lisse.
Filamenteuse, s'enroulant à l'extrémité libre.
4. *Distance des dorsales*, plus grande que la base de la première.
5. *Barbillon*, complètement *absent*.
6. *Anus*, immédiatement au *devant* de l'*anale*.
7. *Espaces nus*, entièrement *absents* au voisinage de l'anus.

III. *Espèce*: *Cynomacrurus Piriei*, Dollo, 1909.—Enfin, nous avons, pour la diagnose spécifique :

B. 6. D¹. 10. P. 21. V. 12.

1. *Museau* (région de la tête en avant du bord antérieur des orbites), court (ne mesurant guère plus de 1.5 fois le diamètre de l'orbite, qui est petite), tronqué en avant.
2. *Rostre* (région de la tête en avant de la limite antérieure de la bouche), tout petit, obtus, sans tubercule médian, terminé en avant par les préorbitaires, qui débordent.*
3. *Bouche*, large, latérale, dépassant le bord postérieur de l'orbite du quart du diamètre de celle-ci.
4. *Narines*, situées immédiatement au devant de l'orbite, bien développées, contiguës.
Prénarine, tubuleuse, à contour étranglé en son milieu et à grand diamètre horizontal.
Postnarine, simple ouverture mytiliforme, mais beaucoup plus grande que la prénarine, et à grand diamètre vertical.
5. *Œil*, petit, entouré d'une large membrane orbitaire, qui forme une bordure égale au tiers du diamètre de l'œil.
Orbite, contenue un peu moins de 7 fois dans la longueur de la tête.
Œil, contenu 11 fois dans la longueur de la tête.
Espace interorbitaire, plat, large, égal à 2 fois le diamètre de l'orbite.
6. *Préopercule*, à angle se projetant en arrière en un lobe arrondi, dentelé et écailleux.
7. *Cavités mucipares*, très développées et débouchant à la surface de la tête par d'énormes ouvertures, particulièrement derrière l'orbite.
8. *Anus*, dont la distance à l'isthme est moindre que les $\frac{3}{4}$ de la longueur de la tête.
Verticale de l'anus passant en arrière de la première dorsale et en avant de la deuxième, et aussi en avant du milieu de la distance qui sépare les deux dorsales (dans le premier $\frac{1}{8}$ de cette distance, en arrière de la première dorsale).

* Nous avons donc, chez les *Macruridæ*, deux sortes de rostres :

- | | |
|--|-----------------------|
| 1. Avec terminaison ethmoïdale | <i>Cœlorhynchus</i> . |
| 2. Avec terminaison préorbitaire | <i>Cynomacrurus</i> . |

Ce dernier étant, en tout petit, ce dont *Peristedion* nous montre l'extrême exagération, chez les *Triglidæ*, par un phénomène de Convergence.

9. *Écailles*, caduques, presque toutes tombées.
 Sur la tête, épineuses, avec 3, 4, ou 5 rangées d'épines.
 Sur le dos, des deux côtés de la deuxième dorsale, épineuses aussi, avec 1, 2, ou 3 rangées d'épines plus fortes.
 Sur le ventre, inermes, mais avec des plis marquant les rangées d'épines disparues, montrant donc que ce sont des écailles épineuses dégénérées.*
 Nombre d'écailles entre la ligne latérale et la deuxième épine dorsale : 8.
10. *Première dorsale*, avec une base contenue à peu près 3·5 fois dans la longueur de la tête.
11. *Deuxième dorsale*, commençant à une distance de la première plus grande que la base de celle-ci (pas beaucoup moindre que le double de cette base) et sensiblement égale à la moitié de la longueur de la tête.
12. *Queue*, géphyrocerque (homocerque) filamenteuse.†
13. *Pectorales*, mutilées, faibles, dont l'extrémité libre n'atteignait certainement pas la verticale de l'anús,‡ et, par conséquent, beaucoup plus courtes que la région post-orbitaire de la tête.
14. *Ventrales*, mutilées, avec rayon externe prolongé en un filament atteignant l'anús, quoiqu'il soit plus court que la moitié de la longueur de la tête ($\frac{4}{11}$ seulement).
15. *Coloration* : les écailles étant presque toutes tombées, le poisson a une teinte générale violacée.
 Par contre, la prénarine, le bord de la postnarine, les régions nues de la tête, le rayon externe des ventrales, la cavité buccale et la cavité branchiale sont noirs.
16. *Longueur totale* : 0^m·31 environ.

IV. *Type du Cynomacrurus Piriei*.—Le type du genre et de l'espèce est conservé au Scottish Oceanographical Laboratory, à Edimbourg (Ecosse).

IV. BIONOMIE DU CYNOMACRURUS PIRIEI.

I. BIOGÉOGRAPHIE.

Habitat : 71° 50' S. et 23° 30' W.

Mer de Weddell.

Océan Antarctique.

Quadrant Américain.

Station 414.

Scotia.

* C'est-à-dire cycloïdes secondaires, ou *Pseudocycloïdes* (L. Dollo, *Poissons de l'Expédition Antarctique Belge, etc.*, p. 140).

Les traces de l'indestructible passé (*Irréversibilité de l'Evolution*) sont, ici, les plis, derniers vestiges des rangées d'épines disparues (L. Dollo, "Les Lois de l'Evolution," *Bulletin de la Société belge de Géologie*, vol. vii. p. 165, Bruxelles, 1893).

† L. Dollo, "Sur la Phylogénie des Dipneustes," *Bulletin de la Société belge de Géologie*, vol. ix. p. 96, Bruxelles, 1895.

L. Dollo, *Poissons de l'Expédition Antarctique Belge, etc.*, p. 235.

‡ Chez *Malacocephalus*, insérées en arrière de la verticale de l'anús ! (A. Günther, *Deep-Sea Fishes, etc.*, pl. xxxix., fig. B).

II. ETHOLOGIE.

1. *Profondeur*.—2102 fathoms.
2. *Nature du Fond*.—Vase bleue.
3. *Température du Fond*.—31°·5 F.
4. *Température de la Surface*.—29°·1 F.
5. *Densité de l'Eau (Fond)*.—1·02555.
6. *Densité de l'Eau (Surface)*.—1·02520.
7. *Mode de Capture*.—Filet vertical de 2^m·50 d'ouverture, entre 0 et 1000 fathoms.
8. *Date de Capture*.—15 Mars 1904.
9. *Heure de Capture*.—Entre 9 heures du matin et 8 heures du soir.
10. *Nombre d'Individus capturés*.—Un seul.
11. *Compagnons de Pêche*.—5 espèces de Poissons, parmi lesquels le *Prymnothonus Hookeri*, et 15 espèces d'autres animaux.

V. CYNOMACRURUS PIRIEI ET NEMATONURUS LECOINTEI.

I. *Comparaison*.—Il serait, maintenant, intéressant de comparer, entre eux, les deux seuls Macrurides actuellement connus à l'intérieur du Cercle Polaire Antarctique: *

NEMATONURUS LECOINTEI, Dollo, 1900.	CYNOMACRURUS PIRIEI, Dollo, 1909.
1. <i>Dents prémaxillaires</i> , sur deux rangs, mais sans croc.	1. <i>Dents prémaxillaires</i> , sur deux rangs, avec un croc bien marqué.
2. <i>Dents mandibulaires</i> , sur un rang, mais sans grands crocs.	2. <i>Dents mandibulaires</i> , sur un rang, avec grands crocs irréguliers.
3. <i>Deuxième épine dorsale</i> , barbelée.	3. <i>Deuxième épine dorsale</i> , lisse et filamenteuse.
4. <i>Distance des dorsales</i> , plus grande que la base de la première (plus de 3 fois).	4. <i>Distance des dorsales</i> , plus grande que la base de la première (moins de 2 fois).
5. <i>Rostre</i> , à extrémité ethmoïdale.	5. <i>Rostre</i> , à extrémité préorbitaire.
6. <i>Bouche</i> , dont la fente n'atteint pas la verticale du centre de l'œil.	6. <i>Bouche</i> , dépassant le bord postérieur de l'orbite du quart du diamètre de celle-ci.
7. <i>Barbillon</i> , présent.	7. <i>Barbillon</i> , absent.
8. <i>Œil</i> , contenu 5 $\frac{1}{4}$ fois dans la longueur de la tête, avec large membrane orbitaire.	8. <i>Œil</i> , contenu 11 fois dans la longueur de la tête, avec large membrane orbitaire.
9. <i>Orbite</i> , contenue 4 fois dans la longueur de la tête.	9. <i>Orbite</i> , contenue presque 7 fois dans la longueur de la tête.
10. <i>Espace interorbitaire</i> , moindre que le diamètre de l'orbite.	10. <i>Espace interorbitaire</i> , égal à deux fois le diamètre de l'orbite.

* L. Dollo, *Poissons de l'Expédition Antarctique Belge, etc.*, p. 44.

11. *Anus*, presque contre l'anale, mais avec verticale passant, entre les dorsales, en arrière du milieu de la distance de celles-ci.

12. *Longueur totale* : 0^m.43 environ.

1. *Habitat* : 70° 40' S. et 102° 15' W.
Mer de Bellingshausen.
Océan Antarctique.
Quadrant Pacifique.
Belgica.
2. *Profondeur* : 1531 fathoms.

11. *Anus*, immédiatement au devant de l'anale, mais avec verticale passant, entre les dorsales, dans le premier huitième de la distance de celles-ci, en arrière de la première dorsale.

12. *Longueur totale* : 0^m.31 environ.

1. *Habitat* : 71° 50' S. et 23° 30' W.
Mer de Weddell.
Océan Antarctique.
Quadrant Américain.
Scotia.
2. *Profondeur* : 2102 fathoms.

II. *Conclusion*.—Il résulte de ce qui précède que les deux Macrurides de l'intérieur du Cercle Polaire Antarctique sont bien distincts et appartiennent certainement à deux genres différents.

L'un habite, d'ailleurs, le prolongement de l'Atlantique (*Cynomacrurus Piriei*); l'autre, le prolongement du Pacifique (*Nematonurus Lecoointei*).

VI. CYNOMACRURUS PIRIEI ET MACRURUS BERGLAX.

I. *Comparaison*.—Comparons, à présent, les Macrurides de l'intérieur du Cercle Polaire Antarctique au seul Macruride connu de l'intérieur du Cercle Polaire Arctique, en remarquant qu'il suffit de s'occuper, ici, de *Cynomacrurus Piriei* et de *Macrurus berglax*, puisque *Nematonurus Lecoointei* a été étudié en détail dans mon mémoire sur les Poissons de la *Belgica* :*

MACRURUS BERGLAX, Lacépède, 1800.

1. *Dents prémaxillaires*, constituant une bande villiforme.
2. *Dents mandibulaires*, constituant également une bande villiforme.
3. *Deuxième épine dorsale*, barbelée.
4. *Distance des dorsales*, moindre que la base de la première.
5. *Rostre*, à extrémité ethmoïdale.
6. *Bouche*, dont la fente n'atteint pas la verticale du centre de l'œil.

CYNOMACRURUS PIRIEI, Dollo, 1909.

1. *Dents prémaxillaires*, sur deux rangs, avec un croc bien marqué.
2. *Dents mandibulaires*, sur un rang, avec grands crocs irréguliers.
3. *Deuxième épine dorsale*, lisse et filamenteuse.
4. *Distance des dorsales*, plus grande que la base de la première (presque le double).
5. *Rostre*, à extrémité préorbitaire.
6. *Bouche*, dépassant le bord postérieur de l'orbite du quart du diamètre de celle-ci.

* F. A. Smitt, *Scandinavian Fishes, etc.*, p. 587.

7. *Barbillon*, présent.

8. *Œil*, contenu 4 fois dans la longueur de la tête, avec large membrane orbitaire.

9. *Orbite*, contenue moins de 3 fois dans la longueur de la tête.

10. *Espace interorbitaire*, beaucoup moindre que le diamètre de l'orbite (la moitié environ).

11. *Anus*, presque contre l'anale, mais avec verticale en arrière de l'origine de la deuxième dorsale.

12. *Longueur totale* : 1^m.00 environ.

1. *Habitat* : 70° 39' 15" N.

Finmark.

Océan Arctique.

Quadrant Européen.

Enigheten.

2. *Profondeur* : 100 à 300 fathoms.

7. *Barbillon*, absent.

8. *Œil*, contenu 11 fois dans la longueur de la tête, avec large membrane orbitaire.

9. *Orbite*, contenue presque 7 fois dans la longueur de la tête.

10. *Espace interorbitaire*, égal à deux fois le diamètre de l'orbite.

11. *Anus*, immédiatement au devant de l'anale, mais avec verticale passant, entre les dorsales, presque contre la première de celles-ci.

12. *Longueur totale* : 0^m.31 environ.

1. *Habitat* : 71° 50' 00" S.

Mer de Weddell.

Océan Antarctique.

Quadrant Américain.

Scotia.

2. *Profondeur* : 2102 fathoms.

II. *Conclusion*.—Il résulte donc, également, de ce qui précède que le Macruride de l'intérieur du Cercle Polaire Arctique est bien distinct de ceux de l'intérieur du Cercle Polaire Antarctique et qu'il appartient certainement aussi à un genre différent.

Circonstance encore défavorable à la Théorie de la Bipolarité.

Mais je reviendrai prochainement là-dessus.

VII. MORPHOLOGIE DES BARBILLONS INFRAMANDIBULAIRES.

I. *Macrurinae*.—La *longueur* du barbillon est fort variable dans cette sous-famille : *

1. *Hymenocephalus longibarbis* { $\frac{2}{3}$ de la longueur de la tête.
Fidji, 315 fathoms.

2. *Cetonus crassiceps* { $\frac{1}{4}$ de la longueur de la tête.
Kermadec, 520 fathoms.

Cependant, sa *présence* est très constante, car je ne connais que deux cas, sur plus de 125 espèces, où il manque tout à fait :

1. *Hymenocephalus lethonemus* Japon, 120 à 265 fathoms.†

2. *Cynomacrurus Piriei* Mer de Weddell, 2102 fathoms.‡

* A. Günther, *Deep-Sea Fishes, etc.*, pl. xviii., fig. C et pl. xxxvii.

† D. S. Jordan and E. C. Starks, *List of Fishes, etc.*, p. 615.

‡ Bien que le *Cynomacrurus Piriei* ait été pêché entre 0 et 1000 fathoms, je prends ici, comme plus haut, la profondeur maximum au point de capture, puisque les Macrurides sont adaptés à la Vie Benthique, vu leur Queue Géphyrocerque : ce sont donc des Poissons de Fond.

II. *Morphologie*.—Les recherches comparatives faites à cette occasion m'ont fourni le petit tableau suivant sur la Morphologie des Barbillons inframandibulaires :

1. Cutané	<i>Macrurus</i> .*
2. Rayons branchiostèges	<i>Mullus</i> .†
3. Rayons pectoraux	<i>Rhodichthys</i> .‡
4. Rayons ventraux	<i>Ophidium</i> .§

VIII. LA MEMBRANE ORBITAIRE.

I. *Macrurides polaires*. — Les Macrurides de l'intérieur des Cercles Polaires :

1. *Macrurus berglax*, Lacépède, 1800—Arctique,
2. *Nematonurus Lecoointei*, Dollo, 1900—Antarctique,
3. *Cynomacrurus Piriei*, Dollo, 1909—Antarctique,

ont l'œil entouré d'une large Membrane orbitaire.

II. *Répartition*.—Examinons si on peut trouver dans la Biogéographie, dans la Bathymétrie, ou dans la Grosseur de l'Œil, une cause à cette structure : ||

1. Biogéographie	{	<i>Cynomacrurus Piriei</i> : Antarctique.	M. orb. large.
		<i>Coryphænoïdes altipinnis</i> : M. du Japon.	„ „
2. Bathymétrie	{	<i>Cynomacrurus Piriei</i> : 2102 fathoms.	„ „
		<i>Macrurus berglax</i> : 100 à 300 fathoms.	„ „
3. Gros Yeux	{	<i>Cœlorhynchus fasciatus</i> : Orb. = $\frac{2}{5}$ tête.	„ absente.
		<i>Macrurus berglax</i> : Orb. > $\frac{1}{3}$ tête.	„ large.
4. Petits Yeux	{	<i>Chalinurus liocephalus</i> : Orb. = $\frac{1}{6}$ tête.	„ absente.
		<i>Cynomacrurus Piriei</i> : Orb. < $\frac{1}{6}$ tête.	„ large.

Nous avons donc, d'une part, une large Membrane orbitaire à l'intérieur et en dehors des Cercles Polaires, puis à de faibles et à de grandes Profondeurs.

Et, d'autre part, nous avons de Gros Yeux et de Petits Yeux avec ou sans Membrane orbitaire.

* F. A. Smitt, *Scandinavian Fishes, etc.*, p. 587.

† S. Lo Bianco, "L'origine dei barbigli tattili nel genere *Mullus*," *Atti della Reale Accademia dei Lincei*, vol. xvi. p. 577, Rome, 1907.

‡ A. S. Jensen, "On Fish-Otoliths in the Bottom-Deposits of the Sea (I. Otoliths of the Gadus-Species deposited in the Polar Deep)," *Meddelelser fra Kommissionen for Havundersøgelser (Fiskeri)*, vol. i. (No. 7), pp. 5 et 6, Copenhague, 1905.

§ G. A. Boulenger, "Teleostei (Systematic Part)," *Cambridge Natural History (Fishes, etc.)*, vol. vii. p. 713, Londres, 1904.

|| A. Günther, *Deep-Sea Fishes, etc.*, pp. 138, 139 et 145.

Ce n'est, par conséquent, ni par la Biogéographie, ni par la Bathymétrie, ni par la Grosseur de l'Œil, que nous arriverons à éclaircir l'origine de cette disposition.

III. *Signification.* — Mais la signification nous en sera, sans doute, donnée par l'observation suivante : *

“In all fishes the general integument of the head passes over the eye, and becomes transparent where it enters the orbit; sometimes it simply passes over the orbit, sometimes it forms a circular fold. The anterior and posterior portions may be especially broad and the seat of an adipose deposit (*adipose eyelids*), as in *Scomber*, *Caranx*, *Mugil*, etc. In many of these fishes the extent of these eyelids varies with the seasons; during the spawning season they are so much loaded with fat as nearly to hide the whole eye.”

D'autant plus que, si le spécimen de *Malacocephalus laevis* figuré par M. Günther : †

Pernambouc (Brésil), 10 Septembre 1873, 350 fathoms,

ne montre pas la moindre trace de membrane orbitaire, celui représenté par Smitt : ‡

Lysekil (Gothebourg, Suède), 10 Novembre 1852, rivage,

permet de constater l'existence d'une Membrane orbitaire déjà fort accentuée.

Il s'agit, dès lors, très probablement, d'une disposition saisonnière, et vraisemblablement en rapport avec l'époque de la reproduction.

* A. Günther, *An Introduction to the Study of Fishes*, p. 112, Edimbourg, 1880.

A. Günther, “Fische der Südsee,” *Journal des Museum Godeffroy*, vol. ii. pl. 84, 85, 86 et 120, Hambourg, 1873–75.

† A. Günther, *Deep-Sea Fishes, etc.*, pl. xxxix. fig. B.

‡ F. A. Smitt, *Scandinavian Fishes, etc.*, p. 594.

XIX.—On Lagrange's Equations of Motion, and on Elementary Solutions of Gyrostatic Problems. By Professor Andrew Gray, F.R.S.

(MS. received January 12, 1909. Read February 15, 1909.)

1. It is now well known that Lagrange's equations of motion for a system of connected particles are not applicable to certain cases of motion—for example, to a rigid sphere rolling without sliding on a given surface. In his *Principien der Mechanik*, Hertz has referred at considerable length to this subject, and has applied the adjective "holonomous" to those systems to which the equations are applicable, and has called all others "non-holonomous." These adjectives correspond to distinct characteristics of the systems as regards the constraints to which they are subject. Holonomous systems are those in which the constraints are expressed, or can be expressed, by finite equations; in non-holonomous systems, on the other hand, these conditions, or some of them at least, are expressed by differential relations, which do not fulfil the conditions of integrability.

2. Long before Hertz's book appeared, attention had been called to the subject; for instance, Ferrers pointed out in the *Quarterly Journal of Mathematics*, vol. xii. (1871-73), that in the case of a hoop rolling on a horizontal plane, while the equation for the inclination of the hoop to the vertical could be obtained from the expressions for the kinetic and potential energies by Lagrange's method, that method failed to give the equations corresponding to the other co-ordinates. Erroneous solutions of the hoop problem have, however, since that time been published by more than one writer who had not perceived the fact noticed by Ferrers. An oversight of this kind in a solution of this problem given by another mathematician seems to have led Appell to his theorem (*Comptes Rendus*, 1899) by which the equations of motion for holonomous and non-holonomous systems alike are obtained by what has since been called the "kinetic energy of the accelerations of the system."

3. When Lagrange's equations are deduced from the principle of least action, or by means of Hamilton's characteristic function, the effect of the nature of the connections of the system is left more or less obscure. Some observations were made by Hertz on the subject, but these were far from conclusive, and it was first shown in 1896 by Hölder (*Gött. Nachr.*, 1896),

by an examination of the logic of the process of deduction, that the varied motion was not in all cases what it was tacitly assumed to be, in itself a possible motion—that is, one consistent with the kinematical conditions of the system. What we are entitled to assume is only that if the motion be slightly varied from a configuration C in the actual succession of configurations to a corresponding configuration C' in a neighbouring succession, the variation from C to C' must be consistent with the conditions of the system. Hölder examined the cases for which, as Hertz pointed out, the method of least action seemed to give erroneous results, and showed how, by a stricter logical process than that usually employed, they could be brought likewise under the scope of the principle.

4. To make clear how the method fails, I shall consider the process of derivation of Lagrange's equations from the equations of motion of a free particle—a process first employed, so far as I am aware, by Lord Kelvin. Exact equations of the kind first indicated by Ferrers, applicable to all cases of motion in which the co-ordinates are such as to explicitly define the configuration at any instant, will be obtained; and then I shall inquire how the method of ignoration of co-ordinates, and the gyrostatic equations, first given by Lord Kelvin, are to be modified when this is done. This inquiry is the principal object of the present paper, and it will be seen leads to a simple result. Solutions will be added of a few problems illustrative of the methods arrived at, and of the sources of errors that may easily arise in their application.

5. The relations between the x, y, z co-ordinates of a representative particle and the independent parameters q_1, q_2, \dots, q_k may be written in the form

$$\left. \begin{aligned} \delta x &= a_1 \delta q_1 + a_2 \delta q_2 + \dots + a_k \delta q_k \\ \delta y &= b_1 \delta q_1 + b_2 \delta q_2 + \dots + b_k \delta q_k \\ \delta z &= c_1 \delta q_1 + c_2 \delta q_2 + \dots + c_k \delta q_k \end{aligned} \right\} \quad \dots \quad (1).$$

for a displacement possible at time t .

In the case of finite equations of condition the coefficients $a_1, a_2, \dots, b_1, b_2, \dots, c_1, c_2, \dots$ are partial differential coefficients of functions of the co-ordinates q_1, q_2, \dots, q_k : in every case they are functions of the co-ordinates. The real displacements for an interval of time dt are

$$\left. \begin{aligned} dx &= a_1 dq_1 + a_2 dq_2 + \dots + a_k dq_k + a dt \\ dy &= b_1 dq_1 + b_2 dq_2 + \dots + b_k dq_k + b dt \\ dz &= c_1 dq_1 + c_2 dq_2 + \dots + c_k dq_k + c dt \end{aligned} \right\} \quad \dots \quad (2),$$

where a, b, c are zero if the constraints do not depend on the time. It is supposed that the coefficients are so chosen that equations of condition ex-

pressing variations $\delta q_{k+1}, \dots, \delta q_{k+m}$ of m other parameters connected by m kinematical equations with the variations of the parameters q_1, q_2, \dots, q_k are taken account of. We shall also suppose for the present that the kinematical equations do not involve the time t explicitly.

6. From (2) the values of $\dot{x}, \dot{y}, \dot{z}$ can be found, and from these again the values of $\ddot{x}, \ddot{y}, \ddot{z}$. The latter include, besides terms in $\ddot{q}_1, \ddot{q}_2, \dots, \ddot{q}_k$, also terms involving the time-rates of variation of the a, b, c coefficients. If, then, we substitute in the expression $\frac{1}{2}\Sigma\{m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\}$, we get a transformed expression of which part, which we may call S , is an explicit function of $\ddot{q}_1, \ddot{q}_2, \dots, \ddot{q}_k$. If then we write

$$\Sigma(a_i X + b_i Y + c_i Z) = Q_i,$$

the generalised force according to the usual specification, we obtain

$$\frac{\partial S}{\partial \ddot{q}_1} = Q_1, \quad \frac{\partial S}{\partial \ddot{q}_2} = Q_2, \quad \dots \dots \dots \quad (3),$$

which are Appell's equations.

7. Consider now the equations of motion

$$m\ddot{x} = X, \quad m\ddot{y} = Y, \quad m\ddot{z} = Z \quad \dots \dots \dots \quad (4),$$

of a free particle, and from the equations of this form for the particles of the system construct

$$\left. \begin{aligned} \Sigma\{m(a_1\ddot{x} + b_1\ddot{y} + c_1\ddot{z})\} &= \Sigma(a_1 X + b_1 Y + c_1 Z) \\ \Sigma\{m(a_2\ddot{x} + b_2\ddot{y} + c_2\ddot{z})\} &= \Sigma(a_2 X + b_2 Y + c_2 Z) \\ \dots \dots \dots \end{aligned} \right\} \dots \dots \dots \quad (5).$$

The quantities on the right-hand sides in (5) are the generalised forces Q_1, Q_2, \dots of the Lagrangian equations.

It will be observed that since any Q is the coefficient of δq in the expression $Q\delta q$ for the work done in a possible variation of the parameter q , Q does not include any of what may be called the *non-active* forces—that is, forces such as those due to guides and constraints which are invariable. We have then to consider the equation

$$\Sigma\{m(a\ddot{x} + b\ddot{y} + c\ddot{z})\} = Q \quad \dots \dots \dots \quad (6).$$

Lord Kelvin's process was as follows:—Writing $\partial x/\partial q, \partial y/\partial q, \partial z/\partial q$ instead of a, b, c , for the system was tacitly supposed to be holonomous, he obtained (in a slightly different notation)

$$\ddot{x} \frac{\partial x}{\partial q} = \frac{d}{dt} \left(\dot{x} \frac{\partial x}{\partial q} \right) - \dot{x} \frac{d}{dt} \frac{\partial x}{\partial q};$$

and then proved that

$$\frac{\partial x}{\partial q} = \frac{\partial \dot{x}}{\partial \dot{q}}, \quad \frac{d}{dt} \frac{\partial x}{\partial q} = \frac{\partial \dot{x}}{\partial q};$$

so that

$$\dot{x} \frac{\partial x}{\partial q} = \frac{d}{dt} \left(\dot{x} \frac{\partial x}{\partial \dot{q}} \right) - \dot{x} \frac{\partial \dot{x}}{\partial q}$$

Hence we obtain at once, putting

$$T = \frac{1}{2} \Sigma \{ m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \},$$

and supposing \dot{x} , \dot{y} , \dot{z} , replaced by their values from (2), the typical Lagrangian equation of motion

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = Q \quad \dots \dots \dots \quad (7).$$

That

$$\frac{d}{dt} \frac{\partial x}{\partial q} = \frac{d\dot{x}}{\partial q}$$

follows from the equation

$$\frac{d}{dt} \frac{\partial x}{\partial q} = \frac{\partial}{\partial q_1} \frac{\partial x}{\partial q} \cdot \dot{q}_1 + \frac{\partial}{\partial q_2} \frac{\partial x}{\partial q} \cdot \dot{q}_2 + \dots + \frac{\partial}{\partial q_k} \frac{\partial x}{\partial q} \cdot \dot{q}_k + \frac{\partial}{\partial t} \frac{\partial x}{\partial q};$$

for here $\partial x/\partial q$ is supposed to be the partial derivative with respect to q of an explicit function of q_1, q_2, \dots, q_k, t , and therefore

$$\frac{\partial}{\partial q_1} \frac{\partial x}{\partial q} = \frac{\partial}{\partial q} \frac{\partial x}{\partial q_1}, \quad \frac{\partial}{\partial q_2} \frac{\partial x}{\partial q} = \frac{\partial}{\partial q} \frac{\partial x}{\partial q_2}, \quad \dots \dots \dots \quad (8).$$

8. But if the system is non-holonomous this process is no longer applicable, and another must be sought. In discussing this question I shall no longer refer to the Cartesian co-ordinates of the particles composing the system, As a rule, we are given only equations connecting the quantities by which the kinetic (and potential) energy is primarily expressed with the generalised co-ordinates, and there must be as many such equations as are required to express the configuration of the system at any instant in terms of these co-ordinates.

Let the kinetic energy be primarily expressed by the time-rates of change $\dot{u}, \dot{v}, \dot{w}, \dots$, of quantities u, v, w, \dots which fulfil equations.

$$\left. \begin{aligned} \delta u &= a_1 \delta q_1 + a_2 \delta q_2 + \dots + a_i \delta q_i + e_1 \delta s_1 + e_2 \delta s_2 + \dots + e_j \delta s_j \\ \delta v &= b_1 \delta q_1 + b_2 \delta q_2 + \dots + b_i \delta q_i + f_1 \delta s_1 + f_2 \delta s_2 + \dots + f_j \delta s_j \\ &\dots \dots \dots \end{aligned} \right\} \quad (9).$$

For a rigid body \dot{u}, \dot{v}, \dots , may be taken as the product of the square roots $\sqrt{M}, \sqrt{N}, \dots$, of inertia coefficients M, N, \dots , into velocity components of the centroid, or the products of angular velocities of the body about given axes by the square roots of the proper inertia coefficients for this case; and so for other systems. Thus if T be the kinetic energy we shall have

$$T = \frac{1}{2} (\dot{u}^2 + \dot{v}^2 + \dot{w}^2 + \dots) \quad \dots \quad (10).$$

The parameters are divided in (9) into two sets—the q 's and the s 's—for a purpose which will appear later in connection with ignorance of co-ordinates. By (9),

$$\left. \begin{aligned} \dot{u} &= a_1\dot{q}_1 + a_2\dot{q}_2 + \dots + a_i\dot{q}_i + e_1\dot{s}_1 + e_2\dot{s}_2 + \dots + e_j\dot{s}_j \\ \dot{v} &= b_1\dot{q}_1 + b_2\dot{q}_2 + \dots + b_i\dot{q}_i + f_1\dot{s}_1 + f_2\dot{s}_2 + \dots + f_j\dot{s}_j \\ &\dots \dots \dots \end{aligned} \right\} \dots \dots (11),$$

if we suppose, as we do for the present, that t does not appear in the kinematical equations.

9. Also we have, by the signification of \dot{u}, \dot{v}, \dots , equations of motion

$$\ddot{u} = U, \quad \ddot{v} = V, \quad \dots \dots,$$

and therefore obtain a series of equations of the form

$$\left. \begin{aligned} a_1\ddot{u} + b_1\ddot{v} + \dots &= a_1U + b_1V + \dots = Q_1 \\ a_2\ddot{u} + b_2\ddot{v} + \dots &= a_2U + b_2V + \dots = Q_2 \\ \dots \dots \dots &\dots \dots \dots \end{aligned} \right\} \dots \dots (12),$$

where Q_1, Q_2, \dots , are generalised forces according to the meanings of \dot{u}, \dot{v}, \dots . These are $i + j$ such equations, since there are now supposed to be $i + j$ independent parameters $q_1, q_2, \dots, s_1, s_2, \dots$.

By (10) and (11) we have

$$\left. \begin{aligned} \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_1} - (\dot{a}_1\dot{u} + \dot{b}_1\dot{v} + \dots) &= Q_1 \\ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_2} - (\dot{a}_2\dot{u} + \dot{b}_2\dot{v} + \dots) &= Q_2 \\ \dots \dots \dots &\dots \dots \dots \end{aligned} \right\} \dots \dots (13).$$

These are equivalent to the equations which, as Ferrers showed, must be substituted for the ordinary Lagrangian equations as typified by (6). They are applicable in all cases, whether the system is holonomous or not, provided always that the co-ordinates chosen are capable of expressing the configuration of the system, or position of the body at any instant.

10. It must, however, be remembered that different modes of breaking up the kinetic energy into a sum of squares according to (10) are in general *not* equivalent, but involve different sets of forces. For example, the term $\frac{1}{2}A(\dot{\theta}^2 + \sin^2 \theta \cdot \dot{\phi}^2)$, which occurs in the expression for the kinetic energy of a gyrostatic pendulum, or of any kinetically symmetrical body rotating with one point fixed, and which is already a sum of two squares, may also be written in the form $\frac{1}{2}A(\dot{\theta} \cos \phi + \dot{\phi} \sin \theta \sin \phi)^2 + \frac{1}{2}A(\dot{\theta} \sin \phi - \dot{\phi} \sin \theta \cos \phi)^2$. The forces involved in the two cases are different, and hence, unless the appropriate corrections necessary on this account are introduced, the distribution of the kinetic energy into a series of squares is not arbitrary. Correct results may, however, always be obtained if the

motion of the system is referred to fixed axes, and the "velocities" \dot{u}, \dot{v}, \dots chosen are such as correspond to *applied forces* U, V, \dots really applied. For instance, it is always correct to express T in the form $\frac{1}{2} \Sigma \{m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\}$, and in the case of a rigid body to insert for $\dot{x}, \dot{y}, \dot{z}$ their values in terms of the motion of the centroid, and of the angular velocities of the body about the chosen axes, taking care that the proper applied forces, whether derived from the potential energy or otherwise assigned, are employed.

11. For the sake of what follows, we compare equations (13) with the usual form a little in detail. We have here

$$\left. \begin{aligned} \dot{a}_1 &= \frac{\partial a_1}{\partial q_1} \dot{q}_1 + \frac{\partial a_2}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial a_1}{\partial s_1} \dot{s}_1 + \frac{\partial a_1}{\partial s_2} \dot{s}_2 + \dots \\ \dot{b}_1 &= \frac{\partial b_1}{\partial q_1} \dot{q}_1 + \frac{\partial b_2}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial b_1}{\partial s_1} \dot{s}_1 + \frac{\partial b_1}{\partial s_2} \dot{s}_2 + \dots \\ &\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \end{aligned} \right\} \dots \dots (14).$$

Also

$$\frac{\partial T}{\partial q_1} = \dot{u} \frac{\partial \dot{u}}{\partial q_1} + \dot{v} \frac{\partial \dot{v}}{\partial q_1} + \dots;$$

or, by (11),

$$\begin{aligned} \frac{\partial T}{\partial q_1} &= \dot{u} \left(\frac{\partial a_1}{\partial q_1} \dot{q}_1 + \frac{\partial a_2}{\partial q_1} \dot{q}_2 + \dots + \frac{\partial e_1}{\partial q_1} \dot{s}_1 + \frac{\partial e_2}{\partial q_1} \dot{s}_2 + \dots \right) \\ &+ \dot{v} \left(\frac{\partial b_1}{\partial q_1} \dot{q}_1 + \frac{\partial b_2}{\partial q_1} \dot{q}_2 + \dots + \frac{\partial f_1}{\partial q_1} \dot{s}_1 + \frac{\partial f_2}{\partial q_1} \dot{s}_2 + \dots \right) \\ &+ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \end{aligned} \quad (15).$$

Thus, by (14) and (15), we get for the first of (13)

$$\begin{aligned} \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} - \dot{u} \left\{ \dot{q}_1 \left(\frac{\partial a_1}{\partial q_1} - \frac{\partial a_1}{\partial q_1} \right) + \dot{q}_2 \left(\frac{\partial a_1}{\partial q_2} - \frac{\partial a_2}{\partial q_1} \right) + \dot{q}_3 \left(\frac{\partial a_1}{\partial q_3} - \frac{\partial a_3}{\partial q_1} \right) + \dots \right. \\ \left. + \dot{s}_1 \left(\frac{\partial a_1}{\partial s_1} - \frac{\partial e_1}{\partial q_1} \right) + \dot{s}_2 \left(\frac{\partial a_1}{\partial s_2} - \frac{\partial e_2}{\partial q_1} \right) + \dots \right\} \\ - \dot{v} \left\{ \dot{q}_1 \left(\frac{\partial b_1}{\partial q_1} - \frac{\partial b_1}{\partial q_1} \right) + \dot{q}_2 \left(\frac{\partial b_1}{\partial q_2} - \frac{\partial b_2}{\partial q_1} \right) + \dot{q}_3 \left(\frac{\partial b_1}{\partial q_3} - \frac{\partial b_3}{\partial q_1} \right) + \dots \right. \\ \left. + \dot{s}_1 \left(\frac{\partial b_1}{\partial s_1} - \frac{\partial f_1}{\partial q_1} \right) + \dot{s}_2 \left(\frac{\partial b_1}{\partial s_2} - \frac{\partial f_2}{\partial q_1} \right) + \dots \right\} \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \end{aligned} \quad (16).$$

The other equations of (13) may of course be written down in a similar form by symmetry.

If the conditions

$$\left. \begin{aligned} \frac{\partial a_1}{\partial q_2} &= \frac{\partial a_2}{\partial q_1}, & \frac{\partial a_1}{\partial q_3} &= \frac{\partial a_3}{\partial q_1}, & \dots \dots \dots \\ \frac{\partial b_1}{\partial q_2} &= \frac{\partial b_2}{\partial q_1}, & \frac{\partial b_1}{\partial q_3} &= \frac{\partial b_3}{\partial q_1}, & \dots \dots \dots \\ \dots \dots \dots & \dots \dots \dots & \dots \dots \dots & \dots \dots \dots & \dots \dots \dots \end{aligned} \right\} \dots \dots \dots (17)$$

are fulfilled, (16) reduces to

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} = Q_1 \quad \dots \quad (18).$$

12. Equations (17) form one set of the conditions which make equations (9) the derivatives of a set of finite equations; and if all these conditions are fulfilled, the equations of motion take the usual form typified by (18). The fulfilment of each set involves the validity of a corresponding equation of motion in the usual form.

It has been remarked by Appell (*Mécanique Rationnelle*, t. ii., art. 462) that if one set, say (17), of these conditions holds, we can write (using our present notation)

$$\left. \begin{aligned} \delta u &= \delta F + \alpha_2 \delta q_2 + \alpha_3 \delta q_3 + \dots \\ \delta v &= \delta G + \beta_2 \delta q_2 + \beta_3 \delta q_3 + \dots \\ &\dots \end{aligned} \right\} \dots \quad (19);$$

where $\delta F, \delta G$, are perfect differentials.

If, then, we notice when the fundamental equations are written down that the terms corresponding to any co-ordinate are thus expressed, we know that the equation of motion corresponding to that co-ordinate can be found by the ordinary Lagrangian process.

13. In what follows we shall, for brevity, adopt the notation

$$\left. \begin{aligned} \phi_1 T &= \dot{a}_1 \dot{u} + \dot{b}_1 \dot{v} + \dots \\ \phi_2 T &= \dot{a}_2 \dot{u} + \dot{b}_2 \dot{v} + \dots \\ &\dots \end{aligned} \right\} \dots \quad (20).$$

If the form of T be modified in any way, for example, by the substitution of the values of s_1, s_2, \dots in terms of the momenta $\partial T / \partial s_1, \partial T / \partial s_2, \dots$, and the other velocities $\dot{q}_1, \dot{q}_2, \dots$, then if T' be the modified form of T , $\phi_1 T', \phi_2 T', \dots$ will be understood to denote the operations indicated in (20), but performed with the new values which are then given to the coefficients of $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_k$.

14. It follows from what has been stated that, as has already been pointed out, the operations indicated in (20) cannot be performed without reference to the fundamental equations from which that expression has been derived. For example, two terms in T might be $\frac{1}{2} A \dot{\theta}^2 + \frac{1}{2} B \dot{\psi}^2$. These might be derived either from $\dot{u} = \sqrt{A} \dot{\theta}, \dot{v} = \sqrt{B} \dot{\psi}$, or from $\dot{u} = \sqrt{A} \sin \theta \cdot \dot{\theta} + \sqrt{B} \cos \theta \cdot \dot{\psi}, \dot{v} = \sqrt{A} \cos \theta \cdot \dot{\theta} - \sqrt{B} \sin \theta \cdot \dot{\psi}$. The former mode of derivation would satisfy the conditions of integrability so far as these terms are concerned, the second would not. It is possible, in fact, to specify two distinct cases of motion which have precisely the same expressions for the kinetic and potential energies, but which have not the

same equations of motion. An example will be given at the end of the present paper.

15. Let now the form of T be modified by the substitutions indicated above. Our object is to inquire what modification is required in the process of "ignorance of co-ordinates" by the non-integrability of the relations between the generalised co-ordinates and the functions of these co-ordinates and their velocities from which the kinetic energy is derived. We shall suppose, therefore, that the co-ordinates s_1, s_2, \dots , are absent from the kinetic energy, and from the function V of the co-ordinates from which the forces are derived, if that function exists. Writing them for a moment $r_1 = \partial T / \partial \dot{s}_1$, $r_2 = \partial T / \partial \dot{s}_2, \dots$, we see that if the fundamental relations were integrable r_1, r_2, \dots , would be constants, since then we should have

$$\frac{\partial T}{\partial s_1} = 0, \quad \frac{\partial T}{\partial s_2} = 0, \quad \dots \dots$$

on the supposition that either

$$\frac{\partial V}{\partial s_1} = 0, \quad \frac{\partial V}{\partial s_2} = 0, \quad \dots \dots$$

or no generalised forces corresponding to s_1, s_2, \dots , exist. The equations of motion are now, however,

$$\left. \begin{aligned} \frac{d}{dt} \frac{\partial T}{\partial \dot{s}_1} - \chi_1 T &= 0 \\ \frac{d}{dt} \frac{\partial T}{\partial \dot{s}_2} - \chi_2 T &= 0 \\ \dots \dots \dots \end{aligned} \right\} \dots \dots \dots (21),$$

where

$$\left. \begin{aligned} \chi_1 T &= \dot{e}_1 \dot{u} + \dot{f}_1 \dot{v} + \dots \dots \\ \dots \dots \dots \end{aligned} \right\} \dots \dots \dots (22);$$

so that χ_1, χ_2, \dots , are the operators for the s co-ordinates that ϕ_1, ϕ_2, \dots , are for the q co-ordinates.

The conditions for constancy of the momenta $\partial T / \partial \dot{s}_1, \partial T / \partial \dot{s}_2, \dots$, are therefore now

$$\chi_1 T = 0, \quad \chi_2 T = 0, \quad \dots \dots \dots (22').$$

These conditions are fulfilled by (22) when $\dot{e}_1, \dot{f}_1, \dots$, are zero, which is the case in various problems of the motions of tops and gyrostats, where none of the coefficients $e_1, f_1, \dots, e_2, f_2, \dots$, contains the time or any of the co-ordinates q_1, q_2, \dots . We shall not assume, unless it is so stated, that $e_1, f_1, \dots, e_2, f_2, \dots$, are absolute constants.

16. We shall assume, however, that

$$\frac{\partial T}{\partial \dot{s}_1} = c_1, \quad \frac{\partial T}{\partial \dot{s}_2} = c_2, \quad \dots \dots \dots (23);$$

where c_1, c_2, \dots are constants. These conditions are fulfilled in a large number of problems regarding rotating fly-wheels in which the co-ordinates q_1, q_2, \dots determining the positions of the axes of rotation have no influence on the momenta $\partial T/\partial s_1, \dots$. When the system is holonomous the constancy of these momenta is secured by the fact that the differential equations (22') become $\partial T/\partial s_1 = 0, \dots$. Equations (23) extended are

$$\left. \begin{aligned} (e_1 a_1 + f_1 b_1 + \dots) \dot{q}_1 + (e_1 a_2 + f_1 b_2 + \dots) \dot{q}_2 + \dots + (e_1^2 + f_1^2 + \dots) \dot{s}_1 \\ + (e_1 e_2 + f_1 f_2 + \dots) \dot{s}_2 + \dots = c_1 \\ (e_2 a_1 + f_2 b_1 + \dots) \dot{q}_1 + (e_2 a_2 + f_2 b_2 + \dots) \dot{q}_2 + \dots + (e_1 e_2 + f_1 f_2 + \dots) \dot{s}_1 \\ + (e_2^2 + f_2^2 + \dots) \dot{s}_2 + \dots = c_2 \\ \dots \end{aligned} \right\} \quad (24).$$

The coefficients of $\dot{q}_1, \dot{q}_2, \dots, \dot{s}_1, \dot{s}_2, \dots$, are those of the products $\dot{q}_1 \dot{s}_1, \dot{q}_2 \dot{s}_1, \dots, \dot{s}_1 \dot{s}_1, \dot{s}_1 \dot{s}_2, \dots$, in the first line, and of $\dot{q}_1 \dot{s}_2, \dot{q}_2 \dot{s}_2, \dots, \dot{s}_1 \dot{s}_2, \dot{s}_2 \dot{s}_2, \dots$, in the second. Denoting these coefficients as in the scheme

$$\begin{aligned} (q_1, s_1), (q_2, s_2), \dots, (s_1, s_1), (s_1, s_2), \dots \\ (q_1, s_2), (q_2, s_2), \dots, (s_1, s_2), (s_2, s_2), \dots \end{aligned}$$

we can write equations (24) as follows:—

$$\left. \begin{aligned} (s_1, s_1) \dot{s}_1 + (s_2, s_1) \dot{s}_2 + \dots = c_1 - (q_1, s_1) \dot{q}_1 - (q_2, s_1) \dot{q}_2 - \dots \\ (s_1, s_2) \dot{s}_1 + (s_2, s_2) \dot{s}_2 + \dots = c_2 - (q_1, s_2) \dot{q}_1 - (q_2, s_2) \dot{q}_2 - \dots \\ \dots \end{aligned} \right\} \quad (25).$$

From these $\dot{s}_1, \dot{s}_2, \dots$, can be determined in terms of c_1, c_2, \dots , and $\dot{q}_1, \dot{q}_2, \dots$. These values then substituted in the expression for T convert it into a function of $\dot{q}_1, \dot{q}_2, \dots, c_1, c_2, \dots$, so that all trace of the variables s_1, s_2, \dots is now removed. We have to inquire what form the equations of motion take when this substitution is made. First we form expressions for $\dot{s}_1, \dot{s}_2, \dots$. Let $(c_1, c_1), (c_1, c_2), \dots$, denote the ratios of the consecutive first minors of the determinant of equations (25) to that determinant, and put

$$K = \frac{1}{2} \{ (c_1, c_1) c_1^2 + 2(c_1, c_2) c_1 c_2 + \dots \} \quad (26).$$

Then

$$\left. \begin{aligned} \dot{s}_1 = \frac{\partial K}{\partial c_1} - (\dot{q}_1 A_1 + \dot{q}_2 B_1 + \dots) \\ \dot{s}_2 = \frac{\partial K}{\partial c_2} - (\dot{q}_1 A_2 + \dot{q}_2 B_2 + \dots) \\ \dots \end{aligned} \right\} \quad (27);$$

where

$$\left. \begin{aligned} A_1 = (c_1, c_1)(q_1, s_1) + (c_1, c_2)(q_1, s_2) + \dots, \quad A_2 = (c_2, c_1)(q_1, s_1) \\ + (c_2, c_2)(q_1, s_2) + \dots \\ B_1 = (c_1, c_1)(q_2, s_1) + (c_1, c_2)(q_2, s_2) + \dots, \quad B_2 = (c_2, c_1)(q_2, s_1) \\ + (c_2, c_2)(q_2, s_2) + \dots \\ \dots \end{aligned} \right\} \quad (28).$$

17. Let now T' denote the expression for the kinetic energy when $\dot{s}_1, \dot{s}_2, \dots$ have been expressed as in (27), and T denote the former expression. Since $\dot{q}_1, \dot{q}_2, \dots$, appear in T' exactly as they did before the substitutions, and now appear besides in consequence of the substitutions, we have

$$\frac{\partial T'}{\partial \dot{q}_1} = \frac{\partial T}{\partial \dot{q}_1} + c_1 \frac{\partial \dot{s}_1}{\partial \dot{q}_1} + c_2 \frac{\partial \dot{s}_2}{\partial \dot{q}_2} + \dots, \quad \frac{\partial T'}{\partial \dot{q}_2} = \frac{\partial T}{\partial \dot{q}_2} + c_1 \frac{\partial \dot{s}_1}{\partial \dot{q}_1} + c_2 \frac{\partial \dot{s}_2}{\partial \dot{q}_2} + \dots, \quad (29);$$

where for $\dot{s}_1, \dot{s}_2, \dots$, their values from (27) are supposed to have been inserted. Thus

$$\frac{d}{dt} \frac{\partial T'}{\partial \dot{q}_1} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_1} + \frac{d}{dt} \left\{ \Sigma \left(c \frac{\partial \dot{s}}{\partial \dot{q}_1} \right) \right\} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_1} - \Sigma \left(c \frac{dA}{dt} \right), \dots$$

by (27), so that

$$\left. \begin{aligned} \frac{d}{dt} \frac{\partial T'}{\partial \dot{q}_1} + \Sigma \left(c \frac{dA}{dt} \right) &= \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_1} \\ \frac{d}{dt} \frac{\partial T'}{\partial \dot{q}_2} + \Sigma \left(c \frac{dB}{dt} \right) &= \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_2} \\ \dots \dots \dots \end{aligned} \right\} \dots \dots \dots (30).$$

Now from the original expression for T we have

$$\left. \begin{aligned} \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_1} - \phi_1 T &= Q_1 \\ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_2} - \phi_2 T &= Q_2 \\ \dots \dots \dots \end{aligned} \right\} \dots \dots \dots (31).$$

But if we substitute in $\phi_1 T$, for example, the values of $\dot{s}_1, \dot{s}_2, \dots$, from (27) we obtain after reduction

$$\begin{aligned} -\phi_1 T &= -[\dot{a}_1 \{a_1 - \Sigma(eA)\} + \dot{b}_1 \{b_1 - \Sigma(fA)\} + \dots] \dot{q}_1 \\ &\quad - [\dot{a}_1 \{a_2 - \Sigma(eB)\} + \dot{b}_1 \{b_2 - \Sigma(fB)\} + \dots] \dot{q}_2 \\ &\quad \dots \dots \dots \\ &\quad - [\dots] \dot{q}_k \end{aligned} \quad (32).$$

If T' be subjected to the same operation ϕ_1 , with the values of the coefficients of $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_k$ as they exist after the substitution for $\dot{s}_1, \dot{s}_2, \dots$, we get

$$\begin{aligned} -\phi_1 T' &= -\frac{d}{dt} \{a_1 - \Sigma(eA)\} [\{a_1 - \Sigma(eA)\} \dot{q}_1 + \{a_2 - \Sigma(eB)\} \dot{q}_2 + \dots] \\ &\quad - \frac{d}{dt} \{b_1 - \Sigma(fA)\} [\{b_1 - \Sigma(fA)\} \dot{q}_1 + \{b_2 - \Sigma(fB)\} \dot{q}_2 + \dots] \\ &\quad \dots \dots \dots \end{aligned} \quad (33).$$

Collecting on the right of this expression the coefficients of $\dot{q}_1, \dot{q}_2, \dots$, we see by (32) that

$$-\phi_1 T' = -\phi_1 T + \Sigma \left(e \frac{dA}{dt} + A \frac{de}{dt} \right) \dot{u} + \Sigma \left(f \frac{dA}{dt} + A \frac{df}{dt} \right) \dot{v} + \dots$$

Now

$$\left(e_1 \dot{u} + f_1 \dot{v} + \dots \right) \frac{dA_1}{dt} = \frac{\partial T}{\partial \dot{s}_1} \frac{dA_1}{dt} = c_1 \frac{dA_1}{dt} \quad \dots \quad (34).$$

Also by (22)

$$\dot{u} \frac{de}{dt} + \dot{v} \frac{df}{dt} + \dots = 0;$$

and therefore

$$-\phi_1 T' = -\phi_1 T + \Sigma \left(c \frac{dA}{dt} \right),$$

or

$$-\phi_1 T' - \Sigma \left(c \frac{dA}{dt} \right) = -\phi_1 T.$$

But by (30) and (31) this gives

$$\left. \begin{aligned} \frac{d}{dt} \frac{\partial T'}{\partial \dot{q}_1} - \phi_1 T' &= Q_1 \\ \frac{d}{dt} \frac{\partial T'}{\partial \dot{q}_2} - \phi_2 T' &= Q_2 \\ \dots \dots \dots \end{aligned} \right\} \dots \dots \dots (35).$$

Hence we have the very simple rule: modify the expression for T by substituting for $\dot{s}_1, \dot{s}_2, \dots$, their values from (27), and then proceed as if no velocities of co-ordinates s_1, s_2, \dots , had ever entered into the expression for the kinetic energy. The equations of motion obtained are of course applicable also to holonomous systems.

18. To verify the results obtained, we write the first of (35) in the form

$$\frac{d}{dt} \frac{\partial T'}{\partial \dot{q}_1} - \frac{\partial T'}{\partial q_1} - \left(\phi_1 T' - \frac{\partial T}{\partial q_1} \right) = Q_1 \quad \dots \quad (36),$$

and consider what it becomes when the system is made holonomous. We have

$$\begin{aligned} -\phi_1 T' &= -\phi_1 T + \Sigma \left(c \frac{dA}{dt} \right) \\ &= -(\dot{a}_1 \dot{u} + \dot{b}_1 \dot{v} + \dots) + \Sigma \left(c \frac{dA}{dt} \right) \quad \dots \quad (37); \end{aligned}$$

and since we have also now

$$\begin{aligned} u &= \{a_1 - \Sigma(eA)\} \dot{q}_1 + \{a_2 - \Sigma(eB)\} \dot{q}_2 + \dots \\ v &= \{b_1 - \Sigma(fA)\} \dot{q}_1 + \{b_2 - \Sigma(fB)\} \dot{q}_2 + \dots \\ &\dots \dots \dots \end{aligned}$$

we obtain

$$\begin{aligned} \frac{\partial T}{\partial q_1} &= \dot{u} \left[\left\{ \frac{\partial a_1}{\partial q_1} - \frac{\partial}{\partial q_1} \Sigma(eA) \right\} \dot{q}_1 + \left\{ \frac{\partial a_2}{\partial q_1} - \frac{\partial}{\partial q_1} \Sigma(eB) \right\} \dot{q}_2 + \dots \right] \\ &+ \dot{v} \left[\left\{ \frac{\partial b_1}{\partial q_1} - \frac{\partial}{\partial q_1} \Sigma(fA) \right\} \dot{q}_1 + \left\{ \frac{\partial b_2}{\partial q_1} - \frac{\partial}{\partial q_1} \Sigma(fB) \right\} \dot{q}_2 + \dots \right] \\ &+ \dots \dots \dots \quad (38). \end{aligned}$$

19. As a first and very simple example, we take the motion of a particle of mass m in a plane curve. If at time t the radius-vector drawn from a fixed point be of length r , and make an angle θ with an axis of x drawn from the same origin, the co-ordinates of the particle are

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Hence for the kinetic energy T we have

$$2T = m\{(\dot{r} \cos \theta - r\dot{\theta} \sin \theta)^2 + (\dot{r} \sin \theta + r\dot{\theta} \cos \theta)^2\} \quad (44),$$

or

$$2T = m(\dot{r}^2 + r^2\dot{\theta}^2) \quad (45).$$

In applying (13) to the problem of finding the r , θ , equations of motion of the particle, we have to take the first expression for the kinetic energy. We obtain

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{r}} = m\ddot{r}.$$

By (13) we have to subtract from this

$$m(\dot{r} \cos \theta - r\dot{\theta} \sin \theta) \frac{d}{dt}(\cos \theta) + m(\dot{r} \sin \theta + r\dot{\theta} \cos \theta) \frac{d}{dt}(\sin \theta);$$

that is, $m r \dot{\theta}^2$. The same result would, of course, be obtained by calculating $\partial T / \partial r$. Hence the r -equation of motion is

$$m(\ddot{r} - r\dot{\theta}^2) = R;$$

where R is the applied force in the outward direction along r .

For the θ -equation we have

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} = m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}).$$

By (13) we have to subtract from this

$$-m(\dot{r} \cos \theta - r\dot{\theta} \sin \theta) \frac{d}{dt}(r \sin \theta) + m(\dot{r} \sin \theta + r\dot{\theta} \cos \theta) \frac{d}{dt}(r \cos \theta)$$

or zero. thus the θ -equation of motion is

$$m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = \Theta;$$

where Θ is the applied force perpendicular to the radius-vector.

This method, if it had been applied to the value of T in (45), would have failed. T is here a sum of squares referred to a set of axes so specialised that in the formation of T the quantities $\cos \theta$, $\sin \theta$ have taken the special values 1 and 0; and unless we go back to the fundamental expressions, for the velocities along the unspecialised axes Ox , Oy , it is not apparent how the process is to be carried out.

It will be observed that from (44) we have

$$\frac{\partial}{\partial \theta}(\cos \theta) = -\frac{\partial}{\partial r}(r \sin \theta), \quad \frac{\partial}{\partial \theta}(\sin \theta) = \frac{\partial}{\partial r}(r \cos \theta)$$

so that the integrability conditions are fulfilled. Thus it is possible to proceed in the ordinary way by calculating $\partial T/\partial r$ and subtracting it from $m\ddot{r}$. The function of r involved in (44) and (45) is the same, and so in the latter case the ordinary process remains applicable, though then, *apparently*, the integrability conditions seem unfulfilled. This explains why in many cases, *e.g.* in the next example, when specialised axes are taken, the ordinary method is applicable, while the other, set forth in § 9 above, is not. The latter can only be applied when the values of \dot{u} , \dot{v} , are perfectly general.

20. As a first example we take the gyrostatic pendulum problem referred to above. The pendulum as ordinarily made is a rigid body symmetrical about a longitudinal axis, and containing a fly-wheel with its axis of rotation along the axis of symmetry. The suspension is by a Hooke's joint, or by means of a piece of steel wire so short that it may be taken as untwistable, while yielding equally freely to bending forces in all vertical planes containing the wire.

Let θ be the inclination of the axis of symmetry to the vertical, ϕ the angle which the vertical plane through this axis makes with a fixed plane through the vertical containing the point of support, ψ the angle which a plane containing the axis of the fly-wheel, and fixed in the wheel, makes with an axial plane fixed in the pendulum. We shall not suppose in the first instance that the pendulum, apart from the fly-wheel, is symmetrical, but take C as its moment of inertia about the axis of the wheel, which we shall suppose to be a principal axis of moment of inertia, and A and B as the other two principal axes for the point of support. We shall also denote the moment of inertia of the fly-wheel about its axis by C' , and its moment of inertia about any axis at right angles to this through the point of support by A' . It is easy to show that the angular velocity of the pendulum (apart from the fly-wheel) about the axis of symmetry is $-\dot{\phi}(1 - \cos \theta)$. That of the fly-wheel about the same axis is $\dot{\psi} - \dot{\phi}(1 - \cos \theta)$.

21. We suppose now that the principal axis about which the moment of inertia is A is inclined at the instant under consideration at an angle ϕ to the vertical plane containing the axis of the fly-wheel, so that if without other change of position of that plane the axis of rotation of the wheel were brought to the vertical this principal axis would lie in the plane from which ϕ is measured. The pendulum is turning with angular velocity $\dot{\theta}$ about an axis perpendicular to the vertical plane through the axis of the fly-wheel, and with angular velocity $\dot{\phi} \sin \theta$ about an axis in that plane and at right angles to the fly-wheel axis. The angular velocities about the axes of A and B (taken as related to the third axis as the

usual axes Ox , Oy , are to Oz) are therefore $\dot{\theta} \sin \phi + \dot{\phi} \sin \theta \cos \phi$ and $-\dot{\theta} \cos \phi + \dot{\phi} \sin \theta \sin \phi$ respectively.

The expression for the kinetic energy can now be written down, and is

$$T = \frac{1}{2} \{ A(\dot{\theta} \sin \phi + \dot{\phi} \sin \theta \cos \phi)^2 + B(-\dot{\theta} \cos \phi + \dot{\phi} \sin \theta \sin \phi)^2 + C(1 - \cos \theta)^2 \dot{\phi}^2 \} \\ + \frac{1}{2} \{ A'(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + C'(\dot{\psi} - \dot{\phi}(1 - \cos \theta))^2 \} \quad . \quad . \quad . \quad (46).$$

From this, by the ordinary process, since the co-ordinates of any particle of the body are connected with the angles here specified by finite relations, we can obtain the equations of motion; and this, of course, is the simplest mode of proceeding. All the data for forming the equations in this manner will be found worked out in Thomson and Tait, § 330. The foregoing specification is only given here for the sake of the following new discussion of the problem.

22. We refer the motion of any particle of mass m in the body to fixed axes coinciding for the instant under consideration with the principal axes A , B , C , and denote the angular velocities about these axes by p , q , r . The corresponding angular velocities for the fly-wheel are p , q , r' . If x , y , z , be the co-ordinates with reference to these axes of a particle of mass m in the pendulum apart from the fly-wheel, and x' , y' , z' , those of a particle of mass m' in the fly-wheel, we have

$$\dot{x} = qz - ry, \quad \dot{y} = rx - pz, \quad \dot{z} = py - qx, \\ \dot{x}' = qz' - r'y', \quad \dot{y}' = r'x' - pz', \quad \dot{z}' = py' - qx'.$$

Hence

$$2T = \Sigma [m \{ (qz - ry)^2 + (rx - pz)^2 + (py - qx)^2 \}] \\ + \Sigma [m' \{ (qz' - r'y')^2 + (r'x' - pz')^2 + (py' - qx')^2 \}] \quad . \quad . \quad . \quad (47);$$

where the second line refers to the fly-wheel and the first to the rest of the pendulum.

From this we obtain

$$\frac{\partial T}{\partial p} = \Sigma [m \{ -(rx - pz)z + (py - qx)y \}] + \Sigma [m' \{ -(r'x' - pz')z' + (py' - qx')y' \}].$$

Calculating the total time-rate of variation of this, and taking account of the fact that the axes of reference coincide with the principal axes of the body, we get

$$\frac{d}{dt} \frac{\partial T}{\partial p} = \Sigma [m \{ -\dot{z}(rx - pz) - z(-\dot{p}z + r\dot{x} - p\dot{z}) \}] \\ + \Sigma [m \{ \dot{y}(py - qx) + y(\dot{p}y + p\dot{y} - q\dot{x}) \}] \\ + \text{similar expressions for fly-wheel} \quad . \quad . \quad . \quad (48)$$

From this we have to subtract, according to (13) above,

$$\Sigma [m \{ -\dot{z}(rx - pz) + \dot{y}(py - qx) \}] + \Sigma [m' \{ -\dot{z}'(r'x' - pz') + \dot{y}'(py' - qx') \}].$$

With the substitutions $\dot{y} = rx - pz$, $\dot{z} = py - qx$, $\dot{y}' = r'x' - pz'$, $\dot{z}' = py' - qx'$, $Bq = \Sigma\{m(\dot{x}z - \dot{z}x)\}$, $A'q = \Sigma\{m(\dot{x}'z' - \dot{z}'x)\}$, the result becomes

$$(A + A')\dot{p} - (B - C + A' - C')qr + Cq(r' - r).$$

This is to be equated to the moment L of external forces round the axis of A. Hence we get

$$(A + A')\dot{p} - (B + A' - C - C')qr + C'q(r' - r) = L \quad (49).$$

If now we suppose the angle ϕ to be $\pi/2$, and insert (see § 21) $\ddot{\theta} - \dot{\phi}^2 \sin \theta$ for \dot{p} , $\dot{\phi} \sin \theta$ for q , $-\dot{\phi}(1 - \cos \theta)$ for r , $\dot{\psi} - \dot{\phi}(1 - \cos \theta)$ for r' , we get the θ -equation of motion. Similarly the other equations can be obtained.

If we suppose the pendulum to be symmetrical about the axis of the fly-wheel, (49) becomes, with the above substitutions,

$$A\ddot{\theta} - \{(C + C')(1 - \cos \theta) \sin \theta + A \sin \theta \cos \theta\} \dot{\phi}^2 + C' \sin \theta \dot{\psi} \dot{\phi} = L \quad (50);$$

where A is now used in the sense assigned above to $A + A'$.

It will be observed that the process indicated in (13) could not be applied to the terms in (46) of which the kinetic energy is there made up. But in that expression for the kinetic energy the functions of the co-ordinates θ , ϕ , are exactly the same as those which would appear in the expression to which (13) is applicable.

23. If $A' = 0$, $C' = 0$, that is, if there be no fly-wheel, (49) becomes the first of the set of three equations

$$\left. \begin{aligned} A\dot{p} - (B - C)qr &= L \\ B\dot{q} - (C - A)rp &= M \\ C\dot{r} - (A - B)pq &= N \end{aligned} \right\} \quad (51),$$

which can all be obtained by the same process. The angular velocities p , q , r , are about fixed axes with which, at the instant under consideration, the principal axes of moment of inertia coincide. They are, of course, Euler's well-known equations of motion for a rigid body one point of which is fixed.

24. As another example, we take the equations of motion of the pendulum and of the fly-wheel about the axis. It is clear, by applying the third of (51), remembering that N is zero and $A = B$ in each case, or by making the calculation indicated in (13) and then substituting as in § 21 that we have

$$C\dot{r} = 0, \text{ or } Cr = c,$$

for the pendulum, and

$$C'\dot{r}' = 0, \text{ or } C'r' = c',$$

for the fly-wheel.

Substituting c/C for r and c'/C' for r' in (49), we see that since the

terms involving r and r' go out on account of the vanishing of the products of inertia, no change in the process which gives (49) is effected.

Equations (49) and (50) have their notation changed. The latter becomes

$$A(\ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2) + (c + c') \sin \theta \cdot \dot{\phi} = L \quad \dots \quad (52).$$

25. An example of two distinct cases of motion in which the expressions for the kinetic and potential energy are the same, has been given by Appell (*Mécanique Rationnelle*, tome ii., art. 469). The first is a disk or hoop of mass m , which is symmetrical about an axis perpendicular to its plane, and rolls without sliding along a horizontal plane, on a sharp edge which forms the terminating circle of the plane drawn through the centroid perpendicular to the axis. In the other case the disk or hoop rests with the edge on a horizontal plane without friction, while its centroid is constrained to move along a vertical circle of which the radius is the same as that of the circular edge. By properly choosing the moments of inertia in the second case, the kinetic energy can be made the same as in the former case, and since the mass of the body is taken the same in both cases, the potential energy is identical also with that in the former.

In the latter case, Lagrange's equations are directly applicable in their ordinary form; in the former case, they are not.

I shall here sketch the modified Lagrangian solution of the problem of the hoop or disk, in order to compare it with an elementary mode of solving gyrostatic problems which I have found readily applicable even in cases of very considerable complication. The Lagrangian solution does not differ, except in notation, procedure, and arrangement, from that contained in the paper by Ferrers referred to above.

26. We refer the motion of the centroid to rectangular axes Ox , Oy , Oz , in and perpendicular to the plane in which the hoop rolls, with origin O at the point of contact, and denote the co-ordinates of the centroid by x , y , z . Let ϕ be the angle which the vertical plane through the axis of the hoop makes with the fixed vertical plane containing Ox , and ψ be the angular velocity of the hoop relatively to the former plane. The angular velocity of the hoop about its axis of figure is thus $\dot{\psi} + \dot{\phi} \cos \theta$.

A consideration of the geometry of the problem shows that the following equations hold:—(1) If the inclination, θ say, of the axis of the hoop to the vertical remain unaltered, and the hoop roll through an angle $\delta\chi$,

$$\delta x_1 = -y \delta\chi, \quad \delta y_1 = y \cot \phi \cdot \delta\chi.$$

(2) Due to the alteration of θ we have

$$\delta x_2 = -a \sin \theta \cos \phi \cdot \delta\theta, \quad \delta y_2 = -a \sin \theta \sin \phi \cdot \delta\theta, \quad \delta z = a \cos \theta \delta\theta.$$

Combining these, we get

$$\left. \begin{aligned} \dot{x} &= -a(\dot{\psi} + \dot{\phi} \cos \theta) \sin \phi - a \sin \theta \cos \phi \cdot \dot{\theta} \\ \dot{y} &= a(\dot{\psi} + \dot{\phi} \cos \theta) \cos \phi - a \sin \theta \sin \phi \cdot \dot{\theta} \\ \dot{z} &= a \cos \theta \cdot \dot{\theta} \end{aligned} \right\} \dots \dots \dots (53),$$

which are the kinematical conditions.

The kinetic energy of the motion of the centroid is

$$\begin{aligned} T_c &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &= \frac{1}{2}ma^2[(\sin \theta \cos \phi \cdot \dot{\theta} + \cos \theta \sin \phi \cdot \dot{\phi} + \sin \phi \cdot \dot{\psi})^2 + (-\sin \theta \sin \phi \cdot \dot{\theta} \\ &\quad + \cos \theta \cos \phi \cdot \dot{\phi} + \cos \phi \cdot \dot{\psi})^2 + \cos^2 \theta \cdot \dot{\theta}^2] \end{aligned} \quad (54).$$

This reduces to

$$T_c = \frac{1}{2}ma^2(\dot{\theta}^2 + \cos^2 \theta \cdot \dot{\phi}^2 + \dot{\psi}^2 + 2 \cos \theta \cdot \dot{\phi}\dot{\psi}) \dots \dots \dots (54'),$$

but for our present purpose it is necessary to leave it in the expanded form.

Along with this we have the kinetic energy of rotation

$$T_r = \frac{1}{2}\{A(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + C(\dot{\psi} + \dot{\phi} \cos \theta)^2\} \dots \dots \dots (55);$$

where C is the moment of inertia of the body about its axis of symmetry, and A that about any other axis at right angles to the axis of symmetry and passing through the centroid. Then

$$T = T_c + T_r \dots \dots \dots (56).$$

27. It will be seen from an examination of the expression for T_c above, that the integrability conditions are fulfilled as between θ and ϕ , and θ and ψ . As regards T_r , the relative co-ordinates are integral functions of θ, ϕ, ψ ; and the ordinary methods apply. Hence the θ -equation for the hoop or disk can be found in the ordinary way by the equation

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = - \frac{\partial V}{\partial \theta} \dots \dots \dots (57);$$

where $V (= mga \sin \theta)$ is the potential energy.

The ϕ and ψ equations are, however,

$$\left. \begin{aligned} \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T_r}{\partial \phi} + ma \left\{ \dot{x} \frac{d}{dt}(\cos \theta \sin \phi) - \dot{y} \frac{d}{dt}(\cos \theta \cos \phi) \right\} &= 0 \\ \frac{d}{dt} \frac{\partial T}{\partial \dot{\psi}} + ma \left\{ \dot{x} \frac{d}{dt}(\sin \phi) + \dot{y} \frac{d}{dt}(\cos \phi) \right\} &= 0 \end{aligned} \right\} \dots \dots \dots (58),$$

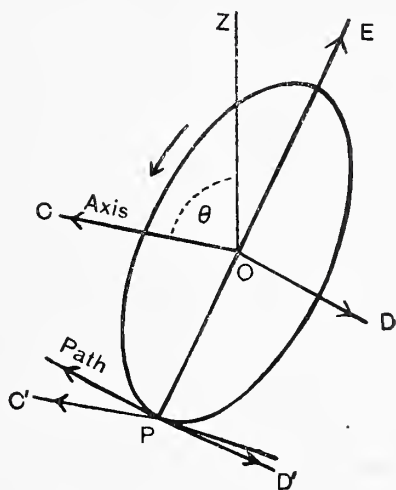
since $\partial T_r / \partial \psi = 0$. The last two equations may be written out in full by the reader. The ψ -equation reduces to

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\psi}} - ma^2 \sin \theta \cdot \dot{\theta} \dot{\phi} = 0.$$

The θ -equation (57) worked out has the form

$$(A + ma^2)\ddot{\theta} + (C + ma^2 - A)\dot{\phi}^2 \sin \theta \cos \theta + (C + ma^2)\dot{\phi}\dot{\psi} \sin \theta = - mga \cos \theta \quad (59).$$

28. The last equation is written down for the sake of the comparison of the Lagrangian method, modified or not, with the following elementary, and I think instructive, method of solving gyrostatic and other rotational problems. It depends on the single principle, which can be demonstrated in a moment, that if a directed quantity L be associated with an axis, OA say, which is turning towards a second axis, OB say, at right angles to the first, with angular velocity ω , there is, as a consequence, a rate of growth of the quantity in the direction OB of amount $L\omega$. If there is already associated with this second direction, at the instant considered, a component of the same quantity M , which is changing at rate \dot{M} , the total rate of growth of the vector associated with the instantaneous direction of OB is $\dot{M} + L\omega$. [It is to be understood that \dot{M} is taken following the direction OB as that revolves keeping itself at right angles to OA .] This, of course, is a particular case of the proposition that if $L_x = L_1 \cos \theta_1 + L_2 \cos \theta_2 + L_3 \cos \theta_3$, then $\dot{L}_x = \dot{L}_1 \cos \theta_1 + \dot{L}_2 \cos \theta_2 + \dot{L}_3 \cos \theta_3 - L \sin \theta_1 \cdot \dot{\theta}_1 - \dots$. But it forms by itself a simple rule, immediately evident, and applicable to all kinds of rotational problems.



The simplest possible example of this is a particle of mass m moving in a circle of radius r with speed v . At a point P on the circle the particle has momentum mv in the forward direction along the tangent. But the tangential direction of motion of the particle is turning towards the radius at P with angular velocity v/r . Hence the rate of growth of momentum in the direction from P to the centre is $mvv/r = mv^2/r$.

If, as there may be, there are two different directions OA , perpendicular to one another and to OB , with which are associated directed quantities of the same kind L , L' , and these are turning towards OB with angular velocities ω , ω' , there will exist in consequence a rate of growth of the quantity along OB of amount $L\omega + L'\omega'$ in addition to \dot{M} .

29. Now apply this principle to the problem of the hoop. Take axes through the point of contact P with the horizontal plane on

which the hoop rolls—one parallel to the axis of figure, one tangential to the hoop in the plane of rolling, and one at right angles to these through the centroid. These axes are represented by PC' , PD' , PE in the figure (p. 345). About PC' the angular velocity is $\dot{\psi} + \dot{\phi} \cos \theta$, and the angular momentum is $(C + ma^2)(\dot{\psi} + \dot{\phi} \cos \theta)$; and in consequence of the turning at angular velocity $\dot{\phi} \sin \theta$ about PE there is a rate of growth $(C + ma^2)(\dot{\psi} + \dot{\phi} \cos \theta)\dot{\phi} \sin \theta$ of angular momentum about the instantaneous position of PD' . Again, in consequence of the turning with angular velocity $\dot{\phi} \cos \theta$ about PC' the axis PE is moving away from PD' , and angular momentum about PD' is growing from this cause at rate $-A\dot{\phi}^2 \sin \theta \cos \theta$. The angular velocity $\dot{\theta}$ about PD is changing at rate $\ddot{\theta}$ as PD' moves, and from this cause there is a rate of growth $(A + ma^2)\ddot{\theta}$ about PD' .

The total rate of growth of angular momentum about PD' , or, more strictly, about the fixed direction with which PD' instantaneously coincides, is thus

$$(A + ma^2)\ddot{\theta} + (C + ma^2 - A)\dot{\phi}^2 \sin \theta \cos \theta + (C + ma^2)\dot{\phi}\dot{\psi} \sin \theta;$$

and this is equal to the moment of applied forces $-mga \cos \theta$, so that we get the same equation of motion as before [(59) above]. In the same manner the $\dot{\phi}$ and $\dot{\psi}$ equations might be established.

30. As another example, we may apply this elementary method to the gyrostatic pendulum. Let O be the point of support, and refer to axes OC , that of symmetry of the arrangement of mass, OE at right angles to OC in the vertical plane through the axis of symmetry, and another OD at right angles to the plane just specified. Then find the rate of growth of angular momentum about OD due to the three moving axes. The rate of turning about OD is $\dot{\theta}$, whether we regard OD as a moving axis or as the fixed axis with which at the instant the moving axis coincides. On the other hand, the rate of alteration of $\dot{\theta}$, associated with the moving axis, is $\ddot{\theta}$. At time dt after the instant considered, the angular momentum about the altered position of OD is $A(\dot{\theta} + \ddot{\theta}dt)$, and hence the component of angular momentum about the old direction of OD contributed by OD in its new position is $A(\dot{\theta} + \ddot{\theta}dt) \cos(\dot{\theta}dt)$ or $A(\dot{\theta} + \ddot{\theta}dt)$. The rate of growth of angular momentum about the instantaneous position of OD at time t is thus $A\ddot{\theta}$.

There are two other sources of growth of angular momentum about this direction. If the motion of the vertical plane containing the axis of symmetry, as regarded by an eye placed below the pendulum, be in the counter-clockwise direction, and the axes OD , OE , OC , taken in this order, form a usual system of axes, the axis OC is, in consequence of the motion

of the system with angular velocity $\dot{\phi} \sin \theta$ about OE, turning towards the instantaneous position of OD. But associated with OC is angular momentum $-C(1 - \cos \theta)\dot{\phi} + C'\{\dot{\psi} - \dot{\phi}(1 - \cos \theta)\}$, where (as explained above, § 20) C refers to the pendulum apart from the fly-wheel, and C' to the fly-wheel. Hence this motion gives a growth of angular momentum about that instantaneous position at rate

$$[-C(1 - \cos \theta)\dot{\phi} + C'\{\dot{\psi} - \dot{\phi}(1 - \cos \theta)\}]\dot{\phi} \sin \theta.$$

Again, the axis OE is moving away from the instantaneous position of OD with angular velocity $\dot{\phi} \cos \theta$, and there is associated with OE, in this position, angular momentum $A\dot{\phi} \sin \theta$. Hence angular momentum about the instantaneous direction of OD is growing from this cause at rate $-A\dot{\phi}^2 \sin \theta \cos \theta$.

From all these causes combined, the rate of growth of angular momentum about the fixed direction with which OD coincides at the instant is

$$A\ddot{\theta} - \{A \cos \theta + (C + C')(1 - \cos \theta)\} \sin \theta \cdot \dot{\phi}^2 + C'\dot{\phi}\dot{\psi} \sin \theta;$$

and this equated to L, the moment of the applied forces about OD, gives the θ -equation of motion. It will be seen that this result agrees with (50) above.

In precisely the same manner the other equations of motion of the pendulum could be obtained.

It will be noticed that the terms which (apart from $A\ddot{\theta}$) are obtained in these two examples from the motion of the axes are those contributed in the Lagrangian method by the term $-\partial T/\partial \theta$. This affords an interpretation of the term in question, and of corresponding terms in the equations of motion in other cases. [See also § 19.]

31. I may notice here the fact, which I have pointed out elsewhere, that this simple principle enables Euler's equations of motion for a rigid body turning about a fixed point to be established intuitively and without analysis. Refer to principal axes passing through the fixed point and turning with the body. Let A, B, C, be the moments of inertia, and $\omega_1, \omega_2, \omega_3$, the angular velocities about these axes. Angular momentum is growing about the first axis at rate $A\dot{\omega}_1$, in consequence of the rate of change of ω_1 . But as the body moves the third axis, about which the angular momentum is $C\omega_3$, is turning towards the instantaneous position of the first with angular velocity ω_2 about the second, and hence angular momentum is growing about the first direction at rate $C\omega_2\omega_3$. Also the second axis, about which the angular momentum is $B\omega_2$, is turning away from the instantaneous position of the first owing to the turning with angular velocity ω_3 about the third, and hence angular momentum is

growing about the first direction at rate $-B\omega_2\omega_3$. Thus, equating the total rate of growth of angular momentum about the first axis arising from these causes to the moment L of applied forces, we get

$$A\dot{\omega}_1 - (B - C)\omega_2\omega_3 = L.$$

Similarly the other equations are obtained.

The same kind of intuitive process is exemplified by the quaternion or vector treatment of motion; but the discussion above seems to me to give a "common sense" view of the subject, unconnected with any formal process of analysis. It certainly enables a large number of problems of rotation—tops, gyrostats, etc.—to be solved with great ease and directness.

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XX.—On Energy Accelerations and Partition of Energy.

By C. W. Follett. *Communicated by Professor W. PEDDIE.*

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(1) IN the volume of the *Arch. Néerlandaises* for 1900, Dr Bryan discussed the investigation of the distribution of the velocities of a dynamical system for a stationary state from probability considerations.

He suggested the method of obtaining expressions for the second differential coefficients with respect to the time of squares and products of velocities such as enter into the expression for the kinetic energy, and finding from them equations of energy equilibrium, as well as conditions of stability for the stationary state.

He applied the method to a few simple examples, and indicated in these the preponderance of conditions for non-equipartition of energy.

Dr Peddie, in two papers (*Proc. Roy. Soc. Edin.*, vol. xxvi., 1905-6, and vol. xxvii., 1906-7), arrived at similar conclusions by an entirely different method. Dr Bryan in his paper advanced the view that the conditions of stability may possibly afford the true clue to the partition of energy problem, while the limitations required may explain the failure of Maxwell's Law of Equipartition to account for many physical phenomena.

The present paper contains an extension of Dr Bryan's work to the case of two particles moving in a field of force and under the influence of their own attraction (or repulsion), and applications of the results obtained to certain illustrative examples. It is necessary to make certain hypotheses which will be defined in the course of the work, as the complications which ensue remove the possibility of treating the problem in its complete generality; the assumptions which will be made cannot, for instance, be justified in the case of a Newtonian field of force.

The method adopted should have an important application to the theory of gases especially; the two particles may, for instance, be considered to form a di-atomic gas-molecule.

We shall use the term molecule to denote the system of two particles under consideration; the two particles themselves may be defined as atoms.

(2) The probability that the co-ordinates of the system shall lie between assigned limits is supposed to be given, and is of course a function of the

controllable co-ordinates; the probability must depend on the state of the body, its crystalline structure, chemical composition, etc.

Dr Bryan, in the paper to which I have referred, says: "The distribution of co-ordinates at any instant is not theoretically really independent of the distribution of velocities at that instant. As, however, it is physically impossible to control or observe the motions of individual molecules, we may assume that the proper measure of the probable distribution of co-ordinates, estimated according to the best of our knowledge of the state of a body, is one which does not depend on the motions of the molecules, and remains independent of the time so long as the energy and controllable co-ordinates of the system are constant." This assumption is found convenient for mathematical calculation.

Let us suppose that the co-ordinates of any system are a, b, c, \dots and the corresponding generalised momenta are p, q, \dots

A quantity of the form $F(a, b, \dots, p, q, \dots) da db dc \dots dp dq \dots$ is the probability that the co-ordinates and momenta of the system may be between the limits a and $a+da$, b and $b+db$, \dots , p and $p+dp$, q and $q+dq$, \dots , as defined in Appendix C of a paper read by Dr Bryan at the Oxford meeting of the British Association (1894).

If we multiply a function of the co-ordinates and momenta by this quantity and integrate, we obtain its mean value.

(3) The system (or molecule) is defined as consisting of two particles at A and B of masses m_1 and m_2 . Let G, the centre of mass, be (x, y, z) , and let $AB = \gamma$, and for the six co-ordinates of the system let us take x, y, z, r, θ, ϕ ; so that the co-ordinates x_1, y_1, z_1 of A and (x_2, y_2, z_2) of B are given by

$$\begin{aligned} x_1 &= x + r_1 \sin \theta \cos \phi \\ y_1 &= y + r_1 \sin \theta \sin \phi \\ z_1 &= z + r_1 \cos \theta \\ x_2 &= x + r_2 \sin \theta \cos \phi \\ y_2 &= y + r_2 \sin \theta \sin \phi \\ z_2 &= z + r_2 \cos \theta, \end{aligned}$$

where

$$m_1 r_1 = -m_2 r_2, \quad r_1 = \frac{m_2 r}{m_1 + m_2}, \quad r_2 = \frac{-m_1 r}{m_1 + m_2}.$$

θ and ϕ are the angular co-ordinates of the straight line AB.

$$\begin{aligned} \dot{x}_1 &= \dot{x} + \dot{r}_1 \sin \theta \cos \phi + r_1 \cos \theta \cos \phi \dot{\theta} - r_1 \sin \theta \sin \phi \dot{\phi}, \\ \dot{y}_1 &= \dot{y} + \dot{r}_1 \sin \theta \sin \phi + r_1 \cos \theta \sin \phi \dot{\theta} + r_1 \sin \theta \cos \phi \dot{\phi}, \\ \dot{z}_1 &= \dot{z} + \dot{r}_1 \cos \theta - r_1 \sin \theta \dot{\theta}. \end{aligned}$$

$\dot{x}_2, \dot{y}_2,$ and \dot{z}_2 are obtained similarly.

The kinetic energy T

$$\begin{aligned} &= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) \\ &= \frac{1}{2}(m_1 + m_2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}\frac{m_1m_2}{m_1 + m_2} (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) \\ &= \frac{1}{2}(m_1 + m_2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2), \end{aligned}$$

where

$$M = \frac{m_1m_2}{m_1 + m_2}$$

The Force Function is specified as follows:—

Let the interior force of the system be an attractive force between the masses, and let $f(r)$ be the corresponding potential energy.

The exterior forces are supposed due to the action of other molecules or to external disturbances; if $\Phi(\xi, \eta, \zeta)$ be the potential of the field at any point (ξ, η, ζ) , we may suppose the potential energy due to this cause to be $\mu_1\Phi(x_1, y_1, z_1) + \mu_2\Phi(x_2, y_2, z_2)$.

The whole potential energy V is therefore equal to

$$\mu_1\Phi(x_1, y_1, z_1) + \mu_2\Phi(x_2, y_2, z_2) + f(r).$$

Forming Lagrange's equations, we have

$$\left. \begin{aligned} (m_1 + m_2)\ddot{x} &= -\frac{\partial V}{\partial x} \\ (m_1 + m_2)\ddot{y} &= -\frac{\partial V}{\partial y} \\ (m_1 + m_2)\ddot{z} &= -\frac{\partial V}{\partial z} \\ M \left\{ \ddot{r} - r(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right\} &= -\frac{\partial V}{\partial r} \\ M \left\{ \frac{1}{r} \frac{d}{dt}(r^2\dot{\theta}) - r \sin \theta \cos \theta \dot{\phi}^2 \right\} &= -\frac{1}{r} \frac{\partial V}{\partial \theta} \\ \frac{M}{r \sin \theta} \frac{d}{dt}(r^2 \sin^2 \theta \dot{\phi}) &= -\frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \end{aligned} \right\} \quad (1).$$

(4) Let the assumption be made that for any given position of either of the particles all directions of translation of the molecule are equally probable.

The actual motion is unknown, and we will apply the theory of probability to the probable motion.

Let us suppose that the probability that a particle is moving with a given velocity u_1 is equal to the probability that it is moving with the reverse velocity; in other words, that the probability of u_1 lying between \bar{u}_1 and $\bar{u}_1 + d\bar{u}_1$ is equal to the probability of its lying between $-\bar{u}_1$ and $-\bar{u}_1 - d\bar{u}_1$ in any direction.

We have assumed, therefore, that the probability that the motion is along Ox is the same as the probability that the motion is along any other axis, and that all transverse velocities of either particle perpendicular to AB are equally probable.

Let

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = u^2$$

and

$$r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 = q^2 ;$$

then, using square brackets to denote mean values, we have

$$[\dot{x}^2] = [\dot{y}^2] = [\dot{z}^2] = \frac{1}{3}[u^2]$$

and

$$[r^2 \dot{\theta}^2] = [r^2 \sin^2 \theta \dot{\phi}^2] = \frac{1}{2}[q^2],$$

where the mean values are formed according to the method laid down in § 2.

Also we must clearly have

$$\left[\frac{d}{dt} \left(\frac{1}{2} \dot{x}^2 \right) \right] = 0, \text{ etc.},$$

and the mean values of the six momenta and the fifteen products of momenta taken two together all obviously vanish.

The assumptions made, when interpreted physically, represent the property that the molecule under consideration is assumed to belong to an isotropic substance.

Let similar assumptions be made with regard to the force.

The functions $\Phi_1(x_1, y_1, z_1)$ and $\Phi_2(x_2, y_2, z_2)$ are constantly changing owing to the motions of the molecules, and it will be necessary to finally express our results in terms of the mean values of the differential coefficients of these functions with respect to the co-ordinates of the points.

Let us assume that our external field of force is an equally probable one; in other words, that the probability that the force has a certain value is the same as the probability that it has the reverse value, and that the probable force in one direction is equal to the probable force in another direction; so that

$$\left[\frac{\partial \Phi_1}{\partial x_1} \right] = 0 \quad \text{and} \quad \left[\frac{\partial \Phi_2}{\partial x_2} \right] = 0,$$

and since the axes might be taken in any directions,

$$\left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] = \left[\left(\frac{\partial \Phi_1}{\partial y_1} \right)^2 \right] = \left[\left(\frac{\partial \Phi_1}{\partial z_1} \right)^2 \right]$$

$$\left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] = \left[\frac{\partial^2 \Phi_1}{\partial y_1^2} \right] = \left[\frac{\partial^2 \Phi_1}{\partial z_1^2} \right],$$

also

$$\left[\frac{\partial^2 \Phi_1}{\partial x_1 \partial y_1} \right] = 0, \quad \text{but} \quad \left[\frac{\partial^2 \Phi_1}{\partial x_1}, \frac{\partial \Phi_2}{\partial x_2} \right] \neq 0 \text{ necessarily.}$$

Similar relations are, of course, true for $\Phi_2(x_2, y_2, z_2)$, and the mean values of $\frac{\partial\Phi_1}{\partial x_1} \frac{\partial\Phi_1}{\partial y_1}$ and $\frac{\partial\Phi_2}{\partial x_2} \frac{\partial\Phi_2}{\partial y_2}$ clearly vanish.

Also

$$\begin{aligned} \left[\left(\frac{\partial V}{\partial x} \right)^2 \right] &= \left[\left(\frac{\partial V}{\partial y} \right)^2 \right] = \left[\left(\frac{\partial V}{\partial z} \right)^2 \right] \\ \left[\frac{\partial^2 V}{\partial x^2} \right] &= \left[\frac{\partial^2 V}{\partial y^2} \right] = \left[\frac{\partial^2 V}{\partial z^2} \right], \\ \left[\frac{\partial^2 V}{\partial x \partial y} \right] &= 0, \quad \left[\frac{\partial^2 V}{\partial \theta \partial \phi} \right] = 0, \quad \left[\frac{\partial^2 V}{\partial x \partial \theta} \right] = 0, \text{ etc.}, \end{aligned}$$

but $\left[\frac{\partial V}{\partial r} \right] \neq 0$ in general, although $\left[\frac{\partial V}{\partial x} \right]$ and the other mean force components vanish.

(5) Let us form what Dr Bryan calls the accelerations of the energy components in the *Arch. Néerlandaises* to which I have referred. From Lagrange's equations we obtain

$$\frac{d}{dt} \left\{ \frac{1}{2}(m_1 + m_2)\dot{x}^2 \right\} = -\dot{x} \frac{\partial V}{\partial x}.$$

Differentiating again with respect to the time, we have

$$\frac{d^2}{dt^2} \left\{ \frac{1}{2}(m_1 + m_2)\dot{x}^2 \right\} = \frac{1}{m_1 + m_2} \left\{ \frac{\partial V}{\partial x} \right\}^2 - \dot{x} \frac{d}{dt} \left(\frac{\partial V}{\partial x} \right),$$

where $\frac{d}{dt} \left(\frac{\partial V}{\partial x} \right)$ represents the total change due to the motion of the particles and the variation of the field; and thus

$$\frac{d}{dt} \left(\frac{\partial V}{\partial x} \right) = \left(\frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z} + \dot{r} \frac{\partial}{\partial r} + r \dot{\theta} \frac{\partial}{\partial \theta} + r \sin \theta \dot{\phi} \frac{\partial}{\partial \phi} \right) \frac{\partial V}{\partial x},$$

where $\frac{\partial}{\partial t}$ represents the part due to the variation of the field alone. As, however, we are considering a statistically stationary state, positive and negative values of $\frac{\partial}{\partial t} \left(\frac{\partial V}{\partial x} \right)$ are equally probable independently of the value of x , and consistently with our assumed conditions the mean value of this term is to be put equal to zero. Therefore, taking mean values, we have in accordance with our assumptions

$$\left[\frac{d^2}{dt^2} \frac{1}{2}(m_1 + m_2)\dot{x}^2 \right] = \frac{1}{m_1 + m_2} \left[\left(\frac{\partial V}{\partial x} \right)^2 \right] - [\dot{x}^2] \left[\frac{\partial^2 V}{\partial x^2} \right]. \quad (2),$$

and the corresponding equations in y and z will give the same result.

Also, from Lagrange's equations

$$\frac{d}{dt} \left\{ \frac{1}{2} M \dot{r}^2 \right\} = M r \dot{r} (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - \dot{r} \frac{\partial V}{\partial r}.$$

Therefore $\frac{d^2}{dt^2} (\frac{1}{2} \dot{r}^2)$

$$\begin{aligned} &= r \ddot{r} (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) + \dot{r}^2 (\ddot{\theta}^2 + \sin^2 \theta \ddot{\phi}^2) + 2 r \dot{r} (\dot{\theta} \ddot{\theta} + \sin^2 \theta \dot{\phi} \ddot{\phi} + \sin \theta \cos \theta \dot{\theta} \dot{\phi}^2) \\ &\quad - \frac{1}{M} \frac{\partial V}{\partial r} \ddot{r} - \frac{\dot{r}}{M} \left(\frac{\partial}{\partial t} \frac{\partial V}{\partial r} + \dot{r} \frac{\partial^2 V}{\partial r^2} + \dot{r} \frac{\partial^2 V}{r \partial \theta \partial r} + r \sin \theta \dot{\phi} \frac{\partial^2 V}{r \sin \theta \partial \phi \partial r} \right. \\ &\quad \quad \quad \left. + \dot{x} \frac{\partial^2 V}{\partial x \partial r} + \dot{y} \frac{\partial^2 V}{\partial y \partial r} + \dot{z} \frac{\partial^2 V}{\partial z \partial r} \right) \\ &= \left\{ r (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - \frac{1}{M} \frac{\partial V}{\partial r} \right\}^2 - 3 \dot{r}^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - \frac{\dot{r}}{M} \left(\dot{r} \frac{\partial^2 V}{\partial r^2} + r \dot{\theta} \frac{\partial^2 V}{r \partial \theta \partial r} + \dots \right) \\ &\quad - \frac{2}{M} \dot{r} \dot{\theta} \frac{\partial V}{\partial \theta} - \frac{2}{M} \dot{r} \sin \theta \dot{\phi} \frac{\partial V}{r \sin \theta \partial \phi}. \end{aligned}$$

Taking mean values, we have

$$\begin{aligned} \left[\frac{d^2}{dt^2} (\frac{1}{2} \dot{r}^2) \right] &= \frac{1}{M^2} \left[\left(\frac{\partial V}{\partial r} \right)^2 \right] + [r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)^2] - 3 [\dot{r}^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)] - \frac{1}{M} \left[\dot{r}^2 \frac{\partial^2 V}{\partial r^2} \right] \\ &\quad - \frac{2}{M} \left[r (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \frac{\partial V}{\partial r} \right]. \end{aligned}$$

The Lagrangian equations referring to the angular co-ordinates will now be considered.

We have

$$M \frac{d}{dt} (r^2 \sin^2 \theta \dot{\phi}) = - 2 \dot{\phi} \frac{\partial V}{\partial \phi} - M \dot{\phi}^2 \frac{d}{dt} (r^2 \sin^2 \theta).$$

Therefore

$$M \frac{d^2}{dt^2} (r^2 \sin^2 \theta \dot{\phi}) = - 2 \dot{\phi} \frac{d}{dt} \left(\frac{\partial V}{\partial \phi} \right) - 2 M \dot{\phi} \frac{d}{dt} (r^2 \sin^2 \theta) \ddot{\phi} - 2 \ddot{\phi} \frac{\partial V}{\partial \phi} - M \dot{\phi}^2 \frac{d^2}{dt^2} (r^2 \sin^2 \theta).$$

Substituting in Lagrange's equations for \ddot{r} , $\ddot{\theta}$, and $\ddot{\phi}$, and simplifying, we obtain for $M \frac{d^2}{dt^2} (r^2 \sin^2 \theta \dot{\phi})$ the expression

$$\begin{aligned} &- \frac{\dot{\phi}}{r^2 \sin^2 \theta} \frac{d}{dt} (r^2 \sin^2 \theta) \frac{\partial V}{\partial \phi} - 2 \dot{\phi} r \sin \theta \frac{d}{dt} \left(\frac{\partial V}{r \sin \theta \partial \phi} \right) \\ &+ \frac{2}{M r^2 \sin^2 \theta} \left\{ \frac{\partial V}{\partial \phi} + M \dot{\phi} \frac{d}{dt} (r^2 \sin^2 \theta) \right\}^2 - 2 r^2 \sin^2 \theta M \dot{\phi}^4 - 2 r^2 \cos^2 \theta M \dot{\theta}^2 \dot{\phi}^2 \\ &- 2 M \dot{r} \dot{\phi}^2 \frac{d}{dt} (r \sin^2 \theta) + 2 r \dot{\phi}^2 \sin^2 \theta \frac{\partial V}{\partial r} + 2 \dot{\phi}^2 \sin \theta \cos \theta \frac{\partial V}{\partial \theta}. \end{aligned}$$

Also we have

$$\frac{d}{dt}(r^2\dot{\theta}^2) = -2r\dot{r}\dot{\theta}^2 + 2r^2\dot{\theta}^2 \sin \theta \cos \theta \dot{\phi}^2 - 2\dot{\theta} \frac{1}{M} \frac{\partial V}{\partial \theta},$$

upon substituting for the accelerations from Lagrange's equations, as before.

Then

$$\begin{aligned} \frac{d^2}{dt^2}(r^2\dot{\theta}^2) = & -2r^2\ddot{\theta}^2 - 2r\dot{r}\ddot{\theta}^2 - 4r\dot{r}\dot{\theta}\ddot{\theta} - 2\ddot{\theta} \frac{1}{M} \frac{\partial V}{\partial \theta} - \frac{2\dot{\theta}}{M} \frac{d}{dt} \left(\frac{\partial V}{\partial \theta} \right) \\ & + 2 \sin \theta \cos \theta \dot{\phi}^2 \frac{d}{dt}(r^2\dot{\theta}) + r^2\dot{\theta}^2 (2 \cos \theta \dot{\phi}^2 \dot{\theta} + 2 \sin \theta \dot{\phi} \ddot{\phi}). \end{aligned}$$

Substitute for \ddot{r} , etc., from Lagrange's equations, as before,

$$\begin{aligned} \frac{d^2}{dt^2}(r^2\dot{\theta}^2) = & 6r^2\dot{\theta}^2 - 2r^2\dot{\theta}^4 - 2(\cos^2 \theta + 2)r^2\dot{\theta}^2\dot{\phi}^2 + 2r\dot{\theta}^2 \frac{1}{M} \frac{\partial V}{\partial r} + \frac{2}{r^2} \frac{1}{M^2} \left(\frac{\partial V}{\partial \theta} \right)^2 \\ & + \frac{8}{r} \dot{r}\dot{\theta} \frac{\partial V}{\partial \theta} \frac{1}{M} - \frac{4r}{M} \sin \theta \cos \theta \dot{\phi}^2 \frac{\partial V}{r\partial \theta} - 12r\dot{r}\dot{\theta}\dot{\phi}^2 \sin \theta \cos \theta - 2\dot{\theta} \frac{1}{M} \frac{d}{dt} \left(\frac{\partial V}{\partial \theta} \right) \\ & - 4\dot{\theta}\dot{\phi} \frac{1}{M} \cos \theta \frac{\partial V}{\partial \phi} + 2r^2 \sin^2 \theta \cos^2 \theta \dot{\phi}^4. \end{aligned}$$

Upon taking mean values, we have

$$\begin{aligned} \left[\frac{1}{2} \frac{d^2}{dt^2}(r^2\dot{\theta}^2) \right] = & 3[r^2\dot{\theta}^2] - [r^2\dot{\theta}^4] - [(\cos^2 \theta + 2)r^2\dot{\theta}^2\dot{\phi}^2] + \frac{1}{M^2} \left[\frac{1}{r^2} \left(\frac{\partial V}{\partial \theta} \right)^2 \right] \\ & - \frac{1}{M} \left[\dot{\theta}^2 \frac{\partial^2 V}{\partial \theta^2} \right] + [r^2 \sin^2 \theta \cos^2 \theta \dot{\phi}^4] + \frac{1}{M} \left[\dot{\theta}^2 r \frac{\partial V}{\partial r} \right] \end{aligned}$$

and

$$\begin{aligned} \left[\frac{1}{2} \frac{d^2}{dt^2}(r^2 \sin^2 \theta \dot{\phi}^2) \right] = & -\frac{1}{M} \left[\dot{\phi}^2 \frac{\partial^2 V}{\partial \phi^2} \right] + \frac{1}{M^2} \left[\frac{1}{r^2 \sin^2 \theta} \left\{ \frac{\partial V}{\partial \phi} + M\dot{\phi} \frac{d}{dt}(r^2 \sin^2 \theta) \right\}^2 \right] \\ & - [r^2 \sin^2 \theta \dot{\phi}^4] - [r^2 \cos^2 \theta \dot{\theta}^2 \dot{\phi}^2] - [r^2 \dot{\phi}^2 \sin^2 \theta] + \frac{1}{M} \left[r \dot{\phi}^2 \sin^2 \theta \frac{\partial V}{\partial r} \right]. \end{aligned}$$

The three equations determining the mean accelerations of the energy components corresponding to r , ϕ , θ are therefore

$$\begin{aligned} \left[\frac{d^2}{dt^2} \left(\frac{1}{2} r^2 \right) \right] = & \frac{1}{M^2} \left[\left(\frac{\partial V}{\partial r} \right)^2 \right] + [r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)] - 3[r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)] \\ & - \frac{1}{M} \left[r^2 \frac{\partial^2 V}{\partial r^2} \right] - \frac{2}{M} \left[q^2 \frac{df}{rdr} \right] \quad (3) \end{aligned}$$

$$\begin{aligned} \left[\frac{d^2}{dt^2} \left(\frac{1}{2} r^2 \sin^2 \theta \dot{\phi}^2 \right) \right] = & -\frac{1}{M} \left[r^2 \sin^2 \theta \dot{\phi}^2 \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left(\frac{\partial V}{r \sin \theta \partial \phi} \right) \right] + \frac{1}{M^2} \left[\left(\frac{\partial V}{r \sin \theta \partial \phi} \right)^2 \right] \\ & + 3[r^2 \dot{\phi}^2 \sin^2 \theta] + 3[r^2 \dot{\theta}^2 \cos^2 \theta \dot{\phi}^2] - [r^2 \sin^2 \theta \dot{\phi}^4] + \frac{1}{2M} \left[q^2 \frac{df}{rdr} \right] \quad (4) \end{aligned}$$

$$\begin{aligned} \left[\frac{d^2}{dt^2} \left(\frac{1}{2} r^2 \dot{\theta}^2 \right) \right] = & 3[r^2\dot{\theta}^2] - [r^2\dot{\theta}^4] - [(\cos^2 \theta + 2)r^2\dot{\theta}^2\dot{\phi}^2] + \frac{1}{M^2} \left[\left(\frac{\partial V}{r\partial \theta} \right)^2 \right] \\ & + [r^2 \sin^2 \theta \cos^2 \theta \dot{\phi}^4] - \frac{1}{M} \left[r^2 \dot{\theta}^2 \frac{\partial}{r\partial \theta} \left(\frac{\partial V}{r\partial \theta} \right) + \frac{1}{2M} \left[q^2 \frac{df}{rdr} \right] \right] \quad (5). \end{aligned}$$

(6) In accordance with the assumptions laid down, the last two equations must be exactly the same.

Expressing this condition, we have

$$3[r^2 \cos^2 \theta \dot{\theta}^2 \dot{\phi}^2] - [r^2 \sin^2 \theta \dot{\phi}^4] = [r^2 \sin^2 \theta \cos^2 \theta \dot{\phi}^4] - [(\cos^2 \theta + 2)r^2 \dot{\theta}^2 \dot{\phi}^2] - [r^2 \dot{\theta}^4] \quad (6).$$

In order to evaluate these mean values, let us put

$$\left. \begin{aligned} q \cos \psi &= r \dot{\theta} \\ q \sin \psi &= r \sin \theta \dot{\phi} \end{aligned} \right\} \quad (7).$$

Then

$$[r^2 \dot{\theta}^2 \dot{\phi}^2 \sin^2 \theta] = \left[\frac{q^4}{r^2} \sin^2 \psi \cos^2 \psi \right] = \left[\frac{q^4}{4r^2} \sin^2 2\psi \right].$$

Now

$$[\sin^2 2\psi] = \frac{1}{2\pi} \int_0^{2\pi} \sin^2 2\psi d\psi = \frac{1}{2}.$$

$$\therefore [r^2 \dot{\theta}^2 \dot{\phi}^2 \sin^2 \theta] = \frac{1}{8} \left[\frac{q^4}{r^2} \right].$$

Likewise

$$[r^2 \sin^4 \theta \dot{\phi}^4] = \frac{3}{8} \left[\frac{q^4}{r^2} \right]$$

and

$$[r^2 \dot{\theta}^4] = \frac{3}{8} \left[\frac{q^4}{r^2} \right];$$

for these depend merely on the evaluation of $\int_0^{2\pi} \sin^4 \psi d\psi$ and $\int_0^{2\pi} \cos^4 \psi d\psi$.

By differentiating the equations (7), squaring and adding, and using the relation (6), we can obtain also the relations

$$\text{and } \left. \begin{aligned} [r^2 \dot{\theta}^2 \dot{\phi}^2] &= \frac{1}{16} \left[\frac{q^4}{r^2} \right] \\ [r^2 \sin^2 \theta \dot{\phi}^4] &= \frac{5}{16} \left[\frac{q^4}{r^2} \right] \end{aligned} \right\} \quad (8);$$

but these relations will not be required.

(7) Expressions such as $\left[\frac{d^2}{dt^2}(r\dot{\theta}) \right]$ and $\left[\frac{d^2}{dt^2}(r^2 \sin \theta \dot{\phi}) \right]$ when evaluated are found to all vanish identically in accordance with the hypotheses. Now let $ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$, so that ds is a small element of length perpendicular to the straight line AB; and let us add together both sides of equations (4) and (5).

We obtain

$$\begin{aligned} & \left[\frac{d^2}{dt^2} \frac{1}{2} (r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) \right] \\ &= -\frac{1}{M} \left[r^2 \dot{\theta}^2 \frac{\partial}{r \partial \theta} \left(\frac{\partial V}{r \partial \theta} \right) + r^2 \sin^2 \theta \dot{\phi}^2 \frac{\partial}{r \sin \theta \partial \phi} \left(\frac{\partial V}{r \sin \theta \partial \phi} \right) \right] \\ &+ \frac{1}{M^2} \left[\left(\frac{\partial V}{r \sin \theta \partial \phi} \right)^2 + \left(\frac{\partial V}{r \partial \theta} \right)^2 \right] + 3 \left[\frac{\dot{r}^2}{r^2} (r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) \right] - 2 [r^2 \dot{\theta}^2 \dot{\phi}^2 \sin^2 \theta] \\ &- [r^2 \sin^4 \theta \dot{\phi}^4] - [r^2 \dot{\theta}^4] + \frac{1}{M} \left[q^2 \frac{df}{r dr} \right], \end{aligned}$$

OR

$$\begin{aligned} \left[\frac{d^2}{dt^2} \left(\frac{1}{2} q^2 \right) \right] &= -\frac{1}{M} \left[q^2 \frac{\partial^2 V}{\partial s^2} \right] + \frac{1}{M^2} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] + 3 \left[\frac{q^2 \dot{r}^2}{r^2} \right] \\ &- 2 [r^2 \dot{\theta}^2 \dot{\phi}^2 \sin^2 \theta] - [r^2 \sin^4 \theta \dot{\phi}^4] - [r^2 \dot{\theta}^4] + \frac{1}{M} \left[q^2 \frac{df}{r dr} \right], \end{aligned}$$

OR

$$\left[\frac{d^2}{dt^2} \frac{1}{2} (q^2) \right] = -\frac{1}{M} \left[q^2 \frac{\partial^2 V}{\partial s^2} \right] + \frac{1}{M^2} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] + 3 \left[\frac{q^2 \dot{r}^2}{r^2} \right] - \left[\frac{q^4}{r^2} \right] + \frac{1}{M} \left[q^2 \frac{df}{r dr} \right].$$

Similarly we obtain

$$\left[\frac{d^2}{dt^2} \left(\frac{1}{2} \dot{r}^2 \right) \right] = \frac{1}{M^2} \left[\left(\frac{\partial V}{\partial r} \right)^2 \right] + \left[\frac{q^4}{r^2} \right] - 3 \left[\frac{\dot{r}^2 q^2}{r^2} \right] - \frac{1}{M} \left[\dot{r}^2 \frac{\partial^2 V}{\partial r^2} \right] - \frac{2}{M} \left[q^2 \frac{df}{r dr} \right].$$

In accordance with our hypotheses we may suppose that

$$\begin{aligned} \left[\dot{x}^2 \frac{\partial^2 V}{\partial x^2} \right] &= [\dot{x}^2] \left[\frac{\partial^2 V}{\partial x^2} \right] \\ \left[\dot{r}^2 \frac{\partial^2 V}{\partial r^2} \right] &= [\dot{r}^2] \left[\frac{\partial^2 V}{\partial r^2} \right] \\ \left[q^2 \frac{\partial^2 V}{\partial s^2} \right] &= [q^2] \left[\frac{\partial^2 V}{\partial s^2} \right], \text{ etc.} \end{aligned}$$

(8) We have thus obtained three equations which determine the mean energy acceleration components corresponding to translation, vibration, and rotation; namely,

$$\left. \begin{aligned} \left[\frac{d^2}{dt^2} \left(\frac{1}{2} \dot{x}^2 \right) \right] &= \frac{1}{(m_1 + m_2)^2} \left[\left(\frac{\partial V}{\partial x} \right)^2 \right] - \frac{1}{m_1 + m_2} [\dot{x}^2] \left[\frac{\partial^2 V}{\partial x^2} \right] \\ \left[\frac{d^2}{dt^2} \left(\frac{1}{2} \dot{r}^2 \right) \right] &= \frac{1}{M^2} \left[\left(\frac{\partial V}{\partial r} \right)^2 \right] + \left[\frac{q^4}{r^2} \right] - 3 \left[\frac{\dot{r}^2 q^2}{r^2} \right] - \frac{1}{M} \left[\dot{r}^2 \frac{\partial^2 V}{\partial r^2} \right] - \frac{2}{M} \left[q^2 \frac{df}{r dr} \right] \\ \left[\frac{d^2}{dt^2} \left(\frac{1}{2} q^2 \right) \right] &= \frac{1}{M^2} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] - \left[\frac{q^4}{r^2} \right] + 3 \left[\frac{\dot{r}^2 q^2}{r^2} \right] - \frac{1}{M} [q^2] \left[\frac{\partial^2 V}{\partial s^2} \right] + \frac{1}{M} \left[q^2 \frac{df}{r dr} \right] \end{aligned} \right\} \quad (9).$$

The conditions for a stationary distribution require that the mean acceleration of energy shall vanish, just as in statics the conditions of equilibrium require that the accelerations of the bodies of the system shall vanish.

The equations representing these conditions will be called equations of energy equilibrium, as in the *Arch. Néerlandaises* to which reference has been made. The equations of energy equilibrium are therefore

$$\left. \begin{aligned} \left[\left(\frac{\partial V}{\partial x} \right)^2 \right] &= (m_1 + m_2) \left[\dot{x}^2 \right] \left[\frac{\partial^2 V}{\partial x^2} \right] \\ 3 \left[\frac{\dot{r}^2}{r^2} q^2 \right] - \left[\frac{q^4}{r^2} \right] &= \frac{1}{M^2} \left[\left(\frac{\partial V}{\partial r} \right)^2 \right] - \frac{1}{M} \left[\dot{r}^2 \frac{\partial^2 V}{\partial r^2} \right] - \frac{2}{M} \left[q^2 \frac{df}{r dr} \right] \\ &= \frac{1}{M} \left[q^2 \frac{\partial^2 V}{\partial s^2} \right] - \frac{1}{M^2} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] - \frac{1}{M} \left[q^2 \frac{df}{r dr} \right] \end{aligned} \right\} \quad (10).$$

(9) In the case of a Newtonian field $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$, and therefore our assumptions would require that

$$\left[\frac{\partial^2 V}{\partial x^2} \right] = \left[\frac{\partial^2 V}{\partial y^2} \right] = \left[\frac{\partial^2 V}{\partial z^2} \right] = 0.$$

The first of the equations of energy equilibrium shows that these assumptions are impossible in this case, and likewise in the case where $\left[\frac{\partial^2 V}{\partial x^2} \right]$ is negative; so that we must exclude this latter case and that of a Newtonian external field of force.

(10) It is next necessary to obtain expressions for $\left[\frac{\partial^2 V}{\partial x^2} \right]$, $\left[\left(\frac{\partial V}{\partial s} \right)^2 \right]$, etc.

We know that

$$V = \mu_1 \Phi_1(x_1, y_1, z_1) + \mu_2 \Phi_2(x_2, y_2, z_2) + f(r).$$

Therefore,

$$\begin{aligned} \frac{\partial V}{\partial x} &= \mu_1 \frac{\partial \Phi_1}{\partial x} + \mu_2 \frac{\partial \Phi_2}{\partial x_2} \\ \frac{\partial V}{\partial y} &= \mu_1 \frac{\partial \Phi_1}{\partial y_1} + \mu_2 \frac{\partial \Phi_2}{\partial y_2} \\ \frac{\partial V}{\partial z} &= \mu_1 \frac{\partial \Phi_1}{\partial z_1} + \mu_2 \frac{\partial \Phi_2}{\partial z_2} \\ \frac{\partial V}{\partial r} &= \frac{df}{dr} + \frac{1}{m_1 + m_2} \left\{ \left(m_2 \mu_1 \frac{\partial \Phi_1}{\partial x_1} - m_1 \mu_2 \frac{\partial \Phi_2}{\partial x_2} \right) \sin \theta \cos \phi \right. \\ &\quad \left. + \left(m_2 \mu_1 \frac{\partial \Phi_1}{\partial y_1} - m_1 \mu_2 \frac{\partial \Phi_2}{\partial y_2} \right) \sin \theta \sin \phi + \left(m_2 \mu_1 \frac{\partial \Phi_1}{\partial z_1} - m_1 \mu_2 \frac{\partial \Phi_2}{\partial z_2} \right) \cos \theta \right\} \\ \frac{\partial V}{r \partial \theta} &= \frac{1}{m_1 + m_2} \left\{ \left(m_2 \mu_1 \frac{\partial \Phi_1}{\partial x_1} - m_1 \mu_2 \frac{\partial \Phi_2}{\partial x_2} \right) \cos \theta \cos \phi \right. \\ &\quad \left. + \left(m_2 \mu_1 \frac{\partial \Phi_1}{\partial y_1} - m_1 \mu_2 \frac{\partial \Phi_2}{\partial y_2} \right) \cos \theta \sin \phi - \left(m_2 \mu_1 \frac{\partial \Phi_1}{\partial z_1} - m_1 \mu_2 \frac{\partial \Phi_2}{\partial z_2} \right) \sin \theta \right\} \\ \frac{\partial V}{r \sin \theta \partial \phi} &= - \frac{1}{m_1 + m_2} \left(m_2 \mu_1 \frac{\partial \Phi_1}{\partial x_1} - m_1 \mu_2 \frac{\partial \Phi_2}{\partial x_2} \right) \sin \phi \\ &\quad + \frac{1}{m_1 + m_2} \left(m_2 \mu_1 \frac{\partial \Phi_1}{\partial y_1} - m_1 \mu_2 \frac{\partial \Phi_2}{\partial y_2} \right) \cos \phi \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 V}{\partial x^2} &= \mu_1^2 \frac{\partial^2 \Phi_1}{\partial x_1^2} + \mu_2^2 \frac{\partial^2 \Phi_2}{\partial x_2^2} \\ \frac{\partial^2 V}{\partial r^2} &= \frac{m_2^2 \mu_1}{(m_1 + m_2)^2} \left\{ \frac{\partial^2 \Phi_1}{\partial x_1^2} \sin^2 \theta \cos^2 \phi + \frac{\partial^2 \Phi_1}{\partial y_1^2} \sin^2 \theta \sin^2 \phi + \frac{\partial^2 \Phi_1}{\partial z_1^2} \cos^2 \theta \right. \\ &\quad \left. + \text{terms of the type } 2 \frac{\partial^2 \Phi_1}{\partial y_1 \partial z_1} \sin \theta \cos \theta \sin \phi \right\} \\ &\quad + \frac{m_1^2 \mu_2}{(m_1 + m_2)^2} \left\{ \frac{\partial^2 \Phi_2}{\partial x_2^2} \sin^2 \theta \cos^2 \phi + \frac{\partial^2 \Phi_2}{\partial y_2^2} \sin^2 \theta \sin^2 \phi + \frac{\partial^2 \Phi_2}{\partial z_2^2} \cos^2 \theta \right. \\ &\quad \left. + \text{terms which vanish when the mean values are taken} \right\} + \frac{d^2 f}{dr} \\ \frac{\partial}{r \sin \theta \partial \phi} \left(\frac{\partial V}{r \sin \theta \partial \phi} \right) &= \frac{m_2^2 \mu_1}{(m_1 + m_2)^2} \left\{ \sin^2 \phi \frac{\partial^2 \Phi_1}{\partial x_1^2} + \cos^2 \phi \frac{\partial^2 \Phi_1}{\partial y_1^2} - 2 \sin \phi \cos \phi \frac{\partial^2 \Phi_1}{\partial x_1 \partial y_1} \right\} \\ &\quad + \frac{m_1^2 \mu_2}{(m_1 + m_2)^2} \left\{ \sin^2 \phi \frac{\partial^2 \Phi_2}{\partial x_2^2} + \cos^2 \phi \frac{\partial^2 \Phi_2}{\partial y_2^2} - 2 \sin \phi \cos \phi \frac{\partial^2 \Phi_2}{\partial x_2 \partial y_2} \right\} \\ &\quad - \frac{\mu_1 m_2}{r \sin \theta (m_1 + m_2)} \left\{ \cos \phi \frac{\partial \Phi_1}{\partial x_1} + \sin \phi \frac{\partial \Phi_1}{\partial y_1} \right\} \\ &\quad + \frac{\mu_2 m_1}{r \sin \theta (m_1 + m_2)} \left\{ \cos \phi \frac{\partial \Phi_2}{\partial x_2} + \sin \phi \frac{\partial \Phi_2}{\partial y_2} \right\}. \end{aligned}$$

Now, if $\sin \theta \cos \phi = l$, $\sin \theta \sin \phi = m$, and $\cos \theta = n$,

$$[l^2] = [m^2] = [n^2] = \frac{1}{3}[l^2 + m^2 + n^2] = \frac{1}{3}.$$

Therefore

$$[\sin^2 \theta \cos^2 \phi] = [\sin^2 \theta \sin^2 \phi] = [\cos^2 \theta] = \frac{1}{3}.$$

We obtain also the relations

$$[\sin^2 \theta] = \frac{2}{3}, \quad [\cos^2 \theta \cos^2 \phi] = \frac{1}{6}, \quad [\cos^2 \theta \sin^2 \phi] = \frac{1}{6},$$

and

$$[\cos^2 \phi] = [\sin^2 \phi] = \frac{1}{2}.$$

If we take the mean values, we have, in accordance with our assumptions,

$$\left. \begin{aligned} \left[\frac{\partial V}{\partial r} \right] &= \left[\frac{df}{dr} \right] \\ \left[\left(\frac{\partial V}{\partial x} \right)^2 \right] &= \mu_1^2 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] + \mu_2^2 \left[\left(\frac{\partial \Phi_2}{\partial x_2} \right)^2 \right] + 2\mu_1 \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \\ \left[\left(\frac{\partial V}{\partial r} \right)^2 \right] &= \left[\left(\frac{df}{dr} \right)^2 \right] + \frac{m_2^2 \mu_1^2}{(m_1 + m_2)^2} \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] + \frac{m_1^2 \mu_2^2}{(m_1 + m_2)^2} \left[\left(\frac{\partial \Phi_2}{\partial x_2} \right)^2 \right] - \frac{2m_1 m_2 \mu_1 \mu_2}{(m_1 + m_2)^2} \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \\ \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] &= \frac{2m_2^2 \mu_1^2}{(m_1 + m_2)^2} \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] + \frac{2m_1^2 \mu_2^2}{(m_1 + m_2)^2} \left[\left(\frac{\partial \Phi_2}{\partial x_2} \right)^2 \right] - \frac{4m_1 m_2 \mu_1 \mu_2}{(m_1 + m_2)^2} \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \\ \left[\frac{\partial^2 V}{\partial x^2} \right] &= \mu_1 \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] + \mu_2 \left[\frac{\partial^2 \Phi_2}{\partial x_2^2} \right] \\ \left[\frac{\partial^2 V}{\partial r^2} \right] &= \left[\frac{d^2 f}{dr^2} \right] + \frac{m_2^2 \mu_1}{(m_1 + m_2)^2} \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] + \frac{m_1^2 \mu_2}{(m_1 + m_2)^2} \left[\frac{\partial^2 \Phi_2}{\partial x_2^2} \right] \\ \left[\frac{\partial^2 V}{\partial s^2} \right] &= \frac{m_2^2 \mu_1}{(m_1 + m_2)^2} \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] + \frac{m_1^2 \mu_2}{(m_1 + m_2)^2} \left[\frac{\partial^2 \Phi_2}{\partial x_2^2} \right] \end{aligned} \right\} (11).$$

It is also necessary to determine the mean values of the differential coefficients of V with respect to $x_1, x_2, y_1, y_2, z_1, z_2$, in terms of the mean values of the coefficients with respect to x, y, z, r, θ, ϕ .

Now

$$r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2,$$

$$\sin \theta \cos \phi = \frac{x_1 - x_2}{r}, \quad \sin \theta \sin \phi = \frac{y_1 - y_2}{r}, \quad \cos \theta = \frac{z_1 - z_2}{r},$$

$$\frac{\partial x}{\partial x_1} = \frac{m_1}{m_1 + m_2}, \quad \frac{\partial r}{\partial x_1} = \frac{x_1 - x_2}{r} = \sin \theta \cos \phi, \quad \frac{r \partial \theta}{\partial x_1} = \cos \theta \cos \phi,$$

and

$$\frac{r \sin \theta \partial \phi}{\partial x_1} = -\sin \phi.$$

Let us differentiate with respect to x_1 and x_2 , and we easily obtain, upon taking mean values, the relations

$$\left. \begin{aligned} \mu_1^2 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] &= \frac{m_1^2}{(m_1 + m_2)^2} \left[\left(\frac{\partial V}{\partial x} \right)^2 \right] + \frac{1}{3} \left[\left(\frac{\partial V}{\partial r} \right)^2 \right] + \frac{1}{3} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] - \frac{1}{3} \left[\left(\frac{df}{dr} \right)^2 \right] \\ \mu_2^2 \left[\left(\frac{\partial \Phi_2}{\partial x_2} \right)^2 \right] &= \frac{m_2^2}{(m_1 + m_2)^2} \left[\left(\frac{\partial V}{\partial x} \right)^2 \right] + \frac{1}{3} \left[\left(\frac{\partial V}{\partial r} \right)^2 \right] + \frac{1}{3} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] - \frac{1}{3} \left[\left(\frac{df}{dr} \right)^2 \right] \\ \mu_1 \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] &= \frac{m_1 m_2}{(m_1 + m_2)^2} \left[\left(\frac{\partial V}{\partial x} \right)^2 \right] - \frac{1}{3} \left[\left(\frac{\partial V}{\partial r} \right)^2 \right] - \frac{1}{3} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] + \frac{1}{3} \left[\left(\frac{df}{dr} \right)^2 \right] \end{aligned} \right\} \quad (12).$$

If the second differential coefficients be written down and mean values be taken, we have

$$\left. \begin{aligned} \mu_1 \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] &= \frac{m_1^2}{(m_1 + m_2)^2} \left[\frac{\partial^2 V}{\partial x^2} \right] + \frac{1}{3} \left[\frac{\partial^2 V}{\partial r^2} \right] + \frac{2}{3} \left[\frac{\partial^2 V}{\partial s^2} \right] - \frac{1}{3} \left[\frac{d^2 f}{dr^2} \right] \\ \mu_2 \left[\frac{\partial^2 \Phi_2}{\partial x_2^2} \right] &= \frac{m_2^2}{(m_1 + m_2)^2} \left[\frac{\partial^2 V}{\partial x^2} \right] + \frac{1}{3} \left[\frac{\partial^2 V}{\partial r^2} \right] + \frac{2}{3} \left[\frac{\partial^2 V}{\partial s^2} \right] - \frac{1}{3} \left[\frac{d^2 f}{dr^2} \right] \end{aligned} \right\} \quad (13).$$

Also, since $\frac{\partial^2 \Phi_1}{\partial x_2 \partial x_1} = 0$,

$$0 = \frac{m_1 m_2}{(m_1 + m_2)^2} \left[\frac{\partial^2 V}{\partial x^2} \right] - \frac{1}{3} \left[\frac{\partial^2 V}{\partial r^2} \right] + \frac{1}{3} \left[\frac{d^2 f}{dr^2} \right] - \frac{2}{3} \left[\frac{\partial^2 V}{\partial s^2} \right] \quad (14).$$

From (12) we obtain the relation

$$m_2 \mu_1^2 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] - m_1 \mu_2^2 \left[\left(\frac{\partial \Phi_2}{\partial x_2} \right)^2 \right] - (m_1 - m_2) \mu_1 \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] = 0 \quad (15)$$

when the masses are unequal; and using the equations (11) we have

$$\left. \begin{aligned} \left[\left(\frac{\partial V}{\partial x} \right)^2 \right] &= \frac{m_1 + m_2}{m_1 - m_2} \mu_1^2 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] - \frac{m_1 + m_2}{m_1 - m_2} \mu_2^2 \left[\left(\frac{\partial \Phi_2}{\partial x_2} \right)^2 \right] \\ \frac{1}{2} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] &= -\frac{m_2^2 \mu_1^2}{m_1^2 - m_2^2} \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] + \frac{m_1^2 \mu_2^2}{m_1^2 - m_2^2} \left[\left(\frac{\partial \Phi_2}{\partial x_2} \right)^2 \right] \end{aligned} \right\} \quad (16),$$

when $m_1 \neq m_2$. Also

$$\left[\left(\frac{\partial V}{\partial r} \right)^2 \right] = \left[\left(\frac{df}{dr} \right)^2 \right] + \frac{1}{2} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right].$$

Again, using the equations (13) and (14), we have

$$\left. \begin{aligned} m_2\mu_1\left[\frac{\partial^2\Phi_1}{\partial x_1^2}\right] &= m_1\mu_2\left[\frac{\partial^2\Phi_2}{\partial x_2^2}\right] = \frac{m_1m_2}{m_1+m_2}\left[\frac{\partial^2V}{\partial x^2}\right] \\ &= (m_1+m_2)\left[\frac{\partial^2V}{\partial s^2}\right] \end{aligned} \right\} \dots \dots (17),$$

when $m_1 \neq m_2$; so that we have the important relation

$$\left[\frac{\partial^2V}{\partial x^2}\right] = \frac{(m_1+m_2)^2}{m_1m_2}\left[\frac{\partial^2V}{\partial s^2}\right] \dots \dots \dots (18).$$

Also, from the equations (9),

$$\left[\frac{\partial^2V}{\partial r^2}\right] = \left[\frac{d^2f}{dr^2}\right] + \left[\frac{\partial^2V}{\partial s^2}\right] \dots \dots \dots (19).$$

(11) The three equations of energy equilibrium finally take the forms

$$\left. \begin{aligned} [\dot{x}^2] &= \frac{m_1}{m_1^2 - m_2^2} \frac{\mu_1^2\left[\left(\frac{\partial\Phi_1}{\partial x_1}\right)^2\right] - \mu_2^2\left[\left(\frac{\partial\Phi_2}{\partial x_2}\right)^2\right]}{\mu_1\left[\frac{\partial^2\Phi_1}{\partial x_1^2}\right]} \\ 3\left[\frac{\dot{r}^2}{r^2}q^2\right] - \left[\frac{q^4}{r^2}\right] &= \left(\frac{m_1+m_2}{m_1m_2}\right)^2\left[\left(\frac{df}{dr}\right)^2\right] - \frac{2(m_1+m_2)}{m_1m_2}\left[q^2\frac{df}{rdr}\right] - \frac{m_1+m_2}{m_1m_2}\left[\dot{r}^2\frac{d^2f}{dr^2}\right] \\ &\quad - \frac{\mu_1}{m_1}\left[\dot{r}^2\frac{\partial^2\Phi_1}{\partial x_1^2}\right] - \frac{m_1+m_2}{m_1-m_2}\left\{\frac{\mu_1^2}{m_1^2}\left[\left(\frac{\partial\Phi_1}{\partial x_1}\right)^2\right] - \frac{\mu_2^2}{m_2^2}\left[\left(\frac{\partial\Phi_2}{\partial x_2}\right)^2\right]\right\} \\ &= \frac{\mu_1}{m_1}\left[q^2\frac{\partial^2\Phi_1}{\partial x_1^2}\right] - \frac{m_1+m_2}{m_1m_2}\left[q^2\frac{df}{rdr}\right] \\ &\quad + \frac{2(m_1+m_2)}{m_1-m_2}\left\{\frac{\mu_1^2}{m_1^2}\left[\left(\frac{\partial\Phi_1}{\partial x_1}\right)^2\right] - \frac{\mu_2^2}{m_2^2}\left[\left(\frac{\partial\Phi_2}{\partial x_2}\right)^2\right]\right\} \end{aligned} \right\} (20)$$

when the masses are unequal.

They may also be written in the forms

$$\left. \begin{aligned} [\dot{x}^2] &= \frac{\mu_1\left[\left(\frac{\partial\Phi_1}{\partial x_1}\right)^2\right] + \mu_2\left[\left(\frac{\partial\Phi_1}{\partial x_1}\frac{\partial\Phi_2}{\partial x_2}\right)\right]}{(m_1+m_2)\left[\frac{\partial^2\Phi_1}{\partial x_1^2}\right]} \\ 3\left[\frac{\dot{r}^2}{r^2}q^2\right] - \left[\frac{q^4}{r^2}\right] &= \left(\frac{m_1+m_2}{m_1m_2}\right)^2\left[\left(\frac{df}{dr}\right)^2\right] - \frac{2(m_1+m_2)}{m_1m_2}\left[q^2\frac{df}{rdr}\right] - \frac{m_1+m_2}{m_1m_2}\left[\dot{r}^2\frac{d^2f}{dr^2}\right] \\ &\quad - \frac{\mu_1}{m_1}\left[\dot{r}^2\frac{\partial^2\Phi_1}{\partial x_1^2}\right] + \frac{\mu_1(m_1+m_2)}{m_1m_2}\left\{\frac{\mu_1}{m_1}\left[\left(\frac{\partial\Phi_1}{\partial x_1}\right)^2\right] - \frac{\mu_2}{m_2}\left[\frac{\partial\Phi_1}{\partial x_1}\frac{\partial\Phi_2}{\partial x_2}\right]\right\} \\ &= \frac{\mu_1}{m_1}\left[q^2\frac{\partial^2\Phi_1}{\partial x_1^2}\right] - \frac{m_1+m_2}{m_1m_2}\left[q^2\frac{df}{rdr}\right] \\ &\quad - \frac{2\mu_1(m_1+m_2)}{m_1m_2}\left\{\frac{\mu_1}{m_1}\left[\left(\frac{\partial\Phi_1}{\partial x_1}\right)^2\right] - \frac{\mu_2}{m_2}\left[\frac{\partial\Phi_1}{\partial x_1}\frac{\partial\Phi_2}{\partial x_2}\right]\right\} \end{aligned} \right\} (21),$$

and these forms can be used in the case when the masses are equal.

MAXWELL'S LAW OF EQUIPARTITION.

(12) Maxwell's Law of Partition of Kinetic Energy requires that, if the kinetic energy of the system be expressed as a sum of squares, the mean values of these squares taken over a large number of systems distributed according to a certain permanent or stationary law shall be equal.

In the case of this system, the kinetic energy is equal to

$$\frac{1}{2}(m_1 + m_2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2).$$

Therefore, since

$$[r^2 \dot{\theta}^2] = [r^2 \sin^2 \theta \dot{\phi}^2] = \frac{1}{2}[q^2],$$

it is necessary that

$$(m_1 + m_2)[\dot{x}^2] = \frac{m_1 m_2}{m_1 + m_2} [\dot{r}^2] = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} [q^2]$$

or

$$(m_1 + m_2)^2 [\dot{x}^2] = m_1 m_2 [r^2] = \frac{1}{2} m_1 m_2 [q^2],$$

if Maxwell's Law of Partition holds good. In this case, the equations of energy equilibrium show that

$$\frac{1}{M^2} \left[\left(\frac{df}{dr} \right)^2 \right] - \frac{1}{M} \left[q^2 \frac{df}{rdr} \right] + \frac{3\mu_1}{M} \left\{ \frac{\mu_1}{m_1} \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] - \frac{\mu_2}{m_2} \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\} - \frac{1}{M} \left[\dot{r}^2 \frac{d^2 f}{dr^2} \right] - \frac{\mu_1}{m_1} \left[(q^2 + \dot{r}^2) \frac{\partial^2 \Phi_1}{\partial x_1^2} \right] = 0$$

or

$$\frac{1}{M^2} \left[\left(\frac{df}{dr} \right)^2 \right] - \frac{2}{M} \left[\frac{df}{rdr} \right] \frac{(m_1 + m_2)^2}{m_1 m_2} [\dot{x}^2] + \frac{3\mu_1}{M} \left\{ \frac{\mu_1}{m_1} \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] - \frac{\mu_2}{m_2} \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\} - \frac{1}{M} \frac{(m_1 + m_2)^2}{m_1 m_2} [\dot{x}^2] \left[\frac{d^2 f}{dr^2} \right] - \frac{3\mu_1}{m_1} \frac{(m_1 + m_2)^2}{m_1 m_2} [\dot{x}^2] \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] = 0$$

or

$$\frac{1}{M^2} \left[\left(\frac{df}{dr} \right)^2 \right] + \frac{3\mu_1}{M} \left\{ \frac{\mu_1}{m_1} \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] - \frac{\mu_2}{m_2} \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\} - \frac{(m_1 + m_2)^2}{m_1 m_2} [\dot{x}^2] \left\{ \frac{2}{M} \left[\frac{df}{rdr} \right] + \frac{1}{M} \left[\frac{d^2 f}{dr^2} \right] + \frac{3\mu_1}{m_1} \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] \right\} = 0$$

or

$$\frac{1}{M} \left[\left(\frac{df}{dr} \right)^2 \right] + 3\mu_1 \left\{ \frac{\mu_1}{m_1} \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] - \frac{\mu_2}{m_2} \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\} - \frac{1}{\left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right]} \left\{ \mu_1 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] + \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\} \left\{ \frac{2}{M} \left[\frac{df}{rdr} \right] + \frac{1}{M} \left[\frac{d^2 f}{dr^2} \right] + \frac{3\mu_1}{m_1} \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] \right\} = 0$$

or

$$\left\{ \left[\left(\frac{df}{dr} \right)^2 \right] - 3\mu_1 \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\} \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] = \left\{ \mu_1 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] + \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\} \left[\frac{d^2 f}{dr^2} + 2 \frac{df}{rdr} \right] \quad (22).$$

This is the condition, therefore, that Maxwell's Law holds good in the general case.

The occurrence of the term involving $[q^4]$ makes it very difficult, if not practically impossible, to obtain another condition for equipartition of energy by this process.

Now

$$\left[\left(\frac{\partial V}{\partial x} \right)^2 \right] = \frac{(m_1 + m_2)\mu_1}{m_1} \left\{ \mu_1 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] + \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\}.$$

Therefore, since the square of a real quantity is necessarily positive,

$$\mu_1^2 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] + \mu_1 \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right]$$

is essentially positive; so that, if Maxwell's Law holds good, the condition for equipartition of energy which has been obtained shows that

$$\frac{\left[\left(\frac{df}{dr} \right)^2 \right] - 3\mu_1 \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right]}{\left[\frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr} \right]}$$

must be positive, in the case when the denominator does not vanish.

(13) In the general case, when the masses are unequal and when Maxwell's Law may or may not hold good, our investigations give at once certain simple inequalities. It does not matter which is the greater of the two masses; let us suppose that $m_1 > m_2$.

It has been shown that

$$\left[\left(\frac{\partial V}{\partial x} \right)^2 \right] = \frac{m_1 + m_2}{m_1 - m_2} \left\{ \mu_1^2 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] - \mu_2^2 \left[\left(\frac{\partial \Phi_2}{\partial x_2} \right)^2 \right] \right\}$$

and

$$\frac{1}{2} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] = \frac{m_1^2 \mu_2^2}{m_1^2 - m_2^2} \left[\left(\frac{\partial \Phi_2}{\partial x_2} \right)^2 \right] - \frac{m_2^2 \mu_1^2}{m_1^2 - m_2^2} \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right].$$

This shows that, if our assumptions are justifiable, it is necessary that

$$\mu_1^2 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] > \mu_2^2 \left[\left(\frac{\partial \Phi_2}{\partial x_2} \right)^2 \right]$$

and

$$m_2^2 \mu_1^2 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] < m_1^2 \mu_2^2 \left[\left(\frac{\partial \Phi_2}{\partial x_2} \right)^2 \right].$$

Now

$$m_2 \mu_1^2 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] - m_1 \mu_2^2 \left[\left(\frac{\partial \Phi_2}{\partial x_2} \right)^2 \right] = (m_1 - m_2) \mu_1 \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right];$$

and $m_2 \mu_1^2 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right]$ may be greater or less than $m_1 \mu_2^2 \left[\left(\frac{\partial \Phi_2}{\partial x_2} \right)^2 \right]$; so that our equations do not require that $\mu_1 \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right]$ should be necessarily positive or necessarily negative.

Also

$$\frac{1}{2M} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] = \frac{\mu_1^2}{m_1} \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] - \frac{\mu_1 \mu_2}{m_2} \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right],$$

so that we have also

$$\frac{\mu_1^2}{m_1} \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] > \frac{\mu_1 \mu_2}{m_2} \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right].$$

Other inequalities which of necessity hold good are

$$\mu_1 \left\{ \mu_1 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] + \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\} > 0$$

and

$$\mu_1 \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] > 0.$$

STABILITY IN THE GENERAL CASE.

(14) In discussing the stability of the energy equilibrium let us first consider the translational mean energy.

Let

$$\left[\frac{1}{2} \dot{x}^2 \right] = T_0 + \epsilon,$$

where T_0 is a third of the mean translational kinetic energy determined by the equation of energy equilibrium, and ϵ is a small variation which may be due to initial disturbance. We thus obtain, by substituting in equations (9),

$$\ddot{\epsilon} = - \frac{2\epsilon}{m_1 + m_2} \left[\frac{\partial^2 V}{\partial x^2} \right].$$

For stability the variations in ϵ must be periodic, and this condition is satisfied if $\left[\frac{\partial^2 V}{\partial x^2} \right]$ is positive; and this is precisely what has been assumed, in order that our hypotheses may allow energy equilibrium to be possible.

The term involving $[q^4]$ in the equations of energy equilibrium makes it very difficult, at any rate, to deal with the question of stability in the general case, as far as the mean vibrational and rotational kinetic energies are concerned.

Let us consider, however, the question of stability, when we suppose Maxwell's Law to hold good.

From the equations which give the average energy accelerations it is clear that

$$\begin{aligned} \left[\frac{d^2}{dt^2} \frac{1}{2} (q^2 + \dot{r}^2) \right] &= \frac{1}{M^2} \left[\left(\frac{\partial V}{\partial r} \right)^2 + \left(\frac{\partial V}{\partial s} \right)^2 \right] - \frac{1}{M} \left[q^2 \frac{\partial^2 V}{\partial s^2} + \dot{r}^2 \frac{\partial^2 V}{\partial r^2} \right] - \frac{1}{M} \left[q^2 \frac{df}{r dr} \right] \\ &= \frac{1}{M^2} \left[\left(\frac{df}{dr} \right)^2 \right] - \frac{1}{M} \left[q^2 \frac{df}{r dr} \right] + \frac{3}{2M^2} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] - \frac{1}{M} \left[\dot{r}^2 \frac{d^2 f}{dr^2} \right] \\ &\quad - \frac{1}{M} \left[(q^2 + \dot{r}^2) \frac{\partial^2 V}{\partial s^2} \right]. \end{aligned}$$

If Maxwell's Law holds good, we have

$$\frac{3}{2} \frac{d^2}{dt^2}[\frac{1}{2}q^2] = \frac{1}{M^2} \left[\left(\frac{df}{dr} \right)^2 \right] - \frac{3}{2M} [q^2] \left[\frac{\partial^2 V}{\partial s^2} \right] - \frac{1}{2M} [q^2] \left[\frac{d^2 f}{dr^2} \right] - \frac{1}{M} [q^2] \left[\frac{df}{rdr} \right] + \frac{3}{2M^2} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] \quad (23).$$

Let $[\frac{1}{2}q^2] = T_2 + \eta$,

where T_2 corresponds to energy equilibrium, and η is a small variation in the rotational kinetic energy. Then

$$\ddot{\eta} = -\frac{2\eta}{3M} \left[3 \frac{\partial^2 V}{\partial s^2} + \frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr} \right].$$

For stability it is necessary that the average value of

$$3 \frac{\partial^2 V}{\partial s^2} + \frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr}$$

should be positive.

Now, since

$$\left[\frac{\partial^2 V}{\partial s^2} \right] = \frac{m_1 m_2}{(m_1 + m_2)^2} \left[\frac{\partial^2 V}{\partial x^2} \right],$$

$\left[\frac{\partial^2 V}{\partial s^2} \right]$ must be positive; for our assumptions require that $\left[\frac{\partial^2 V}{\partial x^2} \right]$ should be positive.

The question of stability will be considered in particular examples.

In the general case, suppose that we put

$$\frac{1}{2}[\dot{r}^2] = T_1 + \eta_1$$

$$\frac{1}{2}[q^2] = T_2 + \eta_2$$

and

$$[q^4] = [q^4]_0 + \xi,$$

where $[q^4]_0$ is the value of $[q^4]$ corresponding to energy equilibrium, and

$$T_1 = [\frac{1}{2}\dot{r}^2]_0$$

$$T_2 = [\frac{1}{2}q^2]_0$$

with the corresponding rotation.

We have the equations

$$\left[\frac{d^2}{dt^2} \frac{1}{2} \dot{r}^2 \right] = \frac{1}{M^2} \left[\left(\frac{\partial V}{\partial r} \right)^2 \right] + \left[\frac{q^4}{r^2} \right] - 3 \left[\frac{\dot{r}^2 q^2}{r^2} \right] - \frac{1}{M} \left[\dot{r}^2 \frac{\partial^2 V}{\partial r^2} \right] - \frac{2}{M} \left[q^2 \frac{df}{rdr} \right]$$

and

$$\left[\frac{d^2}{dt^2} \frac{1}{2} q^2 \right] = \frac{1}{M^2} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] - \left[\frac{q^4}{r^2} \right] + 3 \left[\frac{\dot{r}^2 q^2}{r^2} \right] - \frac{1}{M} \left[q^2 \frac{\partial^2 V}{\partial s^2} \right] + \frac{1}{M} \left[q^2 \frac{df}{rdr} \right].$$

Now

$$\begin{aligned} [\dot{r}^2 q^2] &= [\dot{r}^2][q^2] \\ &= 4(T_1 + \eta_1)(T_2 + \eta_2) \\ &= \{q^2 \dot{r}^2\}_0 + 4\eta_1 T_2 + 4\eta_2 T_1, \end{aligned}$$

neglecting terms of the second order.

Therefore, substituting for $[q^4]$, $[q^2]$, and $[r^2]$ in the equations, we obtain at once the equations

$$\begin{aligned}\ddot{\eta}_1 &= \xi \left[\frac{1}{r^2} \right] - 12T_1 \left[\frac{1}{r^2} \right] \eta_2 - 12T_2 \left[\frac{1}{r^2} \right] \eta_1 - \frac{2\eta_1}{M} \left[\frac{\partial^2 V}{\partial r^2} \right] - \frac{4\eta_2}{M} \left[\frac{df}{rdr} \right] \\ \ddot{\eta}_2 &= -\xi \left[\frac{1}{r^2} \right] + 12T_1 \left[\frac{1}{r^2} \right] \eta_2 + 12T_2 \left[\frac{1}{r^2} \right] \eta_1 - \frac{2\eta_2}{M} \left[\frac{\partial^2 V}{\partial s^2} \right] + \frac{2}{M} \eta_2 \left[\frac{df}{rdr} \right].\end{aligned}$$

It is next necessary to evaluate ξ .

We know that

$$[q^2] = 2[r^2\dot{\theta}^2].$$

Now

$$r^2\dot{\theta}^2 = \{r^2\dot{\theta}^2\}_0 + \epsilon_2, \text{ suppose,}$$

where ϵ_2 is small, and $\{r^2\dot{\theta}^2\}_0$ denotes a value of $r^2\dot{\theta}^2$ corresponding to energy equilibrium; so that, if we take mean values,

$$\eta_2 = [\epsilon_2].$$

Squaring both sides of the above relation, we have

$$r^4\dot{\theta}^4 = \{r^4\dot{\theta}^4\}_0 + 2\epsilon_2\{r^2\dot{\theta}^2\},$$

and when we take mean values this becomes

$$[r^4\dot{\theta}^4] = [r^4\dot{\theta}^4]_0 + 2\eta_2 T_2$$

or

$$\frac{3}{8}[q^4] = \frac{3}{8}[q^4]_0 + 2\eta_2 T_2,$$

for we have proved that

$$[r^4\dot{\theta}^4] = \frac{3}{8}[q^4];$$

so that

$$[q^4] = [q^4]_0 + \frac{16}{3}\eta_2 T_2,$$

whence

$$\xi = \frac{16}{3}\eta_2 T_2.$$

We thus have

$$\begin{aligned}\ddot{\eta}_1 &= \frac{16}{3}T_2 \left[\frac{1}{r^2} \right] \eta_2 - 12T_1 \left[\frac{1}{r^2} \right] \eta_2 - 12T_2 \left[\frac{1}{r^2} \right] \eta_1 - \frac{2\eta_1}{M} \left[\frac{\partial^2 V}{\partial r^2} \right] - \frac{4\eta_2}{M} \left[\frac{df}{rdr} \right] \\ \ddot{\eta}_2 &= -\frac{16}{3}T_2 \left[\frac{1}{r^2} \right] \eta_2 + 12T_1 \left[\frac{1}{r^2} \right] \eta_2 + 12T_2 \left[\frac{1}{r^2} \right] \eta_1 - \frac{2\eta_2}{M} \left[\frac{\partial^2 V}{\partial s^2} \right] + \frac{2\eta_2}{M} \left[\frac{df}{rdr} \right].\end{aligned}$$

The energy equilibrium is stable, if the equation

$$\begin{vmatrix} p^2 - 12T_2 \left[\frac{1}{r^2} \right] - \frac{2}{M} \left[\frac{\partial^2 V}{\partial r^2} \right], & \frac{16}{3}T_2 \left[\frac{1}{r^2} \right] - 12T_1 \left[\frac{1}{r^2} \right] - \frac{4}{M} \left[\frac{df}{rdr} \right] \\ 12T_2 \left[\frac{1}{r^2} \right], & p^2 - \frac{16}{3}T_2 \left[\frac{1}{r^2} \right] + 12T_1 \left[\frac{1}{r^2} \right] - \frac{2}{M} \left[\frac{\partial^2 V}{\partial s^2} \right] + \frac{2}{M} \left[\frac{df}{rdr} \right] \end{vmatrix} = 0 \quad (24),$$

treated as a quadratic in p^2 , has positive roots.

This requires two complicated conditions, namely, that

$$\left\{ 6T_2 \left[\frac{1}{r^2} \right] + \frac{1}{M} \left[\frac{\partial^2 V}{\partial r^2} \right] \right\} \left\{ \frac{8}{3} T_2 \left[\frac{1}{r^2} \right] - 6T_1 \left[\frac{1}{r^2} \right] + \frac{1}{M} \left[\frac{\partial^2 V}{\partial s^2} \right] - \frac{1}{M} \left[\frac{df}{rdr} \right] \right\}$$

$$- 3T_2 \left[\frac{1}{r^2} \right] \left\{ \frac{16}{3} T_2 \left[\frac{1}{r^2} \right] - 12T_1 \left[\frac{1}{r^2} \right] - \frac{4}{M} \left[\frac{df}{rdr} \right] \right\}$$

and

$$\frac{5}{3} T_2 \left[\frac{1}{r^2} \right] - 12T_1 \left[\frac{1}{r^2} \right] + \frac{4}{M} \left[\frac{\partial^2 V}{\partial s^2} \right] + \frac{2}{M} \left[\frac{d^2 f}{dr^2} - \frac{df}{rdr} \right]$$

should be positive.

(15) Having now obtained the general equations of energy equilibrium, let us proceed to apply the results to certain special cases.

Let us consider first the case of two oppositely electrified equal masses.

If we put $\mu_2 = -\mu_1 = -\mu$ and $m_2 = m_1 = m$, the following simplifications appear:—

$$\left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] = \left[\left(\frac{\partial \Phi_2}{\partial x_2} \right)^2 \right]$$

$$\left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] = - \left[\frac{\partial^2 \Phi_2}{\partial x_2^2} \right] = \frac{1}{2\mu} \left[\frac{\partial^2 V}{\partial x^2} \right]$$

$$= \frac{2}{\mu} \left[\frac{\partial^2 V}{\partial s^2} \right]$$

$$\left[\left(\frac{\partial V}{\partial x} \right)^2 \right] = 2\mu^2 \left\{ \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] - \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\}$$

and

$$\left[\left(\frac{\partial V}{\partial s} \right)^2 \right] = \mu^2 \left\{ \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] + \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\}.$$

The equations of energy equilibrium are

$$\left[\dot{x}^2 \right] = \frac{\mu}{2m} \frac{\left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 - \frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right]}{\left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right]},$$

$$3 \left[\frac{r^2}{r^2} \dot{q}^2 \right] - \left[\frac{q^4}{r^2} \right] = \frac{4}{m_2} \left[\left(\frac{df}{dr} \right)^2 \right] - \frac{2}{m} [r^2] \left[\frac{d^2 f}{dr^2} \right] - \frac{4}{m} [q^2] \left[\frac{df}{rdr} \right]$$

$$- \frac{\mu}{m} [r^2] \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] + \frac{2\mu^2}{m^2} \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 + \frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right]$$

$$= \frac{\mu}{m} [q^2] \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] - \frac{4\mu^2}{m^2} \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 + \frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] - \frac{2}{m} [q^2] \left[\frac{df}{rdr} \right]$$

The condition for equipartition, according to Maxwell's Law, becomes

$$\left[\left(\frac{df}{dr} \right)^2 + 3\mu^2 \frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] = \mu \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 - \frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \left[\frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr} \right] \quad (26).$$

THE LAW OF THE INVERSE SQUARE.

(16) The case will now be considered in which the law of force between the particles is that of the inverse square of the distance.

Let $f(r) = -\frac{\lambda}{r}$, where λ is a constant.

Then

$$\frac{df}{dr} = \frac{\lambda}{r^2}$$

$$\frac{d^2f}{dr^2} = -\frac{2\lambda}{r^3},$$

so that

$$\left[\frac{df}{rdr} \right] = \lambda \left[\frac{1}{r^3} \right], \quad \left[\left(\frac{df}{dr} \right)^2 \right] = \lambda^2 \left[\frac{1}{r^4} \right],$$

and

$$\left[\frac{d^2f}{dr^2} \right] = -2\lambda \left[\frac{1}{r^3} \right].$$

The equations of energy equilibrium become

$$\left. \begin{aligned} [\dot{x}^2] &= \frac{\mu_1 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] + \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right]}{(m_1 + m_2) \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right]} \\ 3 \left[\frac{r^2}{r^2} Q^2 \right] - \left[\frac{Q^4}{r^2} \right] &= \frac{\lambda^2}{M^2} \left[\frac{1}{r^4} \right] - \frac{2\lambda}{M} [Q^2] \left[\frac{1}{r^3} \right] + \frac{2\lambda}{M} [r^2] \left[\frac{1}{r^3} \right] \\ &\quad - \frac{\mu_1}{m_1} [r^2] \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] + \frac{\mu_1}{M} \left\{ \frac{\mu_1}{m_1} \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] - \frac{\mu_2}{m_2} \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\} \\ &= \frac{\mu_1}{m_1} [Q^2] \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] - \frac{\lambda}{M} [Q^2] \left[\frac{1}{r^3} \right] - \frac{2\mu_1}{M} \left\{ \frac{\mu_1}{m_1} \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] - \frac{\mu_2}{m_2} \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\} \end{aligned} \right\} \quad (27)$$

Important simplifications occur, however, when we consider Maxwell's Law of Equipartition.

The condition, in the general case, that the Law holds good is

$$\left\{ \left[\left(\frac{df}{dr} \right)^2 \right] - 3\mu_1\mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\} \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] = \left\{ \mu_1 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] + \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\} \left[\frac{d^2f}{dr^2} + \frac{2}{r} \frac{df}{dr} \right].$$

Now, in this case

$$\frac{d^2f}{dr^2} + \frac{2}{r} \frac{df}{dr} = 0,$$

and $\left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right]$ cannot vanish; so that the condition for equipartition becomes simply

$$\left[\left(\frac{df}{dr} \right)^2 \right] = 3\mu_1\mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right]$$

or

$$\lambda^2 \left[\frac{1}{r^4} \right] = 3\mu_1\mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \quad (28).$$

If $\mu_1\mu_2\left[\frac{\partial\Phi_1}{\partial x_1}\frac{\partial\Phi_2}{\partial x_2}\right]$ is negative, Maxwell's Law cannot possibly hold good.

In the special case when $\mu_2 = -\mu_1$, Maxwell's Law cannot possibly hold good if $\left[\frac{\partial\Phi_1}{\partial x_1}\frac{\partial\Phi_2}{\partial x_2}\right]$ is positive; and in the case when $\frac{\mu_1}{m_1} = \frac{\mu_2}{m_2}$, the Law cannot hold good if $\left[\frac{\partial\Phi_1}{\partial x_1}\frac{\partial\Phi_2}{\partial x_2}\right]$ is negative.

In considering the question of stability in the case when Maxwell's Law holds good, we obtained a condition that $3\left[\frac{\partial^2 V}{\partial s^2}\right] + \left[\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}\right]$ should be positive; this becomes a condition that $\left[\frac{\partial^2 V}{\partial s^2}\right]$ should be positive; and this condition is satisfied.

THE LAW OF THE DIRECT DISTANCE.

(17) Let the law of force between the particles be supposed now to be that of the direct distance.

Let $f(r) = \frac{1}{2}\lambda r^2$. Then

$$\frac{df}{rdr} = \lambda, \quad \frac{d^2 f}{dr^2} = \lambda, \quad \text{and} \quad \left(\frac{df}{dr}\right)^2 = \lambda^2 r^2.$$

We obtain at once

$$\frac{1}{M} \left\{ \lambda^2 [r^2] + \frac{3}{2} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] \right\} - [q^2 + \dot{r}^2] \left[\frac{\partial^2 V}{\partial s^2} \right] - \lambda [r^2 + q^2] = 0,$$

whence

$$[q^2 + \dot{r}^2] = \frac{\lambda^2 [r^2] + \frac{3}{2} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right]}{M \left\{ \lambda + \left[\frac{\partial^2 V}{\partial s^2} \right] \right\}}.$$

Also we must have

$$[\dot{x}^2] = \frac{\left[\left(\frac{\partial V}{\partial x} \right)^2 \right]}{(m_1 + m_2) \left[\frac{\partial^2 V}{\partial x^2} \right]}.$$

These equations may be written in the forms

$$[\dot{x}^2] = \frac{\mu_1 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] + \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right]}{(m_1 + m_2) \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right]} \quad \left. \vphantom{[\dot{x}^2]} \right\} \quad (29).$$

and

$$[q^2 + \dot{r}^2] = \frac{\lambda^2 [r^2] + \frac{3\mu_1}{m_1 + m_2} \left\{ \mu_1 m_2 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] - m_1 \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\}}{\frac{m_1 m_2}{m_1 + m_2} \left\{ \lambda + \frac{m_2 \mu_1}{m_1 + m_2} \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] \right\}}$$

These are two of the equations of energy equilibrium in this case; the third is an awkward equation involving $[q^4]$, namely,

$$3 \left[\frac{\dot{r}^2}{r^2} q^2 \right] - \left[\frac{q^4}{r^2} \right] = [q^2] \left\{ \frac{\mu_1}{m_1} \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] - \lambda \frac{m_1 + m_2}{m_1 m_2} \right\} - \frac{2\mu_1(m_1 + m_2)}{m_1 m_2} \left\{ \frac{\mu_1}{m_1} \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] - \frac{\mu_2}{m_2} \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\} \quad (30).$$

The condition that Maxwell's Law should hold good becomes in this case

$$\left\{ \lambda^2 [r^2] - 3\mu_1 \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\} \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] = 3\lambda \left\{ \mu_1 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] + \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\},$$

which may also be put in the form

$$\lambda^2 [r^2] \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] - 3\lambda \mu_1 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] - 3\mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \left\{ \lambda + \mu_1 \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] \right\} = 0 \quad (31).$$

Now, it has been shown that if Maxwell's Law holds good, it is necessary that

$$\frac{\left[\left(\frac{df}{dr} \right)^2 \right] - 3\mu_1 \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right]}{\left[\frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr} \right]}$$

should be positive. In this case, therefore,

$$\lambda [r^2] - \frac{3\mu_1 \mu_2}{\lambda} \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right]$$

must be positive.

If λ is negative, it is clearly necessary that

$$- \frac{3\mu_1 \mu_2}{\lambda} \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right]$$

or that $\mu_1 \mu_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right]$ should be positive, provided that Maxwell's Law holds good. The condition is that $\left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right]$ should be negative in the case of two oppositely electrified equal masses, when λ is negative.

The question of stability will now be considered. From the general results,

$$\frac{d^2}{dt^2} \left[\frac{1}{2} (q^2 + \dot{r}^2) \right] = \frac{1}{M^2} \left\{ \lambda^2 [r^2] + \frac{3}{2} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] \right\} - \frac{1}{M} [q^2 + \dot{r}^2] \left[\frac{\partial^2 V}{\partial s^2} \right] - \frac{\lambda}{M} [q^2 + \dot{r}^2],$$

where we proceed as in the case of the law of the inverse square.

Let $\left[\frac{1}{2} (q^2 + \dot{r}^2) \right] = T_1 + \epsilon$, where T_1 is the value of $\left[\frac{1}{2} (q^2 + \dot{r}^2) \right]$ determined from the equation of energy equilibrium, and ϵ is a small variation in it. Then

$$\ddot{\epsilon} + \frac{2\epsilon}{M} \left\{ \lambda + \left[\frac{\partial^2 V}{\partial s^2} \right] \right\} = 0.$$

For stability $\lambda + \left[\frac{\partial^2 V}{\partial s^2} \right]$ must be positive. Now, in order that energy equilibrium may be possible, it is necessary that

$$\frac{\lambda^2 [\dot{r}^2] + \frac{3}{2} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right]}{M \left\{ \lambda + \left[\frac{\partial^2 V}{\partial s^2} \right] \right\}}$$

should be positive; for it has been shown that this expression is equal to $[q^2 + \dot{r}^2]$, which is obviously positive; that is to say, $\lambda + \left[\frac{\partial^2 V}{\partial s^2} \right]$ must be positive in order that energy equilibrium may be possible; so that the stability condition which we have just obtained is consistent with this. As in the general case, a consideration of the translational mean energy shows that $\left[\frac{\partial^2 V}{\partial x^2} \right]$ must be positive; and this is consistent with our hypotheses.

(18) Let us now consider the case where the attraction between the particles offers very great (in limit infinite) resistance to any variation of the distance AB from a mean value a .

Let $r = a + \xi$, where ξ is small, and let us assume for the law of force $f(r) = \frac{K}{2}(r - a)^2$, where K is a very large constant which may be supposed infinite.

$$\frac{df}{dr} = K(r - a) = K\xi.$$

Let us suppose that

$$\text{Lt.}_{r=a} K(r - a) = A, \text{ a constant.}$$

It is clear that \dot{r}^2 is an infinitesimal, so that Maxwell's Law cannot hold good in this case.

Now

$$\begin{aligned} \frac{df}{rdr} &= \frac{A}{a} + \text{powers of } \xi \\ \frac{d^2f}{dr^2} &= K \\ \left(\frac{df}{dr} \right)^2 &= A^2 \text{ in the limit,} \end{aligned}$$

and \dot{r}^2 is of the same order as ξ .

Let $\dot{r}^2 = B\xi$ when ξ is very small; then $\left[\dot{r}^2 \frac{d^2f}{dr^2} \right]$ becomes AB in the limit, where B is a finite constant.

The two last equations of energy equilibrium are

$$\begin{aligned} 3\left[\frac{\dot{r}^2}{r^2}q^2\right] - \left[\frac{q^4}{r^2}\right] &= \frac{1}{2M^2}\left[\left(\frac{\partial V}{\partial s}\right)^2\right] + \frac{1}{M^2}\left[\left(\frac{df}{dr}\right)^2\right] - \frac{1}{M}\left[\dot{r}^2\frac{\partial^2 V}{\partial s^2}\right] - \frac{1}{M}\left[\dot{r}^2\frac{d^2 f}{dr^2}\right] - \frac{2}{M}\left[q^2\frac{df}{rdr}\right] \\ &= \frac{1}{M}\left[q^2\frac{\partial^2 V}{\partial s^2}\right] - \frac{1}{M^2}\left[\left(\frac{\partial V}{\partial s}\right)^2\right] - \frac{1}{M}\left[q^2\frac{df}{rdr}\right]. \end{aligned}$$

In this case they become

$$\begin{aligned} -\frac{1}{a^2}[q^4] &= \frac{A^2}{M^2} - \frac{AB}{M} - \frac{2A}{aM}[q^2] + \frac{1}{2M^2}\left[\left(\frac{\partial V}{\partial s}\right)^2\right] \\ &= \frac{1}{M}[q^2]\left[\frac{\partial^2 V}{\partial s^2}\right] - \frac{1}{M^2}\left[\left(\frac{\partial V}{\partial s}\right)^2\right] - \frac{A}{aM}[q^2]. \end{aligned}$$

From the first equation,

$$\left[\left(\frac{q^2}{a} - \frac{A}{M}\right)^2\right] + \frac{1}{2M^2}\left[\left(\frac{\partial V}{\partial s}\right)^2\right] = \frac{AB}{M},$$

which shows that AB must be positive.

These two equations can be easily solved for $[q^2]$ and $[q^4]$. We obtain

$$[q^2] = \frac{\frac{3}{2}\left[\left(\frac{\partial V}{\partial s}\right)^2\right] + A^2 - ABM}{A + a\left[\frac{\partial^2 V}{\partial s^2}\right]} \frac{a}{M}$$

or

$$[q^2] = \frac{a}{M} \frac{A^2 - ABM + 3M\mu_1 \left\{ \frac{\mu_1}{m_1}\left[\left(\frac{\partial\Phi_1}{\partial x_1}\right)^2\right] - \frac{\mu_2}{m_2}\left[\frac{\partial\Phi_1}{\partial x_1}\frac{\partial\Phi_2}{\partial x_2}\right] \right\}}{A + \frac{a\mu_1 m_2}{m_1 + m_2}\left[\frac{\partial^2\Phi_1}{\partial x_1^2}\right]}.$$

Hence

$$\left[\frac{q^2}{a} - \frac{A}{M}\right] = \frac{\frac{3}{2}\left[\left(\frac{\partial V}{\partial s}\right)^2\right] - ABM - Aa\left[\frac{\partial^2 V}{\partial s^2}\right]}{M\left[A + a\frac{\partial^2 V}{\partial s^2}\right]} \quad \dots \quad (32),$$

also

$$\left[\left(\frac{q^2}{a} - \frac{A}{M}\right)^2\right] = \frac{AB}{M} - \frac{1}{2M^2}\left[\left(\frac{\partial V}{\partial s}\right)^2\right] \quad \dots \quad (33);$$

it is much simpler to use these expressions than the corresponding ones for $[q^2]$ and $[q^4]$.

$$\begin{aligned} \frac{3}{2}\left[\left(\frac{\partial V}{\partial s}\right)^2\right] + A^2 - ABM &= \frac{3}{2}\left[\left(\frac{\partial V}{\partial s}\right)^2\right] + A^2 - M^2\left[\left(\frac{q^2}{a} - \frac{A}{M}\right)^2\right] - \frac{1}{2}\left[\left(\frac{\partial V}{\partial s}\right)^2\right] \\ &= \left[\left(\frac{\partial V}{\partial s}\right)^2\right] + A^2 - \left[\left(A - \frac{Mq^2}{a}\right)^2\right]; \end{aligned}$$

and this is a positive quantity, if we suppose that A and a are of the same sign.

It is necessary that AB should be greater than $\frac{1}{2M} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right]$ in order that energy equilibrium may be possible, if the hypotheses are justifiable.

Let us consider the question of stability. From the equations which we have obtained for the general case

$$\begin{aligned} \left[\frac{d^2}{dt^2} \frac{1}{2}(q^2 + r^2) \right] &= \frac{1}{M^2} \left[\left(\frac{df}{dr} \right)^2 \right] - \frac{1}{M} \left[q^2 \frac{df}{r dr} \right] + \frac{3}{2M^2} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] \\ &\quad - \frac{1}{M} \left[r^2 \frac{d^2 f}{dr^2} \right] - \frac{1}{M} [q^2 + r^2] \left[\frac{\partial^2 V}{\partial s^2} \right], \end{aligned}$$

so that in this case

$$\left[\frac{d^2}{dt^2} \frac{1}{2} q^2 \right] = \frac{3}{2M^2} \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] + \frac{A^2}{M^2} - \frac{AB}{M} - \frac{A}{Ma} [q^2] - \frac{1}{M} [q^2] \left[\frac{\partial^2 V}{\partial s^2} \right].$$

Let $[\frac{1}{2}q^2] = T_0 + \eta$, where T_0 is the value of $[\frac{1}{2}q^2]$ corresponding to energy equilibrium, and η is a small variation in it.

Then

$$\ddot{\eta} + \frac{2}{M} \left\{ \frac{A}{a} + \left[\frac{\partial^2 V}{\partial s^2} \right] \right\} = 0.$$

For stability $\frac{A}{a} + \left[\frac{\partial^2 V}{\partial s^2} \right]$ must be positive. This condition is necessarily satisfied if we suppose that A and a are of the same sign.

(19) *The case where the external force follows the law of proportion of mass to weight.*—In this case the equations of energy equilibrium are scarcely simplified, but we have the following relations, if we suppose that $\frac{\mu_1}{m_1} = \frac{\mu_2}{m_2} = k$:—

$$\left. \begin{aligned} m_1 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] - m_2 \left[\left(\frac{\partial \Phi_2}{\partial x_2} \right)^2 \right] &= (m_1 - m_2) \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \\ \left[\left(\frac{\partial V}{\partial x} \right)^2 \right] &= k^2 (m_1 + m_2) \left\{ m_1 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] + m_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\} \\ \left[\left(\frac{\partial V}{\partial s} \right)^2 \right] &= \frac{2k^2 m_1 m_2}{m_1 + m_2} \left\{ m_1 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] - m_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] \right\} \\ \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] &= \left[\frac{\partial^2 \Phi_2}{\partial x_2^2} \right] \\ \left[\frac{\partial^2 V}{\partial x^2} \right] &= k(m_1 + m_2) \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] \\ \left[\frac{\partial^2 V}{\partial s^2} \right] &= \frac{km_1 m_2}{m_1 + m_2} \left[\frac{\partial^2 \Phi_1}{\partial x_1^2} \right] \end{aligned} \right\} \quad (34).$$

and

The following inequalities must hold good:—

$$m_1^2 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] > m_2^2 \left[\left(\frac{\partial \Phi_2}{\partial x_2} \right)^2 \right]$$

$$\left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] < \left[\left(\frac{\partial \Phi_2}{\partial x_2} \right)^2 \right]$$

$$\left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] > \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right]$$

and

$$m_1 \left[\left(\frac{\partial \Phi_1}{\partial x_1} \right)^2 \right] + m_2 \left[\frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_2}{\partial x_2} \right] > 0,$$

where it is supposed, for convenience, that m_1 is the greater of the two masses.

CONCLUSIONS.

(20) A purely dynamical problem has been considered, which may throw additional light on the theory of gas molecules.

Equations have been obtained which should determine the law of partition of energy for a system composed of two attracting particles moving in any field of force, when certain hypotheses concerning the probable motion have been made; these hypotheses require that the mean value of $\nabla^2 V$ should be positive, where V is the potential energy due to all the forces of the system; so that the field has been assumed not to be a Newtonian field. There are three such equations, which I have called, for obvious reasons, the equations of translation, vibration, and rotation.

From these equations I have, by equating to zero the three component mean energy accelerations, obtained three equations of energy equilibrium—in other words, equations which should determine the mean values of the squares of the velocities of the system corresponding to a stationary state; but a term involving the mean value of the square of what I have called the rotational energy appears to be an unfortunate hindrance to the work in the general case.

A condition has been obtained that Maxwell's Law of Equipartition of energy should hold good, but there is apparently no reason why it should be satisfied. This condition is quite independent of the masses of the particles, and becomes a very simple one when the law of force between the particles is that of the inverse square of the distance.

In none of the special cases considered is equipartition found to necessarily exist; and in the particular case in which the attraction between the particles offers very great resistance to any variation of the distance between them the Law cannot possibly hold good.

The stability of the energy equilibrium has been investigated. There should be three conditions for stability, corresponding to the three equations of translation, vibration, and rotation. The condition corresponding to the mean translational energy is very simply obtained, and is necessarily satisfied; but the other two conditions are so complicated as to be apparently useless.

In certain special cases, a second simple stability condition has been obtained, *i.e.* in the case in which Maxwell's Law holds good; when the law of force between the particles is that of the direct distance, this second condition is found to be necessarily satisfied if the hypotheses are justifiable.

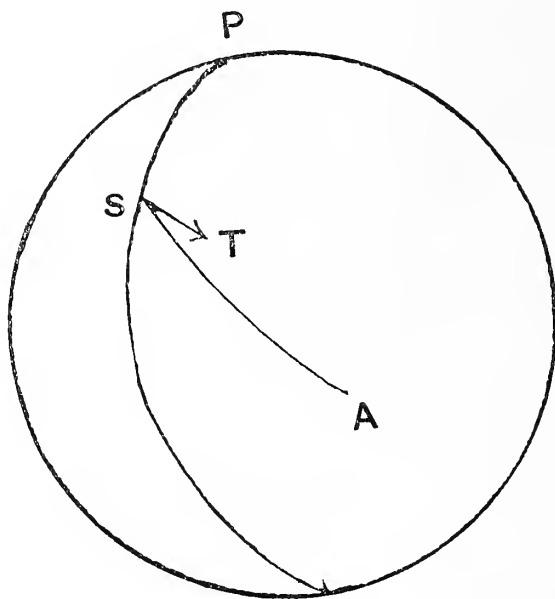
(Issued separately May 15, 1909.)

XXI.—The Systematic Motions of the Stars. (Second Paper.)
By Professor Dyson.

(MS. received February 16, 1909. Read March 1, 1909.)

1. IN a previous paper * the hypothesis propounded by Kapteyn that the stars are moving in two streams was examined by considering the proper motions of stars which were moving more than $20''$ a century. By a graphical method the directions of these two streams were found with results in fair agreement with those found by Kapteyn and Eddington by other methods and with different material. It was evident from that paper that it would be possible in a large percentage of cases to say with tolerable certainty to which stream individual stars of large proper motion belonged.

The most satisfactory way of doing this seemed to be to compare



the direction of each star's proper motion with the direction of the two streams already determined. Thus, in the diagram, if P be the north pole, S the position of any star, A the direction of Stream I, B that of Stream II, and ST the direction of a star's proper motion, the nearness of ST to SA or SB furnishes a criterion as to whether the star belongs to Stream I or Stream II.

The determination of the angles PSA and PSB was made with very great ease by Mr W. B. Blaikie's beautiful device for solving spherical triangles. In each case SA and SB were also determined. Although Mr

* *Proc. Royal Soc. Edin.*, vol. xxviii. pp. 231–238.

Blaikie's "spherical slide rule" admits of more accuracy, these angles were only determined to the nearest degree.

2. The material used differs slightly from that of the former paper. The proper motions of Professor Porter's Catalogues, Nos. 13 and 14 of the Cincinnati Observatory Publications, were substituted for those of No. 12. Additional proper motions were obtained from two papers by Dr Ristenpart in the *Astronomische Nachrichten*, Nos. 4245 and 4276, and from a list of proper motions of the stars observed by Carrington, determined at Greenwich, and published in the *Monthly Notices of the Royal Astronomical Society*, vol. lxxviii. p. 48, and a few from other sources. To facilitate arrangement of the stars in different groups a card catalogue was made.

3. In order to show clearly the method pursued, a table is given of the stars whose proper motion equals or exceeds 100" a century. For each star the name, magnitude, type of spectrum, and approximate R.A. and Dec. for 1900 are given. The magnitudes are taken from the "Revised Harvard Photometry" (vol. 1, *Harvard Annals*), the "Harvard Photometry" (vol. xiv.), or the "Harvard Photometric Durchmusterung" (vol. xlv.), in the order given. The magnitudes of fainter stars are mainly derived from the Catalogues of the *Astronomische Gesellschaft* and the Cape Catalogue of Astrographic Reference Stars. The type of spectrum is obtained from the "Revised Harvard Photometry" or the "Draper" Catalogue. The amount and direction of the centennial proper motion are next given; then the angle AST (positive when $PSA > PST$) and the angle SA under the headings A, a ; then the angles BST and SB.

The assumed positions of A and B are—

A . . .	R.A. 90°	Dec. -10°	}
B . . .	„ 255°	„ -60°	}

These agree closely with the mean of the positions determined by Kapteyn, Eddington, and myself, and quoted at the end of my previous paper.

[TABLE.]

TABLE SHOWING DIVERGENCE OF STARS OF P.M. > 100" FROM THE DIRECTIONS OF THE TWO STREAMS.

No.	Name of Star.	Mag.	Type of Spectrum.	Approx.		Proper Motion.	A. a.		B. b.		
				R.A.	Dec.						
1	Groomb. 34	8.3	...	h m	+43.5	285 82	+ 18	95	+140	134	A
2	ζ Toucani	4.3	II	0 15	-65.5	208 55	+ 36	79	+168	34	A
3	β Hydri	2.9	II	0 21	-77.8	225 82	+ 5	79	+151	36	A
4	Lac. 147	5.7	II	0 32	-25.3	139 90	+ 6	78	+118	79	A
5	Cordoba 617	5.8	II	0 36	-60.0	100 64	+ 23	76	+152	50	A
6	η Cassiop.	3.8	II	0 43	+57.3	121 113	- 9	92	+125	148	A
7	Lal. 1299	5.9	II	0 43	+ 4.8	135 147	- 46	80	+ 60	98	.8
8	μ Cassiop.	5.4	II	1 2	+54.4	376 115	- 7	90	+121	148	A
9	τ Ceti	3.7	II	1 39	-16.5	194 296	+157	63	- 93	93	Neither
10	Lac. 661	6.3	II	2 6	-51.3	227 73	- 3	63	+129	63	A
11	W.B. II. 95	8.8	...	2 9	- 1.7	100 95	+ 6	58	+106	110	A
12	δ Triang.	5.1	II	2 11	+33.8	116 101	+ 17	70	+112	142	A
13	Piazzi II. 123	5.9	II	2 32	+ 6.4	234 57	+ 50	54	+143	119	A
14	Lal. 5490	7.0	...	2 56	+61.3	102 137	- 3	80	+127	165	A
15	ι Persei	4.2	II	3 2	+49.2	132 94	+ 38	71	+132	160	A
16	ζ Reticuli	5.5	II	3 16	-63.0	146 64	- 16	60	+132	55	A
17	ε Eridani	4.3	II	3 16	-43.5	315 76	- 17	49	+117	74	A
18	Lal. 6888-9	8.4	...	3 40	+41.2	134 154	- 14	61	+ 53	157	A
19	Lal. 7443	8.4	...	3 56	+35.0	222 128	+ 14	54	+ 70	153	A
20	40 Eridani	4.7	II	4 11	- 7.8	405 212	-114	28	- 25	112	B
21	W.B. IV. 1189	6.5	II	4 56	- 5.9	124 152	- 47	16	+ 29	114	.2
22	C. Ast. 1734	8.7	...	5 8	-45.0	880 130	-108	36	+ 49	75	B
23	W.B. V. 592	8.5	...	5 26	- 3.7	224 162	- 36	11	+ 13	116	B
24	π Mensæ	5.7	II	5 45	-80.5	112 11	- 7	70	+160	39	A
25	Cordoba 6886	8.5	...	5 46	-70.2	130 344	+ 20	60	-172	49	A
26	α Canis Maj.	-1.3	I	6 41	-16.6	131 203	+100	12	- 36	100	B
27	α Canis Min.	0.5	II	7 34	+ 5.5	124 213	+ 24	28	- 53	118	.6
28	Lac. 2957	5.4	II	7 42	-34.0	172 353	- 43	34	+178	80	A
29	Lal. 15290	8.5	...	7 47	+30.9	150 152	+ 65	48	+ 7	140	B
30	Lal. 15565	7.0	...	7 54	+29.5	117 188	+ 30	48	- 39	138	.8
31	Lal. 16304	6.0	II	8 14	-12.3	101 194	+ 76	33	- 36	98	B
32	Lac. 3386	6.5	II	8 29	-31.2	133 303	- 9	41	-147	79	A
33	Lal. F. 1384	8.5	...	8 46	+71.2	141 256	- 34	86	-162	155	A
34	Lal. F. 1457	6.7	II	9 8	+53.1	170 249	- 21	75	-122	147	A
35	θ Ursæ Maj.	3.3	II	9 26	+52.1	109 239	- 6	76	-109	144	A
36	W.B. IX. 154	9	...	9 46	-11.8	198 143	+123	53	+ 9	89	B
37	Groomb. 1618	6.5	II	10 5	+50.0	144 250	- 9	80	-116	138	A
38	Lal. 21185	7.5	...	10 58	+36.7	473 187	+ 66	84	- 43	121	B
39	Lal. 21258	8.5	...	11 1	+44.0	450 282	- 30	86	-141	127	A
40	O.A. (N) 11677	9.1	...	11 15	+66.4	309 274	- 18	95	-147	142	A
41	Brad. 1584	6.1	II	11 30	-33.3	109 321	- 55	78	-176	59	A
42	Lac. 4887	5.0	II	11 42	-39.9	155 285	- 19	79	-143	52	A
43	Groomb. 1830	6.5	II	11 47	+38.4	701 145	+115	94	+ 2	117	B
44	Lac. 4955	7.0	...	11 53	-27.1	128 242	+ 20	84	- 95	60	A
45	Lal. 23223	8.0	...	12 20	- 3.7	134 261	- 1	95	-110	77	A
46	43 Comæ	4.3	II	13 7	+28.4	118 319	- 50	109	-165	100	A
47	61 Virginis	4.8	II	13 13	-17.8	151 225	+ 30	104	- 74	58	A
48	Berlin 4999	8.8	...	13 40	+18.4	190 172	+ 96	117	- 15	88	B
49	Lal. 25372	8.5	...	13 41	+15.4	230 129	+128	117	+ 28	85	B
50	α Bootis	0.2	II	14 11	+19.7	228 209	+ 62	123	- 49	87	B

TABLE SHOWING DIVERGENCE OF STARS—*continued*.

No.	Name of Star.	Mag.	Type of Spectrum.	Approx.		Proper Motion.	A.	a.	B.	b.		
				R.A.	Dec.							
51	Berlin B. 5072-3	9.0	...	h m	+24°1	140 143	+132	126	+ 18	90		B
52	α^2 Centauri	0.2	II	14 33	-60.4	374 281	- 49	99	-176	18	A	
53	Lal. 27026	7.5	...	14 46	-23.9	101 243	- 3	122	- 87	43	A	
54	Piazzi XIV. 212	5.8	II	14 52	-21.0	200 148	+ 93	125	+ 10	45		B
55	Lal. F. 2544	7.5	...	14 53	+54.1	110 297	+ 5	122	-134	63	A	
56	O.A. (N) 14320-2	9.3	...	15 5	-15.9	368 195	+ 49	130	- 33	49	.5	.5
57	O.A. (N) 14318-9	8.8	...	15 5	-16.0	373 195	+ 47	129	- 34	49	.5	.5
58	Lal. 27744	7.0	...	15 9	- 1.0	145 248	+ 7	136	- 83	63	A	
59	C. Ast. 5545	5.7	II	15 5	-47.9	164 260	- 35	111	-121	19	A	
60	Lal. 28607	7.3	...	15 38	-10.6	118 255	- 14	139	- 88	52	A	
61	γ Serpent.	3.9	II	15 52	+16.0	132 167	+110	150	+ 4	77		B
62	Lal. 30044-5	7.5	...	16 26	+ 4.4	144 199	+ 57	157	- 24	65		B
63	Lal. 30694	7.5	...	16 48	+ 0.2	160 207	+ 34	160	- 29	60		B
64	W.B. XVI. 906	8.0	...	16 50	- 8.2	129 226	- 2	155	- 47	52	.7	.3
65	Lal. 31055	7.5	...	17 0	- 4.9	148 221	+ 4	159	- 41	55	.7	.3
66	A Ophiuchi	4.9	II	17 9	-26.5	124 204	- 4	142	- 21	33	.8	.2
67	Brad. 2179	6.7	II	17 10	-26.4	124 203	- 3	142	- 20	33		
68	Lac. 7215	5.9	II	17 12	-34.9	116 97	+ 89	134	+ 87	26	Neither	
69	72 Herculis	5.4	II	17 17	+32.6	105 173	+161	155	+ 9	92		B
70	W.B. XVII. 322	7.5	...	17 21	+ 2.2	135 207	+ 23	168	- 24	62	.2	.8
71	O.A. (N) 17415	9.9	...	17 38	+68.4	132 197	+157	121	- 11	128		B
72	70 Ophiuchi	4.1	II	18 0	+ 2.5	113 167	+ 13	173	+ 21	65		B
73	P.M. 2164	8.7	...	18 42	+59.5	225 330	+ 44	130	-136	121	A	
74	Munich (I) 1810	9.0	...	18 53	+ 5.8	125 190	- 80	166	+ 5	69		B
75	σ Draconis	4.9	II	19 33	+69.5	184 164	-138	129	+ 41	135		B
76	Lac. 8267	6.0	II	19 56	-67.6	130 121	+ 30	98	+152	20	A	
77	δ Pavonis	3.6	II	19 59	-66.4	162 134	+ 16	100	+132	21	A	
78	Lal. 38383	7.5	...	20 0	+23.1	140 229	-158	148	- 28	91		B
79	Lac. 8362	5.3	II	20 4	-36.4	163 163	- 21	125	+ 53	38	A	
80	Lac. 8381	5.7	II	20 9	-27.3	130 110	+ 26	131	+101	46	A	
81	O.A. (N) 20452	8.5	...	20 18	-21.7	122 155	- 25	134	+ 64	52	A	
82	Lac. 8620	6.5	...	20 51	-44.5	110 209	- 77	113	+ 16	37		B
83	C.P.D. - 34°, 8843	8.5	...	20 56	-34.5	130 173	- 44	119	+ 44	46	.7	.3
84	61 Cygni	5.1	II	21 2	+38.3	521 52	+ 16	130	+156	110	A	
85	Lac. 8760	7.5	...	21 11	-39.3	343 252	-125	113	- 32	44		B
86	W.B. XXI. 502	9.1	...	21 24	-12.9	104 115	- 3	124	+ 95	67	A	
87	ϵ Indi	4.8	II	21 56	-57.2	470 123	- 1	97	+131	37	A	
88	γ Indi	5.4	II	22 16	-72.7	147 121	+ 3	88	+130	31	A	
89	Lac. 9352	7.5	...	22 59	-36.4	693 80	+ 27	96	+136	59	A	
90	Brad. 3077	5.6	II	23 8	+56.6	218 82	+ 3	106	+142	137	A	
91	W.B. XXIII. 175	8.0	...	23 12	-14.4	129 201	- 98	99	+ 10	79		B
92	O.A. (N.) 25685	6.6	II	23 27	+58.6	109 86	+ 3	102	+145	140	A	
93	Lal. 46650	8.5	...	23 44	+ 1.9	136 135	- 35	94	+ 75	97	A	
94	85 Pegasi	6.0	II	23 57	+26.6	130 139	- 40	95	+ 75	120	A	
95	Cordoba 32416	8.5	...	23 59	-37.9	623 113	- 15	84	+ 99	65	A	

4. Inspection of the above table shows very clearly the preference in the apparent motions of the stars for the directions of the two apices A and B, and the extent to which they diverge from these directions. For example, Nos. 1-6, which are all at a considerable distance from A (95° , 79° , etc.), are all moving in directions not far from the direction of A, and in directions far from that of B. Nos. 20, 22, 36, etc., are similarly moving approximately towards B. Nos. 9 and 66 do not fall in either stream. No. 23 is taken as belonging to the second stream (moving towards B), as its distance from A is only 11° . There are a number of doubtful cases, such as No. 27. In the first instance doubtful cases were omitted, and the following tables formed giving the numbers, at various distances from the two apices A and B belonging to the two streams, which diverged 0° - 10° , 10° - 20° , etc., from the directions of the two streams.

STREAM I.

60°-120° from A.						
Proper Motion.	0°-9°.	10°-19°.	20°-29°.	30°-39°.	40°-49°.	50°-59°.
> 100"	14	9	5	7	2	2
60"-99"	19	20	10	2	3	1
50 -59	17	12	5	6	1	0
40 -49	24	22	17	9	8	2
35 -39	26	15	14	6	8	1
30 -34	37	21	17	16	2	1
27 -29	26	16	14	10	4	1
24 -26	29	31	19	12	7	2
22 -23	27	21	13	12	6	1
20 -21	19	15	12	16	3	1
18 -19	10	11	7	6	1	1

40°-59° and 121°-140° from A.						
> 100"	6	4	2	0	1	1
60"-99"	6	4	4	1	6	0
50 -59	2	3	7	3	1	1
40 -49	9	4	6	3	3	2
35 -39	5	2	2	1	1	1
30 -34	10	12	3	1	1	1
27 -29	8	3	5	2	1	0
24 -26	9	5	5	4	1	0
22 -23	11	8	3	3	1	1
20 -21	6	5	5	3	1	0
18 -19	3	2	3	0	1	0

STREAM II.

60°-120° from B.						
Proper Motion.	0°-9°.	10°-19°.	20°-29°.	30°-39°.	40°-49°.	50°-59°.
> 100"	5	4	7	1	1	0
60"-99"	9	7	11	4	2	0
50-59	8	8	10	0	0	3
40-49	16	9	6	5	1	1
35-39	9	5	1	11	1	1
30-34	17	13	11	8	4	1
27-29	10	11	13	6	2	0
24-26	10	8	6	11	4	3
22-23	25	19	9	11	7	2
20-21	12	6	9	5	1	4
18-19	14	9	13	3	3	2

40°-59° and 121°-140° from B.						
	0°-9°.	10°-19°.	20°-29°.	30°-39°.	40°-49°.	50°-59°.
> 100"	1	2	1	1	1	0
60"-99"	2	3	0	4	2	0
50-59	2	1	3	1	1	0
40-49	11	5	5	0	0	0
35-39	4	2	3	1	2	0
30-34	7	4	2	3	0	0
27-29	6	1	2	2	0	1
24-26	5	4	1	2	1	0
22-23	1	1	1	1	0	0
20-21	1	2	1	3	2	0
18-19	0	1	2	0	0	0

The numbers are not given separately for the different magnitudes of the stars whose distances from their apices are less than 40°, but the totals are given below with those of the preceding tables.

STREAM I.

Distance from A.	0°-9°.	10°-19°.	20°-29°.	30°-39°.	40°-49°.	50°-59°.
60°-120°	248	193	133	102	45	13
40°-59° and 121°-140°	75	52	43	21	18	7
20°-39° and 141°-160°	30	11	8	13	15	6

STREAM II.

Distance from B.	0°-9°.	10°-19°.	20°-29°.	30°-39°.	40°-49°.	50°-59°.
60°-120°	135	99	96	65	26	17
40°-59° and 121°-140°	40	25	19	17	9	1
20°-39° and 141°-160°	4	7	7	2	1	2

In this way the stars were provisionally divided as follows:—

110 moving in directions more than 60° from both apices.
1023 belonging to Stream I.
574 „ „ Stream II.
217 doubtful.
1924 Total.

Kapteyn and Eddington, using stars of large and small proper motions, found an equal number in the two streams. The above result is not at variance with this, for if Stream I has a larger stream velocity, a greater proportion of stars belonging to Stream I will naturally be obtained by limiting the stars to those of large proper motion.

5. The number of stars at different distances from the two apices, moving in directions within 60° and 30° of the two apices, are as follows:—

Distance from Apex.	Fraction of Area of whole Sphere included.	Number in Stream I within 60°.	Number in Stream II within 60°.	Number in Stream I within 30°.	Number in Stream II within 30°.
60°–120°	·508	736	440	574	330
40°–59° and 120°–140°	·264	216	111	170	84
20°–39° and 141°–160°	·171	71	23	49	18

The relative density of the stars of the two streams at different distances from the apices, moving within 60° and 30° of the directions of the streams, are therefore:—

Distance from Apex.	Within 60°.		Within 30°.	
	I.	II.	I.	II.
60°–120°	145	87	113	65
40°–59° and 121°–140°	82	42	64	32
20°–39° and 141°–160°	41	13	29	10

6. If the velocity of Stream I be v , then at a distance θ from A the resolved part of this perpendicular to the line of sight is $v \sin \theta$. The mean value of this for stars between 60° and 120° from A (more accurately 59½° and 120½°) is $\cdot 92 v$; for stars from 40° to 60° and 120° to 140° from A is $\cdot 82 v$.

If the velocity of Stream II be v' , then at a distance θ' from B the resolved part of this perpendicular to the line of sight is $v' \sin \theta'$. Assuming that the accidental distribution of velocities is the same in the two streams, the distribution of stars in the two streams should be the same when $v \sin \theta = v' \sin \theta'$.

Using this argument, the agreement between the numbers 87 and 82 of the last table and between 65 and 64 leads to an equation

$$\cdot 92 v' = \cdot 82 v;$$

while the agreement between 42 and 41 and between 32 and 29 leads to

$$\cdot 82 v' = \cdot 52 v.$$

7. These results are very rough, and would be altered considerably if the stars whose proper motions are between $18''$ and $23''$ were omitted. The results suggest that, by a more careful division of the stars into the two streams and by better grouping, it will be possible to determine the relative velocities of the streams.

Dividing the figures on p. 381 by $\cdot 508$, $\cdot 264$, and $\cdot 171$, we obtain

Distance from Apex.	$0^\circ-9^\circ$.	$10^\circ-19^\circ$.	$20^\circ-29^\circ$.	$30^\circ-39^\circ$.	$40^\circ-49^\circ$.	$50^\circ-59^\circ$.
I { $60^\circ-120^\circ$	49	38	26	20	9	3
I { $40^\circ-59^\circ$ and $121^\circ-140^\circ$	28	20	16	8	7	3
I { $20^\circ-39^\circ$ and $141^\circ-160^\circ$	18	7	5	8	9	4
II { $60^\circ-120^\circ$	27	20	19	13	5	3
II { $40^\circ-59^\circ$ and $121^\circ-140^\circ$	15	9	7	6	3	0
II { $20^\circ-39^\circ$ and $141^\circ-160^\circ$	2	4	4	1	1	1

With this table as a guide the original assortment of the stars into the two streams was revised, and the doubtful cases decided. *E.g.*, No. 56 in Table I. $A = 49^\circ$ $a = 130^\circ$. $B = 33^\circ$ $b = 49^\circ$. If belonging to Stream I this star will fall in the last group but one of the second line: interpolating with the number in the last column of the second line we get 6. If the star belongs to Stream II it will fall in the 4th column of line 5, and we again have the number 6. The star is equally likely to belong to either stream, and it is treated as if $\cdot 5$ belongs to Stream I and $\cdot 5$ to Stream II.

After this revision a table like Table II. was again formed. The following changes were made, however. Stars of proper motion $18''$ and $19''$ were omitted. These stars were included originally before the proper motions in R.A. and Dec. were combined, so as to make sure of getting all the stars of p.m. $> 20''$.

The stars were also arranged according to the following distances and their supplements from the apices:—

65°-115°	, or sin (dist.)	from	·9 to 1·0
45°-64°	,,	,,	·7 to ·9
30°-44°	,,	,,	·5 to ·7
18°-29°	,,	,,	·3 to ·5
12°-17°	,,	,,	·2 to ·3.

The results are shown in the following tables, the figures being given for stars of different proper motions.

STREAM I.

65°-115°.						
Proper Motion.	0°-9°.	10°-19°.	20°-29°.	30°-39°.	40°-49°.	50°-59°.
> 100"	13	7	4	7	2·8	2
60"-99"	17	17	9	3	2	1·3
50-59	15	10	5	5	1	0
40-49	25	19	17	8	8	2·9
35-39	23	13	14	3	9	3·5
30-34	29	19	11	9	2·7	1·8
27-29	21	16	11	9	4	1·3
24-26	28	27	18	11	6	2·7
22-23	25	24	8	9	4·8	3·2
20-21	15	10	8	13	4·7	4·2
45°-64° and 135°-116°.						
> 100"	5	5	3	0	2·2	1·5
60"-99"	7	5	6	0·9	5	0
50-59	4	5	3	3·9	1	1·3
40-49	8·9	6	7	4·7	5	2·2
35-39	8·8	4	5·4	4·1	3	2·1
30-34	16	14	10·9	8·6	3·7	0·3
27-29	9	5	7	1	0	1·9
24-26	8	9	7·7	2·7	1·5	0·5
22-23	11	9	8	7·9	4	1
20-21	10·8	11	8	4	2·8	0·6
30°-44° and 150°-136°.						
> 100"	2·8	1	0	0	1	0
60"-99"	3	2	1·7	3	2	0·3
50-59	2·9	0	4	1	0	0·3
40-49	2·6	5·4	1·4	1·7	1·7	2·7
35-39	4	2·4	3·3	1	0·4	3·7
30-34	5·4	4·6	2·7	0·6	4·0	2·2
27-29	4·7	0	3·8	2·5	2·4	0·5
24-26	5	3	2·9	5	3	0·5
22-23	6	2·9	0·5	1	1	1
20-21	5	3·7	3·4	1·4	1	0

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STREAM II.

65°-115°.						
Proper Motion.	0°-9°.	10°-19°.	20°-29°.	30°-39°.	40°-49°.	50°-59°.
>100"	4	3	6.8	1	2	0
60"-99"	7	6	8	4	2.7	1
50-59	7	4	10	0	0	3.7
40-49	15	8	5	6	1.8	1.5
35-39	8	3	1	10	1	0.5
30-34	15	10	9	7	3.5	1.6
27-29	9	9	11	5	0.7	0.3
24-26	10	8	5	11.3	3.8	3
22-23	22	15	5	10	5.6	3.8
20-21	11	5	8	3	1	5.1
45°-64° and 135°-116°.						
>100"	1	3	1.8	0.5	2.9	0.7
60"-99"	3.7	3	2.8	3.5	1	0
50-59	3	5	2	2.4	2	0
40-49	8.7	6.9	5	1.4	0.3	1.5
35-39	4	4	1	2.4	1.5	0.4
30-34	10.3	7	5.2	4.4	2.4	0.8
27-29	7.6	2	6.8	3.8	3.7	1.8
24-26	4.6	7	2.2	2.7	2	0.3
22-23	4	5	5.5	1.3	1.3	1.6
20-21	3.5	3	3.6	0.5	3	0.5
30°-44° and 150°-136°.						
>100"	1	1.5	1.2	0	0	0
60"-99"	0	1	0	2.3	1	0
50-59	0	0	1	0.1	0	0
40-49	2.8	3.3	1.4	0	1.0	0
35-39	1	0	1	0	1	0
30-34	4	2	0.3	0.3	0.2	0
27-29	0.5	1	0.6	2.3	0	1.5
24-26	2	1.1	2	1.3	0	0
22-23	0.1	0	2.6	1	0.5	0.3
20-21	0.3	0	1.3	1	0	0.2

9. Taking the sums of the columns in the preceding table and giving the sums for the small groups which are not given in detail, we find—

STREAM I.

Limits of Dist. from Apex.	Fraction of Area of Sphere.	Number of Stars whose Directions diverge.					
		0°-9°.	10°-19°.	20°-29°.	30°-39°.	40°-49°.	50°-59°.
65°-115°	·430	211	162	105	77	45	22·9
45°-64°, 116°-135°	·283	88·5	73	66·3	37·8	28·2	11·1
30°-44°, 136°-150°	·157	41·4	25	23·6	17·2	16·5	10·6
18°-29°, 151°-162°	·084	10·5	5·8	4·1	7·0	6·3	2·5
12°-17°, 163°-168°	·026	1·8	...	3·4	1	1·6	0·2

STREAM II.

65°-115°	·430	108	71	68·8	57·3	22·1	20·5
45°-64°, 116°-135°	·283	50·4	45·9	35·9	27·4	20·1	7·6
30°-44°, 136°-150°	·157	11·7	9·9	11·4	8·3	3·7	2·0
18°-29°, 151°-162°	·084	2·6	5·2	3·9	0·1	2·1	1·2

Dividing by the areas, the proportional number of stars per unit area $\frac{4\pi}{10}$ belonging to the different groups is obtained, the mean value of the sine of the angular distance from the apex being given in first column.

STREAM I.

Sin (dist.).	0°-9°.	10°-19°.	20°-29°.	30°-39°.	40°-49°.	50°-59°.
·957	49·1	27·7	24·4	17·9	10·5	5·3
·818	31·3	25·8	23·4	13·4	10·0	3·9
·606	26·4	16·0	15·1	11·0	10·5	6·8
·404	12·5	6·9	4·9	8·3	8·7	3·0

STREAM II.

·957	25·1	16·5	16·0	13·3	5·2	4·8
·818	17·8	16·2	12·7	9·7	7·1	2·7
·606	7·5	6·3	7·3	5·3	2·4	1·3
·404	3·1	6·2	4·7	0·2	2·5	1·4

The above figures show that stars of Stream I whose mean distance from the apex of Stream I is $\sin^{-1} \cdot 606$ are distributed similarly to the stars of Stream II at mean distance $\sin^{-1} \cdot 957$ from their apex.

10. If the numbers of stars belonging to the two streams are equal, and

their random distribution similar, it follows that the stream velocities are in the ratio—

$$\cdot 957 : \cdot 606 \quad \text{or} \quad 1 : \cdot 63.$$

If the stars within 30° of the directions of the two apices are considered, we find—

Sin (dist.).	I.	II.
$\cdot 957$	111·2	57·6
$\cdot 818$	80·5	46·7
$\cdot 606$	57·5	21·1
$\cdot 404$	24·3	14·0

And the number of stars of Stream I distant $\sin^{-1} \cdot 404$ from A agrees sufficiently well with the number of Stream II distant $\sin^{-1} \cdot 606$ from B to confirm this result, but the weight of this determination is small.

The interest of this determination lies in the difference between this result of approximately 3 : 2 in the relative stream velocities with that of 3 or 4 : 1 found by Eddington using stars of all proper motions.

11. The value of the method here employed depends on the uniformity with which the stars are distributed. It is therefore of interest to examine whether the distribution of magnitude of proper motion varies with the distances from A and B in the two streams. Confining our attention to stars moving within 30° of the directions of A and B respectively, *i.e.* those stars whose velocities are largely determined by the stream velocities, we find between the distances 65° and 115° from A and B the following table for the number of stars whose proper motions are above certain limits : *—

Proper Motion.	I.	II.
> 100"	24	14
> 60	67	35
> 50	97	56
> 40	158	84
> 35	208	96
> 30	267	130
> 27	315	159
> 24	388	182
> 22	445	224
> 20	478	248

* As a p.m. of $> 99\frac{1}{2}''$ is called 100'', the limits 100'', 60'', etc. are more strictly $99\frac{1}{2}''$, $59\frac{1}{2}''$, etc.

Multiplying the numbers 111·2, 80·5, etc. of § 10 by 4·30, so that they may refer to the same fraction of the sphere as the numbers in the above table, we obtain for the density of stars whose proper motions are $>19''\cdot5$ at different distances from the apices of the streams—

θ .	Sin θ .	I.	II.
73°	·957	478	248
55	·818	346	201
37	·606	247	91
24	·404	104	60

Interpolation from the previous table shows that between the distances 65° – 115° from A and B respectively there are belonging to the two streams—

I.	"	II.	"
478 stars p.m.	$> 19\cdot5$	248 stars p.m.	$> 19\cdot5$
346 „	$> 25\cdot2$	201 „	$> 22\cdot4$
247 „	$> 31\cdot0$	91 „	$> 36\cdot6$
104 „	$> 48\cdot5$	60 „	$> 47\cdot5$

Comparison of this table with the previous one shows that there are in Stream I as many stars per unit with p.m. $>25''\cdot2$ at mean distance 73° from the apex, as there are at mean distance 55° with p.m. $19''\cdot2$, and so on.

If the accidental velocity is small compared with the stream velocity and the stars are uniformly distributed, then at a distance θ from A the number of stars per unit area whose p.m. $>v$ will equal the number per unit area 90° from A, whose p.m. $>v \operatorname{cosec} \theta$.

Multiplying by the corresponding values of $\sin \theta$ we obtain—

I.	II.
18·7	18·7
20·6	18·3
18·8	22·2
19·6	19·2

The accordance of these figures may be regarded as a check on the uniformity of the distribution of the stars.

As there are 247 stars of Stream I between 65° and 115° from A with p.m. $>31''\cdot0$, and 248 of Stream II between 65° and 115° from B with p.m. $>19''\cdot5$, the stream velocities are in the ratio of 31 : 19·5 or 3 : 2, confirming the previous result.

12. When the stars are arranged according to their type of spectrum the following table is obtained:—

Limits of Proper Motion.	Stars of Type I.		Stars of Type II.			Stars of unknown Spectra.		
	Stream I.	Stream II.	Stream I.	Stream II.	Neither.	Stream I.	Stream II.	Neither.
> 100"	0	1	33.9	10.1	2	25.6	15.4	0
60"-99"	3	0	35.0	15.0	8	51.9	34.1	3
50-59	0	0	16.2	15.8	4	47.6	24.4	5
40-49	4	0	39.2	17.8	4	93.8	54.2	4
35-39	4	1	32.6	12.4	1	71.7	31.3	7
30-34	4	2	44.7	24.3	3	102.1	56.9	11
27-29	4	1	27.1	13.9	5	75.9	53.1	4
24-26	8	3	43.2	17.8	5	95.2	48.8	9
22-23	6	2	39.8	16.2	7	82.6	70.4	10
21-22	8	4	31.0	12.0	6	70.7	43.3	6
Total	41	14	342.7	155.3	45	717.1	431.9	59

The stars whose type of spectrum is known are in nearly all cases of $6^m.5$ or greater. A large proportion of the remaining stars are between $8^m.0$ and $9^m.0$.

In the above table it may be noticed—

- (i) That no stars of Type I are moving in directions diverging more than 60° from the directions of A and B.
- (ii) That the proportion of stars belonging to Stream I is large for Type I stars. This may be accidental, as there are few stars. Attention is drawn to it, as if it holds generally it will afford a partial explanation of the smaller ratios found for the stream velocities (3 : 2) here, as compared with the ratio (3 : 1) found by Eddington.
- (iii) The number of stars diverging more than 60° from A and B is smaller for the faint stars.
- (iv) A somewhat greater proportion of faint stars belong to Stream II.

None of the differences referred to above are very striking. Generally speaking, the two streams consist of the same kind of stars.

Incidentally, the relative number of stars of large proper motions belonging to the spectral types I and II is given in the above table. The numbers are as follows:—

Proper Motion.	Type I.	Type II.
>100"	1	46
50"-100"	3	94
30"-50"	15	179
20"-30"	36	224

The small number of stars of large proper motions belonging to Type I is a well-known result.

13. In the following tables the stars of Type I belonging to the two streams are given. It will be seen that they diverge less from the general drift of the streams than the other stars. The most divergent stars appear to be such very bright stars as Sirius and β Centauri, etc.

STARS OF TYPE I ARRANGED IN ORDER OF MAGNITUDE OF PROPER MOTION.

STREAM I.

Star's Name.	Mag.	R.A.		Dec.	Proper Motion.	A.	α .
		h	m				
Groomb. 884	7.1	4	44	+45.7	68	+13	58
α Aquilæ	1.0	19	46	+ 8.6	66	+41	154
Lal. 4141	6.9	2	10	+23.8	60	+ 7	65
ι Urs. Maj.	3.1	8	52	+48.4	49	-13	70
ϵ Cephei	4.2	22	11	+56.6	46	-12	113
Groomb. 94	7.4	0	29	+47.4	40	+19	93
δ Capric.	3.0	21	42	-16.6	40	-27	119
ι Centauri	2.9	13	15	-36.2	37	- 4	99
Lal. 305	6.5	0	14	- 8.6	36	- 1	85
α Lyræ	0.1	18	34	+38.7	36	-20	150
α Piscis Aust.	1.4	22	52	-30.2	36	-11	99
β Centauri	0.9	13	56	-59.9	33	+56	95
γ Cor. Aust.	5.5	18	58	-31.2	33	+ 6	136
δ Cassiop.	2.8	1	19	+59.7	31	+17	89
Piazzi I. 131	6.4	1	33	- 9.9	31	+25	65
67 Ursæ Maj.	5.1	11	57	+43.6	31	-19	96
ζ Virginis	3.4	13	30	- 0.1	29	-19	112
Gr. 3042	5.7	20	4	+52.9	28	+ 2	130
θ Pegasi	3.7	22	5	+ 5.7	28	+16	119
Piazzi I. 70	6.3	1	20	+34.1	27	0	80
μ Ceti	4.4	2	40	+ 9.7	26	+14	53
α Pictoris	3.3	6	47	-61.8	25	+ 3	53
α Can. Ven.	2.9	12	51	+38.9	25	-12	106
α Pegasi	2.7	22	59	+14.7	25	+39	107
α Leonis	1.3	10	3	+12.5	24	-18	64
δ Corvi	3.1	12	25	-16.0	24	+29	93
τ Centauri	4.0	12	32	-48.0	24	- 7	88
λ Bootis	4.3	14	13	+46.6	24	-21	120
Lal. 37403	5.5	19	38	-15.7	24	- 5	145
θ Cassiop.	4.6	1	5	+54.6	23	+13	89
ξ Cephei	4.4	22	1	+64.1	23	- 6	112
Groomb. 2296	5.0	15	55	+55.0	23	+17	129
Lal. 1344	5.6	4	2	-27.9	22	- 4	33
π Sextant	5.9	9	47	+ 2.9	22	-45	59
7 Pavonis	4.4	17	59	-63.7	22	0	106
α Androm.	2.4	0	3	+28.5	21	-37	94
Groomb. 706	7.4	3	29	+42.6	21	+ 6	63
χ^1 Hydræ	5.7	11	1	-26.8	21	+ 1	72
30 Draconis	5.2	17	47	+50.8	21	+ 6	139
α Gruis	2.2	22	2	-47.5	21	-28	101
89 Leonis	5.8	11	29	+ 3.6	20	+20	83
γ Centauri	2.4	12	36	-48.4	20	- 7	89
Brad. 2792	5.6	21	22	+46.3	20	- 5	124

STREAM II.

Star's Name.	Mag.	R.A.		Dec.	Proper Motion.	B.	b.
		h	m				
α Canis Majoris . . .	-1.3	6	41	-16.6	131	-36	100
θ Hydræ	3.8	9	9	+ 2.7	35	- 4	106
Piazzi VI. 59	6.0	6	13	-22.7	34	+29	96
80 Piscium	5.7	1	3	+ 5.1	32	-31	108
Lalande 25818	6.4	13	59	+11.3	32	- 3	80
6 Andromedæ	6.0	23	6	+43.0	28	- 9	127
45 Boötis	5.0	15	3	+25.3	26	+27	89
Groombridge 2630	8.0	18	34	+63.6	26	0	125
α Ophiuchi	2.1	17	30	+12.6	25	+31	73
γ Ceti	3.8	2	38	+ 2.8	22	-25	117
Groombridge 717	6.3	3	32	+42.2	22	-27	158
23 Andromedæ	5.6	0	8	+40.5	20	+ 3	122
21 Eridani	6.3	3	34	- 6.0	21	+ 3	112
δ Leonis	2.6	11	9	+21.1	21	+21	107
λ Piscium	4.6	23	37	+ 1.2	20	-16	96

The directions of the 110 stars which did not fall within 60° of the directions of either A or B were examined, but no evidence was found to indicate that they belonged to a third stream.

The proper motions of the stars classified as belonging to Streams I and II were treated by Airy's method of determining the solar apex. The following positions were found for A and B:—

$$A. \text{ R.A. } 93^\circ \text{ Dec. } -7^\circ.$$

$$B. \text{ R.A. } 246^\circ \text{ Dec. } -64^\circ.$$

The position of A agrees almost exactly with that found in my previous paper (R.A. 94° Dec. -7°); that found for B lies between the position found there (R.A. 240° Dec. -72°) and the assumed position (R.A. 255° Dec. -60°) of this paper.

With the above positions of A and B and relative velocities of the two streams in the proportion 3 : 2, the apparent motion in two streams may be resolved into a motion of the solar system towards a point—

$$(\text{Apex}) \text{ R.A. } 283^\circ \text{ Dec. } +44;$$

and a motion of two streams in opposite directions in the direction—

$$(\text{Vertex}) \text{ R.A. } 268^\circ \text{ Dec. } -21^\circ.$$

Corresponding to the velocities 3 and 2 for the two streams, the velocity of the solar motion is 1.55, and the relative velocity of the two streams 4.04.

If instead of using the positions of A and B found by Airy's method, those found in my previous paper are used, the co-ordinates of the apex are found to be R.A. 281° , Dec. $+42^\circ$, and of the vertex R.A. 268° , Dec. -24° .

In the *Astronomische Nachrichten*, No. 4291, M. Beljawsky determines the apex and vertex according to the ellipsoidal hypothesis of the distribution of proper motions which has been put forward and developed by Professor Schwarzschild in the *Nachrichten der k. Gesellschaft der Wissenschaften zu Göttingen*, 1907 and 1908. As M. Beljawsky uses the proper motions given by Professor Porter (*Cinc. Publ.* 12), the material he uses largely coincides with what I have used. He finds the following positions:—

(Apex) R.A. 281° Dec. + 36°.

(Vertex) R.A. 266° Dec. - 24°.

It is of interest to notice that the two-stream hypothesis of Kapteyn and Eddington agrees with the ellipsoidal hypothesis of Schwarzschild both when applied to the Groombridge stars and to the stars of large proper motion; and that the differences found by using different stars are more than those found by using different methods. It is hardly possible to say which representation accords most closely with the facts.

(Issued separately May 15, 1909.)

XXII.—Flexural Vibrations of Thin Rods. By George Green, M.A., B.Sc., Assistant to the Professor of Natural Philosophy in the University of Glasgow. *Communicated by Professor GRAY.*

(MS. received December 14, 1908. Read January 18, 1909.)

§ 1. THE main object of this paper is to point out a method of applying hydrodynamic solutions already obtained to the solution of problems relating to flexural vibrations of thin elastic rods.

The results found apply to rods not subjected to permanent tension, and vibrating so that one principal axis of each transverse section lies in the plane of vibration. The central line of the rod is assumed to remain unaltered in length; and particles lying in a plane transverse section, when undisturbed, remain always in a plane normal to the central line. For convenience in what follows we may here derive the equation of motion of a rod subject to these conditions. Taking ω as the area of each section, $\kappa^2\omega$ its moment of inertia, q as Young's modulus, dx as the length of the central line of a small portion, R its radius of curvature, and N as the total tangential force acting on the cross-section at x , we obtain the equation of angular motion of the element

$$q\kappa^2\omega\frac{\partial}{\partial x}\frac{1}{R} + N = \rho\kappa^2\omega\frac{\partial^3 y}{\partial x\partial t^2} \quad (1),$$

where y is the vertical displacement of the element and ρ is its density. The equation of vertical motion is

$$\frac{\partial N}{\partial x} = \rho\omega\frac{\partial^2 y}{\partial t^2} \quad (2).$$

As pointed out by Lord Rayleigh,* whose notation we have adopted, terms depending on the angular motions of the sections of the bar may be neglected in the above equations; accordingly, by eliminating N from (1) and (2) we obtain finally the equation

$$\frac{\partial^2 y}{\partial t^2} + \kappa^2 b^2 \frac{\partial^4 y}{\partial x^4} = 0 \quad (3),$$

in which we have put $\frac{\partial^2 y}{\partial x^2}$ for $\frac{1}{R}$ and b^2 for $\frac{q}{\rho}$.

§ 2. Consider now frictionless liquid in a straight canal with vertical

* *Theory of Sound*, vol. i.

sides. Take the origin of co-ordinates at point O, at a distance h above the undisturbed level, and draw OX parallel to the canal and OZ vertically downwards. Let the motion be infinitesimal and let ξ , ζ , be the displacement components at any time t of any particle of water whose undisturbed position is x, z . If the motion of the water be started primarily from rest by pressure applied to the free surface, the hydrodynamical equations of motion take the well-known form :

$$\dot{\xi} = \frac{\partial}{\partial x} \dot{\phi}(x, z, t); \quad \dot{\zeta} = \frac{\partial}{\partial z} \dot{\phi}(x, z, t) \quad . \quad . \quad . \quad . \quad (4);$$

from which we obtain by integration

$$\xi = \frac{\partial}{\partial x} \phi(x, z, t); \quad \zeta = \frac{\partial}{\partial z} \phi(x, z, t) \quad . \quad . \quad . \quad . \quad (5).$$

In virtue of the incompressibility of the fluid we have also the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad . \quad . \quad . \quad . \quad (6),$$

which shows that if ϕ be known at all points of the free surface, and if it be zero at points infinitely distant, its value can be determined at all points throughout the fluid. The equation to be fulfilled at the free surface, according to a result given by Cauchy and Poisson, is

$$\left(\frac{\partial \phi}{\partial z}\right)_{z=h} = \frac{1}{g} \left(\frac{\partial^2 \phi}{\partial t^2}\right)_{z=h} \quad . \quad . \quad . \quad . \quad (7),$$

where g denotes gravity.

§ 3. Remembering now that every function derived from ϕ by differentiations or integrations with respect to x, z, t , satisfies all the equations satisfied by the function ϕ , we see that we have

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{1}{g^2} \frac{\partial^4 \phi}{\partial t^4} \quad . \quad . \quad . \quad . \quad (8)$$

at a free, or level, liquid surface. Equations (6) and (8) combined give us the relation

$$\frac{\partial^2 \phi}{\partial x^2} = -\frac{1}{g^2} \frac{\partial^4 \phi}{\partial t^4} \quad . \quad . \quad . \quad . \quad (9);$$

and when we put y in place of ϕ , κb in place of $\frac{1}{g}$, and interchange x and t , this equation may be written in the form

$$\frac{\partial^2 y}{\partial t^2} + \kappa^2 b^2 \frac{\partial^4 y}{\partial x^4} = 0 \quad . \quad . \quad . \quad . \quad (3),$$

which is the equation already found for flexural vibrations of an elastic bar, when terms depending upon the angular motions of the sections of the bar may be neglected. Accordingly equation (9) proves that every

hydrodynamical potential function, with x and t interchanged, is a solution of equation (3) above. (The converse is not always true.) In relation to the flexural waves problems z is in general to be treated as a constant.

§ 4. It may be of some interest to note that, in all solutions for elastic bars obtained by applying the above result, the displacement y at each point of the bar may be regarded as derived from a single function $V(t, z, x)$ such that

$$y = \frac{\partial V}{\partial x} \quad \dots \quad (10).$$

Further, it follows from the relationship established between such a function V and the hydrodynamic potential function, that every function derived from V by differentiations or integrations with respect to t, z, x , is a solution of the differential equation (3). Thus any solution of (3) may be a displacement potential, or a displacement, or a velocity. If the function V represents displacement, we readily find from equation (7) that the curvature at each point x of the bar can be obtained by a single differentiation with respect to z , being given by the equation

$$\frac{1}{R} = \frac{1}{\kappa b} \frac{\partial V}{\partial z} \quad \dots \quad (11),$$

from which the potential energy can easily be obtained.

§ 5. Solutions for vibrations of a rod of finite length l are derived directly from the hydrodynamic solutions for waves in a canal of length l by simple interchange of x and t ; and they are applicable to all cases, whether the rod has its ends free, or clamped, or "supported." In particular, the normal functions are easily obtained in this way.

In the case of an infinite rod, all mathematical results relating to surface waves in a canal infinitely long and infinitely deep become immediately useful; space-curves in the hydrodynamical waves problems becoming time-curves for the flexural waves, and *vice versa*.

§ 6. In this connection it may be useful at a later time to examine some of the numerous hydrodynamical solutions relating to surface waves and groups of waves. A number of curves are shown in papers on Water-Waves* by the late Lord Kelvin, illustrating results derived from particular hydrodynamic solutions comprehended in the following general expression, given in his last Waves paper:—

$$\{\text{RS}\} \text{ or } \{\text{RD}\} \frac{\partial^{j+k+l}}{\partial t^j \partial x^k \partial z^l} \frac{1}{\sqrt{(z+ix)}} e^{-4(z+ix)gt^2}$$

In this, {RS} denotes a realisation by taking half the sum of what follows

* *Proc. Roy. Soc. Edin.*, vol. xxv., Feb. and June 1904; vol. xxvi., Oct. 1906.

it with $\pm i$; {RD} denotes a realisation by taking the difference of what follows it with $\pm i$ divided by $2i$. As an example of flexural waves in an infinite elastic rod, arising from a given initial displacement, we may take the solution

$$y = \left. \begin{aligned} & \{RS\} \frac{1}{\sqrt{(z+it)}} \epsilon^{-\frac{x^2}{4\kappa b(z+it)}} \\ & = \sqrt{\frac{1}{T}} \cos\left(\frac{x^2 t}{4\kappa b T^2} - \frac{1}{2} \tau\right) \epsilon^{-\frac{x^2 z}{4\kappa b T^2}} \end{aligned} \right\} \dots \dots \dots (12).$$

where

$$T = \sqrt{(z^2 + t^2)}, \text{ and } \tau = \tan^{-1}\left(\frac{t}{z}\right)$$

In what follows, z is taken as 1 and κb as $\frac{1}{4}$ in order to allow us to use Lord Kelvin's hydrodynamical results in our present problem.

§ 7. Taking the origin of co-ordinates at the middle of the bar, the initial configuration is given by

$$y = \epsilon^{-\frac{x^2}{4\kappa b}} \dots \dots \dots (13).$$

Almost immediately after the commencement of motion, an infinite number of waves are formed along the bar, with amplitudes diminishing according to the law $\epsilon^{-\frac{x^2}{4\kappa b T^2}}$ and with the distances from zero to zero becoming shorter and shorter as we pass from the middle toward the ends of the bar. The zeros come into existence at the ends of the bar, and begin travelling inwards to the middle. The first zero formed comes almost instantaneously to the middle region, and the others follow it in their order of formation. The inward progress of the zeros soon ceases, the first zero never quite reaching the centre; in a short time they begin to move in the opposite direction and continue to do so for ever, and the amplitudes at any point x ultimately fall off according to $\frac{1}{t^{\frac{1}{2}}}$. The middle point of the rod subsides to its undisturbed position nonvibrationally, while the distance from it to the first zero on either side continually increases after a certain time, being given by the equation $x^2 = 3\kappa b \pi t$.

§ 8. The seven curves given by Lord Kelvin, as space curves for water-waves, on page 191 of his paper, *Proc. Roy. Soc. Edin.*, vol. xxv., Feb. 1904, show the condition of things in our present problem at seven different points near the middle of the bar, as t increases from 0 to ∞ , provided the curves be continued to meet the axis of t asymptotically at infinity (see fig. 1 and § 9 below). These curves show that points very near the origin never pass through their initial positions, but fall back nonvibrationally to them; points farther from the middle rise slightly and then behave in the same

way. At points along the bar more and more distant from its middle point the disturbance consists of a larger and larger number of waves which

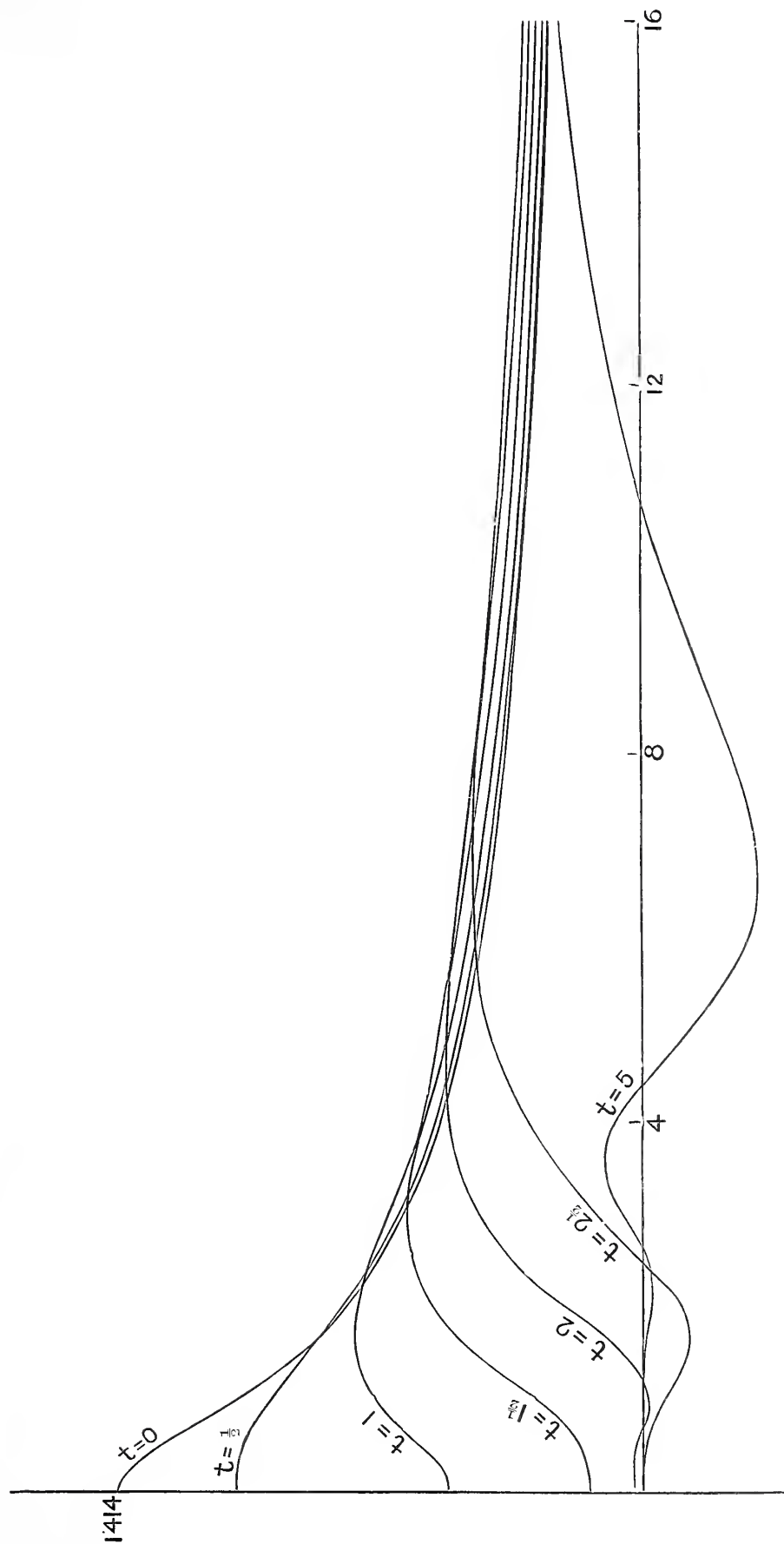


FIG. 1.—Abscissas represent space for water-waves, time for flexural-waves.

travel inwards past the point considered, before $t \doteq 1$, and after that recross the point in the opposite direction one by one. When the first

zero recrosses the point it subsides gradually to its original place of rest, only reaching it, however, after an infinite time. The successive

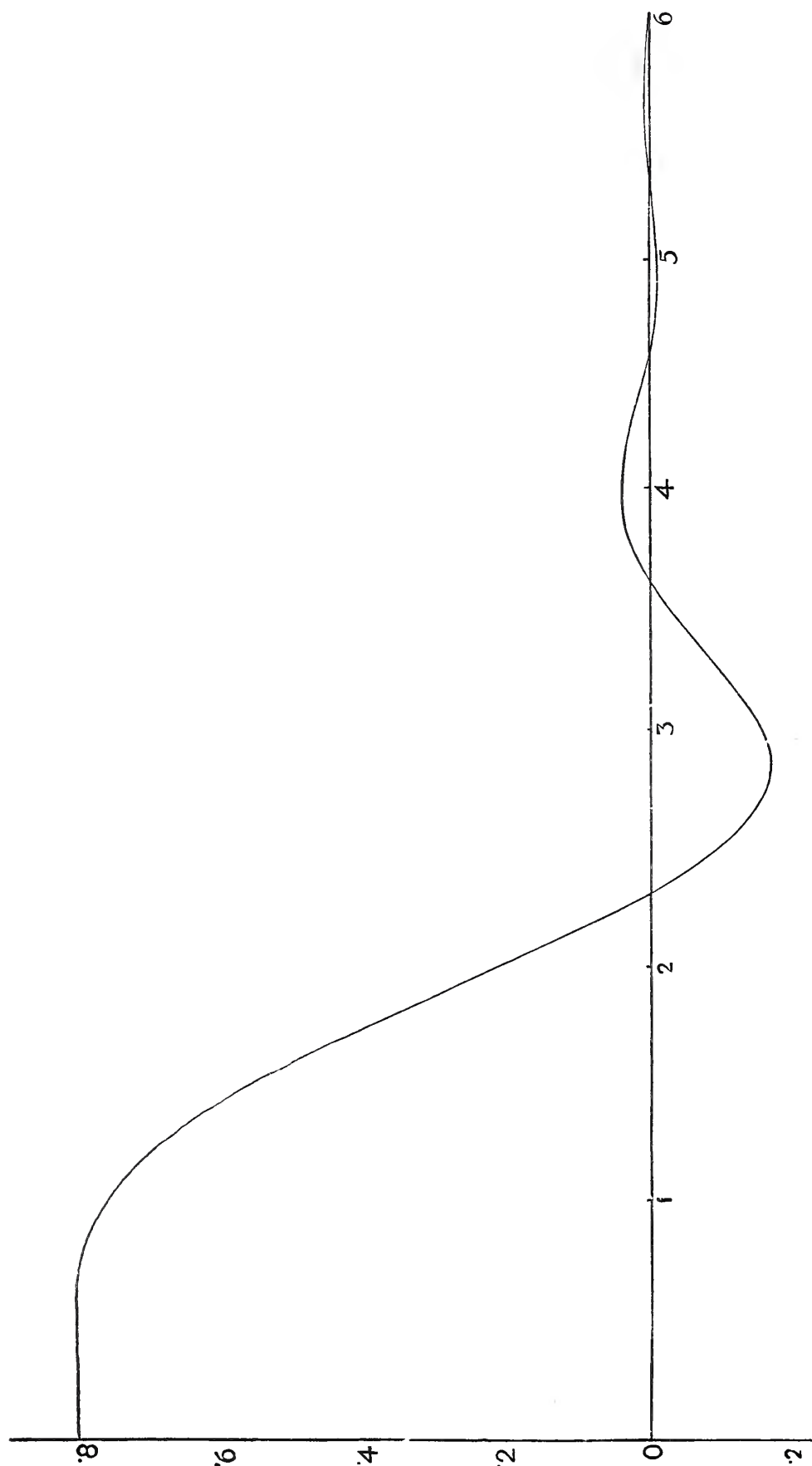


FIG. 2.—Abscissas represent time for water-waves, space for flexural-waves.

maxima of displacement increase very slowly at first, then more quickly, and then diminish finally according to $\frac{1}{t^{\frac{3}{2}}}$ as already stated.

§ 9. The diagrams of figs. 1 and 2 are taken from Lord Kelvin's paper referred to in § 8, and they are reproduced without change of the lettering applicable to them as water-wave diagrams. To make them correspond exactly to the flexural-waves problem solved by equation (12), we must reduce the ordinates in both figures in the ratio $\sqrt{2} : 1$; then in fig. 1 replace t by x on each of the seven curves, and take ordinates as representing displacement and abscissas as representing time. As a water-waves diagram fig. 2 represents the vertical displacement of the water at point $x=2$, from $t=0$ to $t=\infty$; as a flexural-waves diagram it represents the shape of the right-hand half of the rod, $x=0$ to $x=\infty$, at time $t=2$, corresponding to the initial configuration given by equation (13).

These curves are useful chiefly as illustrations of the propagation of waves in dispersive media from a given initial disturbance confined in the main to the neighbourhood of the origin. They show clearly the distinctive features of wave-propagation in the two cases where the wave-velocity varies directly as the square root of the wave-length and inversely as the wave-length respectively, for an infinite succession of regular sinusoidal waves. It is interesting to observe that in both cases the wave-disturbance is ultimately spread throughout the entire medium, but that in the case of water-waves the wave-length of the disturbance at any time increases continuously, and in the case of flexural-waves it diminishes continuously, as we pass outwards from the middle point of the initial disturbance. In the case of water-waves also, Lord Kelvin's investigations show that each individual wave lengthens and increases in speed as it advances, while in the case of flexural-waves each wave lengthens and diminishes in speed, as we shall see by equations (15) and (16) below for the solution we have chosen for illustration.

§ 10. At a moderate time after the motion has commenced, τ may be put equal to $\frac{\pi}{2}$ in equation (12), and the argument of the cosine varies then only with $\frac{x^2 t}{4\kappa b T^2}$, so that it is easy for us to trace the outward progress of any particular maximum or zero of displacement along the bar. Thus the position of any zero is determined by an equation of the form

$$\frac{x^2 t}{4\kappa b T^2} = c = \frac{4n+3}{4} \pi \quad \dots \quad (14),$$

and the velocity of the zero is given by

$$\frac{dx}{dt} = \frac{8\kappa b c t - x^2}{2tx} = \frac{2\kappa b c}{x} \quad \dots \quad (15).$$

When t is large, so that we may put $\frac{t}{T^2} \doteq \frac{1}{t}$, equation (14) enables us to

write (15) in the following approximate form,

$$\frac{dx}{dt} = \frac{x}{2t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15').$$

It can easily be verified that when t is very small and x great the velocity of any zero is given by

$$\frac{dx}{dt} = -\frac{x}{2t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

In each case, what we obtain in these equations is also the wave-velocity for an infinite train of waves of wave-length equal to that maintaining in the immediate neighbourhood of the point x at time t .

§ 11. To obtain the velocity of the group of waves of wave-lengths approximately equal to λ —that maintaining in the neighbourhood of x at time t — we put

$$\frac{x^2 t}{4\kappa b T^2} - \frac{(x - \lambda)^2 t}{4\kappa b T^2} = 2\pi \quad . \quad . \quad . \quad . \quad . \quad (17),$$

which enables us to write down the group-velocity thus:—

$$\frac{dx}{dt} = \frac{8\pi\kappa b t - \lambda x}{\lambda t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

When t is very small and x great, the right-hand side of (18) is approximately equal to $-\left(\frac{x}{t}\right)$. If in (17) we take the wave-length λ as small compared with x , we can obtain the following approximate expression for λ when t is great:—

$$\lambda = \frac{4\pi\kappa b t}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

With this value for λ , the right-hand side of (18) becomes $\left(\frac{x}{t}\right)$.

Equations (15)–(19) show that in the two cases, when t is small and x great, and when t is great, the group-velocity is twice the wave-velocity; which is in accordance with the theory of group-velocity given by Osborne Reynolds and extended by Lord Rayleigh.

XXIII.—A Negative Attempt to detect Fluorescence Absorption.
 By Robert A. Houstoun, M.A., D.Sc., Ph.D., Lecturer on
 Physical Optics in the University of Glasgow. *Communicated*
 by Professor A. GRAY, F.R.S.

(MS. received March 20, 1909. Read July 12, 1909.)

ACCORDING to Kirchhoff's law every body in a state of pure temperature radiation absorbs those rays which it emits, and the ratio of the coefficient of absorption to the coefficient of emission is constant for all bodies. In many cases which do not come under the category of temperature radiation—for example, a sodium flame—the wave-lengths emitted are also absorbed. It is thus possible that a fluorescing substance, in addition to its ordinary absorption, may absorb while fluorescing those wave-lengths that it fluoresces. From a photometric study of fluorescent uranium glass Burke stated he had discovered such an effect. A cube the edges of which measured 1 cm., transmitted while fluorescing only 57 per cent. of the light it transmitted when not fluorescing, the measurements being made, of course, on light of the wave-length emitted during fluorescence. Nichols and Merritt investigated the subject in detail, obtained a similar result for fluorescein, using a Lummer-Brodhun spectrophotometer, and found that the additional absorption produced during fluorescence—or, to give it its usual name, fluorescence absorption—possessed many strange properties; for example, it did not follow the exponential law. Camichel repeated Burke's work with two different experimental arrangements, and found no effect; he also obtained no effect with fluorescein. Miss Wick, using the same apparatus and working in the same laboratory as Nichols and Merritt, confirmed their results for resorufin. For a full description of the work of these observers and an account of the present state of the question, together with the necessary references, the fourth volume of Kayser's *Spectroscopie* (1908), pp. 963-973, should be consulted.

Since that volume was published an article has appeared by R. W. Wood* in which he gives an account of a new and ingenious method of observing the effect directly—without combining three separate

* R. W. Wood, "On a Method of showing Fluorescent Absorption directly if it exists," *Phil. Mag.*, Dec. 1908.

photometric measurements. He applied this method to fluorescein under various conditions, and his results were wholly negative.

The object of the present article is to describe an attempt to detect fluorescence absorption in the case of fluorescein and uranium glass. Two different kinds of uranium glass were used. The method employed was in principle the same as the method of Nichols and Merritt, though the experimental arrangement was somewhat different. The results were wholly negative, and attention is directed to two causes of error which may not have received sufficient care, one of which produces a spurious effect, having one of the properties claimed for fluorescence absorption by Nichols and Merritt. It is true that the analogous reversal of a spectral line cannot be produced very easily; but, making every allowance for the uncertain nature of the phenomenon studied, the author inclines to the belief that all cases of fluorescence absorption hitherto obtained have been due to systematic errors in the photometric arrangement.

The spectrophotometer* used has already been described elsewhere, and proved admirably adapted for the purpose. It consists of a spectroscope before the slit of which is fitted a polarising prism specially designed to divide the field into two halves which are polarised at right angles to one another, and which touch one another without overlapping. The fields are matched by means of a nicol eyepiece. Light enters the special prism by two apertures, one for the upper and one for the lower half of the field, and in obtaining the transmission curve of a piece of coloured glass it is put in turn before each aperture, both apertures being illuminated by a portion of an incandescent mantle 15 cms. away. When a self-luminous piece of glass is used close up to the aperture, if the light emitted by the glass is comparable in intensity with the light transmitted by it, it cannot be placed before either aperture, but must be placed before the lower beam *e*, *f* in fig. 2 in the article above referred to. If it is placed before the upper one, the light in the upper half of the field will not be quite plane polarised, for some of the rays diverge at too great an angle.

The transmission curve of the substance investigated was first of all determined in the ordinary way, the substance being placed in turn before the two apertures, just as if it were an ordinary piece of glass. The fluorescent light can be neglected in comparison with the transmitted light.

Then the arrangement shown in the diagram 1 was used, and the

* R. A. Houstoun, "A New Spectrophotometer of the Hüfner Type," *Phil. Mag.*, Feb. 1908.

fluorescent substance kept before the lower aperture throughout the course of the experiment. An incandescent gas-lamp G was used for exciting the fluorescence. It was placed in a box to cut off all stray light, and its rays were focussed by a lens L on the fluorescing body U. The exciting rays not required were absorbed by a black screen not shown. The fluorescent light from the side of the body entered the instrument. As a comparison source an electric glow lamp was used. This was also placed in a box (not shown), and its intensity reduced to the right strength by placing fogged photographic plates of graded density in front of the opening in the box by which the light issued. S and K are two screens. When K is in position the exciting rays do not reach the substance, and it does not fluoresce. When S is in position the rays from the comparison source do

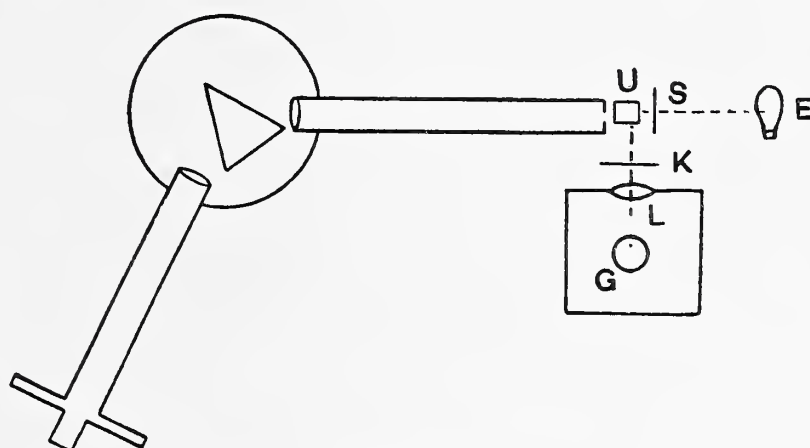


DIAGRAM 1.

not enter the substance and the lower aperture of the instrument. The passage from the comparison source to the upper aperture was always clear, and the upper half of the field was always illuminated by the comparison source alone.

The first measurement made with this arrangement was to obtain the variation of the intensity of fluorescence with the wave-length for the substance. To do this, the screen S was placed in position, the screen K removed, and the comparison source adjusted to a suitable intensity. The lower half of the field was then illuminated solely by fluorescent light. The two halves were matched for different wave-lengths, the relative intensity of the comparison source and the fluorescent light obtained, and thus the shape of the fluorescence curve determined. This was repeated with a piece of cobalt glass in front of the exciting source in the position of K, in order to see whether the shape of the fluorescence curve varied with the nature of the exciting light. The shape did not, although the intensity did.

When the transmission curve and the fluorescence curve of the substance were known, a wave-length was taken on the fluorescence maximum

and the experiment carried out as follows:—First of all the screen K was placed in position and the intensity T of the light transmitted by the substance when not fluorescing obtained. The intensity of the comparison source was made so small that the transmitted light caused no appreciable fluorescence. Then the screen S was placed in position, the screen K removed, and the intensity F of the fluorescent light determined. Both screens were then removed and a determination was made of the light received from the substance when fluorescing and transmitting at the same time, C. If there is fluorescence absorption $T + F > C$.

If α , β , γ be the angles made by the nicol with its zero position, I the intensity of the light received from the comparison source at the upper aperture of the instrument in the above three readings, and s , s' the transmission factors for the instrument for the two halves of the field, s , s' including everything but the effect of the eyepiece nicol, we have in the usual way

$$\begin{aligned} Ts' &= Is \tan^2 \alpha, \\ Fs' &= Is \tan^2 \beta, \\ Cs' &= Is \tan^2 \gamma. \end{aligned}$$

Hence, instead of T, F, C, we may write the square of the tangent of the appropriate angle. The angles α , β , γ were determined by taking readings in two neighbouring quadrants and halving the difference.

The result thus depends on three separate photometric measurements and is subject to three times the error of a single photometric measurement, and that not a very easy one. The space between the sources and the apertures of the instrument was built in so as to keep the room in absolute darkness. Observations were made in the order TFC, CFT so as to eliminate any error due to fluctuation in the relative intensity of the exciting and comparison sources. Instead of using an independent comparison source, light was reflected from the exciting source on to a square of white cardboard and the latter used as comparison source. The two would then vary together. This method was not found so satisfactory on account of greater difficulty in eliminating stray light. Scattered light from particles suspended in the glass or solution was expected and looked for, but did not prove to be an appreciable source of error.

The measurements will now be described in order.

URANIUM GLASS—FIRST SPECIMEN.

This was a pale yellow cube, of unknown manufacture, length of side 11.4 mm., which fluoresced green. The following table gives the

fraction of the incident light transmitted by the glass for different wave-lengths:—

λ		λ	
598 $\mu\mu$	·906	467 $\mu\mu$	·523
560	·829	452	·383
529	·781	438	·195
504	·630	425	·098
484	·570		

As about 8 per cent. of the incident light is lost by reflection at the two faces, this fraction varying inappreciably with the colour, for $\lambda = 598 \mu\mu$ only 1 per cent. of the incident light is absorbed in the glass.

The following table gives the intensity of the fluorescence for different wave-lengths, compared with an electric glow lamp, in arbitrary units, (1) when excited by an incandescent gas-mantle, (2) when the same source was used with a filter of cobalt glass:—

λ	(1)	(2)
606 $\mu\mu$	·147	·068
576	·599	·318
550	1·88	1·04
528	3·92	2·07
509	4·86	2·77
493	2·13	1·09
479	·455	...

This and the preceding table are plotted in the adjoining diagram 2. The fluorescence curves are, of course, not corrected for absorption in the glass, *i.e.* the intensity of the actual fluorescent light emitted by the glass is given, not the intensity we would obtain if none of it were absorbed in the glass in the way out. This latter, or the typical fluorescence curve, as Nichols and Merritt call it, would be more symmetrical.

The fraction of the incident light transmitted by the cobalt glass filter for different wave-lengths is given in diagram 3.

At first, on measuring T, F and C, T+F was always greater than C. A careful investigation, however, showed that this was due to diffuse light reflected from screen S. In diagram 4, H represents the aperture of the instrument. The exciting rays entered in the direction of the arrow, the corners at A and B being screened. The side BD of the cube had black paper gummed on it, to absorb the exciting rays. The green light from the exciting source was roughly one thousand times as strong as the fluorescence from the glass. If one-hundredth of the incident light were diffusely reflected from BD to S, and one-hundredth of that

were reflected into the instrument, it would increase the value of F by one-tenth. Also some of the exciting rays might arrive at S through being scattered on entering the glass or through not being parallel to the

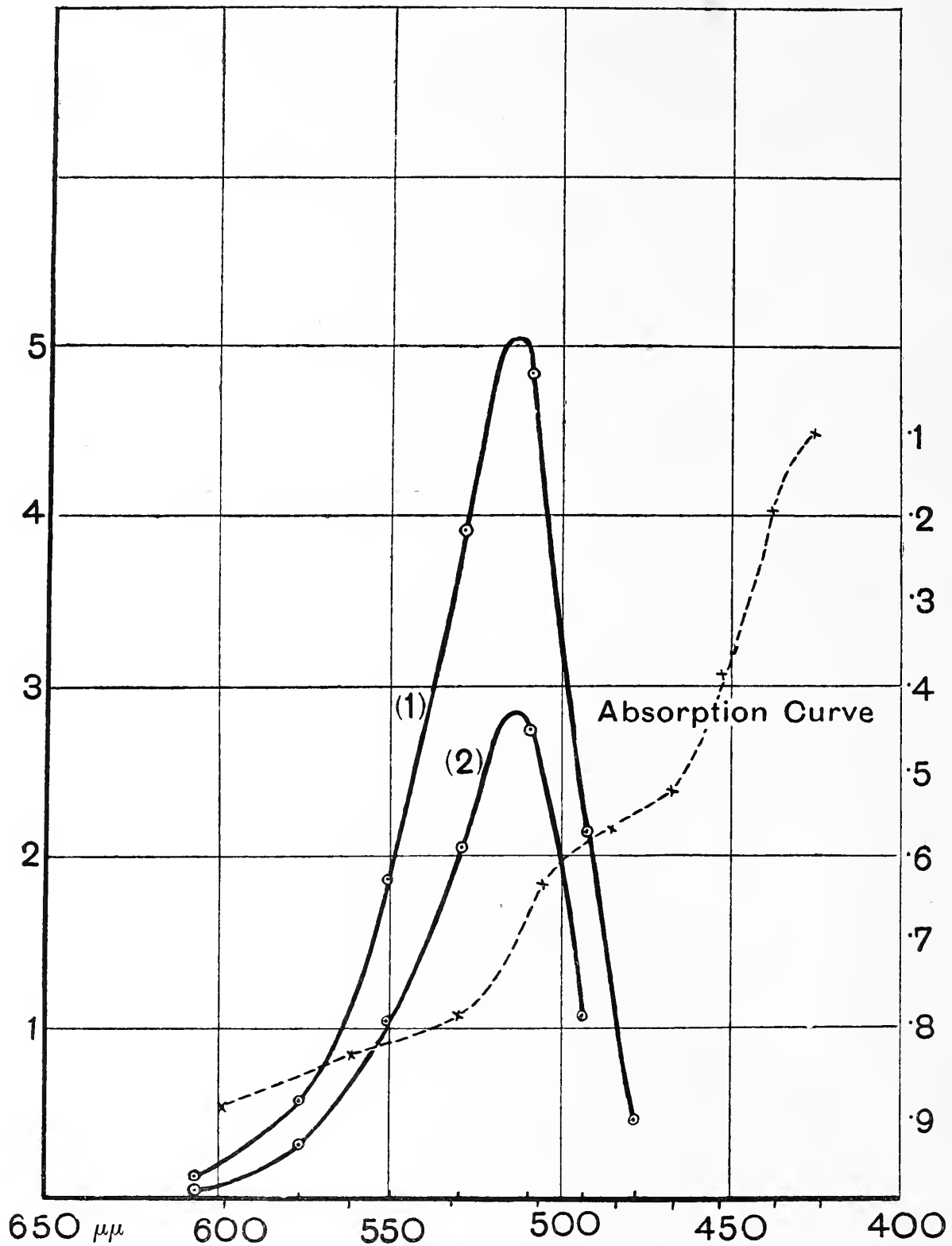


DIAGRAM 2.—Uranium Glass. First specimen.

direction of the arrow. Hence $T+F-C$ would be positive and proportional to F . The nature of this error is shown in an exaggerated form by the following table, which was obtained by replacing the black paper on S by one somewhat greyish:—

T	F	C	T+F-C	$\frac{T+F-C}{T}$
27.86	1.00	28.51	.35	.013
1.45	1.00	2.37	.08	.06
.700	1.00	1.555	.145	.21
.392	1.00	1.178	.214	.55
.143	1.00	.967	.176	1.23

$\frac{T+F-C}{T}$ increases as T decreases. This is a property claimed for fluorescence absorption by Nichols and Merritt. The screen S was not kept in the same place during the above readings. If it had been, the

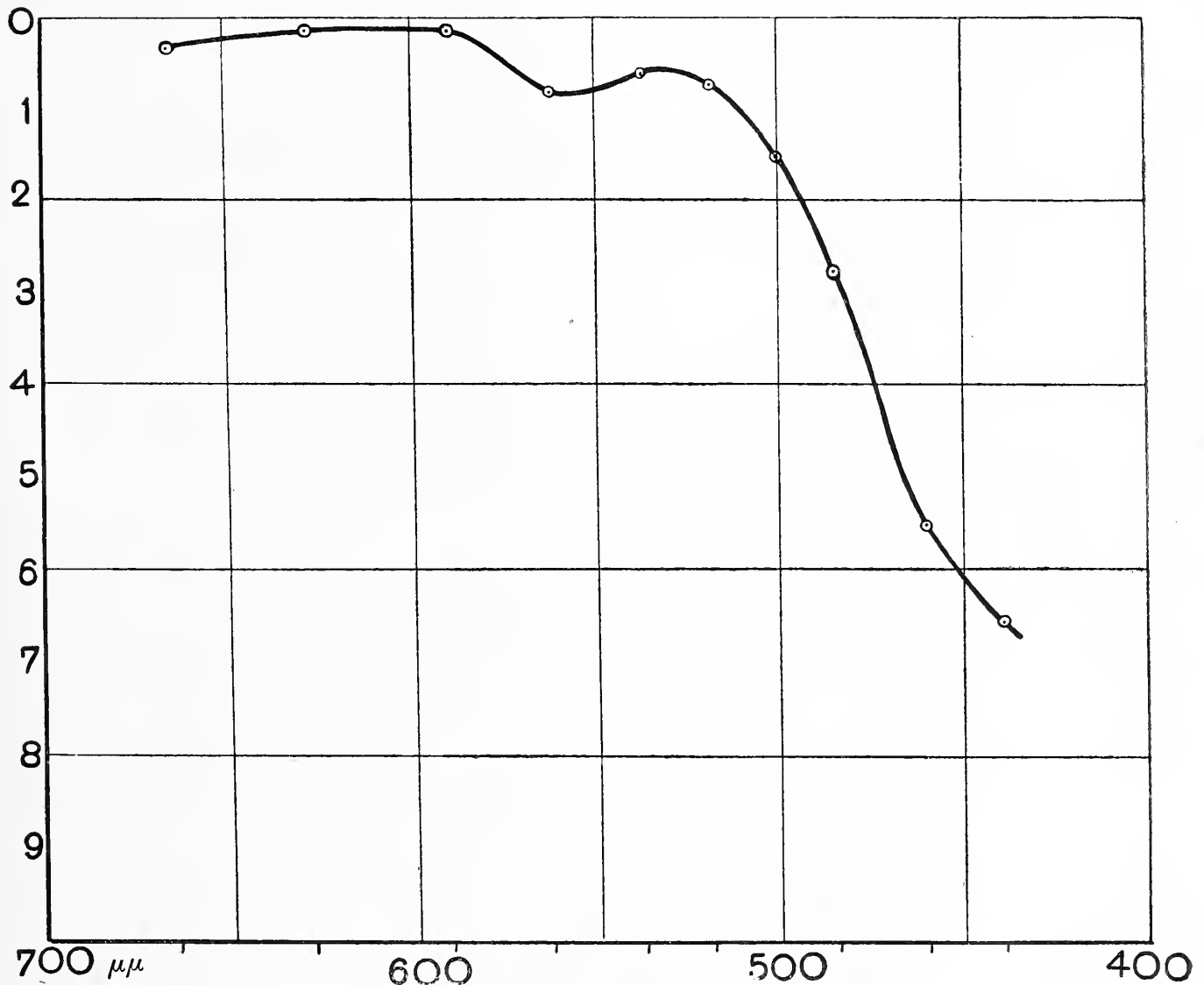


DIAGRAM 3.—Cobalt Glass.

column T+F-C would have been constant. In the similar tables given by Nichols and Merritt, and Miss Wick, T+F-C is constant within the error of observation.

This source of error was removed by replacing the "dead black" paper

by black velvet, the diffuse reflection from which is considerably less. The following table gives some results then obtained:—

T	F	C	T+F-C	Probable Error.
752	3906	4714	- 56	20
825	843	1654	+ 14	12
910	23	952	- 19	9
768	823	1552	+ 39	12
832	623	1428	+ 27	12
590	779	1376	- 7	3
857	260	1175	- 58	8

Each of the above results is the mean of six determinations made at one sitting, which took about two hours. The error is in every case greater than the calculated probable error. F, though represented by different numbers, was approximately the same strength throughout. The

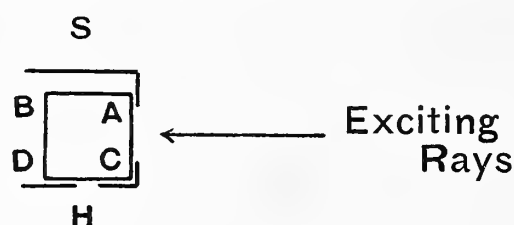


DIAGRAM 4.

numbers are proportional to the squares of the tangents; it was not thought necessary here to reduce them so as to represent F by the same number in each row. In the observations recorded in the sixth row, the cardboard square was used as a comparison source.

A cadmium spark was tried as exciting source, but was not steady enough.

The above determinations were made on the wave-length $\lambda = 554 \mu\mu$. A strip of spectrum $38 \mu\mu$ wide was examined with an equally wide slit. For a given brightness of field, the light is purer when the slit and the strip of the spectrum examined are equally wide. Owing to using this wide slit, a slight difference of tint between the two halves of the field was at first visible in the F measurements. This difference of tint is referred to by Miss Wick and by Burke, who did not resolve his light spectrally at all. In order to test whether the difference of tint made an appreciable error, a piece of green glass was placed before the comparison source. This made the upper half of the field distinctly yellower. It had a distinct effect, giving a value of $(T+F-C)/T = \cdot 11$, and on making the difference of tint more pronounced $(T+F-C)/T$ was obtained = $\cdot 22$.

The very slight difference of tint present originally was removed by

placing a cell with copper nitrate, of a suitable strength, as colour filter before the comparison source. In the measurements made on fluorescein and the other piece of uranium glass, which showed a stronger fluorescence, a narrower slit was worked with and this precaution was not necessary.

In the article above referred to Wood has indicated a method of testing the method employed for systematic error. The test would be improved if a piece of coloured glass, instead of a piece of clear glass, were placed before the slit at an angle of 45° to the axis of the collimator.

Fluorescein.

A standard solution was prepared by dissolving as much fluorescein as possible in water at room temperature, the undissolved particles being removed by filtration. The following table gives the fraction of the incident light transmitted through 1 cm. of this solution:—

λ		λ	
535 $\mu\mu$	·918	484 $\mu\mu$	·0117
515	·710	475	·0381
504	·168	466	·0725
496	·0305	452	·1441
489	·0070	438	·3514

The above table does not include the loss for reflection. Thus for $\lambda = 515 \mu\mu$ ·290 of the light entering the solution would be absorbed in it.

The next table gives the fluorescence curves not corrected for absorption in the solution. In (1) the exciting source had not, in (2) it had the cobalt glass filter. A solution 1 cm. thick and one quarter the strength of the standard solution was used.

λ	(1)	(2)
604 $\mu\mu$	·0187 $\mu\mu$...
574	·137	·038
549	·476	·175
527	1·37	·516
508	2·30	·881
492	1·08	·486

The curves are plotted in diagram 5. The results of two sets of readings made on $\lambda = 508 \mu\mu$ are given:—

T	F	C	T + F - C	Probable Error.
906	290	1173	+ 23	10
935	694	1593	+ 3	20

URANIUM GLASS—SECOND SPECIMEN.

This piece measured $2 \times 1 \times 1$ cms. and was from Schott u. Gen., Jena, the trade name for it being F 3757. The fluorescence was decidedly stronger than in the case of the other specimen. The following table

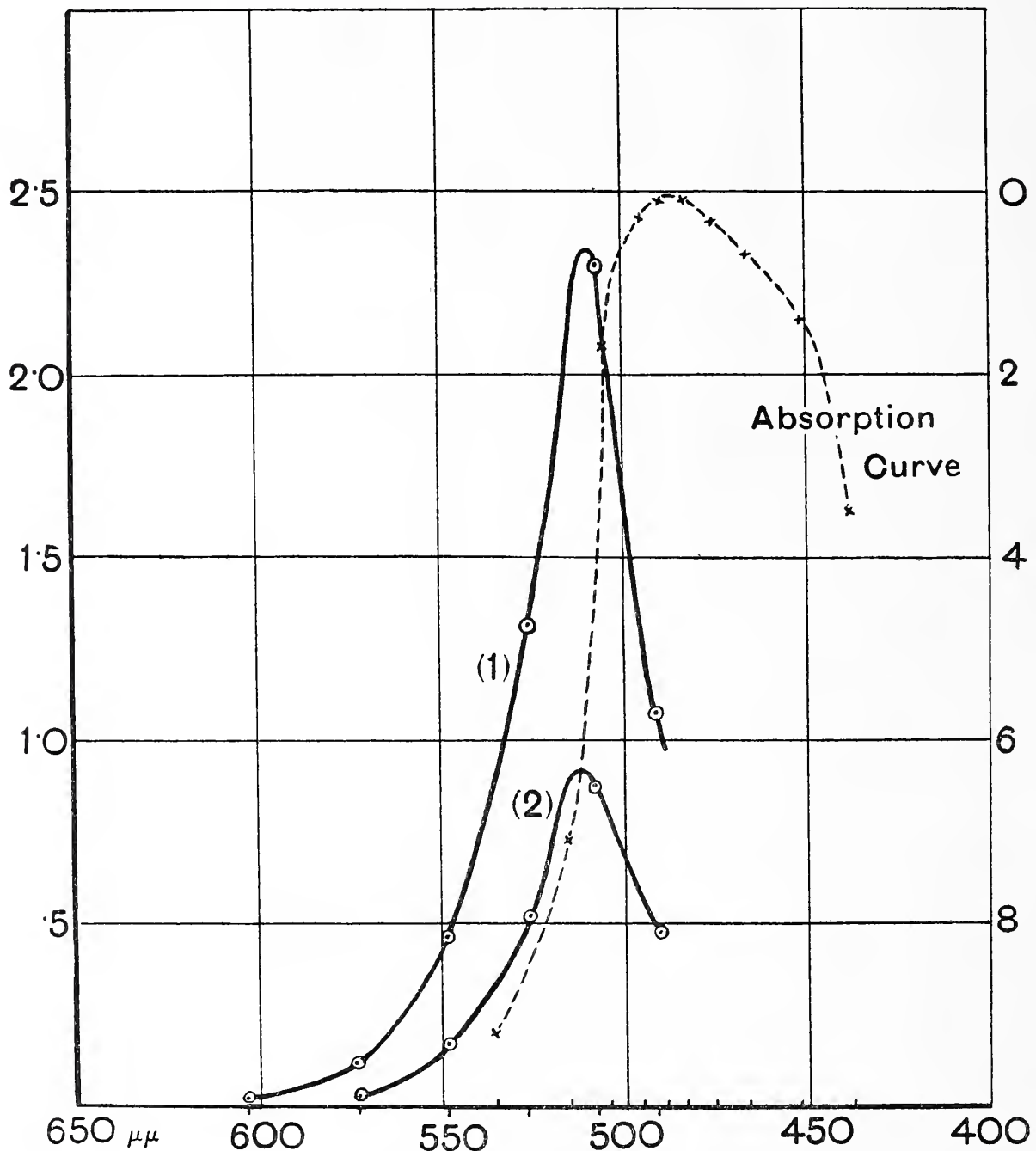


DIAGRAM 5.—Fluorescein.

gives the fraction of the incident light transmitted through 1 cm. of the glass, the loss from reflection being included:—

λ 600 $\mu\mu$	·929	λ 484 $\mu\mu$	·321
562	·914	467	·277
530	·903	452	·131
504	·494		

The next table gives the fluorescence curves as in the two former cases:—

λ	(1)	(2)
620	·089	...
598	·356	·104
578	·946	·310
560	2·82	·875
544	4·86	1·761
529	8·24	2·928
516	9·82	3·202
504	6·13	2·444
494	2·02	·811
484	·194	...

The curves are plotted in diagram 6. It will be noticed that although the absorption of the two glasses is somewhat different, the fluorescence is the same according to the curves. However, when examined with the spectroscope and a narrow slit, the fluorescence maximum of the second glass is seen to consist of four bands, three of which are situated at 516, 535, and 563 $\mu\mu$, the fourth being in the yellow. The first glass did not show bands, although one seemed to be almost visible.

The results of two sets of readings made on $\lambda = 529 \mu\mu$ are given:—

T	F	C	T+F-C	Probable Error.
1821	1055	2810	66	11
766	1291	2016	41	18

In analogy with the anomalous dispersion produced by an absorption band, if fluorescence produces a change in absorption, it should also produce a change in the index of refraction. This was sought for with a Jamin polarisation interferometer.* The two interfering beams are in this instrument about 12 mm. apart. They were passed through the same solution or the same piece of uranium glass and the solution or glass in the path of the one beam made to fluoresce. No shift of the bands was detected. An arc lamp was used as exciting source, but its full intensity could not be utilised, as the fluorescence became so strong that the interference bands could not be seen. The bands were obtained from a sodium or lithium flame.

According to theory, the change should have been too small to detect. For, if we represent the index of refraction in the neighbourhood of an absorption band in the usual way by a complex quantity

$$n(1 - i\kappa),$$

* *Comptes rendus*, 67, 1868, p. 814.

and regard this as a function of λ , it may be shown, according to the theories of Drude, H. A. Lorentz, and others, that the maximum change in n is one-half the maximum value of $n\kappa$, which of course determines the absorption. If $(T+F-C)/T$ were $\cdot 20$ then, in the case of the measurements on fluorescein with the interferometer, $n\kappa$ should alter

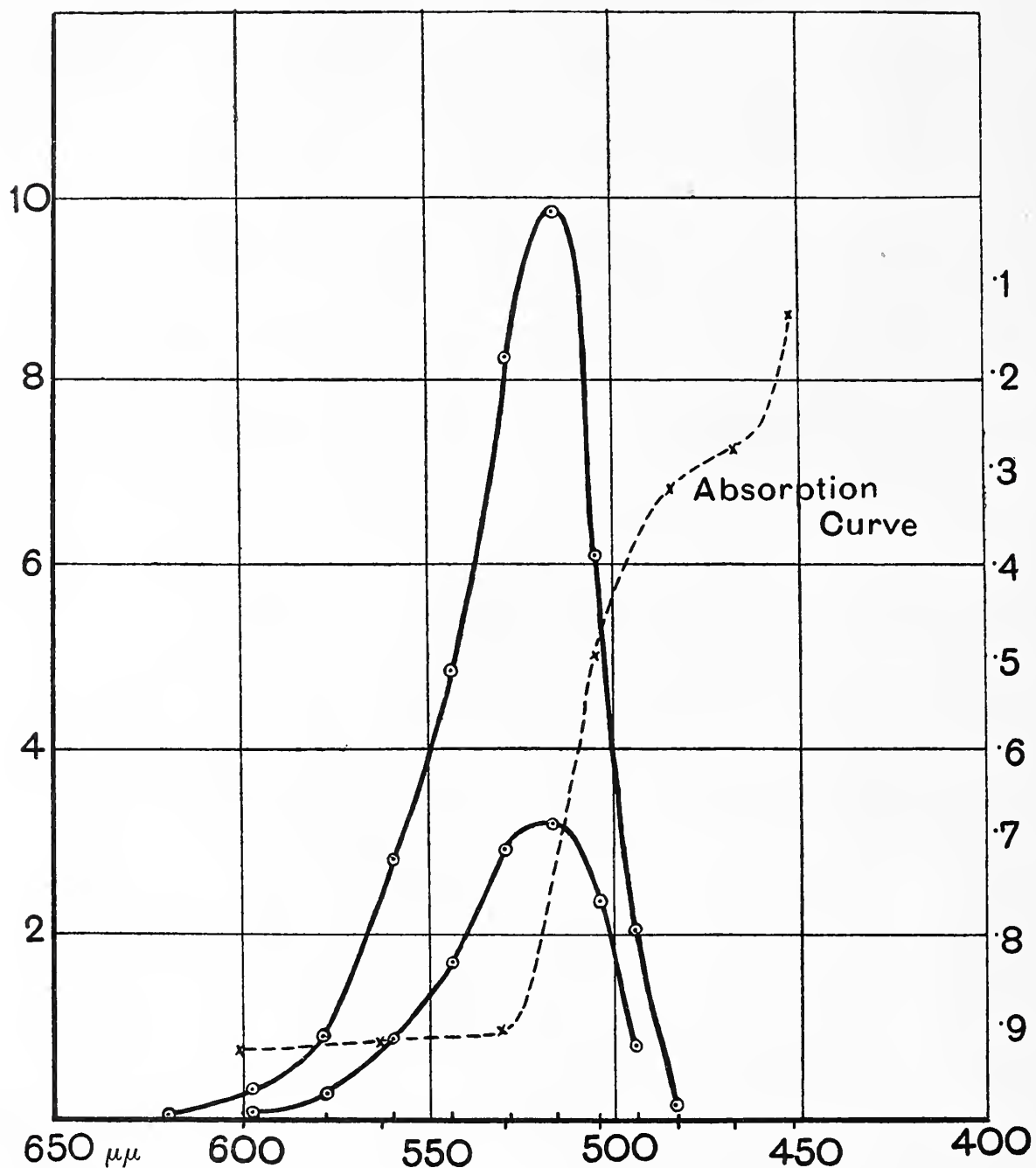


DIAGRAM 6.—Uranium Glass. Second specimen.

by $4.4 \cdot 10^{-7}$ and n by $2.2 \cdot 10^{-7}$, a wholly inappreciable amount on a length of 2 cms.

In conclusion, a very simple construction of the molecule may be suggested to explain fluorescence in such a case as that of fluorescein, where we have a fluorescence band of somewhat longer wave-length than a well-marked absorption band. When a body is fluorescing its molecules are supposed to be in two states. In the first state let us

suppose that a negative electron is moving in an ellipse inside a sphere of positive electricity of uniform density ρ . Its period is then

$$\sqrt{\frac{3\pi m}{\rho e}},$$

e being the charge on the electron and m its mass. Let us suppose that the electron executes forced vibrations under the action of the incident light, the amplitude increasing until it flies out of the sphere. It will then move in an ellipse under the inverse square law, the period being

$$\sqrt{\frac{3\pi m}{\rho e} \frac{r^3}{a^3}}$$

where r is the mean diameter of the path and a the radius of the sphere. The period is now longer. We shall suppose that in this state the electron loses kinetic energy by radiation and finally falls back into the sphere, and that the process is again repeated. We should have in addition to assume the existence of irregular impacts on the sphere due to collision with other spheres, otherwise the electron would remain on its surface. This model has the advantage of explaining why the fluorescent light has a longer wave-length. Unfortunately it does not give it a definite period.

The research described in this article was carried out in the Physical Laboratory of the University of Glasgow. The spectrophotometer employed was the property of the Carnegie Trust for the Universities of Scotland.

(Issued separately July 8, 1909.)

XXIV.—Experiment with the Spark Gap of an Induction Coil.
By Dr Dawson Turner.

IF the electrodes of an induction coil be placed at such a distance that a spark will not pass easily, then the introduction of a dielectric between the poles will greatly facilitate the sparking, provided the dielectric be placed near or against the positive pole, but will not have this effect if placed against the negative pole. Experiments were made with mica, sulphur, glass, and ebonite, etc.

(Issued separately July 8, 1909.)

XXV.—*Strophanthus sarmentosus*: its Pharmacological Action and its Use as an Arrow Poison. By Sir Thomas R. Fraser, M.D., F.R.S.S. L. & E., Professor of Materia Medica in the University of Edinburgh; and Alister T. Mackenzie, M.A., M.B., Ch.B., Carnegie Research Scholar.

(Abstract.)

AN extract of the seeds of *Strophanthus sarmentosus* appears to be an important ingredient of the arrow-poison of Nigeria and other parts of West Africa. While some of the other ingredients of this arrow-poison may possess toxic power, others of them have little or no toxicity and are introduced into the arrow-poison with the object, apparently, of rendering it more viscous and adhesive or with a superstitious intention.

The seeds contain a glucosidal active principle as well as a large quantity of inert oil and other substances soluble in ether.

In order to determine the nature of the pharmacological action, an alcohol extract freed from substances soluble in ether was used. Its minimum-lethal dose by subcutaneous injection per kilogram of animal was found to be, for the frog, 0·0035 gram; for the rat, 0·3 gram; for the rabbit, 0·0015 gram; and for the cat, 0·002 gram; and, by intravenous injection, for the rabbit, 0·0012 gram.

The minimum-lethal dose is, therefore, by subcutaneous injection, almost the same for rabbits and cats, and for frogs it is twice as large, and for rats 200 times as large as for rabbits.

The predominating effects are those produced upon the heart and the skeletal muscles.

Under the influence of *S. sarmentosus*, shortening of skeletal muscles is produced, and, later, fibrillary twitches become conspicuous. When these twitches have disappeared, the contraction of the muscle under stimulation is more gradual, the amount of contraction is less, and the relaxation of the muscle is slower than before; stronger stimuli are needed to produce contraction; and, soon, the strongest direct electrical stimulation fails to cause contraction of the muscle. At this time the muscle is pale, rigid and acid in reaction.

The effects of the extract on the heart are the most important of those

produced. Small doses tend to produce a diastolic type of change, and large doses a systolic type. This applies equally to application of *S. sarmentosus* to the outer surface of the heart and to its administration through the blood-stream.

With small quantities the rate of the heart's contractions is slowed, and the size of the diastolic as well as of the systolic movements of the ventricles and auricles, and the strength of the systolic contractions of both, are increased. These changes are produced when a dilution of even 1 in 500,000 is perfused through the frog's heart. The slowing of the heart's rate is partly due to this increased range of movement, but lengthening of the diastolic pause plays an important part in the retardation.

With large quantities, the heart's contractions may, at first, be modified in the same way as with small quantities, but the diastolic expansions of the ventricles afterwards become reduced, the quantity of blood entering them becomes lessened, and the ventricles finally cease to contract and remain motionless in extreme systole. Previously to the final standstill, the ventricular pulsations are usually limited to small portions of its wall. Paralysis of the vagus by atropine, after the cardiac effects have been developed, does not modify these effects, nor does paralysis of the vagus, before and during the administration of the extract, prevent the occurrence of the changes in the heart which are characteristic of the action of *Strophanthus*.

The cardio-inhibitory function of the vagus nerve is not increased or otherwise appreciably modified by *S. sarmentosus*.

On the blood-vessels, the action of the extract is slight, for very strong solutions cause only an unimportant degree of contraction.

S. sarmentosus does not appear to affect the blood-pressure, excepting through the changes produced in the heart's contractions. No evidence was obtained of a hæmolytic or a blood-clotting effect, *in vivo*.

The lymph hearts of the frog are practically unaffected by this substance.

The chief effects on the respiration are attributable to the action on the heart, though there may also be a direct action on respiration through its medullary centre.

The spinal reflex disappears a considerable time after the administration of lethal doses of the extract, and only after the heart and respirations have been much affected. This disappearance is due not only to reduced blood-supply, but also to a direct action on the cord.

When applied to the skin of frogs, the extract is capable of temporarily abolishing the function of sensory nerves in strong solutions, such as from

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1 in 1000 to 1 in 500. Slight dilatation of the pupil accompanies the anæsthesia produced by placing such solutions on the eyeball of warm-blooded animals.

The function of motor nerves is slowly impaired, owing to the action of the extract on the nerve-ends, and, much later, it is abolished. Before abolition of function, fibrillary twitches occur in the muscles. The occurrence of these twitches is prevented by paralysing the nerve-ends with curara.

It would appear, accordingly, that the action of *S. sarmentosus* is very similar to that of *S. hispidus*.

(Issued separately July 9, 1909.)

XXVI.—On the Histological Changes in the Liver and Kidney after Chloroform administered by Different Channels. By G. Herbert Clark, M.B., D.P.H. (From the Physiological Laboratory of the University of Glasgow.) (With Three Plates.)

(MS. received April 17, 1909. Read May 3, 1909.)

IN a paper published in the *Proceedings* of this Society in 1908, D. Noël Paton (1) showed that chloroform acts very differently upon the metabolism when administered by different channels: that, when given by the respiratory tract, its effect generally is simply to increase the protein metabolism, but that when given by the mouth, it produces a marked disturbance in the distribution of the urinary nitrogen, which he considers to be due to the chloroform acting as an hepatic poison. The action of the drug when administered hypodermically was found to be in the same direction as when given by the mouth. Miss Lindsay in conjunction with D. Noël Paton (2) showed that the rate of elimination varies with the mode of administration, being most rapid when given by the respiratory passages, and slowest when given by the mouth. It was further shown that the chloroform is fixed in the liver to a greater extent when given by the mouth than when given by the respiratory passages. They also, in confirmation of the work of others, recorded the appearance of albumin and of cellular debris and tube casts in the urine, especially after administration by the mouth or hypodermically.

It therefore seemed desirable to study how far the action of chloroform upon the tissues varies with the mode of administration.

PREVIOUS INVESTIGATIONS.

Already a very large amount of work upon the action of chloroform upon the tissues has been recorded. An excellent résumé of the literature is given by Stiles and M'Donald (3) in their paper on delayed chloroform poisoning, and only a general statement of the results of previous investigations is necessary.

It appears to be generally recognised that administration of chloroform is often followed by degenerative changes in various tissues, and the majority of writers consider the change to be of the nature of a fatty degeneration. According to some, droplets of oil are to be seen in the blood-vessels.

Fraenkel (4), Marthen (5), and Cohn (6) further describe the appearance of a yellow pigment in the kidneys and liver of patients who died apparently as a consequence of having been anæsthetised with chloroform administered through the respiratory passages.

There is considerable difference of opinion as to the cause of the degenerative change; some authors (Junkers (7) and Strassmann (8)) considering that it is due to the toxic action of the drug on the cells themselves, others to a primary destructive action on the red corpuscles. (Nothnagel (9) and Ostertag (10)).

In the paper referred to above, Stiles and M'Donald describe in detail the post-mortem appearance of tissues removed from a child who died four days after an operation under chloroform anæsthesia.

The changes consisted in extensive degeneration of liver and kidney tissue. In the case of the liver, the cells throughout the organ were markedly changed and vacuolated, but the most complete degeneration appeared to have taken place in the centre of the lobules. The cells contained droplets of oil, which were clearly demonstrated by staining with Sudan iii. Droplets of oil were also found in the hepatic veins. The kidney showed intense fatty degeneration, which was almost universal throughout the organ.

Stiles and M'Donald then made a series of observations upon rabbits, and to obtain a full action of the drug upon the tissues the chloroform was injected subcutaneously.

Here again they found marked fatty changes in the liver cells, most marked in the central and intermediate zones of the lobules. The cells were seen to be occupied by numerous minute droplets which showed no tendency to coalesce, and many in the centre of the lobules were completely disorganised.

In the kidneys the changes were less marked, and varied from cloudy swelling in the cells of the convoluted tubules and ascending loops of Henle to well-marked fatty change in these tubules and the collecting tubules. No fat was observed in the vessels, and it was noted that the glomeruli showed no change. The authors then examined similar tissues obtained from animals which had inhaled chloroform vapour for varying times. They found the changes to be similar to those observed after injection of the drug, but somewhat less marked.

Doyon (13) also describes the histological changes in the liver after chloroform had been administered experimentally by the mouth and hypodermically. In both instances he found necrosis of the liver cells.

PRESENT INVESTIGATION.

In carrying out this research the tissues used were for the most part those obtained from the rabbits referred to in the papers by D. Noël Paton and Miss Lindsay.

A. CHANGES IN VITRO.

Before considering the changes which occurred in the body, it was thought advisable to find what was the effect of chloroform upon the tissues when it acted upon them in saline solutions kept at the body temperature. This subject was touched upon by D. Noël Paton in 1894 (11).

In the first instance, 0.75 per cent. sodium chloride solution was used, and a small quantity of pure chloroform was introduced into a bottle filled with the saline, and shaken up with it thoroughly for about five minutes. The solution was then set aside for some time, and the supernatant fluid decanted off. The tissue was removed from a newly killed healthy animal, and having been divided into portions of about $1 \times 1 \times \frac{1}{2}$ c.m., these were immersed for varying times in the solution. As controls, portions of the tissue were fixed immediately in 10 per cent. formol-saline solution, and other fragments of similar size placed in a 0.75 per cent. sodium chloride solution, and left for times similar to those during which the chloroform saline acted. The tissues were kept in an incubator at the constant temperature of 35° C. for periods of $\frac{1}{4}$ hour, $\frac{1}{2}$ hour, 1 hour, $1\frac{1}{2}$ hour, etc., to 24 hours. As the time elapsed, the tissue was removed from the incubator and immediately immersed in a 10 per cent. formol-saline solution. After fixation and hardening, the fragments were cut in paraffin, stained with hæmalum and eosin, and examined. The tissues were also stained by osmic acid, Scharlach rot, and Sudan iii. for fat.

It was found that even the fresh tissue fixed in formol-saline containing 0.75 per cent. of sodium chloride immediately after death shows a slight variation from the normal. The cells look swollen, and their borders are not sharp. It seemed probable, therefore, that the saline solution used was not isotonic with the tissue immersed in it. The solution suggested by Castaigne and Rathery (12) ($\Delta = 0.78$), approximately 1.3 per cent. of sodium chloride, was then tried.

Kidney.—In this solution the kidney retains a normal appearance for a considerable length of time, and even after two hours' immersion at a temperature of 35° C. there is but little change in the appearance of the cells. In the later experiments the chloroform was added to this solution.

It was found that necrobiotic changes set in at once in the tissue in

chloroform, but that they do not appear in the control tissue for some time. After five hours the change in the control tissue is quite as great as in the chloroform tissue, and from that time onwards the greatest changes are present in the control tissue. This is due to the action of micro-organisms.

In the kidney the first change which was observed was that the cells of the convoluted tubules seem to lose their definiteness of outline and the protoplasm takes on an appearance like cloudy swelling. The cells appear to be markedly granular, and their free margins rapidly take on a fringed appearance. This is followed by a gradual loss of power of taking on basic stains on the part of the nucleus.

The degenerative changes observed in the tubules, beginning in a cloudy swelling, become more marked as time progresses, until eventually it was difficult to make out the details of structure of the organ. The tubules become filled with granular debris, and little is left beyond the basement membrane mapping out the position which the tubules have occupied.

The glomerular tuft is unaffected in the earlier stages; and, in fact, it is not until after the tissue has been immersed in chloroform-saline solution for three hours that the first marked change appears. This change consists of a slight shrinking of the glomerular tuft away from the capsule of Bowman. In the earlier stages this shrinkage is slight, and leaves a gradually increasing space between the tuft and the capsule, which after longer immersion is found to be occupied by an exudate staining deeply with hæmalum. The same shrinking and exudation appears in the glomeruli of the specimens immersed in saline solution, but some hours later. This confirms *in vitro* the observations of Marthen (5) and Stiles and M'Donald (3).

Liver.—In the case of the liver the influence of the chloroform is well shown, for at an early stage the normally well-defined nucleated liver cells lose their clearness of outline. The finely granular protoplasm becomes coarsely granular, and later becomes broken up and vacuolated. The nuclei soon lose their power of taking on basic stains, and in course of time the cells become so far disintegrated that but little sign of their normal structure persists. After many hours the sections show merely granular debris. The same changes take place in the tissue immersed in saline; but, as in the case of the kidney, this appears after a much longer immersion.

Comparing the whole series of specimens which were examined in these experiments it may be broadly stated that *in vitro* the tissues immersed in chloroform-saline solution show necrobiotic changes at a very much earlier stage than do similar tissues immersed in a similar amount of a pure saline solution of the same concentration.

B. CHANGES IN THE LIVING ANIMAL.

In considering the influence of the drug upon the tissues *in vivo*, two different factors have to be dealt with—the method of administration and the duration of the action of the drug.

ADMINISTRATION BY THE RESPIRATORY TRACT.

In the tissues taken from animals which had chloroform administered through the lungs the amount of change in the liver and kidney was on the whole but small. In some cases, notably those examined some time after anæsthesia, no variation from the normal was observable. The greatest change was in the organs removed from a rabbit which died immediately after the administration. Here the cells lining the ascending and descending tubules of Henle and the convoluted tubules of the kidney showed marked degenerative changes. The liver was much less affected, the cells being in an early stage of albuminous degeneration. It is notable that at the time the animal died the blood contained as much as 77·3 mg. of chloroform per 100 c.c. of blood, and that respiration had stopped during the administration of the drug. The specimens showing the next greatest change were those obtained from an experiment where a small-sized rabbit was anæsthetised for a short time and killed soon after. The blood was found to contain 30·8 mg. chloroform per 100 c.c. The sections of both kidney and liver showed extensive degeneration.

ADMINISTRATION BY THE STOMACH.

When chloroform was administered in oil by the stomach the mortality was very great, and in those animals that survived the administration extensive changes in the organs were found. The kidney tissue had undergone marked degeneration, and in many cases this had gone as far as actual necrosis. The tubules were frequently found to be choked with albuminous debris. In many instances granules which stain bright red with Scharlach rot were observed in the cells and in the debris. The nuclei showed a varying degree of affinity for the basic stain, losing the power to take on the stain as degeneration advances. The glomeruli were in no instance in an advanced stage of degeneration, signs of congestion alone being present. This observation is in accordance with the results obtained by the experiments *in vitro*.

The degree of change in the kidney varies greatly with the length of time after the chloroform is administered. In cases where the animal was

killed within a few hours of this administration, degeneration had not advanced very far. Three hours after the drug had been given the cells showed a considerable degree of cloudy swelling, and here and there there were signs of desquamation in the ascending and descending tubules of Henle.

In a specimen taken at 5½ hours the degeneration had advanced greatly, and the tubules contained a great deal of albuminous material. The cells were frequently found to be vacuolated, and the nuclei had taken on the hæmalum stain badly. When the animal recovered, the kidney tissues apparently began to repair after arriving at this point of degeneration. In specimens taken from animals which were killed two or three days later, apparently recovering from the effects of the drug, the changes were never found to be more marked than those described. On the other hand, when the animal died overnight after the administration or was killed when obviously dying, the kidney was found to be rapidly losing all signs of its original structure; the cells lining the tubules were frequently lost altogether, nothing being left but the basement membrane. Where the cells were still apparent, the nuclei were stained badly and the tubules were choked with debris.

Here and there throughout the organ, particularly in one or two specimens, masses of blood were observed, apparently between the tubules and not in the vessels (figs. 1 and 2). Where this was observed, the cells lining the tubules adjacent to the blood were frequently found to contain small dark-coloured granules similar to those described by Fraenkel (4), Marthen (5), and Cohn (6). These were not observed in any other position in the kidney.

In the liver the degree of change was also found to vary with the length of time after the administration of the drug, and also with the progress towards recovery of the animal.

Examined three hours after administration, the cells at the periphery of the lobules showed but little change—at most a slight degree of cloudy swelling. The cells in the centre of the lobule, on the other hand, had undergone a granular change, and the nuclei had begun to lose their power of taking on the stain.

After five hours this was still more marked; and an hour and a half later some of the cells in the centre of the lobule had completely broken down, leaving granular debris in place of the cells.

When the animal showed evidence of recovering from the effects of the drug, no further change in appearance of the tissues was observed. When the animal was found dead in the morning after the administration, or

showed signs of dying at an early date and was killed in consequence, the degree of change was very much greater. The centre of each lobule was found to be occupied by a granular mass showing neither nuclei nor any appearance of liver tissue. The intermediate zone was frequently also affected, and in the worst cases no sign of liver tissue was seen except a layer of two or three cells thick at the periphery of the lobules (fig. 3). In the granular debris a number of granules staining red with Scharlach rot were seen.

The degenerative change in the liver in the animals where the chloroform was administered by the stomach is generally very great indeed, and in all cases appears much more complete than the change in the kidney. This is possibly due to the action of the chloroform "anchoring" itself to the liver cells.

ADMINISTRATION SUBCUTANEOUSLY.

When chloroform was given in the form of a subcutaneous injection, the mortality among the animals was very great, and a large proportion died during the night after the injection.

On the other hand, the animals that recovered appeared to be quite active a day or two later.

Histologically, the changes in the organs are very similar to those detailed above, the difference being one of degree.

In the kidney, after four hours very little degeneration seems to occur, a slight degree of cloudy swelling being apparent. After five hours, some vacuolation was seen in one of the specimens examined. In one animal killed some hours later, and in a dying condition, the kidney showed an appearance comparable with that observed in some of the worst cases after administration by the stomach. The ascending and descending tubules, the convoluted tubules, and more markedly the collecting tubules showed little or no sign of cellular lining. They were choked and frequently distended with albuminous debris. In all the cases, however, the glomeruli retained an appearance approximating to the normal.

In the animals which died, the appearance of the kidney was similar to those just described. Parts showed tubules denuded of their epithelium, and other parts showed cells in an advanced state of degeneration (fig. 4). Generally there was evidence of congestion of the organ, and occasionally dark granules were observed similar to those referred to above.

In the case of the liver there was generally a slight necrosis in the centre of the lobule even a very few hours after the administration, and as

time advanced this increased in amount (fig. 5). In the worst cases, where the animal was found dead or was killed in a dying condition, the state of the organ was like that described under administration by the stomach, viz., the organ was a honeycomb of cheesy material showing very little sign of the original liver structure.

SUMMARY.

1. When kidney or liver tissue is immersed in a saline solution containing chloroform, degenerative changes take place similar to the normal necrobiotic changes but very much more rapid. In the case of the kidney the glomeruli are not affected for a very considerable time.

2. When chloroform is administered through the respiratory passages a considerable degree of degeneration is only occasionally found in the kidney and liver cells. It is more marked in some cases than in others where a similar amount of chloroform was given to animals of a similar size. This may be associated with the very varying rate at which the drug is eliminated, as shown by Miss Lindsay (*loc. cit.*). The degree of change in the liver was never great. In the kidney there is frequently cloudy swelling, and occasionally desquamation of the epithelium of the ascending and descending tubules.

3. Where the drug is given by the stomach the mortality is great and the changes observed in the organs are marked. In all cases there is evidence of the toxic action of the drug. In the animals most affected, the structure of the liver is almost entirely lost, nothing remaining of the lobules but a shell of liver cells enclosing a cheesy debris.

In the kidney the drug acts in a similar way, the degree of degeneration being somewhat less than in the liver.

The glomeruli are but little affected even in the worst cases.

4. When the drug is given hypodermically the changes are similar to those observed when the drug is given by the stomach. The liver is again more affected than the kidney.

On the whole, however, chloroform does not appear to be quite so destructive to the liver tissue when administered in this form.

5. The marked action of the drug upon the liver, whether administered by the stomach or hypodermically, is probably accounted for by the "anchoring" action referred to by D. Noël Paton (1). It would be interesting to know if there is evidence of a similar action on the part of the kidney cells to account for the extensive degenerative change frequently observed there.

6. The result of these observations helps to explain the different effects of chloroform on hepatic metabolism. When given by the respiratory passages it is rapidly eliminated, produces no marked histological changes, and the metabolic disturbances are slight; but when given by the mouth and hypodermically it is more slowly eliminated, has more time to produce its toxic action, and the metabolic disturbances are pronounced.

In a future paper the action of chloroform upon the blood corpuscles will be dealt with.

A grant was received from the Carnegie Trust to defray the expenses of this research.

REFERENCES.

- (1) D. NOËL PATON, *Proc. Roy. Soc. Edin.*, vol. xxviii., 1907-8, p. 472.
- (2) D. NOËL PATON and Miss LINDSAY, *Proc. Roy. Soc. Edin.*, vol. xxviii., 1907-8, p. 497.
- (3) STILES and M'DONALD, *Scottish Medical and Surgical Journal*, Aug. 1904, p. 97.
- (4) FRAENKEL, *Virchow's Archiv*, Bd. cxxvii. S. 381, and Bd. cxxix. S. 254.
- (5) MARTHEN, *Berlin klin. Woch.*, 1896, No. 10.
- (6) COHN, *Deutsche Zeitsch. f. Chir.*, 1902, Bd. lxiv.
- (7) JUNKERS, *Über fettige Entartung in folge von Chlorof. Inhalat.*, Bonn, 1883.
- (8) STRASSMANN, *Virchow's Archiv*, 1889.
- (9) NOTHNAGEL, *Berlin klin. Woch.*, 1866.
- (10) OSTERTAG, *Virchow's Archiv*, 1889, Bd. cxviii. S. 250.
- (11) D. NOËL PATON, *Phil. Trans. Roy. Soc.*, 1894, vol. clxxxv. p. 248.
- (12) CASTAIGNE and RATHERY, *Arch. de méd. expér. et d'anat. path.*, Sept. 1903.
- (13) DOYON, *Compt. rend. Soc. de biol.*, vol. lviii. pp. 30, 108, 853.

(Issued separately July 9, 1909.)

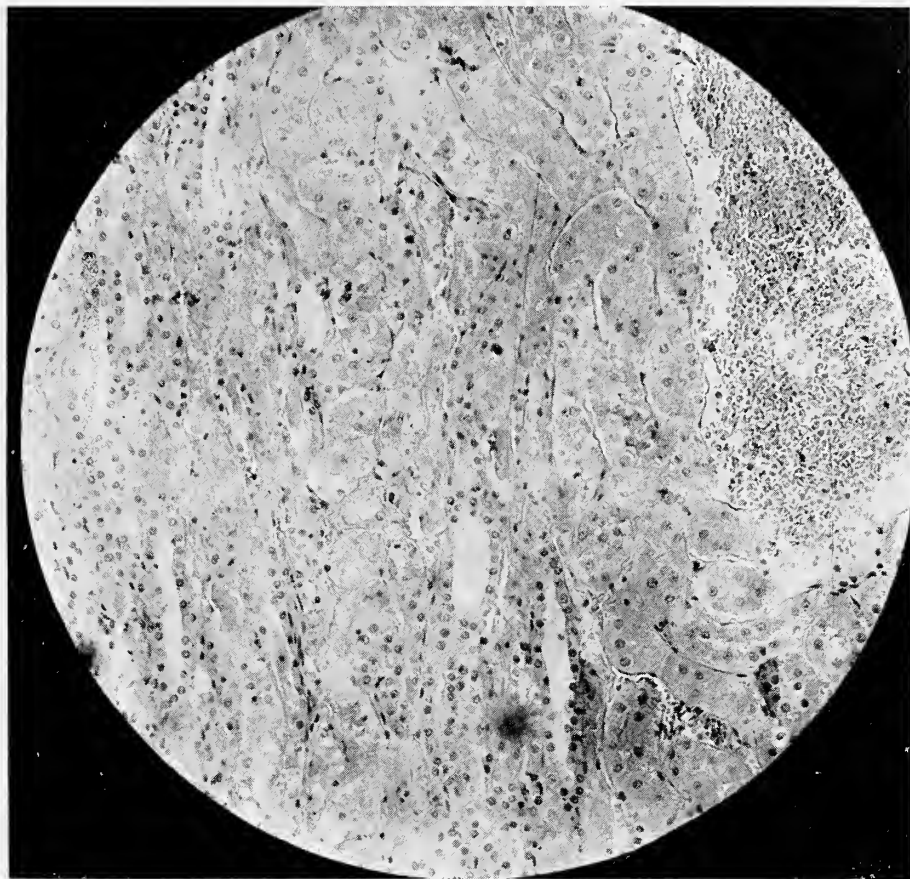


FIG. 1.—Kidney from rabbit, No. 9a of the series, which had 1cc of chloroform administered in oil by the stomach. To the right is seen a quantity of extra-vascular blood. The cells lining the tubules are in various states of degeneration. The nuclei, as a whole, are badly stained. $\times 150$.



FIG. 2.—High power view of the tubules adjacent to the extra-vascular blood. The granules referred to in the text are well shown. $\times 800$.

DR G. HERBERT CLARK.

[Plate I.]

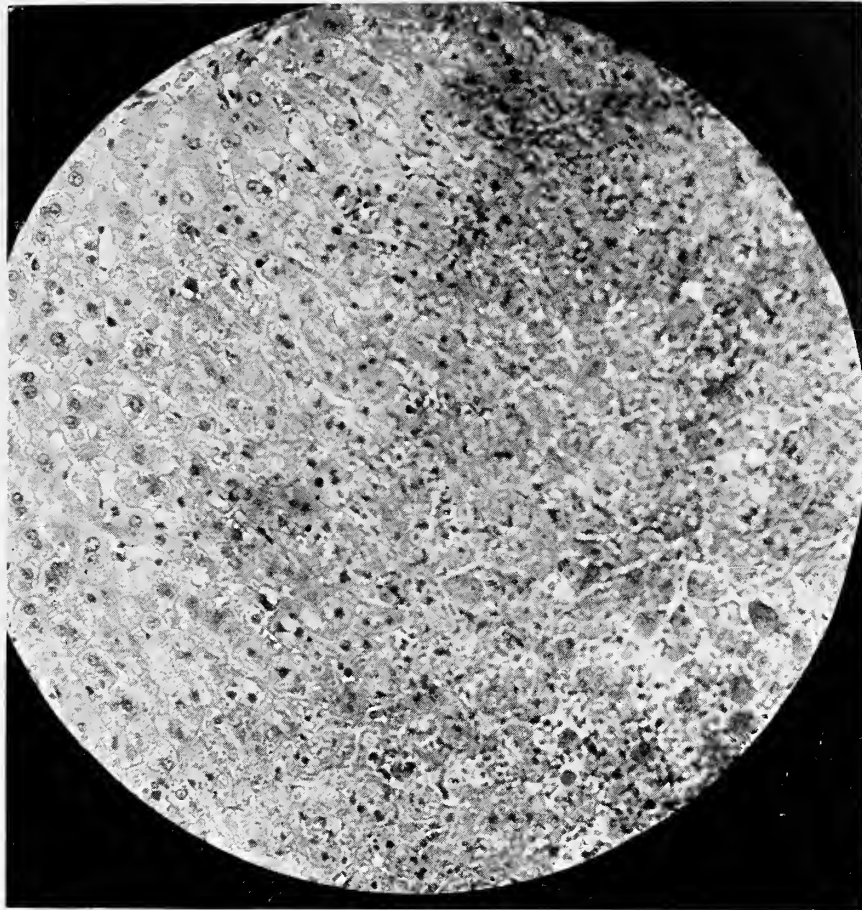


FIG. 3.—Liver from the same rabbit. To the left the centre of the lobule is seen to be completely broken down, while to the right the cells at the periphery appear almost normal. $\times 150$.



FIG. 4.—Medulla of kidney from rabbit, No. 33, which received an injection of 1cc of chloroform subcutaneously. A considerable degree of degeneration is observable, and the nuclei are losing the power of taking on the basic stain. $\times 150$.

DR G. HERBERT CLARK.

[Plate II.

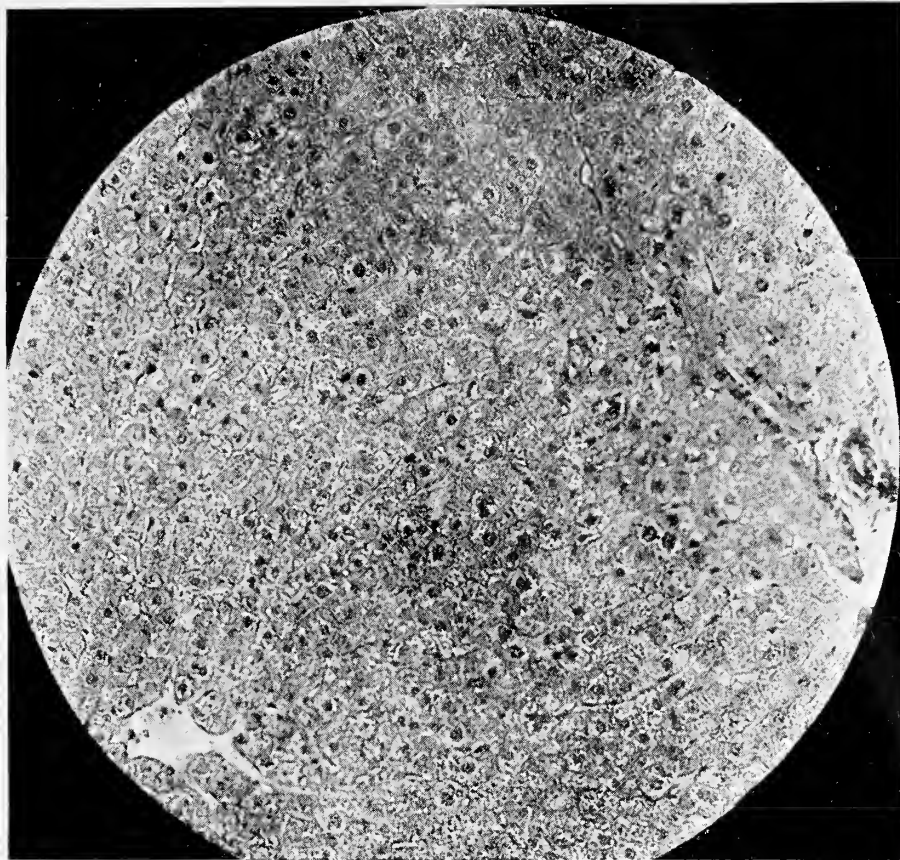


FIG. 5.—Liver from the same animal. The centre of the lobule shows cells considerably degenerated and even vacuolated. The periphery—to the right—shows cells which are almost normal in appearance. $\times 150$.

XXVII.—On the Effect of Internal Friction in Cases of Compound Stress. By G. H. Gulliver, B.Sc., A.M.I.Mech.E., Lecturer in Engineering in the University of Edinburgh.

(MS. received January 19, 1909. Read July 12, 1909.)

IN a previous paper* certain results were deduced for bars supporting simple tensile or compressive loads, on the assumption that materials exhibit an internal resistance to deformation of the nature of a frictional resistance to the sliding of the particles along the cleavage surfaces. These results are easily extended to any system of stress in a solid body.

Let p_1 , p_2 , and p_3 be the principal stresses, acting normally to the three pairs of faces of a small cubic element, p_1 being the maximum tension or minimum compression, and p_3 the minimum tension or maximum compression. If tensions are taken as positive, $p_1 > p_2 > p_3$. Then it is a well-known result that the plane of maximum shear is parallel to the direction of p_2 , and inclined at 45° to both p_1 and p_3 , and that the value of the maximum shearing stress is $\frac{1}{2}(p_1 - p_3)$.

In the same manner, but taking into consideration the effect of internal friction, as already explained, the plane of sliding is parallel to the direction of p_2 , and is inclined at $\alpha = (45^\circ + \phi/2)$ to p_1 , and at $\beta = (45^\circ - \phi/2)$ to p_3 . The minimum resistance to sliding is,

$$\begin{aligned} \mu K &= p_1 \sin \alpha \cos \alpha + \mu p_1 \sin^2 \alpha - p_3 \sin \beta \cos \beta + \mu p_3 \sin^2 \beta \\ &= \sin \alpha \cos \alpha (p_1 - p_3) + \mu (p_1 \sin^2 \alpha + p_3 \cos^2 \alpha) \\ &= \frac{1}{2} \{ (\cos \phi + \mu \sin \phi) (p_1 - p_3) + \mu (p_1 + p_3) \} \\ &= \frac{1}{2} \{ p_1 (\sqrt{1 + \mu^2} + \mu) - p_3 (\sqrt{1 + \mu^2} - \mu) \} \end{aligned}$$

where K is the cohesion of the material as defined before, and $\mu (= \tan \phi)$ is the coefficient of internal friction.

Few investigations of the effect of compound loading upon metal bars have been carried out. The most complete series of experiments are those of Guest; some interesting results have been obtained also by Hancock, by Scoble, and by Goodman. It occurred to the writer that these experiments might afford some evidence for or against the influence of internal friction.

Guest's experiments† were made with ductile materials—steel, copper, and brass—in the form of thin tubes. These tubes were subjected to tension, to torsion, and to internal pressure; and also to tension combined with torsion, to tension combined with internal pressure, and to torsion combined with internal pressure. The loads were so regulated as to obtain

* *Proceedings*, xxviii., 1908, 374.

† *Phil. Mag.*, 1., 1900, 69.

a number of stresses lying between $p, 0, -p$, and $p, 0, p$. Internal pressure causes the intermediate principal stress to have a small negative value (compression), but this is of no importance. The criterion of strength for the steel tubes was taken as the yield-point, and the results of the experiments are given in a series of tables attached to the paper. The conclusion was drawn that the strength of a material is determined by the maximum shearing stress, and not by the maximum principal stress, or by the maximum principal strain.

The writer, by taking Guest's data for the principal stresses, and assuming $\mu = 0.14$ for the steel tubes, found that the numerical values of the minimum resistance to sliding were sometimes less uniform, and sometimes more uniform, than the values of the maximum shearing stress. The figures are not sufficiently interesting to reproduce, but the experiments could not be said to give results more in favour of the minimum resistance theory than of the maximum shear theory.

Another point of interest was the determination of the values of μ which would accord with the experimental results. The values of this coefficient, obtained from the simple tension and simple torsion results only, are given in the table below, from which the amount of variation can be judged. The variations found for each separate bar, under the different conditions of loading, were quite as great. It should be pointed out that small errors in measuring stresses lead to relatively large errors in the determination of μ . No measurements of the direction of sliding at the yield-point seem to have been made by Guest. These experiments, therefore, do not permit one to assign any definite value to μ .

TABLE.

Form of Specimen.	Diameter.	Thickness.	μ .	Remarks.
<i>Guest—</i>				
I. Round tube . . .	1.316	0.029	-0.242	Several different results.
II. " . . .	1.250	0.028	...	
III. " . . .	0.499	0.025	-0.071	
IV. " . . .	1.250	0.025	+0.093	No torsion test.
V. " . . .	1.250	0.025	...	
VI. " . . .	1.250	0.025	+0.015	
VII. " . . .	1.250	0.025	+0.160	
VIII. " . . .	1.250	0.025	+0.090	
IX. " . . .	1.250	0.025	+0.144	
<i>Hancock—</i>				
Round tube . . .	1.000	0.050	+0.146	
" . . .	1.000	0.075	+0.380	
" . . .	1.000	0.250	+0.204	
Round bar . . .	0.500	...	+1.189	

The experiments of Hancock * were performed upon steel tubes and solid round bars, these being submitted to tension and to torsion, both singly and combined in various proportions. Similar calculations, taking μ as 0.14, gave results appreciably more uniform than the maximum shear, especially in the case of the thinnest tube. The divergences in Hancock's results become much greater as the thickness of the tube is increased. The values of μ , calculated from the simple tension and simple torsion experiments, are given also in the above table. The absurd result shown by the solid bar is probably connected with the variation in stress over the cross section, caused by torsion.

Scoble † subjected solid round steel bars to combined bending and torsion. The magnitude of the stress across a section of a bar in such conditions varies considerably, so that it is impossible to determine the yield-point with any degree of accuracy. Recognising this difficulty, Scoble made use of a purely arbitrary point, found by prolonging the plastic curve backwards to meet the elastic curve. This point is certainly not identical with the yield-point, and since it depends upon the individual judgment in extrapolating a curve, which is itself liable to considerable variations in character, the results cannot be considered of great value. As they stand, however, they would give no definite value for μ , as is pointed out by Scoble himself.

Goodman's experiments were confined to the application of combined bending and torsion to solid round bars of cast iron—a brittle material—instead of the ductile substance, steel. The two cases are very different, and the results must be considered in a somewhat different manner.

There is a distinct dissimilarity between ductile and brittle bodies in their behaviour under stress. When broken by simple tension, a ductile substance shows a more or less perfect shearing fracture, but a brittle substance ruptures along a direction approximately perpendicular to the direction of the stress. On the other hand, under a compressive stress brittle bodies give shearing fractures, while ductile materials are simply flattened, but sometimes crack along planes parallel to the direction of the load. Thus the ratio of the tensile strength to the compressive strength for ductile materials is a somewhat indefinite quantity, if strength be defined as the stress necessary to cause rupture. The same ratio for brittle substances is usually small, the compressive strength being much greater than the tensile strength; the ratio is 1/4 to 1/7 for cast iron, and as small as 1/50 for some stones.

When subjected to combined stresses, ductile and brittle pieces may be

* *Phil. Mag.*, xi., 1906, 276 ; xii., 1906, 418.

† *Ibid.*, xii., 1906, 533.

expected, therefore, to differ in their behaviour. A round steel bar, broken by torsion, shears along a surface which approximates to a plane at 5° to the axis of the bar—that is, at 50° to the direction of principal tension, corresponding with the direction of sliding in simple tension. Under similar conditions, a bar of cast iron breaks along a helicoidal surface inclined at 45° to the axis of the bar—that is, perpendicular to the direction of principal tension, agreeing also with the result of the pure tension test. In torsion the principal tension and the principal compression are numerically equal, but in cases where the principal stresses have different values the form of fracture of a brittle body will depend upon the sign of the principal stress which is the immediate cause of breaking. If the principal tension is greater than a certain fraction of the principal compression, the broken surface will be normal to this tension; if the tension bear a less ratio to the compression, the broken surface will be inclined to the direction of principal compression at the angle β (less than 45°), found from the crushing test.

But the manner in which fracture occurs is also of importance. Thus, in a bar of cast iron broken by bending, although the stress upon the transverse section which is most severely loaded may vary from a pure compression to a pure tension, with all intermediate ratios between tension and compression, fracture usually occurs along such a transverse section—that is, in a direction normal both to the principal tension and to the principal compression. Observation shows that the metal begins to separate on the tension side of the bar, and as the crack progresses the stress suffers a redistribution, so that the direction of tension at the extremity of the crack remains approximately constant and normal to the original direction of the crack.

Under combined bending and torsion, therefore, the fracture of cast iron may be expected to take place in a direction normal to the principal tension. In Goodman's experiments* the fractures had a helicoidal form, and the helix due to the intersection of the broken surface with the exterior of each bar was found to be, within narrow limits, perpendicular to the calculated direction of the principal tension. With regard to the determination of the breaking stress, the objections to the use of solid bars in experiments of this kind have been pointed out already, and Goodman's figures for this stress are not of great value—a point which he recognises.

Since the writers mentioned above are all concerned with what is usually called the "equivalent bending moment" for a shaft submitted to simultaneous bending and torsion, it may not be out of place to give the

* *Mechanics Applied to Engineering*, Longmans, 4th ed., 1904, 492.

expression obtained for this moment when internal friction is allowed for. Let M be the bending moment applied to the shaft, and T the simultaneously applied twisting moment. Let M' be the bending moment which would have the same maximum effect upon the shaft as M and T together. Then, assuming that the strength of the shaft is determined by the maximum principal (tensile) stress,

$$M' = \frac{1}{2}(M + \sqrt{M^2 + T^2}).$$

This is the rule given by Rankine, and in general use among engineers. It would appear to be true only for brittle materials. If, as according to Guest, the strength be considered as determined by the maximum shearing stress, the expression becomes,

$$M' = \sqrt{M^2 + T^2}.$$

This seems to represent, at least very closely, the conditions for ductile materials. Finally, if the effect of internal friction is considered, and the strength is supposed to be determined by the minimum resistance to sliding,

$$M' = \frac{1}{\sqrt{1 + \mu^2 + \mu}} \cdot \{\mu M + \sqrt{(1 + \mu^2)(M^2 + T^2)}\}.$$

And, taking $\mu = 0.14$, as on previous occasions,

$$M' = 0.12M + 0.88 \sqrt{M^2 + T^2}.$$

This last expression gives values for M' which are intermediate between those of Rankine and of Guest, though they cannot be said to represent actual conditions for ductile metals more accurately than those of Guest.

SUMMARY.

The minimum resistance to deformation, and the inclination of the surfaces of sliding, are given for any system of stress in a body, supposing internal friction to be operative.

The effect of internal friction in various cases of combined loading, and the value of the coefficient have been calculated from experimental data. The results do not allow of a definite value being assigned to μ for steel.

Certain differences in the behaviour of ductile and of brittle bodies, when loaded, are pointed out.

An expression for the equivalent bending moment of a shaft submitted to simultaneous bending and twisting, when internal friction is allowed for, is given.

XXVIII.—On the Friction at the Extremities of a Short Bar subjected to a Crushing Load, and its Influence upon the Apparent Compressive Strength of the Material. By G. H. Gulliver, B.Sc., A.M.I.Mech.E., Lecturer in Engineering in the University of Edinburgh.

(MS. received November 30, 1908. Read December 21, 1908.)

1. CRUSHING STRENGTH.

IN making crushing tests of various constructional materials the usual form of test-piece employed is that of a small cylinder or prism, having a height of 1 to 2 diameters. The pressure is applied to the bases of the specimen by means of two plates of steel or of cast iron, which are somewhat harder—that is, less easily deformable—than the material under test. The friction between these plates and the ends of the specimen necessitates, for the deformation and rupture of the piece, a load somewhat greater than if the friction did not exist. So far as the writer is aware, no investigation of the effect of this end friction has been published hitherto, and the following is offered therefore as giving an approximate solution to the problem. The method employed is simple. In any bar, two portions about to experience a relative movement along their common plane of sliding are regarded as two rigid bodies in a condition of static equilibrium, and the sum of all the forces acting on each piece is accordingly equated to zero. Since the external forces are known, or can be determined, the internal resistance to sliding, under certain assumptions, may be calculated. The method is admittedly imperfect, and the degree of approximation of the calculated results to actual conditions is uncertain; but these results are not uninteresting, and they show, if not grossly incorrect, the importance of the influence of the end friction upon the apparent crushing strength of a bar.

The assumptions made with regard to internal friction in a previous paper⁽¹⁾ are made here also. The coefficient of friction is supposed to be independent of the load; and a cohesive force, K , is assumed to act normally to the surface of each particle of the metal, giving rise to an internal frictional resistance to sliding, equal to μK . It is assumed, further, that the external frictional resistance to deformation at the loaded ends of the specimen can be regarded as equivalent to a force applied to the piece in a direction tangential to these surfaces. In all cases the bars are supposed to

be so short that any bending is negligible. The results apply only to the yield point, and not to the breaking stress of a substance. In the case of a ductile material, after the yield point is passed, the conditions change greatly, and the test-piece does not rupture by shearing. A brittle bar, however, does usually break by the sliding of some portion over the remainder. A brittle material is often defined as one in which the yield point and the breaking stress are coincident. This is hardly correct, since most brittle substances suffer some permanent deformation before rupture; but it suggests that the calculations may not be very incorrect when

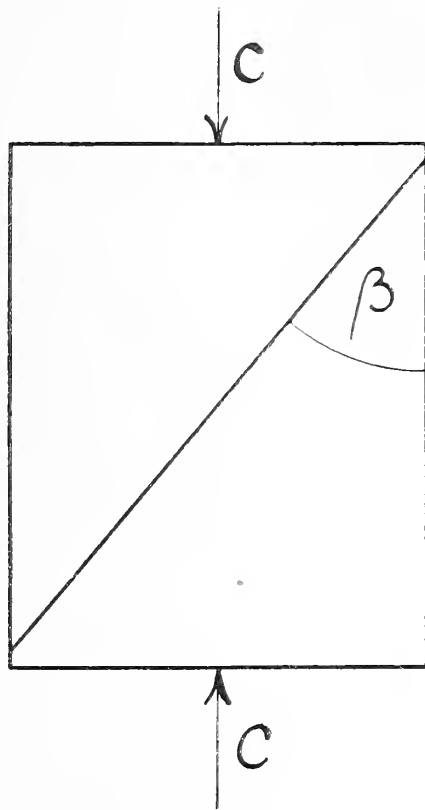


FIG. 1.

applied to the breaking strength of brittle materials in compression. This point is examined more fully in sections 4 and 5.

2. END FRICTION NEGLECTED.

Let the length of the bar be such that the surface of sliding does not cut the extremities, and suppose first that there is no end friction (fig. 1).

Let c be the compression yield point of the material—that is, the stress on a normal cross-section at the moment when sliding commences. Let β be the inclination of the surface of sliding to the axis of the bar, and let μ ($= \tan \phi$) be the coefficient of internal friction for the material.

Then, as has been shown already, the shearing stress along a surface of sliding is $c \sin \beta \cos \beta$, and the resistance to sliding along this surface is

$\mu c \sin^2 \beta + \mu K$, where K is the normal cohesion between the particles. So long as sliding does not occur, $c \sin \beta \cos \beta - \mu c \sin^2 \beta$ must be less than μK ; and since $c \sin \beta \cos \beta$ is greater than $\mu c \sin^2 \beta$,⁽²⁾ the difference between these two quantities increases as c is increased. Sliding takes place as soon as this difference exceeds μK , and the inclination of the surface of sliding, therefore, is such that $c \sin \beta \cos \beta - \mu c \sin^2 \beta$ is a maximum. Hence $\tan 2\beta = 1/\mu$, or $\beta = 45^\circ - \phi/2$.

3. EFFECT OF END FRICTION WITH PLATES HARDER THAN MATERIAL CRUSHED.

Let μ' ($= \tan \phi'$) be the coefficient of friction between the crushing plates and the extremities of the specimen. Let c' be the new compression

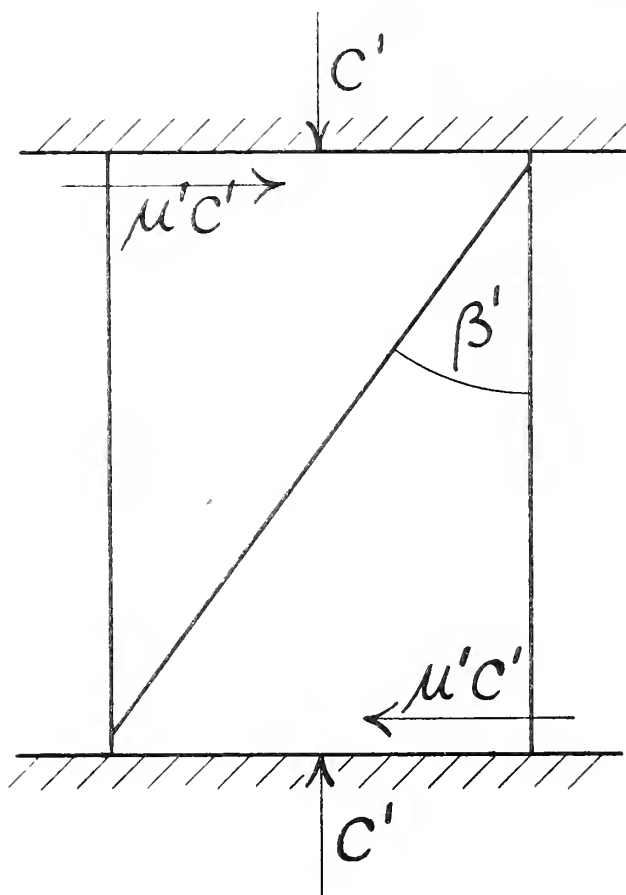


FIG 2.

yield point, and let β' be the inclination of the new surface of sliding (fig. 2).

The friction on the ends of the bar is $\mu'c'$ per unit of area. Resolving this in the same manner as for the stress due to the direct crushing load, the component along the surface of sliding is $\mu'c' \sin \beta'$; and the component normal to this surface is $\mu'c' \cos \beta'$. Since the area of the surface of sliding is equal to the area of the end surface of the bar divided by $\sin \beta'$, the effect

of friction at the extremities may be considered as equivalent to a shearing stress having the value $\mu'c' \sin^2 \beta'$, together with a normal stress equal to $\mu'c' \sin \beta' \cos \beta'$. The new tangential stress acts in the direction opposite to that due to the crushing load, and tends to strengthen the material. The new normal stress increases the pressure between the sliding surfaces, and therefore increases the internal frictional resistance to sliding by the amount $\mu\mu'c' \sin \beta' \cos \beta'$, again tending to strengthen the bar.

The total resistance to sliding is

$$\mu'c' \sin^2 \beta' + \mu K + \mu\mu'c' \sin \beta' \cos \beta' + \mu'c' \sin^2 \beta'.$$

And the inclination of the surface of sliding is such that

$$c' \sin \beta' \cos \beta' - (\mu'c' \sin^2 \beta' + \mu\mu'c' \sin \beta' \cos \beta' + \mu'c' \sin^2 \beta')$$

is a maximum. This occurs when

$$\tan 2\beta' = \frac{1 - \mu\mu'}{\mu + \mu'} = \cot(\phi + \phi');$$

that is, when,

$$\beta' = \left(45^\circ - \frac{\phi + \phi'}{2}\right) = \beta - \phi'/2.$$

Since μ and K are, by assumption, constants for any particular material, the ratio of c' to c is given by

$$\begin{aligned} \frac{c'}{c} &= \frac{\sin \beta \cos \beta - \mu \sin^2 \beta}{\sin \beta' \cos \beta' - \mu \sin^2 \beta' - \mu\mu' \sin \beta' \cos \beta' - \mu' \sin^2 \beta'} \\ &= \frac{\frac{1}{2}\{\cos \phi - \mu(1 - \sin \phi)\}}{\frac{1}{2}[(1 - \mu\mu') \cos(\phi + \phi') - (\mu + \mu')\{1 - \sin(\phi + \phi')\}]} \\ &= \frac{\cos \phi'(1 - \sin \phi)}{1 - \sin(\phi + \phi')} \end{aligned}$$

Or, more conveniently for purposes of calculation,

$$\frac{c'}{c} = \frac{\sqrt{1 + \mu^2} - \mu}{\sqrt{(1 + \mu^2)(1 + \mu'^2)} - (\mu + \mu')}$$

If the length of the bar is relatively less than in fig. 2, sliding takes place as in fig. 3 or fig. 5. The conditions for each portion are the same as in fig. 2, so that the inclination of the sliding surfaces should remain constant. If the relative length of the bar is much greater than in fig. 2, bending occurs, and the above results no longer hold good.

4. CAST IRON.

For cast iron the value of μ , as determined from Morin's experiments on sliding friction, varies from 0.15 to 0.16. The corresponding values of ϕ are $8^\circ 32'$ and $9^\circ 5'$. Hence β varies from $40^\circ 44'$ to $40^\circ 28'$.

If a specimen of cast iron is crushed between cast-iron platens, μ and μ' are equal, or nearly so. In this case

$$\beta = (45^\circ - \phi) = 36^\circ 28' \text{ to } 35^\circ 55'.$$

If the platens are of steel, μ' will be somewhat less than 0.16, and β' consequently slightly greater than the above values. The inclination of the sliding surfaces, measured from broken specimens of cast iron, usually lies between 35° and 37° , and therefore agrees closely with the calculated value of β' .

Taking $\beta' = 36^\circ$ as an average result, the ratio of c' to c is

$$\frac{c'}{c} = \frac{\sqrt{1 + \mu^2} - \mu}{1 + \mu^2 - 2\mu} = 1.2 \text{ (nearly).}$$

In extending this result to the breaking stress of cast iron, it should be noted that bars of this material originally cylindrical or prismatic become

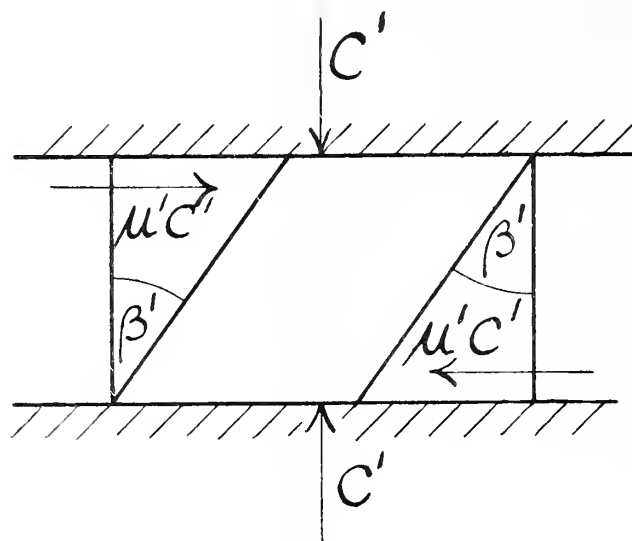


FIG 3

more or less barrel-shaped towards the end of the test. On this account the direction of compressive stress is no longer constant, but varies slightly from point to point within the body. In this case the results are closely represented by saying that the inclination of the surface of sliding to the direction of the compressive stress at each point is constant, and has the same value as already found. The most important effect of the change of shape of the bar is the resulting curvature of the surface of sliding (fig. 4). It may be pointed out that the mere existence of this deformation is evidence that cast iron possesses a yield point lower than its breaking stress; but this yield point, as is well known, cannot be located definitely.

The minimum length of a cast iron specimen necessary to secure a clean shearing fracture is $d \cot 36^\circ$ —that is, about $1.4d$, where d is the diameter of the bar. The length should not be less than this at the moment of

crushing, and must therefore be greater in the original specimen, since the bar is appreciably shortened under the load. The general practice is to make the length from $1\frac{1}{2}$ to 2 times the diameter. Shorter bars fracture as shown in fig. 3.

5. STONE, BRICK, AND CONCRETE.

The coefficient of friction for these materials is extremely variable, and only rough results can be given. For stones, μ usually lies between 0.4 and 0.6, the lower value corresponding with the harder and finer stones; for masonry and brickwork an average value of μ is from 0.6 to 0.65. The

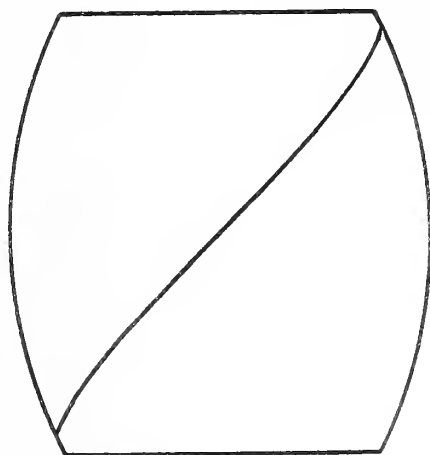


FIG. 4.

value of μ' , the coefficient of friction between stone and steel, or stone and cast iron, may be taken as varying from 0.25 to 0.5. With these extreme values the following results are obtained:—

For

$$\mu = 0.4, \text{ and } \mu' = 0.25 ;$$

$$\phi = 22^\circ, \phi' = 14^\circ, \beta = 34^\circ, \text{ and } \beta' = 27^\circ.$$

For

$$\mu = 0.65, \text{ and } \mu' = 0.5 ;$$

$$\phi = 33^\circ, \phi' = 27^\circ, \beta = 28\frac{1}{2}^\circ, \beta' = 15^\circ.$$

The inclination of sheared surfaces measured from broken stone test-pieces is apt to vary considerably. This is to be expected if one considers the variability and heterogeneity of most kinds of stone, even when samples are taken close together from the same quarry. The bedding of the stone gives rise to surfaces of minimum resistance other than those considered here. A crushed stone is usually much disintegrated, and the angles may be altered by the removal of loose material. Only such surfaces

as show unmistakable signs of sliding should be used for measuring the inclination.

The writer has, from time to time, determined roughly the inclination of surfaces of sliding in a number of specimens of stone, cement, and mortar, and these have been supplemented by measurements made from illustrations of broken specimens given by various authors. Average values of these inclinations are:—

Sandstone	.	.	.	25° to 28°
Limestone	.	.	.	20° to 25°
Cement	.	.	.	20° to 30°
Mortar	.	.	.	15° to 20°

Angles greater than these are not infrequent, but are certainly due in many cases to disintegration, and also to irregularities in the material. On the other hand, cracks parallel, or nearly so, to the direction of the applied crushing load are often found. These would appear to be due to the inability of such materials to withstand much lateral extension; they are not caused by any sliding action, but are the result of the low tensile strength of the substance.

The two extreme ratios of c' to c , calculated from the above values of μ and μ' , are 1.5 and 3.0 nearly, showing that the end friction has a very considerable effect, especially on the softer and coarser materials. No values of the yield points of stones are known. Whether these figures apply with any accuracy or not to the ultimate strength of such substances is impossible to say at present.

In order to obtain a clean shear a length of 2 to 4 diameters is necessary. Specimens of stone and concrete required for crushing tests are usually made in the form of cubes, and fracture takes place as shown in fig. 5; bricks are usually 9 inches by $4\frac{1}{2}$ inches, and 3 inches high, but are often halved vertically, across the long edges, before testing, in order to bring them within the crushing capacity of the testing machine.

6. WROUGHT IRON AND MILD STEEL.

These materials, when crushed, do not fracture by shearing, and they are not usually tested in compression except in the form of long columns, in which bending is of supreme importance. The determination of the yield point and the inclination of the lines of Lüders from short pieces, however, should give useful checks on the previous arguments.

According to Morin, the value of μ for wrought iron and steel is 0.14. This corresponds with $\phi = 8^\circ$, $\beta = 41^\circ$, nearly. If the crushing platens are

of steel, $\beta' = 37^\circ$. So small an inclination of the lines of Lüders on compression specimens of mild steel has been seldom found by the writer; it lies more usually between 40° and 45° , and sometimes even above 45° . No attempt is made here to explain this difference between calculated and experimental values.

Taking $\mu = \mu' = 0.14$, the ratio of the apparent compression yield point to the true yield point is 1.18. The ratio of the tension yield point to the compression yield point is 0.76.* Hence the value of this latter ratio, found

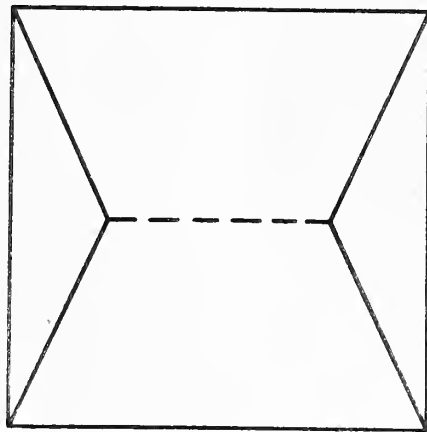


FIG. 5.

from experiments, should be $0.76/1.18$ —that is, 0.64. This last figure is certainly much lower than any obtained hitherto.

7. EFFECT OF END FRICTION WITH PLATES SOFTER THAN MATERIAL CRUSHED.

In sections 3 to 6 the material forming the plates between which the specimens are crushed has been supposed harder than the substance under test. It was a common practice at one time, in making crushing tests of stone, to place a sheet of lead between each end of the specimen and the corresponding platen of the testing machine, with the object of securing a uniform distribution of load. Experiments soon revealed that the effect of the lead, instead of being beneficial, caused a great reduction in the apparent strength of the stone, amounting on an average to one-half of the strength obtained when the stone was in direct contact with the platens. The reason of this is not far to seek. As the load is applied the soft lead is deformed to a greater extent than the stone. When the compressive stress reaches a certain value—generally known as the pressure of fluidity, and estimated variously at $\frac{3}{4}$ ton to 3 tons per square inch^(3, 4)—the lead

* This ratio was given previously as 0.70, ϕ having been taken as 10° . The higher value of t/c would seem to be nearer the correct figure, but experiments suggest one still higher.

behaves like a viscous fluid, and flows laterally over the loaded faces of the test-piece. The friction between the lead and the stone, instead of tending to increase the apparent strength of the specimen, tends to decrease it.

Let c'' be the apparent yield point of the stone under these conditions, and let β'' be the inclination of the surfaces of sliding. Let $\mu'' (= \tan \phi'')$

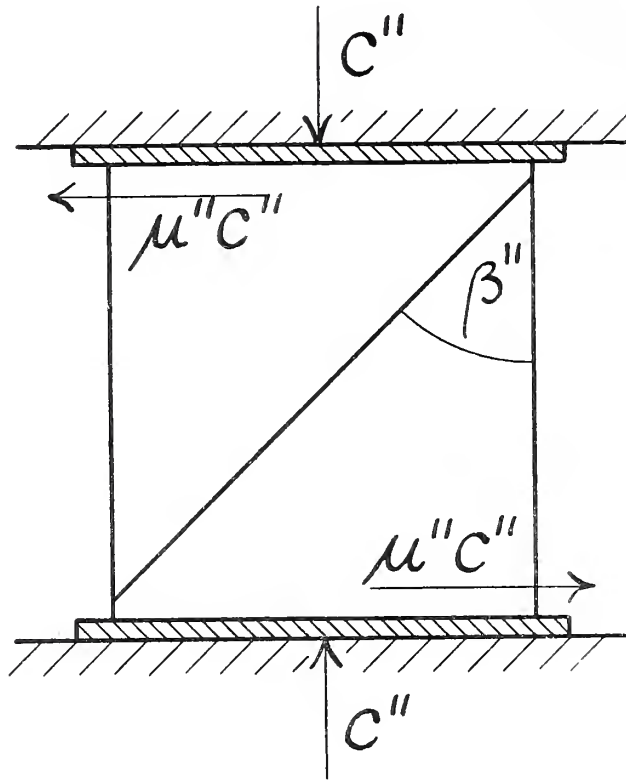


FIG. 6.

be the coefficient of friction between the stone and the lead (fig. 6).

The total resistance to shearing in this case is

$$\mu c'' \sin^2 \beta'' + \mu K - \mu \mu'' c'' \sin \beta'' \cos \beta'' - \mu'' c'' \sin^2 \beta'',$$

and the angle β'' is such that

$$c'' \sin \beta'' \cos \beta'' - \mu c'' \sin^2 \beta'' + \mu \mu'' c'' \sin \beta'' \cos \beta'' + \mu'' c'' \sin^2 \beta''$$

is a maximum. Hence

$$\tan 2\beta'' = \frac{1 + \mu\mu''}{\mu - \mu''} = \cot(\phi - \phi''),$$

and

$$\beta'' = 45^\circ - \frac{1}{2}(\phi - \phi'') = \beta + \phi''/2.$$

As a rule, the stone does not rupture by shearing, but splits up in a different manner, discussed below. In the case of soft stones and concrete the pressure of fluidity of the lead may never be reached, and the interposition of the metal may have no appreciable effect upon the strength of the specimen. On the other hand, for materials with crushing loads far

above this pressure of fluidity, the whole of the lead is squeezed out long before rupture occurs, and the metal has again very little influence upon the maximum load carried by the piece.

The value of the ratio c''/c , though of little use, is not uninteresting:—

$$\begin{aligned} \frac{c''}{c} &= \frac{\sin \beta \cos \beta - \mu \sin^2 \beta}{\sin \beta'' \cos \beta'' - \mu \sin^2 \beta'' + \mu \mu'' \sin \beta'' \cos \beta'' + \mu'' \sin^2 \beta''} \\ &= \frac{\cos \phi'' (1 - \sin \phi)}{1 - \sin (\phi - \phi'')} = \frac{\sqrt{1 + \mu^2} - \mu}{\sqrt{(1 + \mu^2)(1 + \mu''^2)} - (\mu - \mu'')} \end{aligned}$$

The numerical values of μ for stone have been given already on page 437, but the coefficient of friction of lead upon stone does not seem to have been determined. In order to obtain some idea of the importance of the action of the lead, let μ'' be taken as 0.5, corresponding with $\phi'' = 27^\circ$, roughly.

$$\text{For } \mu = 0.4, \text{ and } \mu'' = 0.5; \quad \beta'' = 47\frac{1}{2}^\circ, \text{ and } c''/c = 0.52.$$

$$\text{For } \mu = 0.65, \text{ and } \mu'' = 0.5; \quad \beta'' = 42^\circ, \text{ and } c''/c = 0.46.$$

Though no great accuracy is claimed for these results, on account of the uncertainty as to the value of μ'' they show very clearly the weakening effect produced by the lateral flow of the lead; this effect will be greater or less than the above according as the true value of μ'' is greater or less than 0.5.

A few measurements made upon blocks of cement and of mortar which had been crushed between sheets of lead showed that the inclination of the sliding surfaces was very little greater than when no lead was interposed. The pieces, especially those of cement, failed chiefly by splitting vertically into small fragments, as described below; but a few surfaces of shearing were unmistakable.

As already mentioned, stone specimens crushed between sheets of lead or other soft and plastic material do not usually give way by shearing. The frictional drag of the moving lead gives rise to a tensile stress in a direction normal to the crushing load; rupture seems to be caused chiefly by this tension, and the stone splits into a number of vertical prisms. A little consideration shows that the tensile stress is a maximum on two vertical planes passing through the axis of the block in directions parallel with the vertical faces, and that its value on these planes approximates to $\frac{1}{2}\mu''c''$ if the specimen is cubical. Experiments show that the initial cracks occur, as a rule, in the middle of the vertical sides of the piece, and that the cracks soon extend inwards to the centre⁽⁵⁾. The stone does not give way immediately, but continues to break up into smaller and smaller prisms as the load is increased, until complete failure occurs. The breaking load averages only about one-half of the value obtained without lead, but what

determines the exact crushing stress is not clear. The ratio of c''/c' is smaller the harder the stone; experimental values vary from 0·65, to 0·43^(6,7,8), c' having been measured for blocks in direct contact with the crushing platens, or previously faced with thin layers of plaster of Paris, which are easily scraped flat and parallel, and behave as integral parts of the specimen. From the values of $\frac{c'}{c}$ and $\frac{c''}{c}$ already given, the calculated values of $\frac{c''}{c'}$ are found to lie between 0·35 and 0·15, so that the actual effect of the lead is less serious than it would be according to the assumed conditions.

8. EFFECT OF LENGTH OF SPECIMEN UPON ITS APPARENT STRENGTH.

From the results given in section 3 it would appear that the crushing strength of a material should not be altered by the length of the test-piece, provided that this is short enough for no bending to take place. Published experimental results indicate that a greater load is required to cause rupture as the length of the specimen is diminished. Thus Bauschinger found, from careful tests on a specially fine and uniform sandstone, that a cube is about 20 per cent. weaker than a piece of half the height, and about 7 per cent. stronger than a piece $1\frac{1}{2}$ times the height⁽⁹⁾. Similarly Hodgkinson, in a series of tests of small cast-iron cylinders, found that with a height equal to $1\frac{1}{2}$ diameter the piece was 3 per cent. weaker than when the height was only 1 diameter, 19 per cent. weaker than with a height of $\frac{1}{2}$ diameter, and 30 per cent. weaker than a cylinder with a height equal to $\frac{1}{4}$ diameter⁽¹⁰⁾.

If crushing strength is calculated as the total crushing load divided by the original area of cross-section of the piece, it is undoubtedly increased by diminishing the length of the specimen; but if, as is more rational, the increase in the area of the material under load is allowed for, the case is different. This increase in area is greater the shorter the specimen. One might expect that the friction of the platens, when the specimen is relatively long, would not have so great an effect in the middle of the length of the piece as when this is short. Thus the amount of lateral extension of the middle of the specimen, relatively to that of the ends, would be greater with longer specimens. In crushing cast iron it is noticeable that a piece $1\frac{1}{2}$ diameter long assumes a very distinct barrel shape towards the end of the test (fig. 4), while a piece only $\frac{1}{2}$ diameter long, though its dimensions change, remains nearly cylindrical. The final maximum diameter is greater in short specimens than in longer ones.

The following table gives numerical results, not previously published, obtained from six cast-iron bars tested some time ago for another purpose.

The strength, as calculated for the actual area at the middle of the bar, shows a decrease for the shorter specimens; though the nominal strength, calculated for the original area, shows the usual increase.

Original Ratio Length/Diameter or Length/Side.	Ratio $\frac{\text{Original Area}}{\text{Final Area}}$.	Crushing Stress, Tons per Square Inch.	
		Calculated on Original Area.	Calculated on Final Area.
ROUND BARS, ORIGINALLY 1 INCH DIAMETER.			
1.44	1.24	40.5	32.7
1.01	1.39	38.9	28.0
0.51	1.47	44.2	30.1
SQUARE BARS, ORIGINALLY 1 INCH SQUARE.			
1.56	1.13	37.8	33.5
1.01	1.33	42.0	31.6
0.51	1.48	45.3	30.6

9. PRACTICAL DETERMINATION OF COMPRESSIVE STRENGTH FROM CRUSHING TESTS.

It would appear from the foregoing discussion that, in order to obtain the true compressive strength of a material by means of a crushing test it is necessary to reduce the friction at the ends of the specimen to zero. This result could be obtained only if the surfaces in contact were deformed at the same rate in contiguous parts throughout their whole area. The application of lubricants to the surfaces, for the purpose of reducing the end friction and of obtaining a closer approximation to the true crushing strength, is not to be recommended. For with specimens of high crushing strength the lubricant is soon squeezed out, and if the lubricant is possessed of a sufficiently high viscosity to allow a soft material to be crushed, it exerts a weakening influence upon the specimen. Graphite, boric acid, and similar solids possessed of lubricating properties are possible exceptions to this statement, but the writer has not yet investigated the effects of these.

SUMMARY.

The effect of the friction of the crushing plates upon the yield point of short compression specimens has been investigated.

With plates harder than the material under test, the end friction causes an increase in the apparent yield point. This increase is calculated as 18 per cent. for wrought iron and mild steel, 20 per cent. for cast iron, and

50 to 200 per cent. for stones, bricks, and concrete. These figures, except the first, may apply almost equally well to the crushing strength, but they require experimental support. The corresponding inclinations of the surfaces of shearing are 37° for wrought iron and steel, 36° for cast iron, and 27° to 15° for stones, etc. The first value is seldom obtained, but the others agree fairly well with average experimental results.

When the crushing plates are of material softer than that under test, the lateral flow of the former diminishes the apparent strength of the specimen. With stones crushed between lead plates the strength is calculated as 0.35 to 0.15 of that obtained when iron or steel plates are employed. Experiments give 0.65 to 0.43 as the value of this ratio, but the specimens do not rupture by shearing in the manner contemplated.

The total crushing load of a short specimen of cast iron is increased by diminishing the length of the piece, but the crushing stress per unit of area is simultaneously decreased.

REFERENCES.

- (1) GULLIVER, *Proc. Roy. Soc. Edin.*, vol. xxviii., 1908, p. 374.
- (2) *Ibid.*, p. 378.
- (3) KICK, *Proc. Inst. Civ. Eng.*, vol. l. 1877, p. 188.
- (4) UNWIN, *Testing of Materials of Construction* (Longmans, 1899), p. 393.
- (5) PACE, *Atti Coll. Ingeg. Archi.*, Palermo, 1880, iv.
- (6) UNWIN, *Brit. Assoc. Report*, 1887, p. 879.
- (7) BEARE, *Proc. Inst. Civ. Eng.*, vol. cvii., 1891, sect. ii. p. 341.
- (8) PACE, Int. Assoc. Test. Materials, Brussels Congress, 1906, Paper B 10e, p. 7.
- (9) BAUSCHINGER, *Mitt. Tech. Hoch.*, München, 1876, vi.
- (10) See (4), p. 259.

The values of μ used throughout the paper are taken chiefly from Trautwine, *Civil Engineer's Pocket-Book* (Wiley, New York, 1900), p. 373.

(Issued separately July 16, 1909.)

XXIX.—On Group-Velocity and on the Propagation of Waves in a Dispersive Medium. By George Green, M.A., B.Sc., Assistant to the Professor of Natural Philosophy in the University of Glasgow. Communicated by Professor A. GRAY, F.R.S.

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§ 1. THE theory of group-velocity has been developed by Stokes, Osborne Reynolds, Lord Rayleigh, and later by Professor Lamb. The application of the theory to light-waves was made by Gouy and Lord Rayleigh, and its importance in this connection was emphasised by Professor Schuster in his paper "On Interference Phenomena," *Phil. Mag.*, vol. xxxvii., 1894.

The question of the velocity of a group of waves, as distinct from the velocity of a wave, arose from the well-known observation that when a large group of waves advances into smooth water, each separate crest travels through the group towards the front, where it gradually dies down and disappears, while similar crests meantime have in turn taken its place in the group, so that the main group moves forward at a velocity less than that of the separate waves.

§ 2. It seems that Stokes was the first to prove that the phenomenon was capable of being dealt with analytically, by showing that when two infinite trains of waves, of equal amplitudes and nearly equal wave-lengths are superposed, we obtain an infinite succession of wave-groups, each of which advances with the half-wave-velocity. None of the groups maintains its outline constant for any interval of time, however short; but the whole disturbance periodically returns to the configuration it had at any particular instant if the periods of the superposed trains be commensurable, and the effect is the same as if each group had moved forward in the interval, without change of shape, at the half-wave-velocity.

A most important contribution to the theory of group-velocity was made by Osborne Reynolds in *Nature*, Aug. 23, 1877, where he gave a dynamical explanation of the fact that the regular part of a large group of waves of equal wave-lengths advances with only half the velocity of the separate waves. He proved that the energy propagated across a plane, when a regular train of waves is passing, is just sufficient to feed a regular procession of waves, travelling with half the corresponding wave-velocity.

§ 3. Lord Rayleigh pointed out in his article "On Progressive Waves" (*Theory of Sound*, vol. i., Appendix) that the theorem given by Osborne

Reynolds for the case of a regular procession of water-waves could be extended to apply to all kinds of waves. His result may be briefly stated as follows. When a large regular group of waves is advancing in any dispersive medium, the energy propagated in one wave-period across a plane at right angles to the direction of the wave-motion is equal to the energy contained in one wave-length of the group multiplied by the ratio of group-velocity to wave-velocity.

§ 4. It may be here remarked that the above results all refer to a regular group of waves, or rather to the regular part of a group which is irregular at the front or rear, or to a regular procession of mutually supporting groups, as in the explanation of group-velocity according to interference principles given by Stokes and Lord Rayleigh. The difficulty of the subject lies in the application of the theory to the case of a finite group of waves, or to any irregular disturbance, and doubt has arisen as to whether the theory of group-velocity can be useful in determining the circumstances existing at any time in the actual front of a finite or infinite group of initially regular waves invading undisturbed space. Lord Kelvin, after examining mathematically the front of a large procession of initially regular waves advancing into smooth water, remarks:* “The whole investigation shows how very far from finding any definite ‘group-velocity’ we are, in any initially given group of two, three, four, or any number, however great, of waves.” In his last paper on “Deep Sea Waves,” *Proc. R.S.E.*, 1906, he examined more particularly the case of a finite group of initially regular waves, and came to the same conclusion (on finding that the fronts of the groups extend forward indefinitely to greater and greater distances as time goes on). With reference to diagrams which he gives of the water surface at different stages (reproduced on page 466 below), he says: “The perceptible fronts of these two groups extend rightwards and leftwards from the end of the initial single static group, far beyond the ‘hypothetical fronts,’ supposed to travel at half the wave-velocity, which (according to the dynamics of Osborne Reynolds and Rayleigh in their important and interesting consideration of the work required to feed a uniform procession of water-waves) would be the actual fronts *if* the free groups remained uniform. How far this *if* is from being realised is illustrated by the diagrams of fig. 35, which show a great extension outwards in each direction far beyond distances travelled at half the wave-velocity.”

§ 5. From these investigations it seems clear that the dynamical interpretation of group-velocity cannot be effectively applied to the circumstances of a finite or irregular group of waves. The kinematical interpre-

* Lord Kelvin, *Proc. Roy. Soc. Edin.*, vol. xxv., 1904.

tation of group-velocity given by Lord Rayleigh (*Scientific Papers*, vol. i. p. 540) has been presented with greater generality by Professor Lamb, who gives an interesting application of the theory to the wave-system arising from an impulse applied at a single point of a water surface.* Three separate presentations of the kinematical group-velocity are given by Professor Lamb. The first is contained in the argument from the interference of two infinite trains of waves of equal amplitudes and nearly equal wave-lengths, referred to in § 2 above. Here the groups considered are mutually supporting, the shape and motion of each depending on the presence of the others, and we cannot regard this case of group-velocity as providing a satisfactory explanation suitable to all cases. The more general treatment of the matter given in the two later demonstrations entirely completes the investigation analytically. The second and third presentations are, however, in a sense distinct, one giving a phase relation between the Fourier trains into which any group of waves can be analysed, the other referring to the actual wave-disturbance; and the question arises as to whether the demonstrations cannot be correlated with each other in such a way as to show that the same fundamental idea underlies all three. In neither of the two later demonstrations is the "group" a regular procession of waves such as that required by Osborne Reynolds and Lord Rayleigh in their dynamical theorems, and in neither is it clear that a "group" of waves has any definite group-velocity; that is, if we understand by "group" "a long succession of waves in which the distance between successive crests and the amplitude vary very slightly." † The two accounts of the matter agree if we understand "group-velocity" to mean the velocity of a point which moves so as to always coincide with the point of the wave-system where a particular wave-length singled out for observation is to be found. This is indeed the definition of group-velocity adopted by Professor Lamb; but fuller investigation seems to be necessary to make clear the fundamental relation between the arguments used by Stokes, Osborne Reynolds, and Lord Rayleigh, and those given by Professor Lamb.

§ 6. The object of this paper is to present the idea of group-velocity in the way in which it is used by Lord Kelvin in his paper of 1887, "On the Waves produced by a Single Impulse in Water of any Depth, or in a Dispersive Medium," ‡ but with greater fulness, in order to show that Lord Kelvin's paper provides the explanation that is required. The group-velocity of this paper is essentially the principle of "stationary phase" used

* Lamb, *Hydrodynamics*, §§ 234-238.

† Lamb, "On Group-Velocity," *Proc. Lond. Math. Soc.*, vol. i., 1903-4.

‡ Sir William Thomson, *Phil. Mag.*, March 1887.

by Professor Lamb in his investigation of Ship-Waves (*Hydrodynamics*, § 253), but applied to the Fourier trains which constitute any wave-disturbance. When this view is accepted, the difficulties raised by Lord Kelvin are removed, as it is consistent with the dynamical theory given by Osborne Reynolds and Lord Rayleigh and with calculated results shown in Lord Kelvin's diagrams, which are reproduced below for the sake of illustration. The whole investigation may be useful in drawing attention to the manner in which group-velocity is concerned in the modification of an initially regular group of waves, or of any disturbance initially confined to a finite portion of a dispersive medium; and in showing thereby that the idea of group-velocity contains the explanation of the *modus operandi* of dispersion.

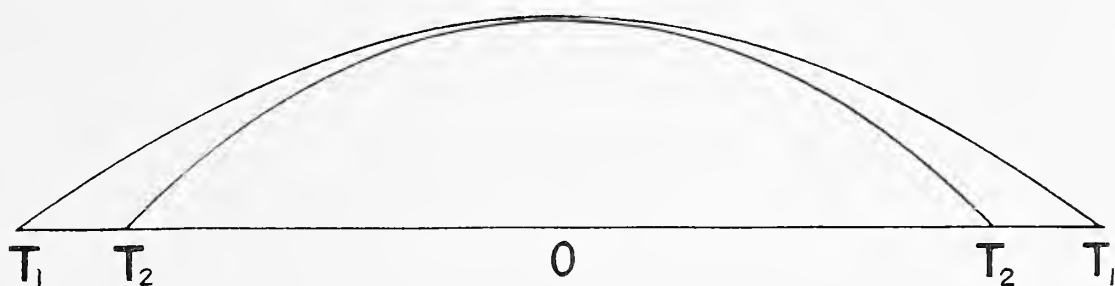
§ 7. Following Lord Kelvin's paper above referred to, let us consider, as being fundamental, the case of an infinitely intense disturbance confined to a point of the dispersive medium which is taken as the origin of co-ordinates. Let V be the wave-velocity of an infinite train of waves of period $2\pi/kV$ and wave-length λ ; then we have $k=2\pi/\lambda$, and, since the medium is such that the wave-velocity varies with the wave-length, we have also $V=f(k)$. The period and the velocity corresponding to λ are each functions of k ; and for convenience in what follows we refer to period $F(k)$ simply as period k . According to Fourier's theorem, the displacement ξ at point x and time t is given by the equation

$$\xi = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \cos k\{x - tV\} \quad . \quad . \quad . \quad . \quad (1).$$

This means that the initial disturbance may be regarded as due to the superposition of the effects of an infinite number of trains of regular waves of equal amplitudes, all of which agree in phase at the origin. Not only is the total number of trains infinite, but the number of trains whose wave-periods lie between k and $k + \delta k$ is also infinite, no matter how small δk may be. At the origin the displacement is sensible; at all other points of the medium, on account of disagreement in phase, the various trains interfere and produce zero displacement. At any time after the commencement of motion, the effect at any point is got by summing the effects due to all the trains, supposing each to have travelled in the interval a distance corresponding to its wave-velocity. By applying this we can get an idea of the manner in which the initial disturbance is propagated, reasoning as follows.

§ 8. Since all the trains initially agree in phase at phase zero, and since each train moves with a velocity corresponding to its wave-length, it is

clear that we cannot again have agreement in phase of all the trains at any single point. But, on the other hand, it can be shown that if at any time we have an infinite number of trains agreeing in phase at any point, and these trains move at nearly the same velocity, there will always be a point at which an infinite number of these trains, though not the whole number, will continue to agree in phase at a different phase. For example, taking into consideration only trains whose wave-lengths and wave-velocities are nearly equal, the space separating points of equal phase on any two of the trains increases continuously as we pass away on either side from the point where all the trains are in agreement of phase. Consequently, on one side of the point of agreement of phase all points on the trains having any specified phase must be approaching each other, while on the other side all points of equal phase are moving more and more apart. This is illustrated in the accompanying diagram, which shows two



trains T_1T_1 and T_2T_2 agreeing in phase at phase zero at point O. Each train moves rightwards at its own wave-velocity. If the wave-velocity for the trains gradually increases with increasing wave-length, then all points of equal phase on the left-hand side of O come in turn into coincidence as time goes on; and if the wave-velocity for the trains gradually diminishes with increasing wave-length, then all points of equal phase on the right-hand side of O come in turn into coincidence, and then continue to move farther and farther apart forever. It follows, therefore, that for the infinite number of trains which we are considering, all nearly of the same velocity and wave-length, there will always be a point at which an infinite number of them will agree in phase, provided one such point exists initially. In general, the phase at which the agreement occurs, and the point of the medium at which the agreement occurs, alter continuously with the time.

§ 9. Now when an infinite number of trains of nearly equal velocity are in agreement of phase at any point, the sum of their effects must determine very approximately the displacement of the medium at that point. For the remaining trains are infinite in number and of all possible phases, and we shall therefore assume for the present that their effects counterbalance each other, as is the case initially with all the trains at

every point except the origin; leaving our assumption to be tested afterwards by results. This is equivalent to saying that the main effect at any point is produced by a small portion of the total number of trains, whose wave-velocities are nearly equal and whose points of equal phase coincide with the point of the medium considered. Other trains, whose phases at the point considered are nearly equal to that of the trains whose phases coincide there, contribute effectively to the resultant disturbance; but the effects of the remaining trains disappear by mutual interference. As we pass from point to point of the medium at any time, the mean period of the effective trains at each point, and the phase at which their main agreement occurs, vary continuously. We can therefore speak of a certain wave-period which is the mean period of all the trains whose coincidence of phase at a given point determines the displacement of the medium at that point, as the predominant period at that point; and our problem is to determine at what point of the medium any specified wave-period will be the predominant period at any time.

§ 10. If k be the mean period of an infinite number of trains of waves whose velocities are nearly equal to the wave-velocity corresponding to k , we require to know at what point of the medium these trains will agree in phase at any time. The equation to any particular phase may be written in the form

$$k\{x - tf(k)\} = c \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where c is a constant; and the distance δx between a point of phase c on any of the neighbouring trains and the point x is given by the equation

$$k\delta x + \{x - tf(k) - tkf'(k)\}\delta k = 0 \quad . \quad . \quad . \quad . \quad (3).$$

From this we see that the distance δx between points of equal phase is zero for all values of δk less than a certain value, provided

$$x - tf(k) - tkf'(k) = 0 \quad . \quad . \quad . \quad . \quad . \quad (4).$$

This equation therefore determines the place at which k is the predominant period at time t ; and it may be written in the form

$$x = \{f(k) + kf'(k)\}t = Ut \quad . \quad . \quad . \quad . \quad (5),$$

where U is the group-velocity corresponding to the wave-period k . We may define the group-velocity as a function of the wave-length λ which determines the velocity of a point coinciding at each instant with the point of agreement of phase of the infinite number of trains of wave-lengths very slightly differing from λ . The group-velocity for the mean wave-length of a very large number of trains of nearly equal wave-lengths might more accurately be termed the velocity of their coincident phase, or simply the

coincident-phase-velocity or the stationary-phase-velocity for that wave-length.

§ 11. At present we have established an equation relating to the Fourier trains effective at each point of the medium, but nothing definite regarding the resultant wave-form of ξ . Indeed, it is evident from the above discussion that the resultant wave-form may be something differing considerably in each part from the constituent wave-trains which predominate at different points along it. It is to the constituent Fourier trains that the idea of group-velocity primarily applies, and we cannot speak of a group-velocity with reference to an endless succession of regular periodic waves; that is, to a single train. We may remark that, so far as the group-velocity relates to the resultant wave-system it refers essentially to each single point of the system and not to an extended succession of waves. Before we can arrive at the conditions under which several consecutive wave-lengths of the resultant disturbance have the same group-velocity, it is necessary to find the relation between the predominating wave-period and the displacement ξ at each point of the medium.

§ 12. The above presentation of group-velocity is effective in showing the intimate relation existing between group-velocity and dispersion. Dispersion is in fact the result of the gradual separation of the points of predominance of trains of nearly equal wave-length and wave-velocity. This is clearly illustrated in our problem of § 7. Initially, the points of predominance of all the trains coincide at the origin; immediately after, the points of predominance of very quickly moving trains occupy the most distant points of the medium, each having travelled out at its own coincident-phase-velocity. The separation out of the closely packed predominant points from the neighbourhood of the origin becomes continuously more and more complete for the less quickly moving trains, and the same process of separation goes on in the front and rear of each predominant point as it moves uniformly forward at its coincident-phase-velocity. Returning to equation (4) above, we see that the points of predominance corresponding to the two wave-periods k and $k + \delta k$ are continually separating from each other at the rate v given by the equation

$$v = \frac{dU}{dk} \delta k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

which shows that the points of predominance of wave-periods intermediate to k and $k + \delta k$ are being spread over an ever-increasing length of the medium. The extent of the medium occupied by these points at time t is given by

$$vt = t \frac{dU}{dk} \delta k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7):$$

an amount which becomes appreciable as time goes on, no matter how small δk may be, provided $\frac{dU}{dk}$ is not zero. If $\frac{dU}{dk}$ is zero, then all wave-periods whose points of predominance are initially in coincidence or almost in coincidence maintain the same relation to each other throughout all time, and there is consequently no dispersion. This will be illustrated later for media for which $V = a + b/k$, where a and b are constants.

§ 13. Now, according to the principle of stationary phase adopted in §§ 8–10 above, only trains whose points of predominance are in the immediate neighbourhood of a point x contribute effectively to the resultant displacement ξ at x ; hence we arrive at the conclusion that when the process of dispersion just described is sufficiently far advanced we can obtain the resultant displacement at any point by considering only the effects of trains differing infinitely little in period from the predominant period at the point. In the early stages of dispersion, when points of predominance of widely differing wave-periods are very near one another, the resultant wave-curve ξ near any point x will in general only very roughly correspond in wave-length and wave-period to the wave-period of the Fourier trains effective in the neighbourhood. It will now be shown that the correspondence becomes more and more pronounced as time goes on; and we shall arrive at an understanding as to how group-velocity is to be applied to the resultant wave-curve. We shall find that the process described in § 12 leads ultimately to the following result referred to by Professor Lamb: “In a medium such as we are considering, where the wave-velocity varies with the frequency, a limited initial disturbance gives rise in general to a wave-system in which the different wave-lengths, travelling with different [constant] velocities, are gradually sorted out,”* and arrive at any given point in the order corresponding to their group-velocities. Each separate crest or trough, however, moves with continually increasing or continually diminishing speed.

§ 14. Returning now to equation (1), we assume that the dispersion is so far advanced that the phases of the effective Fourier trains in the neighbourhood of a point x are determined with sufficient accuracy by two terms of a Taylor’s series. Thus, taking k_0 as the predominant period at x at time t , we have by equation (4)

$$x - tf(k_0) - tk_0 f'(k_0) = 0 \quad . \quad . \quad . \quad . \quad (8);$$

and by Taylor’s theorem we have

$$k\{x - tf(k)\} = k_0\{x - tf(k_0)\} + (k - k_0)\frac{d}{dk_0}k_0\{x - tf(k_0)\} + \frac{(k - k_0)^2}{2}\frac{d^2}{dk_0^2}k_0\{x - tf(k_0)\} \quad (9).$$

* *Hydrodynamics*, 3rd ed., § 234.

Using now equation (8) in this, we may write it in the form

$$k\{x - tf(k)\} = k_0\{x - tf(k_0)\} + \frac{(k - k_0)^2}{2}\{-2f'(k_0) - k_0f''(k_0)\}t \quad . \quad (10).$$

In general, for the evaluation of the integral in equation (1) according to the method used by Lord Kelvin in his paper of 1887 our assumption is that the dispersion is exceedingly far advanced, and t therefore so great that the term $\frac{(k - k_0)^2}{2}\{2f'(k_0) + k_0f''(k_0)\}t$ is very large for the greatest and least values of k considered. If the dispersive medium is such that equation (10) is satisfied for all values of k , it is unnecessary to assume t to be very large, as $(k - k_0)$ may be taken as large as we please positively and negatively. In either case, therefore, when we transform equation (1) by the substitution

$$z^2 = \mp \frac{(k - k_0)^2}{2}\{2f'(k_0) + k_0f''(k_0)\}t \quad . \quad . \quad . \quad (11),$$

where the sign taken is such as to make z^2 positive, the limits of z are practically $+\infty$ and $-\infty$, and the value of ξ is given by

$$\xi = \frac{2^{\frac{1}{2}} \int_{-\infty}^{+\infty} dz \cos [k_0\{x - tf(k_0)\} \pm z^2]}{2\pi t^{\frac{1}{2}} [\mp \{2f'(k_0) + k_0f''(k_0)\}]^{\frac{1}{2}}} \quad . \quad . \quad . \quad (12).$$

By means of the integrals

$$\int_{-\infty}^{+\infty} dz \cos z^2 = \int_{-\infty}^{+\infty} dz \sin z^2 = \sqrt{\frac{\pi}{2}} \quad . \quad . \quad . \quad (13)$$

the above equation ultimately reduces to

$$\xi = \frac{\cos [k_0\{x - tf(k_0)\}] \mp \sin [k_0\{x - tf(k_0)\}]}{2\pi^{\frac{1}{2}} t^{\frac{1}{2}} [\mp \{2f'(k_0) + k_0f''(k_0)\}]^{\frac{1}{2}}} = \frac{\cos \left[k_0\{x - tf(k_0)\} \pm \frac{\pi}{4} \right]}{2^{\frac{1}{2}} \pi^{\frac{1}{2}} t^{\frac{1}{2}} [\mp \{2f'(k_0) + k_0f''(k_0)\}]^{\frac{1}{2}}} \quad (14),$$

which has been verified by Lord Kelvin for water-waves in his paper, and proved to be in agreement with Cauchy and Poisson's result for t very great compared with x . It is easy to verify that the result holds for all values of t and x in the case of flexural waves in an elastic bar.

§ 15. In this result the terms depending upon k_0 vary very slowly in the neighbourhood of x , and the ξ curve is nearly a simple sine curve there. Equation (14) therefore proves that in the case which we are considering, where the dispersion is fairly well advanced, the resultant displacement curve near any point of the medium at any time is a curve of exactly the same

wave-length and period as the effective Fourier trains which predominate in the neighbourhood. It is clear also, from our equation or from the considerations of § 12, that the correspondence of wave-lengths referred to becomes more and more perceptible and the approach to a regular sine curve in each neighbourhood more and more close as time goes on; for the predominant period of the Fourier trains varies more and more slowly as we pass along the ξ curve at later and later times. In this case therefore the demonstration of a coincident-phase-velocity for the constituent wave-trains carries with it a demonstration of group-velocity for each part of the resultant wave-curve where a definite wave-length is observed. If, following Professor Lamb, we understand by "group" in this connection "a long succession of waves in which the distance between successive crests and the amplitude vary very slightly," our result would mean that, whereas a certain wave-length λ is observable in the group at place x at time t , this particular wave-length will be found at any later time t' at a place x' given by the equation

$$x' = x + U(t' - t) \quad . \quad . \quad . \quad . \quad . \quad (15),$$

where U is the group-velocity corresponding to the wave-length λ . In this sense we can speak of a group-velocity with reference to each small part of the original "group," each part having a slightly different group-velocity from contiguous parts; but the "group" as a whole has not any definite "group-velocity," unless we define its group-velocity to be the mean value of the group-velocities of its parts. Such a "group," however, quickly loses definite marks by which it could be recognised on account of its continual extension by the process of § 12.

§ 16. It is important to observe from equation (14) that each wave of the resultant displacement curve moves at each instant with the velocity corresponding to its length. Thus in his paper of 1887, referred to in § 6 above, Lord Kelvin remarks: "The result of our work will show us that the velocity of progress of a zero, or maximum, or minimum, in any part of a varying group of waves is equal to the velocity of progress of periodic waves of wave-length equal to a certain length, which may be defined as the wave-length in the neighbourhood of the particular point looked to in the group (a length which will generally be intermediate between the distances from the point considered to its next neighbour corresponding points on the preceding and following waves)." To illustrate this for any medium in which the group-velocity is positive, let us take $V = Ak^n$, where n may be positive or negative, but not less than -1 . Equation (8) becomes in this case

$$x = (1 + n)Ak_0^n t = Ut \quad . \quad . \quad . \quad . \quad . \quad (16);$$

and when we eliminate k_0 from the argument of the cosine in equation (14) by means of this equation, the argument θ may be written in the form

$$\theta = B \frac{x^{\frac{n+1}{n}}}{t^{\frac{1}{n}}} \pm \frac{\pi}{4} : B = \frac{n}{(1+n)^{\frac{n+1}{n}} A^{\frac{1}{n}}} \quad \dots \quad (17),$$

where the negative sign is taken when n lies between 0 and -1 . If we follow the crest of a wave whose equation is given by

$$\theta = 2r\pi \quad \dots \quad (18),$$

we can find the velocity of this crest by differentiating (17). Thus we have

$$V = \frac{dx}{dt} = \frac{1}{n+1} \cdot \frac{x}{t} \quad \left. \vphantom{\frac{dx}{dt}} \right\} \quad \dots \quad (19);$$

$= Ak_0^n$ by (16)

Since the phase varies by 2π as we advance one wave-length along the ξ curve, the length λ of the wave whose crest we are following is given approximately at any time t by the equation

$$\frac{B}{t^{\frac{1}{n}}} \left\{ \left(x + \frac{\lambda}{2} \right)^{\frac{n+1}{n}} - \left(x - \frac{\lambda}{2} \right)^{\frac{n+1}{n}} \right\} = 2\pi \quad \dots \quad (20).$$

With λ small compared with x , this gives finally by means of equation (18)

$$\lambda = \frac{n}{n+1} \cdot \frac{x}{\left(r \mp \frac{1}{8} \right)} \quad \dots \quad (21).$$

These equations show that each wave lengthens as it proceeds, and that its velocity alters accordingly. If the wave-velocity increases with increasing wave-length, each wave is continually accelerated; if the wave-velocity diminishes with increasing wave-length, each wave-crest moves with constantly diminishing speed. The individual waves which at any time constitute the part of the disturbance which has a certain wave-length λ immediately pass ahead of this part, if the medium is one in which the wave-velocity exceeds the group-velocity, or fall behind it if the group-velocity exceeds the wave-velocity for that wave-length. In the former case succeeding waves in turn become of length λ ; in the latter case, preceding waves in turn become of length λ . A point which moves so as to coincide at each instant with the point of the wave-system at which the wave-length is λ travels at the coincident-phase-velocity corresponding to that wave-length.

§ 17. At this stage it is interesting to examine the coincident-phase-velocity with reference to a disturbance which can be analysed into a limited number of wave-trains. The most interesting case is that of two

interfering trains, which was dealt with first by Stokes, and afterwards by Lord Rayleigh in § 191 of his *Theory of Sound*, vol. i. In Lord Rayleigh's notation the two interfering trains are represented by

$$\cos 2\pi\left(\frac{t}{\tau} - \frac{x}{\lambda}\right), \quad \text{and} \quad \cos 2\pi\left(\frac{t}{\tau'} - \frac{x}{\lambda'}\right);$$

where τ, τ' are the wave-periods, and λ, λ' the wave-lengths of the trains. At the origin, which may be chosen at the point of maximum displacement of any one of the wave-groups which constitute the initial disturbance, the phases of the two trains are initially zero; and the point at which agreement of phase occurs at any later time is given by the equation

$$\frac{t}{\tau} - \frac{x}{\lambda} = \frac{t}{\tau'} - \frac{x}{\lambda'} \quad \dots \quad (22),$$

which corresponds to equation (4) above. It may be written in the form

$$x\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = t\left(\frac{1}{\tau} - \frac{1}{\tau'}\right) \quad \dots \quad (23),$$

from which we obtain the coincident-phase-velocity U , as

$$\left(\frac{1}{\tau} - \frac{1}{\tau'}\right) \div \left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) \quad \dots \quad (24).$$

When the trains are of nearly the same wave-period and wave-length, this becomes $\delta\left(\frac{1}{\tau}\right) / \delta\left(\frac{1}{\lambda}\right)$, which is the form given by Lord Rayleigh.

§ 18. With reference to this case, the view of the matter here adopted frees us from what is in general a misleading conception in connection with group-velocity, namely, the reappearance of the entire group of waves at regular intervals. Thus Professor Schuster, referring to an extended succession of waves, in his article "On Interference Phenomena," remarks: "We can speak of definite rate of propagation U of the group, because at definite intervals τ the group takes up the same shape, displaced through a distance $U\tau$." Our investigation proves beyond doubt that the reappearance of the same shape at regular intervals in a dispersive medium is confined to the particular case of a disturbance represented by two interfering trains. The original shape of any disturbance may maintain itself unchanged, or may reappear at regular intervals, in a medium for which the coincident-phase-velocity is constant or zero for all wave-periods; but, as we have found in § 12, in such media there is no dispersion. This will be illustrated in a later paper. The process of separation of predominant points referred to in § 12 above essentially involves continuous non-periodic change in the shape of any initially given group of waves.

§ 19. The case of three, four, or any finite number of interfering trains

follows easily from the case of two trains; for the coincident-phase-velocity differs for each pair of trains, and, except for occasional agreement of phase of three or more trains, no new features occur. When the number of trains is exceedingly large, and their wave-lengths and wave-periods vary very slightly, as in the explanation of group-velocity given by Gouy and Lord Rayleigh,* each wave-train has its own particular point of predominance in the wave-system; which point moves at its own coincident-phase-velocity. The argument is, in fact, exactly the same for a disturbance represented by

$$\xi = \sum C \cos k\{x - tf(k) + \epsilon\} \quad . \quad . \quad . \quad . \quad (25)$$

as that given in §§ 8-10 above; the point of the medium at which the wave-period k predominates being again found from equation (4). Here the number of terms is exceedingly large, and C and k vary continuously. The general features of propagation of the disturbance are the same as in the case of a single initial impulse. The wave-system is continually being modified by the process of § 12, and the resultant curve ultimately takes a shape in each neighbourhood corresponding in wave-length to the component Fourier trains which predominate there. As time goes on, the rate of change of the wave-length as we pass along the resultant curve becomes more and more gradual, and the whole disturbance occupies an ever-increasing extent of the medium on account of the difference in the coincident-phase-velocities maintaining throughout the wave-system.

§ 20. The argument regarding group-velocity contained in §§ 7-14 above has been confined to the particular case of the wave-system arising from an infinitely intense disturbance at a single point of a dispersive medium, but it is evident throughout the later parts of the discussion that the results arrived at are of much more general application. Group-velocity has been shown to depend upon the principle of "stationary phase," which, it seems reasonable to assume (see § 9), can be applied to any infinite combination of Fourier trains. It is practically Huygens' principle in optics. The agreement in wave-length between the effective Fourier trains and the resultant displacement curve at any point of the medium, on which the application of the theory of group-velocity to the resultant curve depends (§§ 14, 15), requires only that the predominant wave-period should vary continuously in the neighbourhood of the point considered. The condition that t be very great is required in § 14 to enable us to completely evaluate the integral in equation (1), but is unnecessary to prove the correspondence of wave-lengths referred to, which follows at once from considerations of continuity by Taylor's theorem. In general, at only a short interval after a disturbance the

* Lamb's *Hydrodynamics*, 3rd ed., § 234.

separation of predominant points will be sufficiently advanced to allow us to recognise a distinct group-velocity in the various parts of the wave-system.

This makes it possible to apply the theory of group-velocity to any wave-system, provided the word "group" is understood to mean any part of the whole disturbance which has a specified wave-length. In general, the "group," as here defined, is confined to the immediate neighbourhood of a single point, which moves uniformly forward; but when the wave-length varies very slowly from point to point, the group may be taken in a modified sense to refer to an extended portion of the wave-system where the wave-length is nearly constant, provided we remember that such a group would be continually increasing in length as it proceeds, and that its recognition soon ceases to be useful.

§ 21. We can now obtain from the theory of group-velocity a useful general understanding of the way in which any disturbance initially confined to a small portion of a dispersive medium is propagated into regions initially undisturbed. We may arrive at the effect of any initial disturbance by summing the effects due to point displacements applied at each point of the disturbed region; but it is more convenient from our present standpoint to regard any initial displacement of the medium as due to a certain distribution of predominant points. Since the initial disturbance is limited in extent, it is clear that the Fourier trains into which it can be analysed must be infinite in number and of all possible wave periods, and we suppose that their points of predominance are arranged irregularly but continuously within the disturbed part of the medium. At any very short time after motion has commenced, an effect will have been produced in the most distant parts of the medium by the very quickly moving trains whose points of predominance have moved out in the interval. Near the place of the original disturbance the slowly moving trains still agree in phase, but irregularities in amplitude and wave-length must occur owing to the predominant points of the quicker trains overtaking and out-stripping those of the slower. Indeed, it seems certain that irregularities in amplitude will persist, according to the manner in which the *energy* is distributed initially among the effective Fourier trains.† But, as time advances, the various trains sort themselves out according to their coincident-phase-velocities, and the wave-length of the resultant curve ultimately varies regularly and continuously as we pass outwards on either side from the place of the initial disturbance. At this stage the continuity of the disturbance allows us to regard the distance between corresponding points on two consecutive half-waves as equal to a half-wave-length of the train which has its point

† Burnside, *Proc. Lond. Math. Soc.*, t. xx. p. 22, 1888.

of predominance somewhere between them. Thus if we call twice the distance between any two consecutive zeros of the resultant wave-system the perceptible wave-length in their neighbourhood, the perceptible wave-length will agree very closely with that of the wave-train which predominates at the point of maximum displacement between the zeros observed; and the further progress of the dispersion may be described as a gradual increase in the extent of the medium sensibly disturbed, accompanied by a continual diminution in the rate of change of the perceptible wave-length from point to point of the system, owing to the process indicated in § 12. This is exactly the result described by Professor Lamb, quoted in § 13 above. More detailed description would require a knowledge of the initial distribution of energy and of the law of diffusion of energy throughout the medium.

§ 22. The case of an initial disturbance consisting of a large succession of equal, regular waves of wave-length λ , with undisturbed space in front and rear, is interesting in connection with the dynamical interpretation of group-velocity given by Osborne Reynolds and Lord Rayleigh, and on account of its importance in relation to sequences of light-waves in dispersive media. The mathematical expression representing such a group of waves would again involve all possible wave-periods, and we may state our initial conditions by saying that the wave-length λ predominates in the part of the medium initially occupied by the group. According to the theory of group-velocity presented above, at any time t after the commencement of motion there will be a succession of waves of wave-length λ displaced on either side from the place of the original disturbance a distance Ut , where U is the group-velocity corresponding to wave-length λ ; but in front of the waves of length λ there will have appeared a continuous wave-disturbance, of which the wave-length increases continuously as we pass farther and farther ahead if the medium is such that the coincident-phase-velocity of the wave-trains increases with increasing wave-length, and of which the wave-length diminishes continuously if the coincident-phase-velocity diminishes with increasing wave-length. Behind the main group of waves of length λ there will also have appeared a continuous wave-disturbance of which the wave-length either decreases or increases continuously as we pass farther and farther behind the group of waves of wave-length λ , according as the wave-length in front increases or diminishes. Each perceptible wave-length in the front or rear of the main group is always to be found in the wave-system at any time in advance of the place where it was observed at an earlier time by a distance corresponding to the group-velocity for that wave-length. The history of the individual waves is again that given in § 16. The process according

to which the disturbance is propagated throughout the medium is the process of separation of points of predominance of the constituent wave-trains, so that we can at once say that the disturbance which we are considering gives rise to a wave-system consisting of a succession of waves of all possible wave-lengths, each wave-length appearing in the succession according to the order of its group-velocity. The energy of the initial disturbance is ultimately diffused throughout the entire medium.

The "group," defined as the part of the whole disturbance which has the same wave-length as the original group, moves along at the group-velocity corresponding to its wave-length, being distinct for a time owing to the energy it retains. But the regularity of its shape cannot be maintained as it proceeds, for it must supply the energy necessary to feed the ultimately infinite succession of waves of greater and less wave-lengths which constitute its front and rear. As time goes on, therefore, a falling off from sinusoidality, which proceeds inwards from the front and rear of the group, must become more and more evident as the front or rear increases in importance; and the amplitude of the sensibly regular central part must in time diminish. Thus it would seem that we cannot expect perfect regularity to be maintained in any part of a finite group for any time, however short. It is to the part of the group which remains sensibly regular that the dynamical theorem given by Lord Rayleigh applies. The law of diffusion of energy towards the front or rear of the wave-system essentially involved in the process of dispersion described in § 12 does not seem to be easily derived from the dynamical theory of group-velocity as hitherto developed. Unless we can arrive at the law of falling off from regularity of the main group in the front and rear, it seems impossible to follow the distribution of energy throughout the entire system. The kinematical group-velocity theory, however, accounts satisfactorily for the speed at which a group of waves of sensibly the same length advances, and explains in a general way the process by which any initial disturbance is modified in invading undisturbed space.

§ 23. The formation and development of the front and rear of a large initially regular procession of waves in deep water, in accordance with the theory of group-velocity given above, is illustrated by the diagrams of Lord Kelvin's paper, *Proc. Roy. Soc. Edin.*, vol. xxxv., 1904, figs. 9 and 10. For water-waves, the group-velocity $\frac{1}{2}\sqrt{\frac{g\lambda}{2\pi}}$ increases with increasing wave-length, so that at any time long wave-lengths appear in the front and short wave-lengths in the rear of the main group, which, in the case treated by Lord Kelvin, consists initially of a large procession of waves of

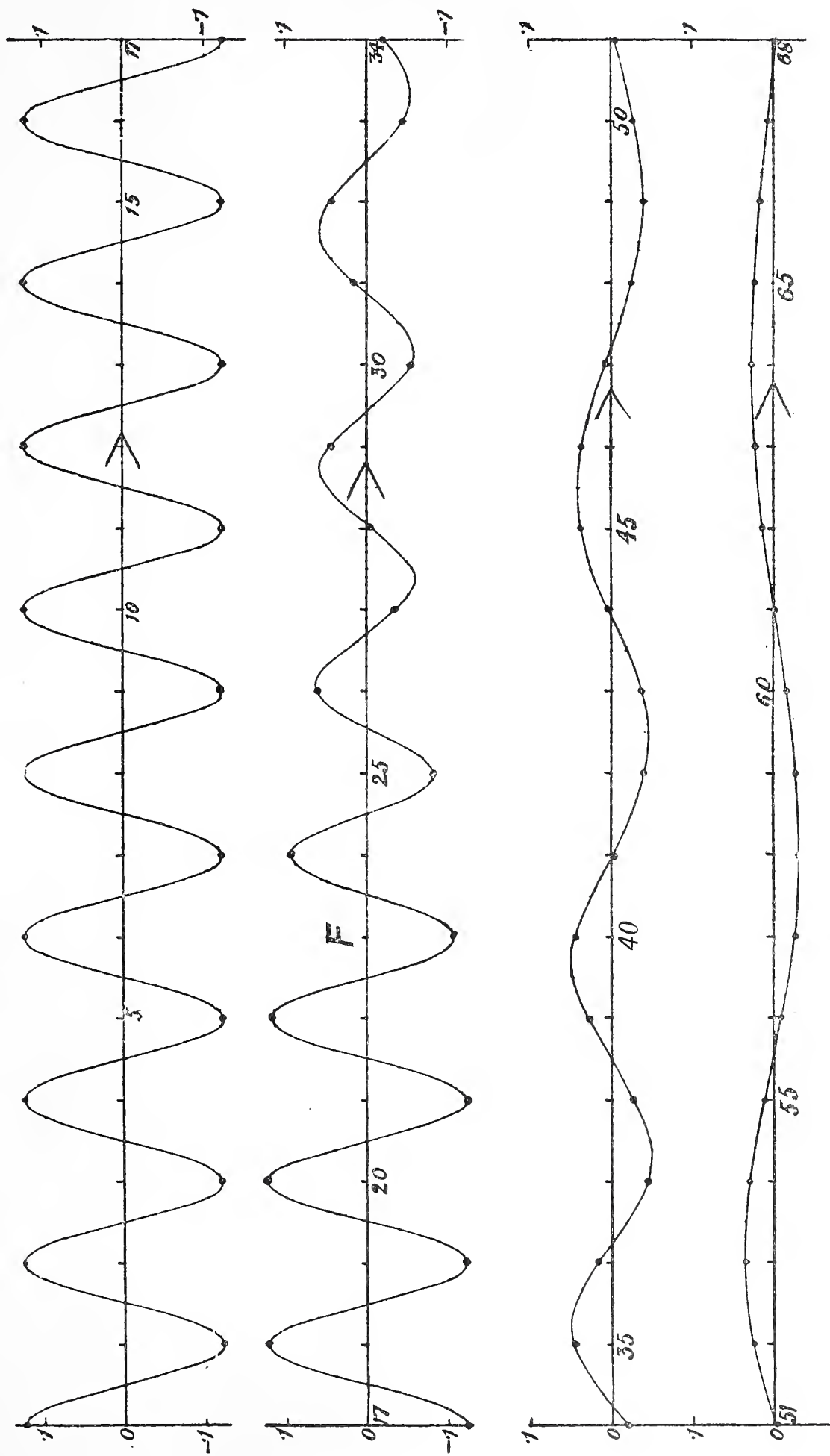


FIG. 9 ; Head and front of rightward procession.

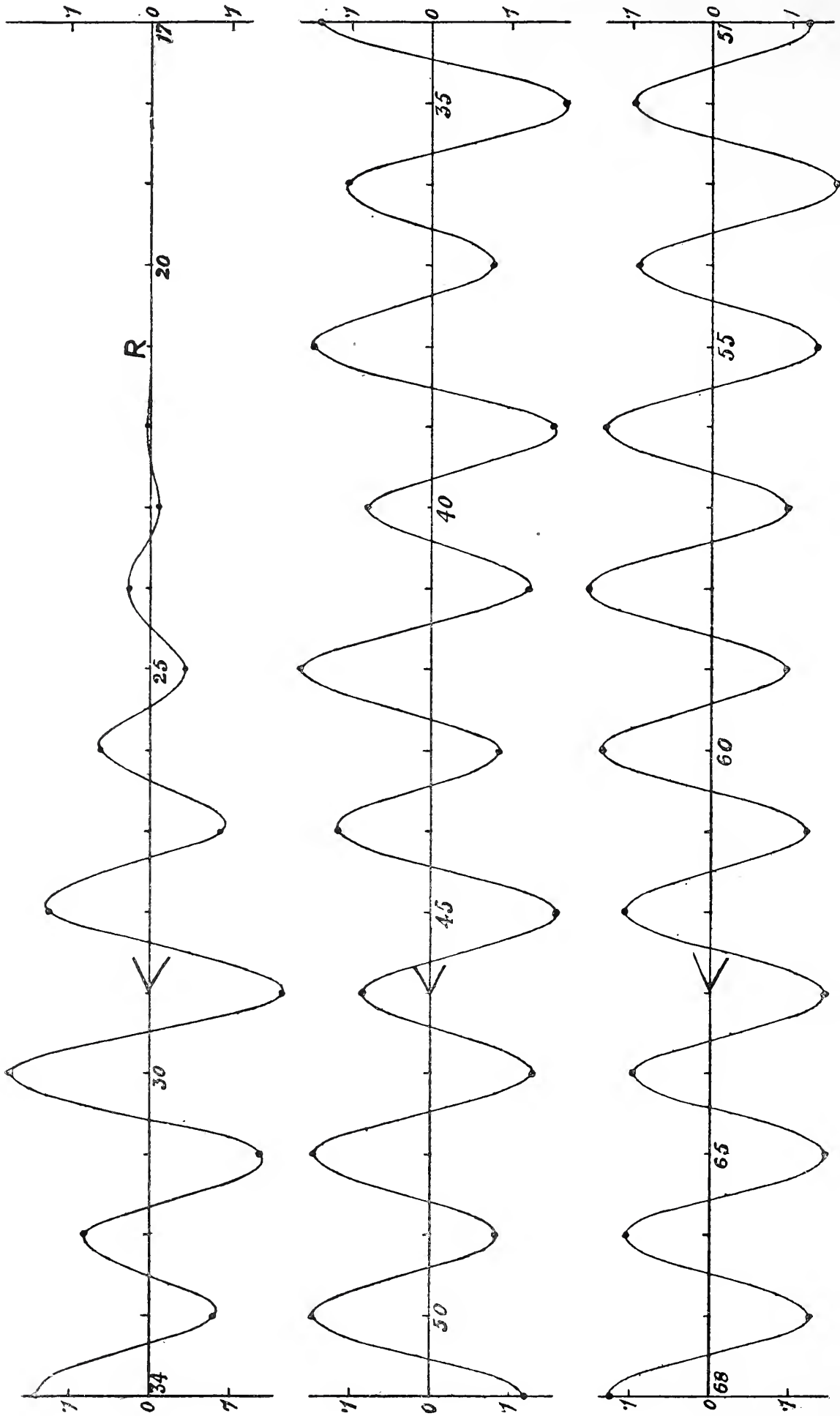


FIG. 10 ; Tail and rear of leftward procession.

wave-length 2 and period $\sqrt{\pi}$. Figs. 9 and 10 show the wave-system at time $t=25\sqrt{\pi}$. Fig. 9 shows the front of the wave-system which has formed while the original group of wave-length 2 has travelled from the origin to point 25 on the diagram, which is exactly the distance required by group-velocity theory, g being taken equal to 4 in Lord Kelvin's calculations. As an example of the application of group-velocity theory to the front of the wave-system, we may take the case of wave-length 6. The place in the wave-system where this wave-length, initially near the origin, should be observable at time $25\sqrt{\pi}$ is given by $x=25\sqrt{\pi}\times\frac{1}{2}\sqrt{\frac{4\times 6}{2\pi}}=43\cdot 3$, which agrees well with fig. 9. The individual waves forming the front initially belonged to the main group, and their places in the group have been taken by other waves, so that somewhere in the rear of the whole system, not indicated in the diagrams, fresh waves must be continually forming and then advancing with increasing length and speed towards the extreme front of the system. The point F on the diagram marks the place where the regularity of the main group perceptibly begins to fail. As we should expect from the continual advance of the individual waves through the group, the perceptible front is much more extensive than the perceptible rear, which is shown in fig. 10. The irregularity of the main group in the rear consists in the main in a variation in amplitude, without any falling off in the wave-length till we reach the extreme perceptible rear at R, beyond which we have a large number of imperceptible waves of continually diminishing wave-length. From fig. 10 we see that the rearmost wave of wave-length 2 at the time of the diagram has just reached the point 25, starting from point 0, which is in accordance with group-velocity theory. The perceptible rear never extends far beyond the last wave-length of the main group of wave-length 2, but the number of the perceptible waves in the front increases with the time, and the increase in wave-length from the regular waves forward becomes more and more gradual owing to the gradual increase in the group-velocity as we go towards the extreme front of the wave-system. We hope to give the law of falling off from regularity, as time goes on, at the front of a large group of sinusoidal waves, in a later paper dealing with the wave-system arising from a given distribution of pressure moving steadily over the surface of infinitely deep water.

§ 24. The diagrams of figs. 34 and 35 are taken from Lord Kelvin's last Waves paper, referred to in § 4 above. Fig. 34 deals with the case of a disturbance mainly confined to the neighbourhood of the origin; and all the diagrams are outlines of the water surface at the times indicated below each, $\sqrt{\pi}$ being the period of an infinite train of waves of wave-length 2

and velocity $2/\sqrt{\pi}$. The zeros are numbered in the order of their coming into existence, so that we can study the history of each perceptible wave, or of each group defined as the part of the whole disturbance which has a given wave-length. According to the argument of §§ 20, 21 above, the wave-trains are initially in agreement of phase near the origin, and, as their points of predominance separate from each other, the wave-length of the resultant curve in this case increases continuously as we pass from the middle outwards. It is clear from the diagrams that we can take twice the perceptible half-wave between any two zeros as the wave-length predominating somewhere near the point of maximum displacement between the zeros, this being a very close approximation when the wave-length does not vary quickly in the neighbourhood. The following table of numbers, calculated from the numerical results from which the diagrams were drawn wherever convenient, or taken direct from the diagrams themselves, shows that each perceptible wave-length of the disturbance appears at each time of observation at a distance from the place of the original disturbance corresponding to its group-velocity. Column 1 gives the diagram observed in each case; column 2 gives the zeros twice whose distance apart is used as the wave-length; column 3 gives the actual position of the maximum or minimum between the zeros, from the original calculations or from the diagrams; and column 4 gives its position approximately by the group-velocity theory of § 10, *i.e.* column 4 gives group-velocity multiplied by the time.

Diagram.	Zeros.	Position of Maximum.	
		On the Diagram.	By Group-Velocity.
2	1 and 2	1·3	1
3	1 „ 2	3	3·4
4	1 „ 2	5	7·3
4	2 „ 3	2	2·3
6	4 „ 5	19·8	19·9
6	5 „ 6	15·1	15·2
6	6 „ 7	12·2	12·2
6	7 „ 8	10·3	10·3
6	8 „ 9	8·8	8·8
6	9 „ 10	8·0	8·0

An examination of the curves makes it clear that the agreement is practically perfect where the change of wave-length between the zeros is slight; and when the change of wave-length is considerable, as in the foremost

Figure 34

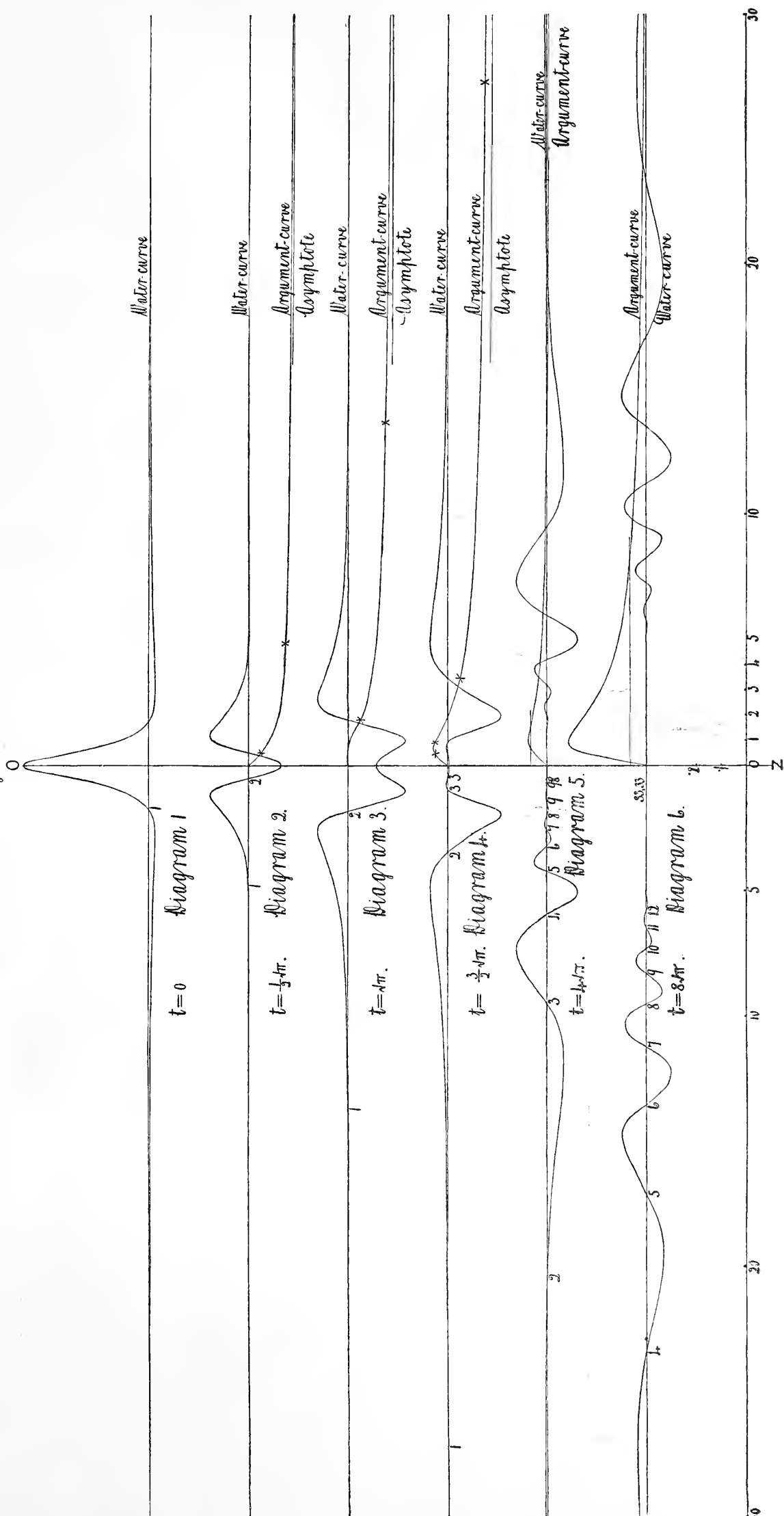
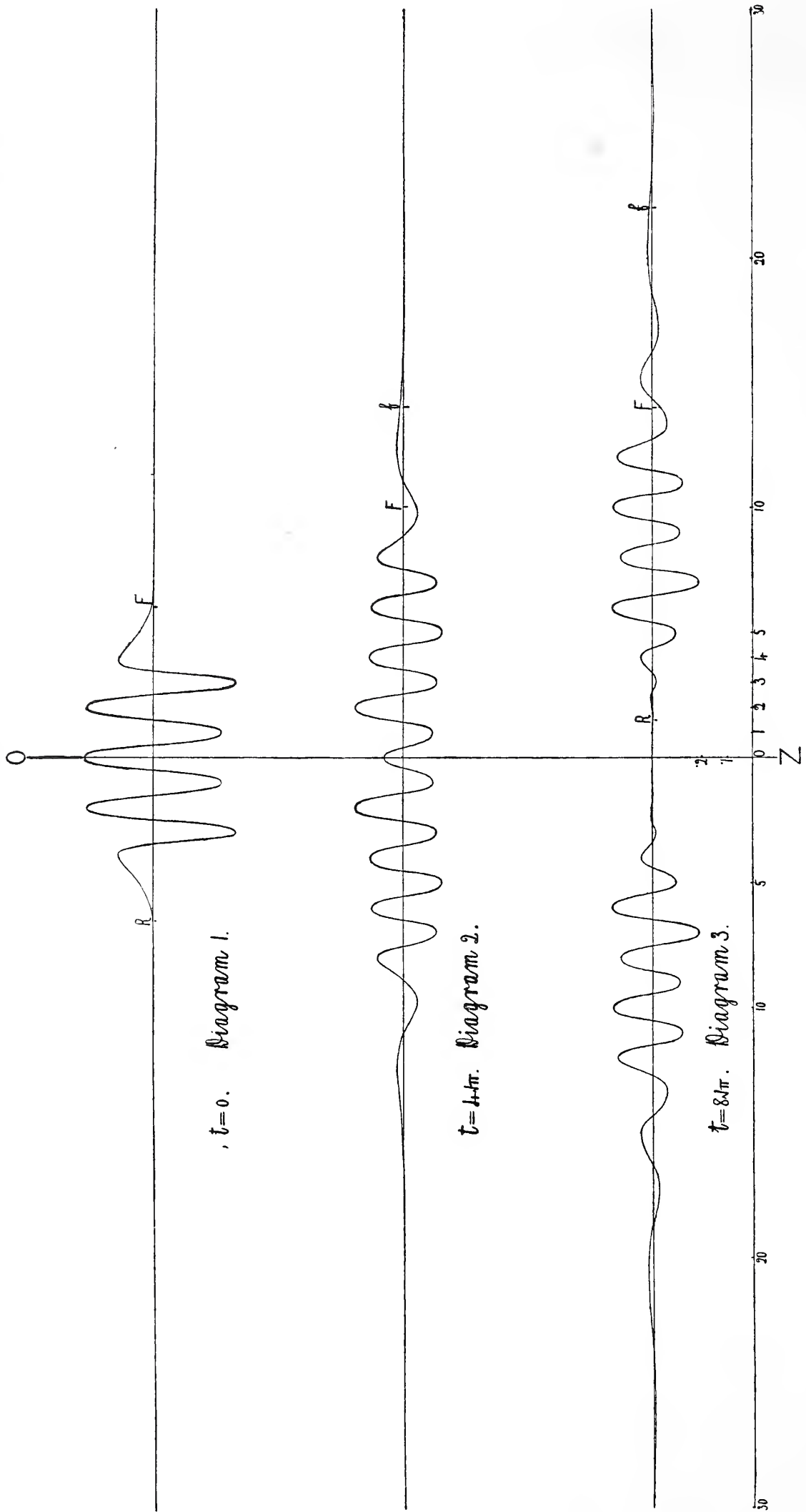


Figure 35.



Initial group of five elevations and four depressions emerging as two groups travelling in opposite directions.

waves at each time, the perceptible wave-length predominates at a point slightly beyond the maximum, which is what is to be expected. The whole progress of the disturbance is in entire agreement with the process described in § 21.

§ 25. Fig. 35 shows the manner in which a finite group of equal waves in deep water is modified as it proceeds, owing to the different coincident-phase-velocities of the component Fourier trains. The group of waves of wave-length 2 comprising the main "group" in diagram 1 can be observed in the two later diagrams displaced from the central position by an amount corresponding exactly to its group-velocity; while it is clear that the individual waves forming this group are continually passing forward to form the front. Each wave originates near the origin and moves through the whole system, lengthening as it proceeds, and carrying some of the energy of the original group forward to the front, which continually increases in importance relatively to the main group, and becomes perceptible at greater and greater distances beyond it. "While there is this great extension of the fronts outward from the middle, we see that the two groups, after emergence from coincidence in the middle, travel with their rears leaving a widening space between them of water not perceptibly disturbed, but with very minute wavelets in ever augmenting number following slower and slower in the rear of each group."* It is easy to verify from the diagrams that each perceptible wave-length appears in the wave-system at any time, displaced from the position at which it is observable at an earlier time by the distance corresponding to the group-velocity for that wave-length. As an example we may take the wave-length at the maximum point of the wave marked f in diagram 2. The particular wave at f in diagram 2 has disappeared beyond the diagram limits at the time of diagram 3, but the new wave at f in diagram 3 is of only slightly shorter wave-length and will suit for comparison. From the diagrams, the distance between the maximum points is about 8.4 units of the scale shown. The value of U for the perceptible wave-length at f is given by $U = \frac{1}{2} \sqrt{\frac{g\lambda}{2\pi}} = \frac{1}{2} \sqrt{\frac{4 \times 9.4}{2\pi}} = \frac{2.17}{\sqrt{\pi}}$, which gives $U(8\sqrt{\pi} - 4\sqrt{\pi}) = 8.7$.

Thus the wave-length at the maximum near f in diagram 2 corresponds to the wave-length appearing slightly beyond the maximum in diagram 3, as is to be expected from its slightly greater wave-length.

§ 26. The general features of all these water-wave disturbances are the gradual increase in wave-length of the disturbance as we pass outward from the place of the original disturbance, and the continually increasing importance of the front as compared with the rear. It seems certain that

* Lord Kelvin, *Proc. Roy. Soc. Edin.*, 1906.

similar features will appear in the case of any medium in which the wave-velocity is greater than the group-velocity for each wave-length and increases with increasing wave-length ($V = Ak^{-n}$, $n < 1$). But it is shown in an earlier paper (see *Proc. Roy. Soc. Edin.*, vol. xxix., 1909) that all the curves shown for water-waves may be used as illustrations of waves in an elastic rod, which is a medium in which the group-velocity exceeds the wave-velocity and the wave-velocity diminishes with increasing wave-length. Our diagrams therefore illustrate the process of dispersion for two distinct laws of dispersion, namely: $V = A_1 k^{-\frac{1}{2}}$, $U = \frac{1}{2} A_1 k^{-\frac{1}{2}}$; and $V = A_2 k$, $U = 2A_2 k$. For the first, each diagram shows the displacement at each point of the medium at a given time; for the second, each diagram shows the displacement at a given point of the medium from $t = 0$ to $t = \infty$. It is sufficient to point out the distinguishing features of the second case illustrated by the diagrams, which may be taken as typical of any medium in which $V = Ak^n$, $n > 0$. These are: the continuous increase in wave-length of the disturbance at each point of the medium as time goes on, and the continuously increasing importance of the rear of any disturbance as compared to the front, owing to the individual waves lagging behind the main group and retaining part of its energy. One interesting point of difference between the two cases illustrated lies in the manner of formation of additional waves. Thus in fig. 34 we see from the numbering of the zeros that they are continually being formed in pairs near the point $x = 1$: the outer zero travels outwards with constantly increasing speed, and the inner zero travels inwards with constantly diminishing speed. From $x = 0$ to $x = 1$ we have an ever increasing number of waves, each moving *inwards* with diminishing length and therefore also diminishing speed, and from $x = 1$ to $x = \infty$ we have an equal number of waves moving outward with increasing length and speed; each wave in the entire system moving at each instant with the velocity corresponding to its length. Only at $t = \infty$ is the number of water-waves infinite. The illustrations when applied to waves in an elastic rod, however, show that each zero is formed at infinity; and an infinite number are formed in quick succession at the commencement. For a short time the waves formed all move *inwards* from infinity towards the part of the medium initially disturbed, lengthening and slowing down as they come inwards. Very soon the inward motion ceases, and all continue for ever afterwards moving outwards with increasing length and diminishing speed. At all times the wave-length diminishes from the middle outwards, and at each point it increases continuously with the time: in both respects exactly the reverse occurs in the case of water-waves. In both cases the flow of energy is outwards from the place of the original disturbance at all times and places, even when the

individual waves and wave-groups travel inwards—an interesting result in connection with the dynamical theory of group-velocity.

§ 27. Lord Kelvin's investigation of 1887 therefore shows that the mathematical theory of group-velocity explains the *modus operandi* of dispersion. In relation to sequences of light-waves,—perhaps one of the most important applications,—where we are entirely concerned with the wave-length, and not with the individual waves, the theory emphasises the importance of group-velocity in all cases of refraction. Lord Kelvin's diagrams not only illustrate the manner in which group-velocity is concerned in the formation of a “front” and “rear,” and in the propagation of an initially finite disturbance, but also show how the velocity of the individual waves, according as it exceeds or falls short of the corresponding group-velocity, influences the distribution of the energy of the initial disturbance among the various wave-lengths of the disturbance throughout the medium. This is of importance whether we are concerned with the maintenance of distinct marks by which a group of waves may for a time be recognised, or merely with the energy contained in the part of a disturbance having a particular wave-length, which is of interest in connection with the theory of radiation. The extreme smallness of the amount of energy contained in the very short waves is a feature of all the diagrams. Thus in fig. 34 the energy of the initial disturbance would need to be very great if we are to regard the energy of even the largest imperceptible wave in diagram 6 as sensible.

§ 28. It is not intended in this paper to enter into a discussion of particular applications of the theory of group-velocity presented above, but it may be useful to point out in conclusion one application of importance in the theory of radiation, which is suggested by § 27. This is to the case of a black body, say a plate of metal, which is slowly heated on one side. At first only long heat-waves are observed, but shorter and shorter heat-waves are emitted as the temperature rises. At about temperature 525° C. the first visible rays appear, and the plate becomes red, owing to the long light-waves it sends out. As the temperature is further raised, shorter and shorter wave-lengths appear in appreciable amount, and the colour of the plate changes from red to yellow, and finally to white.

The points of importance in relation to the diagrams are that the various wave-lengths are emitted in the order corresponding to their group-velocities, and that the energy of the disturbance required to make each wave-length perceptible is greater the shorter the wave-length; exactly as in the continuous *water-spectra* exhibited in the diagrams of fig. 34. It is interesting to find the idea underlying this application expressed in the

following passage, quoted from Professor Schuster's article "On Interference Phenomena" referred to above:—"We cannot help speculating as to the ultimate cause which renders the regularity of vibration a function of the temperature only, and independent of the natural periods of the molecules. Perhaps the solution of the difficulty will be found in the fact that our observations tell us nothing directly as to the vibrations of the atoms or molecules. What we observe is the disturbance of the medium; and the distribution of energy in the spectrum of an incandescent black body which is in thermal equilibrium may indicate a property of the medium rather than that of matter. That is to say, the motion of vibration in the molecule may be perfectly irregular, but the medium may take up and propagate some vibrations quicker than others."

§ 29. In a later paper I hope to illustrate further the process of dispersion by showing the wave-systems arising from the same initial disturbance in several media having different dispersive qualities.

(Issued separately July 16, 1909.)

XXX.—On a Simple Radioscope and a Radiometer for showing and measuring Radioactivity. By Dr John Aitken, F.R.S.

(MS. received May 1, 1909. Read May 17, 1909.)

WHILE working at another subject I evolved a very simple piece of apparatus which on trial was found to be very sensitive to the penetrating rays of radioactive substances, so I have recently diverted my work from its original direction to study and improve this piece of apparatus. The principle of the action of the instrument is not new, as it depends on the fact discovered by Professor C. T. R. Wilson,* that ions in air supersaturated with water vapour become centres of condensation when the supersaturation is sufficiently great.

Before going further it may be as well that a few remarks be made on the condensation of water in supersaturated air. If ordinary air saturated with water vapour be cooled by expansion, the vapour, it is well known, condenses on the dust particles in the air and a fog is formed. This fog is the denser the more numerous the dust particles, and if the particles be few only a rainlike condensation results on expansion; and if no dust particles be present, then no condensation takes place, unless the expansion be great. In my early experiments on this subject it was shown † that only a very slight expansion was necessary to make all the dust particles active as centres of condensation, an expansion of $\frac{1}{50}$ being sufficient to cause condensation to take place on even the smallest of them. It was further shown that higher expansions than $\frac{1}{50}$ might be made without any condensation taking place, but if the expansion was great, and caused to take place very rapidly, then condensation took place in dustless air; but the subject was not further investigated. Here the matter rested till 1897, when C. T. R. Wilson took up the investigation and by means of most ingenious apparatus, in which the air was very rapidly expanded, he showed that there were always nuclei in moist air that became active centres of condensation when the supersaturation was great enough. If v_1 be the initial volume of the air and V_2 the final volume after expansion, he showed that condensation began in dustless air when $v_2/v_1 = 1.250$. When the expansion is that amount the condensation is slight; that is, only a few cloud particles are seen falling as a fine rain, and that as the

* *Trans. Roy. Soc.*, Series A, 1897.† *Trans. Roy. Soc. Edin.*, vol. xxxv. part i., 1888.

expansion is increased the number of drops increases slightly. This increase goes on slowly till a certain degree of expansion is reached, when the appearance of the condensation changes somewhat suddenly, the rainy condensation changing to dense fog; that is, with the lower expansions the particles are few, large, and fall quickly, while in the second stage they are extremely numerous, very small, and fall very slowly. This sudden change in the nature of the condensation takes place when $v_2/v_1 = 1.38$. With still higher expansions the number of particles becomes greater; that is, the fog gets denser.

In Wilson's apparatus, as the expansion is extremely rapid, the degree of expansion may be taken as a measure of the supersaturation, and as the different instruments used by him give almost the same values for v_2/v_1 , we may take it that they are correct. In the instruments here to be described the values of v_2/v_1 are different from Wilson's, and they differ with each other owing to the conditions not being so simple as in Wilson's. But any difficulties one may feel about this will disappear if we will keep in mind that it is the degree of supersaturation and not the amount of expansion which determines the result. Supersaturation appears to depend greatly on the quickness of the expansion. A very slight slowing of the velocity of expansion, even though to the eye there may seem to be no change, requires a decided increase in the expansion. Part of this is due to heat exchanges between the air and the walls of the vessel and to radiation; but in these experiments with high supersaturations a greater part is due to the manner in which the nuclei relieve the vapour tension. If the expansion be not instantaneous, then some of the nuclei, being better centres of condensation than others, come into action before the others: even in these infinitesimally small particles there seems, as elsewhere, to be no equality. The result of this is that the best nuclei begin condensing before the others, and if there is time given they rob the surrounding air of its vapour owing to the high supersaturation, so that if the other nuclei are to come into action time must not be given for this condensation to take place. We see from this that the quicker the expansion the greater the number of nuclei that will become active, while with a slow expansion an odd nucleus here and there will be able to relieve the tension. This suggests that the gradual increase in the number of drops observed with increase of expansion during both rainy and foggy stages may be as much due to increase in rapidity of expansion as to increase in its amount.

These experiments on condensation in supersaturated air refer to the condensation which takes place on nuclei always present in saturated air. When X-rays and radioactive bodies acted on the air in Wilson's

apparatus so as to produce ions in the moist air, he showed that these ions became active centres of condensation; and he also showed that the supersaturation necessary to make them centres of condensation was the same as that required for the nuclei always present in the air. He further showed that these ions were electrically charged, some with positive and some with negative electricity, and that they could be removed by an electric field, but, so far as his experiments went, there was no evidence to show that the nuclei always in the air carried any charge.

Turning now to the new instruments, fig. 1 shows the arrangement of the radioscope which will be described first. A is a U-shaped glass tube about 2 cm. internal diameter, in which the air undergoes the compression and expansion necessary for causing condensation in the dustless air. The U tube is partly filled with water, and the upper part B is the expansion chamber in which the observations are made. The other side of the U tube is connected by means of the india-rubber tube C with the india-rubber ball D. This ball can easily be obtained in any chemist's; its fittings are removed, and the short piece of metal tube E put in their place. The tube E is fixed to the board F, so as to keep the ball in its place in the centre of the hollow cut out in the surface of F. The lever G turns on the hinges H. When G is pressed down it compresses the air in D and in the chamber B of the U tube. K is a strip of spring brass bent into a segment of a circle. On K slides the stop J, which can be fixed at any point on K by means of the pinching screw I. On the outer end of the lever G is fixed a small piece of metal L, and when the lever G is pressed down to compress the air in the ball D it engages with the stop J and keeps the ball compressed until it is desired to expand the air in the test chamber B, when the spring K is drawn back, and the lever G being disengaged the ball at once expands and relieves the pressure in B. M is an india-rubber cord for preventing the lever G being forced too far over.

This arrangement of compressing and allowing the air to expand is not the usual one in experiments of this kind, as it is customary to expand the air by means of a pump or similar arrangement. The method shown, however, has some advantages over the pump. There is no leakage in the working parts, and the simple act of pressing down the lever G is all that is required, while the catch J keeps the pressure on till the temperature and vapour pressures in the expansion chamber have adjusted themselves. It has, however, one defect due to the heat developed by the compression. On the first compression being made the air in the ball and tube gets heated, and some of this heat escapes before expansion is made, and unless considerable time is given before the next compression, it will be made at a

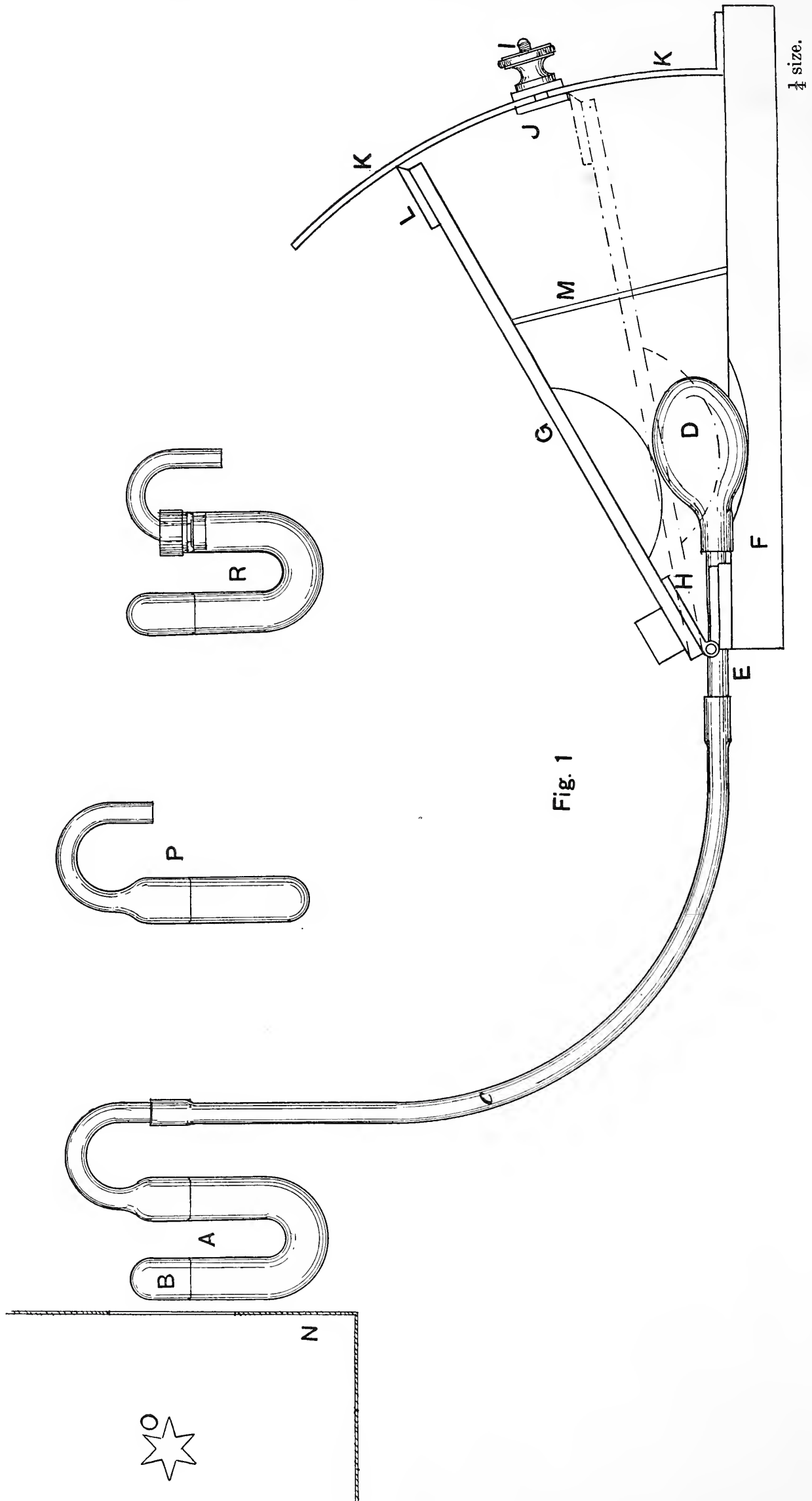


Fig. 1

$\frac{1}{4}$ size.

slightly lower temperature, and therefore not so great as the first; but this does not seem appreciably to affect the results. After a few compressions and expansions things get to a uniform condition, especially if the compressions and expansions are made at equal intervals of time.

For illuminating the test chamber B the lantern N is used. A common gas-jet aided with a lens is required for some experiments, but for most cases an incandescent gas light may be employed and the lens omitted, as without it one gets a more uniform illumination. These experiments are best made in a dark room, and only as much light allowed to come from the lantern as is necessary. A vertical adjustable slit placed in front of the lantern works well. This slit should only be opened about 1 cm. for tubes of 2 cm. diameter, as the narrow strip of light prevents the disagreeable and distracting reflections from the glass surfaces. When well arranged with black background the tube should show nearly black. A magnifying lens held in an adjustable support will be found useful. The lens should be set so as to look rather downwards, to avoid the reflections from the surface of the tube, and across but rather towards the light.

The method of operating with the apparatus is as follows:—The U tube A, after being thoroughly cleaned, is filled with water to about the amount shown in fig. 1. For this purpose a fine india-rubber tube, small enough to pass through the opening in the U tube, will be found useful; by sucking out the air it is quickly filled with water, and by blowing in air it is rapidly emptied. The tube, after washing and filling, is connected by means of the tube C with the compressor D. The light is then adjusted to show the interior of the tube, and as little light reflected by the glass as possible. The lever G is now pressed down for a short time and released; and on now examining the test chamber B it will be found to be full of fog formed by the vapour condensing on the dust in the air. The magnifying lens should now be adjusted so as to show each fog particle clearly. A number of compressions and expansions are next made by pressing and releasing the lever G. This must be done till no fog particles appear in B; the last of them will be seen falling like small rain-drops. After all condensation ceases on slow expansion, the screw I is loosened and the stop J slipped up to a point which will require a little compression for the catch L to engage with the stop J. The lever is then pressed down till L engages with J. After a short time the spring K is drawn back and the catch L released, when the lever at once springs up and the air in B is suddenly expanded. The air in B should be watched while this is taking place. If no drops appear, then the expansion has not been enough to give the supersaturation necessary to cause condensation on the nuclei present. If

this be so, then I is unscrewed and the stop J lowered to give a greater compression. When this is done it will be found that at a certain compression a few drops appear in B. If we go on pushing J down by stages so as gradually to increase the compression, it will be found that the number of drops increases. These drops are formed on some nuclei which Wilson has shown are always present in moist air. These nuclei for convenience we will call natural nuclei. If we go on increasing the compression it will be found that when it exceeds a certain amount that the condensation somewhat suddenly changes its appearance from the rainy form to the foggy, and this fog becomes denser with further increase in the compression.

In the apparatus shown in fig. 1 we cannot tell without trial what amount of compression will be necessary to give the supersaturation required to cause condensation on the natural nuclei, as the action of the instrument is complicated. Part of the expansion in that form of apparatus is due to the compression of the ball D, but part is also due to this expansion being carried on by the impetus given by the expanding air to the water in the tube. That is, some of the energy put into the water at the beginning of the expansion is given out by it at the end. The result of this is that the compression necessary for this form of apparatus is less than that which Wilson found to be necessary with his apparatus for causing condensation in the absence of dust. As has been stated, Wilson found that when the expansion was as great as $v_2/v_1=1.250$ that condensation took place in dustless air, and for all his instruments this figure was fairly constant, but in the instrument above described this is far from being the case. Any slight alteration in the arrangement of the apparatus alters the degree of compression required. For instance, any change in the amount of water in the U tube alters the compression necessary, and any alteration in the tube connecting the U tube with the compression ball has a like effect. In Wilson's apparatus a compression corresponding to a column of about 19 cm. of mercury would, when expanded, give the necessary supersaturation, but with the U tube a pressure of only from 14 to 16 cm. is required according to the conditions. In Wilson's apparatus the foggy stage of the condensation is arrived at when the compression is about equal to 28.8 cm., while the U tube requires only from 19 to 20.5 cm. It is impossible to give these pressures with greater definiteness, as they vary in all the instruments here described with but slight alterations in their arrangement, but the pressures remain constant while the conditions are constant.

Other forms of the compression chamber were tried. The one shown at P, fig. 1, for instance, requires much greater compression to cause condensation

than A, because it entirely depends on the expansion of the air. As this form of instrument requires to be frequently removed from its stand and turned upside down for the purpose of wetting the interior walls, and possesses no advantages, its use was abandoned. This form of compression chamber required a pressure of 20 cm. to give the first or rainy stage of condensation, and 30 cm. to give the foggy stage. The reason for it requiring a greater compression than Wilson's, is due to the resistance offered to free expansion by the tube C and the ball D.

The arrangement shown at R, fig. 1, was also tried. In this case the tube is similar to A, but the expansion is stopped when the air has expanded to its original volume; that is, the movement of the water is stopped before expanding its energy in expanding the air. This was done by the arrangement shown in R. A floating valve rose and fell with the water in the right arm of the U tube, so that, after compression and release, when the water rose to its original level the valve suddenly closed the outlet. As no advantage was observed in its action, it was not further experimented with. The reason for making this instrument was that, theoretically, the best conditions of the expansions in these condensation experiments would appear to be that the expansion be done as quickly as possible. During the first part of the expansion quickness does not matter so much, because no condensation is taking place on the nuclei, but the quicker the expansion is made after a certain stage is reached the more nuclei will become active. Now, the arrangement shown at A, one would imagine, would rather tend to make the expansion slow towards the end, and it was thought that by stopping it when at its maximum rate that an advantage might be derived—giving it, of course, the same amount of expansion as before; but, so far as the purpose for which the instrument is intended, no advantage was observed. This instrument required a compression of 21.5 cm. to give the rainy stage and 32 cm. to give the foggy condition, but the pressure required was variable according to the setting. If the valve did not float just at the level of its seat—that is, floated slightly below it—then, as might be expected, the pressure required was less, as the moving water is allowed to do a little expansion.

When working with the expansion chamber as shown at A, fig. 1, the water after expansion rushes back into the chamber B and by splashing develops a few nuclei, but these are so few they can hardly be said to interfere with the results; and this splashing has the great practical advantage of keeping the inside walls of the tube wet.

Returning now to the apparatus shown in fig. 1, suppose the compression is adjusted so as to give only an odd drop or two falling in the chamber B. If while in this condition any radioactive substance is brought

close to B, an increase in the number of drops will take place, it may amount to a slight shower or a dense shower or a fog, according to the strength of the radioactivity. These new drops are condensed on the ions formed by the radioactive body. Wilson finds that the nuclei always present in moist air, and these ions formed by the radioactive substance, require the same expansion to cause them to become nuclei. My experience, however, does not fully confirm this. This difference in our conclusions may be due to the difference in our apparatus. There, however, may be a different explanation. It has been observed that if the apparatus here described be left for some time, that on making the first expansion after the rest that the number of drops on the natural nuclei is greater than that given before stopping work, and greater than that obtained on subsequent expansions; but as the degree of expansion with this apparatus may alter with time, the conclusion is not certain. In Wilson's experiments it is probable there were longer intervals between the expansions, and this would bring the expansion required by the natural nuclei near to that required by the ions. If this explanation be correct, then it looks as if these natural nuclei became larger or at least better centres of condensation with time.

With the apparatus shown in fig. 1 it is found that if the compression is reduced till there is hardly a drop visible, that the instrument is quite as sensitive to radioactive substances as with higher compressions. Of course, there are not so many drops formed, because with higher expansions the drops produced on the natural nuclei are added to those produced by the radioactivity. But, making allowance for these natural nuclei, there does not seem to be more due to the ions than when the lower compression was given. The question might be asked here, Does the radioactivity act on the natural nuclei and increase their condensing power and cause them to become active with lower supersaturations? It is evident that the whole of the nuclei due to radioactivity are not produced in this way, because, with comparatively feeble radiation, the number of drops far exceeds the densest shower given by the natural nuclei.

The action of the radioactivity is to produce ions in the air in the chamber B, and these ions become centres of condensation on supersaturation. The ions produced by the radiations have a very short life. If we remove the radioactive substance before the expansion is made the density of the condensation is greatly reduced. If we allow two seconds to pass before the expansion is made, only a slight effect remains. After five seconds all the ions are gone, having combined with each other. The continued action of the radioactive substance does not make the ions

larger, or, at least, not better nuclei to any perceptible extent. We can understand why this should be so: the life of an ion being so short, there is no time for any cumulative action of the rays. An interesting point, however, was observed. If after forming a dense, cloudy, or a foggy condensation we leave it alone to settle for a short time, and if while there are still a good number of particles present we compress the air and cause the particles to evaporate, on again expanding the air slowly it will be found that many of the cloud particles have left a dust-like nucleus, as a slow and slight expansion makes them active and a shower results. But, on the other hand, if we compress and evaporate the particles immediately after they are formed, it will be found that hardly any of the dust-like nuclei are present, only an odd drop appearing on slow expansion. Under the first conditions the cloud particles had time to differentiate in size, some growing larger, and it is probable that these larger cloud particles do not thoroughly evaporate but leave a nucleus large enough to be active with but slight supersaturation.

To give an idea of the sensitiveness of this apparatus to radioactivity, it may be mentioned that any of the following substances, when held near the tube, give considerable increase in the condensation: pitchblende, radioactive mud, uranium, and, of course, any feeble radioactive salts or very minute quantities of them. For instance, the radium on a Crooke's Spintaroscope gives quite a dense shower. To get more decided effects with weak radioactive substances, an aluminium window in the tube B has been occasionally used.

A number of gases and vapours were mixed with the air in the test chamber to see if any of them would improve its action. Amongst the substances tried were sulphurous acid, peroxide of hydrogen, sulphuric, nitric, and hydrochloric acids, and chlorine gas. None of these seemed to improve the action of the instrument as a detector of radioactivity. The most interesting of these tests were made with chlorine mixed with more or less air which acted differently from the others. With chlorine, after the tube had been freshly filled, there was always a great difficulty in getting the air cleared of the dust-like nuclei, and the action of the radioactivity was to manufacture great numbers of these dust-like nuclei, as it took a great number of slow expansions to clear out the nuclei after a test had been made with some radioactive substance. If while there was chlorine in the test chamber it was exposed to the light of burning magnesium, it gave dense condensation if expanded while the light was still burning, but all effect was gone if allowed to stand half a minute before expanding. On the other hand, an incandescent gas flame seems to

cause the nuclei to grow to dust size, as a less expansion gives condensation with incandescent light than with gas flame; and if the air was exposed to the incandescent light for fifteen minutes, it gave condensation on very slight expansion, just like dust, while the strongest radium salt I possessed gave no indication of being able to increase the size of the nuclei.

Some tests were also made with methylated alcohol in the U tube, but beyond the lower expansions required to cause condensation on the ions, no advantage was observed.

Another form of instrument was tried which seems to have some advantages over the U tube owing to the expansion being quicker and more suddenly stopped. In this instrument the compression chamber is a short length of an ordinary test tube, connected directly with a pear-shaped india-rubber compression ball. The test tube was connected with the ball by means of an ordinary $\frac{3}{4}$ -inch brass union joint, one end of the union joint being cemented into the neck of the ball while the test tube was cemented into the other end. This joint is necessary for cleaning the tube and filling it with water. The compressing apparatus had to be modified to meet the new conditions. The compressor shown in fig. 1 is turned upside down, and has an opening made in the base-board large enough for the passage of the test tube and union joint, while the hinged board in this case presses on the bottom of the compression ball. By this arrangement the compression chamber projects upwards through the compressor, and the compressions and expansions are made in the same manner as in the other instrument.

THE RADIOMETER.

The principal defect of the radioscope above described is that it only gives a rough indication of the strengths of the radioactive substances brought near it. It, however, seemed possible, if we were to adopt the principle already in use in the dust-counter, to make an instrument that would give numerical values of the strengths of the different radioactive substances, as by this arrangement we would be able to count the number of drops formed in the test chamber, and thus obtain definite figures of the strengths. To carry out this idea, the instrument shown in fig. 2 was constructed. B is the condensation chamber, in which the air is compressed and expanded. This chamber is made of brass, and has an opening E in the top and another opening F in the bottom. These openings are closed with glass plates, the upper opening being covered with clear glass, and the lower one with a glass micrometer ruled with cross lines 1 mm. apart. The chamber B is connected with the compression apparatus shown in

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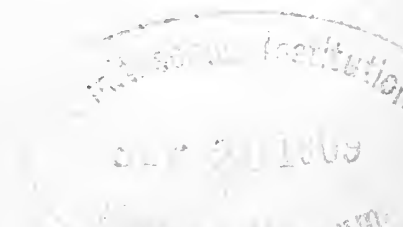
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If an author finds it inconvenient to furnish such drawings, the Society will have the figures re-drawn at his expense; but this will cause delay.

When the illustrations are to form plates, a scheme for the arrangement of the figures (in quarto plates for the Transactions, in octavo for the Proceedings) must be given, and numbering and lettering indicated.

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[Continued on page iii of Cover.]

fig. 1 by means of the tube C. The inside of the chamber, where possible, is covered with blotting-paper, which is kept wet for the purpose of saturating the air. The tube C is attached to the compression chamber B by means of the union joint D. This opening is required for the purpose of cleaning the cover glass and the micrometer, and also for wetting the interior of B. After the air has been compressed and expanded the drops fall on the micrometer and are counted with the aid of the lens L, while the illumination is effected by means of the spot-mirror M, either daylight

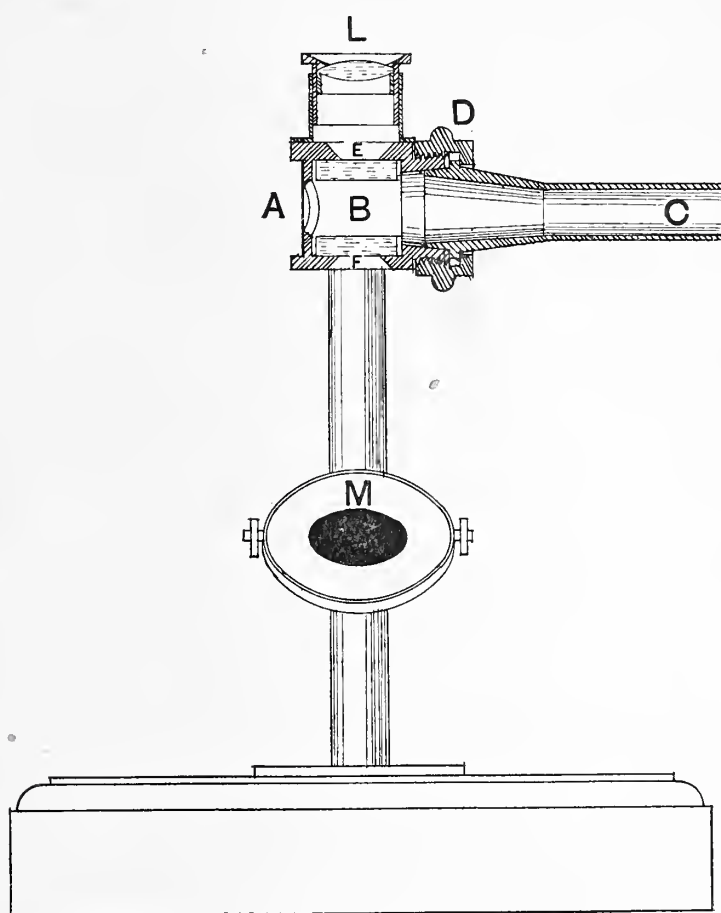


Fig. 2 $\frac{1}{2}$ size.

or artificial light being used. As the thick walls of the chamber B offer considerable obstruction to the rays from the radioactive substances, a window A is cut out on one side of the chamber and covered with aluminium 0.25 mm. thick.

Fig. 3 shows the arrangement of the apparatus when fitted together. The part R, however, may be omitted, as it is for a special purpose, and will be described later. Between the compressor and the chamber B is introduced a pressure gauge P. This gauge is, however, not necessary, but is convenient, as it shows whether all the joints are air-tight or not and the compression keeping constant. The gauge is not necessary for setting the amount of compression required, as that is obtained by trial. After the glass surfaces in B have been cleaned and the sides wetted and all con-

nected up, the first thing to be done is to get rid of the dust particles in B. This is done by pressing the lever G and releasing it, when a fog will be seen slowly settling in B: time is given for it to settle, then another compression and expansion is made, when more cloud particles appear. This process of compression and expansion is continued till all condensation ceases in B with slight compression. The stop J on the compressor is now fixed at any point on K and the lever G pressed down till L engages with the stop J. The compression is left on for a time, and then the trigger K is pulled back, the lever L released, and the air in B expanded. While the air is expanding the eye of the observer should be watching the

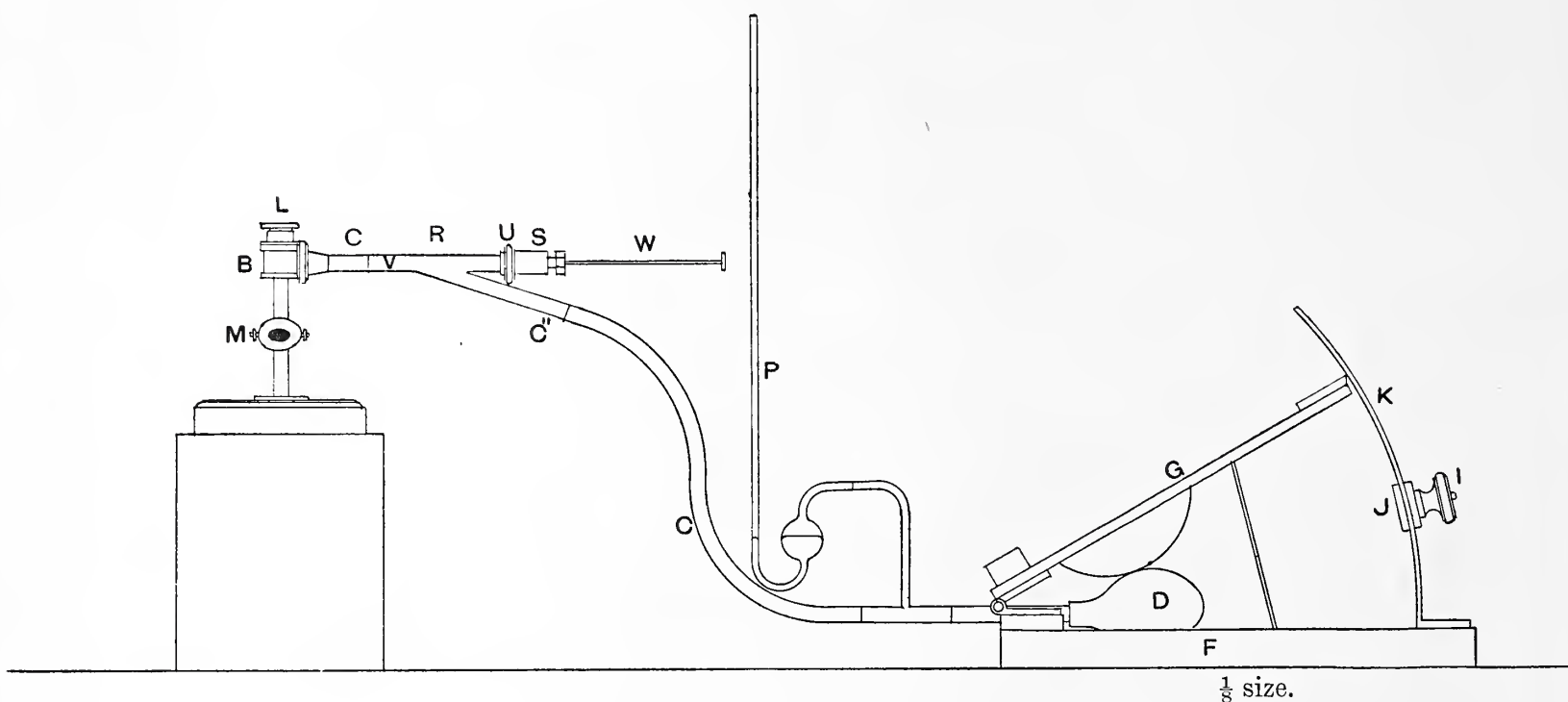


Fig. 3

micrometer through the lens. If no drops appear, then the expansion has not been enough to produce condensation in dustless air: the stop J must therefore be pushed further down and another trial made, and so on till some drops appear on the micrometer. Suppose, on the other hand, that on first trial there should appear a number of drops, then the compression must be reduced. It will be found that the best degree of compression for these experiments is that which gives just an odd drop or two visible over the whole field. In that condition, though the expansion is not enough to bring down all of what we have called the natural nuclei in the air, yet it seems to be enough to bring down all the ions formed by the radioactivity. It will, of course, only bring down the negative ions, as Wilson has shown that while negatively charged ions become active nuclei when the expansion is represented by $V_2/V_1 = 1.250$, while the expansion must be as great as $V_2/V_1 = 1.31$ to bring down the positive ions. If the expansion is in-

creased beyond what is just necessary to bring down a few of the natural nuclei, then we get more drops on the micrometer with the same radioactivity, but there does not seem to be more due to the radioactivity; the increase seems to be due to the greater number of the natural nuclei brought down.

The method generally adopted was to make the compression necessary to give the odd drops, and then bring to a fixed point some radioactive substance which is kept as a standard test and note the number of drops. Repeat the test with slightly varying compressions, but always with the standard radioactive body, and note results. It will be found that the most satisfactory tests are made when the number of drops on the natural nuclei is small, and in that condition practically all the drops are due to the radioactivity. If higher compressions are used, we have to deduct from the total number of drops those due to the natural nuclei. If at any time there is any suspicion that the test is not correct owing to some imperfection in the apparatus, all we have to do is to present the standard radioactive substance and see if we get the correct number of drops.

For the measurement of strong radioactive substances we must either interpose screens or put them at a distance from the instrument, because the number of drops given by them is far too great to be counted.

Reference has already been made to the importance of the quickness of the expansion in these condensation experiments, and the question naturally arose, Was the arrangement used the best for the purpose? or would some other form of release not give better results? Some experiments were accordingly made with other forms of release which promised to make the expansion quicker. The apparatus shown in fig. 4 was tried; this was interposed between the compression chamber and the compressor, taking the place of R, fig. 3. V is a tube open at both ends and provided with a branch tube on which is a stopcock. The end of the tube C' is connected with C, fig. 2, by means of a small piece of india-rubber tube, while the branch tube C'' is connected with the compressor. S is a thick piece of sheet india-rubber which closes the end of the tube V, the india-rubber being fixed on the lever T, which is hinged at its lower end, and held at its upper end by the catch U. The lever requires considerable force to compress the india-rubber and engage with U. The object of this is to make the india-rubber act as a spring as well as an air-tight valve to the end of the tube V, so that when the lever is released from the catch U it will be forced rapidly away, so opening the end of V quickly, and thus giving the air in the compression chamber a free outlet. After the compression has been made, the stopcock on the tube C'' is closed, and

the expansion is made by releasing the lever T, when the rubber valve springs from its seat and has considerable velocity before the air begins to escape, thus ensuring a quick and full opening of the outlet.

Another method of release was also tried. As has been already stated,

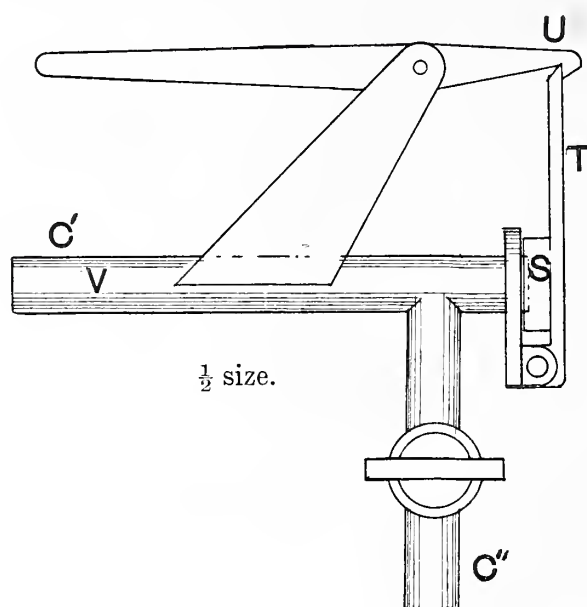


Fig. 4

the ideal form of expansion is the quickest that can be accomplished, but the final part of the expansion is the more important, a slight slowing of the initial part not being likely to have any bad effect. Accordingly, the apparatus shown in fig. 5 was made to meet these conditions. As

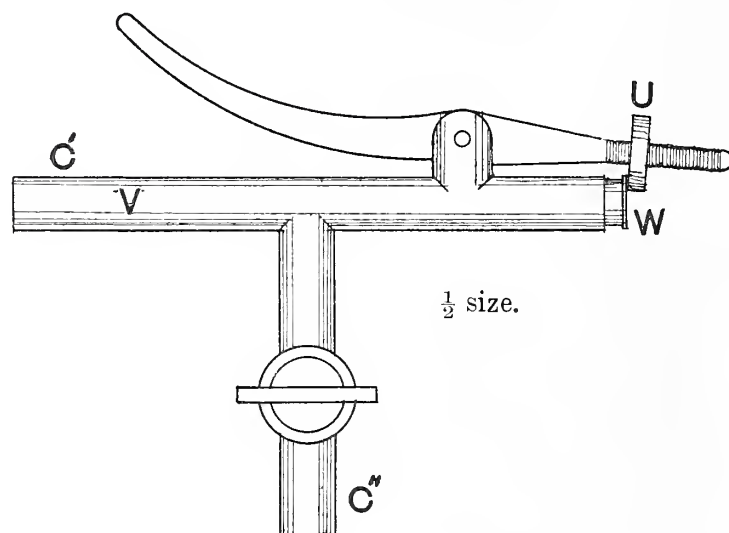


Fig. 5

before, the tube V is connected at C' with the expansion chamber B, fig. 2, and at C'' with the compressor. The release valve in this case consists of a plunger W, which is very accurately ground into the end of the pipe V. This plunger, though fitting quite easily, is air-tight with the aid of a little glycerine or oil. U is a catch for preventing the plunger from being blown out by the pressure of the air. The action of this release valve is as

follows:—When the catch U is lifted the air in the expansion chamber at once begins to expand, but it is slightly retarded by the resistance of the plunger; but by the time the plunger clears the end of the tube it is shot out with considerable velocity, being in fact a very quickly opening valve, and in this way a very free passage is given for the latter part of the expansion. The catch U is in the form of a nut screwed on to the end of the lever. The object of this was to enable tests to be made of the effects of different lengths of stroke of the plunger. With a long stroke the first part of the expansion will be retarded most, but the plunger will acquire a greater velocity, opening the pipe quicker, and the latter part of the expansion be more rapid. A short stroke of the plunger, while quickening the first part, will slightly retard the latter, as it does not open so quickly. After a number of tests neither of these additions seemed to improve the working of the instrument to an extent sufficient to make up for the greater trouble given by their use.

Both the radioscope and the radiometer have the defect that they only test the more penetrating rays—that is, the β - and γ -rays; while the α -rays, which have most of the energy of the radiation, are stopped by the walls of the expansion chamber. It was thought, therefore, that if the radioactive body could be introduced into the expansion chamber, it would be capable of measuring much smaller degrees of radioactivity. When working with the apparatus shown in figs. 4 and 5, it was found to be possible to open one end of the expansion chamber and allow dusty air to enter that end without it in any way interfering with the action of the instrument, the reason for this being that the dusty air only penetrates a short distance into the tube, and the compressions and expansions make the dust particles into cloud particles and deposit them before they penetrate into the expansion chamber. Taking advantage of this, it seemed quite easy to introduce the radioactive body into the expansion chamber. The apparatus shown at R, fig. 3, was therefore constructed. V is the connecting tube as in figs. 4 and 5, and C'' is the branch pipe to the compressor. In this case the branch pipe is connected at a small angle to the tube V to allow as free a passage as possible for the expanding air. The open end of the tube V is fitted with a union joint U and a stuffing box S, through which passes the fine steel wire W. The end of the wire inside the tube is provided with a screw on which may be fitted a small pair of forceps, or other arrangement that may be suitable for the particular experiment.

When we wish to introduce a small piece of any substance into the chamber B the union joint U is unscrewed and the forceps or other

arrangement screwed on to the wire W. The substance being firmly secured, the union joint is replaced and screwed up, then a few compressions and expansions are made to throw down any dust clinging to the sample, after which the wire is pushed forward into the apparatus till the forceps is just visible on the edge of the field seen through the lens. Compression and expansion are now made and result noted. It may be mentioned that a piece of pitchblende about the size of a pin-head gives copious showers under these conditions. Its action, however, does not seem to be constant, as the shower may sometimes permit of counting the drops, while the next shower will be so dense it is impossible to do so. Salts of radium, however, seem to give showers of constant density. For instance, a small disc of brass about 2 mm. diameter was wetted with a very weak solution of a radium salt and allowed to dry. The solution used for this purpose was very weak, being made of a small speck of a very weak salt dissolved in half an ounce of water. On testing this small quantity of radium inside the chamber it gave very dense showers about 50 drops per sq. mm., while through the aluminium window it only gave about 1 per mm. It may be mentioned that this disc did not give a scintillation of action when tested with the phosphorescent screen.

A similar disc of brass was put into a bottle in which was a weak radium salt to test the effects of the emanations from the radium. After the disc had been in the bottle five hours to collect radium A—the active deposit of the emanations—it was put inside the expansion chamber and gave at first about 30 drops per sq. mm. Another disc put in the same bottle with radium for one hour gave nearly the same number. In both cases the number remained high for some time, but in a few hours both ceased to act. As the experiments were made simply for testing the radiometer, no notes were taken of the rates of decay. In collecting the radioactive deposit care was taken that the disc did not touch the bottle containing the radium. The disc was fitted to a new cork which had been fitted to the bottle, and so arranged that the cork supported the disc in the middle of the space over the radium, and all metal parts connected with the disc were further cleaned with emery cloth to remove any chance of contamination.

In using this method of working great care has to be exercised that the body introduced into the expansion chamber does not interfere with the expansion of the air. As already pointed out, any alteration in the tube between the expansion chamber and the compressor alters the results. In making these experiments it was found necessary to test the forceps to see if their introduction did not interfere with the results, when it was found

that if the forceps were at a distance from the chamber—that is, in the tube—they did interfere with the expansion, but that if they were in the expanding part of the chamber they did not. Again, when testing the radium salt on disc, another similar disc was made but kept free from all radioactive matter and its action on the expansion tested before the disc with the radium was tried. It is possible that some other method of introducing the substance to be tested into the expansion chamber might be devised which would give less trouble, but as yet none has been tried.

When the substance to be tested is introduced into the expansion chamber it is evident there are risks of the instrument becoming contaminated either by some of the substance tested escaping from the forceps with the rush of air during expansion, or by the emanations given off by the bodies tested. Care has therefore to be taken that the bodies tested are firmly secured, and the tests quickly made, so that as short a time as possible may be given for the emanations to collect.

(Issued separately July 21, 1909.)

XXXI.—*Nematonurus Lecointei*, Poisson abyssal de la "Belgica" retrouvé par l'Expédition Antarctique Nationale Ecossoise. Note préliminaire, par Louis Dollo, Sc.D. (Cantab.), Ph.D. (Giessen), Min. et Géol. D. (Utrecht), à Bruxelles (Musée). Présentée par M. R. H. TRAQUAIR, M.D., LL.D., F.R.S., V.P.R.S.E.

(MS. received June 7, 1909. Read June 7, 1909.)

I. INTRODUCTION.

I.—Nous avons vu, récemment, que les *Macruridæ*, dans l'état actuel de nos connaissances, sont représentés par une espèce à l'intérieur du Cercle Polaire Arctique et par deux espèces à l'intérieur du Cercle Polaire Antarctique.*

II.—Maintenant, si nous considérons la Zone située entre 60° N. et le Cercle Polaire Arctique, nous constatons que les *Macruridæ* n'y figurent pas pour moins de 8 espèces, appartenant à 6 genres et à 2 sous-familles.†

Toutes ces espèces sont groupées dans le bassin de l'Atlantique,—et on comprend l'absence de Macrurides, poissons essentiellement abyssaux, dans la région correspondante du Pacifique, car la Mer de Behring atteint rarement 100 fathoms entre 60° et le Cercle Polaire,—tandis que les profondeurs de 100 fathoms et moins occupent un espace très restreint dans l'Atlantique entre 60° N. et le Cercle Polaire Arctique.

III.—Passant, ensuite, à la Zone située entre 60° S. et le Cercle Polaire Antarctique, nous remarquons que, jusqu'aujourd'hui, pas un seul *Macruridæ* n'y était signalé.‡

En effet, il n'y a que trois Expéditions Antarctiques, en dehors de la Scotia (1902–1904), qui aient publié des Macrurides :

1. *Erebus* and *Terror* (1839–1843).—Provenance: Côtes de l'Australie Méridionale, par conséquent bien au Nord de 60° S. §

* L. Dollo, "Cynomacrus Piriei, Poisson abyssal nouveau recueilli par l'Expédition Antarctique Nationale Ecossoise," *Proceedings of the Royal Society of Edinburgh*, vol. xxix, p. 316, Edimbourg, 1909.

† Voir, plus loin, l'énumération détaillée.

‡ A. Brauer, "Die Tiefsee-Fische (I. Systematischer Teil)," *Wissenschaftliche Ergebnisse der deutschen Tiefsee-Expedition auf dem Dampfer "Valdivia" 1898–1899*, vol. xv., pl. xvii., Iéna, 1906.

§ J. Richardson, "Fishes," *Zoology of H.M.S. "Erebus" and "Terror," under the command of Captain Sir James Clark Ross, R.N., F.R.S., during the years 1839 to 1843*, p. 53 Londres, 1846.

2. *Belgica* (1897-1899).—Provenance: Intérieur du Cercle Polaire Antarctique, par conséquent au delà de la Zone qui nous intéresse ici.*

3. *Antarctic* (1901-1903).—Provenance: Détroit de Bransfield, par conséquent, cette fois, entre 60° S. et le Cercle Polaire Antarctique, mais le poisson périt avant d'avoir pu être déterminé d'une manière précise, dans le désastre, à jamais déplorable, qui engloutit le navire.†

Quant aux Expéditions Océaniques qui ont recueilli des Macrurides et qui ont pénétré entre 60° S. et le Cercle Polaire Antarctique,—c'est-à-dire celles du *Challenger* (1873-1876) et du *Valdivia* (1898-1899),—elles n'en ont pris qu'au Nord de 60° S. ‡

Ce qui est d'autant plus étonnant que les profondeurs considérables abondent dans la Zone limitée par 60° S. et par le Cercle Polaire Antarctique.

IV.—Il était donc encore réservé à l'*Expédition Antarctique Nationale Ecossaise* de rapporter le premier *Macruridæ* de la Zone située entre 60° S. et le Cercle Polaire Antarctique.

Ce Macruride, comme je vais le montrer, n'est autre que le *Nematonurus Lecointei*, découvert par la *Belgica* à l'intérieur du Cercle Polaire Antarctique.

Rien de trop surprenant à cela, d'ailleurs, puisque le *Macrurus berglax* se trouve, à la fois, à l'intérieur du Cercle Polaire Arctique et dans la Zone située entre 60° N. et ce Cercle Polaire. §

II. IDENTIFICATION DU SPÉCIMEN DE LA "SCOTIA."

I. *Concordances*.—1. *Sous-Famille*.—Par sa première fente branchiale réduite, le Macruride subantarctique de la *Scotia* appartient à la sous-famille des *Macrurinae*.

2. *Genre*.—Par ses dents prémaxillaires sur deux rangs, ses dents mandibulaires sur un seul, sa deuxième épine dorsale barbelée et ses nageoires dorsales largement séparées, notre poisson vient se ranger dans le genre *Nematonurus*.

* L. Dollo, "Poissons de l'Expédition Antarctique Belge," *Résultats du Voyage du S.Y. "Belgica" en 1897, 1898, 1899, sous le commandement de A. de Gerlache de Gomery*, p. 44, Anvers, 1904.

† E. Lönnberg, "The Fishes of the Swedish South Polar Expedition," *Wissenschaftliche Ergebnisse der schwedischen Südpolar-Expedition 1901-1903 unter Leitung von Dr. Otto Nordenskjöld*, p. 50, Stockholm, 1905.

‡ A. Günther, "Report on the Deep-Sea Fishes," *Voyage of H.M.S. "Challenger" during the years 1873-76: Zoology*, vol. xxii. p. 122, Edimbourg, 1887.

A. Brauer, *Die Tiefsee-Fische, etc.*, p. 256.

§ A. Brauer, *Die Tiefsee-Fische, etc.*, pl. xvii.

3. *Espèce*.—Par les écailles du corps inermes et caduques, les écailles de la tête armées de cinq rangées d'épines, la distance des dorsales plus grande que les trois quarts de la longueur de la tête, la verticale de l'anوس passant en arrière du milieu de cette distance et les ventrales avec dix rayons dont l'externe plus grand que la moitié de la longueur de la tête n'atteint pas l'anوس, l'animal considéré s'identifie avec le *Nematonurus Lecoointei*.*

II. *Discordances*.—A part de minimes différences dans les dimensions de la fente buccale, dans le diamètre de l'orbite, dans la largeur de l'espace interorbitaire, dans la longueur du museau, ou d'autres semblables qu'on pourrait trouver, et qui n'ont pas, pour moi, de valeur spécifique, je ne mentionnerai, ici, que trois discordances à expliquer :

1. *Première Dorsale*.—Elle a 10 rayons, au lieu de 9 : mais nous rencontrons un cas analogue de variation, 13 à 14 rayons, dans la première dorsale du *Malacocephalus laevis*.†

2. *Distance des Dorsales*.—Au lieu d'être presque égale à $3\frac{1}{2}$ fois la base de la première dorsale, elle n'atteint pas $2\frac{1}{2}$ fois cette base : mais, comme la distance en question est, quand même, plus grande que les $\frac{3}{4}$ de la longueur de la tête, on voit que c'est la base de la première dorsale qui est plus développée que d'ordinaire, ce qui est en rapport avec le rayon supplémentaire, et non la distance des dorsales qui est réduite.

3. *Pectorales*.—Elles ont 20 rayons, au lieu de 21 : variation individuelle de même ordre chez *Macrurus berglax* (18 à 19) et de plus forte amplitude chez *Nematonurus armatus* (18 à 20).‡

III. *Conclusion*.—Il convient de regarder le Macruride subantarctique de la *Scotia* comme un nouvel exemplaire de *Nematonurus Lecoointei*, Dollo, 1900.

III. BIONOMIE DU NEMATONURUS LECOINTEI.

I. *Données*.—Comparons, maintenant, au point de vue bionomique, le *Nematonurus Lecoointei* de la *Belgica* et celui de la *Scotia* : §

"BELGICA."	"SCOTIA."
I. <i>Biogéographie</i> .	I. <i>Biogéographie</i> .
<i>Habitat</i> : 70° 40' S. et 102° 15' W.	<i>Habitat</i> : 62° 10' S. et 41° 20' W.
Mer de Bellingshausen.	S.E. Orcades du Sud.
Océan Antarctique.	Océan Atlantique.
Quadrant Pacifique.	Quadrant Américain.
Numéro 873.	Station 313.

* L. Dollo, *Poissons de l'Expédition Antarctique Belge, etc.*, p. 44.

† A. Günther, *Deep-Sea Fishes, etc.*, p. 148.

‡ *Ibid.*, pp. 130 et 150.

§ L. Dollo, *Poissons de l'Expédition Antarctique Belge, etc.*, p. 45.

II. *Ethologie*.

1. *Profondeur*.—1531 fathoms.
2. *Nature du Fond*.—Vase et roches erratiques.
3. *Température du Fond*.—33° F.
4. *Température de la Surface*.—28°·2 F.
5. *Densité de l'Eau (Fond)*.—Inconnue.
6. *Densité de l'Eau (Surface)*.—Inconnue.
7. *Mode de Capture*.—Chalut à étrier de cinq pieds.
8. *Date de Capture*.—14 mars 1899.
9. *Heure de Capture*.—Entre 5½ et 6 heures du soir.
10. *Nombre d'Individus capturés*.—Deux, pris ensemble.
11. *Compagnons de Pêche*.—Aleyonaires, Astérides, Ophiurides, Bryozoaires, Lamellibranches.

II. *Ethologie*.

1. *Profondeur*.—1775 fathoms.
2. *Nature du Fond*.—Vase bleue et blocs de plus d'un quintal.
3. *Température du Fond*.—31° F.
4. *Température de la Surface*.—29°·6 F.
5. *Densité de l'Eau (Fond)*.—1·02560.
6. *Densité de l'Eau (Surface)*.—1·02477.
7. *Mode de Capture*.—Chalut.
8. *Date de Capture*.—18 mars 1903.
9. *Heure de Capture*.—Entre 8 heures du matin et 8 heures du soir.
10. *Nombre d'Individus capturés*.—Un seul.
11. *Compagnons de Pêche*.—Crinoïdes, etc.

II. *Discussion*.—1. *Latitude*.—D'après notre détermination, le *Nematonurus Lecoointei* se rencontrerait à l'intérieur du Cercle Polaire Antarctique, et dans la Zone située entre 60° S. et ce Cercle Polaire :

<i>Belgica</i>	70° 40' S.	Antarctique.
<i>Scotia</i>	62° 10' S.	Atlantique.
Différence	8° 30'	

Mais le *Macrurus berglax*, de l'intérieur du Cercle Polaire Arctique, se rencontre également dans la Zone située entre 60° N. et ce Cercle Polaire,—et même bien plus au Sud, jusqu'à la Nouvelle-Angleterre, donc avec une différence de latitude beaucoup plus considérable :

<i>Olga</i>	74° 56' N.	Spitzberg.*
<i>Albatross</i>	41° 47' N.	Massachusetts.†
Différence	33° 09'	

* E. Ehrenbaum, "Die Fische der 'Olga'-Expedition," *Wissenschaftliche Meeresuntersuchungen, herausgegeben von der Kommission zur Untersuchung der deutschen Meere in Kiel und der biologischen Anstalt auf Helgoland*, vol. vii. p. 62, Oldenbourg, 1905.

† G. B. Goode and T. H. Bean, "Oceanic Ichthyology," *Smithsonian Contributions to Knowledge*, vol. xxx. p. 391, Washington, 1895.

D'autre part, le *Nematonurus armatus* nous offre une variation de latitude encore plus forte :

<i>Challenger</i>	36° 10' N.	Pacifique central.*
<i>Challenger</i>	53° 55' S.	Indique austral.†
Total	<u>90° 05'</u>	

2. *Longitude*.—Parallèlement, nous avons :

<i>Belgica</i>	102° 15' W.	Antarctique.
<i>Scotia</i>	41° 20' W.	Atlantique.
Différence	<u>60° 55'</u>	

pour le *Nematonurus Lecoointei*.

Puis :

<i>Enigheten</i>	23° 40' 00" E.	Finmark.‡
<i>Albatross</i>	65° 37' 30" W.	Massachusetts.§
Total	<u>89° 17' 30"</u>	

pour le *Macrurus berglax*.

Et :

<i>Challenger</i>	45° 31' E.	Indique austral.
<i>Challenger</i>	151° 34' W.	Pacifique central.¶
Total	<u>197° 05'</u> ,	ou 162° 55'

pour le *Nematonurus armatus*.

3. *Bathymétrie*.—Enfin, nous noterons les chiffres suivants :

<i>Scotia</i>	1775 fathoms.	Atlantique.
<i>Belgica</i>	1531 „	Antarctique.
Différence	<u>244</u> „	

pour le *Nematonurus Lecoointei*.

* J. Murray, "Summary of Results," *Voyage of H.M.S. "Challenger" during the years 1873-76*, p. 961, Edimbourg, 1895.

† *Ibid.*, p. 505.

‡ R. Collett, "Meddelelser om Norges Fiske i Aarene 1884-1901 (3die Hoved-Supplement til 'Norges Fiske'), II.," *Forhandlinger i Videnskabs-Selskabet i Christiania*, Aar 1903, No. 9, p. 76, Christiania, 1904.

§ G. B. Goode and T. H. Bean, *Oceanic Ichthyology, etc.*, p. 391.

|| J. Murray, *Summary of Results, etc.*, p. 443.

¶ *Ibid.*, p. 1034.

Puis :

<i>Ingolf</i>	1931 fathoms.	Groenland.*
<i>Olga</i>	218	„ Spitzberg.†
					<u>1713</u>	„
				Différence		

pour le *Macrurus berglax*.

Et :

<i>Challenger</i>	2425 fathoms.	Pacifique central.‡
<i>Challenger</i>	400	„ N. Zélande.§
					<u>2025</u>	„
				Différence		

pour le *Nematonurus armatus*.

4. *Conclusion*.—Notre détermination est, dès lors, justifiée aussi biologiquement.

IV. LES MACRURIDÆ ET LA BIPOLARITÉ.

I. *Bipolarité*.—Il reste toujours utile de s'occuper de cette théorie, à laquelle nous avons déclaré, plusieurs fois, ne pouvoir nous rallier, ni en principe, ni en fait.||

Car, il y a quelques mois à peine (1909), M. E. Vanhöffen, Conservateur au Musée royal d'Histoire naturelle de Berlin, écrivait encore :

“In der dritten Arbeit wies ich auf die ausgesprochene Bipolarität der Lucernariden hin.”¶

Puis, à propos des Radiolaires :

“Eine ganze Anzahl von Arten zeigte auch auffallende Uebereinstimmung mit nordischen Formen und vier von ihnen werden als Beispiele für Bipolarität besonders hervorgehoben, da sie im nördlichen und südlichen Kaltwassergebiet häufig anzutreffen sind, während sie im ganzen Warm-

* C. Lütken, “The Ichthyological Results,” *The Danish “Ingolf”-Expedition*, vol. ii. p. 24, Copenhagen, 1899.

† E. Ehrenbaum, *Die Fische der “Olga”-Expedition, etc.*, p. 62.

‡ J. Murray, *Summary of Results, etc.*, p. 1034.

§ A. Günther, *Deep-Sea Fishes, etc.*, p. 150.

|| L. Dollo, *Poissons de l'Expédition Antarctique Belge, etc.*, pp. 191-207.

L. Dollo, “*Notolepis Coatsi*, Poisson pélagique nouveau recueilli par l'Expédition Antarctique Nationale Ecossaïse,” *Proceedings of the Royal Society of Edinburgh*, vol. xxviii. p. 61, Edimbourg, 1908.

¶ E. Vanhöffen, “Vorwort,” *Deutsche Südpolar-Expedition 1901-1903*, vol. x. p. v, Berlin, 1909.

wassergebiet nach den bisherigen Forschungen fehlen und es ausgeschlossen erscheint, dass diese grossen und charakteristischen Arten übersehen wurden." *

II. *Macruridae*.—1. *Données*.—Or, nous avons, ici :

1.—A l'intérieur du Cercle Polaire : †

		Océans :‡	
{	Arctique . . .	<i>Macrurus berglax</i>	A.
	Antarctique . . .	{ <i>Cynomacrurus Piriei</i> . <i>Nematonurus Lecoointei</i> .	A. P.

2.—Entre 60° de Latitude et le Cercle Polaire :

{	Arctique . . .	<i>Chalinura simula</i> .§	A.
		<i>Cælorhynchus cælorhynchus</i> .	A.
		<i>Coryphænoides rupestris</i> .¶	A.
		<i>Macrurus æqualis</i> .**	A.
		„ <i>berglax</i> .††	A.
		„ <i>Ingolfi</i> .§	A.
	<i>Nematonurus Goodei</i> .§	A.	
<i>Trachyrhynchus Murrayi</i> .**	A.		
Antarctique . . .	<i>Nematonurus Lecoointei</i> .‡‡	A.	

* E. Vanhöffen, *Vorwort*, etc., p. vi.

† L. Dollo, *Cynomacrurus Piriei*, etc., p. 316.

‡ A=Atlantique, I=Indique, P=Pacifique, ou leurs prolongements à l'intérieur des Cercles Polaires.

§ C. Lütken, *Ichthyological Results*, etc., pp. 24, 26, 27, 28.

|| R. Collett, "Norges Fiske," *Forhandlinger i Videnskabs-Selskabet i Christiania*, Aar 1874 (Tillægsh.), p. 129, Christiania, 1875.

¶ J. E. Gunnerus, "Efterretning om Berglaxen, en rar norsk Fisk, som kunde kaldes *Coryphænoides rupestris*," *Trondhjemske Selskabs Skrifter*, vol. iii. p. 50, Copenhagen, 1765.

** R. Collett, "Fiske indsamlede under 'Michael Sars' s Togter i Nordhavet 1900-1902," *Report on Norwegian Fishery and Marine-Investigations*, vol. ii., No. 3, pp. 55, 56, 58, 62, Bergen, 1905.

†† R. Collett, *Meddelelser*, etc., p. 75.

‡‡ Voir, plus haut, p. 490.

3.—Entre 50° et 60° de Latitude :

Nord	{	<i>Albatrossia pectoralis</i> .*	P.
		<i>Bogoslovius Clarki</i> .*	P.
		„ <i>firmisquamis</i> .*	P.
		<i>Chalinura serrula</i> .*	P.
		„ <i>simula</i> .†	A.
		„ <i>filifera</i> .‡	P.
		<i>Cœlorhynchus cœlorhynchus</i> .§	A.
		„ <i>Talismani</i> .	A.
		<i>Coryphænoides rupestris</i> .§	A.
		<i>Macrurus acrolepis</i> .*	P.
		„ <i>æqualis</i> .§	A.
		„ <i>berglax</i> .¶	A.
		„ <i>cinereus</i> .*	P.
		„ <i>Güntheri</i> .	A.
		„ <i>lepturus</i> **	P.
		<i>Malacocephalus lævis</i> .§	A.
		<i>Nematonurus cyclolepis</i> .‡	P.
„ <i>Goodei</i> .†	A.		
„ <i>suborbitalis</i> .††	P.		
<i>Trachyrhynchus Murrayi</i> .‡‡	A.		
Sud	{	<i>Cœlorhynchus fasciatus</i> .§§	P.
		<i>Lionurus filicauda</i> .§§	I.
		<i>Macruronus novæ-zelandiæ</i> .§§	P.
		<i>Nematonurus armatus</i> .§§	I.

* B. W. Evermann and E. L. Goldsborough, "The Fishes of Alaska," *Bulletin of the Bureau of Fisheries*, vol. xxvi. pp. 224, 349, Washington, 1907.

† G. B. Goode and T. H. Bean, *Oceanic Ichthyology, etc.*, pp. 407, 412.

‡ C. H. Gilbert, "The Ichthyological Collections of the steamer 'Albatross' during the years 1890 and 1891," *United States Commission of Fish and Fisheries (Commissioner's Report, 1893)*, p. 458, Washington, 1895.

§ E. W. L. Holt and W. L. Calderwood, "Survey of Fishing-Grounds, West Coast of Ireland, 1890-1891: Report on the Rarer Fishes," *Scientific Transactions of the Royal Dublin Society*, vol. v. pp. 451, 455, 463, 472, Dublin, 1895.

|| R. Collett, "Fiske indsamlede under 'Michael Sars' s Togter i Nordhavet 1900-1902," *Report on Norwegian Fishery and Marine Investigations*, vol. ii., No. 3, pp. 55, 56, 58, 62, Bergen, 1905.

¶ C. Lütken, *Ichthyological Results, etc.*, pp. 24, 26, 27, 28.

** D. S. Jordan and C. H. Gilbert, "The Fishes of Bering Sea," *The Fur Seals and Fur-Seal Islands of the North Pacific Ocean*, part 3, p. 487, Washington, 1899.

†† T. Gill and C. H. Townsend, "Diagnoses of New Species of Fishes found in Bering Sea," *Proceedings of the Biological Society of Washington*, vol. xi. p. 234, Washington, 1897.

‡‡ A. Günther, *Deep-Sea Fishes, etc.*, p. 153.

T. H. Tizard and J. Murray, "Exploration of the Faroe Channel, during the summer of 1880, in H.M.'s hired ship 'Knight Errant,'" *Proceedings of the Royal Society of Edinburgh*, vol. xi. p. 660, Edimbourg, 1882.

§§ A. Günther, *Deep-Sea Fishes, etc.*, pp. 129, 141, 150, 157.

Il est impossible, pour le moment, de savoir exactement ce qu'est le Macruride du Canal

2. *Résultats*.—Condensant et comparant, nous obtenons :

		S. Familles.	Genres.	Espèces.	
En tout :	A l'intérieur du Cercle Polaire :	{ Arctique : 1	1	1	
		{ Antarctique : 1	2	2	
	Entre 60° Lat. et le Cercle Polaire :	{ Arctique : 2	6	8	
		{ Antarctique : 1	1	1	
	Entre 50° et 60° de Latitude :	{ Nord : 2	9	20	
{ Sud : 2		4	4		
En commun, au Nord et au Sud :	Espèces :	{ A l'intérieur du Cercle Polaire	0	0	
		{ Entre 60° Lat. et le Cercle Polaire	0	0	
		{ Entre 50° Lat. et 60° Lat.	0	0	
	Genres : *	{ A l'intérieur du Cercle Polaire	0	0	
		{ Entre 60° Lat. et le Cercle Polaire	$\frac{1}{6}$	$\frac{1}{6}$	
		{ Entre 50° Lat. et 60° Lat.	$\frac{2}{11}$	$\frac{2}{11}$	
Genres dominants :	Au Nord :	{ Arctique	×	
		{ Atlantique	×	
		{ Pacifique	×	
		<i>Macrurus</i> :	{ Indique	×
			{ Antarctique	0
	Au Sud :	{ Arctique	0	
		{ Atlantique	×	
		{ Pacifique	×	
		<i>Nematonurus</i> :	{ Indique	×
			{ Antarctique	×

III. *Conclusions*.—1. *Macruridae*.—Il résulte de ce qui précède que les Macrurides, comme les autres Poissons,† ou même comme les autres du Beagle (Terre de Feu) recueilli par l'Expédition Antarctique Suédoise (E. Lönnberg, *Fishes of the Swedish South Polar Expedition, etc.*, p. 9).

Quant au *Macruronus magellanicus* (E. Lönnberg, "Fische," *Ergebnisse der Hamburger Magalhaensischen Sammelreise*, p. 15, Hambourg, 1907), je ne vois pas de raisons suffisantes pour le séparer du *M. novæ-zelandiæ*.

Ce que confirme, biogéographiquement, la distribution de *Cælorhynchus fasciatus*, *Genypterus blacodes*, *Macruronus novæ-zelandiæ* et *Neophrynichthys latus* (L. Dollo, *Poissons de l'Expédition Antarctique Belge, etc.*, pp. 214, 215 ; A. R. M'Culloch, "Fishes and Crustaceans from Eight Hundred Fathoms, The Expedition of the 'Woy Woy,' The Results of Deep-Sea Investigation in the Tasman Sea," *Records of the Australian Museum*, vol. vi. p. 348, Sydney, 1907).

* Les deux genres communs au Nord et au Sud, *Cælorhynchus* et *Nematonurus*, sont d'ailleurs, cosmopolites.

† L. Dollo, *Poissons de l'Expédition Antarctique Belge, etc.*, pp. 199, 205.

L. Dollo, *Notolepis Coatsi, etc.*, p. 61.

Vertébrés,* sont défavorables à la théorie de la Bipolarité,—puisqu'il n'en existe pas une Espèce commune aux deux Zones situées entre 50° de Latitude et le Pôle correspondant, ni non plus un Genre commun à l'intérieur des deux Cercles Polaires.

2. *Méduses*.—Comment, dès lors, expliquer les cas cités par M. Vanhöffen ?

Mais, en ce qui concerne les Méduses, ce Zoologiste ne retient comme polaires que les espèces suivantes : †

ARCTIQUES :	ANTARCTIQUES :
1. <i>Halicyathus lagena</i> ,	0
2. <i>Lucernaria quadricornis</i> ,	0
3. <i>Nausithoë polaris</i> ,	0
4. <i>Cyanea arctica</i> ,	0
5. <i>Aurelia limbata</i> ,	0
0	1. <i>Lucernaria australis</i> ,
0	2. <i>Lucernaria sp.</i> ,
0	3. <i>Desmonema Chierchiana</i> ,
0	4. <i>Ulmaropsis Drygalskii</i> ,

et il n'y a pas une seule Espèce commune au Nord et au Sud dans sa liste entière, qui en comprend cependant 31.

3. *Radiolaires*.—Quant aux Radiolaires, M. le Docteur A. Popofsky, de Magdebourg, interprète les 4 Espèces communes au Nord et au Sud par la Continuité en Profondeur.‡

Ce serait donc du Cosmopolitisme.

4. *Bipolarité*.—Or, comme je l'ai déjà dit,§—en dehors des constatations de fait, à condition qu'elles soient établies d'une manière indiscutable,—je ne me refuse pas à admettre, théoriquement, la possibilité d'Espèces

* L. Dollo, *Poissons de l'Expédition Antarctique Belge, etc.*, pp. 199, 205.

L. Dollo, *Notolepis Coatsi, etc.*, p. 61.

† E. Vanhöffen, "Die Lucernariden und Skyphomedusen," *Deutsche Südpolar-Expedition, 1901-1903*, vol. x. pp. 48, 49, Berlin, 1908.

‡ A. Popofsky, "Die Radiolarien der Antarktis," *Deutsche Südpolar-Expedition, 1901-1903*, vol. x. pp. 197, 198, Berlin, 1908.

§ L. Dollo, *Poissons de l'Expédition Antarctique Belge, etc.*, p. 207.

M. G. Enderlein, Conservateur du Musée de Stettin, a traité, récemment, en détail, la question de la Bipolarité et celle de l'Antarctide (G. Enderlein, "Die biologische Bedeutung der Antarktis," *Deutsche Südpolar-Expedition, 1901-1903*, vol. x. p. 352, Berlin, 1909), en s'appuyant principalement sur les Insectes, mais,—bien que je me sois occupé des mêmes sujets, d'une manière approfondie, et bien que le Zoologiste allemand fasse aussi usage des Poissons pour ses déductions,—mon mémoire de la *Belgica* lui est resté complètement inconnu,—quoiqu'il ait paru dans un recueil uniquement consacré aux problèmes antarctiques.

bipolaires sporadiques, même avec Aires de Dispersion discontinues, à cause des *Migrations en Profondeur*.

Ce que je ne puis accepter, c'est que les *Faunes polaires* soient des *Reliquats* à peine modifiés de la *Faune universelle* de la *Shallow-Water* (G. Pfeffer) ou de la *Mud-Line* (J. Murray).*

J'ai donné mes raisons jadis, et les faits nouveaux qui sont venus depuis à ma connaissance, notamment sur les *Macruridæ*, non plus que ceux relevés par M. Vanhöffen, ne sont pas de nature à modifier mon opinion.

Je garde donc vis-à-vis de la Bipolarité la même attitude qu'en 1904.*

* L. Dollo, *Poissons de l'Expédition Antarctique Belge, etc.*, p. 207.

XXXII.—The Theory of Jacobians in the Historical Order of Development up to 1860. By Thomas Muir, LL.D.

(MS. received March 22, 1909. Read June 7, 1909.)

MY last communication in reference to the history of Jacobians dealt with the period 1815-1841 (*Proc. Roy. Soc. Edin.*, xxiv. pp. 151-195). The present paper continues the history up to 1860.

JACOBI, C. G. J. (1844-1845).

[Theoria novi multiplicatoris systemati æquationum differentialium vulgarium applicandi. *Crelle's Journ.*, xxvii. pp. 199-268; xxix. pp. 218-279, 333-376: or *Math. Werke* (1846), i. pp. 47-226: or *Gesammelte Werke*, iv. pp. 317-509.]

The portion of this long memoir which is of interest to us in the present connection is the first section (pp. 201-209) of the first chapter, the heading being "Lemma fundamentale eiusque varii usus: de determinantibus functionalibus partialibus." Passing over the treatment of the first two cases of the lemma we come upon the general enunciation of it, which is—

If A, A_1, A_2, \dots, A_n be the cofactors of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}$ in the

determinant $\sum \left(\pm \frac{\partial f}{\partial x} \frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} \dots \frac{\partial f_n}{\partial x_n} \right)$, then

$$\frac{\partial A}{\partial x} + \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \dots + \frac{\partial A_n}{\partial x_n} = 0.$$

Preparatory for the proof it is pointed out that since

$$\frac{\partial f}{\partial x} A + \frac{\partial f}{\partial x_1} A_1 + \dots + \frac{\partial f}{\partial x_n} A_n = \sum \left(\pm \frac{\partial f}{\partial x} \frac{\partial f_1}{\partial x_1} \dots \frac{\partial f_n}{\partial x_n} \right)$$

an alternative form of the lemma is

$$\frac{\partial(fA)}{\partial x} + \frac{\partial(fA_1)}{\partial x_1} + \dots + \frac{\partial(fA_n)}{\partial x_n} = \sum \left(\pm \frac{\partial f}{\partial x} \frac{\partial f_1}{\partial x_1} \dots \frac{\partial f_n}{\partial x_n} \right).$$

Then calling the given determinant R , and noting that A, A_1, \dots, A_n are themselves functional determinants, A_i being the determinant of f_1, f_2, \dots, f_n with respect to $x, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$, Jacobi seeks to prove the lemma true in the case of R from assuming it true in the case of A_i . To be able to formulate it in the latter case he takes each element of the first

row of R along with each element of the second row, thus forming $(n+1)^2$ products whose cofactors in the determinant he denotes by

$$\begin{matrix} (00) & (01) & (02) & \dots & (0n) \\ (10) & (11) & (12) & \dots & (1n) \\ (20) & (21) & (22) & \dots & (2n) \\ \dots & \dots & \dots & \dots & \dots \\ (n0) & (n1) & (n2) & \dots & (nn) \end{matrix}$$

—that is to say, he puts

$$(i\kappa) \text{ for the cofactor of } \frac{\partial f}{\partial x_i} \frac{\partial f_1}{\partial x_\kappa} \text{ in R,}$$

a notation which necessitates

$$(ik) = -(ki) \quad \text{and} \quad (ii) = 0.$$

It follows on this that the cofactors of

$$\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial x_1}, \dots, \frac{\partial f_1}{\partial x_{i-1}}, \frac{\partial f_1}{\partial x_{i+1}}, \dots, \frac{\partial f_1}{\partial x_n}$$

in A_i are $(i, 0), (i, 1), \dots, (i, n)$, and thus the assumption above made takes the form

$$\frac{\partial(i, 0)}{\partial x} + \frac{\partial(i, 1)}{\partial x_1} + \dots + \frac{\partial(i, n)}{\partial x_n} = 0,$$

or
$$\frac{\partial\{(i, 0)f_1\}}{\partial x} + \frac{\partial\{(i, 1)f_1\}}{\partial x_1} + \dots + \frac{\partial\{(i, n)f_1\}}{\partial x_n} = A_i.$$

Since, however, $(i, k)f_1 = -(k, i)f_1$ and $(i, i)f_1 = 0$, we can apply to the latter the general proposition that if a_{ik} be any quantities whatever such that $a_{ik} = -a_{ki}$, $a_{ii} = 0$, and H_i stand for

$$\frac{\partial a_{i0}}{\partial x} + \frac{\partial a_{i1}}{\partial x_1} + \dots + \frac{\partial a_{in}}{\partial x_n}$$

then

$$\frac{\partial H_i}{\partial x} + \frac{\partial H_1}{\partial x_1} + \dots + \frac{\partial H_n}{\partial x_n} = 0,*$$

as desired. There only then remains to show that the lemma holds in the case of two variables, and this is unnecessary because it is then identical with the familiar proposition

$$\frac{\partial f_1}{\partial x \partial y} = \frac{\partial f_1}{\partial y \partial x}.$$

In addition to this gradational proof Jacobi gives one of a different

* The reason for this, of course, is that $\frac{\partial \frac{\partial a_{ik}}{\partial x_k}}{\partial x_i} + \frac{\partial \frac{\partial a_{ki}}{\partial x_i}}{\partial x_k} = 0.$

kind. Since A_i , he says, does not involve differential-coefficients with respect to x_i , it follows that $\frac{\partial A_i}{\partial x_i}$ and $\sum \frac{\partial A_i}{\partial x_i}$ cannot involve differential-coefficients taken twice with respect to any one variable. Further, second differential-coefficients taken with respect to different variables x_i, x_k cannot occur anywhere save in*

$$\frac{\partial A_i}{\partial x_i} + \frac{\partial A_k}{\partial x_k}.$$

All we have got to show therefore is that the cofactor of $\frac{\partial^2 f_m}{\partial x_i \partial x_k}$ in $\frac{\partial A_i}{\partial x_i} + \frac{\partial A_k}{\partial x_k}$ vanishes. To do this we express A_i in terms of the elements of one column and their cofactors, say

$$A_i = \alpha_1 \frac{\partial f_1}{\partial x_k} + \alpha_2 \frac{\partial f_2}{\partial x_k} + \dots + \alpha_n \frac{\partial f_n}{\partial x_k},$$

and thus know as above that

$$A_k = -\alpha_1 \frac{\partial f_1}{\partial x_i} - \alpha_2 \frac{\partial f_2}{\partial x_i} - \dots - \alpha_n \frac{\partial f_n}{\partial x_i},$$

where $\alpha_1, \alpha_2, \dots$ involve no differential-coefficients taken with respect to x_i or with respect to x_k .

The observation made in the course of the first proof that A, A_1, \dots, A_n are themselves functional determinants leads Jacobi to the conception of "partial functional determinants" on the analogy of partial differential quotients. The fundamental lemma then becomes viewable as the analogue of

$$\frac{\partial \frac{\partial f_1}{\partial y}}{\partial x} - \frac{\partial \frac{\partial f_1}{\partial x}}{\partial y} = 0,$$

or, in Jacobi's words, "gravissimam manifestat analogiam determinantium functionalium et quotientium differentialium partialium."

Apparently this recalls to Jacobi another analogy of the same kind, which he had omitted to draw attention to in his paper of 1830, when the first two cases of the lemma had been originally enunciated by him. The proposition involving the said analogy he now generalises thus:—*If f, f_1, f_2, \dots, f_n be expressible as series the terms of which involve only powers of the variables x, x_1, x_2, \dots, x_n , the functional determinant does not*

* It would have been well to make clear here that every term of the final expansion of $\sum \frac{\partial A_i}{\partial x_i}$ contains one and only one second differential-coefficient.

involve a term in $x^{-1} x_1^{-1} x_2^{-1} \dots x_n^{-1}$. In support of it he has only to point out that the functional determinant is equal to

$$\frac{\partial(fA)}{\partial x} + \frac{\partial(fA_1)}{\partial x_1} + \dots + \frac{\partial(fA_n)}{\partial x_n},$$

and that the development of the k^{th} term of this expansion cannot contain a term in $\frac{1}{x_{k-1}}$.

After referring to a possible application of the lemma in connection with definite multiple integrals, Jacobi concludes § 2 by returning to the lemma itself and throwing it into a third form originally announced in 1841 (*De determ. funct.* § 9). Viewing x, x_1, x_2, \dots, x_n as functions of f, f_1, f_2, \dots, f_n he obtains of course (*De determ. funct.* § 8)

$$\frac{\partial x}{\partial f} = \frac{A}{R}, \quad \frac{\partial x_1}{\partial f} = \frac{A_1}{R}, \quad \dots, \quad \frac{\partial x_n}{\partial f} = \frac{A_n}{R},$$

so that by substitution the lemma becomes

$$\begin{aligned} 0 &= \frac{\partial \left(R \frac{\partial x}{\partial f} \right)}{\partial x} + \frac{\partial \left(R \frac{\partial x_1}{\partial f} \right)}{\partial x_1} + \dots + \frac{\partial \left(R \frac{\partial x_n}{\partial f} \right)}{\partial x_n}, \\ &= \left. \begin{aligned} &\frac{\partial x}{\partial f} \frac{\partial R}{\partial x} + \frac{\partial x_1}{\partial f} \frac{\partial R}{\partial x_1} + \dots + \frac{\partial x_n}{\partial f} \frac{\partial R}{\partial x_n} \\ &+ R \frac{\partial}{\partial x} \left(\frac{\partial x}{\partial f} \right) + R \frac{\partial}{\partial x_1} \left(\frac{\partial x_1}{\partial f} \right) + \dots + R \frac{\partial}{\partial x_n} \left(\frac{\partial x_n}{\partial f} \right) \end{aligned} \right\}, \\ &= \frac{\partial R}{\partial f} + R \left\{ \frac{\partial}{\partial x} \frac{\partial x}{\partial f} + \frac{\partial}{\partial x_1} \frac{\partial x_1}{\partial f} + \dots + \frac{\partial}{\partial x_n} \frac{\partial x_n}{\partial f} \right\}, \end{aligned}$$

or

$$0 = \frac{\partial \log R}{\partial f} + \frac{\partial}{\partial x} \frac{\partial x}{\partial f} + \frac{\partial}{\partial x_1} \frac{\partial x_1}{\partial f} + \dots + \frac{\partial}{\partial x_n} \frac{\partial x_n}{\partial f},$$

where firstly R and the x 's have to be viewed as functions of the f 's and all differentiated with respect to f , and secondly the differential-coefficients thus obtained have to be expressed in terms of the x 's preparatory to performing the final set of differentiations.

Here the consideration of functional determinants would have come to an end in the present memoir, had it not been that a theorem on the subject which had been given incorrectly in 1841 (*De determ. funct.* § 14) was now wanted in § 3 for use. Two and a half pages (pp. 215–217) of matter are consequently intercalated in order to enunciate the theorem

correctly, to prove it, and to elucidate it by a commentary. The enunciation is—If $f_1 = a_1, f_2 = a_2, \dots, f_n = a_n$, where the a 's are constants, the functional determinant

$$\sum \left(\pm \frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} \dots \frac{\partial f_n}{\partial x_n} \right)$$

will not be altered in value if before performing the differentiations every function f_i be transformed in any way whatever by means of the equations $f_{i+1} = a_{i+1}, f_{i+2} = a_{i+2}, \dots, f_n = a_n$. On looking back it will be seen that Jacobi had previously not excluded the use of the first $i-1$ equations in making transformation of f_i . His proof now depends on taking the a 's to the same side as the f 's; denoting the resulting equations by $\phi_1 = 0, \phi_2 = 0, \dots, \phi_n = 0$; applying his theorem (*De determ. funct.* § 10) to obtain the desired determinant in the form

$$(-1)^n \frac{\sum \left(\pm \frac{\partial \phi_1}{\partial x_1} \frac{\partial \phi_2}{\partial x_2} \dots \frac{\partial \phi_n}{\partial x_n} \right)}{\sum \left(\pm \frac{\partial \phi_1}{\partial a_1} \frac{\partial \phi_2}{\partial a_2} \dots \frac{\partial \phi_n}{\partial a_n} \right)};$$

and then showing that the denominator of this is $(-1)^n$. His commentary closes by assuming as allowable that

$$\phi_i = \lambda_1^{(i)} f_1 + \lambda_2^{(i)} f_2 + \dots + \lambda_n^{(i)} f_n,$$

and having thus obtained

$$\sum \left(\pm \frac{\partial \phi_1}{\partial x_1} \frac{\partial \phi_2}{\partial x_2} \dots \frac{\partial \phi_n}{\partial x_n} \right) = \sum \pm \lambda_1^{(1)} \lambda_2^{(2)} \dots \lambda_n^{(n)} \cdot \sum \left(\pm \frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} \dots \frac{\partial f_n}{\partial x_n} \right)$$

he concludes that the condition for the equality of the two functional determinants is that the determinant of the λ 's shall be equal to 1.

HESSE (1844).

[Ueber die Elimination der Variabeln aus drei algebraischen Gleichungen vom zweiten Grade mit zwei Variabeln. *Crelle's Journ.*, xxviii. pp. 68-96: or *Werke*, pp. 89-122.]

Having demonstrated that for the elimination of x_1, x_2, x_3 from the three quadrics f_1, f_2, f_3 it was important to discover a function of the third degree possessing certain properties, Hesse proceeds (§ 11) to show how such a function may be obtained. From a well-known theorem regarding the differentiation of homogeneous functions he has, on putting $u_\kappa^{(\lambda)}$ for $\frac{\partial f_\kappa}{\partial x_\lambda}$,

$$\left. \begin{aligned} x_1 u_1^{(1)} + x_2 u_1^{(2)} + x_3 u_1^{(3)} &= 2f_1, \\ x_1 u_2^{(1)} + x_2 u_2^{(2)} + x_3 u_2^{(3)} &= 2f_2, \\ x_1 u_3^{(1)} + x_2 u_3^{(2)} + x_3 u_3^{(3)} &= 2f_3, \end{aligned} \right\};$$

and, ϕ being Jacobi's determinant of f_1, f_2, f_3 with respect to x_1, x_2, x_3 , there thus follows

$$(a) \quad x_1 \phi = 2f_1(u_2^{(2)}u_3^{(3)} - u_2^{(3)}u_3^{(2)}) + 2f_2(u_3^{(2)}u_1^{(3)} - u_3^{(3)}u_1^{(2)}) + 2f_3(u_1^{(2)}u_2^{(3)} - u_1^{(3)}u_2^{(2)}),$$

with similar expressions for $x_2 \phi, x_3 \phi$; so that any set of values of x_1, x_2, x_3 which makes f_1, f_2, f_3 vanish will make ϕ vanish also. The formal enunciation of this result is then given, and it is pointed out that the like theorem holds when there are n homogeneous functions all of n variables and of the r^{th} degree.

From (a) by differentiation there is next obtained

$$\begin{aligned} & x_1 \frac{\partial \phi}{\partial x_1} + \phi \\ &= 2u_1^{(1)}(u_2^{(2)}u_3^{(3)} - u_2^{(3)}u_3^{(2)}) + 2u_2^{(1)}(u_3^{(2)}u_1^{(3)} - u_3^{(3)}u_1^{(2)}) + 2u_3^{(1)}(u_1^{(2)}u_2^{(3)} - u_1^{(3)}u_2^{(2)}) \\ &+ 2f_1 \frac{\partial}{\partial x_1} (u_2^{(2)}u_3^{(3)} - u_2^{(3)}u_3^{(2)}) + 2f_2 \frac{\partial}{\partial x_1} (u_3^{(2)}u_1^{(3)} - u_3^{(3)}u_1^{(2)}) + 2f_3 \frac{\partial}{\partial x_1} (u_1^{(2)}u_2^{(3)} - u_1^{(3)}u_2^{(2)}), \end{aligned}$$

and thence

$$\begin{aligned} & x_1 \frac{\partial \phi}{\partial x_1} - \phi \\ &= 2f_1 \frac{\partial}{\partial x_1} (u_2^{(2)}u_3^{(3)} - u_2^{(3)}u_3^{(2)}) + 2f_2 \frac{\partial}{\partial x_1} (u_3^{(2)}u_1^{(3)} - u_3^{(3)}u_1^{(2)}) + 2f_3 \frac{\partial}{\partial x_1} (u_1^{(2)}u_2^{(3)} - u_1^{(3)}u_2^{(2)}), \end{aligned}$$

so that any set of values of x_1, x_2, x_3 which makes the ternary quadrics f_1, f_2, f_3 vanish, will make the first differential quotients of the determinant of f_1, f_2, f_3 vanish also,—a second theorem which is asserted to hold when the number of homogeneous functions is n , the number of variables n , and each function of the r^{th} degree in the said variables.

Combining the two results, and using later phraseology, we may say that *When n n -ary n -thics vanish, their Jacobian and each of its first differential-quotients will vanish also.*

The connection of this with the problem of elimination can be indicated in a few words. Jacobi's determinant ϕ being of the third degree in x_1, x_2, x_3 , its first differential-quotients are like f_1, f_2, f_3 linear in $x_1^2, x_2^2, x_3^2, x_2x_3, x_3x_1, x_1x_2$; and consequently the resultant is at once obtained as a six-line determinant.

CAYLEY, A. (1847, February).

[On the differential equations which occur in dynamical problems. *Cambridge and Dub. Math. Journ.*, ii. pp. 210-219: or *Collected Math. Papers*, i. pp. 276-284.]

This is a short exposition of Jacobi's elaborate memoir of 1844 with considerable variation in the details. The portion (§ 1) which concerns us is of course that referring to the "fundamental lemma." This is established in its third form, the proof, like that originally given by Jacobi, being dependent on the theorem

$$\frac{\partial R}{\partial \frac{df_i}{dx_k}} = R \frac{\partial x_k}{\partial f_i},$$

but differing in appearance, mainly because of the use of differentials.

BERTRAND, J. (1851, February).

[Mémoire sur le déterminant d'un système de fonctions. *Journ. (de Liouville) de Math.*, xvi. pp. 212-227: abstract in *Comptes Rendus . . . Acad. des. Sci. (Paris)*, xxxii. pp. 134-135.]

Recalling how Jacobi had insisted on the marked analogy between a functional-determinant and a differential-coefficient, Bertrand at once intimates the adoption of a new definition of the former which in his opinion makes the analogy still more striking, and from which the properties of the determinant are deducible like mere corollaries.

Save that Δ and δ are used where Bertrand without distinction uses d , the following is the definition:—If f_1, f_2, \dots, f_n be functions of x_1, x_2, \dots, x_n , and the latter receive n distinct sets of increments

$$\begin{array}{cccc} \Delta_1 x_1 & \Delta_1 x_2 & \dots & \Delta_1 x_n \\ \Delta_2 x_1 & \Delta_2 x_2 & \dots & \Delta_2 x_n \\ \dots & \dots & \dots & \dots \\ \Delta_n x_1 & \Delta_n x_2 & \dots & \Delta_n x_n \end{array}$$

with the result that the corresponding increments of the functions are

$$\begin{array}{cccc} \Delta_1 f_1 & \Delta_1 f_2 & \dots & \Delta_1 f_n \\ \Delta_2 f_1 & \Delta_2 f_2 & \dots & \Delta_2 f_n \\ \dots & \dots & \dots & \dots \\ \Delta_n f_1 & \Delta_n f_2 & \dots & \Delta_n f_n, \end{array}$$

then the limiting value of the ratio of the determinant of the second array to the determinant of the first array as the elements of the latter array are indefinitely diminished is called the determinant of the n functions. Since in the circumstances mentioned

$$\Delta_k f_i = \frac{\partial f_i}{\partial x_1} \Delta_k x_1 + \frac{\partial f_i}{\partial x_2} \Delta_k x_2 + \dots + \frac{\partial f_i}{\partial x_n} \Delta_k x_n$$

for all values of k and i not greater than n , it follows from the multiplication-theorem that the aforesaid limiting value is equal to

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{vmatrix};$$

and, this determinant being independent of the increments given to the independent variables, it is held that the definition is legitimised. It might have been added that the name assigned to the limiting value is also thereby justified.

The more important of Jacobi's results, eight or nine in number, are then re-established, precedence being given to those regarding the vanishing of the determinant. Supposing, first, that the functions are independent of one another, he asserts that x_1, x_2, \dots, x_n may be conceived as expressed in terms of f_1, f_2, \dots, f_n ; and, the latter being viewed as independent variables, the determinant of their increments can be considered as completely arbitrary and can thus have a value different from zero. Further, in respect to this determinant the determinant of the increments of x_1, x_2, \dots, x_n cannot be infinitely great, because the terms of both determinants have the same number of infinitesimal factors of the first order. It thus follows that their quotient—that is, the functional determinant—is not zero. Next, supposing that the functions are not all independent of one another, but that $f_{p+1}, f_{p+2}, \dots, f_n$ are functions of f_1, f_2, \dots, f_p , and that the latter alone are mutually independent, Bertrand asserts that we may suppose

$$\Delta_1 f_1 = 0, \quad \Delta_1 f_2 = 0, \quad \dots, \quad \Delta_1 f_p = 0,$$

this in fact being possible in an infinite number of ways, because only p relations are thereby established between the increments $\Delta_1 x_1, \Delta_1 x_2, \dots, \Delta_1 x_n$. It will then result that the increments of $f_{p+1}, f_{p+2}, \dots, f_n$ being

sums of multiples of the increments of f_1, f_2, \dots, f_p will also be zero; and thus the whole of the first row

$$\Delta_1 f_1 \quad \Delta_1 f_2 \quad \dots \quad \Delta_1 f_n$$

will be composed of zeros, and the determinant to which it belongs will vanish. On the other hand the determinant of the increments of the x 's can at the same time be made different from zero, the increments in the first row being not necessarily all zero and those in the other rows being what we please. The ratio of the two determinants therefore vanishes.

Following this are given the theorem regarding the relation of

$$\sum \left(\pm \frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} \dots \frac{\partial f_n}{\partial x_n} \right) \text{ to } \sum \left(\pm \frac{\partial x_1}{\partial f_1} \frac{\partial x_2}{\partial f_2} \dots \frac{\partial x_n}{\partial f_n} \right):$$

the theorem for finding the functional determinant when the f 's are given *mediately* as functions of the x 's, that is to say, as functions of $\phi_1(x_1, x_2, \dots, x_n), \phi_2(x_1, x_2, \dots, x_n), \dots, \phi_p(x_1, x_2, \dots, x_n)$: and the corresponding theorem when the functions are only *implicitly* given, that is to say, by means of connecting equations

$$F_1(x_1, x_2, \dots, x_n, f_1, f_2, \dots, f_n) = 0, \dots, F_n(x_1, x_2, \dots, x_n, f_1, f_2, \dots, f_n) = 0.$$

The mode of treatment will be readily guessed from what has gone before.

The same cannot, however, be confidently affirmed in connection with the theorem which expresses the functional determinant as a single product. This is found grouped under the heading "Diverses formes que l'on peut donner à un déterminant" (fonctionnel), the said forms being obtainable on varying the systems of increments assigned to the variables. In the first example, the array of increments of the x 's is taken to be

$$\begin{array}{cccc} \Delta_1 x_1 & 0 & \dots & 0 \\ 0 & \Delta_2 x_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \Delta_n x_n \end{array}$$

which necessitates the other array to be

$$\begin{array}{cccc} \frac{\partial f_1}{\partial x_1} \Delta_1 x_1 & \frac{\partial f_2}{\partial x_1} \Delta_1 x_1 & \dots & \frac{\partial f_n}{\partial x_1} \Delta_1 x_1 \\ \frac{\partial f_1}{\partial x_2} \Delta_2 x_2 & \frac{\partial f_2}{\partial x_2} \Delta_2 x_2 & \dots & \frac{\partial f_n}{\partial x_2} \Delta_2 x_2 \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_1}{\partial x_n} \Delta_n x_n & \frac{\partial f_2}{\partial x_n} \Delta_n x_n & \dots & \frac{\partial f_n}{\partial x_n} \Delta_n x_n \end{array}$$

and the ratio of the determinants of the arrays to be

$$\sum \left(\pm \frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} \dots \frac{\partial f_n}{\partial x_n} \right).$$

To this there is added "C'est l'expression donnée comme définition par M. Jacobi,"—a remark, however, equally applicable when, as at the outset, the increments of the x 's were taken in their most general form. Preparatory to the next example it is pointed out that any n of the variables

$$x_1, x_2, \dots, x_n, f_1, f_2, \dots, f_n$$

may have arbitrary values, the other n being then determinable; and that therefore if $n - 1$ of them be taken to be invariable, the ratios of the increments of the others may be considered known. We thus see that the two arrays

$$\begin{array}{cccccccccc} \Delta_1 x_1 & \Delta_1 x_2 & \Delta_1 x_3 & \dots & \Delta_1 x_n & \Delta_1 f_1 & 0 & 0 & \dots & 0 \\ 0 & \Delta_2 x_2 & \Delta_2 x_3 & \dots & \Delta_2 x_n & \Delta_2 f_1 & \Delta_2 f_2 & 0 & \dots & 0 \\ 0 & 0 & \Delta_3 x_3 & \dots & \Delta_3 x_n & \Delta_3 f_1 & \Delta_3 f_2 & \Delta_3 f_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \Delta_n x_n & \Delta_n f_1 & \Delta_n f_2 & \Delta_n f_3 & \dots & \Delta_n f_n \end{array}$$

of the second example are simultaneously possible, the n independent variables in the case of the first row being $x_1, f_2, f_3, \dots, f_n$, in the case of the second row $x_1, x_2, f_3, \dots, f_n$, and so on: and we consequently learn that the functional determinant may be written in the form

$$\left(\frac{\partial f_1}{\partial x_1} \right) \left(\frac{\partial f_2}{\partial x_2} \right) \dots \left(\frac{\partial f_n}{\partial x_n} \right)$$

on the understanding that the brackets enclosing $\partial f_r / \partial x_r$ imply that f_r is there viewed as a function of $x_1, x_2, \dots, x_r, f_{r+1}, f_{r+2}, \dots, f_n$. A third pair of possible arrays is

$$\begin{array}{cccccccccc} \Delta_1 x_1 & 0 & 0 & \dots & 0 & \Delta_1 x_{m+1} & \Delta_1 x_{m+2} & \dots & \Delta_1 x_n \\ 0 & \Delta_2 x_2 & 0 & \dots & 0 & \Delta_2 x_{m+1} & \Delta_2 x_{m+2} & \dots & \Delta_2 x_n \\ 0 & 0 & \Delta_3 x_3 & \dots & 0 & \Delta_3 x_{m+1} & \Delta_3 x_{m+3} & \dots & \Delta_3 x_n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \Delta_m x_m & \Delta_m x_{m+1} & \Delta_m x_{m+2} & \dots & \Delta_m x_n \\ 0 & 0 & 0 & \dots & 0 & \Delta_{m+1} x_{m+1} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \Delta_{m+2} x_{m+2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \Delta_n x_n \end{array}$$

and

$$\begin{array}{cccccccc}
 \Delta_1 f_1 & \Delta_1 f_2 & \Delta_1 f_3 & \dots & \Delta_1 f_m & 0 & 0 & \dots & 0 \\
 \Delta_2 f_1 & \Delta_2 f_2 & \Delta_2 f_3 & \dots & \Delta_2 f_m & 0 & 0 & \dots & 0 \\
 \Delta_3 f_1 & \Delta_3 f_2 & \Delta_3 f_3 & \dots & \Delta_3 f_m & 0 & 0 & \dots & 0 \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 \Delta_m f_1 & \Delta_m f_2 & \Delta_m f_3 & \dots & \Delta_m f_m & 0 & 0 & \dots & 0 \\
 \Delta_{m+1} f_1 & \Delta_{m+1} f_2 & \Delta_{m+1} f_3 & \dots & \Delta_{m+1} f_m & \Delta_{m+1} f_{m+1} & \Delta_{m+1} f_{m+2} & \dots & \Delta_{m+1} f_n \\
 \Delta_{m+2} f_1 & \Delta_{m+2} f_2 & \Delta_{m+2} f_3 & \dots & \Delta_{m+2} f_m & \Delta_{m+2} f_{m+1} & \Delta_{m+2} f_{m+2} & \dots & \Delta_{m+2} f_n \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 \Delta_n f_1 & \Delta_n f_2 & \Delta_n f_3 & \dots & \Delta_n f_m & \Delta_n f_{m+1} & \Delta_n f_{m+2} & \dots & \Delta_n f_n
 \end{array}$$

from which the functional determinant is obtained in the form

$$\sum \left[\pm \left(\frac{\partial f_1}{\partial x_1} \right) \left(\frac{\partial f_2}{\partial x_2} \right) \dots \left(\frac{\partial f_m}{\partial x_m} \right) \right] \cdot \sum \left(\pm \frac{\partial f_{m+1}}{\partial x_{m+1}} \frac{\partial f_{m+2}}{\partial x_{m+2}} \dots \frac{\partial f_n}{\partial x_n} \right)$$

where, from looking as before at corresponding rows of the two arrays, we see that in the first determinant f_1, f_2, \dots, f_m are to be viewed as functions of $x_1, x_2, \dots, x_m, f_{m+1}, f_{m+2}, \dots, f_n$, and in the second determinant $f_{m+1}, f_{m+2}, \dots, f_m$ are to be viewed as functions of x_1, x_2, \dots, x_n .

The next section is still more interesting, as it concerns the proposition which Jacobi stated incorrectly in his original memoir of 1841 and returned to in 1844. The data according to Bertrand are the usual n functions f_1, f_2, \dots, f_n each dependent on x_1, x_2, \dots, x_n , with the addition that the said functions when expressed in terms of $x_1, x_2, \dots, x_n, f_1, f_2, \dots, f_n$ become $\phi_1, \phi_2, \dots, \phi_n$: and the problem he sets himself is to find the relation between

$$\sum \left(\pm \frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} \dots \frac{\partial f_n}{\partial x_n} \right) \text{ and } \sum \left(\pm \frac{\partial \phi_1}{\partial x_1} \frac{\partial \phi_2}{\partial x_2} \dots \frac{\partial \phi_n}{\partial x_n} \right),$$

the differentiations in the latter determinant being performed on the understanding that the f 's there occurring in the ϕ 's are to be viewed as constants. The equations

$$\phi_1 - f_1 = 0, \quad \phi_2 - f_2 = 0, \quad \dots, \quad \phi_n - f_n = 0,$$

may be held to give implicitly the f 's as functions of the x 's, and therefore by a previous result

$$\sum \left(\pm \frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} \dots \frac{\partial f_n}{\partial x_n} \right) = (-1) \sum \pm \left(\frac{\partial \phi_1}{\partial x_1} \frac{\partial \phi_2}{\partial x_2} \dots \frac{\partial \phi_n}{\partial x_n} \right)$$

$\frac{\partial \phi_1}{\partial f_1} - 1$	$\frac{\partial \phi_1}{\partial f_2}$	\dots	$\frac{\partial \phi_1}{\partial f_n}$
$\frac{\partial \phi_2}{\partial f_1}$	$\frac{\partial \phi_2}{\partial f_2} - 1$	\dots	$\frac{\partial \phi_2}{\partial f_n}$
\dots	\dots	\dots	\dots
$\frac{\partial \phi_n}{\partial f_1}$	$\frac{\partial \phi_n}{\partial f_2}$	\dots	$\frac{\partial \phi_n}{\partial f_n} - 1$

which is the relation desired. If ϕ_1 does not involve f_1 , ϕ_2 does not involve f_1 or f_2 , ϕ_3 does not involve f_1 or f_2 or f_3 , and so on,* the determinant in the denominator takes the value $(-1)^n$, and the relation becomes one of equality.

The last section deals with the theorem regarding the change of variables in multiple integrals,—a theorem which in the ten years from Jacobi's memoir to Bertrand's had been discussed by Boole † and Dienger ‡

SPOTTISWOODE, W. (1851, 1853).

[ELEMENTARY THEOREMS RELATING TO DETERMINANTS, . . . viii+63 pp., London. Second edition as an article in *Crelle's Journ.*, li. pp. 209–271, 328–381.]

Spottiswoode has a special chapter (§ x., pp. 51–57) headed “On Functional Determinants,” its contents being a selection of Jacobi's theorems in unimproved form and a reprint of the first three paragraphs of Cayley's paper of 1847. In the second edition (§ ix., pp. 338–343) there is no change, save that the extract from Cayley is left out.

SYLVESTER, J. J. (1853, June).

[On a theory of the syzygetic relations *Philos. Trans. R. Soc. London*, cxliii. pp. 407–548: or *Collected Math. Papers*, i. pp. 429–586.]

The only interest of this long and important memoir in the present connection lies in the fact that Sylvester at page 476 of it uses for the first time the term *Jacobian* and the symbolism $J(f, g)$. His words are “J indicates the Jacobian of the given functions f, g, \dots meaning thereby the functional determinant of Jacobi.”

DONKIN, W. F. (1854, February).

[On a class of differential equations, including those which occur in dynamical problems, Part I. *Philos. Trans. R. Soc. London*, cxliv. pp. 71–113.]

It is only the first four pages of Donkin's memoir that concern us, these being introductory and referring to properties of a set of n functions

* Or if f_n be not involved in the ϕ_n , neither f_n nor f_{n-1} involved in ϕ_{n-1} , and so on.

† *Cambridge Math. Journ.*, iv. (1843), pp. 20–28.

‡ *Archiv d. Math. u. Phys.*, x. (1847), pp. 417–421.

of n variables without any regard to possible connections with dynamics. Drawing attention at the outset to the analogy remarked on by Jacobi and Bertrand, he proposes to signalise it by denoting the determinant of f_1, f_2, \dots, f_n with respect to x_1, x_2, \dots, x_n by

$$\frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x_1, x_2, \dots, x_n)}.$$

Further, he views the numerator and denominator here as standing for the determinants of Bertrand's arrays of differences, remarking pointedly that the fraction indicated "is a real fraction, provided its numerator and denominator be interpreted in a manner exactly analogous to that in which the numerator and denominator of an ordinary total or partial differential-coefficient are interpreted."

Having thus explained his notation he proceeds to generalise the proposition that

$$\frac{\partial f_r}{\partial x_1} \frac{\partial x_1}{\partial f_s} + \frac{\partial f_r}{\partial x_2} \frac{\partial x_2}{\partial f_s} + \dots + \frac{\partial f_r}{\partial x_n} \frac{\partial x_n}{\partial f_s} = 1 \text{ or } 0$$

according as r and s are equal or unequal. He recalls Jacobi's theorem (*De determin. funct.* §11) that if u_1, u_2, \dots, u_m be functions of y_1, y_2, \dots, y_n , n being greater than m , and the y 's be functions of x_1, x_2, \dots, x_n , then

$$\frac{\partial(u_{\alpha_1} u_{\alpha_2} \dots u_{\alpha_m})}{\partial(x_{\gamma_1} x_{\gamma_2} \dots x_{\gamma_m})} = \sum_{\beta} \left\{ \frac{\partial(u_{\alpha_1} u_{\alpha_2} \dots u_{\alpha_m})}{\partial(y_{\beta_1} y_{\beta_2} \dots y_{\beta_m})} \cdot \frac{\partial(y_{\beta_1} y_{\beta_2} \dots y_{\beta_m})}{\partial(x_{\gamma_1} x_{\gamma_2} \dots x_{\gamma_m})} \right\}$$

where $\alpha_1, \alpha_2, \dots, \alpha$ and $\gamma_1, \gamma_2, \dots, \gamma_m$ are fixed sets of m integers chosen from $1, 2, \dots, n$, and $\beta_1, \beta_2, \dots, \beta_m$ as any such set whatever. Taking then what he considers to be a particular case of this, namely,

$$\frac{\partial(y_{\alpha_1} y_{\alpha_2} \dots y_{\alpha_m})}{\partial(y_{\gamma_1} y_{\gamma_2} \dots y_{\gamma_m})} = \sum_{\beta} \left\{ \frac{\partial(y_{\alpha_1} y_{\alpha_2} \dots y_{\alpha_m})}{\partial(x_{\beta_1} x_{\beta_2} \dots x_{\beta_m})} \cdot \frac{\partial(x_{\beta_1} x_{\beta_2} \dots x_{\beta_m})}{\partial(y_{\gamma_1} y_{\gamma_2} \dots y_{\gamma_m})} \right\}$$

he points out that if the sets $\alpha_1, \alpha_2, \dots, \alpha_m$ and $\gamma_1, \gamma_2, \dots, \gamma_m$ be identical the determinant on the left is equal to 1, and that on the other hand if even one of the γ 's be not included among the α 's the determinant will have a column of zeros and therefore be zero itself. The generalisation aimed at thus is, that *If y_1, y_2, \dots, y_n be functions of x_1, x_2, \dots, x_n , and m be less than n , then*

$$\sum_{\beta} \left\{ \frac{\partial(y_{\alpha_1} y_{\alpha_2} \dots y_{\alpha_m})}{\partial(x_{\beta_1} x_{\beta_2} \dots x_{\beta_m})} \cdot \frac{\partial(x_{\beta_1} x_{\beta_2} \dots x_{\beta_m})}{\partial(y_{\gamma_1} y_{\gamma_2} \dots y_{\gamma_m})} \right\} = 1 \text{ or } 0$$

according as the α set of m integers chosen from $1, 2, \dots, n$ is identical

or not with the γ set. The illustrative example taken is the case where $n=2$, and the mode of stating it is that

$$\sum \left\{ \left(\frac{\partial y_n}{\partial x_i} \frac{\partial y_q}{\partial x_j} - \frac{\partial y_p}{\partial x_j} \frac{\partial y_q}{\partial x_i} \right) \left(\frac{\partial x_i}{\partial y_\alpha} \frac{\partial x_j}{\partial y_\beta} - \frac{\partial x_i}{\partial y_\beta} \frac{\partial x_j}{\partial y_\alpha} \right) \right\} = 1 \text{ or } 0$$

according as $\alpha, \beta = p, q$ or not, it being understood that i, j is in succession

$$\begin{array}{c} 1, 2; 1, 3; \dots; 1, n \\ 2, 3; \dots; 2, n \\ \dots \dots \dots \\ n-1, n. \end{array}$$

DONKIN, W. F. (1854, February).

[Demonstration of a theorem of Jacobi's relative to functional determinants. *Cambridge and Dub. Math. Journ.*, ix. pp. 161-163.]

The theorem or identity in question is that of the year 1844. The functions being u_1, u_2, \dots, u_n and the independent variables x_1, x_2, \dots, x_n , Donkin says that the functional determinant may be represented by

$$\begin{vmatrix} \frac{\partial_1}{\partial x_1} & \frac{\partial_1}{\partial x_2} & \dots & \frac{\partial_1}{\partial x_n} \\ \frac{\partial_2}{\partial x_1} & \frac{\partial_2}{\partial x_2} & \dots & \frac{\partial_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial_n}{\partial x_1} & \frac{\partial_n}{\partial x_2} & \dots & \frac{\partial_n}{\partial x_n} \end{vmatrix} u_1 u_2 \dots u_n,$$

it being understood that each symbol of differentiation is operative only upon that one of the functions which has the same suffix as the upper ∂ of the symbol. As a consequence of this he considers that the non-zero member of the identity sought to be established would be

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \dots & \frac{\partial}{\partial x_n} \\ \frac{\partial_2}{\partial x_1} & \frac{\partial_2}{\partial x_2} & \dots & \frac{\partial_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial_n}{\partial x_1} & \frac{\partial_n}{\partial x_2} & \dots & \frac{\partial_n}{\partial x_n} \end{vmatrix} u_2 u_3 \dots u_n \tag{A}$$

where the upper ∂ 's of the first row being now without a suffix are supposed to be no longer restricted in their effect. As, however, the

unrestricted symbol $\frac{\partial}{\partial x_i}$ is held to be equivalent to

$$\frac{\partial_2}{\partial x_i} + \frac{\partial_3}{\partial x_i} + \dots + \frac{\partial_n}{\partial x_i}$$

the determinant operating on $u_2 u_3 \dots u_n$ has the first element of each column equal to the sum of all the other elements of the column, and therefore vanishes. The identity is thus thought to be established.

In regard to this so-called demonstration we need only remark in passing (1) that the subject operated on is written in too product-like a form; (2) that an appropriate substitute for it would be (u_1, u_2, \dots, u_n) , this being explained to be such that

$$\frac{\partial_r}{\partial x_s}(u_1, u_2, \dots, u_n) = \frac{\partial u_r}{\partial x_s},$$

and

$$\frac{\partial}{\partial x_s}(u_1, u_2, \dots, u_n) = \frac{\partial u_1}{\partial x_s} + \frac{\partial u_2}{\partial x_s} + \dots + \frac{\partial u_n}{\partial x_s};$$

(3) that the assertion (A) is unsubstantiated, the fact that

$$|a_1 b_2 c_3| = a_1 |b_2 c_3| - a_2 |b_1 c_3| + a_3 |b_1 c_2|$$

being nothing more than a suggestion that

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

may be a suitable *abridged notation* for

$$\frac{\partial}{\partial x_1} |b_2 c_3| - \frac{\partial}{\partial x_2} |b_1 c_3| + \frac{\partial}{\partial x_3} |b_1 c_2|.$$

BRIOSCHI, F. (1854, March).

[LA TEORICA DEI DETERMINANTI,; viii + 116 pp., Pavia. French translation by Edouard Combescure, ix + 216 pp., Paris, 1856. German translation by Schellbach, vii + 102 pp., Berlin, 1856.]

Like Spottiswoode, Brioschi devotes his tenth chapter or section (§ 10) to “determinanti delle funzioni”; but his exposition is much more extensive (pp. 84-106), and, although of course he follows in Jacobi’s footsteps, he does so less closely than Spottiswoode.

Thus the fact that the cofactor of $\frac{\partial f_i}{\partial x_k}$ in R is $R \frac{\partial x_k}{\partial f_i}$, a fact which we

may write temporarily in the form $\left[\frac{\partial f_i}{\partial x_k} \right] = R \frac{\partial x_k}{\partial f_i}$, he obtains by solving $n+1$ sets of equations like

$$\left. \begin{aligned} \frac{\partial x}{\partial f} \frac{\partial f}{\partial x} + \frac{\partial x}{\partial f_1} \frac{\partial f_1}{\partial x} + \dots + \frac{\partial x}{\partial f_n} \frac{\partial f_n}{\partial x} &= 1 \\ \frac{\partial x}{\partial f} \frac{\partial f}{\partial x_1} + \frac{\partial x}{\partial f_1} \frac{\partial f_1}{\partial x_1} + \dots + \frac{\partial x}{\partial f_n} \frac{\partial f_n}{\partial x_1} &= 0 \\ \dots &\dots \\ \frac{\partial x}{\partial f} \frac{\partial f}{\partial x_n} + \frac{\partial x}{\partial f_1} \frac{\partial f_1}{\partial x_n} + \dots + \frac{\partial x}{\partial f_n} \frac{\partial f_n}{\partial x_n} &= 0 \end{aligned} \right\};$$

and by using the same sets of equations after row-by-row multiplication he obtains

$$\sum \left(\pm \frac{\partial x}{\partial f} \frac{\partial x_1}{\partial f_1} \dots \frac{\partial x_n}{\partial f_n} \right) \cdot \sum \left(\pm \frac{\partial f}{\partial x} \frac{\partial f_1}{\partial x_1} \dots \frac{\partial f_n}{\partial x_n} \right) = 1,$$

or, say, $SR = 1.$

Further, he notes that as a consequence of these two theorems there results

$$\left[\frac{\partial f_i}{\partial x_k} \right] \cdot \left[\frac{\partial x_i}{\partial f_k} \right] = \frac{\partial x_k}{\partial f_i} \cdot \frac{\partial f_k}{\partial x_i},$$

which he might well have generalised by changing the i, k of the second factor of both sides into r, s .

In dealing with the "fundamental lemma" his order of procedure is the reverse of Jacobi's, that is to say, he deduces the form of 1844 from the original form of 1841. Thus, using the latter in regard to S , he has

$$\frac{\partial S}{\partial f_k} = S \cdot \sum_{i=0}^{i=n} \frac{\partial \frac{\partial x_i}{\partial f_k}}{\partial x_i},$$

whence, because of R being the reciprocal of S , he obtains

$$\frac{\partial R}{\partial f_k} + R \cdot \sum_{i=0}^{i=n} \frac{\partial \frac{\partial x_i}{\partial f_k}}{\partial x_i} = 0;$$

so that on substituting

$$\sum_{i=0}^{i=n} \frac{\partial R}{\partial x_i} \frac{\partial x_i}{\partial f_k} \quad \text{for} \quad \frac{\partial R}{\partial f_k}$$

there results

$$\sum_{i=0}^{i=n} \frac{\partial \left(R \frac{\partial x_i}{\partial f_k} \right)}{\partial x_i} = 0,$$

which on further substituting $\left[\frac{\partial f_k}{\partial x_i} \right]$ for $\frac{\partial R}{\partial f_k}$ becomes the form desired.

The next fresh paragraph (p. 91) appears, although unnecessarily, as an addendum to Jacobi's solution of a set of simultaneous linear equations whose determinant is a functional determinant (*De determ. funct.* § 8). If the square of R be obtained by row-by-row multiplication, and the square of S by column-by-column multiplication it is easily verified that

$$\begin{aligned} (h^{\text{th}} \text{ row of } S^2) \times (k^{\text{th}} \text{ column of } R) &= \frac{\partial x_k}{\partial f_h}, \\ \text{i.e.} \quad &= (k, h)^{\text{th}} \text{ element of } S, \end{aligned}$$

thus incidentally giving $S^2R = S$ as it should do. From this it is deduced that

$$(h^{\text{th}} \text{ row of } S^2) \times (k^{\text{th}} \text{ row of } R^2) = 1 \text{ or } 0$$

according as h and k are equal or unequal,* and that therefore R^2 and S^2 as just defined are in the matter of their primary minors related as R and S have been shown to be.

In the remaining fourteen pages (pp. 92-106) the only matter calling for attention concerns Jacobi's theorem

$$\sum (\pm bb_1^{(1)} \dots b_n^{(m)}) = B^m \cdot \sum \left(\pm \frac{\partial f}{\partial x} \frac{\partial f_1}{\partial x_1} \dots \frac{\partial f_{n+m}}{\partial x_{n+m}} \right)$$

where

$$B = \sum \left(\pm \frac{\partial f}{\partial x} \frac{\partial f_1}{\partial x_1} \dots \frac{\partial f_{n-1}}{\partial x_{n-1}} \right) \quad \text{and} \quad b_k^{(i)} = \sum \left(\pm \frac{\partial f}{\partial x} \frac{\partial f_1}{\partial x_1} \dots \frac{\partial f_{n-1}}{\partial x_{n-1}} \frac{\partial f_{n+i}}{\partial x_{n+k}} \right).$$

From this Brioschi, by taking the f 's to be linear functions of the x 's, obtains Sylvester's theorem of March 1851 regarding a compound determinant.

BELLAVITIS, G. (1857, June).

[Sposizione elementare della teorica dei determinanti. *Memorie Istituto Veneto* vii. pp. 67-144.]

To the subject of a "Determinante formato colle derivate-prime di alquante funzioni di altrettanti variabili" Bellavitis devotes nine and a half pages (pp. 52-61, §§ 65-78), that is to say, about the same as Spottiswoode, though his selection of theorems is not quite the same. In substance he gives nothing fresh. His symbolism for the determinant of u, v, \dots with respect to x, y, \dots resembles Cauchy's of 1841, being

$$|D_x u, D_y v, \dots|;$$

other changes made by him in notation are less satisfactory.

* Viewing R and S as matrices of which the conjugates are \bar{R} and \bar{S} we have as an equivalent of this

$$\begin{aligned} \bar{S}S \cdot \bar{R}R &= \bar{S} \cdot SR \cdot \bar{R} \\ &= \bar{S}\bar{R} = \bar{R}S = 1. \end{aligned}$$

BALTZER, R. (1857).

[THEORIE UND ANWENDUNG DER DETERMINANTEN, vi+129 pp., Leipzig. French translation by J. Houel, xii+235 pp., Paris, 1861.]

“Die Functional-determinante” is the heading of Baltzer’s thirteenth chapter or section (§ 13, pp. 61–72). Though the exposition is neither so full nor so fresh as Brioschi’s, it has the advantage in arrangement, concision and clearness. Jacobi’s last theorem (*De determ. funct.* § 18), expressing the determinant as a single product,

$$\left(\frac{\partial f}{\partial x}\right) \left(\frac{\partial f_1}{\partial x_1}\right) \left(\frac{\partial f_2}{\partial x_2}\right) \dots \left(\frac{\partial f_n}{\partial x_n}\right),$$

Baltzer makes his first, the proof being readily altered to suit. This change enables him to deal very effectively with the proposition regarding the vanishing of the determinant. For then he can assert that as the determinant vanishes, one of the factors of the said product must vanish; and thence step-by-step can infer the vanishing of the succeeding factors including the last,—a conclusion which entails f_n being expressible in terms of the other f ’s.

A footnote recalls the fact, which we should have noted before this, that Möbius had given in *Crelle’s Journ.*, xii. p. 116, in the year 1834, the equation

$$(t_x u_y - t_y u_x)(v_t v_u - v_u v_t)(x_v y_w - x_w y_v) = 1$$

where t_x stands for $\partial t / \partial x$.

SALMON, G. (1859).

[LESSONS INTRODUCTORY TO THE MODERN HIGHER ALGEBRA. xii+147 pp., Dublin.]

Salmon gives little, and certainly nothing fresh, on the subject; but his unreserved adoption of Sylvester’s word “Jacobian” (§§ 53, 54; p. 37) doubtless helped greatly to spread the usage.

LIST OF AUTHORS

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(Issued separately August 6, 1909.)

XXXIII.—Motion of Neptune's Satellite. By David Gibb, M.A., B.Sc.
Communicated by Prof. DYSON.

(MS. received 10th June 1909. Read 12th July 1909.)

1. NEPTUNE'S satellite was discovered by Mr Lassell of Liverpool in 1846. This satellite differs from the other planets and satellites, except those of Uranus, in the direction of its motion, which is from east to west. After the satellite had been observed for a few years, it was pointed out by Mr Marth that the node and the inclination of its orbit were slowly changing. These changes were explained by Tisserand and Newcomb as arising from the spheroidal shape of Neptune. The observations prior to 1892 have been discussed by Dr Struve in the *Mémoires de l'Académie Impériale des Sciences de St Pétersbourg*, vii^e série, tome xlii. No. 4. Since then a large number of valuable observations have been made at American observatories, especially by Prof. Barnard, and at Greenwich Observatory. The former were obtained visually with a position micrometer, the latter from photographs. Some of the American observations have already been discussed by Mr Hall (*A.J.*, xii. 22, xix. 65), Mr Brown (*A.J.*, xx. 134), and Prof. See (*A.N.*, 153), and the Greenwich observations are discussed in the *Monthly Notices* (vols. lxxv. and lxxviii.) by Prof. Dyson and Mr Edney. In this paper I propose to discuss some of the visual observations made since 1892. Since 1899 the *Connaissance des Temps* has given tables which facilitate the calculation of the ephemeris of the satellite. These tables are based on the elements of Dr Struve. As no such tables are published previous to the above date, I have formed them from 1892 to 1898 inclusive, using the same elements.

2. The elements given by Dr Struve are :—

Epoch 1890·0.

$$a = 16''\cdot271$$

$$u = 234^{\circ}\cdot42$$

$$n = 61^{\circ}\cdot25748$$

$$N = 185^{\circ}\cdot15 + 0^{\circ}\cdot148(t - 1890)$$

$$I = 119^{\circ}\cdot35 - 0^{\circ}\cdot165(t - 1890),$$

where a is the distance of the satellite from the planet for the mean distance of the planet from the Earth.

$$\left. \begin{array}{l} n = \text{mean daily motion} \\ u = \text{longitude of satellite} \\ N = \text{longitude of node} \\ I = \text{inclination of orbit} \end{array} \right\} \begin{array}{l} \text{referred to the Earth's} \\ \text{equator and the equinox} \\ \text{of date.} \end{array}$$

The small eccentricity found by Dr Struve is neglected. From these elements the auxiliary quantities B, P, and U are found by means of the equations:

$$\begin{aligned}\cos B \sin P &= -\cos(\alpha - N) \sin I \\ \cos B \cos P &= \sin \delta \sin(\alpha - N) \sin I + \cos \delta \cos I \\ \sin B &= \cos \delta \sin(\alpha - N) \sin I - \sin \delta \cos I \\ \cos B \sin U &= -\cos \delta \sin(\alpha - N) \cos I - \sin \delta \sin I \\ \cos B \cos U &= \cos \delta \cos(\alpha - N),\end{aligned}$$

where

P = position angle of the pole of the orbit of the satellite

B = planetocentric latitude of the Earth with reference to the orbit of the satellite

180 - U = planetocentric longitude of the Earth with reference to the orbit of the satellite.

With these auxiliary quantities the distance s and the position angle p are found from the equations:

$$\begin{aligned}s \sin(p - P) &= r \sin(u + U) \\ s \cos(p - P) &= r \sin B \cos(u + U),\end{aligned}$$

where

$$\begin{aligned}r &= a(\rho)/\rho' \\ \rho' &= (\rho)(1 + a \cos B \cos(u + U) \sin I'')\end{aligned}$$

and (ρ) is the mean distance of Neptune from the Earth.

The formulæ employed in obtaining the corrections to the elements are those deduced by Mr Marth, viz.:

$$\begin{aligned}s \sin dp &= r \sin \tau \sin du + (r \sin \tau \cos I + r \cos \tau \cos u \sin I) \sin dN \\ &\quad - r \cos \tau \sin u \sin dI - r \sin \tau \cos u \cdot 2e \sin Q \\ &\quad + r \sin \tau \sin u \cdot 2e \cos Q \\ ds &= r \cos \sigma \cos \tau \sin du + r \cos \sigma \sin p \cos \delta \sin dN \\ &\quad + r \cos \sigma \sin \tau \sin u \sin dI \\ &\quad - \left(r \cos \sigma \cos \tau \cos u + \frac{s}{2} \sin u \right) \cdot 2e \sin Q \\ &\quad + \left(r \cos \sigma \cos \tau \sin u - \frac{s}{2} \cos u \right) \cdot 2e \cos Q + s \frac{da}{a},\end{aligned}$$

where

e = eccentricity

Q = longitude of periastron measured from the node of the satellite's orbit on the Earth's equator

$$\sin \tau = \frac{r}{s} \sin B$$

$$\cos \tau = \frac{r}{s} \cos B \sin(u + U)$$

$$\cos \sigma = \cos B \cos(u + U).$$

3. The following table gives a list of places where most of the observations made since 1892 may be found:—

Observatory.	Epoch.	Source.	Vol.	Page.	Observatory.	Epoch.	Source.	Vol.	Page.	
Lick	1892-3	A. J.	XIII	10	Washington	1905-6	A. J.	XXV	100	
	1893-4		XIV	9		*1897-8		XX	134	
	1894-5		XV	26, 42		*1899-1900		A. N.	153	257
	1895-6		XVII	62		1902-3		A. J.	XXIII	144
	1896		XVII	62		**1904-5		XXIV	188	
	1897		XVIII	168		**1905-6		XXV	93	
	1898		XX	71		Lowell		1896-7	XVII	132
Yerkes	1898-9	A. N.	149	374	1898	XX	30			
	*1897-8		A. J.	XIX	27, 65	Greenwich	*1902	M. N.	LXII	624
	1898-9	XX		41	*1902-3	LXIII	504			
	1899-1900	XXII		27	*1903-4	LXIV	835			
	1900-1	XXII		29	*1904-5	LXVI	10			
	1901-2	XXIII		105	*1905-6	LXVII	92			
	1902-3	XXIII		107	*1906-7	LXVIII	33			
	1903-4	XXV		41	*1907-8	LXVIII	586			
	1904	XXV		42						

A. J. = *Astronomical Journal*.

A. N. = *Astronomische Nachrichten*.

M. N. = *Monthly Notices of the Royal Astronomical Society*.

* Elements are deduced from these measures.

** Measures are compared with the ephemeris, but the corrections to the elements are not deduced.

4. In the next table are given, for each series, the observer, the mean date, the times of observation, and the values of $d s$ and $s dp$ in the sense Tabular—Observed, obtained by differencing the observed distances and angles with those I have computed from Dr Struve's elements:—

LICK OBSERVATORY—PROF. BARNARD. 1893·0.

Paris Mean Time.				$d s$.	$s dp$.	Paris Mean Time.				$d s$.	$s dp$.
1892	H.	M.	S.			1892	H.	M.	S.		
Nov. 13,	15	8	2	-0"25	+0"08	Dec. 16,	12	23	44	-0"07	+0"02
18,	18	32	58	+0"01	-0"14	1893					
20,	17	41	21	+0"37	-0"07	Jan. 8,	15	8	30	-0"27	+0"37
Dec. 9,	14	49	2	+0"03	-0"11	13,	15	20	32	+0"05	+0"18
						20,	12	24	6	-0"17	+0"10

LICK OBSERVATORY—PROF. BARNARD. 1894·0.

1893				$d s$.	$s dp$.	1894				$d s$.	$s dp$.
Nov. 12,	H.	M.	S.	-0"05	-0"08	Jan. 10,	H.	M.	S.	+0"02	-0"20
13,	18	33	42	+0"32	-0"21	21,	14	34	7	-0"08	-0"02
Dec. 3,	14	8	45	+0"30	+0"46	22,	12	42	19	-0"11	+0"07
10,	14	47	9	-0"21	-0"01	28,	11	32	29	+0"06	-0"02
1894						Feb. 26,	11	53	43	+0"02	+0"10
Jan. 7,	14	49	38	-0"30	-0"07						

LICK OBSERVATORY—PROF. BARNARD. 1895·0.

Paris Mean Time.				<i>d s.</i>	<i>s dp.</i>	Paris Mean Time.				<i>d s.</i>	<i>s dp.</i>
1894	H.	M.	S.			1895	H.	M.	S.		
Nov. 18,	19	32	12	+0 ^{''} 21	-0 ^{''} 12	Feb. 5,	11	31	5	+0 ^{''} 08	+0 ^{''} 27
19,	18	53	16	+0 ^{''} 05	+0 ^{''} 12	17,	12	43	41	-0 ^{''} 07	+0 ^{''} 21
Dec. 23,	13	45	19	-0 ^{''} 04	0 ^{''} 00	18,	13	32	48	+0 ^{''} 01	-0 ^{''} 51
31,	15	0	39	0 ^{''} 00	+0 ^{''} 04	24,	12	3	28	-0 ^{''} 05	-0 ^{''} 43
1895						25,	11	19	9	0 ^{''} 00	+0 ^{''} 17
Jan. 6,	13	14	5	-0 ^{''} 05	+0 ^{''} 07	Mar. 3,	11	45	22	-0 ^{''} 27	-0 ^{''} 11
28,	13	31	34	+0 ^{''} 25	+0 ^{''} 09	4,	11	41	1	-0 ^{''} 16	-0 ^{''} 07
Feb. 4,	11	10	43	+0 ^{''} 14	-0 ^{''} 24						

LICK OBSERVATORY—PROF. SCHAEBERLE. 1895·0.

1894	H.	M.	S.			1895	H.	M.	S.		
Dec. 13,	12	33	37	+0 ^{''} 30	+0 ^{''} 07	Jan. 30,	11	55	16	+0 ^{''} 09	+0 ^{''} 17
1895						31,	12	51	50	+0 ^{''} 36	+0 ^{''} 08
Jan. 10,	11	56	47	-0 ^{''} 10	-0 ^{''} 13	Feb. 2,	12	17	7	-0 ^{''} 21	+0 ^{''} 02
25,	14	30	39	+0 ^{''} 04	+0 ^{''} 27	3,	12	7	4	+0 ^{''} 05	-0 ^{''} 09
26,	13	49	6	-0 ^{''} 04	-0 ^{''} 06	6,	11	53	32	+0 ^{''} 14	+0 ^{''} 17
29,	14	19	2	-0 ^{''} 17	+0 ^{''} 07						

LICK OBSERVATORY—PROF. SCHAEBERLE. 1896·0.

1895	H.	M.	S.			1895	H.	M.	S.		
Oct. 27,	15	54	28	-0 ^{''} 06	-0 ^{''} 07	Dec. 7,	13	41	12	-0 ^{''} 01	+0 ^{''} 25
28,	15	46	28	+0 ^{''} 03	+0 ^{''} 16	9,	13	23	28	-0 ^{''} 12	+0 ^{''} 12
29,	15	56	53	-0 ^{''} 08	+0 ^{''} 20	31,	12	24	6	+0 ^{''} 12	+0 ^{''} 25
Nov. 9,	15	46	51	-0 ^{''} 01	+0 ^{''} 11	1896					
20,	14	25	3	-0 ^{''} 09	+0 ^{''} 23	Jan. 2,	13	7	59	+0 ^{''} 20	-0 ^{''} 02
						Feb. 21,	11	59	42	+0 ^{''} 15	+0 ^{''} 14

LICK OBSERVATORY—PROF. SCHAEBERLE. 1896·8.

1896	H.	M.	S.			1896	H.	M.	S.		
Oct. 16,	17	5	54	+0 ^{''} 37	+0 ^{''} 07	Oct. 29,	15	34	57	-0 ^{''} 15	+0 ^{''} 02
17,	17	4	18	-0 ^{''} 20	+0 ^{''} 19	30,	17	8	17	+0 ^{''} 31	-0 ^{''} 26
27,	16	7	9	+0 ^{''} 22	-0 ^{''} 01	Nov. 6,	17	33	31	-0 ^{''} 18	+0 ^{''} 35
28,	15	34	35	+0 ^{''} 11	+0 ^{''} 03						

LICK OBSERVATORY—PROF. SCHAEBERLE. 1897·8.

1897	H.	M.	S.			1897	H.	M.	S.		
Sept. 13,	19	2	43	-0 ^{''} 35	+0 ^{''} 33	Nov. 15,	15	14	20	-0 ^{''} 17	-0 ^{''} 02
18,	19	41	56	+0 ^{''} 05	+0 ^{''} 10	27,	15	49	34	+0 ^{''} 15	-0 ^{''} 07
Oct. 16,	16	43	59	+0 ^{''} 00	+0 ^{''} 15	29,	14	43	59	-0 ^{''} 04	+0 ^{''} 25
18,	16	39	21	-0 ^{''} 10	+0 ^{''} 22	Dec. 3,	15	47	10	-0 ^{''} 12	+0 ^{''} 11
29,	16	53	28	-0 ^{''} 03	+0 ^{''} 10	11,	12	50	18	-0 ^{''} 23	+0 ^{''} 32
30,	15	40	39	-0 ^{''} 13	+0 ^{''} 14	24,	12	26	22	+0 ^{''} 17	-0 ^{''} 03
Nov. 1,	16	5	30	-0 ^{''} 10	+0 ^{''} 10	25,	14	50	2	+0 ^{''} 03	-0 ^{''} 07
12,	15	47	5	-0 ^{''} 18	+0 ^{''} 20	27,	12	21	4	-0 ^{''} 29	+0 ^{''} 13

LICK OBSERVATORY—MR AITKEN. 1899·0.

Paris Mean Time.				<i>d s.</i>	<i>s dp.</i>	Paris Mean Time.				<i>d s.</i>	<i>s dp.</i>
1898	H.	M.	S.			1898	H.	M.	S.		
Oct. 14,	19	36	32	-0·08	-0·20	Nov. 20,	15	25	20	+0·44	+0·15
	16,	18	22	+0·10	+0·03	Dec. 2,	14	45	39	+0·23	+0·06
	23,	17	31	+0·45	-0·25		3,	15	18	+0·45	-0·20
Nov. 5,	17	41	47	-0·10	+0·52		9,	14	56	+0·25	-0·11
	11,	14	49	-0·21	+0·47		11,	14	21	-0·26	+0·34
	12,	17	21	-0·49	-0·17	1899					
	15,	17	38	+0·15	-0·12	Feb. 11,	16	25	18	+0·22	-0·05

LICK OBSERVATORY—MR HUSSEY. 1899·0.

1898	H.	M.	S.			1898	H.	M.	S.		
Oct. 27,	17	38	22	+0·03	-0·17	Dec. 23,	13	17	14	-0·27	+0·39
	28,	17	32	-0·09	-0·08	1899					
Nov. 17,	16	34	34	+0·13	+0·21	Jan. 5,	12	14	19	+0·26	+0·21
Dec. 7,	12	11	47	+0·46	+0·11		12,	12	20	-0·37	+0·12
	15,	19	54	-0·23	+0·40		20,	12	38	-0·16	-0·06
	16,	12	49	-0·53	-0·02	Feb. 10,	15	16	12	+0·09	+0·02
							16,	14	35	-0·19	+0·00

YERKES OBSERVATORY—PROF. BARNARD. 1899·0.

1898	H.	M.	S.			1899	H.	M.	S.		
Aug. 29,	17	32	28	-0·24	+0·21	Jan. 18,	10	33	0	+0·01	+0·06
	30,	17	14	-0·13	+0·15		24,	10	10	-0·09	+0·02
	31,	17	15	+0·02	-0·03		30,	12	15	+0·39	-0·24
Sept. 1,	16	37	10	+0·50	-0·08		31,	12	47	-0·03	+0·03
	2,	16	34	+0·06	-0·01	Feb. 6,	9	34	43	+0·39	-0·30
	3,	16	57	-0·11	+0·03		7,	10	54	-0·10	+0·03
	14,	15	59	+0·17	+0·09		9,	12	1	+0·28	-0·01
	20,	16	34	+0·08	-0·16		10,	9	27	-0·01	+0·17
	22,	19	9	-0·04	-0·13		11,	10	48	+0·29	-0·13
	26,	16	21	+0·02	-0·07		12,	10	40	+0·23	-0·12
	27,	16	22	+0·00	+0·30		13,	9	7	+0·20	-0·10
Oct. 10,	15	17	17	-0·10	+0·11		20,	11	22	+0·13	+0·14
	11,	17	4	-0·14	-0·23		28,	9	58	-0·18	+0·12
Nov. 7,	14	26	46	-0·28	+0·04	Mar. 13,	11	11	12	-0·20	-0·23
	14,	20	18	+0·24	-0·38		18,	9	35	+0·17	-0·06
	15,	12	50	-0·06	-0·06		19,	9	53	+0·11	-0·02
	22,	17	51	+0·19	-0·15		28,	10	14	-0·05	+0·03
	24,	11	44	+0·08	+0·05		29,	9	39	+0·18	+0·08
	26,	11	57	-0·02	+0·26		30,	10	20	+0·43	-0·04
	29,	14	46	+0·24	-0·19	Apr. 3,	10	21	53	-0·21	+0·20
Dec. 3,	12	20	32	+0·12	-0·16		4,	9	46	+0·14	+0·03
	6,	15	56	+0·04	+0·02		7,	10	17	-0·04	+0·23
	10,	11	45	+0·48	-0·16		17,	9	45	+0·03	-0·16
	11,	11	11	+0·04	+0·14		18,	9	44	-0·07	+0·28
	12,	11	35	-0·02	+0·11		19,	9	42	-0·14	+0·15
	13,	11	18	-0·54	+0·06						

YERKES OBSERVATORY—PROF. BARNARD. 1900·0.

Paris Mean Time.				<i>d s.</i>	<i>s dp.</i>	Paris Mean Time.				<i>d s.</i>	<i>s dp.</i>
1899	H.	M.	S.			1899	H.	M.	S.		
Aug. 13,	17	54	33	+0 ^{''} 11	-0 ^{''} 16	Oct. 23,	15	0	54	+0 ^{''} 24	-0 ^{''} 05
14,	17	53	33	+0 ^{''} 30	-0 ^{''} 03	28,	16	14	44	+0 ^{''} 12	+0 ^{''} 05
15,	17	59	42	+0 ^{''} 02	-0 ^{''} 07	29,	14	14	48	-0 ^{''} 02	+0 ^{''} 05
18,	17	48	48	+0 ^{''} 02	+0 ^{''} 25	30,	15	45	11	+0 ^{''} 01	-0 ^{''} 03
19,	17	32	13	+0 ^{''} 19	+0 ^{''} 18	Nov. 4,	14	22	31	+0 ^{''} 34	+0 ^{''} 10
20,	17	48	32	-0 ^{''} 12	-0 ^{''} 06	5,	15	54	48	+0 ^{''} 06	+0 ^{''} 10
21,	17	50	22	+0 ^{''} 21	+0 ^{''} 18	6,	14	2	34	+0 ^{''} 08	+0 ^{''} 20
22,	17	28	25	-0 ^{''} 15	-0 ^{''} 06	11,	13	27	6	+0 ^{''} 06	+0 ^{''} 01
26,	17	29	7	+0 ^{''} 23	-0 ^{''} 14	12,	15	52	25	-0 ^{''} 10	-0 ^{''} 01
27,	17	51	53	+0 ^{''} 14	+0 ^{''} 18	18,	14	10	13	-0 ^{''} 19	+0 ^{''} 10
28,	17	43	7	-0 ^{''} 08	+0 ^{''} 39	19,	14	11	23	+0 ^{''} 03	+0 ^{''} 17
29,	17	40	0	-0 ^{''} 13	+1 ^{''} 15	25,	12	39	39	-0 ^{''} 20	+0 ^{''} 29
Sept. 3,	17	53	52	-0 ^{''} 31	+0 ^{''} 14	26,	12	53	25	+0 ^{''} 09	+0 ^{''} 06
4,	17	28	7	-0 ^{''} 37	+0 ^{''} 40	27,	13	2	0	+0 ^{''} 22	-0 ^{''} 05
6,	16	57	53	+0 ^{''} 54	-0 ^{''} 69	Dec. 4,	12	0	46	-0 ^{''} 43	-0 ^{''} 08
8,	17	54	50	+0 ^{''} 13	+0 ^{''} 05	1900					
10,	17	58	36	+0 ^{''} 05	+0 ^{''} 18	Mar. 30,	10	23	14	-0 ^{''} 03	+0 ^{''} 01
11,	17	48	43	+0 ^{''} 14	+0 ^{''} 01	31,	9	46	18	-0 ^{''} 10	+0 ^{''} 21
12,	17	47	28	+0 ^{''} 25	+0 ^{''} 09	April 2,	9	58	54	-0 ^{''} 10	+0 ^{''} 19
18,	16	51	38	+0 ^{''} 03	-0 ^{''} 29	3,	9	45	55	-0 ^{''} 09	+0 ^{''} 11
24,	16	0	27	+0 ^{''} 02	+0 ^{''} 16	4,	10	16	7	+0 ^{''} 06	+0 ^{''} 10
25,	16	58	26	-0 ^{''} 20	-0 ^{''} 11	6,	9	31	12	+0 ^{''} 08	+0 ^{''} 28
26,	16	40	3	-0 ^{''} 09	+0 ^{''} 27	7,	9	35	50	-0 ^{''} 18	+0 ^{''} 27
30,	16	49	9	-0 ^{''} 11	-0 ^{''} 10	9,	9	29	56	-0 ^{''} 42	-0 ^{''} 02
Oct. 1,	16	28	51	+0 ^{''} 09	-0 ^{''} 07	10,	9	35	23	+0 ^{''} 15	-0 ^{''} 01
2,	17	0	11	-0 ^{''} 11	+0 ^{''} 31	18,	9	54	43	+0 ^{''} 20	+0 ^{''} 37
7,	16	21	45	+0 ^{''} 02	+0 ^{''} 01	19,	10	40	46	-0 ^{''} 51	-0 ^{''} 29
8,	17	1	32	-0 ^{''} 07	+0 ^{''} 28	22,	10	8	41	+0 ^{''} 14	-0 ^{''} 72
9,	17	25	30	+0 ^{''} 14	+0 ^{''} 05	24,	9	51	35	+0 ^{''} 28	+0 ^{''} 13
14,	18	0	3	-0 ^{''} 23	+0 ^{''} 31	26,	9	50	9	-0 ^{''} 04	+0 ^{''} 04
15,	14	27	39	+0 ^{''} 14	+0 ^{''} 59	27,	9	48	27	-0 ^{''} 34	-0 ^{''} 13
17,	15	44	53	+0 ^{''} 13	+0 ^{''} 00	30,	9	53	2	-0 ^{''} 08	-0 ^{''} 02
21,	17	22	18	+0 ^{''} 26	+0 ^{''} 06	May 4,	9	56	5	+0 ^{''} 35	-0 ^{''} 24

YERKES OBSERVATORY—PROF. BARNARD. 1900·9.

1900	H.	M.	S.			1900	H.	M.	S.		
Sept. 6,	17	5	55	-0 ^{''} 08	-0 ^{''} 40	Oct. 17,	13	46	54	-0 ^{''} 11	+0 ^{''} 51
10,	17	30	51	-0 ^{''} 22	-0 ^{''} 07	18,	16	37	4	+0 ^{''} 24	-0 ^{''} 08
11,	17	43	56	-0 ^{''} 31	-0 ^{''} 01	25,	14	25	14	-0 ^{''} 34	-0 ^{''} 01
13,	17	43	8	+0 ^{''} 16	+0 ^{''} 09	26,	14	48	7	+0 ^{''} 02	+0 ^{''} 00
19,	16	53	1	+0 ^{''} 28	-0 ^{''} 05	27,	16	15	29	-0 ^{''} 15	-0 ^{''} 22
24,	15	45	50	+0 ^{''} 15	-0 ^{''} 34	30,	16	46	15	+0 ^{''} 08	+0 ^{''} 25
25,	16	3	20	+0 ^{''} 23	+0 ^{''} 19	Nov. 1,	18	1	39	-0 ^{''} 03	+0 ^{''} 02
Oct. 2,	16	20	40	-0 ^{''} 28	+0 ^{''} 32	2,	18	9	13	+0 ^{''} 11	-0 ^{''} 19
3,	16	24	53	-0 ^{''} 11	+0 ^{''} 19	3,	14	30	24	+0 ^{''} 21	+0 ^{''} 06
4,	16	3	10	-0 ^{''} 39	-0 ^{''} 21	4,	13	35	41	+0 ^{''} 10	-0 ^{''} 05
5,	17	21	13	+0 ^{''} 13	+0 ^{''} 03	5,	14	59	31	+0 ^{''} 31	+0 ^{''} 10
8,	17	5	6	-0 ^{''} 08	+0 ^{''} 16	8,	17	58	24	+0 ^{''} 18	-0 ^{''} 27
9,	17	4	27	+0 ^{''} 10	+0 ^{''} 04	13,	14	36	43	-0 ^{''} 33	+0 ^{''} 00
10,	16	11	55	+0 ^{''} 07	-0 ^{''} 14	22,	11	25	50	+0 ^{''} 24	-0 ^{''} 38
11,	17	18	11	+0 ^{''} 18	-0 ^{''} 40	26,	11	59	37	-0 ^{''} 45	-0 ^{''} 43
16,	15	28	25	+0 ^{''} 00	-0 ^{''} 10	Dec. 8,	13	27	20	-0 ^{''} 11	-0 ^{''} 05

YERKES OBSERVATORY—PROF. BARNARD. 1900·9—*continued.*

Paris Mean Time.				$d s.$	$s dp.$	Paris Mean Time.				$d s.$	$s dp.$	
1900	H.	M.	S.			1901	H.	M.	S.			
Dec. 11,	11	20	30	+0 ^{''} 06	+0 ^{''} 06	Jan. 14,	15	15	54	+0 ^{''} 01	-0 ^{''} 06	
	18,	11	40	+0 ^{''} 04	+0 ^{''} 15		16,	11	4	18	-0 ^{''} 18	+0 ^{''} 16
	19,	14	35	+0 ^{''} 17	-0 ^{''} 06		19,	10	19	19	+0 ^{''} 24	+0 ^{''} 00
	28,	15	13	-0 ^{''} 43	-0 ^{''} 09		21,	13	39	51	+0 ^{''} 01	+0 ^{''} 14
	29,	12	40	-0 ^{''} 04	+0 ^{''} 20		25,	11	14	21	+0 ^{''} 17	+0 ^{''} 00
	31,	17	6	-0 ^{''} 23	-0 ^{''} 12		28,	10	44	30	-0 ^{''} 16	+0 ^{''} 01
1901						Feb. 5,	10	52	48	+0 ^{''} 11	-0 ^{''} 20	
Jan. 1,	10	33	8	+0 ^{''} 46	-0 ^{''} 48							

YERKES OBSERVATORY—PROF. BARNARD. 1902·0.

1901	H.	M.	S.			1902	H.	M.	S.			
Aug. 27,	18	3	35	-0 ^{''} 12	+0 ^{''} 32	Jan. 10,	11	48	58	-0 ^{''} 16	+0 ^{''} 05	
Sept. 3,	17	48	51	-0 ^{''} 15	+0 ^{''} 18		12,	10	38	12	+0 ^{''} 42	+0 ^{''} 00
	16,	17	33	+0 ^{''} 27	-0 ^{''} 13		13,	10	17	40	-0 ^{''} 02	-0 ^{''} 03
	22,	17	42	+0 ^{''} 43	-0 ^{''} 13		18,	10	5	40	+0 ^{''} 17	+0 ^{''} 09
	23,	17	44	+0 ^{''} 17	+0 ^{''} 03		24,	13	56	39	-0 ^{''} 09	+0 ^{''} 20
	24,	17	16	+0 ^{''} 26	+0 ^{''} 00		27,	10	0	1	-0 ^{''} 45	+0 ^{''} 03
Oct. 1,	17	25	35	-0 ^{''} 46	+0 ^{''} 33		31,	9	45	49	+0 ^{''} 21	+0 ^{''} 18
	20,	18	47	-0 ^{''} 39	-0 ^{''} 26	Feb. 2,	9	42	54	-0 ^{''} 26	+0 ^{''} 06	
	21,	15	50	+0 ^{''} 05	+0 ^{''} 15		7,	9	28	2	-0 ^{''} 01	+0 ^{''} 09
	29,	15	31	+0 ^{''} 16	-0 ^{''} 16		8,	9	3	7	-0 ^{''} 30	+0 ^{''} 20
Nov. 12,	18	33	21	-0 ^{''} 32	+0 ^{''} 02		15,	9	12	16	-0 ^{''} 24	-0 ^{''} 02
	18,	16	58	+0 ^{''} 14	+0 ^{''} 03		17,	8	56	41	+0 ^{''} 06	+0 ^{''} 20
	19,	13	12	+0 ^{''} 09	-0 ^{''} 03		24,	9	0	7	+0 ^{''} 07	+0 ^{''} 13
	26,	17	37	-0 ^{''} 15	+0 ^{''} 48		25,	8	54	52	+0 ^{''} 02	-0 ^{''} 03
Dec. 10,	13	43	58	+0 ^{''} 27	+0 ^{''} 26	Mar. 17,	10	42	45	+0 ^{''} 37	-0 ^{''} 39	
	15,	12	0	+0 ^{''} 04	-0 ^{''} 25		18,	10	27	52	+0 ^{''} 01	+0 ^{''} 19
	16,	13	42	-0 ^{''} 29	+0 ^{''} 12		24,	9	51	34	-0 ^{''} 03	+0 ^{''} 11
	17,	12	26	-0 ^{''} 24	+0 ^{''} 10		25,	10	5	23	-0 ^{''} 36	+0 ^{''} 10
	22,	10	31	+0 ^{''} 20	+0 ^{''} 44	April 6,	9	47	43	-0 ^{''} 45	-0 ^{''} 12	
	23,	11	50	+0 ^{''} 00	-0 ^{''} 02		8,	10	0	10	-0 ^{''} 06	+0 ^{''} 05
	29,	12	11	-0 ^{''} 26	+0 ^{''} 16		13,	9	56	4	-0 ^{''} 38	-0 ^{''} 10
	30,	10	36	-0 ^{''} 30	+0 ^{''} 22		14,	9	41	18	-0 ^{''} 15	+0 ^{''} 13
1902							15,	9	49	2	+0 ^{''} 10	-0 ^{''} 25
Jan. 2,	10	28	35	+0 ^{''} 24	+0 ^{''} 27							
	5,	11	58	+0 ^{''} 04	-0 ^{''} 21							

YERKES OBSERVATORY—PROF. BARNARD. 1903·0.

1902	H.	M.	S.			1902	H.	M.	S.			
Aug. 25,	18	12	6	+0 ^{''} 25	+0 ^{''} 14	Dec. 1,	17	7	45	+0 ^{''} 13	+0 ^{''} 23	
Sept. 1,	17	27	14	-0 ^{''} 39	+0 ^{''} 35		30,	11	40	10	-0 ^{''} 35	-0 ^{''} 30
	8,	18	30	-0 ^{''} 31	+0 ^{''} 14	1903						
	9,	17	29	+0 ^{''} 05	-0 ^{''} 12	Jan. 12,	12	10	49	+0 ^{''} 13	+0 ^{''} 05	
	15,	17	30	-0 ^{''} 24	-0 ^{''} 24		19,	11	44	34	+0 ^{''} 25	+0 ^{''} 04
	16,	17	19	+0 ^{''} 21	-0 ^{''} 15		20,	10	5	21	+0 ^{''} 24	+0 ^{''} 17
	18,	17	49	+0 ^{''} 00	+0 ^{''} 09	Feb. 9,	11	41	38	+0 ^{''} 12	-0 ^{''} 10	
Oct. 6,	17	47	33	-0 ^{''} 30	+0 ^{''} 28		16,	11	1	46	+0 ^{''} 32	+0 ^{''} 33
	7,	15	29	-0 ^{''} 15	+0 ^{''} 36		17,	10	46	41	+0 ^{''} 23	+0 ^{''} 08
	13,	17	14	-0 ^{''} 40	+0 ^{''} 25		23,	10	55	10	+0 ^{''} 04	+0 ^{''} 07
	14,	16	59	-0 ^{''} 44	-0 ^{''} 15		24,	11	52	1	+0 ^{''} 02	-0 ^{''} 11
	27,	19	19	-0 ^{''} 38	+0 ^{''} 28	Mar. 2,	10	28	18	-0 ^{''} 17	-0 ^{''} 05	
Nov. 24,	13	17	57	-0 ^{''} 08	-0 ^{''} 10		25,	10	29	27	-0 ^{''} 07	+0 ^{''} 00
							30,	12	7	41	-0 ^{''} 06	+0 ^{''} 06

YERKES OBSERVATORY—PROF. BARNARD. 1904·0.

Paris Mean Time.				<i>d s.</i>	<i>s dp.</i>	Paris Mean Time.				<i>d s.</i>	<i>s dp.</i>
1903	H.	M.	S.			1903	H.	M.	S.		
Aug. 31,	18	41	40	-0 ^{''} 38	+0 ^{''} 39	Dec. 21,	13	53	39	-0 ^{''} 25	+0 ^{''} 27
Sept. 21,	17	13	3	+0 ^{''} 02	+0 ^{''} 05	22,	11	50	43	-0 ^{''} 51	-0 ^{''} 13
28,	16	47	54	+0 ^{''} 25	-0 ^{''} 11	1904					
Oct. 13,	15	31	21	-0 ^{''} 04	+0 ^{''} 00	Jan. 3,	16	4	17	-0 ^{''} 13	+0 ^{''} 63
19,	16	54	30	-0 ^{''} 19	+0 ^{''} 18	26,	14	4	47	-0 ^{''} 35	+0 ^{''} 08
20,	13	49	9	-0 ^{''} 01	-0 ^{''} 33	Feb. 2,	11	11	28	-0 ^{''} 13	-0 ^{''} 12
26,	17	4	5	+0 ^{''} 13	-0 ^{''} 04	15,	13	48	56	+0 ^{''} 12	+0 ^{''} 33
27,	15	49	23	+0 ^{''} 17	-0 ^{''} 08	16,	13	49	26	-0 ^{''} 16	+0 ^{''} 31
Nov. 9,	15	1	4	+0 ^{''} 00	+0 ^{''} 28	April 4,	10	5	55	-0 ^{''} 19	+0 ^{''} 03
21,	16	25	15	-0 ^{''} 13	+0 ^{''} 02	18,	10	54	4	-0 ^{''} 11	+0 ^{''} 06
24,	14	43	54	-0 ^{''} 25	-0 ^{''} 20	19,	11	24	59	+0 ^{''} 36	-0 ^{''} 42
Dec. 14,	12	50	45	+0 ^{''} 41	+0 ^{''} 19						

YERKES OBSERVATORY—PROF. BARNARD. 1904·9.

1904	H.	M.	S.			1904	H.	M.	S.		
Oct. 15,	16	12	23	-0 ^{''} 12	+0 ^{''} 23	Nov. 21,	14	3	0	-0 ^{''} 27	+0 ^{''} 02
22,	18	45	43	-0 ^{''} 32	+0 ^{''} 22	26,	14	1	56	+0 ^{''} 04	-0 ^{''} 39
29,	15	13	3	-0 ^{''} 05	-0 ^{''} 02	28,	15	30	44	-0 ^{''} 13	+0 ^{''} 08
31,	15	42	22	+0 ^{''} 17	+0 ^{''} 02	Dec. 5,	13	24	30	+0 ^{''} 23	+0 ^{''} 20
Nov. 5,	13	35	43	+0 ^{''} 12	+0 ^{''} 00	10,	13	47	2	-0 ^{''} 22	+0 ^{''} 08
12,	15	34	25	-0 ^{''} 02	-0 ^{''} 32	12,	14	38	15	-0 ^{''} 22	-0 ^{''} 03
14,	15	55	57	-0 ^{''} 49	+0 ^{''} 18	31,	11	35	41	-0 ^{''} 42	+0 ^{''} 20

YERKES OBSERVATORY—PROF. BARNARD. 1906·1.

1905	H.	M.	S.			1906	H.	M.	S.		
Dec. 9,	19	0	22	+0 ^{''} 00	-0 ^{''} 29	Jan. 23,	13	42	48	-0 ^{''} 61	-0 ^{''} 17
19,	13	31	23	-0 ^{''} 54	+0 ^{''} 36	Feb. 6,	14	22	28	+0 ^{''} 14	-0 ^{''} 02
23,	17	37	24	-0 ^{''} 49	-0 ^{''} 24	27,	12	1	9	-0 ^{''} 22	-0 ^{''} 25
26,	13	17	44	-0 ^{''} 38	-0 ^{''} 01	Mar. 20,	11	21	5	+0 ^{''} 22	-0 ^{''} 27
30,	16	1	56	+0 ^{''} 15	-0 ^{''} 01	April 17,	9	30	55	+0 ^{''} 09	-0 ^{''} 06

MR LOWELL'S OBSERVATORY—MESSRS DREW AND COGSHALL. 1897·0.

1896	H.	M.	S.			1897	H.	M.	S.		
Oct. 16,	16	43	57	+0 ^{''} 42	+0 ^{''} 54	Feb. 21,	13	1	7	+0 ^{''} 09	+0 ^{''} 37
29,	16	14	0	+0 ^{''} 01	-0 ^{''} 41	22,	13	37	30	-0 ^{''} 15	-0 ^{''} 47
30,	16	10	51	+0 ^{''} 09	-0 ^{''} 33	23,	12	53	5	-0 ^{''} 09	+0 ^{''} 24
Nov. 4,	16	48	12	+0 ^{''} 23	+0 ^{''} 17	24,	12	40	27	-0 ^{''} 06	+0 ^{''} 11
6,	16	26	23	+0 ^{''} 69	-0 ^{''} 03	25,	11	30	48	+0 ^{''} 30	+0 ^{''} 12
7,	16	29	41	-0 ^{''} 17	-0 ^{''} 31	27,	13	4	1	-0 ^{''} 18	+0 ^{''} 00
1897						28,	11	56	23	+0 ^{''} 13	-0 ^{''} 36
Jan. 9,	15	17	49	+0 ^{''} 54	+0 ^{''} 06	Mar. 1,	12	2	44	-0 ^{''} 34	-0 ^{''} 13
14,	14	46	54	+0 ^{''} 16	+0 ^{''} 14	2,	11	56	51	-0 ^{''} 49	+0 ^{''} 10
18,	14	50	13	+0 ^{''} 56	-0 ^{''} 20	4,	10	32	34	+0 ^{''} 37	-0 ^{''} 03
28,	15	17	49	+0 ^{''} 00	+0 ^{''} 43	6,	10	36	56	+0 ^{''} 07	-0 ^{''} 36
30,	14	41	11	-0 ^{''} 30	+0 ^{''} 09	7,	10	43	49	+0 ^{''} 47	-0 ^{''} 56
Feb. 6,	13	3	51	-0 ^{''} 17	-0 ^{''} 37	8,	10	36	57	-0 ^{''} 39	+0 ^{''} 17
8,	14	36	5	+0 ^{''} 34	+0 ^{''} 42	10,	10	25	57	+0 ^{''} 05	-0 ^{''} 47
13,	14	30	9	-0 ^{''} 07	-0 ^{''} 17	11,	10	23	10	-0 ^{''} 17	-0 ^{''} 28
18,	13	44	15	-0 ^{''} 47	-0 ^{''} 33	12,	10	29	30	-0 ^{''} 42	-0 ^{''} 33
19,	13	56	25	+0 ^{''} 59	-0 ^{''} 15	19,	9	55	42	-0 ^{''} 55	-0 ^{''} 76
20,	13	48	53	+0 ^{''} 68	+0 ^{''} 10	26,	10	16	44	+0 ^{''} 01	-0 ^{''} 13

MR LOWELL'S OBSERVATORY—MESSRS DREW AND COGSHALL. 1898·8.

Paris Mean Time.				<i>d s.</i>	<i>s dp.</i>	Paris Mean Time.				<i>d s.</i>	<i>s dp.</i>		
1898	H.	M.	S.			1898	H.	M.	S.				
Sept. 9,	18	53	28	+0 ^{''} 07	+0 ^{''} 08	Sept. 27,	18	34	40	-0 ^{''} 15	+0 ^{''} 34		
	12,	18	41	53	-0 ^{''} 24	-0 ^{''} 05		29,	18	15	2	+0 ^{''} 09	+0 ^{''} 33
	14,	18	23	3	-0 ^{''} 62	+0 ^{''} 06	Oct. 17,	18	13	56	-0 ^{''} 36	+0 ^{''} 14	
	18,	18	25	37	-0 ^{''} 30	+0 ^{''} 18		21,	18	12	38	-0 ^{''} 03	+0 ^{''} 10
	19,	18	36	45	-0 ^{''} 18	+0 ^{''} 18	Nov. 9,	19	13	15	+0 ^{''} 44	-0 ^{''} 09	
	22,	18	1	10	-0 ^{''} 16	+0 ^{''} 27	Dec. 7,	16	33	40	-0 ^{''} 33	+0 ^{''} 50	

WASHINGTON OBSERVATORY—MR DINWIDDIE. 1903·0.

1902						1903							
	H.	M.	S.				H.	M.	S.				
Oct. 24,	17	41	57	-0 ^{''} 27	-0 ^{''} 50	Jan. 6,	12	25	59	+0 ^{''} 41	+0 ^{''} 73		
	28,	17	27	56	-0 ^{''} 18	-0 ^{''} 04	Feb. 22,	9	42	52	-0 ^{''} 15	+0 ^{''} 48	
	29,	17	54	49	+0 ^{''} 00	-0 ^{''} 50		25,	9	50	49	-0 ^{''} 38	+0 ^{''} 12
	30,	17	30	11	+0 ^{''} 35	-0 ^{''} 26		26,	9	19	12	+0 ^{''} 13	+0 ^{''} 20
	31,	17	57	48	+0 ^{''} 02	-0 ^{''} 23	Mar. 3,	8	47	16	-0 ^{''} 24	+0 ^{''} 12	
Nov. 1,	17	16	55	-0 ^{''} 37	+0 ^{''} 46		12,	8	39	4	+0 ^{''} 42	+0 ^{''} 20	
	2,	17	18	32	+0 ^{''} 02	+0 ^{''} 21		17,	8	42	48	+0 ^{''} 22	+0 ^{''} 38
	21,	16	0	53	+0 ^{''} 06	+0 ^{''} 00		18,	9	2	13	-0 ^{''} 30	+0 ^{''} 32
Dec. 5,	15	33	0	+0 ^{''} 03	-0 ^{''} 27		26,	8	50	54	-0 ^{''} 07	-0 ^{''} 38	
	7,	14	51	51	-0 ^{''} 32	+0 ^{''} 67							

WASHINGTON OBSERVATORY—MESSRS HAMMOND AND RICE. 1905·0.

1904						1905							
	H.	M.	S.				H.	M.	S.				
Nov. 21,	16	52	53	+0 ^{''} 14	+0 ^{''} 08	Jan. 16,	10	48	34	-0 ^{''} 34	+0 ^{''} 34		
	30,	15	52	4	-0 ^{''} 16	+0 ^{''} 46		27,	10	41	46	-0 ^{''} 50	+0 ^{''} 03
Dec. 14,	12	33	1	+0 ^{''} 67	+0 ^{''} 27	Feb. 7,	9	47	4	-0 ^{''} 18	-0 ^{''} 06		
	16,	12	4	31	-0 ^{''} 24	+0 ^{''} 01		10,	9	35	24	-0 ^{''} 30	+0 ^{''} 22
	18,	12	38	25	-0 ^{''} 13	+0 ^{''} 34		24,	9	35	1	+0 ^{''} 03	-0 ^{''} 05
	19,	11	47	31	-0 ^{''} 22	-0 ^{''} 03	Mar. 10,	10	14	14	-0 ^{''} 21	+0 ^{''} 49	
	29,	11	0	9	-0 ^{''} 51	+0 ^{''} 05		13,	8	54	18	+0 ^{''} 11	+0 ^{''} 24
1905								25,	9	30	11	-0 ^{''} 07	-0 ^{''} 11
Jan. 1,	10	39	24	+0 ^{''} 05	+0 ^{''} 35			31,	9	59	58	+0 ^{''} 42	-0 ^{''} 17

WASHINGTON OBSERVATORY—MR HAMMOND. 1906·0.

1905						1906							
	H.	M.	S.				H.	M.	S.				
Oct. 29,	14	47	16	-0 ^{''} 02	-0 ^{''} 16	Feb. 13,	8	25	32	-0 ^{''} 28	+0 ^{''} 18		
	30,	14	36	7	-0 ^{''} 22	+0 ^{''} 18		13,	11	41	47	-0 ^{''} 04	+0 ^{''} 33
Nov. 1,	13	54	45	-0 ^{''} 16	+0 ^{''} 40			16,	8	54	21	+0 ^{''} 37	+0 ^{''} 43
1906								16,	10	43	8	-0 ^{''} 07	+0 ^{''} 43
Jan. 5,	13	6	29	+0 ^{''} 40	+0 ^{''} 11			17,	8	47	39	+0 ^{''} 12	+0 ^{''} 34
	6,	11	32	41	+0 ^{''} 22	+0 ^{''} 20		19,	8	53	49	+0 ^{''} 00	+0 ^{''} 46
	16,	10	48	40	+0 ^{''} 25	+0 ^{''} 27		23,	8	38	26	-0 ^{''} 01	+0 ^{''} 48
	18,	12	52	55	+0 ^{''} 35	+0 ^{''} 29		23,	11	7	46	-0 ^{''} 20	+0 ^{''} 19
	24,	9	48	17	+0 ^{''} 23	+0 ^{''} 28	Feb. 24,	11	3	3	+0 ^{''} 13	+0 ^{''} 28	
	24,	11	56	44	+0 ^{''} 19	+0 ^{''} 40		25,	8	33	36	-0 ^{''} 28	+0 ^{''} 42
	28,	11	58	7	+0 ^{''} 13	-0 ^{''} 13		25,	10	46	24	+0 ^{''} 43	+0 ^{''} 28
	29,	9	38	58	-0 ^{''} 06	+0 ^{''} 44	Mar. 6,	9	24	12	+0 ^{''} 00	+0 ^{''} 33	
	29,	12	46	48	+0 ^{''} 11	+0 ^{''} 20		10,	9	25	52	+0 ^{''} 02	+0 ^{''} 08
	31,	9	46	54	-0 ^{''} 37	+0 ^{''} 08		20,	17	26	33	-0 ^{''} 19	+0 ^{''} 24
	31,	12	0	32	-0 ^{''} 28	+0 ^{''} 28		23,	17	51	19	+0 ^{''} 33	-0 ^{''} 02
Feb. 11,	8	26	23	+0 ^{''} 19	+0 ^{''} 36								

5. The equations of condition were then formed by using Marth's formulæ on page 518. The following, derived from the 1892 observations of Prof. Barnard, will serve as an example of these:—

POSITION-ANGLE.							
Date.	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	$s \sin dp$.	
1892 Nov.	13.	- 9 ^{''} 55	- 5 ^{''} 90	+ 7 ^{''} 05	+ 8 ^{''} 23	- 4 ^{''} 84 =	+ 0 ^{''} 08
	18.	- 11 ^{''} 7	- 4 ^{''} 51	- 3 ^{''} 01	+ 11 ^{''} 3	+ 2 ^{''} 94 =	- 0 ^{''} 14
	20.	- 15 ^{''} 9	+ 9 ^{''} 11	+ 5 ^{''} 77	- 4 ^{''} 32	- 15 ^{''} 3 =	- 0 ^{''} 07
Dec.	9.	- 10 ^{''} 4	- 6 ^{''} 72	+ 0 ^{''} 549	- 10 ^{''} 4	+ 0 ^{''} 426 =	- 0 ^{''} 11
	16.	- 10 ^{''} 6	+ 0 ^{''} 27	+ 11 ^{''} 9	- 4 ^{''} 48	+ 9 ^{''} 60 =	+ 0 ^{''} 02
1893 Jan.	8.	- 9 ^{''} 48	- 4 ^{''} 62	+ 9 ^{''} 10	- 7 ^{''} 17	+ 6 ^{''} 21 =	+ 0 ^{''} 37
	13.	- 11 ^{''} 6	- 4 ^{''} 91	- 2 ^{''} 62	- 11 ^{''} 3	+ 2 ^{''} 47 =	+ 0 ^{''} 18
	20.	- 9 ^{''} 59	- 3 ^{''} 25	+ 10 ^{''} 3	- 6 ^{''} 29	+ 7 ^{''} 24 =	+ 0 ^{''} 10

DISTANCE.								
Date.	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .	ds .	
1892 Nov.	13.	+ 0 ^{''} 014	- 0 ^{''} 016	+ 0 ^{''} 005	- 4 ^{''} 29	- 7 ^{''} 27	+ 16 ^{''} 9 =	- 0 ^{''} 25
	18.	+ 7 ^{''} 03	- 9 ^{''} 26	- 1 ^{''} 72	- 5 ^{''} 08	- 8 ^{''} 41	+ 13 ^{''} 7 =	+ 0 ^{''} 01
	20.	- 4 ^{''} 78	- 0 ^{''} 711	- 12 ^{''} 2	- 6 ^{''} 22	- 2 ^{''} 21	+ 10 ^{''} 2 =	+ 0 ^{''} 37
Dec.	9.	+ 5 ^{''} 37	- 6 ^{''} 25	+ 0 ^{''} 171	+ 5 ^{''} 69	+ 7 ^{''} 54	+ 15 ^{''} 5 =	+ 0 ^{''} 03
	16.	- 5 ^{''} 97	+ 4 ^{''} 71	- 4 ^{''} 38	+ 4 ^{''} 30	+ 8 ^{''} 62	+ 15 ^{''} 1 =	- 0 ^{''} 07
1893 Jan.	8.	- 1 ^{''} 74	+ 1 ^{''} 63	- 0 ^{''} 776	+ 4 ^{''} 17	+ 7 ^{''} 46	+ 16 ^{''} 7 =	- 0 ^{''} 27
	13.	+ 7 ^{''} 17	- 9 ^{''} 21	+ 1 ^{''} 44	+ 5 ^{''} 55	+ 8 ^{''} 19	+ 13 ^{''} 6 =	+ 0 ^{''} 05
	20.	- 3 ^{''} 25	+ 2 ^{''} 87	- 1 ^{''} 72	+ 4 ^{''} 02	+ 7 ^{''} 78	+ 16 ^{''} 3 =	- 0 ^{''} 17

6. Next the normal equations were formed for each series:—

NORMAL EQUATIONS.

LICK OBSERVATORY—PROF. BARNARD. 1893·0.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .	
Sin du	+ 1220	- 39	- 321	+ 291	- 46	+ 56	= - 1·86
Sin dN		+ 483	- 54	+ 41	- 155	- 211	= - 4·91
Sin dI			+ 607	- 95	+ 106	- 232	= + 1·05
$2e \sin Q$				+ 759	+ 168	+ 158	= - 7·82
$2e \cos Q$					+ 899	+ 349	= + 1·50
da/a						+ 1776	= - 7·51

LICK OBSERVATORY—PROF. BARNARD. 1894·0.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .	
Sin du	+ 1360	+ 173	- 444	- 536	- 140	+ 78	= - 2·23
Sin dN		+ 495	- 145	- 120	+ 194	- 207	= + 1·32
Sin dI			+ 613	+ 90	- 71	- 158	= + 0·08
$2e \sin Q$				+ 916	+ 178	- 178	= - 8·45
$2e \cos Q$					+ 1037	- 316	= + 1·31
da/a						+ 2360	= + 0·09

LICK OBSERVATORY—PROF. BARNARD. 1895·0.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+2509	- 398	- 416	+ 167	+ 416	+ 64	=	+ 8·35
Sin dN		+ 901	+ 152	- 18	- 153	- 408	=	- 9·07
Sin dI			+1171	+ 170	- 117	- 724	=	- 4·02
$2e \sin Q$				+1250	+ 556	+ 2	=	- 8·62
$2e \cos Q$					+1900	- 47	=	- 5·12
da/a						+2694	=	+ 0·92

LICK OBSERVATORY—PROF. SCHAEBERLE. 1895·0.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+1923	- 511	- 392	+ 89	- 167	- 180	=	- 10·19
Sin dN		+ 613	+ 155	- 16	+ 109	- 115	=	+ 2·29
Sin dI			+1113	+ 2	+ 53	- 694	=	- 2·13
$2e \sin Q$				+ 749	+ 326	+ 98	=	- 1·23
$2e \cos Q$					+1621	- 78	=	+14·38
da/a						+1789	=	+ 4·77

LICK OBSERVATORY—PROF. SCHAEBERLE. 1896·0.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+1837	- 322	- 183	+ 214	- 921	+ 127	=	- 15·78
Sin dN		+ 679	+ 156	+ 312	+ 306	- 396	=	- 3·12
Sin dI			+ 776	- 341	+ 213	- 567	=	+ 3·92
$2e \sin Q$				+ 893	+ 479	+ 386	=	- 7·19
$2e \cos Q$					+1428	+ 625	=	+ 8·67
da/a						+1918	=	+ 1·80

LICK OBSERVATORY—PROF. SCHAEBERLE. 1896·8.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+1287	- 158	- 104	- 300	+ 65	+ 91	=	+ 1·74
Sin dN		+ 440	+ 70	- 64	+ 49	- 244	=	- 13·13
Sin dI			+ 509	+ 72	- 76	- 382	=	- 4·29
$2e \sin Q$				+ 637	+ 239	- 107	=	- 0·35
$2e \cos Q$					+1010	- 375	=	+ 1·21
da/a						+1420	=	+ 1·94

LICK OBSERVATORY—PROF. SCHAEBERLE. 1897·8.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+3222	- 717	- 382	+ 420	+1016	- 109	=	- 25·47
Sin dN		+ 981	+ 72	- 26	- 491	- 94	=	- 2·59
Sin dI			+1540	+ 318	- 130	-1140	=	+ 7·17
$2e \sin Q$				+1264	+ 376	+ 4	=	- 16·24
$2e \cos Q$					+2733	+ 459	=	- 13·01
da/a						+3018	=	- 19·25

LICK OBSERVATORY—MR AITKEN. 1899·0.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+2842	- 840	- 455	- 4	+ 947	- 344	=	+ 2·78
Sin dN		+ 740	+ 39	+ 132	- 424	+ 152	=	- 18·45
Sin dI			+1530	+ 234	- 91	- 1119	=	+ 0·35
$2e \sin Q$				+ 861	- 67	- 122	=	- 20·63
$2e \cos Q$					+2359	+ 289	=	+ 10·86
da/a						+2300	=	+ 12·61

LICK OBSERVATORY—MR HUSSEY. 1899·0.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+1887	+ 105	- 268	- 160	+ 169	+ 87	=	- 11·64
Sin dN		+ 608	- 25	- 136	+ 6	- 223	=	- 2·68
Sin dI			+ 609	+ 127	- 4	- 387	=	+ 4·66
$2e \sin Q$				+1143	+ 162	- 88	=	- 7·60
$2e \cos Q$					+1446	- 273	=	- 5·67
da/a						+2804	=	- 13·14

YERKES OBSERVATORY—PROF. BARNARD. 1899·0.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+9823	- 1887	- 943	- 545	+ 705	- 310	=	- 3·15
Sin dN		+2659	+ 386	- 178	- 61	- 399	=	- 2·94
Sin dI			+ 4386	+ 393	- 58	- 3319	=	- 12·79
$2e \sin Q$				+3805	+ 985	- 266	=	+ 14·05
$2e \cos Q$					+8311	- 600	=	+ 50·02
da/a						+9557	=	+ 42·81

YERKES OBSERVATORY—PROF. BARNARD. 1900·0.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+11641	- 1051	- 892	+ 267	- 1432	+ 176	=	- 56·53
Sin dN		+3296	+ 422	+ 429	+ 922	- 929	=	- 16·22
Sin dI			+ 4732	- 285	- 26	- 3439	=	- 0·68
$2e \sin Q$				+5230	+1119	+ 463	=	+ 46·47
$2e \cos Q$					+9643	+ 574	=	+ 44·62
da/a						+12774	=	+ 16·14

YERKES OBSERVATORY—PROF. BARNARD. 1900·9.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+ 9241	- 1674	- 301	+ 576	+ 779	+ 291	=	+ 22·20
Sin dN		+2472	+ 420	+ 149	- 333	- 860	=	- 2·37
Sin dI			+3928	+ 368	+ 214	- 3260	=	- 4·36
$2e \sin Q$				+3540	+ 930	- 39	=	+ 47·64
$2e \cos Q$					+7885	+ 543	=	+ 52·73
da/a						+8748	=	+ 0·90

YERKES OBSERVATORY—PROF. BARNARD. 1902·0.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+8833	-1019	-510	-505	-241	+69	=	-25·68
Sin dN		+2310	+139	-198	+188	-232	=	+1·63
Sin dI			+3505	+2	-10	-2905	=	-11·59
$2e \sin Q$				+3822	+501	+71	=	+16·45
$2e \cos Q$					+7631	-665	=	+69·36
da/a						+9508	=	-32·73

YERKES OBSERVATORY—PROF. BARNARD. 1903·0.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+5348	-1130	+14	+89	-377	+166	=	-15·12
Sin dN		+1316	+197	+117	-19	-202	=	-13·16
Sin dI			+2332	-160	+35	-1990	=	+9·42
$2e \sin Q$				+1882	+419	+57	=	+15·66
$2e \cos Q$					+4692	+209	=	+36·18
da/a						+4751	=	-18·08

YERKES OBSERVATORY—PROF. BARNARD. 1904·0.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+4535	-893	-114	+24	+458	-61	=	-18·14
Sin dN		+1089	+7	+21	-148	+26	=	-3·02
Sin dI			+1992	+208	-35	-1682	=	+9·26
$2e \sin Q$				+1592	+30	-41	=	+14·19
$2e \cos Q$					+3957	+19	=	+39·77
da/a						+4101	=	-21·12

YERKES OBSERVATORY—PROF. BARNARD. 1904·9.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+2850	-459	+187	+74	+65	-14	=	-11·73
Sin dN		+899	+126	-235	+27	+30	=	+7·67
Sin dI			+1012	+8	-15	-895	=	+5·33
$2e \sin Q$				+1315	+45	-248	=	+1·15
$2e \cos Q$					+2263	-51	=	+15·04
da/a						+2913	=	-23·19

YERKES OBSERVATORY—PROF. BARNARD. 1906·1.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+2141	-466	+199	-283	+215	+206	=	+15·36
Sin dN		+493	+81	-114	-130	-224	=	-5·22
Sin dI			+951	-96	-53	-826	=	-3·35
$2e \sin Q$				+724	+185	+49	=	+4·70
$2e \cos Q$					+1885	-359	=	+13·72
da/a						+1858	=	-26·43

MR LOWELL'S OBSERVATORY—MESSRS DREW AND COGSHALL. 1897·0.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+6568	-1230	-962	-229	+1002	-129	=	+66·88
Sin dN		+2080	+324	-112	-399	-587	=	-32·28
Sin dI			+3136	+289	-242	-2166	=	-7·24
$2e \sin Q$				+2827	+1059	-383	=	-15·60
$2e \cos Q$					+5342	-974	=	+29·75
da/a						+6572	=	+25·87

MR LOWELL'S OBSERVATORY—MESSRS DREW AND COGSHALL. 1898·8.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+2226	-352	-103	+29	+529	+237	=	-27·51
Sin dN		+673	+133	-208	-137	-452	=	+0·76
Sin dI			+1021	+155	-147	-720	=	+8·36
$2e \sin Q$				+1071	+428	·230	=	+2·98
$2e \cos Q$					+1803	-112	=	-12·57
da/a						+2229	=	-25·09

WASHINGTON OBSERVATORY—MR DINWIDDIE. 1903·0.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+3174	+120	-105	+92	+65	+107	=	-21·84
Sin dN		+887	+16	+35	-25	-185	=	-5·89
Sin dI			+975	+23	+2	-820	=	+8·75
$2e \sin Q$				+1780	+125	+14	=	-45·28
$2e \cos Q$					+2475	+94	=	-20·11
da/a						+4286	=	+2·30

WASHINGTON OBSERVATORY—MESSRS HAMMOND AND RICE. 1905·0.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+3332	-405	-321	-17	+370	-362	=	-33·79
Sin dN		+821	-174	+14	-163	+432	=	-3·21
Sin dI			+1300	+97	-26	-1126	=	+18·48
$2e \sin Q$				+1407	-297	-73	=	-8·73
$2e \cos Q$					+2796	-44	=	-44·66
da/a						+3472	=	-20·33

WASHINGTON OBSERVATORY—MR HAMMOND.

	Sin du .	Sin dN .	Sin dI .	$2e \sin Q$.	$2e \cos Q$.	da/a .		
Sin du	+5537	-253	+74	-756	+625	+90	=	-96·84
Sin dN		+1445	-53	-133	-294	+69	=	-15·33
Sin dI			+1875	+277	-126	-1641	=	+2·15
$2e \sin Q$				+2724	+35	-23	=	+25·87
$2e \cos Q$					+4422	-802	=	-32·06
da/a						+6434	=	+22·05

7. In the next table are given the solutions of these equations. To these are added the solutions of those series marked with an asterisk on page 519.

LICK OBSERVATORY—36-INCH REFRACTOR.

Epoch.	No. of Obsns.	$du.$	$dN.$	$dI.$	$da.$	$2e \sin Q.$	$2e \cos Q.$	Observations by
1893·0	8	+0°0	-0°63	-0°19	-0"098	-·0099	+·0044	Barnard
1894·0	10	-0·47	0·00	-0·21	-0·012	-·0142	+·0022	"
1895·0	14	+0·13	-0·58	-0·10	-0·028	-·0060	-·0024	"
1895·0	10	-0·39	+0·04	-0·24	-0·002	+·0032	-·0091	Schaerberle
1896·0	10	-0·08	-0·36	-0·42	-0·065	-·0144	+·0141	"
1896·8	7	-0·27	-2·06	+0·85	+0·009	-·0107	+·0071	"
1897·8	16	-0·52	-0·60	-0·02	-0·114	-·0098	-·0008	"
1899·0	13	-0·04	-1·84	+0·83	+0·244	-·0221	-·0117	Aitken
1899·0	12	-0·31	-0·41	+0·20	-0·084	-·0086	-·0033	Hussey

YERKES OBSERVATORY—40-INCH REFRACTOR.

Epoch.	No. of Obsns.	$du.$	$dN.$	$dI.$	$da.$	$2e \sin Q.$	$2e \cos Q.$	Observations by
1898·0	53	-0°32	-0°89	-0°13	+0"047	+·0066	-·0066	Barnard
1899·0	51	-0·03	-0·03	+0·05	+0·084	+·0023	+·0062	"
1900·0	65	-0·31	-0·50	+0·02	+0·005	+·0091	+·0036	"
1900·9	46	+0·06	-0·01	-0·24	-0·027	+·0123	+·0054	"
1902·0	47	-0·18	-0·07	-0·47	-0·087	+·0029	+·0084	"
1903·0	26	-0·34	-0·96	+0·18	-0·055	+·0086	+·0065	"
1904·0	22	-0·38	-0·39	-0·07	-0·092	+·0089	+·0105	"
1904·9	14	-0·16	+0·45	-0·23	-0·147	+·0005	+·0064	"
1906·1	10	+0·78	-0·30	-1·87	-0·514	+·0094	-·0025	"

LOWELL OBSERVATORY—24-INCH REFRACTOR.

Epoch.	No. of Obsns.	$du.$	$dN.$	$dI.$	$da.$	$2e \sin Q.$	$2e \cos Q.$	Observations by
1897·0	35	+0°48	-0°53	+0°36	+0"092	-·0074	+·0061	Drew & Cogshall
1898·8	12	-0·66	-0·69	-0·11	-0·229	+·0009	-·0055	" "

WASHINGTON OBSERVATORY—26-INCH REFRACTOR.

Epoch.	No. of Obsns.	$du.$	$dN.$	$dI.$	$da.$	$2e \sin Q.$	$2e \cos Q.$	Observations by
1898·0	40	-0°38	-0°37	+0°22	+0"001	+·0012	+·0053	Brown
1900·0	65	-0·37	-0·53	-0·03	-0·270	-·0025	+·0053	See
1903·0	19	-0·32	-0·27	+0·77	+0·048	-·0248	-·0064	Dinwiddie
1905·0	17	-0·50	-0·43	+0·47	-0·057	-·0104	-·0160	Hammond & Rice
1906·0	30	-0·98	-0·82	+0·25	+0·071	+·0037	-·0049	Hammond

GREENWICH OBSERVATORY—26-INCH PHOTOGRAPHIC REFRACTOR.

Epoch.	No. of Obsns.	$du.$	$dN.$	$dI.$	$da.$	$2e \sin Q.$	$2e \cos Q.$
1902·1	51	-0°·95	-0°·42	-0°·12	+0°·087	+·0005	-·0004
1903·1	63	-0°·67	-0°·40	-0°·30	+0°·092	+·0005	+·0016
1904·1	51	-0°·83	-0°·72	-0°·20	+0°·033	-·0017	+·0002
1905·1	57	-0°·82	-0°·95	-0°·37	+0°·069	+·0038	+·0053
1906·1	59	-0°·90	-1°·17	+0°·12	+0°·085	-·0043	-·0027
1907·2	29	-1°·02	-0°·93	-0°·20	-0°·008	+·0024	+·0009
1908·2	25	-0°·64	-0°·44	+0°·11	+0°·244	-·0110	+·0032

8. The next table has been formed by taking the means of these results, giving to each set the weight represented by the number of observations. The last line is the mean of the other five, the weights being given by the number of observations as before.

Epoch.	No. of Obsns.	$du.$	$dN.$	$dI.$	$da.$	$2e \sin Q.$	$2e \cos Q.$	Observations at
1896·4	100	-0°·22	-0°·61	+0°·07	-0°·016	-·0104	-·0012	Lick
1901·7	334	-0°·17	-0°·34	-0°·16	-0°·027	+·0068	+·0039	Yerkes
1897·5	47	+0°·19	-0°·57	+0°·24	+0°·010	-·0053	+·0031	Lowell
1901·4	171	-0°·49	-0°·50	+0°·22	-0°·090	-·0038	+·0001	Washington
1904·7	335	-0°·83	-0°·73	-0°·16	+0°·080	-·0008	+·0010	Greenwich
1901·9	987	-0°·44	-0°·54	-0°·05	+0°·0013	-·0001	+·0022	

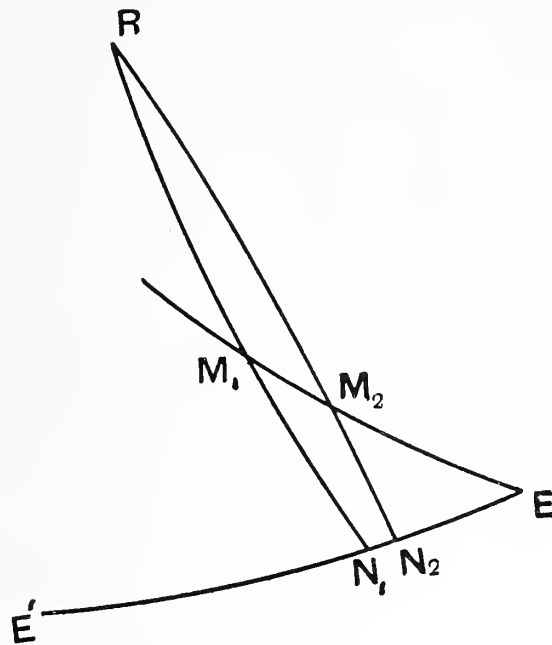
9. *The Eccentricity of the Orbit.*—The values of $2e \sin Q$ and $2e \cos Q$ just found show that the eccentricity of the orbit is extremely small. Dr Struve gave ·01 as its maximum value; the present observations give ·001.

10. *The Mean Distance of the Satellite and the Mass of Neptune.*—The value found for the mean distance is 16·270, which differs very little from that of Dr Struve, 16·271. The corresponding value of the mass of Neptune, $\frac{1}{M} = 19396$, is the same as Dr Struve's. It is interesting to notice that the photographic results give a smaller value for the distance of satellite.

11. *The Movement of the Plane of the Satellite's Orbit.*—The values $dN = -0°·54$, $dI = -0°·05$ give for the epoch 1901·9, $N = 187°·45$, $I = 117°·44$. The following table shows the changes in the node and the inclination since the discovery of the satellite:—

Epoch.	N.	I.	Epoch.	N.	I.
1849·9	179°·12	126°·01	1896·4	186°·71	118°·22
1864·0	181·49	124·20	1897·5	186·83	117·87
1875·8	183·10	121·46	1901·4	187·34	117·25
1883·0	184·36	120·08	1901·7	187·22	117·58
1890·4	185·27	119·16	1904·7	188·06	117·08.

These changes, as already stated, are due to the spheroidal shape of Neptune. In consequence of the spheroidal figure the orbit will preserve a constant inclination to the equator of Neptune, and the node will revolve uniformly on Neptune's equator. The problem then arises of determining the direction of Neptune's axis, and the inclination of the pole of the orbit to this axis.



In the figure, let $E'E$ be the Earth's equator, RM_1N_1 the plane of the satellite's orbit for 1874.0, RM_2N_2 the plane of the orbit for 1896.2, and M_1M_2E the plane of Neptune's equator.

Let $\gamma = N_1M_1E$ = the inclination of the orbit of the satellite to Neptune's equator.

θ_1 = longitude of this node on Neptune's equator at the epoch 1874.0.

$\psi_1 = M_1N_1$ = the distance between the nodes of the orbit on the two planes of Neptune's equator and the Earth's equator.

The differential relations between θ , ψ , γ and N , I , the node and inclination, are given by the equations:

$$\begin{aligned}\sin \gamma d\theta &= -\cos \psi \sin I dN + \sin \psi dI \\ d\gamma &= -\sin \psi \sin I dN - \cos \psi dI.\end{aligned}$$

Putting

$$\frac{d\gamma}{dt} = 0 \quad \text{and} \quad \frac{d\theta}{dt} = \text{const.},$$

we obtain

$$\tan \psi = -\frac{dI}{\sin I dN}.$$

Using these formulæ Dr Struve finds for the epoch 1874·0, $\psi_1 = 52^\circ\cdot6$ and $\sin \gamma d\theta = 0^\circ\cdot208$, giving the period of revolution of the pole of the orbit round the pole of Neptune equal to $1734 \sin \gamma$ years.

Comparing the present observations with Dr Struve's in 1890, the changes of N and I in this interval and the value of ψ for the mean date 1896·2 are found.

(Dr Struve)	1890·4	$N = 185^\circ\cdot27$	$I = 119^\circ\cdot16$
	1901·9	$N = 187^\circ\cdot45$	$I = 117^\circ\cdot44$

Hence in 11·5 years $dN = +2^\circ\cdot18$, $dI = -1^\circ\cdot72$; and the annual changes of N and I are

$$dN = +0^\circ\cdot190 \quad dI = -0^\circ\cdot150$$

for 1896·2, Dr Struve's result for 1874·0 being

$$dN = +0^\circ\cdot148 \quad dI = -0^\circ\cdot165.$$

Thus, for the mean date 1896·2,

$$\begin{aligned} N &= 186^\circ\cdot36 & I &= 118^\circ\cdot30 \\ \psi_2 &= 41^\circ\cdot9 & \sin \gamma d\theta &= \cdot225, \end{aligned}$$

and the time of revolution = $1600 \sin \gamma$ years.

Comparison of the values of ψ for 1874·0 and 1896·2 may be used to determine the value of γ and of the position of Neptune's equator.

We have

$$\begin{aligned} M_1 N_1 &= 52^\circ\cdot6 & N_2 M_2 &= 41^\circ\cdot9 \\ M_1 N_1 N_2 &= 122^\circ\cdot0 & M_2 N_2 N_1 &= 61^\circ\cdot7 \\ N_1 N_2 &= 3^\circ\cdot58. \end{aligned}$$

Solving the triangles, we find

$$\begin{aligned} EM_2 N_2 &= \gamma = 21^\circ\cdot2 \\ M_2 EN_2 &= 47^\circ\cdot2 \\ N_2 E &= 19^\circ\cdot2. \end{aligned}$$

Thus the inclination of the orbit of the satellite to Neptune's equator is $21^\circ\cdot2$, while the longitude of the node and the inclination of Neptune's equator to the Earth's equator are $205^\circ\cdot6$ and $132^\circ\cdot8$.

This value of the inclination implies a rotation of the pole of the satellite's orbit in about 580 years.

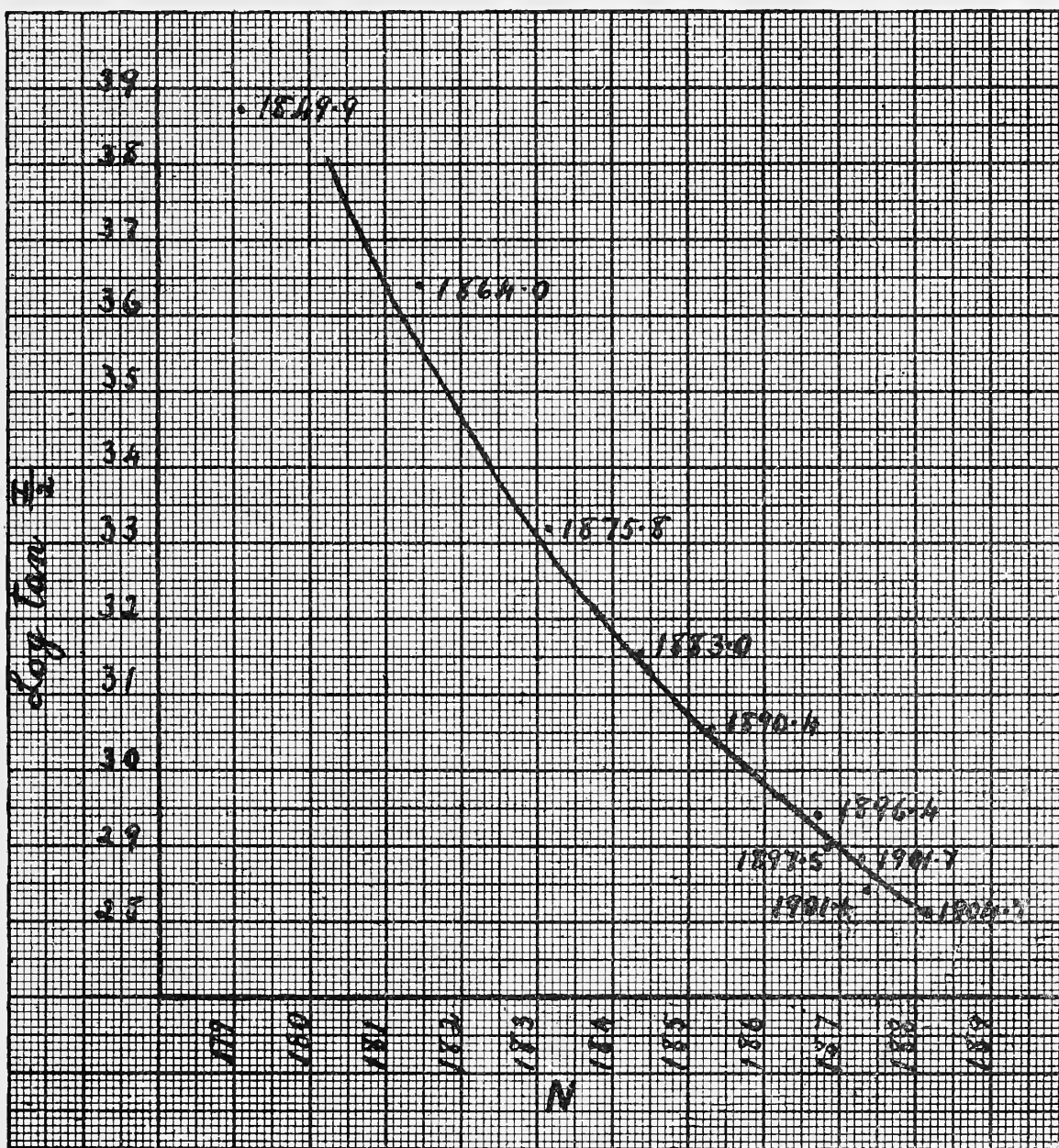
It follows that the inclination of Neptune's equator to the ecliptic, which agrees closely with Neptune's orbit round the Sun, is about 27° .

These results are in good agreement with those obtained from the Greenwich observations by Prof. Dyson and Mr Edney (*M.N.*, vol. lxxv.).

12. *Graphical Solution.* — The equation $\tan \psi = -\frac{dI}{\sin I dN}$ may be written in the form

$$\tan \psi = -\frac{d\left(\log \tan \frac{I}{2}\right)}{dN}.$$

Plotting the values of $\log \tan \frac{I}{2}$ and N as ordinates and abscissæ, we obtain the following graph, in which the curvature is evident:—



The values of $\frac{d\left(\log \tan \frac{I}{2}\right)}{dN}$ derived from this graph for the epochs 1874.0 and 1890.4 are 1.40 and .94, which are the tangents of the angles $54^{\circ}.5$ and $43^{\circ}.2$.

Comparing these with the values of ψ already calculated, we have:

	1874.0.	1890.4.
(Graphic)	$54^{\circ}.5$	$43^{\circ}.2$
(Calculated)	52.6	41.9

The agreement is as close as could be expected. The curvature confirms the result already obtained, viz. that ψ , the distance between the nodes of the orbit on Neptune's and on the Earth's equators, is becoming smaller.

In conclusion, I desire to express my best thanks to Professor Dyson for the interest he has taken in the investigation and for his kind assistance.

(Issued separately August 23, 1909.)

XXXIV.—The Pathogenesis of *Micrococcus melitensis*. By J. Eyre, M.D., Bacteriologist to Guy's Hospital, Member Advisory Board of Mediterranean Fever Commission, and Chairman of the 1906 Working Party in Malta.

(MS. received February 26, 1909. Read May 3, 1909.)

VIRULENCE OF *M. MELITENSIS*.

M. MELITENSIS is virulent to a greater or less degree for all the usual laboratory animals — guinea-pigs, rabbits, rats, mice, dogs, and monkeys — and by means of suitable passages its virulence for any particular species can be considerably exalted.

Bruce's early experiments were carried out upon the monkey, in which animal he was able, even when using comparatively small doses, to reproduce the clinical thermometric symptoms of *M. melitensis* septicæmia, previously observed in man. The virulence naturally possessed by *M. melitensis* for rodents is, however, exceedingly low, and in order to produce a *fatal* infection in these animals it is necessary to introduce enormous doses of a recently isolated strain subcutaneously or intraperitoneally: even then the infection follows such a protracted course that weeks or even months may elapse before death takes place—a fact which sufficiently explains Bruce's remarks in the *Practitioner*, 1888, vol. xl. pp. 241-9. Speaking of guinea-pigs which he had inoculated subcutaneously, he says: "After keeping them for two months I could perceive no difference in their general condition, so consider them refractory to the disease." A few years later (*Annales de l'Institut Pasteur*, 1893, vol. vii. pp. 289-304) Bruce further states: "One only obtains negative results on inoculating small quantities of pure culture of this micrococcus (*melitensis*) under the skin of mice, guinea-pigs, and rabbits." Hughes merely repeats Bruce's conclusions. Babes, too (*Handbuch für path. Mik.*), was unable to obtain any evidence of pathogenic action on rodents. This was not surprising, as he was working with a strain of the coccus obtained from Kräl, probably much attenuated from long sojourn in the laboratory.

Attenuation of Virulence.

Artificial cultivations of *M. melitensis*, especially when kept at room temperature (18-22° C.), undergo a progressive but exceedingly slow

diminution in virulence—least marked in nutrose and agar cultivations, most marked in milk cultivations. So gradual a process is this loss of virulence that the minimal lethal dose of a subcultivation from a year-old culture may only be twice as large as that originally noted. Rapid subcultivations upon artificial media effect a more rapid diminution in virulence, especially when combined with incubation at or near the upper limits of the temperature range of the organism. It is, I believe, to this tenacity with which *M. melitensis* retains its virulence, almost as much as to the absence of specific bactericidal substances in the normal blood serum, that the bulk of the laboratory infections in research workers must be attributed.

Exaltation of Virulence.

Durham (*Journ. Path. and Bact.*, 1899, vol. v. pp. 377–88) and myself (*Rep. Med. Fev. Com.*, 1905, vol. ii. pp. 67–80) showed that after intracerebral passages the virulence of *M. melitensis* became so exalted for rabbits and guinea-pigs that acute and rapidly fatal infections could readily be produced by this method of inoculation. Continuing my experiments, I found that a strain of *M. melitensis* thus exalted in virulence for the guinea-pig was practically as virulent for the rabbit when introduced into the brain of this rodent; but although highly virulent (in proportion to the number of passages) for the guinea-pig intraperitoneally and subcutaneously, further passages through the brain of the rabbit were necessary to render it highly virulent for the rabbit by intraperitoneal or subcutaneous methods of inoculation. In like manner I found that *M. melitensis* thus exalted was highly virulent for the white rat and the mouse when injected intracerebrally; but again passages through the tissues of the central nervous system of these animals respectively were needed before the coccus would provoke a fatal infection when introduced into the peritoneal cavity or the subcutaneous tissues of these small rodents.

Carbone, too, by a series of intravenous passages (five in number) through young rabbits, was able to so exalt the virulence of a strain he had isolated as to render it highly pathogenic for the guinea-pig when injected intraperitoneally; and this observer was the first to note the suppurative inflammation of the tunica vaginalis which follows intraperitoneal injections of the organism when a small male guinea-pig is employed—a phenomenon quite comparable to that observed under similar conditions when inoculating *B. mallei*. Previous to this I had attempted to infect the male of the common goat, with, as I considered, negative results; subsequently, however, Zammit showed that the goat was a susceptible animal, and Horrocks,

Kennedy, and I fully confirmed this fact. The dog and the horse have also been infected experimentally, and there is very definite evidence to show that the cat, the cow, and the mule occasionally become infected under natural conditions, though none of these animals have so far been experimentally inoculated.

Investigation by various members of the Mediterranean Fever Commission of domestic animals bred and maintained in Malta yielded evidence, by reason of the existence of specific agglutinins in the blood-serum, in support of the view that infection sometimes occurred naturally, and the results obtained may be conveniently tabulated here:—

Animal.	No. Examined.	Serum Reaction at least 1 : 30 in	<i>M. melitensis</i> recovered from.
Horse	100	21	nil.
Mule	94	8	nil.
Cow	33	10	2
Goat	2137	847	238
Dog	162	4	1
Cat	22	5	1
Sewer rat . . .	129	4	nil.

But it is interesting to note that the guinea-pig, which forms an article of diet among the lower classes in Malta and is bred for its flesh, does not appear to become infected naturally, as no evidence of disease could be detected in any of the 78 examined.

PATHOGENESIS.

The pathogenic effects noted in the various animals already enumerated, after inoculation with pure cultivations of *M. melitensis*, may now be considered seriatim, and as my own observations in this connection have been fairly comprehensive, the descriptions are entirely derived therefrom.

PATHOGENIC EFFECTS PRODUCED BY THE INOCULATION OF *M. MELITENSIS* INTO VARIOUS ANIMALS.

RODENTIA.

Guinea-pig, Rabbit.

The course of the infection in various rodents presents no material differences; hence where an experimentally inoculated guinea-pig is mentioned as illustrating a particular point, it must be understood that the behaviour of an experimentally inoculated rabbit would, under comparable conditions, be identical, and *vice versa*.

M. melitensis is pathogenic for these animals whether injected intracerebrally, intravenously, intraperitoneally, or subcutaneously. Speaking generally, the period that elapses between infection and death is shorter when the intracerebral and the intravenous methods of inoculation are practised, and longer when the intraperitoneal or the subcutaneous is chosen, though the course of the infection may, under different conditions, be acute or chronic after any form of injection, and I therefore propose to consider my experiments under these two headings.

A. *Intracerebral Inoculation*.—Infection after this method of inoculation falls under the headings of (1) Acute and (2) Chronic, according to whether death is caused in a few hours or days, or is delayed for from one week to two or three months.

1. *Acute Infection*.—An animal dying within a few days of inoculation with a moderate dose of a highly virulent cultivation, or a large dose of a less virulent one introduced into the brain substance in accordance with the technique I have previously described (*Med. Fever Reports*, part ii. p. 71), supplies the type for this form of *M. melitensis* infection.

A short incubation period, varying in duration from two or three to twenty-four hours, follows the inoculation; and during this time the animal appears to be in normal health and eats well, although the progressive loss of weight which is the marked characteristic of the infection begins within a few hours of inoculation. A stage of irritation follows the incubation period, and lasts for about twenty-four hours: it is marked by convulsions, at first localised and produced in response to direct stimuli, and not necessarily bearing any relationship to the area of cortex injured by the inoculating needle; afterwards becoming generalised, tonic in character, and occurring at frequent and irregular intervals. Progressive muscular weakness is a marked feature of this stage, throughout which the animal is obviously ill and stupid, and refuses food. The stage of irritation passes gradually into one of coma, with paresis or even paralysis affecting first the hind legs, afterwards involving the fore limbs also. Handling or even touching will at first rouse the animal and provoke general convulsions; later, the guinea-pig falls on its side, becomes insensible, and in fact appears moribund. In this condition, however, the animal may remain for twenty-four or even thirty-six hours, and during the latter part of this period no rectal temperature can be recorded by the ordinary clinical thermometer, for 30° C. is hardly ever exceeded. Death is sometimes preceded by convulsions, but usually no such warning is given. To illustrate the train of symptoms and post-mortem findings in these acute infections I cannot do better than cite in full the clinical history of one of my experimental

animals, which is quite typical of this form of infection in either guinea-pig or rabbit, and the post-mortem results that are usually obtained:—

GUINEA-PIG, MALE.

11·2	4 p.m.	A.C.E. was administered, and a 6 mm. trephine circle was cut from left parietal bone. Four (4) loops of 3-day old agar cultivations of <i>M. melitensis</i> from spleen of guinea-pig 18, emulsified in 0·2 c.c. sterile saline solution, injected into substance of left cerebral hemisphere. Disc of bone replaced, also periosteum, skin incision sutured, and wound sealed with cotton-wool and collodion.
12·2	12 p.m. 9 a.m.	Appears quite well. Has eaten well since inoculation. Is huddled up in one corner of cage; is not eating; hair dull and standing on end; is obviously ill. Has lost 60 grammes in weight.
13·2	„ 10.15 a.m.	Condition apparently unchanged. Has lost a further 60 grammes in weight. Is now grinding teeth, moves slowly, and, if turned on back, rights itself very slowly.
	10.30 a.m.	Generalised spasms result if touched; convulsive movements occur from time to time even in the absence of obvious stimuli.
	1 p.m.	Much worse; marked paresis of hind quarters.
	2 p.m.	Convulsive "circus" movements occur from time to time, the animal dragging itself round "clockwise" by means of its fore-paws.
14·2	9 p.m. 9 a.m.	Quiet in corner of cage; breathing laboured. Apparently unconscious, lying on side; if placed on legs is unable to stand, and falls down after feeble convulsive movements; once more becomes still. Breathing shallow and slow.
	11.30 a.m.	Condition unaltered.
	11.45 a.m.	Still in same position as when last looked at. Is now dead.

Post-mortem Examination.—*Scalp Incision.*—Scalp wound healthy in appearance, lips of incision healing by primary union; no signs of pus visible in wound; no stitch abscesses.

Subcutaneous tissue occupied by œdematous and jelly-like exudation marked here and there with small hæmorrhages.

Bone.—The disc of bone is firmly fixed *in situ* by serous exudation in which the periosteum is also involved. On raising the disc of bone no protrusion of meninges, etc., occurs.

General congestion and injection of the vessels of the dura mater, the site of inoculation being marked out by an area of bloody lymph roughly corresponding in size and shape to the circle of bone removed from the calvarium. On removing the meninges a thick layer of yellowish lymph is seen adhering to the surface of the convolutions in the left parieto-occipital region (this microscopically consists of a dense mass of large mononuclear leucocytes, permeated throughout by masses of cocci), cerebral vessels greatly engorged, numerous petechial hæmorrhages visible on the surface of the brain.

The cerebro-spinal fluid is more or less increased in amount and contains numerous micrococci, free and also included in cells.

On section there are numerous small hæmorrhages scattered throughout the brain substance, whilst the *velum interpositum* is so congested as to resemble a clot of blood. Elsewhere the brain appears unaffected to the naked eye. Agar tubes inoculated with brain substances from any portion of cerebrum or cerebellum or from cerebro-spinal fluid, either cerebral or spinal, yield a copious and pure growth of *M. melitensis* within thirty-six hours.

Thoracic Cavity.—Slight enlargement of anterior mediastinal and of bronchial glands. Small quantity of clear serous effusion in the pleural cavity. Cultivations from this fluid remain sterile. Few hæmorrhages on the surface of lungs. Pericardium distended with clear serous fluid, also sterile. Agar plate cultivations prepared with blood removed from right side of heart yield on an average some 35 colonies per cubic millimetre. The agglutination titre of the serum = 1 : 600.

Peritoneal Cavity.—Excess of clear, blood-stained fluid in peritoneal cavity. Gall-bladder distended with clear bile. Liver, spleen, kidneys, dark and engorged with blood—spleen being distinctly enlarged.

Omentum injected: a few large mesenteric glands noted. Bladder distended with turbid urine.

Cultivations from liver, gall-bladder, and spleen give good growth of *M. melitensis* within forty-eight hours. Kidney pulp yields only a few scattered colonies of *M. melitensis*.

Cultivations prepared from the centrifugalised deposit of the few cubic centimetres of urine contained in the bladder remain sterile.

Cultivations from the bone marrow from practically all the long bones yield a more luxuriant growth of *M. melitensis* than from other organs, with perhaps the exception of the spleen and brain.

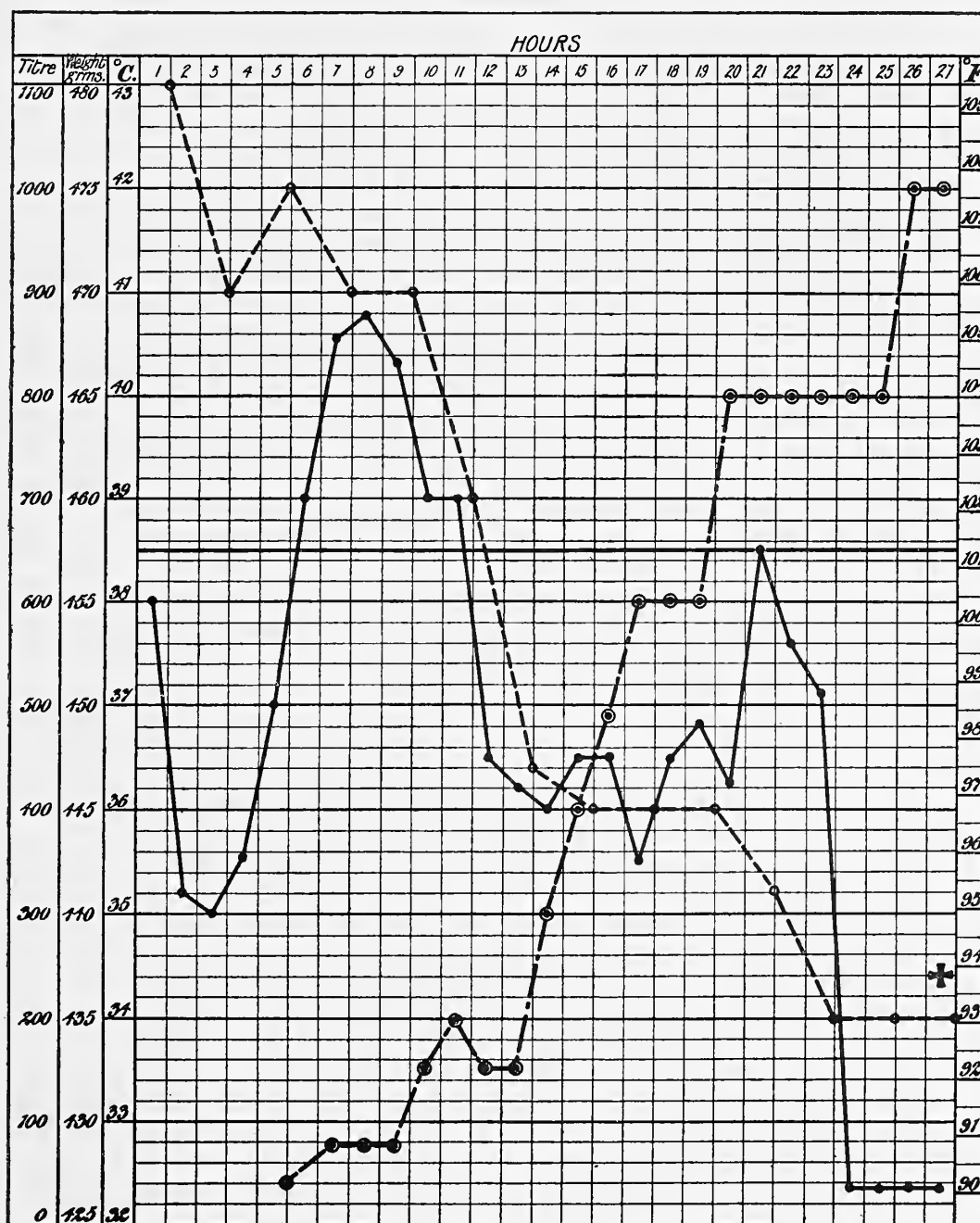
The accompanying chart (No. 1), showing the hourly temperature, weight, and sedimentation value of the serum in a more acute case still, where death occurs in twenty-seven hours, exhibits clearly the characteristic features of the infection.

2. *Chronic Infection.*—After intracerebral inoculation with a very minute dose of a highly virulent culture or a fair-sized dose of a less virulent one, the infection pursues an extremely chronic course, and beyond progressive emaciation and profound anæmia presents no very marked or characteristic symptoms. The early symptoms resemble those of the more acute infection above described, but are much less severe in character. For instance, the incubation period is usually prolonged to two or three days,

and is followed by a period extending over from three to six days, during which the animal is distinctly ill and refuses its food; remains huddled up

NOTE.—In all the following charts the temperature is shown by the continuous line: the thick interrupted line ---○--- records the titre of the serum, and the thinner line ---- the weight of the experimental animal.

CHART 1.



INTRACEREBRAL INOCULATION.

Animal guinea-pig
 Weight 480 grams
 Sex male
 Dose 1.0 loop
 Duration of infection 27 hours

in one corner of its cage; loses weight rapidly, and becomes extremely weak. Convulsions of a mild type can usually be provoked at the beginning of this stage by handling the animal or by turning it on to its back.

The animal then gradually recovers, eats well—even ravenously—and, although the emaciation may be arrested for a while, the original weight is not usually entirely recovered. After an interval extending over weeks or even months, during which, except for emaciation, the animal appears in perfect health, death suddenly takes place.

More rarely the animal is obviously ill for two or three days before death, refuses food, and becomes comatose just before the end.

Post-mortem Appearances.—*Seat of Inoculation.*—The site of the skin incision is occupied by a firm linear scar, usually adherent to the periosteum and bone beneath; the disc of bone, if it has been replaced, has usually united completely.

Cranial Cavity.—Slight injection of meningeal and cerebral vessels usually present; brain substance appears normal.

Cultures from brain substance and cerebro-spinal fluid yield only a scanty growth of *M. melitensis* or remain sterile.

Thoracic Cavity.—Lungs usually anæmic; otherwise normal. Cultures from heart blood remain sterile. The sedimentation value of the serum varies within very wide limits; it may be as low as 1:20, or as high as 1:10,000.

Peritoneal Cavity.—Peritoneum and intestines blanched and anæmic; no subperitoneal, omental, or mesenteric fat visible. Spleen often hypertrophied. Otherwise viscera normal.

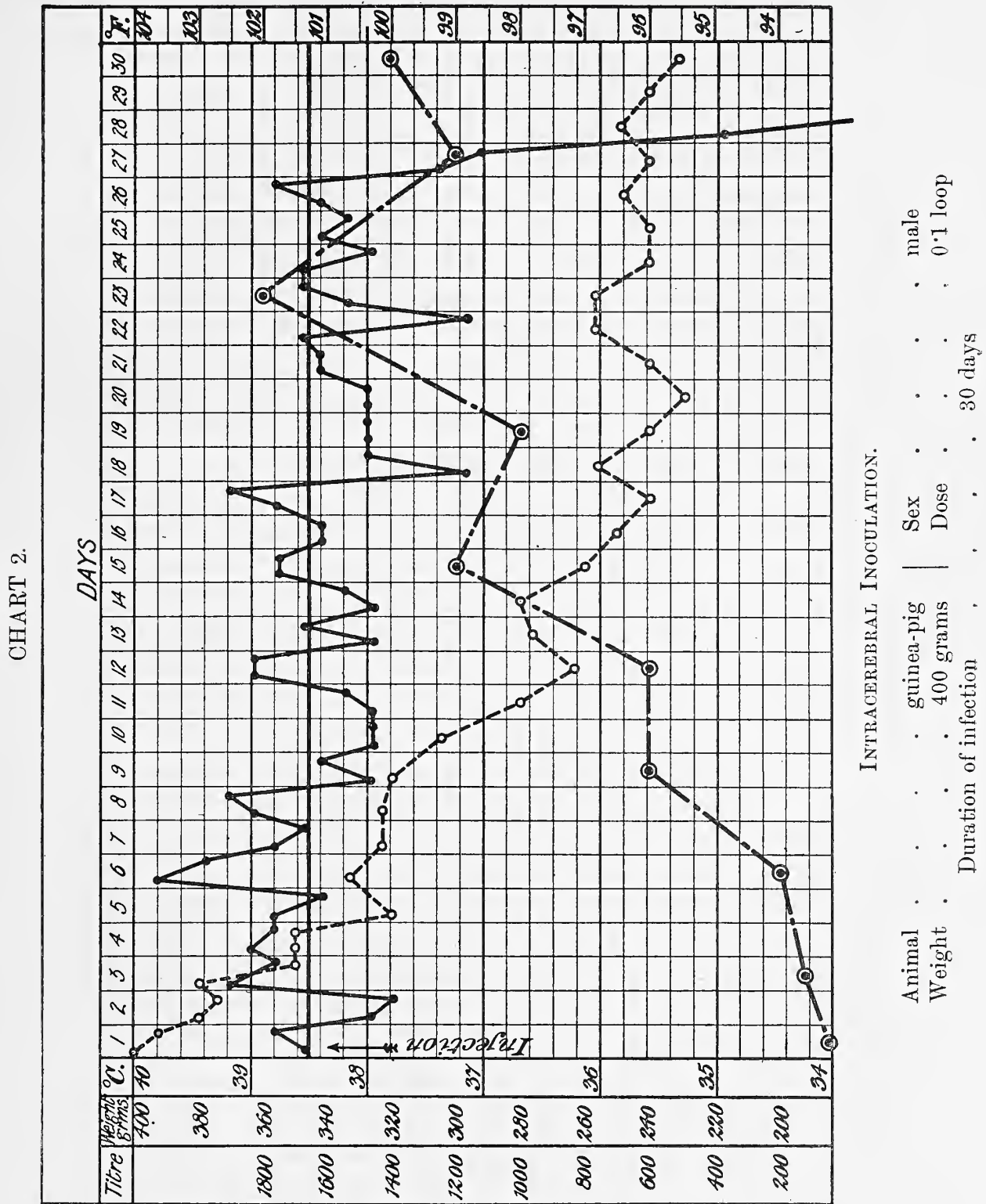
Bone Marrow.—In chronic infections the constitution of this tissue shows marked alterations from the normal. Nucleated red cells and *lymphoid cells*, that is, giant cells, mononuclear cells, and lymphocytes are markedly increased in number; *granular cells*, *i.e.* myelocytes and polymorphonuclear leucocytes, are considerably diminished, the whole forming a typical lympho-erythroblastic bone marrow, and is in marked contrast to the leucoblastic marrow associated with, for instance, pneumococcic infections of these rodents. The following table gives counts averaged from ten intracranial inoculations in the guinea-pig.

Type of Cell.	Normal.	<i>M. melitensis</i> Septicæmia.
Lymphoid cells	43·3	61·1
Granular cells	56·2	38·4
Nucleated red cells	8·0	18·0

Cultivations from liver usually remain sterile; those from spleen and bone marrow may or may not yield a scanty growth; on the other hand

those established from the kidney show a scanty growth, and those from the centrifugised deposit of the turbid urine or from the urine itself generally give rise to a fairly good growth.

The next chart (No. 2) records the observations relating to the



temperature, weight, and serum reaction of an experimental infection of this type.

B. *Intravenous Inoculation*.—1. *Acute*.—The intravenous injection of large amounts of cultivation of highly virulent strains of *M. melitensis* into small rabbits gives rise to an acute septicæmia. The experimental animal

becomes dull and apathetic; sits huddled up in one corner of the cage, and refuses food. The temperature rises rapidly to 41°, 42°, or 43° C.; the body weight falls rapidly, and the animal finally becomes dyspnoëic, death supervening in from two to four days. Post-mortem, beyond engorgement of the viscera and dilatation of small blood-vessels, nothing abnormal can be detected. The agglutination titre of the serum is, as a rule, low—not more than 1:50—rarely it may be as high as 1:200 or 1:400. Cultivations from all the organs and from the heart blood give rise to luxuriant growth of *M. melitensis*. Chart 3 is typical of this infection.

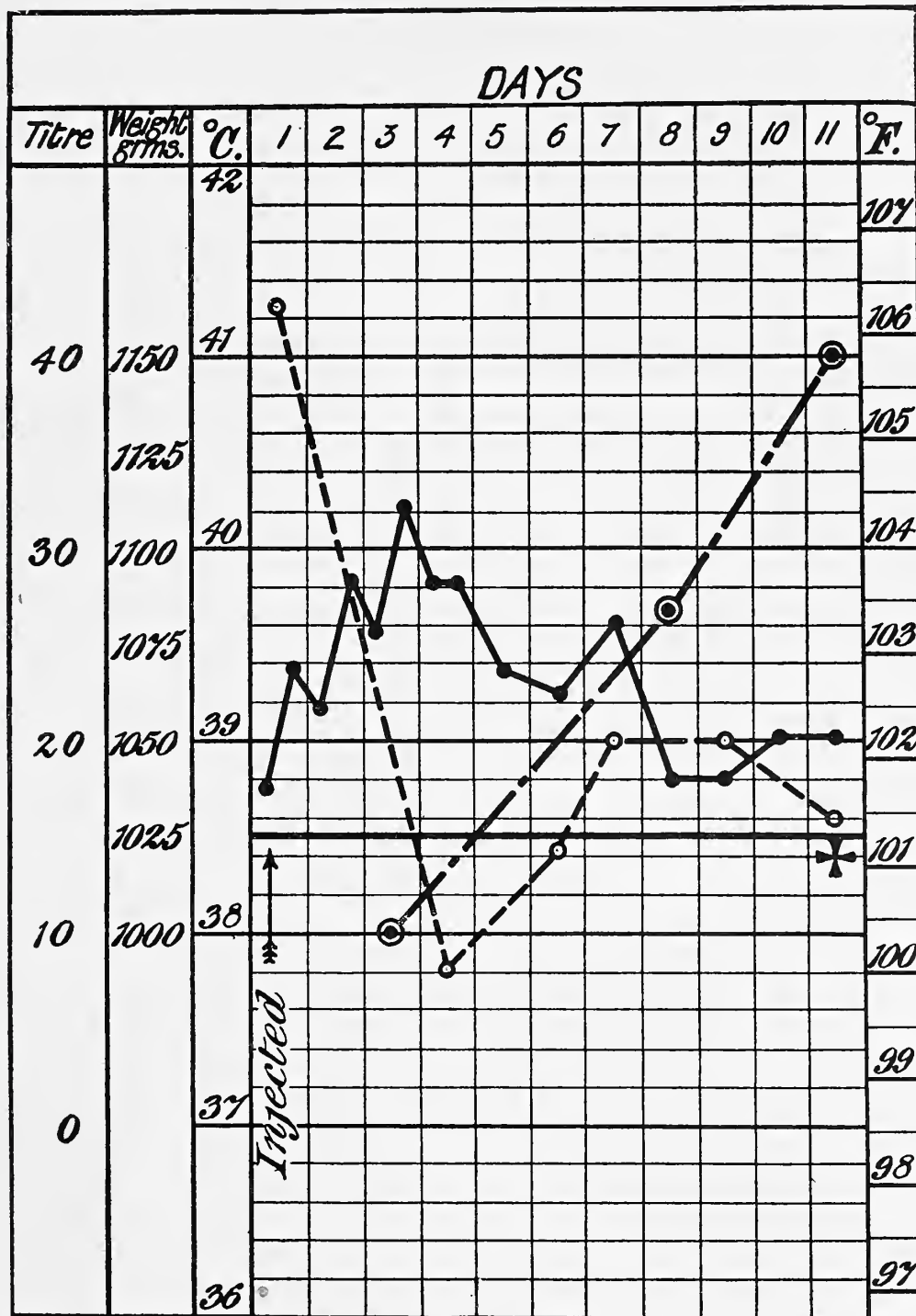
2. *Chronic*.—The converse of the conditions of inoculation enumerated above give rise to the chronic form of infection. Here the clinical phenomena detailed under the acute infection are less marked, although the same sequence of events is at first noted. The animal refuses food, is obviously uncomfortable; the temperature rises in the same way, but rarely above 40° C.; the weight falls, but as a rule not more than about 10 per cent. of the body weight is lost. After three or four days the general condition improves; the animal takes its food eagerly, even ravenously, and commences to put on weight again, but usually never regains its original weight. After a week or two it again loses weight. Suddenly, at the end of seventeen to twenty days, or even after a lapse of a month or two, the animal dies.

Post-mortem, no abnormal conditions can be detected, and cultivations from the heart blood and various organs, including the glands, remain sterile. Cultures from the centrifugalisated deposit of the urine may give rise to a sparse growth of *M. melitensis*. Occasionally, too, if large pieces of kidney tissue are planted in flask broth cultures, a scanty growth of the micrococcus can be obtained. Such a case is shown in chart 4.

C. Intraperitoneal Inoculation.—Intraperitoneal inoculations of recently isolated strains of *M. melitensis*, as a rule, only provoke an infection characterised by extreme chronicity, even when large doses are injected into young animals. Intravenous passages certainly increase the virulence of *M. melitensis* for rodents by all methods of inoculation to a very marked degree, but the only really satisfactory and certain method of so increasing the virulence of this organism that acute intraperitoneal and subcutaneous infections can readily be produced, consists in a number of preliminary passages through brain tissue. This selective affinity of *M. melitensis* for the nerve cell undoubtedly has some bearing upon nerve symptoms so frequently observed in *M. melitensis* septicæmia as it occurs naturally in man, and is a subject that affords ample scope for future experimental investigation.

1. *Acute*.—Intraperitoneal injections of *M. melitensis* thus artificially exalted in virulence may prove rapidly fatal, but the clinical phenomena

CHART 3.

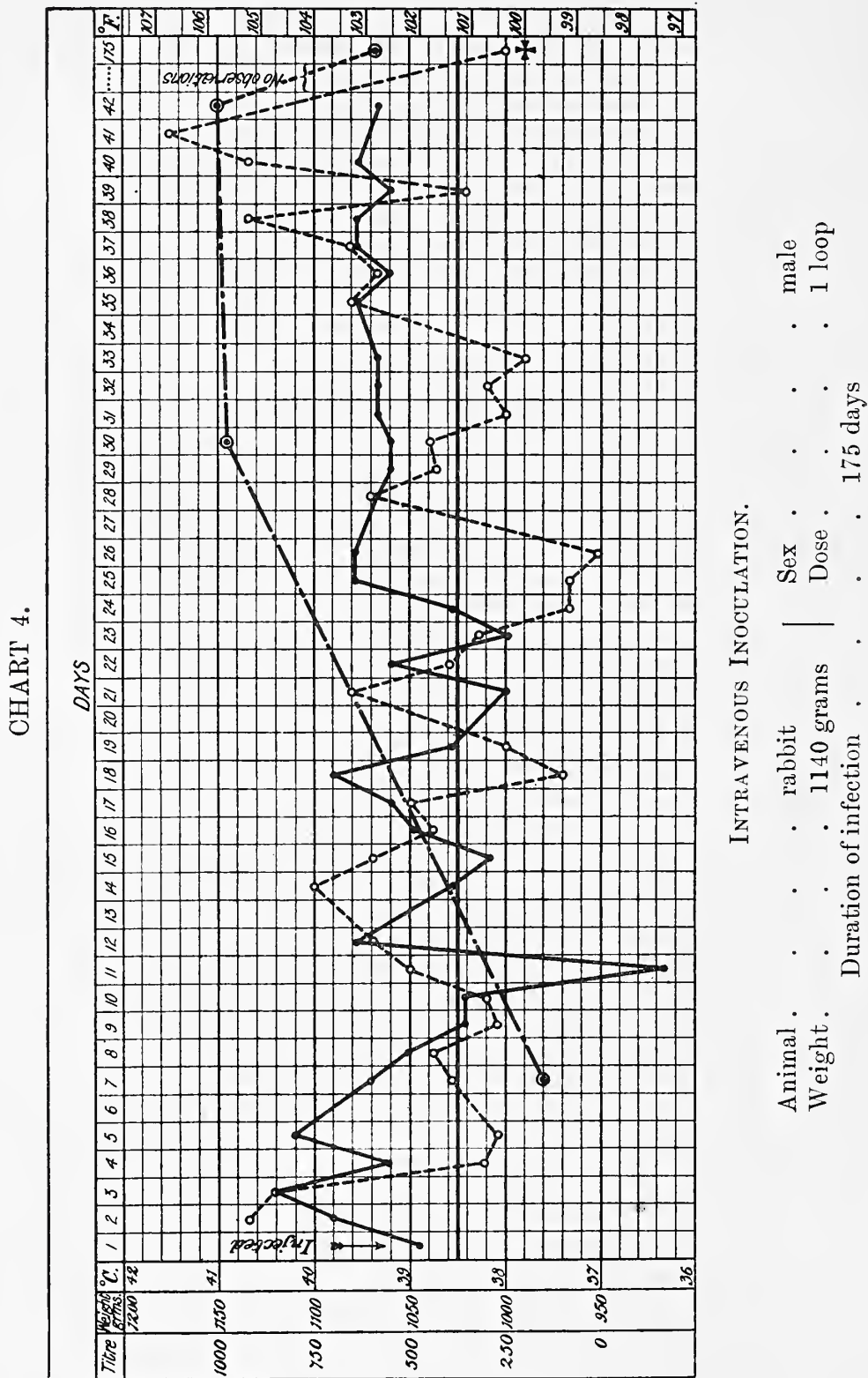


INTRAVENOUS INOCULATION.

Animal rabbit
 Weight 1200 grams
 Sex male
 Dose entire agar culture
 Duration of infection 11 days

observed in the experimental animal are few and slight. For the first twenty-four hours after injection the animal may refuse food; subsequently it appears normal in this respect. The temperature is usually but only

slightly raised, and but very rarely goes above 40° C. The weight does not fall with that astonishing rapidity which is noted after intracerebral and intravenous injections, and death takes place suddenly and without premonitory symptoms in from three to five days. Post-mortem, the



mesentery is found to be rolled up on itself, and appears quite red from dilatation of its vessels and from the presence of numerous hæmorrhagic areas. The intestinal walls are markedly injected and are sticky, have lost their shining lustre, and the coils of the intestines are matted together by the peritoneal exudate, which is for the most part pink in colour and

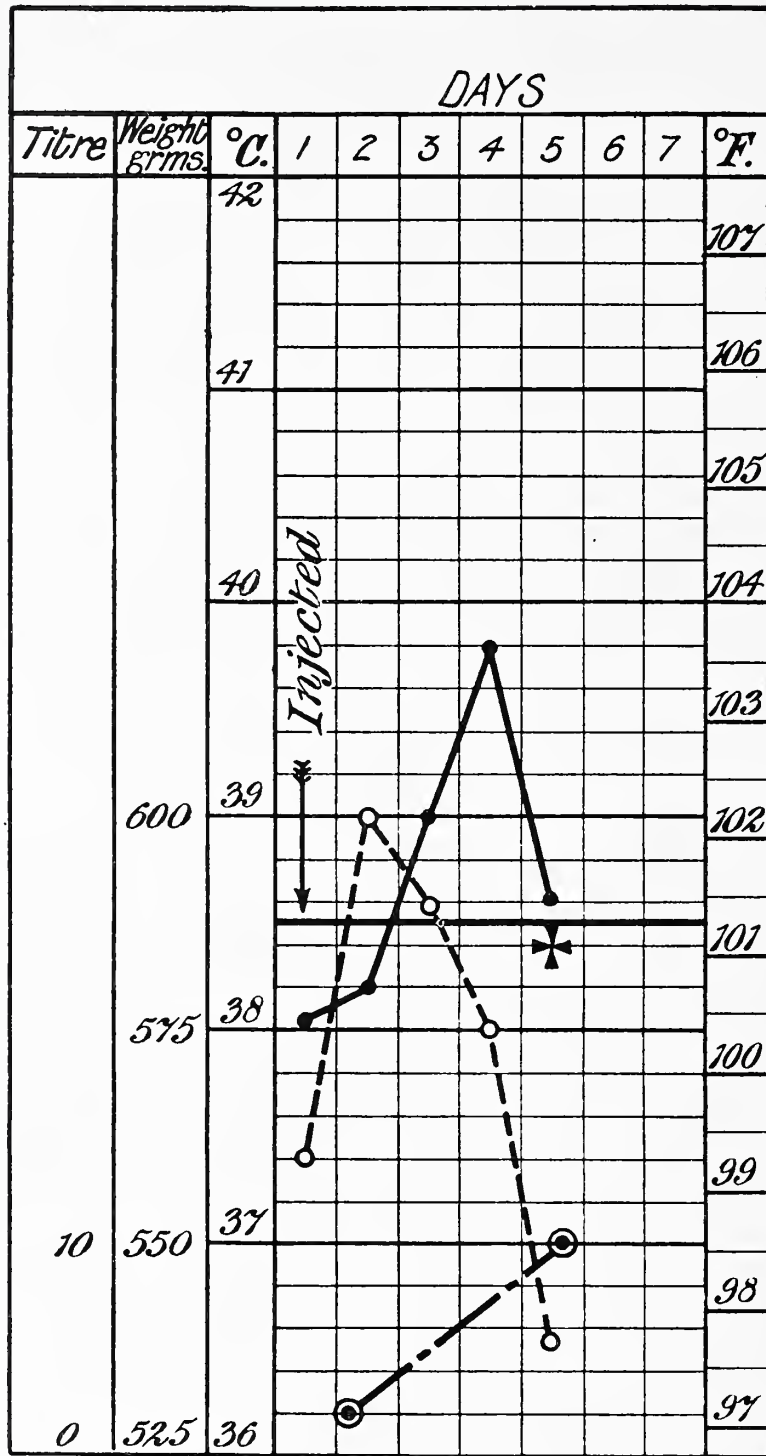
glutinous, and not present in very large amount; while numerous yellow flakes consisting of polymorphonuclear leucocytes and micrococci are scattered all through the peritoneal cavity. In the case of the male animal the epididymis and vasa at first sight appear like clots of blood, from the excessive engorgement of their vessels. The testis itself is swollen and red, and on incising the organ the tunica vaginalis is found to be adherent to the body of the testis, bound down by sticky gelatinous pus, the interstices between the adhesions being occupied by a thin sero-pus containing flakes of fibrin. In the inspissated pus or in the sero-purulent fluid *M. melitensis* is particularly abundant. The spleen is large, generally of a dark red colour; the Malpighian corpuscles are very prominent; the organ itself is firm and tense owing to the pressure within the capsule, but on incising the organ the spleen pulp is frequently almost fluid. Cultures from all the organs yield copious growth of *M. melitensis*, the heart blood is full of cocci, and occasionally the urine contains a few, which have probably gained access through the peritoneum covering the walls of the ureter and the bladder. The temperature, weight, and agglutination curves of such an infection are shown in chart 5.

2. *Chronic*.—The course of a chronic intraperitoneal infection does not differ in any material respect from that of the chronic intravenous infection (see chart 6) beyond the length of time that usually elapses between inoculation and death. In the former case this period may extend to six or seven months; in one of my cases thirteen months intervened. Post-mortem, no macroscopical lesions are present, the peritoneal cavity and the abdominal viscera are normal in appearance, and as a rule cultures of *M. melitensis* can only be obtained from the centrifugalised deposit of the bladder contents, or from the kidney tissue if large pieces of that organ are planted in broth. Occasionally all cultivations remain sterile, and the presence of specific agglutinins in the blood and changes in the histological structure of the blood and bone marrow form the only evidence of infection.

D. *Subcutaneous Inoculation*.—1. *Acute*.—No amount of experimental exaltation of the virulence of *M. melitensis* has yet enabled me to produce an acute infection of the rodent as the result of subcutaneous inoculation—the infection, so far as relates to the fatal termination, is always and extremely chronic. Again, unless the organism has been previously exalted in virulence by means of passages through animals of the same species, pus formation is the exception rather than the rule, and it is often difficult to obtain any evidence of successful infection, other than the presence of specific agglutinins in the blood-serum, if the process is allowed to pursue its natural course.

2. *Chronic*.—The small swelling at the site of inoculation which marks the introduction of the emulsion of *M. melitensis* into the subcutaneous

CHART 5.



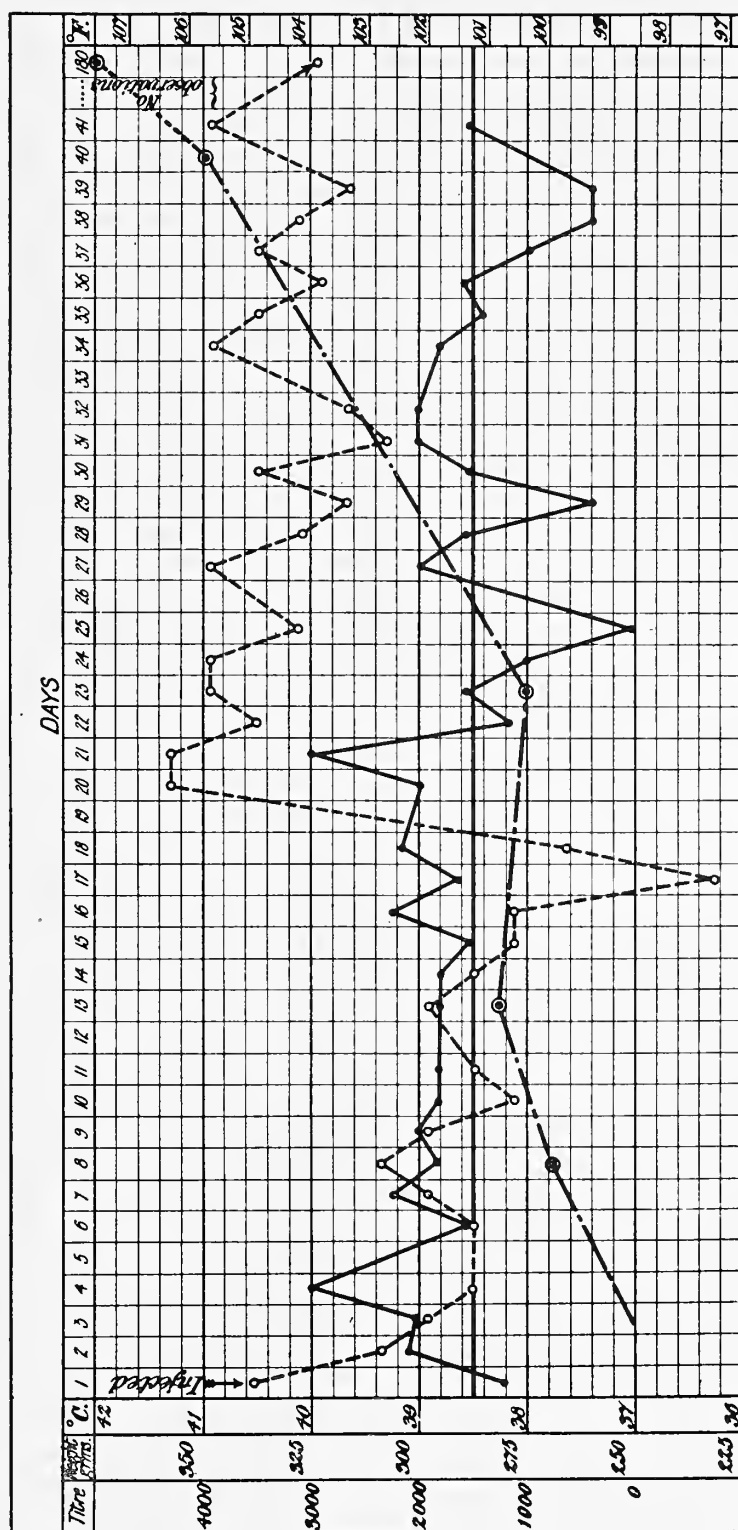
INTRAPERITONEAL INOCULATION.

Animal rabbit
 Weight 560 grams
 Sex male
 Dose 0.3 agar culture
 Duration of infection . . . 5 days

tissues rapidly disappears, and a few hours after injection the animal appears quite normal. Twenty-four to forty-eight hours later, the seat of inoculation is marked by a large, hard, circumscribed tumour. The animal

appears distressed, and the temperature rises to nearly 40° C. During the next few days the local lesion becomes soft, and fluctuation can be obtained. A few days later necrosis of the skin covering the tumour takes place, a slough separates, and thick creamy pus crowded with *M. melitensis* exudes

CHART 6.



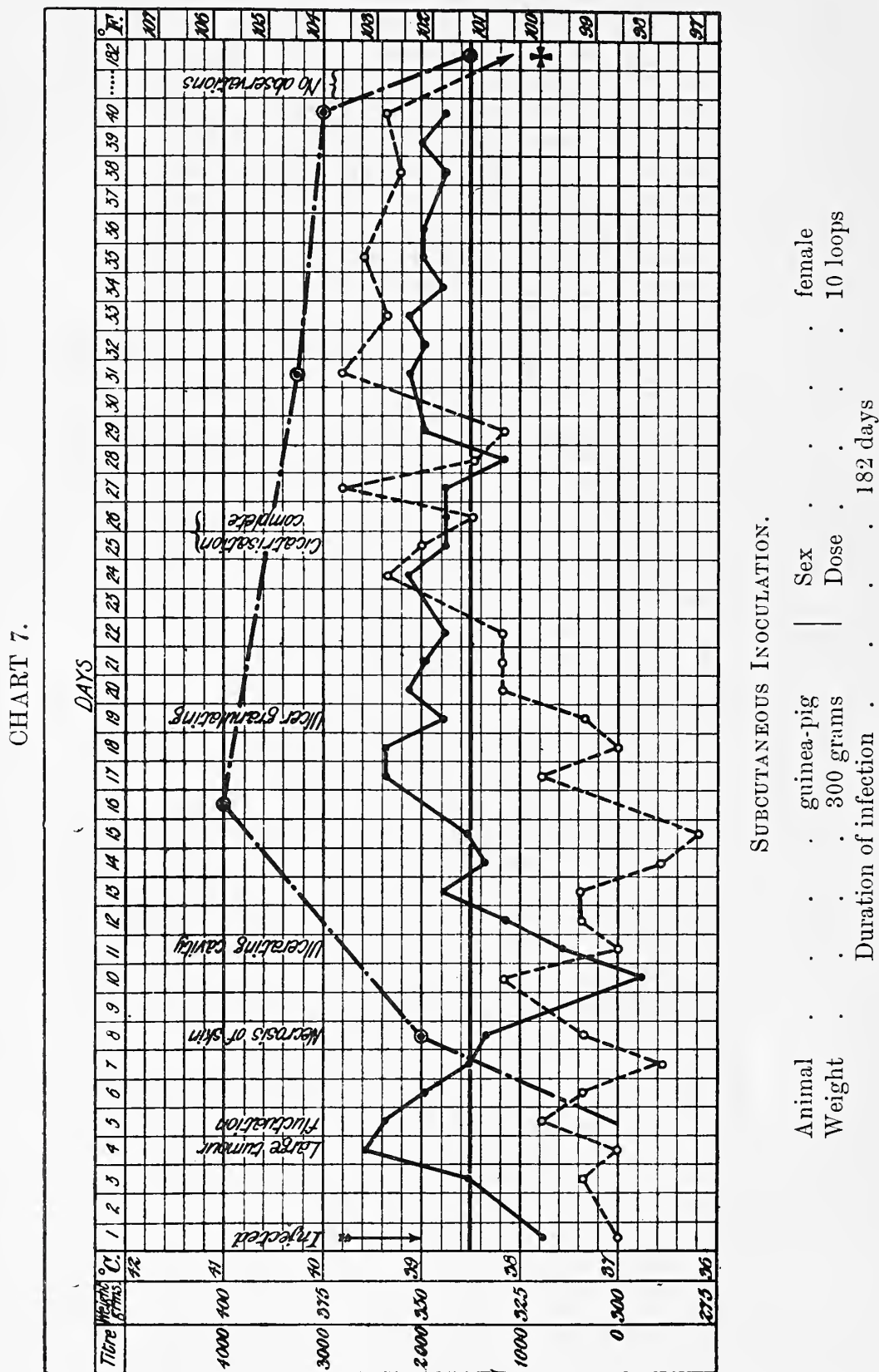
INTRAPERITONEAL INOCULATION.

Animal guinea-pig | Sex female
 Weight 335 grams | Dose 1 loop
 Duration of infection 180 days

from the abscess cavity, leaving an extensive ulcerating cavity. After a week or two, during which the exudation gradually becomes less in amount, healthy-looking granulations spring up, cicatrisation proceeds, and in about three weeks the animal appears perfectly normal as to temperature, and may even put on weight; but in the course of the following weeks or

months a decrease in weight may again be noted, and finally the animal dies (see chart 7).

Post-mortem, beyond the cicatricial tissue at the seat of inocula-



tion no pathological appearances can be found. Cultivations from the heart blood and various organs remain sterile, but usually *M. melitensis*—though often in very small numbers—can be isolated from the urine.

Rat, Mouse.

No serious observations as to the susceptibility of these small rodents appear to have been made until I first took up the subject on my arrival in Malta in April 1906. Speaking generally, the pathogenic effects observed in these animals after inoculation with *M. melitensis* are identical with those already observed in the larger rodents—rabbits and guinea-pigs—when once the organism has adapted itself to life in the body tissues of the rat and the mouse, and has acquired a certain degree of virulence for them—with this single but noteworthy difference, viz. that the rat and the mouse appear to be ill adapted for the elaboration of specific agglutinins to *M. melitensis*. These anti-bodies are consequently formed in very small amounts or not at all. Out of the scores of experimental rats and mice that I have examined, one white rat alone gave evidence of the presence of agglutinin, and even in this animal the titre of the serum did not rise above 1 : 40.

Experiments were first undertaken in connection with the sewer rat. A black sewer rat, captured two days previously, apparently healthy and whose blood failed to yield evidence of the presence of specific agglutinins when tested against *M. melitensis*, was injected subcutaneously at the root of the tail with half of a two-days-old culture (emulsified in 0.75 c.c. normal saline solution). The strain of *M. melitensis* employed had been recently isolated from a fatal case of Malta fever in man by one of my colleagues.

The animal was not obviously affected, but on the fourth morning after inoculation was found dead. Cultivations from the heart blood yielded a few colonies of *M. melitensis*; from the spleen, numerous colonies; and from the urine, innumerable colonies.

With the culture thus obtained another sewer rat was inoculated intracerebrally (1 loopful of two-days agar culture in 0.1 c.c. normal saline). This animal was found dead the following morning, and a pure culture of *M. melitensis* was recovered from the spleen.

These two experiments are somewhat discounted by the fact that sewer rats usually die off quickly in captivity—that is, in two or four days—and the successful recovery of the coccus from the cadaver might only have been possible by reason of post-mortem multiplication.

Consequently, white rats were substituted for the sewer rats, and a series of intracerebral passages (always using spleen cultures) to the number of four were carried out. The experimental animals all died within twenty-four hours. The micrococcus was successfully isolated from the spleen of

each, but no specific agglutinins were demonstrable in the blood. The only noteworthy feature of the post-mortems was the appearance of the spleen (enormously hypertrophied, and dark red or even black in colour; and the pulp very friable) and the general glandular enlargement. With the culture derived from the spleen of the last rat of the series other rats were inoculated intraperitoneally and subcutaneously.

Rats 5 and 5a each received intraperitoneally 0·2 loop of a twenty-four-hour-old agar culture emulsified in normal saline solution. Death occurred in twenty-four hours in each case. Post-mortem, the heart blood and spleen were crowded with *M. melitensis*. Serum reaction nil.

Rat 5b received 0·2 loop subcutaneously over abdomen from the same culture emulsified in normal saline solution. Four days later an abscess had formed at the seat of inoculation, containing pus, which was full of *M. melitensis*. Death ensued in twenty-one days. Post-mortem showed the presence of a pocket of pus in right thorax, suppurative peritonitis and vaginalitis, and in each situation the pus was crowded with *M. melitensis*. The blood, spleen, axillary glands, and vas (left) were also crowded with the micrococcus. Serum reaction again nil.

These experiments were repeated and confirmed during 1907 and 1908, and the same method of exalting the virulence of *M. melitensis* was applied to the mouse with equally successful results. Thus white and black mice inoculated intracerebrally with 0·1 of a loop of twenty-four-hour-old agar culture of the *rat* strain of *M. melitensis* died within three days in the earlier experiments; but after three intracerebral passages death ensued with the standard dose within twenty-four hours.

When this point was reached, intraperitoneal inoculation of a like dose caused death also within twenty-four hours, and subcutaneous inoculation was followed by pus formation and death in from seven to twenty-one days.

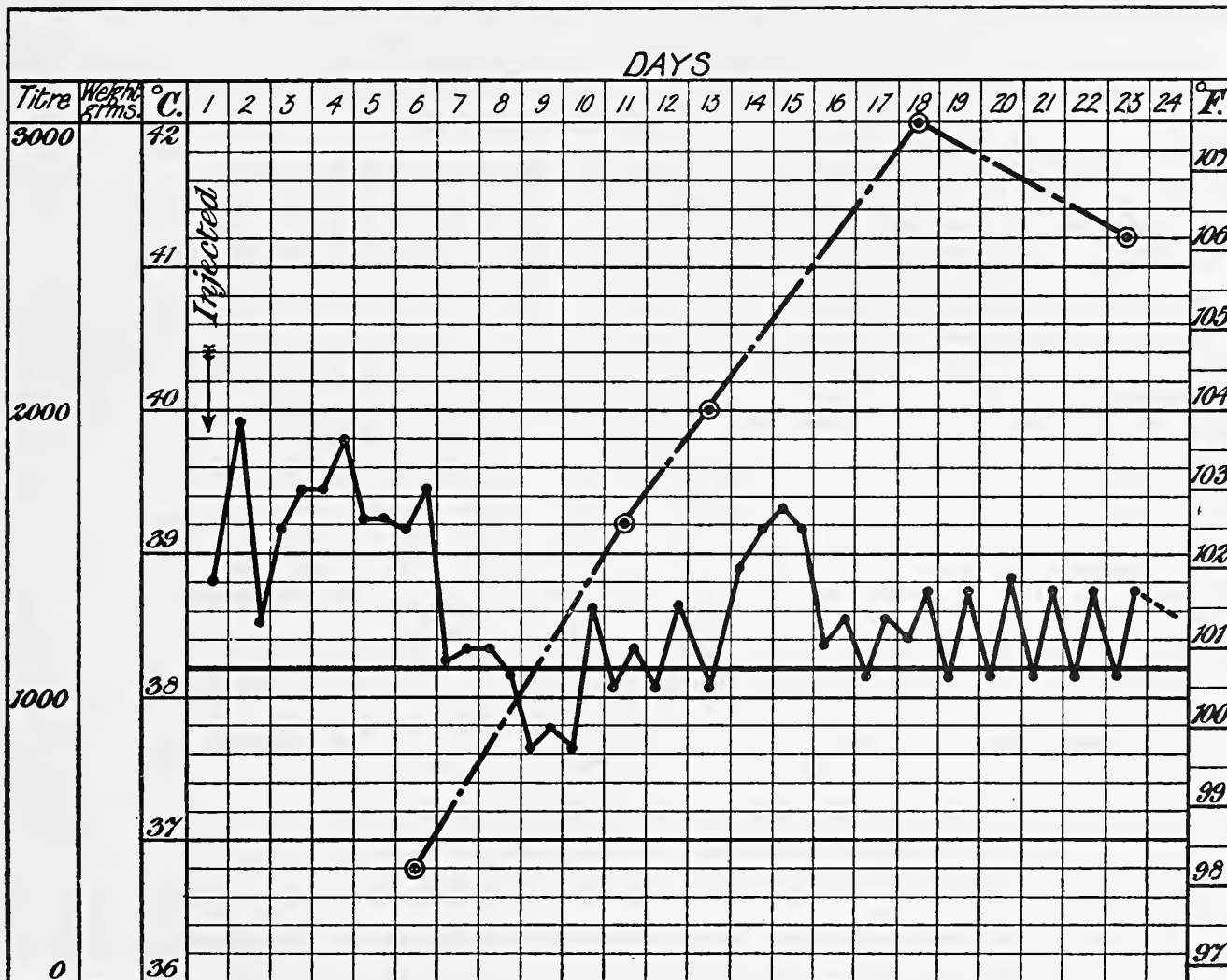
CARNIVORA.

Dog.

Having dealt at some length with the results of the various experimental infections of the commoner laboratory animals, the dog can be very shortly dismissed, more especially as I have not carried out any observations relating to the exaltation of virulence of *M. melitensis* in dogs by means of intracerebral passages, and the subsequent injection of the highly virulent strains thus obtained into other tissues; but I have no doubt that, judging from the results of my experiments upon other species of mammals, effects similar to those already described should be obtained.

The dog is susceptible to infection by intracerebral, intravenous, intra-peritoneal, and subcutaneous inoculation with living cultivations of *M. melitensis*—probably also by feeding with infective material, although I have not personally conducted any experiments in this direction, nor do I know of any recorded observations on this method of infection in the dog.

CHART 8.

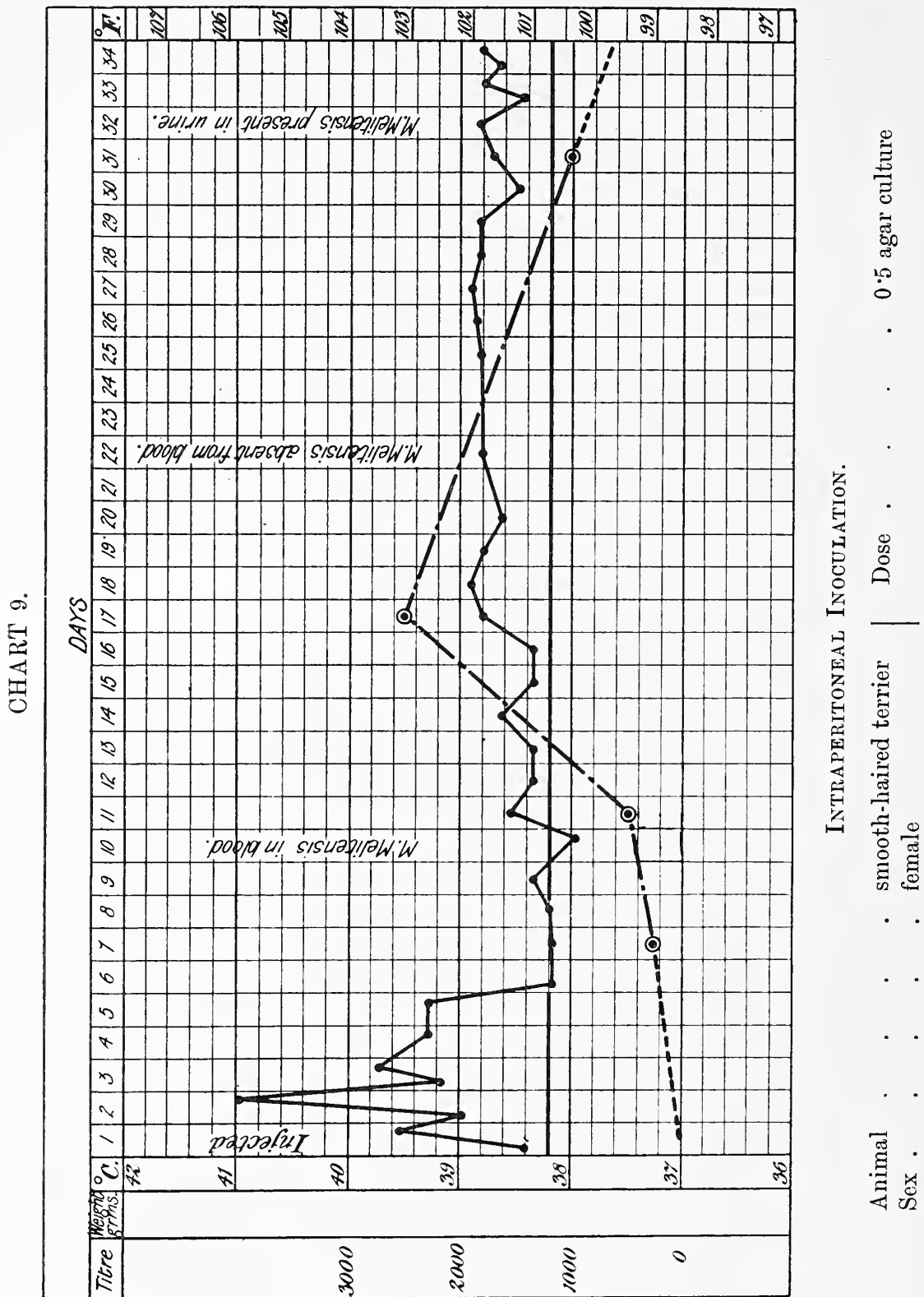


INTRAVENOUS INOCULATION.

Animal smooth-haired terrier
 Sex male
 Dose 0.5 agar culture
 Killed 30th day of infection.

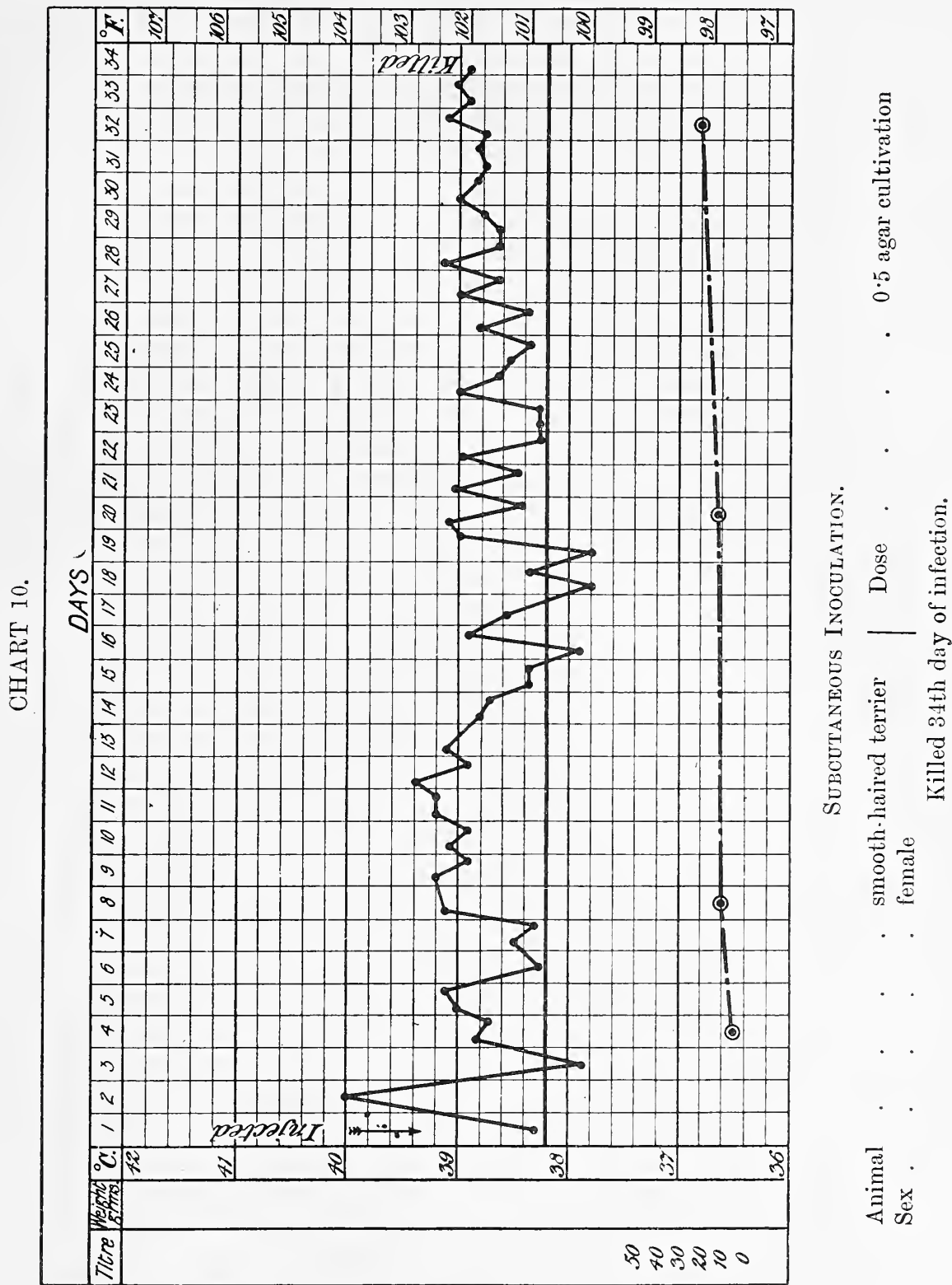
Clinical symptoms produced as a direct result of the intravenous, intra-peritoneal, or subcutaneous inoculation of dogs with living cultures of *M. melitensis* are conspicuous by their absence. The experimental animal may or may not be listless and “off food” for a day or two, and the temperature may rise a degree or so, but usually no suggestion of illness can be detected, and the animal appears to be totally unaffected; while the infection apparently has no deleterious influence in the direction of shorten-

ing life or influencing the course of a pregnancy. Occasionally, however, and chiefly after intravenous inoculation, irregularities of temperature may persist for a week or two. The success of the infection can readily be demonstrated after any form of inoculation by the isolation of the micrococcus from the peripheral blood during the early stage, and later on from



the urine, and in the female from the milk; also by the presence of specific agglutinins, often in large amount, in the blood serum. If the animal is destroyed later on, the micro-organism can be recovered from the various organs—in numbers and from a variety of situations, decreasing directly

with the length of time that has elapsed from the inoculation. The tempera-



ture chart and serum reactions of some typical infections are here inserted (charts 8, 9, 10).

UNGULATA.

Horse, Mule, Cow, Goat, Sheep.

A considerable amount of evidence was collected by the Mediterranean Fever Commission during 1905 and 1906, proving that various members of

this family were the subjects of *M. melitensis* infection naturally acquired. Thus the horse and mule, by reason of the presence of often large amounts of specific agglutinin in the blood serum, were considered to be subject to natural infection, though the specific organism was never recovered from the tissues. The cow, goat, and sheep yielded stronger evidence, as the microorganism was successfully isolated from the milk of some of the females in each species. The number of cows upon the island was too few to give any reliable information as to the percentage incidence of *M. melitensis* septicæmia; but in the case of the goat, which in Malta is the commonest quadruped, and numbers some 20,000 head, this point was studied by means of a careful examination of a large number of individuals, and from the results obtained it is estimated that nearly 30 per cent. are or have been the subject of *M. melitensis* septicæmia naturally acquired, while fully 10 per cent. yield absolute proof of present, though chronic, infection by the presence of the micrococcus in their milk. Having had no personal experience of the infection of mules, cows, or sheep by laboratory methods, I shall confine my remarks to the experimental inoculation of the horse and of the goat.

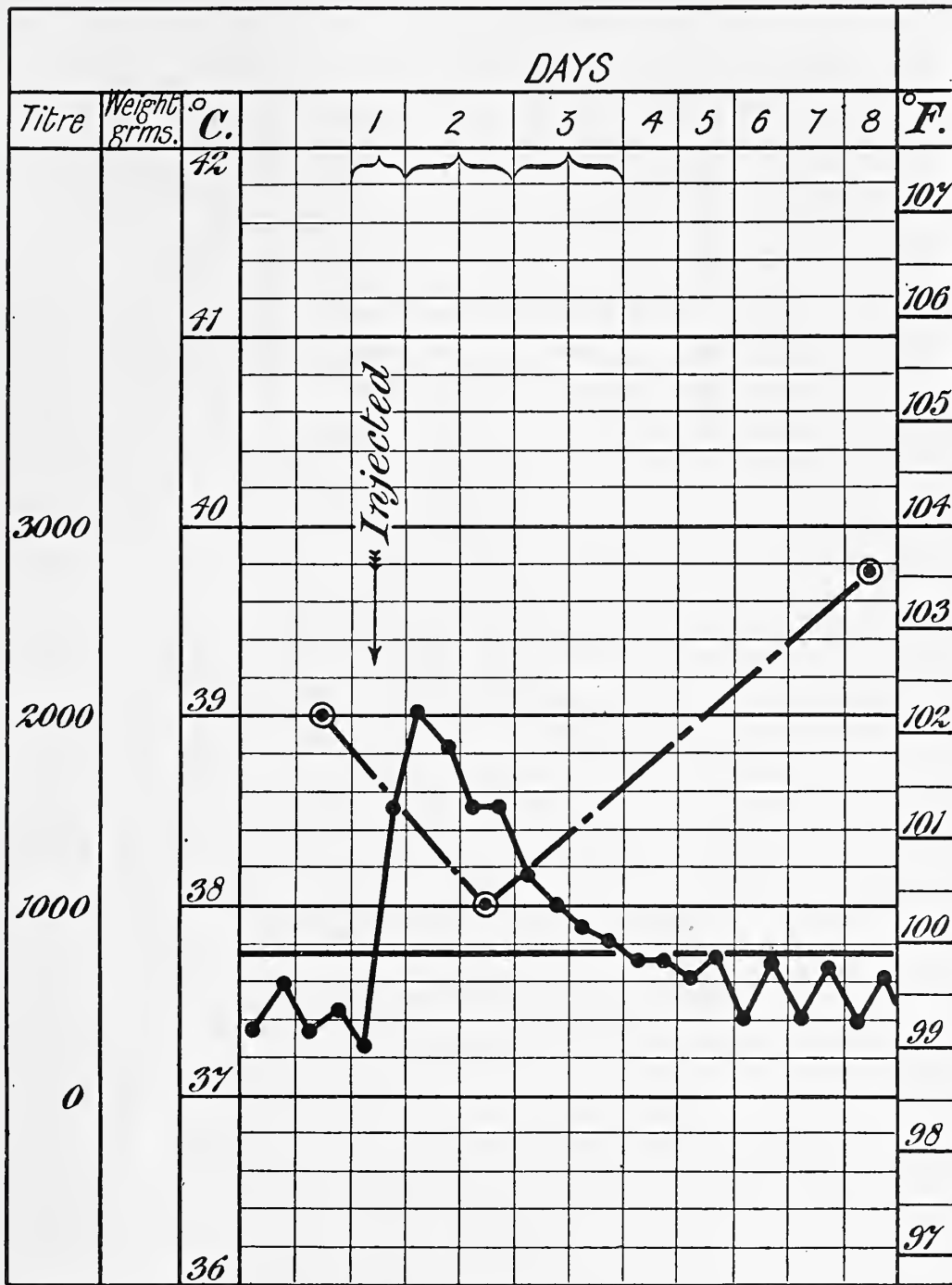
Horse.—The horse I have infected by means of intravenous injections in the course of some immunising experiments whilst attempting to prepare an anti-serum. The clinical phenomena observed after inoculation of this animal—a chestnut mare—were remarkably few. The temperature rose within a few hours, but rarely more than 1.5° C. to 2° C., and the mare was “off her feed” for perhaps twenty-four to thirty-six hours. The temperature rapidly returned to normal, the coat remained smooth and glossy, and the animal appeared to be perfectly normal. Examination of the blood, however, showed the presence of specific agglutinins, and the cocci were present in the general circulation for between three and four weeks after each injection (chart 11).

The goat, so far as my observation carries me, is never the subject of an acute infection, though it is true I have never employed the intracranial method of inoculation of this animal. The animal is susceptible to infection as the result of intravenous (see chart 12), intraperitoneal, subcutaneous, and cutaneous injection of *M. melitensis*, and also by feeding methods. By whatever method the infective material is introduced into the tissues, the clinical phenomena are similar. The animal rarely appears to be adversely affected. Its appetite is as vigorous as ever, and beyond a few days' pyrexia, during which the coat may or may not lose a little of its sheen, nothing can be detected by the ordinary clinical methods of observation.

Examination of the blood from time to time shows the presence in the serum of specific agglutinins, first demonstrable in from seven to twenty-one

days after infection; while examination of the blood in the early stages of infection, of the urine later on, and later still, in the case of milch goats, of the milk, yields absolute evidence of the reality of infection, by the isolation

CHART 11.



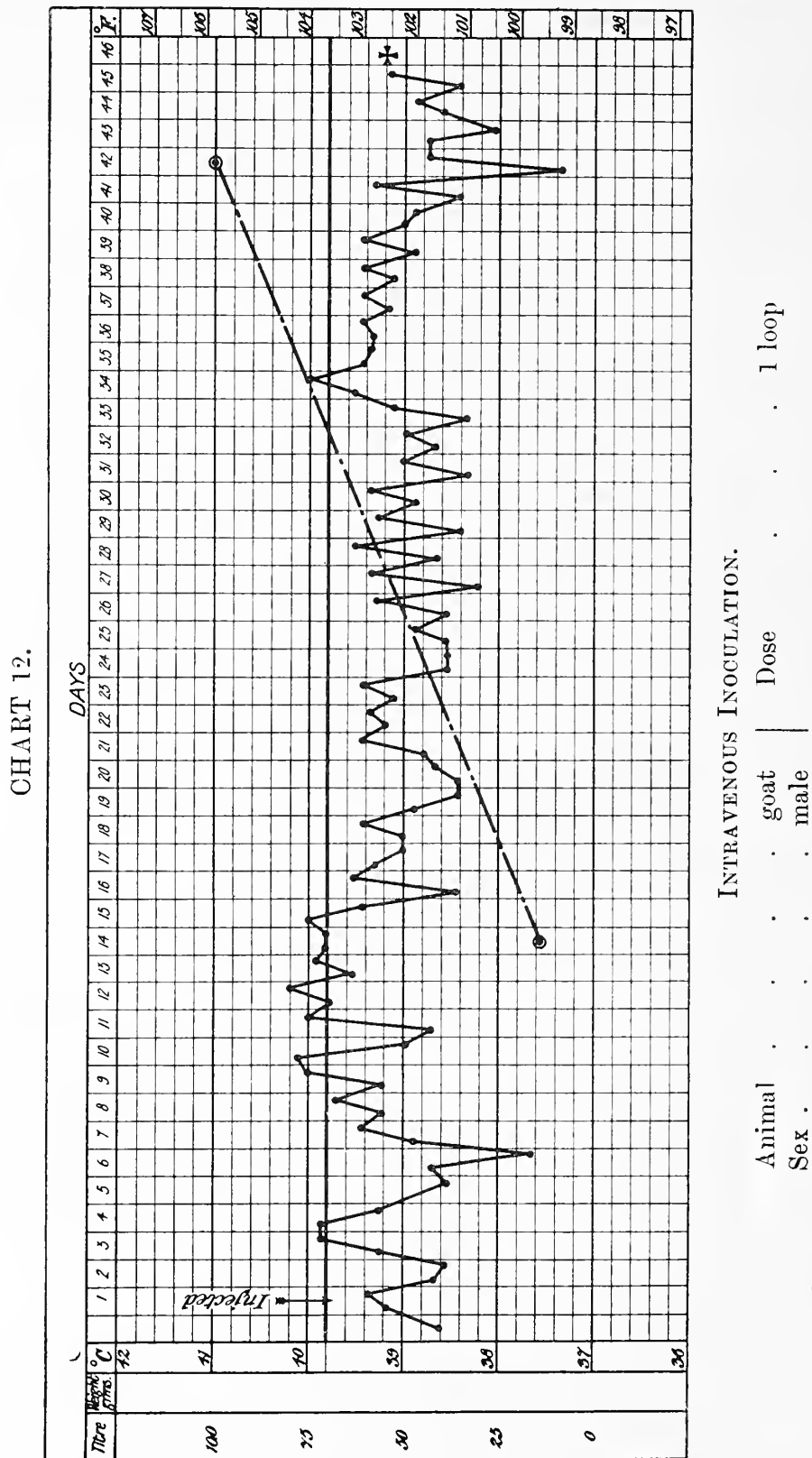
INTRAVENOUS INOCULATION.

Animal horse
 Sex female
 Dose 25 milligrammes agar culture

of the specific micro-organism. None of the experimental animals died as the direct result of the artificial inoculation. Consequently, I shall refer in detail only to some subcutaneous and cutaneous experiments which are of particular interest.

Subcutaneous Infection.—Four normal healthy half-grown goats were

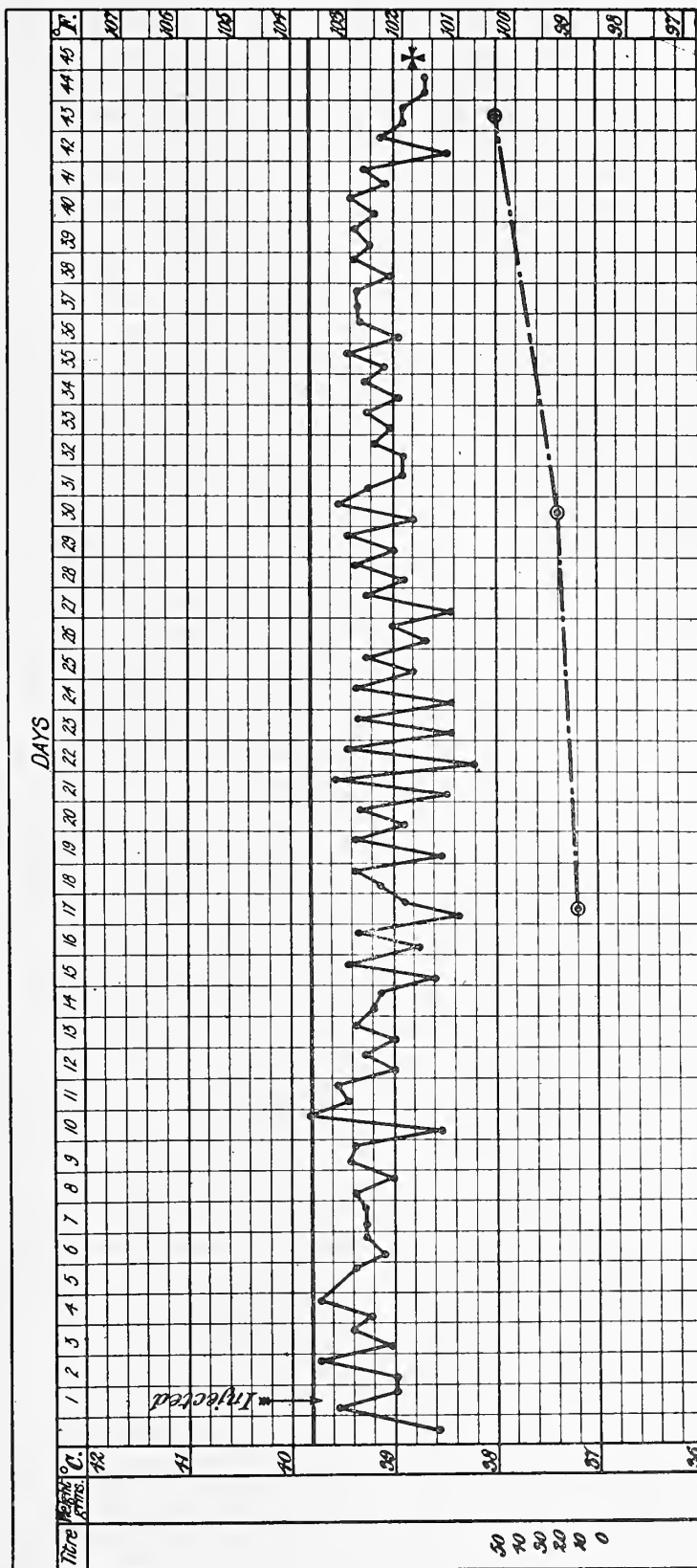
selected, and injected subcutaneously, each with a different-sized dose of infective material. Thus one loopful of the growth from a forty-eight-hours agar culture was emulsified in 10 c.c. of sterile saline solution, and the four



goats were inoculated subcutaneously at the root of the left ear with 1 c.c., 0.1 c.c., 0.01 c.c., and 0.001 c.c. respectively of the emulsion. Portions of the remainder of the emulsion, after suitable dilution, were plated out and incubated. Enumeration of the resulting colonies showed that the emulsion contained 10 million cocci per cubic centimetre.

Consequent upon these injections, all four goats became infected, as shown by the appearance of specific agglutinins in the blood serum, but from the clinical point of view there was little to record. At no time

CHART 13.



SUBCUTANEOUS INOCULATION.

Animal	goat	Dose	10,000 cocci
Sex	female		
		Killed 45th day of infection.	

during the course of the experiment did either of the goats appear to be ill: their coats were in good condition, and the animals fed well, as usual. The temperature after an immediate but transitory rise was somewhat irregular, but not markedly so (see chart 13). All the animals were killed

at the end of six weeks, and at the post-mortem inspection ample evidence of *M. melitensis* infection was available in the recovery of the organism from the spleen in all four goats, and in three of them from other organs as well; but the point of interest to be noted in this experiment is the relationship that exists between the size of the dose of infective material to the date of onset of signs of infection. Thus the first two goats, which received 10 and 1 million cocci respectively, gave evidence of reaction to the infection by the appearance of specific agglutinins in the blood within a week of inoculation, and a full ten days earlier than the other two goats, which had each received less than a million cocci.

No. of Goat.	No. of Cocci Injected.	Serum Reaction.		Titre of Serum at 44 days.
		Day of First Appearance.	Titre.	
No. 10	10,000,000	7	1 : 20	1 : 200
„ 9	1,000,000	7	1 : 20	1 : 100
„ 5	100,000	17	1 : 20	1 : 20
„ 4	10,000	17	1 : 10	1 : 20

Cutaneous Infection.—The suspicion that had been aroused in my mind with reference to the probability of infection being carried from goat to goat by way of the goatherds' hands when soiled with infective milk was strengthened by the knowledge that such a comparatively small amount of infective material as that just mentioned injected subcutaneously was sufficient to produce infection.

The technique adopted by the Maltese goatherd closely resembles that of his English confrère, and consists in lubricating his own hands and the outside of the goat's udder with some of the foremilk. When a number of goats have to be milked in rapid succession, the lubricant obtained from the first goat will serve for perhaps some half-dozen goats; with the seventh goat a fresh supply of milk is taken for the same purpose, and so on. Now given that goat No. 1 or goat No. 7 is passing *M. melitensis* in its milk, it is obvious that at any rate goat No. 2 or goat No. 8 stands a very good chance of becoming infected by a process of subcutaneous inoculation; I therefore investigated the possibility of this occurrence in the following manner.

A healthy female goat, nearly full grown, was selected, cast on an operating table, and securely held by assistants. A fairly large area of skin over the left mammary gland was shaved somewhat roughly, in such

a manner as to remove in many places the superficial layer of epithelium as well as the hair, but care was taken to avoid drawing blood. Thus the shaved area replaced the scratches, abrasions, and small ulcers that are so frequently seen on the udder and teats of the milch goat. Next, the hands of the operator being protected by a pair of sterilised indiarubber gloves, four drops of freshly drawn milk (amounting in total bulk to 0·2 c.c.) from a goat known to be excreting *M. melitensis* were delivered into the palm of the right hand from a sterile capillary pipette, and then thoroughly rubbed into the shaved area with movements similar to those practised by goatherds as a preliminary to milking. The quantity of the milk used was so small that the skin surface rapidly dried, and the goat was then isolated in a stall, apart from the other animals. Immediately after the experiment was concluded, a sample of the milk that had been used was carefully plated out (after suitable dilution) and found to contain 24,800 *M. melitensis* per cubic centimetre; the approximate number of cocci therefore that came into contact with the prepared area of skin amounted to 5600. Samples of blood were taken from a vein in the ear of this animal, and examined from day to day for the presence of specific agglutinins, which first made their appearance on the fifteenth day (dilutions 1 in 10 and 1 in 20). Three weeks after inoculation the goat was killed and a careful post-mortem examination carried out, with the result that the specific organism was recovered from the spleen and inguinal glands.

The post-mortem observations upon animals killed after periods of observation varying in duration from three to at least fifty-three weeks are extremely interesting as showing that within a very few weeks of infection the micrococcus disappears from the general circulation, though still present in the spleen to about the end of nine months. Later still, the coccus cannot usually be recovered from this organ, but may be found in the mesenteric glands and the inguinal glands. In one animal where the organism was recovered from the tissues of the mammary glands it could not be detected in any other situation in the body; in another successful recovery from the gland substance the coccus was also detected in the neighbouring inguinal glands and nowhere else. These results are more readily appreciated when arranged in tabular form:—

[TABLES.

RESULTS OF POST-MORTEM EXAMINATIONS OF GOATS EXPERIMENTALLY INFECTED.

No.	Sex.	Remarks.	Duration of Infection or Period of Observation. Weeks.	Presence of <i>M. melitensis</i> noted During Life. Milk.	Titre of Serum.	Post-mortem Findings. <i>M. melitensis</i> present in					
						Heart Blood.	Spleen.	Kidney.	Glands.		Udder.
									Inguinal.	Mesenteric.	
1	F	Cutaneous inoculation .	3	...	1 : 100	-	+	-	+	-	0
2	M	Subcutaneous inoculation .	6	-	1 : 200	-	+	-	+	-	-
3	M	" "	6	-	1 : 100	-	+	-	+	+	-
4	F	" "	6	...	1 : 20	-	+	-	-	-	0
5	F	" "	6	-	1 : 20	-	+	-	-	-	0
6	F	" "	21	-	1 : 1500	-	+	-	-	+	0
7	F	Feeding	26	-	1 : 100	-	+	-	+	-	0
8	F	"	37	At 7 weeks	1 : 20	-	-	-	-	-	-
9	F	Subcutaneous inoculation .	39	-	1 : 20	-	+	-	-	-	0
10	F	Feeding	41	-	1 : 10	-	-	-	-	-	0
11	F	"	42	-	?	-	+	-	-	-	0
12	F	"	45	At 13 weeks	1 : 10	-	-	-	-	-	0
13	F	"	45	At 20 "	1 : 150	-	-	-	+	-	+
14	F	"	49	At 20 "	1 : 20	-	-	-	-	-	+
15	F	"	49	-	1 : 50	-	-	-	-	-	-
16	F	"	53	At 43 weeks till killed.	1 : 160	-	-	-	-	-	0

- = *M. melitensis* not recovered. + = *M. melitensis* recovered. 0 = Not examined.

POST-MORTEM EXAMINATION OF 5 NATURALLY INFECTED MILCH GOATS.

Period of Observation. Weeks.	<i>M. melitensis</i> recovered During Life from			<i>M. melitensis</i> recovered Post-mortem from				
	Urine.	Milk.	Titre of Serum.	Heart Blood.	Spleen.	Glands.		Udder.
						Mesenteric.	Inguinal.	
19	-	-	1 : 20	-	-	-	+	0
19	-	+	1 : 20	-	-	-	-	+
19	-	+	1 : 10	-	-	-	+	0
43	-	+	1 : 160	-	-	-	+	+
56	+	+	1 : 20	-	-	-	-	0

- = *M. melitensis* not recovered. + = *M. melitensis* recovered. 0 = not examined.

The point of greatest practical importance that emerges from the study of laboratory infections of the goat is the appearance of *M. melitensis* in the milk. When the fluid has been systematically examined in experimental milch goats the appearance of the coccus has invariably been a late phenomenon. One goat, for instance, commenced to pass the micrococcus in

its milk seven weeks after infection, another three months after infection. A third showed no signs of the coccus up to four months after inoculation, when it ran "dry"; six months later still—that is, ten months after experimental inoculation—this goat dropped two kids, and three days later the coccus was first detected in the milk.

A series of daily observations during the summer of 1906 showed that although the excretion of *M. melitensis* in the milk during some stages of the infection of the milch goat is persistent, it is by no means constant or even consistent; nor was it possible to detect any correlation between the atmospheric temperature curve and the number of cocci excreted in the milk.

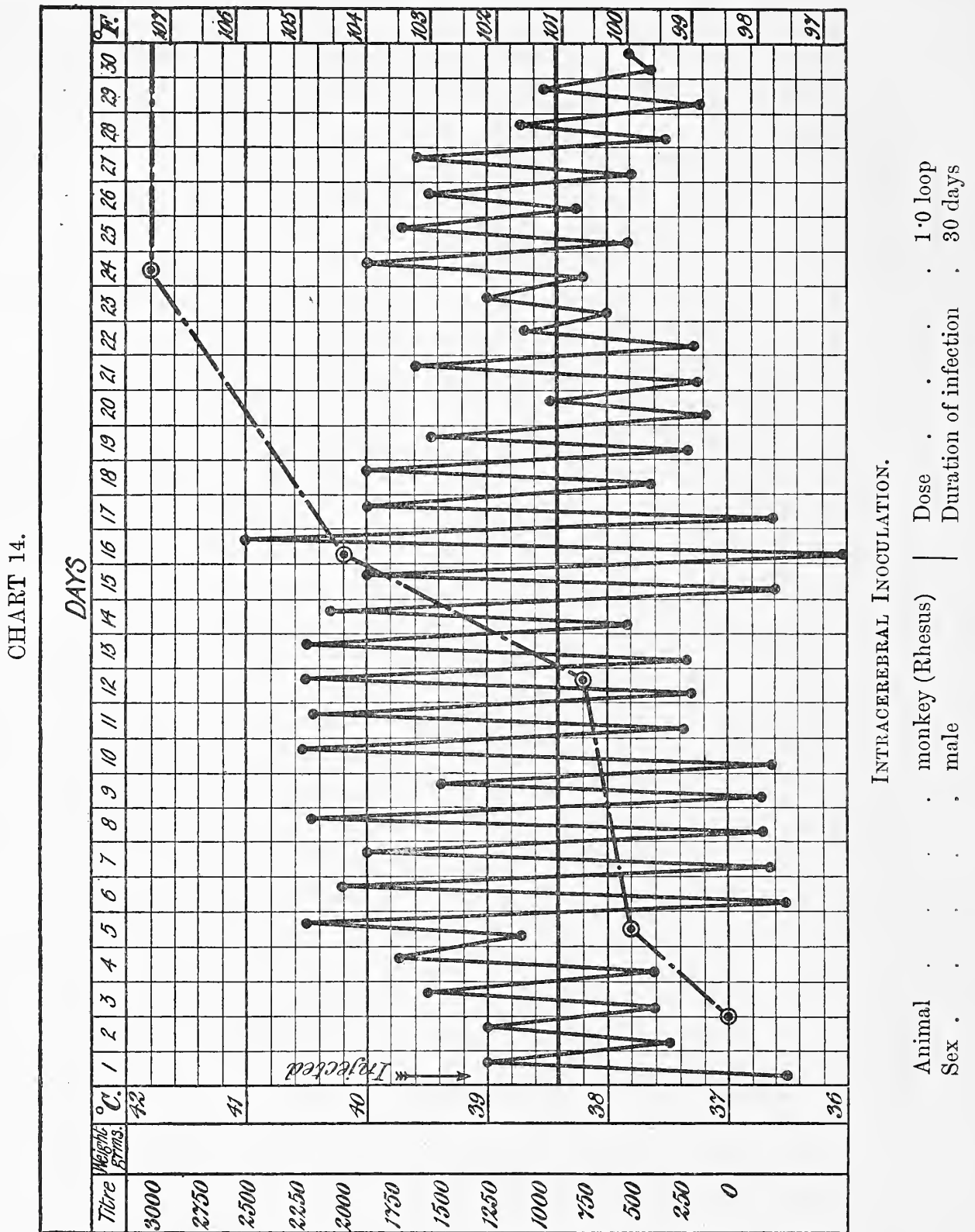
So that for the moment the only explanation that can be offered of the day-to-day variations in the number of cocci present in the milk is that the micro-organism, lodged in a suitable soil and richly supplied with a medium of high nutritive value, multiplies rapidly in the interstices between and upon the surface of the gland-epithelium cells. This multiplication proceeds up to a certain point, when, owing perhaps to the mechanical irritation set up by the mere presence of the coccus, a flushing process is carried out by the milk itself, which removes the excess of cocci and leaves behind in the gland tissue only those cocci which are in intimate relationship with the gland cells. A certain interval is then necessary for further multiplication of those cocci left behind, when the process is repeated again and again.

PRIMATES.

Monkey, Man.

Both the Rhesus and Bonnet monkeys resemble man in that they are susceptible to all the methods of experimental inoculation already enumerated, and in addition to infection through apparently intact mucous membranes. In these animals, as in man, the infection is usually of the subacute or chronic type, though occasionally acute infections of a rapidly fatal course are observed. In fact, the introduction of living cultures into the tissues of the monkey, or the administration of infective food, is followed by an attack of fever strictly comparable in general symptomatology, and in the course of the temperature curve, to one or other of the various types of the disease clinically distinguished in man. The multiplication of the organism in the peripheral circulation and the production of specific agglutinins in the serum are similarly demonstrable, although the disease usually runs a shorter course, and in the majority of the experimental infections would end in complete recovery.

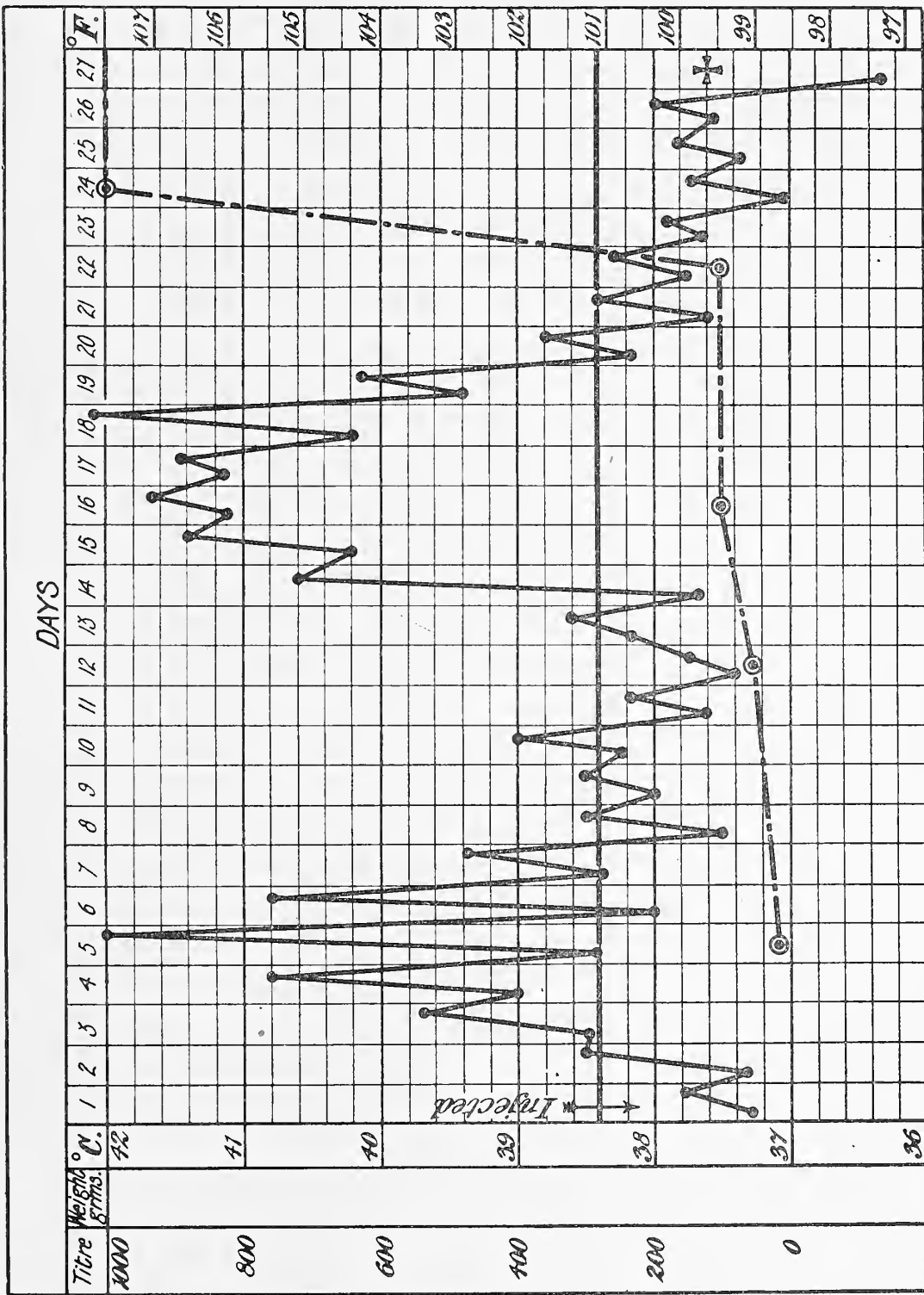
Intracerebral inoculations generally produce a more acute and more fatal infection than that immediately to be described (see chart 14), while intracerebral passages result in a very marked exaltation in the virulence of *M. melitensis* for the monkey.



All other Forms of Infection.—Unlike the rodents, carnivora, and ungulates previously referred to, in which the temperature commences to rise within a few hours of inoculation, the monkey, like man, exhibits an incubation period of from three to ten or even twenty days from the time of infection, during which interval the animal appears perfectly well and

the temperature remains normal. At the end of this latent period the temperature commences to rise higher in the afternoon and evening, when the animal is listless, dull and apathetic, and refuses food; and to remit in the morning, when the animal appears quite lively and in normal health—so marked is the alteration in the animal's behaviour as its temperature

CHART 15.

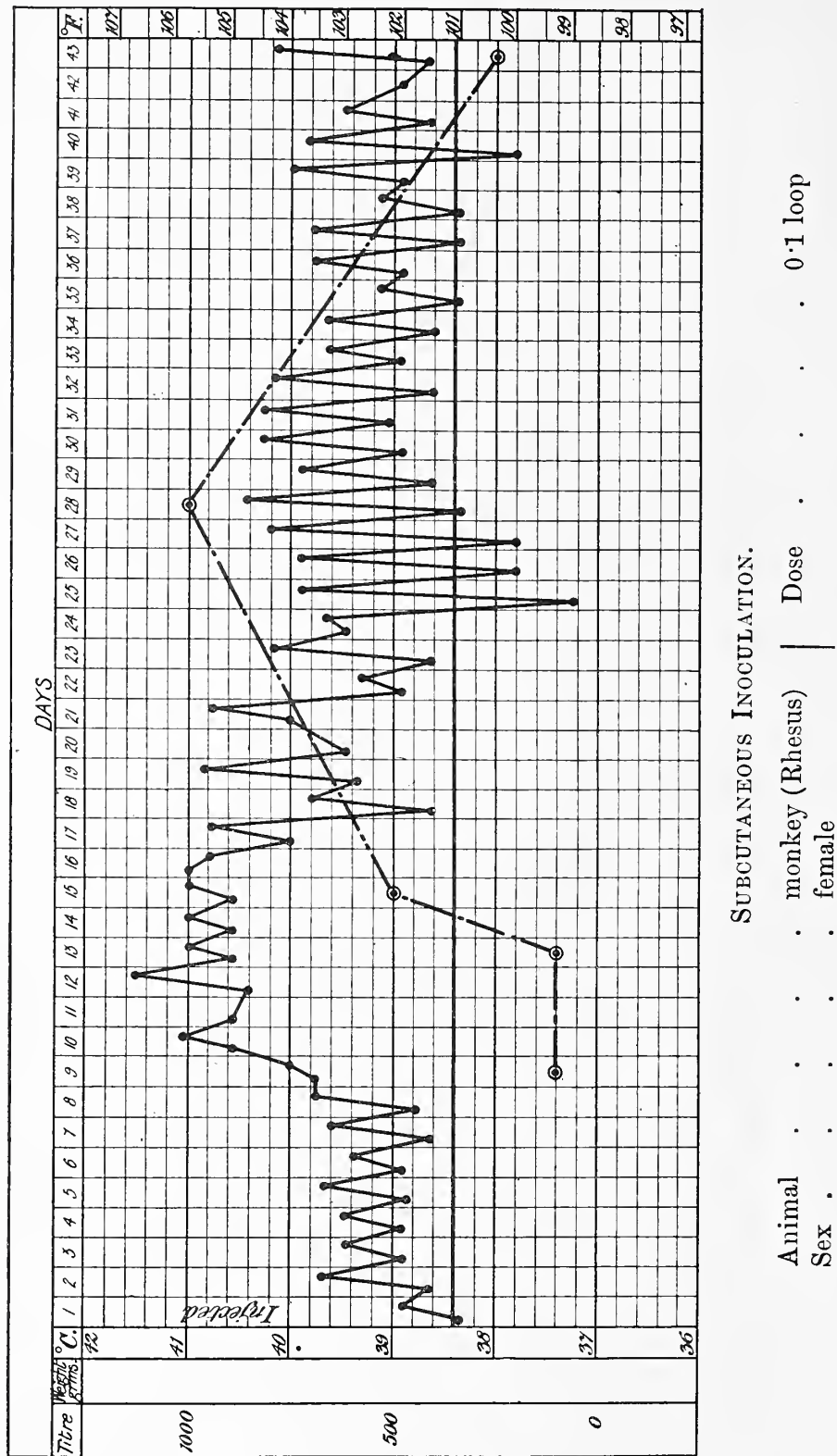


INTRAVENOUS INOCULATION.
 Animal monkey (Rhesus) | Dose 2 loops
 Sex male | Duration of infection 27 days

risers, that it is sufficient to glance at its cage to realise when the hour of noon approaches. The pyrexia lasts usually from five to ten days, diarrhoea and loss of weight being, apart from the pyrexia, the most noticeable feature of the infection. A short period of slight pyrexia, normal or even subnormal temperature, may be succeeded by a second attack of high fever

(see chart 15); but this is quite the exception. Usually it is directly followed by a period of intermittent pyrexia of slight range and short duration, the temperature gradually returning to normal and complete recovery taking

CHART 16.



place, although death may occur without warning at almost any period of the infection.

Post-mortem, *M. melitensis* can be isolated from all the tissues and organs, though the spleen, which is hypertrophied and hard and dark in colour, and the lymphatic glands, which are enlarged and often present soft, almost diffluent areas in their interior, are the only organs of abnormal appearance.

In some cases of extreme chronicity, where the infection has pursued a very protracted course, and the temperature, though very irregular, has rarely risen above 40° C., the specific organism can often be recovered only from the lymphatic glands and urine (see chart 16).

Infection by Feeding.—In a series of experimental feedings with infective goats' milk in the summer of 1905, 93 per cent. of the monkeys were successfully infected, which sufficiently demonstrates their susceptibility to this form of infection. On the clinical aspect of the infection produced by feeding experiments it is unnecessary to dwell at length, but a few points of interest may be briefly mentioned:—

Temperature.—Speaking generally, the infection appears to be mild, as judged by the course of the experimental animals' temperature and supported by their general appearance and behaviour during the six or seven weeks some were under observation, but opinions based upon clinical symptoms were rudely contradicted by the result of the post-mortem examinations.

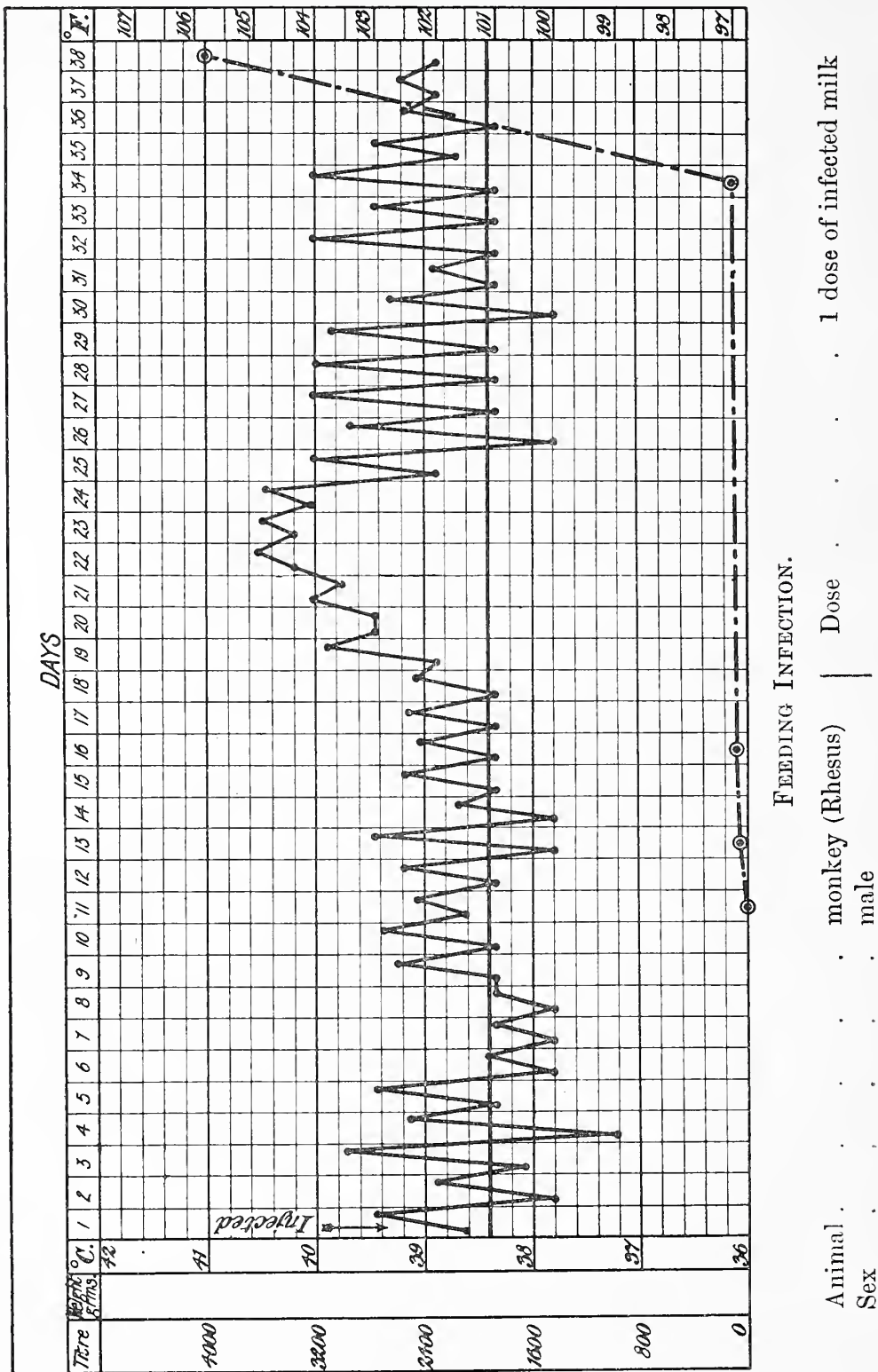
Again, speaking generally, the temperature chart of the Rhesus infected with *M. melitensis*, except in the case of very severe infections such as follow intracranial and intravenous injections of the micrococcus, shows but one period of pyrexia, followed by an intermittent temperature of slight range and short duration. A second period of pyrexia, or "wave" as it is colloquially termed, is quite the exception. The remittent type of pyrexia does, however, occur in the monkey; also this animal sometimes exhibits a type of temperature absolutely comparable to the one obtaining in man when the subject of what Shaw has designated the "ambulatory" type of *M. melitensis* septicæmia.

All these three types are met with in the course of experimental feedings, as will be seen in the three accompanying charts (17, 18, and 19), which I have selected from the many in my possession. In these three animals the severity of the infection must have been of nearly equal intensity in these cases, judging by the results of the bacterioscopic examination, which showed that the blood and all the organs of each of these animals were literally teeming with the *M. melitensis*.

Agglutination Reaction.—The repeated examinations that were made of blood from each of the infected monkeys showed that for a day or two, or even several days, before a definite reaction was obtainable a 1:10 dilution of the serum produced microscopically what is regarded as an "incomplete" reaction—that is, the micrococci ceased to exhibit active vibratory movement and adhered together in small bunches; but large clumps and masses were not formed, and the general field was made up of discrete cocci. Then

a good reaction, large clumps in a perfectly clear fluid, readily visible with the two-thirds lens or, indeed, the unaided eye, would be produced by a low dilution of the serum—1:10 or 1:20. Very often, even at this stage, the macroscopical reaction in the sedimentation tube was absent. Next, the

CHART 17.

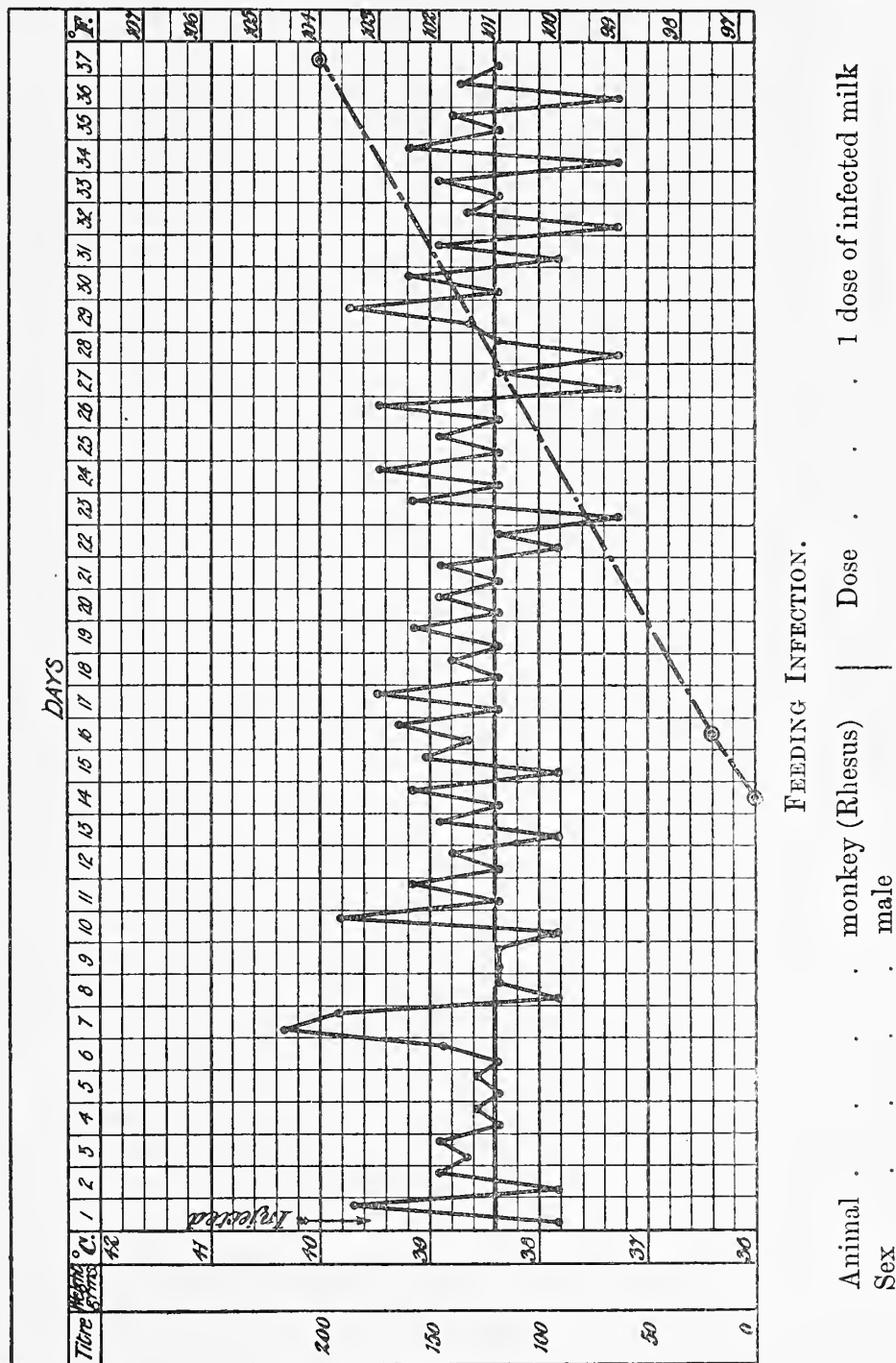


microscopical reaction would often disappear for a day or two, or even longer; finally, it would become firmly established, and obtainable in the majority of cases in considerably higher dilutions, and the micro- and macroscopical reactions would control and confirm each other with absolute precision. The exigencies of experiment, necessitating the destruction of

animals early in the course of the disease, are responsible for the fact that but few examples of the development of a very high agglutinative power in the serum were noted.

Man.—Man is susceptible to infection by subcutaneous inoculation, to infection through apparently intact mucous membranes, and the administra-

CHART 18.

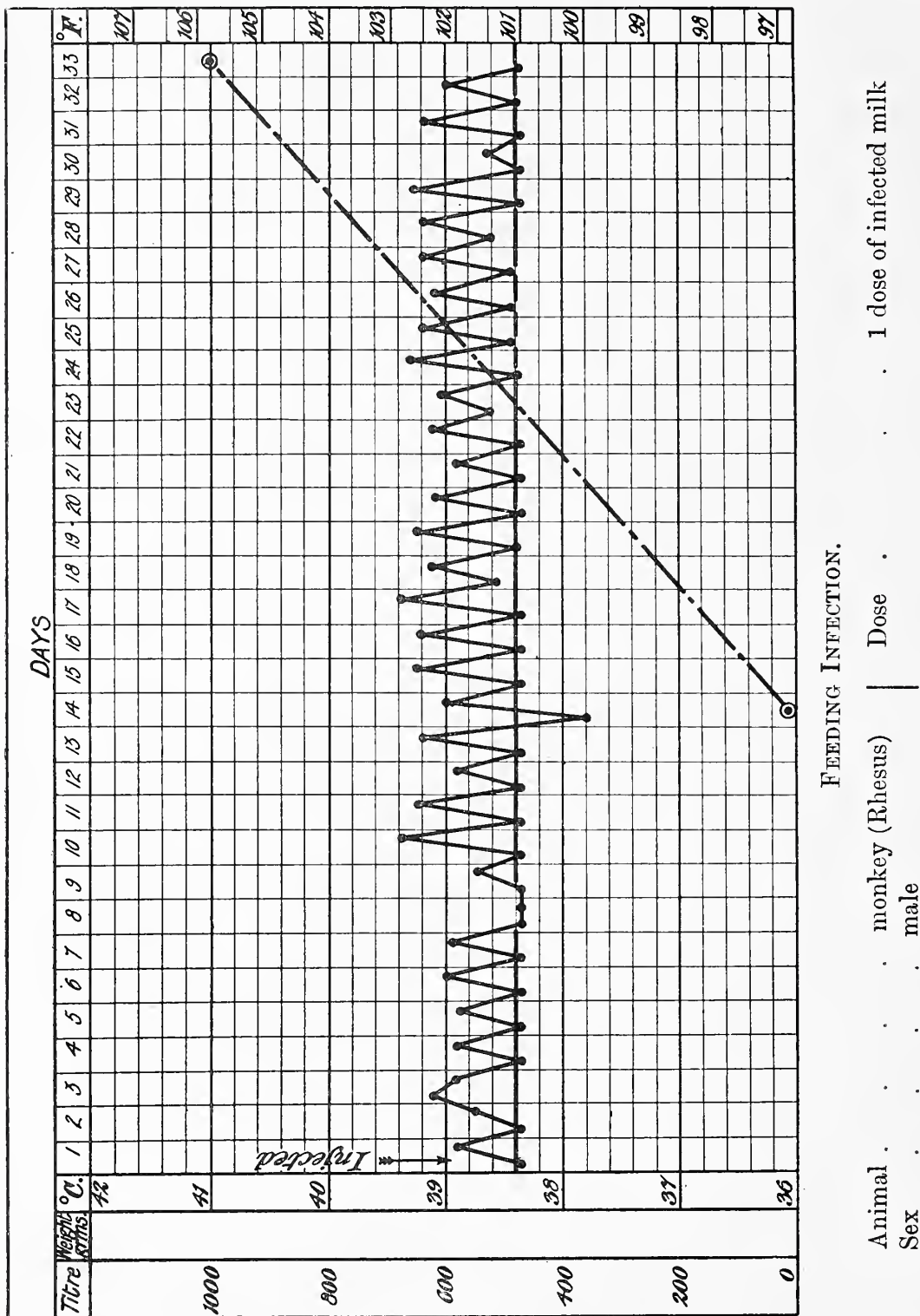


tion of infective food. Laboratory inoculations in which man has served as the experimental animal, though not intentional, are sufficiently numerous to enable us to say that the type of infection which results compares absolutely with that naturally acquired within endemic areas, and also enable us to fix within fairly narrow limits the period of incubation for subcutaneous infections. Thus of eight cases of laboratory infection where

the exact date of inoculation could be fixed, the shortest period of incubation was 5 days, the longest 16 days, the average being just over 8 days (the exact duration being 5, 5, 5, 6, 8, 15, 15, and 16 days respectively).

In those cases where the incubation period extended to 15 days, the

CHART 19.

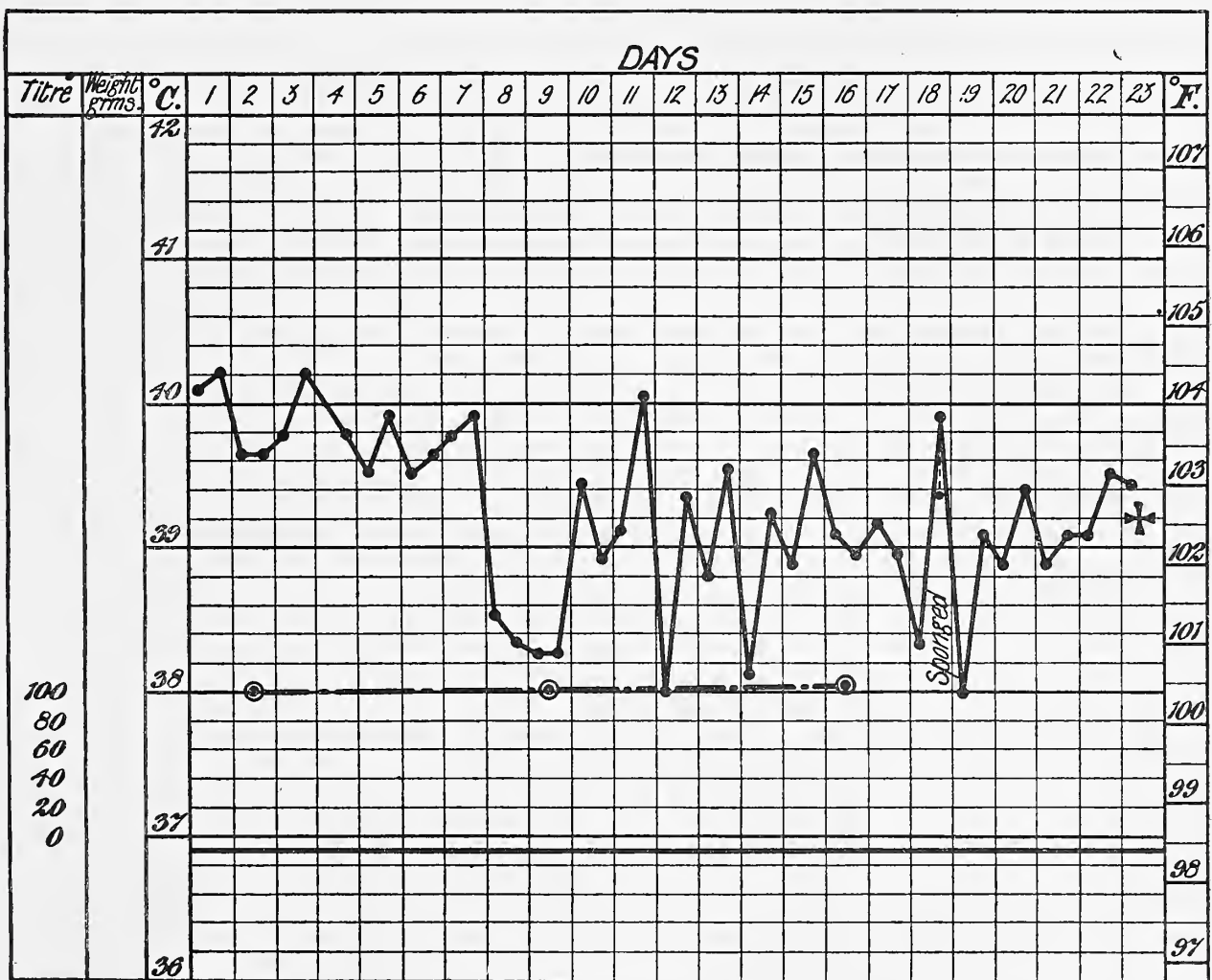


accident which caused the infection was noted at the time, and strenuous efforts, by the liberal application of antiseptics, were made to reduce the probability of an attack of *M. melitensis* septicæmia. The lengthy incubation period (16 days) in one case previously treated with *M. melitensis* vaccine would appear to indicate that some protection—although obviously inadequate—had been conferred as a result of the treatment.

Cases of *M. melitensis* septicæmia in man present different features with almost every individual attacked, but may be grouped under one or other of the headings acute, subacute, and ambulatory, and may be briefly described as follows:—

(1) *Acute Form*.—This type is extremely acute from the onset, and is initiated in the previously healthy by rigors, accompanied by a temperature of 38·5°, 40°, or 41·5° C.; severe headache, often limited to the back of the

CHART 20.



? FOOD INFECTION IN MAN.

Type of infection acute
 Result death 23rd day of illness

eyeball; indefinite pains about the trunk and limbs, particularly in the back; and general malaise. The face is flushed, the dorsum of the tongue is thickly coated with white fur, but pink and moist at the sides and tip, or more rarely dry, brown, glazed and cracked, and the breath offensive. Diarrhoea is not infrequently present during the first few days of the attack, but soon gives place to constipation. The pulse is strong and increased in frequency, though not usually in proportion to the temperature. The urine is diminished in amount, high in colour, and contains large quantities of uric acid and urates. This type of fever sometimes passes into the "typhoid"

state, and death results from cardiac failure (see chart 20), or, more rarely, hyperpyrexia supervenes. Sometimes a crisis occurs and recovery takes place, but usually the temperature gradually falls to or near normal, and the case assumes the subacute type.

(2) *The subacute form*, on the other hand, is often slow and gradual in onset. For some days slight headache, thirst, constipation and gastric disturbances, pains in the back, neck, and limbs, usually described as "rheumatic," accompanied by insomnia, mental anxiety, and general depression, combine to produce a marked, but at the same time indefinite, feeling of ill-health. Next follows a steady and gradually increasing rise of evening temperature, with morning remission, until 39.5° to 41.5° C. is reached, followed by a similar and almost equally gradual fall until the morning temperature becomes practically normal. The remissions of temperature are almost invariably accompanied by profuse perspirations. The duration of the initial pyrexial attack varies in different cases from one to five weeks; then, after an apyrexial interval lasting from five to ten days or a fortnight, during which the temperature remains at or about normal, a relapse sets in, similar in all respects to the first attack, but often distinctly shorter and less severe. This sequence of events is repeated again and again, the duration of the disease varying from six weeks to six or nine months (see charts 21 and 22). I have seen several cases where the duration of the disease has exceeded two years, and one where the fever had existed with typical pyrexial attacks at irregular intervals for three years.

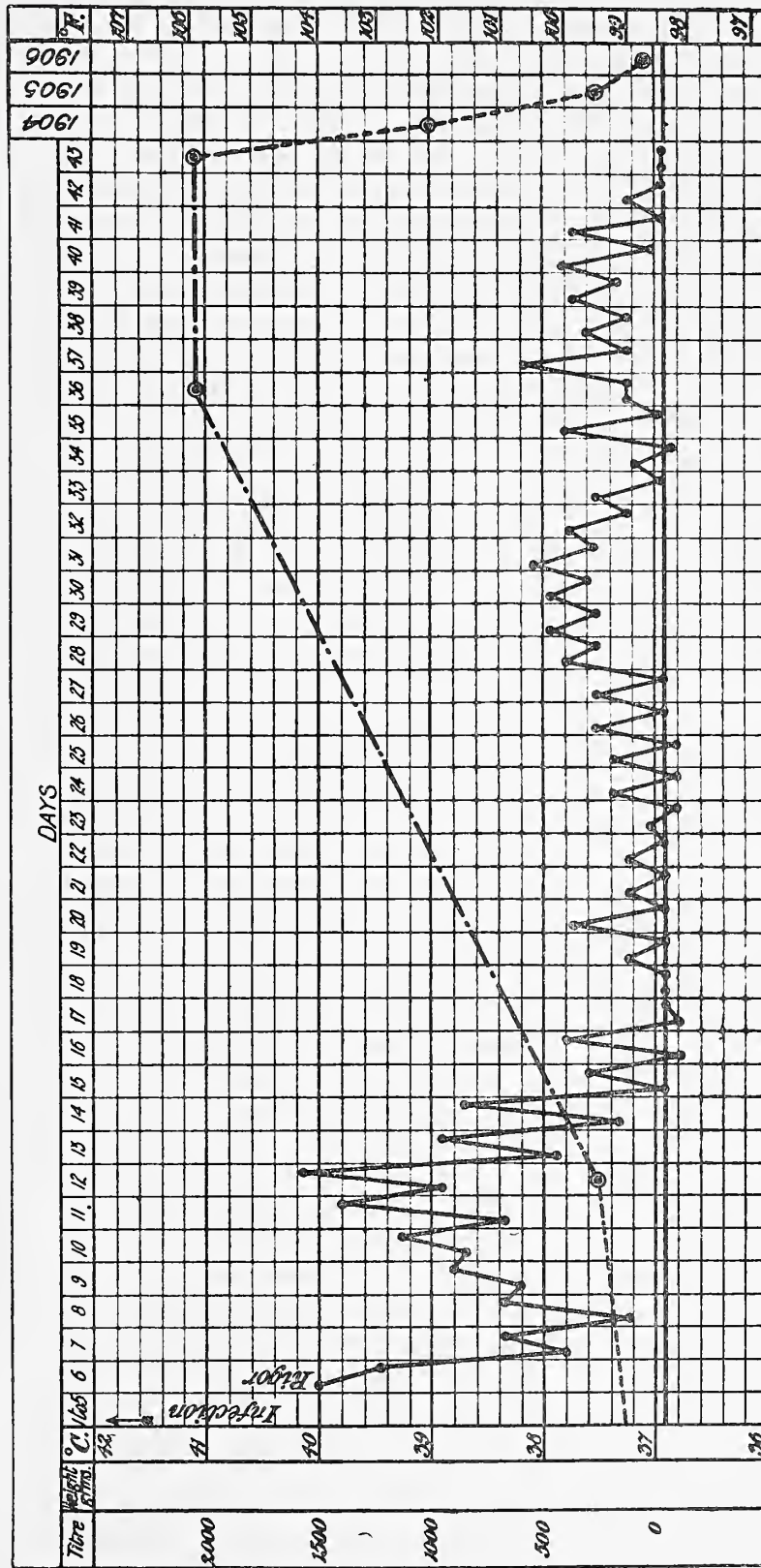
(3) *Ambulatory Form*.—Finally, mention must be made of the ambulatory type of case described by Shaw, in which symptoms are entirely absent, or are limited to a few days' fever (37.5° C.), and the only proof of the existence of the disease is the presence of agglutinins in the blood, and occasionally of the *M. melitensis* itself also; while in the urine (although normal in appearance) the specific organism is often present in enormous numbers (22,000 per c.c.) and in a highly virulent condition. Marston in 1863, it should be noted, was also familiar with this ambulatory type of case, for he says: ". . . upon the other [hand], so mild may the symptoms appear, that the patient may never be confined to bed, and be all the while supposed to be labouring under a peculiar form of dyspepsia."

IMMUNISATION AND ANTIBODIES.

The immunisation of rodents is a matter of considerable difficulty, and the amount of immunity attained is totally inadequate to the length of time and the labour involved—in fact, one is almost led to the conclusion

that in rodentia at any rate an active immunity to *M. melitensis* is never acquired. Full-grown rabbits and guinea-pigs occasionally appear resistant to small doses of the micrococcus or doses of cocci of low virulence; but this

CHART 21.

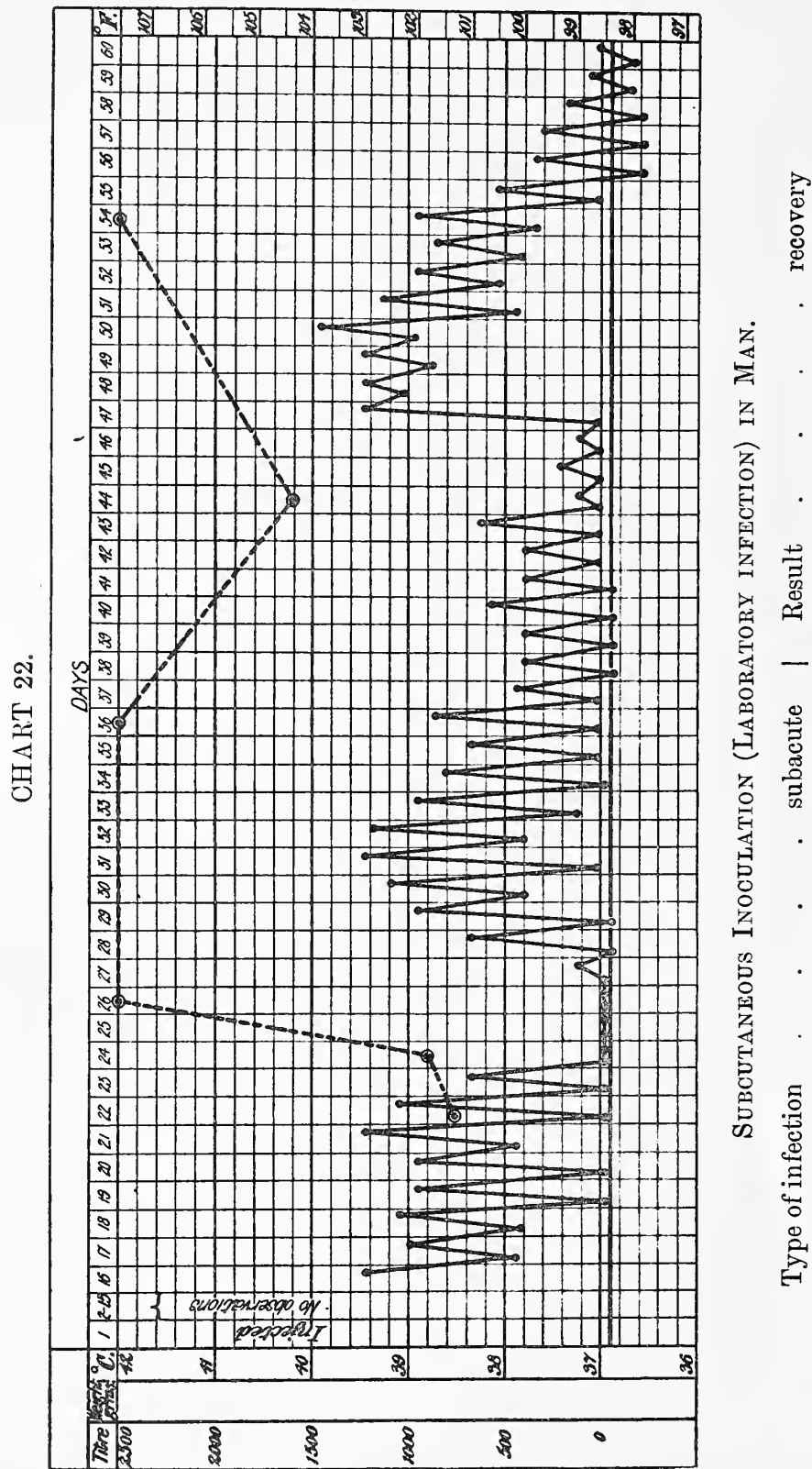


SUBCUTANEOUS INOCULATION (LABORATORY INFECTION) IN MAN.

Type of infection subcutaneous | Result recovery

immunity is apparent rather than real, and a larger dose, or a similar-sized dose of a more virulent strain, or inoculation by some other channel, will always produce a successful infection. Hence the observed absence of bactericidal substances from the serum of healthy or infected rodents is not

a matter for surprise, and affords a ready explanation of the fact that after most careful treatment of an experimental animal by any of the approved methods of immunisation, the administration of a suitable quantum of



M. melitensis culture is invariably followed by infection. These remarks hold good in the case of adult animals of all other orders of mammals, including monkeys—despite Shaw’s opinion to the contrary, which was based upon experiments made whilst he was still unaware of the fact that a type of infection could be produced in the monkey resembling the ambulatory

form of *M. melitensis* septicæmia in man, in which, though the blood and tissues contain the micrococcus, no pyrexia is produced—and probably man.

But although specific bacteriolysin is absent from the serum of the inoculated animal, specific agglutinin is readily formed, and this aspect of resistance to the *M. melitensis* and its products has been exhaustively studied by many observers, particularly by Durham and myself.

Agglutination Reaction.—Since the “serum reaction” test of Gruber and Durham was applied to the diagnosis of *M. melitensis* septicæmia by Wright in 1897 this method has been extensively employed; and it has been shown that the specific agglutinin may be present in the blood of infected animals, including man, from the fifth day onwards (Aldridge, Bassett-Smith, Gilmour)—exceptionally it is present on the first day, sometimes its appearance is delayed until weeks after the disease is well established. Usually it is present in large amount, giving to the serum a titre of 1 : 100 or 1 : 1000 even during the first week of the disease, whilst subsequently sera reacting in much higher dilutions, even up to 1 : 500,000, have been observed.

Moreover, it must be noted that in man the agglutinins persist in the blood long after recovery, and the serum may react in dilutions of 1 : 50 and 1 : 100 for from three to seven or even ten years after naturally acquired infection.

Finally, sera from healthy animals and from those suffering from infections other than those due to *micrococcus melitensis* never yield a complete reaction when tested in dilutions of 1 : 10 (Birt and Lamb), or a partial reaction when tested in dilutions of 1 : 20.

Specific agglutinins are elaborated in response to injections of the dead bodies of *M. melitensis* as well as to living cultivations.

Experiments have so far failed to confer active immunity upon any of the ordinary laboratory animals. After long-continued treatment of rodents by repeatedly injecting suitable doses of killed cultures in the attempt to produce some degree of immunity and the establishment and maintenance of a high agglutinin content of the blood serum, the introduction of even comparatively small amounts of living virulent cultures almost invariably caused the death of the animal.

Animal.	Weight in Grams.	Period of Treatment.	Titre of Serum.	Dose of Living Cultivation.	Method of Inoculation.	Result.
Rabbit 8 .	3000	7 months	1 : 1500	1 loop	Intracerebral	† in 10 days
Rabbit 25 .	2500	7 „	1 : 5000	0·01 „	„	† in 60 „
Rabbit 73 .	2800	7 „	1 : 4000	0·0001 „	Subcutaneous	Recovery
Guinea-pig 50	600	4 „	1 : 5000	1·0	Intracerebral	† in 3 days

Passive acquired immunity is likewise conferred to a very slight degree only, for the serum of treated rodents, even when it possesses a very high agglutination titre, fails to protect normal rabbits and guinea-pigs against infection, or even to modify the course of such infection. Agglutinating serum from larger animals appears to be no more potent.

On the other hand, a certain degree of immunity is undoubtedly transmitted from the infected maternal parent to her offspring, for in both the infected dog and goat, although the micrococcus melitensis, minute though it is, does not traverse the placenta and so no actual infection occurs, an appreciable amount of specific agglutinin is transmitted to the pups and kids. But beyond a doubt other antibodies are also carried over, for although these animals are readily susceptible to infection with *M. melitensis* *via* the alimentary tract, and although the infected parent is secreting milk teeming with the micrococcus, which is taken by the suckling young, infection does not result during the period of suckling. The inference is therefore justified that the young are supplied with other antibodies besides the specific agglutinins which protect them from infection during the period of suckling. These statements are based, it is true, upon a limited number of observations, but at my suggestion Professor Zammit is continuing these experiments, with results which go far to support my conclusions.

Animal.	Examination of Parent.		Examination of Offspring.		
	Titre of Serum.	<i>M. melitensis</i> in Milk.	Length of Life.	Titre of Serum.	Recovery of <i>M. melitensis</i> from Tissues.
Dog 8	1 : 200	Yes			
Pup <i>a</i>	1 day	1 : 10	nil.
" <i>b</i>	1 day	1 : 10	"
" <i>c</i>	14 days	1 : 10	"
" <i>d</i>	14 days	1 : 20	"
Dog 28	1 : 20	Yes			
Pup	5 days.	1 : 10	nil.
" <i>f</i>	5 "	1 : 10	"
" <i>g</i>	5 "	1 : 5	"
Goat 3	1 : 100	Yes			
Kid <i>a</i>	17 days.	1 : 50	nil.
" <i>b</i>	17 days.	1 : 20	"

This immunity, however, is only partial and temporary, being limited to protection against infection *via* the alimentary tract; and to the period of suckling, for the young are still susceptible to subcutaneous, intravenous,

or intraperitoneal infection during the suckling period, and at a later period of life to infection by feeding methods also.

MORBID ANATOMY.

For the study of the morbid anatomy and morbid histology of *M. melitensis* infections, laboratory animals of course afford ample material, but the case mortality of *M. melitensis* septicæmia in man is extremely low. Writing in 1903, Hayat (38) could only collect details of 76 autopsies, derived from Malta (67), Palermo (5), Netley (2), Naples (1), and Padua (1), for study. In Malta, during 1904, four, and in 1905 nine, fatal cases were carefully investigated by the Mediterranean Fever Commission. Personally, whilst in Malta in 1906 I only had the opportunity of assisting at the post-mortem examination of one case of *M. melitensis* septicæmia that died in the Military Hospital, Valletta, and of being present at some twenty autopsies at the Civil Hospital, Floriana. The pathological appearances produced by *M. melitensis* septicæmia in fatal infections, based upon observations upon post-mortem examinations of lower animals and of man, I have already described elsewhere (*Lancet*, June 20, 1908).

MORBID HISTOLOGY.

Observations upon the morbid histology of *M. melitensis* septicæmia do not appear to have been numerous—chiefly, I imagine, because the changes that take place in the various organs and tissues are few, generally insignificant, and in no way to be distinguished from those noted in septicæmia due to any other micro-organism. This obtains no matter whether the material is collected from acute cases killed shortly after infection or from chronic cases where long intervals have elapsed between infection and death, or whether in lower animals or in man. I have carefully studied a large number of sections during the past seven or eight years, collected from many different mammals, and I must confess that up to the present I have been unable to demonstrate any extensive deviations from the normal—except in those cases where a laboratory inoculation has been planned to produce suppurative lesions such as subcutaneous abscess, suppurative peritonitis, or suppuration within the tunica vaginalis. Such lesions as have been observed are best considered in the order of the various organs, etc., adopted for the morbid anatomy.

Heart.—Hæmorrhages between the muscle fibres is the only noticeable feature: the myocardium itself appears normal.

Lungs.—Here and there at the bases, foci of congestion show desquamation of the alveolar epithelium and collections of extravasated red cells and fibrin filaments within the alveoli.

Alimentary Canal.—In some situations, particularly in the duodenum, extravasations of red cells are to be found, limited to the mucous membrane immediately below the glandular layer, and to the submucosa; in other spots, denudation of the surface epithelium and small-celled infiltration at the base of the ulcer formed by the destruction of the mucous cells. In still other areas, marked proliferation of the cellular elements may be observed.

Liver.—The capillaries are dilated with red cells, very few polymorphonuclear leucocytes being visible; and small-celled infiltrations may occasionally be noted around the perihepatic vein. Numerous intracellular capillaries are obvious throughout the sections. No changes are apparent in the hepatic cells themselves, though occasionally the hepatic trabeculæ seem to be attenuated.

Spleen.—Arteries, capillaries, and venous sinuses are dilated and filled with red-blood cells. In acute cases the splenic pulp is crowded with red cells, but in cases of long duration there is considerable increase of lymphoid tissue and of Malpighian corpuscles. The elastic fibres and reticulum are normal.

Kidney.—Hyperæmia is the chief change noted in this organ: the loops of the glomeruli are distended with blood; the interlobular arteries and vasa erecta are also dilated. No thickening of the capsule and no change in the epithelium of either tubules or glomeruli can be made out.

Mesenteric glands are enlarged, not so much by reason of hyperplasia of the follicles and trabeculæ, as by hyperæmia, as is shown by the dilatation of the sinuses, which are filled with blood cells. The follicles are not, as a rule, particularly rich in lymphocytes.

Bone-marrow.—In chronic infections this tissue shows marked alterations from the normal in its constitution. Nucleated red cells and *lymphoid cells*, that is, giant cells, mononuclear cells, and lymphocytes, are markedly increased in number; *granular cells*, *i.e.* myelocytes and polymorphonuclear leucocytes, are considerably diminished: the whole forming a typical lymphoerythroblastic bone-marrow, and is in marked contrast to the leucoblastic marrows associated with, for instance, pneumococcic infections.

Blood.—During the early stages of *M. melitensis* infection the micrococcus can be detected in the peripheral blood by cultural methods, but microscopical examination of the blood direct usually, except in the very acute and severe infections, fails to demonstrate its presence. In man the

coccus has been isolated (by cultural methods) from the second to the three-hundredth day of the disease, and in numbers varying from less than one per cubic centimetre to over 10,000, by various observers.

The histological structure of the blood in the first few days of infection is characterised by a relative polymorphonuclear leucocytosis. This, however, soon gives place to a relative or an absolute leucocytosis, chiefly affecting the non-granular cells—the lymphocytes and mononuclear leucocytes.

Brain.—No abnormal appearances can usually be observed. A human brain from a fatal case of *M. melitensis* septicæmia, which had during life shown unmistakable symptoms of cerebral irritation, I submitted to Dr F. W. Mott; but he informed me that, apart from obviously post-mortem changes, the microscopical appearances of this organ were perfectly normal.

In two monkeys injected intracerebrally with small doses of *M. melitensis*, of a strain that had not been exalted in virulence for this animal, abscess formation was observed localised to the cortex at the spot entered by the syringe needle. But in the majority of the intracerebral inoculations I have performed there has been no attempt at localisation of the micrococcus to the cerebral tissues.

Some of the Italian observers—Caracciolo, Carbone, and Trambusti—record much more definite changes in the organs removed from *M. melitensis* septicæmia cases. Most of these I am rather inclined to attribute to post-mortem influences, which in Italy and Sicily so rapidly produce marked alterations in tissue cells. But one point on which all are agreed as being the most striking feature of sections not only of spleen but also of liver and kidney, is the presence of numerous cells which they term globuliferous cells, derived from the endothelium lining blood sinuses, and containing in their interior from one to fifteen or twenty red-blood discs. In the spleen of a case examined by Carbone, he says they were absolutely innumerable; those containing a few blood cells only were rare, those containing from eight to ten were common, and some were seen containing very many more. These observations I have not been able to confirm.

XXXV.—The Development of the Auditory Ossicles in the Horse, with a Note on their possible Homologues in the Lower Vertebrata. By Ray F. Coyle, B.S. (From the Zoological Department of the University of Edinburgh.) *Communicated* by Professor J. C. EWART, M.D., F.R.S. (With Six Plates.)

(MS. received May 25, 1909. Read July 5, 1909.)

I.

THE discussion as to the homologies existing between the sound-conducting apparatus of the mammalia and certain elements of the lower jaw and branchial skeleton of fishes has occupied the attention of numerous authors. In fact the question may be considered as a classic, dating as it does to at least 1778, when Geoffroy published his *Dissertations sur l'organe de l'ouïe de l'homme, des reptiles, et des poissons*. From that time until 1898—from Geoffroy to Gaupp—the contributions to the literature upon this subject constituted an enormous amount. Inasmuch as Gaupp has given a detailed and masterly review of the literature up to the time of his work (1898), and since Fuchs thoroughly covered the ground seven years later, I shall not undertake that task in the present communication. Moreover, as this paper is in the nature of a preliminary report upon the entire developmental history of the skull in the horse, I shall not here review in detail the contributions to the literature since the time of Fuchs' account (1905).

A specialised apparatus for the conduction of sound to the auditory organ first makes its appearance as the *stapedial plate* (operculum) and *columella auris* of urodele amphibians. Amongst many Anura the spiracular cleft, which disappears entirely during the development of the Urodela and Gymnophiona, becomes "modified to form a *Eustachian tube*, opening into the pharynx and leading into a tympanic cavity. The latter is closed externally by a tympanic membrane supported by a cartilaginous ring (*annulus tympanicus*), and to it the distal end of the columella is attached" (Wiedersheim). In the Mammalia the sound-conducting apparatus consists of an osseous malleus, articulated with the tooth-like incus. The incus articulates by its long arm (*crus longum incudis*) with the stapes (fig. 5).

Many investigators have assigned the stapedial plate and accessory cartilages of the amphibia to the second visceral arch, making it thus possibly homologous to the hyomandibula of the fishes. Fuchs, however, has clearly shown that the stapedial plate is developed from the auditory

capsule itself, which hence makes that theory untenable. Fuchs would go further, and, since he conceived it to be plainly established that the mammalian stapes could have no relationship with the hyoid arch, derive the mammalian stapes also from the auditory capsule, and thus a homologue of the amphibian operculum and stylus.

Our problem is, then, to determine the process of development of the mammalian sound-conducting apparatus, and to compare the relationships of its various components with the relationships existing between those parts having a similar function or a similar position in the lower vertebrates, in order to determine, if possible, the exact homologies.

Before, however, discussing the actual development it is fitting to define an exact position with regard to one or two points. Fuchs has rightly emphasised Reichert's distinction between the terms *visceral arch* (Visceralbogen) and visceral bar (Visceralstreifen). A visceral arch consists of an axis of mesenchyme with an epithelial covering, which is endodermal on the inner side and ectodermal on the outer. Each visceral arch lies between a pair of visceral clefts, and in the arch is developed the visceral bar. Each visceral arch may give rise to one or more visceral cartilages, *i.e.* in general the first arch gives rise to Meckel's cartilage and the quadrate; the second arch gives rise to the hyoid or Reichert's cartilage (often composed of various elements) and the hyomandibula. I have used the term "visceral bar" to refer only to Meckel's or the mandibular cartilage, Reichert's or the hyoidean cartilage and the remaining branchial cartilages. Thus it will be seen that there is a distinction between the quadrate and the first visceral bar and between the hyomandibula and the second visceral bar. Further, it is clear that the visceral arch is an embryonic structure in which certain skeletal elements later develop, while in the lower vertebrates the visceral bar is a permanent skeletal element. I shall try to show that, contrary to Salensky and some other authors, the only part of the ossicular auditory apparatus to have any connection with a visceral bar is the malleus.

The term visceral cleft comprises two distinct kinds of structures, one invaginated from the exterior, the other from the pharynx. The former will be known as gill furrows, the latter as pharyngeal pouches. The first pharyngeal pouch persists as the Eustachian tube, and the first gill furrow as the external auditory meatus.*

* It has again been recently asserted that the Eustachian tube is not the first visceral cleft but the second (M. T. Sudler, "The Development of the Nose and the Pharynx and its Derivation in Man," *Amer. Jour. of Anat.*, i. 1901-1902). However, as the designation of the Eustachian tube as first or second pharyngeal pouch does not affect its actual relations as a structure lying between and separating the Meckelian and Reichertian cartilages, a detailed examination of that point is foreign to the present discussion.

II.

My investigations were carried out upon a series of horse embryos which Professor Ewart kindly allowed me to study. The series comprised individuals from three weeks until twelve weeks, at weekly intervals, and then various stages up to full term. Of these one, aged four weeks, length 12 mm., was already sectioned and mounted. Models were made of the auditory region of the five-weeks embryo and of a seven-weeks embryo.

The three-weeks embryo was found to be so young, showing no skeletal development other than the chorda, as to be of no concern in the present discussion.

The four-weeks embryo is important as showing the primitive condition of the visceral clefts. Fig. 6 shows a drawing of a section of the head of this individual passing through the region of the first and second visceral clefts. The first pharyngeal pouch (I. P.P.) is seen to extend laterally and to almost open into the first gill furrow, always, however, with a separating membrane between. From this point the first pharyngeal pouch is seen to be re-bent and to extend medially and *dorsally*. The second pharyngeal pouch (II. P.P.) extends caudo-laterally until it almost meets the second gill furrow (not shown in figure), being also separated from the latter by a membrane. The ideal condition of the pharyngeal pouches I have shown in a diagram, substantially the same as that employed by Fuchs (text fig. 1, A). The conditions met with in this embryo are different only in that there is an extension of the first pharyngeal pouch medially (marked with an asterisk in figures). This re-bent portion plays an important part in later topographical relationships, for the stapes is developed medial and dorsal to it. The mesenchyme is arranged in dense masses both cranial and caudal to the first pharyngeal pouch, representing the first two visceral *arches*, and cranial nerves are found running through the mesenchyme.

Fuchs has contended that the mandibular arch is lateral to the first pharyngeal pouch. This statement is indefinite, because the first pharyngeal pouch has become extended in one portion while retaining primitive relationships in others. In the ideal condition it is enough to state that a structure lies cranial or caudal to the first pharyngeal pouch to immediately define its position with regard to the visceral arches. But in the four-weeks stage there has already arisen a new complication, and we must give the relative position of a structure with regard not only to the pharyngeal pouch, but also to the prominent flexure of the pouch. Thus it will be seen that in the present stage the mandibular arch lies lateral to the re-bent portion of the first pharyngeal pouch, and also cranial to the point of

flexure. Thus to define the mandibular arch region as being separated from the region of the hyoid arch by a line passing from the extreme distal end of the first pharyngeal pouch to the first gill furrow is obviously a mistake. The plane of separation in this case passes through the prominent point of flexure of the first pharyngeal pouch and through the first gill furrow.

Although this embryo shows no skeletal elements which may be definitely identified as such, it is important to further note the condition of the future auditory capsule.

In fig. 7 a section is shown through the auditory sac. This appears as a hollow subspherical mass of ectodermal tissue, lying in the mesenchyme between the seventh and ninth nerves. The fifth and eleventh nerves are seen cranially and caudally respectively. Around the auditory sac the mesenchyme is seen to be in a condensed condition, which is an indication of future cartilage formation. Thus we have here the anlage of the auditory capsule, in a very primitive condition. In view of the contention of Fuchs that the stapes is developed from the wall of the auditory capsule, the position of the capsule as it here appears is very important.

The seventh nerve runs latero-caudally from the ganglion shown in fig. 7, and crosses the vena capitis lateralis (not shown in the figures) and comes to lie between the latter structure and the outer surface of the head. It then runs caudally and ventrally, coming to lie in the position shown in fig. 6. The vena capitis lateralis shows at this stage the same relations as in the later embryos, that is, it runs dorsal and lateral to the auditory region, being crossed by the seventh nerve, until it reaches the region of the ganglion of the fifth nerve, where it bifurcates and loses its importance in this discussion.

Figs. 1, 2 and 3 show three drawings of a model of the auditory region of a five-weeks horse embryo, length 15.5 mm. The skull shows no sign of cartilage formation except in the posterior cranial portion (chorda and trabeculæ) and in the first two visceral bars. The cartilage in these latter structures extends as far dorsal as to the area to be described as the anlage of the malleus, and is shown coloured blue in fig. 8. It is here scarcely yet in the condition of true cartilage, but it is in a very advanced stage of pro-cartilage. More dorsal to the point figured in fig. 8, cartilage disappears entirely except for a small area in the anlagen of the malleus and incus, to be described later. It will thus be seen that the dorsal (proximal) portions of the first two visceral bars are less advanced than their ventral (distal) ends towards cartilage formation. Lateral and somewhat dorsal to the first

third of the first pharyngeal cleft, the mesenchymic area known as the first visceral arch thickens medio-laterally. Moreover, it begins to extend slightly cranially and strongly caudally, enclosing the chorda tympani, until it almost merges with the hyoid arch. Now the more cranial portion extends in two directions, one dorso-laterally, the other dorso-medially. We have here the first anlage of the incus, with its crus longum and crus breve. The long cranio-caudal area of mesenchyme is the anlage of the malleus. At this time the mesenchymic mass *encloses both structures*, although their later forms are to be distinguished by certain cell differentiations. This distinction is important, and it is on this basis that I have modelled this embryo. The model does not show cartilage and pro-cartilage as distinct from the surrounding mesenchyme—but rather the whole mesenchymal extent has been modelled.

Kastchenko, together with some other investigators, has declared that the external auditory meatus has nothing to do with the spiracular or first visceral cleft. But inasmuch as the present discussion depends little upon whether the external auditory meatus be the first gill furrow or not, it is of no good to enter into that discussion here, except merely to state that the view of Kölliker, Gaupp, and others, that the external auditory meatus is derived from the first gill furrow, seems to have vastly more weight than the contrary hypothesis.

The first pharyngeal pouch of the five-weeks embryo will be seen to arise somewhat caudal from the first gill furrow and to extend cranio-laterally for two-thirds of its length. Then it bends sharply, and for the remainder of its length extends medio-cranially. This medio-cranial flexure has apparently escaped Fuchs' notice. It is marked in the diagram with an asterisk. Now it is evident that in the ideal condition, to speak of a structure as cranial to the first pharyngeal pouch would throw it at once into the territory of the mandibular arch. But as Fuchs rightly points out, to say that a structure is cranial to the first pharyngeal pouch when the relations are those of this stage, does not of necessity make it lie in mandibular territory, for the mandibular arch in this case lies *lateral* to the first pharyngeal pouch. Moreover, the first pharyngeal pouch is inclined somewhat dorsally, as is shown in fig. 1, and consequently it is conceivable that a structure (*i.e.* stapes anlage in diagrams) might have become so displaced by pressure and unequal growth as to lie medial and dorsal to the first pharyngeal pouch and still retain its ontogenetic relation with the first visceral arch.

Reference to text fig. 1, B, and to figs. 1 to 3, will show that the mesenchyme has grown between and separated the first pharyngeal pouch

from the first gill furrow by a space of considerable extent. We have here a condition similar to that found in the four-weeks embryo, except that the first pharyngeal pouch is thus separated from the first gill furrow. Moreover, the mesenchyme in extending down between the two structures might be conceived to have carried with it structures derived either from the mandibular or hyoidean arch. But it will be possible to determine from which arch these structures come, provided that they retain some close connection with their respective arches. If a structure lies both cranial to the flexure of the first pharyngeal pouch and lateral to the re-bent portion, that

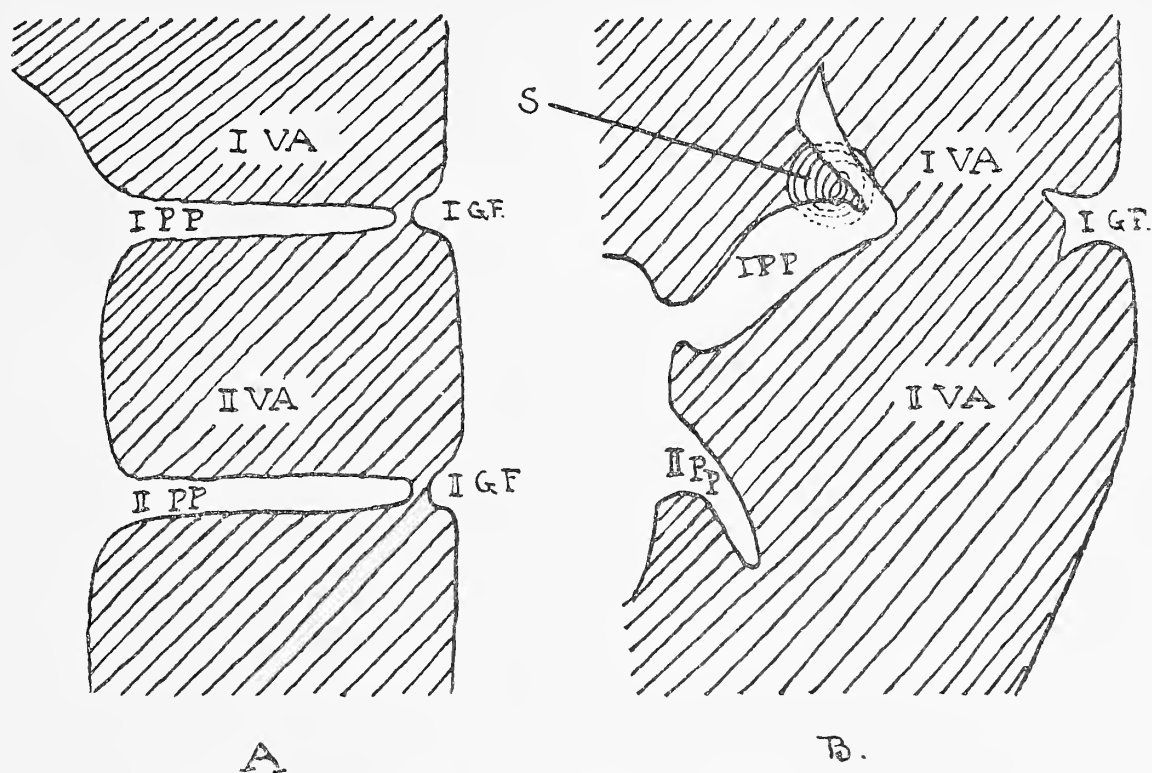


FIG. 1. —V.A., visceral arch; G.F., gill furrow; P.P., pharyngeal pouch; S., anlage of the stapes.

structure is clearly in the region which is without question mandibular and which is, in fact, the same region which we have defined as the mandibular arch in the four-weeks embryo. It is in this position that the anlage of the incus lies, and about the ontogenetic relations of the incus there can be then no question. It is clearly developed from the first visceral arch.

Medial to the incus anlage a spherical mass of mesenchyme arises, perforated by a minute branch of the carotis interna. This, the anlage of the stapes, lies medial and dorsal to the first pharyngeal pouch and cranial to the prominent point of flexure. Fuchs states, in contradiction to Broman (who stated that the stapes arises caudal to the first visceral pouch), that the stapes lies medial to the first pharyngeal pouch, and with this view I have, in an earlier portion, concurred. But this is not entirely exact. The

stapes anlage lies medial *and dorsal* to the first pharyngeal pouch. The stapes mesenchyme, moreover, extends to the crus longum incudis, and there is no point at which the *incus and stapes may be said to be distinct*.* This relation is shown in fig. 10. The stapes anlage at one point lies very close to the auditory capsule, so close, in fact, as to be almost indistinguishable from it, but always we may find a line of demarcation, whereas the connection between stapes and incus is exceedingly close, the mesenchyme having at all points an equal density. Moreover, the crus longum and the stapes show no signs of differentiation into cartilage.

The hyoid bar as cartilage or pro-cartilage extends somewhat farther dorsal than the mandibular cartilage. It then contracts and sends a more or less broad band of mesenchyme to meet the stapes anlage. This is the hyostapedial ligament of Huxley and others, shown in the figures marked with a double asterisk. It is important to note that the entire auditory ossicular apparatus is at this time represented by a more or less homogeneous mass of mesenchyme. It is impossible to distinguish sharply between the proximal ends of the hyoid cartilage, Meckel's cartilage, the malleus, incus and stapes. Moreover, it is extremely difficult to draw a line sharply between the stapes and the auditory capsule—although apparently the stapes is much less intimately connected with the latter than with the other elements.

I have spoken of the mesenchyme anlagen of the auditory ossicles as being more or less homogeneous. There are, however, within the area certain differentiations which it is important to notice.

Fig. 8 (M) shows the Meckelian cartilage beginning to elongate cranio-caudally. This is the first indication of the malleus. The mandibular bar is carried up and into the malleus by a region of advanced pro-cartilage. The malleus is scarcely yet in the condition of pro-cartilage, but its outlines are fairly easy to distinguish by a somewhat denser arrangement of the mesenchyme. Proximal to the malleus, indications of pro-cartilage are observed to occupy a somewhat different position. Fig. 9 is a higher power view of a section distal to fig. 8, and shows the indication of an articulation between the malleus and incus. It will be seen, then, that pro-cartilage is found in the articular portion of the incus and in the distal end of the malleus; that pro-cartilage indications are found in the malleus, and pro-cartilage is present in the connecting rod between the malleus and the

* I found in examining sections of rabbit of seventeen days (length 13·5 mm.) and a mole (length 10·5 mm.) which Dr Beard kindly allowed me to use, that this connection between stapes and incus was less marked in the mole than in the horse, and lacking entirely in the rabbit. However, both individuals (mole and rabbit) were further developed than the four-weeks horse.

cartilaginous portion of Meckel's cartilage—and in no other portion of the auditory apparatus.

Inasmuch as there is, even in this early condition, an evidence of a separate development of the incus and malleus, the former can have no possible connection with the first *visceral bar*. For the first visceral bar is essentially a *cartilaginous* element and the incus is *at all times* a separate and distinct element. But, on the other hand, the incus is a structure derived from the first visceral arch—a condition shown by its position, and by its relations with other structures.

The incus lies lateral to the first visceral pouch and cranial to the point of flexure, in the territory peculiar to the first visceral arch. It lies also between the seventh and fifth nerves, the course and position of which I will describe later. Moreover, the articulation with the malleus is at this time a simple hinge articulation—of exactly the type that would be expected in a primitive jaw articulation.

It is therefore evident that the malleus may be a product of the upper end of the Meckelian bar but that the incus is not. Fuchs has attempted to show that the connection between malleus and the mandibular bar is secondary, finding the mandibular articulation distal to the malleus. This would tend to corroborate the view of Bardeleben and some others that the mammalian lower jaw is a complete homologue of the lower jaw of lower forms. That the connection between Meckel's cartilage and the malleus is not secondary and that the malleus is indeed developed from the proximal end of the first visceral bar, I shall now attempt to show.

I have already called attention to the fact that the connection between Meckel's cartilage and the malleus anlage was more advanced toward true cartilage than any more proximal portions of the products of the first visceral arch. Fuchs, in a diagram on page 152 of *Arch. für Anatomie und Physiologie*, Jahrgang 1905, Supplement-Band, has figured this Meckelian-malleus connection as in an earlier condition of development toward cartilage than either the malleus-incus anlage or Meckel's cartilage. My investigations do not afford evidence of anything approaching a similar condition. It is true that this connection is in an earlier stage than the more distal portion of the Meckelian bar, but it is *more advanced* than the malleus-incus portion or than the stapes. Now if the connection between the Meckelian bar and the malleus be fully formed in pro-cartilage before the malleus has reached a similar stage in its development, it seems impossible to consider that connection as secondarily acquired. Moreover, Fuchs has figured in the same diagram an anlage of the true mandibular

articulation *distal* to the malleus. I have failed to find this structure in any one of the four earlier embryos—the oldest of which shows the ossicular apparatus formed of true cartilage. Now if the mallear-incus articulation appears as the only mandibular articulation even at the time when the structures are formed of true cartilage, it follows that the later mandibular articulation must be both phylogenetically and ontogenetically acquired very much more recently. Therefore it seems clear that we must accept the reasoning of Gaupp and consider the quadrato-mandibular articulation as represented in the mammalia by the inco-mallear articulation. Further, the older view of Salensky, which Hertwig has followed, that both incus and malleus are split off, as it were, from the Meckelian bar, must be given up. The incus appears as a separate and distinct skeletal element developed from the proximal portion of the first visceral arch. The malleus appears at first, and continues to a very late period to be, a dependent portion of the first visceral bar.

Before considering the relations of the stapes it appears advantageous to give a brief account of the courses of the main branches of those cranial nerves pertinent to our discussion, leaving, however, the elucidation of the facts rather largely to an examination of the figures.

There are, of course, three nerves which bear an important relation to the auditory ossicles—the trigeminus, facial, and the vagus group, indicated in the figures by the numerals V., VII., and IX. The facial nerve exhibits the most important relations. The first branch (large superficial petrosal) is given off soon after the nerve leaves the ganglion. This branch runs medially and ventrally, and is of small moment here. The main trunk then bends slightly dorsal, along with the vena capitis lateralis, and runs almost directly caudal till it reaches the dorsal end of the hyoid bar. It is, moreover, dorsal to the tympanic cavity, so that all of the auditory ossicles must be ventral to it. At the proximal end of the hyoid it curves sharply latero-ventrally, sending off a small twig to the external ear and a larger branch—the chorda tympani. The chorda tympani skirts the hyoid bar and plunges into the mesenchymic mass surrounding the malleus and incus at the latero-caudal end, thence it runs cranio-medio-ventrally to emerge from the mesenchyme at the medio-caudal side of Meckel's cartilage. Beyond the juncture of the main trunk and the chorda tympani the main nerve runs latero-ventrally and becomes unimportant in the present discussion. Thus it will be seen that the auditory ossicles come to lie in a flexure of the seventh nerve—but that, nevertheless, they also lie in a position between the fifth and seventh nerves.

It has been shown that the stapes arises medial and dorsal to the first

pharyngeal pouch. Fuchs has nowhere figured or modelled the exact shape of this pouch as I have found it in the horse. The cranial reflexed portion which I have marked with an asterisk does not appear in his work. But the stapes anlage is in a very important position with regard to this portion of the first pharyngeal pouch. This position I have attempted to show in text fig. 4.

It will be recalled that the stapes is related through mesenchyme to two distinct structures, the hyoid cartilage and the incus. This connection between the hyoid and stapes represents the hyostapedial ligament of Huxley. Fig. 4 shows a model of the auditory ossicles of an embryo of seven weeks. It will be here seen that there is no such connection (marked ** in figures). In the stage represented in fig. 4 the entire auditory apparatus was cartilaginous. In an embryo of six weeks the same parts are of very young cartilage, and no hyostapedial connection is evident. It therefore seems plain that this connection never reaches the state of cartilage. Moreover, as Fuchs has shown, the distal end of the hyoid cartilage, as shown in figs. 1, 2, and 3, is not the true distal end, but the cartilage grows somewhat more dorsal than that point. The connection between hyoid and stapes, then, is to be considered as never attaining the dignity of a skeletal element. Moreover, a study of the drawings of the model and of fig. 11 will show that the stapes lies between the seventh and fifth nerves, while the hyoid arch lies between the seventh and ninth. The trigeminus ganglion has been omitted from the drawings of the models for the sake of simplicity, but it is shown clearly in fig. 11. In other words, the connection between stapes and the hyoid bar crosses the seventh nerve. This is shown in figs. 1 and 3.

But that being true, there is no other possibility than to regard, with Fuchs, the hyostapedial ligament as a structure whose relation to the auditory ossicles must be purely secondary. It appears the more evident when certain histological characters are considered. An examination of the figures will show that the area marked in the figures ** is histologically different from the surrounding mesenchyme. This point has been amply dealt with by Fuchs.

From the foregoing it is evident that the stapes has nothing at all to do with the second visceral bar. It is now pertinent to examine it with regard to its relations with the second visceral arch. We have seen that in its topographical relations the stapes shows characteristics which would seem to preclude the possibility of its being a product of the second arch—in that it lies medio-dorsal to the first visceral pouch, and between the seventh and fifth nerves.

Hertwig, following Rabl, has given as his main reason for considering the stapes to be in part a derivation of the hyoid that the innervation of the *M. stapedialis* is from the seventh nerve. Moreover, the stapes is perforated by a branch of the carotis interna. Kingsley has stated that the malleus and incus are obtruded into the tympanic cavity from in front, *i.e.* are pre-spiracular, while he leaves one to infer that the stapes is obtruded from the back. Disregarding the lack of accuracy involved in such a statement, it appears that in the position of the stapes and in its relationship with the visceral arches we have to deal with a problem which is complicated by an enormous amount of shifting of position that has occurred during the growth of the embryo, and that such a statement is not an answer to our question. Fuchs has come to the conclusion that the stapes is derived from the wall of the auditory capsule. With that view, which was the earlier view of Parker, Gaupp agreed in his earlier papers. More recently he has come to doubt the strength of that position. Hertwig concludes that the stapes is in part derived from the auditory capsule, and in part from the hyoid arch.

Opposed to the arguments derived from the position of the stapes are the arguments of Rabl as to the nerves, and the argument drawn from the position of the *arteria stapedialis*. Miss Platt has stated clearly that muscles are phylogenetically older than the skeletal elements. The primitive condition, then, shows the muscles inserted into the integument, at which points cartilaginous bars were later developed. Now, in none of the embryos of from four to seven weeks is there any indication of a stapedius muscle. The first appearance that I have found of this muscle is in an embryo of about three months (fig. 12). I have studied no individuals of ages between seven weeks and three months, so that I do not know the exact time when the muscle appears. But inasmuch as the muscle appears very much later than the skeletal elements, and since even at four weeks the muscles of the neck are indicated, it would seem that as a diagnostic character the muscle is of small value—at least, not to be compared with arguments from relative position.

Inasmuch, then, as the *stapedialis* muscle is innervated from the seventh nerve, and since the whole argument hinges on this point, I believe that, unless we have stronger corroboration, the fact that the stapes and *N. facialis* are related in this way must be regarded in the light of secondary evidence.

The internal carotid may be seen in figs. 1, 2 and 3, just skirting the auditory capsule, running cranio-caudally between the capsule and the first pharyngeal pouch, ventral to the latter. Just as it is about to pass the

cranial border of the first pharyngeal pouch, a minute branch is given off which runs cranio-laterally and perforates the stapes anlage, thence proceeding cranio-laterally between the vena capitis lateralis and the anlage of the malleus and incus. Now, were the arteria stapediales clearly the second aortic arch, one would be justified in considering that the perforation of the stapes by it was strong evidence in support of the claim that the stapes anlage were closely related to the second visceral arch. Tandler has shown that in the case of the rat and of man the arteria stapediales is derived from three different structures. From the point at which it arises from the aorta dorsalis *up to* the stapes, it is the dorsal portion of the second aortic arch. From that point on the artery with its rami consists of structures which are either developed secondarily or from the first aortic arch. Thus the portion which actually perforates the stapes anlage is not developed from the second aortic arch. Moreover, the stapes of monotremes and of certain marsupials is not perforated by an artery. So then, if the stapes is developed similarly throughout the mammalia, the relations between it and the arteria stapediales cannot be regarded as indications of its derivation.

Thus it is seen that there is at present no evidence weighty enough to overbalance that drawn from considerations of relative position, and we must therefore conclude that the stapes is related neither to the second visceral bar nor to the second visceral arch.

Fuchs arrived at this same conclusion, but went further and declared the stapes to be derived from the auditory capsule. We have seen that there is difficulty in visually separating the stapes from the auditory capsule at all points. Now let us agree that the stapes is developed from the mesenchyme of the capsule, and with that in mind examine some more fully developed stages.

Text fig. 2 shows a section through the stapes and incus of an embryo of six weeks. The stapes and incus are just between a condition of pro-cartilage and true cartilage, and the cochlear portion of the auditory capsule is in the same developmental stage. The fenestra ovalis is clearly to be seen closed by a mesenchymous membranous indication. The vestibular portion of the capsule is still undifferentiated mesenchyme. The stapes is seen to be *impinged against the closing membrane* of the fenestra. Text fig. 3 shows a section through the same region of a seven-weeks embryo. Unfortunately the sections are not upon exactly the same plane, so that the one shows the incus, the other the vestibular portion of the capsule. However, the relations between fenestra ovalis and stapes are similarly shown. All the parts are now cartilaginous. The stapes is

obtruded *into* the *fenestra ovalis*. By referring to fig. 11 one sees the obtrusion of the stapes into the fenestra carried a step farther, and the portion enclosed in the fenestra is expanding to form a plate. Contrary to Hertwig, the plate and ring portion of the stapes are not developed separately. Moreover, we have a gradual shifting of the relative posi-

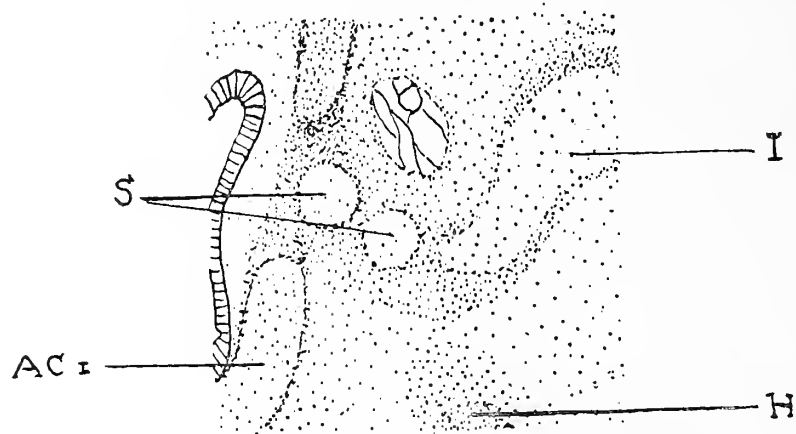


FIG. 2.

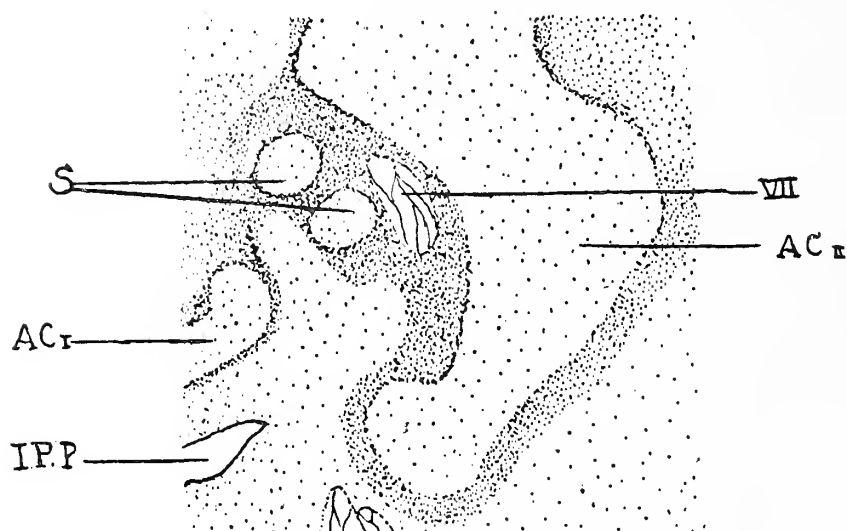


FIG. 3.

FIGS. 2 and 3.—S., stapes; A.Cr., auditory capsule, cochlear portion; A.Ch., idem-vestibular portion; VII, facial nerve; I. P.P., first pharyngeal pouch; I., incus; H., hyoid.

tions of stapes and capsule, the direction of movement being toward the latter.

If Fuchs is right the stapes must first be separated off from the auditory capsule and then afterwards push its way back into the fenestra.

Consider again that in the early anlage of the auditory capsule there is no indication of a fenestra. The inference is that the capsule in chondrifying simply leaves a space closed by mesenchyme, which later becomes the fenestra ovalis. There are a few sections which might indicate, as Fuchs

has shown diagrammatically, that the stapes is cut off from a protuberance of the capsule, but in the model this slight swelling is seen to form the caudal boundary of the fenestra ovalis and to *continue* beyond the point of contact of the stapes. Moreover in the four-weeks embryo the vena capitis lateralis lies between the external surface of the head and the auditory capsule, and at this stage the auditory capsule shows no indication of any mesenchymic differentiation. In the five-weeks embryo the auditory capsule is quite distinct from the stapes, except for a very limited area. And the stapes at this stage is seen to lie not medial to the vena capitis lateralis, but dorsal to it.

It will be recalled that in the four-weeks embryo the auditory capsule anlage lay between the seventh and ninth nerves. As development has proceeded, it is evident that the seventh and fifth nerves have come to lie more closely together, with a resultant shifting of the position of the capsule. But, as has been shown, the stapes arises in the area between the seventh and fifth nerves.

I regret my inability to examine additional stages between those represented by the two embryos whose ages are four weeks and five weeks, but it is highly improbable that there would have been any great relative changes in the position of the various anlagen. To derive the stapes from the second arch would involve carrying it around the first visceral pouch. To derive it from the auditory capsule would imply a complicated process which would have certainly resulted in some more definite indication than we have in my second (five-weeks) stage. It is not at all surprising that the mesenchyme seemingly connects the capsule with the stapes, inasmuch as whenever two structures are developed closely together, the mesenchyme of their anlagen tends to become very difficult of visual separation. Moreover, if the auditory capsule and the stapes were from the same anlage, the stages of their development towards cartilage and later bone ought to bear a certain amount of relation one to the other. But this is not borne out by the facts.

I have therefore come to the conclusion that the stapes cannot be developed from the auditory capsule. But if it is developed from neither the auditory capsule nor the second visceral arch, the question arises—Whence does it come?

The fact that the stapes lies medial and cranial to the first pharyngeal pouch, while it excludes the second arch, does not exclude the first arch from the formation of the stapes. For the stapes lies also *dorsal* to the first pharyngeal pouch, and *cranial to the point of flexure*.

Fig. 1 shows also that the incus, which is undoubtedly derived from the

first arch, also lies slightly dorsal to the first pharyngeal pouch. Again, the stapes is clearly in the area between the fifth and seventh nerves. It would appear from the model that the stapes were between the large superficial petrosal branch of the seventh and the main trunk which bends around the hyoid. But by reference to fig. 11 the stapes will be seen to lie between the facial nerve and the trigeminal ganglion just as truly as does the incus. Moreover, the stapes is connected to the incus very closely, as is shown in fig. 10.

In view of the evidence submitted I consider it to be clear that—

1. The whole ossicular chain is developed from the first visceral arch.

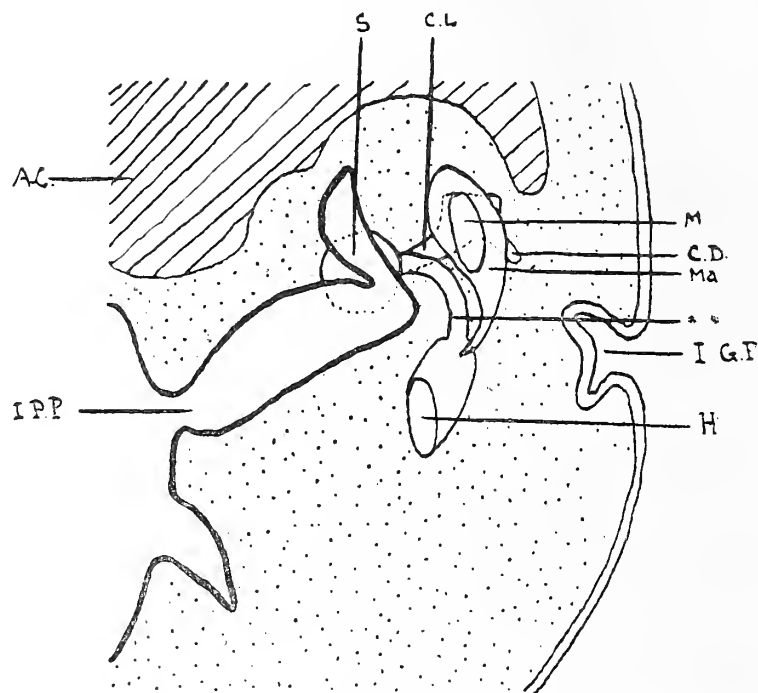


FIG. 4.—A.C., auditory capsule; C.L., crus longum incudis; G.I., gill furrow; C.B., crus breve incudis; H., hyoid; M., Meckel's cartilage; Ma., malleus; P.P., pharyngeal pouch.

2. That the malleus alone is related to the first visceral bar. These relations are shown in text fig. 4.

III.

There remains to discuss the possible homologies of the auditory ossicles.

Reichert's view that the malleus is the homologue of the articulare and the incus of the quadrate, so admirably fills all the conditions of an exact homology that one is tempted to let the discussion end there. But clearly the articulare can be but a portion of the malleus, owing to the fact that the former is a covering bone and the latter is not so entirely. With this view the majority of authors substantially agree, as Parker (II.), Gradenigo, Gengenbaur, Salensky, Hertwig, Kingsley, Wiedersheim, Dollo,

Rabl, Gaupp, and many others. As we have seen, the malleus and incus are derivatives of the first arch; their articulation satisfies the proper conditions; they are so similarly related to the chorda tympani and to the seventh nerve that I feel it unnecessary to argue at great length that they are homologous with the articulare and quadrate. Moreover, Gaupp's discussion of that question is admirable, and may be considered final. The hypothesis held by Gadow, Albrecht, in part by Fraser and some others, that either one of these two elements is a derivative of the second arch, will not bear close analysis.

Parker, followed by Gaupp's first position, and Fuchs, together with many other investigators, have believed the stapes to be derived from the auditory capsule either wholly or in part, thus making it homologous with the sauropsidan stapedia apparatus. Others, as Gadow, Albrecht, Wiederheim, and many more, have believed the stapes to be homologous with the hyomandibular of fishes. As to the conditions met with in the Amphibia, I know them only through the literature. Miss Platt gives for *necturus* the fact that the operculum arises independently of the auditory capsule, and Killian for *axolotl*, whilst Stöhr for *triton* and *siredon* derives the operculum from the anterior boundary of the fenestra. For the *Anura* he derives the operculum from a different portion of the auditory capsule. Fuchs, however, has I think clearly shown that the amphibian stapes is derived from the auditory capsule and the extra-columella from the hyoid arch. The conclusion is at once drawn that the mammalian stapes is on no account homologous with the sauropsidan operculum plus the accessory parts. Peter has shown in the development of the skull of *Ichthyophis glutinosus* that the stapes is closely connected to the quadrate. But the stapes-quadrato connection is in the vicinity of the quadrato-mandibular articulation, whilst the mammalian stapedo-quadrato connection is at an entirely different region.

Unless it can be shown, then, that the Amphibia develop two types of accessory auditory apparatus, it is impossible to consider the mammalian stapes as a representation of any structure found in the Amphibia and reptiles.

And inasmuch as the hyomandibula of fishes is a derivative of the second arch, it is impossible to conclude that the mammalian stapes is homologous with it; and the stapes cannot be a portion of the quadrate, because it is chondrified from a separate centre. Moreover, there are no other possible homologues, and therefore one is forced to conclude that the mammalian stapes has no homologue in the lower Vertebrata.

IV.

A brief summary of the principal conclusions to which I think my preparations lead is as follows:—

1. (*a*) The malleus, stapes, and incus are derivatives of the first visceral arch. In the case of the malleus and incus, this is evident from their position and relations with other structures. In the case of the stapes, the exact correlation of parts is more obscure, but it has been shown that, owing to twisting and compression, the stapes anlage has become shifted so that it lies dorsal and medial to the re-bent portion of the first pharyngeal pouch, but that it lies also cranial to the point of flexure. Inasmuch as in an early stage this point of flexure is a cardinal point in the topography of the region, the position of the stapes with regard to it, in addition to the relationship shown by the stapes with other structures, precludes the possibility of the stapes having been derived from the hyoid arch. But there is nothing in the position of the stapes nor in its other relationships to preclude the possibility of its having been derived from the first arch. And as the stapes has been shown to be independent of the auditory capsule, it is evident that it must be a derivative of the mandibular arch.

(*b*) The stapes and incus are at all times structures distinct from the Meckelian bar, being chondrified independently. The malleus, on the other hand, is continuous with the proximal end of Meckel's cartilage, and the connection between them is of a primary nature.

2. The stapes cannot be homologous with any structure found in the lower Vertebrata, for it is derived from the first arch, whilst the amphibian and reptilian stapes is either derived from the auditory capsule or from the second arch, and the hyomandibula of fishes is also derived from the second arch. Thus the mammalian stapes must be regarded as a structure peculiar to that group. The malleus and incus are, however, homologous with the articulare (in part) and the quadrate of the lower groups.

My thanks are due to Professor Cossar Ewart for the material for these investigations, and for many helpful suggestions, and it is from his laboratory that this communication is offered. It is a great pleasure to acknowledge my indebtedness to him and to Drs Beard and Ashworth of the Edinburgh University Zoological Department.

A grant towards the expenses of the investigation has been made from the Earl of Moray Endowment Fund of the University of Edinburgh.

LITERATURE.

1883. ALBRECHT, P., *Sur la valeur morphologique de l'articulation mandibulaire du cartilage de Meckel et des Osselets de l'ouïe*, Bruxelles.

1905. BARDELEBEN, v. K., "Der Unterkiefer der Säugetiere, besonders des Menschen," *Anat. Anzeiger*, vol. xxvi.

1905. BARDELEBEN, v. K., "Die Homologie des Unterkiefers in der Wirbeltierreiche," *Anat. Anzeiger Ergeb.*

1907. BARDELEBEN, v. K., "Zur vergleichenden Anatomie besonders Paläontologie des Unterkiefers der Wirbeltiere," *idem*, 1907.

1905. FUCHS, HUGO, "Bemerkungen ueber die Herkunft und Entwicklung der Gehörknöchelchen bei Kaninchen-Embryonen, etc.," *Archiv für Anat. und Physiologie*, Anat. Abthlg. (Supplement).

1906. FUCHS, HUGO, "Nachtrag zu meiner Arbeit. Bemerkung über die Herkunft, etc.," *idem*.

1907. FUCHS, HUGO, "Ueber die Entwicklung des Operculums der Urodelen, und des Distalidimus (*Columella auris*) einiger Reptilien," *Anat. Anz. Ergeb.*, 1907, p. 8, 34.

1888. GADOW, "On the Modifications of the First and Second Visceral Arches, etc.," *Phil. Trans.*, 1888, vol. clxxix., B.

1898. GAUPP, E., "Ontogenese und Phylogenese des schallleitenden Apparates bei den Wirbeltieren," *Merkle und Bonnet's Ergebnisse*, p. 990.

1905. GAUPP, E., "Die Nicht-Homologie des Unterkiefers in der Wirbelthierreiche," *Anat. Anz. Ergeb.*

1907. GAUPP, E. (Discussion of Fuchs' paper in 1907), *idem*.

1872. GEGENBAUR, *Das Kopfskelet der Selachier, etc.*, Leipzig, 1872.

1898. GÖPPERT, E., "Der Kehlkopf der Amphibien und Reptilien," *Morpholog. Jahrbuch*.

1887. GRADENIGO, G., "Die Embryonale Anlagens des Mittelohres; die morphologische Bedenkung der Gehörknöchelchen," *Mitth. a. d. Embryol. Inst. d. Univ. Wien*, Heft 1887.

1869. HUXLEY, "On the Representations of the Malleus and Incus of the Mammalia in the other Vertebrata," *Proceed. Zool. Soc.*, 1869.

1905. HERTWIG, O., *Embryology of Man and Mammals*, translated from the third German edition by E. L. Mark. London, Swan Sonnenschein & Co., 1905.

1880. HIS, *Anatomie Menscher Embryonen*, Leipzig, 1880.

1890. KILLIAN, G., "Die Ohrmuskeln des Krokodils, nebst vorläufigen Bemerkungen ueber die Homologie des *Musculus stapedius* und des *Stapes*," *Jenaische Zeitschrift für naturwissenschaften*, Bd. xxiv., N.F.B. xvii., 1890.

1890. KILLIAN, G., "Zur vergleichenden Anatomie und Entwicklungsgeschichten der Ohrmuskeln," *Anat. Anz.*, Bd. v.

1904-05. KINGSLEY, J. S., "The Mammalian Lower Jaw," *Proceed. American Soc. Anatomists*, 1904-05 (in outline, *Amer. Journ. Anat.*, vol. iv.).

1906. KINGSLEY, J. S., *Vertebrate Zoology*. New York, Henry Holt & Co.

1871-85. PARKER, W. K., Various papers in *Phil. Trans.*, in *Trans. Linn. Society*, vol. ii., and in *Trans. Zool. Soc.*, vol. ii.

1898. PLATT, JULIA B., "The Development of the Cartilaginous Skull and of the Branchial and Hypoglossal Musculature in Necturus," *Morph. Jahr.*

1898. PETER, K., "Die Entwicklung und funktionelle Gestaltung des Schädels von *Ichthyophis glutinosus*," *Morph. Jahr.*

1837. REICHERT, C., "Ueber die Visceralbogen der Wirbelthiere in Allgemeinen und deren Metamorphose bei den Vögeln und Säugethieren," *Archiv für Anat. u. Physiol. und Wissenschaftliche Medicin*, 1837.

1887. RABL, K., "Ueber das Gebiet des Nervus facialis," *Anat. Anz.*, 1887, Abt. 2.

1879. STÖHR, PH., "Zur Entwicklungsgeschichte des Urodelenschädels," *Zeitschrift f. Wissen. Zool.*, Bd. xxxiii.

1881. STÖHR, PH., "Zur Entwicklungsgeschichte des Anurenschädels," *idem*, Bd. xxxvi.

1880. SALENSKY, W., "Beitrage zur Entwicklungsgeschichte der knorpeligen Gehörknöchelchen bei Säugethieren," *Morph. Jahr.*, Bd. vi.

1902. TAUDLER, J., "Zur Entwicklungsgeschichte der Kopffartieren bei den Mammalia," *Morph. Jahr.*, Bd. xxx.

1877. WIEDERSHEIM, R., "Das Kopfskelett den Urodelen," *Morph. Jahr.*, Bd. iii.

1907. WIEDERSHEIM, R., *Comparative Anatomy of Vertebrates*, trans. by W. N. Parker. 3rd edition, founded on the 6th German edition. London, Macmillan & Co. Ltd., 1907.

EXPLANATION OF FIGURES.

A.C. Auditory capsule. 1, cochlear, and 2, vestibular portion.	H. Hyoid cartilage.
A.S. Arteria stapediales.	I. Incus.
Br. Barin.	Ma. Malleus.
C.B. Crus breve incudis.	M. Meckel's cartilage.
C.I. Carotis interna.	Ph. Pharynx.
C.L. Crus longum incudis.	P.G. Processus gracilis.
Ch. T. Chorda tympani.	P.P. Pharyngeal pouch.
E.A. External auditory meatus.	S. Stapes.
G.F. Gill furrow.	V.C.L. Vena capitis lateralis.
	Y. Eye.
A. Auditory pit.	

Figs. 1, 2, and 3. Model of the auditory region of a five-weeks horse embryo (length 15.5 mm.), magnified 41 diameters. The auditory capsule and the ventral portion of the first pharyngeal pouch are left incomplete. This model shows the mesenchyme areas, and not pro-cartilage or cartilage.

Fig. 4. Model of the auditory ossicles of a horse embryo of seven weeks' development (length 21 mm.), magnified 30 diameters. In this model the auditory ossicles only are shown complete.

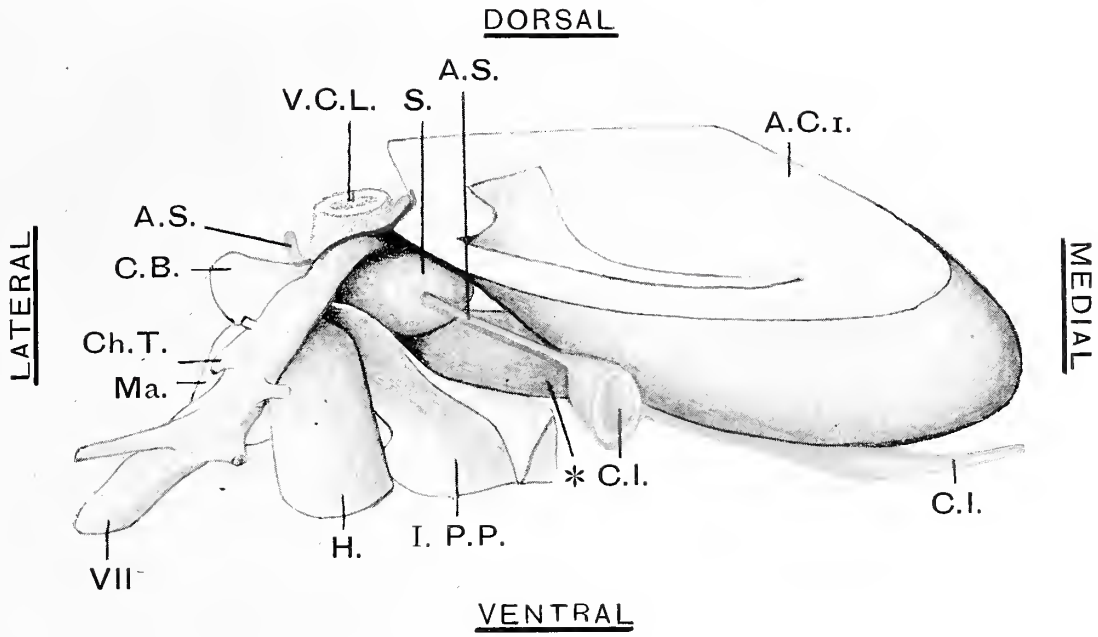


FIG. 1.

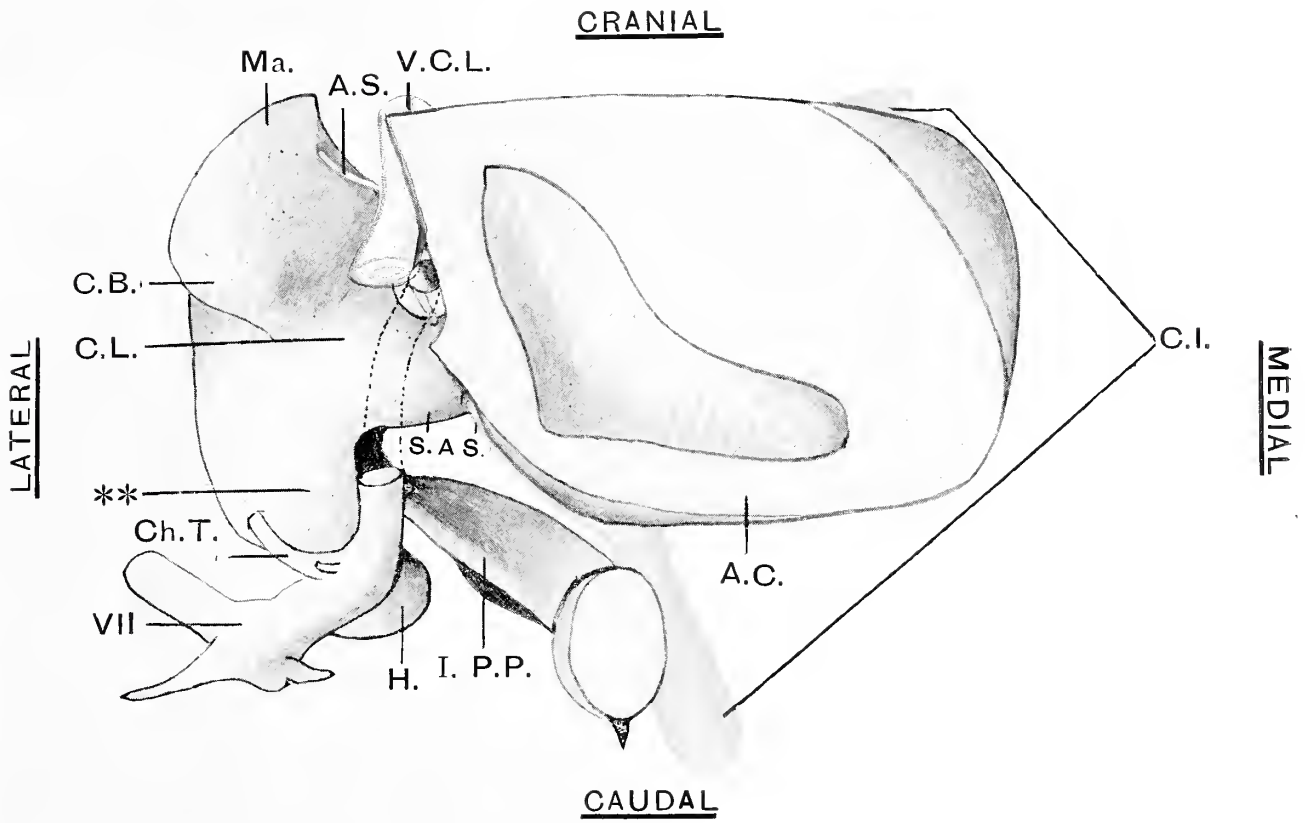


FIG. 2.

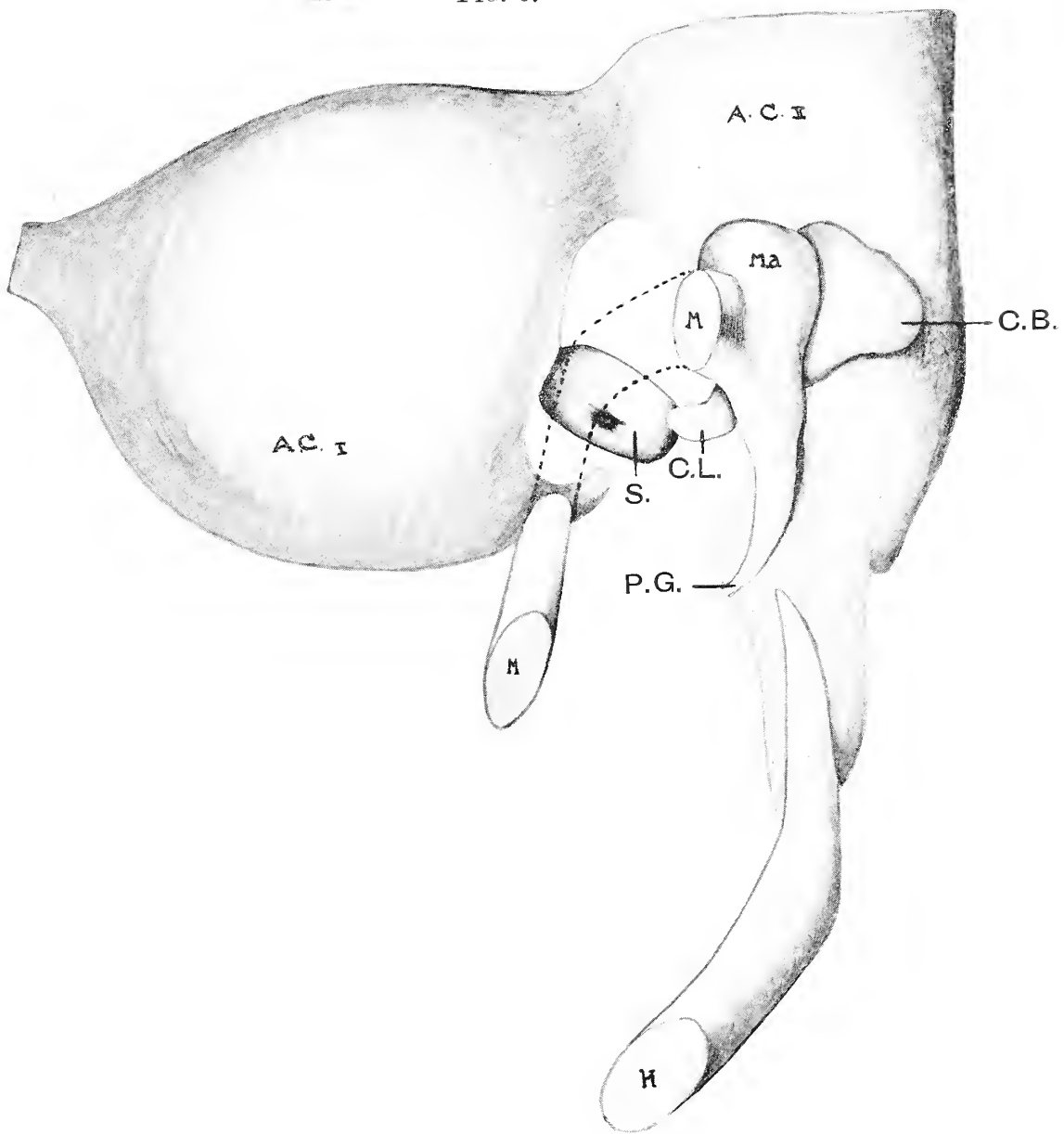
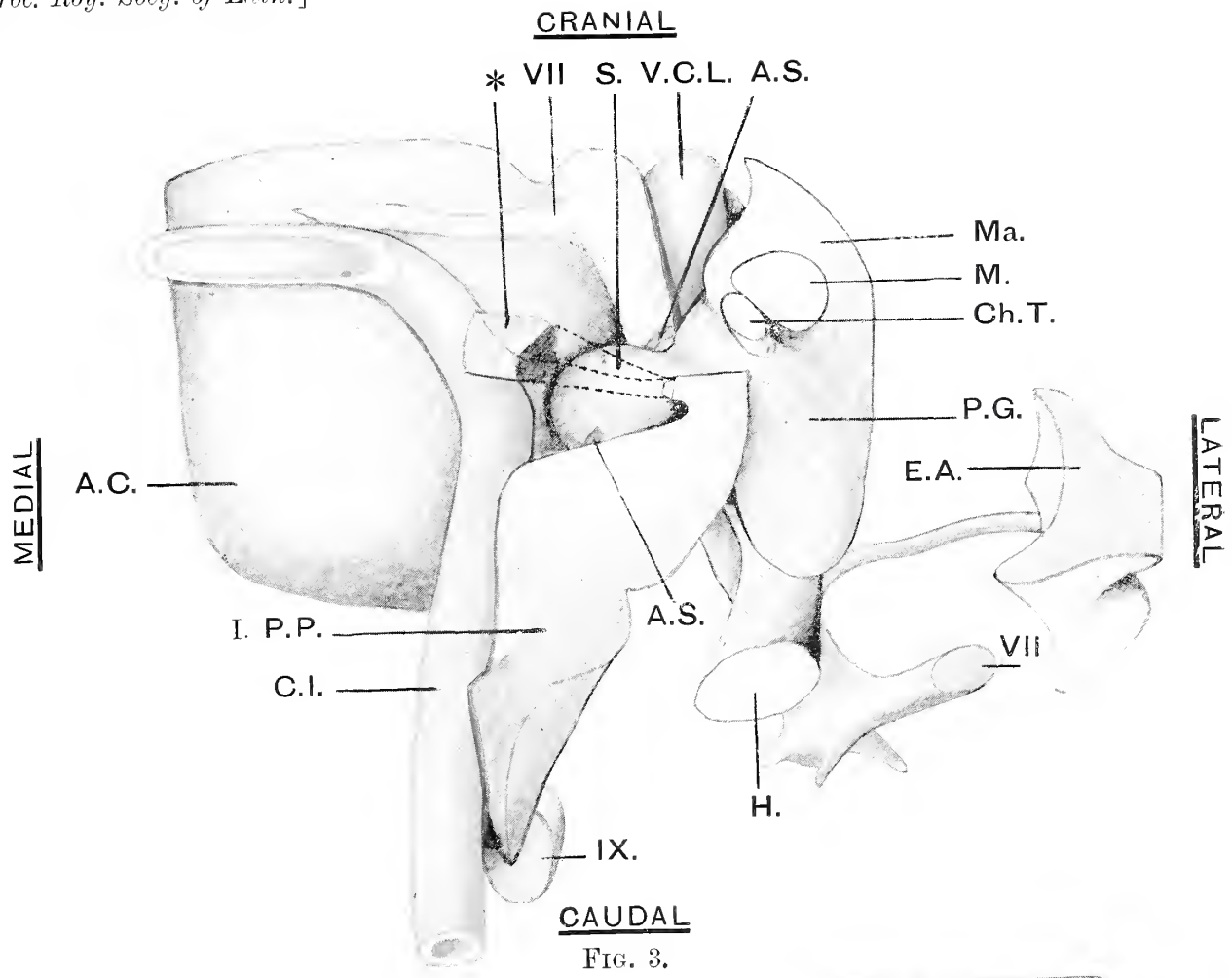


FIG. 4.

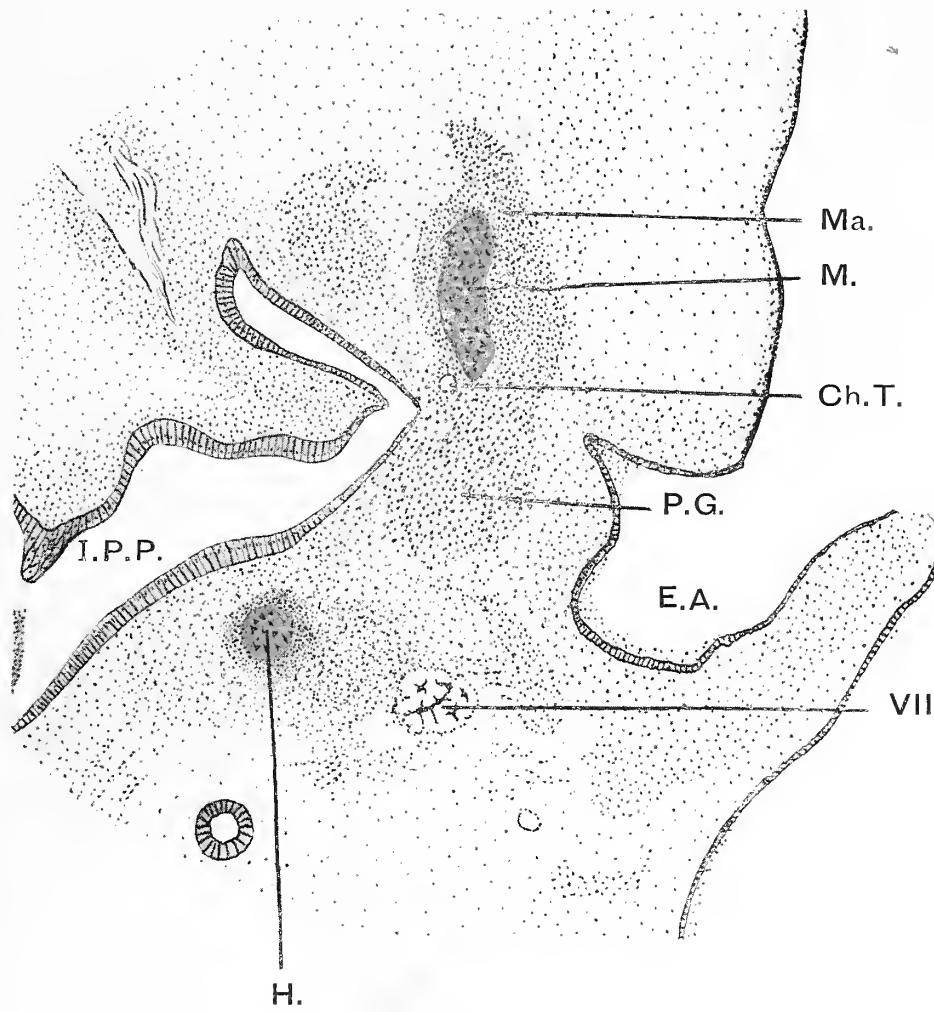


FIG. 8.

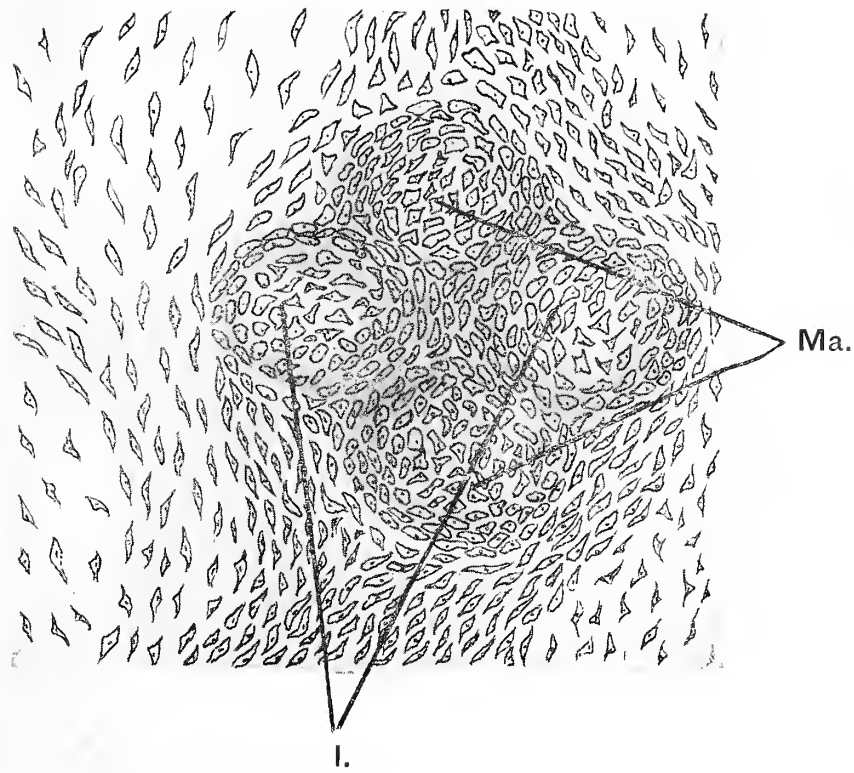


FIG. 9.

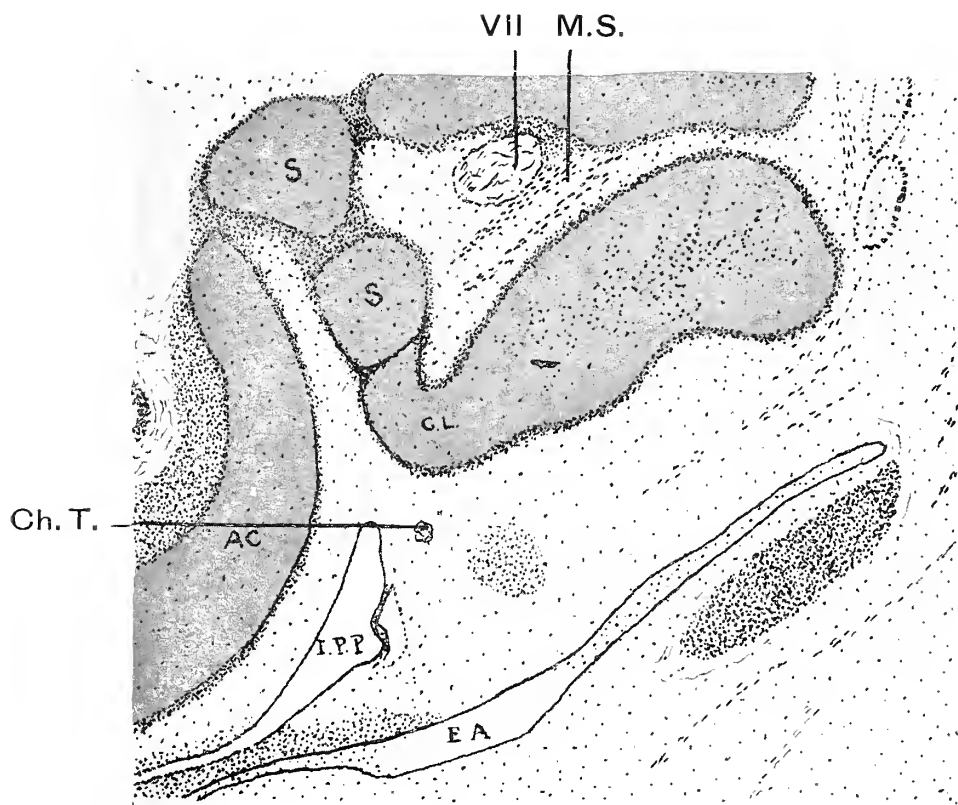


FIG. 12.

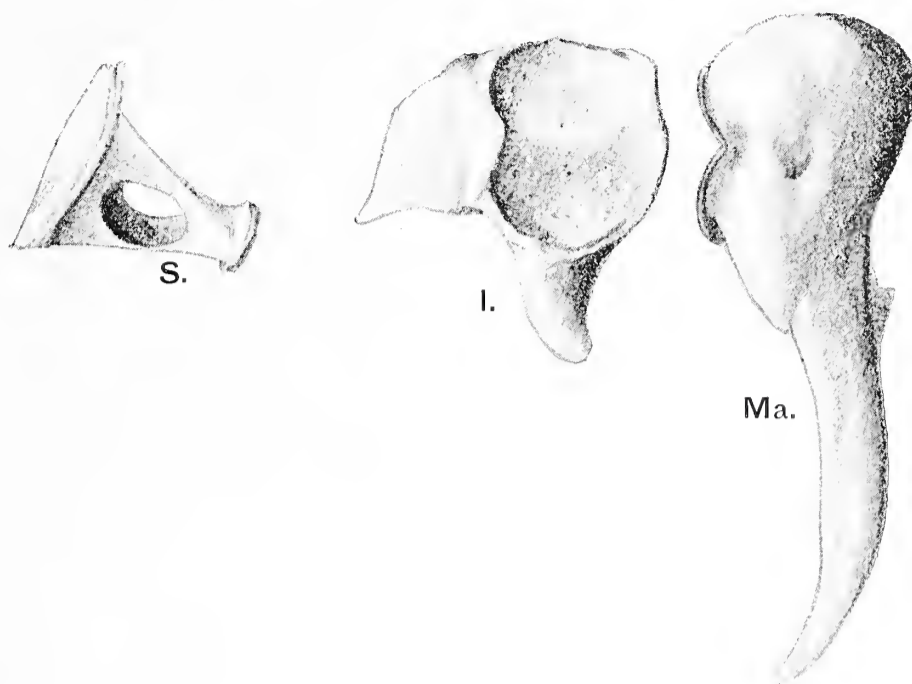


FIG. 5.

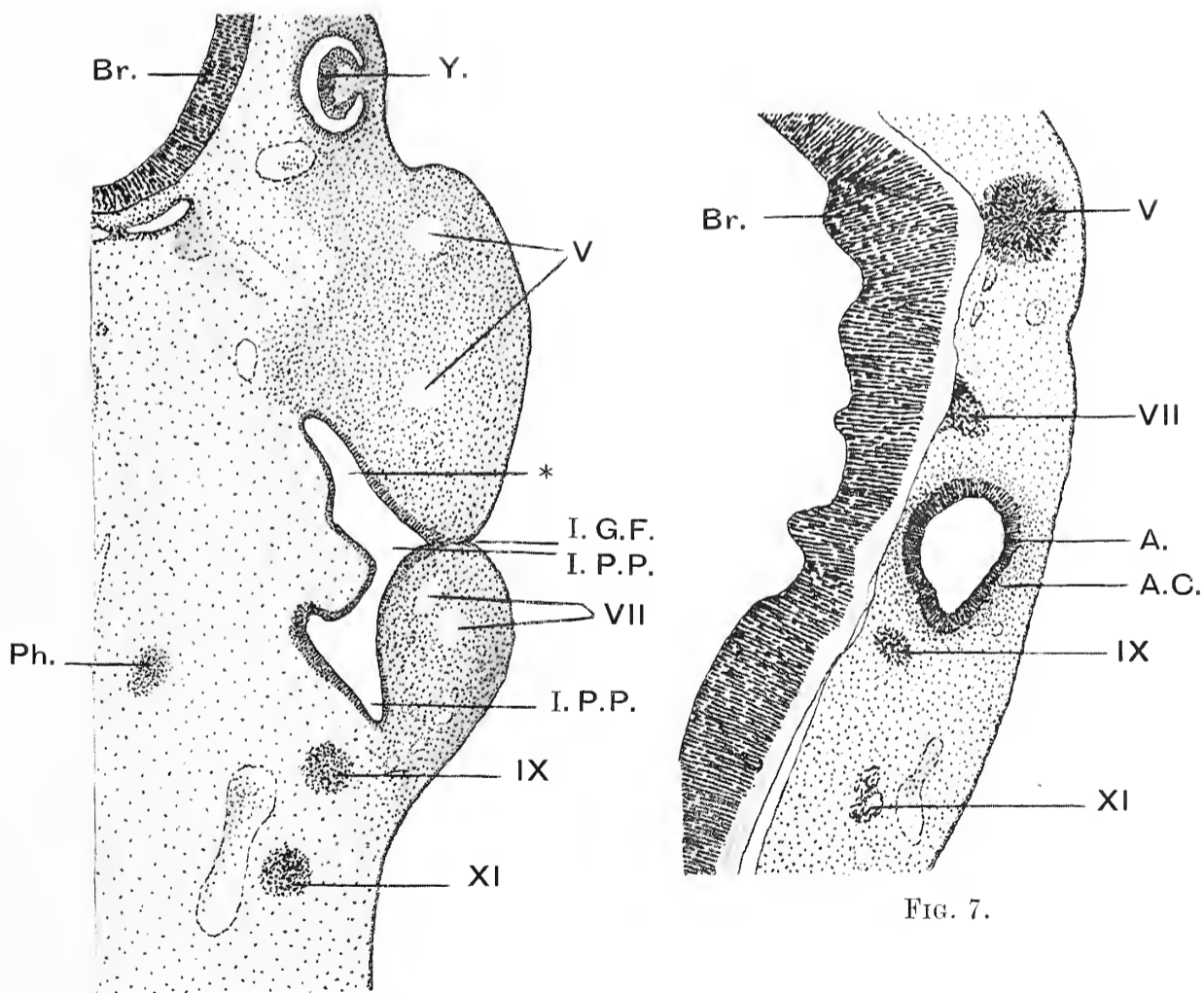


FIG. 6.

FIG. 7.

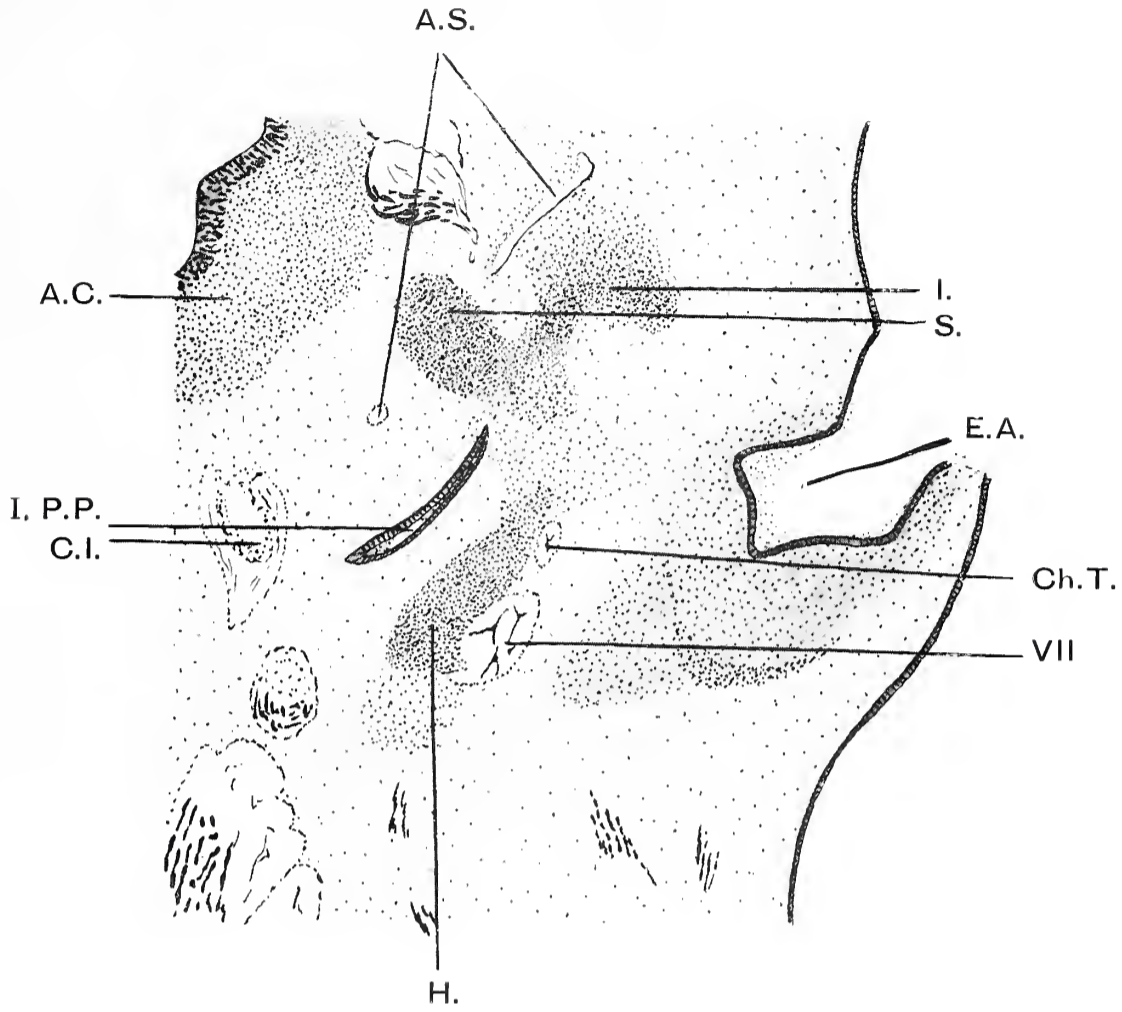


FIG. 10.

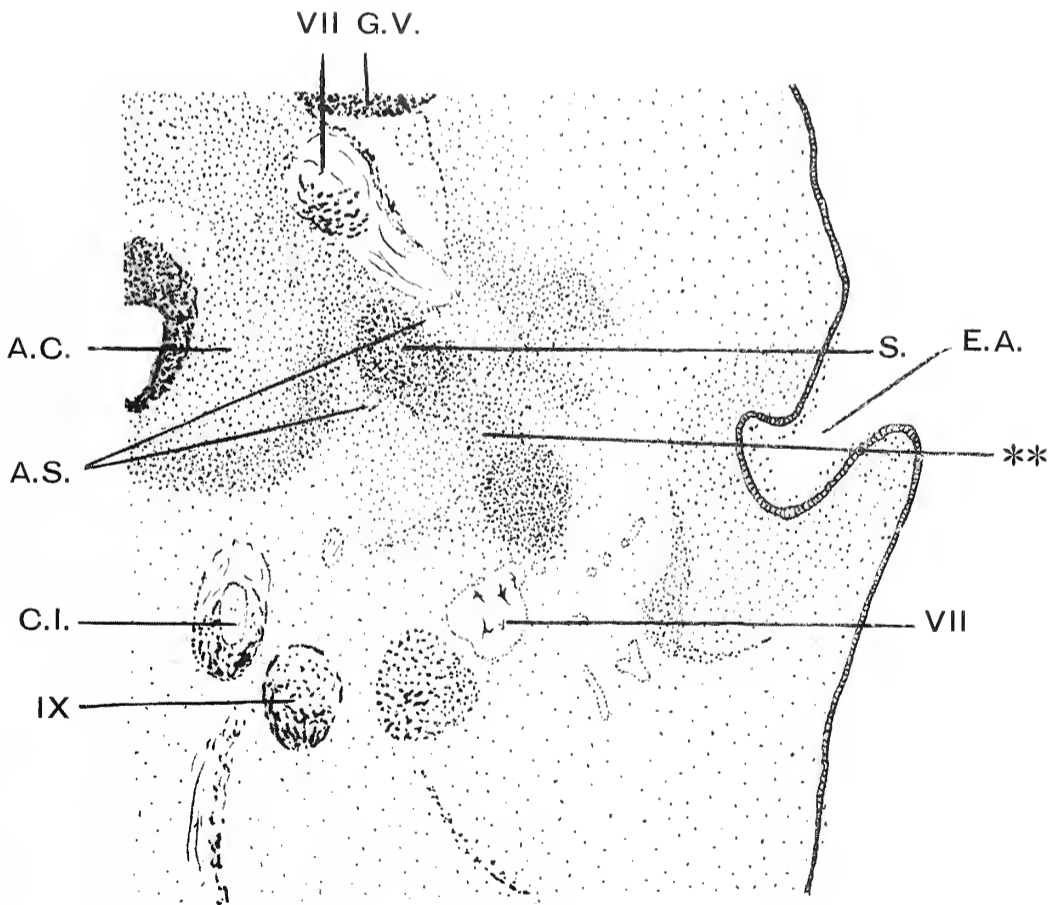


FIG. 11.

Fig. 5. Auditory ossicles of a horse at term (the bones are here fully ossified), magnified three diameters.

Fig. 6. Section through the first and second pharyngeal pouches and the first gill furrow of an embryo aged five weeks (length 12 mm.). $\times 32$ diam.

Fig. 7. Section through the auditory sac of the same embryo. $\times 54$ diam.

Fig. 8. Section through the proximal end of Meckel's cartilage and the anlage of the malleus of a horse embryo aged five weeks. Blue represents pro-cartilage. $\times 66$ diam.

Fig. 9. Section through the maleo-incus articulation of the same embryo. $\times 208$ diam.

Fig. 10. Section through the stapes and incus anlagen of the same embryo. $\times 42$ diam.

Fig. 11. Section dorsal to fig. 10 through the stapes anlage of the same embryo, showing the arteria stapediales. $\times 36$ diam.

Fig. 12. Section through the stapes and incus (crus longum) of an embryo aged about three months. The cartilage is shown blue. The stapes, incus, and vestibular portion of the auditory capsule show the beginnings of ossification. $\times 42$ diam.

(Issued separately September 9, 1909.)

XXXVI.—Dr O. Pettersson's Observations on Deep Water Oscillations. By E. M. Wedderburn, W.S. (With a Plate.)

(MS. received July 5, 1909. Read July 5, 1909.)

DR PETTERSSON has recently published a paper on "Gezeitenaehnliche Bewegungen des Tiefenwassers" (*Publications de Circonstance*, No. 47), in which he describes certain movements in the deep waters of the Skagerak observed in January, February, and March 1909 at the Swedish station Bornö, in the Gullmarfjord, lat. $58^{\circ} 24' N.$, long. $11^{\circ} 33' E.$ The observations were taken daily at 12.30 p.m. by means of soundings made through the ice. The diagram accompanying this paper is a reproduction of a diagram prepared by Dr Pettersson, and shows the observed variations in the position of the isotherms and isohalines. No observations appear to have been made on 7th, 8th, 10th, and 11th February, nor from 15th to 18th March, so that there are large interpolations on the diagram. Fig. 1 is a map of the Skagerak, the point of observation being marked by a cross.

There is a marked discontinuity in the density of the water of the Skagerak at this time of year at a depth which averages about 20 metres. The presence of this discontinuity is shown in all the observations. The following set of observations for 10th March is an example:—

Depth, Metres.	Temp. Celsius.	Density.
0	·05	1·00212
10	·42	1·02315
20	·85	1·02387
25	3·98	1·02675
30	5·41	1·02670
40	5·88	1·02701
49·5	5·95	1·02701

The observations show an oscillation in the level of the discontinuity with a period of about fourteen days, which Dr Pettersson seeks to explain as a long-period tide in the lower dense layer, and states that the moon seems to have a tendency to accumulate the deep water of the Skagerak against the eastern coast whenever it attains a high northerly or southerly declination, and suggests that a yearly oscillation may, in the same way, be produced by the influence of the sun. He justly adds, however, "It is difficult to imagine how this attraction can act in a different manner upon

two water-layers (the surface water and the deep water) in the same part of the ocean. Likewise, the idea of tidal waves of so long periodicity as those described in the annexed diagram existing in such a limited part of the ocean as Skagerak presents almost insurmountable difficulties, whether they are assumed to have the character of forced or free waves. On the contrary

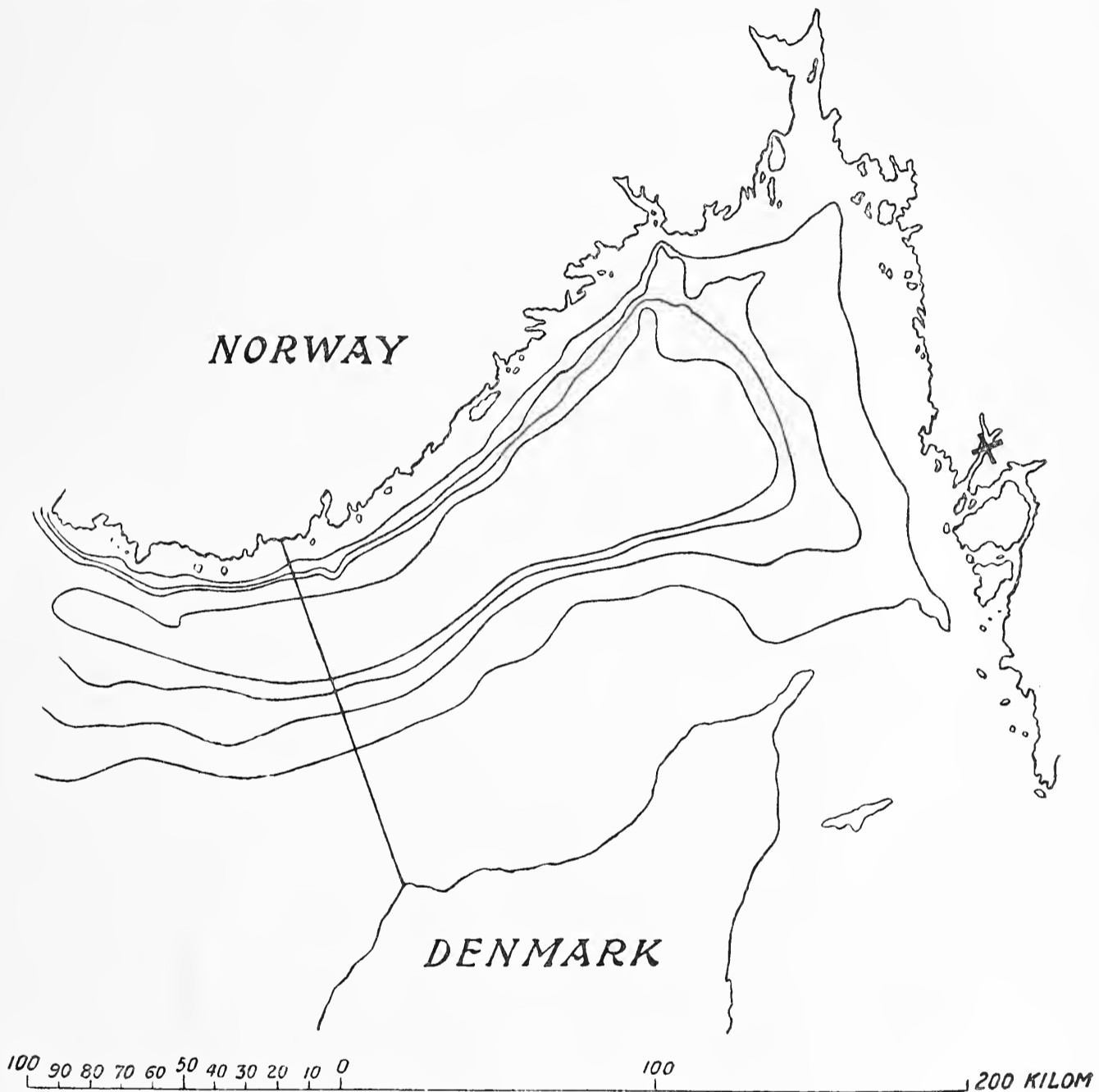


FIG. 1.
The contours show depths every 50 fathoms.

the whole character of the phenomenon bears out the idea that it is essentially an oscillation in the deep water of the ocean, the origin of which must be left to future investigation to discover."

The observations of the Scottish Lake Survey have shown the presence of oscillations in the lower water of deep lakes during the autumn of the year, when there is a layer of warm water of uniform temperature superimposed on the colder abysmal waters. When the boundary between the

upper and lower layers is sufficiently distinct, an internal (or temperature) seiche takes place in the lower layer, the period of which can be approximately calculated from the formula

$$t = \frac{2l}{\sqrt{\frac{g(\rho - \rho')}{\frac{\rho}{h} + \frac{\rho'}{h'}}}},$$

where ρ and ρ' , h and h' are respectively the densities and depths of the upper and lower layers, and l the length of the lake.* In Loch Ness the period of this internal seiche was about three days, in Loch Garry about twelve hours.

We also know from the elaborate observations on secondary undulations of oceanic tides carried out for the Earthquake Investigation Committee of Japan by K. Honda and others (*Journal of the College of Science, Tōkyō*, vol. xxiv., 1908), that in deep bays secondary undulations or seiches are of frequent occurrence, with a period given by the formula

$$t = \frac{4l}{\sqrt{gh}},$$

the node of the seiche being at the mouth of the bay, and the loop being at the end of the bay.

It was suggested to me by Sir John Murray that the oscillations observed by Pettersson might be analogous to those observed by the Lake Survey, and it is not a very large stretch of imagination to suppose that Pettersson's observations really show the presence, not of a long-period tide, but of a temperature seiche, having its node at the mouth of the Skagerak and its loop at the point of observation. A glance at the map will show that the basin is deep and well-defined, and that all the conditions favour an oscillation of the lower dense layer of water.

The period of such an oscillation should be given by the formula

$$t = \frac{4l}{\sqrt{\frac{g(\rho - \rho')}{\frac{\rho}{h} + \frac{\rho'}{h'}}}}.$$

The line drawn on the map shows what may be taken as the mouth of the Skagerak, and the length of the basin is about 200 km. The Japanese observers found that a mouth correction had to be applied in calculating the period of a seiche in a bay; and in the case of a basin of the breadth of the

* Cp. *Geogr. Journal*, vol. xxiv. p. 430; *Trans. Roy. Soc. Edin.*, vol. xlv. p. 420; *Proc. Roy. Soc. Edin.*, vol. xxviii. p. 1; *ibid.*, vol. xxix. p. 98.

Skagerak this correction is about 25 per cent. of the length,* making the effective length of the basin 250 km.

In calculating the period of a temperature seiche in the Skagerak by the above formula, I assumed as the depth of the upper layer 20 metres and as its density 1.023. For the depth of the lower layer, I took first a depth of 100 metres and then a depth of 200 metres. The mean depth of the basin, calculated by taking the square of the mean of the square roots of the depths marked on the chart, was about 180 metres, which gives for the depth of the lower layer 160 metres. It is therefore thought that the true period for the basin should lie between the two calculated periods. These were respectively 14.2 days and 13.9 days.

This period is in very close agreement with the period observed by Dr Pettersson, and to my mind there is little doubt that the oscillations are not tidal, but are analogous to the oscillations observed in Loch Ness and Loch Garry, and also in the Wolfgangsee by Dr Exner.

Dr Pettersson also gives a few observations made at Revsnoes, in the Great Belt, in July 1908, which indicate an oscillation of the deeper water with a period of about a day. They are too few to state the period of the oscillation with any certainty, but a calculation similar to the above shows that in a basin about 25 km. long there might be an oscillation with a 24-hour period. The basin of the Great Belt is very irregular, and it is difficult to know what may be the length of the basin in which an oscillation would take place. It seems likely, however, that the July observations also record the presence of an internal or temperature seiche. On several occasions the observations made by Sir John Murray in the sea-lochs of the West of Scotland showed oscillations in the bottom dense layer of water; and although the observations were not continued for a sufficient length of time to show whether the oscillations were periodic or aperiodic, it is quite likely that they, too, were due to an internal seiche.†

Pettersson's observations were discussed at a meeting of the "Challenger" Society held on 30th June 1909, and those present were practically unanimous in considering the tidal effect as a minor element. Dr Everdingen, however, pointed out that if there was a great interchange of water with the diurnal tides, similar curves would result from the fact that all the observations were made at the same hour each day, and thus at a later phase of the tide which would recur in about fourteen days. It is most unlikely, however, that the daily tides could produce anything like a range of 100 metres in the level of the lower layer of water; and the feeling of the meeting was that

* *Op. cit.*, p. 60.

† *E.g.* see observations in Loch Etive in 1888, *Proc. Roy. Soc. Edin.*, vol. xviii. p. 158.

the suggestion of an internal seiche, communicated by Sir John Murray, was the correct one.*

It is, of course, possible that the oscillations may be to some degree forced by tidal influences, and there is nothing new in the idea of a forced seiche, as Dr Pettersson seems to think. †

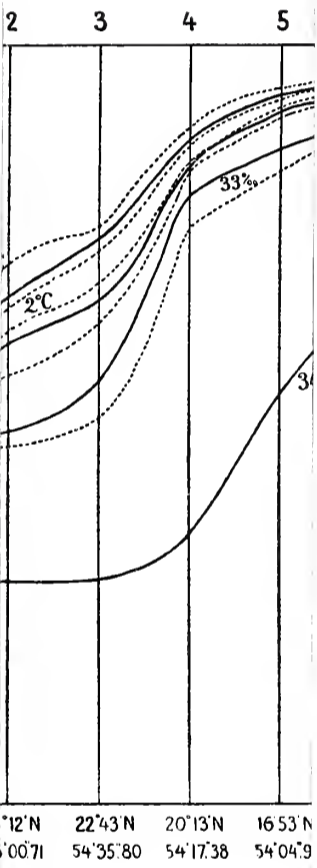
Pettersson adds, that whatever may be the main cause of the phenomena, there can be no doubt of their influence upon the fish-life and the fisheries of our seas. In 1877, G. Ekman discovered that herring shoals vanished suddenly from the Swedish fjords and from the coast bank whenever the ice-cold Baltic water accumulated there. In 1909, the greatest fish catches were made from 1st to 6th February and from 20th to 23rd February, when the warm and salt deep water had its periods of flow. In the ebb periods the fishery was almost nil, and ceased altogether about 27th February.

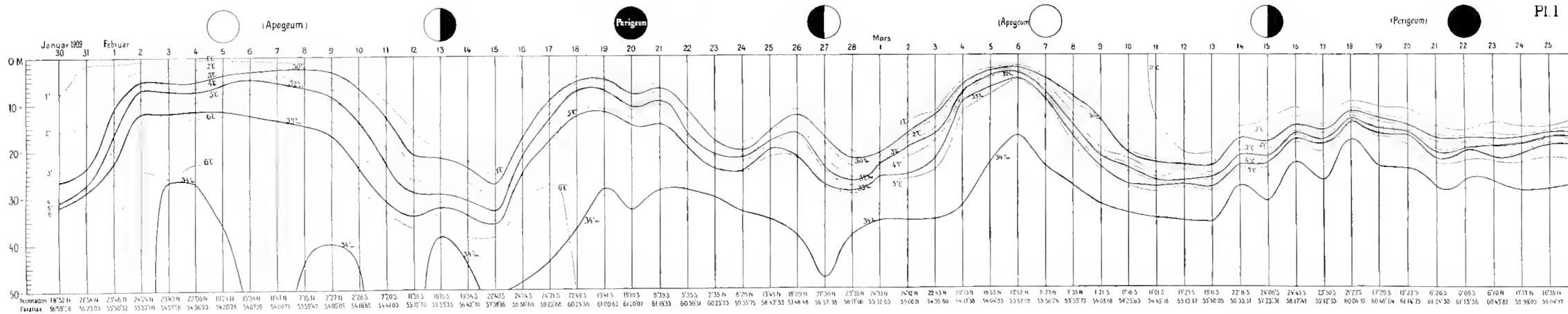
It may be some satisfaction to people who think that scientific work should not be undertaken without a practical end in view, to know that the temperature observations made by the Scottish Lake Survey may bear fruit in a better understanding of the movements of herring shoals.

* I am indebted to Dr H. R. Mill for a report of this meeting.

† See Chrystal, "Hydrodynamical Theory of Seiches," *Trans. Roy. Soc. Edin.*, vol. xli., part iii., p. 608, 1905 ; also *ibid.*, vol. xlvi., part iii., p. 514.

(*Issued separately August 14, 1909.*)





XXXVII.—Mendelian Action on Differentiated Sex. By D. Berry Hart, M.D., F.R.C.P.E., Lecturer on Midwifery, Surgeons' Hall, Edinburgh; Carnegie Research Fellow. (From the Laboratory of the Royal College of Physicians.)

(ABSTRACT.)

It has long been known that the male and the female human genital tract contain not only organs characteristic of their sex proper, but also certain parts of the opposite sex in a less developed but yet perfectly definite form. Thus the female genital tract is made up of, not only its characteristic organs, the ovaries, tubes, uterus, etc., but also the epoophoron (parovarium) and its duct, the equivalent of the epididymis and ductus epididymis of the testis. In the same way, the human male has his characteristic sexual organs and also the appendix testis and prostatic utricle, the representatives of the fimbriated end of the Fallopian tube and of the lower end of the vaginal tract (hymen mainly, but varying).

The significance of these facts has not been hitherto definitely investigated, and it occurred to me, in the course of a study of Mendelism in relation to the nature of sex, that the genital elements already defined might be considered as potent and non-potent, or as dominant and recessive in Mendel's terminology, and as elements on which Mendelian action took place and could thus be studied.

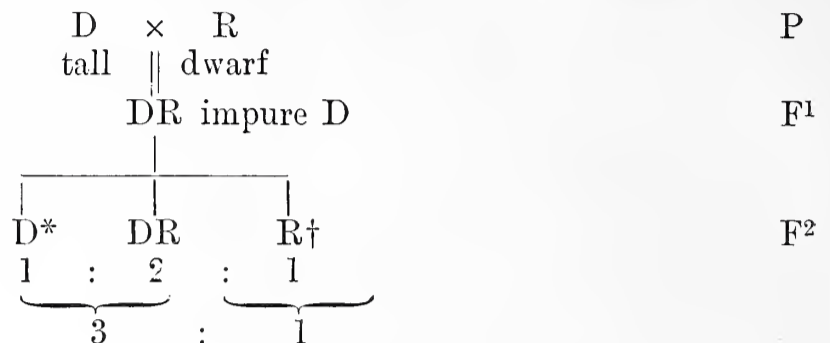
Mendel showed in his variation experiments with the eating-pea, that if peas with each one contrasted character,* such as tallness and dwarfness, the other characters being common, were crossed, all the plants were tall in F^1 , while in the subsequent selfed generations the tallness and dwarfness segregated in the ratio of $D:DR:R$ as $1:2:1$. This means that of the plants one quarter bred true to tallness (D), one quarter to dwarfness (R), while one half (DR) gave always $DR:R$ as $3:1$. It was found in the crossing experiments that it made no difference in the results which plant, tall or dwarf, was used for pollination.

Mendel described the tallness as a dominant unit-character, D ; the dwarfness as recessive, R ; while the tall plant giving tall and dwarfs $3:1$ is termed an impure dominant, DR .

* Mendel chose seven characters, of which I select the one given above for illustration.

The results can therefore be tabulated as follows:—

If D be the term for dominant, R that for recessive, DR that for impure dominant, P for the first crossing plants, and F¹–Fⁿ for the subsequent generations, we have



Mendel established, therefore, that the qualities which make up a plant may be considered as unit-characters, and that the ones selected for study, whether single or coupled, were autonomous, that is, did not blend, and ultimately segregated into D:DR:R as 1:2:1. DR always gives D:R as 3:1.

Thus in the tall-dwarf crossing, the plants in the subsequent generations were not intermediate in height between tallness and dwarfness, but segregated the parental selected unit-characters practically unaltered.

To explain this, Mendel advanced his theory of gametic segregation. He supposed that his unit-characters of tallness and dwarfness were segregated pure in the gametes of the tall and the dwarf plant respectively. He stated his theory in the simplest case as follows. If *A* be the dominant character, *a* the recessive, and *Aa* the impure dominant (Mendel's hybrid), then $A + 2Aa + a$ gives the terms in the series for the progeny of the impure dominants of two differentiating characters. $(A + a)(A + a) = A^2 + 2Aa + a^2$, i.e. $A + 2Aa + a$; the square being neglected as not numerically efficient. In the same way, where more than one unit-character was used, the results of the crossing agreed with the algebraic expectation.

I wish now to show that in human fertilisation we have a Mendelian action, but one, in my opinion, not in strict accordance with the views of Mendel. I must first define certain terms to be used.

The ovum and the spermatozoon are termed gametes; and, as I hold there are two kinds of each, I speak of a male sex gamete, a male non-sex gamete, a female sex gamete, and a female non-sex gamete. For the adult or developed plant or animal I use the term holophyte. The fertilised ovum is spoken of as the zygote; and, as sex is considered to be determined as soon

* Breeds true to tallness, and is an extracted dominant.

† Breeds true to dwarfness, and is an extracted recessive.

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[Continued on page iii of Cover.]

as fertilisation is completed, I speak of a male zygote and a female zygote. The proof for each of these statements will come up presently.

I further employ Weismann's terminology and views to a certain extent. Weismann has shown that the development-results in the adult (its determinates) are represented causally in the gametes and zygote by "determinants," and that there is continuity of the germ plasma. His term "id" for the group of determinants necessary in the gametes or zygote to produce a holophyte I also find convenient. Mendel's and Weismann's terminologies enable us to state and discuss this question—a most important point—and the terminology must be considered as analogous to an algebraic one. In the human race sex has differentiated. The zygote necessarily contains the determinants for the dominant (potent) and recessive (non-potent) sexual determinants, and this makes it an impure dominant, as in F^1 , *i.e.* it contains D and R determinants combined, and never normally segregated, I hold, in subsequent development. The human zygote being an impure dominant and produced by the union of male and female gametes, we have to consider the number and natures of the gametes producing such a zygote. For long it has been considered that one gamete from each parent was necessary, but the existence of dimorphic male and female gametes has shaken this belief. Many of the animals lower in the scale than man have two kinds of male gametes, one of them having in insects an extra chromosome (McClung), and some have dimorphic female gametes. In man there are dimorphic spermatozoa, but it is disputed as to whether the ova are dimorphic (Russo, Heape).

It has also long been held that sex was not determined at fertilisation, but later in the development of the embryo. This, however, is erroneous, and there is no evidence that the human zygote is hermaphrodite or indifferent in sex at any early stage.

It has been urged by some able observers (Beard, Castle, and many others) that there are male and female eggs: this, however, makes an organism hermaphrodite, and the male gamete contain no sexual determinants, and is incompatible with the true view as to the origin of the gametes.

For reasons that will presently appear, I consider there is the highest probability that in human fertilisation a sex and non-sex male gamete, and a sex and non-sex ovum are concerned: that for the formation of the male zygote a sex male gamete and non-sex female gamete have united; for the formation of the female zygote a sex female gamete and non-sex male gamete. This makes sex determined at fertilisation, and necessarily gives the 50 per cent. of the two sexes.

The question of the origin of the gametes has now to be considered. This has been most studied as to the origin of the female gamete, the ovum. The opinion currently held is that it originates from a germ epithelial cell. One admits, of course, that this view has done good service as a stop-gap, but there is really no fact in its favour. It makes a gamete originate from a somatic cell whose fellows, not so transformed, become ordinary protective epithelium. No satisfactory intermediate stages have been noted, and in view of the great fact that each gamete contains a certain proportion of heredity determinants, it is at present impossible to see how a gamete, the most highly organised cell in the body as to function, could arise from a germ epithelial cell. This view of its origin, further, gives no clue to the understanding of how a gamete becomes so specially endowed.

A more promising view of the origin of the gametes is that they are ultimately derived from the primitive germ-cell mass from which the primitive germ cells are first formed. These primitive germ cells become reduced to gametes—spermatogenesis and oogenesis. The P.G.C. mass is set aside from the zygote at its very earliest period. The zygote is thus divided into a somatic portion containing one id, and the P.G.C. mass, necessarily containing thousands of ids. The former is thus the somatic portion of the zygote, giving rise to the individual, while the latter is the propagative part, the stirp in Galton's phrase. The holophyte is thus merely the trustee of the propagative part. In the stages of zygotic development subsequent to the formation of the P.G.C. mass, the P.G.C.'s travel back into the somatic part *viâ* the body-stalk (yolk-stalk) and blastoderm before the embryo is differentiated as such, and mobilise on the Wolffian ridges, thus ultimately forming in each case the sexual gland, ovary or testis.

Much detail on this point has yet to be worked out. Most important deductions can be made from this view :

1. The P.G.C. mass is really an unreduced part of the zygote, and is thus zygotic.
2. Each P.G.C. is potentially a zygote.
3. The gamete is a reduced P.G.C., and this reduction normally prevents zygotic development of the P.G.C.
4. To regain the power of zygotic development, gametes of separate adults must unite, the male gametes with the female gametes; and thus fertilisation is gametic variation.
5. The gametes contain certain of the heredity determinants, because they are derived directly and by reduction from an unreduced part of the zygote, the P.G.C. mass.
6. The male zygote in all likelihood also gives off a primitive sperm-cell

mass, just as the female zygote gives off the primitive germ-cell mass; and from these, as stated in 3, the gametes arise.

The history of the development of this view of the zygotic origin of the gametes and its long neglect are of great interest.

Owen in 1849 made a remarkable observation. In his well-known paper on parthenogenesis he stated that "not all the progeny of the primary impregnated germ cells are required for the formation of the body of all animals: certain of the derivative germ cells may remain unchanged and become included in the body which has been composed of their metamorphosed and diversely combined or confluent brethren: so included, any derivative germ cell or the nucleus of such may commence to repeat the same process of growth." This was a striking observation, and it is the irony of investigation that this epoch-making observation should have been neglected for more than half a century while his contribution to parthenogenesis, still an obscure and doubtful theory, should alone have attracted great attention and been constantly quoted. It was only in 1891 that Eigenmann independently described the same phenomenon in *Cymatogaster segregatus* and thus confirmed Owen's work. Since then observations have been fairly numerous and important. Balbiani, Boveri, Beard, F. A. Woods, Ingalls, and others have substantiated and extended Owen's discovery in their views of the continuity of the germ cells. Ingalls's observation is of importance, as the primitive germ cells in a 4.9 mm. human embryo were found, not on the Wolffian ridges, but *en route* for them and arrested at the root of the mesentery beneath the coelomic epithelium. Beard especially has demonstrated and fought for this view.

This origin of the germ cells first shown by Owen may be combined with Weismann's great generalisation of the continuity of the germ plasma under the term of the Owen-Weismann law, or continuity of the germ cells. This means that the gametes are a reduced part of the primitive germ cells, these arising from the primitive germ-cell mass, a non-reduced portion of the original zygote. The gametes are thus directly zygotic in their origin, are not derived from somatic cells, and they thus carry on the race pure and not influenced by the "soma" proper.

This being the nature of the gametes, we have now to consider if we can in any way find out how the determinants of heredity, using this term in a general sense, are allotted to the dimorphic gametes. What contribution does each gamete bring to the zygote it helps to form?

For this purpose a source of information is present which has been ignored by almost all observers with the exception of Wilms and Beard, viz. ovarian and testicular dermoids and the dermoids found elsewhere in

the body. I take up this question in regard to ovarian dermoids with which I am familiar, but shall add some remarks in regard to the less common testicular dermoids.

The ovarian dermoid is a quite common and familiar tumour. It is usually cystic, and contains a creamy fluid often like salad dressing, and may have hair, teeth, brain, brain tissue, alimentary canal—in fact, samples, as it were, of all the germ layers, and not mainly of the ectoderm, as is usually urged. There is occasionally a more or less solid form, the teratoma; and this, as Wilms pointed out, is usually an imperfect *anterior* part of an embryo. Wilms and others have shown that the tissues are normal in structure. A very important teratoma removed by operation by Cullingworth and examined by Shattock is a most remarkable specimen. It is preserved in the Hunterian Museum, London, and is well worth a visit. Two limbs are present, and a peritoneal cavity with a blind coil of intestine. There is a short rudimentary spinal column with a pelvic girdle, and in each of the lower limbs there is an osseous element filled with fatty marrow. Between the limbs are two distinct labia, between these a distinct depression, and behind them a perineal raphe.

I must next draw attention to a very important fact as to all dermoids, viz., they contain no evident genital organs proper, and, so far as any observer has as yet noted, no microscopical elements pointing to their distinct existence. There is no doubt as to such teratomata not being the product of immediate fertilisation, as they are found in virgin women, and I have, like other operators, removed them from children before puberty. How, then, do we get such remarkable productions? I have already stated that the primitive germ cell is a zygote owing to its origin, and must contain an id: the gamete is a reduced primitive germ cell. A primitive germ cell can therefore form an embryo, but never does so, owing to its apparently constant reduction to a gamete and its loss of the power of zygotic development. If, however, a gamete has a certain number of determinants and is imperfectly reduced, it may retain the power of zygotic development and thus form part of an embryo.

Now in the ova I have suggested that we have two kinds, a sex ovum and a non-sex ovum; and for the reasons already given we may consider a teratoma as a non-sex ovum which by an imperfect reduction has retained the power of zygotic development not normally possessed by a gamete. The teratoma never contains all the parts of the anterior part of an embryo; and this is significant, as we shall see.

The same applies to testicular teratoma. In a recent paper Okhubo analyses the cases in the Prag collection, 11 in number, and also 107

other cases in literature, and to his valuable paper I must refer those interested.

Before stating an explanation of dermoids, I may mention that chorio-epitheliomata, the malignant tumours associated with the hydatid mole, have been found in teratomata by Schlagenhauser, Ritchie, and others.

Many views have been brought forward as to the origin of teratomata, and the one I advise as a working theory is that it is derived from a non-sex male or female gamete which has retained its power of zygotic development. The facts already stated as to the primitive germ cells passing after their formation from the P.G.C. mass by the yolk-sac and blastoderm into the embryo and their occasional arrest *en route*, helps one to understand their rare position outside the ovary and testis.

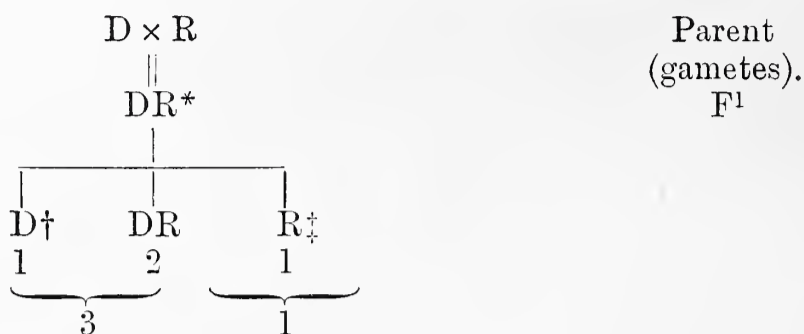
If thus a teratoma like Shattock's is the anterior part of an embryo, *i.e.*, in the main somatopleuric, and if it arises from a non-sex ovum, this makes the non-sex ovum contain the determinants of the somatopleuric part and the sex ovum the determinants of the genital organs and other parts of the splanchnopleure. This supports the popular belief that boys get brain and body from and through the mother, their sex and splanchnopleure through the father. The converse holds good for girls. A teratoma is only *part* of the anterior part of an embryo; but this is not inconsistent with the above supposition, and indeed boys do not follow the mother and girls the father fully in the respects defined above. This means that the father also contributes in part to the boy's somatopleuric element, the mother to the girl's.

I think this line of inquiry as to teratomata very promising, and that a particularly full examination should be made of all suitable specimens.

The human zygote, male and female, contains the determinants, *inter alia*, for the potent and non-potent organs. Thus, sex has differentiated from a hermaphrodite condition away far back in the invertebrata. Mendelism acts on differentiated sex, and is evident owing to the difference between the potent and non-potent organs.

The potent and non-potent determinants are, however, combined and inseparable in the human zygote, and thus in twinning of one zygote we always get an equal division of sex determinants.

It is different in some animals, and in black cattle we get male twins, one potent and the other sterile—the free-martin. In the potent twins the potent sex determinants have been segregated; in the free-martin, the non-potent ones. Thus we may arrange the relations of sex differentiation to the Mendelian scheme as follows:—



I am accumulating facts showing that Mendelism acts in producing sex in bees. How extraordinary current explanations are in this question of bee sex may be understood when I state that the drone, a perfect male, is supposed to arise from an unfertilised egg of the queen—and this in view of the fact that in crossing with a black (English) queen and an Italian drone (yellow-striped) some of the progeny show yellow stripes. The view of the origin of the drone from an unfertilised egg is neither a fact nor a theory, but a *reductio ad absurdum*.

The following are the main conclusions of this inquiry:—

1. The human zygote is an impure dominant of F^1 in Mendel's scheme.
2. Two varieties of gamete—male and female, sex and non-sex—are required to produce it.
3. The dominant and recessive determinants of sex are united in the sex gamete, and do not segregate normally.
4. It is probable that the somatopleuric determinants are in the main present in the non-sex gamete; the sexual and splanchnopleuric ones in the sex gamete.
5. A teratoma most probably arises from an imperfectly reduced non-sex gamete.
6. The free-martin is not a sterile cow when the potent twin is a bull, but a sterile bull with the recessive sexual determinants segregated in it. It is an extracted recessive.
7. Dominance of a character probably means that it is expressed in the soma, while at the same time the recessive character is secluded in the propagative part of the same plant, but is not expressed in the soma. It appears in the soma in a later generation; and when the plant breeds true, the recessive or dominant character is present pure both in the propagative and somatic part.
8. The theory of gametic segregation is doubtful.
9. The gametes are derived, not from the germ or sperm epithelium,

* The equivalent of the human zygote.

† The equivalent of the potent twin in black cattle (extracted dominant).

‡ The equivalent of the free-martin (extracted recessive).

but from the primitive germ-cell mass—a part of the zygote, and zygotic therefore. Thus the P.G.C.'s are also zygotes, and the gametes reduced P.G.C.'s. This generalisation I call the Owen-Weismann law.

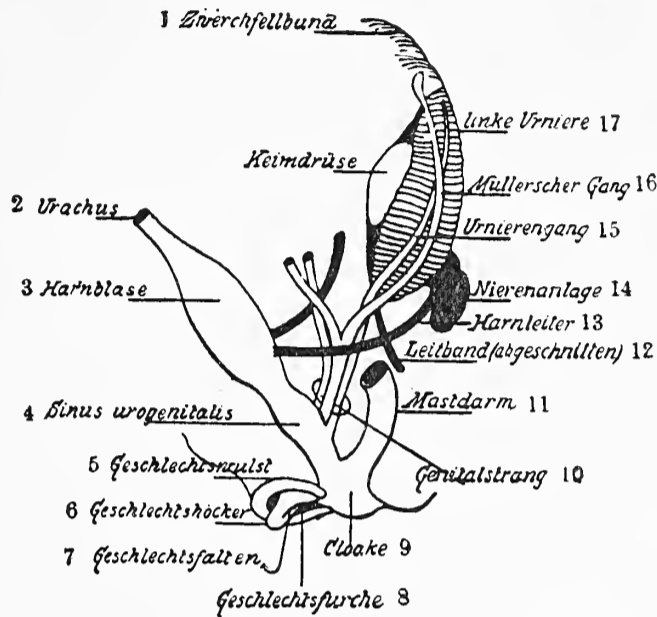


FIG. 1.—Diagram of urogenital system in type common to both sexes (Bonnet).

- | | | |
|----------------------------|-------------------------|---|
| 1. Diaphragmatic ligament. | 7. Sexual folds. | 13. Ureter. |
| 2. Urachus. | 8. Sexual furrow. | 14. Kidney rudiment. |
| 3. Bladder. | 9. Cloaca. | 15. Wolffian duct. |
| 4. Urinogenital sinus. | 10. Genital cord. | 16. Müller's duct. |
| 5. Prepuce. | 11. Rectum. | 17. Left Wolffian body with sexual gland. |
| 6. Sexual eminence. | 12. Gubernaculum (cut). | |

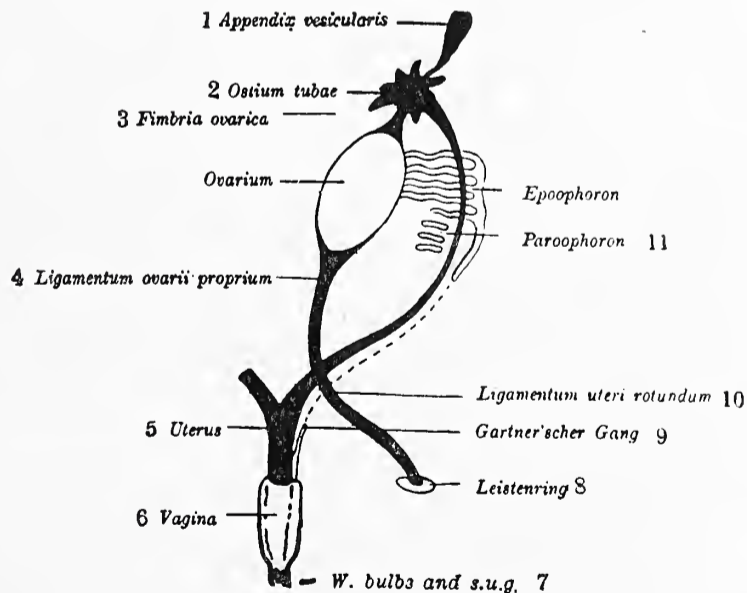


FIG. 2.—Diagram of female urogenital organs, showing ovary, tube, uterus, vagina, ovarium, and round ligaments, *Dominant*: epoophoron and paroophoron, *Recessive* and on male type (modified from Bonnet).

- | | | |
|-----------------|----------------------|-------------------|
| 1. Hydatid. | 3. Ovarian fimbria. | 8. Internal ring. |
| 2. Ostium tubæ. | 4. Ovarian ligament. | |

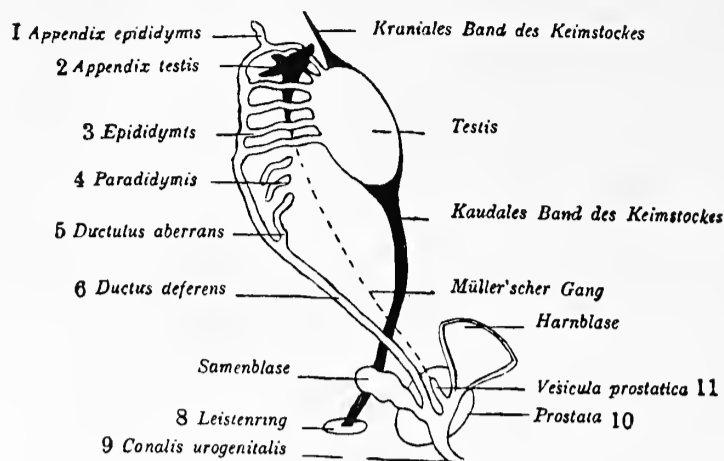


FIG. 3.—Diagram of male urogenital organs, showing testis, caudal ligament, gubernaculum, prostate, *Dominant*: appendix testis (Müller's duct), and prostatic utricle, *Recessive* and on female type (Bonnet).

6. Vas deferens.

In 2 and 3 the differentiated sexual organs are shown, arising from a hypothetical common type in fig. 1.

LITERATURE.

ADAMI, J. G., "The Principles of Pathology," *Oxford Medical Publications*, 1909, Oxford, Henry Frowde.

Adami remarks, *apropos* of the Differentiation of Sex: "The existence in the normal male or female of useless rudiments of parts characteristic of the opposite sex, must not be taken as an indication that man is descended from an originally hermaphrodite ancestry. . . . Rather such rudiments are, in Mendelian terminology, recessive features, due to the origin of the fertilised ovum from both male and female germ-plasma" (pp. 257-8).

AHLFELD, *Die Missbildungen des Menschen*, I. Abschnitt, Leipzig, Gronow, 1880.

BALLANTYNE, J. W., *Antenatal Pathology and Hygiene, the Fœtus and Embryo, A Manual of*, Edinburgh, Green & Sons, 1902 and 1904. Two vols.

BATESON, W., *Mendel's Principles of Heredity*, Cambridge, University Press, 1909.

BATESON, *Materials for the Study of Variation*, London, Macmillan & Co., 1894.

BATESON, *The Methods and Scope of Genetics*, Cambridge, 1908.

In discussing Doncaster's results as to the crossing of varieties of the currant-moth, Professor Bateson remarks: "If we are right, as I am strongly inclined to believe, we get a glimpse of the significance of the popular idea that in certain respects daughters are apt to resemble their fathers, and sons their mothers—a phenomenon which is certainly sometimes to be observed" (p. 44, *Genetics, etc.*).

BEARD, "The Germ-cells," Part I., Raja Batis, Part I. contd., *Journ. of Anat. and Phys.*, 1904, pp. 82 and 341.

BOVERI, "Befruchtung," *Merkel und Bonnet's Ergebnisse*, vol. i., 385-585.

CAREY, "Report of Two Testicular Teratomata, with a Review of the Recent Literature," *Johns Hopkins Hosp. Bull.*, vol. xiii., No. 140, 1902.

CASTLE, W. E., "The Heredity of Sex," *Bull. Mus. Comp. Zool. Harvard*, vol. xl., No. 4, 1903.

CUNNINGHAM, J. T., "The Heredity of Secondary Sexual Characters in relation to Hormones: A Theory of the Heredity of Somatogenic Characters," *Arch. für Entwicklungsmechanik der Organismen* (Roux'), Bd. xxvi., Heft 3. Castle's theory is considered and discussed.

DARBISHIRE, "Mendelism," *Science Progress*, No. 7, January 1908. Castle's theory is explained and discussed.

DARWIN, C., *Origin of Species and Variation of Animals and Plants under Domestication*, London, Murray.

DONCASTER, "Sex Inheritance in the Moth *Abraxas grossulariata* and its var. *lacticolor*," *Rep. Evol. Comm.*, iv. p. 41, 1908.

DURHAM and MARRYAT, D. C. E., "Inheritance of Sex in Canaries," *Rep. Evol. Comm.*, iv., 1908.

EIMER, *Organic Evolution*, London, Macmillan & Co., 1890, J. T. Cunningham's translation.

EIGENMANN, "The Precocious Segregation of the Sex-cells *Cymatogaster aggregatus*," *Journ. of Morph.*, vol. v. pp. 481-492.

GALTON, *Natural Inheritance*, London, Macmillan & Co., 1889.

GEDDES and THOMSON, *Evolution of Sex*, London, Scott, 1901.

HART, D. BERRY, "Nature and Cause of Descent of Testes," *Journ. of Anat. and Phys.*, 1909; also "Morphology of the Human Urinogenital Tract," *Journ. of Anat. and Phys.* and *E.O.S. Trans.*, 1900-1901.

HARTUNG, *Ueber einem Fall von Mamma accessoria*, Erlangen, 1875, Druck der Univ. Buch Druckerei, E. Th. Jacob.

INGALLS, "Beschreibung eines menschlichen Embryos von 4·9 mm.," *Arch. für Nuk. v. Anat.*, Bd. lxx.

KEMPE, v., *Schwalbe's Jahresbericht*, xxi. p. 632, for a short summary.

LEICHTENSTERN, *Virch. Arch. f. Anat. und Path.*, 1878, p. 222.

LOCK, *Recent Progress in the Study of Variation, Heredity, and Evolution*, London, Murray, 1906.

LOTSY, J. P., *Vorlesungen über Descendenztheorien*, Jena, Fischer, 1906. This is a most comprehensive and valuable work.

M'CLUNG, "The Accessory Chromosome-Sex Determinant?" *Biol. Bull.*, iii., 1902, p. 43.

MITCHELL-BRUCE, *Journ. of Anat. and Phys.*, 1879, xiii. p. 425.

MENDEL, "Versüchle über Pflanzen Hybriden verh. naturf. Verein in Brünn" (English translation by Bateson in *Jour. Roy. Hort. Soc.*, xxvi., 1901).

OHKUBO, SAKAYE, "Zur Kenntniss der Embryome des Hodens," *Roux' Archiv f. Entwicklungsmechanik der Organism.*, Bd. xxvi. Heft. 4.

OWEN, *On Parthenogenesis*, London, Van Voorst, 1849.

PRZIBRAM, *Experimental Zoology*, Part I., "Embryogeny," Cambridge, University Press, 1908. A most valuable monograph.

PUNNETT, *Mendelism*, Cambridge, Bowes & Bowes, 1907.

QUAIN, *Anatomy*, vol. i., Bryce on "Embryology," London, 1908.

RITCHIE, J., "A Case of Embryoma occurring in the Mediastinum," *Jour. of Obst. and Gynæc. of the British Empire*, iv. p. 65.

SABIN, *Amer. Jour. of Anat.*, vol. i.

SCHLAGENHAUFER, *Wien. klin. Woch.*, 1902, Nos. 22 and 23.

SHATTOCK, E. G., "Ovarian Teratoma," Erasmus Wilson Lecture, 1908, *Lancet*, i., 1908, p. 479.

SUTTON, J. BLAND, *Evolution and Disease*, London, Walter Scott, 1890.

THOMSON, J. ARTHUR, *Heredity*, London, John Murray, 1908. A great storehouse of facts.

WEISMANN, *The Evolution Theory* (J. A. Thomson's trans.), London, Arnold, 1904.

WHETHAM, *The Recent Development of Physiological Science*, London, Murray, 1904.

WIEDERSHEIM, *Vergleichende Anat. der Wirbelthiere*, Sechste Auflage, Jena, 1906.

WOOD JONES, "The Nature of the Malformations of the Rectum and Urogenital Passages," *Brit. Med. Journal*, Dec. 17, 1904.

WOODS, F. A., "Origin and Migration of the Germ-cells in Acanthias," *Amer. Jour. of Anat.*, i. 307. His conclusion is "that the germ-cells in the dog-fish are not developed from somewhat specialised cells of the body, but that a few undifferentiated cells of the earliest type are taken out and passed on until the new individual is formed" (p. 318).

(Issued separately September 29, 1909.)

XXXVIII.—Observations with a Current Meter in Loch Ness.

By E. M. Wedderburn, W.S., and W. Watson, M.A., B.Sc.*

(MS. received July 5, 1909. Read July 5, 1909.)

ONE of the authors having made an experimental investigation on the currents produced in a trough of water by a blast of air driven along the surface of the water, † it was desired to test the correctness of his deductions by actual observations in a large lake. Loch Ness was chosen on account of its length and uniformity of basin, as it was thought that the length and narrowness of the loch would lead to clearly defined currents being set up in the lake. The sequel showed, as in the case of observations on seiches, that it would have been better to confine attention to a smaller lake, for a two-fold reason, (1) because in a large lake the difficulties of observations are much greater than in a small lake during stormy weather, and in very deep lakes the difficulties in the way of obtaining a fixed point from which to use the current meter are formidable, and (2) because it would seem from a few observations made in Loch Garry (Ness Basin) that currents are more defined and more regular in small than in great lakes.

Various current meters have been designed, and after inquiry it was decided that the one best suited for observation in lakes was the instrument designed by Dr Ekman and described by him in *Publications de Circonstance* No. 24, and the choice was justified by the satisfactory way in which the instrument worked under adverse conditions. Fig. 1 gives a photograph of the instrument. The current is measured by the number of revolutions of a very light screw propeller, which is kept directed against the current by a vane set at right angles to the propeller. By an ingenious arrangement, at every thirty-three revolutions of the propeller a small metal ball is allowed to drop into a cup in the centre of a compass needle, after which the ball runs down a groove on the north leg of the needle into the compass box, which is rigidly attached to the meter. The compass box is divided into 36 divisions,

* The cost of the current meter and the necessary gear with which the observations described were carried out was partly defrayed by a grant from the Moray Bequest of the University of Edinburgh. The cost of carrying on the observations was partly borne by a grant from the Carnegie Trustees. We have also to acknowledge indebtedness to Professor D'Arcy Thompson for advice, and for supplying a small but strong sounding machine for use with the current meter, and to Mr J. Davidson, Superintendent of the Caledonian Canal, for the use of a buoy and anchor and for assistance in preparing the necessary gear.

† E. M. Wedderburn, "An Experimental Investigation of the Temperature Changes occurring in Fresh-water Lochs," *Proc. Roy. Soc. Edin.*, xxviii. p. 2.

each division corresponding to 10 degrees of the compass. The division into which a ball falls, therefore, shows the direction in which the vane is pointing

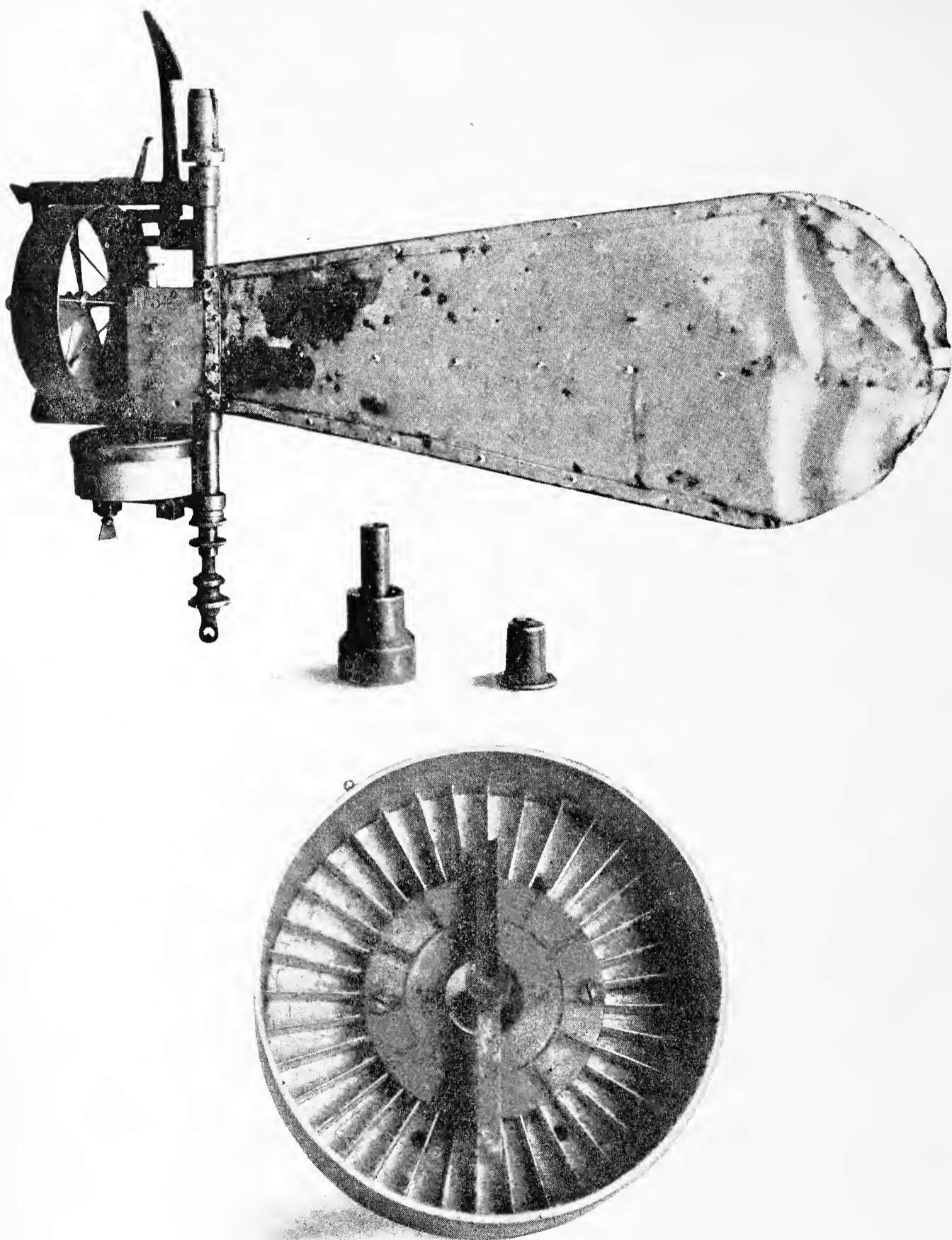


FIG. 1.

when the ball is allowed to fall, and in this way the direction of the current is determined. When an observation is to be made a small receptacle in the

instrument is filled with the balls or shot which are to indicate the direction of the current, and the meter is lowered to the desired depth by means of a sounding line. One messenger is then sent down which releases the propeller, and the time at which the messenger reaches the meter is the time at which the observation begins. When it is desired to end the observation a second messenger is sent down which stops the propeller. The meter is then brought to the surface and the number of revolutions which the propeller has made read on the dials of the meter, the direction of the current being ascertained by the position of the balls in the compass box. The meter was constructed by the Central Laboratory for the International Study of the Sea, Christiania, under the superintendence of Dr Ekman, and was carefully calibrated before being sent out, the rate of the current in centimeter seconds being given by the formula $v = .8 + .424n$, where n is the number of revolutions of the propeller per minute. The instrument was not supposed to be accurate for currents which produced fewer than 5 revs. per minute, but in our observations great accuracy was not aimed at and only qualitative results were desired. Owing to the conditions under which the observations were carried out the instrument could not always be kept in perfect adjustment, and for this reason, at least for the slower currents which were measured, the above formula is probably not correct. As will be seen in the sequel, most of the currents which were measured in Loch Ness were slower than 5 revs. per minute; but as the meter very easily took up the direction of the current, the directions were measured with much greater accuracy than the velocities of the current.

It was the exception rather than the rule that the compass balls indicated a steady current. The changes of direction were often very great during the time occupied by a single observation, and for this reason it would have been of great advantage had the balls been distinguishable from one another by numbers or otherwise, as then some idea could have been obtained of the manner in which the direction of the current varied. Such an arrangement was suggested by Dr Ekman, and also by our boatman, Mr Wm. Macdonald, of Fort Augustus.

It is, of course, necessary to have a fixed point from which to suspend the current meter during the observations. The method adopted was that used by Helland-Hansen in his current measurements in Norwegian fiords in 1906 (*Bergens Museums Aarbog*, 1907, No. 15). A buoy was moored at the point of observation by two grapnels, to each of which was attached a line of a length about twice the depth of water in which the buoy was to be moored. The grapnels were put as far apart as the length of the line would allow in the direction of the axis of the lake. The lines were then

tightened as much as possible, so that the buoy was moored under a considerable strain, and any variation in its position while so fixed was very small. At first the buoy was moored at a depth of nearly 700 feet about a mile to the north-east of Invermoriston by hemp ropes, but owing to the exposed position in the lake in which the buoy was fixed, little but disaster attended the observations which were made in that position. Even in a moderate breeze the waves in Loch Ness are of considerable size, and in a small rowing boat it was found very difficult and not a little dangerous to observe in the centre of the lake. The observations were put a stop to for a time by the loss, in some unexplained manner, of the compass box; and during the interval which occurred before the lost box could be replaced the buoy broke from its moorings and took a journey of some sixteen miles, leaving about 2000 feet of rope at the bottom of Loch Ness. It was then decided to moor the buoy in a less exposed position in the lake, and in a less depth. Such a position was found in about 300 feet of water 300 yards off Invermoriston pier. At first the buoy was moored as before with ropes, and latterly with ordinary galvanised fencing wire, which was found more satisfactory and much cheaper. Mr J. Murray Grant of Glenmoriston kindly granted the use of his boathouse for housing the boats and apparatus, and thus gave material aid to the work. One or two observations were also made near Fort Augustus in the position where the Lake Survey yacht "Rhoda" was formerly anchored (*Trans. Roy. Soc. Edin.*, vol. xlv. p. 410), and also at a point about 400 yards off Glendoe pier. (See Sketch Map, fig. 2, for positions.) For the observations of short duration the meter was lowered from a rowing boat attached to the moored buoy. Even when all precautions were taken it was found that in strong winds the boat swayed a little with the wind, especially if the wind blew broadside on to the boat, and to a certain extent the observations are vitiated by this; but the swaying of the boat was slight and slow, and may be neglected in view of the complexity of the observations. The variations in direction and strength of the currents which were observed were much greater than could be explained by the swaying of the boat. Observations were also frequently made by attaching the meter directly to the buoy and leaving it overnight; and in such observations, as the buoy did not present a large surface to the wind, the effect of swaying with the wind was much reduced. The earlier observations were made by Mr E. M. Wedderburn, including a few observations in Loch Garry (*Proc. Roy. Soc. Edin.*, vol. xxix. p. 98), and those in August and September by Mr W. Watson. During the succeeding winter a number of observations were made by Mr Wm. Macdonald, who acted as boatman throughout, and whose inventive faculty overcame many difficulties.

The observations in Loch Garry were so satisfactory, and showed so clearly the existence of a return current to supply the place of water carried along at the surface by the wind, that the Loch Ness observations were looked forward to with confidence. But the Loch Ness observations have proved so complicated that the authors do not pretend to understand them fully, and in what follows they have endeavoured to select from the observa-

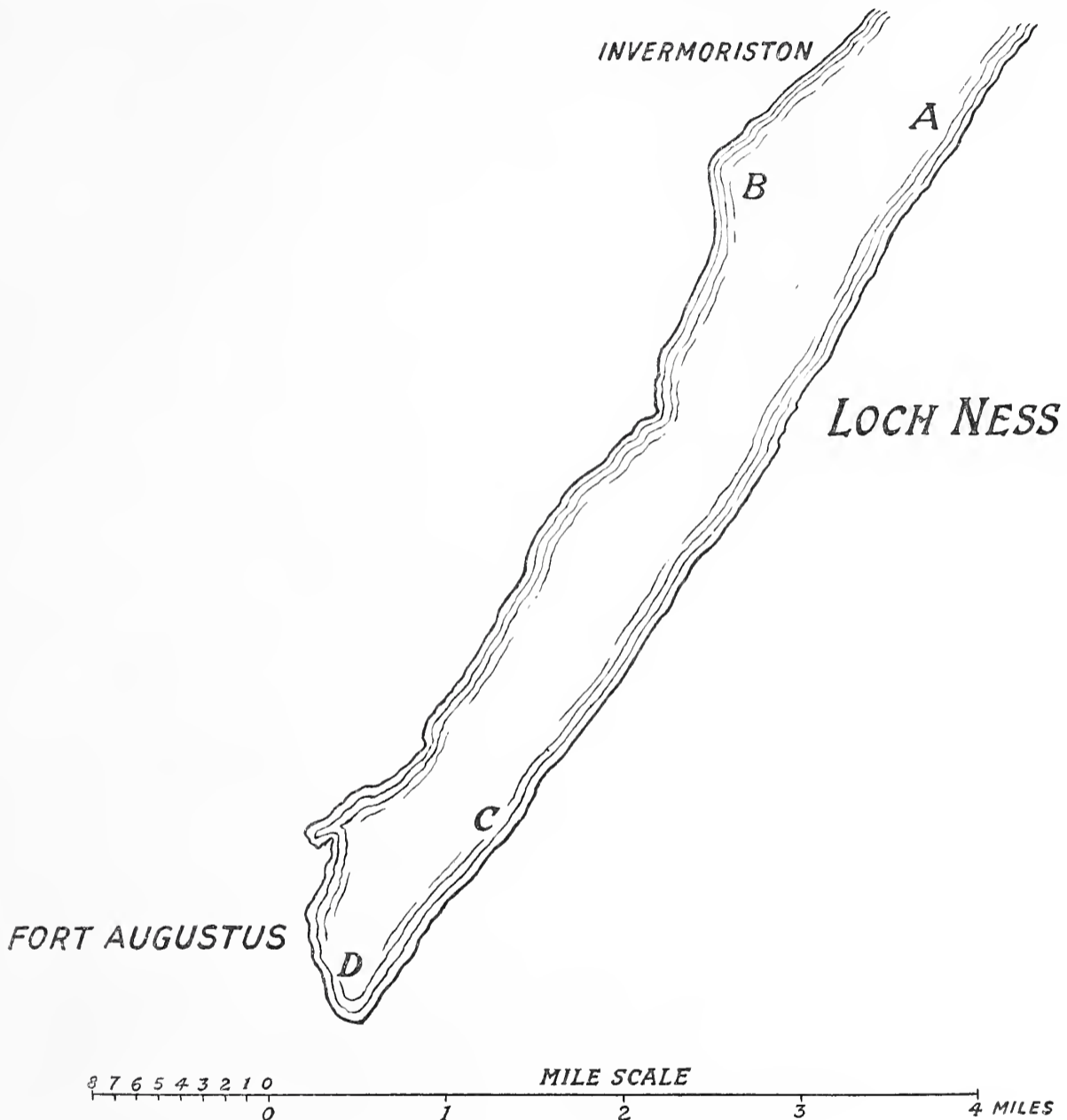


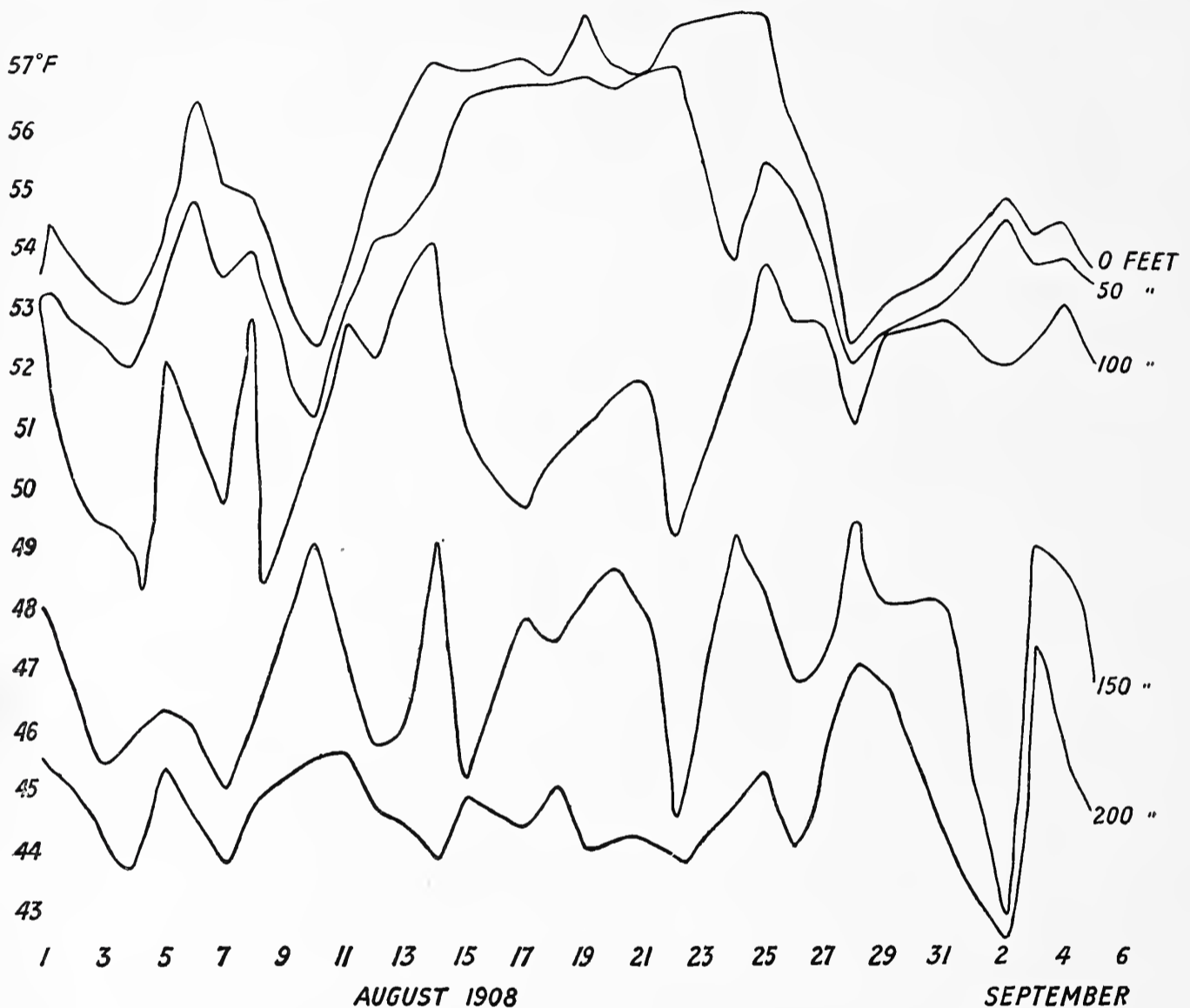
FIG. 2.

tions such as they think they are able to explain. Most of the observations are tabulated in the Appendix to this paper, as they will be of interest for comparison in the event of current measurements being made in other lakes.

Numerous temperature observations were also made in Loch Ness to correlate with the current observations. These also are given in the Appendix. The Invermoriston temperature observations for every fifty feet are shown graphically in fig. 3. The temperature changes at the surface and 50 feet are evidently chiefly due to the changes of wind; but the observations at

100, 150, and 200 feet show a well-marked, if somewhat irregular, temperature seiche. The rapidity of the change of temperature from 2nd to 3rd September is especially notable, and will be referred to later.

Before proceeding to discuss the current observations it is necessary to point out that very considerable currents must accompany the temperature seiche, apart from the currents and return currents produced by winds. A



INVERMORISTON - LOCH NESS

FIG. 3.

rough calculation of the magnitude of these currents has been made as follows:—

The mean depth of Loch Ness is about 138 metres (450 feet), and it was assumed that for the purposes of calculation the lake could be replaced by a rectangular basin of this depth and with a length of 36 km. ($22\frac{1}{2}$ miles). It was further assumed that there was a sharply defined discontinuity at a depth of 46 metres (150 feet). Then for a seiche with an amplitude of 46 metres the quantity of water both in the upper and in the lower layer trans-

ferred from one side of a node to the other is (as the node is at the centre of the lake) $1800000 \times 2300 \times b = 414b \times 10^7$ cub. cm., where b is the breadth of the lake. All this water must pass through the nodal section, and it was assumed that in the upper and lower layers respectively this transference was accomplished by a current uniformly spread over the nodal section whose area in the upper layer is $46b \times 10^2$ sq. cm. and in the lower layer $92b \times 10^2$ sq. cm. The transference takes place in half the period of the seiche, which for purposes of computation was taken as forty hours. The average rate of the current at the node on the foregoing assumptions is, then, in the upper layer

$\frac{414 \times 10^7}{46 \times 10^2 \times 40 \times 3600}$ cm. sec., or approximately 6 cm. sec., and similarly in

the lower layer 3 cm. sec. The currents in the two layers are of course in opposite directions. If it is further assumed that the motion of the water particles is harmonic, the maximum velocities of these currents are found to be about 10 cm. sec. and 5 cm. sec. respectively. A current of 10 cm. sec. would produce over 20 revs. per minute of the propeller of the current meter.

In the tables of observations in the Appendix, the first entry gives the rate of the current in centimetres per second calculated according to the formula supplied with the instrument. The second entry shows the directions of the current. The prefix gives the number of balls which have fallen into the compass box, and the suffix the variation of the directions in degrees. The centre figure gives the number of degrees east or west of north or south which denote the mean direction for the current. The third entry gives the time of commencing and sending the observation in hours (numbered from 1 to 24) and minutes. A bar over the time of ending the observation indicates that the observation was not ended until the next day, and that the hour refers to the following day. At the top of each column appears the date on which the observation was commenced, and at the left hand the depth at which the observations were made are shown. Thus the entry for 150 feet on 12th August: $1\cdot2_{12}N7E_{180}17\cdot32 - \overline{10\cdot28}$, means that the average velocity for the currents for the period of observation was 1.2 cm. secs.; that there were twelve indications of direction, that the average direction was 7° east of north, that the variation in the direction of the current was 180° ; that the observation started at 17 hours (5 p.m.) 32 minutes on the 12th and ended at 10 hours 28 minutes on 13th August.

All temperatures are measured in degrees Fahrenheit, and all depths are given in feet. Wind force is estimated on the scale 1 to 10.

A glance at the tabulated observations will show the great complexity of the results and will give some idea of the difficulty of correlating them.

In their discussion we have dealt separately with the results obtained at Station B, Invermoriston (see Sketch Map), and at Station C, Fort Augustus.

PRELIMINARY OBSERVATIONS.

A few observations were made at Station A (Invermoriston) in April and May when the water was of practically uniform temperature, about 42° F. On only one occasion was a current rate of over 5 cm. secs. observed, and at a depth of only 24 feet. Currents were indicated at both 500 and 600 feet on the two occasions on which observations were obtained at these depths, although the winds during the period of observation were not very strong. The winds were also variable, so that no importance can be attached to the direction of the currents. For the observation at 500 feet the meter was left immersed for a week and the current directions varied from N. 30 W. to S. 40 E. When the meter was brought to the surface at the close of the 600-foot observation it was found that the compass box had disappeared, so that the directions were not recorded.

INVERMORISTON OBSERVATIONS.

During the period from 30th July to 5th September the observations show currents with a velocity over 5 cm. sec. in a direction opposite to that of the wind on seven occasions, viz. :

Date.	Velocity. Cm. sec.	Depth. Feet.	Direction	
			Of Current.	Of Wind.
Aug. 7	8·4	6	N. 51 E.	S.W.
„ 7	7·1	15	N. 59 E.	„
„ 10	5·1	2	N. 30 E.	„
„ 27	7·4	15	N. 45 E.	„
„ 28	11·2	15	N. 80 E.	„
„ 28	12·3	30	N. 60 E.	„
Sept. 3	6·8	15	S. 10 W.	N.E.

On all these occasions it is seen that the return current takes place near the surface, both during S.W. and N.E. winds, and it is to be noted that the place of observation is during a S.W. wind near the windward end of the lake, and during a N.E. wind near the lee end. The observations on 3rd September are of interest, as although the wind was light the currents were strong. On 2nd September there was also little wind, and there were no appreciable currents. But it will be noted that on 3rd September there was a very strongly marked temperature seiche, and the currents recorded

may be due to this. The temperature was rapidly changing at the hour of observation, and, as already shown, the currents due to this cause may be of considerable amplitude.

On 6th August a strong S.W. wind was blowing, and also on 8th and 9th August. On 7th and 10th August the wind was of moderate strength, and possibly the strong currents observed are the result of the previous day's storms.

On 27th and 28th August also the wind was very strong. The discontinuity in temperature was at this time at a depth of 125 feet to 150 feet.

The only other days during this period on which the wind was of considerable strength and on which observations were made were 3rd and 22nd August. On the 3rd no observations were made between 6 and 50 feet, but at 50 feet there was a current from N. 52 W. of 8 cm. secs. On the 22nd there was at 15 feet a current of 2·2 cm. sec. from N. 60 E.; 1·6 cm. secs. from N. 40 E. at 75 feet and 2·9 cm. secs. from N. 80 E. at 90 feet. This was after a week of prevailing N.E. winds, and the currents are probably due to the progress of a reversion of the temperature distribution from the N.E. to the S.W. type. On 21st August the currents were very undecided in direction.

Frequently cross currents were observed at right angles to the direction of the wind, and during the strong S.W. winds on 3rd and 8th they were well marked, viz. :—

On 3rd August 8·0 cm. sec. from N. 52 W. at 50 feet.

On 8th August 9·7 cm. sec. from N. 42 W. at 15 feet.

In fact, the observations on 8th August show directions at various depths all round the compass; *e.g.* :—

S. 88 W. near surface.

N. 42 W. at 15 feet.

S. 23 E. „ 30 „

N. 45 E. „ 60 „

On 28th August a high-current velocity of 15·2 cm. sec. was recorded at 60 feet. Unfortunately on this occasion the compass needle was found to be off its pivot when the meter was brought to the surface, so that the direction of the current is uncertain. The observation is nevertheless interesting, as the velocity is the greatest recorded during this period at any depth with the exception of an observation on 4th August of 15·5 cm. sec. at 24 feet, and on 5th August 16·9 at 6 feet.

A contrast between the observations on 5th August during a N.E. wind

with those on 28th August during a S.W. wind is instructive. At Invermoriston the N.E. winds have much more "gather" than the S.W. winds. As a matter of fact, during strong N.E. winds observation was impossible, owing to the size and character of the waves. On the 5th, then, it is seen that although the wind was not strong (force 1 on scale of 1 to 10) the velocities of the currents measured ranged from 3·2 to 16·9 cm. secs. from the surface down to 60 feet, all with a north-easterly direction. At 100 feet and 200 feet the directions were N. 8 W. and N. 10 W. respectively. On the morning of the 6th, a day of variable breezes, there was a current of 3·9 cm. sec. from S. 39 W. at the surface, and the current at 30 feet was very variable. On the 28th there was a very strong S.W. wind, so that waves were breaking into the boat, yet as near the surface as 15 feet we have a current velocity of 11·2 cm. sec. from the N.E. and of 12·3 cm. sec. at 30 feet from the same quarter. The surface was too rough for an observation to be made there; but a curious effect was noticed during a temperature observation: when the thermometer was sunk to a considerable depth the displacement of the line from the vertical was very marked in a direction opposite to that of the wind.

The observation on 5th August above referred to was made during an isolated day of N.E. winds. From 14th to 20th August, however, there were continuous easterly winds, variable in strength. In no case is there a trace of a south-westerly current nearer the surface than 30 feet, and, save for a single observation on 17th August, none nearer than 90 feet. All night observations from 14th to 15th August show an average current of 1·5 cm. sec. from N. 40 E. at 120 feet, and from 15th to 16th of 1·0 cm. sec. from N. 31 E. at 180 feet. But it is evident that the latter observation was made on the upper margin of the return current, for while ten balls indicated a N.E. current, there were also three indications of currents N. 60 W., S., S. 40 W. Two days later, on the 17th, there are indications of a return current from 30 to 60 feet. The rising of the return current during the continuance of N.E. winds is clearly shown by the overnight observations commenced on 15th and 17th August, viz. :—

Date.	Depth.	Average Current Velocity.	Directions.
Aug. 15-16	180 feet	1·0 cm. sec.	$_{10}$ N. 31 E. $_{70}$; $_1$ N. 60 W. $_1$ S. ; $_1$ S. 40 W.
„ 17-18	150 „	1·2 „	$_{11}$ S. 10 W. $_{60}$; $_1$ N. 20 E. ; $_1$ N. 20 W. ; $_1$ N. 40 E.

The loch was nearly calm on the evening of the 17th, but the wind rose again the next day, and north-easterly currents of 11·2 cm. sec. at the surface, 8 cm. sec. at 30 feet, 3·5 cm. sec. at 80 feet, and 7·4 cm. sec. at 90 feet, were observed. The directions at 90 feet were not, however, very consistent, and at 96 feet a southerly current of 1 cm. sec. was observed, and at 102 feet a N.W. current of 1·6 cm. sec.

The general conclusions to be drawn from these observations during N.E. winds would seem to be that at the station of observation, *i.e.* near the lee end of the lake, for moderate winds the return current confines itself to the deeper water and takes place in the neighbourhood of the temperature discontinuity. The return current seldom penetrates to the cold water below the discontinuity. The effect of very strong winds can only be guessed at, as observation was impossible with the boats and apparatus at our disposal.

On one occasion the meter happened to be suspended at 180 feet from the buoy during a strong N.E. wind, *viz.* on 1st September, when a N.E. gale was sandwiched between two days of calm. The overnight observation showed a slow current (1 cm. sec.) with variable direction. (There were thirteen balls indicating direction, and while one was in the N.E. quadrant the remaining twelve were equally distributed over the other three quadrants.) This isolated observation is not inconsistent with the above conclusion. It should, however, be mentioned that although the return current does not appear to descend below the discontinuity, direct currents do appear to penetrate to considerable depths at the commencement of the north-east winds, and only after the wind has continued for some time does the return current displace the direct current and force it to be confined to a less depth.

An examination of the results of south-westerly winds does not appear to give the same conclusions, but they are not inconsistent therewith. The general rule is that strong return currents are met with very near the surface, no matter what is the strength of the wind. This may be due either to the fact that the observation station was near the windward end of the lake, or to the sheltered position of the buoy. The position was slightly sheltered by Portclair Point, which forms the western side of Invermoriston Bay. It was more than once suggested that the return current might be felt very deep on the exposed side of the station, and that Portclair Point caused subsidiary return currents near the surface. Whatever the cause, there is not the same consistency about the results during S.W. winds as during N.E. winds. But it is clear, especially with strong winds, that the return current at the point of observation took place near the surface (*cp.* observations on 27th and 28th August). With a S.W. wind succeeding a day of calm or N.E. winds, S.W. currents are set up at considerable depths—*e.g.*, at

90 feet on 12th August and 12th September—but this deep direct current does not continue long. With a moderate breeze the return current makes its appearance with small velocity comparatively near the surface. With a strong wind it is met with very near the surface, and sometimes with a considerable velocity.

The period from 30th July to 5th September contained no continuous spell of steady S.W. winds, so that deductions from the results are difficult. But from 12th to 19th September there was a fairly steady wind. Starting on the 12th there is a S.W. current down to 90 feet at least, and a balance in favour of a N.E. current at 210 feet. By the 14th the N.E. current is seen at 30 feet and strongly at 60 feet. On the 15th a N.E. current is registered at 25 feet. On the 17th there was a strong wind, and, as on 27th–28th August, there is a strong N.E. return current just below the surface. The loch was too stormy to use the current meter at the surface, but the velocity of the surface current was estimated from the drift of particles floating in the water at about 13 cm. sec.

On the 17th the indications of direction are rather variable, but they indicate—

At the surface a current from S.W. of 13 cm. sec.					
From 2 to 30 feet	„	„	N.E.	„	4 to 5 „
„ 30 „ 150	„	„	N.E.	„	3 „ 4 „

Below 150 feet the tendency of the current is from the S.W. This may be due to the existence of a secondary current in the same direction as the wind below the return current. There are also indications of this on 12th and 19th September, but the evidence is too slender to draw any sure inference.

A point of difference between the observations from 12th to 19th September and the earlier observations is the greater strength of currents in deep water. Velocities of from 2 to 4·5 cm. sec. were not met with in deep water in August. This may either be due to the prevalence of a S.W. wind or to the fact that the temperature discontinuity was then at a greater depth.

FORT AUGUSTUS OBSERVATIONS.

The observations from 23rd September till 21st October 1908 were taken at a point near the head of Loch Ness, about 1 mile from Fort Augustus, and 400 yards from the east shore, nearly opposite Glendoe pier (C on Sketch Map).

As the canal authorities considered that mooring a large buoy in that position was a menace to shipping, we had recourse to other means, and a

series of empty petrol tins, etc., proved very successful, and was probably more efficient than the large buoy, which had a considerable area exposed to the wind, and swayed slightly in consequence. The depth of the loch at the point of observation was found by sounding to be 465 feet, or about twice the depth of the Invermoriston station.

During the period of observation at Fort Augustus the prevailing wind was westerly, and as a matter of fact continuous easterly winds of any considerable strength would have made observation impossible.

The currents observed seemed very complicated, and this may be attributed to a number of causes. With a westerly wind Fort Augustus is the windward end of the lake, and small variations in the direction of the wind are more felt than further down the lake, for the steep mountains which rise all along the lake tend to stereotype the direction of the wind. Cross currents may therefore be set up more easily at Fort Augustus than elsewhere. When an easterly wind is blowing, the depth of the lake is rapidly decreasing in the direction of the direct surface current, and this must have a disturbing effect on the currents setting in towards the shallowing shore. As evidence of the confusion which exists, two points may be mentioned: (1) the great variation in the direction of the currents registered during observations of only an hour's duration, which means that in a large number of cases there was no steady direction for even an hour on end; (2) compared with Invermoriston the velocities measured were small.

The following are all the occasions on which velocities above 5 cm. sec. were observed:—

Depth in Feet.	Velocity. Cm. secs.	Date.	Direction.	Wind.
0	8·0	Sept. 24	N.E.	N.E. 1-2.
0	5·0	" 24	N.E.	N.E. 1-2.
0	9·3	" 29	S.W.	S.W. 3-4.
30	9·0	" 29	S.W.	S.W. 3-4.
120	5·0	" 29	N.E.	S.W. 3-4.
0	5·1	Oct. 1	S.W.	S.W. 0-1.
30	5·1	" 1	S.W.	Variable.
0	5·1	" 3	S.W.	S.W. 1-4.
90	5·6	" 9	S.W.	S.W. 1-4.
120	6·7	" 10	S.W.	S.W. 3-4.
0	10·0	" 13	N.E.	S.W. 1-3.
0	9·8	" 16	N.E.	N.E. 3-4.
30	5·1	" 16	"	N.E. 3-4.
0	5·1	" 19	S.W.	S.W. 0-1.
0	7·4	" 21	S.E.	Variable.

Here there are fifteen cases, and in nine of these the observation is at the surface. The most curious fact of all is that the largest velocity recorded was

only 10·0 cm. sec., and that was a return current at the surface, setting in as a S.W. gale died away. At the more sheltered position at Invermoriston velocities were registered up to 17 cm. sec. on several occasions, and in a considerable number of cases there were observations of velocities exceeding 10 cm. sec. The slowness of the currents is probably due to the proximity of the head of the lake and to the shallowing of the basin, for there must obviously be at each end of a lake a certain amount of water which does not partake of the general circulation of the lake.

The comparative shallowness of the water at Invermoriston may also, by confining the depth through which the currents could move, have increased their velocity; but this is unlikely, as it is really the depth of the temperature discontinuity which determines the depth to which the currents descend.

No velocity above 5 cm. sec. was observed below 120 feet, but at that depth there are two observations of a current velocity of 5 cm. sec., both during S.W. gales. Curiously, one was a direct current (10th October) and the other a return current (29th September).

There are only observations on two days with N.E. winds from which reliable conclusions can be drawn, viz. 24th September and 15th October. On the former date the easterly current was decidedly felt at 180 feet about midday, but three hours later the return current has had time to set in, and, with practically the same velocity as the direct current had at 180 feet, it appears at 150 feet. On the 15th October, when the easterly wind had been blowing for not more than twelve hours, the return current is taking place steadily between 60 and 120 feet. On the following day the currents are again confused. There is a tendency for the return current to appear near the surface, while direct currents are also recorded at considerable depths. In consequence, there are indications of currents of all directions at 60 feet. Unfortunately in the other observations the compass needle was displaced.

In spite of the variable conditions there are a sufficient number of observations taken on days when the currents were comparatively steady to warrant our drawing the same general conclusions as regards S.W. winds as at Invermoriston, viz., that the return current tends to appear very near the surface. The return current appeared at 30 feet on the 3rd, 7th, and 9th October, at 60 feet on 28th September and 19th October, and at 90 feet on 29th September. The observations on 7th October are especially clear. They show a S.W. current at the surface, a N.E. current from 30 to 120 feet, and a S.W. current at 300 feet. These observations strongly resemble the Invermoriston observations of 17th September in showing a deep water current in the same direction as the surface current. A similar observation made on 20th October, in which some method of distinguishing the order in

which the balls reached the compass, would have been helpful. The observation on afternoon of the 20th shows a N.E. return current at 150 feet, with a velocity of 3·5 cm. sec. An all-night observation beginning immediately after shows a small current with directions N. 80 E., $\frac{1}{4}$ S. 50 W.₁₀₀. Had the balls been distinguishable, and had it been seen that the N. 80 E. ball was the first indication of direction, less doubt would have been felt in making the statement that there is a tendency for the return current to rise towards the surface, and for this secondary current to appear.

It was mentioned in connection with the Invermoriston observations that as the season progressed currents penetrated to greater depths. The observations at Fort Augustus show the same thing, and all the observations go to show that the surface current and the return current are chiefly confined to the water above the temperature discontinuity. There are small currents deeper down, but they are negligible in comparison to the upper currents. During the season of the year within which the observations were made the temperature discontinuity is gradually sinking, and therefore it is found (1) currents are appreciable to greater depths, and (2) the velocities in individual cases are less, owing to the larger mass of water affected. This confinement of the appreciable currents to the upper layer may be seen at once from the Fort Augustus observations. From 23rd September till 21st October there are eighteen observations at depths below 250 feet, and in only one case did the velocity exceed 1 cm. sec., viz. during a N.E. wind on 24th September. In several of the other observations the meters gave no indication at all.

In this connection it is interesting to refer to Forel's description of "les eaux troubles du Rhône" (*Le Léman*, vol. ii. p. 280). During a strong north wind when the surface of Lake Geneva is viewed from a height the lake appears to be separated into two compartments. For a mile or two from the shore of the lake the water is greenish, while the rest of the lake appears a deep blue. The line of demarcation is quite distinct but does not run parallel to the shore, and varies from time to time. Examined closely it is seen that the blue water is quite clear, while the greenish water is opaline with a low transparency, which is not due to matter in suspension. This greenish water Forel thinks is the water from the depths of the lake brought to the surface by the return current.

The extent of the lake which takes on this greenish appearance indicates that the rise of the return current may be felt at a considerable distance from the windward end of the lake, and it is not surprising that one mile from Fort Augustus, or even at Invermoriston, there should appear to be inconsistencies in the current systems during S.W. winds.

For with different strengths of wind the return current must come to the surface at different points.

Fig. 4 shows the typical direction of isotherms in a thermally stratified lake during a moderately strong wind whose direction is shown by the arrow, and it is easy to imagine from it that the return current may take place very near the surface at the windward end of the lake, and also that slight variations in the intensity of the wind, or the rise and fall of the temperature seiche, may cause variations, not only in the strength of the return current, but also in the depth at which it is found.

To gain further knowledge of the behaviour of currents when the lake was of uniform temperature, arrangements were made for Mr Wm.

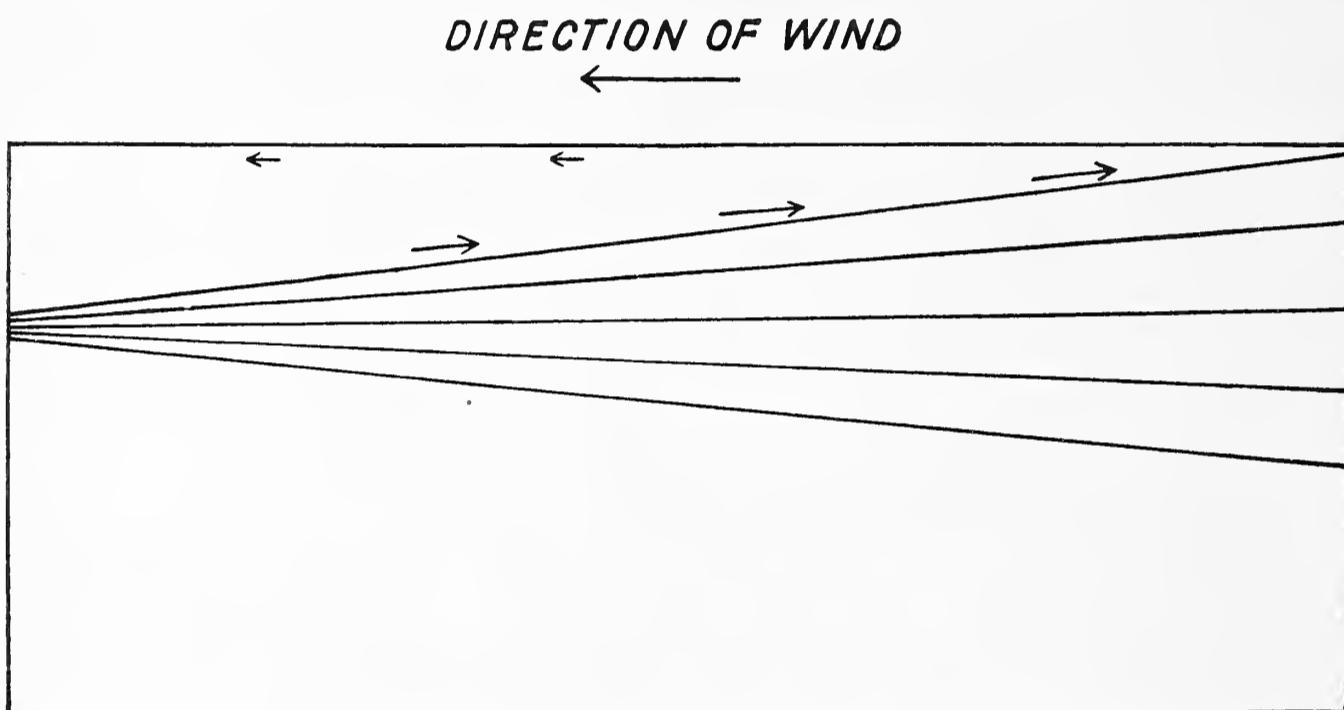


FIG. 4.

Macdonald to observe during January and February 1909. The observations were not taken with any very definite object in view. Mr Macdonald was instructed to make observations at stated depths and intervals. Had there been any change of wind during the period he worked, better results might have been obtained. As it was, the wind blew persistently from the S.W. of all strengths to a gale, and in practically all cases the observations show greater velocities in the very deep water than at intermediate depths, and persistently in a S.W. direction, with numerous cross-currents. The following examples will serve to show how consistently S.W. the currents in the deep water were. This must have been an accident, due to the point at which the observations were made. The position of the return current was not detected at all, and the presumption is that it has been forced up into shallow water by the shallowing of the lake, as indicated on page 631.

The examples referred to above are as follows:—

Date.	Velocity. Cm. sec.	Depth Feet.	Directions.
1909			
Jan. 11	3·7	300	¹⁵ S. 54 W. ^{·50}
„ 18	3·3	150	⁶ S. 41 W. ^{·90}
„ 20	3·1	450	¹⁸ S. 50 W. ^{·120}
„ 21	3·3	450	{ ¹⁰ S. 80 W. ^{·190} ¹ N. 60 E.
„ „	4·2	200	¹⁵ S. 31 W. ^{·110}
„ 23	7·7 (?)	400	⁸ S. 55 W. ^{·20}
„ 25	3·7	400	{ ¹⁰ S. 61 W. ^{·160} ¹ N. 60 E.
„ 26	3·7	465	¹³ S. 50 W. ^{·80}
„ 27	4·6	465	¹⁶ S. 48 W. ^{·40}
„ „	4·4	400	{ ¹⁵ S. 70 W. ^{·110} ⁸ S. 52 W. ^{·140}
„ „	3·1	300	¹ N. 60 E.
Feb. 12	6·4	465	²² S. 55 W. ^{·60}

SUMMARY OF RESULTS.

From observations so complicated and, at times, apparently conflicting, it is difficult and dangerous to draw general conclusions, but it is thought that the following deductions are justified with regard to the currents in Loch Ness:—

1. When the lake is of uniform temperature, then the direct current produced by wind is felt to considerable depths, and the return current is also felt in the deepest parts of the lake.

2. When the lake has become stratified and the temperature discontinuity has appeared, the return current is nearly always found to be above the discontinuity, although there are indications of secondary currents in the same direction as the surface current below the discontinuity.

3. When a wind follows a calm, or when the direction of the wind changes, the direct surface current is felt to a considerable depth; but when the wind has been blowing for about twelve hours the return current asserts itself and the surface current is restricted to a narrower zone.

4. When a calm follows a strong wind the isotherms tend to assume a horizontal position, and in consequence there is in the upper layers a current in an opposite direction to that of the wind which has previously been blowing.

5. Towards the windward end of the lake the return current may take place very close to the surface. Towards the lee end it is found in deeper water.

6. At the windward end of the lake there is complexity in the current coming to the surface.

7. There are indications of secondary currents in the same direction as the wind below the temperature discontinuity when the lake is stratified.

8. Cross currents are frequent and form part of the circulation of the lake, reducing the strength of the main return current.

TEMPERATURE OBSERVATIONS AT INVERMORISTON.

Date.	Hour.	Surface.	5	7	10	25	50	75	100	125	150	175	200	225	250	300	700
1908.																	
Apr. 4	14.30	41.7	41.4	41.4	...	41.4	41.3
" 11	15.0	42.2	41.9	41.9	...	41.5	...	41.5	...	41.5
May 11	13.0	44.7	44.3	44.3	44.0	42.7	42.6	...	42.3	...	42.2	...	42.0
July 29	17.30	53.0	52.0	...	49.6	...	47.6	47.3
" 30	11.0	50.2	50.2	46.6	46.4	...	46.1
" 31	11.0	51.5	51.0	50.5	50.2	...	47.7	...	44.8
" 31	17.0	51.1	51.2	50.4	50.1	49.7	48.9	45.2	44.5
Aug. 1	12.0	53.5	51.2	53.1	53.0	53.1	49.8	48.0	46.1	45.5
" 1	17.0	54.4	54.0	53.6	53.6	50.4	49.3	47.9	45.5	44.9
" 3	11.0	53.3	52.9	52.2	51.4	49.4	47.0	45.4	45.0	44.1
" 4	12.0	53.0	52.8	52.0	51.4	48.9	47.0	45.9	44.0	43.7
" 4	17.0	52.8	52.4	51.0	50.1	48.3	46.6	45.9	45.8	45.7
" 5	11.0	54.0	53.7	53.4	53.3	52.0	49.8	46.3	45.8	45.3	44.2
" 5	16.0	54.5	54.2	54.2	54.1	50.4	48.6	46.7	45.6	44.5	44.4
" 6	11.0	56.2	54.8	54.7	53.8	50.9	47.4	46.0	45.2	44.5	43.4
" 6	16.0	56.4	54.7	54.2	53.4	49.2	46.8	45.5	45.0	44.1	43.0
" 7	12.0	55.0	53.8	53.5	53.2	49.6	46.3	45.0	44.2	43.8	43.0
" 7	17.0	55.0	53.8	53.6	53.0	49.7	46.8	45.0	44.1	43.3	43.0
" 8	11.0	54.9	54.8	53.8	52.9	52.8	47.2	46.2	45.4	44.8	44.3
" 8	16.0	54.0	53.3	53.0	51.1	48.4	46.8	45.7	45.3	44.4	43.6
" 10	16.0	52.3	51.4	51.1	50.9	50.7	50.4	49.0	46.4	45.5	44.4
" 11	12.0	53.4	53.1	52.9	52.9	52.8	52.3	47.3	46.0	45.6	44.7
" 11	16.0	53.9	53.4	53.0	53.0	52.7	51.0	46.3	45.7	45.0	44.5
" 12	Noon	55.1	55.0	54.1	53.2	52.1	48.0	45.7	45.1	44.6	42.8
" 12	16.0	55.2	55.0	54.0	53.0	52.3	49.2	45.6	44.8	43.4	43.2
" 13	11.0	56.1	54.5	54.2	53.8	53.4	48.0	45.9	45.0	44.4	43.2
" 13	17.0	57.0	57.0	54.9	54.5	52.3	49.2	45.6	44.4	44.0	43.9
" 14	Noon	57.0	56.4	55.0	54.8	54.0	52.2	49.1	46.0	43.9	43.3
" 15	11.0	56.8	56.5	56.4	54.3	50.9	46.0	45.1	45.0	44.8	4.44
" 15	17.0	57.0	56.9	56.9	56.6	51.7	47.3	46.1	44.7	44.4	44.0
" 17	11.0	57.1	56.8	56.8	56.3	49.7	48.9	47.8	46.0	44.4	43.5
" 17	15.0	57.0	57.0	56.9	56.5	49.8	48.6
" 18	11.0	56.8	56.7	56.7	56.7	50.5	48.5	47.4	46.8	45.0	44.1
" 18	17.0	57.0	56.8	56.8	56.8	54.7	49.8	48.0	47.2	45.9	44.3
" 19	11.0	58.0	56.7	56.7	56.5	51.0	50.2	48.1	46.0	44.0	43.1
" 19	16.0	57.8	57.7	56.8	51.7	50.5	49.1	46.8	44.0	43.8	43.4
" 20	Noon	57.0	56.9	51.5	50.2	48.6	45.8	44.1	44.2
" 20	16.0	57.0	56.9	56.9	56.9	52.0	50.9	49.4	48.3	45.1	43.9
" 21	11.0	56.9	56.7	56.6	56.5	51.8	50.2	47.9	45.2	44.1	43.2
Aug. 21	16.0	57.7	57.0	56.8	56.6	51.8	49.1	46.3	44.2	43.7	43.2
" 21			57.0	56.8	56.6	51.8	49.1	46.3	44.2	43.7	43.2

22	11.0	57.6	57.0	56.9	52.0	49.1	46.7	44.5	44.0	43.8	43.3	...
"	16.0	57.0	57.0	56.7	53.0	52.0	49.0	45.6	44.0	43.5	43.5	...
"	11.0	58.2	55.1	53.8	52.4	52.0	50.5	49.1	46.2	44.7	43.5	...
"	16.0	58.3	55.0	53.5	52.2	52.0	49.0	46.6	45.8	44.3	43.9	...
"	11.0	57.9	57.0	55.4	54.0	53.6	53.0	48.2	46.2	45.3	44.5	...
"	16.0	58.0	56.1	54.7	53.9	53.0	51.8	47.5	46.0	44.8	43.0	...
"	11.0	56.1	55.4	55.0	54.0	52.8	51.0	46.7	45.0	44.0	43.2	...
"	16.0	56.0	55.2	55.0	53.8	52.3	49.4	46.2	44.5	43.1	43.1	...
"	11.0	54.8	54.5	53.7	53.0	52.7	52.0	47.0
"	11.0	52.4	52.2	52.0	51.7	51.0	49.8	49.4	48.0	47.0	45.0	...
"	11.0	53.0	52.5	52.5	52.5	52.5	50.1	48.0	47.3	46.7	44.7	...
"	17.0	51.4	48.0	46.6	45.0	44.4	...
"	11.0	53.6	53.4	53.0	52.9	52.8	52.0	48.0	45.2	44.3	44.2	...
"	16.0	54.0	53.5	53.4	53.1	53.0	52.0	50.1	47.0	43.8
Sept. 2	Noon	54.7	54.6	54.4	53.9	52.0	46.0	42.9	42.8	42.5
"	11.0	54.2	54.0	53.7	53.0	52.4	50.0	49.0	48.3	47.4	46.6	...
"	16.0	54.2	54.0	54.0	54.0	53.5	53.0	52.9	52.0	48.0	46.4	...
"	11.0	54.4	53.6	53.7	53.6	53.0	52.4	48.7	46.7	45.5	44.5	...
"	16.0	54.6	53.8	53.6	53.6	53.0	52.0	46.4	46.0	45.7	45.3	...
"	11.0	53.6	52.5	52.0	49.0	46.7	46.2	44.0	43.5	...
"	11.0	52.7	52.0	51.5	48.5	47.0	46.2	46.0	...
"	Noon	52.2	50.5	50.3	49.7	48.7	46.8	44.4	...
"	11.0	52.5	51.9	51.6	51.5	51.3	51.0	51.0	48.1	48.0	46.5	...
"	16.0	52.4	51.4	50.7	49.8	47.4	45.7	44.5	...
"	11.0	52.0	52.0	51.8	51.7	51.7	51.1	49.3	47.1	45.8	44.5	...
"	16.0	51.7	51.4	51.1	49.5	47.3	46.0	44.2	...
"	11.0	51.5	51.3	50.2	49.7	49.1	48.0	47.3	45.1	44.0	43.4	...
"	16.0	51.3	49.0	48.8	46.4	45.4	44.7	43.4	...
"	11.0	51.3	51.8	51.8	51.7	51.7	51.7	51.6	51.6	49.3	45.2	...
"	16.0	52.0	52.0	52.0	52.0	51.9	49.1	47.0	...
"	11.0	52.5	52.5	52.0	52.0	52.0	51.3	50.4	49.4	47.8	45.0	...

AT FORT AUGUSTUS.

Sept. 22	13.0	53.0	53.0	52.9	52.8	52.2	...	49.4	45.1	44.4	43.1	43.0	...
"	15.0	53.0	53.0	52.8	52.8	52.8	...	52.5	51.0	44.7	43.3	43.0	...
"	Noon	52.9	52.9	52.7	52.7	52.7	...	52.3	47.3	44.7	43.2	43.0	...
"	15.0	53.0	53.0	52.9	52.9	52.9	...	50.0	47.0	44.5	...	43.0	...
"	13.0	53.0	53.0	52.9	52.9	52.9	...	52.6	48.1	46.4	44.7	43.4	...
"	11.0	53.3	53.3	52.0	52.0	52.0	49.7	48.0	49.5	46.6	43.8	43.0	...
"	16.0	52.1	52.1	50.4	50.4	50.4	...	46.7	44.1	43.4	43.0	42.8	...
"	Noon	53.2	53.2	51.3	51.3	51.3	...	50.0	49.2	47.4	43.8	43.0	...
"	11.0	53.2	53.2	51.0	51.0	51.0	...	50.1	49.5	45.6	43.3	43.0	...
"	16.0	52.1	52.1	50.4	50.4	50.4	...	46.7	44.1	43.4	43.0	42.8	...
"	9.0	51.5	51.5	50.1	50.1	50.1	...	49.4	46.3	44.1	43.0	42.9	...

INVERMORISTON.—STATION A.

	4th April 1908.			11th April 1908.			23rd April 1908.			5th May 1908.			11th May 1908.			
	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	
Surf.	7.2	N. 60 E.	15.9- 15.14	3.3	₇ S. 40 W. _{.90}	12.32- 13.18	2.9	₂ S. 50 W. _{.0}	14.19- 14.29		
6	1.2	₁ S. 60 W.	15.32- 15.57		
24	5.0	₅ S. 62 E. _{.50}	16.11- 16.30		
50	1.9	₁ S. 50 E.	15.40- 16.4	1.9	...	14.35- 15.25		
100	1.4	...	14.53- 15.3	1.6	₃ N. 47 W. _{.10}	14.7- 14.57	1.3	₁ S. 20 E.	14.33- 14.53		
400	
5009	₁₂ S. 65 W. _{.190}	17.25- 17.12 (30th)	
600	
700	.0	...	12.58- 14.38	1.3	...	16.16- 11.45 (16th)		
	N. E. squalls.	2-4.	Variable breezes.	Variable breezes.	Variable breezes.	Variable breezes.	
																* Compass lost.

STATION B.

29th July 1908.			30th July 1908.			31st July 1908.			1st August 1908.		
Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.
Surf. 3.9	₅ N. 70 W. ²⁰	17.18-17.50
3	3 { 3.3
6	3 { 11.0	₅ S. 82 E. ²⁰	16.42-17.15
30	13.1	₄ N. 88 E. ¹⁰	17.57-18.8
50	1.2	...	11.12-11.42	...	₁₃ N. 65 E. ³⁰	17.23-17.50
66	1.0
100	3.3	₆ N. 25 E. ¹¹⁰	11.47-12.27	10.52-11.22
120	11.35-11.57
180	10.40-12.1
200	1.9	₈ N. 60 E. ²⁸⁰	13.0 -18.7
S. W. squalls. S. W. 3.											
S. W. 1. S. W. 1-2.											
3rd August 1908.			4th August 1908.			5th August 1908.			6th August 1908.		
Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.
Surf. 6.5	₅ S. 44 W. ⁹⁰	10.45-11.5	...	₂ S. 40 E. ¹⁴⁰	11.14-11.38	9.3	₁₁ N. 60 E. ²⁰	17.31-17.46	3.9	₉ S. 39 W. ²⁰	10.20-11.7
24	1.6	...	17.30-17.50	16.9	₁₄ N. 80 E. ²⁰	10.48-11.13
30	15.5	...	11.45-12.16
48	4.9	₁₀ S. 15 E. ⁹⁰	16.43-17.19	6.9	₁₁ N. 44 E. ⁸⁰	11.23-11.56	1.6	₄ N. 78 W. ¹⁸⁰	15.50-17.30
50	7.9
60	₁₃ N. 52 E. ⁶⁰	11.12-11.55
90	₆ S. 32 W. ¹³⁰	12.8 -12.55	3.2	₅ N. 22 E. ³⁰	12.7 -12.43
103	16.8 -16.36	12.53-15.22	11.22-15.0
200	1.1	1.9	₁₁ N. 8 W. ⁸⁰	15.51-17.23
S. W. 3-4. S. W. 1-2. N. E. 1.											
Variable S. W. 2.											

STATION B—continued.

Surf.	7th August 1908.			8th August 1908.			10th August 1908.			11th August 1908.					
	Velo- city	Direction.	Hour.	Velo- city.	Direction.	Hour.	Velo- city.	Direction.	Hour.	Velo- city.	Direction.	Hour.			
2.0	₃ N. 17 W. ₃₀	10.59-11.30	6.9	₁₁ S. 88 W. ₇₀	10.28-10.58	₁ N. 40 W.	10.54-11.32	1.6	₁ N. 40 W.	...			
1.2			
8.4	₁₂ N. 51 E. ₄₀	17.36-17.58	₉ N. 30 E. ₇₀	16.40-17.10			
7.1	₁₃ N. 59 E. ₆₀	11.37-12.29			
3.6	₁₁ N. 33 E. ₉₀	12.41-15.25	9.7	₄ N. 42 W. ₅₀	11.4-11.34			
3.3	₁₂ N. 20 E. ₁₁₀	15.47-17.29	2.5	₃ S. 23 E. ₂₀	11.43-12.12	₁ N. 40 W.	11.36-12.21			
...	1.1	₂ N. 45 E. ₁₁₀	15.51-17.2			
...	1.6	₄ N. 35 W. ₁₈₀ *	12.19-15.24			
180	9	...	15.32-16.32	16.50-10.20			
	S. W. 1.—Variable.						W. 0-2.						Variable.		
	* Compass needle off.														
Surf.	12th August 1908.			13th August 1908.			14th August 1908.			15th August 1908.					
	Velo- city	Direction.	Hour.	Velo- city.	Direction.	Hour.	Velo- city.	Direction.	Hour.	Velo- city.	Direction.	Hour.			
1.3	₁ S. 30 W.	11.26-11.55	14.0	...	10.44-11.4	12.5	₁₂ N. 63 E. ₃₀	11.40-11.55			
4.5	₄ S. 60 W. ₄₀	16.32-16.49	₁₀ N. 60 E. ₈₀	15.30-15.51			
1.4	₁ S. 60 W.	10.41-11.22	₅ N. 44 E. ₅₀	15.56-16.23			
1.4	₂ S. 45 W. ₁₀	17.12-17.27	₃ N. 13 E. ₁₀₀	16.28-17.22			
1.3	₁ S. 80 E.	12.0-12.31	{ 7.8	₁₀ N. 55 E. ₃₀	11.12-11.31	...	{ 2.2	₁ N. 80 W.	12.7-12.24	3.1	₄ N. 35 E. ₂₀	12.1-12.28			
2.5	₁₂ S. 36 W. ₃₀	12.36-15.8	{ 8.4	...	12.3-12.23	...	{ 2.3	...	12.31-12.50	{ 1.0			
2.6	₅ S. 28 W. ₄₀	15.20-16.21	1.9	...	11.31-12.2	{ 1.3	₄ N. 80 E. ₈₀	10.47-11.35			
...	1.3	₁ S. 20 W.	15.47-16.32	₁₀ N. 27 E. ₄₀	10.42-11.27	12.33-14.56			
...	1.7	₃ N. 66 E. ₂₀₀	16.40-17.36	...	{ 1.9	...	12.55-15.17	15.44-17.44			
...	1.4	₆ S. 80 E. ₁₁₀	12.32-15.33	₁₀ N. 40 E. ₇₀	17.28-10.28			
1.2	₁₂ N. 7 E. ₁₉₀	17.32-10.28	1.1	₁₄ N. 11 E. ₁₁₀	17.45-10.22			
180	(Save for one ball, divergence 80°.)	₁₀ N. 31 E. ₇₀ *	17.54-10.20			
	W. 1.						N. N. W. 1.						N. E. 1.		
													* Also 1 N. 60 W., 1 S., 1 S. 40 W.		

STATION B--continued.

Surf.	5th September 1908.		12th September 1908.		14th September 1908.		15th September 1908.		
	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.
25	10.13-10.34	1.4	₁ S. 70 W.	16.27-16.45	2.8	₃ S. 70 W. ^{.20}	10.43-11.3
30	1.6	₁ N. 40 E.	12.26-12.51
50
60	10.37-11.21	7.5	$\begin{cases} \text{N. 59 E.}^{.70} \\ \text{S. 50 W.} \end{cases}$	11.11-11.36	1.5	₂ N. 70 E. ^{.20}	11.47-12.20
90	13.4-15.28
120	2.4	Var.	11.11-11.34	1.5	...	11.44-12.18	15.34-16.10
130	11.27-13.0
150	1.9	₁₂ N. 72 E. ^{.40}	12.58-15.19
210	1.0	Var.	18.34-10.46	...	Var.	...	1.1	₃ S. 10 E. ^{.100}	16.22-10.21
225
S.W. 1-2 dying away [on 13th S.W.] * Compass needle off.									
Surf.	16th September 1908.		17th September 1908.		18th September 1908.		19th September 1908.		
	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.
2	2.9	...	12.4 -12.20	10.40-11.40	3.2	₇ S. 44 W. ^{.70}	11.4-11.32
15	1.3	₁ S. 10 W.	15.36-16.5	5.5	₆ N. 23 E. ^{.100}
30	2.2	₂ S. 15 E. ^{.30}	11.36-12.0	4.0	₅ N. 20 E. ^{.170}	11.46-12.6
60	4.0	₅ N. 70 E. ^{.50}	11.21-11.41
90	3.7	$\begin{cases} \text{S. 70 W.} \\ \text{N. 80 E.}^{.50} \end{cases}$	10.44-11.14
120	1.2	₅ S. 18 W. ^{.60}	16.11-16.26
150	12.24-15.18
180	4.5	Var.	16.30-10.27	3.2	$\begin{cases} \text{S. 10 W.} \\ \text{N. 57 E.}^{.130} \end{cases}$	12.13-15.57
210
235	16.46-10.1	1.2	Var.
S.W. 1-2.									
S.W. 3-4 strong over-night.									
Light westerly airs.									
S.W. 0-1									

STATION B—continued. FORT AUGUSTUS—STATION C.

20th September 1908.			23rd September 1908.			24th September 1908.			25th September 1908.		
Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.
Surf.	2·5	₃ S. 80 W. ₂₀	15.15-15.30	{ 8	₃ N. 85 E. ₄₀	15.20-15.30	3·3	₇ S. 65 W. ₁₈₀	11.55-12.40
15	{ 5	₃ N. 60 E. ₂₀	15.30-15.40	2·1	Var.	12.45-13.15
30	2·4	₂ N. 35 E. ₁₃₀
60	1·6	₁ N. 20 W.	12.55-13.10
150	2·8	₅ S. 20 W. ₁₀₀	12.30-12.50
180	14.20-15.0	3·1	₇ N. 65 E. ₁₃₀	14.40-15.10
210	...	12.15-10.22	1·2	₂ S. 5 W. ₁₀	11.35-12.20	1·8	₁₀ N. 85 W. ₁₈₀	13.20-15.20
300	₄ S. 65 W. ₁₅₀	13.15-14.30
400	Var.	E. to N.E. 2.	16.0 -11.25	16.5 -10.59
Calm to Southerly airs.											
26th September 1908.			28th September 1908.			29th September 1908.			30th September 1908.		
Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.
Surf.	...	11.10-11.20	4·0	₄ S. 70 W. ₉₀	11.20-11.38	9·3	₁₄ S. 80 W. ₆₀	9.55-10.55	1·5	...	8.10-9.50
30	2·5	₂ N. 35 W. ₉₀	13.5-13.22	9·0	...	11.15-12.15
60	3·9	₇ N. 15 E. ₁₁₀	11.42-12 15	4·1	var. { ₂ N. 30 W. ₄₀ ₁₁ S. 12 E. ₁₃₀ }	12.40-13.40
90	3·9	var. { ₃ S. 73 W. ₁₁₀ ₁₀ N. 79 E. ₁₆₀ }	14.0-15.0
120	5	var. { ₂ S. 60 W. ₂₀ ₁₂ N. 75 E. ₁₈₀ }	15.20-16.20
135	₁ E.	11.45-12.30
150	₁ N. 30 W.	12.35-14.30
180	...	11.25-11.40
200	3·0	var. { ₄ N. 50 W. ₄₀ ₂ S. 50 E. ₈₀ }	12.23-12.57
250	2·1	₄ S. 100	13.28-15.40
260	₁ N.	16.2-9.35
300	₁ S. 50 W.	14.38-11.10
Calm, with rain.											
S.W. 2-3, ending in rain, and calmer.											
S.W. 3-4.											
S. W.											

STATION C—continued.

10th October 1908.			12th October 1908.			14th October 1908.			15th October 1908.		
Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.
Surf. 2.7	8 S. 5 W. _{.120}	9 6-10.6	10.0	14 N. 60 E. _{.40}	7.3-8.3	1.2	1 N. 40 W.	8.22-9.22	4.8	15 N. 65 E. _{.150}	7.0-8.0
30	1.3	1 N. 50 E.	8.15-9.15	1.5	3 S. 50 E. _{.20}	9.36-10.36	2.9	6 N. 80 E. _{.80}	8.12-9.0
60	1.8	3 S. 55 E. _{.70}	10.22-11.22	1.5	?	10.44-11.44	2.0	10 S. 30 W. _{.70}	9.17-11.6
90	1.2	1 N.	11.33-12.33	1.5	?	12.24-13.24	3.9	13 S. 40 W. _{.100}	11.22-12.22
120 6.7	15 S. 50 W. _{.190}	1.10-2.10	1.2	2 N. _{.40}	12.42-13.42	2.6	5 S. _{.90}	13.40-14.40	4.0	14 S. 65 W. _{.150}	12.34-13.34
150	1.1	1 N. 10 E.	13.53-14.53	3.7	12 S. 75 E. _{.200}	14.55-15.55	2.4	10 S. 70 W. _{.180}	14.15-15.37
180	3.0	10 S. 60 W. _{.130}	15.15-16.15	2.2	...	16.0-17.0
250	1.1	1 N. 20 W.	16.33-17.33
300
350	2 S. 20 W. _{.60}	17.50-7.56	8	Var.	16.24-6.40	8	Var.	17.17-9.20
400	16.35-9.1
S.W. 3-4. S.W. gales on 11th.			S.W. 1-3 to calm. S.W. 1-3 to calm.			S.W. 2-3. S.W. 2-3.			N.E. 1-3. * Compass needle off.		
16th October 1908.			17th October 1908.			19th October 1908.			20th October 1908.		
Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.
Surf. 9.8	13 N. 70 E. _{.90}	9.45-10.45	5.1	13 S. 15 W. _{.40}	9.55-10.55	3.2	9 S. 15 W. _{.90}	8.50-9.50
30 5.1	...	11.9-12.9	2.2	5 N. 5 W. _{.10}	10.36-11.36	3.8	12 S. 70 W. _{.10}	11.17-12.17	2.6	7 W. _{.40}	10.30-11.30
60 4.4	Var.	12.25-13.25	2.9	10 S. 5 E. _{.10}	11.44-15.45	1	4 N. 55 E. _{.20}	12.52-15.13	1.4	1 N. 50 W.	11.53-12.53
90	2.4	7 S. 20 W. _{.0}	15.30-16.30	3.2	8 N. 20 W. _{.30}	13.10-14.10
120 1.3	...	13.45-16.45	2.4	Var.	16.41-17.41	2.5	6 S. 25 W. _{.60}	14.23-15.23
150	3.5	11 N. 80 E. _{.100}	15.38-16.38
300	8	4 S. 45 E. _{.10}	17.52-8.30	8	4 S. 50 W. _{.100}	17.1-6.52
350	8	Light var. winds.	17.0-7.6
N.E. 3-4. * Compass needle off.			Light var. winds.			S.W. 1 unsettled. S.W. 1 unsettled.			S.W. 1. S.W. 1.		

STATION C—continued.

	21st October 1908.			22nd October 1908.			23rd October 1908.			24th October 1908.		
	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.	Velo-city.	Direction.	Hour.
Surf.	7.4	₂₁ S. 75 E. ₁₅₀	8.0-9.0	1.2	₂ N. 80 W. ₄₀	9.20-10.20	.8	...	9.55-10.55	1.0	...	10.16-11.16
30	1.2	{ ₁ S. ... ₁ N. 80 E. }	15.50-16.50
90	1.5	₂ N. 30 W. ₁₀₀	10.32-11.32	5.7	₇ S. 60 W. ₁₀	11.2-12.2	2.0	...	11.22-13.17
120	1.0	...	14.57-16.57
150	3.9	{ ₉ S. 70 W. ₅₀ ₄ N. 35 E. ₅₀ }	9.19-10.19	.8	...	11.55-12.55
180	1.0	₁ S. 60 W.	12.17-13.17	.8	...	13.48-14.48
200	1.6	₃ S. 80 E. ₁₀	15.45-16.45	1.0	₁ N. 20 E.	13.10-15.10
225	2.3	₁₄ N. 80 E. ₁₂₀	10.38-12.38	.8	...	15.25-16.55
250	1.8	₃ N. 60 W. ₃₀	13.0-14.08	₁ N. 80 E.	13.35-15.35
3008	₁ S. 50 W.	17.0-9.30
400	.8	...	17.15-8.40
			Gusty var. winds to 4.		S. W. 2-3.				Calm to S. W. 1-2 to calm.			Calm to northerly airs.

(Issued separately October 1, 1909.)

XXXIX.—Hydrolysis of Salts of Amphoteric Electrolytes. By Miss Heather Henderson Beveridge, B.Sc., Carnegie Research Scholar. Communicated by Professor JAMES WALKER.

(MS. received June 12, 1909. Read June 21, 1909.)

DIFFERENT experimenters on the degree of hydrolysis of amphoteric electrolytes in acid solution have obtained very discordant results, according to the method they employed.

Winkelblech,* for instance, from his work on the rate of catalysis of methyl acetate by ortho-amidobenzoic acid hydrochloride at 25° C., estimates the hydrolysis constant $\frac{k_b}{K}$ at 120.

From the electrical conductivity of the same solution at $v=16$ he gives the value 180 for $\frac{k_b}{K}$.

The calculation of the hydrolysis constant in these two cases rests on quite a different basis. The rate of catalysis depends simply on the concentration of hydrion in the solution, and therefore should give as directly as possible the amount of free acid present, and so the hydrolysis.

The conductivity of the solution, on the other hand, is the sum of the molecular conductivities of all the substances present. It depends, therefore, on various factors, any of which may conceivably exert a disturbing influence on the others.

There are present in the solution—

(1) The Salt, of the general formula $HRCl$, probably largely dissociated into its ions, HR and Cl .

(2) Hydrochloric acid, formed by the hydrolytic action of water, and also highly ionised.

(3) The amphoteric base $HROH$, equivalent in amount to the free acid. It probably contributes but little to the total conductivity, as it is only slightly dissociated.

(4) Water practically un-ionised.

It was thought, in view of these results, that some effort should be made to find out what values are given by methods independent of these, in order to be in a position to judge which of the original methods is at fault, and, if possible, for what reason.

* *Zeitschrift für Physikal. Chem.*, vol. xxxvi. p. 546.

Ortho-amidobenzoic acid was chosen as the most convenient example of an amphoteric base whose hydrolysis in acid solution was neither too great nor too small at the temperature and dilutions examined, and therefore most typical of the class.

ELECTRICAL CONDUCTIVITY.

The conductivity experiments were all carried out in a thermostat at 25° C. The water used was distilled in the open air, using a block tin condenser, and its conductivity at 25° varied from .8 to 1.3×10^{-6} .

The degree of hydrolysis, x , was obtained from the formula $x = \frac{M_v - \mu_v}{\mu_{\text{HCl}} - \mu_v}$,* where M_v is the observed molecular conductivity and μ_v the molecular conductivity of the unhydrolysed salt.

The values for μ_v were taken from Winkelblech. His numbers in Siemens mercury units were recalculated into reciprocal ohms.

v .	μ_v .
16	102.6
32	106.8
64	109.9
128	113.2
256	115.4

The values for μ_{HCl} were found by experiment. The original $\frac{N}{16}$ solution for each set of observations was made up by weighing out the required amount of o-amidobenzoic acid, adding the corresponding quantity of $\frac{N}{8}$ HCl, of which a large stock was kept, and making up to $\frac{N}{16}$ with conductivity water. The subsequent dilutions were sometimes carried out by means of Arrhenius' pipettes, sometimes by mixing equal quantities of water and solution in resistance-glass bottles.

v .	μ_{HCl} .	μ_v .	M_v .	x .	$\frac{k_b}{K} = \frac{(1-x)v}{x^2}$.	Winkelblech.
16	392	102.6	173.6	.245	201	
32	400	106.8	207.2	.34	180	177
64	407	109.9	246.7	.46	163	164
128	413	113.2	288.0	.58	158	155
256	417	115.4	327.1	.70	155	150

* Bredig, *Zeit. Physikal. Chem.*, vol. xiii. p. 321.

The conductivity was also taken of a series of solutions in which the concentration of acid was kept constant, but the concentration of the base was varied. The hydrolysis in this case was calculated from the specific conductivity. In the case of the $\frac{N}{16}$ solution, for example, the calculation was as follows:—

$$S = \text{Specific conductivity of salt} = \frac{102.6}{16} = 6.4.$$

$$H = \text{,, ,, of HCl} = \frac{392}{16} = 24.5.$$

M = Observed specific conductivity.

x = Amount of base hydrolysed.

When acid and base are in equivalent proportions, $M = (1 - x) S + xH$.

When the concentration of base is half that of the acid, $M = (.5 - x) S + (.5 + x) H$.

When the concentration of base is .25, that of the acid $M = (.25 - x) S + (.75 + x) H$.

The hydrolysis is of course at a maximum when acid and base are equivalent; and it will also be noted, although it does not follow from the foregoing, that $\frac{k_b}{K}$ is at a minimum with equivalent proportions of acid and base. The variation from the mean becomes less and less as the acid used is more dilute.

(1) *Constant Concentration of Acid* = $\frac{N}{16}$.

Concentration of Base.	M_v .	Specific Conductivity.	x .	$\frac{k_b}{K}$.
16	173.1	10.82	.244	200
32	262.2	16.4	.053	256

With smaller concentrations of base, $\frac{k_b}{K}$ becomes still greater.

(2) *Constant Concentration of Acid* = $\frac{N}{32}$.

Concentration of Base.	M_v .	Specific Conductivity.	x .	$\frac{k_b}{K}$.
16	141.5	4.421	.1180	213.9
32	206.5	6.452	.3398	182.7
64	284.0	8.876	.2086	200.9
			Mean	199.2

(3) *Constant Concentration of Acid* = $\frac{N}{64}$.

Concentration of Base.	M_v .	Specific Conductivity.	x .	$\frac{k_b}{K}$.
32	178.0	2.781	.2290	175.1
64	246.2	3.848	.4590	164.3
128	310.6	4.852	.3508	175.4
256	353.9	5.530	.2856	190.5
			Mean	176.3

(4) *Constant Concentration of Acid* = $\frac{N}{128}$.

Concentration of Base.	M_v .	Specific Conductivity.	x .	$\frac{k_b}{K}$.
32	170.0	1.328	.1913	169.5
64	222.3	1.737	.3678	160.8
128	284.7	2.224	.5780	161.7
256	335.6	2.622	.4994	171.1
			Mean	165.8

In order to be sure that the nitrate of anthranilic acid would have acted in the same way as the hydrochloride, its hydrolysis was calculated both from its electrical conductivity and rate of catalysis. The results are seen to differ in just the same way as those of the hydrochloride.

CONDUCTIVITY OF ANTHRANILIC NITRATE.

μ_v was taken as being roughly the same as for the hydrochloride, and the conductivity of nitric acid was taken for μ_{HNO_3} .

v .	μ_v .	μ_{HNO_3} .	M_v .	x .	$\frac{k_b}{K}$.
8	96	379.4	138.7	.1507	299
16	102	385	169	.2352	221

RATE OF CATALYSIS OF METHYL ACETATE BY ANTHRANILIC NITRATE.

For comparison, a solution was catalysed at the same time by nitric acid of the same strength approximately as the nitrate solution.

20 c.c. $\frac{N}{8}$ Anthranilic Nitrate. 1 c.c. Methyl Acetate.				5 c.c. $\frac{N}{8}$ HNO ₃ , 15 c.c. $\frac{N}{8}$ KNO ₃ . 1 c.c. Methyl Acetate.		
<i>t.</i>	<i>x.</i>	<i>A - x.</i>	<i>k.</i>	<i>x.</i>	<i>A - x.</i>	<i>k.</i>
16.3	3.65	19.45	102.7	3.68	18.22	(140.0)
88.3	14.0	9.1	104.9	14.9	8.0	119.0
144.0	17.9	5.2	103.1	18.8	4.1	119.3
210.0	20.6	2.5	105.5	20.8	2.1	113.6
		Mean	. 104.0		Mean	. 117.3

$$\text{Hydrolysis at } \frac{N}{8} = \frac{104}{117.3} \times .25 = .2222.$$

$$\therefore \frac{k_b}{K} = 126.$$

It was thought possible that the apparently anomalous behaviour was not caused entirely by the amphoteric character of the base, but was due perhaps to the fact that in the example taken the hydrolysis was neither very great nor very small, and therefore more affected by errors in calculation caused by flaws in the theory. It was therefore necessary to test a non-amphoteric base also, with about the same degree of hydrolysis. The most convenient one answering to this description was thiazole,* C₃H₃NS, and accordingly experiments were made on the rate of catalysis of methyl acetate and conductivity of thiazole hydrochloride.

Conductivity.—In order to obtain a value for μ^v —the molecular conductivity of the unhydrolysed salt—the conductivity was taken of a solution in which thiazole was molecular-normal, and hydrochloric acid sixteenth-normal. In this way the hydrolysis of the sixteenth-normal hydrochloride present was reduced very considerably. (This device † could not be adopted in the case of anthranilic hydrochloride on account of the very slight solubility of the base.) But Bredig's plan is not applicable here in its simple form, because the hydrolysis is so great

* Walker, *Zeit. Physikal. Chem.*, vol. iv. p. 332.

† Bredig, *loc. cit.*, p. 214.

that it cannot be reduced to a negligible quantity merely by the presence of an excess of base.

As the presence of so much thiazole would increase the viscosity of the solution quite appreciably, a check experiment was carried out, using KCl instead of HCl, in order to find the approximate correction which must be applied on this account.

$$\frac{N}{16} \text{ KCl in N. thiazole } \mu_v = 112.7$$

$$\frac{N}{16} \text{ KCl in water } = 129.7$$

$$\frac{N}{16} \text{ HCl in thiazole } = 91.1$$

μ_v for $\frac{N}{16}$ thiazole hydrochloride in pure water instead of in thiazole will therefore be approximately $91.1 \times \frac{129.7}{112.7} = 104.8$.

But the amount of hydrolysis still not eliminated must be corrected for by a double approximation. Using the value $\mu_v = 104.8$ we find the hydrolysis in $\frac{N}{16}$ solution to be—

$$\frac{M_v - \mu_v}{\mu_{\text{HCl}} - \mu_v} = \frac{176 - 104.8}{392 - 104.8} = .248.$$

We must therefore subtract for the conductivity of the HCl present (supposing the hydrolysis to be reduced to one-sixteenth of what it would be in equivalent $\frac{N}{16}$ solution)—

$$(392 - 104.8) \frac{.248}{16} = 4.4.$$

This gives us for the second approximation to μ_v $104.8 - 4.4 = 100.4$.

The hydrolysis will now be—

$$\frac{M_v - \mu_v}{\mu_{\text{HCl}} - \mu_v} = \frac{176 - 100.4}{392 - 100.4} = .26.$$

Using this value in finding the correction for the HCl present, we have

$$(392 - 100.4) \frac{.26}{16} = 4.7.$$

The most probable value for μ_v at $v = 16$ is therefore $104.8 - 4.7 = 100.1$.

Now, assuming μ_v to increase with dilution at the same rate as Winkelblech's values for o-amidobenzoic acid, we have $t = 25.0^\circ \text{ C}$.

$v.$	$\mu_v.$	$\mu_{\text{HCl.}}$	$M_v.$	$x.$	$\frac{k_b}{K}$
16	100	392	176.1	.26	174.8
32	104	400	201.4	.33	198.2

RATE OF CATALYSIS BY THIAZOLE HYDROCHLORIDE.

$t = 25.0^\circ \text{C. } 20 \text{ c.c. } \frac{N}{16} \text{ Thiazole Hydrochloride.}$ 1 c.c. Methyl Acetate.					5 c.c. $\frac{N}{16} \text{ HCl, } 15 \text{ c.c. } \frac{N}{16} \text{ KCl.}$ 1 c.c. Methyl Acetate.				
$t.$	$x.$	$A - x.$	$k.$	Another Catalysis.	$t.$	$x.$	$A - x.$	$k.$	Another Catalysis.
360	.85	21.60	1067	1041	360	.82	21.63	1028	1067
1500	3.25	19.20	1041	1059	1500	3.12	19.33	995	1009
3090	6.17	16.28	1039	1051	3090	5.95	16.50	995	1021
5990	10.53	11.92	1055	1097	5990	10.27	12.18	1020	1023
7260	12.20	10.25	1079	1086	7260	11.72	10.73	1016	1023
Mean = 1060					Mean = 1015				

Hydrolysis at $v = 16$ is $\frac{1060}{1015} \times .25 = .2160 \therefore \frac{k_b}{K} = 173.$

20 c.c. $\frac{N}{32} \text{ Thiazole Hydrochloride.}$ 1 c.c. Methyl Acetate.				6.7 c.c. $\frac{N}{32} \text{ HCl, } 13.3 \text{ c.c. } \frac{N}{32} \text{ KCl.}$ 1 c.c. Methyl Acetate.			
$t.$	$x.$	$A - x.$	$k.$	$t.$	$x.$	$A - x.$	$k.$
1500	2.13	20.32	6639	2880	3.90	18.55	6610
2880	3.95	18.50	6707	4050	5.25	17.20	6569
6840	8.38	14.07	6825	6840	8.65	14.20	6688
8620	9.95	12.50	6784	8620	9.90	12.55	6739
Mean = 6739				Mean = 6707			

Hydrolysis at $v = 32$ is $\frac{6739}{6707} \times .3 = .3348 \therefore \frac{k_b}{K} = 190.$

The values for the hydrolysis constant of a non-amphoteric electrolyte as obtained from the two methods are thus seen to agree comparatively well with each other.

$v.$	$\frac{k_b}{K}$ Conductivity.	$\frac{k_b}{K}$ Catalysis.
16	174.8	173.6
32	198.2	190

Evidently the theory holds good so long as the base is non-amphoteric, and either method is equally good as a gauge of the hydrolysis.

The Ratio of Distribution of a base between two solvents has been used by Farmer* as a method of estimating the basic constant. His value for o-amidobenzoic acid at 25° corresponds very well with that obtained from the rate of catalysis of methyl acetate, viz. $\frac{k_b}{K} = 118$.

The Solubility of the amphoteric base in water and in acid of different concentrations was then investigated, as giving a measure of the hydrolysis. Bottles of resistance-glass were used for these estimations. They were almost filled with the solvent, the air above which was replaced by nitrogen in order to preclude oxidation. The liquids were shaken up with recrystallised anthranilic acid in a thermostat at 25° C. At intervals of some hours portions of the liquids were siphoned off through a filter into weighed flasks and titrated against twentieth-normal caustic soda solution.

In the acid solutions it is assumed that the concentration of free base is the same as the concentration of the saturated aqueous solution, and that the extra base dissolved is therefore all in the form of combined hydrochloride. The values obtained in this way for $\frac{k_b}{K}$ are not constant for the different strengths of acid used, but at $v = 16$ especially seems to approximate most closely to the catalysis value.

The solutions containing hydrochloric acid became saturated comparatively quickly, but the o.-amidobenzoic acid dissolved in the water very slowly, the acidity still increasing slightly after several days' shaking.

* *Journal Chem. Soc.*, vol. 79, p. 863 ; vol. 85, p. 1713.

SOLUBILITY IN WATER AT 25·2°.

I.					
		Water.	In	$\frac{N}{16}$ HCl.	
Total acidity =		·04137 N.		·1554 N.	
Free base =		·04137		·04137	
	∴ Salt =			·0515	
	Free HCl =	Total HCl - Salt =		·0110 N.	
	$\frac{k_b}{K} =$	$\frac{\cdot0515}{\cdot0110 \times \cdot04137}$		= 113·2.	

II.			III.		
	Water.	$\frac{N}{16}$ HCl.	Water.	$\frac{N}{24}$ HCl.	$\frac{N}{16}$ HCl.
Total acidity	·04315 N.	·1551 N.	·03723	·1106	·1493
Free base	...	·04315	...	·03723	·03723
Salt	...	·05110	...	·0317	·04957
Free HCl	...	·00975	...	·01007	·01293
$\frac{k_b}{K}$...	121·4	...	85·2	103·0

$\frac{N}{32}$ HCl was also used in this way and gave still lower values of $\frac{k}{K}$.

The *Rate of Catalysis of Diazo-acetic ester* by nitric acid is a possible method of estimating the concentration of hydrion, and therefore the hydrolysis in a solution.* It was found on investigation to give quite reliable results, and was therefore used to estimate the hydrolysis of anthranilic nitrate solution. The hydrochloride could not be used for this catalysis on account of secondary reactions with the ester.

About ·1 of a gram of diazo-acetic ester was weighed out and dissolved in a known volume of water in a small flask. A small quantity of the acid or nitrate solution was put into the flask in a small test-tube to avoid any mixing of the liquids. The flask was immersed in a thermostat at 25° C. and connected to a gas-burette in a water-jacket at room temperature.

When the volume of the contained air was steady, the liquids in the flask were mixed, and readings of the volume were taken every few minutes. The velocity constant was calculated for a reaction of the first degree. In the nitric acid catalysis, enough potassium nitrate was dissolved in the water to reproduce the conditions in the solution of anthranilic nitrate.

* Bredig and Fraenkel, *Zeit. für Electrochemie*, vol. xi. p. 525.

I.—CATALYSIS BY $\frac{N}{320}$ NITRIC ACID.

37 c.c. Water, 2 c.c. $\frac{N}{16}$ KNO ₃ , 1 c.c. $\frac{N}{8}$ HNO ₃ in tube.				
·100 gm. Ester.				·102 gm. Ester.
<i>t.</i>	<i>x.</i>	<i>A - x.</i>	<i>k.</i>	<i>k.</i>
0	0	20·2		
5	8·4	11·8	(·1074)	
6	9·8	10·4	·1105	1096
7	10·9	9·3	·1107	1090
8	11·9	8·3	·1110	1074
9	12·8	7·4	·1115	1076
11	14·0	6·2	(·1073)	(1023)
13	15·1	5·1	(·1058)	
		Mean	· 1109	1084

II.—CATALYSIS BY $\frac{N}{160}$ ANTHRANILIC NITRATE.

38 c.c. Water, 2 c.c. $\frac{N}{8}$ Anthranilic Nitrate in tube.				
·100 gm. Ester.				·102 gm. Ester.
<i>t.</i>	<i>x.</i>	<i>A - x.</i>	<i>k.</i>	<i>k.</i>
6	12·0	8·2	1501	1537
7	13·2	7·0	1512	1545
8	14·2	6·0	1516	1522
9	15·1	5·1	1528	(1555)
10	15·9	4·3	(1545)	(1567)
		Mean	· 1514	1534

Hydrolysis at $v = 160$ is $\frac{1514}{1109} \times \cdot 5 = \cdot 683$.

$$\therefore \frac{k_b}{K} = 108\cdot 7.$$

The constant obtained in this way agrees fairly well with that obtained from the ordinary catalysis of methyl acetate by anthranilic nitrate $\frac{k_b}{K} = 126$.

As another means of determining the hydrolysis of anthranilic hydrochloride, the hydron concentration in its solution was estimated by the *electro-motive force method*. The difference of potential between a hydrogen electrode immersed in the solution and a normal calomel electrode was measured. Then a solution of hydrochloric acid giving nearly the same E.M.F. was substituted for the hydrochloride, and the hydron concentration in the latter calculated from the formula $d = 0.0572 \log \frac{c_1}{c_2}$, where c is the hydron concentration of the two solutions, and d the difference between the observed electro-motive forces.*

A moving-coil galvanometer with mirror and scale was used as zero instrument, and by an arrangement of resistances at each end of a metre bridge the length of wire available for measurement was about 10 metres.

The solutions whose difference of potential was being measured were immersed in a thermostat at 25° C., and were connected by side tubes dipping into saturated ammonium nitrate solution. This has been found by Cumming † to be the most effectual eliminator of liquid surface potential.

The hydrogen used was generated from zinc and sulphuric acid, passed through permanganate acidulated with sulphuric acid, and then washed with pure water. It was led into the liquid in the cell through the rubber stopper, and the excess escaped by a side tube with a water-trap.

The hydrogen-platinum electrode was prepared by coating a glass tube with a film of burnt-in platinum, which was then lightly platinised.

The hydrogen had to be passed through the liquid for about twenty minutes before the E.M.F. exhibited any degree of constancy. The current was then stopped and a reading immediately taken, the stream of hydrogen again passed for five minutes and another reading taken, and so on till the E.M.F. did not vary more than 0.2 of a millivolt.

The potential of a normal calomel electrode was taken in this way alternately against hydrochloric acid and anthranilic hydrochloride solutions. After every reading of the unknown E.M.F., that of a standard cadmium cell was measured.

The values for $\frac{k_b}{K}$ obtained in this way are seen to agree fairly well with the results of catalysis experiments.

* Nernst, *Zeit. Physikal. Chem.*, vol. iv. p. 129 (1889).

† *Trans. Faraday Soc.*, ii. p. 213 (1907).

E.M.F. of the combination :—

N. Calomel. $\frac{N}{64}$ HCl/H ₂ Pt.	N. Calomel. $\frac{N}{16}$ Anthranilic Hydrochloride/H ₂ Pt.	Difference.
volt.		
.3832 } .3803 }	.3817	.3773 volt.
.3822 } .3820 }	.3821	.3789 ,,
.3851 } .3839 }	.3845	.3794 ,,
.3834 } .3834 }	.3834	.3777 ,,
.3834 } .3832 }	.3833	.3792 ,,
	Mean difference	.0045 volt.

The ratio $\frac{c_1}{c_0}$ corresponding to this difference is calculated from the formula $d = .0572 \log \frac{c_1}{c_0}$. For a difference of .0045 volt $\frac{c_1}{c_0} = 1.199$. c_0 is the concentration of hydrion in the hydrochloric acid solution, therefore—

$$c_1 = (.015625 \times .9692) \times 1.199 = .01816 \text{ N.}$$

$$\therefore \text{Hydrolysis at } v = 16 \text{ is } \frac{.01816}{.0625} = .2904.$$

$$\text{Whence } \frac{k_b}{K} = 134.6.$$

E.M.F. of combination :—

N. Calomel. $\frac{N}{64}$ HCl/H ₂ Pt.	N. Calomel. $\frac{N}{32}$ Anthranilic Hydrochloride/H ₂ Pt.	Difference.
volt.		
.3839 } .3833 }	.3836	.3898 volt.
.3833 } .3835 }	.3834	.3894 ,,
	Mean difference	.0061 volt.

$$\frac{c_0}{c_1} \text{ for } \cdot 0061 \text{ volt} = 1\cdot 127.$$

$$\therefore c_1 = (\cdot 015625 \times \cdot 9692) \times \frac{1}{1\cdot 27} = \cdot 01192.$$

$$\therefore \text{Hydrolysis at } v = 32 \text{ is } 32 + \cdot 01192 = \cdot 3815,$$

$$\text{and } \frac{k_b}{K} = 136\cdot 0.$$

E.M.F. of combination :—

N. Calomel.	N. Calomel.	Difference.
$\frac{N}{128} \text{ HCl/H}_2\text{Pt.}$	$\frac{N}{64} \text{ Anthranilic Hydrochloride/H}_2\text{Pt.}$	
$\cdot 4001 \text{ volt.}$	$\cdot 3996 \text{ volt.}$	$\cdot 0005 \text{ volt.}$

$$\frac{c_1}{c_0} \text{ for } \cdot 0005 \text{ volt} = 1\cdot 02.$$

$$c_1 = (\cdot 0078125 \times \cdot 9835) \times 1\cdot 02 = \cdot 007839 \text{ N.}$$

$$\therefore \text{Hydrolysis at } v = 64 \text{ is } 64 \times \cdot 007839 = \cdot 5017,$$

$$\text{and } \frac{k_b}{K} = 127\cdot 6.$$

These values for the hydrolysis constant are much nearer the results of catalysis than those of conductivity experiments.

The Depression of the Freezing-Point by solutions of o-amidobenzoic acid as compared with the depression caused by the same concentration of hydrochloric acid was then considered as a measure of the number of molecules in the solution.

The degree of ionisation of the hydrochloride was assumed to be the same as that of the same concentration of hydrochloric acid. Any extra depression must therefore be caused by the free base present, which was considered to be un-ionised. Its concentration, and thus the hydrolysis of the solution, was then easily obtained from the normal molecular depression caused by an un-ionised substance.

The freezing-point of o-amidobenzoic acid itself proved to be quite normal :—

Concentration.	Depression.	Corresponding Molecular Depression.
$\cdot 01717 \text{ N.}$	$\cdot 032^\circ$	$1\cdot 864^\circ$
$\cdot 0214$	$\cdot 040$	$1\cdot 869$
$\cdot 02205$	$\cdot 041$	$1\cdot 860$
	Mean	$1\cdot 864^\circ \text{ C.}$

For one set of experiments the method of Walker and Robertson* was used.

About 80 c.c. of distilled water were cooled to zero in a Dewar bulb surrounded by a jacket of ice and water. The mouth of the inside vessel was closed by a rubber stopper with three holes, for the thermometer, stirring tube, and filter tube respectively. A block of ice was frozen from distilled water and then planed finely. 50-60 grams of this wet ice were put into the cooled water in the vessel and the apparatus closed. When the zero of the thermometer had been determined, the water was siphoned off by the filter tube without moving the thermometer, and the same quantity of solution cooled to zero was introduced through the stirring tube and rinsed down with cooled distilled water. When the mercury was again steady, the solution was filtered off and analysed.

Hydrochloric Acid:—

Concentration.	Depression.	Depression calculated for .05 N.
·05359 N.	·1935°	·1805° C.
·03954	·1410	For .04 N. ·1427° C.

Anthranilic Hydrochloride:—

Concentration.	Depression.	Hydrolysis.	$\frac{k_b}{K}$
·04891 N.	·195°	·2033	394
·05545	·218	·1735	495
·04282	·173	·2335	328

A variation of this method was tried by adding a few c.c. of concentrated solution to the water instead of siphoning it off and adding dilute solution.

Hydrochloric Acid:—

Concentration.	Depression.	Depression for .05 N.
·0494 N.	·1725°	·1756°
·05417	·190	·1772
·0562	·194	·1728
·0619	·213	·1721

* *Proc. Roy. Soc. Edin.*, vol. xxiv. p. 363.

Anthranilic Hydrochloride:—

Concentration.	Depression.	Hydrolysis.	$\frac{k_b}{K}$.
·0412 N.	·1615°	·226	367
·0426	·1630	·177	617
·04636	·1800	·207	399
·0537	·2035	·158	628
·05453	·2055	·152	673
·05586	·2140	·186	421
·0596	·2240	·146	672
·0598	·2250	·148	650

The convergence temperature method of Nernst and Abegg* was also used for some freezing-point determinations. The apparatus consisted of a Jena glass beaker stirred mechanically by a circular silver disc just fitting into it, with a central hole for the thermometer. The beaker was supported inside a brass cylinder fitted with a lid, and this, again, was surrounded by a freezing mixture of ice and salt solution.

The convergence temperature of the apparatus was found, and the outside bath kept at the required distance below zero. 200 c.c. of water were overcooled in the beaker 2–2·5° C. It was then put in position and the lids closed. When the temperature (observed through a reading telescope) had become steady, the stirring was commenced, and this froze out the ice while the temperature rose to zero. The freezing-point was noted, and enough of a cooled concentrated solution added to make the whole quantity about twentieth-normal, when the new freezing-point could be observed.

Hydrochloric Acid:—

Concentration.	Depression.	Depression for ·05 N.
·04818 N.	·171°	·1775°
·04822	·172	·1783
·04827	·171	·1771
·04827	·173	·1792
	Mean	·1780°

* *Zeit. Physikal. Chem.*, vol. xv. p. 681.

Anthranilic Hydrochloride:—

Concentration.	Depression.	Hydrolysis.	$\frac{k_b}{K}$.
·0438 N.	·173°	·211	404
·0450	·177	·202	434
·04456	·173	·1733	618
(·0490	·187	·139	909)
·0500	·198	·216	336
·05082	·202	·223	307

The values for $\frac{k_b}{K}$ obtained in this way are consistently very much higher even than the results of the conductivity method. For a fair comparison with them, the conductivity was measured at zero, and the hydrolysis constant calculated for that temperature. The value for M_v at 25° C. had also to be reduced to zero, and in order to find its approximate coefficient the conductivity of cinchonin hydrochloride was taken at 25° and at 0° C. Cinchonin was chosen as a base whose positive ion had about the same speed as the ion of anthranilic acid.*

CONDUCTIVITY OF CINCHONIN HYDROCHLORIDE.

25° C.	$v=16$	$\mu=76\cdot4$
0° C.	$v=16$	$\mu=39\cdot9$

The temperature coefficient was taken as being approximately ·5, and μ_v for anthranilic hydrochloride at zero calculated from the formula—

$$\begin{aligned} \mu_0 &= \mu_{25} \{1 - (25 \times \cdot 05k)\} \\ \therefore \text{ at } v=16 \mu_0 &= 102\cdot6 (1 - \cdot 625) \\ &= 38\cdot62. \end{aligned}$$

CONDUCTIVITY OF ANTHRANILIC HYDROCHLORIDE AT ZERO.

v .	μ_{HCl}	μ_v .	M_v .	x .	$\frac{k_b}{K}$.
16	247	38·6	93·5	·2634	169·8
20	249	39·0	98·3	·2824	180·0

We see that the depression of the freezing-point gives values for the hydrolysis constant which differ very widely from the conductivity values,

* Bredig, *loc. cit.*, p. 231.

and still more from the results of catalysis. We find, however, that those methods of ascertaining the hydrolysis of *o*-amidobenzoic salts which depend directly on the concentration of hydrion in the solution agree fairly well with each other, *i.e.* catalysis of methyl acetate by the hydrochloride, and of diazoacetic ester by the nitrate, and the electro-motive force. The solubility and distribution between two solvents also point to the same value being correct. The question is, then, why should not the more indirect methods—conductivity and freezing-point—lead to the same result?

The molecular conductivity of a sixteenth normal solution of anthranilic hydrochloride is 173. The amount contributed to this by the hydrion present is, according to the catalysis value for the hydrolysis, $\cdot 313 \times \frac{392}{420} \times 352 = 102\cdot 7$. Then, assuming the chloride to be ionised to about the same extent, the conductivity supplied by chloridion is $\cdot 9 \times 68 = 61\cdot 2$. Thus there is left for the anthranilic ion $M = 173 - (102\cdot 7 + 61\cdot 2) = 9$, which is not nearly enough to correspond with 68·7 per cent. of chloridion.

Or we can account for the specific conductivity of the solution in this way. The total specific conductivity is $\frac{173}{16} = 10\cdot 8$. The hydrochloric acid present supplies $\cdot 313 \times \frac{392}{16} = 7\cdot 6$. This leaves a specific conductivity for the salt of 3·2, *i.e.* a molecular conductivity of 51·2—not nearly enough for the 68·7 per cent. of anthranilic hydrochloride.

Evidently the method of calculating the hydrolysis from conductivity and freezing-point data is not legitimate, on account of some factor being overlooked, or perhaps unknown. We assume, for example, that the degrees of ionisation of the acid and salt, and the speeds of their ions, are quite normal, and that the water itself acts quite normally even in presence of the large quantities of ions formed from the strong electrolytes present.

In accounting for the conductivity, the small amount left for the hydrochloride seems to point to its ionisation being abnormally small. If this were so in the comparatively dilute sixteenth-normal solution, it would show up much more in a more concentrated, say half-normal, solution, where the ionisation would probably be reduced to zero, and the hydrolysis, as calculated from the conductivity at that concentration, would therefore appear to differ still more from the catalysis value.

CONDUCTIVITY OF $\frac{N}{2}$ ANTHRANILIC HYDROCHLORIDE.

$$\begin{aligned}
 M_v \text{ observed} &= 122.7. \\
 \mu_{\text{HCl}} \text{ ,,} &= 332.4. \\
 \mu_v \text{ approximately} &= 95. \\
 \alpha &= \frac{M_v - \mu_v}{\mu_{\text{HCl}} - \mu_v} = 0.117. \quad \frac{k_b}{K} = 129.0.
 \end{aligned}$$

From this result, however, we see that the ionisation cannot diverge very widely from the normal value, as the hydrolysis in half-normal solution is quite what one would expect it to be.

The speed of the ions is another factor in the calculation which is assumed to be quite normal without any particular justification. The viscosities of sixteenth- and of half-normal anthranilic solutions were therefore compared with those of solutions of hydrochloride acid at the same dilution.

Time taken in seconds for the same volume of liquid to run through the capillary. Thermostat 24.9° C.:—

	Water.	$\frac{N}{16}$ HCl.	$\frac{N}{16}$ Anthranilic Hydrochloride.
	125.5	128.0	127.2
	126.6	128.0	128.5
	126.0	127.6	127.8
	125.4	127.4	127.0
Mean	125.9	127.75	127.6

	Water.	$\frac{N}{2}$ HCl.	$\frac{N}{2}$ Anthranilic Hydrochloride.
	124.6	127.2	143.0
	124.8	126.0	142.5
	124.8	127.0	
	124.8	127.0	
Mean	124.8	126.8	142.7

Even in half-normal solution we see that the difference is not nearly enough to account for the great reduction in the conductivity. It may

therefore be assumed in calculation that the speed of the ions is normal.

Another possibility which was considered was that the ionising power of the water might be altered in presence of the strong electrolytes in the solution.

To test this supposition, the conductivity of a series of solutions of hydrochloric acid diluted with $\frac{N}{8}$ acetic acid was compared with the conductivity of a series diluted with water. With the same object, solutions of KCl were diluted with water and NH_4OH , and solutions of NaCl with water and dimethylamine, respectively, and the conductivity of the resulting solutions taken. The pairs of substances were chosen with positive ions of as nearly as possible the same speed, so that double decomposition alone would not affect the conductivity.

I.—24.9° C. CONDUCTIVITY OF HYDROCHLORIC ACID, DILUTED WITH—

<i>c.</i>	(1) Water.		(2) $\frac{N}{8}$ Acetic Acid.	
16	391.7	392.3	387.4	389.3
32	401	399.2	398.5	397.8
64	407.7	405.8	403.4	404.8
128	413.6	412.2	410.6	410.1

Molecular conductivity of the acetic acid = 4.65.

II.—25° C. $\frac{N}{8}$ POTASSIUM CHLORIDE DILUTED TO $\frac{N}{16}$ WITH—

(1) Water.

(2) $\frac{N}{8}$ NH_4OH .

$\mu = 131.6$.

135.2.

III.— $\frac{N}{8}$ SODIUM CHLORIDE DILUTED TO $\frac{N}{16}$ WITH—

(1) Water.

(2) Dimethylamine .0578 N.

110.2.

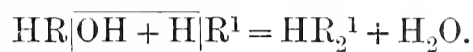
131.5.

Molecular conductivity of the dimethylamine = 21.

In every case we see the resulting conductivity to be just what would be expected from the ordinary dissociation theory, so that no abnormality need be looked for in this respect.

We are forced to conclude that there are fewer molecules in the solution of the salt of an amphoteric electrolyte than there would be in the

case of a non-amphoteric substance. This may be accounted for by some kind of association of molecules or ions. A molecule of free base may associate with the positive ion of the salt to form a complete ion—



From the normal depression of the freezing-point, we see that in solutions of anthranilic acid itself there is no such association, but it is quite possible that the ion may have a more strongly associative tendency than the complete molecule. For, in the formation of the ion HR^1 from a molecule of base R or HROH , a hydroxyl group is taken away, leaving a still possible hydrogen ion. This makes the ion HR of an acidic nature, and it would naturally tend to form an internal salt with a fresh basic molecule.

(Issued separately October 14, 1909.)

XL.—The Superadjugate Determinant and Skew Determinants having a Univarial Diagonal. By Thomas Muir, LL.D.

(MS. received June 21, 1909. Read July 12, 1909.)

1. THE object of the present paper is to make some advance in the theory of those skew determinants that have the elements of the diagonal all equal. A few results on determinants of more general form and on related subjects are given incidentally.

2. It is recalled as a preliminary that Cayley's first paper on skew determinants (*Crelle's Journ.*, xxxii., 1846, pp. 119–123) really concerned something else, namely, the construction of a general orthogonant. Taking a skew determinant $|a_{11}a_{22} \dots a_{nn}|$, or Δ say, having 1 for each of its diagonal elements, he formed a square array with

$$\frac{2a_{ss}A_{rs}}{\Delta} \text{ in the } (r,s)^{\text{th}} \text{ place,}$$

$$\text{and } \frac{2a_{rr}A_{rr}}{\Delta} - 1 \text{ in the } (r,r)^{\text{th}} \text{ place,}$$

and affirmed that these n^2 elements satisfied the conditions laid down for the coefficients of an orthogonal substitution. As a result it followed, of course, that

$$\begin{vmatrix} 2a_{11}A_{11} - \Delta & 2a_{22}A_{12} & 2a_{33}A_{13} & \dots \\ 2a_{11}A_{21} & 2a_{22}A_{22} - \Delta & 2a_{33}A_{23} & \dots \\ 2a_{11}A_{31} & 2a_{22}A_{32} & 2a_{33}A_{33} - \Delta & \dots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \Delta^n.$$

$a_{rs} = -a_{sr}$
 $a_{rr} = 1$

Now, there is nothing in Cayley's construction that limits its application to a unit-axial skew determinant, and it is therefore suggested to inquire what the construction leads to when the elements of the originating determinant $|a_{11}a_{22} \dots a_{nn}|$ have no conditions whatever attached to them. In other words, the determinant just written at length, and found in the particular case specified to be equal to Δ^n , is like the adjugate a derivative of *any* determinant, and as such is worthy of investigation apart altogether from the peculiar circumstances under which it first made its appearance. In fact, as each element of it is a degree higher than each element of the

adjugate, it may be approximately spoken of as the *superadjugate*. It may also be convenient to denote it by S , and its $(p,q)^{th}$ element by s_{pq} .

3. *The product of the h^{th} column of the superadjugate by the k^{th} column of the original determinant Δ is $-a_{hk}\Delta$ if h be different from k , and is $+a_{hh}\Delta$ if h and k be the same.* (i.)

For

$$\begin{aligned} & (s_{1h}, s_{2h}, \dots, s_{nh} \checkmark a_{1k}, a_{2k}, \dots, a_{nk}) \\ &= 2a_{hh}(A_{1h}, A_{2h}, \dots, A_{nh} \checkmark a_{1k}, a_{2k}, \dots, a_{nk}) - a_{hk}\Delta \\ &= \begin{cases} 2a_{hh} \cdot 0 - a_{hk}\Delta & \text{if } h \neq k, \\ 2a_{hh} \cdot \Delta - a_{hk}\Delta & \text{if } h = k. \end{cases} \end{aligned}$$

From this it follows that

$$S \cdot \Delta = \begin{vmatrix} a_{11}\Delta & -a_{12}\Delta & -a_{13}\Delta & \dots \\ -a_{21}\Delta & a_{22}\Delta & -a_{23}\Delta & \dots \\ -a_{31}\Delta & -a_{32}\Delta & a_{33}\Delta & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

and thence we conclude that *The superadjugate of any determinant Δ is equal to $\Delta^{n-1}\bar{\Delta}$, where $\bar{\Delta}$ is what Δ becomes on changing the sign of all the non-diagonal elements.* (ii.)

If, then, Δ be any skew determinant, that is to say, a skew determinant with any kind of diagonal whatever, $\bar{\Delta}$ is obtainable from Δ by merely changing rows into columns. We are thus led to affirm that *The superadjugate of any skew determinant Δ is equal to Δ^n .* (iii.)

The case of this where the diagonal elements of Δ are all 1 is the property of Cayley's orthogonant with which we started. It may be noted that this particular case can also be established independently by multiplying S and Δ together *row-wise*.

4. The condition that the superadjugate may be the n^{th} power of the original is seen from (ii.) to be $\Delta = \bar{\Delta}$. But by expressing Δ as a series arranged according to products of the diagonal elements there is obtained

$$\begin{aligned} \Delta = & a_{11}a_{22} \dots a_{nn} + \sum (a_{11}a_{22} \dots a_{n-2, n-2} \cdot | a_{n-1, n-1} a_{nn} |_0) \\ & + \sum (a_{11}a_{22} \dots a_{n-3, n-3} \cdot | a_{n-2, n-2} a_{n-1, n-1} a_{nn} |_0) \\ & + \dots \dots \dots \\ & + | a_{11}a_{22} \dots a_{nn} |_0, \end{aligned}$$

where the subscript zero is used to indicate that the determinant to which it is attached has had all its diagonal elements changed to 0: and treating $\bar{\Delta}$ in the same way we have

$$\begin{aligned} \bar{\Delta} = & a_{11}a_{22} \dots a_{nn} + \sum (a_{11}a_{22} \dots a_{n-2, n-2} \cdot | a_{n-1, n-1} a_{n, n} |_0) \\ & - \sum (a_{11}a_{22} \dots a_{n-3, n-3} \cdot | a_{n-2, n-2} a_{n-1, n-1} a_{nn} |_0), \\ & + \dots \dots \dots \\ & + (-1)^n | a_{11}a_{22} \dots a_{nn} |_0, \end{aligned}$$

the two expansions thus reached differing only in the signs of the terms which involve zero-suffixed determinants of odd order. The consequence is that for equality of Δ and $\bar{\Delta}$ as a condition we may substitute the vanishing of the sum of the said terms. Hence *The superadjugate of Δ will be Δ^n when and only when*

$$\sum(|a_{11}a_{22}a_{33}|_0 \cdot a_{44}a_{55}a_{66} \dots) + \sum(|a_{11}a_{22}a_{33}a_{44}a_{55}|_0 \cdot a_{66} \dots) + \dots = 0. \quad (\text{iv.})$$

Thus, the superadjugate of $|a_1b_2c_3|$ is $|a_1b_2c_3|^3$ if

$$|a_1b_2c_3|_0 = 0, \quad \text{i.e. if } a_2b_3c_1 + a_3b_1c_2 = 0;$$

and the superadjugate of $|a_1b_2c_3d_4|$ is $|a_1b_2c_3d_4|^4$ if

$$|a_1b_2c_3|_0 d_4 + |a_1b_2d_4|_0 c_3 + |a_1c_3d_4|_0 b_2 + |b_2c_3d_4|_0 a_1 = 0,$$

i.e. if

$$(a_2b_3c_1 + a_3b_1c_2)d_4 + (a_2b_4d_1 + a_4b_1d_2)c_3 + (a_3c_4d_1 + a_4c_1d_3)b_2 + (b_3c_4d_2 + b_4c_2d_3)a_1 = 0.$$

5. Since the p^{th} row of $|a_{11}a_{22} \dots a_{nn}|$ is

$$a_{p1}, a_{p2}, \dots, a_{pp}, \dots, a_{pn},$$

and the p^{th} column is

$$a_{1p}, a_{2p}, \dots, a_{pp}, \dots, a_{np},$$

it follows that *In any skew determinant the sum of any row and the corresponding column is twice their common element,* (v.)

or, say,

$$\text{row}_p + \text{col}_p = (0, 0, \dots, 2a_{pp}, 0, \dots).$$

From this we can prove that *If Δ be skew, any element s_{pq} of the superadjugate of Δ is*

$$(\text{row}_p \text{ of adjug. of } \Delta \text{ } \checkmark \text{ col}_q \text{ of } \Delta). \quad (\text{vi.})$$

For from (v.) we have

$$\begin{aligned} & (\text{row}_q \text{ of } \Delta + \text{col}_q \text{ of } \Delta \text{ } \checkmark \text{ row}_p \text{ of adjug. of } \Delta) \\ &= (0, 0, \dots, 2a_{qq}, 0, \dots \checkmark A_{p1}, A_{p2}, \dots, A_{pn}) \\ &= 2a_{qq}A_{pq}; \end{aligned}$$

and from the fundamental property of the adjugate

$$(\text{row}_q \text{ of } \Delta \text{ } \checkmark \text{ row}_p \text{ of adjug. of } \Delta) = \begin{cases} 0 & \text{if } p \neq q, \\ \Delta & \text{if } p = q; \end{cases}$$

therefore by subtraction

$$(\text{col}_q \text{ of } \Delta \text{ } \checkmark \text{ row}_p \text{ of adjug. of } \Delta) = \begin{cases} 2a_{qq}A_{pq} & \text{if } p \neq q, \\ 2a_{qq}A_{qq} - \Delta & \text{if } p = q, \end{cases}$$

as was to be proved.

6. From (vi.) it follows immediately that the superadjugate of Δ , when Δ is skew, is the product of Δ by its adjugate, and therefore is Δ^n as we have already seen in (iii.). Further, we have only to think of the case of

this where Δ besides being skew is unit-axial to see that *Cayley's orthogonal is resolvable into determinant factors*—that, in fact, *it is the row-by-column product of Δ^{-1} and Δ , where Δ is the basic determinant, and Δ^{-1} is in the form which is got from the adjugate of Δ by dividing each element by Δ .* (vii.)

7. Confining ourselves now to skew determinants which have a univariar diagonal, let us denote the repeated element of the diagonal by a , so that, Δ being $| a_{11} a_{22} \dots a_{nn} |$, it is understood that $a_{rs} = -a_{sr}$ and $a_{rr} = a_{ss} = \dots = a$.

The first property we note is that *In any skew determinant with univariar diagonal the product of any two rows is the same as the product of the corresponding columns.* (viii.)

This is seen on writing the p^{th} and q^{th} rows

$$\begin{array}{l} a_{p1}, a_{p2}, \dots, a_{pp}, \dots, a_{qq}, \dots, a_{pn} \\ a_{q1}, a_{q2}, \dots, a_{qp}, \dots, a_{qq}, \dots, a_{qn} \end{array}$$

and the p^{th} and q^{th} columns

$$\begin{array}{l} a_{1p}, a_{2p}, \dots, a_{pp}, \dots, a_{qp}, \dots, a_{np} \\ a_{1q}, a_{2q}, \dots, a_{pq}, \dots, a_{qq}, \dots, a_{nq} \end{array}$$

and then noting (1) that $a_{pr} \cdot a_{qr} = a_{rp} \cdot a_{rq}$ because of the skewness, and (2) that

$$a_{pp} \cdot a_{qp} + a_{pq} \cdot a_{qq} = a_{pp} \cdot a_{pq} + a_{qp} \cdot a_{qq}$$

because each of them $= a(a_{pq} + a_{qp}) = 0$.

8. *In any skew determinant with a univariar diagonal the conjugate of any m -line minor is got from the latter by changing the signs of the elements of it that belong to the original diagonal and prefixing to it the sign $(-1)^m$.* (ix.)

For we can multiply the given minor by $(-1)^m$ by changing the signs of all its elements, and we can change all the said signs by altering the elements of the form a_{pq} into a_{qp} , and those of the form a into $-a$, that is to say, by taking the conjugate minor and altering the signs of the elements that belong to the original diagonal.

The number of 2-line minors of Δ that are independent of the diagonal element is $3C_{n,4}$. (x.)

For there are $\frac{1}{2}n(n-1)$ pairs of rows to be considered, each pair having $\frac{1}{2}(n-2)(n-3)$ minors of the kind mentioned, and each minor occurring twice.

9. *If Δ be skew and have a univariar diagonal, the product of the p^{th} row of the superadjugate of Δ by the q^{th} row of Δ is $a_{pq}\Delta$ whether p and q be different or the same.* (xi.)

This is supplementary to (i.) and can be proved in the same way. From the two theorems it follows that, when Δ is skew and has a univariar diagonal, the product of the p^{th} row of S by the q^{th} row of Δ differs only in sign from the product of the corresponding columns if p and q be different, and does not differ at all if p and q be the same.

10. If $|a_{11}a_{22}\dots a_{nn}|$, or Δ say, be a skew determinant with univariar diagonal, the product of the p^{th} row of Δ by the sum of the q^{th} row and q^{th} column of the adjugate of Δ is $2aA_{pq}$. (xii.)

This theorem is due to Torelli (see *Giornale di Mat.*, iii., 1864, pp. 7–10). It may be written :

$$(a_{p1}, a_{p2}, \dots, a_{pn} \text{ } \text{X} \text{ } A_{q1} + A_{1q}, A_{q2} + A_{2q}, \dots, A_{qn} + A_{nq}) = 2aA_{pq},$$

and as, because of the skewness of Δ ,

$$(a_{p1} + a_{1p}, a_{p2} + a_{2p}, \dots, a_{pn} + a_{np} \text{ } \text{X} \text{ } A_{q1} + A_{1q}, A_{q2} + A_{2q}, \dots, A_{qn} + A_{nq}) \\ = 2a_{pp}(A_{qp} + A_{pq})$$

it follows by subtraction that

$$(a_{1p}, a_{2p}, \dots, a_{np} \text{ } \text{X} \text{ } A_{q1} + A_{q1}, A_{q2} + A_{2q}, \dots, A_{qn} + A_{nq}) = 2aA_{qp};$$

in other words, if for “ p^{th} row” in (xii.) we substitute “ p^{th} column,” we must at the same time alter $2aA_{pq}$ into $2aA_{qp}$.

Further, since from (v.) we have

$$(A_{p1}, A_{p2}, \dots, A_{pn} \text{ } \text{X} \text{ } a_{q1} + a_{1q}, a_{q2} + a_{2q}, \dots, a_{qn} + a_{nq}) = 2a_{qq}A_{pq},$$

there results by comparison with (xii.) the curious fact that

$$(a_{p1}, a_{p2}, \dots, a_{pn} \text{ } \text{X} \text{ } A_{q1} + A_{1q}, A_{q2} + A_{2q}, \dots, A_{qn} + A_{nq})$$

is not altered by interchanging the a 's and A 's. (xiii.)

11. It will be convenient to insert here two properties of general determinants, for which no proof is necessary. *The product of the p^{th} row of the adjugate of an n -line determinant Δ by any n elements whatever is the determinant obtained from Δ by substituting for its p^{th} row the said new elements.* (xiv.)

The product of the p^{th} and q^{th} rows of the adjugate of a determinant Δ is obtained from Δ by substituting for its p^{th} row the q^{th} row of the adjugate, or for its q^{th} row the p^{th} row of the adjugate. (xv.)

12. If $|a_{11}a_{22}\dots a_{nn}|$, or Δ say, be a skew determinant with univariar diagonal, then

$$(A_{p1}, A_p, \dots, A_{pn} \text{ } \text{X} \text{ } A_{q1}, A_{q2}, \dots, A_{qn}) = \frac{1}{2}(A_{pq} + A_{qp})\frac{\Delta}{a}$$

whether p and q be different or the same. (xvi.)

This was originally announced for the case where $a = 1$ by Spottiswoode in 1853 (see *Crelle's Journ.*, li. p. 261) as a deduction from the corre-

sponding property of Cayley's orthogonalant. Following Torelli, we have from (xii.), on putting $p = 1, 2, \dots, n$,

$$\left. \begin{aligned} a_{11}(A_{1q} + A_{q1}) + a_{12}(A_{2q} + A_{q2}) + \dots + a_{1n}(A_{nq} + A_{qn}) &= 2aA_{1q} \\ a_{21}(A_{1q} + A_{q1}) + a_{22}(A_{2q} + A_{q2}) + \dots + a_{2n}(A_{nq} + A_{qn}) &= 2aA_{2q} \\ \dots &\dots \\ a_{n1}(A_{1q} + A_{q1}) + a_{n2}(A_{2q} + A_{q2}) + \dots + a_{nn}(A_{nq} + A_{qn}) &= 2aA_{ns} \end{aligned} \right\}$$

and solving we obtain, with the help of (xv.),

$$A_{pq} + A_{qp} = 2a(A_{1q}, A_{2q}, \dots, A_{nq} \text{ } \checkmark \text{ } A_{1p}, A_{2p}, \dots, A_{np}) \div \Delta,$$

which is essentially what was to be proved.

13. If, instead of using (xii.) in the preceding paragraph, we were to use the altered form of it referred to in § 10, we should obtain for $A_{pq} + A_{qp}$ another expression entitling us to assert that *In the case of a skew determinant with a univariial diagonal the product of any two rows of the adjugate is the same as the product of the corresponding columns.* (xvii.)

In conjunction with this it is important to recall (viii.); also careful note should be taken of the complete diversity of the reasoning now employed to establish the same property.

14. Taking the square of the adjugate of Δ , we have from (xvi.)

$$(\Delta^{n-1})^2 = \begin{vmatrix} A_{11} & \frac{1}{2}(A_{12} + A_{21}) & \dots & \frac{1}{2}(A_{1n} + A_{n1}) \\ \frac{1}{2}(A_{12} + A_{21}) & A_{22} & \dots & \frac{1}{2}(A_{2n} + A_{n2}) \\ \dots & \dots & \dots & \dots \\ \frac{1}{2}(A_{1n} + A_{n1}) & \frac{1}{2}(A_{2n} + A_{n2}) & \dots & A_{nn} \end{vmatrix} \cdot \left(\frac{\Delta}{a}\right)^n$$

whence, as Cremona noted,

$$\begin{vmatrix} A_{11} + A_{11} & A_{12} + A_{21} & \dots & A_{1n} + A_{n1} \\ A_{12} + A_{21} & A_{22} + A_{22} & \dots & A_{2n} + A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} + A_{n1} & A_{2n} + A_{n2} & \dots & A_{nn} + A_{nn} \end{vmatrix} = (2a)^n \Delta^{n-2}. \quad \text{(xviii.)}$$

By dividing every row of the determinant on the left here by $2a$ we obtain an expression for Δ^{n-2} , that is to say, for $\Delta, \Delta^2, \Delta^3, \dots$ in the case of the 3rd, 4th, 5th, \dots orders respectively; and as these expressions for $\Delta, \Delta^2, \Delta^3, \dots$ are not the ordinary expressions, there results by comparison a series of interesting identities. For example:

$$x^3 \begin{vmatrix} x & a & b \\ -a & x & c \\ -b & -c & x \end{vmatrix} = \begin{vmatrix} a^2 + x^2 & ab & ac \\ ab & b^2 + x^2 & bc \\ ac & bc & c^2 + x^2 \end{vmatrix},$$

$$\begin{vmatrix} x & a & b & c \\ -a & x & d & e \\ -b & -d & x & f \\ -c & -e & -f & x \end{vmatrix}^2 = \begin{vmatrix} x^2 + d^2 + e^2 + f^2 & -bd - ce & ad - cf & ae + bf \\ -bd - ce & x^2 + b^2 + c^2 + f^2 & -ab - ef & -ac + df \\ ad - cf & -ab - ef & x^2 + a^2 + c^2 + e^2 & -bc - de \\ ae + bf & -ac + df & -bc - de & x^2 + a^2 + b^2 + d^2 \end{vmatrix}.$$

As in the latter example the non-diagonal elements on the right hand are

$$\begin{array}{ccc} -r_1r_2, & -r_1r_3, & -r_1r_4, \\ & -r_2r_3, & -r_2r_4, \\ & & -r_3r_4, \end{array}$$

(r_q being used for the q^{th} row), we should expect the diagonal elements to be

$$\begin{array}{cccc} -r_1r_1, & -r_2r_2, & -r_3r_3, & -r_4r_4, \\ \text{i.e. } -(x^2 + a^2 + b^2 + c^2), & -(x^2 + a^2 + d^2 + e^2), & \dots \end{array}$$

The explanation of the apparent anomaly is that the determinant which is squared on the left being equal to

$$x^4 + x^2(a^2 + b^2 + c^2 + d^2 + e^2 + f^2) + (af - be + cd)^2$$

is not altered by the interchange

$$\begin{pmatrix} a & b & c \\ f & e & d \end{pmatrix};$$

for if this change be made on the right we have only got further to alter the signs in the 1st and 3rd rows and then in the 2nd and 4th columns in order to evolve the familiar result.

15. Denoting the p^{th} row of Δ by r_p and the p^{th} row of the adjugate by R_p , we know (§ 11, xv.) that $R_p R_q$ is expressible as a determinant which is obtainable from Δ by deleting r_q and inserting R_p . As a result of multiplying this determinant by Δ we obtain

$$R_p R_q \cdot \Delta = \begin{vmatrix} r_1r_1 & r_1r_2 & \dots & r_1r_p & \dots & r_1r_n \\ r_2r_1 & r_2r_2 & \dots & r_2r_p & \dots & r_2r_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ r_{q-1}r_1 & r_{q-1}r_2 & \dots & r_{q-1}r_p & \dots & r_{q-1}r_n \\ \dots & \dots & \dots & \Delta & \dots & \dots \\ r_{q+1}r_1 & r_{q+1}r_2 & \dots & r_{q+1}r_p & \dots & r_{q+1}r_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ r_n r_1 & r_n r_2 & \dots & r_n r_p & \dots & r_n r_n \end{vmatrix} = (-1)^{p+q} \Delta \cdot (\Delta_{(q)\Delta_{(p)}},$$

where $\Delta_{(p)}$ means the array got from Δ by deleting the p^{th} row. But from (xvi.) we have

$$R_p R_q = \frac{1}{2}(A_{pq} + A_{qp}) \frac{\Delta}{a};$$

and thus by comparison there results

$$\Delta_{(p)\Delta_{(q)}} = (-1)^{p+q} \cdot \frac{1}{2}(A_{pq} + A_{qp}) \frac{\Delta}{a};$$

so that *In the case of a skew determinant with univariial diagonal the product of any $n-1$ rows by the same or any other $n-1$ rows contains the determinant as a factor.* (xix.)

16. For the purposes of (xviii.) the most suitable expression for A_{pq} is as a series of terms arranged according to powers of a , it being then possible to make, by the application of (x.), an immediate simplification of the expression for $\frac{1}{2}(A_{pq} + A_{qp})$. To obtain the former expression let us pass the p^{th} and q^{th} rows over all the others so as to become the 1st and 2nd, and then do the like with the p^{th} and q^{th} columns. The result

$$| a_{pp}a_{qq}a_{11}a_{22} \dots a_{p-1,p-1}a_{p+1,p+1} \dots a_{q-1,q-1}a_{q+1,q+1} \dots a_{nn} | ,$$

which is known to be still a skew determinant with univariial diagonal, has the element a_{pq} the place (1, 2), and its cofactor A_{pq} is seen to be

$$- | a_{qp}a_{11}a_{22} \dots a_{p-1,p-1}a_{p+1,p+1} \dots a_{q-1,q-1}a_{q+1,q+1} \dots a_{nn} | .$$

We thus learn that *If* $| a_{11}a_{22} \dots a_{nn} |$ be a skew determinant with univariial diagonal, any primary minor of it is expressible as the result of "bordering" a skew determinant of like kind; or, more definitely, is the determinant obtained by prefixing to the secondary coaxial minor

$$| a_{11}a_{22} \dots a_{p-1,p-1}a_{p+1,p+1} \dots a_{q-1,q-1}a_{q+1,q+1} \dots a_{nn} |$$

the row

$$a_{qp}, a_{q1}, a_{q2}, \dots, a_{q,p-1}, a_{q,p+1}, \dots, a_{q,q-1}, a_{q,q+1}, \dots, a_{qn}$$

and the column

$$a_{qp}, a_{1p}, a_{2p}, \dots, a_{p-1,p}, a_{p+1,p}, \dots, a_{q-1,p}, a_{q+1,p}, \dots, a_{np}$$

and making the sign —.

(xx.)

From this it follows that in A_{pq} the terms containing the product of all the original diagonal elements is

$$- a_{qp} \cdot a^{n-2}, \text{ i.e. } a_{pq} \cdot a^{n-2};$$

and that the cofactor of the product of any other number of the said elements is a "bordered" zero-axial skew determinant, and therefore is expressible as the product of two Pfaffians. Thus the cofactor of

$$a_{33} \dots a_{p-1,p-1}a_{p+1,p+1} \dots a_{q-1,q-1}a_{q+1,q+1} \dots a_{nn}$$

is

$$- \begin{vmatrix} a_{qp} & a_{q1} & a_{q2} \\ a_{1p} & \cdot & a_{12} \\ a_{2p} & a_{21} & \cdot \end{vmatrix} \text{ i.e. } a_{12}(a_{1p}a_{q2} - a_{2p}a_{q1} + a_{qp}a_{21});$$

and the cofactor of

$$a_{44} \dots a_{p-1,p-1}a_{p+1,p+1} \dots a_{q-1,q-1}a_{q+1,q+1} \dots a_{nn}$$

is

$$- \begin{vmatrix} a_{qp} & a_{q1} & a_{q2} & a_{q3} \\ a_{1p} & \cdot & a_{12} & a_{13} \\ a_{2p} & a_{21} & \cdot & a_{23} \\ a_{3p} & a_{31} & a_{32} & \cdot \end{vmatrix} \text{ i.e. } - \begin{vmatrix} a_{q1} & a_{q2} & a_{q3} \\ a_{12} & a_{13} & \\ a_{23} & & \end{vmatrix} \cdot \begin{vmatrix} a_{p1} & a_{p2} & a_{p3} \\ a_{12} & a_{13} & \\ a_{23} & & \end{vmatrix} .$$

In the “umbral” notation the former of these is

$$[12][pq12],$$

and the latter

$$-[q123][p123].$$

The rather striking result to which after some difficulty we are finally led is that *If* $|a_{11}a_{22} \dots a_{nn}|$ be a skew determinant with univari-
al diagonal, then

$$A_{pq} = a^{n-2} \cdot [pq] - a^{n-3} \cdot \sum [p\alpha][q\alpha] + a^{n-4} \cdot \sum [pq\alpha\beta][\alpha\beta] - a^{n-5} \cdot \sum [p\alpha\beta\gamma][q\alpha\beta\gamma] + \dots,$$

where $\alpha, \beta, \gamma, \dots$ are any of the integers 1, 2, 3, ..., n other than p and q. (xxi.)

Thus, when $n=5$ we have

$$A_{23} = a^3[23] - a^2([21][31] + [24][34] + [25][35]) + a([2314][14] + [2315][15] + [2345][45]) - [2145][3145];$$

so that if $|a_{11}a_{22} \dots a_{55}|$ be given in the form

$$\begin{vmatrix} x & a_2 & a_3 & a_4 & a_5 \\ -a_2 & x & b_3 & b_4 & b_5 \\ -a_3 & -b_3 & x & c_4 & c_5 \\ -a_4 & -b_4 & -c_4 & x & d_5 \\ -a_5 & -b_5 & -c_5 & -d_5 & x \end{vmatrix}$$

we know that

$$B_3 = x^3 \cdot b_3 - x_2(a_2a_3 + b_4c_4 + b_5c_5) + x \left\{ \begin{matrix} a_4 | a_2 & a_3 & a_4 \\ & b_3 & b_4 \\ & & c_4 \end{matrix} \right. + a_5 \left| \begin{matrix} a_2 & a_3 & a_5 \\ & b_3 & b_5 \\ & & c_5 \end{matrix} \right. + d_5 \left| \begin{matrix} b_3 & b_4 & b_5 \\ & c_4 & c_5 \\ & & d_5 \end{matrix} \right\} - \left| \begin{matrix} a_2 & a_4 & a_5 \\ & b_4 & b_5 \\ & & d_5 \end{matrix} \right| \cdot \left| \begin{matrix} a_3 & a_4 & a_5 \\ & c_4 & c_5 \\ & & d_5 \end{matrix} \right|.$$

Further, it deserves to be noted that an alternative for the coefficient of a^{n-3} in (xxi.) is $r_p r_q$; also that when n is odd there appear in the development n Pfaffians of the highest possible degree, and that when n is even there is only one Pfaffian of highest degree. The latter unique Pfaffian may be conveniently denoted by F , and the former set of n by F_1, F_2, \dots, F_n , where F_r is got by deleting the r^{th} frame-line of

$$\begin{matrix} a_{12} & a_{13} & \dots & a_{1n} \\ & a_{23} & \dots & a_{2n} \\ & & \dots & \dots \\ & & & a_{n-1, n} \end{matrix}$$

Thus the above expression for B_3 may also be written

$$b_3 x^3 - r_2 r_3 x^2 + (a_4 F_5 + a_5 F_4 + d_5 F_1) x - F_2 F_3.$$

17. If $|a_{11} a_{22} \dots a_{nn}|$ be a skew determinant with univariar diagonal, then

$$\frac{1}{2}(A_{pq} + A_{qp}) = -a^{n-3} \cdot \sum [pa][qa] - a^{n-5} \cdot \sum [pa\beta\gamma][qa\beta\gamma] - \dots$$

or

$$= a^{n-2} \cdot \sum [pq] + a^{n-4} \cdot \sum [pq\alpha\beta][\alpha\beta] + \dots$$

according as n is odd or even.

(xxii.)

This follows at once from (xxi.) and (ix.).

18. If Δ be a skew determinant with a univariar diagonal, the product of the p^{th} and q^{th} rows of the superadjugate of Δ is 0 when p and q are different, and is Δ^2 when p and q are the same.

(xxiii.)

For when p and q are different we have

$$(s_{p1}, s_{p2}, \dots, s_{pn}) \checkmark (s_{q1}, s_{q2}, \dots, s_{qn}) \\ = 4a^2(A_{p1}, A_{p2}, \dots, A_{pn}) \checkmark (A_{q1}, A_{q2}, \dots, A_{qn}) - \Delta(2aA_{qp} + 2aA_{pq}),$$

and therefore by (xvi.)

$$= 4a^2 \cdot \frac{1}{2}(A_{pq} + A_{qp}) \frac{\Delta}{a} - 2a\Delta(A_{qp} + A_{pq}) = 0;$$

and when p and q are the same the product

$$= 4a^2(A_{p1}^2 + A_{p2}^2 + \dots + A_{pn}^2) - 4aA_{pp}\Delta + \Delta^2$$

and therefore by (xvi.)

$$= 4a^2 \cdot A_{pp} \frac{\Delta}{a} - 4aA_{pp}\Delta + \Delta^2 \\ = \Delta^2,$$

as was to be proved.

The theorem is Cayley's for the case where $a=1$.

From it we have

$$S^2 = \begin{vmatrix} \Delta^2 & 0 & 0 & \dots \\ 0 & \Delta^2 & 0 & \dots \\ 0 & 0 & \Delta^2 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = \Delta^{2n},$$

which agrees with (iii.) and § 6.

19. If $|a_{11} a_{22} \dots a_{nn}|$, or Δ say, be a skew determinant with univariar diagonal, $r_p r_q$ the product of its p^{th} and q^{th} rows, and F the Pfaffian of the elements on the right of the diagonal, then

$$\begin{vmatrix} r_1 r_1 - a^2 & r_1 r_2 & \dots & r_1 r_n \\ r_2 r_1 & r_2 r_2 - a^2 & \dots & r_2 r_n \\ \dots & \dots & \dots & \dots \\ r_n r_1 & r_n r_2 & \dots & r_n r_n - a^2 \end{vmatrix} = \begin{cases} 0 & \text{when } n \text{ is odd.} \\ F^4 & \text{when } n \text{ is even.} \end{cases} \quad \text{(xxiv.)}$$

By way of proof we have only to note that the subtraction of a^2 from the diagonal elements of the determinant $|r_1r_1 \ r_2r_2 \ \dots \ r_nr_n|$ is equivalent to putting $a=0$ in the said determinant, and that

$$|r_1r_1 \ r_2r_2 \ \dots \ r_nr_n|_{a=0} = (\Delta^2)_{a=0} = (\Delta_{a=0})^2.$$

20. If $|a_{11}a_{22} \ \dots \ a_{nn}|$, or Δ say, be a skew determinant with univariaral diagonal, then

$$\begin{vmatrix} A_{11} - \Delta/a & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} - \Delta/a & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} - \Delta/a \end{vmatrix} = \begin{cases} 0 & \text{when } n \text{ is odd.} \\ \Delta^{n-1}F^2/(-a)^n & \text{when } n \text{ is even.} \end{cases} \quad (\text{xxv.})$$

This may be proved in two ways. We may either multiply $\Delta_{a=0}$ by the determinant proposed for investigation, when we shall obtain

$$\begin{vmatrix} \dots & -a_{21}\Delta/a & -a_{31}\Delta/a & \dots \\ -a_{12}\Delta/a & \dots & -a_{32}\Delta/a & \dots \\ -a_{13}\Delta/a & -a_{23}\Delta/a & \dots & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix},$$

which evidently

$$\begin{aligned} &= \left(-\frac{\Delta}{a}\right)^n \cdot \Delta_{a=0} \\ &= \frac{\Delta^n}{(-a)^n} \cdot \begin{cases} 0 & \text{when } n \text{ is odd,} \\ F^2 & \text{when } n \text{ is even,} \end{cases} \end{aligned}$$

whence the desired result at once follows. Or, taking a hint from the form of the result, we may multiply the adjugate of Δ by $\Delta_{a=0}$ with equally good effect.

21. If $|a_{11}a_{22} \ \dots \ a_{nn}|$, or Δ say, be a skew determinant with univariaral diagonal, then

$$\begin{vmatrix} A_{11} - \Delta/a & \frac{1}{2}(A_{12} + A_{21}) & \dots & \frac{1}{2}(A_{1n} + A_{n1}) \\ \frac{1}{2}(A_{21} + A_{12}) & A_{22} - \Delta/a & \dots & \frac{1}{2}(A_{2n} + A_{n2}) \\ \dots & \dots & \dots & \dots \\ \frac{1}{2}(A_{n1} + A_{1n}) & \frac{1}{2}(A_{n2} + A_{2n}) & \dots & A_{nn} - \Delta/a \end{vmatrix} = \begin{cases} 0 & \text{when } n \text{ is odd.} \\ \Delta^{n-2}F^4/(-a)^n & \text{when } n \text{ is even.} \end{cases} \quad (\text{xxvi.})$$

This is obtained by squaring both sides of the result of § 20 and then dividing by $(-\Delta/a)^n$.

22. Skew determinants of odd order have properties to which there is nothing analogous in the case of those of even order. To one or two of these attention will now be directed.

As has already been seen, there are then associated with the determinant n Pfaffians of the degree $\frac{1}{2}(n-1)$. These are connected with one another by n linear relations, the exact proposition being that *If F_1, F_2, \dots, F_n be the n primary minors of the quasi-Pfaffian of an odd-ordered zero-axial skew determinant, the product of $F_1, -F_2, F_3, -F_4, \dots$ by any row of the determinant vanishes.* (xxvii.)

The reason for this lies in the fact that the product in question is itself expressible as a Pfaffian of the next higher degree, having two frame-lines so related as to produce evanescence. Thus, the determinant being

$$\begin{vmatrix} \cdot & a_2 & a_3 & a_4 & a_5 \\ -a_2 & \cdot & b_3 & b_4 & b_5 \\ -a_3 & -b_3 & \cdot & c_4 & c_5 \\ -a_4 & -b_4 & -c_4 & \cdot & d_5 \\ -a_5 & -b_5 & -c_5 & -d_5 & \cdot \end{vmatrix}$$

the five signed Pfaffians are

$$\begin{vmatrix} b_3 & b_4 & b_5 \\ c_4 & c_5 \\ d_5 \end{vmatrix}, \quad - \begin{vmatrix} a_3 & a_4 & a_5 \\ c_4 & c_5 \\ d_5 \end{vmatrix}, \quad \begin{vmatrix} a_2 & a_4 & a_5 \\ b_4 & b_5 \\ d_5 \end{vmatrix}, \quad - \begin{vmatrix} a_2 & a_3 & a_5 \\ b_3 & b_5 \\ c_5 \end{vmatrix}, \quad \begin{vmatrix} a_2 & a_3 & a_4 \\ b_3 & b_4 \\ c_4 \end{vmatrix},$$

and the product of them by the row

$$-a_3, \quad -b_3, \quad 0, \quad c_4, \quad c_5$$

is the Pfaffian obtained by prefixing this row to the quasi-Pfaffian of the determinant, namely,

$$\begin{vmatrix} -a_3 & -b_3 & \cdot & c_4 & c_5 \\ & a_2 & a_3 & a_4 & a_5 \\ & & b_3 & b_4 & b_5 \\ & & & c_4 & c_5 \\ & & & & d_5 \end{vmatrix},$$

which being the square root of

$$\begin{vmatrix} \cdot & -a_3 & -b_3 & \cdot & c_4 & c_5 \\ a_3 & \cdot & a_2 & a_3 & a_4 & a_5 \\ b_3 & -a_2 & \cdot & b_3 & b_4 & b_5 \\ \cdot & -a_3 & -b_3 & \cdot & c_4 & c_5 \\ -c_4 & -a_4 & -b_4 & -c_4 & \cdot & d_5 \\ -c_5 & -a_5 & -b_5 & -c_5 & -d_5 & \cdot \end{vmatrix}$$

vanishes because of the equality of the 1st and 4th rows.

If we take the full set of five relations and eliminate $F_1, -F_2, F_3, -F_4, F_5$, an already well-known result is obtained.

23. Using also the full set of five results in the case where the columns are the multipliers, we have, whatever the x 's may be,

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \cdot & a_2 & a_3 & a_4 & a_5 & F_1 \\ -a_2 & \cdot & b_3 & b_4 & b_5 & -F_2 \\ -a_3 & -b_3 & \cdot & c_4 & c_5 & F_3 \\ -a_4 & -b_4 & -c_4 & \cdot & d_5 & -F_4 \\ -a_5 & -b_5 & -c_5 & -d_5 & \cdot & F_5 \end{array} = 0;$$

and substituting for the row of x 's any row of the determinant, say the 3rd, we have, on taking the alternative development of the bipartite, the interesting result

$$r_3 r_1 \cdot F_1 - r_3 r_2 \cdot F_2 + r_3 r_3 \cdot F_3 - r_3 r_4 \cdot F_4 + r_3 r_5 \cdot F_5 = 0,$$

i.e.

$$\begin{aligned} & (-a_2 b_3 + a_4 c_4 + a_5 c_5)(b_3 d_5 - b_4 c_5 + b_5 c_4) - (a_2 a_3 + b_4 c_4 + b_5 c_5)(a_3 d_5 - a_4 c_5 + a_5 c_4) \\ & + \dots + (a_3 a_5 + b_3 b_5 - c_4 d_5)(a_2 c_4 - a_3 b_4 + a_4 b_3) = 0. \end{aligned}$$

This result is not greatly altered by substituting an a for the zero of each row, because the two terms of $r r$ which vanish separately on account of the zeros will in the four cases where p and q are different cancel each other when the zeros are replaced by a 's. In the 5th case where q is the same as p there is an additional term, namely, $a^2 F_p$, with the sign $+$ or $-$ according as p is odd or even. We thus have the theorem that *In the case of an odd-ordered skew determinant with univariar diagonal*

$$r_p r_1 \cdot F_1 - r_p r_2 \cdot F_2 + r_p r_3 \cdot F_3 - \dots = (-1)^{p-1} a^2 F_p. \quad (\text{xxviii.})$$

Putting p in this equal to 1, 2, ..., n , we have n equations from which F_1, F_2, \dots, F_n may be eliminated, the result being a corroboration of (xxiv.)

24. *If $|a_{11} a_{22} \dots a_{nn}|$, or Δ say, be an odd-ordered skew determinant with univariar diagonal, then*

$$(A_{p1}, A_{p2}, \dots, A_{pn} \text{ } \checkmark \text{ } F_1, -F_2, F_3, -F_4, \dots) = (-1)^{p-1} F_p \cdot \frac{\Delta}{a}. \quad (\text{xxx.})$$

The way to establish this which naturally occurs to one is to substitute for the A 's on the left the expressions obtained in § 16 (xxi.) and then perform the simplifications necessary to produce

$$(-1)^{p-1} F_p \{ a^{n-1} + a^{n-3} \cdot \Sigma \phi_1^2 + a^{n-5} \cdot \Sigma \phi_2^2 + \dots \},$$

where $\Sigma \phi_n^2$ means the sum of the squares of all Pfaffians of degree h . This, though not impracticable, is lengthy and troublesome to set forth; at the

same time it has the advantage of bringing to light a series of Pfaffian identities. Thus, to take the case where $n = 5$ and $p = 2$, we have from (xxi.)

$$\begin{aligned} A_{21} &= a_{21} \cdot a^3 - r_2 r_1 \cdot a^2 + (-a_{34} F_5 - a_{35} F_4 - a_{45} F_3) \cdot a - F_2 F_1, \\ A_{22} &= a^4 + (a_{13}^2 + a_{14}^2 + a_{15}^2 + a_{34}^2 + a_{35}^2 + a_{45}^2) a^2 + F_2^2, \\ A_{23} &= a_{23} \cdot a^3 - r_2 r_3 \cdot a^2 + (a_{14} F_5 + a_{15} F_4 + a_{45} F_1) \cdot a - F_2 F_3, \\ A_{24} &= a_{24} \cdot a^3 - r_2 r_4 \cdot a^2 + (-a_{13} F_5 + a_{15} F_3 - a_{35} F_1) \cdot a - F_2 F_4, \\ A_{25} &= a_{25} \cdot a^3 - r_2 r_5 \cdot a^2 + (-a_{13} F_4 - a_{14} F_3 + a_{34} F_1) \cdot a - F_2 F_5; \end{aligned}$$

so that in

$$A_{21} F_1 - A_{22} F_2 + A_{23} F_3 - A_{24} F_4 + A_{25} F_5$$

the cofactor of a^4 is $-F_2$; the cofactor of a^3 , as we see from § 22 (xxvii.), is 0; the cofactor of a^2 is

$$-r_2 r_1 \cdot F_1 - (a_{13}^2 + a_{14}^2 + \dots + a_{45}^2) F_2 - r_2 r_3 \cdot F_3 - r_2 r_4 \cdot F_4 - r_2 r_5 \cdot F_5$$

which, if increased by $r_2 r_2 \cdot F_2 - r_2 r_2 \cdot F_2$, is seen from (xxviii.) to be

$$\begin{aligned} &= a^2 F_2 - r_2 r_2 \cdot F_2 - (a_{13}^2 + a_{14}^2 + \dots + a_{45}^2) F_2 \\ &= -(r_2 r_2 - a^2 + a_{13}^2 + a_{14}^2 + \dots + a_{45}^2) F_2 \\ &= -F_2 \cdot \Sigma \phi_1^2; \end{aligned}$$

the cofactor of a^1 is 0; the term independent of a is $-F_2(F_1^2 + F_2^2 + \dots + F_5^2)$; and the whole expression is

$$-F_2(a^4 + a^2 \cdot \Sigma \phi_1^2 + \Sigma \phi_2^2), \quad \text{i.e.} \quad -F_2 \cdot \frac{\Delta}{a}.$$

It may be noted as a corollary that by putting $p = 1, 2, \dots, n$ in (xxx.) we obtain n equations from which F_1, F_2, \dots, F_n may be eliminated, the result obtained corroborating (xxv.).

25. The general property of Pfaffians which enables us to deal with the cofactors of the different powers of a in the foregoing method of proving theorem xxx. is that *If F_1, F_2, \dots, F_n be the primary minors which pertain to the quasi-Pfaffian [123 . . . n] when n is odd, and*

$$pq_1, pq_2, pq_3, \dots$$

be used as abbreviations for the coefficients

$$[pq], \quad \sum [pa][qa], \quad \sum [pq\alpha\beta][\alpha\beta], \quad \dots$$

of theorem (xxi.), then

$$\begin{aligned} &(p1_m, p2_m, \dots, pn_m \times F_1, -F_2, F_3, -F_4, \dots) + (-1)^p \cdot pp_m \cdot F_p \\ &= \begin{cases} 0 & \text{when } m \text{ is odd.} \\ \sum [pa_1 a_2 \dots a_{m-1}]^2 & \text{when } m \text{ is even.} \end{cases} \quad (\text{xxxi.}) \end{aligned}$$

In the left-hand member, it should be noted, there is no term containing F_p , the means taken to indicate this being first to include a term

$(-1)^{p-1} \cdot p p_m \cdot F_p$, as one of a group, and then annex a cancelling term $(-1)^p \cdot p p_m \cdot F_p$. Thus, taking the case where $n=7, p=2, m=4$, what we have to simplify is

$$21_4 \cdot F_1 + 23_4 \cdot F_3 - 24_4 \cdot F_4 + 25_4 \cdot F_5 - 26_4 \cdot F_6 + 27_4 \cdot F_7,$$

where $21_4 = \sum [2\alpha\beta\gamma][1\alpha\beta\gamma]$ and where generally the cofactor of each F is the sum of 10 products of two Pfaffians of the 2nd degree, the first Pfaffian of each product having p (*i.e.* 2) as an umbra, that is to say, being of the form $[2\alpha\beta\gamma]$. As there are only 20 (*i.e.* 6.5.4/1.2.3) Pfaffians of this form, it is clear that each of them must occur thrice. For example, a little examination will show that $[2134]$ cannot occur in $21_4 \cdot F_1, 23_4 \cdot F_3$, or $24_4 \cdot F_4$, but occurs in each of the other terms of the given expression, its full cofactor being

$$[5134] \cdot F_5 - [6134] \cdot F_6 + [7134] \cdot F_7.$$

Now we know from (xxvii.) that

$$[21][5671] - [51][2671] + [61][2571] - [71][2561] = 0,$$

from which there follows by "extension"

$$[2134][567134] - [5134][267134] + [6134][257134] - [7134][256134] = 0,$$

i.e.
$$-[2134] \cdot F_2 + [5134] \cdot F_5 - [6134] \cdot F_6 + [7134] \cdot F_7 = 0,$$

so that the cofactor of $[2134]$ above found is seen to be $[2134]F_2$. In like manner it can be shown that every other Pfaffian of the form $[2\alpha\beta\gamma]$ has $[2\alpha\beta\gamma] \cdot F_2$ as cofactor; and thus we conclude that

$$21_4 \cdot F_1 + 23_4 \cdot F_3 - 24_4 \cdot F_4 + 25_4 \cdot F_5 - 26_4 \cdot F_6 + 27_4 \cdot F_7 = F_2 \cdot \sum [2\alpha\beta\gamma]^2,$$

where α, β, γ are any three of the six integers 1, 3, 4, 5, 6, 7.

26. From theorem xiv. we know that the left-hand member of (xxx.) is equal to the determinant got by substituting $F_1, -F_2, F_3, -F_4, \dots$ in place of the p^{th} row of Δ . Multiplying this determinant row-wise by Δ , and utilising theorem xxvii., we obtain the determinant which, on being divided by $(-1)^{p-1}a$, is the determinant got by substituting $F_1, -F_2, F_3, -F_4, \dots$ in place of the p^{th} row of Δ^2 . There is thus reached the curious theorem that *If $|a_{11}a_{22} \dots a_{mm}|$, or Δ say, be an odd-ordered skew determinant with univariar diagonal, the determinant obtained by inserting $F_1, -F_2, F_3, -F_4, \dots$ in place of the p^{th} row of Δ^m is*

$$\left\{ (-1)^{p-1} \frac{\Delta}{a} \right\}^m \cdot F_p. \tag{xxxii.}$$

This could also be established by proving that *the product of $F_1, -F_2, F_3, -F_4, \dots$ by the p^{th} row of Δ^m is $(-1)^{p-1}a^m F_p$* ,—a theorem which has for its first two cases theorem xxviii. and a corollary to theorem xxvii.

27. We have already used (§ 24) the fact that if Δ be an n -line skew determinant with univariial diagonal

$$\Delta = a^n + a^{n-2} \cdot \sum [\alpha\beta]^2 + a^{n-4} \cdot \sum [\alpha\beta\gamma\delta]^2 + \dots,$$

where $a\beta$ is any pair of the first n integers, $a\beta\gamma\delta$ any set of four, and so on. By adding to Σ the suffix $-p$ to signify that a, β, γ, \dots are not to be taken from $1, 2, \dots, n$, but from $1, 2, \dots, p-1, p+1, \dots, n$, we can readily give expression to the like development of any primary coaxial minor of Δ : for example,

$$A_{11} = a^{n-1} + a^{n-3} \cdot \sum_{-1} [\alpha\beta]^2 + a^{n-5} \cdot \sum_{-1} [\alpha\beta\gamma\delta]^2 + \dots$$

With the same notation we have

$$r_1 r_1 = a^2 + \sum_{-1} [1a]^2,$$

and therefore by multiplication

$$\begin{aligned} \frac{1}{a} A_{11} \cdot r_1 r_1 &= a^n + a^{n-2} \left\{ \sum_{-1} [\alpha\beta]^2 + \sum_{-1} [1a]^2 \right\} \\ &+ a^{n-4} \left\{ \sum_{-1} [\alpha\beta\gamma\delta]^2 + \sum_{-1} [1a]^2 \cdot \sum_{-1} [\alpha\beta]^2 \right\} \\ &+ \dots \end{aligned}$$

Comparing this development with that of Δ , we see that on subtraction the coefficients of a^n and a^{n-2} would vanish, that the coefficient of a^{n-4} would be

$$\sum_{-1} [\alpha\beta\gamma\delta]^2 + \sum_{-1} [1a]^2 \sum_{-1} [\alpha\beta]^2 - \sum_{-1} [\alpha\beta\gamma\delta]^2,$$

i.e.
$$\sum_{-1} [1a]^2 \sum_{-1} [\alpha\beta]^2 - \sum_{-1} [1a\beta\gamma]^2,$$

and that as a collective result we should have

$$\begin{aligned} \frac{1}{a} A_{11} \cdot r_1 r_1 - \Delta &= a^{n-4} \left\{ \sum_{-1} [1a]^2 \sum_{-1} [\alpha\beta]^2 - \sum_{-1} [1a\beta\gamma]^2 \right\} \\ &+ a^{n-6} \left\{ \sum_{-1} [1a]^2 \sum_{-1} [\alpha\beta\gamma\delta]^2 - \sum_{-1} [1a\beta\gamma\delta\epsilon]^2 \right\} \\ &+ \dots \end{aligned} \tag{xxxiii.}$$

28. The simplification, however, does not end here, there being connected with the coefficients on the right a rather remarkable theorem which enables us to substitute for each of them a sum of squares.

To illustrate the nature of this, let us examine the coefficient of a^{n-4} when $n=5$. The first part of it is

$$\left\{ [23]^2 + [24]^2 + [25]^2 + [34]^2 + [35]^2 + [45]^2 \right\} \left\{ [12]^2 + [13]^2 + [14]^2 + [15]^2 \right\},$$

which by multiplication is expressible as the sum of 24 squares, 12 being of the type $[1\alpha]^2[\alpha\beta]^2$ and 12 of the type $[1\alpha]^2[\beta\gamma]^2$. The second part is

$$[1234]^2 + [1235]^2 + [1245]^2 + [1345]^2,$$

i.e.

$$\left\{ [12] \cdot [34] - [13][24] + [14][23] \right\}^2 + \left\{ [12][35] - [13][25] + [15][23] \right\}^2 + \dots$$

and therefore is expressible as the sum of 12 squares, all of the type $[1\alpha]^2[\beta\gamma]^2$, together with 12 other terms of the type $-2[1\alpha][1\beta][\alpha\gamma][\beta\gamma]$. The excess of the first part over the second is thus

$$\sum [1\alpha]^2[\alpha\beta]^2 + \sum 2[1\alpha][1\beta][\alpha\gamma][\beta\gamma],$$

which is readily found to be equal to

$$(13 \cdot 32 + 14 \cdot 42 + 15 \cdot 52)^2 + (12 \cdot 23 + 14 \cdot 43 + 15 \cdot 53)^2 \\ + (12 \cdot 24 + 13 \cdot 34 + 15 \cdot 54)^2 + (12 \cdot 25 + 13 \cdot 35 + 14 \cdot 45)^2,$$

or, as it may also be written,

$$(r_1 r_2)^2 + (r_1 r_3)^2 + (r_1 r_4)^2 + (r_1 r_5)^2.$$

The general theorem here illustrated is

$$\sum_{-1} [1\alpha]^2 \cdot \sum_{-1} [\alpha\beta]^2 - \sum_{-1} [1\alpha\beta\gamma]^2 = \sum_{-1} ([1\alpha][\alpha r] + [1\beta][\beta r] + [1\gamma][\gamma r] + \dots)^2,$$

where $\alpha, \beta, \gamma, \dots$ on the right is any set of $n-2$ integers taken from $2, 3, \dots, n$, and r is the remaining integer. Although this is the most appropriate form in the present connection, it is better for general purposes to transfer $\sum_{-1} [1\alpha\beta\gamma]^2$ to the right-hand side, when it will be seen that we may formulate the result by saying that, *A triangular number of elements*

$$\begin{array}{cccc} a_{12} & a_{13} & \dots & a_{1n} \\ & a_{23} & \dots & a_{2n} \\ & & \dots & \dots \\ & & & a_{n-1, n} \end{array}$$

being given, the product of the sum of the squares of those in the first row by the sum of the squares of all the others is expressible as a sum of $\frac{1}{6}(n-1)(n^2-5n+12)$ squares, namely, $\frac{1}{6}(n-1)(n-2)(n-3)$ squares of the form $(a_{1\alpha}a_{\beta\gamma} - a_{1\beta}a_{\alpha\gamma} + a_{1\gamma}a_{\alpha\beta})^2$ and $n-1$ squares of the form $(a_{1\alpha}a_{\alpha r} + a_{1\beta}a_{\beta r} + a_{1\gamma}a_{\gamma r} + \dots)^2$. (xxxiv.)

29. With regard to the cofactor of a^{n-6} there is the analogous theorem

$$\sum_{-1} [1\alpha]^2 \cdot \sum_{-1} [\alpha\beta\gamma\delta]^2 - \sum_{-1} [1\alpha\beta\gamma\delta\epsilon]^2 \\ = \sum_{-1} \left\{ [1\alpha][\alpha rst] + [1\beta][\beta rst] + \dots \right\}^2 \quad (\text{xxxv.})$$

where $\alpha, \beta, \gamma, \dots$ on the right is a set of $n-4$ integers taken from $2, 3, \dots, n$, and r, s, t are those remaining.

Similarly we have

$$\sum_{-1} [1\alpha^2] \cdot \sum_{-1} [\alpha\beta\gamma\delta\epsilon\zeta]^2 - \sum_{-1} [1\alpha\beta\gamma\delta\epsilon\zeta\eta]^2 = \sum_{-1} \left\{ [1\alpha][arstuv] + [1\beta][\beta rstuv] + \dots \right\}^2 \quad (\text{xxxvi.})$$

where on the right $\alpha, \beta, \gamma, \dots$ is a set of $n-6$ integers forming with r, s, t, u, v the full set $2, 3, \dots, n$; and so on.

The general proposition thus derivable regarding quadrate numbers is that *The product of the sum of $n-1$ squares by the sum of $C_{n-1, 2m}$ squares is expressible as the sum of $C_{n-1, 2m+1} + C_{n-1, 2m-1}$ squares.* (xxxvii.)

30. Returning to the result of § 27 and making the substitutions now possible, we have the important theorem that *If $|a_{11}a_{22} \dots a_{nn}|$, or Δ_n say, be a skew determinant with a univariial diagonal, then*

$$\begin{aligned} \frac{1}{a} A_{11} \cdot r_1 r_1 - \Delta_n &= a^{n-4} \cdot \sum \left\{ [1\alpha][ar] + [1\beta][\beta r] + \dots \right\}^2 \\ &+ a^{n-6} \cdot \sum \left\{ [1\alpha][arst] + [1\beta][\beta rst] + \dots \right\}^2 \\ &+ a^{n-8} \cdot \sum \left\{ [1\alpha][arstuv] + [1\beta][\beta rstuv] + \dots \right\}^2 \\ &+ \dots \dots \dots \end{aligned}$$

where, under the first Σ , r is any one integer taken from $2, 3, \dots, n$, and α, β, \dots are those remaining; under the second Σ , rst is any set of three integers taken from $2, 3, \dots, n$, and α, β, \dots are those remaining, and so on. (xxxviii.)

Since the number of integers α, β, \dots under the first Σ is $n-2$, under the second Σ $n-4$, and so on, the number of terms in the expressions to be squared in any case is 2 more than the index of the attached power of a . When n is even the last term is independent of a , and is a sum of squares of binomials for which there is an alternative mode of expression. This is due to the fact that the cofactors of $[1\alpha], [1\beta], \dots$ are then primary minors of $[123 \dots n]$ and can be denoted by $A_{12}, -A_{13}, A_{14}, -A_{15}, \dots$, the term thus being

$$(a_{12}A_{13} - a_{13}A_{12})^2 + (a_{12}A_{14} - a_{14}A_{12})^2 + \dots$$

For example, putting $n=6$ we have

$$\frac{1}{a} A_{11} \cdot r_1 r_1 - \Delta_6 = a^2 \left\{ (r_1 r_2)^2 + \dots + (r_1 r_6)^2 \right\} + \left\| \begin{matrix} a_{12} a_{13} \dots a_{16} \\ A_{12} A_{13} \dots A_{16} \end{matrix} \right\|^2$$

When n is odd, the last term is

$$\frac{1}{a} \sum \left\{ [1\alpha][arst \dots] \right\}^2,$$

where $rst \dots$ is any set of $n-2$ integers taken from $2, 3, \dots, n$, and a is the one remaining. The term therefore is

$$\frac{1}{a} \left\{ [12]^2[23 \dots n]^2 + [13]^2[324 \dots n]^2 + \dots \right\},$$

i.e.

$$\frac{1}{a} [23 \dots n]^2 \cdot \left\{ a_{12}^2 + a_{13}^2 + \dots + a_{1n}^2 \right\}.$$

Thus, taking $n=7$ we have

$$\begin{aligned} \frac{1}{a} A_{11 \cdot r_1 r_1} - \Delta_7 &= x^3 \left\{ (r_1 r_2)^2 + \dots + (r_1 r_7)^2 \right\} \\ &+ x \left\{ \left([12][2567] + [13][3567] + [14][4567] \right)^2 \dots \right\} \\ &+ \frac{1}{x} [23 \dots 7]^2 (a_{12}^2 + a_{13}^2 + \dots + a_{17}^2). \end{aligned}$$

From (xxxviii.) it follows directly that *If* $|a_{11} a_{22} \dots a_{nn}|$, or Δ say, be a skew determinant with univariial diagonal, then when n is odd $A_{11 \cdot r_1 r_1}$ and $a\Delta$ are both positive, and when n is even $\frac{1}{a} A_{11 \cdot r_1 r_1}$ and Δ are both positive, the former in each case being the greater. (xxxix.)

(Issued separately October 15, 1909.)

XLI.—The Skeleton of a Sowerby's Whale (*Mesoplodon bidens*) stranded at St Andrews, and the Morphology of the Manus in *Mesoplodon*, *Hyperoodon* and the *Delphinidæ*.
By Sir Wm. Turner, K.C.B., D.C.L., F.R.S., President of the Society.

(Read July 5, 1909. MS. received July 28, 1909.)

IN May 1908 an adult female Sowerby's whale, *Mesoplodon bidens*, was stranded in St Andrews Bay, about a mile from the clubhouse. Its capture and external characters were recorded by Professor W. C. M'Intosh in the *Annals and Magazine of Natural History*, December 1908. The skeleton was obtained by him for the Gatty Marine Laboratory, and at his request I have examined and prepared this report on its characters.

The first observation on this interesting cetacean as a Scottish species was made by Mr James Sowerby * on an animal stranded in the Moray Firth near Elgin in 1800; no record existed of another example in Scotland until 1872, when I described † the skull of a female in the Museum of Science and Art, now the Royal Scottish Museum. In 1881 I obtained from Messrs Anderson of Hillswick an imperfect skeleton ‡ of a male stranded in April in Urafirth Voe, North Mavine, Shetland, and in May 1885 the same gentlemen sent me the carcass of another male § captured at Voxter Voe, Delting, Shetland. In October 1888 a male was stranded in Dalgety Bay, on the north shore of the Firth of Forth, the characters of which I described at the time. || In April 1895 another male was captured in the Firth of Forth, at Morrison's Haven, Prestonpans, and its skull, some bones, and a limb were secured by my late assistant, Mr James Simpson, ¶ for the Anatomical Museum of the University. In drawing up the notes on the St Andrews animal, I have compared the skull and skeleton with the 1881, 1885, 1888 and 1895 specimens in the Anatomical Museum.

* *The British Miscellany*, 1804-1806, vol. i., pl. i.

† *Trans. Roy. Soc. Edin.*, vol. xxvi. p. 759, 1872.

‡ *Journal of Anat. and Phys.*, vol. xvi., April 1882; *Proc. Roy. Soc. Edin.*, Jan. 1882.

§ "The Anatomy of a Second Specimen of Sowerby's Whale," *Journal of Anat. and Phys.*, vol. xx. p. 144, October 1885; *Proc. Roy. Soc. Edin.*, vol. xiii. p. 279.

|| *Proc. Roy. Phys. Soc. Edin.*, vol. x., 1888-1889.

¶ *Annals of Scottish Natural History*, October 1895, p. 250.

MEASUREMENTS OF SKULLS.

	Royal Scottish Museum, 1872, ♀.	Shetland, 1881, ♂.	Shetland, 1885, ♂.	Dalgety, Firth of Forth, 1888, ♂.	Morrison's Haven, 1895, ♂.	St Andrews, 1908, ♀.
Greatest length of skull in straight line	75 cm.	broken	76	77·3	broken	82·8
Length of rostrum	49	broken	51·2	51	51	59
Height from vertex to ptery- goids	24	26	27	27	broken	25·4
Breadth between upper mid- orbital borders	28·5	28·5	28·4	29	...	29·1
Breadth across occipital condyls	11	10	11·1	10·3	10·5	11·2
Breadth between ant-orbital notches	20	18·5	17·3	17·5	20·5	19·8
Premaxillæ, width behind anterior nares	13	11·5	11·8	12·3	12·5	12·3
Premaxillæ, width in front of anterior nares	10	10	11	9·2	10·5	9·9
Premaxillæ, width opposite anterior nares	11	10	10·4	10	10·6	11
Width of anterior nares	4·5	5·5	5·6	5·4	4·8	6·2
Mandible, length of	69	65	67	65·5	broken	73·8
" " symphysis	24	broken	23·3	21	22·8	27·6
" height of ramus	11	10	10·7	11	broken	11·2

Skull.—The skull of the St Andrews specimen had the characteristic elongated, slender beak, and other general characters of the species. It was 82·8 cm. ($32\frac{1}{2}$ inches) in maximum length, and was the longest skull in the Table of measurements. The rostrum was not broken, and the slender tips of the superior and premaxillary bones as well as the mesial cartilage of the beak reached its free end. The length of the beak was 59 cm. ($23\frac{1}{4}$ inches). The mes-ethmoid septum was prolonged into the upper end of the medio-rostral gutter for 8 cm., and was embraced anteriorly by the medio- (meso-) rostral bone, which occupied the gutter for 21 cm. This bone was divided on its upper surface into two lateral halves by a longitudinal groove, and in front of it the unossified medio-rostral cartilage extended to the tip of the beak.

In the 1881 specimen from Shetland, the mes-ethmoid was embraced by the medio-rostral bone, which was divided by a longitudinal groove into two distinct lateral halves as far as 24 mm. from the anterior pointed end, near which the surface of the bone was tuberculated. In the skull from Morrison's Haven the halves were fused together for 154 mm. from the pointed end, and the surface of the bone was smooth. The free end of the rostrum of this skull was somewhat curved to the right, and a

similar deflection was present in the tip of the mandible. In the Dalgety Bay cranium the longitudinal groove was present in the middle third and the lateral halves were distinct, the posterior and anterior thirds were not grooved, the surface of the medio-rostral bone was smooth and the anterior end was pointed.

In a paper on the development of the rostrum in *Mesoplodon*, H. O. Forbes regarded* the meso-rostral consolidation as an upgrowth formed by the proliferation of the osseous tissue of part of the vomer and perhaps of the premaxillaries, and not as an ossification of the meso-rostral cartilage. The presence of a longitudinal groove, and the consequent indication of two lateral halves to the medio-rostral bone, favour to some extent, as regards its sides, this view, but I think that ossification of the mesial cartilage also participates in the production of the consolidated structure which occupies the medio-rostral gutter both in *Ziphius* and *Mesoplodon*.

In the female skull which I described in 1872 the medio-rostral gutter did not contain the corresponding bone, which I thought might be a female character, but its extensive ossification in the adult female now described showed that the absence of the bone in the previous specimen was an age, and not a sexual feature.

The two halves of the St Andrews mandible were not fused at the symphysis; the alveolus for the tooth was situated immediately in front of the hinder end of the symphysis, and the apex of the tooth projected for only 9 mm. beyond the alveolus; the retention of the crown within the alveolus indicated the female sex. In the mandible of the skull from Morrison's Haven the two halves were in process of fusion, the teeth were lost, but the large sockets extended for about half their extent behind the symphysis.

Ear Bones.—In the St Andrews specimen the tympanic bullæ and petrous bones were lodged in the hollow near the mastoid. The bulla was bilobed inferiorly and posteriorly, characteristic of the genus *Mesoplodon*.†

Hyoid Apparatus.—The hyoid proper consisted of a body with which the two great cornua were fused. At its anterior border was a notch bounded by a pair of short processes, each with an articular facet, to which the cerato-hyals had doubtless been attached. A well-marked pair of stylo-hyals was anterior to and separated by an interval from the thyro-hyals.

Spine.—As the epiphyses were fully anchylosed to their respective

* *Proc. Zool. Soc. Lond.*, 28th Feb. 1893.

† See my account of the tympano-petrous bones in the *Odontoceti*, *Proc. Roy. Soc. Edin.*, vol. xxiv. p. 423, 1903.

vertebræ, the specimen was an adult. The column consisted of forty-seven vertebræ.

Cervical vertebræ from the 1st to the 4th were fused together, and formed a massive bone, but the laminæ of the 4th were distinct and did not meet to form a neural spine. The 5th, 6th and 7th were separate and with flattened bodies; in the 5th and 6th the neural arches were incomplete and without spines, but the 7th had a short spine. The transverse processes were distinct; in the atlas and axis no foramen was present at the root of each process; in the 3rd to the 6th the transverse process was formed by the junction of the diapophysis and the parapophysis, between which was the vertebrarterial foramen; in the 7th these two processes had not joined and the boundary of the foramen was incomplete.

Dorsal Vertebræ.—The 1st and the 5th to the last were separate bones, but the bodies of the 2nd, 3rd and 4th were tied together by a strong bar of bone fused with the ventral surface of their bodies. The body of the 1st dorsal was flattened, as in the last cervicals, but behind it the bodies gradually increased in antero-posterior diameter until the last, which measured 10·9 cm. and possessed a median ridge on its ventral surface. From the lower part of the side of the body of the 1st dorsal a tubercle projected which resembled the parapophysis of a cervical. The laminæ and spines were complete in all the dorsals and increased in size from before backwards. The 1st dorsal, where it gave rise to the parapophysis (inferior tubercle), had a large costal facet for the head of the 1st rib, on the posterior surface of the tubercle which extended on to the body, but no costal facet was seen on the short pointed transverse process. The body of each vertebra from the 2nd to the 7th had a costal facet on each side at its junction with the pedicle. From the 1st to the 8th vertebra, and close to the anterior zygapophysis, a transverse process sprang from the neural arch, which, except in the 1st, had a large costal facet for the tubercle of a rib. In the terminal dorsals the transverse process did not project from the neural arch, but from the anterior part of the side of the body; zygapophyses were present in the 1st to the 8th dorsal; strong metapophyses projected from the laminæ of the terminal dorsals which overlapped the laminæ of the vertebra immediately anterior, and short metapophyses were present in the 7th and 8th vertebræ. In this spine and in those of the two Shetland and the Dalgety Bay animals the 7th vertebra was the last to show costal facets for both head and tubercle; the dorsal vertebræ behind it had a facet only on the transverse process for the tubercle of the rib.

Lumbar vertebræ had no facets for articulation with the chevron bones. They were the largest of the vertebræ, and were characteristic of the

region. The body was keeled on its ventral surface; the transverse processes were spatulate and sprang from about the middle of the sides of the bodies; the spines were long, flattened plates of bone.

Caudal Vertebrae.—Eighteen vertebrae followed the last lumbar. They diminished in size from before backwards; the eight most posterior consisted only of a body, and that of the terminal vertebra measured only 20 mm. in its antero-posterior and 18 mm. in its transverse diameter. The ten anterior had facets on the ventral surface for the nine chevron bones, which had articulated with them and with the intervertebral discs.

Ribs.—There were ten pairs of ribs. The 1st and the 10th were the shortest; they increased in length from the 1st (32 cm.) to the 6th (62 cm.) and then diminished to the last (16 cm.). The 1st, the broadest and most flattened, had a facet on the head and one on the tubercle; the right was marked by an oblique roughened groove on its surfaces, as if it had been fractured and afterwards repaired. The 2nd to the 7th had also vertebral facets on the head and tubercle, but the 8th and 9th had no head and articulated only with a vertebral transverse process. The 10th was an elongated flat bone 30 mm. in greatest width, pointed at both ends, and without head and tubercle.

The opportunity which I have had through the courtesy of Professor M'Intosh of studying the skeleton of the St Andrews Mesoplodon has led me to re-examine the skeletons of the two Shetland and the Dalgety Bay animals and to reconsider the mode of articulation of the ribs with the spinal column. In each skeleton the head or capitular process of the 1st rib was jointed to the body of the 1st dorsal vertebra, the transverse process of which was so slender and pointed that the large articular facet on the tubercle of the rib could not be adapted to it. In approximating the 1st rib to the spine its tubercle came into contact with the transverse process of the 2nd dorsal, with which it obviously had articulated. It consequently followed that from the 1st to the 7th rib the tubercle articulated with the transverse process of the vertebra immediately behind the body to which the head was jointed, and with which it was in numerical correspondence. As the tubercle of the 7th rib therefore articulated with the transverse process of the 8th dorsal, the 8th, 9th and 10th ribs, which had no capitular processes, articulated with the transverse processes of the 9th, 10th and 11th post-cervical vertebrae. It required therefore eleven vertebrae to articulate with ten pairs of ribs. In the St Andrews skeleton the transverse processes of the 11th post-cervical vertebra were broken, but they were entire in the corresponding vertebra in the other three skeletons, in each of which the free outer border

was marked by a distinct facet, so that provision existed for the articulation of the 10th rib with the 11th post-cervical vertebra.

Vertebral Formula.—The St Andrews skeleton had 47 vertebræ, and a similar number was found in the 1885 Shetland and the Dalgety Bay specimens, but the spines in the 1881 and 1895 animals were incomplete. The complete skeletons had 7 cervicals; 11 post-cervicals articulating with 10 ribs which were therefore dorsal vertebræ; 29 lumbo-caudal vertebræ, of which 11 may be termed lumbar and 18 caudal. The formula may be written $C_7D_{11}L_{11}Cd_{18}=47$. Variations in the relative number of lumbar and caudal vertebræ may without doubt from time to time occur.

The length of the spine in the St Andrews skeleton, without allowing for the thickness of the intervertebral discs, was 12 feet 5 inches, and the length of the skull was 2 feet $8\frac{1}{2}$ inches, together 15 feet $3\frac{1}{2}$ inches. Professor McIntosh estimated that the carcass was over 16 feet, and the length of the skeleton, allowance being made for the thickness of the discs, corresponded with this measurement. The Mesoplodon stranded in Dalgety Bay was 15 feet 1 inch in length; but, as the epiphysial plates of the vertebræ were not completely fused with the bodies, the animal was not fully grown, and although a male, the mandibular teeth projected only 2.6 cm. beyond the alveolus. The Shetland specimen, 1885, measured 15 feet 8 inches, the epiphysial plates were fused with the bodies, and the mandibular teeth projected 4.5 cm. beyond the alveolus. About 16 feet may therefore be regarded as the usual length of the adult Sowerby's whale.

Sternum.—It consisted of five transverse segments with four large intersegmental holes. The 1st, much the largest, 22 cm. long, was a broad plate, with two cornua in front, between which was a deep presternal notch. The two lateral halves of the 2nd segment were united mesially by a suture; the 5th or terminal segment was incomplete, and the hole between it and the 4th had an imperfect boundary behind; the 3rd and 4th segments were completely fused in the middle line. Each lateral border had articular facets for five costal cartilages: that for the 1st cartilage marked the anterior part of the 1st segment; those for 2nd, 3rd, 4th and 5th were opposite the line of articulation between the 1st and 2nd, 2nd and 3rd, 3rd and 4th, 4th and 5th sternal segments.

Pelvic Bones.—These bones were rudimentary; each consisted of a slender bar, 61 mm. long and 10 mm. in greatest width, faintly curved at the ends, which were attenuated. There was no sign of a rudimentary femur.

Anterior Extremity.—The *Scapulae* were plate-like, 32 cm. in diameter between the anterior and posterior angles, and 22 cm. between the glenoid

and vertebral borders. The coracoid was 10.6 cm. in length. The post-spinous fossa formed so large a proportion of the outer surface that the pre-spinous fossa was a mere groove. The spine was a sharp low ridge, but the large plate-like acromion was 11.2 cm. long and 4.6 cm. wide.

The *Humerus* was a short thick bone, only 15.5 cm. long. The head, neck and tuberosity were distinct; the shaft was somewhat flattened on its two surfaces, on the inner of which was a roughened depression, stronger in the left bone; the lower end had definite facets separated by a sharp ridge for the radius and ulna, and a concave surface on the inner border of the bone received the articular facet of the olecranon process.

The *Radius* and *Ulna* were parallel and not movable on each other. The radius was 16.5 cm. long, the shaft was 4.5 cm. wide at the middle, its surfaces were flattened. The humeral epiphysis was blended with the shaft; the carpal one was ossified, though not fused with the shaft. The ulna was almost the same length as the radius, the shaft was 3.2 cm. wide; the carpal epiphysis was ossified but not fused; the olecranon epiphysis was ossified but movable on the shaft, and in this respect it differed from the bone in the 1885 and 1888 specimens, in which the olecranon was completely fused with the ulna, though in them the carpal epiphysis was distinct.

Manus.—The manus was pentadactylous. It consisted of carpus, metacarpus and phalanges enclosed in a common tegumentary covering. The *Carpus* had a proximal row, *procarpus*, a distal row, sometimes called *mesocarpus*, a pisiform and an os centrale or ossa centralia. In the proximal and distal rows the bones were flattened on the palmar and dorsal surfaces and the cartilage between them was thin. The *procarpus* consisted of the three bones usually found in this row of the cetacean carpus, the relative position of which is indicated by their names, radiale, the smallest; intermedium, the largest; and the ulnare. The intermedium had a longitudinal groove on the palmar surface for the flexor digitorum ulnaris, and its upper border sent an angular prolongation between the carpal epiphyses of the radius and ulna.

In my memoir on the Sowerby's whale captured in 1885,* I described its carpus and showed that the distal carpalia were only three in number, which I designated as follows: the smallest was carpale₁, next to which was a large bone which represented carpalia₂₊₃, and on the ulnar side a large bone which represented carpalia₄₊₅. In the St Andrews example a similar arrangement existed. C₁, 21 mm. in transverse diameter, articulated distally with the metacarpal of the pollex, M_I, also with M_{II}, and with the radiale, centrale and the conjoined C₂+C₃. Carpalia₂₊₃ formed a

* *Journal of Anat. and Phys.*, vol. xx. p. 180.

bone 37 mm. in transverse diameter, which was grooved for the flexor digitorum, and articulated with M_{II} on one side of the groove and with M_{III} on the other, also with C_1 , centrale, intermedium and C_{4+5} . Carpalia₄₊₅ formed a bone 42 mm. in transverse diameter, which articulated by its distal border with M_{IV} and M_V ; opposite the interval between these metacarpals its distal and proximal borders and the dorsal surface were notched as if the two halves of the bone were imperfectly united, so that

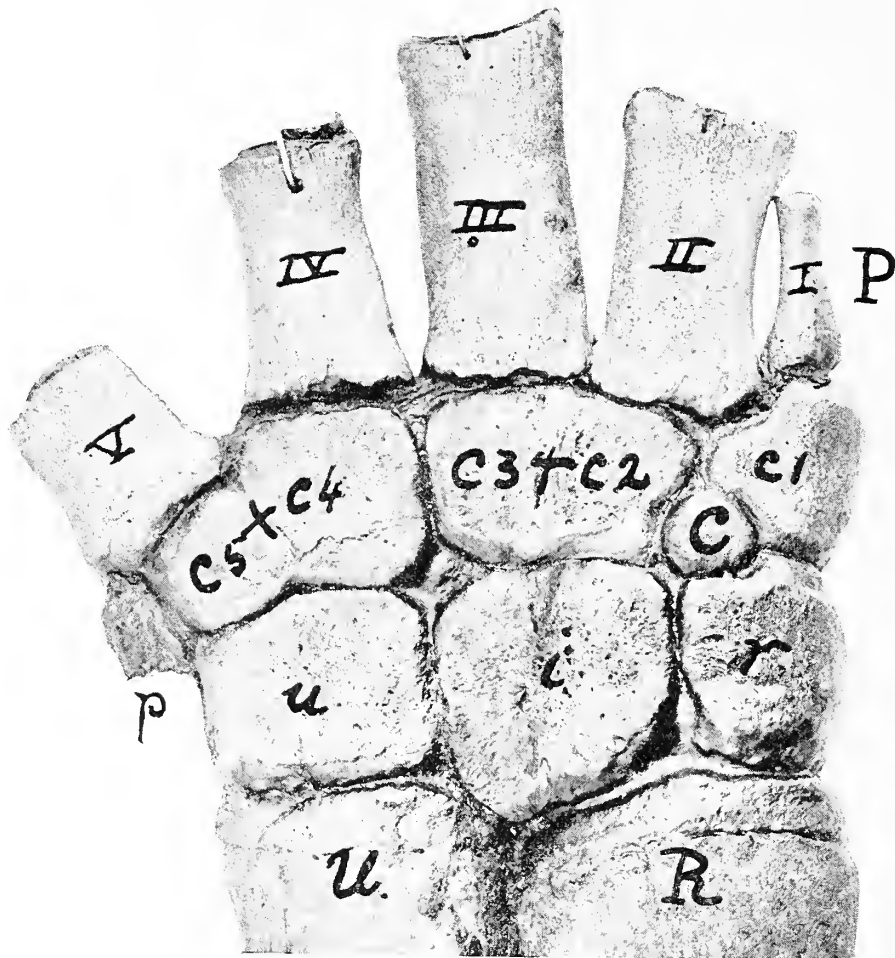


FIG. 1.—Right manus, palmar surface, Sowerby's Whale, St Andrews.

In this and other figures R means radius; U, ulna; *r*, radiale; *i*, intermedium; *u*, ulnare; *c* is a distal carpal, and the numeral associated with it is its number in the distal row; C is os centrale; the roman numerals are the metacarpal bones; P is pollex; *p* is the pisiform element.

it appeared that one part belonged to M_{IV} , the other to M_V ; it also articulated with C_3 , ulnare and the pisiform cartilage. Both in the St Andrews and 1885 animals the three distal carpalia represented the five distinct elements found in *Hyperoodon*, in which a distal carpal occurs for the metacarpal bone of each digit. It is likely that the two elements which formed the bone C_{2+3} , as well as those of C_{4+5} had, as in *Hyperoodon*, been developed independently in the carpal cartilage, but thereafter became conjoined. In the St Andrews animal a narrow bar of cartilage formed the ulnar border of the carpus and a nodule of bone, the

size of a small pea, situated in its mid-part, represented the pisiform. In the interval between the proximal and distal rows a bone on the palmar surface of each manus articulated with the radiale, intermedium, C_1 and a part of the adjoining bone which represented C_2 . It was 13 mm. in transverse and 12 mm. in antero-posterior diameter, and was from its position the os centrale (C, fig. 1), but it was not visible on the dorsal aspect. The carpus of the right manus contained also a second os centrale, 8 mm. in transverse and 9 mm. in antero-posterior diameter, which articulated with

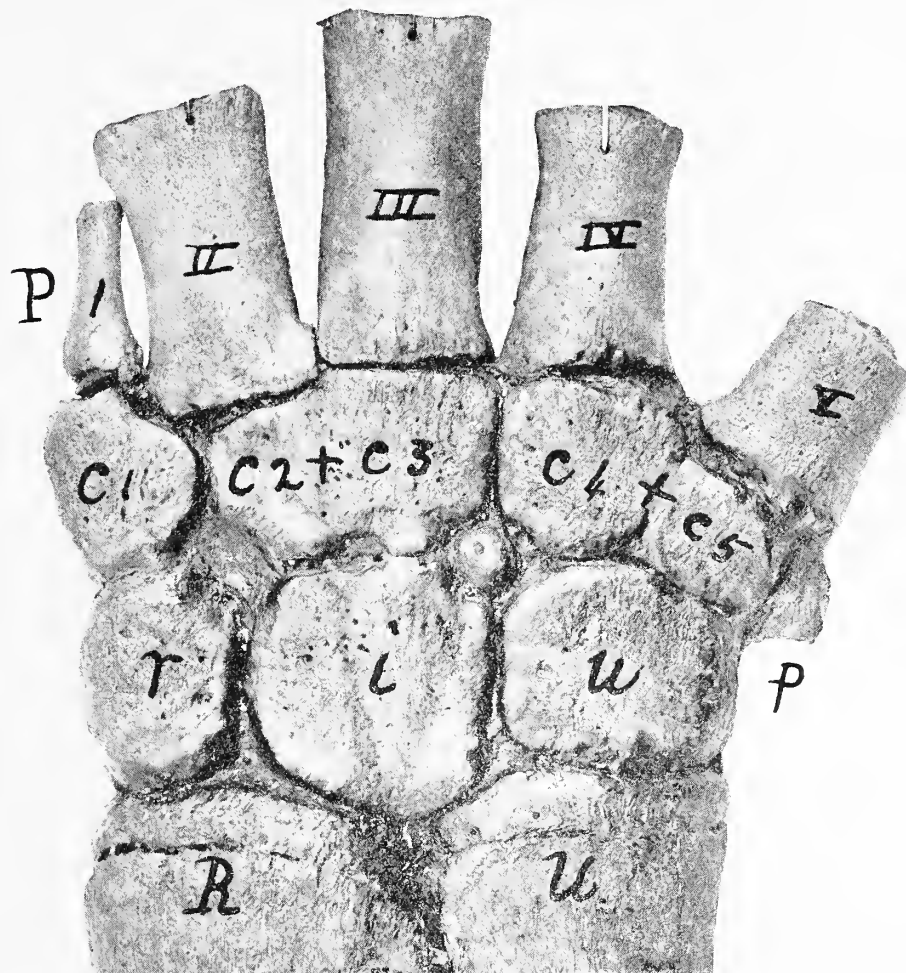


FIG. 2.—Right manus, dorsal surface, Sowerby's Whale, St Andrews.

the angles of the intermedium, ulnare, C_3 and C_4 , and was more visible on the dorsal aspect (fig. 2). In its constitution the carpus closely corresponded with that of the 1885 specimen previously described, with the addition in the right manus of a second os centrale.

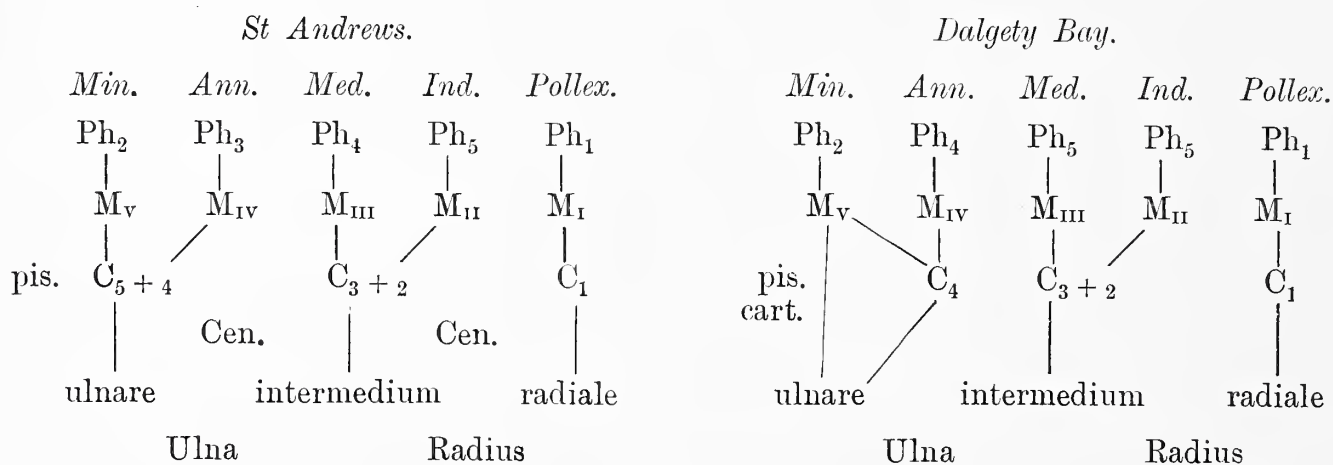
The carpus of the *Mesoplodon* from Dalgety Bay differed in the following points from those above described:—No os centrale existed as a separate bone, but the angle of the right radiale, which projected towards the bone designated $C_2 + C_3$, was partially separated by a constriction which might indicate an os centrale fused with the radiale. The transverse diameter of the distal carpalia was as follows: C_1 , 22 mm.; C_{2+3} , 36 mm.; whilst the internal carpale was only 25 mm., considerably less therefore than in

the St Andrews and 1885 specimens. Owing to its small transverse diameter the internal carpale articulated by its distal border only with M_{IV} , whilst M_V was jointed to its ulnar border and to the ulnare in the proximal row through an intermediate band of pisiform cartilage. The internal carpale represented perhaps only C_4 , whilst C_5 was not differentiated in the carpal cartilage, nor was a pisiform bone ossified in the cartilaginous inner border of the carpus.

Digits.—Each of the five digits of the St Andrews animal consisted of a metacarpal bone M , and one or more phalanges, Ph . The metacarpal of the pollex M_I was a slender cylindriform bone 26 mm. long; at the free end was a nodule of cartilage which represented the phalanx. The metacarpals of the other digits were flattened; M_{III} was the longest, 51 mm., M_V was only 25 mm. long, and was set at an angle to and articulated with the distal border of the carpale, which represented C_5 . As some of the phalanges had been lost their exact number cannot be stated, but they were probably the same as in the 1885 specimen, viz.: D_{II} , five phalanges; D_{III} , four; D_{IV} , three; D_V , two, and the longest digit was the index.

In the Dalgety Bay specimen, however, D_{III} had five phalanges in one manus though only four in the other, and D_{IV} had four phalanges, of which the terminal was no bigger than a small shot.

The constitution of the manus in the St Andrews and Dalgety Mesoplodons may be expressed by the following formulæ, the number of phalanges in the St Andrews specimen being taken from the 1885 specimen:—



In addition to the specimens of Sowerby's whale found on the coast of Scotland described in this communication, we owe to Mr Wm. Taylor of Lhanbryde notes of other examples. In 1897 he obtained the skull with other bones of a male, about 15 feet long, stranded about $1\frac{1}{2}$ mile east of the harbour of Nairn.* In September 1899 two females, mother and a

* *Annals of Scottish Natural History*, 1897, p. 42; and in a paper read at a meeting of Literary and Scientific Societies at Nairn, July 1898, p. 4.

young, were stranded near Nairn. The mother was nearly 16 feet long, the young one about 9½ feet. Mr Taylor stated* that in the upper jaw of the young animal 8 rudimentary teeth were concealed in the gum on each side, and in the lower jaw 17 on each side behind the normal tooth. Some years ago I described† a somewhat similar series of rudimentary teeth in the gum of the upper and lower jaws of a Hyperoodon upwards of 20 feet long. In March 1904 Mr Taylor obtained the head of a male specimen, 14 feet in length, stranded at Fraserburgh.‡ It is obvious, therefore, that Sowerby's whale is not an uncommon species in the Scottish seas.

In my notice of the capture of Sowerby's whale in Dalgety Bay§ the only English specimen to which I could refer was one recorded by Messrs Thomas Southwell and Eagle Clarke caught on the Yorkshire coast near Spurn Head in September 1885.|| Since then, Messrs Southwell and Harmer have described a gravid female, 16 feet long, containing a fœtus 5 feet long, captured in December 1892 at Overstrand, near Cromer, and purchased by the Hon. Walter Rothschild for his museum.¶ In my previous memoirs on Sowerby's Whale, I referred to specimens caught on the French coast and to the descriptions by Reinhardt, Malm and Aurivillius of specimens stranded on the coast of Scandinavia. More recently J. A. Greig has described** two specimens obtained in 1895 in the Scandinavian seas, one at the island Karmö, the other at the adjoining island Fæø. Mr Glover M. Allen has also recorded†† two recent specimens captured off the coast of North America, one at Annisquam, Massachusetts, in 1898; the other at Long Branch, New Jersey, in 1905. In a recent note (*Science*, vol. xxvi., 1907) F. W. True stated that, in his opinion, three species of *Mesoplodon* occur on the U.S. Atlantic coast, *bidens*, *europæus*, *densirostris*.

Morphology of Manus in Hyperoodon and in the Delphinidæ.

Since publishing in October 1885 my memoir on the Anatomy of Sowerby's Whale, in which I dwelt on the morphology of the manus in this cetacean, and compared it with the manus of *Hyperoodon* and *Globicephalus*, I have had the opportunity of dissecting several other specimens of the *Odontoceti*

* The same *Annals*, p. 66, April 1900. The skeleton of the mother and the skull of the young animal are now in the Royal Scottish Museum.

† *Proc. Roy. Phys. Soc. Edin.*, vol. ix. p. 25, 1885-86.

‡ *Annals of Scottish Natural History*, p. 186, July 1904.

§ *Proc. Roy. Phys. Soc.*, vol. x., 1888-9.

|| *Ann. and Mag. Nat. Hist.*, ser. 5, vol. xxii. p. 53, 1886.

¶ *Ann. and Mag. Nat. Hist.*, ser. 6, vol. xi. pp. 275, 439, 1893.

** *Bergens Museums Aarboog*, 1897, No. v., plates 1, 11, animal and skull.

†† *American Naturalist*, vol. xl. p. 357, 1906.

and of studying in fresh specimens the hand with the bones *in situ*, and not artificially articulated. I propose therefore to continue my account of the morphological constitution of the manus in the toothed whales.*

HYPEROODON ROSTRATUS.

A young male $20\frac{1}{2}$ feet long was stranded near Dunbar in November 1885. I secured the specimen and mounted the skeleton in the Anatomical Museum of the University. As the vertebral plates and the epiphyses of the long bones were not fused with their diaphyses the animal had not reached maturity. The carpal end of both radius and ulna had a well-marked epiphysial cartilage partially ossified. The carpalia were flattened on the surfaces and consisted of osseous nodules in masses of cartilage.

The *procarpus* or proximal row contained radiale, intermedium and ulnare. The radiale showed an osseous nodule 31 mm. in transverse diameter surrounded by cartilage which articulated with radius, intermedium and ulnare. In the intermedium the osseous nodule was 41 mm. wide, and with its surrounding cartilage articulated with radius, ulna, radiale, ulnare and carpalia_{2, 3, 4}. The osseous nodule in the ulnare was 32 mm. wide, and with its cartilage articulated with ulna, intermedium, carpalia_{4, 5}, and with the unossified pisiform cartilage, which formed the inner border of the carpus and was connected with the cartilage of the ulnare, carpale₅ and metacarpal_v.

The distal row, *mesocarpus*, consisted of five distinct elements, three of which were partially ossified and two were unossified. Carpale₁ had an osseous nodule 17 mm. wide, and articulated with radiale, C₂ and M. Carpale₂ was unossified and articulated with radiale, intermedium, C₁, C₃ and M_{II}. Carpale₃ had an osseous nodule 24 mm. wide, and articulated with intermedium, C₂, C₄ and M_{III}. Carpale₄ had an osseous nodule 16 mm. wide, and articulated with intermedium, ulnare, C₃, C₅ and M_{IV}. Carpale₅ was unossified and articulated with ulnare, pisiform cartilage, C₄ and M_v. The carpal elements in this Hyperoodon, though imperfectly ossified, corresponded in number and position with the bony carpus of the adult female previously described and a figure † of which is now reproduced on the next page; they showed that ossification did not proceed uniformly in the bones, for a needle passed through C₂ and C₅ failed to detect bone in the substance of their cartilage, whilst in C₁, C₃ and C₄ the osseous nodule was visible on the surface. No os centrale was present in either carpus.

The hand was pentadactylous, the metacarpal of each digit articulated

* In the *Proc. Roy. Soc. Edin.*, 1891-92, vol. xix. p. 70, I have discussed the morphology of the manus in the whalebone whale, *Balænoptera rostrata*.

† *Journ. of Anat. and Phys.*, October 1885, vol. xx. p. 184.

with its corresponding carpale, which belonged therefore to the same digit. M_I had an unossified phalanx; M_{II} had five phalanges in process of ossification; M_{III} had also five phalanges; M_{IV} had three phalanges and M_V had only two. The formula of the hand in *Hyperoodon* is as follows:—

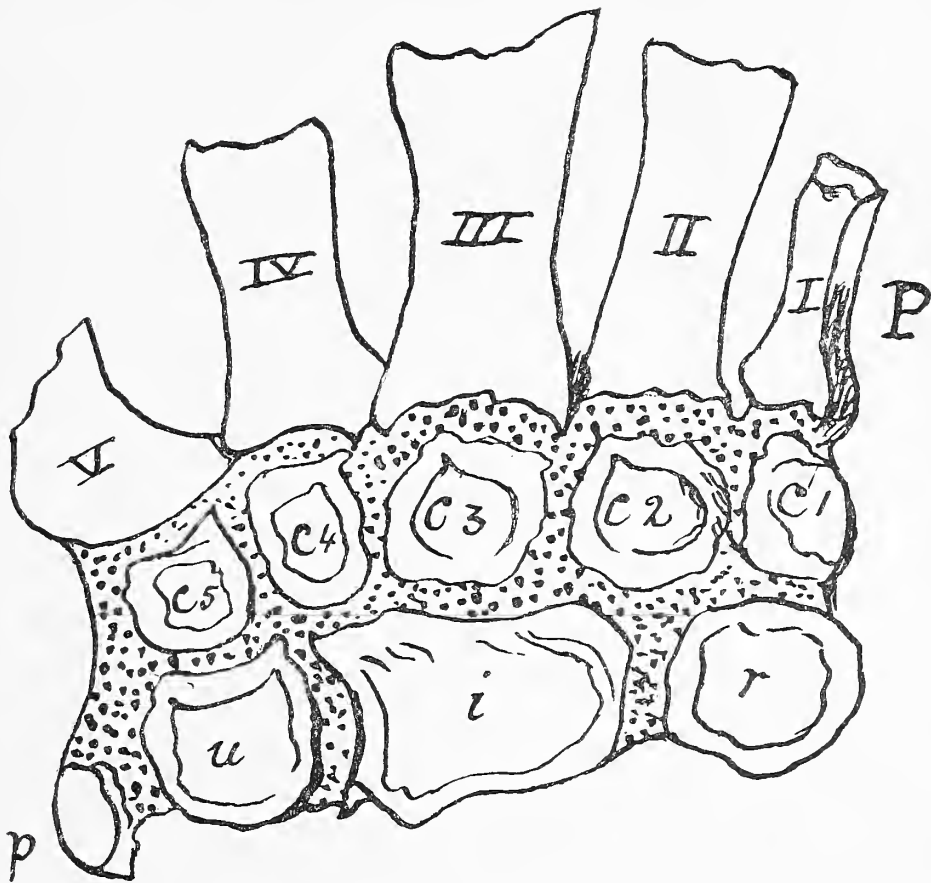
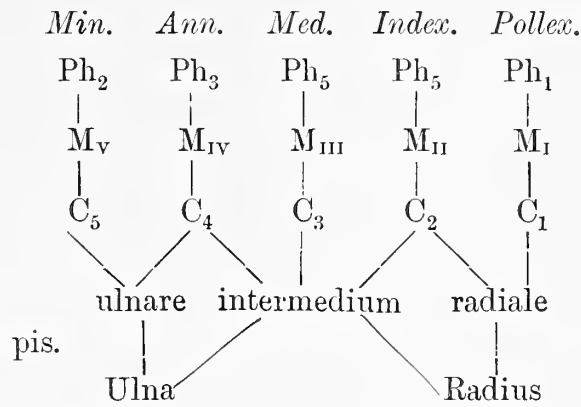


FIG. 3.—Carpo-metacarpal region of *Hyperoodon rostratus* enlarged from the figure in *Journ. Anat. and Phys.*, October 1885, vol. xx. p. 184. The dotted area represents cartilage.

The study of the manus in the younger animal confirmed the opinion, previously formed, from the examination of the *Hyperoodon* described and figured in my account of the Shetland Mesoplodon, that the distal row of the carpus in *Hyperoodon* possessed five distinct carpalia, one for each digit. In that memoir I referred to the generalised type of carpus described by Gegenbaur* in the reptilian form *Chelydra serpentina*, in which five distinct distal carpalia were present. He contrasted this with the arrange-

* Carpus and Tarsus, 1864.

ment in the mammalian carpus, in which the 4th and 5th metacarpals had together only one distal carpal, the os hamatum (unciform), a condition which he regarded as characteristic of the mammalia. My recognition of five distinct distal carpalia in *Hyperoodon* furnished, however, an example in a mammal of conformity in this respect with the earlier reptilian type.

In a valuable memoir, "Die Hand der Cetaceen,"* the first part of which appeared in 1889, W. Kükenthal described the hand of a young *Hyperoodon* in which five distinct distal carpalia, "corresponding," he said, "with the carpus described by Turner," were present. They were approximately of equal size; one was associated with the radiale, one with the ulnare, and three with the intermedium; each carpale carried the metacarpal of a digit. Only the intermedium, radiale and carpale₁ possessed ossific nuclei. In an older animal five distal carpalia were seen, each with an osseous nucleus. In it an os centrale was also present. In the second part of the memoir, 1893, Kükenthal described a section of the hand of *Hyperoodon* in the Museum of the Royal College of Surgeons, London, in the carpus of which were three proximal carpals, five distal carpalia, and an os centrale. He had also two foetuses of this animal 55 and 15·8 cm. long respectively, in both of which carpale₅ was distinct from carpale₄; in the larger foetus carpale₂ was partially fused with carpale₃ and an os centrale was present.

The hands in the two specimens which I have personally examined and the carpal regions in the five animals described by Kükenthal prove conclusively that *Hyperoodon* possessed five distal carpalia in a large proportion of individuals. At the same time it should be stated that a smaller number of distal carpalia has been described in some skeletons. Thus Vrolik's example had four bones; Van Bambeke spoke of four bones, C₁ and C₂ separate, C₃₊₄ conjoined; Max Weber stated that in one specimen in Amsterdam four distal carpals were found, viz. C₁ to C₄, C₅ not being represented; † whilst in a smaller animal only three bony distal carpalia had been developed, the fourth probably not having reached the stage of ossification.

Delphinidæ.

DELPHINUS DELPHIS—*The Common Dolphin.*

I have dissected two females of this dolphin; one, 5 feet 5½ inches in a straight line, was shot in the Firth of Forth in February 1887; ‡ the other,

* *Vergleich. Anat. Entwicklungsgeschichte untersucht an Wahlthieren*, Jena, Erster Theil 1889, Zweiter Theil 1893; also in *Denksch. der medic.-natur. wiss. Gesellsch. zu Jena*, Dritter Band, 1889 and 1893.

† *Morphologisches Jahrbuch*, May 1888, Bd. xiii., Taf. xxvii. fig. 4, p. 626.

‡ *Proc. Roy. Phys. Soc. Edin.*, 1887, p. 346.

taken in Shetland in February 1895,* was 5 feet 8 inches long. The smaller specimen was not adult, as the epiphyses were only partially ossified, but in the longer animal the radial and ulnar epiphyses were ossified, though not fused with the shafts. The hand was pentadactylous. The *procarpus* consisted of an osseous radiale 15 mm. wide, an intermedium 20 mm. wide, and an ulnare 16 mm. wide in the longer, though with smaller osseous nodules and more cartilage in the younger animal. The radiale articulated with radius, intermedium, carpale₁ and C₂; the intermedium with radius, ulna, radiale, ulnare and two distal carpalia. The ulnare articulated with ulna, intermedium, a third carpale and metacarpal_v.

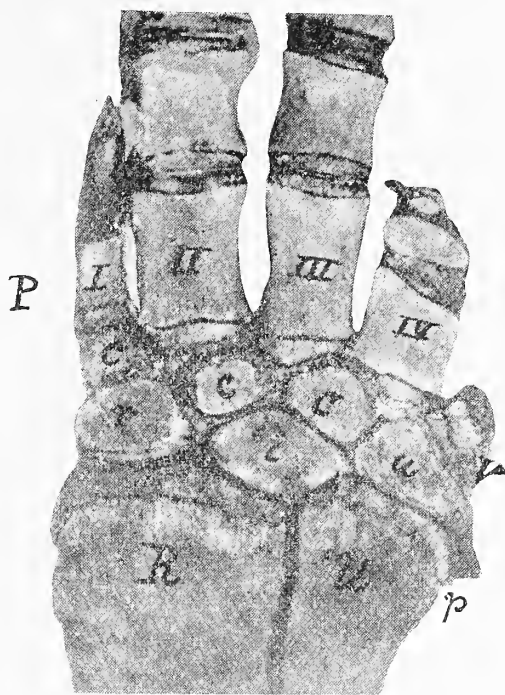


FIG. 4.—Dorsal surface, right manus, *Delphinus delphis*.

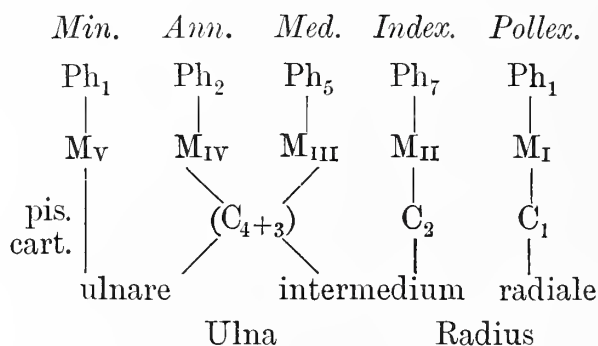
Only three bones were present in the distal row. At the radial end of this row was the bone which I have designated carpale₁, though some writers regard it as the metacarpal of the pollex. In the older animal it was 9 mm. in transverse diameter and articulated with radiale, largely with metacarpal_I and slightly with metacarpal_{II}; it obviously belonged to the pollex. The bone next to it I name carpale₂; it was 13 mm. wide and articulated with radiale, intermedium, largely with M_{II} and slightly with M_{III}; it belonged to the index digit, and may be regarded therefore as carpale₂. The third bone in the row, 13 mm. wide, probably represented two carpalia; it articulated with intermedium, ulnare, and in almost equal proportions with M_{III} and M_{IV}; it seemed to belong, therefore, to both medius and annularis, and was probably formed by the fusion of carpalia₃ and₄. No other osseous

* I am indebted to Mr Thos. Anderson of Hillswick for this specimen.

nodule was in the distal row, but in the younger specimen a piece of cartilage intervened on one border between C_{3+4} and the ulnare, and on the other between C_{3+4} , M_{IV} and M_V : it possibly represented an unossified carpale₅. Neither specimen showed an os centrale. The pisiform cartilage extended on the border of the carpus from the ulna to the fifth metacarpal. In the digits the metacarpals and phalanges had both proximal and distal epiphyses. M_I the pollex had one phalanx; M_{II} had seven phalanges; M_{III} five phalanges; M_{IV} , two, and M_V only one phalanx; owing to the absence of carpale₅ M_V was connected through the intermediate cartilage with the ulnare and was rudimentary.

Max Weber in his very suggestive memoir "Ueber den Carpus der Cetaceen" * figured the manus of this dolphin; he considered that only two distal carpalia were present, which he numbered C_{2+3} and C_4 ; they corresponded to the carpalia that I have named C_2 and C_{3+4} , whilst the bone named in my description C_1 he regarded as the metacarpal of the pollex. Kükenthal inclined to the same view, though he thought this bone might be carpale₁, or that C_1 might have blended with M_I , or with radiale, or had perhaps disappeared.

The formula of the hand of *Delphinus delphis* is as follows:—



DELPHINUS ACUTUS (*Lagenorhynchus leucopleurus*).

In April 1906 a white-sided dolphin was stranded near Dunrobin Castle, and the external characters and skeleton were described by me in June of the same year.† I may refer to that description for details regarding the animal, and I reproduce a radiograph of one flipper (fig. 5).

The animal was a female and was not adult, for the skeleton was imperfectly ossified, and the radio-ulnar epiphyses were not fused with the shafts. The *procarpus* consisted of radiale, intermedium and ulnare. Two relatively large bones were present in the middle of the mesocarpus, one of which articulated with M_{II} and M_{III} , and obviously represented $C_2 + C_3$,

* *Morphologisches Jahrbuch*, May 1888, Taf. xxvii. fig. 7.

† *Proc. Roy. Soc. Edin.*, vol. xxvi. p. 310, 1906.

the other articulated with M_{III} and M_{IV} , and represented without doubt C_4 , though possibly also C_5 was fused with it, as no separate C_5 was present. On the radial side of $C_2 + C_3$ a bone which I regarded as C_1 was in series with the radiale and the pollex. The pisiform cartilage was unossified, and there was no sign of an os centrale. The metacarpals were five in number, and that of the fifth articulated with the ulnare. The metacarpals and phalanges had proximal and distal epiphyses, but the phalanges of the pollex and minimus were not ossified.

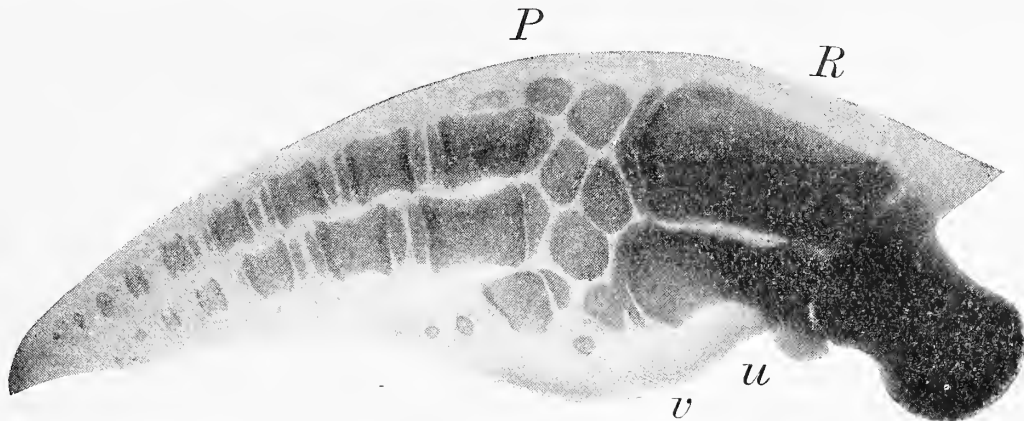
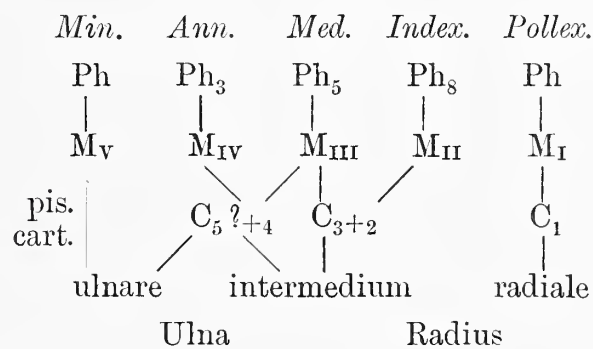


FIG. 5.—Radiogram of flipper of *Delphinus acutus*.

Kükenthal has described an embryo of this dolphin 45.5 cm. long, in which he recognised two cartilaginous distal carpalia comparable with the relatively large bones above described. A third cartilage distal to the radiale similar to my $carpale_1$ was present, but Kükenthal left it an open question whether it should be regarded as $carpale_1$ or the metacarpal of the pollex; in the latter case $carpale_1$ had either disappeared or become fused with the radiale.

I suggest the following formula for the hand of *Delphinus acutus*:—



LAGENORHYNCHUS ALBIROSTRIS (*Delphinus albirostris*).

In 1888 I obtained an adult female white-beaked dolphin, and had the skeleton mounted in the Anatomical Museum. The manus was carefully dissected. The radial and ulnar epiphyses and the metacarpal and phalangeal epiphyses were fused with their respective shafts. The carpal

bones were fully ossified. In the *procarpus* the radiale, with a transverse diameter 25 mm., articulated with radius, with a distal carpal which I regard as C_{2+3} , and with metacarpals I and II. The intermedium 30 mm. wide articulated with radius, ulna, ulnare and the two distal carpalia. The ulnare 24 mm. wide articulated with ulna, intermedium, distal carpal₄, with the metacarpal of digit V, but was separated from M_{IV} by an interval occupied by cartilage.

The distal row of the carpus contained only two bones. The larger was wedged between and separated the radiale and intermedium from each

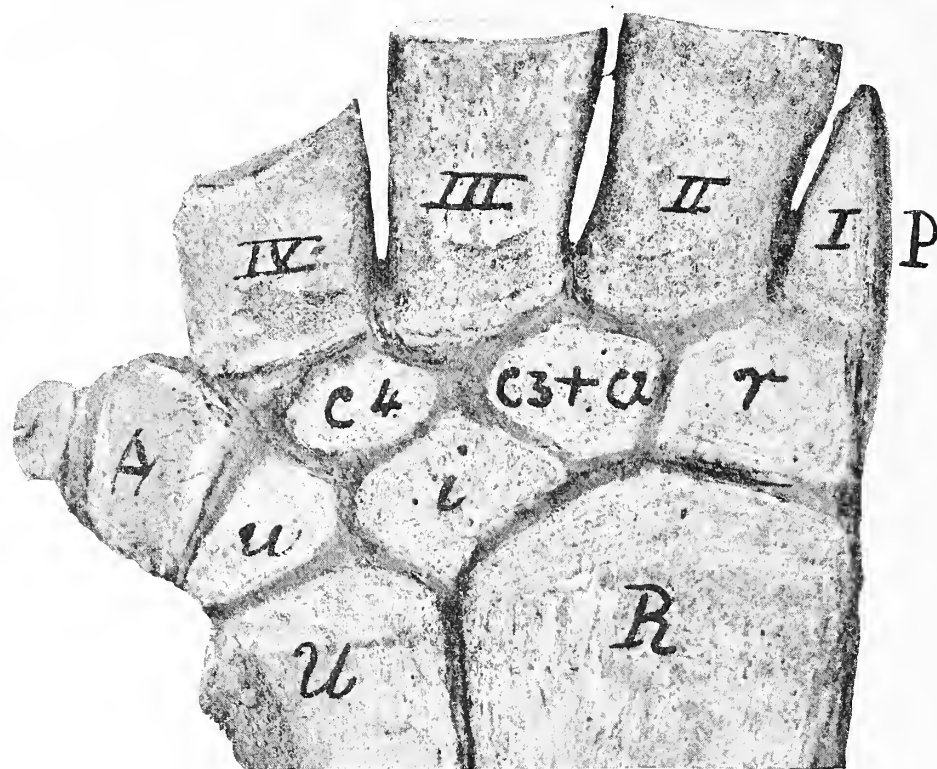


FIG. 6.—Left manus, dorsal surface, *Lagenorhynchus albirostris*.

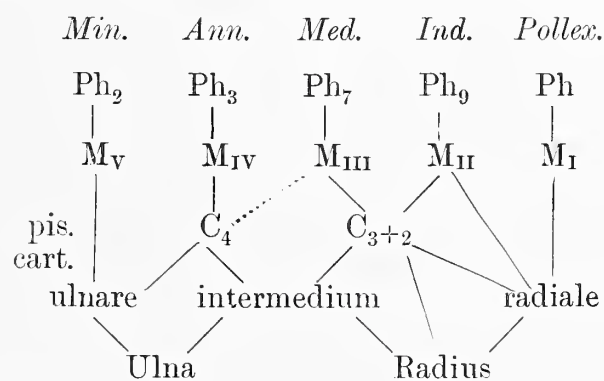
other; it was 27 mm. wide, and articulated with both these bones, slightly with the carpal end of the radius, extensively with metacarpals II and III, and also with the smaller disto-carpal; it was obviously carpal₂₊₃ fused together. The intermedium and the larger carpal were marked by a shallow groove on both palmar and dorsal surfaces, which was in line with the interosseous space in the forearm and with the interval between M_{II} and M_{III} ; the groove was doubtless for a tendon, or an artery. The smaller carpal was 24 mm. wide and articulated with intermedium, ulnare, almost equally with M_{III} and M_{IV} , and with the larger disto-carpal; to its ulnar side a piece of unossified cartilage separated it from M_V ; it is marked C_4 in fig. 6, and the unossified cartilage may represent an undeveloped C_5 . A plate of pisiform cartilage was at the ulnar border of the carpus, and extended from the lower end of the ulna

to the base of M_V . No ossific nodule had been formed in it. The carpus did not contain an os centrale.

In digit I, the pollex, the metacarpal was elongated and somewhat conical; it articulated directly with the radiale, for $carpale_1$ was not developed, and also with $carpale_{2+3}$; the phalanx was not ossified. In the other digits the metacarpals were flattened. Metacarpal $_{II}$ articulated for more than a third with radiale and the rest with $carpale_{2+3}$; it had nine osseous phalanges. Metacarpal $_{III}$ articulated almost equally with $carpalia_{2+3}$ and C_4 ; it had seven osseous phalanges. Metacarpal $_{IV}$ articulated with $carpale_4$ and with the cartilage between it and the ulnare; it had three osseous phalanges. Metacarpal $_V$ was marked by a groove on the dorsal surface which passed obliquely between the two lateral borders; the bone projected at the ulnar border of the manus, and was set at an angle to M_{IV} ; it articulated with ulnare and M_{IV} and it joined the pisiform cartilage; two rudimentary osseous phalanges were embedded in cartilage.

Max Weber gave figures of the manus in two specimens of *Lagenorhynchus albirostris*. In the older the three bones of the procarpus were represented, whilst only two disto-carpalia were figured, which he interpreted as $C_2 + C_3$ and C_4 , giving a significance like that which I have also attached to them. His conception of the metacarpus and phalanx of digit V corresponded with the conclusion to which I have subsequently arrived. From the figure of the carpus in the younger specimen it would seem that he recognised in it three distinct disto-carpalia, whilst a fourth was to be found in a bone occupying the position of the ulnare: the ulnare proper had either disappeared or become blended with the intermedium.

I suggest the following formula for *L. albirostris*:—



PHOCÆNA COMMUNIS (*Delphinus phocæna*).

The manus of the common porpoise was dissected in an adult female in December 1892, and has also been examined in other specimens.

In the adult the radial and ulnar carpal epiphyses were united with the

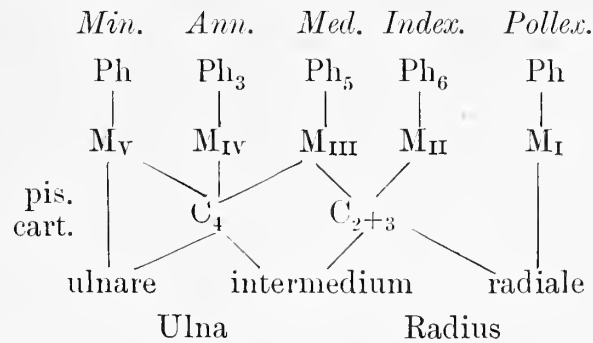
shafts. The *procarpus* contained the three customary bones. The radiale, 14 mm. in transverse diameter, articulated with radius, intermedium, the carpal cartilage associated with the pollex and with M_{II} . The intermedium was 14 mm. wide, grooved on the palmar surface, and articulated with radius, ulna, radiale, ulnare and the two distal carpalia. The ulnare was 7 mm. wide and articulated with ulna, intermedium, the more ulnar distal carpale, M_V , and with the pisiform cartilage. This cartilage was only 5 mm. wide and 7 mm. long, and extended from the ulnar epiphysis to M_V ; it did not contain an ossified nucleus. No os centrale was seen.

Two distal-carpalia were situated in the mesocarpus. The more radial was a small bone 9 mm. wide, with a groove in line with that on the intermedium opposite the interval between M_{II} and M_{III} , with both of which the nodule articulated, as well as with the radiale, intermedium, and the more ulnar distal carpal. The last-named bone was 12 mm. wide, and articulated with ulnare, intermedium, the radial distal carpal, and M_{III} , M_{IV} and M_V . The more radial bone apparently represented C_{2+3} , and the more ulnar C_4 , or it may be C_{4+5} . Both C_1 and C_5 were absent as independent units.

The hand was pentadactylous. Digit I, the pollex, was so close to the radial border and so slender that it might easily have been overlooked. It was connected with a band of cartilage on the radial border of the carpus extending to the radiale, which, though no ossific nodule was found in it, might perhaps represent carpale₁. M_I , 12 mm. long, was very slender, and had at its distal end an unossified phalanx. Digit II, the longest, consisted of M_{II} , 21 mm. long, and six small osseous phalanges, the terminal of which was no bigger than a very small shot; it was connected with the carpal cartilage of the pollex, the radiale, and the outermost distal carpal. Digit III had a metacarpal 20 mm. long and five small phalanges; M_{III} articulated with both distal carpals. Digit IV had a metacarpal 13 mm. long and three small phalanges. Digit V projected from the ulnar side of the carpus; M_V , 8 mm. long and 11 mm. wide, was connected with M_{IV} , the ulnar distal carpal, the ulnare and the pisiform cartilage.

The presence of three bones in the procarpus of the porpoise and of only two disto-carpalia has been generally described by anatomists. Both Weber and Kükenthal numbered them as C_1 and C_2 . Kükenthal considered that C_1 , as described in this memoir on the radial border of the carpus, was either absent or blended with M_I , or with the radiale; his C_1 would therefore represent my $C_2 + C_3$ conjoined, whilst his C_2 might be the conjoined $C_4 + C_5$, though more likely from its position only C_4 , whilst C_5 had either not developed or had blended with the ulnare.

The following formula is given for *Phocæna communis*:—



GLOBICEPHALUS MELAS (*Pilot Whale*).

In my previous memoir on *Mesoplodon*,* so frequently referred to, I gave a short description of the carpus of *Globicephalus melas*. I have again studied this specimen, compared it with the descriptions which were subsequently published by Max Weber and by Kükenthal, and have been led to modify the opinion which I had previously expressed on the morphology of certain of its disto-carpalia.

In the two hands of my specimen the radial and ulnar epiphyses were ossified, but not fused with the shafts. The bony elements of the carpus were well formed and dominated the cartilaginous matrix. The radiale was 47 mm. in transverse diameter, the intermedium 40 mm. and the ulnare 24 mm.

The bone on the radial border of the carpus, which I had previously spoken of as carpale₂, I now regard as carpale₁, which necessarily modifies the numerical order of the other disto-carpalia, so that the former carpale₃ now becomes C₂, and the former C₄ becomes carpale₃. Again, the bone situated in the plane of the proximal ends of the metacarpals, which from its rounded form and roughened surface I had regarded as distal carpal C₁, is now described as the metacarpal of the pollex. This change in opinion of the morphology of these bones is based on a fuller recognition of the position of the os centrale, which in the original description was regarded as placed between carpalia₂ and ₃, whereas its true position is between carpale₁, C₂, the radiale and the intermedium. The present description follows, therefore, the amended view of the morphology of the bones.

Carpale₁, 34 mm. in transverse diameter, articulated with radiale, an os centrale in right carpus, and about equally with M_I and M_{II}: carpale₂, the bone of which was 33 mm. wide, articulated with intermedium, os centrale in right carpus, radiale in left, carpale₃, and about equally with M_{II} and M_{III}: carpale₃, the bony part 31 mm. wide, articulated in right carpus with inter-

* *Journ. Anat. and Phys.*, Oct. 1885, vol. xx. p. 185.

medium, ulna, ulnare, carpale₂, and about equally with M_{III} and M_{IV}; in this carpus it was distinct from the ulnare, but in the left hand it was fused with it and formed a conjoined bone (compare fig. 7 with fig. 8). No separate carpale₄ or ₅ existed, though possibly C₃ may contain the element C₄, and carpale₅ may potentially be present as a part of M_V, for the ulnare was relatively small, so that C₅ could scarcely be regarded as fused with it.

In the interval between carpale₁, C₂, the radiale and the intermedium was shrunken cartilage containing in the right carpus a small osseous nodule, which may be regarded as an os centrale, but in the left carpus the corresponding cartilage was unossified. Unossified and shrunken pisiform cartilage

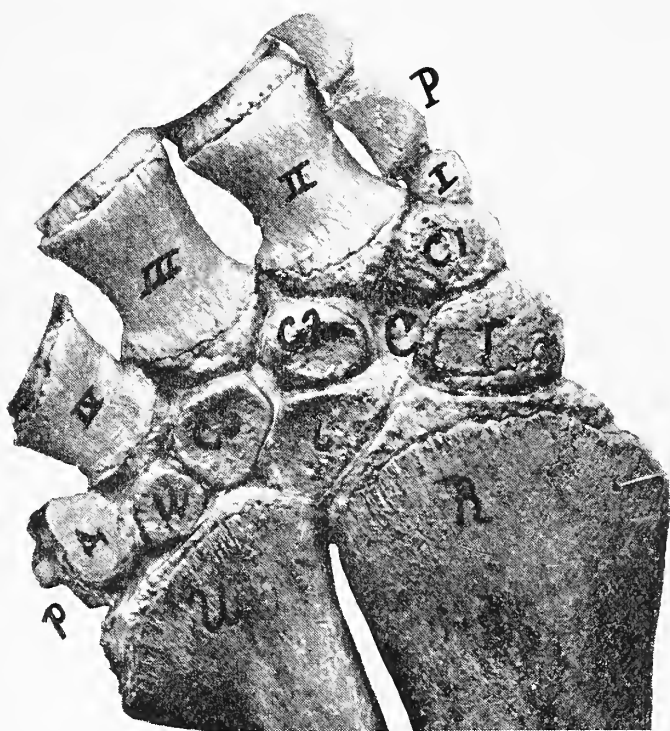


FIG. 7.—The right carpus of *G. melas*.

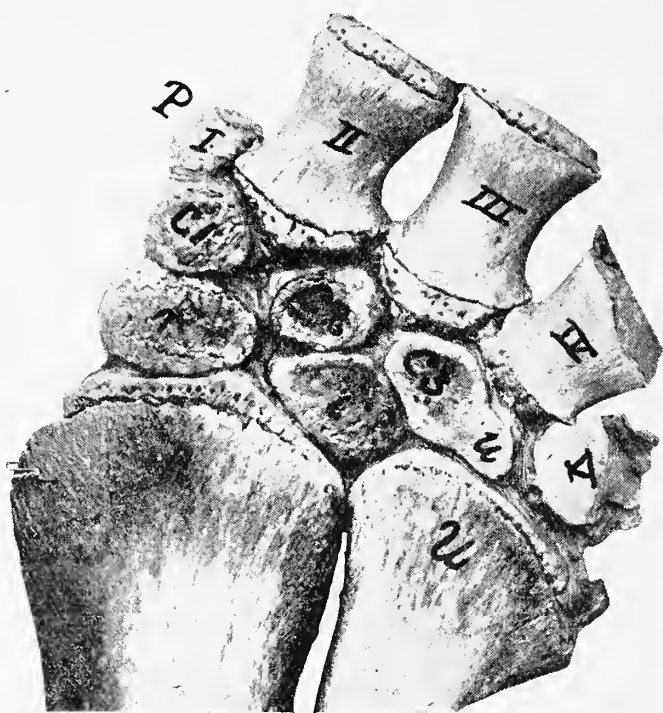


FIG. 8.—The left carpus of *G. melas*.

formed the ulnar border of the carpus. In their articulations the metacarpals were arranged as follows:—M_I, the rounded bone with roughened surface for the pollex above referred to, articulated with carpale₁: M_{II} with C₁ and C₂ about equally, and with the cartilage of the os centrale: M_{III} with C₂ and C₃ about equally: the carpal ends of M_{II} and M_{III} were bifaceted, the facets separated by a mesial ridge were inclined laterally, so that the corresponding distal carpalia were not opposite to the ridge, but to the lateral facets and to the interval between two metacarpals: M_{IV} with carpale₃ and the ulnare: M_V with ulnare, the ulna and the pisiform cartilage.

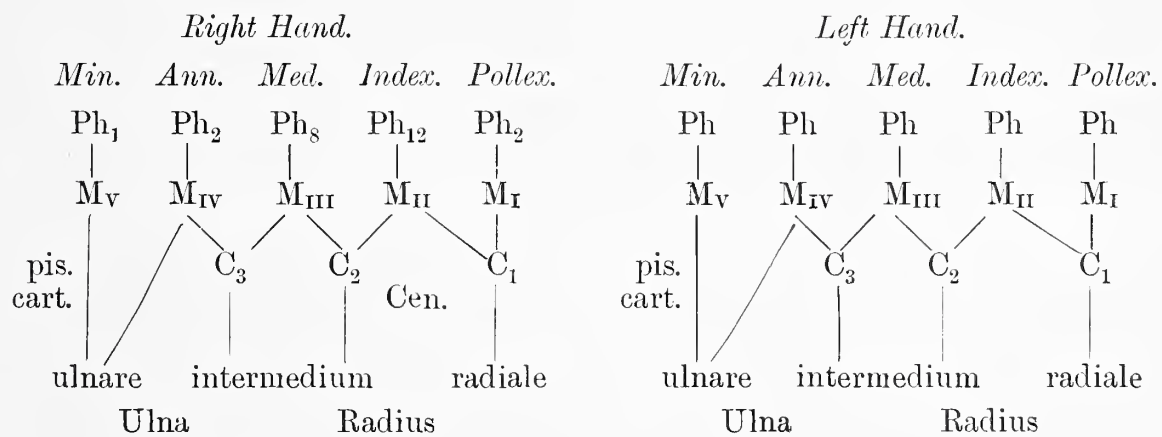
In Max Weber's figure of the hand of *Globicephalus melas* three distinct disto-carpalia were recognised:—carpale₁, the most radial, jointed with M_I and M_{II}: a bone which he regarded as representing carpale₂₊₃ jointed with the bases of M_{II} and M_{III}: next to it a bone lettered C₄ which articulated

with M_{III} and M_{IV} . No carpale₅ was present, and both M_{IV} and M_V articulated with the ulnare and the latter also with the ulna. An os centrale occupied the position described above in my specimen.

Kükenthal had the opportunity of studying several embryos of *G. melas*. He recognised only three separate distal carpalia. Carpale₁ in the larger embryos was the biggest and had a short metacarpal_I; carpale₂ was associated with M_{II} and M_{III} , though principally with M_{II} ; carpale₃ with M_{III} and M_{IV} ; whilst carpale₅ was directly fused with the ulnare with which M_V articulated. He referred to an os centrale described by me as present in the well-grown *G. melas*, but in the majority of the hands of his embryos the centrale was not visible. He stated that the foetal carpus corresponded in the number and position of its elements with that of the well-grown animal. The pisiform cartilage was well formed in the embryos, and in all a small, rounded piece of cartilage projecting from the radial border of the radiale represented the præpollex of von Bardeleben.

As is well known, *G. melas* is distinguished by the number of the phalanges in the second and third digits and by the great length of the digits, the index being the longest.

The formulæ of the carpus in my specimen are appended, and that in the right hand corresponds with formula No. 2 in Kükenthal's memoir, p. 34.



GRAMPUS GRISEUS (*Risso's Dolphin*).

In September 1899 a school of Risso's dolphin was captured near Hillswick, Shetland, and specimens were sent to me by Dr Charles Anderson.* I dissected the hands in two animals. From the state of the ossification neither had reached maturity, though one was more advanced than the other: the radio-ulnar epiphyses were ossified, though not fused with the shafts.

* See my account in *Proc. Roy. Phys. Soc. Edin.*, vol. xi. p. 192, 1891-92; also my description of the Viscera in *Journ. Anat. and Phys.*, vol. xxvi. p. 258, 1892. Two of the skeletons are mounted in the Anatomical Museum of the University.

In the more advanced specimen the carpal bones were well ossified, but in the other the cartilage was more abundant. The *procarpus* had radiale, intermedium and ulnare. In the older animal the radiale was 33 mm. in transverse diameter, the intermedium 40 mm. and the ulnare 28 mm. The radiale articulated with radius, intermedium, carpalia ₁ and ₂. The intermedium was grooved on its palmar surface; its proximal border mostly articulated with radius and only slightly with ulna; it also articulated with radiale, C₂ and C₃₊₄, the latter of which intervened between intermedium

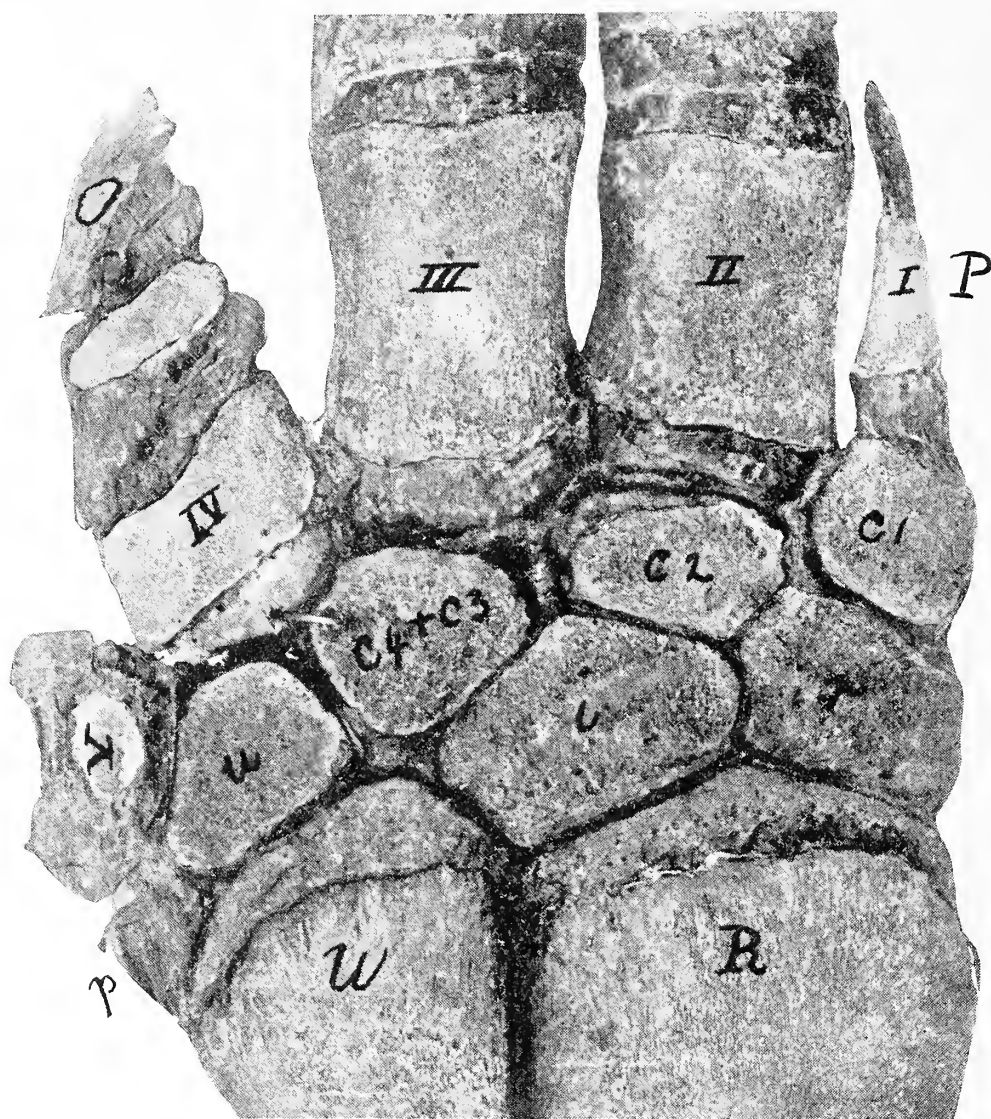


FIG. 9.—Dorsal surface, left manus, *Grampus griseus*.

and ulnare. The ulnare articulated with the inner half of the ulnar epiphysis, with C₃₊₄, with M_{IV} and M_V, and with the pisiform cartilage, which contained no osseous nodule.

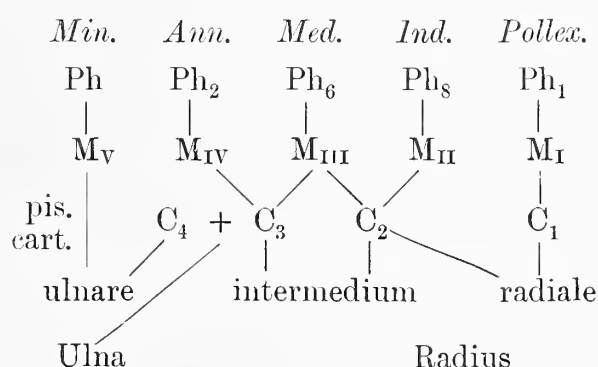
The distal row had three separate carpalia. Carpale₁ was 25 mm. wide and articulated with radiale, C₂ and the epiphysis of M_I. Carpale₂ was 31 mm. wide and articulated with radiale, intermedium, C₃, largely with M_{II} and slightly with M_{III}. Carpale₃, 29 mm. wide, was grooved on palmar surface, and articulated with intermedium, ulnare, epiphysis of ulna, largely

with M_{III} and less so with M_{IV} ; carpale₄ was probably fused with it. No os centrale was present, and the pisiform cartilage extended from ulna to 5th metacarpal.

The manus was pentadactylous. Digit I had its metacarpal 25 mm. long, somewhat conical and articulating by its proximal epiphysis with carpale₁, whilst distally it was continuous with an unossified phalanx. The other metacarpals were flattened bones, which as well as the phalanges had proximal and distal epiphyses. The second metacarpal articulated with C_2 and very slightly with C_1 ; its digit was the longest and had eight phalanges. The third articulated with carpalia₂ and ₃, and had six phalanges. The fourth, whilst 31 mm. in transverse diameter, was only 20 mm. long; it articulated with carpale₃ and with the ulnare; its two phalanges were short and flattened. The fifth metacarpal consisted of a bony nodule 24 mm. wide and 13 mm. long, which was imbedded in cartilage continuous with the pisiform, and there was no bony phalanx.

A feature in this manus was the rudimentary condition of the fifth digit, the absence of carpale₅, and no separate C_4 , though this element may have blended with the ulnare, or rather with carpale₃ (fig. 9), which articulated with both M_{III} and M_{IV} . From the close relation of the fifth metacarpal with the ulnare, C_5 had either never been developed or had disappeared very early, either by atrophy, or by fusion with the ulnare or with M_V . So rudimentary was the fifth digit that in the manus of the younger of the two dolphins it was scarcely recognisable, and the manus almost seemed as if it were tetradactylous.

The formula in the well-ossified manus of Risso's dolphin was as follows:—

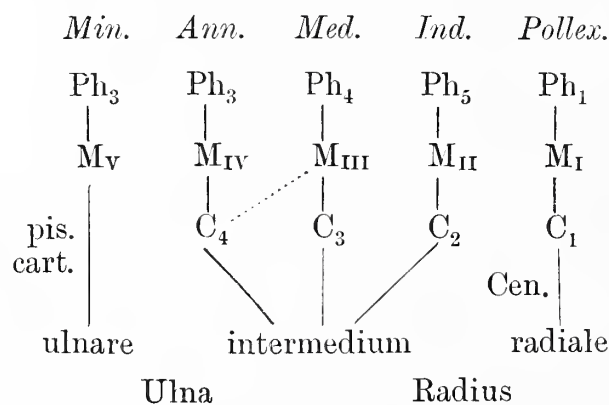


DELPHINAPTERUS LEUCAS (*Beluga or White Whale*).

In his memoir "On the Hand in the Cetacea," W. Kükenthal described and figured the characters in an adult Beluga and in a number of embryos at various stages of development. The *procarpus* was typical. The *mesocarpus* was variable. In the youngest embryo five cartilaginous distocarpalia C_1 to C_5 were recognisable. Carpale₅ however disappeared early,

so that at a later stage only four cartilaginous distal carpalia were present. A fusion of carpale₃ with C₄ might also occur and the number be reduced to three carpalia, C₁, C₂, C₃₊₄. He thought that carpale₅ probably became a part of the ulnare. He saw sometimes two centralia, one of which might remain, though usually both fused with adjoining bones.

In an important memoir on the Anatomy of a Beluga the late Sir John Struthers gave a detailed account with figures* of the structure of the manus. The procarpus consisted of the three customary bones, whilst the other bones were regarded as four distal carpalia C₁ to C₄, a small os centrale, and a pisiform, the ulnare, centrale, C₃ and pisiform not being ossified. As Sir John Struthers had presented to me for the Anatomical Museum this and other specimens from his collection of cetacea, I have examined the manus, and I concur in his interpretation of the carpal elements. Thus carpale₁ belonged to the pollex: C₂ was mainly for M_{II}, though with a slight articulation with M_{III}: C₃ was for M_{III}, though C₄ was divided equally between M_{III} and M_{IV} and its distal angle intervened between these two metacarpals: no independent C₅ existed, for M_V articulated directly with the ulnare, which was relatively large and probably included, though still in the cartilaginous stage, C₅, and its dorsal angle also reached the base of M_{IV}. The area of cartilage interposed between radiale, intermedium, C₁ and C₂, which Struthers regarded as an os centrale, holds the position of that carpal element and may be regarded as its unossified representative. The formula of this carpus corresponds with No. 4 of the formulæ of Beluga given by Kükenthal on p. 28 of his memoir, except that there is only one os centrale.



MONODON MONOCEROS (*Narwhal*).

In 1887 and 1889 Professor Leboucq of Ghent published an account of his researches on the morphology of the hand in marine mammals,† and described the constitution of the carpus in four foetuses of the Narwhal.

* *Journ. Anat. and Phys.*, vol. xxx., Oct. 1895.

† *Anatomischer Anzeiger*, 16th March 1887; *Archives de Biologie*, t. ix. p. 571, 1889.

In all the specimens the three cartilaginous proximal elements, radiale, intermedium, and ulnare, were present. In one the intermedium and ulnare were in process of fusion with each other, but the pisiform cartilage was recognised in only the two smallest. In three specimens (B, C, D) five disto-carpalia were seen, of which C_1 was blended with the base of M_I , but the fusion was not sufficient to obliterate the primitive separation; C_2 articulated with M_{II} ; C_3 , usually the largest, articulated with M_{III} ; C_4 with M_{IV} ; C_5 for M_V was distinct in specimen (C), but in B and D it was in process of fusion with $carpale_4$. In the smallest foetus (A) C_1 was distinct from M_I , but C_5 was not differentiated, and possibly had not been developed as an independent unit. A and B had an os centrale in the interval between the radiale, intermedium, and carpalia $_2$ and $_3$, which in B had the form of a rounded nodule continuous with the distal border of the intermedium. In the largest foetus (D) the centrale was most distinct and in process of fusion with C_2 , though the original line of separation was apparent. In a paper on the carpus of Beluga, published in 1893, Kükenthal referred to an embryo of *Monodon monoceros*, 24 cm. long, in which he saw two centralia.*

I may also state that Leboucq in his memoir described three foetuses of Beluga, in each of which he found a well-formed centrale; in one it was free between the radiale, intermedium, and C_2 ; in another it was in process of fusion with C_2 ; in the third it was partially fused with the radiale.

In the collection of cetacean limbs presented to me by Sir John Struthers were three specimens of the Narwhal, in one of which, an adult, the radio-ulnar epiphyses were fused with their respective bones; but in the other two the ossification was not so complete. The adult was dissected by Professor Struthers, who, so far as I know, did not publish a description of it.† Radiograms of the three hands were taken for me by Mr Ernest Henderson. That of the adult has greatly assisted me to determine the constitution of the carpus: in it the bony radiale was 26 mm. in transverse diameter; the intermedium, deeply grooved on both surfaces, was 37 mm.; the ulnare was 26 mm. wide. In the distal row a piece of cartilage separated the radiale from M_I ; in the radiogram it was seen to have in its substance a small nodule of bone, and it probably represented C_1 , which was

* *Morphologisches Jahrbuch*, vol. xix. p. 63, 1893.

† At the Aberdeen meeting of the British Association, September 1885, Professor Struthers exhibited the carpal bones and cartilages of several cetacea, including the Narwhal, but his description was not published. See Report of meeting, p. 1056. In the *Journal of Anat. and Phys.*, vol. vi. p. 115, 1872, he referred in a footnote to the fibrous arrangements replacing muscles in the hand of the Narwhal, but did not speak of the bones.

only partially ossified; so that this element of the carpus was not fused with M_I , but resembled the foetus A in Leboucq's collection, in which C_1 formed an independent unit. To its ulnar side was a well-defined bony carpale, 25 mm. wide, which articulated with the cartilage of C_1 , with the intermedium, os centrale, C_4 , M_{II} and M_{III} ; it should be regarded as C_2 conjoined with C_3 . C_4 , 21 mm. wide, articulated with intermedium, ulnare, C_{2+3} , M_{III} , and M_{IV} , also with a piece of cartilage on its disto-ulnar border

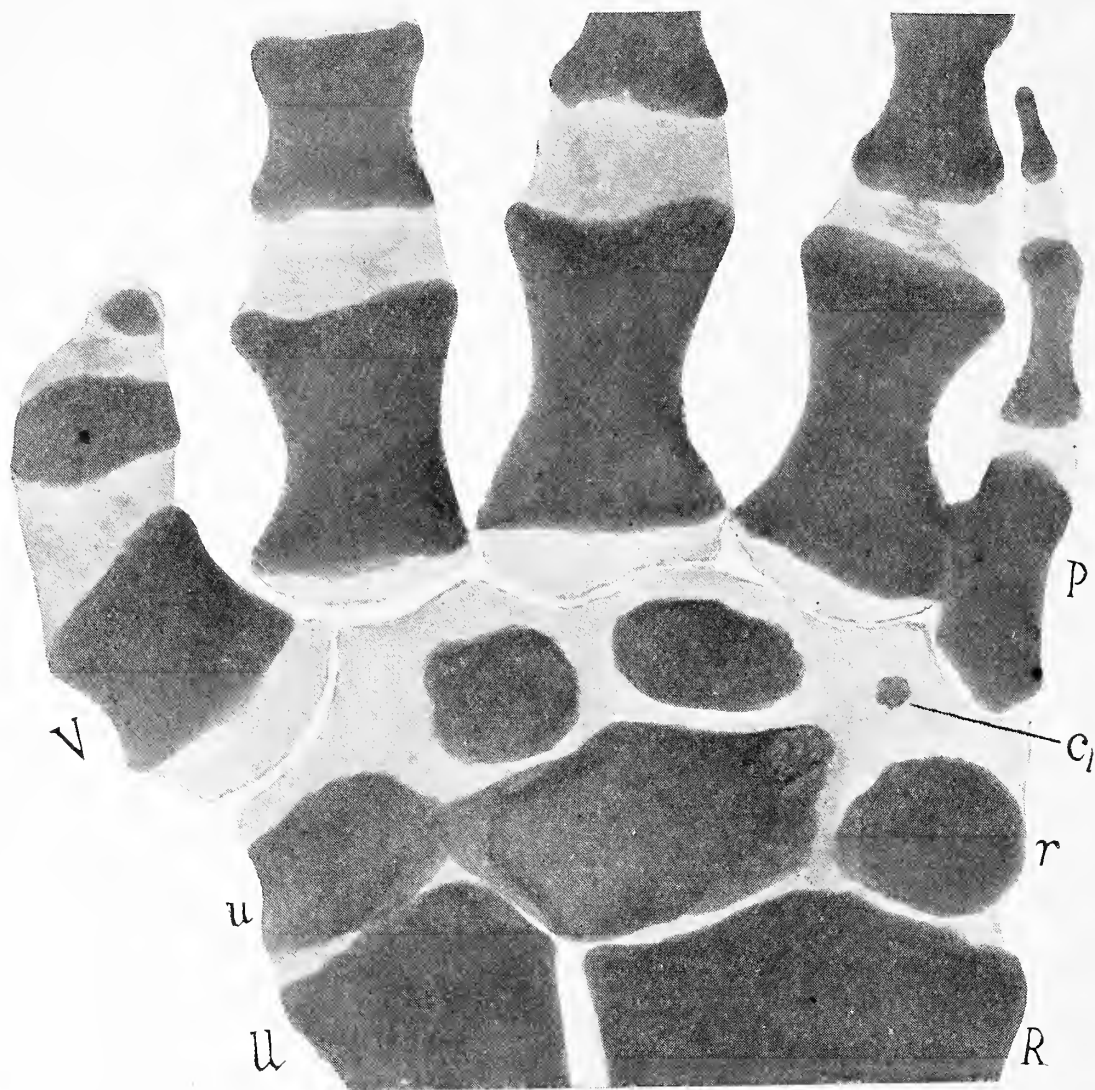


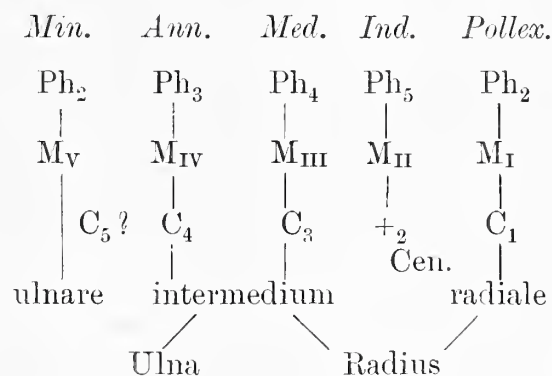
FIG. 10.—Radiogram of manus of adult *Monodon monoceros*.

which was situated between the ulnare, M_{IV} and M_V , and may possibly represent an unossified C_5 . Projecting from the disto-radial border of the intermedium, and continuous with it by a narrow neck, was a nodule of bone 7 mm. in transverse diameter, which articulated with the radiale, C_{2+3} , and the cartilage with its nodule described as C_1 ; it was obviously an os centrale fused with the intermedium; in its relations and intimate association with that bone it corresponded with the arrangement in the foetus B in the Leboucq collection. M_I articulated with the probable C_1 and with the radiale; M_{II} with C_1 and C_{2+3} ; M_{III} with C_{2+3} and C_4 ;

M_{IV} with C_{IV} and the cartilage of the possible C_V ; M_V slightly with that cartilage, but mostly with the ulnare.

The two less advanced specimens had in the distal row the bony carpalia which I have designated C_{2+3} and C_4 ; no ossific nodule was seen in the radiograms in the cartilage between the radiale and M_I , or in that between the ulnare and M_V . In both a process of bone projected from the disto-radial border of the intermedium, like that which I have interpreted in the adult as an os centrale.

The formula of *Monodon monoceros* is as follows:—



MORPHOLOGICAL SUMMARY.

It will have been seen from the foregoing description that in the Ziphioid whales, *Hyperoodon* and *Mesoplodon*, and in the Delphinidæ, whilst the three bones of the proximal row of the carpus, and the pisiform cartilage either unossified or only partially so, are constant in number and correspond generally in their arrangement, yet that the distal carpalia vary in number, and in their articulations with the procarpus and with the metacarpal bones; also that the ossa centralia are inconstant. *Hyperoodon* should, as I stated in 1885, be regarded as the type in which each of the five digits has its corresponding distal carpal, for articulation with the metacarpal of the digit to which it belongs. In other genera and species, however, a smaller number of distal carpalia are present, and the question naturally arises which members of the type-number have remained and which have disappeared as independent units. The range in number varies from five in *Hyperoodon* to four, or three, or even two in other species. In my memoir on the Shetland *Mesoplodon*, 1885, I discussed the question of the diminution in the type-number from five distal carpalia, and stated that it "may be due either to one or more centres of ossification not having formed in the carpal cartilage, or to the fusion with each other of ossific nuclei which were distinct in the younger condition of the same carpus." The terminology of the carpal bones introduced

by Gegenbaur, and their substitution for the older descriptive terms (adopted from human anatomy in the writings of anatomists as eminent as Owen, van Beneden and Flower), have greatly facilitated the description of this region; the graphic formulæ employed by Leboucq, Max Weber, and Kükenthal, which I have also used in this memoir, enable the eye to follow the description and assist one in recognising the morphology of the carpal bones in the manus of the Cetacea.

The guide to the solution of this problem is to be found in a manus in which the number of disto-carpalia corresponds with that of the digits; whilst in specimens in which some carpalia are wanting, it is important to study the articulations of those which are present with the metacarpal bones.

As previously stated, Hyperoodon provides us with the necessary key, for in this carpus each metacarpal, in the majority of specimens examined, has a definite disto-carpal for articulation with it. The researches of Kükenthal and Leboucq into the development of embryos of several species of Cetacea have shown that, in the early stages of Beluga and Monodon, a fifth differentiated cartilage existed in the distal row, which was not represented by a bony C_5 in the fully formed carpus, so that in the progress towards ossification this carpal element had disappeared.* A similar defect has also been observed in the carpus of a few specimens of Hyperoodon; the missing bone was carpale_5 , and the formula of the distal row was reduced in them to $\text{carpalia}_{1, 2, 3, 4}$.

Are we to assume that either five carpalia constitute in the Cetacea generally the normal number of elements in the distal row in the very early stages of development; or that a carpus may be formed at its initial development in which the elements of the disto-carpalia are fewer than five? In the latter instance the diminution in number would be a fundamental developmental defect, and could only be satisfactorily determined by the study in a given species of a sufficient number of specimens at the commencement of and in the early stages of cartilaginous differentiation in the carpus. In the former case the deficiency would be due to the disappearance of the cartilaginous precursors of the bones, either by atrophy in the early stages of development, or by fusion between adjoining cartilages or bones at somewhat later stages. The fusion might take place: *a*, between cartilages or bones in the same row; *b*, between cartilages or bones in the distal with those in the proximal row; *c*, between the disto-carpalia and the metacarpals.

* Gervais, in his part of the great *Ostéographie des Cétacés* by Van Beneden and himself, figured the manus of a foetal Hyperoodon in which five cartilages were present in the distal row, but he regarded the fifth of these as a pisiform.

Examples of fusion in the same row are not uncommon in the cetacean carpus, and their articulation with the metacarpals should be carefully noted in determining which of the bones are fused together. Of the species described in this communication there can be no doubt that the rule in *Mesoplodon bidens* is for C_2 to fuse with C_3 , and to articulate with M_{II} and M_{III} ; usually also C_4 fuses with C_5 and articulates with M_{IV} and M_V . The fusion of C_2 and C_3 constitutes the *ziphius* type of carpus of Kükenthal. In the Dalgety Bay *Mesoplodon* the small size of the most ulnar disto-carpal led me to think that it represented only C_4 for articulation with M_{IV} ; whilst M_V was displaced from the proper distal border of C_4 , and articulated with the ulnare; C_5 was, on this view, possibly absent.

In *Lagenorhynchus albirostris* C_2 and C_3 were fused, and the conjoined bone articulated with M_{II} and M_{III} ; C_4 was small, articulated with M_{IV} and not with M_V ; C_1 and C_5 were absent as separate bones or were perhaps combined with others.

In *Grampus griseus* C_3 and C_4 were fused and articulated with M_{III} and M_{IV} , whilst C_2 remained as a separate bone for M_{II} , and C_1 articulated with the pollex.

In *Delphinus acutus* and *Monodon monoceros* C_2 and C_3 were fused and articulated with M_{II} and M_{III} ; C_4 was present, and articulated principally with M_{IV} ; C_1 was also present, but C_5 did not exist as a separate bone. *Phocæna communis* had only two disto-carpals, being the minimum number found in the Odontoceti; of these one apparently represented C_2 fused with C_3 , whilst the other was C_4 ; C_1 and C_5 were not visible as independent units.

As a distinct example of fusion of a distal with a proximal carpal I may refer to fig. 8, the left carpus of *Globicephalus melas*, in which C_3 had fused with the ulnare. The question of fusion between bones of the distal and proximal rows, or of the distal with the metacarpals, arises also in connection with the condition of C_1 and C_5 of the distal carpalia, which bones may in several species be absent as independent units; the possibility of fusion either with a bone of the proximal row or with a metacarpal has to be considered. C_5 is the element which most frequently has no separate representative, and which cannot definitely be regarded as fused with C_4 . In *Delphinus delphis* and *acutus*, *Grampus griseus*, *Lagenorhynchus albirostris*, *Globicephalus melas*, *Phocæna communis*, *Delphinapterus leucas* (Beluga), *Monodon monoceros*, M_V articulated directly with the ulnare without the interposition of a separate disto-carpale. Kükenthal regarded this as the *Beluga* type of carpus common in the Odontoceti. It might arise from fusion of the cartilaginous C_5 with the ulnare, or from

fusion of C_5 with M_V , or from an early atrophy and disappearance of the cartilaginous C_5 .

The question whether C_1 is present or absent in some species of cetacea has been a matter of discussion. There can be no doubt of its presence in Mesoplodon, Hyperoodon, Beluga and Grampus. In Lagenorhynchus and Phocœna it was absent, and M_I articulated directly with the radiale, with which C_1 might have fused; though, as Leboucq has shown in *Monodon monoceros*, the fusion of C_1 with M_I does at times undoubtedly take place. In Delphinus, again, the question has arisen whether the bone immediately distal to the radiale is to be regarded as M_I or C_1 . If the former, then the pollex would possess two phalanges, and its metacarpal would articulate directly with the radiale. If the latter, C_1 would be interposed between the radiale and M_I , and the carpus would possess three disto-carpals. I have adopted the latter interpretation and applied it also to the carpus of Globicephalus.

As regards the os centrale, Mesoplodon and Globicephalus furnished examples of the presence of an os centrale as an independent bone; in the right manus of the St Andrews Mesoplodon a second centrale had been developed, and in that from Dalgety Bay an os centrale had fused with the radiale. In Monodon the centrale was fused with the intermedium. In the Struthers collection of cetacea, already referred to, is the dissected hand of an adult cetacean which, though not marked, is I believe that of Hyperoodon.* The carpus consisted of ten bones and cartilages, three in the proximal row; five separate disto-carpalia, of which C_1 and C_4 were well ossified, C_2 and C_3 partially so, whilst C_5 and the pisiform were unossified. An almond-shaped unossified os centrale, 21 mm. in transverse and 14 mm. in vertical diameter, was intercalated on the palmar surface between the radiale, intermedium, carpalia $_1$ and $_2$. In the presence of an os centrale this specimen corresponded with specimens of Hyperoodon, two well grown and a fœtus, as described by Kükenthal. In Beluga an unossified piece of cartilage probably represented the centrale. Kükenthal came to the conclusion from his observations on embryos that centralia are not unfrequently present in the early stages, but that they commence to disappear when other elements in the carpus lose their

* This limb probably belonged to the Hyperoodon which Professor Struthers obtained in 1871, and the finger muscles of which he described in the *Journal of Anat. and Phys.*, vol. vi. p. 115. At the Aberdeen meeting of the British Association, 1885, he exhibited the carpus of Hyperoodon along with those of other cetacea, but no description was published. *Report*, p. 1056. The radio-ulnar epiphyses were fused with their shafts. The radiale, intermedium, and ulnare were respectively 43, 58 and 45 mm. wide; the disto-carpalia 1 to 5 were 39, 33, 35, 27, 30 mm. respectively.

independence. The disappearance may perhaps in some cases be due to atrophy of the cartilaginous centrale, or, as in *Mesoplodon* and *Monodon*, to its fusion with an adjoining carpal bone.

The pisiform element of the carpus seems to be constant in the cartilaginous stage, but in my specimens it was undergoing ossification only in *Hyperoodon* and the St Andrews *Mesoplodon*.

From the observations of Leboucq and Kükenthal on the development of the phalanges, their number in the adult is less than in that of the embryo of the same species, apparently by fusion with each other of pieces originally distinct, a condition which applies also to diminution in the number of pieces originally present in the carpus.

In this memoir I have purposely restricted myself to the consideration of the carpus in those *Odontoceti* that I have personally examined. The principles which have guided me in ascertaining the morphology of the bones can be applied to their determination in other species of whales; but as this paper has reached a length more than I had originally intended, I must leave for another occasion their further application.

As supplementary to my description and figures of the carpus of *Mesoplodon* I append a radiogram of the manus of that animal, which shows

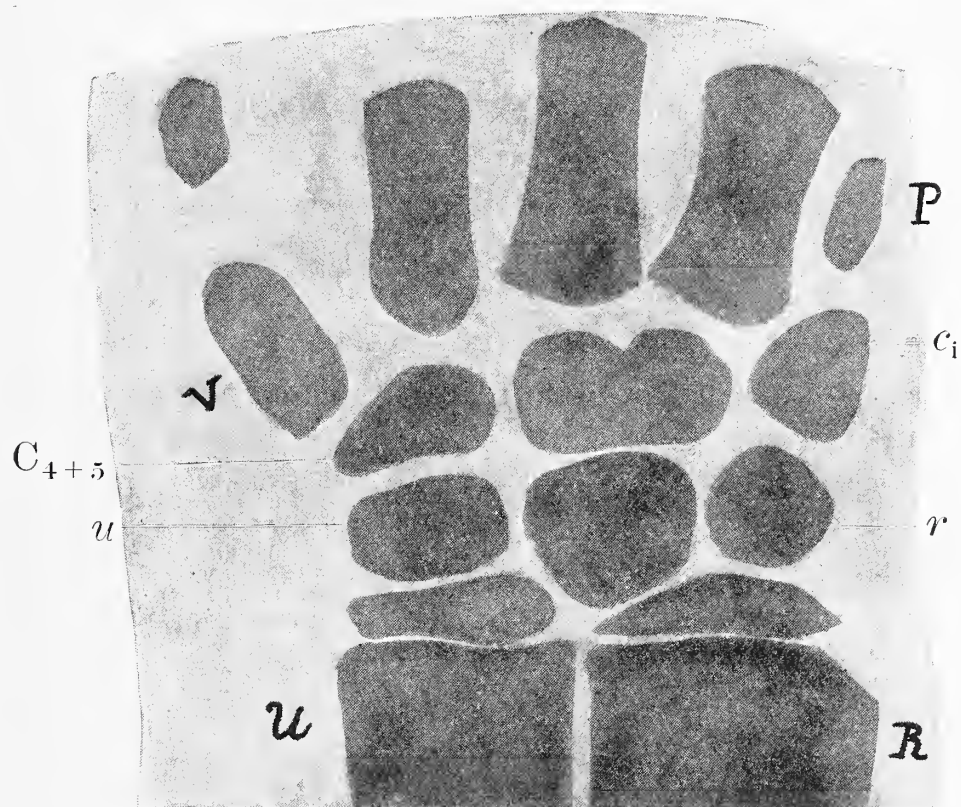


FIG. 11.—Radiogram of hand of *Mesoplodon bidens* from Morrison's Haven.

distinctly, in the undissected carpus, carpal₁ as a separate bone associated with the pollex; carpal₂ fused with C₃, as indicated by the notch opposite the interval between M_{II} and M_{III}; carpal₄ fused with C₅ for M_{IV} and M_V.

The radio-ulnar epiphyses, though ossified, are not fused with the shafts of their bones.

All the illustrations, with the exception of fig. 3, are from photographs of the specimens made in the Anatomical Department of the University by the Museum assistants, Mr Ernest Henderson and Mr William Gill, to whom I would express my indebtedness.

For convenience of reference the lettering of the figures is as follows:— R, radius; U, ulna; *r*, radiale; *i*, intermedium; *u*, ulnare; *c*, with arabic numerals disto-carpalia; the roman numerals are the metacarpals; C, the os centrale; P, pollex; *p*, the pisiform element.

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Communicated by Professor MACGREGOR.

(MS. received March 10, 1909. Read June 7, 1909.)

TWENTY-TWO years have passed since the atomic weight of platinum was studied by Dittmar and M'Arthur.* This is the last investigation recorded which has been concerned with the value of this constant.

During the intervening period, as the result of many investigations carried on by Professor T. W. Richards and his students,† the accuracy with which many of the manipulations incident to atomic weight investigations can be executed has been greatly increased. Perhaps of still greater importance, methods for the preparation of many compounds of undoubted purity have been devised, and the principles underlying the preparation of pure substances have been clearly set forth.‡

In the case of the atomic weight of platinum, the different investigations, which are comparatively few in number, have seldom given results which show even fair agreement; and even in these cases, as will be shown later, the agreement between the mean values obtained by different ratios leaves much to be desired. It seems unfortunate that many of the ratios used in calculating the value at present accepted were obtained by weighing a salt, which, after being precipitated from a water solution, was heated to a temperature of only 150° for the purpose of expelling the water.

The observations of Precht § have shed a great deal of light upon the difficulties involved in the preparation of pure halogen compounds of platinum.

In view of the above considerations, and particularly as the atomic weight of platinum is constantly being used in the estimation of potash, a

* *Trans. Roy. Soc. Edin.*, xxxiii. 561 (1887).

† Richards, *Proc. Am. Acad.*, xxii. 342 (1887), xxiii. 177 (1888), xxv. 195 (1890), xxvi. 240 (1891), xxviii. 1 (1893), xxix. 369 (1894); Richards and Rogers, *ibid.*, xxviii. 200 (1893); Richards and Parker, *ibid.*, xxxii. 55 (1896); Richards and Cushman, *ibid.*, xxxiii. 97 (1897); Richards and Baxter, *ibid.*, xxxiii. 115 (1897); Richards and Marigold, *ibid.*, xxxvii. 365 (1902); Richards and Archibald, *ibid.*, xxxviii. 443 (1903); Richards and Wells, *Jour. Am. Chem. Soc.*, xxvii. 475 (1905); Richards, Staehler, Forbes, Mueller, and Jones, *Carnegie Inst. of Washington, Publication* 69.

‡ Richards, *Proc. Amer. Phil. Soc.*, xlii. 28 (1903).

§ Precht, *Zeit. Anal. Chem.*, xviii. 509 (1879).

critical study of the constant in question seemed altogether desirable. The work recorded in the following pages has extended over a period of four years.

HISTORICAL.

Berzelius* was the first to work upon the atomic weight of platinum. As early as 1826 he determined the per cent. of platinum in platinous chloride, and from these analyses a value of 193·16 is obtained for the atomic weight. Two years later,† by decomposing potassium chloroplatinate in hydrogen, he obtained data from which the atomic weight of platinum can be calculated by several ratios. The values calculated from these ratios vary between 195·4 and 196·6, giving a mean of 195·90.

A report of some work by Andrews‡ appeared in 1852, but a detailed statement of his experiments is lacking. He worked with potassium chloroplatinate, decomposing it by means of zinc and water. After dissolving the excess of zinc and filtering, the chlorine in the filtrate was estimated and the platinum was dried and weighed. From three determinations we have the values 197·9, 197·7, and 198·1 for the atomic weight, or a mean value of 197·9.

Previous to the publication of Seubert's§ determinations in 1881 the atomic weight of platinum was supposed to lie above that of gold. This investigation, however, showed that platinum had an atomic weight several units below that of gold. He studied both potassium and ammonium chloroplatinate, preparing these salts with great care, and decomposing them in a stream of hydrogen. The resulting hydrochloric acid was in some cases absorbed in water or silver nitrate solution, and the chlorine estimated as silver chloride. From the weight of the original salt, the weight of platinum residue, of potassium chloride, and of chlorine lost upon ignition, several independent ratios are obtained, from which the value sought can be calculated.

Three different samples of ammonium chloroplatinate gave the following mean values, reducing the weight to the vacuum standard, and assuming that chlorine = 35·46, nitrogen = 14·01, hydrogen = 1·008, bromine = 79·92, silver = 107·88, and potassium = 39·11.

I.	From the ratio	$(\text{NH}_4)_2\text{Cl}_6$: Pt	.	.	.	195·12
II.	„	„	$(\text{NH}_4)_2\text{Cl}_6$: Pt	.	.	194·47
III.	„	„	$(\text{NH}_4)_2\text{Cl}_6$: Pt	.	.	195·47

* *Poggend. Annalen*, viii. 177 (1826).

† *Ibid.*, xiii. 468 (1828).

‡ *British Assoc. Report*, 1852.

§ *Ber. Deutsch. Chem. Gesell.*, xiv. 865 (1881).

Here is a difference of one unit in the mean values. Another series of analyses where the total chlorine was determined as silver chloride gave :

From the ratio	$(\text{NH}_4)_2\text{Cl}_6$:	Pt	194.62
"	"		6AgCl	:	Pt	.	.	195.80
"	"		6AgCl	:	$(\text{NH}_4)_2\text{PtCl}_6$.	.	197.20

It seems very likely that the high result in the case of the last ratio was due to the presence of water in the salt weighed.

For the potassium chloroplatinate we have the following mean values :

From the ratio	K_2Cl_6	:	Pt	194.81
"	"		2KCl	:	Pt	.	.	194.82
"	"		2KCl	:	K_2PtCl_6	.	.	195.00

From another series of analyses, where the chlorine lost upon ignition was determined as well as the platinum, we have :

From the ratio	K_2Cl_6	:	Pt	194.72
"	"		4AgCl	:	Pt	.	.	195.05
"	"		4AgCl	:	K_2PtCl_6	.	.	195.51

In 1884 Halberstadt* published the results of a very complete study of the chloroplatinates of ammonium and potassium. He also made some analyses of platinic bromide. The platinum was estimated both by weighing the metal left after reducing the salt in hydrogen and by weighing the metal electrolytically. The mean values calculated from his ratios are given below.

From the ratio	Br_4	:	Pt	194.51
"	"		$(\text{NH}_4)_2\text{Br}_6$:	Pt	.	.	194.82
"	"		K_2Br_6	:	Pt	.	.	195.05
"	"		2KBr	:	Pt	.	.	195.19
"	"		2KBr	:	PtBr_4	.	.	195.89
"	"		$(\text{NH}_4)_2\text{Cl}_6$:	Pt	.	.	195.05
"	"		K_2Cl_6	:	Pt	.	.	194.75
"	"		2KCl	:	Pt	.	.	194.90
"	"		2KCl	:	Pt	.	.	195.33

The extreme values among these results can be explained if we assume that the original salt in each case contained some water which had not been driven out before it was weighed. Thus in the ratio Br_4 : Pt any water in the PtBr_4 weighed makes the weight for the Br_4 too great; likewise in the ratio 2KBr : PtBr_4 the water in the original K_2PtBr_6 weighed gives too high a value to the PtBr_4 . It should be remembered that the above

* *Ber. Deutsch. Chem. Gesell.*, xvii. 2962 (1884).

are all average values from a number of determinations; individual determinations would show considerably greater variation.

In 1887 Dittmar and M'Arthur* published an account of a critical examination of potassium chloroplatinate. They concluded that the salt is seldom if ever prepared pure; that hydrolysis invariably takes place, some chlorine being replaced by hydroxyl, and also that some potassium is always replaced by hydrogen. In the light of these considerations, they apply corrections to a number of determinations which they made of the ratio $2\text{KCl}:\text{Pt}$. The results which they found give a mean value of 195.50.

These authors at the same time criticised the work of Seubert, and concluded that if the proper corrections were applied to his results a value above 195.00 would be obtained.

Dittmar and M'Arthur have made some timely suggestions regarding the preparation of pure potassium chloroplatinate. The strong tendency of the salt to hydrolyse in water solutions seems to have been generally disregarded by others, while the difficulty of getting rid of nitric acid by evaporating with excess of hydrochloric acid after dissolving the platinum in aqua regia was not fully appreciated.

Seubert † replied to the criticisms of Dittmar and M'Arthur, claiming that the close agreement of the values calculated from the different ratios given by his analyses showed that the impurities mentioned by these authors could not have been present in the material he worked with. In view of the values shown above, this claim should be more or less modified.

The different investigations mentioned above give us the following values for the atomic weight of platinum. From the work of

Berzelius	195.90
Andrews	197.88
Seubert	195.22
Halberstadt	195.05
Dittmar and M'Arthur	195.50

When selecting a method for the determination of the atomic weight of platinum, one naturally turns to the analysis of the double salts of potassium chloride, or ammonium chloride and platinic chloride. These salts, when pure, are among the most stable salts of platinum. From their analysis they allow of several ratios being formed, from which not only can the atomic weight of platinum be calculated, but one can at the same time gain a most complete knowledge of the purity of the salts. In addition to these points, the determination of chlorine as silver chloride is among the

* *Loc. cit.*

† *Ber. Deutsch. Chem. Gesell.*, xxi. 2179 (1888).

most accurate of our analytical operations. While it would be very interesting and most instructive, if it were feasible, to determine the ratio of platinum to oxygen or to silver directly, a few analyses, by means of a reliable method, are of more value than any number carried out according to methods which have been insufficiently studied to reveal their possibly large constant errors.

The present investigation covers a study of the potassium and ammonium salts of chloroplatinic acid, and the corresponding salts of bromoplatinic acid. The method of analysis necessitated the preparation of the pure salt, the determination of its weight in a perfectly dry condition, its reduction in a stream of hydrogen, with the absorption in water of the hydrochloric acid formed, and subsequently the weighing of the platinum residue, and the determination of the potassium chloride left behind, and the hydrochloric acid formed, by precipitating and weighing the chlorine of each separately as silver chloride.

PREPARATION OF PURE MATERIALS.

Potassium Chloride.—Chemically pure potassium chloride was twice precipitated from an almost saturated solution with gaseous hydrogen chloride, prepared by boiling a solution of pure hydrochloric acid. After each of these precipitations, the salt was washed and dried in a centrifugal apparatus, as recommended by Richards.* This treatment was usually resorted to for the purpose of removing the mother liquor from a mass of crystals. The value of the process of crystallisation as a means of purifying a substance is greatly enhanced by removing the mother liquor from the crystals as completely as possible. Consequently this point received a great deal of attention throughout this work. The potassium chloride from the second precipitation was once recrystallised from water. The above operations were carried out in platinum vessels. The product thus obtained was used in precipitating some platinum as potassium chloroplatinate. This platinum was very pure, having been through all but the last stage of its purification. The platinum salt was now reduced in pure hydrogen gas at a low temperature. The potassium chloride set free was dissolved in water, and twice precipitated in platinum vessels by means of gaseous hydrogen chloride. This product was employed in the preparation of the potassium chloroplatinate used in the analyses described below.

Ammonium Chloride.—The method of preparing pure ammonium chloride was analogous to that used in the case of the potassium salt. After

* Richards, *Jour. Am. Chem. Soc.*, xxvii. 104 (1905).

precipitating from the acid solution and recrystallising from water, the salt was added to pure chloroplatinic acid solution. The chloroplatinate formed was reduced in pure hydrogen gas, and the ammonium chloride resulting dissolved in water. After precipitating the salt twice with hydrogen chloride gas and thoroughly washing in each case, it was ready to be used in the preparation of the pure ammonium chloroplatinate.

It should be noted that the precipitation of these chlorides from a strongly acid solution should remove all foreign acid radicles, while other metallic radicles would be most effectually removed by the precipitation as the chloroplatinate.

Potassium Bromide.—In preparing pure potassium bromide the method adopted by Richards and Mueller* was used. A specimen of potassium oxalate, already very pure, was four times recrystallised in platinum, the crystals being well washed each time. This oxalate was then converted into bromide by treating it with pure bromide in a quartz dish. After recrystallising the bromide from an acid solution, it was used in the preparation of the pure potassium bromoplatinate.

Ammonium Bromide.—This salt was prepared by distilling some pure hydrobromic acid directly into redistilled ammonium hydroxide contained in a platinum vessel.

Bromine and Hydrobromic Acid.—Chemically pure bromine of commerce was redistilled three times, rejecting in each case the first and last portions of the distillate. This treatment would remove all but a trace of chlorine present. The bromine was now dissolved in pure calcium bromide and distilled from this solution. The solution and distillation from a fresh portion of the calcium bromide was carried out a second time. The hydrobromic acid was prepared from this pure bromine by means of thoroughly washed red phosphorus. The solution of hydrobromic acid formed was then redistilled twice, after adding an excess of pure bromine, rejecting each time the first and last parts of the distillate. Every care was taken to exclude chlorine or hydrochloric acid from the laboratory where the bromine and hydrobromic acid were being prepared.

Hydrochloric and Nitric Acids.—These acids were prepared from the chemically pure acids by redistilling from platinum and condensing the vapours in platinum. The first and last portions of the distillate were always rejected. The absence of chlorine from the nitric acid was further assured by testing a portion of each sample in the nephelometer† before using it.

* *Loc. cit.*, p. 29.

† Richards, *Proc. Am. Acad.*, xxx, 385 (1894); Richards and Wells, *Am. Chem. Jour.*, xxxi, 235 (1904).

Chloroplatinic Acid.—Four samples of chloroplatinic acid were prepared. The material for three of these was platinum scrap. The scrap was first boiled with hydrochloric and nitric acids separately to remove surface impurities. It was then dissolved in aqua regia. The nitric acid was then almost, if not entirely, removed by repeated evaporation with hydrochloric acid. The solution was now diluted to a concentration corresponding to a 5 per cent. solution of platinum. One-third of this solution was set aside for separate treatment. In order to remove the iridium and any iron present, the remaining two-thirds were treated with enough of a dilute solution of pure ammonium chloride to precipitate about 98 per cent. of the platinum. The precipitate was thoroughly washed and dried. It was then reduced in a current of pure hydrogen. The ammonium chloride set free was washed out of the platinum black, and the latter was boiled with successive portions of concentrated hydrochloric acid to dissolve out a possible trace of iron. The platinum was now ready to be again dissolved in aqua regia and subjected to the above treatment a second time.

It was found very difficult to remove the last traces of iron from the platinum. Treating it repeatedly with concentrated hydrochloric acid failed to do it, but after a number of precipitations as ammonium chloroplatinate, followed each time with the boiling in the presence of hydrochloric acid, no iron could be found in the platinum.

The elimination of the iridium could be plainly followed from the colour of the residue left upon evaporating the mother liquor from a precipitation. The characteristic dark red colour of the iridium compounds is in evidence when only a very small amount of iridium is present. All indications of iridium had vanished after the above operations had been repeated three times.

After the platinum ammonium salt had been precipitated and reduced in hydrogen for the fifth time, a portion of the platinum was set aside, which eventually was used to prepare sample I. of the potassium chloroplatinate. The remainder was redissolved and the above procedure of precipitation and reduction repeated four times more. This treatment gave the platinum which was used to prepare sample II. of the potassium chloroplatinate.

The portion of chloroplatinic acid set aside at the beginning of the purification process was treated for the removal of other platinum metals, as well as any iron, according to the method of Schneider and Seubert.* The precipitations were carried out in very dilute solutions in the presence of some alcohol, the more effectually to remove any gold that might be present. The reductions of the platinum salt were effected in the case

* Graham Otto's *Lehrbuch*, 5th ed., iv. 1153.

of this portion with ammonium formate, prepared by passing ammonia vapour, from a platinum still and condenser, directly into freshly distilled formic acid.

This portion of platinum was subjected to five precipitations and reductions. It was then used in the preparation of the third sample of potassium chloroplatinate.

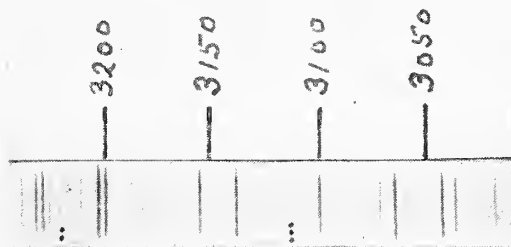
The fourth sample of chloroplatinic acid was prepared from 150 grams of osmo-iridium ore supplied by Baker and Company of Newark, N.J. This ore contained about 35 per cent. of platinum. It was first boiled with concentrated hydrochloric acid to dissolve a small per cent. of iron present. This treatment was repeated several times. The ore was then well washed, and afterward boiled with successive portions of nitric acid. To dissolve out the free platinum, the ore was boiled with successive portions of aqua regia until nothing more appeared to go into solution. These several platinum solutions were now combined and evaporated with a large excess of hydrochloric acid. After the nitric acid had been expelled in this way, the solution was diluted to correspond to a 5 per cent. solution of platinum, a considerable amount of alcohol was added, and the platinum then precipitated with ammonium chloride. The mother liquor from this precipitation showed, upon evaporation, that an appreciable amount of iridium had been dissolved along with the platinum. The precipitate of ammonium chloroplatinate, after being thoroughly washed, was reduced with ammonium formate. The finely divided platinum was now boiled with nitric and hydrochloric acid separately to remove any metal soluble in these acids. It was found practically impossible to remove all the iron in this way, as, after being treated with eight different portions of 50 c.c. each of hydrochloric acid, the boiling being continued for two hours in each case, iron was still found in the last portion. The platinum was now dissolved in aqua regia and subjected to the treatment recommended by Schneider and Seubert,* precipitated, and again reduced. The treatment with nitric and hydrochloric acid was now repeated. This time only a trace of iron was found. The process of precipitation and reduction was repeated until the platinum had been precipitated four times. The salt was then reduced and dissolved as usual in aqua regia, and this solution was electrolysed, at as a low a voltage as possible, until about 90 per cent. of the platinum had been deposited. The platinum anode used here was thickly coated with electrolytic platinum before being used. As, according to Classen,† the iridium requires a much higher voltage for its

* *Loc. cit.*

† *Ber. Deutsch. Chem. Gesell.*, xvii. 2467 (1884).

deposition than the platinum, this treatment should free the platinum most effectually from iridium. That the pure metal might not again be contaminated with iridium or iron from the electrode only about 75 per cent. of the deposit was dissolved off. This was again precipitated as the ammonium double salt, and after being reduced, was ready to be used in preparing sample IV. of potassium chloroplatinate.

Professor F. A. Saunders has very kindly photographed the arc spectrum from the orange to the end of the ultra-violet of a portion of sample No. I. of the platinum, using a concave grating, and finds no indication of the presence of iridium, osmium, palladium, Rhodium, Ruthenium, or iron in the platinum. The strongest lines of iridium are entirely absent, as indicated in a portion of the spectrum, enlarged and reproduced below. If the iridium has been so effectually removed from this sample of platinum, we are surely warranted in assuming the other samples which were subjected to even more severe treatment for purposes of purification to be equally pure. I wish here to record my appreciation of the kindness of Professor Saunders in making this test.



Small part of platinum arc spectrum (in ultra-violet) photographed with concave grating. Enlarged $\times 2$. Place marked : strong Ir line might occur ; ; Pd line ditto, but does not.

It has been shown by Precht,* that it is almost impossible to remove the last trace of nitric acid from a solution of chloroplatinic acid prepared by dissolving platinum in aqua regia. This point has also been emphasised by Noyes and Weber,† and, according to the latter's statement, they never succeeded in preparing perfectly pure potassium chloroplatinate from acid made by this method. It was thought safer, in the present instance, to avoid the use of nitric acid altogether in the preparation of the chloroplatinic acid, from which the potassium chloroplatinate for the analyses was to be prepared. In order to dissolve the platinum, recourse was had to the electrolytic method described by Weber.‡ According to this method, the platinum to be dissolved rests upon the platinum anode B in the tube A, fig. 1; the anode being in

* *Loc. cit.*

† *Jour. Am. Chem. Soc.*, xxx. 13 (1908).

‡ *Jour. Am. Chem. Soc.*, xxx. 29 (1908).

turn supported by glass beads which fill the lower part of the tube. The cathode is of platinum, and is placed inside the porous cell D. The tube is now filled to the point *e* with concentrated hydrochloric acid. Upon passing

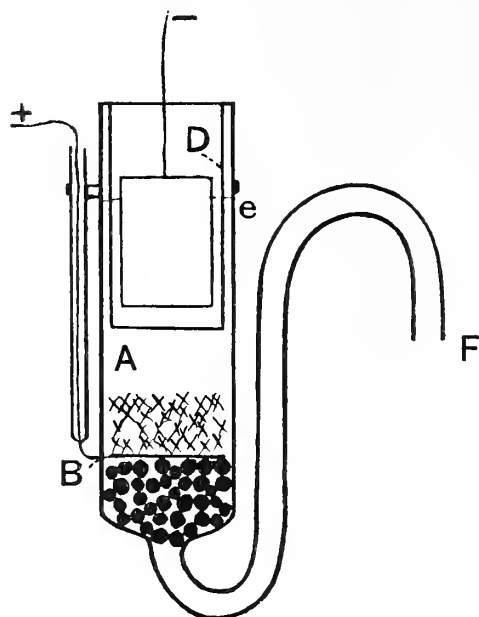


FIG. 1.

a current of, say, 8 amperes, through the cell from the regular 110 volt circuit, the platinum goes rapidly into solution, and the chloroplatinic acid formed can be siphoned off at F.

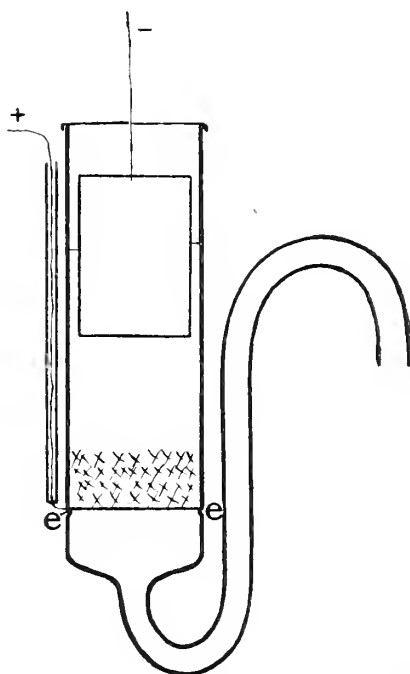


FIG. 2.

This apparatus was somewhat modified for the present purpose. In the first place, as it was desirable to have as small a surface of glass as possible in contact with the solution, the glass beads as a support for the anode were not used. Instead of these, several depressions were made in the tube at the points *e e*, as shown in fig. 2, and the anode supported by these. Secondly, it was found impossible to obtain a porous cup from which some impurities

would not be dissolved by the concentrated hydrochloric acid. The cleanest appearing ones to which we had access, and which appeared perfectly white, after being boiled with hydrochloric acid for weeks, still gave off traces of iron to a fresh portion of concentrated acid. The porous cup diaphragm was therefore discarded.

In the next place, as it seemed possible that traces of the platinum anode might be dissolved, along with the finely divided platinum in contact with it, and as this might result in the contamination of the solution, a thick coating of platinum was deposited electrolytically upon the anode, from a solution of pure platinum.

This apparatus was steamed out for several hours before it was used. In order to still further diminish the action of the solution upon the glass, the tube, while being used, was immersed in a bath of cold water. The current used was a weak one, never more than 4 amperes.

Potassium Chloroplatinate.—After each portion of platinum had been separately dissolved in this apparatus, by means of the electric current, chlorine generated from pure hydrochloric acid and potassium permanganate was passed through each solution. If a considerable excess of hydrochloric acid was not already present, more was added in order to neutralise any tendency to hydrolyse on the part of the platinic chloride. The several solutions were now diluted to a concentration corresponding to about $1\frac{1}{2}$ per cent. of platinum, and each was precipitated separately by adding it slowly, with constant agitation, to a dilute solution of pure potassium chloride. The precipitates were thoroughly washed with water and alcohol, and after drying as completely as possible, were kept in porcelain over calcium chloride. In this manner samples of potassium chloroplatinate Nos. I., II., III., and IV. were prepared from the above samples of platinum.

Ammonium Chloroplatinate.—For the preparation of the ammonium double salt, the platinum resulting from the analysis of samples I. and III. of the potassium salt was combined and dissolved in the manner described above; the platinum from sample IV. was likewise dissolved separately, and the platinum in each precipitated in exactly the same way as in the case of the potassium salt. These precipitates gave the samples of ammonium chloroplatinate I. and II.

Potassium and Ammonium Bromoplatinates.—The platinum which remained from the analysis of the chloroplatinates was used in preparing the corresponding salts of bromoplatinic acid. It was dissolved in pure hydrobromic acid, by aid of the electric current, in the apparatus described above. The potassium and ammonium bromide solutions used in pre-

precipitating the platinum were both very dilute, and the platinum solution was added as before to the bromide solution slowly and with constant stirring.

The several solutions from which the above platinates were precipitated were so dilute, and precipitation took place so slowly, that it was thought to be impossible for an appreciable amount of potassium chloride or bromide, or the corresponding ammonium salt, to be occluded by the precipitate. That considerable amounts of these are carried down, as contended by Dittmar and M'Arthur,* when the platinum salts are formed in solutions at all concentrated, has been found for a number of cases. When carrying out some preliminary work on these salts, it was found that as much as 0.5 per cent. in excess of the potassium salt may be present in a precipitate formed in a 25 per cent. solution of potassium bromide.

Silver.—The mode of preparing pure silver followed closely the admirable methods developed by Professor Richards, which have proved so satisfactory. Briefly stated, C.P. silver nitrate was dissolved in pure water and precipitated, from a very dilute solution, with pure hydrochloric acid. This precipitate was well washed and reduced, in a strongly alkaline solution, with invert sugar. The sugar used here had been dissolved, filtered, and recrystallised in the laboratory. The caustic soda used had been electrolysed, in order to remove a possible trace of iron. The reduced silver, after a thorough washing, was dissolved in pure nitric acid, and this solution was divided into three portions; these portions giving eventually three samples of silver. The first portion was again precipitated with hydrochloric acid. Reduction by means of invert sugar followed, and the reduced metal, after washing, was fused on pure lime before the blowpipe. This lime was prepared from calcium nitrate, which had been recrystallised several times, and then precipitated as carbonate with pure ammonium carbonate. The lumps of silver obtained from the fusion were now electrolysed, with a very weak current, in a solution of nitrate prepared from one of these pellets of silver and pure nitric acid. The crystals of silver deposited were well washed, and were then fused on a boat of pure lime, in an atmosphere of hydrogen, at a pressure of 40 mm.

The second portion of silver nitrate solution was recrystallised twice from platinum. It was then reduced with ammonium formate prepared by passing ammonia vapor, from platinum still and condenser, directly into freshly distilled formic acid. The reduced metal, after being washed, was fused on lime and electrolysed as in the case of the first portion; this was

* *Loc. cit.*

followed also by fusion in hydrogen on a lime boat under a pressure of 40 mm.

The treatment of the third portion of silver nitrate solution was the same in principle as in the case of the second sample, but here the silver solutions were not allowed at any time to come in contact with glass vessels. The reductions were carried out in a silver dish, while the electrolysis was effected in a quartz basin. The final fusion took place under the same conditions as in the above cases.

The pellets of silver resulting from the fusions in hydrogen were now scrubbed with sand and cut into pieces of the proper size with a clean piece of hard steel. They were then warmed with diluted nitric acid until well etched. After washing and drying, they were kept in a desiccator until used. The above preparations gave samples of silver Nos. I., II., and III.

The arc spectrum, from the orange to the ultra-violet, of a portion of sample II. of the silver, has also been photographed by Professor Saunders, under very favourable conditions, using the concave grating. The results indicate the entire absence of copper and iron from this sample of silver.

Considerable attention was paid to the water used in this research. It was twice distilled, the second time in the presence of a trace only of sodium hydroxide and potassium permanganate. It was condensed in block tin, received and stored in Jena glass stoppered bottles. Every care was taken to exclude dust, and it was always used soon after distilling. To make certain of the absence of chlorine ions, every sample was tested in the nephelometer before using.

The balance used was made by Christian Becker of New York. It was procured especially for this work, and was not used for any other purpose. It is a gold-plated short-arm balance sensitive to 0.02 mg. with a load of 40 gms. This was looked upon as being quite accurate enough for the purpose in hand. It cannot be too strongly emphasised that even in atomic weight work the errors of a chemical nature are likely to far exceed the errors made in weighing. There is little to be gained in weighing to a few thousandths of a mg., when there may be several hundredths of a mg. of unknown impurity in the portion of substance weighed.

The weights used were new ones of the Sartorius make, gold-plated. They were standardised according to the method suggested by Richards,* and have not been used in any other work.

* *Jour. Am. Chem. Soc.*, xxii. 144 (1900).

THE ANALYSIS OF POTASSIUM CHLOROPLATINATE.

From a complete analysis of potassium chloroplatinate, one should be able to gain considerable information concerning the purity of the compound. If the salt at the beginning can be weighed free from water, and if, after being reduced in hydrogen, the platinum left behind can be weighed and the potassium chloride and the chlorine set free estimated separately, we can from the relations between these constituents tell which, if any, has been present in excess, or has been replaced by some impurity. Only with a pure salt, having the exact composition represented by the formula, will the several ratios found between these constituents give the same value for the atomic weight. Unfortunately, we are here confronted at once with the knowledge that as the salt has been precipitated from a water solution, it must inevitably have enclosed within the crystals some of the mother liquor, and complete expulsion of this water will only be brought about, as pointed out by Richards,* by the entire disintegration of the crystal. It remains to be considered whether or not the salt in question can be freed, before weighing, from all but an inappreciable quantity of moisture.

In the first place, we are dealing here with a salt which can be obtained, as a precipitate, in an exceedingly fine state of division. As it is very soft, it can be ground to a still finer powder without danger of contaminating it with pieces of the pestle or mortar. After such treatment, it would seem as if all but the most minute of the crystal cells must be broken. Further, as pointed out below, we are able to heat the salt in this fine state of division to a temperature but little short of 400° C. and bottle it in a current of pure dry air. It does not seem as if, under these conditions, a very large amount of moisture can be left in the salt. Nevertheless, although the results of the weighings of the salt are given, and are useful as a check upon its purity, the atomic weights calculated from ratios in which the weight of original salt appears are not used in determining the final value.

The analytical procedure is now about as follows:—Two weighing-bottles, not differing in weight by more than two or three tenths of a gram, and two porcelain boats fulfilling the same condition, are selected: the one bottle to hold the boat containing the salt to be weighed, the other bottle and boat to act as a tare; thus doing away with the necessity of making buoyancy corrections, except for the salt weighed. We can also assume, as the bottles are made of the same kind of glass, that the moisture on the surface of one, when being weighed, is practically equal to the moisture on

* *Proc. Am. Phil. Soc.*, xlii. 28 (1903).

the surface of the other. After carefully drying and desiccating for a long time, the difference in weight between these two sets of apparatus is accurately determined. One boat is now filled with the salt to be analysed, and placed in the combustion tube of the Richards* bottling apparatus. The bottle and stopper belonging to this boat are also placed in position. A current of pure dry air (dried by passing over 18 inches of phosphorous pentoxide) is passed through the apparatus, and the combustion tube, which passes through an asbestos oven, is carefully heated. The apparatus through which the air passes, while being purified and dried, is entirely of glass, the pure air not coming in contact with any rubber whatever.

The temperature of the asbestos oven is now gradually raised to 380°-400°. By passing the escaping air through a solution of silver nitrate, and then examining this solution in the nephelometer, it was shown that a little hydrochloric acid was given off between 100° and 150°. All the hydrochloric acid seemed to be expelled between these temperatures. The observation of Noyes and Weber,† in regard to the stability of pure potassium chloroplatinate at high temperatures was confirmed, as it was found that the salt could be heated to 400° in dry air without any decomposition taking place. The salt was kept at a temperature between 380° and 400° for about two hours. The boat and contents were then allowed to cool, and with the stream of air still passing through the apparatus, they were bottled and transferred to the desiccator. After desiccating for two hours the salt is weighed in its bottle and is now ready to be reduced.

In order to make sure that nothing more could be expelled from the salt by further heating, it was upon several occasions reheated, bottled, desiccated and weighed as before; but no change in weight was found to have taken place. The results from one such experiment will suffice.

Weight of salt after first heating	2.20469
„ after heating again for three hours	2.20468

The silver nitrate solution, through which the escaping air was bubbling, showed no sign of the presence of any chlorine ions. It is evident that nothing appreciable is volatilised from the salt, while it is being heated at this high temperature.

The weighed salt is now placed in the combustion tube of the reducing apparatus and carefully heated in a stream of pure hydrogen. The hydrogen for this purpose was prepared by electrolysing a solution of

* *Proc. Am. Acad.*, xxxii. 55 (1896).

† *Loc. cit.*

barium hydroxide. The gas first passed over heated platinum sponge, where any oxygen would be burned. It was then passed through an 18-inch tube, closely packed with fused potassium hydroxide, then through a tube filled with pure phosphorous pentoxide to the combustion tube. As the apparatus was of glass throughout, the hydrogen did not come in contact anywhere with rubber. The hydrochloric acid formed upon reducing the salt passes first into the tube A, fig. 3, which contains about $\frac{1}{2}$ a c.c. of water. The purpose of the water here is to absorb any potassium chloride which might be volatilised during the reduction. From here the gas passes to two bottles placed in series, containing water, where the hydrochloric acid is completely absorbed.

The temperature at which the reduction takes place in hydrogen is comparatively low for the more exposed portions of salt. It always required a much higher temperature, however, to carry the reaction to

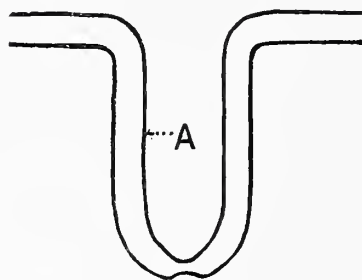


FIG. 3.

completion. This temperature was about 350° for the potassium salt. Even at this temperature a very little potassium chloride was usually volatilised and condensed on the upper parts of the tube. It was never necessary to continue the heating long enough to change any of the platinum black to spongy platinum. It was of course much more easily washed in the more porous form of platinum black than in the form of sponge.

When the reduction of the salt appeared to be complete, the mass of platinum black and potassium chloride was allowed to cool with the stream of hydrogen still passing through the apparatus. While cooling, the platinum will of course absorb a great deal of the hydrogen. After it had cooled, it was reheated until a great deal of this hydrogen had been expelled. This treatment ensures complete reduction of the salt, as it brings the hydrogen into intimate contact with the whole mass, and at the same time serves as it were to wash the platinum black free from any gaseous hydrogen chloride, that might otherwise be retained even at a high temperature.

After this treatment has ensured the complete reduction of the chloro-

platinite, and the removal of all hydrogen chloride, the water in the bulb A is carefully evaporated into the absorbers, and any residue left is cautiously heated.

The platinum black has now to be washed free from potassium chloride, and this proved to be one of the most difficult parts of the whole analysis. The platinum and chloride were washed into a platinum dish, and here heated to boiling with successive portions of distilled water. As already pointed out by Seubert,* if the platinum has been reduced at the lowest possible temperature, it will be in so fine a condition as scarcely to be retained by the filter; on the other hand we find that when heated to such a temperature that it clings together and filters readily, it parts with the potassium chloride so reluctantly that a great many treatments with the boiling water are necessary. If, again, this treatment is continued too long, the platinum becomes more or less colloidal and will not be retained by the filter. It was found that the presence of a little potassium nitrate in the wash water greatly retarded this formation of colloidal platinum, and permitted the boiling with water to be carried on to a much greater extent.

When the platinum has been washed free from potassium chloride in this way, it is put in a tared platinum boat, and this is placed in the combustion tube of the apparatus in which it had been reduced. There it is heated in a stream of hydrogen to a red heat, which converts it entirely into the form of spongy platinum. After cooling in the hydrogen it is desiccated and weighed. This ignition and conversion of the platinum black into sponge in the atmosphere of hydrogen was thought to be necessary, in order to make certain that no oxidation of the platinum in this finely divided state took place. At higher temperatures, it seems likely that platinum even in its most coherent form is slowly oxidised, and in a finely divided condition this would probably take place at a much lower temperature.

A second weighing of the platinum was made after it had been heated in the air, to the highest heat of the bunsen burner. These two weighings always gave identical results, as the following data will show:—

No. 1.	After heating in hydrogen	1st weighing	add to tare	0·85094	
	„	„	in air to approximately 950° C.	„ 0·85092	
No. 2.	„	„	in hydrogen	1st weighing	„ 0·61483
	„	„	in air to 950° C.	„ 0·61483	

From an examination of the work already done on the occlusion of gases by the different forms of platinum, it appears by no means certain that an

* *Loc. cit.*

amount of gas, appreciable in atomic weight work, might not be absorbed by platinum sponge cooled at atmospheric pressure. A great many experimenters have investigated this phenomenon. The work of Graham,* Smith,† Gladstone and Tribe,‡ Berthelot,§ Traube,|| Berliner,¶ Neumann and Streintz,** Wilm †† and Mond, Ramsay and Shields ‡‡ should be cited. The latter authors, Mond, Ramsay and Shields, concluded from a very careful piece of work, that platinum black upon being heated to a red heat in a vacuum gave up all of the occluded gases. In order to ascertain whether it would be necessary to heat and weigh the platinum sponge in a vacuum, the following experiments were carried out.

A few grams of platinum sponge were placed in a platinum boat and carefully weighed. The boat and sponge were then placed in a hard glass tube, and this connected by means of a stopper and a mercury seal with a Töpler pump. The tube was then evacuated to a few tenths of a mm. pressure, and the tube containing the boat and sponge heated until the glass began to soften. The pump was again evacuated while the tube was being heated, and all the gas which could be driven off was pumped away. The boat and sponge were now allowed to cool, and the tube sealed off at a point which had been drawn out. After desiccating a long while, the tube containing the boat and sponge was weighed, using as tare a tube of the same kind of glass, and of the same diameter, as the tube which had been heated, and sealed off under the same pressure.

The first tube was now broken under conditions which ensured no loss of glass, at a scratch made on it before heating; the platinum boat and sponge were withdrawn and the glass weighed. As the tube used as tare had been broken for this weighing, only a small buoyancy correction has to be applied if the two tubes have been carefully chosen. We have now all the data necessary for finding the difference in weight of the platinum when heated, cooled and weighed in air, and when the same operations are carried out in a vacuum. Several such experiments were carried out, and they all gave negative results, showing that the difference in weight of the platinum under these different conditions was not larger than the experimental error of the determination.

* *Proc. Roy. Soc.*, xvi. 422 (1868).

† *Chem. News*, xxxi. 55 (1875).

‡ *Jour. Chem. Soc.*, xxxv. 172 (1879).

§ *U. R.*, xciv. 1377 (1882).

|| *Ber. Deutsch. Chem. Gesell.*, xv. 2854 (1882).

¶ *Ann. der. Phys. (Pogg.)*, [2] xxxiii. 289 (1888).

** *Monatsh. Chem.*, xii. 642 (1891).

†† *Jour. Russ. Chem. Soc.*, xxiv. I. 241 (1892).

‡‡ *Proc. Roy. Soc.*, lviii. 242 (1895).

In one case it was found that—

Platinum boat and sponge cooled in air weighed	4.20520
„ „ „ and weighed in vacuum	4.20516

As we wish to know the amount of silver required for the precipitation, as well as the amount of silver chloride formed, two portions of silver are carefully weighed out, one for the precipitation of the chlorine in the hydrochloric acid, the other for the precipitation of the chlorine in the potassium chloride. A slight excess of silver, amounting to two or three milligrams, is used in each case. The silver is now dissolved in nitric acid, taking every care to prevent any of the silver being carried away mechanically with the escaping vapours.

All the solutions are diluted to a low concentration before mixing, and the silver nitrate is added very slowly and with constant agitation to the chloride solutions. When precipitation is complete, the silver chloride is shaken up vigorously with the solution for perhaps half an hour, and then allowed to stand for at least ten hours before filtering. Actinic light is of course excluded entirely from the silver chloride.

The solutions were filtered on platinum Gooch crucibles, using a good grade of Italian asbestos as a mat. Needless to say that the asbestos had been boiled in nitric and hydrochloric acids, and had been thoroughly washed. After washing, the precipitates were dried first for one hour at 100°, and then for three hours at 150° in an electric drying oven.

A platinum crucible, almost equal in weight to the crucibles used in filtering, and containing about the same amount of asbestos as one employs for a filtering mat, was always used as a tare when weighing the Gooch crucibles.

The moisture which, as shown by Richards and Wells,* is always retained by silver chloride, even when dried at 150°, was determined by placing the pellet of silver chloride, after it had been weighed in the Gooch crucible, without any adhering asbestos, in a tared porcelain crucible; weighing this combination, then heating until the mass was fused and weighing again. The loss in weight is applied as a correction to the weight of silver chloride previously determined.

The small shreds of asbestos which were carried away from the mat while filtering, were collected on a small ashless filter and ignited; and the weight of these was applied as another correction to the weight of silver chloride. Obviously this is opposite in sign to the previous correction.

Unfortunately silver chloride is appreciably soluble in water, and therefore the amount which has dissolved in the water used in washing

* *Jour. Am. Chem. Soc.*, xxvii. 475 (1905).

the precipitate must be determined. The silver chloride in the wash water, as well as the excess of silver over that required for the precipitation, was estimated by means of the nephelometer. The instrument used in this research was supplied by Kny & Scherrer (and proved very satisfactory). Particular attention was paid to having the standard solution correspond as nearly as possible, in every particular, to the solution that was being analysed, and sufficient time was always allowed to elapse before the opalescence of the two solutions was finally compared.

In order to reduce the weights of the different substances to the vacuum standard, the specific gravities must be known. For silver, silver chloride and bromide, these constants have been determined with sufficient accuracy. The specific gravity of platinum sponge was redetermined, and a slightly lower value than those already recorded was obtained. In the case of the platinum salts, new measurements were made using toluol as the liquid to be displaced. Kahlbaum's toluol was dried over sodium and redistilled. 21.5570 grams of this liquid at 24° C. were found to occupy the same volume as 25.0120 grams of water at 4° C. This gives a specific gravity of 0.8618 for the toluol at 24° C. referred to water at 4° C. It was shown by digesting a portion of each salt with the toluol, and evaporating some of the liquid in a platinum dish, that the salts were not at all soluble in the toluol. The values found are tabulated below, together with the correction to be applied for all the substances weighed. In the case of the platinum salts, each result is the mean of two determinations.

TABLE I.—SPECIFIC GRAVITIES.

Salt.	Weight of Salt.	Weight of Toluol displaced.	Sp. Gr. of Salt.	Correction to be applied to 1 gram. of Sub.
K_2PtCl_6	1.6712	0.4116	3.499	+0.00020
$(NH_4)_2PtCl_6$	1.4563	0.4137	3.034	+0.00026
K_2PtBr_6	1.4787	0.2736	4.658	+0.00012
$(NH_4)_2PtBr_6$	1.3210	0.2669	4.265	+0.00015
Pt	21.16	-0.00009
Ag	10.49*	-0.00003
AgCl	5.55†	+0.00007
AgBr	6.473‡	+0.00004

The results from the analysis of potassium chloroplatinate, together with the ratios obtained from these data, are set forth in the tables below. These are all the analyses made of the potassium salt, after the preliminary

* Richards and Wells, *Jour. Am. Chem. Soc.*, xxvii. 459 (1905).

† Richards and Wells, *ibid.* ‡ Baxter and Hines, *Amer. Chem. Jour.*, xxxi. 220 (1904).

work was finished. Parts of experiments Nos. 5 and 7 are wanting, as these were lost by accident. The atomic weight values for chlorine, silver and potassium used in these calculations were those given above.

TABLE II.—RESULTS OF THE ANALYSIS OF POTASSIUM CHLOROPLATINATE.

No. of Experiment.	Preparation.		Weight of K_2PtCl_6 in vacuum.	Weight of Pt in vacuum.	Weight of 4AgCl in vacuum.	Weight of 2AgCl in vacuum.	Weight of 4Ag in vacuum.	Weight of 2Ag in vacuum.
	K_2PtCl_6 .	Silver.						
1	I.	II.	1.43605	0.57667	1.69324	0.84690	1.27475	0.63722
2	"	I.	1.69914	0.68226	2.00402	1.00172	1.50834	0.75401
3	"	"	2.11830	0.85062	2.49836	1.24894	1.88046	0.93993
4	II.	"	2.49734	1.00287	2.94462	1.47249	2.21626	1.10841
5	"	II.	...	0.86012	...	1.26271	...	0.95030
6	"	"	2.20619	0.88588	2.60135	1.30106	1.95842	0.97909
7	III.	III.	1.70600	0.68486	2.01201	1.00580
8	"	"	1.74397	0.70018	2.05691	1.02820	1.54816	0.77402
9	"	II.	2.06137	0.82789	2.43096	1.21526	1.82982	0.91481
10	IV.	"	2.34095	0.93991	2.76105	1.38034	2.07759	1.03868
11	"	"	1.54787	0.62150	1.82560	0.91266	1.37391	0.68702
12	"	III.	1.95944	0.78694	2.31070	1.15522	1.73902	0.86967
18	"	"	2.28366	0.91697	2.69304	1.34636	2.02640	1.01338
24	"	II.	2.27441	0.91320	2.68244	1.34093	2.01870	1.00924

TABLE III.—RATIOS OBTAINED FROM THE ANALYSIS OF POTASSIUM CHLOROPLATINATE. COMPARISON OF PLATINUM WITH POTASSIUM CHLOROPLATINATE AND WITH SILVER CHLORIDE.

Experiment.	Parts of Pt = 100.00 parts of K_2PtCl_6 .	Atomic Weight of Platinum.	Parts of Pt = 100.00 parts of 4AgCl.		Atomic Weight of Platinum.	Parts of Pt = 100.00 parts of 2AgCl.		Atomic Weight of Platinum.	Parts of K_2PtCl_6 = 100.00 parts of 4AgCl.		Atomic Weight of Platinum.	Parts of K_2PtCl_6 = 100.00 parts of 2AgCl.		Atomic Weight of Platinum.
			Parts of Pt = 100.00 parts of 4AgCl.	Atomic Weight of Platinum.		Parts of Pt = 100.00 parts of 2AgCl.	Atomic Weight of Platinum.		Parts of K_2PtCl_6 = 100.00 parts of 4AgCl.	Atomic Weight of Platinum.		Parts of K_2PtCl_6 = 100.00 parts of 2AgCl.	Atomic Weight of Platinum.	
1	67.103	195.25	34.057	195.27	68.092	195.20	84.811	195.29	169.57	195.13	169.57	195.13		
2	67.094	195.22	34.045	195.19	68.109	195.25	84.787	195.15	169.62	195.29	169.62	195.29		
3	67.101	195.24	34.047	195.21	68.107	195.25	84.788	195.16	169.61	195.25	169.61	195.25		
4	67.105	195.26	34.058	195.27	68.107	195.24	84.810	195.29	169.60	195.16	169.60	195.16		
5	67.118	195.29	68.117	195.27	169.61	195.25	169.61	195.25		
6	67.096	195.23	34.055	195.25	68.089	195.19	84.810	195.28	169.57	195.14	169.57	195.14		
7	67.068	195.15	34.039	195.16	68.091	195.20	84.791	195.18	169.62	195.28	169.62	195.28		
8	67.081	195.18	34.040	195.17	68.098	195.22	84.786	195.15	169.61	195.26	169.61	195.26		
9	67.118	195.29	34.056	195.26	68.125	195.29	84.797	195.21	169.62	195.29	169.62	195.29		
10	67.087	195.20	34.042	195.18	68.093	195.20	84.785	195.14	169.59	195.21	169.59	195.21		
11	67.090	195.21	34.044	195.19	68.098	195.22	84.787	195.15	169.60	195.23	169.60	195.23		
12	67.116	195.29	34.056	195.26	68.120	195.28	84.799	195.22	169.62	195.28	169.62	195.28		
18	67.094	195.22	34.050	195.22	68.107	195.25	84.799	195.22	169.62	195.28	169.62	195.28		
24	67.087	195.20	34.045	195.19	68.102	195.23	84.789	195.18	169.62	195.27	169.62	195.27		
Average	67.097	195.23	34.049	195.21	68.104	195.24	84.795	195.20	169.61	195.24	169.61	195.24		

TABLE IV.—RATIOS OBTAINED FROM THE ANALYSIS OF POTASSIUM CHLOROPLATINATE.
COMPARISON OF SILVER WITH PLATINUM AND WITH POTASSIUM CHLOROPLATINATE.

Experi- ment.	Parts of Pt = 100·00 parts of 4Ag.	Atomic Weight of Pla- tinum.	Parts of Pt = 100·00 parts of 2Ag.	Atomic Weight of Pla- tinum.	Parts of K_2PtCl_6 = 100·00 parts of 4Ag.	Atomic Weight of Pla- tinum.	Parts of K_2PtCl_6 = 100·00 parts of 2Ag.	Atomic Weight of Pla- tinum.
1	45·238	195·21	90·498	195·25	112·65	195·15	225·36	195·27
2	45·233	195·19	90·484	195·23	112·65	195·14	225·35	195·24
3	45·235	195·20	90·498	195·26	112·65	195·13	225·37	195·28
4	45·251	195·27	90·478	195·22	112·68	195·28	225·31	195·16
5	90·510	195·29	225·36	195·27
6	45·234	195·20	90·480	195·22	112·65	195·15	225·33	195·20
8	45·227	195·16	90·460	195·18	112·65	195·13	225·31	195·17
9	45·244	195·24	90·499	195·26	112·65	195·16	225·33	195·21
10	45·240	195·22	90·488	195·24	112·68	195·19	225·37	195·29
11	45·236	195·20	90·463	195·18	112·66	195·19	225·30	195·14
12	45·252	195·27	90·487	195·24	112·68	195·25	225·31	195·16
18	45·251	195·25	90·486	195·23	112·70	195·29	225·35	195·25
24	45·237	195·21	90·484	195·23	112·67	195·21	225·36	195·26
Average	45·240	195·22	90·486	195·23	112·66	195·19	225·34	195·22

It appears from the foregoing results that the different methods adopted for the purification of the platinum salt have yielded products of a considerable degree of purity, or else some particular impurity is present to the same extent in each sample—a rather unlikely supposition. The different samples are probably not all equally pure, but the amount of impurity is not large enough to affect the value for the atomic weight beyond what may be regarded as the experimental error. This would seem to be true as well for the different preparations of silver.

If we compare the values obtained for different ratios, it does not appear that any particular ratio gives consistently a higher or a lower value than any other, and this seems to be the best argument for the purity of the salt and the exactness of the analysis. If, for instance, potassium chloride was occluded in appreciable amounts during precipitation, the ratio $2AgCl : Pt$ would always give a low result, or if $PtCl_4$ was occluded, a high result would be obtained. Again, if water still remained in the salt after heating to 380° , the ratio $K_2PtCl_6 : Pt$ would give a low value for the atomic weight. If some of the chlorine had been replaced by hydroxyl, this ratio would give a high value while the ratio $4AgCl : Pt$ would be still higher. On the other hand, if, during the analysis, the potassium chloride had not all been washed out of the platinum, the ratio $2AgCl : Pt$ would

give a doubly high result, on account of a too small amount of silver chloride, as well as a too great weight for the platinum. If some of the potassium chloride had been driven, during the heating, into the hydrochloric acid absorbers, the ratio $2\text{AgCl} : \text{Pt}$ would give a high value, while the value from the ratio $4\text{AgCl} : \text{Pt}$ would be correspondingly low.

THE ANALYSIS OF AMMONIUM CHLOROPLATINATE AND BROMOPLATINATE.

The analysis of the ammonium salts was carried out in much the same way as in the case of the potassium salt, except that here the salt could not be heated to as high a temperature for the purpose of expelling the occluded water. The purest sample that I have been able to prepare would decompose slightly above 185° . All the hydrochloric or hydrobromic acid seemed to be driven off below 150° . Nevertheless, as the salt might still contain an appreciable amount of water, the values for the atomic weight obtained from ratios in which the weight of original salt appears are not used in estimating the final mean value. On account of the low temperature at which the ammonium chloride or bromide volatilises, and the consequent difficulty of preventing some of this salt from being carried over into the absorption bottles, with the halogen acid, it was thought safer to estimate the ammonium chloride or bromide, and the acid formed, together. The results given by these analyses, together with the ratios obtained, are set forth in the tables below. The atomic weight values for nitrogen, hydrogen, and bromine used in the calculations were those given above.

TABLE V.—RESULTS OF THE ANALYSIS OF AMMONIUM CHLOROPLATINATE.

Experi- ment.	Preparation.		Weight of $(\text{NH}_4)_2\text{PtCl}_6$ in vacuum.	Weight of Platinum in vacuum.	Weight of 6AgCl in vacuum.	Weight of 6Ag in vacuum.
	$(\text{NH}_4)_2\text{PtCl}_6$.	Silver.				
13	I.	II.	1.75088	0.76976	3.39181	2.55181
14	„	„	1.36500	0.59997	2.64317	1.99014
15	II.	I.	1.15060	0.50585	2.22810	1.67695
16	„	„	1.27475	0.56049	2.46936	1.85794
17	„	„	2.54096	1.11688	4.92047	3.70420

TABLE VI.—RATIOS OBTAINED FROM THE ANALYSIS OF AMMONIUM CHLOROPLATINATE. COMPARISON OF PLATINUM AND AMMONIUM CHLOROPLATINATE WITH SILVER AND SILVER CHLORIDE.

Experiment.	Parts of Pt = 100·00 parts of $(\text{NH}_4)_2\text{Cl}_6$.	Atomic Weight of Platinum.	Parts of Pt = 100·00 parts of 6AgCl.	Atomic Weight of Platinum.	Parts of $(\text{NH}_4)_2\text{PtCl}_6$ = 100·00 parts of 6AgCl.	Atomic Weight of Platinum.	Parts of Pt = 100·00 parts of 6Ag.	Atomic Weight of Platinum.	Parts of $(\text{NH}_4)_2\text{PtCl}_6$ = 100·00 parts of 6Ag.	Atomic Weight of Platinum.
13	78·457	195·22	22·695	195·18	51·621	195·12	30·165	195·25	68·613	195·29
14	78·424	195·14	22·699	195·22	51·643	195·31	30·147	195·14	68·588	195·13
15	78·457	195·28	22·703	195·25	51·640	195·29	30·165	195·25	68·613	195·29
16	78·471	195·26	22·698	195·21	51·623	195·14	30·167	195·27	68·611	195·28
17	78·428	195·15	22·699	195·22	51·641	195·29	30·152	195·17	68·597	195·19
Average	78·447	195·21	22·699	195·22	51·634	195·23	30·159	195·22	68·604	195·24

TABLE VII.—RESULTS OF THE ANALYSIS OF AMMONIUM BROMOPLATINATE.

Experiment.	Sample of Silver used.	Weight of $(\text{NH}_4)_2\text{PtBr}_6$ in vacuum.	Weight of Pt in vacuum.	Weight of 6AgBr in vacuum.	Weight of 6Ag in vacuum.
19	I.	1·83860	0·50497	2·91430	1·67448
20	II.	2·31057	0·63437	3·66269	2·10379
21	„	2·33965	0·64272	3·70900	2·13049

TABLE VIII.—RATIOS OBTAINED FROM THE ANALYSIS OF AMMONIUM BROMOPLATINATE. COMPARISON OF PLATINUM AND AMMONIUM BROMOPLATINATE WITH SILVER AND SILVER BROMIDE.

Experiment.	Parts of Pt = 100·00 parts of $(\text{NH}_4)_2\text{Br}_6$.	Atomic Weight of Platinum.	Parts of Pt = 100·00 parts of 6AgBr.	Atomic Weight of Platinum.	Parts of $(\text{NH}_4)_2\text{PtBr}_6$ = 100·00 parts of 6AgBr.	Atomic Weight of Platinum.	Parts of Pt = 100·00 parts of 6Ag.	Atomic Weight of Platinum.	Parts of $(\text{NH}_4)_2\text{PtBr}_6$ = 100·00 parts of 6Ag.	Atomic Weight of Platinum.
19	37·865	195·23	17·327	195·25	63·089	195·28	30·157	195·20	109·801	195·13
20	37·846	195·13	17·320	195·16	63·084	195·23	30·154	195·18	109·829	195·30
21	37·876	195·29	17·329	195·26	63·080	195·19	30·168	195·27	109·817	195·22
Average	37·862	195·25	17·325	195·22	63·084	195·23	30·160	195·21	109·819	195·22

THE ANALYSIS OF POTASSIUM BROMOPLATINATE.

It was found that the potassium bromoplatinate was even more stable than the chloroplatinate. It could be heated for an indefinite length of time, in pure dry air, at as high a temperature as 400° C. without undergoing the least decomposition. Consequently, all but inappreciable amounts of water must have been expelled from the salt before it was weighed. This salt was somewhat more difficult to decompose in hydrogen than the chloroplatinate, and the reduced platinum had to be washed many times with the occluded hydrogen, as described above, before all the hydrobromic acid was expelled. The method of analysis was very similar to that used for the other compounds. The bromine remaining as potassium bromide was estimated separately from that forming hydrobromic acid. Here, too, the silver required for the precipitations, as well as the amount of silver bromide produced, was determined.

The advantages of silver bromide as an ideal precipitate to work with are well known. Its low solubility in water allows one to estimate the amount dissolved by the wash water even more accurately than in the case of the chloride.

The results obtained, together with the ratios which may be found from these, are set forth below.

TABLE IX.—RESULTS OF THE ANALYSIS OF POTASSIUM BROMOPLATINATE.

Experiment.	Sample of Silver used.	Weight of K_2PtBr_6 in vacuum.	Weight of Platinum in vacuum.	Weight of 4AgBr in vacuum.	Weight of 2AgBr in vacuum.	Weight of 4Ag in vacuum.	Weight of 2Ag in vacuum.
22	II.	2.19076	0.56779	2.18543	1.09273	1.25544	0.62770
23	„	2.42094	0.62766	2.41510	1.20758	1.38761	0.69378
25	„	1.78705	0.46344	1.78284	0.89156	1.02416	0.51214
26	I.	1.81840	0.47156	1.81430	0.90703	1.04228	0.52105
27	„	2.47056	0.64063	2.46507	1.23246	1.41572	0.70800
28	„	2.19017	0.56787	2.18525	1.09260	1.25530	0.62756

TABLE X.—RATIOS OBTAINED FROM THE ANALYSIS OF POTASSIUM BROMOPLATINATE. COMPARISON OF PLATINUM WITH POTASSIUM BROMOPLATINATE AND WITH SILVER BROMIDE.

Experiment.	Parts of Pt = 100·00 parts of K ₂ Br ₆ .	Atomic Weight of Platinum.	Parts of Pt = 100·00 parts of 4AgBr.	Atomic Weight of Platinum.	Parts of Pt = 100·00 parts of 2AgBr.	Atomic Weight of Platinum.	Parts of K ₂ PtBr ₆ = 100·00 parts of 4AgBr.	Atomic Weight of Platinum.	Parts of K ₂ PtBr ₆ = 100·00 parts of 2AgBr.	Atomic Weight of Platinum.
22	34·985	195·12	25·981	195·17	51·961	195·16	100·244	195·29	200·485	195·28
23	35·007	195·21	25·989	195·23	51·976	195·22	100·242	195·28	200·479	195·26
25	35·013	195·29	25·995	195·27	51·981	195·25	100·236	195·24	200·441	195·13
26	35·012	195·28	25·991	195·25	51·990	195·27	100·226	195·16	200·478	195·26
27	35·008	195·25	25·988	195·22	51·980	195·24	100·223	195·13	200·458	195·19
28	35·004	195·20	25·990	195·24	51·974	195·21	100·225	195·15	200·455	195·17
	35·005	195·21	25·989	195·23	51·977	195·22	100·233	195·21	200·466	195·22

TABLE XI.—COMPARISON OF SILVER WITH PLATINUM AND WITH POTASSIUM BROMOPLATINATE.

Experi- ment.	Parts of Pt = 100·00 parts of 4Ag.	Atomic Weight of Pla- tinum.	Parts of Pt = 100·00 parts of 2Ag.	Atomic Weight of Pla- tinum.	Parts of K ₂ PtBr ₆ = 100·00 parts of 4Ag.	Atomic Weight of Pla- tinum.	Parts of K ₂ PtBr ₆ = 100·00 parts of 2Ag.	Atomic Weight of Pla- tinum.
22	45·226	195·16	90·456	195·17	174·50	195·27	349·01	159·29
23	45·233	195·19	90·470	195·20	174·47	195·14	348·95	195·15
25	45·251	195·26	90·491	195·25	174·49	195·21	348·94	195·13
26	45·243	195·24	90·502	195·27	174·46	195·13	348·99	195·24
27	45·251	195·27	90·485	195·23	174·51	195·30	348·95	195·15
28	45·238	195·23	90·489	195·24	174·47	195·16	349·00	195·26
	45·240	195·22	90·482	195·23	174·49	195·20	348·97	195·20

If we now collect the several average values, rejecting those in which the weight of original salt is concerned, we have—

For K ₂ PtCl ₆	from the ratio of Pt to 4AgCl . . .	195·21
“ “	“ “ Pt „ 2AgCl . . .	195·24
“ “	“ “ Pt „ 4Ag . . .	195·22
“ “	“ “ Pt „ 2Ag . . .	195·23
“ (NH ₄) ₂ PtCl ₆	“ “ Pt „ 6AgCl . . .	195·22
“ “	“ “ Pt „ 6Ag . . .	195·22
“ (NH ₄) ₂ PtBr ₆	“ “ Pt „ 6AgBr . . .	195·22
“ “	“ “ Pt „ 6Ag . . .	195·21
“ K ₂ PtBr ₆	“ “ Pt „ 4AgBr . . .	195·23
“ “	“ “ Pt „ 2AgBr . . .	195·22
“ “	“ “ Pt „ 4Ag . . .	195·22
“ “	“ “ Pt „ 2Ag . . .	195·23
	Final Mean Value . . .	195·23

SUMMARY.

The results set forth in the foregoing pages may be briefly summarised as follows:—

1. The platinum salts analysed by previous investigators must have contained appreciable amounts of impurities, as very divergent results are obtained when the weight of original salt is used in the calculation of the atomic weight of platinum.

2. Very pure platinum metal can be obtained by the precipitation of the platinum as ammonium chloroplatinate.

3. The platinum metal has been converted into the salts of bromoplatinic and chloroplatinic acid, without the introduction of any nitric acid, under conditions which ensure the absence of any appreciable amounts of impurity from the utensils employed.

4. The potassium salts of chloro- and bromoplatinic acid, and the corresponding ammonium salts, have been heated to a temperature of 400° and 175° respectively, for the purpose of expelling the absorbed and occluded moisture.

5. Essentially the same results are obtained whether the platinum be heated and cooled in hydrogen and weighed at atmospheric pressure, or heated, cooled, and weighed in a vacuum.

6. The ratios obtained from the analysis of samples of platinum salts obtained from different sources, and purified by different methods, lead to essentially the same value for the atomic weight.

7. Assuming the values given by the International Committee for the atomic weights concerned in the calculation, the atomic weight of platinum cannot be far from 195.23.

CHEMICAL LABORATORY,
SYRACUSE UNIVERSITY,
SYRACUSE, N.Y., U.S.A.

(Issued separately December 2, 1909.)

XLIII.—On the Development of Mixed Cultures of Bacteria and Lower Fungi in Liquid and Solid Media. By Emil Westergaard, Lecturer on Technical Mycology, Heriot-Watt College, Edinburgh.

[*Preliminary Notice.*]

AN investigation was undertaken in order to ascertain to what extent and in what manner the individual species and varieties of lower fungi and bacteria present in liquid and solid media affect the simultaneous development of other similar organisms. The preliminary experiments were carried out with a culture of streptococcus lactis acidi, which was added in varying proportions to cow's milk immediately after milking, and gelatine plates were made at once, partly from the milk to which the cultures had been added and partly from the natural milk. The gelatine employed contained either glucose and peptone (Witte), or in addition an excess of calcium carbonate. The proportion in which the culture was added to the milk varied from 1 c.c. in 500 c.cs. of milk to one drop of the culture in half a gallon of milk. The results were in all cases practically the same, namely, that the plates infected with the normal milk gave an abundant development of bacteria and were in all cases rapidly liquefied, whereas the plates made from the infected milk showed a far less varied bacterial contents and a total suppression of the liquefying species.

The results obtained with the plates containing an excess of calcium carbonate were the same as those obtained with the plates which did not contain this, showing that the observed results are not due to any formation of free lactic acid. The most probable explanation of the facts just mentioned seems to be the production of anti-enzymes which, by binding the proteoclastic enzymes of the liquefying species, prevent the development of the latter. This view seems to find some support in the fact that spontaneously soured milk as a rule ultimately undergoes proteoclastic decomposition (*i.e.* after the lactic acid bacteria have been paralysed by the free acid).

The experiments are now being continued with a number of pure cultures of well-known bacteria and lower fungi, and an endeavour is being made to demonstrate the presence of the anti-enzymes referred to.

(*Issued separately December 2, 1909.*)

O B I T U A R Y N O T I C E.

John Hardy, Librarian R.S.E.

Minute of Council, Royal Society, 25th June 1909.

“The Council of the Royal Society of Edinburgh desire to place on record their high appreciation of the services of their Librarian, Mr John Hardy, who died on the 25th May 1909.

“After a preliminary training in the Library of the University of Edinburgh, Mr Hardy became Assistant Librarian to the Society under the late Mr Gordon on 13th June 1887, and, on Mr Gordon’s retirement on 21st February 1902, was appointed Librarian. He was therefore twenty-two years in the service of the Society. The great improvement of the Library, begun by Mr Gordon, was carried so far by Mr Hardy that it is now one of the best Scientific Libraries in the United Kingdom. He was most zealous and efficient in the conduct of the general business of the Society. Many of the excellent arrangements now in use, which have contributed so much to the promptness and perfection of our publications, were devised by Mr Hardy; and in practice all of them owed much to his high intelligence and unremitting attention.

“A man of somewhat retiring disposition, he was nevertheless possessed of much knowledge of men and affairs, of great tact and judgment, of never-failing courtesy and discretion. The Fellows of the Society who knew him, and visitors who used the Library during his term of office, will remember with kindly regret his pleasant smile and obliging attentions: and the Officials of the Society will not soon forget the thorough knowledge of the Society’s affairs, and the admirable business capacity which did so much to lighten their work for the Society. During the removal of the Society to its new building Mr Hardy’s services were invaluable. The scheme for the rearrangement of the new building was to a very large extent due to him; and thanks to this scheme and the able training and supervision of his Chief, Mr Stewart the Sub-Librarian was able to complete the transference of the Library in a reasonable time with very little friction. Mr Hardy’s interest in the affairs of the Society did not flag even during his last illness, but continued keen till the day of his death; and it is a matter of much regret to the Council that he did not live to enjoy the improved accommodation now provided for the Librarian, and otherwise to reap the fruit of his faithful labour in the interest of the Royal Society.”

APPENDIX.

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PROCEEDINGS OF THE STATUTORY GENERAL MEETING.
The 126th Session.

At the Annual Statutory Meeting, Monday, 26th October 1908,

Dr R. H. Traquair in the Chair,

the SECRETARY explained that owing to commencement of the operations for the removal of the Society's Library and other effects to the new premises in George Street, the Lecture Room in the Royal Institution was not available for this Statutory Meeting, nor for the Ordinary Meetings of the Society for the reading of Papers to be held in the months of November, December, and possibly January.

By the courtesy of the University Court and of the Professor of Engineering it had been arranged that the Statutory Meeting and, until further notice, the Ordinary Meetings should take place in the class-room of the Engineering Laboratory, High School Yards, Infirmary Street.

On the motion of Dr PEACH, seconded by Dr KNOTT, Mr D. B. DOTT and Mr G. H. GULLIVER were appointed Scrutineers, and the ballot for the New Council commenced.

The TREASURER'S Accounts for the past year were submitted. These, with the Auditor's Report, were read and approved.

On the motion of Dr KNOTT, seconded by Mr D. B. DOTT, the following changes in the Laws of the Society, proposed by the Council, were unanimously adopted:—

- (1) That Law VIII. read:—“All Ordinary Fellows of the Society who are not in arrear of their Annual Contributions shall be entitled to receive, gratis, copies of the parts of the Transactions of the Society which shall be published subsequent to their admission, upon application, either personally or by an authorised agent, to the Librarian, provided they apply for them within five years of the date of publication of such parts.”
“Copies of the parts of the Proceedings shall be distributed to all Fellows of the Society, by post or otherwise, as soon as may be convenient after publication.”
“That the second paragraph of Law XV. be transferred to its natural place at the end of Law XVII.”

The Scrutineers reported that the following New Council had been duly elected:—

Principal Sir WM. TURNER, K.C.B., D.C.L., F.R.S.,	President.	
Professor ANDREW GRAY, M.A., LL.D., F.R.S.,	} Vice-Presidents.	
RAMSAY H. TRAQUAIR, M.D., LL.D., F.R.S.,		
Professor A. CRUM BROWN, M.D., LL.D., F.R.S.,		
Professor J. C. EWART, M.D., F.R.S.,		
JOHN HORNE, LL.D., F.R.S.,		
JAMES BURGESS, C.I.E., LL.D.,	} Secretaries to Ordinary Meetings.	
Professor GEORGE CHRYSTAL, LL.D.,		General Secretary.
Professor D. J. CUNNINGHAM, M.D., LL.D., F.R.S.,		
CARGILL G. KNOTT, D.Sc.,		
JAMES CURRIE, M.A.,		Treasurer.
JOHN S. BLACK, M.A., LL.D.,	Curator of Library and Museum.	

COUNCILLORS.

Professor E. A. SCHÄFER, LL.D., F.R.S.	O. CHARNOCK BRADLEY, M.D., D.Sc.
The Hon. LORD M'LAREN, LL.D.	CHARLES TWEEDIE, M.A., D.Sc.
Professor F. O. BOWER, M.A., D.Sc., F.R.S.	Professor J. W. GREGORY, D.Sc., F.R.S.
Professor T. HUDSON BEARE, M.Inst.C.E.	A. P. LAURIE, M.A., D.Sc.
Professor F. W. DYSON, M.A., F.R.S., Astro- nomer Royal for Scotland.	Professor WM. PEDDIE, D.Sc.
Professor D'ARCY W. THOMPSON, C.B.	Professor H. M. MACDONALD, M.A., F.R.S.

On the motion of Dr BLACK, seconded by Professor DYSON, thanks were voted to the Scrutineers.

On the motion of Dr WEDDERBURN, seconded by Dr CARSE, the Auditors were thanked and reappointed.

On the motion of Mr W. A. P. TAIT, seconded by Professor BEARE, thanks were voted to the Chairman.

WM. TURNER,
President.

PROCEEDINGS OF THE ORDINARY MEETINGS,
Session 1908-1909.

FIRST ORDINARY MEETING.

Monday, 2nd November 1908.

Professor Gray, F.R.S., Vice-President, in the Chair.

Dr H. DRINKWATER signed the Roll, and was formally admitted a Fellow of the Society.

The following Communications were read :—

1. Temperature Observations on Loch Garry. By E. M. WEDDERBURN, M.A., LL.B. (*With Lantern Illustrations.*)
2. On the Discharge of Water from Circular Weirs and Orifices. By G. H. GULLIVER, B.Sc.
3. Dissymmetrical Separations in the Zeeman Effect in Tungsten and Molybdenum. By ROBERT JACK, M.A., B.Sc., Ph.D., 1851 Exhibition Research Scholar, Glasgow University. Communicated by Professor GRAY.
4. On a Question in Absorption Spectroscopy. By ROBERT A. HOUSTON, M.A., Ph.D., D.Sc., assisted by ALEXANDER S. RUSSELL, M.A. Communicated by Professor GRAY.
5. Preliminary Note on the Action of Nitric Anhydride on Mucic Acid. By G. E. GIBSON, B.Sc. Communicated by Professor A. CRUM BROWN, F.R.S.

SECOND ORDINARY MEETING.

Monday, 16th November 1908.

Dr James Burgess, C.I.E., Vice-President, in the Chair.

Mr J. HANNAY THOMPSON signed the Roll, and was duly admitted a Fellow of the Society.

The following Communications were read :—

1. An Investigation of the Seiches of Loch Earn by the Scottish Lake Survey. Parts III.-V. By Professor CHRYSTAL.
2. Notes on Hydrodynamics : chiefly on Vortex Motion. By Professor ANDREW GRAY, F.R.S.

THIRD ORDINARY MEETING.

Monday, 7th December 1908.

Dr Horne, F.R.S., Vice-President, in the Chair.

The following Communications were read :—

1. A Monograph on the General Morphology of the Myxinoid Fishes, based on a study of Myxine. Part III.—Further Observations on the Skeleton. By FRANK J. COLE, B.Sc., Professor of Zoology, University College, Reading. Communicated by Dr R. H. TRAQUAIR, F.R.S.
2. An Investigation of the Seiches of Loch Earn by the Scottish Lake Survey. Parts III.-V. (*With Lantern Illustrations.*) By Professor CHRYSTAL. (*Continuation.*)

FOURTH ORDINARY MEETING.

Monday, 21st December 1908.

Professor Crum Brown, LL.D., Vice-President, in the Chair.

The following Communications were read :—

1. A Photographic Apparatus for the purpose of Automatically Recording the Readings of the Scale and Vernier of any Instrument. By Dr J. R. MILNE. (*The Apparatus will be exhibited.*)
2. On the Friction at the extremities of a Short Bar subjected to a crushing load, and its influence upon the apparent compressive strength of the Material. By G. H. GULLIVER, B.Sc.

Mr WILLIAM GENTLE, B.Sc., was balloted for, and declared duly elected a Fellow of the Society.

FIFTH ORDINARY MEETING.

Monday, 4th January 1909.

Dr R. H. Traquair, F.R.S., Vice-President, in the Chair.

The following Communications were read :—

1. On the Fossil Osmundaceæ. Part III. By R. KIDSTON, LL.D., F.G.S., and D. T. GWYNNE-VAUGHAN, M.A.

Supplementary Report on the Hydroids of the Scottish National Antarctic Expedition. By JAMES RITCHIE, M.A., B.Sc. Communicated by Dr W. S. BRUCE.

SIXTH ORDINARY MEETING.

Monday, 18th January 1909.

Professor A. Gray, F.R.S., Vice-President, in the Chair.

Mr W. GENTLE signed the Roll, and was duly admitted a Fellow of the Society.

The following Communications were read :—

1. The Influence of Castration upon Bone-Growth and Giantism. By A. CAMPBELL GEDDES, M.D., Demonstrator of Anatomy, University of Edinburgh. Communicated by Professor CUNNINGHAM. (*With Lantern Illustrations.*)

2. Flexural Vibrations of Thin Rods. By GEORGE GREEN, M.A., B.Sc. Communicated by Professor A. GRAY.

The following Gentlemen were balloted for, and declared duly elected Fellows of the Society :— JAMES G. GRAY, B.Sc., HENRY R. KENWOOD, M.B., ALEXANDER D. ROSS, M.A., B.Sc., and C. R. MARSHALL, M.D., M.A.

SEVENTH ORDINARY MEETING.

Monday, 1st February 1909.

Dr James Burgess, Vice-President, in the Chair.

Messrs J. G. GRAY, B.Sc., A. D. ROSS, M.A., B.Sc., and C. R. MARSHALL, M.D., signed the Roll, and were duly admitted Fellows of the Society.

The following Communications were read :—

1. Magnetic Quality in the Homogeneous Hexagonal Arrangement of Molecular Magnets. By Professor W. PEDDIE.

2. On an Improved Form of Magnetometer and Accessories for the Testing of Magnetic Materials at Different Temperatures. Communicated by Professor A. GRAY, F.R.S. By JAMES G. GRAY, B.Sc., Lecturer on Physics in the University of Glasgow, and ALEXANDER D. ROSS, M.A., B.Sc., Assistant to the Professor of Natural Philosophy in the University of Glasgow.

3. On the Conditions for the Reversibility of the Order of Partial Differentiation. By W. H. YOUNG, Sc.D., F.R.S. Communicated by J. H. MACLAGAN WEDDERBURN, D.Sc.

EIGHTH ORDINARY MEETING.

Monday, 15th February 1909.

Professor A. Gray, LL.D., Vice-President, in the Chair.

The following Communications were read :—

1. The Electro-motive Force of Iodine Concentration Cells with one Electrode saturated with Iodine. By Principal A. P. LAURIE, D.Sc.

2. On the Magnetic Properties of certain Copper Alloys. By ALEXANDER D. ROSS, M.A., B.Sc., Assistant to the Professor of Natural Philosophy in the University of Glasgow, and ROBERT C. GRAY, Thomson Experimental Scholar in the University of Glasgow.

3. Some Low Temperature Experiments in Magnetism. By JAMES G. GRAY, B.Sc., Lecturer in Physics in the University of Glasgow, and HUGH HIGGINS, M.A., Thomson Experimental Scholar in the University of Glasgow.

4. On Lagrange's Equations of Motion and on Elementary Solutions of Gyrostatic Problems. By Professor ANDREW GRAY, F.R.S.

The following Candidates for Fellowship were balloted for, and duly elected Fellows of the Society :—JOHN BROWNLEE, M.A., M.D., D.Sc., BERNARD LANGLEY MILLS, M.D., F.R.C.S. E. and L., ARTHUR GEO. SYDNEY MITCHELL, REGINALD L. A. E. WESTERGAARD.

NINTH ORDINARY MEETING.

Monday, 1st March 1909.

Dr R. H. Traquair, F.R.S., Vice-President, in the Chair.

Mr ANDREW W. KERR, Dr J. H. HARVEY PIRIE, and Mr R. L. A. E. WESTERGAARD signed the Roll, and were duly admitted Fellows of the Society.

The following Communications were read:—

1. The Systematic Motion of the Stars (Second Paper), By Professor DYSON, F.R.S.
2. *Cynomacrus Piriei*, Poisson nouveau abyssal, recueilli par l'Expédition Antarctique Nationale Écossaise. Note préliminaire, par LOUIS DOLLO, Sc.D. (Cantab.), For. Mem. G.S., C.M.Z.S. Présentée par M. R. H. TRAQUAIR, M.D., LL.D., F.R.S.

TENTH ORDINARY MEETING.

Monday, 15th March 1909.

Dr John Horne, F.R.S., Vice-President, in the Chair.

Dr B. L. MILLS signed the Roll, and was admitted a Fellow of the Society.

The following Communications were read:—

1. The Glacial Deposits of Western Carnarvonshire. By THOS. J. JEHU, M.A., M.D. (*With Lantern Illustrations.*)
2. The Glenboig Fireclay: its Halloysite and Sideroplesite. By Professor J. W. GREGORY, F.R.S.
3. Tuesite: a Scotch variety of Halloysite. By the same.

The following Candidate was balloted for, and declared duly elected:—PETER COMRIE, M.A.

ELEVENTH ORDINARY MEETING.

Monday, 3rd May 1909.

Professor Cossar Ewart, F.R.S., Vice-President, in the Chair.

Mr PETER COMRIE signed the Roll, and was formally admitted a Fellow of the Society.

The following Communications were read:—

1. *Strophanthus Sarmentosus*: its Pharmacological Action and use as an Arrow-Poison. By Professor Sir THOMAS FRASER, M.D., and ALISTER P. MACKENZIE, M.B., Carnegie Research Scholar.
2. On the Histological Changes in the Liver and Kidney after Chloroform administered by different Channels. By Dr G. HERBERT CLARK. Communicated by Professor D. NOËL PATON.
3. The Pathogenesis of *Micrococcus melitensis*. By Dr J. EYRE.
4. Life and Chemical Work of ARCHIBALD SCOTT COUPER. By RICHARD ANSCHÜTZ, Ph.D., LL.D. Communicated by Professor CRUM BROWN, LL.D.

TWELFTH ORDINARY MEETING.

Monday, 17th May 1909.

Dr John Horne, F.R.S., Vice-President, in the Chair.

The following Communications were read:—

1. On a Simple Radioscope and a Radiometer for Showing and Measuring Radioactivity. By Dr JOHN AITKEN, F.R.S.
2. Mendelian Action and Differentiated Sex. By Dr D. BERRY HART.

The following Candidates for Fellowship were balloted for, and declared duly elected Fellows of the Society:—THOMAS MORRISON CLAYTON, M.D., D.Hy., B.Sc., D.P.H., AUCLAND C. GEDDES, M.D.

THIRTEENTH ORDINARY MEETING.

Monday, 7th June 1909.

Professor A. Crum Brown, LL.D., Vice-President, in the Chair.

The following Communications were read:—

1. Scottish National Antarctic Expedition: The Anatomy of the Weddell Seal. By Professor DAVID HEPBURN.
2. Lower Palæozoic Hyolithidæ from Girvan. By F. R. COWPER REED, M.A., F.G.S. Communicated by Dr JOHN HORNE, F.R.S.
3. The Atomic Weight of Platinum. By Professor E. H. ARCHIBALD. Communicated by Professor J. G. MACGREGOR.
4. On Group Velocity, and on the Propagation of Waves in a Dispersive Medium. By Mr GEORGE GREEN. Communicated by Professor A. GRAY, F.R.S.
5. The Theory of Jacobians in the Historical Order of Development up to 1860. By Dr THOMAS MUIR.
6. *Nematonurus Lecoointei*, Poisson abyssal de la "Belgica" retrouvé par l'Expédition Antarctique Nationale Ecossaise. Note préliminaire, par LOUIS DOLLO, Sc.D. (Cantab.), Phil.D. (Giessen), C.M.Z.S., à Bruxelles (Musée). Présentée par M. R. H. TRAQUAIR, M.D., LL.D., F.R.S.
7. An Experiment with the Spark Gap of an Induction Coil. By Dr DAWSON TURNER.

FOURTEENTH ORDINARY MEETING.

Monday, 21st June 1909.

Dr Jas. Burgess, Vice-President, in the Chair.

The following Communications were read:—

1. The Pharmacological Action of Protocatechyltropheine. By Professor C. R. MARSHALL. (*With Lantern Illustrations.*)
2. The Toot-poison of New Zealand: an Investigation into its Pharmacological Action. By Professor C. R. MARSHALL. (*With Lantern Illustrations.*)
3. Hydrolysis of Salts in Amphoteric Electrolytes. By Miss H. H. BEVERIDGE, B.Sc., Carnegie Research Scholar. Communicated by Professor JAMES WALKER.
4. Seismic Radiations.—Part II. By Dr C. G. KNOTT.

FIRST SPECIAL MEETING.

Monday, 28th June 1909.

Dr R. H. Traquair, F.R.S., Vice-President, in the Chair.

The following Address was delivered in French:—

Sur les gigantesques Reptiles Éteints de la Belgique. Par M. LOUIS DOLLO, Sc.D., Phil.D., Professeur à l'Université et Conservateur au Musée royal d'Histoire naturelle à Bruxelles. (*With Lantern Illustrations.*)

FIFTEENTH ORDINARY MEETING.

Monday, 5th July 1909.

Professor Cossar Ewart, F.R.S., Vice-President, in the Chair.

Dr A. C. GEDDES signed the Roll, and was duly admitted a Fellow of the Society.

The following Communications were read:—

1. Notes on the Skeleton of a Sowerby's Whale (*Mesoplodon bidens*) stranded at St Andrews, and on the Morphology of the Manus in Hyperoodon, and in the Delphinidæ. By Sir WM. TURNER, K.C.B., President of the Society.
2. Current and Temperature Observations in Loch Ness. By E. M. WEDDERBURN, W.S., and W. WATSON, M.A., B.Sc.
3. Pettersson's Observations on Deep-water Oscillations. By E. M. WEDDERBURN, W.S.
4. A Carboniferous Fauna from Nowaja Semlja. Collected by Dr W. S. BRUCE during a voyage in Major Andrew Coats' yacht *Blencathra*. By Dr G. W. LEE. Communicated by Dr JOHN HORNE.

5. Note on the Flight of Nigerian Arrows. By Dr C. G. KNOTT.

6. The Development of the Auditory Ossicles in the Horse, with a Note on their possible Homologs in the Lower Vertebrata. By RAY F. COYLE. Communicated by Professor COSSAR EWART.

HUGH STEUART GLADSTONE, M.A., M.B.O.U., F.Z.S., was balloted for, and declared duly elected Fellow of the Society.

SECOND SPECIAL MEETING.

Monday, 12th July 1909.

Dr Horne, F.R.S., Vice-President, in the Chair.

The following Communications were read:—

1. A Further Contribution to a Comparative Study of the Dominant Phanerogamic and Higher Cryptogamic Flora of Aquatic Habit in Scottish Lakes. (Scottish Lake Survey). By GEORGE WEST, Esq. Communicated by Sir JOHN MURRAY, K.C.B.

2. Osteology of Antarctic Seals. (Scottish National Antarctic Expedition.) By R. B. THOMSON, M.B., Ch.B. Communicated by Dr W. S. BRUCE.

3. A Negative Attempt to detect Fluorescence Absorption. By Dr R. A. HOUSTON. Communicated by Professor A. GRAY.

4. On the Effect of Internal Friction in Cases of Compound Stress. By G. H. GULLIVER, B.Sc.

5. A New Experimental Method of Investigating certain Systems of Stress. By G. H. GULLIVER, B.Sc.

6. Motion of Neptune's Satellite. By DAVID GIBB, M.A., B.Sc. Communicated by Professor DYSON.

7. The Monsoons of the Chilian Littoral. By R. C. MOSSMAN.

8. The Superadjugate Determinant and Skew Determinants having a Univariial Diagonal. By Dr THOMAS MUIR.

LAWS OF THE SOCIETY,

As revised 26th October 1908.

[By the Charter of the Society (printed in the *Transactions*, vol. vi. p. 5), the Laws cannot be altered, except at a Meeting held one month after that at which the Motion for alteration shall have been proposed.]

I.

THE ROYAL SOCIETY OF EDINBURGH shall consist of Ordinary and Honorary Fellows. Title.

II.

Every Ordinary Fellow, within three months after his election, shall pay Two Guineas as the fee of admission, and Three Guineas as his contribution for the Session in which he has been elected; and annually at the commencement of every Session, Three Guineas into the hands of the Treasurer. This annual contribution shall continue for ten years after his admission, and it shall be limited to Two Guineas for fifteen years thereafter.* Fellows may compound for these contributions on such terms as the Council may from time to time fix. The fees of Ordinary Fellows residing in Scotland.

III.

All Fellows who shall have paid Twenty-five years' annual contribution shall be exempted from further payment. Payment to cease after 25 years.

IV.

The fees of admission of an Ordinary Non-Resident Fellow shall be £26, 5s., payable on his admission; and in case of any Non-Resident Fellow coming to reside at any time in Scotland, he shall, during each year of his residence, pay the usual annual contribution of £3, 3s., payable by each Resident Fellow; but after payment of such annual contribution for eight years, he shall be exempt from any further payment. In the case of any Resident Fellow ceasing to reside in Scotland, and wishing to continue a Fellow of the Society, it shall be in the power of the Council to determine on what terms, in the circumstances of each case, the privilege of remaining a Fellow of the Society shall be continued to such Fellow while out of Scotland. Fees of Non-Resident Ordinary Fellows. Case of Fellows becoming Non-Resident.

* A modification of this rule, in certain cases, was agreed to at a Meeting of the Society held on the 3rd January 1831.

At the Meeting of the Society, on the 5th January 1857, when the reduction of the Contributions from £3, 3s. to £2, 2s., from the 11th to the 25th year of membership, was adopted, it was resolved that the existing Members shall share in this reduction, so far as regards their future annual Contributions.

V.

Defaulters. Members failing to pay their contributions for three successive years (due application having been made to them by the Treasurer) shall be reported to the Council, and, if they see fit, shall be declared from that period to be no longer Fellows, and the legal means for recovering such arrears shall be employed.

VI.

Privileges of Ordinary Fellows. None but Ordinary Fellows shall bear any office in the Society, or vote in the choice of Fellows or Office-Bearers, or interfere in the patrimonial interests of the Society.

VII.

Numbers unlimited. The number of Ordinary Fellows shall be unlimited.

VIII.

Fellows entitled to Transactions and Proceedings. All Ordinary Fellows of the Society who are not in arrear of their Annual Contributions shall be entitled to receive, gratis, copies of the parts of the Transactions of the Society which shall be published subsequent to their admission, upon application, either personally or by an authorised agent, to the Librarian, provided they apply for them within five years of the date of publication of such parts.

Copies of the parts of the Proceedings shall be distributed to all Fellows of the Society, by post or otherwise, as soon as may be convenient after publication.

IX.

Mode of Recommending Ordinary Fellows. Candidates for admission as Ordinary Fellows shall make an application in writing, and shall produce along with it a certificate of recommendation to the purport below,* signed by at least *four* Ordinary Fellows, two of whom shall certify their recommendation from personal knowledge. This recommendation shall be delivered to the Secretary, and by him laid before the Council, and shall be exhibited publicly in the Society's rooms for one month, after which it shall be considered by the Council. If the Candidate be approved by the Council, notice of the day fixed for the election shall be given in the circulars of at least two Ordinary Meetings of the Society.

X.

Honorary Fellows, British and Foreign. Honorary Fellows shall not be subject to any contribution. This class shall consist of persons eminently distinguished for science or literature. Its number shall not exceed Fifty-six, of whom Twenty may be British subjects, and Thirty-six may be subjects of foreign states.

* "A. B., a gentleman well versed in science (*or Polite Literature, as the case may be*), being "to our knowledge desirous of becoming a Fellow of the Royal Society of Edinburgh, we hereby "recommend him as deserving of that honour, and as likely to prove a useful and valuable "Member."

XI.

Personages of Royal Blood may be elected Honorary Fellows, without regard to Royal Personages. the limitation of numbers specified in Law X.

XII.

Honorary Fellows may be proposed by the Council, or by a recommendation (in Recommendation of Honorary Fellows. the form given below*) subscribed by three Ordinary Fellows; and in case the Council shall decline to bring this recommendation before the Society, it shall be competent for the proposers to bring the same before a General Meeting. The election shall be by ballot, after the proposal has been communicated *viva voce* from Mode of election. the Chair at one Meeting, and printed in the circulars for Two Ordinary Meetings of the Society, previous to the day of election.

XIII.

The election of Ordinary Fellows shall take place only at one Afternoon Ordinary Election of Ordinary Fellows. Meeting of each month during the Session. The election shall be by ballot, and shall be determined by a majority of at least two-thirds of the votes, provided Twenty-four Fellows be present and vote.

XIV.

The Ordinary Meetings shall be held on the first and third Mondays of each Ordinary Meetings. month from November to March, and from May to July, inclusive; with the exception that when there are five Mondays in January, the Meetings for that month shall be held on its second and fourth Mondays. Regular Minutes shall be kept of the proceedings, and the Secretaries shall do the duty alternately, or according to such agreement as they may find it convenient to make.

XV.

The Society shall from time to time publish its Transactions and Proceedings. The Transactions. For this purpose the Council shall select and arrange the papers which they shall deem it expedient to publish in the Transactions of the Society, and shall superintend the printing of the same.

XVI.

The Transactions shall be published in parts or *Fasciculi* at the close of each How Published. Session, and the expense shall be defrayed by the Society.

* We hereby recommend _____
for the distinction of being made an Honorary Fellow of this Society, declaring that each of us from our own knowledge of his services to (*Literature or Science, as the case may be*) believe him to be worthy of that honour.

(To be signed by three Ordinary Fellows.)

To the President and Council of the Royal Society
of Edinburgh.

XVII.

The Council.

That there shall be formed a Council, consisting—First, of such gentlemen as may have filled the office of President ; and Secondly, of the following to be annually elected, viz. :—a President, Six Vice-Presidents (two at least of whom shall be Resident), Twelve Ordinary Fellows as Councillors, a General Secretary, Two Secretaries to the Ordinary Meetings, a Treasurer, and a Curator of the Museum and Library.

The Council shall have power to regulate the private business of the Society. At any Meeting of the Council the Chairman shall have a casting as well as a deliberative vote.

XVIII.

Retiring
Councillors.

Four Councillors shall go out annually, to be taken according to the order in which they stand on the list of the Council.

XIX.

Election of
Office-Bearers

An Extraordinary Meeting for the election of Office-Bearers shall be held annually on the fourth Monday of October, or on such other lawful day in October as the Council may fix, and each Session of the Society shall be held to begin at the date of the said Extraordinary Meeting.

XX.

Special
Meetings ; how
called.

Special Meetings of the Society may be called by the Secretary, by direction of the Council ; or on a requisition signed by six or more Ordinary Fellows. Notice of not less than two days must be given of such Meetings.

XXI.

Treasurer's
Duties.

The Treasurer shall receive and disburse the money belonging to the Society, granting the necessary receipts, and collecting the money when due.

He shall keep regular accounts of all the cash received and expended, which shall be made up and balanced annually ; and at the Extraordinary Meeting in October, he shall present the accounts for the preceding year, duly audited. At this Meeting, the Treasurer shall also lay before the Council a list of all arrears due above two years, and the Council shall thereupon give such directions as they may deem necessary for recovery thereof.

XXII.

Auditor.

At the Extraordinary Meeting in October, a professional accountant shall be chosen to audit the Treasurer's accounts for that year, and to give the necessary discharge of his intrusions.

XXIII.

General
Secretary's
Duties.

The General Secretary shall keep Minutes of the Extraordinary Meetings of the Society, and of the Meetings of the Council, in two distinct books. He shall, under the direction of the Council, conduct the correspondence of the Society, and superintend its publications. For these purposes he shall, when necessary, employ a clerk, to be paid by the Society.

XXIV.

The Secretaries to the Ordinary Meetings shall keep a regular Minute-book, in which a full account of the proceedings of these Meetings shall be entered; they shall specify all the Donations received, and furnish a list of them, and of the Donors' names, to the Curator of the Library and Museum; they shall likewise furnish the Treasurer with notes of all admissions of Ordinary Fellows. They shall assist the General Secretary in superintending the publications, and in his absence shall take his duty.

XXV.

The Curator of the Museum and Library shall have the custody and charge of all the Books, Manuscripts, objects of Natural History, Scientific Productions, and other articles of a similar description belonging to the Society; he shall take an account of these when received, and keep a regular catalogue of the whole, which shall lie in the hall, for the inspection of the Fellows.

XXVI.

All articles of the above description shall be open to the inspection of the Fellows at the Hall of the Society, at such times and under such regulations as the Council from time to time shall appoint.

XXVII.

A Register shall be kept, in which the names of the Fellows shall be enrolled at their admission, with the date.

XXVIII.

If, in the opinion of the Council of the Society, the conduct of any Fellow is unbecoming the position of a Member of a learned Society, or is injurious to the character and interests of this Society, the Council may request such Fellow to resign; and, if he fail to do so within one month of such request being addressed to him, the Council shall call a General Meeting of the Fellows of the Society to consider the matter; and, if a majority of the Fellows present at such Meeting agree to the expulsion of such Member, he shall be then and there expelled by the declaration of the Chairman of the said Meeting to that effect; and he shall thereafter cease to be a Fellow of the Society, and his name shall be erased from the Roll of Fellows, and he shall forfeit all right or claim in or to the property of the Society.

THE KEITH, MAKDOUGALL-BRISBANE, NEILL, AND
GUNNING VICTORIA JUBILEE PRIZES.

The above Prizes will be awarded by the Council in the following manner:—

I. KEITH PRIZE.

The KEITH PRIZE, consisting of a Gold Medal and from £40 to £50 in Money, will be awarded in the Session 1909–1910 for the “best communication on a scientific subject, communicated,* in the first instance, to the Royal Society during the Sessions 1907–1908 and 1908–1909.” Preference will be given to a paper containing a discovery.

II. MAKDOUGALL-BRISBANE PRIZE.

This Prize is to be awarded biennially by the Council of the Royal Society of Edinburgh to such person, for such purposes, for such objects, and in such manner as shall appear to them the most conducive to the promotion of the interests of science; with the *proviso* that the Council shall not be compelled to award the Prize unless there shall be some individual engaged in scientific pursuit, or some paper written on a scientific subject, or some discovery in science made during the biennial period, of sufficient merit or importance in the opinion of the Council to be entitled to the Prize.

1. The Prize, consisting of a Gold Medal and a sum of Money, will be awarded at the commencement of the Session 1910–1911, for an Essay or Paper having reference to any branch of scientific inquiry, whether Material or Mental.

2. Competing Essays to be addressed to the Secretary of the Society, and transmitted not later than 8th July 1910.

3. The Competition is open to all men of science.

4. The Essays may be either anonymous or otherwise. In the former case, they must be distinguished by mottoes, with corresponding sealed billets, superscribed with the same motto, and containing the name of the Author.

5. The Council impose no restriction as to the length of the Essays, which may be, at the discretion of the Council, read at the Ordinary Meetings of the Society.

* For the purposes of this award the word “communicated” shall be understood to mean the date on which the manuscript of a paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

They wish also to leave the property and free disposal of the manuscripts to the Authors; a copy, however, being deposited in the Archives of the Society, unless the paper shall be published in the Transactions.

6. In awarding the Prize, the Council will also take into consideration any scientific papers presented * to the Society during the Sessions 1908-09, 1909-10, whether they may have been given in with a view to the prize or not.

III. NEILL PRIZE.

The Council of the Royal Society of Edinburgh having received the bequest of the late Dr PATRICK NEILL of the sum of £500, for the purpose of "the interest thereof being applied in furnishing a Medal or other reward every second or third year to any distinguished Scottish Naturalist, according as such Medal or reward shall be voted by the Council of the said Society," hereby intimate :

1. The NEILL PRIZE, consisting of a Gold Medal and a sum of Money, will be awarded during the Session 1909-1910.

2. The Prize will be given for a Paper of distinguished merit, on a subject of Natural History, by a Scottish Naturalist, which shall have been presented * to the Society during the two years preceding the 23rd October 1909,—or failing presentation of a paper sufficiently meritorious, it will be awarded for a work or publication by some distinguished Scottish Naturalist, on some branch of Natural History, bearing date within five years of the time of award.

IV. GUNNING VICTORIA JUBILEE PRIZE.

This Prize, founded in the year 1887 by Dr R. H. GUNNING, is to be awarded quadrennially by the Council of the Royal Society of Edinburgh, in recognition of original work in Physics, Chemistry, or Pure or Applied Mathematics.

Evidence of such work may be afforded either by a Paper presented to the Society, or by a Paper on one of the above subjects, or some discovery in them elsewhere communicated or made, which the Council may consider to be deserving of the Prize.

The Prize consists of a sum of money, and is open to men of science resident in or connected with Scotland. The first award was made in the year 1887.

In accordance with the wish of the Donor, the Council of the Society may on fit occasions award the Prize for work of a definite kind to be undertaken during the three succeeding years by a scientific man of recognised ability.

* For the purposes of this award the word "presented" shall be understood to mean the date on which the manuscript of a paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

AWARDS OF THE KEITH, MAKDOUGALL - BRISBANE,
NEILL, AND GUNNING VICTORIA JUBILEE PRIZES.

I. KEITH PRIZE.

- 1ST BIENNIAL PERIOD, 1827-29.—Dr BREWSTER, for his papers “on his Discovery of Two New Immiscible Fluids in the Cavities of certain Minerals,” published in the Transactions of the Society.
- 2ND BIENNIAL PERIOD, 1829-31.—Dr BREWSTER, for his paper “on a New Analysis of Solar Light,” published in the Transactions of the Society.
- 3RD BIENNIAL PERIOD, 1831-33.—THOMAS GRAHAM, Esq., for his paper “on the Law of the Diffusion of Gases,” published in the Transactions of the Society.
- 4TH BIENNIAL PERIOD, 1833-35.—Professor J. D. FORBES, for his paper “on the Refraction and Polarization of Heat,” published in the Transactions of the Society.
- 5TH BIENNIAL PERIOD, 1835-37.—JOHN SCOTT RUSSELL, Esq., for his researches “on Hydrodynamics,” published in the Transactions of the Society.
- 6TH BIENNIAL PERIOD, 1837-39.—Mr JOHN SHAW, for his experiments “on the Development and Growth of the Salmon,” published in the Transactions of the Society.
- 7TH BIENNIAL PERIOD, 1839-41.—Not awarded.
- 8TH BIENNIAL PERIOD, 1841-1843.—Professor JAMES DAVID FORBES, for his papers “on Glaciers,” published in the Proceedings of the Society.
- 9TH BIENNIAL PERIOD, 1843-45.—Not awarded.
- 10TH BIENNIAL PERIOD, 1845-47.—General Sir THOMAS BRISBANE, Bart., for the Makerstoun Observations on Magnetic Phenomena, made at his expense, and published in the Transactions of the Society.
- 11TH BIENNIAL PERIOD, 1847-49.—Not awarded.
- 12TH BIENNIAL PERIOD, 1849-51.—Professor KELLAND, for his papers “on General Differentiation, including his more recent Communication on a process of the Differential Calculus, and its application to the solution of certain Differential Equations,” published in the Transactions of the Society.
- 13TH BIENNIAL PERIOD, 1851-53.—W. J. MACQUORN RANKINE, Esq., for his series of papers “on the Mechanical Action of Heat,” published in the Transactions of the Society.
- 14TH BIENNIAL PERIOD, 1853-55.—Dr THOMAS ANDERSON, for his papers “on the Crystalline Constituents of Opium, and on the Products of the Destructive Distillation of Animal Substances,” published in the Transactions of the Society.
- 15TH BIENNIAL PERIOD, 1855-57.—Professor BOOLE, for his Memoir “on the Application of the Theory of Probabilities to Questions of the Combination of Testimonies and Judgments,” published in the Transactions of the Society.
- 16TH BIENNIAL PERIOD, 1857-59.—Not awarded.
- 17TH BIENNIAL PERIOD, 1859-61.—JOHN ALLAN BROWN, Esq., F.R.S., Director of the Trevandrum Observatory, for his papers “on the Horizontal Force of the Earth’s Magnetism, on the Correction of the Bifilar Magnetometer, and on Terrestrial Magnetism generally,” published in the Transactions of the Society.
- 18TH BIENNIAL PERIOD, 1861-63.—Professor WILLIAM THOMSON, of the University of Glasgow, for his Communication “on some Kinematical and Dynamical Theorems.”
- 19TH BIENNIAL PERIOD, 1863-65.—Principal FORBES, St Andrews, for his “Experimental Inquiry into the Laws of Conduction of Heat in Iron Bars,” published in the Transactions of the Society.

- 20TH BIENNIAL PERIOD, 1865-67.—Professor C. PIAZZI SMYTH, for his paper “on Recent Measures at the Great Pyramid,” published in the Transactions of the Society.
- 21ST BIENNIAL PERIOD, 1867-69.—Professor P. G. TAIT, for his paper “on the Rotation of a Rigid Body about a Fixed Point,” published in the Transactions of the Society.
- 22ND BIENNIAL PERIOD, 1869-71.—Professor CLERK MAXWELL, for his paper “on Figures, Frames, and Diagrams of Forces,” published in the Transactions of the Society.
- 23RD BIENNIAL PERIOD, 1871-73.—Professor P. G. TAIT, for his paper entitled “First Approximation to a Thermo-electric Diagram,” published in the Transactions of the Society.
- 24TH BIENNIAL PERIOD, 1873-75.—Professor CRUM BROWN, for his Researches “on the Sense of Rotation, and on the Anatomical Relations of the Semicircular Canals of the Internal Ear.”
- 25TH BIENNIAL PERIOD, 1875-77.—Professor M. FORSTER HEDDLE, for his papers “on the Rhombohedral Carbonates,” and “on the Felspars of Scotland,” published in the Transactions of the Society.
- 26TH BIENNIAL PERIOD, 1877-79.—Professor H. C. FLEEMING JENKIN, for his paper “on the Application of Graphic Methods to the Determination of the Efficiency of Machinery,” published in the Transactions of the Society; Part II. having appeared in the volume for 1877-78.
- 27TH BIENNIAL PERIOD, 1879-81.—Professor GEORGE CHRYSAL, for his paper “on the Differential Telephone,” published in the Transactions of the Society.
- 28TH BIENNIAL PERIOD, 1881-83.—THOMAS MUIR, Esq., LL.D., for his “Researches into the Theory of Determinants and Continued Fractions,” published in the Proceedings of the Society.
- 29TH BIENNIAL PERIOD, 1883-85.—JOHN AITKEN, Esq., for his paper “on the Formation of Small Clear Spaces in Dusty Air,” and for previous papers on Atmospheric Phenomena, published in the Transactions of the Society.
- 30TH BIENNIAL PERIOD, 1885-87.—JOHN YOUNG BUCHANAN, Esq., for a series of communications, extending over several years, on subjects connected with Ocean Circulation, Compressibility of Glass, etc.; two of which, viz., “On Ice and Brines,” and “On the Distribution of Temperature in the Antarctic Ocean,” have been published in the Proceedings of the Society.
- 31ST BIENNIAL PERIOD, 1887-89.—Professor E. A. LETTS, for his papers on the Organic Compounds of Phosphorus, published in the Transactions of the Society.
- 32ND BIENNIAL PERIOD, 1889-91.—R. T. OMOND, Esq., for his contributions to Meteorological Science, many of which are contained in vol. xxxiv. of the Society's Transactions.
- 33RD BIENNIAL PERIOD, 1891-93.—Professor THOMAS R. FRASER, F.R.S., for his papers on *Strophanthus hispidus*, Strophanthin, and Strophanthidin, read to the Society in February and June 1889 and in December 1891, and printed in vols. xxxv., xxxvi., and xxxvii. of the Society's Transactions.
- 34TH BIENNIAL PERIOD, 1893-95.—Dr CARGILL G. KNOTT, for his papers on the Strains produced by Magnetism in Iron and in Nickel, which have appeared in the Transactions and Proceedings of the Society.
- 35TH BIENNIAL PERIOD, 1895-97.—Dr THOMAS MUIR, for his continued communications on Determinants and Allied Questions.
- 36TH BIENNIAL PERIOD, 1897-99.—Dr JAMES BURGESS, for his paper “on the Definite Integral $\frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt$, with extended Tables of Values,” printed in vol. xxxix. of the Transactions of the Society.
- 37TH BIENNIAL PERIOD, 1899-1901.—Dr HUGH MARSHALL, for his discovery of the Persulphates, and for his Communications on the Properties and Reactions of these Salts, published in the Proceedings of the Society.
- 38TH BIENNIAL PERIOD, 1901-03.—Sir WILLIAM TURNER, K.C.B., LL.D., F.R.S., &c., for his memoirs entitled “A Contribution to the Craniology of the People of Scotland,” published in the Transactions of the Society, and for his “Contributions to the Craniology of the People of the Empire of India,” Parts I., II., likewise published in the Transactions of the Society.
- 39TH BIENNIAL PERIOD, 1903-05.—THOMAS H. BRYCE, M.A., M.D., for his two papers on “The Histology of the Blood of the Larva of *Lepidosiren paradoxa*,” published in the Transactions of the Society within the period.
- 40TH BIENNIAL PERIOD, 1905-07.—ALEXANDER BRUCE, M.A., M.D., F.R.C.P.E., for his paper entitled “Distribution of the Cells in the Intermedio-Lateral Tract of the Spinal Cord,” published in the Transactions of the Society within the period.

II. MAKDOUGALL-BRISBANE PRIZE.

- 1ST BIENNIAL PERIOD, 1859.—Sir RODERICK IMPEY MURCHISON, on account of his Contributions to the Geology of Scotland.
- 2ND BIENNIAL PERIOD, 1860–62.—WILLIAM SELLER, M.D., F.R.C.P.E., for his “Memoir of the Life and Writings of Dr Robert Whytt,” published in the Transactions of the Society.
- 3RD BIENNIAL PERIOD, 1862–64.—JOHN DENIS MACDONALD, Esq., R.N., F.R.S., Surgeon of H.M.S. “Icarus,” for his paper “on the Representative Relationships of the Fixed and Free Tunicata, regarded as Two Sub-classes of equivalent value; with some General Remarks on their Morphology,” published in the Transactions of the Society.
- 4TH BIENNIAL PERIOD, 1864–66.—Not awarded.
- 5TH BIENNIAL PERIOD, 1866–68.—Dr ALEXANDER CRUM BROWN and Dr THOMAS RICHARD FRASER, for their conjoint paper “on the Connection between Chemical Constitution and Physiological Action,” published in the Transactions of the Society.
- 6TH BIENNIAL PERIOD, 1868–70.—Not awarded.
- 7TH BIENNIAL PERIOD, 1870–72.—GEORGE JAMES ALLMAN, M.D., F.R.S., Emeritus Professor of Natural History, for his paper “on the Homological Relations of the Cœlenterata,” published in the Transactions, which forms a leading chapter of his Monograph of Gymnoblasic or Tubularian Hydroids—since published.
- 8TH BIENNIAL PERIOD, 1872–74.—Professor LISTER, for his paper “on the Germ Theory of Putrefaction and the Fermentive Changes,” communicated to the Society, 7th April 1873.
- 9TH BIENNIAL PERIOD, 1874–76.—ALEXANDER BUCHAN, A.M., for his paper “on the Diurnal Oscillation of the Barometer,” published in the Transactions of the Society.
- 10TH BIENNIAL PERIOD, 1876–78.—Professor ARCHIBALD GEIKIE, for his paper “on the Old Red Sandstone of Western Europe,” published in the Transactions of the Society.
- 11TH BIENNIAL PERIOD, 1878–80.—Professor PIAZZI SMYTH, Astronomer-Royal for Scotland, for his paper “on the Solar Spectrum in 1877–78, with some Practical Idea of its probable Temperature of Origination,” published in the Transactions of the Society.
- 12TH BIENNIAL PERIOD, 1880–82.—Professor JAMES GEIKIE, for his “Contributions to the Geology of the North-West of Europe,” including his paper “on the Geology of the Faroes,” published in the Transactions of the Society.
- 13TH BIENNIAL PERIOD, 1882–84.—EDWARD SANG, Esq., LL.D., for his paper “on the Need of Decimal Subdivisions in Astronomy and Navigation, and on Tables requisite therefor,” and generally for his Recalculation of Logarithms both of Numbers and Trigonometrical Ratios, —the former communication being published in the Proceedings of the Society.
- 14TH BIENNIAL PERIOD, 1884–86.—JOHN MURRAY, Esq., LL.D., for his papers “On the Drainage Areas of Continents, and Ocean Deposits,” “The Rainfall of the Globe, and Discharge of Rivers,” “The Height of the Land and Depth of the Ocean,” and “The Distribution of Temperature in the Scottish Lochs as affected by the Wind.”
- 15TH BIENNIAL PERIOD, 1886–88.—ARCHIBALD GEIKIE, Esq., LL.D., for numerous Communications, especially that entitled “History of Volcanic Action during the Tertiary Period in the British Isles,” published in the Transactions of the Society.
- 16TH BIENNIAL PERIOD, 1889–90.—Dr LUDWIG BECKER, for his paper on “The Solar Spectrum at Medium and Low Altitudes,” printed in vol. xxxvi. Part I. of the Society’s Transactions.
- 17TH BIENNIAL PERIOD, 1890–92.—HUGH ROBERT MILL, Esq., D.Sc., for his papers on “The Physical Conditions of the Clyde Sea Area,” Part I. being already published in vol. xxxvi. of the Society’s Transactions.
- 18TH BIENNIAL PERIOD, 1892–94.—Professor JAMES WALKER, D.Sc., Ph.D., for his work on Physical Chemistry, part of which has been published in the Proceedings of the Society, vol. xx. pp. 255–263. In making this award, the Council took into consideration the work done by Professor Walker along with Professor Crum Brown on the Electrolytic Synthesis of Dibasic Acids, published in the Transactions of the Society.
- 19TH BIENNIAL PERIOD, 1894–96.—Professor JOHN G. M’KENDRICK, for numerous Physiological papers, especially in connection with Sound, many of which have appeared in the Society’s publications.
- 20TH BIENNIAL PERIOD, 1896–98.—Dr WILLIAM PEDDIE, for his papers on the Torsional Rigidity of Wires.
- 21ST BIENNIAL PERIOD, 1898–1900.—Dr RAMSAY H. TRAQUAIR, for his paper entitled “Report on Fossil Fishes collected by the Geological Survey in the Upper Silurian Rocks of Scotland,” printed in vol. xxxix. of the Transactions of the Society.

- 22ND BIENNIAL PERIOD, 1900-02.—Dr ARTHUR T. MASTERMAN, for his paper entitled “The Early Development of *Cribrella oculata* (Forbes), with remarks on Echinoderm Development,” printed in vol. xl. of the Transactions of the Society.
- 23RD BIENNIAL PERIOD, 1902-04.—Mr JOHN DOUGALL, M.A., for his paper on “An Analytical Theory of the Equilibrium of an Isotropic Elastic Plate,” published in vol. xli. of the Transactions of the Society.
- 24TH BIENNIAL PERIOD, 1904-06.—JACOB E. HALM, Ph.D., for his two papers entitled “Spectroscopic Observations of the Rotation of the Sun,” and “Some Further Results obtained with the Spectroheliometer,” and for other astronomical and mathematical papers published in the Transactions and Proceedings of the Society within the period.
- 25TH BIENNIAL PERIOD, 1906-08.—D. T. GWYNNE-VAUGHAN, M.A., F.L.S., for his papers, 1st, “On the Fossil Osmundaceæ,” and 2nd, “On the Origin of the Adaxially-curved Leaf-trace in the Filicales,” communicated by him conjointly with Dr R. Kidston; the committee having read a letter from Dr Kidston explaining their joint authorship.

III. THE NEILL PRIZE.

- 1ST TRIENNIAL PERIOD, 1856-59.—Dr W. LAUDER LINDSAY, for his paper “on the Spermogones and Pycnides of Filamentous, Fruticulose, and Foliaceous Lichens,” published in the Transactions of the Society.
- 2ND TRIENNIAL PERIOD, 1859-61.—ROBERT KAYE GREVILLE, LL.D., for his Contributions to Scottish Natural History, more especially in the department of Cryptogamic Botany, including his recent papers on Diatomaceæ.
- 3RD TRIENNIAL PERIOD, 1862-65.—ANDREW CROMBIE RAMSAY, F.R.S., Professor of Geology in the Government School of Mines, and Local Director of the Geological Survey of Great Britain, for his various works and memoirs published during the last five years, in which he has applied the large experience acquired by him in the Direction of the arduous work of the Geological Survey of Great Britain to the elucidation of important questions bearing on Geological Science.
- 4TH TRIENNIAL PERIOD, 1865-68.—Dr WILLIAM CARMICHAEL M'INTOSH, for his paper “on the Structure of the British Nemerteans, and on some New British Annelids,” published in the Transactions of the Society.
- 5TH TRIENNIAL PERIOD, 1868-71.—Professor WILLIAM TURNER, for his papers “on the Great Finner Whale; and on the Gravid Uterus, and the Arrangement of the Foetal Membranes in the Cetacea,” published in the Transactions of the Society.
- 6TH TRIENNIAL PERIOD, 1871-74.—CHARLES WILLIAM PEACH, Esq., for his Contributions to Scottish Zoology and Geology, and for his recent contributions to Fossil Botany.
- 7TH TRIENNIAL PERIOD, 1874-77.—Dr RAMSAY H. TRAQUAIR, for his paper “on the Structure and Affinities of *Tristichopterus alatus* (Egerton),” published in the Transactions of the Society, and also for his contributions to the Knowledge of the Structure of Recent and Fossil Fishes.
- 8TH TRIENNIAL PERIOD, 1877-80.—JOHN MURRAY, Esq., for his paper “on the Structure and Origin of Coral Reefs and Islands,” published (in abstract) in the Proceedings of the Society.
- 9TH TRIENNIAL PERIOD, 1880-83.—Professor HERDMAN, for his papers “on the Tunicata,” published in the Proceedings and Transactions of the Society.
- 10TH TRIENNIAL PERIOD, 1883-86.—B. N. PEACH, Esq., for his Contributions to the Geology and Palæontology of Scotland, published in the Transactions of the Society.
- 11TH TRIENNIAL PERIOD, 1886-89.—ROBERT KIDSTON, Esq., for his Researches in Fossil Botany, published in the Transactions of the Society.
- 12TH TRIENNIAL PERIOD, 1889-92.—JOHN HORNE, Esq., F.G.S., for his Investigations into the Geological Structure and Petrology of the North-West Highlands.
- 13TH TRIENNIAL PERIOD, 1892-95.—ROBERT IRVINE, Esq., for his papers on the Action of Organisms in the Secretion of Carbonate of Lime and Silica, and on the solution of these substances in Organic Juices. These are printed in the Society's Transactions and Proceedings.
- 14TH TRIENNIAL PERIOD, 1895-98.—Professor COSSAR EWART, for his recent Investigations connected with Telegony.
- 15TH TRIENNIAL PERIOD, 1898-1901.—Dr JOHN S. FLETT, for his papers entitled “The Old Red Sandstone of the Orkneys” and “The Trap Dykes of the Orkneys,” printed in vol. xxxix. of the Transactions of the Society.

- 16TH TRIENNIAL PERIOD, 1901-04.—Professor J. GRAHAM KERR, M.A., for his Researches on *Lepidosiren paradoxa*, published in the Philosophical Transactions of the Royal Society, London.
- 17TH TRIENNIAL PERIOD, 1904-07.—FRANK J. COLE, B.Sc., for his paper entitled “A Monograph on the General Morphology of the Myxinoid Fishes, based on a study of Myxine,” published in the Transactions of the Society, regard being also paid to Mr Cole’s other valuable contributions to the Anatomy and Morphology of Fishes.

IV. GUNNING VICTORIA JUBILEE PRIZE.

- 1ST TRIENNIAL PERIOD, 1884-87.—Sir WILLIAM THOMSON, Pres. R.S.E., F.R.S., for a remarkable series of papers “on Hydrokinetics,” especially on Waves and Vortices, which have been communicated to the Society.
- 2ND TRIENNIAL PERIOD, 1887-90.—Professor P. G. TAIT, Sec. R.S.E., for his work in connection with the “Challenger” Expedition, and his other Researches in Physical Science.
- 3RD TRIENNIAL PERIOD, 1890-93.—ALEXANDER BUCHAN, Esq., LL.D., for his varied, extensive, and extremely important Contributions to Meteorology, many of which have appeared in the Society’s Publications.
- 4TH TRIENNIAL PERIOD, 1893-96.—JOHN AITKEN, Esq., for his brilliant Investigations in Physics, especially in connection with the Formation and Condensation of Aqueous Vapour.
- 1ST QUADRENNIAL PERIOD, 1896-1900.—Dr T. D. ANDERSON, for his discoveries of New and Variable Stars.
- 2ND QUADRENNIAL PERIOD, 1900-04.—Sir JAMES DEWAR, LL.D., D.C.L., F.R.S., etc., for his researches on the Liquefaction of Gases, extending over the last quarter of a century, and on the Chemical and Physical Properties of Substances at Low Temperatures: his earliest papers being published in the Transactions and Proceedings of the Society.
- 3RD QUADRENNIAL PERIOD, 1904-08.—Professor GEORGE CHRYSAL, M.A., LL.D., for a series of papers on “Seiches,” including “The Hydrodynamical Theory and Experimental Investigations of the Seiche Phenomena of Certain Scottish Lakes.”

THE COUNCIL OF THE SOCIETY,

October 1909.

PRESIDENT.

SIR WILLIAM TURNER, K.C.B., M.B., F.R.C.S.E., LL.D., D.C.L., D.Sc.Dub., F.R.S.,
Principal of the University of Edinburgh.

VICE-PRESIDENTS.

RAMSAY H. TRAQUAIR, M.D., LL.D., F.R.S., F.G.S., late Keeper of the Natural History
Collections in the Royal Scottish Museum, Edinburgh.
ALEXANDER CRUM BROWN, M.D., D.Sc., F.R.C.P.E., LL.D., F.R.S., Emeritus Professor of
Chemistry in the University of Edinburgh.
JAMES COSSAR EWART, M.D., F.R.C.S.E., F.R.S., F.L.S., Regius Professor of Natural
History in the University of Edinburgh.
JOHN HORNE, LL.D., F.R.S., F.G.S., Director of the Geological Survey of Scotland.
JAMES BURGESS, C.I.E., LL.D., M.R.A.S.
T. HUDSON BEARE, M.Inst.C.E., Professor of Engineering in the University of Edinburgh.

GENERAL SECRETARY.

GEORGE CHRYSTAL, M.A., LL.D., Professor of Mathematics in the University of Edinburgh.

SECRETARIES TO ORDINARY MEETINGS.

CARGILL G. KNOTT, D.Sc., Lecturer on Applied Mathematics in the University of Edinburgh.
ROBERT KIDSTON, LL.D., F.R.S., F.G.S.

TREASURER.

JAMES CURRIE, M.A.

CURATOR OF LIBRARY AND MUSEUM.

JOHN SUTHERLAND BLACK, M.A., LL.D.

COUNCILLORS.

FRANK WATSON DYSON, M.A., F.R.S., Astronomer-Royal for Scotland, and Pro- fessor of Astronomy in the University of Edinburgh.	A. P. LAURIE, M.A., D.Sc., Principal of the Heriot-Watt College, Edinburgh.
D'ARCY W. THOMPSON, C.B., B.A., F.L.S., Professor of Natural History in University College, Dundee.	WM. PEDDIE, D.Sc., Professor of Natural Philosophy in University College, Dundee.
O. CHARNOCK BRADLEY, M.D., D.Sc.	HECTOR MUNRO MACDONALD, M.A., F.R.S., Professor of Mathematics in the University of Aberdeen.
CHARLES TWEEDIE, M.A., B.Sc., Lecturer on Mathematics in the University of Edinburgh.	D. NOËL PATON, M.D., B.Sc., F.R.C.P.E., Professor of Physiology in the University of Glasgow.
JOHN WALTER GREGORY, D.Sc., F.R.S., Professor of Geology in the University of Glasgow.	WILLIAM S. BRUCE, LL.D.
	F. G. BAILY, M.A., Professor of Applied Physics, Heriot Watt College, Edinburgh.
	J. G. BARTHOLOMEW, LL.D., F.R.G.S.

ALPHABETICAL LIST OF THE ORDINARY FELLOWS
OF THE SOCIETY,

Corrected to October 1909.

N. B.—*Those marked * are Annual Contributors.*

B. prefixed to a name indicates that the Fellow has received a Makdougall-Brisbane Medal.

K. " " " Keith Medal.

N. " " " Neill Medal.

V. J. " " " the Gunning Victoria Jubilee Prize.

C. " " " contributed one or more Communications to the
Society's TRANSACTIONS or PROCEEDINGS.

Date of Election.			
1898	C.	* Abercromby, The Hon. John, 62 Palmerston Place	
1898		Adami, Prof. J. G., M.A., M.D. Cantab., F.R.S., Professor of Pathology in M'Gill University, Montreal	
1896		* Affleck, Jas. Ormiston, M.D., LL.D., F.R.C.P.E., 38 Heriot Row	
1871		Agnew, Sir Stair, K.C.B., M.A., Registrar-General for Scotland, 22 Buckingham Terrace	
1875	C. K.	Aitken, John, LL.D., F.R.S., Ardenlea, Falkirk	5
	V. J.		
1895		* Alford, Robert Gervase, Memb. Inst. C.E., 1 Windmill Hill, Hampstead, London, N. W.	
1889		* Alison, John, M.A., Headmaster, George Watson's College, Edinburgh	
1894		Allan, Francis John, M.D., C.M. Edin., M.O.H., City of Westminster, Westminster City Hall, Charing Cross Road, London	
1888	C.	* Allardice, R. E., M.A., Professor of Mathematics in Stanford University, Palo Alto, Santa Clara Co., California	
1878		Allchin, Sir William H., M.D., F.R.C.P.L., Senior Physician to the Westminster Hospital, 5 Chandos Street, Cavendish Square, London	10
1906		Anderson, Daniel E., M.D., B.A., B.Sc., 121 Avenue des Champs Elysées, Paris, France	
1893		Anderson, J. Macvicar, Architect, 6 Stratton Street, London	
1883		Anderson, Sir Robert Rowand, LL.D., 16 Rutland Square	
1905		Anderson, William, F.G.S., 52 Lancis Buildings, Loveday Street, Johannesburg, Transvaal, South Africa	
1905		* Anderson, William, M.A., Head Science Master, George Watson's College, Edinburgh, 29 Lutton Place	15
1903		Anderson-Berry, David, M.D., C.M. Edin., F.S.A. Scot., West Brow, St Leonards-on-Sea	
1905		* Andrew, George, M.A., B.A., H.M.I.S., Glenhuntly, Hyndford Road, Lanark	
1881	C.	Anglin, A. H., M.A., LL.D., M.R.I.A., Professor of Mathematics, Queen's College, Cork	
1906		Appleton, Arthur Frederick, F.R.C.V.S., Lieut.-Colonel, Army Veterinary Department, Heworth Croft, York	
1899		Appleyard, James R., Royal Technical Institute, Salford, Manchester	20
1893		* Archer, Walter E., 17 Sloan Court, London	
1907		* Archibald, James, M.A., Headmaster, St Bernard's School, 1 Leamington Terrace, Edinburgh	
1907		* Badre, Muhammad	
1894		* Bailey, Frederick, Lieut.-Col. (<i>late</i>) R.E., 7 Drummond Place	
1896		* Baily, Francis Gibson, M.A., Professor of Applied Physics, Heriot-Watt College	25
1877	C.	Balfour, I. Bayley, M.A., Sc.D., M.D., LL.D., F.R.S., F.L.S., King's Botanist in Scotland, Professor of Botany in the University of Edinburgh and Keeper of the Royal Botanic Gardens, Inverleith House	

Alphabetical List of the Ordinary Fellows of the Society. 773

Date of Election.			
1905		Balfour-Browne, William Alexander Francis, M.A., Barrister-at-Law, Claremont, Holywood, Co. Down, Ireland	
1892		* Ballantyne, J. W., M.D., F.R.C.P.E., 24 Melville Street	
1902	C.	Bannerman, W. B., M.D., D.Sc., Lt.-Colonel, Indian Medical Service, Director, Bacteriological Laboratory, Parel, Bombay, India	
1889		* Barbour, A. H. F., M.A., M.D., F.R.C.P.E., 4 Charlotte Square	30
1886		* Barclay, A. J. Gunion, M.A., 729 Great Western Road, Glasgow	
1872		Barclay, George, M.A., 17 Coates Crescent	
1883	C.	Barclay, G. W. W., M.A., 91 Union Street, Aberdeen	
1903		Bardswell, Noël Dean, M.D., M.R.C.P. Ed. and Lond., King Edward VII. Sanatorium, Midhurst	
1882	C.	Barnes, Henry, M.D., LL.D., 6 Portland Square, Carlisle	35
1904		Barr, Sir James, M.D., F.R.C.P. Lond., 72 Rodney Street, Liverpool	
1874		Barrett, William F., F.R.S., M.R.I.A., Prof. of Physics, Royal College of Science, Dublin	
1889		Barry, T. D. Collis, M.R.C.S., F.I.C., Lt.-Colonel I.M.S., Chemical Analyser to the Government of Bombay, and Prof. of Chemistry and Medical Jurisprudence, Grant Medical College, Bombay	
1887		* Bartholomew, J. G., LL.D., F.R.G.S., The Geographical Institute, Dalkeith Road	
1895	C.	Barton, Edwin H., D.Sc., A.M.I.E.E., Memb. Phys. Soc. of London, Professor of Experimental Physics, University College, Nottingham	40
1904		* Baxter, William Muirhead, 2A Merchiston Place	
1888		* Beare, Thomas Hudson, B.Sc., Memb. Inst. C.E., Professor of Engineering in the University of Edinburgh (VICE-PRESIDENT)	
1897	C.	* Beattie, John Carruthers, D.Sc., Professor of Physics, South African College, Cape Town	
1892		Beck, J. H. Meining, M.D., M.R.C.P.E., Rondebosch, Cape Town	
1893	B. C.	* Becker, Ludwig, Ph.D., Regius Professor of Astronomy in the University of Glasgow, The Observatory, Glasgow	45
1882	C.	Beddard, Frank E., M.A. Oxon., F.R.S., Prosector to the Zoological Society of London, Zoological Society's Gardens, Regent's Park, London	
1887		* Begg, Ferdinand Faithfull, Bartholomew House, London	
1886		* Bell, A. Beatson, 17 Lansdowne Crescent	
1906		Bell, John Patrick Fair, F.Z.S., Fulforth, Witton Gilbert, Durham	
1874		Bell, Joseph, M.D., F.R.C.S.E., 2 Melville Crescent	50
1900		* Bennett, James Bower, Memb. Inst. C.E., 42 Frederick Street	
1887		* Bernard, J. Mackay, of Dunsinnan, B.Sc., Dunsinnan, Perth	
1893	C.	* Berry, George A., M.D., C.M., F.R.C.S., 31 Drumsheugh Gardens	
1897	C.	Berry, Richard J., M.D., F.R.C.S.E., Professor of Anatomy in the University of Melbourne, Victoria	
1904		* Beveridge, Erskine, LL.D., St Leonards Hill, Dunfermline	55
1880	C.	Birch, De Burgh, M.D., Professor of Physiology in the University of Leeds, 16 De Grey Terrace, Leeds	
1900		* Bisset, James, M.A., F.L.S., F.G.S., 9 Greenhill Park	
1907		* Black, Frederick Alexander, Solicitor, 59 Academy Street, Inverness	
1884		Black, John S., M.A., LL.D. (CURATOR OF LIBRARY AND MUSEUM), 6 Oxford Terrace	
1897		* Blaikie, Walter Biggar, The Loan, Colinton	60
1904	C.	* Bles, Edward J., M.A., D.Sc., The Mill House, Iffley, Oxford	
1898	C.	* Blyth, Benjamin Hall, M.A., Memb. Inst. C.E., 17 Palmerston Place	
1894		* Bolton, Herbert, F.G.S., F.L.S., Curator of the Bristol Museum, Queen's Road, Bristol	
1884		Bond, Francis T., B.A., M.D., M.R.C.S., Gloucester	
1872	C.	Bottomley, J. Thomson, M.A., D.Sc., LL.D., F.R.S., F.C.S., 13 University Gardens, Glasgow	65
1886		* Bower, Frederick O., M.A., D.Sc., F.R.S., F.L.S., Regius Professor of Botany in the University of Glasgow, 1 St John's Terrace, Hillhead, Glasgow	
1884	C.	Bowman, Frederick Hungerford, D.Sc., F.C.S. (Lond. and Berl.), F.I.C., Assoc. Inst. C.E., Assoc. Inst. M.E., M.I.E.E., &c., 4 Albert Square, Manchester	
1901		Bradbury, J. B., M.D., Downing Professor of Medicine, University of Cambridge	
1903	C.	* Bradley, O. Charnock, M.D., D.Sc., Royal Veterinary College, Edinburgh	
1886		* Bramwell, Byrom, M.D., F.R.C.P.E., 23 Drumsheugh Gardens	70
1907		* Bramwell, Edwin, M.B., F.R.C.P.E., M.R.C.P. Lond., 23 Drumsheugh Gardens	

Date of Election.			
1895		* Bright, Charles, Assoc. Memb. Inst. C.E., Memb. Inst. E.E., F.R.A.S., F.G.S., London, 26 Devonshire Terrace, Hyde Park, and Caxton House, Westminster, London, S.W.	
1893		Brock, G. Sandison, M.D., 6 Corso d'Italia, Rome, Italy	
1901	C.	* Brodie, W. Brodie, M.B., Thaxted, Essex	
1907		Brown, Alexander, M.A., B.Sc., Professor of Applied Mathematics, South African College, Cape Town	75
1864	C. K. B.	Brown, Alex. Crum, M.D., D.Sc., F.R.C.P.E., LL.D., F.R.S. (VICE-PRESIDENT), Emeritus Professor of Chemistry in the University of Edinburgh, 8 Belgrave Crescent	
1898		* Brown, David, F.C.S., F.I.C., Willowbrae House, Midlothian	
1883	C.	Brown, J. J. Graham, M.D., F.R.C.P.E., 3 Chester Street	
1885	C.	Brown, J. Macdonald, M.D., F.R.C.S., 2 Frognal, London, N.W.	
1909	C.	* Brownlee, John, M.A., M.D., D.Sc., Ruchill Hospital, Bilsland Drive, Glasgow	80
1883	K. C.	Bruce, Alexander, M.A., M.D., LL.D., F.R.C.P.E., 8 Ainslie Place	
1906		* Bruce, William Speirs, LL.D., Antarctica, Joppa, Midlothian	
1898	K. C.	* Bryce, T. H., M.A., M.D. (Edin.), 2 Granby Terrace, Glasgow	
1870	C. K.	Buchanan, John Young, M.A., F.R.S., Christ's College, Cambridge	
1902		* Buchanan, Robert M., M.B., F.F.P.S.G., Corindi, Scotstounhill, Glasgow	85
1882		Buchanan, T. R., M.A., M.P., 12 South Street, Park Lane, London, W.	
1887	C.	* Buist, J. B., M.D., F.R.C.P.E., 1 Clifton Terrace	
1905		Bunting, Thomas Lowe, M.D., Scotswood, Newcastle-on-Tyne	
1902		* Burgess, A. G., M.A., Mathematical Master, Edinburgh Ladies' College, 64 Strathearn Road	
1894	C. K.	* Burgess, James, C.I.E., LL.D., Hon. A.R.I.B.A., F.R.G.S., Hon. M. Imp. Russ. Archaeol. Soc., and Amer. Or. Soc., M. Soc. Asiat. de Paris, M.R.A.S., H. Corr. M. Batavian Soc. of Arts and Sciences, and Berlin Soc. Anthropol., H. Assoc. Finno-Ugrian Soc. (VICE-PRESIDENT), 22 Seton Place	90
1902		* Burn, The Rev. John Henry, B.D., The Parsonage, Ballater	
1887		* Burnet, John James, Architect, 18 University Avenue, Hillhead, Glasgow	
1888		* Burns, Rev. T., D.D., F.S.A. Scot., Minister of Lady Glenorchy's Parish Church, Croston Lodge, Chalmers Crescent	
1896		* Butters, J. W., M.A., B.Sc., Rector of Ardrossan Academy	
1887	C.	* Cadell, Henry Moubray, of Grange, B.Sc., Bo'ness	95
1897		* Caird, Robert, LL.D., Shipbuilder, Greenock	
1893	C.	Calderwood, W. L., Inspector of Salmon Fisheries of Scotland, South Bank, Canaan Lane, Edinburgh	
1894		* Cameron, James Angus, M.D., Medical Officer of Health, Firhall, Nairn	
1905	C.	Cameron, John, M.D., D.Sc., M.R.C.S. Eng., Anatomy Department, Middlesex Hospital Medical School, London	
1904		* Campbell, Charles Duff, 21 Montague Terrace, Inverleith Row	100
1908		* Campbell, Lt.-Colonel John, Westwood, Cupar, Fife	
1899	C.	* Carlier, Edmund W. W., M.D., B.Sc., Prof. of Physiology in Mason College, Birmingham	
1906		Carruthers, John Bennett, F.L.S., Assoc. R.B.S., Dept. of Agriculture, Port of Spain, Trinidad	
1905	C.	* Carse, George Alexander, M.A., D.Sc., Lecturer on Natural Philosophy, University of Edinburgh, 120 Lauriston Place	
1901		Carslaw, H. S., M.A., D.Sc., Professor of Mathematics in the University of Sydney, New South Wales	105
1905		Carter, Joseph Henry, F.R.C.V.S., Rowley Hall, Burnley, Lancashire	
1898		* Carter, Wm. Allan, Memb. Inst. C.E., 32 Great King Street	
1898		Carus-Wilson, Cecil, F.R.G.S., F.G.S., 16 Waldegrave Park, Strawberry Hill, Middlesex	
1908		Cavanagh, Thomas Francis, M.D., 396 Eccleshall Road, Sheffield	
1882		Cay, W. Dyce, Memb. Inst. C.E., 39 Victoria Street, Westminster, London	110
1890		Charles, John J., M.A., M.D., C.M., late Prof. of Anatomy and Physiology, Queen's College, Cork, Karlsruhe, Port Stewart, Co. Derry	
1899		Chatham, James, Actuary, 7 Belgrave Crescent	
1874		Chiene, John, C.B., M.D., LL.D., F.R.C.S.E., Professor of Surgery in the University of Edinburgh, 21 Alva Street	
1880	C. K. V. J.	Chrystal, George, M.A., LL.D., Professor of Mathematics in the University of Edinburgh (GENERAL SECRETARY), 5 Belgrave Crescent	
1891		* Clark, John B., M.A., Head Master of Heriot's Hospital School, Lauriston, Garleffin, Craiglea Drive	115
1903		* Clarke, William Eagle, F.L.S., Keeper of the Natural History Collections in the Royal Scottish Museum, Edinburgh, 35 Braid Road	

Alphabetical List of the Ordinary Fellows of the Society. 775

Date of Election.			
1909		Clayton, Thomas Morrison, M.D., D.Hy., B.Sc., D.P.H., Medical Officer of Health, Gateshead, 13 The Crescent, Gateshead-on-Tyne	
1875		Clouston, T. S., M.D., LL.D., 26 Heriot Row	
1887		* Cockburn, John, F.R.A.S., The Abbey, North Berwick	
1904	C.	Coker, Ernest George, M.A., D.Sc., Professor of Mechanical Engineering and Applied Mechanics, City and Guilds Technical College, Finsbury, Leonard Street, City Road, London, E.C.	120
1904		Coles, Alfred Charles, M.D., D.Sc., York House, Poole Road, Bournemouth, W.	
1888	C.	Collie, John Norman, Ph.D., F.R.S., F.C.S., Professor of Organic Chemistry in the University College, Gower Street, London	
1904	C.	* Colquhoun, Walter, M.A., M.B., 18 Walmer Crescent, Ibrox, Glasgow	
1909		* Comrie, Peter, M.A., B.Sc., Head Mathematical Master, Boroughmuir Junior Student Centre, 19 Craighouse Terrace	
1886		Connan, Daniel M., M.A.	125
1872		Constable, Archibald, LL.D., 11 Thistle Street	
1894		Cook, John, M.A., 30 Hermitage Gardens, Late Principal, Central College, Bangalore, Director of Meteorology in Mysore, and Fellow, University of Madras, India	
1891		* Cooper, Charles A., LL.D., 41 Drumsheugh Gardens	
1905		* Corrie, David, F.C.S., Nobel's Explosives Company, Polmont Station	
1908		Craig, James Ireland, M.A., B.A., Director of the Computation Office, Survey Department, Egypt, Mataria, Egypt	130
1875		Craig, William, M.D., F.R.C.S.E., Lecturer on Materia Medica to the College of Surgeons, 71 Bruntsfield Place	
1907		* Cramer, William, Ph.D., Lecturer in Physiological Chemistry in the University of Edinburgh, Physiological Department, The University	
1903		Crawford, Lawrence, M.A., D.Sc., Professor of Mathematics in the South African College, Cape Town	
1887		* Crawford, William Caldwell, 1 Lockharton Gardens, Colinton Road	
1870		Crichton-Browne, Sir Jas., M.D., LL.D., F.R.S., Lord Chancellor's Visitor and Vice-President of the Royal Institution of Great Britain, 72 Queen's Gate, and Royal Courts of Justice, Strand, London	135
1886		* Croom, Sir John Halliday, M.D., F.R.C.P.E., Professor of Midwifery in the University of Edinburgh, Vice-President, Royal College of Surgeons, Edinburgh, 25 Charlotte Square	
1898		* Cullen, Alexander, F.S.A. Scot., Millburn House, by Hamilton	
1898		* Currie, James, M.A. Cantab. (TREASURER), Larkfield, Goldenacre	
1904		* Cuthbertson, John, Secretary, West of Scotland Agricultural College, 4 Charles Street, Kilmarnock	
1885		Daniell, Alfred, M.A., LL.B., D.Sc., Advocate, The Athenæum Club, Pall Mall, London	140
1884		Davy, R., F.R.C.S. Eng., Consulting Surgeon to Westminster Hospital, Burstone House, Bow, North Devon	
1894		* Denny, Archibald, Braehead, Dumbarton	
1869	C. V. J.	Dewar, Sir James, M.A., LL.D., D.C.L., D.Sc. Dub., F.R.S., F.C.S., Jacksonian Professor of Natural and Experimental Philosophy in the University of Cambridge, and Fulleren Professor of Chemistry at the Royal Institution of Great Britain, London	
1905		* Dewar, James Campbell, C.A., 27 Douglas Crescent	
1906		* Dewar, Thomas Wm., M.D., F.R.C.P., Kincairn, Dunblane	145
1904		Dickinson, Walter George Burnett, F.R.C.V.S., Boston, Lincolnshire	
1884		Dickson, The Right Hon. Charles Scott, K.C., LL.D., 22 Moray Place	
1888	C.	* Dickson, Henry Newton, M.A., D.Sc., The Lawn, Upper Redlands Road, Reading	
1876	C.	Dickson, J. D. Hamilton, M.A., Fellow and Tutor, St Peter's College, Cambridge	
1885	C.	Dixon, James Main, M.A., Litt. Hum. Doctor, Professor of English, University of Southern California, Wesley Avenue, Los Angeles, California, United States	150
1897		* Dobbie, James Bell, F.Z.S., 12 South Inverleith Avenue	
1904	C.	* Dobbie, James Johnston, M.A., D.Sc., LL.D., F.R.S., Director of the Royal Scottish Museum, Edinburgh, 27 Polwarth Terrace	
1881	C.	Dobbin, Leonard, Ph.D., Lecturer on Chemistry in the University of Edinburgh, 6 Wilton Road	
1867	C.	Donaldson, Sir James, M.A., LL.D., Principal of the University of St Andrews, St Andrews	
1896		* Donaldson, William, M.A., Viewpark House, Spylaw Road	155
1905		* Donaldson, Rev. Wm. Galloway, The Manse, Forfar	

Date of Election.			
1882		Dott, David B., F.I.C., Memb. Pharm. Soc., Ravenslea, Musselburgh	
1901		* Douglas, Carstairs Cumming, M.D., D.Sc., Professor of Medical Jurisprudence and Hygiene, Anderson's College, Glasgow, 2 Royal Crescent, Glasgow	
1866		Douglas, David, 22 Drummond Place	
1908	C.	Drinkwater, Harry, M.D., M.R.C.S. (Eng.), Grosvenor Lodge, Wrexham, North Wales	160
1901		* Drinkwater, Thomas W., L.R.C.P.E., L.R.C.S.E., Chemical Laboratory, Surgeons' Hall	
1878		Duncanson, J. J. Kirk, M.D., F.R.C.P.E., 22 Drumsheugh Gardens	
1904		* Dunlop, William Brown, M.A., 7 Carlton Street	
1903		* Dunstan, John, M.R.C.V.S., 1 Dean Terrace, Liskeard, Cornwall	
1892	C.	Dunstan, M. J. R., M.A., F.I.C., F.C.S., Principal, South-Eastern Agricultural College, Wye, Kent	165
1899		* Duthie, George, M.A., Inspector General of Education, Salisbury, Rhodesia	
1906	C.	* Dyson, Frank Watson, M.A., F.R.S., Astronomer Royal for Scotland, and Professor of Astronomy in the University of Edinburgh, Royal Observatory, Blackford Hill, Edinburgh	
1893		Edington, Alexander, M.D., 20 Kilmaurs Road	
1904		* Edwards, John, 4 Great Western Terrace, Kelvinside, Glasgow	
1904		* Elder, William, M.D., F.R.C.P.E., 4 John's Place, Leith	170
1875		Elliot, Daniel G., Curator of Department of Zoology, Field Columbian Museum, Chicago, U.S.	
1906	C.	* Ellis, David, D.Sc., Ph.D., Lecturer in Botany and Bacteriology, Glasgow and West of Scotland Technical College, Glasgow	
1897	C.	* Erskine-Murray, James Robert, D.Sc., 77 Kingsfield Road, Watford, Herts	
1884		Evans, William, F.F.A., 38 Morningside Park	
1879	C. N.	Ewart, James Cossar, M.D., F.R.C.S.E., F.R.S., F.L.S., Regius Professor of Natural History, University of Edinburgh (VICE-PRESIDENT), Craigiefield, Penicuik, Midlothian	175
1902		* Ewen, J. T., B.Sc., Memb. Inst. Mech. E., H.M.I.S., 104 King's Gate, Aberdeen	
1878	C.	Ewing, James Alfred, M.A., B.Sc., LL.D., Memb. Inst. C.E., F.R.S., Director of Naval Education, Royal Naval College, Greenwich	
1900	C.	Eyre, John W. H., M.D., M.S. (Dunelm), D.P.H. (Camb.), Guy's Hospital (Bacteriological Department), London	
1875		Fairley, Thomas, Lecturer on Chemistry, 8 Newton Grove, Leeds	
1907	C.	Falconer, John Downie, M.A., D.Sc., F.G.S., Director, Mineral Survey of Northern Nigeria, The Limes, Little Berkhamstead, Hertford, and Imperial Institute, London	180
1888	C.	* Fawsitt, Charles A., 9 Foremount Terrace, Dowanhill, Glasgow	
1883	C.	Felkin, Robert W., M.D., F.R.G.S., Fellow of the Anthropological Society of Berlin, 47 Bassett Road, North Kensington, London, W.	
1899		* Fergus, Andrew Freeland, M.D., 22 Blythswood Square, Glasgow	
1907		* Fergus, Edward Oswald, 12 Clairmont Gardens, Glasgow	
1904		* Ferguson, James Haig, M.D., F.R.C.P.E., F.R.C.S.E., 7 Coates Crescent	185
1888		* Ferguson, John, M.A., LL.D., Professor of Chemistry in the University of Glasgow	
1868	C.	Ferguson, Robert M., Ph.D., LL.D. (SOCIETY'S REPRESENTATIVE ON GEORGE HERIOT'S TRUST), 5 Douglas Gardens	
1898		* Findlay, John R., M.A. Oxon, 27 Drumsheugh Gardens	
1899		* Finlay, David W., B.A., M.D., LL.D., F.R.C.P., D.P.H., Professor of Medicine in the University of Aberdeen, Honorary Physician to His Majesty in Scotland, 2 Queen's Terrace, Aberdeen	
1906		* Fleming, Robert Alexander, M.D., F.R.C.P.E., Assistant Physician, Royal Infirmary, 10 Chester Street	190
1900	C. N.	* Flett, John S., M.A., D.Sc., Geological Survey Office, 28 Jermyn Street, London	
1880		Flint, Robert, D.D., Corresponding Member of the Institute of France, Corresponding Member of the Royal Academy of Sciences of Palermo, Emeritus Professor of Divinity in the University of Edinburgh, 5 Royal Terrace	
1872	C.	Forbes, Professor George, M.A., Memb. Inst. C.E., Memb. Inst. E.E., F.R.S., F.R.A.S., 11 Little College Street, Westminster, S.W.	
1904		Forbes, Norman Hay, F.R.C.S.E., L.R.C.P. Lond., M.R.C.S. Eng., Corres. Memb. Soc. d'Hydrologie médicale de Paris, Druminnor, Church Stretton, Salop	
1892		* Ford, John Simpson, F.C.S., 4 Nile Grove	195
1858		Fraser, A. Campbell, Fellow of the British Academy, Hon. D.C.L. Oxford, LL.D., Litt. D., Emeritus Professor of Logic and Metaphysics in the University of Edinburgh, Gorton House, Hawthornden	
1896		* Fraser, John, M.B., F.R.C.P.E., one of H.M. Commissioners in Lunacy for Scotland, 13 Heriot Row	

Alphabetical List of the Ordinary Fellows of the Society. 777

Date of Election.			
1867	C.	Fraser, Sir Thomas R., M.D., LL.D., F.R.C.P.E., F.R.S., Professor of Materia Medica in the University of Edinburgh, Honorary Physician to the King in Scotland, 13 Drumsheugh Gardens	
1891	K. B.	* Fullarton, J. H., M.A., D.Sc., Brodick, Arran	
1891		* Fulton, T. Wemyss, M.D., Scientific Superintendent, Scottish Fishery Board, 417 Great Western Road, Aberdeen	200
1907		* Galbraith, Alexander, Organiser of Continuation Classes in Science, Glasgow and West of Scotland Technical College, 4 Maxwell Square, Pollokshields, Glasgow	
1888	C.	* Galt, Alexander, D.Sc., Keeper of the Technological Department, Royal Scottish Museum, Edinburgh	
1901		Ganguli, Sanjiban, M.A., Principal, Maharaja's College, and Director of Public Instruction, Jaipur State, Jaipur, India	
1899		Gatehouse, T. E., Assoc. Memb. Inst. C.E., Memb. Inst. M.E., Memb. Inst. E.E., Tulse Hill Lodge, 100 Tulse Hill, London	
1867		Gayner, Charles, M.D., F.L.S.	205
1900		Gayton, William, M.D., M.R.C.P.E., Ravensworth, Regent's Park Road, Finchley, London, N.	
1909	C.	* Geddes, Auckland C., M.D., Professor of Anatomy, Royal College of Surgeons in Ireland, Dublin	
1880	C.	Geddes, Patrick, Professor of Botany in University College, Dundee, and Lecturer on Zoology, Ramsay Garden, University Hall, Edinburgh	
1861	C. B.	Geikie, Sir Archibald, K.C.B., D.C.L. Oxf., D.Sc. Camb. Dub., LL.D. St And., Glasg., Aberdeen, Edin., Ph.D. Upsala, Pres. R.S., Pres. G.S., Foreign Member of the Reale Accad. Lincei, Rome, of the National Acad. of the United States, of the Academies of Stockholm, Christiania, Göttingen, Corresponding Member of the Institute of France and of the Academies of Berlin, Vienna, Munich, Turin, Belgium, Philadelphia, New York, &c., Shepherd's Down, Haslemere, Surrey	
1871	C. B.	Geikie, James, LL.D., D.C.L., F.R.S., F.G.S., Professor of Geology in the University of Edinburgh, Kilmorie, Colinton Road	210
1909		* Gentle, Wm., B.Sc., 2 Blackwood Crescent, Edinburgh	
1881	C.	Gibson, George Alexander, D.Sc., M.D., LL.D., F.R.C.P.E., 3 Drumsheugh Gardens	
1890		* Gibson, George A., M.A., LL.D., Professor of Mathematics in the University of Glasgow, 8 Sandyford Place, Glasgow	
1877	C.	Gibson, John, Ph.D., Professor of Chemistry in the Heriot-Watt College, 16 Woodhall Terrace, Juniper Green	
1900		Gilchrist, Douglas A., B.Sc., Professor of Agriculture and Rural Economy, Armstrong College, Newcastle-upon-Tyne	215
1880		Gilruth, George Ritchie, Surgeon, 53 Northumberland Street	
1907		Gilruth, John Anderson, M.R.C.V.S., Professor, University, Melbourne, Australia	
1909		* Gladstone, Hugh Steuart, M.A., M.B.O.U., F.Z.S., Capenoch, Thornhill, Dumfriesshire	
1898		* Glaister, John, M.D., F.F.P.S. Glasgow, D.P.H. Camb., Professor of Forensic Medicine in the University of Glasgow, 3 Newton Place, Glasgow	
1901		Goodwillie, James, M.A., B.Sc., Liberton, Edinburgh	220
1899		* Goodwin, Thomas S., M.B., C.M., F.C.S., 1 Heron Terrace, St Margaret's, Middlesex	
1897		Gordon-Munn, John Gordon, M.D., 34 Dover Street, London, W.	
1891		* Graham, Richard D., 11 Strathearn Road	
1898	C.	* Gray, Albert A., M.D., 14 Newton Terrace, Glasgow	
1883		Gray, Andrew, M.A., LL.D., F.R.S., Professor of Natural Philosophy in the University of Glasgow	225
1909	C.	* Gray, James Gordon, B.Sc., Lecturer in Physics in the University of Glasgow, 11 The University, Glasgow	
1886		* Greenfield, W. S., M.D., F.R.C.P.E., Professor of General Pathology in the University of Edinburgh, 7 Heriot Row	
1897		Greenlees, Thomas Duncan, M.D. Edin., Amana, Tulse Hill, London	
1905	C.	* Gregory, John Walter, D.Sc., F.R.S., Professor of Geology in the University of Glasgow, 4 Park Quadrant, Glasgow	
1906		Greig, Edward David Wilson, M.D., B.Sc., Captain, H.M.'s Indian Medical Service, Byculla Club, Bombay, India	230
1905		Greig, Robert Blyth, F.Z.S., Fordyce Lecturer in Agriculture, University of Aberdeen, Torloisk, Cults, Aberdeenshire	
1899		* Guest, Edward Graham, M.A., B.Sc., 5 Church Hill	
1907	C.	* Gulliver, Gilbert Henry, B.Sc., A.M.I. Mech. E., Lecturer in Experimental Engineering in the University of Edinburgh, 10 Stanley Street, Portobello	
1888	C.	Guppy, Henry Brougham, M.B., Rosario, Salcombe, Devon	

Date of Election.			
1905	B. C.	* Halm, Jacob E., Ph.D., Chief Assistant Astronomer, Royal Observatory, Cape Town, Cape of Good Hope	235
1899		Hamilton, Allan M'Lane, M.D., 44 East Twenty-ninth Street, New York	
1876	C.	Hannay, J. Ballantyne, Cove Castle, Loch Long	
1896		* Harris, David, Fellow of the Statistical Society, Lyncombe Rise, Prior Park Road, Bath	
1896	C.	* Harris, David Fraser, B.Sc. (Lond.), M.D., F.S.A. Scot., The Physiological Department, The University, Birmingham	
1888	C.	* Hart, D. Berry, M.D., F.R.C.P.E., 5 Randolph Cliff	240
1869		Hartley, Sir Charles A., K.C.M.G., Memb. Inst. C.E., 26 Pall Mall, London	
1877	C.	Hartley, W. N., D.Sc., F.R.S., F.I.C., Prof. of Chemistry, Royal College of Science for Ireland, Dublin	
1881		Harvie-Brown, J. A., of Quarter, F.Z.S., Dunipace House, Larbert, Stirlingshire	
1880	C.	Haycraft, J. Berry, M.D., D.Sc., Professor of Physiology in the University College of South Wales and Monmouthshire, Cardiff	
1892	C.	* Heath, Thomas, B.A., Assistant Astronomer, Royal Observatory, Edinburgh	245
1893		Hehir, Patrick, M.D., F.R.C.S.E., M.R.C.S.L., L.R.C.P.E., Surgeon-Captain, Indian Medical Service, Principal Medical Officer, H.H. the Nizam's Army, Hyderabad, Deccan, India	
1890	C.	Helme, T. Arthur, M.D., M.R.C.P.L., M.R.C.S., 3 St Peter's Square, Manchester	
1900		Henderson, John, D.Sc., Assoc. Inst. E.E., Kinnoul, Warwick's Bench Rd., Guildford, Surrey	
1908		* Henderson, William Dawson, M.A., B.Sc., Ph.D., Assistant Professor, Zoological Department, University College, Dundee	
1890	C.	* Hepburn, David, M.D., Professor of Anatomy in the University College of South Wales and Monmouthshire, Cardiff	250
1881	C. N.	Herdman, W. A., D.Sc., F.R.S., Pres. L.S., Prof. of Natural History in the University of Liverpool, Croxteth Lodge, Ullet Road, Liverpool	
1908		* Hewat, Archibald, F.F.A., F.I.A., 13 Eton Terrace	
1894		Hill, Alfred, M.D., M.R.C.S., F.I.C., Valentine Mount, Freshwater Bay, Isle of Wight	
1902		* Hinxman, Lionel W., B.A., Geological Survey Office, 33 George Square	
1904		Hobday, Frederick T. G., F.R.C.V.S., 6 Berkeley Gardens, Kensington, London	255
1885		Hodgkinson, W. R., Ph.D., F.I.C., F.C.S., Prof. of Chem. and Physics at the Royal Military Acad. and Royal Artillery Coll., Woolwich, 89 Shooter's Hill Road, Blackheath, Kent	
1881	C. N.	Horne, John, LL.D., F.R.S., F.G.S., Director of the Geological Survey of Scotland (VICE-PRESIDENT), 33 George Square, Edinburgh	
1896		Horne, J. Fletcher, M.D., F.R.C.S.E., The Poplars, Barnsley	
1904		* Horsburgh, Ellice Martin, M.A., B.Sc., Lecturer in Technical Mathematics, University of Edinburgh, 11 Granville Terrace	
1897		Houston, Alex. Cruikshanks, M.B., C.M., D.Sc., 19 Fairhazel Gardens, South Hampstead, London, N.W.	260
1893		Howden, Robert, M.A., M.B., C.M., Professor of Anatomy in the University of Durham, 14 Burdon Terrace, Newcastle-on-Tyne	
1899		Howie, W. Lamond, F.C.S., 26 Neville Court, Abbey Road, Regent's Park, London, N.W.	
1883	C.	Hoyle, William Evans, M.A., D.Sc., M.R.C.S., Crowland, Llandaff, Wales	
1886		Hunt, Rev. H. G. Bonavia, Mus.D. Dub., Mus.B. Oxon., The Vicarage, Burgess Hill, Sussex	
1887	C.	* Hunter, James, F.R.C.S.E., F.R.A.S., Rosetta, Liberton, Midlothian	265
1887	C.	* Hunter, William, M.D., M.R.C.P.L. and E., M.R.C.S., 54 Harley Street, London	
1908		Hyslop, Theophilus Bulkeley, M.D., M.R.C.P.E., Senior Physician, Bethlem Royal Hospital, London, S.E.	
1882	C.	Inglis, J. W., Memb. Inst. C.E., 26 Pitt Street	
1906		* Innes, Alexander Taylor, LL.D., M.A., Advocate, 48 Morningside Park	
1904	C.	Innes, R. T. A., Director, Government Observatory, Johannesburg, Transvaal	270
1904		* Ireland, Alexander Scott, S.S.C., 2 Buckingham Terrace	
1875		Jack, William, M.A., LL.D., Professor of Mathematics in the University of Glasgow	
1894		Jackson, Sir John, LL.D., 48 Belgrave Square, London	
1889		* James, Alexander, M.D., F.R.C.P.E., 14 Randolph Crescent	
1882		Jamieson, Prof. A., Memb. Inst. C.E., 16 Rosslyn Terrace, Kelvinside, Glasgow	275
1901		* Jardine, Robert, M.D., M.R.C.S. Eng., F.F.P. and S. Glas., 20 Royal Crescent, Glasgow	

Alphabetical List of the Ordinary Fellows of the Society. 779

Date of Election.			
1906	C.	* Jehu, Thomas James, M.A., M.D., F.G.S., Lecturer in Geology, University of St Andrews, Strathmartine, Hepburn Gardens, St Andrews	
1900		* Jerdan, David Smiles, M.A., D.Sc., Ph.D., Temora, Colinton, Midlothian	
1895		Johnston, Lieutenant-Colonel Henry Halcro, C.B., R.A.M.S., D.Sc., M.D., F.L.S., Orphir House, Kirkwall, Orkney	
1903	C.	* Johnston, Thomas Nicol, M.B., C.M., Corstorphine House, Corstorphine	280
1902		Johnstone, George, Lieut. R.N.R., late Marine Superintendent, British India Steam Navigation Co., 26 Comiston Drive	
1874		Jones, Francis, M.Sc., Lecturer on Chemistry, Beaufort House, Alexandra Park, Manchester	
1888		Jones, John Alfred, Memb. Inst. C.E., Fellow of the Univ. of Madras, Sanitary Engineer to the Government of Madras, c/o Messrs Parry & Co., 70 Gracechurch St., London	
1905		Jones, George William, M.A., B.Sc., LL.B., Scottish Tutorial Institute, Edinburgh and Glasgow, 25 North Bridge, Coraldene, Kirk Brae, Liberton	
1907		* Kemp, John, M.A., Headmaster, High School, Kelso	285
1909		Kenwood, Henry Richard, M.B., Chadwick Professor of Hygiene in the University of London, 150 Bethune Road, Amherst Park, London, N.	
1908		* Kerr, Andrew William, F.S.A. Scot., Royal Bank House, St Andrew Square	
1892		* Kerr, Rev. John, M.A., Manse, Dirleton	
1903	C. N.	* Kerr, John Graham, M.A., Professor of Zoology in the University of Glasgow	
1891		Kerr, Joshua Law, M.D., Biddenden Hall, Cranbrook, Kent	290
1908		Kidd, Walter Aubrey, M.D., F.Z.S., 12 Montpelier Row, Blackheath, London	
1886	C. N.	* Kidston, Robert, LL.D., F.R.S., F.G.S. (SECRETARY), 12 Clarendon Place, Stirling	
1907		* King, Archibald, M.A., B.Sc., Rector of the Academy, Castle-Douglas, Hazeldene, Castle-Douglas, Kirkcudbrightshire	
1877		King, Sir James, of Campsie, Bart., LL.D., 115 Wellington Street, Glasgow	
1880		King, W. F., Lonend, Russell Place, Trinity	295
1883		Kinnear, The Rt. Hon. Lord, one of the Senators of the College of Justice, 2 Moray Place	
1878		Kintore, The Right Hon. the Earl of, M.A. Cantab., LL.D., Cambridge, Aberdeen and Adelaide, Keith Hall, Inverurie, Aberdeenshire	
1901		* Knight, The Rev. G. A. Frank, M.A., St Leonard's United Free Church, Perth	
1907		* Knight, James, M.A., D.Sc., F.C.S., F.G.S., Headmaster, St James School, Glasgow, The Shielling, Uddingston, by Glasgow	
1880	C. K.	Knott, C. G., D.Sc., Lecturer on Applied Mathematics in the University of Edinburgh (late Prof. of Physics, Imperial University, Japan) (SECRETARY), 42 Upper Gray Street, Edinburgh	300
1886		* Laing, Rev. George P., 17 Buckingham Terrace	
1907		* Lanchester, William Forster, M.A., Den of Gryffe, Kilmalcolm	
1878	C.	Lang, P. R. Scott, M.A., B.Sc., Professor of Mathematics, University of St Andrews	
1885	C.	Laurie, A. P., M.A., D.Sc., Principal of the Heriot-Watt College, Edinburgh	
1894	C.	* Laurie, Malcolm, B.A., D.Sc., F.L.S., Royal College of Surgeons, Edinburgh	305
1905		* Lawson, David, M.A., M.D., L.R.C.P. and S.E., Druimdarroch, Banchory, Kincardineshire	
1903		* Leighton, Gerald Rowley, M.D., Sunnyside, Russell Place	
1874	C. K.	Letts, E. A., Ph.D., F.I.C., F.C.S., Professor of Chemistry, Queen's College, Belfast	
1905		* Lightbody, Forrest Hay, 56 Queen Street	
1889		* Lindsay, Rev. James, D.D., B.Sc., F.G.S., M.R.A.S., Corresponding Member of the Royal Academy of Sciences, Letters and Arts, of Padua, Associate of the Philosophical Society of Louvain, Annick Lodge, Irvine	310
1870	C. B.	Lister, The Right Hon. Lord, O.M., P.C., M.D., F.R.C.S.L., F.R.C.S.E., LL.D., D.C.L., F.R.S., Foreign Associate of the Institute of France, Emeritus Professor of Clinical Surgery, King's College, Surgeon Extraordinary to the King, 12 Park Crescent, Portland Place, London	
1903		Liston, William Glen, M.D., Captain, Indian Medical Service, c/o Grindlay Groom & Co., Bombay, India	
1903		* Littlejohn, Henry Harvey, M.A., M.B., B.Sc., F.R.C.S.E., Professor of Forensic Medicine in the University of Edinburgh, 11 Rutland Street	
1898		* Lothian, Alexander Veitch, M.A., B.Sc., Glendoune, Manse Road, Bearsden, Glasgow	
1884		Low, George M., Actuary, 11 Moray Place	315
1888		* Lowe, D. F., M.A., LL.D., late Head Master of Heriot's Hospital School, Lauriston, 19 George Square	

Date of Election.			
1904		* Lowson, Charles Stewart, M.B., C.M., Captain, Indian Medical Service, c/o Messrs Thomas Cook & Son, Bombay, India.	
1900		Lusk, Graham, Ph.D., M.A., Prof. of Physiology, Univ. and Bellevue Medical College, N.Y.	
1894		* Mabbott, Walter John, M.A., Rector of County High School, Duns, Berwickshire	
1887		M'Aldowie, Alexander M., M.D., Glengarriff, Leckhampton, Cheltenham	320
1907		MacAlister, Donald Alexander, A.R.S.M., F.G.S., 20 Hanover Square, London, W.	
1891		Macallan, John, F.I.C., 3 Rutland Terrace, Clontarf, Dublin	
1888	C.	M'Arthur, John, F.C.S., 196 Trinity Road, Wandsworth Common, London	
1883		M'Bride, P., M.D., F.R.C.P.E., 16 Chester Street	
1903		* M'Cormick, W. S., M.A., LL.D., 13 Douglas Crescent	325
1899		* M'Cubbin, James, B.A., Rector of the Burgh Academy, Kilsyth	
1905		* Macdonald, Hector Munro, M.A., F.R.S., Professor of Mathematics, University of Aberdeen, 52 College Bounds, Aberdeen	
1894		* Macdonald, James, Secretary of the Highland and Agricultural Society of Scotland, 2 Garscube Terrace	
1897	C.	* Macdonald, James A., M.A., B.Sc., H.M. Inspector of Schools, Glengarry, Dingwall	
1904		* Macdonald, John A., M.A., B.Sc., High School, Stellenbosch, Cape Colony	330
1886		* Macdonald, The Rt. Hon. Sir J. H. A., K.C.B., K.C., LL.D., F.R.S., M.I.E.E., Lord Justice-Clerk, and Lord President of the Second Division of the Court of Session, 15 Abercromby Place	
1904		Macdonald, William, B.Sc., M.Sc., Agriculturist, Editor <i>Transvaal Agricultural Journal</i> , Department of Agriculture, Pretoria Club, Pretoria, Transvaal	
1886		* Macdonald, William J., M.A., 15 Comiston Drive	
1901	C.	* MacDougal, R. Stewart, M.A., D.Sc., 13 Archibald Place	
1888	C.	* M'Fadyean, Sir John, M.B., B.Sc., LL.D., Principal, and Professor of Comparative Pathology in the Royal Veterinary College, Camden Town, London	335
1878	C.	Macfarlane, Alexander, M.A., D.Sc., LL.D., Lecturer in Physics in Lehigh University, Pennsylvania, Gowrie Grove, Chatham, Ontario, Canada	
1885	C.	Macfarlane, J. M., D.Sc., Professor of Botany and Director of the Botanic Garden, University of Pennsylvania, Philadelphia, Pennsylvania, U.S.A.	
1897		* MacGillivray, Angus, C.M., M.D., South Tay Street, Dundee	
1878		M'Gowan, George, F.I.C., Ph.D., 21 Montpelier Road, Ealing, Middlesex	
1886		* MacGregor, The Very Rev. James, D.D., 3 Eton Terrace	340
1880	C.	MacGregor, James Gordon, M.A., D.Sc., LL.D., F.R.S., Prof. of Natural Philosophy in the University of Edinburgh, 24 Dalrymple Crescent	
1903		* M'Intosh, D. C., M.A., B.Sc., 3 Glenisla Gardens	
1869	C. N.	M'Intosh, William Carmichael, M.D., LL.D., F.R.S., F.L.S., Professor of Natural History in the University of St Andrews, 2 Abbotsford Crescent, St Andrews	
1895	C.	* Macintyre, John, M.D., 179 Bath Street, Glasgow	
1882		Mackay, John Sturgeon, M.A., LL.D., late Mathematical Master in the Edinburgh Academy, 69 Northumberland Street	345
1873	C. B.	M'Kendrick, John G., M.D., F.R.C.P.E., LL.D., F.R.S., Emeritus Professor of Physiology in the University of Glasgow, Maxieburn, Stonehaven	
1900	C.	* M'Kendrick, John Souttar, M.D., F.F.P.S.G., 2 Buckingham Terrace, Glasgow	
1894		* Mackenzie, Robert, M.D., Napier, Nairn	
1898		Mackenzie, W. Cossar, D.Sc., Alderston, Haddington	
1904		* Mackenzie, W. Leslie, M.A., M.D., D.P.H., Medical Member of the Local Government Board for Scotland, 1 Stirling Road, Trinity	350
1905		Mackenzie, William Colin, M.D., F.R.C.S., Demonstrator of Anatomy in the University of Melbourne, Elizabeth Street North, Melbourne, Victoria	
1904		* Mackintosh, Donald James, M.V.O., M.B., Supt. Western Infirmary, Glasgow	
1869	C.	Maclagan, R. C., M.D., F.R.C.P.E., 5 Coates Crescent	
1869	C.	M'Laren, The Hon. Lord, LL.D., Edin. & Glasg., F.R.A.S., one of the Senators of the College of Justice, 46 Moray Place	
1899		Maclean, Ewan John, M.D., M.R.C.P. London, 12 Park Place, Cardiff	355
1888	C.	* Maclean, Magnus, M.A., D.Sc., Memb. Inst. E.E., Prof. of Electrical Engineering in the Glasgow and West of Scotland Technical College, 51 Kerrsland Terrace, Hillhead, Glasgow	
1876		Macleod, Very Rev. Norman, D.D., 74 Murrayfield Gardens	
1876		Macmillan, John, M.A., D.Sc., M.B., C.M., F.R.C.P.E., 48 George Square	
1893		* M'Murtrie, The Very Rev. John, M.A., D.D., 13 Inverleith Place	

Alphabetical List of the Ordinary Fellows of the Society. 781

Date of Election.			
1906		* Macnair, Duncan Scott, Ph.D., B.Sc., H.M. Inspector of Schools, 67 Braid Avenue	360
1907		* Macnair, Peter, Curator of the Natural History Collections in the Glasgow Museums, Kelvingrove Museum, Glasgow	
1890		* M'Vail, John C., M.D., LL.D., 20 Eton Place, Hillhead, Glasgow	
1898	C.	Mahâlanobis, S. C., B.Sc., Professor of Physiology, Presidency College, Calcutta, India	
1908		Mallik, Devendranath, B.A., B.Sc., Professor of Physics and Mathematics, Patna College, Bankipur, Bengal, India	
1880	C.	Marsden, R. Sydney, M.D., C.M., D.Sc., M.R.I.A., F.I.C, F.C.S., Rowallan House, Cearns Road, and Town Hall, Birkenhead	365
1909	C.	* Marshall, C. R., M.D., M.A., Professor of Materia-Medica and Therapeutics, University of St Andrews, Fairmount, Blackness Avenue, Dundee	
1882	C.	Marshall, D. H., M.A., Professor of Physics in Queen's University and College, Kingston, Ontario, Canada	
1901	C.	* Marshall, F. H. A., M.A., D.Sc., Lecturer on Agricultural Physiology in the University of Cambridge, Christ's College, Cambridge	
1888	C. K.	* Marshall, Hugh, D.Sc., F.R.S., Professor of Chemistry in the University College, Dundee	
1892		* Martin, Francis John, W.S., 17 Rothesay Place	370
1903		Martin, Nicholas Henry, F.L.S., F.C.S., Ravenswood, Low Fell, Gateshead	
1885	C.	Masson, Orme, D.Sc., F.R.S., Professor of Chemistry in the University of Melbourne	
1898	C. B.	* Masterman, Arthur Thomas, M.A., D.Sc., Inspector of Fisheries, Board of Agriculture, Whitehall, London	
1906		* Mathieson, Robert, F.C.S., Rillbank, Innerleithen	
1902		Matthews, Ernest Romney, Assoc. Memb. Inst. C.E., F.G.S., Bessemer Prizeman, Soc. Engineers, Bridlington, Yorkshire	375
1901		* Menzies, Alan W. C., M.A., B.Sc., F.C.S., Kent Chemical Laboratory, University, Chicago, U.S.A.	
1888		* Methven, Cathcart W., Memb. Inst. C.E., F.R.I.B.A., Durban, Natal, S. Africa	
1902	C.	Metzler, William H., A.B., Ph.D., Corresponding Fellow of the Royal Society of Canada, Professor of Mathematics, Syracuse University, Syracuse, N.Y.	
1885	C. B.	Mill, Hugh Robert, D.Sc., LL.D., 62 Camden Square, London	
1908		* Miller, Alexander Cameron, M.D., F.S.A. Scot., Craig Linnhe, Fort-William, Inverness-shire	380
1905		* Miller-Milne, C. H., M.A., Rector, The High School, Arbroath, 8 Dalhousie Place, Arbroath	
1909		Mills, Bernard Langley, M.D., F.R.C.S.E., M.R.C.S.L., D.P.H., Lt.-Col. R.A.M.C., late Army Specialist in Hygiene, 4 Palmerston Road, Broomhill, Sheffield	
1905		* Milne, Archibald, M.A., B.Sc., Lecturer on Mathematics and Science, Edinburgh Provincial Training College, 108 Comiston Drive	
1904	C.	* Milne, James Robert, D.Sc., 56 Manor Place	
1886		* Milne, William, M.A., B.Sc., 70 Beechgrove Terrace, Aberdeen	385
1899		* Milroy, T. H., M.D., B.Sc., Professor of Physiology in Queen's College, Belfast, Thomlea, Malone Park, Belfast	
1889	C.	Mitchell, A. Crichton, D.Sc., Director of Public Instruction in Travancore, India	
1897		* Mitchell, George Arthur, M.A., 9 Lowther Terrace, Kelvinside, Glasgow	
1900		* Mitchell, James, M.A., B.Sc., 4 Manse Street, Kilmarnock	
1899		* Mitchell-Thomson, Sir Mitchell, Bart., 6 Charlotte Square	390
1906	C.	Moffat, The Rev. Alexander, M.A., B.Sc., Professor of Physical Science, Christian College, Madras, India	
1890	C.	Mond, R. L., M.A., Cantab., F.C.S., The Poplars, 20 Avenue Road, Regent's Park, London	
1887	C.	Moos, N. A. F., L.C.E., B.Sc., Professor of Physics, Elphinstone College, and Director of the Government Observatory, Colaba, Bombay	
1896		* Morgan, Alexander, M.A., D.Sc., Principal, Edinburgh Provincial Training College, 1 Midmar Gardens	
1892		Morrison, J. T., M.A., B.Sc., Professor of Physics and Chemistry, Victoria College, Stellenbosch, Cape Colony	395
1901		Moses, O. St John, M.D., B.Sc., F.R.C.S.E., Captain, Indian Medical Service, 8 Lansdowne Road, Calcutta, India	

Date of Election.			
1892	C.	Mossman, Robert C., Superintendent of Publications, Argentine Meteorological Office, Cuyo 947, Buenos Ayres	
1874	C. K.	Muir, Thomas, C.M.G., M.A.; LL.D., F.R.S., Superintendent-General of Education for Cape Colony, Education Office, Cape Town, and Mowbray Hall, Rosebank, Cape Colony	
1888	C.	* Muirhead, George, Commissioner to His Grace the Duke of Richmond and Gordon, K.G., Speybank, Fochabers	
1907		Muirhead, James M. P., Bredisholm, Clarendon, near Cape Town, Cape Colony	400
1887		Mukhopādhyay, Asútosh, M.A., LL.D., F.R.A.S., M.R.I.A., Professor of Mathematics at the Indian Association for the Cultivation of Science, 77 Russa Road North, Bhowanipore, Calcutta	
1894		* Munro, J. M. M., Memb. Inst. E.E., 136 Bothwell Street, Glasgow	
1891	C.	* Munro, Robert, M.A., M.D., LL.D., Hon. Memb. R.I.A., Hon. Memb. Royal Soc. of Antiquaries of Ireland, Elmbank, Largs, Ayrshire	
1896		* Murray, Alfred A., M.A., LL.B., 20 Warriston Crescent	
1892	C.	* Murray, George Robert Milne, F.R.S., F.L.S., 32 Market Square, Stonehaven	405
1907	C.	* Murray, James, Park Road, Maxwelltown, Dumfries	
1877	C.	Murray, Sir John, K.C.B., LL.D., D.C.L., Ph.D., D.Sc., F.R.S., Member of the Prussian Order <i>Pour le Mérite</i> , Director of the Challenger Expedition Publications. Office, Villa Medusa, Boswell Road. House, Challenger Lodge, Wardie, and United Service Club	
	B. N.		
1907		* Musgrove, James, M.D., F.R.C.S. Edin. and Eng., Bute Professor of Anatomy, University of St Andrews, 56 South Street, St Andrews	
1887		Muter, John, M.A., F.C.S., South London Central Public Laboratory, 325 Kennington Road, London	
1902		Mylne, The Rev. R. S., M.A., B.C.L. Oxford, F.S.A. Lond., Great Amwell, Herts	410
1888		Napier, A. D. Leith, M.D., C.M., M.R.C.P.L., 28 Angas Street, Adelaide, S. Australia	
1897		Nash, Alfred George, C.E., B.Sc., Engineer, Department of Public Works, Jamaica, Belretiro, Mandeville, Jamaica, W.I.	
1906		* Newington, Frank A., Memb. Inst. C.E., Memb. Inst. E.E., 4 Osborne Terrace	
1898		Newman, George, M.D., D.P.H. Cambridge, Lecturer on Preventive Medicine, St Bartholomew's Hospital, University of London: Dene, Hatch End, Middlesex	
1884		Nicholson, J. Shield, M.A., D.Sc., Professor of Political Economy in the University of Edinburgh, 3 Belford Park	415
1880	C.	Nicol, W. W. J., M.A., D.Sc., 15 Blacket Place	
1878		Norris, Richard, M.D., M.R.C.S. Eng., 3 Walsall Road, Birchfield, Birmingham	
1906		* O'Connor, Henry, C.E., Assoc. Memb. Inst. C.E., 1 Drummond Place	
1888		* Ogilvie, F. Grant, C.B., M.A., B.Sc., Principal Assistant Secretary for Science, Art, and Technology, Board of Education, South Kensington, London	
1888		* Oliphant, James, M.A., 11 Heathfield Park, Willesden, London	420
1886		Oliver, James, M.D., F.L.S., Physician to the London Hospital for Women, 18 Gordon Square, London	
1895	C.	Oliver, Sir Thomas, M.D., LL.D., F.R.C.P., Professor of Physiology in the University of Durham, 7 Ellison Place, Newcastle-upon-Tyne	
1884	C. K.	Omond, R. Traill, 3 Church Hill	
1908		Page, William Davidge, F.C.S., F.G.S., M. Inst. M.E., 10 Clifton Dale, York	
1905		Pallin, William Alfred, F.R.C.V.S., Captain in the Army Veterinary Department, c/o Messrs Holt & Co., 3 Whitehall Place, London	425
1892		Parker, Thomas, Memb. Inst. C.E., Severn House, Iron Bridge, Salop	
1901		* Paterson, David, F.C.S., Lea Bank, Rosslyn, Midlothian	
1886	C.	* Paton, D. Noël, M.D., B.Sc., F.R.C.P.E., Professor of Physiology in the University of Glasgow, University, Glasgow	
1889		* Patrick, David, M.A., LL.D., c/o W. & R. Chambers, 339 High Street	
1892		* Paulin, Sir David, Actuary, 6 Forres Street	430
1881	C. N.	Peach, Benjamin N., LL.D., F.R.S., F.G.S., late District Superintendent and Acting Palæontologist of the Geological Survey of Scotland, 72 Grange Loan	
1907		* Pearce, John Thomson, B.A., B.Sc., School House, Tranent	
1904		* Peck, James Wallace, M.A., Principal Assistant to Executive Officer (Education) of the London County Council, Stanfield House, High Street, Hampstead, London	

Alphabetical List of the Ordinary Fellows of the Society. 783

Date of Election.			
1889		* Peck, William, F.R.A.S., Town's Astronomer, City Observatory, Calton Hill, Edinburgh	
1887	C. B.	* Peddie, Wm., D.Sc., Professor of Natural Philosophy in University College, Dundee, Rosemount, Forthill Road, Broughty Ferry	435
1900		Penny, John, M.B., C.M., D.Sc., Great Broughton, near Cockermouth, Cumberland	
1893		Perkin, Arthur George, F.R.S., 8 Montpellier Terrace, Hyde Park, Leeds	
1889		* Philip, R. W., M.A., M.D., F.R.C.P.E., 45 Charlotte Square	
1907	C.	Phillips, Charles E. S., Castle House, Shooter's Hill, Kent	
1905		* Pinkerton, Peter, M.A., D.Sc., Head Mathematical Master, George Watson's College, Edinburgh, 36 Morningside Grove	440
1908	C.	* Pirie, James Hunter Harvey, B.Sc., M.D., M.R.C.P.E., 13 Alva Street	
1906		Pitchford, Herbert Watkins, F.R.C.V.S., Bacteriologist and Analyst, Natal Government, The Laboratory, Pietermaritzburg, Natal	
1886		Pollock, Charles Frederick, M.D., F.R.C.S.E., 1 Buckingham Terrace, Hillhead, Glasgow	
1888		Prain, David, Lt.-Col., Indian Medical Service, M.A., M.B., LL.D., F.L.S., F.R.S., Hon. Memb. Soc. Lett. ed Arti d. Zelanti, Acireale; Corr. Memb. Pharm. Soc. Gt. Britain, etc.; Director, Royal Botanic Gardens, Kew (late Director, Botanical Survey of India, Calcutta), Botanic Gardens, Kew	
1902		* Preller, Charles Du Riche, M.A., Ph.D., Assoc. Memb. Inst. C.E., 61 Melville Street	445
1892		* Pressland, Arthur, J., M.A. Camb., Edinburgh Academy	
1875	C.	Prevost, E. W., Ph.D., Weston, Ross, Herefordshire	
1908		* Pringle, George Cossar, M.A., Rector of Peebles Burgh and County High School, Bloomfield, Peebles	
1885		Pullar, J. F., Rosebank, Perth	
1903		* Pullar, Laurence, The Lea, Bridge of Allan	450
1880		Pullar, Sir Robert, LL.D., M.P. for the City of Perth, Tayside, Perth	
1898		* Purves, John Archibald, D.Sc., 13 Albany Street	
1897		* Rainy, Harry, M.B., C.M., F.R.C.P. Ed., 16 Gt. Stuart Street	
1899		* Ramage, Alexander G., 8 Western Terrace, Murrayfield	
1884		Ramsay, E. Peirson, M.R.I.A., F.L.S., C.M.Z.S., F.R.G.S., F.G.S., Fellow of the Imperial and Royal Zoological and Botanical Society of Vienna, Curator of Australian Museum, Sydney, N.S.W.	455
1891		* Rankine, John, M.A., LL.D., K.C., Professor of the Law of Scotland in the University of Edinburgh, 23 Ainslie Place	
1904		Ratcliffe, Joseph Riley, M.B., C.M., c/o The Librarian, The University, Birmingham	
1900		Raw, Nathan, M.D., 66 Rodney Street, Liverpool	
1883	C.	Readman, J. B., D.Sc., F.C.S., Staffield Hall, Kirkoswald, R.S.O., Cumberland	
1889		Redwood, Sir Boverton, D.Sc. (Hon.), F.I.C., F.C.S., Assoc. Inst. C.E., Wadham Lodge, Wadham Gardens, London	460
1902		Rees-Roberts, John Vernon, M.D., D.Sc., D.P.H., Barrister-at-Law, National Liberal Club, Whitehall Place, London	
1902		Reid, George Archdall O'Brien, M.B., C.M., 9 Victoria Road South, Southsea, Hants	
1908	C.	* Rennie, John, D.Sc., Lecturer on Parasitology, and Assistant to the Professor of Natural History, University of Aberdeen, 60 Desswood Place, Aberdeen	
1908		Richardson, Linsdall, F.G.S., F.L.S., Director, Cheltenham School of Science and Technology, 10 Oxford Parade, Cheltenham	
1875		Richardson, Ralph, W.S., 10 Magdala Place	465
1906	C.	* Ritchie, William Thomas, M.D., F.R.C.P.E., 9 Atholl Place	
1898	C.	Roberts, Alexander William, D.Sc., F.R.A.S., Lovedale, South Africa	
1880		Roberts, D. Lloyd, M.D., F.R.C.P.L., 23 St John Street, Manchester	
1900		* Robertson, Joseph M'Gregor, M.B., C.M., 26 Buckingham Terrace, Glasgow	
1896		* Robertson, Robert, M.A., 25 Mansionhouse Road	470
1902	C.	* Robertson, Robert A., M.A., B.Sc., Lecturer on Botany in the University of St Andrews	
1896	C.	* Robertson, W. G. Aitchison, D.Sc., M.D., F.R.C.P.E., 2 Mayfield Gardens	
1881		Rosebery, The Right Hon. the Earl of, K.G., K.T., LL.D., D.C.L., F.R.S., Dalmeny Park, Edinburgh	
1909	C.	* Ross, Alex. David, M.A., B.Sc., Assistant to the Professor of Natural Philosophy in the University of Glasgow, 7 Queen's Terrace, Glasgow	
1880		Rowland, L. L., M.A., M.D., President of the Oregon State Medical Society, and Professor of Physiology and Microscopy in Willamette University, Salem, Oregon	475

Date of Election.			
1906		* Russell, Alexander Durie, B.Sc., Mathematical Master, Falkirk High School, Dunaura, Heugh Street, Falkirk	
1902	C.	* Russell, James, 12 Argyll Place	
1880		Russell, Sir James A., M.A., B.Sc., M.B., F.R.C.P.E., LL.D., Woodville, Canaan Lane	
1904		Sachs, Edwin O., Architect, 7 Waterloo Place, Pall Mall, London, S.W.	
1906		Saleeby, Caleb William, M.D., 13 Greville Place, London	480
1903		* Samuel, John S., 8 Park Avenue, Glasgow	
1903		* Sarolea, Charles, Ph.D., D.Litt., Lecturer on French Language, Literature, and Romance Philology, University of Edinburgh, 21 Royal Terrace	
1891		Sawyer, Sir James, Knt., M.D., F.R.C.P., F.S.A., J.P., Consulting Physician to the Queen's Hospital, 31 Temple Row, Birmingham	
1900	C.	* Schäfer, Edward Albert, M.R.C.S., LL.D., F.R.S., Professor of Physiology in the University of Edinburgh	
1885	C.	Scott, Alexander, M.A., D.Sc., F.R.S., The Davy-Faraday Research Laboratory of the Royal Institution, London	485
1880		Scott, J. H., M.B., C.M., M.R.C.S., Prof. of Anatomy in the University of Otago, New Zealand	
1905		Scougal, A. E., M.A., LL.D., H.M. Senior Chief Inspector of Schools and Inspector of Training Colleges, 1 Wester Coates Avenue	
1902		Senn, Nicholas, M.D., LL.D., Professor of Surgery, Rush Medical College, Chicago, U.S.A.	
1897		* Shepherd, John William, Carrickarden, Bearsden, Glasgow	
1894		* Shield, Wm., Memb. Inst. C.E., 33 Old Queen Street, Westminster, London	490
1871		Simpson, Sir A. R., M.D., Emeritus Professor of Midwifery in the University of Edinburgh, 52 Queen Street	
1908		* Simpson, George Freeland Barbour, M.D., F.R.C.P.E., F.R.C.S.E., 43 Manor Place	
1900	C.	* Simpson, James Young, M.A., D.Sc., Professor of Natural Science in the New College, Edinburgh, 25 Chester Street	
1900		Sinhjee, Sir Bhagvat, G.C.I.E., M.D., LL.D. Edin., H.H. The Thakur Sahib of Gondal, Gondal, Kathiawar, Bombay	
1903		* Skinner, Robert Taylor, M.A., Governor and Headmaster, Donaldson's Hospital, Edinburgh	495
1901		* Smart, Edward, B.A., B.Sc., Tillyloss, Tullylumb Terrace, Perth	
1891	C.	* Smith, Alexander, B.Sc., Ph.D., Professor of General Chemistry, University of Chicago, Ills., U.S.	
1882	C.	Smith, C. Michie, B.Sc., F.R.A.S., Director of the Kodaikânal and Madras Observatories, The Observatory, Kodaikânal, South India	
1885		Smith, George, F.C.S., Polmont Station	
1871	C.	Smith, John, M.D., F.R.C.S.E., LL.D., 11 Wemyss Place	500
1904		* Smith, William Charles, K.C., M.A., LL.B., Advocate, 10 Doune Terrace	
1907	C.	Smith, William Ramsay, D.Sc., M.B., C.M., Permanent Head of the Health Department, South Australia, Winchester Street, East Adelaide, South Australia	
1880		Smith, William Robert, M.D., D.Sc., Barrister-at-Law, Professor of Forensic Medicine in King's College, 74 Great Russell Street, Bloomsbury Square, London	
1899		Snell, Ernest Hugh, M.D., B.Sc., D.P.H. Camb., Coventry	
1880		Sollas, W. J., M.A., D.Sc., LL.D., F.R.S., late Fellow of St John's College, Cambridge, and Professor of Geology and Palæontology in the University of Oxford	505
1889	C.	Somerville, Wm., M.A., D.Sc., D.Oec., Sibthorpean Professor of Rural Economy in the University of Oxford, 121 Banbury Road, Oxford	
1882		Sorley, James, F.I.A., F.F.A., C.A., 82 Onslow Gardens, London	
1896		* Spence, Frank, M.A., B.Sc., 25 Craiglea Drive	
1874	C.	Sprague, T. B., M.A., LL.D., Actuary, 29 Buckingham Terrace	
1906		Squance, Thomas Coke, M.D., Physician and Pathologist in the Sunderland Infirmary, 15 Grange Crescent, Sunderland	510
1891		* Stanfield, Richard, Professor of Mechanics and Engineering in the Heriot-Watt College	
1886	C.	* Stevenson, Charles A., B.Sc., Memb. Inst. C.E., 28 Douglas Crescent	
1884		Stevenson, David Alan, B.Sc., Memb. Inst. C.E., 45 Melville Street	
1888	C.	* Stewart, Charles Hunter, D.Sc., M.B., C.M., Professor of Public Health in the University of Edinburgh, 9 Learmonth Gardens	
1904		* Stewart, Thomas W., M.A., B.Sc., Science Master, Edinburgh Ladies' College, 29 Bruntsfield Gardens	515

Alphabetical List of the Ordinary Fellows of the Society. 785

Date of Election.			
1877		Stirling, William, D.Sc., M.D., LL.D., Brackenbury Professor of Physiology and Histology in Owens College and Victoria University, Manchester	
1902		* Stockdale, Herbert Fitton, Clairinch, Upper Helensburgh, Dumbartonshire	
1889		* Stockman, Ralph, M.D., F.R.C.P.E., Professor of Materia Medica and Therapeutics in the University of Glasgow	
1906		Story, Fraser, Lecturer in Forestry, University College, Bangor, North Wales	
1907		* Strong, John, B.A., Rector of Montrose Academy, 11 Union Place, Montrose	520
1903		Sutherland, David W., M.D., M.R.C.P. Lond., Captain, Indian Medical Service, Professor of Pathology and Materia Medica, Medical College, Lahore, India	
1896		* Sutherland, John Francis, M.D., Dep. Com. in Lunacy for Scotland, Scotsburn Road, Tain, Ross-shire	
1905		Swithinbank, Harold William, Denham Court, Denham, Bucks	
1885	C.	Symington, Johnson, M.D., F.R.C.S.E., F.R.S., Prof. of Anatomy in Queen's College, Belfast	
1904		* Tait, John W., B.Sc., Rector of Leith Academy, 18 Netherby Road, Leith	525
1898	C.	Tait, William Archer, B.Sc., Memb. Inst. C.E., 38 George Square	
1895		Talmage, James Edward, D.Sc., Ph.D., F.R.M.S., F.G.S., Professor of Geology, Univ. of Utah, Salt Lake City, Utah	
1890	C.	Tanakadate, Aikitu, Prof. of Nat. Phil. in the Imperial University of Japan, Tokyo, Japan	
1870		Tatlock, Robert R., F.C.S., City Analyst's Office, 156 Bath Street, Glasgow	
1899		* Taylor, James, M.A., Mathematical Master in the Edinburgh Academy, Edinburgh Academy	530
1892		Thackwell, J. B., M.B., C.M.	
1885	C.	Thompson, D'Arcy W., C.B., B.A., F.L.S., Professor of Natural History in University College, Dundee	
1907		* Thompson, John Hannay, M. Inst. C.E., M. Inst. Mech. E., Engineer to the Dundee Harbour Trust, Earlville, Broughty Ferry	
1905		* Thoms, Alexander, 7 Playfair Terrace, St Andrews	
1887		* Thomson, Andrew, M.A., D.Sc., F.I.C., Rector, Perth Academy, Ardenlea, Pitcullen, Perth	535
1896		* Thomson, George Ritchie, M.B., C.M., Cumberland House, Von Brandis Square, Johannesburg, Transvaal	
1903		Thomson, George S., F.C.S., Dairy Commissioner for Queensland, Department of Agriculture, Brisbane, Queensland	
1906		* Thomson, Gilbert, C.E., 164 Bath Street, Glasgow	
1887	C.	* Thomson, J. Arthur, M.A., Regius Prof. of Natural History in the Univ. of Aberdeen	
1906		Thomson, James Stuart, F.L.S. (Assistant Professor of Zoology, South African College, Cape Town), 24 Brückfeldstrasse, Bern, Switzerland	540
1880		Thomson, John Millar, LL.D., F.R.S., Prof. of Chem. in King's College, Lond., 9 Campden Hill Gardens, London	
1899		* Thomson, R. Tatlock, F.C.S., 156 Bath Street, Glasgow	
1870		Thomson, Spencer C., Actuary, 10 Eglinton Crescent	
1882		Thomson, Wm., M.A., B.Sc., LL.D., Registrar, University of the Cape of Good Hope, University Buildings, Cape Town	
1876	C.	Thomson, William, Royal Institution, Manchester	545
1874	C. B. N.	Traquair, R. H., M.D., LL.D., F.R.S., F.G.S., late Keeper of the Natural History Collections in the Royal Scottish Museum, Edinburgh (VICE-PRESIDENT), The Bush, Colinton	
1874		Tuke, Sir J. Batty, M.D., D.Sc., LL.D., F.R.C.P.E., M.P. for the Universities of Edinburgh and St Andrews, 20 Charlotte Square	
1888		* Turnbull, Andrew H., Actuary, The Elms, Whitehouse Loan	
1905		* Turner, Arthur Logan, M.D., F.R.C.S.E., 27 Walker Street	
1906	C.	* Turner, Dawson F. D., B.A., M.D., F.R.C.P.E., M.R.C.P. Lond., Lecturer on Physics, Surgeon's Hall, and Physician in charge of Electrical Department, Royal Infirmary, Edinburgh, 37 George Square	550
1861	K. N. C.	Turner, Sir William, K.C.B., M.B., F.R.C.S.E., LL.D., D.C.L., D.Sc. Dub., F.R.S., Principal of the University of Edinburgh (PRESIDENT), 6 Eton Terrace	
1895		Turton, Albert H., M.I.M.M., 18 Harrow Road, Bowenbrook, Birmingham	
1898	C.	* Tweedie, Charles, M.A., B.Sc., Lecturer on Mathematics in the University of Edinburgh, 40 Gillespie Crescent	
1889		Underhill, T. Edgar, M.D., F.R.C.S.E., Dunedin, Barnt Green, Worcester-shire	

Date of Election.			
1906		Vandenbergh, William J., Barrister-at-Law, S.S.C., F.R.S.L., F.R.M.S., 29-32 Exchange Buildings, Pirie Street, Adelaide, S. Australia	555
1888		Walker, James, Memb. Inst. C.E., Engineer's Office, Tyne Improvement Commission, Newcastle-on-Tyne	
1891	C. B.	* Walker, James, D.Sc., Ph.D., LL.D., F.R.S., Professor of Chemistry in the University of Edinburgh, 5 Wester Coates Road	
1873	C.	Walker, Robert, M.A., LL.D., University, Aberdeen	
1902		* Wallace, Alexander G., M.A., 56 Fonthill Road, Aberdeen	
1886	C.	* Wallace, R., F.L.S., Professor of Agriculture and Rural Economy in the University of Edinburgh	560
1898		Wallace, Wm., M.A., Belvedere, Alta, Canada	
1891		* Walmsley, R. Mullineux, D.Sc., Prin. of the Northampton Inst., Clerkenwell, London	
1907		Waters, E. Wynston, Medical Officer, H.B.M. Administration, E. Africa, Malindi, British East Africa Protectorate, <i>via</i> Mombasa	
1901	C.	* Waterston, David, M.A., M.D., F.R.C.S.E., Lecturer on Regional Anatomy in the University of Edinburgh, 1 Coates Place	
1904		* Watson, Charles B. Boog, Huntly Lodge, 1 Napier Road	565
1862	C.	Watson, Rev. Robert Boog, B.A., LL.D., F.L.S., Past President of the Concho- logical Society, 11 Strathearn Place	
1900		* Watson, Thomas P., M.A., B.Sc., Principal, Keighley Institute, Keighley	
1907		* Watt, Andrew, M.A., Secretary to the Scottish Meteorological Society, 6 Woodburn Terrace	
1896		Webster, John Clarence, B.A., M.D., F.R.C.P.E., Professor of Obstetrics and Gynæcology, Rush Medical College, Chicago, 706 Reliance Buildings, 100 State Street, Chicago	
1907	C.	* Wedderburn, Ernest Maclagan, M.A., LL.B., 6 Succoth Gardens	570
1903	C.	* Wedderburn, J. H. Maclagan, M.A., D.Sc., 11 Alexander Street, Princeton, N.J., U.S.A.	
1904		Wedderspoon, William Gibson, M.A., LL.D., Indian Educational Service, Senior Inspector of Schools, Burma, The Education Office, Rangoon, Burma	
1896		Wenley, Robert Mark, M.A., D.Sc., D.Phil., Litt.D., LL.D., Professor of Philosophy in the University of Michigan, Ann Arbor, Mich., U.S.A.	
1909	C.	* Westergaard, Reginald Ludovic Andreas Emil, Lecturer in Technical Mycology, Heriot Watt College, 6 Suffolk Road, Edinburgh	
1896	C.	White, Philip J., M.B., Prof. of Zoology in University College, Bangor, North Wales	575
1890		White, Sir William Henry, K.C.B., Memb. Inst. C.E., LL.D., F.R.S., late Assistant Controller of the Navy, and Director of Naval Construction, Cedarscroft, Putney Heath, London	
1881		Whitehead, Walter, F.R.C.S.E., late Professor of Clinical Surgery, Owens College and Victoria University, Birchfield, Rusholme, Manchester	
1894		Whymper, Edward, F.R.G.S., Holmwood, Waldegrave Road, Teddington, Middlesex	
1879		Will, John Charles Ogilvie, of Newton of Pitfodels, M.D., 17 Bon-Accord Square, Aberdeen	
1897		* Williams, W. Owen, F.R.C.V.S., Professor of Veterinary Medicine and Surgery, University of Liverpool, The Veterinary School, The University, Liverpool	580
1908		* Williamson, Henry Charles, M.A., D.Sc., Naturalist to the Fishery Board for Scotland, 28 Polmuir Road, Aberdeen	
1900		Wilson, Alfred C., F.C.S., Voewood Croft, Stockton-on-Tees	
1879		Wilson, Andrew, Ph.D., F.L.S., Lecturer on Zoology and Comparative Anatomy, 110 Gilmore Place	
1902		* Wilson, Charles T. R., M.A., F.R.S., Glencorse House, Peebles, and Sidney Sussex College, Cambridge	
1895		Wilson-Barker, David, F.R.G.S., Captain-Superintendent Thames Nautical Training College, H.M.S. "Worcester," Greenhithe, Kent	585
1882		Wilson, George, M.A., M.D., LL.D., 7 Avon Place, Warwick	
1891		* Wilson, John Hardie, D.Sc., University of St Andrews, 39 South Street, St Andrews	
1902		Wilson, William Wright, F.R.C.S.E., M.R.C.S. Eng., Cottesbrook House, Acock's Green, Birmingham	
1908		* Wood, Thomas, M.D., Eastwood, 182 Ferry Road, Bonnington, Leith	
1886	C.	* Woodhead, German Sims, M.D., F.R.C.P.E., Professor of Pathology in the University of Cambridge	590
1884		Woods, G. A., M.R.C.S., Eversleigh, 1 Newstead Road, Lee, Kent	
1890		* Wright, Johnstone Christie, Northfield, Colinton	

Alphabetical List of the Ordinary Fellows of the Society. 787

Date of Election.	
1896	* Wright, Robert Patrick, Professor of Agriculture, West of Scotland Agricultural College, 6 Blythswood Square, Glasgow
1882	Young, Frank W., F.C.S., H.M. Inspector of Science and Art Schools, 32 Buckingham Terrace, Botanic Gardens, Glasgow
1892	Young, George, Ph.D., 79 Harvard Court Mansions, Honeybourne Road, West Hampstead, London, N.W. 595
1896	C. * Young, James Buchanan, M.B., D.Sc., Dalveen, Braeside, Liberton
1900	* Young, J. M'Lauchlan, F.R.C.V.S., Lecturer on Veterinary Hygiene, University of Aberdeen
1904	Young, R. B., M.A., B.Sc., Transvaal Technical Institute, Johannesburg, Transvaal

LIST OF HONORARY FELLOWS OF THE SOCIETY

At October 1909.

HIS MOST GRACIOUS MAJESTY THE KING.

FOREIGNERS (LIMITED TO THIRTY-SIX BY LAW X.).

Elected

1897 Alexander Agassiz,	<i>Cambridge (Mass.).</i>
1897 E.-H. Amagat,	<i>Paris.</i>
1900 Arthur Auwers,	<i>Berlin.</i>
1900 Adolf Ritter von Baeyer,	<i>Munich.</i>
1905 Waldemar Chr. Brögger,	<i>Christiania.</i>
1897 Stanislao Cannizzaro,	<i>Rome.</i>
1905 Moritz Cantor,	<i>Heidelberg.</i>
1902 Jean Gaston Darboux,	<i>Paris.</i>
1905 Paul Ehrlich,	<i>Frankfurt-a.-M.</i>
1908 Emil Fischer,	<i>Berlin.</i>
1902 Albert Gaudry,	<i>Paris.</i>
1905 Paul Heinrich Groth,	<i>Munich.</i>
1888 Ernst Haeckel,	<i>Jena.</i>
1883 Julius Hann,	<i>Graz.</i>
1908 George William Hill,	<i>New York.</i>
1879 Jules Janssen,	<i>Paris.</i>
1908 Friedrich Wilhelm Georg Kohlrausch,	<i>Charlottenburg.</i>
1897 Gabriel Lippmann,	<i>Paris.</i>
1895 Éleuthère-Élie-Nicolas Mascart,	<i>Paris.</i>
1895 Carl Menger,	<i>Vienna.</i>
1897 Fridtjof Nansen,	<i>Christiania.</i>
1908 Henry Fairfield Osborn,	<i>New York.</i>
1908 Iwan P. Pawlov,	<i>St Petersburg.</i>
1905 Eduard Pflüger,	<i>Bonn.</i>
1895 Jules Henri Poincaré,	<i>Paris.</i>
1889 Georg Hermann Quincke,	<i>Heidelberg.</i>
1908 Gustaf Retzius,	<i>Stockholm.</i>
1908 Augusto Righi,	<i>Bologna.</i>
1897 Giovanni V. Schiaparelli,	<i>Milan.</i>
1905 Eduard Suess,	<i>Vienna.</i>
1908 Louis Joseph Troost,	<i>Paris.</i>
1905 Wilhelm Waldeyer,	<i>Berlin.</i>
1905 Wilhelm Wundt,	<i>Leipzig.</i>
1897 Ferdinand Zirkel,	<i>Bonn am Rhein.</i>

Total, 34.

BRITISH SUBJECTS (LIMITED TO TWENTY BY LAW X.).

Elected

1889 Sir Robert Stawell Ball, Kt., LL.D., F.R.S., M.R.I.A., Lowndean Professor of Astronomy in the University of Cambridge,	<i>Cambridge.</i>
1892 Colonel Alexander Ross Clarke, C.B., R.E., F.R.S.,	<i>Redhill, Surrey.</i>
1897 Sir George Howard Darwin, K.C.B., M.A., LL.D., F.R.S., Plumian Professor of Astronomy in the University of Cambridge,	<i>Cambridge.</i>
1900 David Ferrier, M.D., LL.D., F.R.S., Professor of Neuro- Pathology, King's College, London,	<i>London.</i>
1900 Andrew Russell Forsyth, D.Sc., F.R.S., Sadlerian Professor of Pure Mathematics in the University of Cambridge,	<i>Cambridge.</i>
1892 Sir David Gill, K.C.B., LL.D., F.R.S., formerly His Majesty's Astronomer at the Cape of Good Hope,	<i>London.</i>
1895 Albert C. L. G. Günther, Ph.D., F.R.S.,	<i>London.</i>
1883 Sir Joseph Dalton Hooker, K.C.S.I., M.D., LL.D., D.C.L., F.R.S., Corresp. Mem. Inst. of France,	<i>London.</i>

1884 Sir William Huggins, K.C.B., LL.D., D.C.L., P.R.S., Corresp. Mem. Inst. of France,	<i>London.</i>
1908 Sir Alexander B. W. Kennedy, LL.D., F.R.S., Past Pres. Inst. C.E.,	<i>London.</i>
1908 Sir Edwin Ray Lankester, K.C.B., LL.D., F.R.S.,	<i>London.</i>
1900 Archibald Liversidge, LL.D., F.R.S., Professor of Chemistry in the University of Sydney,	<i>Sydney.</i>
1908 Sir James A. H. Murray, LL.D., D.C.L., Editor of the Oxford English Dictionary,	<i>Oxford.</i>
1905 Sir William Ramsay, K.C.B., LL.D., F.R.S., Professor of Chemistry in the University College, London,	<i>London.</i>
1886 The Lord Rayleigh, D.C.L., LL.D., D.Sc. Dub., F.R.S., Corresp. Mem. Inst. of France,	<i>London.</i>
1908 Charles S. Sherrington, M.A., M.D., LL.D., F.R.S., Holt Professor of Physiology in the University of Liverpool,	<i>Liverpool.</i>
1905 Sir Joseph John Thomson, D.Sc., LL.D., F.R.S., Cavendish Professor of Experimental Physics, University of Cambridge,	<i>Cambridge.</i>
1900 Thomas Edward Thorpe, D.Sc., LL.D., F.R.S., Principal of the Government Laboratories, London,	<i>London.</i>
1895 Sir Charles Todd, K.C.M.G., F.R.S., Government Astronomer, South Australia,	<i>Adelaide.</i>
Total, 19.	

ORDINARY FELLOWS OF THE SOCIETY ELECTED

During Session 1908-9.

(Arranged according to the date of their election.)

21st December 1908.

WILLIAM GENTLE, B.Sc.

18th January 1909.

JAMES GORDON GRAY, B.Sc.
ALEX. DAVID ROSS, M.A., B.Sc.

HENRY RICHARD KENWOOD, M.B.
C. R. MARSHALL, M.D., M.A.

15th February 1909.

JOHN BROWNLEE, M.A., M.D.
ARTHUR G. S. MITCHELL.

BERNARD LANGLEY MILLS, M.D., F.R.C.S.E.
REGINALD L. A. E. WESTERGAARD.

15th March 1909.

PETER COMRIE, M.A., B.Sc.

17th May 1909.

THOMAS MORRISON CLAYTON, M.D. D.Hy.

AUCLAND CAMPBELL GEDDES, M.D.

12th July 1909.

HUGH STEUART GLADSTONE, M.A.

ORDINARY FELLOWS DECEASED AND RESIGNED

During Session 1908-9.

DECEASED.

GEORGE SETON, M.A.
 Professor THOMAS GRAY, B.Sc.
 D. M. C. L. ARGYLL ROBERTSON, M.D.
 FRANCIS ELGAR, M.Inst. C.E., LL.D.
 Rev. Professor DUNS, D.D.
 WALTER STEWART.
 ROBERT HENRY BOW, C.E.
 Professor D. J. HAMILTON, LL.D.

Em. Professor S. S. LAURIE, M.A., LL.D.
 GEORGE BROADRICK, Memb. Inst. C.E.
 Professor D. J. CUNNINGHAM, M.D., LL.D.
 JOHN A. W. DOLLAR, M.R.C.V.S.
 ANDREW FULLER HARGREAVES, F.C.S.
 Em. Professor HUGH BLACKBURN, M.A., LL.D.
 Sir ARTHUR MITCHELL, K.C.B.
 PATRICK DOYLE, C.E.

RESIGNED.

T. G. NASMYTH, M.D.

HENRY COATES.

FOREIGN HONORARY FELLOWS DECEASED.

SIMON NEWCOMB (Washington, U.S.A.).

ANTON DOHRN (Naples).

BRITISH HONORARY FELLOW DECEASED.

EDWARD CAIRD, LL.D. (Oxford).

ABSTRACT

OF

THE ACCOUNTS OF JAMES CURRIE, ESQ.,*As Treasurer of the Royal Society of Edinburgh.*

SESSION 1908-1909.

I. ACCOUNT OF THE GENERAL FUND.

CHARGE.

1. Arrears of Contributions at 1st October 1908		£219	9	0
2. Contributions for present Session :—				
1. 162 Fellows at £2, 2s. each	£340	4	0	
133 Fellows at £3, 3s. each	418	19	0	
	£759	3	0	
<i>Less included in payments in lieu of future contributions</i>		2	2	0
	£757	1	0	
2. Commutation Fee in lieu of Future Contributions of one Fellow	21	0	0	
3. Fees of Admission and Contributions of nine new Resident Fellows at £5, 5s. each	47	5	0	
4. Fees of Admission of three new Non-Resident Fellows at £26, 5s. each	78	15	0	
				904 1 0
3. Interest received—				
Interest, less Tax	£372	4	8	
Annuity from Edinburgh and District Water Trust, less Tax	49	16	7	
				422 1 3
4. Transactions and Proceedings sold				143 2 0
5. Annual Grant from Government				600 0 0
				£2288 13 3
Amount of the Charge				

DISCHARGE.

1. RENT OF SOCIETY'S APARTMENTS for Half-year to Martinmas 1908, less Tax		£142	10	0
2. TAXES, INSURANCE, COAL AND LIGHTING :—				
Property Tax	£7	10	0	
Insurance	11	1	6	
Coal	13	8	9	
Gas	0	13	9	
Electric Light	2	9	7	
				35 3 7
3. SALARIES :—				
General Secretary	£100	0	0	
Librarian	100	0	0	
Assistant Librarian	62	10	0	
Office Keeper (S. Heddle)	65	0	6	
Do. (Mrs Hay)	8	14	5	
Doorkeeper	5	15	8	
Treasurer's Clerk	25	0	0	
Allowance to Widow of Mr Hardy, late Librarian	62	10	0	
				429 10 7
Carry forward				£607 4 2

792 Proceedings of the Royal Society of Edinburgh.

	Brought forward	. . .	£607 4 2
4. EXPENSES OF TRANSACTIONS:—			
Neill & Co., Ltd., Printers		£262 1 2	
<i>Less</i> Received from Scottish National Antarctic Expedition, per Whitson & Methuen		25 0 0	
		<hr/>	
		£237 1 2	
Neill & Co., Ltd., Printers, for illustrations		34 14 0	
Do. (Ben Nevis)	£155 2 11		
<i>Less</i> Received from the Carnegie Trust per the Meteorological Society of Scotland	100 0 0		
		<hr/>	
		55 2 11	
M'Farlane & Erskine, Lithographers	£134 13 0		
<i>Less</i> Received from the Scottish National Antarctic Expedition, per Whitson & Methuen	58 0 0		
		<hr/>	
		76 13 0	
James Green, Lithographer		33 12 0	
G. Waterston & Sons, do.		5 7 6	
J. Bartholomew & Co., do.		3 5 0	
Hislop & Day, Engravers		17 16 0	
Orrock & Son, Bookbinders		70 2 6	
		<hr/>	
			533 14 1
5. EXPENSES OF PROCEEDINGS:—			
Neill & Co., Ltd., Printers		£603 6 11	
Do. (for illustrations)		56 18 6	
M'Farlane & Erskine, Lithographers		4 5 6	
Hislop & Day, Engravers		3 19 0	
		<hr/>	
			668 9 11
6. BOOKS, PERIODICALS, NEWSPAPERS, ETC.:—			
Otto Schulze & Co., Booksellers		£119 2 2	
James Thin, do.		52 3 9	
R. Grant & Son, do.		7 6 4	
Wm. Green & Sons, do.		0 15 6	
International Catalogue of Scientific Literature		17 0 0	
Robertson & Scott, News Agents		5 6 6	
Egypt Exploration Funds Subscription		3 3 0	
Ray Society do.		1 1 0	
Palæontographical Society do.		1 1 0	
Journal de Conchyliologie		0 15 0	
Orrock & Son, Bookbinders		16 12 3	
		<hr/>	
			224 6 6
7. OTHER PAYMENTS:—			
Neill & Co., Ltd., Printers		£65 8 0	
R. Blair & Son, Confectioners		30 14 2	
S. Duncan, Tailor (uniforms)		11 18 0	
Lantern Exhibitions, etc., at Lectures		6 10 0	
Lindsay, Jamieson & Haldane		6 6 0	
National Telephone Co.		9 10 0	
Tods, Murray & Jamieson, W.S.		11 6 0	
Petty Expenses, Postages, Carriage, etc.		65 16 7	
		<hr/>	
			207 8 9
8. SPECIAL EXPENSES CONSEQUENT ON REMOVAL:—			
A. Dishington, Assistant at Removal		£49 0 0	
J. F. Ellis, do.		30 10 0	
Edinburgh Corporation, Electric Light		12 3 3	
Morison & Co., Upholsterers		21 12 6	
J. & T. Scott, Cabinetmakers		13 6 9	
C. H. Woolford, Artist, Cleaning Pictures		32 10 0	
Aitken, Dott & Son, do.		23 19 6	
Drummond, Young & Watson, Photographers		5 17 0	
J. Cavagnari, Cleaning Busts		3 0 0	
A. & J. M'Nab, Cleaners		2 5 9	
		<hr/>	
			194 4 9
9. IRRECOVERABLE ARREARS of Contributions written off			9 9 0
			<hr/>
	Carry forward	. . .	£2444 17 2

Abstract of Accounts.

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	Brought forward	£2444 17 2
10. ARREARS of CONTRIBUTIONS outstanding at 1st October 1909 :—		
Present Session, per list	£102 18 0	
Previous Session, per list	166 19 0	
	269 17 0	
Amount of the Discharge	£2714 14 2	
Amount of the Discharge	£2714 14 2	
Amount of the Charge	2288 13 3	
Excess of the Discharge	£426 0 11	
FLOATING BALANCE DUE BY THE SOCIETY at 1st October 1908	£202 7 3	
<i>Add</i> Excess of Discharge as above	426 0 11	
Floating Balance due by the Society at 1st October 1909	£628 8 2	
<i>Being—</i>		
Balance due to Union Bank on Current Account	£628 8 2	

II. ACCOUNT OF THE KEITH FUND

To 1st October 1909.

CHARGE.

1. BALANCE due by Union Bank at 1st October 1908	£24 18 10
2. INTEREST RECEIVED :—	
On £896, 19s.1d. North British Railway Company 3 per cent. Debenture Stock for year to Whitsunday 1909, less Tax	£25 10 11
On £211, 4s. North British Railway Company 3 per cent. Lien Stock for year to Lammas 1909, less Tax	6 0 0
	31 10 11
	£56 9 9

DISCHARGE.

Nil.

1. BALANCE due by Union Bank at 1st October 1909	£56 9 9
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III. ACCOUNT OF THE NEILL FUND

To 1st October 1909.

CHARGE.

1. BALANCE on hand—Uncashed Dividend Warrants, less sum due to the Union Bank at 1st October 1908	£30 16 2
2. INTEREST RECEIVED :—	
On £355 London, Chatham and Dover Railway 4½ per cent. Arbitration Debenture Stock for year to 30th June 1909, less Tax	£15 2 10
<i>Less</i> Interest paid to Union Bank	0 4 9
	14 18 1
	£45 14 3

DISCHARGE.

Nil.

BALANCE due by Union Bank at 1st October 1909	£45 14 3
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IV. ACCOUNT OF THE MAKDOUGALL-BRISBANE FUND

To 1st October 1908.

CHARGE.

1. BALANCE due at 1st October 1908 :—			
By Union Bank of Scotland on Deposit Receipt	£135	0	0
By Union Bank of Scotland on Current Account	46	17	11
			<u>£181 17 11</u>
2. INTEREST RECEIVED on £365 Caledonian Railway Company 4 per cent. Consolidated Preference Stock No. 2 for year to 30th June 1909, less Tax	£13	16	7
On Deposit Receipt with Union Bank	1	9	8
			<u>15 6 3</u>
			<u>£197 4 2</u>

DISCHARGE.

Nil.

3. BALANCE due at 1st October 1909 :—			
By Union Bank of Scotland on Deposit Receipt	£135	0	0
By Union Bank of Scotland on Current Account	62	4	2
			<u>£197 4 2</u>

V. ACCOUNT OF THE MAKERSTOUN MAGNETIC METEOROLOGICAL OBSERVATION FUND

To 1st October 1909.

CHARGE.

SUM on Deposit Receipt with the Union Bank of Scotland at 1st October 1908	£212	10	3
INTEREST thereon	2	6	8
			<u>£214 16 11</u>

DISCHARGE.

Nil.

Above SUM on Deposit Receipt with the Union Bank of Scotland at 1st October 1909	£214	16	11
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VI. ACCOUNT OF THE GUNNING-VICTORIA JUBILEE PRIZE FUND

To 1st October 1909.

(Instituted by Dr R. H. GUNNING of Edinburgh and Rio de Janeiro.)

CHARGE.

1. BALANCE due by Union Bank at 1st October 1908	£102	17	10
2. INTEREST received on £1000 North British Railway Company Consolidated Lien Stock for year to Lammass 1909, less Tax	28	8	5
			<u>£131 6 3</u>

DISCHARGE.

Nil.

BALANCE due by the Union Bank of Scotland on Current Account at 1st October 1909	£131	6	3
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**STATE OF THE FUNDS BELONGING TO THE ROYAL
SOCIETY OF EDINBURGH**

As at 1st October 1909.

1. GENERAL FUND—

1. £2090, 9s. 4d. three per cent. Lien Stock of the North British Railway Company at $82\frac{7}{8}$ per cent., the selling price at 1st October 1909	£1732	7	7
2. £8519, 14s. 3d. three per cent. Debenture Stock of do. at $85\frac{1}{2}$ per cent., do.	7284	7	10
3. £52, 10s. Annuity of the Edinburgh and District Water Trust, equivalent to £875 at 172 per cent., do.	1505	0	0
4. £1811 four per cent. Debenture Stock of the Caledonian Railway Company at $113\frac{3}{4}$ per cent., do.	2060	2	0
5. £35 four and a half per cent. Arbitration Debenture Stock of the London, Chatham and Dover Railway Company at $117\frac{1}{8}$ per cent., do.	40	19	10
6. Arrears of Contributions, as per preceding Abstract of Accounts	269	17	0
		<hr/>	
	£12,892	12	5
<i>Deduct</i> Floating Balance due by the Society, as per preceding Abstract of Accounts		628	8
		<hr/>	
AMOUNT	£12,264	4	3

Exclusive of Library, Museum, Pictures, etc., Furniture of the Society's Rooms at George Street.

2. KEITH FUND—

1. £896, 19s. 1d. three per cent. Debenture Stock of the North British Railway Company at $85\frac{1}{2}$ per cent., the selling price at 1st October 1909	£766	18	8
2. £211, 4s. three per cent. Lien Stock of do. at $82\frac{7}{8}$ per cent., do.	175	1	2
3. Balance due by the Union Bank of Scotland	56	9	9
		<hr/>	
AMOUNT	£998	9	7

3. NEILL FUND—

1. £355 four and a half per cent. Arbitration Debenture Stock of the London, Chatham and Dover Railway Company at $117\frac{1}{8}$ per cent., the selling price at 1st October 1909	£415	15	5
2. Balance due by Union Bank of Scotland	45	14	3
		<hr/>	
AMOUNT	£461	9	8

4. MAKDOUGALL-BRISBANE FUND—

1. £365 four per cent. Consolidated Preference Stock No. 2 of the Caledonian Railway Company at $104\frac{1}{2}$ per cent., the selling price at 1st October 1909	£381	8	6
2. Sum on Deposit Receipt with the Union Bank of Scotland	135	0	0
3. Balance due by do. on Current Account	62	4	2
		<hr/>	
AMOUNT	£578	12	8

5. MAKERSTOUN MAGNETIC METEOROLOGICAL OBSERVATION FUND—

Sum on Deposit Receipt with the Union Bank of Scotland at 1st October 1909	£214	16	11
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6. GUNNING-VICTORIA JUBILEE PRIZE FUND—Instituted by Dr Gunning of Edinburgh and Rio de Janeiro—

1. £1000 three per cent. Consolidated Lien Stock of the North British Railway Company at $82\frac{7}{8}$ per cent., the selling price at 1st October 1909	£828	16	0
2. Balance due by the Union Bank of Scotland on Current Account	131	6	3
		<hr/>	
AMOUNT	£960	2	3

EDINBURGH, 20th October 1909.—We have examined the six preceding Accounts of the Treasurer of the Royal Society of Edinburgh for the Session 1908–1909, and have found them to be correct. The securities of the various Investments at 1st October 1909, as noted in the above Statement of Funds, have been exhibited to us.

LINDSAY, JAMIESON & HALDANE,
Auditors.

**VIDIMUS of ESTIMATED INCOME of THE GENERAL FUND FOR
SESSION 1909-1910.**

I. INTEREST:—

On £8519, 14s. 3d. Railway Debenture Stock at 3 per cent.	£255 11 10
On £2090, 9s. 4d. Railway Lien Stock at 3 per cent.	62 14 4
On £1811 Railway Debenture Stock at 4 per cent.	72 8 8
On £35 Railway Debenture Stock at 4½ per cent.	1 11 6
	£392 6 4
2. ANNUITY from the Edinburgh and District Water Trust	52 10 0
	£444 16 4
<i>Deduct</i> Income Tax at 1s. 2d. per £	25 18 10
	£418 17 6
3. ANNUAL CONTRIBUTIONS:—	
Of 161 Fellows at £2, 2s. each	£338 2 0
Of 130 Fellows at £3, 3s. each	409 10 0
	747 12 0
4. ANNUAL GRANT from Government	600 0 0
5. SALES of Society's Transactions	30 0 0
	TOTAL ESTIMATED INCOME, £1796 9 6

Exclusive of Fees of Admission and Contributions of New Fellows
who may be admitted during the Year.

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