6

40387

60

PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

PROCEEDINGS

OF

THE ROYAL SOCIETY

OF

EDINBURGH.

VOL. XXXI.

NOVEMBER 1910 TO JULY 1911.

EDINBURGH:
PRINTED BY NEILL AND COMPANY, LIMITED.
MDCCCCXII.

CONTENTS.

PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

SESSION 1910–11.

Part I.] VOL. XXXI. [Pp. 1–192.

CONTENTS.

[*Continued on page* iv *of Cover.*

EDINBURGH:

Published by ROBERT GRANT & SON, 107 Princes Street, and
WILLIAMS & NORGATE, 14 Henrietta Street, Covent Garden, London.

MDCCCCX.

Price Thirteen Shillings and Elevenpence.

REGULATIONS REGARDING THE PUBLICATION OF PAPERS IN THE PROCEEDINGS AND TRANSACTIONS OF THE SOCIETY.

The Council beg to direct the attention of authors of communications to the Society to the following Regulations, which have been drawn up in order to accelerate the publication of the Proceedings and Transactions, and to utilise as widely and as fairly as possible the funds which the Society devotes to the publication of Scientific and Literary Researches.

1. Manuscript of Papers.—As soon as any paper has been passed for publication, either in its original or in any altered form, and has been made ready for publication by the author, it is sent to the printer.

2. Method of Publication.—As soon as the final revise of a Transactions paper has been returned, or as soon as the sheet in which the last part of a Proceedings paper appears is ready for press, a certain number of separate copies or reprints, in covers bearing the title of the paper and the name of the author, are printed off and placed on sale. The date of such separate publication will be printed on each paper.

3. Additions to a Paper after it has been finally handed in for publication, if accepted by the Council, will be treated and dated as separate communications, and may, or may not, be printed immediately after the original paper.

4. Brief Abstracts of Transactions Papers will be published in the Proceedings, provided they are sent along with the original paper.

5. Special Discussion of Papers accepted for Publication.—Where a paper has been accepted for publication, the Council may, with the consent of the author, select this paper for Special Discussion. In the case of such papers advanced proofs will be sent to the members of the Society desiring copies, and copies will be supplied to the author for distribution. A paper selected for Special Discussion will be marked with an asterisk (*) and placed first on the Billet for the day of reading. Any following papers for that day may be adjourned or held as read if the discussion prevents their being read.

6. Communications not submitted for Publication, such as Demonstrations of Experiments, Statement of Scientific Problems, etc., may be received by the Council, and may also be selected for Special Discussion. The Council does not undertake to publish any notice of such communications in the Proceedings or Transactions of the Society.

[*Continued on page* iii *of Cover.*

PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

VOL. XXXI. 1910-11.

I.—Craniological Observations on the Lengths, Breadths, and Heights of a Hundred Australian Aboriginal Crania. By A. W. D. Robertson, M.D. Melb., Government Research Scholar in the Anatomy Department of the University of Melbourne. *Communicated by* Professor R. J. A. Berry.

(MS. received March 30, 1910. Read May 16, 1910.)

The present work forms the first portion of an investigation into the craniological characters of the Australian aboriginal crania at present being conducted by the author, and is part of a general scheme for the osteological investigation of the Australian aboriginal initiated by the Professor of Anatomy in the University of Melbourne.

There are, as is well known, *two methods of treating craniometric statistics,* and which, for convenience, may be designated (1) The Empirical Method; (2) The Rational Method.

The object aimed at in both is to find numbers that shall be characteristic of the race of any country.

"The *Empirical method* places very little value on the length and breadth of a skull except as a means of arriving at their ratio—the cephalic index—'the subdivisions of which,' says Gray (15), 'are all arbitrary.' Some, *e.g.* Retzius, made use of centres; others, *e.g.* Welcker, made use of limits, and there appears to be no special reason for fixing the limits of the groups at one index more than another. The analysis of craniometrical statistics by this method becomes comparatively simple: calculate the cephalic index and other indices for each individual, and find the average or mean index for the whole group of people measured. The highest and lowest index is usually stated as indicating the range of variation on each side of the mean. As the range of variation of the cephalic index is usually quite as great as the range of variation of the absolute dimensions, it is difficult to understand the belief in the value of indices of the Empirical school."

Myers (19), referring to this Empirical method, when reviewing some recent work of Fawcett and Lee, also says: "It is to be hoped that never again will the old school be tolerated which collects a few measurements, dissects them, and publishes ill-founded conclusions.

"The *Rational method* aims at determining in a group of people (*a*) the frequency curve for each dimension; (*b*) the variability about the mean; (*c*) the mode or most frequent dimension; (*d*) the correlation between pairs of dimensions by the application of rigid mathematical analysis; (*e*) the estimation of the deviations (due to random sampling) of observed values from true values."

Quetelet (4) was the founder of this rational method of treating anthropometric statistics, and, as far back as 1846, applied this method to statistics of stature and chest measurements; but his theory was practically neglected by anthropologists till quite recently, when it was revived and extended by Galton (5) and (6), and Pearson (7) and (8).

Fawcett and Lee (1) were the first to apply statistical methods to any series of skull measurements, though Stieda (3) published the first scientific determination on the variabilities of the skull.

That the Rational method is to replace the Empirical or old school methods is borne out by the published opinions of recent investigators; *e.g.* Macdonnell (2) says: "I venture to think that the chief aim of craniologists at present should be to table the means, standard deviations, and correlations of further long series of skulls." And again the same writer says: "Only when that collection is far more complete will it be possible to state general conclusions applying to the whole range of craniology; and when such tables are formed for forty or fifty long series, we shall have far more light, not only on intra-racial, but on inter-racial problems."

Fawcett and Lee (1) state that "the correlation of the mean values of the chief craniological characters in fifty or one hundred races would be a most valuable investigation, breaking practically untrodden ground."

"The first determination of the correlation of any parts of the skull was," say the same authors (1), "made in 1895, and published by Pearson (8), who correlated length and breadth of skull in modern Germans, modern French, and the Naquada crania. In his memoir on Spurious Correlation (9), published in 1896, he gives the correlation values of the length, breadth, and height, and the two cephalic indices."

Further correlation results were published in 1899 by Dr Alice Lee (12) and by Boas (14), both of whom dealt with the correlation of cranial capacity with length, breadth, and height, and with the cephalic index.

Fawcett and Lee (1) state (*a*) "that craniometry cannot in future content itself with either the raw measurements, tables of mere average, or graphical exhibition of correlation results, but must adopt the methods of modern statistical investigation, tabulating means, variabilities, correlations, and their probable errors, in order to draw safe inferences and make racial comparisons. (*b*) The relationship between cranial characters, as exhibited by their coefficients of correlation in the case of Naquada and other races, is seen to be low, and to vary much from race to race. It is therefore very doubtful how far it is legitimate to press results found for individuals of one race upon those of another. We cannot pass from intra-racial to inter-racial conclusions, but we must work towards a knowledge of inter-racial correlation; and the first step in the direction should be to obtain the average values of some forty or fifty characters in fifty or one hundred races, measured on some uniform plan. Only on such inter-racial correlations will it be possible to establish a properly founded statistical theory of race in man."

From the foregoing opinions of the leading and most recent investigators of craniological data it is obviously imperative that all investigations into the characteristics of any group of crania should be conducted upon modern biometric plans, and this work embraces the application of the "Rational method" of dealing with the craniological data of the Australian aboriginal with the following objects:—

1. The location of the position of the Australian aboriginal with the races of man.

2. The recording of the results for future use, as has been suggested by the investigators already named, and by the Wistar Institute of Anatomy and Biology, Philadelphia (21).

3. The comparison of the results obtained with those for other races that have been already worked out.

The *correct sexual separation of skulls* is a matter of great difficulty, and has been referred to by many investigators; *e.g.* Parsons (16) speaks thus: "As long as the sexing of the skulls remains a question of individual judgment, nothing approaching mathematical accuracy can be expected from the results of measurements, though I find that by shifting a block of 100 of more or less doubtful skulls from one sex to the other, I have only succeeded in altering the average length index 0·1 per cent."

Pearson (7) says: "While in observations made on the living, or in the case of hospital data, the sex can be at once distinguished, the determination of sex from the skull is not so straightforward a character."

Warren (23) refers to the matter in connection with the Naquada race: "The determination of sex has been a matter of very great difficulty, and some skeletons were assigned to the sex towards which they possessed a preponderance of characters."

Lee and Pearson (13) say "that there is a greater equality of variation for the two sexes in uncivilised than in civilised races as regards the organs, and the same applies to the skull"; and conclude by stating that "in uncivilised races the sexes are more nearly equal in the matter of size variation and correlation than in the case of civilised races"; and hence these authors would also appear to be convinced of this difficulty.

Klaatsch (17), in referring directly to Australian skulls, says: "In some cases to distinguish sex by the skull is impossible. As regards an isolated skull, I should say that in a great majority of cases the specialist who has devoted his whole attention to the matter would be able to speak with certainty."

Aeby, quoted by Havelock Ellis (22), states that there are no sexual differences in the skull except size.

Virchow, also quoted by Ellis, says, "that amongst non-European races it is extremely difficult to determine the sex from the skulls, as criteria for one race do not hold good for other races, although amongst some savage races the differences may be colossal."

Bartel (25), in his inaugural dissertation, states that "owing, therefore, to the difficulty in separating collections of skulls into male and female, it seems to me that it would always be wise to work out the constants with their probable errors for the skulls as a whole, as well as the constants for male and female of any race."

I can endorse the opinions expressed by the above authors, for I have found the difficulty of determining the sex of the individual Australian aboriginal skulls to be a very real one, and consequently I am inclined to place more reliance and weight on the results obtained for the crania irrespective of their division into sexes. In the present work I have separated them, as far as I am able, into male and female. This has been done with great care, but is still open to revision. The crania which I regarded as doubtful I have placed in that sex to which the majority of the determining features pointed.

One of the greatest problems in all craniological investigations is to arrive at the conclusion as to whether *the skulls under review are those of a pure or homogeneous race, or those of a mixed or heterogeneous race.*

Pearson (10) defines his conception of "pure," as applied to race, thus: "I doubt whether anything corresponding to a pure race exists in man, if by that term is meant a group absolutely without 'Blutmischung' or mixture. Such a view would mean an indefinite number of special creations or independent evolutions of man. The 'purest race,' as I have said elsewhere (11), is for me the one which has been isolated, intrabred, and selected for the longest period. It may well, in the dim past, have been a blend of the most diverse elements."

The purer the race, therefore, the more homogeneous will it be, and *vice versa.*

According to Fawcett and Lee (1), "It is very hard to obtain a homogeneous group of skulls, even fifty in number; and these, again, must be distributed between the two sexes; the probable errors therefore of constants determined from such series are proportionately large."

Macdonnell (2), in commenting on Turner's paper on Scottish Skulls (18), says, "that as they range from quite unknown antiquity to modern specimens from the dissecting-room, they cannot be considered a homogeneous series; and this conclusion is confirmed by an examination of their variability, *e.g.* the standard deviation of the glabello-occipital length in these Scottish skulls is 7·418 for males and 7·151 for females, which shows a much higher variability than is found in fairly contemporary individuals of homogeneous races."

As to the general principles concerning homogeneity or heterogeneity to be deduced from figures, Pearson (11) says: "I think they are these. The heterogeneity of any series, the variability of which for skull lengths exceeds 6·5, or for skull breadths exceeds about the same quantity, should be suspected, and the series subjected to a close examination. If the variability of the skull lengths be less than 5·5, or of the skull breadths less than 3·3, then we must suspect that the series is a rather stringently selected sample. This rule will generally enable us to distinguish between heterogeneity due to a mixture of crania from diverse races and the homogeneity of a single race, which indeed may be the product of a number of generations of cross breeding, such as we may assert of modern English, French, or Germans, but hardly with the same certainty of Ainos or Bengal castes."

According to Macdonnell (2) and Pearson (11) one must conclude, therefore, that if we get a high standard deviation, *e.g.* as in Turner's Scottish skulls (18), the material cannot be looked upon as homogeneous. On the other hand, Fawcett and Lee (1) get a high standard deviation for what they regard as a homogeneous series of French skulls, viz.

the Paris Catacomb skulls (Table III.), but the homogeneity of this series Pearson doubts.

The *hundred Australian aboriginal skulls dealt with in this paper* were obtained from the collections in the recently established Museum of the Anatomical Department of the University of Melbourne and the National Museum, Melbourne, and for access to the former collection I am indebted to the Professor of Anatomy, and for the latter to the Director of the Museum. The crania are numbered serially one to a hundred, and with each one is placed the number allotted to it in the museum in which it is located. It will be observed that the crania in the Anatomical Department of the University of Melbourne are numbered according to the system adopted by Martin (24) for Physical Anthropology and Anthropological Bibliography, and alongside this system number is placed the serial number of the skull.

The crania are those of adult aboriginals from Victoria, New South Wales, South Australia, and Queensland. Of four, the exact State whence they come is not known. The number is therefore as follows:—

Victoria	85
New South Wales	4
South Australia	4
Queensland	3
State not known	4
	100

The measurements have been taken in accordance with the directions laid down by the Monaco International Commission of 1906 (20), by means of the instrument known as Hepburn's Improved Craniometer Callipers, and the readings recorded to the nearest millimetre.

Table I. records the whole of the skulls under review in this paper. They are separated into male and female, and numbered serially 1 to 78 for the males, and 1 to 22 for the females, making a total number of 100.

Opposite the serial and museum numbers of each skull is placed, in tabular form, the corresponding length, breadth, and height.

Table II. furnishes an abstract of the results obtained for the crania under review, and gives the mean, the standard deviation, and the coefficient of variation, with the probable errors of length, breadth, height, cephalic index, height index, and breadth-height index of a hundred unsexed skulls, and of these separated into male and female.

Table III. gives the standard deviation and coefficient of variation for length and breadth of the Australian, and a series from other authors of what are regarded as Homogeneous Crania: to these are added the values obtained by Berry, Robertson, and Cross (26) for Tasmanians and Papuans, and of those obtained by Smith (27) for Melanesians.

Pearson (11) cites the data marked * on this table as being, as far as he is aware, all that have been worked out on skull length and breadth, excepting that he substitutes a fresh series of French skulls with a standard deviation for length of 5·942 and for breadth of 5·214 for the French series of the table, because he says that he suspects these Paris Catacomb series of heterogeneity, owing to their standard deviation being over 6.

An examination of the table shows that the Australian male and unsexed crania give a standard deviation for length greater than the French Catacomb skulls.

Table IV. details the correlation values for length, breadth, and height of Australians, Tasmanians, and Papuans, and these are compared with values for other series abstracted from *Biometrika*, vol. i. p. 457, and with those of Smith (27) for Melanesians.

Lastly, Table V. records the variability of the cephalic index, height index, and breadth-height index of the Australian, and of those other races of which there is a comparative mathematical knowledge.

Wherever results are marked with an asterisk the height is the auricular height. In the case of the Naquada skulls the height is the vertical height, as measured from the basion in a plane at right angles to the horizontal plane of the Frankfort agreement. The point in the vertex of the skull in this measurement does not always coincide with the bregma, and consequently is not the same as the basilo-bregmatic height of the international agreement, as used by the present author, and by Berry, Robertson, and Cross (26) in their work on the Tasmanians and Papuans, and also by Smith (27).

Concerning the lengths, breadths, and heights, a study of Table II. shows that these measurements for the male crania are all greater than those for the female. If they be viewed as an entity without reference to the question of sex, a mean is arrived at which falls in each case between the values obtained for male and female skulls. The means of the indices show that the female crania, relative to length, are broader than the male, but are not so high. The females are therefore more brachycephalic than males. On the other hand, the unsexed crania afford results which fall about midway between those obtained for male and female.

Turning next to the variability of length, breadth, and height, as measured by the *standard deviation*, the males, as judged by the first two, are more variable than the females, but are less so for height.

A consideration of the *indices* shows that for the cephalic index the males are more variable than the females, but less so for the height and breadth-height index.

The results arrived at for the unsexed skulls under all three indices give more variable values than for either sex, excepting for the height index, where the value is less variable.

The coefficient of variation is a percentage measurement of the variability, and, according to Pearson (7), the true criterion for relative variability in both sex and race. On comparing the coefficient of variation for length and breadth, the males are seen to be more variable than the females, but less so for height, as is borne out in each case by the standard deviation, as can be seen from Table II.

The test of homogeneity or purity of race is, as already shown, according to Pearson (11), a standard deviation of length for male skulls of 6·0, and a coefficient of variation of 3·0, and of about the same values for breadth.

The standard deviation and coefficient of variation values for length of the male Australian skull are higher even than those given by Pearson, but the values for breadth fall within his range (Table II.).

The evidence, therefore, from Pearson's standpoint indicates, when the test is applied to length, that the Australian is an "impure" race, but a "pure" one when gauged by the values obtained for breadth.

Turning now to the *coefficient of correlation* (Table IV.), a higher value of correlation is found for unsexed skulls than for either male or female skulls, and it is higher for males than for females.

The most highly correlated pair of dimensions for unsexed and male skulls are length and height, and the least correlated pair are length and breadth; while for female skulls breadth and height are the highest, and length and breadth the least.

If we turn now to a *comparison* of the variability in length and breadth of skull as represented by Table III., which records, as far as I am aware, all the available data for comparison, it will be seen that the standard deviation for the length of the Australian male is higher than that for any other males in the series, while that for Australian females is the lowest, with one exception, viz. that for the Melanesians—positions also borne out by a reference to the coefficient of variation.

The comparison of the standard deviation of breadth places the Australian male between the Aino, Naquada, Tasmanian, and Melanesian on the one hand, and the Bavarian, Whitechapel English, and Paris Catacombs on the other. But, from a study of the coefficients of variation, the Whitechapel English skulls are transferred to the first group. The Australian female for both standard deviation and coefficient of variation falls between the Aino, Tasmanian, and Melanesian on the one side, and Naquada, Bavarian, Paris Catacomb, and Whitechapel English on the other.

We now come to the *comparison of the correlation results* for length, breadth, and height with other races (Table IV.). The comparison of the values for the unsexed crania of Australians with those of Tasmanians and Papuans has already been worked out (26). Of those remaining, only the values of length and breadth are strictly comparable, owing to the differences adopted in estimating the height measurement. The height measurement for Germans and Ainos is the auricular height, and not the total height; while that for the Naquada is in a plane from the basion at right angles to the Frankfort horizontal plane, and does not always coincide with the basilo-bregmatic height, while those marked † in the table are measurements on the living head.

The values obtained for the following races are placed in order of their correlation, the highest being placed first, so that it can be seen at a glance which are more and which less highly correlated than the Australians:—

Unsexed.	Male.	Female.
Tasmanian	Eskimo	Melanesian
Melanesian	Tasmanian	German
Australian	Aino	Aino
Papuan	Naquada	Tasmanian
	Australian	Naquada
	German	French (Catacombs)
	Melanesian	*Australian*
	Sioux Indians	
	French Peasant	
	Badenoer	
	French (Catacombs)	
	Parisians	

The correlation values for male skulls are those usually regarded as correct for comparative purposes, and from these it will be seen that the Australian comes about halfway in the series, and is not as homogeneous or, in other words, not so pure as the Eskimo, Tasmanian, Aino, or Naquada races. If the female crania be of any comparative value, then the Australian becomes still more impure, as they have the least correlation value of the series.

As judged by the standard deviation of the cephalic index, it is very clear that the Australian male is less variable than the highly advanced modern races, but more variable than the more primitive peoples and those which are slightly in advance of the primitive condition.

Conclusions.

1. That the methods of modern statistical investigation must be adopted in all future estimations for making intra-racial and inter-racial comparisons.

2. That, from the difficulty of separating the skulls into sexes, it would be advisable to have resulting values obtained for crania of races without reference to sex.

3. That the Australians are not as homogeneous or pure as some series which are regarded as homogeneous, but are less heterogeneous than some other races, and the relationship of their cranial measurements, as indicated by the coefficient of correlation, is low, but is higher than most modern, though lower than primitive, races.

References.

(1) Fawcett and Lee, "A Second Study of the Variation and Correlation of the Human Skull, with special reference to the Naquada Crania," *Biometrika*, vol. i. p. 408.

(2) Macdonnell, W. R., "A Study of the Variation and Correlation of Human Skulls, with special reference to English Crania," *Biometrika*, vol. iii. p. 190.

(3) Stieda, "Ueber die Anwendung der Wahrscheinlickkeitsrechnung in der Anthropologischen," *Statistik Archiv für Anthropologie*, Bd. xiv. S. 167–182.

(4) Quetelet, *Lettres sur la Théorie des Probabilités*, 1846.

(5) Galton, Francis, *Biometrika*, vol. i.

(6) Galton, Francis, *National Inheritance.*

(7) Pearson, Karl, *The Chances of Death*, vol. i. pp. 256–372.

(8) Pearson, Karl, "Repression, Heredity, and Panmixia," *Phil. Trans.*, vol. clxxxvii. A, pp. 279–281.

(9) Pearson, Karl, "Correlation Values of Length, Breadth, and Height, and Two Cephalic Indices," Spurious Correlation, *Proc. Roy. Soc.*, vol. lx. p. 495.

(10) Pearson, Karl, Note on Dr Keith's Review of Professor Arthur Thomson's "The Ancient Races of the Thebaid," *Man*, vol. v., 1905, No. 65.

(11) Pearson, Karl, "Homogeneity and Heterogeneity in Collections of Crania," Craniological Notes, 11, *Biometrika*, vol. ii. p. 511.

(12) Lee, Alice, "Data for the Problem of Evolution in Man, vi.: A First Study of the Correlation of the Human Skull," *Phil. Trans.*, vol. cxcvi. A, pp. 225–264.

(13) Lee and Pearson, "Mathematical Contributions to the Theory of Evolution, Relative Variation, and Correlation in Civilised and Uncivilised Races," *Phil. Trans.*, 1895, A, vol. clxxxvi.

(14) Boas, Franz, "Heredity in Head Form," *American Anthropologist*, vol. i. p. 453.

(15) Gray, John, "The Analysis of Anthropometric Statistics," *Proc. Anat. and Anthrop. Soc. Aberdeen*, 1902–4, p. 85.

(16) Parsons, F. G., "Report on Hythe Crania," *Journ. Roy. Anthrop. Instit. Great Britain and Ireland*, vol. xxxviii., 1908, pp. 419–424.

(17) Klaatsch, Herman, *Reports from Pathological Laboratory of the Lunacy Department of N.S.W.*, vol. i., pt. iii., 1908.

(18) Turner, Sir Wm., "A Contribution to the Craniology of the People of Scotland," *Trans. Roy. Soc. Edin.*, vol. xl., pt. iii., pp. 547–613.

(19) Myers, C. S., "Review of Fawcett and Lee's Second Study of the Human Skull," *Man*, 1903, No. 13.

(20) *International Agreement for the Unification of Craniometrical and Cephalometrical Measurements*, Monaco, 1906.

(21) *The Anatomical Record*, "On the Preservation of Data used in Biometric Investigations," vol. iii., No. 3, Mar. 1909. Published by the Wistar Institute of Anatomy and Biology, Philadelphia.

(22) Ellis, Havelock, *Man and Woman.*

(23) Warren, Ernst, "An Investigation on the Variability of the Human Skeleton, with especial reference to the Naquada Race," *Phil. Trans.*, B, vol. clxxxix.

(24) Martin, Rud., "System der (Physichen) Anthropologie und Anthropologische Bibliographie." Sonderabdruck aus dem Korrespondenzblatt der Deutschen Anthropologischen Gesellschaft xxxviii., *Jahrgang*, cxc., Nr. 9–12.

(25) Bartel, Paul, "Über Geschlechtsunterschiede am Schädel," Inaug. Diss., Berlin, 1897.

(26) Berry, Robertson, and Cross, "A Biometrical Study of the Relative Degree of Purity of Race of the Tasmanian, Australian, and Papuan," *Proc. Roy. Soc. Edin.*, vol. xxxi., 1910.

(27) Smith, S. A., Unpublished Work on Melanesians, University of Sydney.

[Table.

Table I.—Maximum Length, Breadth, and Height of Seventy-eight Australian Aboriginal Male Crania.

	Mus. No.	L.	B.	H.		Mus. No.	L.	B.	H.
1	·54(94·5) 1	180	132	129					
2	3	193	138	133		Melb. Mus.			
3	4	186	127	130	41	12982	178	127	131
4	6	185	134	132	42	13008A	183	136	133
5	7	169	143	139	43	13010A	189	128	132
6	9	182	139	136	44	12977	183	134	124
7	13	178	127	124	45	12988	181	132	139
8	14	187	127	139	46	12987	179	128	129
9	16	180	130	127	47	12975	178	129	133
10	17	179	134	123	48	13011A	175	125	127
11	18	182	134	135	49	12986	177	133	126
12	19	185	139	121	50	13006A	197	136	144
13	20	180	136	125	51	12994	190	128	136
14	22	196	132	141	52	13023	197	127	131
15	30	189	135	135	53	13026	177	123	123
16	31	176	128	130	54	13005A	180	124	132
17	32	184	129	127	55	12991	179	120	130
18	33	189	124	130	56	13025	189	123	123
19	35	164	128	128	57	12989	195	136	137
20	36	192	133	135	58	13001	181	128	136
21	37	171	123	116	59	13018A	184	127	131
22	38	178	135	126	60	12967	183	134	127
23	40	182	127	128	61	12971	194	136	126
24	42	181	136	133	62	12965	187	130	130
25	43	192	140	137	63	13027	181	133	124
26	44	192	135	139	64	13004A	191	129	133
27	45	189	134	140	65	13003A	182	134	125
28	46	186	126	130	66	13002	195	136	141
29	47	174	129	122	67	12973	182	131	138
30	48	193	133	127	68	12970	190	131	135
31	49	187	127	135	69	12969	189	136	137
32	50	185	136	134	70	12800	170	125	127
33	51	191	134	126	71	12736	182	127	134
34	52	179	127	133	72	12960A	182	131	130
35	53	171	123	129	73	12836	191	128	137
36	·54(94·4) 2	188	128	135	74	12739	182	135	138
37	7	187	126	122	75	12821	176	123	123
38	·54(94·3) 1	164	129	134	76	12822	199	135	130
39	2	176	130	133	77	12738	183	128	130
40	6	177	121	133	78	12964	188	133	135

Maximum Length, Breadth, and Height of Twenty-two Australian Aboriginal Female Crania.

	Mus. No.	L.	B.	H.		Mus. No.	L.	B.	H.
1	·54(94·5) 2	176	127	120	11	12979	170	136	124
2	10	177	132	138	12	12978	179	125	116
3	11	170	135	132	13	12974	170	132	120
4	12	179	132	124	14	12995	177	132	131
5	15	171	131	129	15	12999	173	130	124
6	21	182	128	120	16	12968	180	128	126
7	34	175	125	137	17	12966	169	122	126
8	41	171	125	115	18	12810	172	131	117
					19	12724	179	129	124
	Melb. Mus.				20	12925	189	131	129
9	12980	179	129	123	21	12837	175	124	124
10	12981	176	127	120	22	12963	174	121	120

Table II.—The Variability of the Australian Crania.

	Mean.	Standard Deviation.	Coefficient of Variation.
78 Males.			
Length	183·56 ± ·567	7·42 ± ·401	4·04 ± ·218
Breadth	130·60 ± ·368	4·82 ± ·260	3·69 ± ·200
Height	131·13 ± ·423	5·54 ± ·299	4·23 ± ·228
100 B/L	71·29 ± ·245	3·21 ± ·173	4·51 ± ·244
100 H/L	71·50 ± ·259	3·39 ± ·183	4·74 ± ·256
100 H/B	100·50 ± ·352	4·61 ± ·249	4·59 ± ·248
22 Females.			
Length	175·59 ± ·680	4·73 ± ·481	2·69 ± ·274
Breadth	128·73 ± ·555	3·86 ± ·392	3·00 ± ·305
Height	124·50 ± ·870	6·05 ± ·615	4·86 ± ·494
100 B/L	73·36 ± ·439	3·05 ± ·310	4·16 ± ·423
100 H/L	70·95 ± ·551	3·83 ± ·389	5·40 ± ·549
100 H/B	96·64 ± ·692	4·81 ± ·489	4·98 ± ·506
100 Unsexed.			
Length	181·81 ± ·517	7·67 ± ·366	4·22 ± ·202
Breadth	130·19 ± ·316	4·69 ± ·224	3·60 ± ·172
Height	129·67 ± ·424	6·29 ± ·330	4·86 ± ·232
100 B/L	71·75 ± ·222	3·29 ± ·157	4·59 ± ·212
100 H/L	71·38 ± ·236	3·50 ± ·167	4·90 ± ·234
100 H/B	99·65 ± ·333	4·94 ± ·236	4·96 ± ·237

Table III.—Variability in Length and Breadth of Australian Crania compared with Homogeneous Series of Crania.

Series.	Length Male.		Length Female.		Length Unsexed.	
	Standard Deviation.	Coefficient of Variation.	Standard Deviation.	Coefficient of Variation.	Standard Deviation.	Coefficient of Variation.
Australian .	7·42 ± ·401	4·04 ± ·218	4·73 ± ·481	2·69 ± ·274	7·67 ± ·366	4·22 ± ·202
* Naquada .	5·72	3·09	5·25	2·96	...	...
* Bavarian . .	6·09	3·37	6·20	3·57	...	...
* Aino . .	5·94	3·20	5·45	3·08	...	...
* French . .	7·20	3·97	6·44	3·65	...	...
* English . .	6·45	3·44	6·54	3·66	...	...
† Tasmanian .	5·86 ± ·38	3·18 ± ·21	5·72 ± ·50	3·27 ± ·29	7·04 ± ·36	3·90 ± ·20
† Papuan . .	...	...	...	...	8·08 ± ·28	4·57 ± ·16
‡ Melanesian .	4·98	2·71	3·87	2·23	6·53	3·65
	Breadth Male.		Breadth Female.		Breadth Unsexed.	
Australian	4·82 ± ·260	3·69 ± ·200	3·86 ± ·392	3·00 ± ·305	4·69 ± ·224	3·60 ± ·172
* Naquada .	4·61	3·42	4·50	3·42	...	...
* Bavarian . .	5·85	3·89	4·89	3·39	...	...
* Aino . .	3·90	2·76	3·66	2·68	...	...
* French . .	6·07	4·21	5·06	3·67	...	...
* English . .	4·98	3·55	5·06	3·78	...	...
† Tasmanian .	4·77 ± ·31	3·49 ± ·23	3·57 ± ·31	2·47 ± ·22	4·79 ± ·25	3·54 ± ·18
† Papuan .	...	...	...	...	5·91 ± ·20	4·60 ± ·16
‡ Melanesian .	4·28	3·27	3·59	2·82	4·37	3·37

* Taken from Fawcett and Lee's work on Naquada crania, *Biometrika*, vol. i. p. 424.
† Berry, Robertson, and Cross.
‡ Melanesians. Unpublished work of S. A. Smith.

TABLE IV.—CORRELATION OF LENGTH, BREADTH, AND HEIGHT OF AUSTRALIAN CRANIA AS COMPARED WITH THOSE OF OTHER RACES.

Series.	Length and Height.			Length and Breadth.		
	Male.	Female.	Unsexed.	Male.	Female.	Unsexed.
Australian .	·337 ± ·060	·083 ± ·143	·449 ± ·054	·313 ± ·069	·016 ± ·144	·318 ± ·061
French . .	·294 ± ·022	·132 ± ·036	...	·089 ± ·024	·042 ± ·037	...
German . .	(·096 ± ·067)*	(·314 ± ·061)*	...	·286 ± ·062	·488 ± ·052	...
English (Criminals) .	...	...	...	(·402 ± ·010)†	...	...
English (Middle Class) . .	...	...	...	(·345 ± ·019)†	...	...
Aino . .	(·501 ± ·054)*	(·349 ± 075)*	...	·432 ± ·059	·376 ± ·073	...
Naquada . .	·489 ± ·044	·283 ± ·048	...	·344 ± ·050	·143 ± ·049	...
Sioux Indians .	·36 ± ·08	...	...	·24 ± ·08	...	...
Living Sioux .	...	...	...	(·24 ± ·04)†	...	...
Eskimo . .	...	...	...	·47 ± ·08	...	...
Indians (British Columbia) .	...	...	...	(·08)†	...	...
Shuswap Indians	...	...	..	(·04)†	...	...
Badenoer . .	...	...	...	·09	...	...
Bagda Caste, Bengal . .	...	...	...	(·13)†	...	...
Papuan . .	...	...	·465 ± ·038	...	...	·190 ± ·047
Tasmanian .	·577 ± ·061	·178 ± ·119	·541 ± ·051	·452 ± ·073	·256 ± ·115	·526 ± ·053
Melanesian .	·214 ± ·098	·609 ± ·088	·567 ± ·056	·246 ± ·091	·504 ± ·101	·476 ± ·061
Parisians . .	Pearson, 1896, *Phil. Trans.*, vol. clxxxvii. A, p. 280			·05 ± ·06	(Male or Female not stated)	
French Peasants				·13 ± ·07		

TABLE IV.—*continued.*

Series.	Breadth and Height.		
	Male.	Female.	Unsexed.
Australian	·314 ± ·069	·311 ± ·130	·344 ± ·059
French	·224 ± ·023	·229 ± ·035	...
German	(·072 ± ·067)*	(·276 ± 063)*	...
English (Criminals) . .	...	...	...
English (Middle Class) . .	...	...	...
Aino	(·345 ± .064)*	(·178 ± ·082)*	...
Naquada	·273 ± ·055	·119 ± ·052	...
Sioux Indians . . .	·00 ± ·09	...	...
Living Sioux . . .	...	...	...
Eskimo	...	...	...
Indians (British Columbia) .	...	...	...
Shuswap Indians . . .	...	...	...
Badenoer	...	...	...
Bagda Caste, Bengal . .	...	...	...
Papuan	...	...	·368 ± ·042
Tasmanian	·268 ± ·085	·151 ± ·120	·367 ± ·063
Melanesian	·285 ± ·094	·286 ± ·128	·423 ± ·061
Parisians	...		
French Peasants . . .	...		

* Auricular height, not total height used . . . . not properly comparable.
† Measurements taken on living heads.

Table V.—Variability of the Indices (as indicated by the Standard Deviation).

Series.	Breadth and Length.		
	Male.	Female.	Unsexed.
Australian	3·21 ± ·173	3·05 ± ·310	3·29 ± ·157
*Bavarian	3·50	2·97	...
English	3·31	3·37	...
French	4·43	4·19	...
Naquada	2·80	3·12	...
*Egyptian Mummies	3·35	3·36	...
*Modern Mummies	5·42	5·10	...
Negroes	2·77	3·52	...
*Panjabi Low Caste	2·98	3·75	...
*Ainos	2·41	2·54	...
*Row Grave Germans	2·28	2·35	...
	Height and Length.		
Australian	3·39 ± ·183	3·83 ± ·389	3·50 ± ·167
*Bavarian	...	...	...
English	...	...	...
French	3·53	3·67	...
Naquada	2·73	2·96	...
*Egyptian Mummies	...	...	...
*Modern Mummies	...	...	...
Negroes	...	...	...
*Panjabi Low Caste	...	...	...
*Ainos	...	...	...
*Row Grave Germans	...	...	...
	Height and Breadth.		
Australian	4·59 ± ·248	4·81 ± ·489	4·94 ± ·236
*Bavarian	...		...
English	...		...
French	4·74	4·31	...
Naquada	4·73	4·66	...
*Egyptian Mummies	...	...	...
*Modern Mummies	...	...	...
Negroes	...	...	...
*Panjabi Low Caste	...	...	...
*Ainos	...	...	...
*Row Grave Germans	...	...	...

* Taken from *Biometrika*, vol. i. p. 440.

(*Issued separately November* 28, 1910.)

II.—A Biometrical Study of the Relative Degree of Purity of Race of the Tasmanian, Australian, and Papuan.

By **Richard J. A. Berry**, M.D., **A. W. D. Robertson**, M.D. Melb., and **K. Stuart Cross**, M.Sc. Melb. (From the Anatomy Department of the University of Melbourne.)

(MS. received March 30, 1910. Read May 16, 1910.)

If the marked divergence of ethnological opinion concerning the degree of racial purity or admixture of the *Australian aboriginal* affords evidence of anything at all, it can only be of the fact that the problems of his origin, as of his degree of racial purity, are not yet solved. Not only are there the most conflicting opinions on these points, but the cognate subjects of the origin, and degree of purity of the Tasmanian, and of his relationship, if any, to the Australian, are equally undetermined, whilst there is lastly the debatable point of the relation of both those primitive peoples to the Papuan.

It would be a surprisingly easy thing to give example after example of the divergence of opinion on these points, but a few characteristic specimens must suffice.

For Keane (1) the Australian is an admixture. "Despite a general physical and mental likeness, most observers now recognise two original elements—a black and perhaps a low Caucasic—in the constitution of the Australian aborigines."

Again (2): "In this continent (Australia), of which Tasmania may be regarded as an 'ethnical annexe,' most anthropologists recognise at least two fundamental types beneath a general physical and linguistic uniformity."

For Flower and Lydekker the "Australians are probably not a homogeneous group at all, as supposed by Huxley, but a cross between two already formed stocks."

For Topinard (3) the problem is an unsolved one. He says: "If the Australians are thorough Hindoos as regards their hair, they are Melanesians, or, if you will, New Hebrideans, New Caledonian negroes, in every other respect. The question may therefore be left. We are still in ignorance as to whether the present Australian race took its origin on the spot, with the characters that we admit as belonging to it, or whether, on the contrary, it was altogether constituted in Asia, or whether it is a cross race, and, in that case, of what elements it is composed."

Tylor (4), whilst expressing no opinion as to the homogeneity or otherwise of the Australian, is perfectly clear that he is something quite apart from either the Papuan or the Tasmanian. He says: "In Australia . . . there appears a thin population of roaming savages, strongly distinct from the blacker races of New Guinea at the north, and Tasmania at the south."

Curr (5) is dogmatic on the point of the racial admixture of the Australian. He says: "I shall proceed to show that the Australian is by descent a negro crossed by some other race."

De Quatrefages (6) is almost as emphatic as Curr on the duality of the Australian, for he states that "in Australia there are two distinct types—Australians proper and Australian Neanderthaloids. . . . This fact can be accounted for by presuming that true negroes formerly occupied the whole or a part of Australia; that they were invaded by a black race with straight hair, and that it is to a blood-mixture that the differences in the hair must be attributed. It is very probable that the Tasmanians furnish this negretic element."

Mathew (30), who has devoted much painstaking care to the elucidation and verification of his theory, is of opinion that the Australian represents an admixture of most probably Papuan, Malaysian, and Dravidian elements.

Notwithstanding the eminent character of the authorities just quoted, whose evidence seems to trend towards the duality of the Australian aboriginal, their ideas are simply torn to metaphorical shreds by the German anthropologist, H. Klaatsch (7), who so recently as 1908 writes as follows:—

"Views which are absolutely without any scientific basis are sometimes given expression to with regard to the origin of the Australian race. Australian aboriginals have been claimed to be a mixed race due to the crossing of different present existing types of mankind, a Papuan, a negroid, and a Dravidian element. Such a view is often repeated in Australian ethnographical publications by non-scientifically trained persons, and is supposed to be a fact based on osteological investigations; but whenever I have asked the name of the publication giving authority for the statement, the information has not been forthcoming; . . . they never quote Turner's classical work, where the unitary type of the Australian aboriginal is clearly explained. . . . I agree with Turner in his views of the unitary nature of the Australian race. I do not believe that the variations, many of which are additional to those mentioned by him, have anything to do with the crossing of different races. On the other hand, I consider that the specialisation of various types in the Australian is the result of a development due to the special condition of the Australian continent during the countless ages that it has been inhabited by blacks. My opinion as to the origin of the aboriginal I cannot better explain than by quoting the remarks made by Ch. H. Barton, in his publication on Australian Physiography. . . . Why should it not be coeval with the continent itself? . . . We may, in the absence of rebutting evidence, be content to regard the Australian as truly autochthonous, and his origin as but part of the unsolved problem of the beginning of humanity."

As Klaatsch's ideas on the unitary character of the Australian aboriginal differ so markedly from the dualists already quoted, we have thought it but right, in justice to him, to set forth his views at some length.

Nor is Klaatsch alone in his ideas. Otto Schötensack (8), in an important and well-reasoned work, has put forward the novel idea that Australia is, in all probability, the land where the final transition from the simian ancestor to the genus Homo has taken place, in which case the primary unity of the Australian aboriginal follows as a matter of course.

Having thus indicated the diversity of opinion concerning the Australian aboriginal, we may next point out that concerning the *Papuan* there is not quite so much speculation.

"At present the Papuan domain," says Keane, "is restricted to Melanesia and parts of Fiji, practically the whole of New Guinea with the neighbouring Torres Strait Islands, and most of the smaller groups in East Malaysia as far west as Flores inclusive."

They are a branch of the Oceanic negroes and "are a very old ethnic group, here and there modified on the spot by crossings with populations always on the move (Malays, Bugis, etc.)."

We wish to direct particular attention to this statement of Keane of the modification of the Papuan by crossings, a view which is also shared by Semon (9), who says he thinks it proved that the Papuans represent a mixed race.

Of the *Tasmanian* it is unnecessary for us here to speak, because all authorities are agreed that, whatever his origin may have been, he has lived in a state of absolute isolation ever since the remote times of the separation of Tasmania from the Australian continent, and further, one of us (Berry, 10), has already published an account of a half-caste Tasmanian in which the various views regarding the origin and character of the Tasmanian are fully set forth, views in which a more recent writer (Turner, 11) apparently agrees, as his conclusions are identical with Berry's.

From what has been said, it should therefore be clear that the comparative racial purity of the Tasmanian is accepted, the admixture of the Papuan is proved, and the purity or otherwise of the Australian is debated.

Commencing with these ethnological ideas as a working hypothesis, the *main object of our research is to see what light can be thrown by investigation on the Australian problem*, and with this idea we have devised several independent lines of research all leading up to the one ultimate object.

These lines of research are as follows:—

1. A biometrical examination of certain standard measurements of the Tasmanian, Australian, and Papuan crania—namely, length, breadth, and height.

In this connection it must be remembered that the recent discovery by two of us (Berry and Robertson, 12 and 13) of over forty Tasmanian crania previously unknown to scientists has greatly extended the possible range of observations.

2. A biometrical examination of the same standard observations for the Australian aboriginal, to serve as a basis of comparison for the present work, and also as an independent piece of research.

3. A morphological examination of the "form analysis" of the Tasmanian calvarium on the lines devised by Schwalbe, from which the relative position of the Tasmanian in the anthropoid, homo primigenius, homo fossilis, homo sapiens series is to be deduced.

4. A like examination for the Australian aboriginal calvarium, with the like object.

Of these four lines of investigation the first three are now completed, and the fourth is in active progress.

The first of these investigations is embodied in the present work.

The second line of research is the sole work of Dr Robertson, with mathematical assistance from Mr Cross.

The third and fourth investigations are the work of Professor Berry and Dr Robertson.

As will be easily understood, investigations covering such a large and important field can hardly have been conducted without having given rise to minor and incidental problems for investigation, and these will be undertaken by one or other of the present writers as time permits.

As already indicated, the present work is an application of rigid mathematical formulæ to the standard measurements of length, breadth, and height of 86 Tasmanian crania, 100 Australian crania, and 191 Papuan crania. We have purposely dealt with as large numbers as were obtainable, in order to minimise, so far as possible, errors arising from random sampling: in fact, of the Tasmanian crania, practically every available one in any part of the world has been employed for the investigation.

Concerning the sources from which our material has been derived, they are as follows:—

Of the *Tasmanian* crania, Numbers 1 to 19 inclusive are derived from Sir William Flower's catalogue of the osteological specimens in the Royal College of Surgeons Museum in London (14).

Numbers 20 to 34 are from Barnard Davis's *Thesaurus Craniorum* (15). These crania are now, we believe, housed in the College of Surgeons, London.

Numbers 35 to 36 are from Topinard's measurements on two Tasmanian crania in Paris (16).

Numbers 37 to 42 are the Tasmanian crania in the University of Oxford. Our measurements of these skulls are taken from Turner's memoir (11).

Numbers 43 to 51 are the crania in the University of Edinburgh. For their measurements we are again indebted to Turner (11).

Numbers 52 to 62 are from the Tasmanian Museum, Hobart, and their measurements we owe to Harper and Clarke (17).

With the sole exception of 95, which is one of De Quatrefages and Hamy's measurements, all the remainder are from Berry and Robertson's recent discovery of additional Tasmanian crania (12, 13, and 18).

Of these 95 Tasmanian crania, we have, for the purposes of the present investigation, only made use of 86, for the sufficient reason that in the remainder some one or other of the standard measurements—length, breadth, and height—was not available.

The sources of the *Australian* aboriginal crania are fully set forth by Dr Robertson in his own memoir. Suffice it, therefore, to say here that they are all original, and are derived either from the National Museum, Melbourne, or from the recently established Anatomical and Anthropological Museum in the University of Melbourne, and in about equal numbers from the two institutions. For those from the Museum we have to express our indebtedness to the director, Professor Spencer, and the curator, Mr J. E. Kershaw.

Of the 191 *Papuan* crania, Numbers 1 to 124 are from Gray's "Measurements of Papuan Skulls" (19). Of these crania, Gray tells us that they are from the collection of Mr W. D. Webster, of Streatham. "There is reason to believe that these skulls came from the Prurari Delta and other places on the shores of the Gulf of Papua, except the last six, . . . which came from German New Guinea. All the skulls were carved and blackened, except those from German New Guinea."

Numbers 125 to 164 are from Flower's catalogue (14).

Numbers 165 to 171 and Number 183 are from Sir William Turner's "Decorated and Sculptured Skulls from New Guinea" (20).

Numbers 172, 173, 174, and 184 are from some hitherto undescribed decorated New Guinea skulls in the Anatomical Museum of the University of Melbourne.

Numbers 175 to 182 and 185 to 191 are from Dorsey and Holmes' "Observations on a Collection of Papuan Crania" (21).

In view of the limited number of Tasmanian crania available, we deemed it unnecessary to multiply the numbers of Papuan crania, though it would have been easy to do so, as, for example, by incorporating Meyer's (31) hundred and thirty-five Papuan skulls, and others.

Such being the material, it now becomes necessary to say something of the biometrical methods adopted in its examination. It is unnecessary for us to make any attempt to extenuate the employment of biometrical methods in a craniological investigation, since the magnificent work of Professor Karl Pearson and his school has proved that no other method can now be legitimately employed in such work. Nor is it necessary for us to make any examination of the literature of the method to be found in the pages of *Biometrika*, and also scattered throughout the pages of the *Philosophical Transactions of the Royal Society of London*; for whilst all of that work is important, but little of it has a direct bearing on the subject-matter in hand.

Of the special adaptation of Professor Pearson's methods to the present research, the following is an outline:—

The principal quantities determined are the means, standard deviations, coefficients of variation, and, in particular, the coefficients of correlation, these last being taken as measures of the constancy of type in the respective groups of crania.

In comparing this constancy of type, or purity of any two groups of crania, the coefficients of correlation of a selected pair of measurements, as, for example, length and breadth, are calculated, and the difference between the respective values in the two groups under consideration, together with the probable error of that difference, is determined. A difference greater than its probable error indicates a real difference, and not one due merely to random sampling; and as the ratio of the difference to its probable error increases, so also does the probability that the difference is a real one, but much more rapidly. A discussion as to which pair of characteristics should be chosen for this purpose follows later.

In the actual calculations, provisional means, which are whole numbers, are in all cases initially adopted, to be afterwards suitably corrected to give the true values. In this way whole numbers only have to be manipulated.

Throughout the work, x refers to length, y to breadth, z to height, these symbols themselves standing for the differences of individual measurements from the corresponding true means. Accented letters refer to provisional or uncorrected values: $\bar{x}$, $\bar{y}$, $\bar{z}$ are the differences between the respective provisional and true means.

The total number of crania in any investigation is denoted by N, n being the symbol for the number having any one particular measurement; thus $N=\Sigma(n)$. μ_1 is the average first moment about the mean; μ_2 the average second moment.

One example is given in detail of the determination of each of the quantities. Thereafter only results are shown.

For the benefit of those who are not familiar with this application of modern biometric methods, it may be mentioned that there is an excellent account of the same published in the *Proceedings of the Anatomical and Anthropological Society of the University of Aberdeen* for 1902–1904, by Mr John Gray, under the title of "The Analysis of Anthropometric Statistics" (22); whilst there is also another explanatory paper by Worthington in the *Journal of Anatomy and Physiology*, vol. xxxv. p. 455, entitled "On Professor Pearson's Contributions to Osteology" (23).

The detailed results of the present investigation are set forth in the accompanying tables, and following these is a comparative table showing the results attained for the three races.

Tasmanian.

Table of measurements: Table I.

Crania not separated into sexes. Length: Table II.

In this table is displayed the method of working, the mean length, the standard deviation, the coefficient of variation, and the probable errors.

Crania not separated into sexes. Breadth and height: Table III.

This table shows the results only of the mean breadth and mean height, with their standard deviations, coefficients of variation, and probable errors.

Crania not separated into sexes. Coefficients of correlation: Table IV.

Breadth-height. Height-length. Length-breadth.

Male crania only: Table V.

Length, breadth, height. Coefficients of correlation.

Female crania only: Table VI.

Details as under males.

Australian.

For all tables see Dr Robertson's independent work on the Australian crania.

Papuan.

Table of measurements : Table VII.

Crania not separated into sexes : Table VIII.

Details as under Tasmanian.

Comparative Tables of Tasmanians, Australians, and Papuans.

Tables IX., X., XI., and XII.

Concerning the relative lengths of the crania in the three racial groups examined, it will be noticed from Tables IX. and X. that the Australian has absolutely the longest skull, and the Papuan the shortest.

For breadth, the Tasmanian occupies the highest position, and the Papuan the lowest.

Regarding height, to which, in view of Schwalbe's work (24), very considerable evolutionary importance is to be attached, the Papuan comes first and the Australian last.

Translated into the language of the craniologist, these results show for the breadth-length ratio, that is, the cephalic index, the Australian is the most dolichocephalic with an index of 71·75, the Papuan comes next with an index of 72·54, and the Tasmanian is within a decimal point of being mesaticephalic with an index of 74·94.

Whilst all three groups are, therefore, dolichocephalic, the Australian is most so, almost within the confines of hyperdolichocephaly—in fact, for Krause (25) he actually is so, with an index for the male of 69·7, whilst Broca's figures (26) approximate to our own, namely, 71·49 for 27 crania.

As regards the height index, the skulls of all three races are metriocephalic or orthocephalic. The Papuan with an index of 74·41 approaches most nearly to the hypsicephalic, akrocephalic, or high type of skull; the Tasmanian ranks second with an index of 72·19; whilst the Australian most nearly approaches the chamaecephalic, tapeinocephalic, or low type of skull with an index of 71·38.

The breadth-height index shows that the Papuan alone has a hypsistenocephalic or high, narrow skull, with an index of 102·56. The remaining two possess a platychamaecephalic or wide, low skull, with indices respectively of 99·65 for the Australian and 96·33 for the Tasmanian.

The coefficients of variation for length, breadth, and height for the three races show (see Tables IX. and X.) that as regards length and breadth the minimum figure occurs in the Tasmanian and the maximum in the Papuan, whilst as regards height the maximum figure occurs in the Australian and the minimum in the Papuan—results which in the main show that the greatest uniformity of type occurs in the Tasmanian, with the Papuan at the other end of the scale; or, in other words, that the Tasmanian is the purest of the three races compared, the Papuan the least pure, and the Australian intermediate.

These results are more strikingly brought out by reference to the coefficients of correlation, which we regard as of primary importance in the present investigations, because they are functions of shape only. The relative value of these coefficients as criteria for comparative purposes has next to be considered.

There is not, unfortunately, much information available as to what importance is to be attached to the comparison of breadth-height and height-length, but of the importance of the comparison of length and breadth every craniologist is fully convinced, for it is to the varying relationships of these dimensions that are due the transitions from dolichocephaly to brachycephaly.

This general observation is confirmed by the results of Pearson and Lee's data (27), which is, according to Hatai (28), the only work as yet available for comparisons of the coefficients of correlation of breadth-height and height-length. In general, their work shows that the correlation for length and breadth and for height and length is about the same, but that both are much higher than for breadth and height.

For the albino rat Hatai (28) obtains precisely reverse results; but, as he shows, "the difference is probably due to differences in relative development of the several bones of the cranium, depending on the skull-form characteristic for the two species."

Tschepourkowsky (29) has also shown that, with increase of the body length, head length and head height also increase, but that head breadth and facial breadth decrease. From race to race the breadth diminishes the greater the length formation of the head, but the height increases. His figures finally show that in the different races of mankind the head length is less variable than the breadth, as is also the height, and hence form differences are chiefly due to increased breadth, as may be clearly seen by comparing the ranges of the mean lengths, breadths, and heights in our Tables IX. and X.

It is thus sufficiently clear both on biometric and on morphological grounds that, from the evidence above quoted, the greatest stress must be laid upon the length-breadth coefficient of correlation, and the least upon that for breadth-height.

An examination of our figures under the three heads for the races examined will show that they confirm Pearson and Lee as regards the relative stress to be laid upon the coefficients of correlation for length-breadth, height-length, and breadth-height.

Turning lastly to the examination of these coefficients of correlations for the particular objects with which this research was undertaken, namely, the examination of the relative purity of type of the Tasmanian, Australian, and Papuan, our final results are set forth in Table XII.

In this table the probable error of the difference of any two coefficients of correlation is calculated in the usual way as the square root of the sum of the squares of the probable errors of the coefficients themselves.

In Table XI. of the coefficients of correlation for breadth-height and height-length it is seen that the figures apparently indicate a greater constancy in shape of head for the Papuans than for the Australians, being practically the same as for the Tasmanians on the first count, and intermediate to the Tasmanians and Australians on the second. But these figures must be read in conjunction with their respective probable errors (see Table XII.), when it is at once seen that in all those cases where the differences as taken have negative values they are quite overshadowed by their probable errors, and therefore the chances are that such differences are due to random sampling, and consequently are not indicative of any real difference at all.

With regard to the correlation of breadth-height, the coefficients for all three races are, within the limits of the probable errors, constant.

In the case of height-length the Tasmanians show a coefficient higher than for either of the other two races, the difference in each case being greater than the probable error. On this count, therefore, the figures show that the Tasmanian is of a more uniform type than are either of the other two races.

As might naturally be expected from the evidence already adduced as to the relative values of breadth-height, height-length, and length-breadth, the last-mentioned shows, in our results, the most striking differences. The length-breadth coefficient of correlation for the Tasmanians exceeds that for the Australians by from two to three times the probable error of the excess, and the excess of the Australian coefficient over the Papuan coefficient is considerably greater than its probable error.

It follows from this investigation that, for the three races considered, the breadth-height coefficient of correlation is practically constant, the height-length coefficient a little less so, and the length-breadth coefficient has the greatest range of all. These results confirm those attained mathematically by Pearson and Lee, and also show that morphologically the comparison of length-breadth is the most important, with height-length second in point of importance, and breadth-height a negligible quantity.

Bearing in mind what has already been stated as to the disputed question of the heterogeneity or homogeneity of the Australian aboriginal, our final conclusion must not be misunderstood.

We do not definitely state that the Australian is a dual type: we merely maintain that biometrical investigation proves that the Tasmanian is the purest of the three racial types here compared, the Papuan is the least pure, and the Australian is about midway between the two; this, however, is evidence which those ethnologists who maintain the duality of the Australian aboriginal will doubtless welcome and make the most of it. For ourselves, we regard this as one link in the chain of evidence concerning the heterogeneity of the Australian as contrasted with the homogeneity of the Tasmanian.

Table I.—Measurements of Tasmanian Crania.

	Length (mm.).	Breadth (mm.).	Height (mm.).
Flower's Catalogue (Coll. Surg.).			
1. No. 1096, M.	183	138	134
2. „ 1097, F.	175	133	119
3. „ 1098, M.	180	137	139
4. „ 1099, M.	181	139	133
5. „ 1100, M.	188	135	140
6. „ 1101, M.	183	137	129
7. „ 1102, M.	183	140	127
8. „ 1103, M.	173	135	124
9. „ 1104, M.	188	133	137
10. „ 1105, F.	169	135	126
11. „ 1106, F.	175	127	127
12. „ 1107, F.	176	132	125
13. „ 1108, F.	163	130	123
14. „ 1109, F.	175	135	124
15. „ 1109 (?)	175	138	129
16. „ 1110 (? Tasm.), F.	164	131	122
17. „ 1111, M. (?)	177	136	130
18. „ 1112 (young)	175	132	130
19. „ 1113A (? Tasm.), M.	171	138	130

Table I.—*Continued.*

	Length (mm.).	Breadth (mm.).	Height (mm.).
Barnard Davis, *Thesaurus Craniorum* (Coll. Surg.).			
20. No. 1481, M.	180	130	135
21. „ 1482, F.	178	137	127
22. „ 1761, M.	193	142	132
23. „ 1763, F.	180	135	130
24. „ 860, M.	180	137	132
25. „ 861, M.	190	140	140
26. „ 862, F.	178	135	135
27. „ 863, F.	178	135	142
28. „ 867, M.	188	140	137
29. „ 928, M.	198	142	140
30. „ 1054, M.	188	140	140
31. „ 1119, F.	173	135	130
32. „ 1120, F.	173	135	135
33. „ 1121, M.	185	135	140
34. „ 1297, M.	188	142	135
Topinard.			
35. M.	184·5	143	131
36. F.	175	131·5	120·5
University of Oxford (Turner's Paper).			
37. No. 1017, F.	185·5	132	125
38. „ 1019, F.	174	127	133
39. „ 1020, F. (?)	181	129	122
40. „ 1021, M. (?)	181	142	128
41. „ 1021A, M.	174·5	135	128
42. „ 1021B, F.	166	131	126
University of Edinburgh (Turner's Paper).			
43. No. 1, M.	183	130	132
44. „ 2, M.	186	139	129
45. „ 3, F.	178	132	130
46. „ 4, M.	181	128	132
47. „ 5, M.	181	134	133
48. „ 6, M.	181	125	134
49. „ 7, M.	190	141	135
50. „ 8, M.	191	141	
51. „ 10, M.	175	127	125
Tasmanian Museum, Hobart (Harper and Clarke's Paper).			
52. No. 1, M.	184	135	126
53. „ 2, M.	193	142	129
54. „ 3, M.	182	133	129
55. „ 4, M.	192	141	
56. „ 5, M.	180	136	133
57. „ 6, M.	180	135	
58. „ 7, F.	170	132	133
59. „ 8, F.	172	135	117
60. „ 9, F.	175	132	125
61. „ 10, F.	175	133	125
62. „ 11, F. (?)	170	132	123

TABLE I.—*Continued.*

	Length (mm.).	Breadth (mm.).	Height (mm.).
Berry and Robertson.			
63. No. 4302, Hob., M. . .	190	140	140
64. „ 4297, „ M. . .	185	141	
65. „ 4290, „ F. . .	184	137	133
66. „ 4296, „ M. . .	179	135	136
67. „ 4303, „ F. (?) . .	...	137	130
68. „ 1572, „ F. . .	...	141	129
69. Cook, Hob., M. . . .	184	144	127
70. Taylor, Hob., M. . . .	182	138	135
71. „ „ F. . . .	179	137	131
72. Jolly, Launc., M. . . .	178	126	129
73. No. 1201, Launc., M. . .	193	132	137
74. „ 1202, „ F. . .	179	139	123
75. „ 1203, „ M. . .	180	135	
76. „ 1204, „ M. . .	185	145	134
77. „ 1205, „ M. . .	195	141	135
78. Devonport 1, M. . . .	186	131	125
79. „ 2, F. . . .	183	126	127
80. Cotton 1, M.	180	134	123
81. „ 2, M.	188	142	136
82. „ 3, M.	173	137	127
83. „ 4, M.	178	129	126
84. „ 5, M.	182	135	126
85. „ 6, M. (?) . . .	169	125	
86. „ 7, M.	181	143	132
Crowther.			
87. No. 3, M.	182	133	130
88. „ 4, F.	181	129	122
89. „ 5, M.	173	135	125
90. „ 6, M.	190	139	132
91. „ 7, M.	189	140	137
92. „ 8, M.	178	133	128
93. „ 9, M.	186	141	136
94. Melb. Museum 1, F. . .	165	123	128
95. De Quatrefages and Hamy, F. .	175	133	

Table II.—Tasmanian Crania, not Separated into Sexes

N=86.

Length. (mm.).	n.	x'.	nx'.	nx'^2.
163	1	−17	− 17	289
164	1	−16	− 16	256
165	1	−15	− 15	225
166	1	−14	− 14	196
167	...	...	...	...
168	...	...	...	...
169	1	−11	− 11	121
170	2	−10	− 20	200
171	1	− 9	− 9	81
172	1	− 8	− 8	64
173	5	− 7	− 35	245
174	1	− 6	− 6	36
175	10	− 5	− 50	250
176	1	− 4	− 4	16
177	1	− 3	− 3	9
178	7	− 2	− 14	28
179	3	− 1	− 3	3
180	6	...	−225	...
181	8	+ 1	+ 8	8
182	4	+ 2	+ 8	16
183	5	+ 3	+ 15	45
184	3	+ 4	+ 12	48
185	3	+ 5	+ 15	75
186	4	+ 6	+ 24	144
187	...	...	...	...
188	6	+ 8	+ 48	384
189	1	+ 9	+ 9	81
190	4	+10	+ 40	400
191	...	...	...	...
192	...	...	...	...
193	3	+13	+ 39	507
194	...	...	...	...
195	1	+15	+ 15	225
196	...	...	...	...
197	...	...	...	...
198	1	+18	+ 18	324
			+251	
	86	...	+ 26	4276

$$\mu'_1 = \frac{\Sigma(nx')}{N} = \frac{+26}{86} = +0{\cdot}30,$$

$$\mu'^2_1 = 0{\cdot}09\ ;$$

$\therefore$ Mean Length (L) $= 180{\cdot}30 \pm 0{\cdot}51$.

TABLE II.—*Continued.*

$$\mu'_2 = \frac{\Sigma(nx'^2)}{N} = \frac{4276}{86} = 49{\cdot}72\,,$$

$$\mu^2 = \mu'_2 - \mu'^2_1 = 49{\cdot}63\,,$$

and

$$\text{Standard Deviation } (\sigma_x) = \sqrt{49{\cdot}63} = 7{\cdot}04 \pm 0{\cdot}36\,;$$

the Coefficient of Variation (V_x) being $\dfrac{7{\cdot}04 \times 100}{180{\cdot}30} = 3{\cdot}90 \pm 0{\cdot}20.$

The Probable Errors are calculated from the usual formulæ, viz. :—

Probable Error of the Mean

$$= \frac{0{\cdot}6745\,.\,\sigma_x}{\sqrt{N}}$$

$$= \frac{0{\cdot}6745 \times 7{\cdot}04}{\sqrt{86}} = 0{\cdot}51\,;$$

Probable Error of the Standard Deviation

$$= \frac{0{\cdot}6745\,.\,\sigma_x}{\sqrt{2N}}$$

$$= \frac{0{\cdot}6745 \times 7{\cdot}04}{\sqrt{172}} = 0{\cdot}36\,;$$

Probable Error of the Coefficient of Variation

$$= \frac{0{\cdot}6745 \times V\left\{1 + 2\left(\frac{V}{100}\right)^2\right\}}{\sqrt{2N}}$$

$$= \frac{0{\cdot}6745 \times 3{\cdot}90\{1 + 2(0{\cdot}039)^2\}}{\sqrt{172}}$$

$$= 0{\cdot}20.$$

TABLE III.—TASMANIAN CRANIA, NOT SEPARATED INTO SEXES.

Breadth—

Mean Breadth (B)	$= 135{\cdot}14 \pm 0{\cdot}35$
Standard Deviation (σy)	$= 4{\cdot}79 \pm 0{\cdot}25$
Coefficient of Variation (V_y)	$= 3{\cdot}54 \pm 0{\cdot}18$

Height—

Mean Height (H)	$= 130{\cdot}26 \pm 0{\cdot}40$
Standard Deviation (σz)	$= 5{\cdot}49 \pm 0{\cdot}28$
Coefficient of Variation (V_z)	$= 4{\cdot}21 \pm 0{\cdot}22$

Table IV.—Tasmanian Crania, not Separated into Sexes.

Coefficients of Correlation.

Length.—True mean, 180·30 Provisional mean, 180
Breadth.— „ 135·14 „ 135
Height.— „ 130·26 „ 130

$$\bar{x} = +0·30 \qquad \sigma_x = 7·04$$
$$N = 86 \qquad \bar{y} = +0·14 \qquad \sigma_y = 4·79$$
$$\bar{z} = +0·26 \qquad \sigma_z = 5·49$$
$$\therefore \; N\bar{y}\bar{z} = +3·1, \quad N\bar{z}\bar{x} = +6·7, \quad N\bar{x}\bar{y} = +3·6.$$

No.	x'.	y'.	z'.	$y'z'$.	$z'x'$.	$x'y'$.
1	+ 3	+ 3	+ 4	+12	+ 12	+ 9
2	− 5	− 2	−11	+22	+ 55	+ 10
3	...	+ 2	+ 9	+18		
4	+ 1	+ 4	+ 3	+12	+ 3	+ 4
5	+ 8	...	+10	...	+ 80	
6	+ 3	+ 2	− 1	− 2	− 3	+ 6
7	+ 3	+ 5	− 3	−15	− 9	+ 15
8	− 7	...	− 6	...	+ 42	
9	+ 8	− 2	+ 7	−14	+ 56	− 16
10	−11	...	− 4	...	+ 44	
11	− 5	− 8	− 3	+24	+ 15	+ 40
12	− 4	− 3	− 5	+15	+ 20	+ 12
13	−17	− 5	− 7	+35	+119	+ 85
14	− 5	...	− 6	...	+ 30	
15	− 5	+ 3	− 1	− 3	+ 5	− 15
16	−16	− 4	− 8	+32	+128	+ 64
17	− 3	+ 1	...	...	...	− 3
18	− 5	− 3	...	...	...	+ 15
19	− 9	+ 3	...	...	...	− 27
20	...	− 5	+ 5	−25	...	
21	− 2	+ 2	− 3	− 6	+ 6	− 4
22	+13	+ 7	+ 2	+14	+ 26	+ 91
23	...	...	...	...	...	
24	...	+ 2	+ 2	+ 4	...	
25	+10	+ 5	+10	+50	+100	+ 50
26	− 2	...	+ 5	...	− 10	
27	− 2	...	+12	...	− 24	
28	+ 8	+ 5	+ 7	+35	+ 56	+ 40
29	+18	+ 7	+10	+70	+180	+126
30	+ 8	+ 5	+10	+50	+ 80	+ 40
31	− 7					
32	− 7	...	+ 5	...	− 35	
33	+ 5	...	+10	...	+ 50	
34	+ 8	+ 7	+ 5	+35	+ 40	+ 56
35	+ 5	+ 8	+ 1	+ 8	+ 5	+ 40
36	− 5	− 3	− 9	+27	+ 45	+ 15
37	+ 6	− 3	− 5	+15	− 30	− 18
38	− 6	− 8	+ 3	−24	− 18	+ 48
39	+ 1	− 6	− 8	+48	− 8	− 6
40	+ 1	+ 7	− 2	−14	− 2	+ 7
41	− 5	...	− 2	...	+ 10	
42	−14	− 4	− 4	+16	+ 56	+ 56
43	+ 3	− 5	+ 2	−10	+ 6	− 15
44	+ 6	+ 4	− 1	− 4	− 6	+ 24
45	− 2	− 3	...	...	...	+ 6

TABLE IV.—*continued.*

No.	x'.	y'.	z'.	$y'z'$.	$z'x'$.	$x'y'$.
46	+ 1	− 7	+ 2	−14	+ 2	− 7
47	+ 1	− 1	+ 3	− 3	+ 3	− 1
48	+ 1	−10	+ 4	−40	+ 4	− 10
49	+10	+ 6	+ 5	+30	+ 50	+ 60
51	− 5	− 8	− 5	+40	+ 25	+ 40
52	+ 4	...	− 4	...	− 16	...
53	+13	+ 7	− 1	− 7	− 13	+ 91
54	+ 2	− 2	− 1	+ 2	− 2	− 4
56	...	+ 1	+ 3	+ 3	...	...
58	−10	− 3	+ 3	− 9	− 30	+ 30
59	− 8	...	−13	...	+104	...
60	− 5	− 3	− 5	+15	+ 25	+ 15
61	− 5	− 2	− 5	+10	+ 25	+ 10
62	−10	− 3	− 7	+21	+ 70	+ 30
63	+10	+ 5	+10	+50	+100	+ 50
65	+ 4	+ 2	+ 3	+ 6	+ 12	+ 8
66	− 1	...	+ 6	...	− 6	...
69	+ 4	+ 9	− 3	−27	− 12	+ 36
70	+ 2	+ 3	+ 5	+15	+ 10	+ 6
71	− 1	+ 2	+ 1	+ 2	− 1	− 2
72	− 2	− 9	− 1	+ 9	+ 2	+ 18
73	+13	− 3	+ 7	−21	+ 91	− 39
74	− 1	+ 4	− 7	−28	+ 7	− 4
76	+ 5	+10	+ 4	+40	+ 20	+ 50
77	+15	+ 6	+ 5	+30	+ 75	+ 90
78	+ 6	− 4	− 5	+20	− 30	− 24
79	+ 3	− 9	− 3	+27	− 9	− 27
80	...	− 1	− 7	+ 7	...	...
81	+ 8	+ 7	+ 6	+42	+ 48	+ 56
82	− 7	+ 2	− 3	− 6	+ 21	− 14
83	− 2	− 6	− 4	+24	+ 8	+ 12
84	+ 2	...	− 4	...	− 8	...
86	+ 1	+ 8	+ 2	+16	+ 2	+ 8
87	+ 2	− 2	...	...	...	− 4
88	+ 1	− 6	− 8	+48	− 8	− 6
89	− 7	...	− 5	...	+ 35	...
90	+10	+ 4	+ 2	+ 8	+ 20	+ 40
91	+ 9	+ 5	+ 7	+35	+ 63	+ 45
92	− 2	− 2	− 2	+ 4	+ 4	+ 4
93	+ 6	+ 6	+ 6	+36	+ 36	+ 36
94	−15	−12	− 2	+24	+ 30	+180

$$\Sigma(y'z') = +1106 - 272 = +834$$
$$\Sigma(z'x') = +2122 - 317 = +1805$$
$$\Sigma(x'y') = +1774 - 246 = +1528$$

Hence the corrected values are given by—

$$\Sigma(yz) = \Sigma(y'z') - \mathrm{N}yz = +834 - 3\cdot 1 = +830\cdot 9$$
$$\Sigma(zx) = \Sigma(z'x') - \mathrm{N}zx = +1805 - 6\cdot 7 = +1798\cdot 3$$
$$\Sigma(xy) = \Sigma(x'y') - \mathrm{N}xy = +1528 - 3\cdot 6 = +1524\cdot 4$$

Table IV.—*Continued.*

The coefficients of correlation are thus—

For breadth and height—

$$r_{yz} = \frac{\Sigma(yz)}{N.\sigma_y.\sigma_z.} = +0{\cdot}367 \pm 0{\cdot}063$$

Height and length—

$$r_{zx} = \frac{\Sigma(zx)}{N.\sigma_z.\sigma_x.} = +0{\cdot}541 \pm 0{\cdot}051$$

Length and breadth—

$$r_{xy} = \frac{\Sigma(xy)}{N.\sigma_x.\sigma_y.} = +0{\cdot}526 \pm 0{\cdot}053,$$

the probable errors being obtained from the formula—

$$0{\cdot}6745.\frac{1-r^2}{\sqrt{N}}.$$

Table V.—Tasmanian Crania—Males only.

$N = 54$.

Length—

$L = 183{\cdot}44 \pm 0{\cdot}54$
$\sigma_x = 5{\cdot}86 \pm 0{\cdot}38$
$V_x = 3{\cdot}18 \pm 0{\cdot}21$

Breadth—

$B = 136{\cdot}69 \pm 0{\cdot}44$
$\sigma_y = 4{\cdot}77 \pm 0{\cdot}31$
$V_y = 3{\cdot}49 \pm 0{\cdot}23$

Height—

$H = 132{\cdot}11 \pm 0{\cdot}44$
$\sigma_z = 4{\cdot}76 \pm 0{\cdot}31$
$V_z = 3{\cdot}60 \pm 0{\cdot}23$

Coefficients of Correlation—

$r_{yz} = +0{\cdot}268 \pm 0{\cdot}085$
$r_{zx} = +0{\cdot}577 \pm 0{\cdot}061$
$r_{xy} = +0{\cdot}452 \pm 0{\cdot}073$

Table VI.—Tasmanian Crania—Females only.

$N = 30$.

Length—

$L = 175{\cdot}00 \pm 0{\cdot}70$
$\sigma_x = 5{\cdot}72 \pm 0{\cdot}50$
$V_x = 3{\cdot}27 \pm 0{\cdot}29$

Breadth—

$B = 132{\cdot}43 \pm 0{\cdot}44$
$\sigma_y = 3{\cdot}57 \pm 0{\cdot}31$
$V_y = 2{\cdot}47 \pm 0{\cdot}22$

Height—

$H = 126{\cdot}97 \pm 0{\cdot}66$
$\sigma_z = 5{\cdot}34 \pm 0{\cdot}47$
$V_z = 4{\cdot}21 \pm 0{\cdot}37$

Coefficients of Correlation—

$r_{yz} = +0{\cdot}151 \pm 0{\cdot}120$
$r_{zx} = +0{\cdot}178 \pm 0{\cdot}119$
$r_{xy} = +0{\cdot}256 \pm 0{\cdot}115$

Table VII.—Table of Measurements of 191 Papuan Crania.

No.	Length (mm.).	Breadth (mm.).	Height (mm.).	No.	Length (mm.).	Breadth (mm.).	Height (mm.).
1	180	121	132	55	183	134	144
2	180	126	127	56	191	130	135
3	167	118	127	57	167	117	119
4	176	125	126	58	183	131	131
5	179	127	132	59	170	125	134
6	174	130	129	60	181	137	129
7	157	130	124	61	160	125	129
8	170	121	131	62	168	130	129
9	178	124	126	63	171	125	130
10	168	119	118	64	175	131	133
11	160	121	124	65	176	142	126
12	173	123	129	66	178	125	125
13	189	144	141	67	189	134	138
14	177	133	132	68	169	133	135
15	189	126	136	69	176	128	133
16	165	136	128	70	175	135	128
17	180	132	135	71	175	131	132
18	179	130	125	72	177	127	132
19	167	131	127	73	168	128	133
20	185	132	132	74	169	131	130
21	157	121	118	75	188	129	134
22	187	125	141	76	175	120	136
23	183	133	129	77	191	131	130
24	162	131	129	78	179	134	127
25	182	130	134	79	186	124	137
26	184	132	130	80	167	121	125
27	174	129	131	81	168	129	132
28	161	131	137	82	168	132	128
29	178	121	127	83	176	126	136
30	174	125	135	84	185	130	136
31	186	139	141	85	195	131	137
32	185	120	135	86	186	132	133
33	189	131	137	87	167	122	128
34	171	123	130	88	181	128	131
35	196	129	130	89	175	123	126
36	177	128	137	90	194	130	134
37	174	129	129	91	162	133	131
38	181	120	132	92	177	137	135
39	190	125	133	93	167	134	125
40	180	124	132	94	169	129	138
41	169	135	136	95	181	123	137
42	173	125	138	96	184	129	130
43	177	121	135	97	185	129	138
44	178	120	131	98	190	139	137
45	188	136	135	99	182	143	132
46	173	130	131	100	175	134	132
47	184	119	126	101	188	132	137
48	178	125	125	102	180	125	137
49	177	122	135	103	180	125	123
50	175	130	131	104	170	126	127
51	173	123	125	105	184	143	149
52	197	129	134	106	180	122	129
53	162	133	131	107	170	129	135
54	184	126	133	108	163	129	130

Table VII.—*Continued.*

No.	Length (mm.).	Breadth (mm.).	Height (mm.).	No.	Length (mm.).	Breadth (mm.).	Height (mm.).
109	181	123	134	151	164	117	127
110	177	125	133	152	186	133	138
111	168	130	133	153	177	135	135
112	171	122	130	154	185	129	135
113	173	122	126	155	187	127	139
114	174	123	131	156	166	136	120
115	176	140	140	157	166	117	128
116	179	122	133	158	174	127	127
117	175	112	123	159	181	121	133
118	178	133	132	160	190	119	133
119	179	128	137	161	182	125	131
120	185	135	143	162	169	119	123
121	178	134	135	163	185	125	140
122	174	130	133	164	179	119	129
123	173	120	128	165	178	121	128
124	178	130	138	166	166	128	133
125	172	122	129	167	177	134	133
126	181	128	127	168	179	128	140
127	173	130	130	169	171	127	128
128	182	127	133	170	182	133	132
129	166	137	127	171	176	132	129
130	188	132	135	172	172	134	145
131	176	127	130	173	188	132	149
132	176	123	134	174	178	129	134
133	176	117	127	175	191	131	137
134	168	122	129	176	187	140	130
135	176	125	126	177	173	128	130
136	178	137	137	178	174	130	135
137	179	129	130	179	185	133	128
138	182	125	132	180	182	133	134
139	175	134	134	181	179	131	129
140	175	132	136	182	186	122	131
141	183	137	135	183	173	123	121
142	182	129	130	184	184	130	137
143	196	136	138	185	166	120	125
144	167	129	125	186	175	133	125
145	174	138	140	187	176	132	131
146	185	129	135	188	164	126	134
147	176	146	132	189	170	131	128
148	174	127	122	190	170	130	120
149	160	129	128	191	179	118	128
150	168	136	137				

Table VIII.—Papuan Crania, not separated into Sexes.

N = 191.

Length—

$L = 176·96 \pm 0·39$

$\sigma_x = 8·08 \pm 0·28$

$V_x = 4·57 \pm 0·16$

Breadth—

$B = 128·38 \pm 0·29$

$\sigma_y = 5·91 \pm 0·20$

$V_y = 4·60 \pm 0·16$

Height—

$H = 131·68 \pm 0·26$

$\sigma_z = 5·24 \pm 0·18$

$V_z = 3·98 \pm 0·14$

Coefficients of Correlation—

$r_{yz} = +0·368 \pm 0·042$

$r_{zx} = +0·465 \pm 0·038$

$r_{xy} = +0·190 \pm 0·047$

Table IX.—Comparison of the Mean, Standard Deviation, and Coefficient of Variation of Tasmanian, Australian, and Papuan Crania.

		Tasmanian.	Australian.	Papuan.
			Mean.	
Length	Male	183·44 ± ·54	183·56 ± ·57	...
	Female	175·00 ± ·70	175·59 ± ·68	...
	Unsexed	180·30 ± ·51	181·81 ± ·52	176·96 ± ·39
Breadth	Male	136·69 ± ·44	130·60 ± ·37	...
	Female	132·43 ± ·44	128·73 ± ·55	...
	Unsexed	135·14 ± ·35	130·19 ± ·32	128·38 ± ·29
Height	Male	132·11 ± ·44	131·13 ± ·42	...
	Female	126·97 ± ·66	124·50 ± ·87	...
	Unsexed	130·26 ± ·40	129·67 ± ·42	131·68 ± ·26
			Standard Deviation.	
Length	Male	5·86 ± ·38	7·42 ± ·40	...
	Female	5·72 ± ·50	4·73 ± ·48	...
	Unsexed	7·04 ± ·36	7·67 ± ·37	8·08 ± ·28
Breadth	Male	4·77 ± ·31	4·82 ± ·26	...
	Female	3·57 ± ·31	3·86 ± ·39	...
	Unsexed	4·79 ± ·25	4·69 ± ·22	5·91 ± ·20
Height	Male	4·76 ± ·31	5·54 ± ·30	...
	Female	5·34 ± ·47	6·05 ± ·62	...
	Unsexed	5·49 ± ·28	6·29 ± ·33	5·24 ± ·18
			Coefficient of Variation.	
Length	Male	3·18 ± ·21	4·04 ± ·22	...
	Female	3·27 ± ·29	2·69 ± ·27	...
	Unsexed	3·90 ± ·20	4·22 ± ·20	4·57 ± ·16
Breadth	Male	3·49 ± ·23	3·69 ± ·20	...
	Female	2·47 ± ·22	3·00 ± ·31	...
	Unsexed	3·54 ± ·18	3·60 ± ·17	4·60 ± ·16
Height	Male	3·60 ± ·23	4·23 ± ·23	...
	Female	4·21 ± ·37	4·86 ± ·49	...
	Unsexed	4·21 ± ·22	4·86 ± ·23	3·98 ± ·14

Table X.—Table of Differences of Tasmanian, Australian, and Papuan Crania.

Mean Length—

Australian – Tasmanian	=	1·51 ± 0·73
Tasmanian – Papuan	=	3·34 ± 0·64
Australian – Papuan	=	4·85 ± 0·65

Mean Breadth—

Tasmanian – Australian	=	4·95 ± 0·47
Australian – Papuan	=	1·81 ± 0·43
Tasmanian – Papuan	=	6·76 ± 0·45

Mean Height—

Papuan – Tasmanian	=	1·42 ± 0·47
Tasmanian – Australian	=	0·59 ± 0·58
Papuan – Australian	=	2·01 ± 0·49

Coefficient of Variation of Length—

Papuan – Australian	=	0·35 ± 0·26
Australian – Tasmanian	=	0·32 ± 0·28
Papuan – Tasmanian	=	0·67 ± 0·26

Coefficient of Variation of Breadth—

Papuan – Australian	=	1·00 ± 0·23
Australian – Tasmanian	=	0·06 ± 0·25
Papuan – Tasmanian	=	1·06 ± 0·24

Coefficient of Variation of Height—

Australian – Tasmanian	=	0·65 ± 0·32
Tasmanian – Papuan	=	0·23 ± 0·26
Australian – Papuan	=	0·88 ± 0·27

Table XI.—Comparison of the Coefficients of Correlation of Length-Breadth, Length-Height, and Breadth-Height of Tasmanian, Australian, and Papuan Crania.

	Males.	Females.	Unsexed.
		Length and Height.	
Tasmanian	·577 ± ·061	·178 ± ·119	·541 ± ·051
Australian	·337 ± ·060	·083 ± ·143	·449 ± ·054
Papuan	...	...	·465 ± ·038
		Length and Breadth.	
Tasmanian	·452 ± ·073	·256 ± ·115	·526 ± ·053
Australian	·313 ± ·069	·016 ± ·144	·318 ± ·061
Papuan	...	...	·190 ± ·047
		Breadth and Height.	
Tasmanian	·268 ± ·085	·151 ± ·120	·367 ± ·063
Australian	·314 ± ·069	·311 ± ·130	·344 ± ·059
Papuan	...	...	368 ± ·042

Table XII.

Differences in Coefficients of Correlation of Breadth and Height—

Tasmanian	–	Australian	=	+·023	±	·086
Australian	–	Papuan	=	–·024	±	·072
Tasmanian	–	Papuan	=	–·001	±	·076

Differences in Coefficients of Correlation of Height and Length—

Tasmanian	–	Australian	=	+·092	±	·074
Australian	–	Papuan	=	–·016	±	·066
Tasmanian	–	Papuan	=	+·076	±	·064

Differences in Coefficients of Correlation of Length and Breadth—

Tasmanian	–	Australian	=	+·208	±	·081
Australian	–	Papuan	=	+·128	±	·077
Tasmanian	–	Papuan	=	+·336	±	·071

WORKS REFERRED TO IN THE TEXT.

(1) Keane, A. H., *The World's Peoples*, Hutchinson & Co., London, 1908.

(2) Keane, A. H., *Ethnology*, University Press, Cambridge, 1901.

(3) Topinard, P., *Anthropology*, Chapman & Hall, Ltd., London, 1894.

(4) Tylor, E. B., *Anthropology*, Macmillan & Co., London, 1904.

(5) Curr, E. M., *The Australian Race*, vol. i. p. 152 *et seq.*, London, 1886.

(6) De Quatrefages, A., *The Human Species*, London, 1890, 5th edition.

(7) Klaatsch, A., "The Skull of the Australian Aboriginal," *Reports from the Path. Lab. of the Lunacy Dept., N.S.W.*, vol. i. p. 45 *et seq.*

(8) Schötensack, O., "Die Bedeutung Australiens für die Heranbildung des Menschen aus einer niederen Form," *Zeitschr. Ethnol.*, 33. Jhrg., 1901, S. 127 ff.

(9) Semon, R., "Australier und Papua," *Corr. Bl. deutsch. Ges. Anthrop.*, xxxii., 1902, pp. 4, 11, 22, 32.

(10) Berry, R. J. A., "A Living Descendant of an Extinct (Tasmanian) Race," *Proc. Roy. Soc. of Victoria*, vol. xx. (N.S., Part I.), 1907, pp. 1–20.

(11) Turner, Sir Wm., "The Craniology, Racial Affinities, and Descent of the Aborigines of Tasmania," *Trans. Roy. Soc. of Edinburgh*, vol. xlvi., 1908, pp. 365–403.

(12) Berry, R. J. A., and Robertson, A. W. D., "Preliminary Communication on Fifty-three Tasmanian Crania, forty-two of which are now recorded for the first time," *Proc. Roy. Soc. of Victoria*, vol. xxii. (N.S., Part I.), 1909, pp. 47–58.

(13) Berry, R. J. A., and Robertson, A. W. D., "Preliminary Account of the Discovery of Forty-two hitherto unrecorded Tasmanian Crania," *Anatomischer Anzeiger*, Bd. xxxv., 1909, pp. 11–17.

(14) Flower, Sir Wm. H., *Catalogue of Osteological Specimens in the Museum of the Royal College of Surgeons of England*, Part I., 2nd edition, Taylor & Francis, London.

(15) Davis, Barnard, *Thesaurus Craniorum.*

(16) Ling Roth, H., *The Aborigines of Tasmania*, London, 1890.

(17) Harper, W. E., and Clarke, A. H., "Notes on the Measurements of the Tasmanian Crania in the Tasmanian Museum, Hobart," *Papers and Proc. Roy. Soc. Tasmania*, 1897, pp. 97–110.

(18) Berry, R. J. A., and Robertson, A. W. D., "Dioptrographic Tracings in Four Normæ of Fifty-two Tasmanian Crania," *Trans. Roy. Soc. of Victoria*, vol. v., 1910.

(19) Gray, J., "Measurements of Papuan Skulls," *Journ. Anthrop. Inst. of Great Britain and Ireland*, vol. xxxi., 1901, p. 261.

(20) Turner, Sir Wm., "Decorated and Sculptured Skulls from New Guinea," *Proc. Roy. Soc. of Edinburgh*, vol. xxii., 1898–9, p. 553.

(21) Dorsey and Holmes, "Observations on a Collection of Papuan Crania," *Publications of the Field Columbian Museum*, Anthropological Series, vol. ii., No. 1, 1897, p. 1.

(22) Gray, J., "The Analysis of Anthropometric Statistics," *Proc. Anat. and Anthrop. Soc. Aberdeen*, 1902–4, p. 85.

(23) Worthington, R., "On Professor Pearson's Contributions to Osteology," *Journ. Anat. and Phys.*, vol. xxxv., 1901, pp. 455–475.

(24) Schwalbe, G., "Studien über Pithecanthropus erectus Dubois," *Zeitschrift für Morphologie und Anthropologie*, Bd. i., Heft 1, 1899, p. 16 *et seq.*

(25) Krause, W., "Australische Schädel," *Verh. Berlin. Ges. Anthrop.*, 1897, p. 508.

(26) Broca, P., "Sur la classification et la nomenclature d'après les indices céphaliques," *Revue d'Anthropologie*, vol. i., 1872, p. 385.

(27) Pearson, K., and Lee, A., "Data for the Problem of Evolution in Man: A First Study of the Correlation of the Human Skull," *Phil. Trans. Roy. Soc. London*, vol. cxcvi.

(28) Hatai, S., "Studies on the Variation and Correlation of Skull Measurements in both Sexes of Mature Albino Rats," *American Journal of Anatomy*, vol. vii., 1907–8, pp. 423–441.

(29) Tschepourkowsky, "Contributions to the Study of Inter-racial Correlation," *Biometrika*, vol. iv. pp. 286–312.

(30) Mathew, Rev. J., *Eaglehawk and Crow*, London and Melbourne, 1899.

(31) Meyer, A. B., "Über hundert fünf und dreissig Papua-Schädel von Neu Guinea und der Insel Mysore (Geelvinksbai)," *Mittheil. des kgl. Zoolog. Mus. Dresden.*

(*Issued separately December* 8, 1910.)

III.—The Place in Nature of the Tasmanian Aboriginal as deduced from a Study of his Calvarium.—Part I. His Relations to the Anthropoid Apes, Pithecanthropus, Homo primigenius, Homo fossilis, and Homo sapiens. By Richard J. A. Berry, M.D., Professor of Anatomy in the University of Melbourne, and A. W. D. Robertson, M.D. Melb., Government Research Scholar in the Anatomy Department of the University of Melbourne. (With Two Folding Tables.)

(Read March 7, 1910. MS. received March 30, 1910.)

In our communication to the Royal Society of Victoria of the 11th March, 1909 (1), describing our recent discovery of forty-two Tasmanian crania hitherto quite unknown to the world of science, we stated that "one of the earliest purposes to which it is proposed to utilise the present material is the determination of the relationship of the Tasmanian to the anthropoids and primitive man on the one hand, and to the Australian aboriginal on the other hand. Schwalbe's study of *Pithecanthropus erectus* (2) may serve as a basis for the former purpose, and Klaatsch's recent work (3) for the latter, though it must be remembered that innumerable authors have contributed to both subjects." The present work is the fulfilment of the first part of this undertaking, namely, the determination of the relationship of the Tasmanian to the anthropoids and primitive man.

In view of the fact that the whole object of the present investigation is one of comparison, it will be obvious that some previously existing work or works on anthropoids and primitive man had to be taken as the basis upon which the present research was to be reared; and in view of the distinguished character of the work of Professor G. Schwalbe, the Director of the Anatomical Institute of the University of Strassburg, in both of the fields mentioned, it need cause no surprise that his investigational methods of "form analysis" have been selected by us almost exclusively as the basis upon which our examination of the Tasmanian aboriginal calvarium has been made, whilst, for purposes of comparison, we have availed ourselves not only of Schwalbe's work but also of Klaatsch's researches.

We are, of course, aware that Schwalbe's investigational methods of examining the skull roof have not met with universal approval. Cunningham (4), for example, says: "What is the value of this glabello-cerebral index of Schwalbe? Can we rely upon it giving a true and proper idea of the relative extent and degree of projection of the pars glabellaris of the cranium? I do not think that we can, and I look upon the figures given . . . . as being of little value, and in certain respects misleading."

Giufrida-Ruggeri (5) has also expressed dissatisfaction with Schwalbe's glabello-cerebral index, as well as with certain other of Schwalbe's investigational methods (6), such as the bregma angle, the parietal index, the bregma positional index, etc.

These criticisms notwithstanding, it must be borne in mind that even Schwalbe's critics admit the great value of his work. That work further provided us with much of the required data for our comparison of the Tasmanian aboriginal, and, above all, was readily accessible in Melbourne—for all of which weighty reasons our selection of Schwalbe's methods as our investigational basis is amply justified.

The material upon which our investigations have been carried out comprises a series of 52 Tasmanian crania, 41 of which have never been examined by anyone, nor have any of them ever previously been subjected to Schwalbe's methods of "form analysis." The whole of this material we have now made available to scientists in all parts of the world in our recent publication, "Dioptrographic Tracings in Four Normæ of Fifty-two Tasmanian Crania" (7). For reasons already mentioned in our recent works (1 and 8), the present series of observations has been limited to those which can be observed upon the median sagittal tracing, or, in other words, upon a dioptrographic tracing of the norma lateralis. Of such observations we have recorded 27 upon every skull, or at all events upon every skull the preservation of which permitted such observations to be carried out.

These 27 observations are as follow:—

1. The glabella-inion length.
2. The calvarial height.
3. The calvarial height index, that is, the proportion which the calvarial height bears to the glabella-inion length, the latter being taken as 100.
4. The greatest breadth.
5. The calvarial height-breadth index, that is, the proportion which the calvarial height bears to the greatest breadth, the latter being taken as 100.
6. Half the sum of the glabella-inion length and breadth added together.
7. The calvarial height half sum glabella-inion length plus breadth index, that is, the proportion which the calvarial height bears to the half sum of the glabella-inion length plus breadth, the latter being taken as 100.
8. The distance of the foot-point of the calvarial height from the glabella.

9. The calvarial height foot-point positional index, that is, the relation which the distance of the foot-point of the calvarial height from the glabella bears to the glabella-inion length, the latter being taken as 100.

10. The frontal angle.

11. The bregma angle.

12. The distance of the foot-point of the bregma from the glabella.

13. The bregma foot-point positional index, that is, the proportion which the distance of the foot-point of the bregma from the glabella bears to the glabella-inion length, the latter being taken as 100.

14. The length of the frontal arc.

15. The length of the frontal chord.

16. The curvature index of the os frontale, that is, the proportion which the length of the frontal chord bears to the length of the frontal arc, the latter being taken as 100.

17. The angle of frontal curvature.

18. The length of the chord of the pars glabellaris of the os frontale.

19. The length of the chord of the pars cerebralis of the os frontale.

20. The glabella-cerebral chord index, that is, the proportion which the length of the glabellar chord bears to the length of the cerebral chord, the latter being taken as 100.

21. The length of the parietal arc.

22. The length of the parietal chord.

23. The curvature index of the os parietale.

24. The angle of parietal curvature.

25. The parietal frontal arc index, that is, the proportion which the parietal arc bears to the frontal arc, the latter being taken as 100. Schwalbe himself briefly calls this index "der Scheitelbein-Index."

26. The lambda angle.

27. The opisthionic angle.

Our choice of these 27 observational factors has not been altogether haphazard. Many of the curved horizontal measurements we were prevented from taking by the non-arrival of the diagraph; but even if this instument had arrived in time the owners of the crania would not have left their specimens in our hands for a sufficiently long period, as they all, with one exception, refused to allow the specimens to be transported from Tasmania to Melbourne. Our observational field was therefore immediately restricted to the median sagittal plane, but upon this plane we have recorded practically all the observations which Schwalbe introduced in his investigation of Pithecanthropus.

Our choice of a base line was determined, as it was in Schwalbe's work, by the fact that in Dubois's Pithecanthropus (10) the nasion was absent, and consequently our working base line became the glabella-inion length.

Klaatsch (3), too, employs the same base line for the same reason and the like object. He says: "With regard to the Pithecanthropus, Spy, and Neanderthal remains, I followed Professor Schwalbe in taking for a common horizon the plane passing through the glabella and inion." And again, "for purposes of more precise comparative investigations with the fossil fragments, the glabella-inion line is clearly preferable."

Turner (11), on the other hand, in his 1908 work on ten Tasmanian crania, employs a "nasio-tentorial plane drawn from the nasion to the upper border of the groove for the lateral sinus, which divided the cranial cavity into a cerebral part above this plane, and a basal part for the lodgment of the cerebellum, pons, and medulla." His reason for the selection of this plane as opposed to the glabella-inion plane is that "the range of variation in its projection (that is, the glabella), associated in a more or less degree with the development of the frontal sinuses, unfits it to be used for taking the point in front from which to estimate the length of the cerebral part of the cranial cavity."

We agree with Turner that the glabella-inion plane is not the best "from which to estimate the length of the cerebral part of the cranial cavity," for in our own experience we have found the nasio-inion plane coincides more closely with the length of the cerebral part of the cranial cavity than either the glabella-inion or nasio-tentorial planes. The nasio-inion plane is not, however, used in the present investigation, for the sufficiently adequate explanations already given; but in the second part of this work, that is, the comparison of 100 Australian aboriginal crania with the Tasmanian crania and with those of primitive man, both nasio-inion and glabella-inion planes will be employed.

Such being the reasons which determined, for the present investigation, the choice of the glabella-inion plane as a base line, it is next necessary to say something as to the locations of the anterior and posterior ends of the plane, that is, of the glabellar point and the inion.

The glabellar point has been defined by a large number of authors, as is fully set forth and discussed by Schwalbe (2). Klaatsch (3) simply says the "glabella is the most prominent surface of the forehead in the median line between the superciliary ridges."

Schwalbe (2) defines the glabellar point, that is, the anterior end of the glabella-inion plane, in this way: "Da die Glabella auf dem Medianschnitt durch eine Curve dargestellt wird, so ist es nöthig, einen Punkt derselben als Glabellar Punkt zum Ausgangspunkt der Glabella-Inion-Linie zu machen und dies kann natürlich nur der Gipfelpunkt der Glabellacurve sein, welcher den weitesten vertikalen Abstand von der Verbindungslinie NA zwischen Nasion (N) und den oberen Ende der Glabellacurve (A) besitzt. Dieser Punkt bezeichnet also das vordere Ende der Glabella-Inion-Linie."

It is in this way that the glabellar point has been determined in the present work. The glabellar chord is drawn from the nasion to the junction of the glabellar and cerebral parts of the os frontale, and the

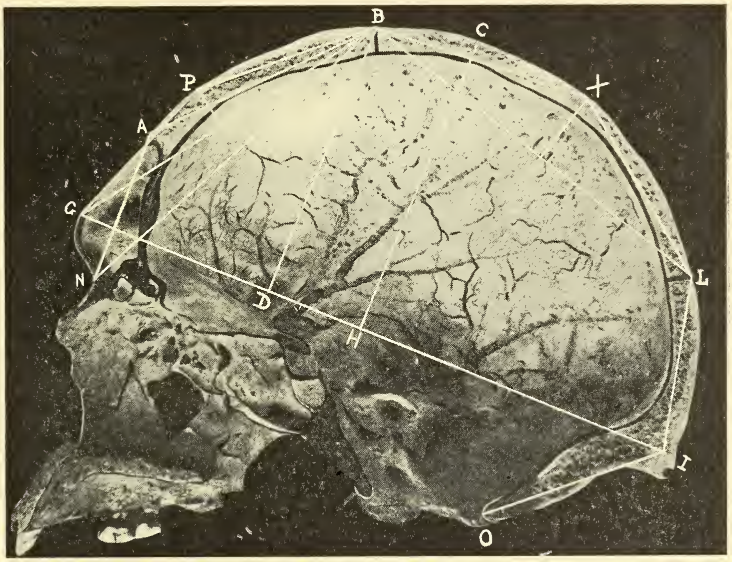


Fig. 1.—Median Sagittal Section through an Adult Male Australian Aboriginal Skull.

N. The nasion.
G. The glabellar point.
A. The upper limit of the glabellar curve.
P. The maximum point of the frontal curvature.
B. The bregma.
C. The maximum point of the calvarial height.
X. The maximum point of the parietal curvature.
L. The lambda.
I. The inion.
O. The opisthion.
D. The bregma foot-point.
H. The calvarial height foot-point.
G I. The glabella-inion length.
C.H. The calvarial height.
G.H. The distance of the calvarial height foot-point from the glabella.
B G.I. The bregma angle.
G.D. The distance of the bregma foot-point from the glabella.
N.B. The frontal chord.
G.P.B. The glabella-bregma frontal angle.
N.A. The glabellar chord.
A.B. The cerebral chord.
B.L. The parietal chord.
B.X.L. The angle of parietal curvature.
L.I.G. The lambda angle.
O.I.G. The opisthionic angle.

glabellar point is the point on the curve of the pars glabellaris which is farthest away from the glabellar chord. (See fig. 1.)

In the Tasmanian crania the difficulty in defining the glabellar point consists in the fact that it is not easy to separate off the glabellar part of the os frontale from its cerebral part, that is, to determine the upper end of the glabellar chord. A glance at column 20 of Table XXIX., which sets forth the relative positions of the crania compared for the glabellar-cerebral chord index, shows that the Tasmanian occupies a surprisingly high position in this respect. He stands above the European. In other words, the Tasmanian has a less well-developed pars glabellaris than even the European; and consequently the difficulty referred to of locating with precision the upper end of the glabellar chord, and thereafter of the glabellar point, is a very real one. It may be thought that there is a fallacy in our results concerning this glabellar cerebral chord index on account of their being based on unsexed data. We thought the same, and we therefore worked out the index on the males only, with the result that the index became a decimal point less for the males than for the unsexed crania. We can therefore only reiterate our statement, surprising though it may sound, that the Tasmanian has a less well-developed pars glabellaris than the European, and that his glabellar point is consequently somewhat difficult of precise location.

As with the glabella, so with the inion. Innumerable definitions have been given from time to time. For Broca (12) and Topinard (13) the inion is the protuberantia occipitalis externa. For Duckworth (14) the inion is "the most prominent point on the external occipital protuberance." According to Klaatsch (3), this is exactly what the inion is not. He says: "By the inion (which is not identical with the external occipital protuberance, a development wanting in the lower races of mankind) must be understood the mid-point between the right and the left part of the torus occipitalis; it can be easily found in almost any skull just above the groove (fossa supra-toralis of Klaatsch) lying midway on the linea nuchæ superior."

Schwalbe (2) discusses the whole question carefully, and concludes thus: "Somit ergiebt sich aus dieser Erörterung, dass ich als Inion entweder den einheitlichen medianen Vereinigunspunkt der Lineæ nuchæ superiores und supremæ bezeichne oder im Falle des Auftretens eines besonderen Tuberculum linearum für die Lineæ nuchæ superiores den oberen der beiden dann verhandenen Höcker oder Vereinigunspunkte als Inion bestimme." It is in this way that the inion has been located in the present investigation, though in the Tasmanian crania we found with the inion, as with the glabellar point, certain difficulties due to the peculiar formation of this part of the skull. In many of the Tasmanian crania the lineæ nuchæ supremæ and the lineæ nuchæ superiores are particularly well developed, with a well-developed intervening torus occipitalis, whilst that part of the squama occipitalis which lies immediately above the lineæ nuchæ supremæ bulges backwards in a most extraordinary and inexplicable manner.

Turner (10) does not seem to regard this suprainial bulging of the squama occipitalis as of any very great importance. He says "the suprainial part of the occipital squama formed a large rounded protuberance in the female skull, a character which may perhaps be regarded as sexual rather than racial."

As we have found the same suprainial bulging in many of the cadavera—all of Caucasian origin—in the ordinary routine work of the dissecting-room, we are inclined to agree with Turner that, whatever the explanation of this suprainial prominence may be, it is certainly not racial.

Having selected a working base line and the points between which that base line was to be drawn, the rest of the observational points presented no difficulties. Schwalbe's procedure and instructions were explicitly followed. and those already mentioned as having been selected by us are illustrated in fig. 1—a median sagittal section of an adult male Australian aborginal skull.

In Table XXVIII. are set forth the individual measurements of the 27 observations made by us upon our series of 52 Tasmanian crania. In this table are recorded in the upper four horizontal lines the serial number of the skulls as used in the present investigation, the present location of the skulls, their original numbers, and their probable sex. In the three left-hand vertical columns are set forth the numbers and names of the observations recorded and the nature of the indices employed. In the vertical columns 1 to 52 inclusive are set forth the individual measurements and indices of each skull. The four vertical columns on the right record the number of observations made, the results worked out at an average, together with the minimum and maximum figures for that particular observation. Thus the number of observations made on the glabella-inion length was 44, giving an average length of 173·1 mm., a minimum of 157 which occurred in skull number 10, and a maximum of 188 in skull 29.

In our previous writings on the discovery of these crania (1 and 8) we pointed out that there was some dubiety as to the purity of one or two of the examples here dealt with. In order to test this point we have indicated by a + or a − the skulls in which the maximum or minimum figure occurred: thus at the junction of the columns 10 and 1 the glabella-inion length 157 is marked with a minus, which thus indicates the minimum figure recorded under this observation, and also shows at a glance in which skull the minimum figure occurred. As this procedure has been adopted uniformly throughout, it makes it perfectly clear that in no part of the table is there a preponderance of plus and minus signs, and consequently we regard this uniformity in the distribution of such signs as an additional argument of the purity of the crania dealt with.

It will easily be understood that it was not possible to record all the observations upon every skull: thus the glabella-inion length could only be recorded upon 44 out of the 52 crania; this for the very sufficient reason that one or other of the two points was absent in the remaining 8 crania, and similarly with the other observations. The fourth column from the right sets forth the number of observations recorded.

It should be further noted that skull number 48 is that of a young subject. In another publication (7) we have estimated the age of this skull as from 20 to 24 years of age; and in the present series of calculations it has been excluded uniformly from all the final results. Column 48 will therefore suffice for a comparison between the average figures obtained for the adult skull and that of the juvenile.

With such an enormous number of calculations we can hardly hope to have escaped all possibility of error, but every precaution was taken to prevent such errors occurring; thus every calculation was made with the sliding rule and checked arithmetically.

Having worked out in this way all those major points in the anatomy of the Tasmanian calvaria to which Schwalbe has aptly applied the term "form analysis," it became necessary to have some data with which to compare the final results attained by us for the Tasmanian. With this object we made use of the following:—

Of the apes we determined to restrict our comparative observations to the nearest anthropoid.

Pithecanthropus erectus was naturally selected in the reasonable expectation that it might afford a transition from the anthropoid to Homo primigenius.

Of Homo primigenius we selected the two Spy skulls and that of Neanderthal. The Krapina remains, which Klaatsch and others include with the Spy-Neanderthal group, were purposely excluded.

The Gibraltar skull, believed by Schwalbe to form a slight advance on the remains of Homo primigenius, was also included in our comparative series.

Of Homo fossilis we selected the remains from Brüx and Galley Hill.

Of the primitive form of Homo sapiens we selected the crania of Brünn, Cro-Magnon, and Cannstatt, the last, as it turned out, not a fortunate choice.

Of modern man we selected two examples of Homo Æthiopicus, namely, the Veddah and the Dschagga negro, one from Homo Mongolicus, the Kalmuck, and, finally, examples of Homo Caucasicus.

Our series, comprising as it did some fourteen objects for comparison, was therefore a sufficiently large one to enable us to determine with some degree of certainty the final position of the Tasmanian with reference to the Anthropoids, Pithecanthropus, Homo primigenius, Homo fossilis, and Homo sapiens both extinct and recent.

Notwithstanding that the Anatomy Department possesses casts of the majority of the skulls of primitive man mentioned, we thought it better to make use of previously existing figures recorded by competent authorities rather than to make fresh ones of our own on our casts, and with this object our comparative figures for the nearest anthropoid ape, *Pithecanthropus erectus,* and the three examples of Homo primigenius, are selected exclusively from Schwalbe's work "Studien über *Pithecanthropus erectus*" (2). In connection with the remains of Homo primigenius, it is important to notice the source whence our comparative data were derived, because in other writings Schwalbe gives different measurements for these crania: thus for the length of the parietal arc of Spy 1, Spy 2, and Neanderthal, Schwalbe gives in his Pithecanthropus work the figures employed in the present investigation, namely, 120, 120, and 119, whereas in his "Zur Frage der Abstammung des Menschen" (15) the figures read 125, 110, and 110, respectively. As this is not the only instance where Schwalbe's different measurements have afforded different results, we regard it as important, in justice to ourselves, that the source whence our comparative data for the Spy-Neanderthal group are derived should be borne in mind.

For all the remainder of our comparative data our figures are derived from one or other of six absolutely reliable sources, namely, Schwalbe's "Studien über *Pithecanthropus erectus*" (2); the same author's "Zur Frage der Abstammung des Menschen" (15); "Das Schädelfragment von Brüx und verwandte Schädelformen" (16); "Das Schädelfragment von Cannstatt" (17); "Ueber die specifischen Merkmale des Neanderthalschädels" (9); or from Klaatsch's excellent work in Merkel and Bonnet's *Ergebnisse der Anatomie und Entwickelungsgeschichte* (18).

As all the indices and data required by us are not furnished by these authors in the works quoted, it will be understood that we have either had to calculate the indices from the actual figures furnished in the above-mentioned works, or have obtained them from the profile curves of Schwalbe and Klaatsch. This last procedure was a comparatively easy one, inasmuch as their drawings are to scale.

The curved measurements, such as the lengths of the frontal and parietal arcs, were more difficult to obtain, and we cannot absolutely guarantee for the primitive crania the authenticity of our work under this heading. It must also be borne in mind that wherever a curved measurement is given upon a skull in which it is known that the part does not exist, such measurement has been made from the reconstruction of either Schwalbe or Klaatsch.

The results attained by this comparison are first of all set forth in Tables I. to XXVII. inclusive, where each table records in a progressive arithmetical series the results of the comparisons for each one of the twenty-seven observations made by us upon the Tasmanian crania. Unless specially mentioned to the contrary, the results are set forth in a progressive series from the lowest figure to the highest, or from the highest to the lowest, according to the scale of evolution; and as it has been impossible for us to obtain all the required data in the objects selected by us for comparison, we have thought it advisable to show in each table not only those with which the comparison is made, but also those which were excluded from the comparison.

Table I.—Comparison of the Glabella-Inion Length.

	Minimum.	Average.	Maximum.
1. An adult male gorilla	...	147	...
2. *Pithecanthropus erectus*	...	181	...
3. Galley Hill	...	201	...
4. Brünn	...	201	...
5. Cro-Magnon	...	202	...
6. Three Spy-Neanderthal	196	198·6	202
7. Gibraltar	...	187	...
8. Brüx	...	185 (180)	...
9. Cannstatt	...	174	...
10. Forty-four Tasmanians, unsexed	157	173·1	188
11. Eight Veddahs	160	169·8	176
12. Thirty-one Europeans, unsexed	155	168	184
13. Twenty-three Dschagga negroes	145	167·4	180
14. Four Kalmucks	157	166·2	174

On the assumption that marked length of skull with dolichocephaly is a primitive characteristic, the figures in this table are arranged in an ascending scale from the nearest anthropoid to the maximum, and thereafter in a diminishing scale. As such an arrangement is purely hypothetical, this table is very properly omitted from the final results obtained in Mr Cross's paper which follows.

Table II.—Comparison of the Calvarial Height (Kalottenhöhe).

	Minimum.	Average.	Maximum.
1. An adult male chimpanzee . . .	...	48·5	...
2. *Pithecanthropus erectus*	...	62	...
3. Gibraltar	...	85	...
4. Brüx	...	85	...
5. Three Spy-Neanderthal . . .	81	85·3	88
6. Four Kalmucks	88	90·7	94
7. Galley Hill	...	97	...
8. Forty-eight Tasmanians, unsexed . .	87	97	108
9. Eight Veddahs	94	99·2	107
10. Thirty-four Europeans, unsexed . .	91	99·9	115
11. Twenty-three Dschagga negroes . .	84	100	115·5
12. Cro-Magnon	...	101	...
13. Brünn	...	103	...
14. Cannstatt	...	105	...

Table III.—Comparison of the Calvarial Height Index (Kalottenhöhen-Index).

	Minimum.	Average.	Maximum.
1. *Pithecanthropus erectus*	...	34·2	...
2. An adult male chimpanzee . . .	...	35·1	...
3. Three Spy-Neanderthal	40·9	44·9	47
4. Gibraltar	...	45·4	...
5. Brüx	...	47·6	...
6. Galley Hill	...	48·2	...
7. Cro-Magnon	...	50	...
8. Brünn	...	51·2	...
9. Four Kalmucks	52·8	54·5	84·9
10. Forty-four Tasmanians, unsexed . .	48·3	56·1	62·7
11. Eight Veddahs	54·6	58·4	62·9
12. Cannstatt	58·9	59·6	60·3
13. Twenty-three Dschagga negroes . .	52·1	59·8	67·1
14. Thirty-two Europeans, unsexed . .	54·4	59·8	66·2

Table IV.—Comparison of the Maximum Breadth.

	Minimum.	Average.	Maximum.
1. An adult male chimpanzee . . .	...	113	...
2. Four Veddahs	123	129·7	135
3. Brüx	...	130	...
4. Galley Hill	...	130	...
5. *Pithecanthropus erectus*	...	133	...
6. Forty-eight Tasmanians, unsexed . .	120	134·7	145
7. Brünn	...	139	...
8. Five Europeans	137	142·4	149
9. Cannstatt	...	146	...
10. Four Kalmucks	140	146	148
11. Gibraltar	...	148	...
12. Three Spy-Neanderthal	146	150·3	153
13. Cro-Magnon	...	151	...

Dschagga negroes absent.

Table V.—Comparison of the Calvarial Height-Breadth Index.

	Minimum.	Average.	Maximum.
1. An adult male chimpanzee . . .	...	42·9	...
2. *Pithecanthropus erectus*	...	46·6	...
3. Three Spy-Neanderthal	55·4	56·7	57·9
4. Gibraltar	...	57·4	...
5. Four Kalmucks	62·1	63·3	64·8
6. Brüx	...	63·3	...
7. Cro-Magnon	...	66·8	...
8. Cannstatt	...	71·9	...
9. Forty-four Tasmanians, unsexed . .	65·9	72·2	79·2
10. Four Europeans, unsexed . . .	69	72·4	76·2
11. Brünn	...	74·1	...
12. Galley Hill	...	74·6	...
13. Four Veddahs	69·6	76·9	82·9

Dschagga negroes absent.

Table VI.—Comparison of Half the Sum of the Glabella-Inion Length plus the Breadth.

	Minimum.	Average.	Maximum.
1. An adult female gorilla	...	129	...
2. Four Veddahs	142·5	149·5	153·5
3. Forty-four Tasmanians, unsexed . .	140·5	154	164·5
4. Four Kalmucks	148·5	154·3	161
5. Brüx	...	155	157·5
6. Five Europeans, unsexed . . .	153	156·8	159
7. *Pithecanthropus erectus*	...	157	...
8. Cannstatt	...	160	162
9. Galley Hill	...	165·5	...
10. Gibraltar	...	167·5	...
11. Brünn	...	170	...
12. Three Spy-Neanderthal	172	174·3	177
13. Cro-Magnon	...	176·5	...

Dschagga negroes absent.

Table VII.—Comparison of the Calvarial Height Half-Sum Glabella-Inion Length plus Breadth Index.

	Minimum.	Average.	Maximum.
1. An adult male chimpanzee . . .	...	38·6	...
2. *Pithecanthropus erectus*	...	39·4	...
3. Three Spy-Neanderthal	47	48·9	50
4. Gibraltar	...	50·7	...
5. Galley Hill	...	50·8	...
6. Brüx	...	54·8	...
7. Cro-Magnon	...	57·2	...
8. Four Kalmucks	57·1	58·7	60·2
9. Brünn	...	60·5	...
10. Forty-four Tasmanians	55·2	63	69·5
11. Cannstatt	...	65·6	...
12. Five Europeans, unsexed . . .	60·9	65·8	69·8
13. Four Veddahs	61·2	66·6	71·5

Dschagga negroes absent.

Table VIII.—Comparison of the Distance of the Foot-Point of the Calvarial Height from the Glabella.

	Minimum.	Average.	Maximum.
1. *Pithecanthropus erectus*	70	80·5	91
2. An adult male gorilla	...	84	...
3. Four Kalmucks	76	86·3	95
4. Forty-one Europeans, unsexed . .	78	95·8	112·5
5. Forty-five Tasmanians, unsexed . .	85	101·9	115·5
6. Cannstatt	...	102·5	...
7. Gibraltar	...	105·7	...
8. Brünn	...	110	...
9. Three Spy-Neanderthal	103	111	123
10. Brüx	...	111	...
11. Galley Hill	...	111	...
12. Cro-Magnon	...	121·5	...

Dschagga negroes and Veddahs absent.

Table IX.—Comparison of the Calvarial Height Foot-Point Positional Index.

	Minimum.	Average.	Maximum.
1. *Pithecanthropus erectus*	38·6	44·4	50·2
2. Four Kalmucks	43·6	52·6	56·8
3. Brünn	...	54·7	...
4. Galley Hill	...	55·2	...
5. Three Spy-Neanderthal	52	55·7	60·8
6. Forty-one Europeans, unsexed . .	48·8	56·2	66·9
7. Gibraltar	...	56·5	...
8. Cannstatt	...	58·3	...
9. Forty-four Tasmanians	53·1	59	64·8
10. Brüx	...	60 (61·6)	...
11. Cro-Magnon	...	60·1	...
12. An adult female gorilla	...	61·8 .	...

Dschagga negroes and Veddahs absent.

Table X.—Comparison of the Frontal Angle.

	Minimum.	Average.	Maximum.
1. *Pithecanthropus erectus*	...	52·5	...
2. An adult male chimpanzee . . .	...	56	...
3. Three Spy-Neanderthal . . .	57·5	64·8	70
4. Gibraltar	...	73	...
5. Brüx	...	74·7	...
6. Brünn	...	75	...
7. Galley Hill	...	82	...
8. Cro-Magnon	...	83	...
9. Four Kalmucks	80	85·2	91
10. Forty-four Tasmanians, unsexed . .	72	86	96
11. Cannstatt	...	90	...
12. Forty Europeans, unsexed . . .	73	92·5	103
13. Twenty-four Dschagga negroes . .	88	100·3	110

Veddahs absent.

Table XI.—Comparison of the Bregma Angle.

	Minimum.	Average.	Maximum.
1. An adult male chimpanzee . . .	...	39·5	...
2. *Pithecanthropus erectus*	34	37·5	41*
3. Three Spy-Neanderthal . . .	45	47·5	50·5
4. Gibraltar	50	50·5	51
5. Brüx	...	51·1	...
6. Galley Hill	...	52	...
7. Brünn	...	54	...
8. Cro-Magnon	...	54	...
9. Forty-five Tasmanians, unsexed . .	51·5	56	64
10. Four Kalmucks	55	56·5	57
11. Twenty-four Dschagga negroes . .	53	58·6	63·5
12. Forty Europeans, unsexed . . .	54	59·9	68
13. Cannstatt	...	60	...

Veddahs absent. * This figure was employed for comparative purposes.

Table XII.—Comparison of the Distance of the Bregma Foot-Point from the Glabella.

	Minimum.	Average.	Maximum.
1. *Pithecanthropus erectus*	72	81·5	91
2. Brüx	...	75	...
3. Three Spy-Neanderthal . . .	67	72·3	81
4. An adult male chimpanzee . . .	...	72	...
5. Galley Hill	...	69	...
6. Brünn	...	67·5	...
7. Gibraltar	...	66	...
8. Cro-Magnon	...	66	...
9. Forty-four Tasmanians, unsexed . .	45	58·7	71·5
10. Cannstatt	...	58	...
11. Four Kalmucks	51·5	54·6	61
12. Twenty-four Dschagga negroes . .	41	53·9	62·5
13. Thirty-five Europeans, unsexed . .	40	51·3	61

Veddahs absent.

Table XIII.—Comparison of the Bregma Foot-Point Positional Index.

	Minimum.	Average.	Maximum.
1. An adult male gibbon	...	63·4	...
2. *Pithecanthropus erectus*	39·7	44·1	50·2
3. Brüx	...	37·3	...
4. Three Spy-Neanderthal	34·8	36·6	40·1
5. Gibraltar	...	35·2	...
6. Galley Hill	...	34·3	...
7. Brünn	...	34	...
8. Forty-four Tasmanians, unsexed . .	26	33·5	40·6
9. Cannstatt	...	33·3	...
10. Four Kalmucks	30·1	32·8	37·4
11. Cro-Magnon	...	32·6	...
12. Twenty-four Dschagga negroes . .	26·6	32·1	37·2
13. Forty-five Europeans, unsexed . .	22·2	30·4	35·7

Veddahs absent.

TABLE XIV.—COMPARISON OF THE LENGTH OF THE FRONTAL ARC.

	Minimum.	Average.	Maximum.
1. An adult male chimpanzee	...	92	...
2. *Pithecanthropus erectus*	100	110	120
3. Four Kalmucks	110	115·2	120
4. Three Spy-Neanderthal	115	124	133
5. Five Europeans, unsexed	121	125·6	130
6. Forty-seven Tasmanians, unsexed	113	126	143
7. Gibraltar	...	126	...
8. Brüx	...	135	...
9. Cannstatt	...	135	...
10. Galley Hill	...	135	...
11. Brünn	...	135	...
12. Cro-Magnon	...	138	...

Dschagga negroes and Veddahs absent.

TABLE XV.—COMPARISON OF THE LENGTH OF THE CHORD OF THE OS FRONTALE.

	Minimum.	Average.	Maximum.
1. An adult male chimpanzee	...	87	...
2. Four Kalmucks	98	103·8	107·5
3. *Pithecanthropus erectus*	96	104	112
4. Fifty Tasmanians, unsexed	97	109·5	120
5. Gibraltar	...	111	...
6. Five Europeans, unsexed	109	112·5	118·5
7. Three Spy-Neanderthal	108	114	119
8. Brüx	...	114	...
9. Cannstatt	...	117	...
10. Galley Hill	...	120	...
11. Brünn	...	123	...
12. Cro-Magnon	...	123	...

Dschagga negroes and Veddahs absent.

TABLE XVI.—COMPARISON OF THE CURVATURE INDEX OF THE OS FRONTALE.

	Minimum.	Average.	Maximum.
1. *Pithecanthropus erectus*	93·3	94·6	96
2. An adult male chimpanzee	...	94·5	...
3. Three Spy-Neanderthal	89·4	92·8	93·9
4. Brünn	...	91·1	...
5. Four Kalmucks	88·3	90·1	92·8
6. Five Europeans, unsexed	87·4	89·5	91·1
7. Cro-Magnon	...	89·1	...
8. Galley Hill	...	88·8	...
9. Gibraltar	...	88	...
10. Forty-seven Tasmanians, unsexed	81·4	87·1	97·5
11. Canstatt	...	86·6	...
12. Brüx	...	84·4	...

Dschagga negroes and Veddahs absent.

Table XVII.—Angle of Frontal Curvature measured on the Glabella-Bregma Chord.

	Minimum.	Average.	Maximum.
1. An adult gibbon	...	160	...
2. Three Spy-Neanderthal	150	153·3	159
3 *Pithecanthropus erectus*	148·5	153·2	158
4. Fifty Tasmanians, unsexed . . .	131·5	139·5	149
5 Seven Europeans, unsexed . . .	127	135·4	148
6. Four Dschagga negroes	122	131·5	136·5

Gibraltar, Brüx, Galley Hill, Brünn, Cannstatt, Cro-Magnon, Veddahs and Kalmucks absent.

Table XVIII.—Comparison of the Length of the Chord of the Pars Glabellaris of the Os Frontale.

	Minimum.	Average.	Maximum.
1. An adult female orang	...	20	...
2. *Pithecanthropus erectus*	...	24	...
3. Three Spy-Neanderthal	30	33·1	37·5
4. Gibraltar	...	36	...
5. Cro-Magnon	...	31·5	...
6. Brünn	...	30	...
7. Five Dschagga negroes	27	26·4	28·5
8. Eleven Europeans, unsexed . . .	19·5	24·5	28
9. Brüx	...	24	...
10. Galley Hill	...	24	...
11. Forty-nine Tasmanians, unsexed . .	18	23·8	29
12. Cannstatt	...	19	...

Kalmucks and Veddahs absent.

On the assumption that great length of the chord of the pars glabellaris is a primitive feature, the figures in this table are arranged in an ascending scale from the nearest anthropoid to the maximum, and thereafter in a descending scale. This table is also omitted from the final results obtained in Mr Cross's paper, and for the same reasons that Table I. is omitted.

Table XIX.—Comparison of the Length of the Chord of the Pars Cerebralis of the Os Frontale.

	Minimum.	Average.	Maximum.
1. An adult female chimpanzee and a gibbon	...	55	...
2. Gibraltar	...	82	...
3. Spy-Neanderthal	77	83·6	87
4. *Pithecanthropus erectus*	80	87·5	95
5. Eleven Europeans, unsexed	87	92·1	101
6. Fifty Tasmanians, unsexed	73	93·7	106·5
7. Galley Hill	...	95	...
8. Five Dschagga negroes	94	95·8	97
9. Brünn	...	96	...
10. Cro-Magnon	...	97·5	...
11. Brüx	...	99	...
12. Cannstatt	...	104	...

Kalmucks and Veddahs absent.

Table XX.—Comparison of the Glabella-Cerebral Chord Index.

	Minimum.	Average.	Maximum.
1. Gibraltar	...	43	...
2. An adult female orang	...	40	...
3. Three Spy-Neanderthal	34·4	39·6	43·1
4. Cro-Magnon	...	32·3	...
5. Brünn	...	31·2	...
6. *Pithecanthropus erectus*	25·2	27·6	30
7. Five Dschagga negroes	23·3	27·4	30·3
8. Eleven Europeans, unsexed	21·4	26·6	31·8
9. Fifty Tasmanians, unsexed	17·6	25·5	35·6
10. Galley Hill	...	25·2	...
11. Brüx	...	24·2	...
12. Cannstatt	...	18·2	...

Kalmucks and Veddahs absent.

Table XXI.—Comparison of the Length of the Parietal Arc.

	Minimum.	Average.	Maximum.
1. An adult female chimpanzee	...	62	...
2. *Pithecanthropus erectus*	93	103	113
3. Brüx	...	108	...
4. Gibraltar	...	111	...
5. Three Spy-Neanderthal	119	119·6	120
6. Forty-eight Tasmanians, unsexed	112	125·8	145
7. Cannstatt	...	130	...
8. Galley Hill	...	132	...
9. One European	...	133	...
10. Cro-Magnon	...	135	...
11. Brünn	...	139·5	...

Dschagga negroes, Veddahs, and Kalmucks absent.

Table XXII.—Comparison of the Length of the Parietal Chord.

	Minimum.	Average.	Maximum.
1. Brüx	...	100·5	...
2. *Pithecanthropus erectus*	...	104	...
3. Three Spy-Neanderthal . . .	104	107·7	113
4. Gibraltar	...	108	...
5. Forty-eight Tasmanians, unsexed . .	99·5	113	127
6. Cannstatt	...	117·5	...
7. Galley Hill	...	120	...
8. Cro-Magnon	...	123	...
9. Brün	...	127·5	...

Dschagga negroes, Veddahs, Kalmucks, and Europeans absent.

Table XXIII.—Comparison of the Curvature Index of the Os Parietale.

	Minimum.	Average.	Maximum.
1. Gibraltar	...	97·2	...
2. Three Spy-Neanderthal	90·4	93·7	96·3
3. Brüx	...	93	...
4. *Pithecanthropus erectus*	...	92 ?	...
5. Brünn	...	91·3	...
6. Cro-Magnon	...	91·1	...
7. Galley Hill	...	90·9	...
8. Cannstatt	...	90·3	...
9. Forty-seven Tasmanians, unsexed . .	83·8	90	97·6

Dschagga negroes, Veddahs, Kalmucks, and Europeans absent.

Table XXIV.—The Angle of Parietal Curvature.

	Minimum.	Average.	Maximum.
Forty-nine Tasmanians, unsexed . . .	125·5	134·3	141·5

No data for comparison are available.

Table XXV.—Comparison of the Parietal Frontal Arc Index.

	Minimum.	Average.	Maximum.
1. Brüx	...	80	...
2. An adult female chimpanzee	...	82·6	...
3. *Pithecanthropus erectus*	71·1	85·8	102·7
4. Gibraltar	...	88	...
5. Cannstatt	...	96·2	...
6. Three Spy-Neanderthal	89·4	96·8	104·3
7. Galley Hill	...	97·7	...
8. Cro-Magnon	...	97·8	...
9. Forty-five Tasmanians, unsexed	85·8	99·7	114·1
10. Brünn	...	103·3	...
11. One European	...	109·9	...

Dschagga negroes, Veddahs, and Kalmucks absent.

Table XXVI.—Comparison of the Lambda Angle.

	Minimum.	Average.	Maximum.
1. Nearest Anthropoids	43	55·5	68
2. *Pithecanthropus erectus*	...	66	...
3. Neanderthal	...	66·5	67
Spy 1	67	68	...
4. Gibraltar	...	69	...
5. Cro-Magnon	...	70	...
6. Galley Hill	...	74	...
7. Brünn	...	78	...
8. Brüx	...	90	...
9. Forty-six Tasmanians, unsexed	74	80·5	88
10. Modern Man	78	81·5	85
11. Cannstatt	...	83	...

Dschagga negroes, Veddahs, and Kalmucks absent.

Table XXVII.—Comparison of the Opisthionic Angle.

	Minimum.	Average.	Maximum.
1. *Pithecanthropus erectus*	...	64	...
2. Nearest anthropoids	50	59·5	69
3. Spy 1	...	54	...
Neanderthal	...	51·5	...
4. Galley Hill	...	42	...
5. Brünn	...	42	...
6. Thirty-eight Tasmanians, unsexed	34·5	40·6	47
7. Gibraltar	...	36	...
8. Recent Man	31	35·5	40
9. Cro-Magnon	...	34	...

Brüx, Cannstatt, Dschagga negroes, Veddahs, and Kalmucks absent.

With the object of combining the foregoing results in a graphic manner we have devised Table XXIX., in which the figs. 1 to 27 in the upper horizontal line correspond to those in Table XXVIII., and indicate the number of observations recorded by us upon the Tasmanian crania together with the comparative data derived from other sources: thus column 1 indicates the glabella-inion length, column 2 the calvarial height, and so on.

The left-hand vertical column in Table XXIX. is numbered from 1 to 14 in an ascending scale from below upwards. These numbers indicate the evolutionary order of the objects compared: thus in vertical column 1 the gorilla occupies the lowest place numbered 1, with a lowest glabella-inion length of 147. In this column, on the assumption that marked length of skull with dolichocephaly is a primitive characteristic, the figures are arranged in an ascending scale from the nearest anthropoid—the gorilla—to the maximum, 202, in the Cro-Magnon cranium, and thereafter in a diminishing scale to the fourteenth place, occupied by the Kalmuck, with a glabella-inion length of 166·2. As such an arrangement is purely hypothetical, these figures are very properly omitted from the final results obtained by Mr Cross, and which are set forth in his paper, which follows.

In column 2, which indicates the calvarial height-length, there is a marked interchange of position as compared with the positions occupied in column 1. These several changes of position can be noted by simply following the lines which connect the objects together: thus the Tasmanian drops from the tenth place in column 1 to the eighth place in column 2. The Cannstatt skull has the greatest calvarial height of any of the compared crania, and rises from the ninth place in column 1 to the fourteenth or top place as regards calvarial height in column 2. The anthropoid ape and Pithecanthropus alone retain the same positions.

Of the several changes in evolutionary position, only one or two of the more interesting and instructive results need any reference.

The position of the nearest anthropoid ape is instructive. In fifteen out of a possible twenty-four observations the ape occupies the lowest position. In seven observations the ape is in the second position; and in one instance, column 12, the distance of the bregma foot-point from the glabella, he occupies the fourth position, that is to say, the bregma foot-point is farther forwards in the chimpanzee than in the crania of *Pithecanthropus erectus*, Brüx, or Spy-Neanderthal.

In column 9, the distance of the calvarial height foot-point from the glabella, the ape actually occupies the highest position of all; which implies

[*To face* p. 60.

RESULTS.

44	*45*	*46*	*47*	*48*	*49*	*50*	*51*	*52*	No.	Min.	Av.	Max.
L. Crowther.							National Museum, Melbourne.					
7	8	9	10	11	12	13	12922	12997				
M.	M.	M.	F.	J.	F.	M.	F.	F.				
183	174	183	164	156	159	...	164	171	44	157	173·1	188
+ 108	98·5	107	95·5	83	91	...	92·5	89	46	87	97	108
59	*56·6*	*58·4*	*58·2*	53·2	*57·2*	...	*56·4*	*52*	44	48·3	56·1	62·7
140	133	141	133	...	122	...	123	135	48	120	134·7	145
77·1	*74*	*75·8*	*71·8*	...	*74·6*	...	*75·2*	*65·9*	44	65·9	72·2	79·2
161·5	153·5	162	148·5	...	140·5	...	143·5	153	44	140·5	154	164·5

TABLE XXVIII.—THE INDIVIDUAL AND GENERALISED RESULTS OF THE EXAMINATION OF FIFTY-TWO TASMANIAN CRANIA.

Berry and Robertson	Serial Number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
	Present Location of Specimen	Tasmanian Museum, Hobart.																
	Original Number of Specimen	4288	4291	1300	4301	4298	4257	4293	4289	3362	4294	4292	4302	4297	4290	4295	4296	4303
Nature of Observation.	Probable Sex	M.	M.	M.	M.	M.	F.	F.	F.	F.	F.	F.	M.	M.	F.	F.	M.	F.
1 Glabella-Inion Length.	...	167	178	172	165	173	165	162	166	160	−157	...	174	169	176	...	173	.
2 Calvarial Height.	...	97	104·5	108	98·5	107	90·5	91	94·5	89	95	..	106·5	101	96	...	96·5	...
3 *Calvarial Height Index.*	Calvarial Height × 100 / Glabella-Inion Length	*58*	*58·7*	*+62·7*	*59·6*	*61·8*	*60·3*	*56·1*	*56·9*	*55·6*	*60·5*	...	*61·2*	*59·7*	*57·8*	...	*55·7*	..
4 Greatest Breadth.	...	135	142	141	136	135	132	135	132	133	132	130	140	141	137	131	135	137
5 *Calvarial Height-Breadth Index.*	Calvarial Height × 100 / Greatest Breadth	*71·8*	*73·5*	*76·6*	*72·4*	*+79·2*	*75·3*	*67·4*	*71·5*	*66·9*	*72*	..	*78*	*76*	*70*	...	*71·5*	.
6 Half the sum of Glabella-Inion Length + Breadth.	...	151	160	156·5	150·5	154	148·5	118·5	149	146·5	144·5	...	157	155	156·5	...	151	...
7 *Calvarial Height Half-Sum (Length + Breadth) Index.*	Calvarial Height × 100 / (Glabella-Inion Length + Breadth)	*64·2*	*65·3*	*69*	*65·4*	*+69·5*	*67*	*61·2*	*63·4*	*60·7*	*65·7*	...	*67·8*	*66·1*	*61·3*	..	*62·6*	...
8 Distance of Foot-Point of Calvarial Height from Glabella	...	97·5	95	110	98	105	95	105	90·5	−85	96	...	106	99	95		109	...
9 *Calvarial Height Foot-Point Positional Index.*	Distance of Foot-Point of Calvarial Height from Glabella × 100 / Glabella-Inion Length	*58·3*	*53·3*	*63·9*	*59·4*	*62·4*	*57·5*	*+64·8*	*54·5*	*−53·1*	*61·1*	.	*60·9*	*58·5*	*54*	.	*63*	...
10 Frontal Angle.	...	84·5	90	87	90	83	92	90	95	83	85·5	..	89	76	88		85·5	
11 Bregma Angle.	...	57	56·5	58	58	54	+64	56	57	52	57	...	58	55·5	53·5	...	56·5	...
12 Distance of Foot-Point of Bregma from Glabella.	...	56	60·5	61	54	63	−43	55	56	65	55	..	57	62·5	65	...	58·5	..
13 *Bregma Foot-Point Positional Index.*	Distance of Foot-Point of Bregma from Glabella × 100 / Glabella-Inion Length	*33·5*	*34*	*35·6*	*32·7*	*36·5*	*−26*	*34*	*33·7*	*+40·6*	*35·1*	...	*32·7*	*37*	*37*	...	*33·8*	...
14 Length of Frontal Arc.	...	123	133	+143	120	125	119	120	126	121	120	127	130	129	134	−113	123	116
15 Length of Frontal Chord.	...	108	114	116·5	117	110	103·5	102·5	106	109	104·5	113·5	113	114	114	98·5	112	107
16 *Curvature Index of Os Frontale.*	Length of Frontal Chord × 100 / Length of Frontal Arc	*87·8*	*85·7*	*−81·4*	*+97·5*	*88*	*87*	*85·4*	*84·1*	*85·5*	*87*	*89·3*	*86·9*	*88·4*	*85*	*87·1*	*91*	*92·2*
17 Angle of Frontal Curvature.	...	140	136	142	133	140	135	136	125	138	139	143	135	144	136	135	140	139
18 Length of the Chord of the Pars Glabellaris.	...	27·5	22·5	28	21	21·2	21	26·5	20	27	25	26	23	24·5	25·5	22	23·5	..
19 Length of the Chord of the Pars Cerebralis.	...	90	101	100·5	91·5	95	88	88	95	91	87	94	97	106·5	97	82·5	95·5	96
20 *Glabellar-Cerebral Chord Index.*	Length of Chord of Pars Glabellaris × 100 / Length of Chord of Pars Cerebralis	*30·5*	*22·2*	*27·8*	*23·9*	*22·3*	*23·8*	*30*	*21*	*28·7*	*18·7*	*27·6*	*23·7*	*23*	*26·2*	*26·6*	*24·6*	...
21 Length of Parietal Arc.	...	121	135	+145	133	121	132	120	122	119	121	120	130	130	127	−112	114	124
22 Length of Parietal Chord.	...	111·5	121	+127	118	109	118	108·5	111	107	105·5	110·5	117·5	115	116	−99·5	103·5	111
23 *Curvature Index of Os Parietale.*	Length of Parietal Chord × 100 / Length of Parietal Arc	*92·3*	*89·6*	*87·5*	*88·7*	*90*	*89·3*	*90·4*	*90·9*	*89·9*	*87·1*	*91·3*	*90·3*	*88·4*	*91·3*	*88·8*	*90·7*	*89·5*
24 Angle of Parietal Curvature.	...	132	133	133	133	133	135	133	139	130·5	133	138	133	133·5	137	138	138	134
25 *Parietal Frontal Arc Index.*	Parietal Arc × 100 / Frontal Arc	*98·3*	*101·5*	*101·5*	*110·8*	*96·8*	*110·9*	*100*	*96·8*	*93·7*	*100·8*	*94·4*	*100*	*100·7*	*94·7*	*99·1*	*102·6*	*106·8*
26 Lambda Angle.	...	−74	78	79·5	82	78	83	82·5	77·5	83	80	...	83	77	82·5		77	...
27 Opisthionic Angle.	...	39	35	..	42	...	42	35	38	44·5	41	...	41	..	40	...	44	...

Nature of Observation.	18	19	20	21	22	23	24	25	26	27	28	29	30	31
Present Location of Specimen	Cook.	Taylor.		Tasmanian Museum, Hobart.			Jolly.	Launceston Museum.					Devonport Town Board.	
Original Number of Specimen	.	1	2	1572	.	..	.	1201	1202	1203	1204	1205	1	2
Probable Sex	M.	M.	F.	F.	F.	M.	M.	M.	F.	M.	M.	M.	M.	F.
1 Glabella-Inion Length.	174	176	172	...	...	...	171	167	168	173	180	+188	183	177
2 Calvarial Height.	106	96	96	98 ×	94	..	94	101	97·5	97	100	104	93	92
3 *Calvarial Height Index.*	*60·9*	*54·9*	*55·8*	...	...	..	*55*	*54*	*58*	*56*	*55·5*	*56·3*	*50·8*	*52*
4 Greatest Breadth.	144	138	137	141		...	126	132	139	136	+145	141	131	126
5 *Calvarial Height-Breadth Index.*	*73·6*	*69·5*	*70*	...	..	...	*74·6*	*76·6*	*70·1*	*71·3*	*68·9*	*73·7*	*71*	*73*
6 Half the sum of Glabella-Inion Length + Breadth.	159	156·5	154·5	..			148·5	159·5	153·5	154·5	162·5	164·5	157	151·5
7 *Calvarial Height Half-Sum (Length + Breadth) Index.*	*66·6*	*61·3*	*62·1*	.	.		*65·3*	*63·3*	*63·5*	*62·7*	*61·4*	*63·2*	*59·2*	*60·7*
8 Distance of Foot-Point of Calvarial Height from Glabella	99	101·5	108	88 ×	90	...	92	108·5	104	103	108	110·5	104·5	107·5
9 *Calvarial Height Foot-Point Positional Index.*	*56·9*	*58*	*62·3*	...	.	...	*53·2*	*58·4*	*62*	*60·8*	*60*	*58·7*	*57·1*	*60·7*
10 Frontal Angle.	90	89·5	85	..	88·5	...	75	90	+96	89	92	88·5	86	81
11 Bregma Angle.	59·5	55·5	57	57 ×	61·5	...	52·5	50	57·5	56·5	56·5	55	52·5	−51·5
12 Distance of Foot-Point of Bregma from Glabella.	58	60	54·5	59 ×	41 ×	...	68	60·5	53·5	60	62	64·5	62	66
13 *Bregma Foot-Point Positional Index.*	*33·3*	*34·2*	*31·6*	...	...	...	*39·7*	*32·3*	*31·8*	*34·6*	*33·4*	*34·3*	*33·9*	*37·2*
14 Length of Frontal Arc.	133	129	121	...	...	139	130	130	122	127	134	138	126	127
15 Length of Frontal Chord.	115·5	110	106	115	...	117	113	116	105	111·5	115	116	109	109
16 *Curvature Index of Os Frontale.*	*56·8*	*85·2*	*87·6*	...	...	*84·1*	*86·9*	*89·2*	*86*	*87·8*	*85·8*	*84*	*86·6*	*85·8*
17 Angle of Frontal Curvature.	140	134·5	145	136 ×	+140	135	146	139	134	140	136	137·5	141·5	142
18 Length of the Chord of the Pars Glabellaris.	26	20	23	24 ×	26	26	−18	28·5	22·5	24	22·5	+29	25	22·5
19 Length of the Chord of the Pars Cerebralis.	100	99·5	91	99 ×	73	102	102	97	92	98	102	101·5	93	94
20 *Glabellar-Cerebral Chord Index.*	*26*	*20·1*	*25·2*	*24·3 ×*	*35·6*	*25·4*	*17·6*	*29·3*	*24·4*	*24·5*	*22*	*28·5*	*26·8*	*24*
21 Length of Parietal Arc.	128	128	129	120	129	...	118	129	122	126	125	131	131	127
22 Length of Parietal Chord.	117	118·5	118·5	106	...	..	99	119	111	114	111·5	111·5	119	111·5
23 *Curvature Index of Os Parietale.*	*91·3*	*92·5*	*91·8*	*84·1*	...	...	*−83·8*	*92·2*	*90·9*	*90·4*	*91·6*	*85·1*	*90·8*	*87·7*
24 Angle of Parietal Curvature.	134	136	135	138·5	128		134	138	135	134	134·5	139·5	134·5	133
25 *Parietal Frontal Arc Index.*	*96·2*	*99·2*	*106·6*	...	...	...	*90·7*	*99·2*	*100*	*99·2*	*93·2*	*94·9*	*104*	*100*
26 Lambda Angle.	84·5	79·5	84	81·5	86	.	80·5	78	85	83	81	76·5	79	78·5
27 Opisthionic Angle.	−34·5	39·5	42·5	38	...	...	41·5	38	12·5	...	39·5	37	36	...

Nature of Observation.	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
Present Location of Specimen	E. O. Colton.							W. L. Crowther.											
Original Number of Specimen	1	2	3	4	5	6	7	1	3	4	5	6	7	8	9	10	11	12	13
Probable Sex	M.	M.	M.	M.	M.	M.	M.	M.	M.	F.	M.	M.	M.	M.	M.	F.	J.	F.	M.
1 Glabella-Inion Length.	175	187	170	173	180	160	179	172	179	171	172	187	183	174	183	164	156	159	...
2 Calvarial Height.	92	99	94	90	−87	88·5	99	88	101	96·5	92	99	+108	98·5	107	95·5	83	91	..
3 *Calvarial Height Index.*	*52·5*	*52·8*	*55·3*	*55·5*	*−48·3*	*52·3*	*55·2*	*51·1*	*56·4*	*54·8*	*53·4*	*52·9*	*59*	*56·6*	*58·4*	*58·2*	*53·2*	*57·2*	...
4 Greatest Breadth.	134	142	137	129	135	125	143	−120	133	129	135	139	140	133	141	133		122	
5 *Calvarial Height-Breadth Index.*	*68·6*	*69·7*	*68·6*	*74·4*	*64·4*	*70·8*	*69·2*	*73·3*	*75·9*	*74·8*	*68·1*	*71·2*	*77·1*	*74*	*75·8*	*71·8*	...	*74·6*	...
6 Half the sum of Glabella-Inion Length + Breadth.	154·5	164·5	153·5	151	157·5	147	161	146	156	151·5	153·5	163	161·5	153·5	162	145·5		140·5	..
7 *Calvarial Height Half-Sum (Length + Breadth) Index.*	*59·5*	*60·1*	*61·2*	*63·6*	*−55·2*	*60·2*	*61·4*	*60·2*	*64·7*	*63·6*	*59·9*	*60·7*	*66·8*	*64·1*	*66*	*64·8*	...	*64·7*	.
8 Distance of Foot-Point of Calvarial Height from Glabella	107·5	+115·5	100	100·5	105	103	103	105	96	100	96	111·5	111	103	104	94·5	85·5	92·5	
9 *Calvarial Height Foot-Point Positional Index.*	*61·4*	*61·7*	*62·3*	*58*	*58·3*	*61·5*	*57·5*	*61*	*55·6*	*62·6*	*55·8*	*59·6*	*60·7*	*59·1*	*56·8*	*57·6*	*54·8*	*58·1*	.
10 Frontal Angle.	73·5	76·5	83·5	80·5	80	83	83·5	−72	91	86	85·5	82	92·5	84	93	88	102	90	...
11 Bregma Angle.	52	54·5	58	56	52·5	55	56	−51·5	58	56·5	55	52	60	60	58·5	55·5	54	55	...
12 Distance of Foot-Point of Bregma from Glabella.	66	62·5	54·5	61	63·5	56	58·5	59	55·5	53	54·5	+71·5	51·5	49	58·5	49	55	56·5	...
13 *Bregma Foot-Point Positional Index.*	*37·7*	*33·4*	*32*	*35·2*	*35·2*	*33·1*	*32·6*	*34·3*	*31*	*30·5*	*31·6*	*38·2*	*29·8*	*28·1*	*31·9*	*29·8*	*35·2*	*35·5*	.
14 Length of Frontal Arc.	130	125	124	125	126	120	126		123	119	115	138	128	−114	138	113	116	121	.
15 Length of Frontal Chord.	111·5	111	114·5	112	107	102	109	99	108·5	102·5	101	+120	113·5	103	116	−98	97	103·5	107·5
16 *Curvature Index of Os Frontale.*	*85·7*	*88·8*	*92·3*	*89·6*	*84·9*	*85*	*86·6*	..	*88·2*	*86·1*	*87·8*	*86·8*	*88·6*	*90·3*	*84*	*86·7*	*83·0*	*86·5*	..
17 Angle of Frontal Curvature.	146	145	139	145	142	146	139	141	139	115·5	143·5	143·5	111·5	146	135·5	138·5	126	−131·5	139
18 Length of the Chord of the Pars Glabellaris.	27	21·5	22	22	27	23	23	21·5	23	21·5	25·5	26·5	21	22·5	26	18·5	16·5	19	27
19 Length of the Chord of the Pars Cerebralis.	94·5	95·5	90·5	97·5	89	88	93	84·5	94	86·5	80·5	102	100	87·5	102·5	87	88·5	91	91·5
20 *Glabellar-Cerebral Chord Index.*	*28·5*	*22·5*	*24·3*	*22·5*	*30·3*	*26·1*	*24·7*	*29*	*24·4*	*28·4*	*35·4*	*26*	*21*	*25·7*	*25·4*	*21·2*	*18·6*	*20·8*	*29·5*
21 Length of Parietal Arc.	124	130	128	125	122	119	130	...	128	121	122	130	133	135	136	129	111	118	...
22 Length of Parietal Chord.	114·5	116	115	113	114	109	119·5	107·5	125	115	108·5	116·5	120·5	123	123	113	99	107	.
23 *Curvature Index of Os Parietale.*	*92·3*	*89·2*	*89·8*	*90·4*	*93·4*	*−83·8*	*91·9*	...	*+97·6*	*92·7*	*88·9*	*88·8*	*90·6*	*89·1*	*90·4*	*87·6*	*89·1*	*90·6*	.
24 Angle of Parietal Curvature.	138	132	134	133	140	134	131·5	128	132	+111·5	130	133	132	132	134	−125·5	−125·5	133	
25 *Parietal Frontal Arc Index.*	*95·3*	*104*	*103·2*	*100*	*96·8*	*99·1*	*103·1*	...	*104*	*104·2*	*106*	*94·2*	*103·9*	*121*	*98·5*	*+114·1*	*95·6*	*97·5*	...
26 Lambda Angle.	+88	75	83	83	77·5	78·5	82	80	76	78·5	75	79	80	82	83	81	78	+88	
27 Opisthionic Angle.	39	41	38·5	42·5	...	...	14	41	40·5	39	+47	44	38·5	45	43·5	37·5	40	11	...

Nature of Observation.	51	52	No.	Min.	Av.	Max.
Present Location of Specimen	National Museum, Melbourne.					
Original Number of Specimen	12922	12997				
Probable Sex	F.	F.				
1 Glabella-Inion Length.	161	171	44	157	173·1	188
2 Calvarial Height.	92·5	89	46	87	97	108
3 *Calvarial Height Index.*	*56·4*	*52*	44	48·3	56·1	62·7
4 Greatest Breadth.	123	135	48	120	134·7	145
5 *Calvarial Height-Breadth Index.*	*75·2*	*−65·9*	44	65·9	72·2	79·2
6 Half the sum of Glabella-Inion Length + Breadth.	143·5	153	44	140·5	154	161·5
7 *Calvarial Height Half-Sum (Length + Breadth) Index.*	*64·4*	*58·1*	44	55·2	63	69·5
8 Distance of Foot-Point of Calvarial Height from Glabella	94	102·5	45	85	101·9	115·5
9 *Calvarial Height Foot-Point Positional Index.*	*57·3*	*60*	44	53·1	59	61·8
10 Frontal Angle.	88	90	44	72	86	96
11 Bregma Angle.	57·5	−51·5	45	51·5	56	64
12 Distance of Foot-Point of Bregma from Glabella.	64·5	64·5	44	43	58·7	71·5
13 *Bregma Foot-Point Positional Index.*	*33·2*	*37·8*	44	26	34	40·6
14 Length of Frontal Arc.	124	127	47	113	126	143
15 Length of Frontal Chord.	108	110·5	50	97	109·5	120
16 *Curvature Index of Os Frontale.*	*87*	*87*	47	81·4	87·1	97·5
17 Angle of Frontal Curvature.	135·5	135	50	131·5	139·5	149
18 Length of the Chord of the Pars Glabellaris.	24·5	21·5	49	18	23·8	29
19 Length of the Chord of the Pars Cerebralis.	91	96	50	73	93·7	106·5
20 *Glabellar-Cerebral Chord Index.*	*25·8*	*32·4*	50	17·6	25·5	35·6
21 Length of Parietal Arc.	120	109	48	112	125·8	145
22 Length of Parietal Chord.	110	100	48	98·5	113	127
23 *Curvature Index of Os Parietale.*	*91·6*	*91·7*	47	83·8	90	97·6
24 Angle of Parietal Curvature.	131	136	40	125·5	134·3	141·5
25 *Parietal Frontal Arc Index.*	*96·7*	*−85·8*	45	85·8	99·7	114·1
26 Lambda Angle.	61	78	46	74	80·5	88
27 Opisthionic Angle.	44	46·5	38	34·5	40·6	47

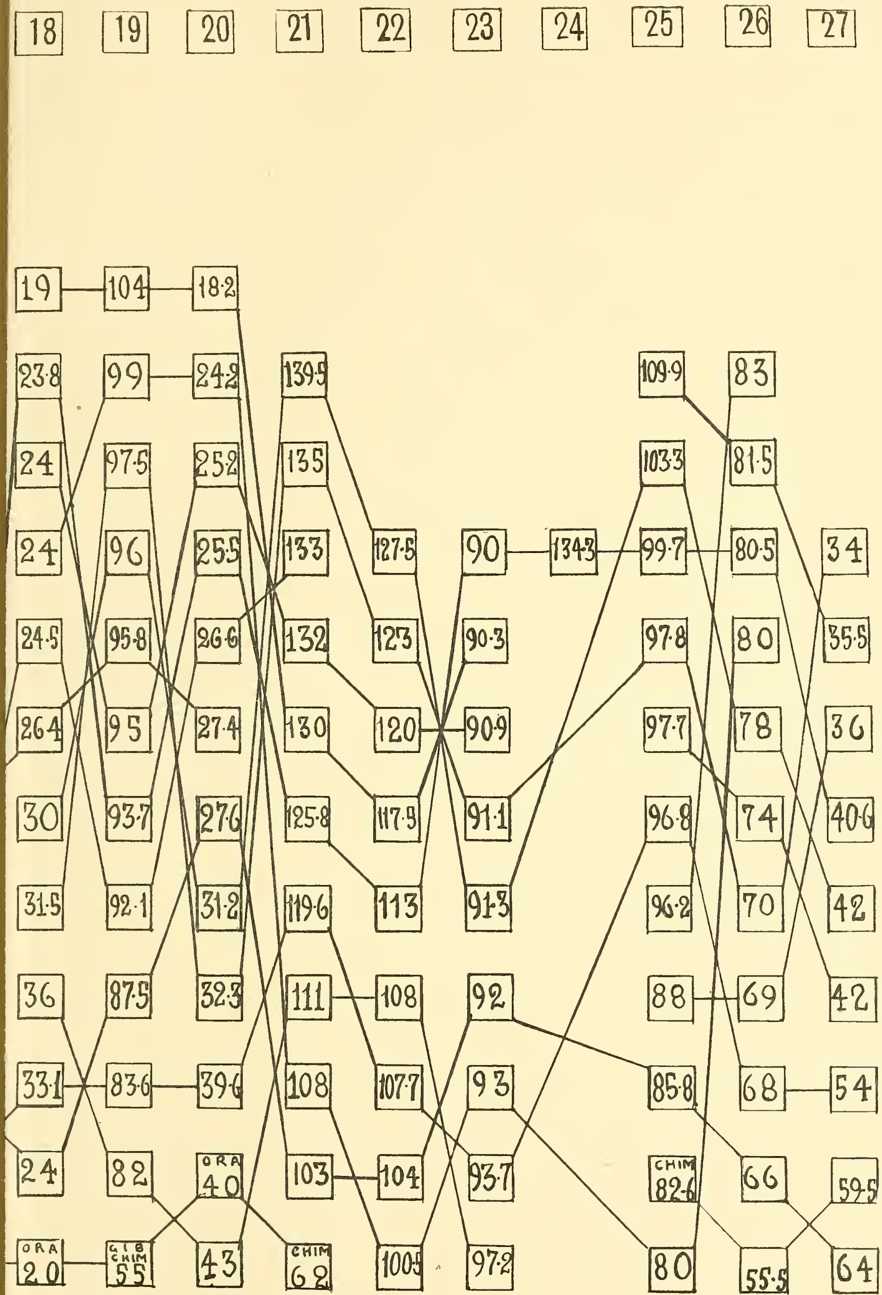

18
19
20
21
22
23
24
25
26
27
19
23·8
24
24
24·5
26·4
30
31·5
36
33·1
24
ORA
20
104
99
97·5
96
95·8
95
93·7
92·1
87·5
83·6
82
GIB CHIM
55
18·2
24·2
25·2
25·5
26·6
27·4
27·6
31·2
32·3
39·6
ORA
40
43
139·5
135
133
132
130
125·8
119·6
111
108
103
CHIM
62
127·5
123
120
117·5
113
108
107·7
104
100·5
90
90·3
90·9
91·1
91·3
92
93
93·7
97·2
134·3
109·9
103·3
99·7
97·8
97·7
96·8
96·2
88
85·8
CHIM
82·6
80
83
81·5
80·5
80
78
74
70
69
68
66
55·5
34
35·5
36
40·6
42
42
54
59·5
64

TABLE XXIX.

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
Kalmucks	14	166·2	105	59·8																								
Dschagga Negroes	13	167·4	103	59·8	151	76·9	176·5	66·6			100·3	60	51·3	30·4														
Europeans	12	168	101	59·6	150·3	74·6	174·3	65·8	121·5	GOR 61·8	92·5	59·9	53·9	32·1	138	123	84·4		19	104	18·2							
Veddahs	11	169·8	100	58·4	148	74·1	170	65·6	111	60·1	90	58·6	54·6	32·6	135	123	86·6		23·8	99	24·2	139·5				109·9	83	
Tasmanians	10	173·1	99·9	56·1	146	72·4	167·5	63	111	60	86	56·5	58	32·8	135	120	87·1		24	97·5	25·2	135				103·3	81·5	
Cannstatt	9	174	99·2	54·5	146	72·2	165·5	60·5	111	59	85·2	56	58·7	33·3	136	117	88		24	96	25·5	133	127·5	90	134·3	99·7	80·5	34
Brüx	8	185	97	51·2	142·4	71·9	160	58·7	110	58·3	83	54	66	33·5	135	114	88·8		24·5	95·8	26·6	132	123	90·3		97·8	80	35·5
Gibraltar	7	187	97	50	139	66·8	157	57·2	105·7	56·5	82	54	66	34	126	114	89·1		26·4	95	27·4	130	120	90·9		97·7	78	36
Spy-Neanderthal	6	198·6	90·7	48·2	134·7	63·3	156·8	54·8	102·5	56·2	75	52	67·5	34·3	126	112·5	89·5	131·5	30	93·7	27·6	125·8	117·9	91·1		96·8	74	40·6
Cro-Magnon	5	202	85·3	47·6	133	63·3	155	50·8	101·9	55·7	74·7	51·1	69	35·2	125·6	111	90·1	135·4	31·5	92·1	31·2	119·6	113	91·3		96·2	70	42
Brünn	4	201	85	45·4	130	57·4	154·3	50·7	95·8	55·2	73	50·5	CHIM 72	36·6	124	109·5	91·1	139·5	36	87·5	32·3	111	108	92		88	69	42
Galley Hill	3	201	85	44·9	130	56·7	154	48·9	86·3	54·7	64·8	47·5	72·3	37·3	115·2	104	92·8	153·2	33·1	83·6	39·6	108	107·7	93		85·8	68	54
Pithecanthropus Erectus	2	181	62	CHIM 35·1	129·7	46·6	149·5	39·4	GOR 8·4	52·6	CHIM 56	41 MAX	75	44·1	110	103·8	CHIM 94·5	153·3	24	82	ORA 40	103	104	93·7		CHIM 82·6	66	59·5
Nearest Anthropoid Ape	1	147	CHIM 48·5	34·2	CHIM 113	CHIM 42·9	GOR 129	CHIM 38·6	80·8	44·4	52·5	CHIM 39·5	81·5	GIB 63·4	CHIM 92	CHIM 87	94·6	GIB 160	ORA 20	GIB CHIM 55	43	CHIM 62	100·5	97·2		80	55·5	64

that in the gorilla the position at which the maximum height of the skull is situated is farther back than it is in either primitive or recent man. Seeing that under this observation *Pithecanthropus erectus* is at one end of the scale, with the anthropoid ape at the other, with primitive and recent man occupying intermediate positions between the two, we can only infer that the precise position at which the calvarial height occurs cannot be a point of any very great evolutionary importance.

It is also important to note that the anthropoid ape which stands nearest to man most frequently in this work is the chimpanzee. Out of a possible twenty-two observations, the chimpanzee ranks next to man in fourteen instances, that is to say, in 63·6 per cent. Of the remaining anthropoid apes, the gorilla occurs four times, the gibbon three times, and the orang twice. The apparent discrepancy in the numbers is due to the fact that in one instance the gibbon and chimpanzee rank equally, and therefore count twice.

These observations are particularly interesting in view of Keith's work as to the anthropoid ape which stands most nearly related to man. Keith (19) has shown that man has 396 structural points in common with the chimpanzee, 385 with the gorilla, 272 with the orang, and only 188 with the gibbon. Our work supports Keith in regarding the chimpanzee as man's nearest relative, with the gorilla second. The paucity of our observations compared with Keith's will account for the slight discrepancy between his observations and ours as to the relative positions occupied by the gibbon and the orang.

No less instructive is an analysis of the relative positions occupied by Pithecanthropus and the nearest anthropoid. In seven instances out of a possible twenty-five, *Pithecanthropus erectus* actually occupies a lower evolutionary position than does the nearest anthropoid ape. These instances are of sufficient importance as to warrant their being specifically mentioned. They are as follow:—

1. The calvarial height index is less in *Pithecanthropus erectus* than in the nearest anthropoid—the chimpanzee.

2. The distance of the foot-point of the calvarial height from the glabella is farther forwards in Pithecanthropus than in the nearest anthropoid—the gorilla.

3. The calvarial height foot-point positional index is markedly less in Pithecanthropus than in the gorilla—a point to which we have already directed attention.

4. The frontal angle is less in Pithecanthropus than in the nearest anthropoid—the chimpanzee.

5. The bregma foot-point is farther back in Pithecanthropus than in the nearest anthropoid—the chimpanzee.

6. The curvature index of the os frontale is less in Pithecanthropus than in the chimpanzee.

7. The opisthionic angle is less in *Pithecanthropus erectus* than it is in the nearest anthropoid ape.

In all the vertical columns in which length or breadth enter as one or both of the concerned factors, there is a very marked interchange of position of the compared objects. This is due to the great dimensions, as compared with those of modern man, of some of the crania of primitive man with which the table deals. Once these dimensional data are excluded, there is not nearly so much interchange in position.

The relative positions occupied by the Tasmanian under the 27 observational counts are respectively 10, 8, 10, 6, 9, 3, 10, 5, 9, 10, 9, 9, 8, 6, 4, 10, 4, 11, 6, 9, 6, 5, 9, 9, 9, 9, 6. He thus maintains a fairly uniform position, but stands nearest the anthropoid apes in column 6, which deals with the comparison of half the sum of the glabella-inion length plus the breadth; and is farthest away from them in column 18, which deals with the length of the chord of the pars glabellaris of the os frontale, a somewhat surprising result, to which we have already directed attention.

It is unnecessary to pursue the comparative data obtained by a study of Table XXIX. farther, because we have devised other methods for the final display of the relative evolutionary positions of the various objects compared.

Our first attempt at such evolutionary placing consisted in taking the position of the Tasmanian wherever he occurs as the zero point, all the objects which lie between the Tasmanian and the anthropoid and the anthropoid himself being regarded as minus points, and all those which lie farther away than the Tasmanian from the anthropoid as plus points. Thus in column 1 of Table XXIX. the Tasmanian is tenth on the list. This is the zero point. The Cannstatt skull is ninth, and is therefore minus 1. The Brüx skull is eighth, and becomes minus 2, and so on down to the anthropoid, which in this column is first in position, and becomes therefore minus 9. In the same column the Veddah is credited with a plus 1, the European with a plus 2, the Dschagga negro with a plus 3, and the Kalmuck with a plus 4.

This procedure was adopted for every one of the compared objects throughout the possible 27 observations. To take the Veddah as an example. In column 1 he is credited with a plus 1, because he stands one point farther away from the anthropoid than does the Tasmanian. In columns 2 and 3 he also receives a plus 1 for each column, for the like reason. In column 4 he receives a minus 3, inasmuch as he stands much nearer the anthropoids; at first sight it might look as though the Veddah ought here to receive a minus 4, inasmuch as he stands fourth down from the zero point—the Tasmanian; but it will be noted that two of the intervening objects, namely Brüx, and Galley Hill, have the same breadth, 130 mm., and we therefore regard them from an evolutionary standpoint as being in this instance on the same scale of evolution, and count them here as 1, which entitles the Veddah to his minus 3 as explained. In column 5 the Veddah stands highest, four places more removed from the anthropoid than the Tasmanian, and therefore he is here credited with a plus 4. In columns 6 and 7 he is credited with a minus 1 and a plus 3 respectively, and thereafter occurs as an object of comparison no more.

If these figures be added up it will be found that the Veddah has received 10 pluses and 5 minuses. The minuses are subtracted from the pluses, which leaves plus 5. This figure is divided by the number of observations made upon the Veddah in this work, namely, 7, with the nett result that the Veddah is credited with plus 0·857.

This procedure has been adopted uniformly for every one of the fourteen objects of comparison. It will be noted that, although original, it is merely an adaptation of modern biometrical methods of analysis, and gives the following interesting though not conclusive results:—

1.	The nearest anthropoid	−5·500
2.	*Pithecanthropus erectus*	−4·923
3.	Spy-Neanderthal	−2·961
4.	Gibraltar	−2·520
5.	Brüx	−1·708
6.	The Kalmucks	−0·875
7.	Galley Hill	−0·720
8.	Brünn	−0·200
9.	Tasmanian	0.
10.	Cro-Magnon	+0·560
11.	Veddah	+0·857
12.	European	+1·125
13.	Cannstatt	+1·541
14.	Dschagga negro	+1·188

As it is clear that our first attempt at a comparative method as just described, and which we may term the "unit interval system of classification," is open to the objection that the spacing between the compared objects is always the same, namely, unity, and that it does not therefore sufficiently allow for marked morphological intervals, we submitted the method to a mathematician, Mr K. S. Cross, M.Sc., who points out that mathematically the "unit interval system of classification affords nothing more than a rough approximation of the relative evolutionary positions of the compared objects, because it simply allots to them successive numerical integral values from unity onwards." He has therefore been good enough to deal with the whole question from a mathematical standpoint in his own paper, which follows, and to which the reader is referred.

Whilst we claim neither infallibility nor finality for these particular applications of biometric methods to scientific results, as suggested by us, and subsequently worked out by Mr Cross, and notwithstanding that there are at first sight one or two apparent singular anomalies, the final results as set forth in fig. 1 of Mr Cross's paper must inevitably afford much ground for thought and reflection.

In the first place, it can be no mere coincidence that our methods place primitive man in almost precisely the same order as given by Schwalbe in his "Das Schädelfragment von Brüx und verwandte Schädelformen." This is so striking that we cannot do better than place the two results side by side, as follows:—

Schwalbe.		Berry, Robertson, and Cross. (See Cross's paper.)
Neanderthal	Homo primigenius.	
Spy		Spy-Neanderthal.
Krapina		
Gibraltar.		Gibraltar.
Abstand.		
Brüx	Homo fossilis, variety of Homo sapiens.	Brüx.
Galley Hill		Galley Hill.
Brünn.		Tasmanian.
Australneger.		Brünn.
Rezenter Mensch.		European.

Not only is the order almost precisely the same in both instances, but in the exact place where Schwalbe hypothecates an interval, that is, between the Gibraltar and Brüx remains, so also does Mr Cross's figure actually demonstrate such an interval.

The position of *Pithecanthropus erectus* is, in view of the difference of opinion which exists as to the human, transitional, or apelike character of the remains, of the greatest interest and importance. From fig. 1 of Mr Cross's paper, it seems to us clear that *Pithecanthropus erectus* is, as Dubois has always maintained, a transitional form, but that he is decidedly nearer the anthropoid apes than to Homo primigenius. This conclusion is also borne out, as we have previously shown, by an examination of the position which Pithecanthropus occupies in the form analysis Table XXIX.

The position of the Cro-Magnon race on the plus side of the Tasmanian need cause no surprise. From our figures it is clear that the cranium of the Cro-Magnon man has attained a slightly higher stage of morphological evolution than has the Tasmanian, but very little. An extract from Keane's *Ethnology* (20) will suffice to prove our contention that the relative position of the Cro-Magnon skull on the plus side of the Tasmanian is justified. Keane, speaking of the Cro-Magnon race, says: "Thus is explained the appearances of low human types (Neanderthal, Spy, Castenedolo) in various parts of Europe during Late Pliocene and Early Pleistocene times. They represent the first waves of migration from North Africa soon after the arrival of Pliocene man in that region. But they were followed later by higher types, such as that of Cro-Magnon, which, radiating from the Vézère district, gradually spread over a great part of Europe, and is by some ethnologists already regarded as the substratum of the present populations of West Europe. De Quatrefages agrees with M. Verneau in identifying the Cro-Magnon race with those groups of tall dolichocephalic Kabyles (Berbers), of fair complexion, and often characterised by blue eyes, who still survive in various parts of Mauritania, and were even represented among the Guanches of the Canary Islands."

Our placing of the neolithic Cro-Magnon man on the morphologically plus side of the eolithic Tasmanian, so far from giving any cause for surprise, is seen, just as it was for Schwalbe's classification, to be nothing more than confirmatory proof of the position hypothetically allotted to him by the ethnologist.

The exceedingly high evolutionary position of the Cannstatt skull number 12 in fig. 1 of Mr Cross's paper, is at first sight an outstanding anomaly, but a closer examination shows that our morphological investigation is once more but a proof of the theories of the ethnologist. Speaking of the Cannstatt skull, Keane (20) says: "No conclusions can certainly be drawn from the skull found at Canstatt nearly two hundred years ago, and somewhat hastily taken as representing a palæolithic 'Cannstatt race.' It is even doubtful whether this skull . . . . is the one actually found, not in a quarternary bed, as was said, but associated with some potsherds in the talus or rainwash at the foot of the cliff, on which is a modern cemetery. It may be quite recent, and probably pathological." Our work simply confirms Keane's view that the Cannstatt skull is not that of primitive man at all, but belongs to a modern European type. Schwalbe (17) is apparently of the same opinion as ourselves; at all events he is perfectly certain that the skull is that of a modern variety of Homo sapiens, for he concludes: "Ich kann zum Schluss nur soviel sagen, dass das Schädelframent von Cannstatt einen Schädel angehort hat, der den bestentwickelten rezenten Schädeln in jeder Beziehung vergleichbar ist. Diesen Schädel als Typus einer niederen Rasse anzusehen, ist vollkommen unhaltbar."

The relative positions of the remaining primitive crania examined by us call for no comment, inasmuch as we have already shown as the results of our investigation that they fall into the very position hypothetically allocated to them by Schwalbe.

Of the specimens of recent man, the Kalmuck, in Mr Cross's figure, comes out lowest, on the minus side of the Tasmanian. He is thus morphologically inferior to the Tasmanian. This result may be due to the much more limited number of the Kalmuck specimens employed in the investigation as compared with the Tasmanian, or it may actually reflect the relative positions of the two races.

Homo Mongolicus, to which the Kalmuck belongs, is a complex type composed of two main stems, the Mongolo-Tartars and the Tibeto-Indo-Chinese (Keane, 20). To the Mongolo-Tartar division belong the Akkads, the Koreans, the Finno-Tartars, the Mongols proper, and the Eskimo. The Kalmuck belongs to the most primitive of these stems, namely, the Mongols proper. Of the Mongols proper, Keane (21) says, "in their own homes they have scarcely anywhere advanced beyond the hunting, fishing, or pastoral states." And again speaking of the racial invasions which have from time to time completely altered the characteristics of many of the Asiatic peoples, Keane says that such invasions had but little effect on the Mongols proper, to which the Kalmucks belong, inasmuch as "these continued and continue to occupy the original camping grounds, as changeless and uniform in their physical appearance, mental characters, and social usages as the Arab Bedouins, and all other inhabitants of monotonous, undiversified steppe lands."

From all this we can only conclude that the final placing of the Kalmuck a little behind the Tasmanian is not, after all, so very much at fault as might at first glance appear.

The Veddah, omitted from Mr Cross's figure for reasons specified in his paper, but in our first "unit interval classification" placed on the immediate plus side of the Tasmanian, is also in all probability correctly placed. Morphologically and ethnologically he is about on the same plane as the Tasmanian. His appearance on the immediate plus side of the Tasmanian, but only to a decimal point, is probably accurately accounted for by supposing that his slight morphological advance has been due to the fact that he has occupied a region of the world where he has been amenable to extraneous influences, whereas the Tasmanian, on the other hand, has lived for many thousand generations in absolute seclusion, and undisturbed by any extraneous crossings or admixtures.

The one point in which our tabular results seem to us to give a contradictory result is in the Dschagga negro attaining a higher plus figure than the modern European. The Europeans, based as they are on Schwalbe's work, have all emanated from the same district, and this may possibly indicate a certain amount of inbreeding, or lack of evolutionary development. Of the Dschagga negro, also taken from Schwalbe's work, we cannot find any ethnological data, so we are consequently unable to explain his apparently anomalous position. Anomalous though it be, it is the only one in the completed result, and should not therefore, be regarded as invalidating the main results, which, as has been shown, are of a very striking character. For further remarks on both the Dschagga negro and the European, see also Mr Cross's paper, in which the final results for the whole series are fully dealt with.

Of the Tasmanian himself, it is clear that our final results place his position in Nature very accurately. Of recent man the Tasmanian stands nearest to Homo fossilis, but morphologically has progressed a very long way from Homo primigenius and the anthropoid apes—very much farther than most writers would seem to believe. We cannot as yet state whether the Australian aboriginal will stand on the evolutionary plus or the minus side of the Tasmanian aboriginal, for the simple reason that our investigations on the Australian are not yet available for these purposes of comparison; but, from our mathematical results (22), as well as from the observations already in progress on the Australian aboriginal, we expect that the latter will stand on the plus side of the Tasmanian.

Quite irrespective of the relationship of the Australian to the Tasmanian, the present results prove that the latter has progressed a very long way from the Spy-Neanderthal men. Our figures show that both the Spy-Neanderthal men and the man (or woman) of Gibraltar stand more nearly related to *Pithecanthropus erectus* and the anthropoid apes than are the Tasmanians. We therefore cannot agree with the statement that "recent experiences show so many connections between

Pithecanthropus and Australian and Tasmanian skulls that I am more inclined than before to accept a very close approximation of Pithecanthropus to the first tribe of human beings."

We have already shown, or rather our final results have demonstrated for us, that Pithecanthropus stands nearer to the anthropoids than he does to Homo primigenius, let alone the modern Australian and the recently extinct Tasmanian, as some authors would have us believe.

A study of the individual figures and measurements recorded by us on the 52 Tasmanian crania with which this work deals also shows that in most points the Tasmanian is well within the range of variation of modern man.

From both these lines of argument we are led to conclude that the morphological evolution of the Tasmanian had progressed, at the time of his extinction, on its own independent lines, and to a higher plane than is generally either admitted or supposed. That his mental culture was on such a primitive plane is fully explained by his complete and total isolation for many countless years, and not to the fact that his physical organisation was incapable of attaining a more improved degree of culture.

That the "lately extinct Tasmanians recall the mental level of eolithic man in Britain" we can quite believe, but that either the Australian or the Tasmanian "carries us back nearly to the Neanderthal physical type" we must, as the result of the present investigation, deny, because the physical construction of the Tasmanian is herein certainly shown to go back only so far as the Galley Hill type at farthest, and more than this cannot be maintained with any degree of scientific certainty.

REFERENCES.

(1) Berry, R. J. A., and A. W. D. Robertson, "Preliminary Communication on Fifty-three Tasmanian Crania, forty-two of which are now recorded for the first time," *Proc. Roy. Soc. of Vict.*, vol. xxii., N.S., pt. i., 1909, p. 47.

(2) Schwalbe, G., "Studien über *Pithecanthropus erectus*, Dubois," *Zeit. für Morph. und Anthrop.*, Bd. i., 1899, pp. 16 *et seq.*

(3) Klaatsch, H., "The Skull of the Australian Aboriginal," *Reports from the Path. Lab. of the Lunacy Dept. N.S.W.*, vol. i., part iii., 1908, pp. 45 *et seq.*

(4) Cunningham, D. J., "The Evolution of the Eyebrow Region of the Forehead, with special reference to the excessive supraorbital development in the Neanderthal race," *Trans. Roy. Soc. Edin.*, vol. xlvi., part ii., 1908, pp. 283–311.

(5) Guifrida-Ruggeri, V., "Nuove Ricerche morfologiche e craniometriche," *Atti della Società Romana di Antrop.*, vol. viii., 1901, pp. 21 *et seq.*

(6) Guifrida-Ruggeri, V., "La posizone del bregma nel cranio del *Pithecanthropus erectus* e la tendenza neomonogenista in Germania," *Atti della Società Romana di Antrop.*, vol. x., fasc. 1.

(7) Berry, R. J. A., and A. W. D. Robertson, "Dioptrographic Tracings in Four Normæ of Fifty-two Tasmanian Crania," *Trans. Roy. Soc. Vict.*, vol. v., 1910.

(8) Berry, R. J. A., and A. W. D. Robertson, "Preliminary Account of the Discovery of Forty-two hitherto unrecorded Tasmanian Crania," *Anat. Anz.*, vol. xxxv., 1909, pp. 11–17.

(9) Schwalbe, G., "Über die specifischen Merkmale des Neanderthalschädels," *Verhand. der anatomischen Ges.*, Bonn, 1901, Pp. 44–61.

(10) Dubois, E., "*Pithecanthropus erectus*, eine Stammform des Menschen," *Anat. Anz.*, vol. xii., 1896, pp. 1–22.

(11) Turner, Sir W., "The Craniology, Racial Affinities, and Descent of the Aboriginals of Tasmania," *Trans. Roy. Soc. Edin.*, vol. xlvi., 1908, pp. 365–403.

(12) Broca, P., "Instructions Craniologiques et Craniométriques," *Mémoires de la Société d'Anthropol.*, vol. xi., 2nd series, Paris, 1875.

(13) Topinard, P., *Anthropology*, Chapman & Hall, Ltd., London, 1904.

(14) Duckworth, W. L. H., *Morphology and Anthropology*, University Press, Cambridge, 1904.

(15) Schwalbe, G., "Zur Frage der Abstammung des Menschen," *Zeit. für Morph. und Anthrop.*, Sonderheft, Mai 1906.

(16) Schwalbe, G., "Das Schädelfragment von Brüx und verwandte Schädelformen," *ibid.*

(17) Schwalbe, G., "Das Schädelfragment von Cannstatt," *ibid.*

(18) Klaatsch, H., "Die Fortschritte der Lehre von den fossilen Knochenresten des Menschen im den Jahren 1900–1903," Merkel und Bonnet's *Ergebnisse der Anatomie und Entwickelungsgeschichte*, vol. xii., 1903, pp. 545–651.

(19) Keith, A., quoted by Schwalbe under reference No. 15.

(20) Keane, A. H., *Ethnology*, University Press, Cambridge, 1901.

(21) Keane, A. H., *Man, Past and Present*, University Press, Cambridge, 1900.

(22) Berry, R. J. A., A. W. D. Robertson, and K. S. Cross, "A Biometrical Study of the Relative Degree of Purity of Race of the Tasmanian, Australian, and Papuan," *Proc. Roy. Soc. Edin.*, vol. xxxi., 1910.

(*Issued separately December* 8, 1910.)

IV.—On a Numerical Determination of the Relative Positions of certain Biological Types in the Evolutionary Scale, and of the Relative Values of various Cranial Measurements and Indices as Criteria. By K. Stuart Cross, M.Sc. Melb. (From the Anatomy Department of the University of Melbourne.) *Communicated by* Professor R. J. A. Berry.

(Read March 7, 1910. MS. received July 5, 1910.)

The present investigation arose in part as an examination into, and an extension of, an attempt by Professor R. J. A. Berry and Dr A. W. D. Robertson * at placing the Tasmanian aboriginal in his relation to various other racial types, extinct and existent. It has resolved itself into a numerical calculation of the approximate relative positions of these types in the evolutionary scale, together with a determination of the relative values of various cranial measurements and indices employed by Schwalbe in his "form analysis" of the calvarium, as criteria is assigning these positions.

I am also indebted to the above authors for the collected data on which the present work is based. These data are set forth in Table I. For an account of the various sources and authorities for the figures there given, reference may be made to the above-mentioned paper. The order of the various titles of measurements and indices in this table is purely one of convenience, being determined by the absence of various measurements for one or other type of cranium, except that the last three occupy their positions at the foot of the list because of their non-inclusion in the subsequent calculations.

Of the various measurements under consideration, the great majority show a general increase or decrease in passing from the anthropoid ape to modern man. Those of the glabella-inion length and the length of the pars glabellaris of the os frontale do not, however, satisfy this condition. They appear to rise irregularly to a maximum and then to recede. As this investigation is based on the assumption that the increase or decrease, where continuous throughout the scale, is proportional to the increase in the degree of development, these two measurements have been rejected. The relative weights they would have carried had they been included have,

* Berry and Robertson, "The Place in Nature of the Tasmanian Aboriginal as deduced from a Study of his Calvarium," *Proceedings of the Royal Society of Edinburgh*, 1910, vol. xxxi.

however, been approximately calculated at a later stage, and found so small as almost to justify their exclusion on that ground alone. Of the angle of parietal curvature, the value for the Tasmanians only is available, so it has also to be excluded. The remaining twenty-four measurements and indices form the basis of the investigation.

Taking each of these in turn, the various crania are arranged in order, according to the respective values of that measurement or index. The lowest award is zero, since the determination of an absolute zero is not possible, whilst the highest is one less than the number of "entries" under that head. The other awards are intermediate to these, and so allotted that the intervals between them are proportional to those between the actual values of the particular measurement or index for the various types of crania, whilst from the above choice of range the average value of the interval between successive awards throughout the series is unity.

In Table II. are shown the proportionate values thus determined for each measurement or index.

The total sum of the awards for each type on all the counts wherein it is represented is then determined, together with the total of the maximum awards obtainable by that type. The ratio of the former to the latter is taken to represent the relative position in the evolutionary scale. Thus this "index of position" has a range between zero and unity. The minimum value would indicate that the type whose index had that value occupied on each count the lowest position in the scale, whereas the maximum value would be obtainable only by a type that attained to the highest position on each count.

The relative positions in the evolutionary scale of the various types of crania, as represented by the above "indices of position," must, however, be regarded only as a first approximation, for, apart from the varying number of types represented, the values of the various measurements and indices as criteria have been taken as equal. Inspection of the differences in the orders as each of these is in turn adopted as the criterion shows this not to be the case. Thus it is necessary to attempt an estimation of the relative weight to be attributed to each of the measurements and indices with which this work is concerned.

This is done by comparing each individual order with the composite order as given by the resultant relative positions in the last line of Table II.

Table III. gives the resultant relative positions adapted for direct comparison with the various individual series of Table II. This adaptation is made necessary by the absence of various measurements in Table I., and the consequent variations in the members and the range of the different series in Table II.

The comparison is then made by calculating the coefficient of correlation between the positions as given by each measurement or index in turn, and the positions as deduced from the composite series.

The maximum value of unity for any particular coefficient would indicate that the corresponding measurement or index would give an order precisely the same as the composite order deduced from the consideration of all these measurements and indices. A zero value would show an absence of relation between variation in the measurement and variation in degree of development. Thus, the nearer the value of the coefficient of correlation for any measurement or index approaches unity, the greater the value of that measurement or index as a criterion in determining the relative evolutionary positions of the various crania.

In the column headed r in Table IV. are given the values of the various coefficients of correlation thus determined.

A second approximation to the evolutionary order is then arrived at by weighting each measurement or index proportionately to the most probable value of the corresponding coefficient of correlation. Thus, the series in Table IV. are obtained by multiplying the various series in Table II. by the corresponding coefficients of correlation. The subsequent procedure is then the same as with the latter series.

A comparison of this new composite order in Table IV. with the first approximation in Table II. will show distinct modifications in the relative intervals, though the order is unaltered, except that Veddahs move up to equality with Europeans, or to a position slightly in advance.

These new values for the composite relative positions necessitate, however, a recalculation of the coefficients of correlation, so as to arrive at a second approximation to the values of the various measurements and indices as criteria.

Table V. gives the necessary adaptations of the new order for this recalculation, these various series being compared with the original series of Table II. and the coefficients of correlation calculated as before.

These new values are given under the heading r in Table VI.

With these new values a further step has been taken in determining a third approximation to the order of the various types. A comparison of this approximation with the second shows only very small variations. Here, then, the process ends.

The high position of the Cannstatt skull is discussed by Berry and

Robertson, who produce evidence to show it to be, in all probability, of a modern European type. The positions of the Dschagga Negroes and Veddahs in advance of the Europeans, as shown in the last line of Table VI., is an apparent anomaly. Remembering, however, that in the case of the Veddahs the values of but six of the twenty-four measurements and indices were known, and only nine for the Dschaggas, it admits of ready explanation.

Thus, confining ourselves to the measurements and indices common to the Europeans and Veddahs, we have the following table:—

Measurement or Index.	Veddahs.	Europeans.	Maximum.
1	11·41	11·57	12·71
2	11·50	12·16	12·16
7	2·75	4·84	6·26
8	11·39	9·88	11·39
9	2·47	3·35	5·73
10	11·21	10·89	11·21
Total	50·73	52·69	59·46
Relative position	0·853	0·886	

We see, then, that as far as these measurements go, the Europeans are higher in the scale than the Veddahs. The apparent anomaly above referred to is due to the fact that, on the other sixteen counts wherein the Europeans are represented, the latter take a comparatively lower position in the scale than do either they or the Veddahs on the six counts wherein they are both represented. It is possible that the relative position of the Veddahs would be similarly altered were their other measurements known.

Comparing the Dschagga Negroes and Europeans in the same way, we have—

	Dschagga Negroes.	Europeans.	Maximum.
Total	79·80	78·77	86·70
Relative position . . .	0·920	0·908	

and thus the apparent interval of 0·050 becomes reduced to 0·012.

Finally, treating the Cannstatt, Dschagga Negroes, and Europeans in the same way, we have—

	Dschagga Negroes.	Cannstatt.	Europeans.	Maximum.
Total . . .	75·01	76·59	74·63	81·91
Relative position .	0·916	0·935	0·911	

Here, then, we see that, as far as they can be compared, the Cannstatt skull is higher than both the Europeans and the Dschagga Negroes, whilst on the eight counts common to the three the Europeans and Dschagga Negroes are practically equal in position.

The above partial tables emphasise the necessity of knowing as many of the cranial measurements as possible before arriving at any conclusion concerning the relative degrees of development of any given crania. Where only a few of the measurements or indices are obtainable for any particular cranium we are, of course, compelled to confine ourselves to these, treating them as we have those of the Veddahs and Dschagga Negroes in this paper; but the above work shows how necessary it is to know as many of the measurements as possible.

In fig. 1, then, which represents graphically the relative positions of the various crania as deduced in this study, the Veddahs and Dschagga Negroes are not included. Their approximate positions relative to the European and Cannstatt crania and to one another are shown in the above partial tables. They might have been altogether omitted from the investigation, but that their presence serves to exemplify the principles mentioned.

It should be stated here that the measurements designated European are those given by Schwalbe of some Alsace crania, and, further, that in many cases the average values adopted have been founded on a very limited number of observations. Thus, a reference to the paper of Berry and Robertson will show that some values are the averages of measurements taken on 8, or in some cases only 4, Veddah crania, 5 European crania, 4 Dschagga crania, and so on. There must, then, in these cases be

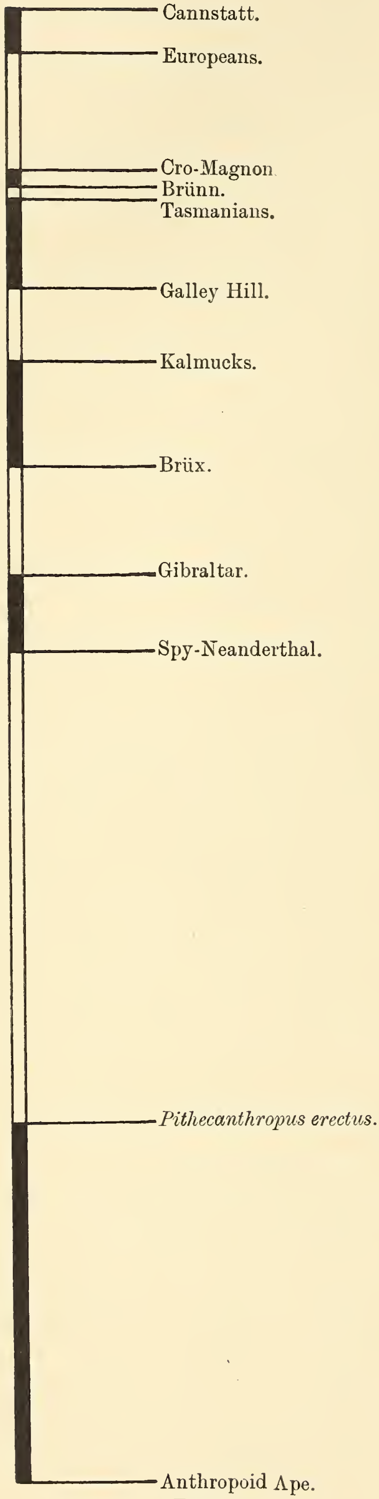


Fig. 1.

large probable errors due to random sampling. Wherever possible, there are needed more extended observations of the various measurements and indices on many of the crania, so that the probable errors of the average values in Table I. may be reduced as far as possible, and the absent measurements supplied. Where the main object is an attempt at placing various crania of extinct type in their correct evolutionary positions between the anthropoid apes on the one hand, and modern man on the other, this is necessarily in many cases impossible. Attempts should, however, be made to determine as accurately as possible certain standard positions in the evolutionary scale. Anthropoid apes and certain highly and lowly developed races of modern man could have their various average measurements determined to a considerable degree of accuracy. Any individual prehistoric cranium would, of course, take its position on the strength of its own individual characteristics. It might or might not be a representative specimen of the race to which it belonged.

It should be pointed out that, were new determinations of the relative positions and the relative values of the measurements and indices as criteria at any time undertaken with data completed as above indicated, and dependent on a much greater amount of material, such determinations would be no more laborious than are the present. In fact, were the empty spaces in Table I. filled in, the calculations of all the resultant values would be lightened, the necessity for Tables III. and V. would vanish, the coefficients of correlation in Tables IV. and VI. being determined by reference in each case to a single standard series, and so on. The values obtained in the present work are those deduced from the available data. They are correct as far, and only as far, as are the available data. More accurate initial average values of the various cranial measurements would, following the same methods of analysis, lead to more accurate final values.

Turning to the *relative values of the various cranial measurements and indices as criteria of position in the evolutionary scale,* we find the calvarial height to be the most reliable single measurement by which to place various types of crania in their proper evolutionary positions. At the top of the scale, where several types are near one another, minor alterations of positions are only to be expected; but, taken as a whole, the close agreement between the evolutionary order as given by this measurement and the composite order will be seen by comparing, say, the first line of Table II. with the corresponding series of Table V.

Following the calvarial height ($r = 0{\cdot}9777$) are, in order (see fig. 2), the frontal curvature ($r = 0{\cdot}9579$), the calvarial height-breadth index

($r = 0·9495$), and the bregma angle ($r = 0·9470$). Based on the data of this investigation, these are the most valuable. Of the remaining measurements and indices, the first five also give values of r greater than 0·9.

The two rejected measurements, the glabella-inion length and the length of the chord of the pars glabellaris of the os frontale, together with one other, also merit some reference. The coefficients of correlation of the orders given by the first two compared with the composite order of Table VI. were determined. That for the length of the chord of the pars glabellaris of the os frontale has the extremely small value of 0·1152 (with a probable error of 0·1922), while that for the glabella-inion length has a value of 0·2374 (with a probable error of 0·1701). The higher value of this latter coefficient is due in great part to the very low value (147 mm.) of the glabella-inion length for the anthropoid ape. Otherwise this measurement is at least as valueless for the present purpose as is the length of the pars glabellaris.

The other measurement referred to is the calvarial height foot-point positional index. Of the measurements retained, it is quite the least valuable, with $r = 0·1827$. Unlike the two measurements just discussed, it shows no indication of approaching and then receding from a maximum or minimum in passing up the evolutionary scale. As far as this investigation goes, it seems practically worthless.

It has already been pointed out that a recalculation from more abundant data will probably alter in detail the relative values of the measurements and indices as given by the numerical values of the coefficients of correlation. When that redetermination comes to be made, the present values will, however, prove useful in arriving more rapidly at a sufficiently close approximation, weights proportional to these present values being assigned to the various series in determining the first of the new approximations.

When the missing values of Table I. are all supplied, or some types at discretion added or omitted, each series may be calculated with a range, say, from zero to unity. Any newly discovered or other cranium, or group of crania, could then be given a new column without in any way disturbing the rest of the table, and its relative position for each measurement and index (multiplied by the assigned weight) determined. The ratio of the total awards to the maximum possible award would give its position. Or a formula could be devised on practically the same principle, the various terms consisting of the product of the respective coefficient of correlation and the ratio of the difference of the observed measurement from the corresponding maximum or minimum (according as to which occurred at the lower end of the scale) to the original total range of value.

Thus, from the values of r in Table VI. and those of the various measurements of Table I., we deduce as the first two terms of such a formula—

$$\frac{1}{n}\left\{0{\cdot}9777 \times \frac{m_1 - 48{\cdot}5}{56{\cdot}5} + 0{\cdot}9356 \times \frac{m_2 - 34{\cdot}2}{25{\cdot}6} + \text{etc.}\right\},$$

or

$$\frac{1}{n}\left\{0{\cdot}01730(m_1 - 48{\cdot}5) + 0{\cdot}03655(m_2 - 34{\cdot}2) + \text{etc.}\right\},$$

where n is the total sum of the various coefficients of correlation involved, m_1 the observed length of the calvarial height in mm., m_2 the calvarial height index, and so on. The numerical value given by this formula would then only have to be directly compared with the numbers giving the composite relative positions of the types already included in the table, in order to place the cranium, or group of crania, between its nearest neighbours on either side, and at its proper relative distance from each.

Calvarial height	0·9777 ± 0·0079
Frontal curvature	0·9579 ± 0·0227
Calvarial height-breadth index	0·9495 ± 0·0184
Bregma angle	0·9470 ± 0·0193
Calvarial height index	0·9356 ± 0·0224
Calvarial height-$\frac{1}{2}$(length + breadth) index	0·9342 ± 0·0238
Length of parietal arc	0·9237 ± 0·0299
Frontal angle	0·9207 ± 0·0285
Bregma foot-point positional index	0·9023 ± 0·0348
Lambda angle	0·8720 ± 0·0487
Opisthionic angle	0·8655 ± 0·0564
Length of frontal arc	0·8551 ± 0·0523
Length of chord of pars cerebralis of os frontale	0·8400 ± 0·0574
Length of chord of os frontale	0·8027 ± 0·0693
Parietal-frontal arc index	0·7186 ± 0·0984
Distance of bregma foot-point from glabella	0·7136 ± 0·0918
Length of parietal chord	0·7041 ± 0·1134
Curvature index of os frontale	0·6944 ± 0·1008
Distance of calvarial height foot-point from glabella	0·5836 ± 0·1284
Glabella-cerebral chord index	0·5450 ± 0·1369
Maximum breadth	0·5216 ± 0·1362
Curvature index of os parietale	0·4794 ± 0·1732
$\frac{1}{2}$(glabella-inion length + breadth)	0·4778 ± 0·1444
Calvarial height foot-point positional index	0·1827 ± 0·1882
Glabella-inion length	0·2374 ± 0·1701
Length of chord of pars glabellaris of os frontale	0·1152 ± 0·1922

Fig. 2.

Table I.

Measurement or Index.	Anthropoid Ape.	*Pithecanthropus erectus.*	Spy-Neanderthal.	Gibraltar.	Brüx.	Kalmucks.	Galley Hill.	Tasmanians.	Brünn.	Cro-Magnon.	Veddahs.	Europeans.	Cannstatt.	Dschagga Negroes.
1. Calvarial height	48·5	62	85·3	85	85	90·7	97	97	103	101	99·2	99·9	105	100
2. Calvarial height index	35·1	34·2	44·9	45·4	47·6	54·5	48·2	56·1	51·2	50	58·4	59·8	59·6	59·8
3. Frontal angle	56	52·5	64·8	73	74·7	85·2	82	86	75	83	...	92·5	90	100·3
4. Bregma angle	39·5	37·5	47·5	50·5	51·1	56·5	52	56	54	54	...	59·9	60	58·6
5. Distance of bregma foot-point from glabella	72	81·5	72·3	66	75	54·6	69	58·7	67·5	66	...	51·3	58	53·9
6. Bregma foot-point positional index	63·4	44·1	36·6	35·2	37 3	32·8	34·3	33·5	34	32·6	...	30·4	33·3	32·1
7. Maximum breadth	113	133	150·3	148	130	146	130	134·7	139	151	129·7	142·4	146	
8. Calvarial height-breadth index	42·9	46·6	56·7	57·4	63·3	63·3	74·6	72·2	74·1	66·8	76·9	72·4	71·9	
9. $\frac{1}{2}$ sum glabella-inion length plus breadth	129	157	174·3	167·5	155	154·3	165·5	154	170	176·5	149·5	156·8	160	
10. Calvarial height- $\frac{1}{2}$ sum glabella-inion length + breadth index	38·6	39·4	48·9	50·7	54·8	58·7	50·8	63	60·5	57·2	66·6	65·8	65·6	
11. Distance of calvarial height foot-point from glabella	84	80·5	111	105·7	111	86·3	111	101·9	110	121·5	...	95·8	102·5	
12. Calvarial height foot-point positional index	61·8	44·4	55·7	56·5	60	52·6	55·2	59·0	54·7	60·1	...	56·2	58·3	
13. Length of frontal arc	92	110	124	126	135	115·2	135	126	135	138	...	125·6	135	
14. Length of chord of os frontale	87	104	114	111	114	103·8	120	109·5	123	123	...	112·5	117	
15. Curvature index of os frontale	94·5	94·6	92·8	88	84·4	90·1	88·8	87·1	91·1	89·1	...	89·5	86·6	
16. Length of chord of pars cerebralis of os frontale	55	87·5	83·6	82	99	...	95	93·7	96	97·5	...	92·1	104	95·8
17. Glabella-cerebral chord index	40	27·6	39·6	43	24·2	...	25·2	25·5	31·2	32·3	...	26·6	18·2	27·4
18. Length of parietal arc	62	103	119·6	111	108	...	132	125·8	139·5	135	...	133	130	
19. Parietal-frontal arc index	82·6	85·8	96·8	88	80	...	97·7	99·7	103·3	97·8	...	109·9	96·2	
20. Lambda angle	55·5	66	66·5	69	80	...	74	80·5	78	70	...	81·5	83	
21. Length of parietal chord	...	104	107·7	108	100·5	...	120	113	127·5	123	...	...	117·5	
22. Curvature index of os parietale	...	92	93·7	97·2	93	...	90·9	90	91·3	91·1	...	...	90·3	
23. Opisthionic angle	59·5	64	54	36	...	...	42	40·6	42	34	...	35·5		
24. Frontal curvature measured on glabella-bregma chord	160	153·2	153·3	...	...	...	...	139·5	...	...	...	135·4	...	131·5
25. Glabella-inion length	147	181	196·6	187	185	166·2	201	173·1	201	202	169·8	168	174	167·4
26. Length of chord of pars glabellaris of os frontale	20	24	33·1	36	24	...	24	23·8	30	31·5	...	24·5	19	26·4
27. Angle of parietal curvature	...	...	...	...	...	...	...	134·3						

TABLE II.

Measurement or Index.	Anthropoid Ape.	*Pithecanthropus erectus.*	Spy-Neanderthal.	Gibraltar.	Brüx.	Kalmucks.	Galley Hill.	Tasmanian.	Brünn.	Cro-Magnon.	Veddahs.	Europeans.	Cannstatt.	Dschagga Negroes.
1	0	3·11	8·47	8·40	8·40	9·71	11·16	11·16	12·54	12·08	11·67	11·83	13·00	11·85
2	0·46	0	5·43	5·69	6·80	10·31	7·11	11·12	8·63	8·02	12·29	13·00	12·90	13·00
3	0·88	0	3·09	5·15	5·57	8·21	7·41	8·41	5·65	7·66	...	10·03	9·41	12·00
4	1·07	0	5·33	6·93	7·25	10·13	7·73	9·87	8·80	8·80	...	11·95	12·00	11·25
5	3·77	0	3·66	6·16	2·58	10·69	4·97	9·06	5·56	6·16	...	12·00	9·34	10·97
6	0	7·02	9·74	10·25	9·49	11·13	10·58	10·87	10·69	11·20	...	12·00	10·95	11·38
7	0	6·32	11·78	11·05	5·37	10·42	5·37	6·85	8·21	12·00	5·27	9·28	10·42	
8	0	1·31	4·87	5·12	7·20	7·20	11·19	10·34	11·01	8·43	12·00	10·41	10·23	
9	0	7·07	11·44	9·73	6·57	6·39	9·22	6·32	10·36	12·00	5·18	7·02	7·83	
10	0	0·34	4·41	5·19	6·94	8·61	5·23	10·46	9·39	7·97	12·00	11·66	11·57	
11	0·94	0	8·18	6·76	8·18	1·56	8·18	5·74	7·91	11·00	...	4·10	5·90	
12	11·00	0	7·14	7·65	9·86	5·18	6·83	9·23	6·51	9·92	...	7·46	8·79	
13	0	4·30	7·65	8·13	10·28	5·55	10·28	8·13	10·28	11·00	...	8·03	10·28	
14	0	5·19	8·25	7·33	8·25	5·13	10·08	6·88	11·00	11·00	...	7·79	9·17	
15	0·11	0	1·94	7·12	11·00	4·85	6·25	8·09	3·77	5·93	...	5·50	8·63	
16	0	7·30	6·42	6·06	9·88	...	8·98	8·69	9·20	9·54	...	8·33	11·00	9·16
17	1·33	6·83	1·51	0	8·34	...	7·90	7·76	5·23	4·75	...	7·27	11·00	6·92
18	0	5·29	7·43	6·32	5·93	...	9·03	8·23	10·00	9·42	...	9·16	8·77	
19	0·87	1·94	5·62	2·68	0	...	5·92	6·59	7·79	5·95	...	10·00	5·42	
20	0	3·82	4·00	4·91	8·91	...	6·73	9·09	8·18	5·27	...	9·45	10·00	
21	...	1·04	2·13	2·22	0	...	5·78	3·70	8·00	6·67	...	...	5·04	
22	...	5·78	3·89	0	4·67	...	7·00	8·00	6·56	6·78	...	...	7·67	
23	1·20	0	2·67	4·47	...	...	5·87	6·24	5·87	8·00	...	7·60		
24	0	1·19	1·18	...	...	...	...	3·60	...	...	...	4·32	...	5·00
Total	21·63	67·85	136·23	140·32	151·47	115·07	178·80	194·43	191·14	199·55	58·41	198·20	209·32	91·53
Poss. max.	242	258	258	253	245	177	253	258	253	253	74	242	245	101
Rel. posn.	0·089	0·263	0·528	0·555	0·618	0·653	0·707	0·754	0·755	0·789	0·789	0·819	0·854	0·906

Table III.

	Anthropoid Ape.	*Pithecanthropus erectus.*	Spy-Neanderthal.	Gibraltar.	Brüx.	Kalmucks.	Galley Hill.	Tasmanians.	Brünn.	Cro-Magnon.	Veddahs.	Europeans.	Cannstatt.	Dschagga Negroes.
Relative positions	0	0·174	0·439	0·466	0·529	0·564	0·618	0·665	0·666	0·700	0·700	0·730	0·765	0·817
" "	...	0	0·265	0·292	0·355	0·390	0·444	0·491	0·492	0·526	0·526	0·556	0·591	0·643
Order for Measurement or Index—														
1 2	0	2·77	6·98	7·41	8·42	8·97	9·83	10·58	10·60	11·14	11·14	11·62	12·17	13·00
3, 4, 5, 6	0	2·56	6·45	6·84	7·77	8·28	9·08	9·77	9·78	10·28	...	10·72	11·23	12·00
7, 8, 9, 10	0	2·73	6·89	7·31	8·30	8·85	9·69	10·43	10·45	10·98	10·98	11·45	12·00	
11, 12, 13, 14, 15	0	2·50	6·31	6·70	7·61	8·11	8·89	9·56	9·58	10·07	...	10·50	11·00	
16, 17	0	2·34	5·91	6·27	7·12	...	8·32	8·95	8·97	9·42	...	9·83	10·30	11·00
18, 19, 20	0	2·27	5·74	6·09	6·91	...	8·08	8·69	8·71	9·15	...	9·54	10·00	
21, 22	...	0	3·59	3·95	4·81	...	6·01	6·65	6·66	7·12	...	...	8·00	
23	0	1·91	4·81	5·11	...	...	6·77	7·29	7·30	7·67	...	8·00		
24	0	1·06	2·69	...	...	...	...	4·07	...	...	...	4·47	...	5·00

TABLE IV.

Measurement or Index.	r.	Maximum.	Anthropoid Ape.	*Pithecanthropus erectus.*	Spy-Neanderthal.	Gibraltar.	Brüx.	Kalmucks.	Galley Hill.	Tasmanians.	Brünn.	Cro-Magnon.	Veddahs.	Europeans.	Cannstatt.	Dschagga Negroes.
1	0·9794	12·73	0	3·05	8·30	8·23	8·23	9·51	10·93	10·93	12·28	11·83	11·43	11·59	12·73	11·61
2	0·9266	12·05	0·43	0	5·03	5·27	6·30	9·55	6·59	10·30	8·00	7·43	11·39	12·05	11·95	12·05
3	0·9188	11·03	0·81	0	2·84	4·73	5·12	7·54	6·81	7·73	5·19	7·04	...	9·22	8·65	11·03
4	0·9425	11·31	1·01	0	5·02	6·53	6·83	9·55	7·29	9·30	8·29	8·29	...	11·26	11·31	10·60
5	0·7039	8·45	2·65	0	2·58	4·34	1·82	7·53	3·50	6·38	3·91	4·34	...	8·45	6·58	7·72
6	0·8967	10·76	0	6·30	8·74	9·19	8·51	9·98	9·49	9·75	9·59	10·04	...	10·76	9·82	10·20
7	0·5426	6·51	0	3·43	6·39	6·00	2·91	5·65	2·91	3·72	4·45	6·51	2·86	5·04	5·65	
8	0·9414	11·30	0	1·23	4·59	4·82	6·78	6·78	10·54	9·73	10·36	7·94	11·30	9·80	9·63	
9	0·5063	6·08	0	3·58	5·79	4·93	3·33	3·24	4·67	3·20	5·25	6·08	2·62	3·55	3·96	
10	0·9228	11·07	0	0·31	4·07	4·79	6·40	7·95	4·83	9·65	8·67	7·35	11·07	10·76	10·68	
11	0·6079	6·69	0·57	0	4·97	4·11	4·97	0·95	4·97	3·49	4·81	6·69	...	2·49	3·59	
12	0·2046	2·25	2·25	0	1·46	1·57	2·02	1·06	1·40	1·89	1·33	2·03	...	1·53	1·80	
13	0·8648	9·51	0	3·72	6·62	7·03	8·89	4·80	8·89	7·03	8·89	9·51	...	6·94	8·89	
14	0·8124	8·94	0	4·22	6·70	5·96	6·70	4·17	8·19	5·59	8·94	8·94	...	6·33	7·45	
15	0·6977	7·67	0·08	0	1·35	4·97	7·67	3·38	4·36	5·64	2·63	4·14	...	3·84	6·02	
16	0·8381	9·22	0	6·12	5·38	5·08	8·28	...	7·53	7·28	7·71	8·00	...	6·98	9·22	7·68
17	0·5379	5·92	0·72	3·67	0·81	0	4·49	...	4·25	4·17	2·81	2·56	...	3·91	5·92	3·72
18	0·9186	9·19	0	4·86	6·83	5·81	5·45	...	8·30	7·56	9·19	8·65	...	8·42	8·06	
19	0·7126	7·13	0·62	1·38	4·01	1·91	0	...	4·22	4·70	5·55	4·24	...	7·13	3·86	
20	0·8678	8·68	0	3·32	3·47	4·26	7·73	...	5·84	7·89	7·10	4·57	...	8·20	8·68	
21	0·7003	5·60	...	0·73	1·49	1·55	0	...	4·05	2·59	5·60	4·67	...	...	3·53	
22	0·4782	3·83	...	2·76	1·86	0	2·23	...	3·35	3·83	3·14	3·24	...	...	3·67	
23	0·8673	6·94	1·04	0	2·32	6·48	...	...	5·09	5·41	5·09	6·94	...	6·59		
24	0·9538	4·77	0	1·14	1·13	...	...	...	...	3·43	...	...	...	4·12	...	4·77
Total			10·18	49·82	101·75	107·56	114·66	91·64	138·00	151·19	148·78	151·03	50·67	158·96	161·65	79·38
Possible maximum .			188·20	197·63	197·63	192·86	185·92	136·35	192·86	197·63	192·86	192·86	59·74	188·20	185·92	86·24
Relative position . .			0·054	0·252	0·515	0·558	0·617	0·672	0·715	0·765	0·771	0·783	0·848	0·845	0·869	0·920

Table V.

	Anthropoid Ape.	*Pithecanthropus erectus.*	Spy-Neanderthal.	Gibraltar.	Brüx.	Kalmucks.	Galley Hill.	Tasmanians.	Brünn.	Cro-Magnon.	Veddahs.	Europeans.	Cannstatt.	Dschagga Negroes.
Relative positions . . .	0	0·198	0·461	0·504	0·563	0·618	0·661	0·711	0·717	0·729	0·794	0·791	0·815	0·866
" " . . .	...	0	0·263	0·306	0·365	0·420	0·463	0·513	0·519	0·531	0·596	0·593	0·617	0·668
Order for Measurement or Index—														
1, 2	0	2·97	6·92	7·57	8·45	9·28	9·92	10·67	10·76	10·94	11·92	11·87	12·24	13·00
3, 4, 5, 6	0	2·74	6·39	6·98	7·80	8·56	9·16	9·85	9·94	10·10	...	10·96	11·29	12·00
7, 8, 9, 10	0	2·92	6·79	7·42	8·29	9·10	9·73	10·47	10·56	10·73	11·69	11·65	12·00	
11, 12, 13, 14, 15	0	2·67	6·22	6·80	7·60	8·34	8·92	9·60	9·68	9·84	...	10·67	11·00	
16, 17	0	2·51	5·85	6·40	7·15	...	8·40	9·03	9·11	9·26	...	10·05	10·35	11·00
18, 19, 20	0	2·43	5·66	6·18	6·91	...	8·11	8·72	8·80	8·94	...	9·70	10·00	
21, 22,	...	0	3·41	3·97	4·73	...	6·00	6·65	6·73	6·88	...	...	8·00	
23	0	2·00	4·66	5·10	...	...	6·69	7·19	7·25	7·37	...	8·00		
24	0	1·14	2·66	...	...	...	...	4·10	...	...	...	4·57	...	5·00

TABLE VI.

Measurement or Index.	r.	Maximum.	Anthropoid Ape.	*Pithecanthropus erectus.*	Spy-Neanderthal.	Gibraltar.	Brüx.	Kalmucks.	Galley Hill.	Tasmanians.	Brünn.	Cro-Magnon.	Veddahs.	Europeans.	Cannstatt.	Dschagga Negroes.
1	0·9777	12·71	0	3·04	8·28	8·21	8·21	9·49	10·91	10·91	12·26	11·81	11·41	11·57	12·71	11·59
2	0·9356	12·16	0·43	0	5·08	5·32	6·36	9·65	6·65	10·40	8·07	7·50	11·50	12·16	12·07	12·16
3	0·9207	11·05	0·81	0	2·84	4·74	5·13	7·56	6·82	7·74	5·20	7·05	...	9·23	8·66	11·05
4	0·9470	11·36	1 01	0	5·05	6·56	6·87	9·59	7·32	9·35	8·33	8·33	...	11·32	11·36	10·65
5	0·7136	8·56	2·69	0	2·61	4·40	1·84	7·63	3·55	6·46	3·97	4·40	...	8·56	6·67	7·83
6	0·9023	10·83	0	6·34	8·79	9·25	8·56	10·04	9·55	9·81	9·65	10·11	...	10·83	9·88	10·27
7	0·5216	6·26	0	3·30	6·14	5·76	2·80	5·43	2·80	3·57	4·28	6·26	2·75	4·84	5·43	
8	0·9495	11·39	0	1·24	4·62	4·86	6·84	6·84	10·63	9·82	10·45	8·00	11·39	9·88	9·71	
9	0·4778	5·73	0	3·38	5·47	4·65	3·14	3·05	4·41	3·02	4·95	5·73	2·47	3·35	3·74	
10	0·9342	11·21	0	0·32	4·12	4·85	6·48	8·04	4·89	9·77	8·77	7·45	11·21	10·89	10·81	
11	0·5836	6·42	0·55	0	4·77	3·95	4·77	0·91	4·77	3·35	4·62	6·42	...	2·39	3·44	
12	0·1827	2·01	2·01	0	1·30	1·40	1·80	0·95	1·25	1·69	1·19	1·81	...	1·36	1·61	
13	0·8551	9·41	0	3·68	6·54	6·95	8·79	4·75	8·79	6·95	8·79	9·41	...	6·87	8·79	
14	0·8027	8·83	0	4·17	6·62	5·88	6·62	4·12	8·09	5·52	8·83	8·83	...	6·25	7·36	
15	0·6944	7·64	0·08	0	1·35	4·94	7·64	3·37	4·34	5·62	2·62	4·12	...	3·82	5·99	
16	0·8400	9·24	0	6·13	5·39	5·09	8·30	...	7·54	7·30	7·73	8·01	...	7·00	9·24	7·69
17	0·5450	6·00	0·72	3·72	0·82	0	4·55	...	4·31	4·23	2·85	2·59	...	3·96	6·00	3·77
18	0·9237	9·24	0	4·89	6·86	5·84	5·48	...	8·34	7·60	9·24	8·70	...	8·46	8·10	
19	0·7186	7·19	0·63	1·39	4·04	1·93	0	...	4·25	4·74	5·60	4·28	...	7·19	3·89	
20	0·8720	8·72	0	3·33	3·49	4·28	7·77	...	5·87	7·93	7·13	4·60	...	8·24	8·72	
21	0·7041	5·63	...	0·73	1·50	1·56	0	...	4·07	2·61	5·63	4·70	...	...	3·55	
22	0·4794	3·84	...	2·77	1·86	0	2·24	...	3·36	3·84	3·15	3·25	...	...	3·68	
23	0·8655	6·92	1·04	0	2·31	6·47	...	...	5·08	5·40	5·08	6·92	...	6·58	...	
24	0·9579	47·9	0	1·14	1·13	...	...	...	...	3·45	...	...	...	4·14	...	4·79
Total			9·97	49·57	100·98	106·89	114·19	91·42	137·59	151·08	148·39	150·28	50·73	158·89	161·41	79·80
Possible maximum .			187·67	197·14	197·14	192·35	185·43	135·57	192·35	197·14	192·35	192·35	59·46	187·67	185·43	86·70
Relative position . .			0·053	0·251	0·512	0·556	0·616	0·674	0·715	0·766	0·771	0·781	0·853	0·847	0·870	0·920

(*Issued separately December* 9, 1910.)

V.—On the Magnetism of the Copper-Manganese-Tin Alloys under varying Thermal Treatment.

By **Alexander D. Ross**, M.A., B.Sc., Lecturer on Natural Philosophy in the University of Glasgow, and **Robert C. Gray**, M.A., Houldsworth Research Scholar, University of Glasgow.

(MS. received June 20, 1910. Read same date.)

Ternary Magnetic Alloys.—In 1905 investigations on magnetic alloys of the Heusler type were commenced in the Physical Institute of the University of Glasgow. The earlier tests * were carried out on an alloy containing about 62·5 per cent. copper and the remainder manganese and aluminium in atomic proportions. Towards the end of 1907 the research was extended to other members of the ternary system, each alloy having, however, the same relative proportions of manganese and aluminium.† From a comparison of all the data obtained, the authors were led to the view that the presence of Cu_3Al played an important part in the magnetic properties of these alloys.‡ This intermetallic compound exhibited effects of the same nature as did the Heusler alloys when subjected to annealing, quenching, or other thermal treatment. It therefore seemed not unlikely that the magnetism of the alloys in the ternary system was due to the entrance of Cu_3Al into solid solutions.

Copper-manganese-tin Alloys.—The production of strongly magnetic materials by alloying non-magnetic elements has such an important bearing on a general theory of magnetism, that it is very desirable to investigate fully the various magnetic ternary systems. Several such groups are known,§ as, for example, alloys formed by the addition of antimony, arsenic, boron, bismuth, or tin to manganese bronze. The authors have now carried out a series of tests on members of the copper-manganese-tin system, and it is proposed to give in this paper a general sketch of the results obtained.

Scheme of Castings.—The object of the research was to ascertain the influence of heat treatment on the magnetism of alloys belonging to this

* A. Gray, *Proc. Roy. Soc.*, lxxvi., A, 271 (1905); A. D. Ross, *Proc. Roy. Soc. Edin.*, xxvii., 88 (1907); J. G. Gray, *ibid.*, xxviii., 403 (1908).

† A. D. Ross and R. C. Gray, *Proc. Roy. Soc. Edin.*, xxix., 274 (1909).

‡ A. D. Ross and R. C. Gray, *Zeit. f. anorg. Chem.*, lxiii., 349 (1909).

§ F. Heusler, *Verh. d. deut. phys. Gesell.*, v., 219 (1903).

group, rather than to study fully the variation of magnetic properties with composition. Only a few castings were therefore necessary. In arriving at a decision as to what castings should be attempted, the authors were influenced by the following considerations. As the melting-points of the three constituent elements were spread over a wide range of temperature,* the simplest procedure was to prepare a manganese-copper alloy and add the tin to it. An alloy of 70 per cent. by weight of copper and 30 per cent. manganese was the most suitable for this purpose, as it has the lowest melting point (viz. about 870° C.) of the copper-manganese system.† An alloy richer in manganese would have been less satisfactory, as such material is apt to be far from homogeneous on casting.‡ Moreover, with increasing manganese-content there is an increasing tendency for the heated metal to absorb carbon, for which element manganese has a strong affinity.§ With the 30 per cent. manganese bronze this tendency is little marked and may be effectively guarded against by suitable precautions when casting. The amount of tin to be added was provisionally fixed from Heusler's observations.|| From these it appeared that the interesting alloys would be those with 15 or 18 per cent. of tin. It was found, however, that alloys much richer in tin were more magnetic, and hence the series finally prepared contained alloys with from 14 to 50 per cent.

Materials Employed.—The materials employed were the purest specimens obtainable. The copper was a high-grade electrolytic copper, and any traces of impurities must have been extremely small, and of negligible influence. The tin, obtained from Kahlbaum, was of a similar high degree of purity. The manganese metal contained about 98·5 per cent. of pure manganese, the remainder consisting chiefly of silicon, iron, and aluminium in diminishing proportions.

Preparation of the 30 per cent. Manganese Bronze.—The preparation of the 30 per cent. manganese copper alloy was carried out in the following manner:—The copper was first melted in a "Salamander" crucible under a layer of barium chloride. The proper amount of manganese was then added, and the heating continued until the metal was completely dissolved. The crucible was now removed from the furnace, the contents carefully stirred, and, while the metal cooled to the desired casting temperature, the surface was skimmed. The alloy was poured into a series of dry sand moulds which

* Melting points:—Manganese 1260°, copper 1084°, tin 227° C.

† S. Schemtschuschny, G. Urasow, and A. Rykowski, *Jour. Russ. Phys. Chem. Soc.*, xxxviii., 1050 (1906); *Zeit. f. anorg. Chem.*, lvii., 253 (1908).

‡ R. Sahmen, *Zeit. f. anorg. Chem.*, lvii., 1 (1908).

§ K. Bornemann, *Metallurgie*, vi., 329 (1909).

|| F. Heusler, *Schriften Naturf. Ges. Marburg*, xiii. [5], 266 (1904

had been lined with whiting. The castings were in the form of cylindrical rods about one inch in diameter, and when cold were removed from the moulds and sawn up into short lengths for use in the subsequent castings.

Preparation of the Ternary Alloys.—A similar procedure was adopted in the preparation of the ternary alloys. The cupro-manganese was first melted and the tin added to it. Some difficulty was experienced in getting satisfactory castings.

If the alloys are allowed to stand before casting there is risk of segregation commencing. On the other hand, if the stirring is continued up to the time of casting, a porous structure may result. When the appearance of the alloy or the chemical tests described below indicated any deviation from homogeneity, the material was again melted up and recast. In view of the apparatus to be employed for the magnetic measurements, the castings were made in the form of rods about nine inches long and a little under half an inch in diameter. The preparation of these alloys was carried out by the authors at the works of Messrs Gray & Caldwell, brassfounders, Paisley.

Chemical Analysis.—While it was found that there was very little loss in the castings, full chemical analysis was made of each alloy. This was considered advisable, as it gave the strongest proof of the homogeneity of the material. The analyses were made on portions of the actual specimens used in the principal tests. Two small portions were selected from different parts of the rod and examined gravimetrically. The tin was estimated in the form of stannic oxide and the copper as copper sulphide. The filtrate from which the tin and copper had been removed was made alkaline with ammonia, and the hydrated peroxide of manganese precipitated by the addition of bromine water. The precipitate was afterwards converted into Mn_3O_4 and the manganese estimated as tetroxide.

Programme of the Magnetic Tests.—The scheme of tests was as follows:—

1. To determine the magnetic quality of the alloys in the condition as cast.

2. To find the simplest and most efficient heat treatment for bringing the alloys into a stable condition with the highest possible magnetic quality.

3. To investigate the changes produced in the magnetic properties by exposure of the alloys to different temperatures, and incidentally to determine the critical temperature and investigate the reversibility of the effects due to the thermal treatment.

4. To examine the effects produced by quenching, re-annealing, and baking the alloys.

Apparatus.—The tests were performed on a Gray-Ross magnetometer.* For the measurement of the magnetic quality at temperatures above the normal, the specimens were placed in an electric furnace of the newest type † for use within the magnetising solenoid. By this means the alloys were raised to temperatures approaching a red heat without danger of oxidation. The temperatures employed in the tests were measured by two thermo-elements consisting of wires of platinum and platinum-iridium enclosed in tubes of fused quartz. The instruments were calibrated before and after the tests by determination of five freezing and boiling points, and were found to have remained unchanged during the interval.

Composition of the Alloys.—The alloys employed were found on subsequent analysis to have the compositions given in Table I. The first column gives the approximate tin content of the alloys, and the various materials

Table I.—Analyses of the Alloys.

Designation of the Alloy.	Percentage Composition.		
	Cu.	Mn.	Sn.
14 per cent. tin	60·2	25·1	14·1
16 ,, ,, ...	58·7	24·5	16·1
18 ,, ,, ...	57·6	23·9	18·0
30 ,, ,, ...	49·2	20·4	29·8
38 ,, ,, ...	43·4	18·1	38·0
48 ,, ,, ...	36·4	15·0	48·1

will hereafter be designated by these names. The amount of iron present in the alloys varies with the manganese content, being introduced as an impurity in the manganese. Its amount is always small, never exceeding a half per cent.

Tests on the Alloys as Cast.—The rods of the alloys in the form of cylinders measuring $20 \times 1·15$ cms. were first of all tested in the condition as cast. No preliminary dressing operations were performed with the exception of the removal of the end portions. As the metal was hard and brittle, this could be done by a tap with a hammer and chisel, the specimen being supported on a second chisel at the point where the fracture was desired. The alloys broke with a clean fracture and the metal was white and lustrous. Table II. gives in outline the results of the magnetic tests, on a specimen of each of the various alloys.

* J. G. Gray and A. D. Ross, *Proc. Roy. Soc. Edin.*, xxix., 182 (1909).

† J. G. Gray and A. D. Ross, *Proc. Roy. Phil. Soc. Glasg.*, xli., 84 (1910).

Table II.—The Alloys as Cast.

Test.	Alloy.					
	14 per cent. tin.	16 per cent. tin.	18 per cent. tin.	30 per cent. tin.	38 per cent. tin.	48 per cent. tin.
I_{20}	17	20	21	?	21	?
I_{50}	35	43	45	?	53	0·4
I_{100}	54	61	66	?	95	1·0
I_{200}	70	80	80	0·7	159	1·9
I_{300}	78	88	88	1·0	208	2·8
I_r	22	13	15	?	11	0·6
C.F.	24	10	14	?	9	ca. 50

The values obtained for the different alloys are set out in six vertical columns. The first five horizontal lines give the intensities of magnetisation, $I_{20}, \ldots, I_{300}$, in c.g.s. units corresponding to effective magnetising fields of 20, 50, 100, 200, and 300 gauss. In calculating these values allowance has been made for the self-demagnetising action of the specimens themselves by employing the factors investigated by Du Bois for iron cylinders.* These corrections have been shown to give satisfactory results when employed in tests on copper-manganese-aluminium alloys.† They will probably apply with like accuracy in the present instance, but in any case the errors introduced in the absolute values given must be small, and quite insignificant in the case of the relative effects which are here of chief importance. The sixth horizontal line in Table II. gives the intensity of the residual magnetisation, I_r, when the specimen is acted on by zero force, that is to say, when the current in the magnetising solenoid is adjusted so as to give a magnetic field equal and opposite to that due to the magnetisation of the specimen. The seventh line exhibits the magnitude of the coercive force, C.F., in c.g.s. units. As this quantity and the residual magnetisation depend to a considerable extent on the maximum field employed in testing, they are taken here and in the subsequent tables to be the values obtained in a cyclic variation of the true effective field between the limits ± 200 gauss. It will be seen that the saturation values of I vary somewhat irregularly with the composition. Thus we have evidence of two groups of magnetic alloys, one group containing the alloys with 14–18 per cent. tin and the other group having the 30 per cent. alloy as a member. The magnetism of the 30 per cent. and 48 per cent. tin alloys is so feeble that the values given in the table must be taken merely as approximations.

* H. du Bois, *Ann. d. Phys. u. Chem.*, xlvi., 485 (1892).
† A. D. Ross, *Proc. Roy. Soc. Edin.*, xxvii., 88 (1907).

Improvement of the Magnetic Quality.—Before proceeding with any thermal treatment which might have a permanent effect on the materials, it was thought desirable to make some tests which could be used later to ascertain whether the heat treatment had probably resulted in change of constitution. The most suitable test for this purpose was to reinvestigate the magnetism of the alloys when they were cooled to the temperature of liquid air. Such a procedure seldom has any influence in altering the constitution of alloys. In cases where transformations do occur between +15° and −190° C., their velocity is so slow that the magnetic test can be carried out and the specimen brought back to room temperature before the constitution has been sensibly affected.

Liquid Air Test on the Cast Specimens.—When the alloys were cooled to liquid air temperature and tested thereat in the usual manner,* the susceptibility showed in most cases a more or less marked increase. The retentivity and coercive force were altered in a like manner. Table III. gives the percentage changes produced in the values of I and C.F. by the cooling.

Table III.—Liquid Air Effects.

Test.	Percentage Change.					
	Alloy.					
	14 per cent. tin.	16 per cent. tin.	18 per cent. tin.	30 per cent. tin.	38 per cent. tin.	48 per cent. tin.
I_{20}	+15	+ 5	+15	...	0	...
I_{50}	+17	+12	+20	...	+ 2	...
I_{100}	+21	+26	+35	...	+ 1	...
I_{200}	+29	+34	+60	...	0	...
I_{300}	+33	+37	+65	+	0	0
I_r	+45	+60	+35	...	0	...
C.F.	+20	+30	+15	...	+10	+

The 38 per cent. and 48 per cent. alloys showed almost no change in susceptibility due to the change in temperature, but the coercive force and consequently the hysteresis loss were increased. In employing the values given in this table one must bear in mind that the changes in I are small when expressed in c.g.s. units of intensity. Hence the percentages may in some cases—especially with the lower fields—be several units in error for the less magnetic alloys. With the 30 and 48 per cent. alloys nothing

* J. G. Gray and A. D. Ross, *Proc. Roy. Soc. Edin.*, xxix., 182 (1909).

further has been attempted than to show the probable nature of the change, if any. In stating the alterations in retentivity and coercive force (which are in all cases comparatively small quantities) the results have been given in round figures, viz. 5, 10, 15, etc. The effects produced by the cooling to $-190°$ C. were temporary. When room temperature was regained the specimens gave the same magnetisation curves as before. In the case of the 18 per cent. alloy there appeared to be a slight permanent improvement in quality, but it was exceedingly small, and did not recur on a further repetition of the process. This change is of still less importance, as it will be shown that the alloy as cast is in a comparatively unstable condition as regards exposure to temperature variations, and that it may easily be brought to a more stable and magnetically superior condition.

Normalising.—It is well known that iron or steel bars are, as a rule, initially in a state of strain as a result of the casting, forging, or rolling operations which they have undergone in the process of manufacture. To remove such strains it is customary to heat the bar to about a dull red heat and cool it down somewhat slowly. The period during which the metal is maintained at the high temperature is very short. Such a process is termed "normalising," and is to be distinguished from "annealing," in which the maximum temperature is maintained for a much longer period. A further treatment is that which is referred to in magnetic testing as "ageing." In this latter process the metal is maintained for prolonged periods at high temperatures with a resultant deterioration in magnetic quality. The coercive force augments, and the hysteresis loss is thereby much increased. These three processes, which act in the case of iron and steel comparatively slowly and only at somewhat high temperatures, rapidly produce similar effects in the magnetic bronzes even when the temperatures employed are relatively low. To bring the alloys to a "normalised" condition we have therefore to expose them for a very brief time to a moderately low heat, and thence bring them back, not too rapidly, to room temperature. That such treatment is a true normalising of the material and not an annealing or baking process involving structural modification, is shown by the fact that the alloys exhibit precisely the same liquid air effects before and after.* The critical temperature of the material is also found to be unaltered. Moreover, there is no appreciable change in the hysteresis, and it is characteristic of these alloys that anything of the nature of continued

* The 18 per cent. tin alloy is to a slight extent an exception. The increase in susceptibility produced by cooling to $-190°$ C. is subsequently a little less, but the coercive force, critical temperature, etc., have not been altered. Here the thermal treatment seems to have been partly normalising and partly annealing, the alloy being very sensitive to heat effects.

exposure to heat, such as we have in true annealing, results in a very rapid growth of hysteresis. It is also evidence of the same fact, that after this normalising the alloys exhibit great regularity of magnetic properties. The process has therefore resulted merely in removing internal strains and differences due to slight variation of conditions during alloying and casting. The alloys were normalised by heating to a temperature of about 180° C., that being found by trial to be the most suitable. The results of the magnetic tests performed after the normalising are given in Table IV. It

Table IV.—The Normalised Alloys.

Test.	Alloy.					
	14 per cent. tin.	16 per cent. tin.	18 per cent. tin.	30 per cent. tin.	38 per cent. tin.	48 per cent. tin.
I_{20}	18	28	24	?	22	?
I_{50}	36	55	52	?	54	0·5
I_{100}	55	77	82	?	96	1·0
I_{200}	72	96	117	0·8	160	2·0
I_{300}	80	105	129	1·2	209	3·0
I_r	21	16	18	?	11	0·5
C.F.	22	10	14	?	9	ca. 40

will be observed that the first three alloys—those which are poor in tin—have been decidedly improved by the treatment. The increase in the saturation value of I is very large in the case of the 18 per cent. tin alloy. The tin rich alloys are little affected. Indeed the 38 per cent. alloy—as will be seen from a comparison of the various tests described in this paper—retains its magnetic properties practically unchanged under widely different conditions. Fig. 1 shows how the normalised 38 per cent. tin alloy—the most magnetic of those tested—compares with other magnetic materials. The four comparison curves are those obtained in the case of cast iron, cobalt, nickel, and the Heusler alloy. It will be at once observed that the copper-manganese-tin alloy does not show distinctly the characteristic shape which one associates with the majority of ferromagnetic substances. In this alloy the susceptibility does not vary so markedly with the field intensity as it does in the case of iron, nickel, or the Heusler alloy. There is even an indication that under the action of very strong magnetising force the intensity of magnetisation might exceed that of the Heusler alloy. This, however, is a feature only of the 38 per cent. tin alloy and the very feebly magnetic 30 and 48 per cent. tin alloys. The alloys with a tin content of 14, 16, and 18 per cent. all show saturation phenomenon. This is illustrated in fig. 2,

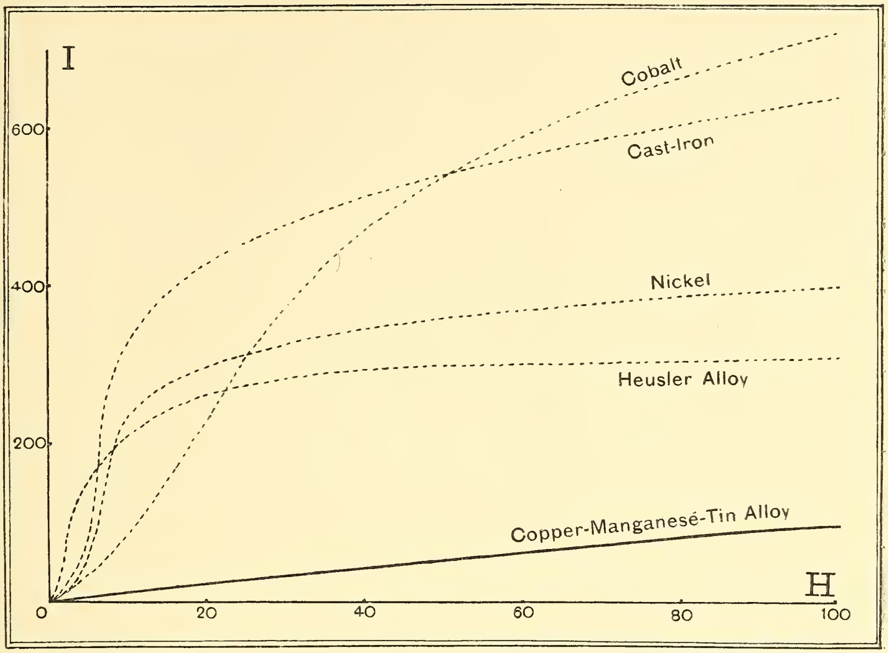


Fig. 1.—I-H curves for various materials.

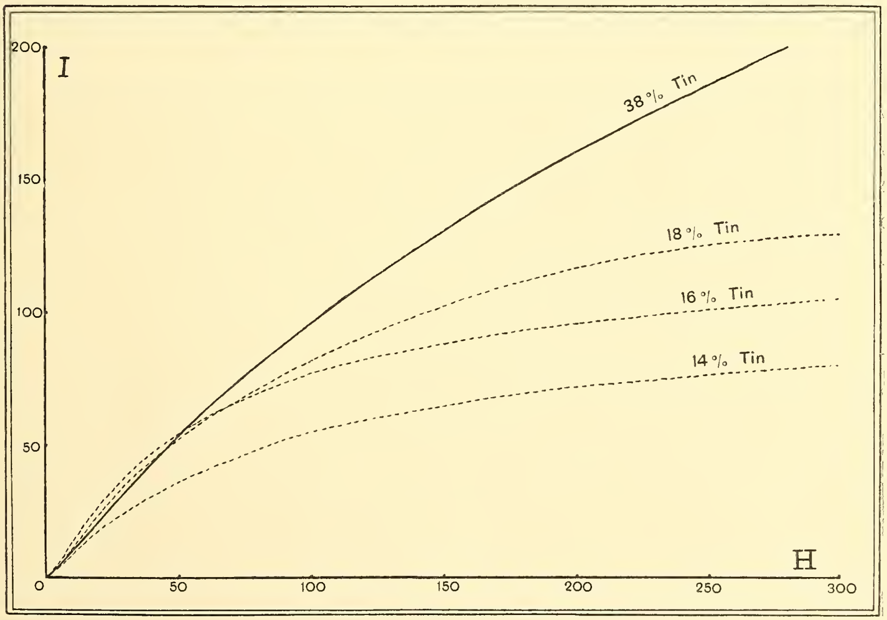


Fig. 2.—I-H curves for Copper-manganese-tin Alloys.

where the magnetisation curves of the four magnetic alloys are drawn to the same scale. Not only, then, are the magnetic 14–18 per cent. tin alloys separated from the strongly magnetic 38 per cent. alloy by comparatively non-magnetic alloys about 30 per cent., but there is a noteworthy dissimilarity in the general magnetic properties of the two divisions.

Effect of Baking.—Investigations were carried out with a view to ascertaining what thermal treatment brought the alloys into the condition of maximum susceptibility. The methods found most suitable vary somewhat with the alloy, but all involve exposure for a longer period to a temperature at least as high as that used in the normalising process. Table V. gives the treatment resulting in maximum susceptibility for a

Table V.—Effect of Baking the Alloys.

Alloy.	Thermal Treatment.	Percentage Increase in I_{200}.	Percentage Increase in Hysteresis Loss.
14 per cent. tin.	7 hours at 200° C.	+3	+10
16 ,,	5 ,, 200°	+3	+15
18 ,,	6 ,, 180°	+2	+10
38 ,,	4 ,, 180°	+	+ 5

magnetising field of 200 units. The efficiency of the procedure is indicated by the percentage increase in I over the value obtained in the normalised specimen. The last column shows the augmentation of hysteresis loss brought about by the process, and shows that, from a practical point of view, the alloys are now inferior in quality. Hence in all the subsequent tests described in this paper the material used was the normalised alloy, exhibiting as it does the most satisfactory magnetic condition. The effect of prolonged baking at moderate temperatures was investigated in the case of the alloy with 18 per cent. of tin. The change produced was very

Table VI.—Baked 18 per cent. Tin Alloy.

Condition.	Percentage Change produced by Cooling to −190° C.						
	I_{20}.	I_{50}.	I_{100}.	I_{200}.	I_{300}.	I_r.	C.F.
Normalised	+15	+20	+35	+60	+65	+35	+15
Baked, 24 hours at 185° C. . .	+ 5	+10	+20	+35	+37	+20	+10

similar to that found by the authors in the case of copper-manganese-aluminium alloys, namely, a steady diminution in susceptibility towards a limiting value for the temperature employed, and an increased coercive

TABLE VII.—TESTS AT VARIOUS TEMPERATURES.

Alloy.	Test.	Temperature of Testing.					
		−190°.	15°.	75°.	125°.	180°.	230° C.
14 per cent. tin	I_{20}	21	18	15	11	8	2
	I_{50}	42	36	30	26	19	3
	I_{100}	67	55	46	37	26	4
	I_{200}	93	72	60	46	34	5
	I_{300}	108	80	67	50	36	5
	I_r	31	21	18	13	8	1
	C.F.	27	22	21	20	18	9
16 per cent. tin	I_{20}	30	28	24	20	11	4
	I_{50}	62	55	46	37	19	7
	I_{100}	96	77	63	48	27	10
	I_{200}	127	96	76	58	31	12
	I_{300}	144	105	84	61	33	13
	I_r	26	16	14	10	5	2
	C.F.	12	10	9	8	7	5
18 per cent. tin	I_{20}	27	24	18	12	6	2
	I_{50}	59	52	36	23	13	4
	I_{100}	102	82	59	35	19	5
	I_{200}	162	117	82	49	27	9
	I_{300}	195	129	89	54	29	11
	I_r	23	18	13	9	5	1
	C.F.	16	14	13	12	10	4
38 per cent. tin	I_{20}	22	22	22	21	21	0
	I_{50}	54	54	52	50	47	0
	I_{100}	96	96	92	88	83	0
	I_{200}	160	160	153	142	130	0
	I_{300}	209	209	201	184	160	0
	I_r	11	11	11	10	8	0
	C.F.	10	9	9	8	7	0

force and hysteresis loss tending towards definite large values of these quantities.* The liquid air effect on the baked material was also considerably different from that on the normalised alloy. This will be seen from Table VI.

Tests on Heating to various Temperatures.—Tests were also carried out on the alloys when the specimens were maintained at various tempera-

* A. D. Ross and R. C. Gray, *Proc. Roy. Soc. Edin.*, xxix., 274 (1909).

tures between the normal room temperature and the critical temperature. In all cases the alloys showed less susceptibility, retentivity, and coercive force at these higher temperatures. This was perhaps to be anticipated from the enhanced susceptibility found in bars cooled in liquid air. In carrying out these investigations the normalised specimen was placed in an electric furnace within the solenoid, raised to the desired temperature, submitted to the action of an alternating field of intensity gradually diminishing to zero (being thereby rendered devoid of previous magnetic history), and put through a hysteresis cycle. The current in the electric furnace was then altered so as to give a new higher temperature, and the process was repeated. Table VII. gives the results obtained in these tests. [The 30 and 48 per cent. tin alloys being so feebly magnetic were not tested.] The values obtained in the tests at $-190°$ C. have been included in the table, and in this way the variation of magnetic quality over the whole range of over 400° C. is shown. It will be observed that the susceptibility of the 38 per cent. tin alloy is fairly constant through the wide variation of temperature from $-190°$ up to about 180° C., but that it very rapidly falls off after this point has been reached. The magnetism of the other alloys diminishes much more evenly with increase of temperature.

Critical Temperatures.—In the course of the furnace tests the critical temperatures of the alloys were ascertained and are set out in Table VIII.

Table VIII.—Critical Temperature.

Alloy.	Critical Temperature.
14 per cent. tin	275° C.
16 "	270°
18 "	255°
38 "	225°

The temperatures of transformation from the magnetic to the non-magnetic state and *vice versa* were found to be practical identical in all cases. Table VIII. shows that the variation of critical temperature with composition is much smaller in the case of these ternary alloys than in the copper-manganese-aluminium system.*

Tests at various Temperatures while Cooling.—After the determination of the critical temperatures, the specimen was gradually brought back to room temperature, tests being carried out again at different stages during

* F. Heusler and F. Richarz, *Zeit. anorg. Chem.*, lxi., 265 (1901); and A. D. Ross and R. C. Gray, *Proc. Roy. Phil. Soc. Glasg.*, lv., 94 (1909).

the process. Little difference was found between the susceptibility corresponding to a given field for any temperature during the heating and that for the same temperature during the cooling, provided the interval between the two determinations was small. If, however, the specimen had been kept heated for some considerable time, a deterioration in quality was observed. Its amount was small, but increased with the time which had elapsed and the temperature employed. A remarkable effect was, however, noted in the case of the 38 per cent. tin alloy which exhibited such constancy of magnetic properties over the wide range from $-190°$ to $180°$ C. If heated up to 225° C. the alloy becomes non-magnetic, and if the heating be continued up to about 330° C. the magnetic quality is not regained on cooling.

Quenching.—The effects of quenching the alloys were next investigated. Two temperatures of quenching were employed, one about 350° and the other about 580° C. The specimens to be experimented upon were raised to the desired temperature in the furnace, and then dropped vertically into cold water. Table IX. gives the results of the tests carried out on the quenched material at 15° and $-190°$ C. with the corresponding values for the normalised condition for the sake of comparison. The effects of quenching are somewhat complex, but in all cases the coercive force is less in the quenched than in the unquenched materials, and there is generally a much greater increase in susceptibility on cooling to $-190°$ C. In the case of the quenching at 580° C. the resulting material is in all cases less magnetic than before. Quenching at 350° slightly improves the alloy with least tin, but the others undergo a deterioration which is larger the greater the tin content. In the 14 and 16 per cent. alloys the material is the less magnetic the higher the temperature of quenching. On the other hand, the 18 and 38 per cent. alloys are more magnetic after quenching at 580° C. than after quenching at 350°. This is similar to the effect observed in a copper-manganese-aluminium alloy which was rendered feebly magnetic by quenching at 610°, but which acquired greater permeability upon subsequent quenching about 700° C.*

Instability of the Quenched Alloy.—In the case of a specimen of the 16 per cent. tin alloy which was quenched at about 580° the value of I_{200} was reduced to about 44 per cent. of its initial value. In the course of two months it was found that this had increased from 44 to nearly 46 per cent., thereby showing that the quenched material undergoes slow transformation even at ordinary room temperature.

* A. D. Ross, *Proc. Roy. Soc. Edin.*, xxvii., 88 (1907).

Annealing of the Quenched Material.—The reannealing of the material was tried. No procedure, however, would reverse completely the deteriorating effect which had been produced in the magnetic quality by the quenching at 580°. Reannealing had little influence in restoring the coercive force to its initial value unless the duration of the annealing process was considerable, and then it is probable that the apparent restoration was really due to ageing of the material. Reannealing of material which had been quenched at 350° with a resultant increase in its susceptibility led to very complex effects. It may be stated, however, that when ageing or baking

Table IX.—Tests on the Quenched Material.

Alloy.	Test.	Normalised Material.		Quenched ~ 350° C.		Quenched ~ 580° C.	
		15°.	- 190°.	15°.	- 190°.	15°.	- 190°.
14 per cent. tin	I_{20}	18	21	17	16	0	0
	I_{50}	36	42	35	40	0	0
	I_{100}	55	67	55	69	0	0
	I_{200}	72	93	72	97	0	0
	I_{300}	80	108	81	112	0	0
	I_r	21	31	21	31	0	0
	C.F.	22	27	18	24	0	0
16 per cent. tin	I_{20}	28	30	28	30	11	12
	I_{50}	55	62	53	63	22	27
	I_{100}	77	96	73	97	32	41
	I_{200}	96	127	91	129	42	59
	I_{300}	105	144	100	146	47	70
	I_r	16	26	15	24	4	8
	C.F.	10	12	10	13	8	13
18 per cent. tin	I_{20}	24	27	21	23	22	12
	I_{50}	52	59	45	53	49	28
	I_{100}	82	102	65	88	77	57
	I_{200}	117	162	81	129	98	107
	I_{300}	129	195	89	146	111	132
	I_r	18	23	9	17	9	18
	C.F.	14	16	6	12	6	22
38 per cent. tin	I_{20}	22	22	6	7	13	15
	I_{50}	54	54	12	14	30	34
	I_{100}	96	96	21	23	51	56
	I_{200}	160	160	33	36	78	85
	I_{300}	209	209	42	45	97	104
	I_r	11	11	3	4	9	10
	C.F.	9	10	6	7	7	7

effects were not superposed, or when an allowance was made for their occurrence, the susceptibility for high fields was lowered almost to its original value, and that for low or moderate fields was reduced to a value equal to or less than the initial amount.

Origin of the Magnetic Properties.—The authors have already suggested that the magnetic properties of the Heusler alloys of copper-manganese-aluminium are to be ascribed to the formation of a series of solid solutions of which the binary compound Cu_3Al is a constituent. A similar explanation may hold in the present instance, and it is proposed to discuss these hypotheses in a subsequent paper when an examination of the micro-structure of polished samples of the alloys has been concluded. The ternary system discussed in the present paper is more complex than the $Cu + Mn + Al$ system, in so far as it has been shown that the magnetic members fall into at least two distinct divisions. It is interesting to note in this connection that manganese and tin form two magnetic compounds Mn_4Sn and Mn_2Sn,* whereas manganese and aluminium give only one magnetic compound Mn_3Al.†

Summary.

1. Alloys were prepared containing 14, 16, 18, 30, 38, and 48 per cent. of tin, and the remainder copper and manganese in the ratio 7 : 3.

2. Tests were carried out at different temperatures on the materials as cast, normalised, annealed, and quenched.

3. At 15° C. in the normalised condition the six alloys gave I respectively equal to 55, 77, 82, 0·4, 96, and 1, for $H = 100$.

4. When cooled to $-190°$ C. the three alloys poor in tin were much more magnetic.

5. Annealing produced no decided improvement.

6. The susceptibility diminishes with increase of temperature, and the critical temperatures range from 225° to 275° C.

7. Quenching produces complex changes in the susceptibility. The quenched materials have less coercive force and exhibit greater improvement on cooling to $-190°$ C.

8. Reannealing tends to restore the initial susceptibility of the quenched material, but has little influence on the coercive force.

9. The magnetic properties are probably due to the formation of solid solutions of certain definite concentrations.

* R. S. Williams, *Zeit. f. anorg. Chem.*, lv., 1 (1907).

† G. Hindricks, *Zeit. f. anorg. Chem.*, lix., 414 (1908).

(*Issued separately December* 22, 1910.)

VI.—Abnormal Bone Growth in the absence of Functioning Testicles. By A. C. Geddes, M.D., Professor of Anatomy at the Royal College of Surgeons in Ireland. (With Three Plates.)

(Read January 18, 1909, under the title of "The Influence of Castration upon Bone Growth." MS. received June 3, 1910.)

Introductory.

At the instigation of the late Professor D. J. Cunningham, and at first working upon his material, the present writer has reinvestigated the problems of giantism and acromegaly.

In the course of the work a group of facts persistently presented themselves which appeared to indicate the existence of a definite antagonism between growth and sexual activity. Side by side with these, and in apparent contradiction of them, was the undoubted fact that the state of pregnancy does not in the young female impede growth, but, on the contrary, favours it. These are not new discoveries. The fact that sexual activity is inimical to growth has not infrequently been demonstrated, more especially by Pirsche (1), Launois and Roy (2) Poncet (3), and Lortet (4). Indeed Pirsche, in the *Thèse de Lyon*, "De l'Influence de la Castration sur le Développement du Squelette," 1902, might almost be said to have established beyond reach of question the fact that castration in youth is followed by abnormal growth of the long bones. Apparently he did not do so, for Professor Hunt Morgan (5), in his book *Experimental Zoology*, 1907, quotes, with approval from Minot (6), as follows:—"It has been asserted by Carpenter, Spencer, and others that the functions of nutrition and reproduction are in principle opposed to one another, because reproduction makes such a demand upon the parent for material that the supply of nutrition and growth of the parent is lessened." Professor Morgan then proceeds:—"Unfortunately for this philosophic generalisation, the premises are wrong: the growing animal is not growing at its maximum of assimilative power. Spencer's 'dogmatic assertions' concerning the opposition of growth and reproduction are open to justly severe criticism."

Any subject upon which such divergent views are expressed by responsible writers is clearly in need of reinvestigation.

In the course of this reinvestigation some facts which appear to be new have been ascertained.

Although the work here reported was primarily undertaken in connection with a general study of giantism, the scope of the present communication is limited to a consideration of those cases of abnormal bone growth which are associated directly with the removal by operation or disease, or with the developmental failure, of functioning testicles. Giantism as a whole is not discussed, neither is physiological giantism, nor any of the numerous class of cases of giantism associated with other disturbances of the metabolism of the body.

The problems which it has been attempted to solve are—

1. What disturbances occur in the orderly development of the skeleton in men and animals who have been deprived of functioning sexual glands by developmental error, operation, or disease?

2. What is the cytological process by which these alterations are brought about?

3. What is the explanation of their varying manifestations?

The record of the search for definite answers to these questions has divided itself naturally into three parts:—

1. Anatomical and Anthropological.
2. Histological and Physiological.
3. Anatomical and Anthropological (*continued*).

These are the titles of the three principal parts of this communication.

SECTION I.—ANATOMICAL AND ANTHROPOLOGICAL.

Observation 1.

Description of the Body of an Anorchid dissected at the Royal College of Surgeons in Ireland.

Subject No. A 9.—*Age* 28. Male (?). *Cause of death*—Pulmonary tuberculosis. *Stature* (measured on autopsy table)—1732 mm.

General Description.—The subject was thin and ill-proportioned. His limbs were long, his trunk short. Within the trunk the thoracic segment was disproportionately long. The cheeks were extremely hollow. The skin was freckled and glabrous. The penis and testicles were small.

Examination of the Body.—The first step in the dissection of the body was to remove the testicles and to have them prepared for microscopic examination.

Thereafter the body was preserved by the intra-vascular injection of formalin 15 per cent. solution.

The examination of the body was divided into three parts:—

A. **Microscopic examination of—**

1. Pituitary body.	6. Kidney.
2. Thyroid gland.	7. Suprarenal.
3. Liver.	8. Prostate.
4. Spleen.	9. Penis.
5. Pancreas.	10. Testicle.

B. **Dissection of the soft tissues.**

C. **Maceration and detailed examination of the bones of the skeleton.**

The following facts were discovered:—

A. Microscopic Examination.

1. The Pituitary.

(*Note.*—Pituitaries obtained from the post-mortem room were used as guides in determining the nature and extent of the structural alterations.)

The size of the organ was in no way abnormal.

Stain.—Mann's long method.

Anterior Lobe.—The sinusoids were small, and contained an unusually small amount of blood. The dark-staining granular cells were more numerous than usual—in some parts much more numerous. There were one or two large colloid spaces.

Intermediate Zone.—The epithelial cells were few in number, and did not extend so far as is usual into the posterior lobe.

Posterior Lobe.—There was very slight vascularity, but in other respects the structure was similar to that of the posterior lobes of the pituitaries used as controls.

2. The Thyroid Gland.

The amount of colloid slightly exceeded the normal; some of the vesicles were unusually large, and in these the lining cells were flattened, resembling stratified epithelium rather than the normal cubical. There was no other departure from the normal structure.

3. Liver. 4. Spleen. 5. Pancreas. 6. Kidney.

Showed no abnormalities of structure beyond such as could be accounted for by the manner of death, viz. mixed septic infection from tuberculous cavities in the lungs.

7. Suprarenal.

The size was considerably less than normal. The structure was absolutely normal, and the cells were apparently quite healthy.

8. Prostate.

The glandular tissue was extremely small in amount.

9. Penis.

The structure was normal.

10. Testicle.

The fibrous tissue was excessive in amount. The straight tubules were normal. The seminiferous tubules were absent, but were represented by groups of epithelial cells lying in the meshes of the fibrous tissue, and here and there by small ducts lined with a simple stratified or columnar epithelium. No healthy tubules were discovered. No spermatozoa were seen.

Summary and Interpretation of Microscopic Observations.

The testicle was functionless. The pituitary and thyroid glands showed departures from the normal, which were interpreted as indicative of an increased activity. This was deduced from the increase in the number of the granular cells in the anterior lobe of the pituitary, and the increase in the amount of colloid in the thyroid. The suprarenal was normal. The condition of the prostate was regarded as a direct result of the failure of the testicles.

B. **Dissection.**

Beyond the small size of the testicles and the small size of the suprarenals, no structural abnormalities of importance were noted, except that there was a very imperfect development of the anterior part of the antrum of Highmore. This caused a deep concavity on the facial surface of the superior maxilla (*vide infra*—Description of Skeleton). Another interesting abnormality, but probably in no way connected with the general condition, was that the left nasal duct opened into the middle meatus of the nose (7).

C. **Description of the Skeleton.**

1. Skull.

The skull of subject A 9, when viewed from the front, is remarkable for the depth of the concavity of the facial surface of the superior maxilla, and for the way in which the roots of the teeth are exposed on the outer aspect of the alveolar process. These two peculiarities are associated with a failure of the anterior part of the antrum of Highmore to develop. When the skull is studied in section it is found that the other air-sinuses are well developed.

The measurements of the skull are given in Table I.

Table I.—Principal Skull Measurement and Indices.

	Subject A 9.	Average of Seven Irish Skulls (Cunningham) (8).
	mm.	mm.
Glabello-occipital length	174	189
Maximum breadth	135	143
Cephalic index	*77*	*75·7*
Horizontal circumference	504	526
Length of foramen magnum	32	36·7
Basi-nasal length	97	103
Basi-alveolar length	93	98·6
Gnathic index	*95·8*	*95·6*
Interzygomatic breadth	122	131
Intermalar breadth	100	111·1
Naso-alveolar length	68	73·4
Facial index	*55·7*	*56*
Nasal height	*36*	*53·7*
Orbital width	37	41
Orbital height	33	35·1
Orbital index	*89*	*85·3*
Naso-mental length	*114*	*120*

Table I. (*continued*).—Dimensions of Mandible.

	Subject A 9.	Average of Three Irish Mandibles.
	mm.	mm.
Symphysial height	34	34
Coronoid height	70	67
Condyloid height	66	70
Gonio-mental length	89	95
Gonio-alveolar length	85	91
Intergonial width	85	100
Breadth of ascending ramus	26	33

A comparison of the measurements of the skull of subject A 9 with the measurements of seven Irish skulls shows that it is small but well-proportioned. The most striking departure from the normal is the extreme shortness of the intermalar breadth.

This is due to the small development of the maxillary antrum, which also reduces the naso-alveolar length.

A comparison of the naso-alveolar length with stature makes this obvious. The figures of this comparison are given in Table II.

Table II.—Naso-alveolar Length compared with Stature.
Stature = 100.

	Naso-alveolar Length.	Stature.	Index.
	mm.	mm.	
Subject A 9 . . .	68	1732	3·9
Seven Irishmen . .	73·4	1710	4·26

This shows that not only was the superior maxilla absolutely smaller than usual in A 9, but also that it was relatively far smaller.

It is interesting to discover in which part of the superior maxilla the growth failure was most marked. This can be sufficiently accurately determined by comparing the nasal height and the naso-alveolar length.

Table III.—Nasal Height compared with Naso-alveolar Length.
Naso-Alveolar Length = 100.

	Nasal Height.	Naso-Alveolar Length.	Index.
	mm.	mm.	
Subject A 9 . . .	45	68	66·1
Seven Irish skulls . .	53·7	73·4	73·1

The lower the index in this table, the greater is the relative depth of the alveolar process of the superior maxilla. The low index, 66·1, shows that in subject A 9 the alveolar process is very deep. But the alveolar process forms not only relatively a large part of the superior maxilla, it is also absolutely much deeper than in the normal Irish skull, for the measurement in A 9 is 23 mm., whereas the average of seven normal skulls is 19·7 mm.

This indicates that in the superior maxilla of A 9 there was a partial failure in the development of the body of the bone, and simultaneously some overgrowth of the alveolar process.

In the mandible the most striking departures from the normal proportions are the relatively great symphysial depth and coronoid height.

The preponderance of the mandibular element in the facial height is shown by comparing the symphysial height with the naso-alveolar length.

TABLE IV.—MANDIBULAR SYMPHYSIAL HEIGHT COMPARED WITH NASO-ALVEOLAR HEIGHT.

Naso-Alveolar Height = 100.

	Symphysial Height.	Naso-Alveolar Height.	Index.
	mm.	mm.	
Subject A 9 . . .	34	68	50
Three Irish mandibles .	34	73·4	46·5

The shortness of the naso-alveolar height in subject A 9 accentuates the symphysial massiveness. Its amount is more truly indicated in the following tables, in which the condylar height is compared with the cranial circumference, and then with the symphysial height.

TABLE V.—CONDYLAR HEIGHT COMPARED WITH CRANIAL CIRCUMFERENCE.

Cranial Circumference = 100.

	Condylar Height.	Cranial Circumference.	Index.
	mm.	mm.	
Subject A 9 . . .	66	504	13
Three Irish mandibles .	70	526	13·3

TABLE VI.—SYMPHYSIAL HEIGHT COMPARED WITH CONDYLAR HEIGHT.

Condylar Height = 100.

	Symphysial Height.	Condylar Height.	Index.
	mm.	mm.	
Subject A 9 . . .	34	66	51·5
Three Irish mandibles .	34	70	48·5

Table V. shows that in relation to cranial circumference the condylar height of the mandible of A 9 is normal; Table VI., that in relation to condylar height the symphysis of A 9 is of unusual depth.

Summary of Analysis of Skull Measurements.

The skull is smaller than the average Irish skull, although the stature of the individual is slightly greater than the average stature. Its proportions are, however, nearly normal, the chief points in which it diverges from the average condition being—

1. The small size of the body of the superior maxilla.

2. The unusually great size of the alveolar process of the superior maxilla.

3. The relatively large size of the mandibular symphysis.

4. The great length of the coronoid process.

It is interesting to note that, the condition of the body of the superior maxilla excepted, the skulls of many giants show strictly analogous but much more intense departures from the normal. In giants' skulls the development of the antrum of Highmore is often enormous (8, 9).

2. Vertebral Column.

The usual number of vertebræ are present. There is no disturbance of the usual vertebral formula, but the thoracic segment is unusually long. This is shown by the measurements in Table VII. and the percentage proportions of the segments of the column in Table VIII.

Table VII.—Measurements of Vertebral Column and Segments of Vertebral Column in Subject A 9.

Total Length of Column, excluding sacrum . . .	596 mm.	Length of Cervical Segment .	120 mm.
		„ „ Thoracic „ .	291 mm.
		„ „ Lumbar „ .	185 mm.
Total .	596 mm.	Total .	596 mm.

Table VIII.—Proportions of the Different Segments of the Vertebral Column.
Length of Spine = 100.

	Spine of Subject No. A 9.	Average of Six Irish Spines (10).
Cervical Segment . .	20·1	21·8
Thoracic Segment . .	48·8	46·5
Lumbar Segment . .	31·1	31·7

The cervical segment, therefore, is relatively considerably shorter, the lumbar segment slightly shorter, and the thoracic segment considerably longer than is normal in Irishmen.

That a similar departure from the normal proportions is found in the vertebral columns of giants was shown by the late Professor D. J. Cunningham (8).

The proportionate lengths of the regional segments of the vertebral columns of the giants Magrath and Byrne are shown in Table IX.

Table IX.—Percentage Proportions of the Segments of the Vertebral Column in the Giants Magrath and Byrne (8).

	Magrath.	Byrne.
Cervical Segment . .	19·3	21·4
Thoracic Segment . .	51	52·2
Lumbar Segment . .	29·7	26·4

These sets of indices show that these giants' vertebral columns and the column of subject A 9 are marked by similar departures from the normal, and that in the case of the giants the departure was greater.

It is particularly noteworthy that this "giant" type of vertebral column is not a reproduction or persistence of a fœtal or infantile condition. The age-changes in the percentage lengths of the segments of the normal column are shown in the following table.

Table X.—Percentage Lengths of the Cervical Thoracic and Lumbar Regions (11).

	New-born Child.	Five-year-old Child.	Adult (German).
Cervical Segment . .	25·6	20·3	22·1
Thoracic Segment . .	47·5	45·6	46·6
Lumbar Segment . .	26·8	34·2	31·3

Not only is A 9's column unusual in its proportions, but also in its curvatures. This is well shown in fig. 1. The lower part of the lumbar curve is well marked; the upper part entirely absent. The thoracic segment is inclined forward, and is almost straight. The cervical segment is placed vertically, and is straight. The cause of the posture of the thoracic segment is clearly to be associated with the shape of the lowest (11th and 12th) vertebral bodies. These are wedge-shaped, being much deeper behind than in front. The extent of the difference between the anterior and posterior vertebral heights of the body of the 11th thoracic vertebra and the corresponding measurements of normal vertebræ are shown in Table XI. (see also fig. 2).

In the skeleton of the giant Magrath, Professor Cunningham found that the thoracic segment of the vertebral column was convex anteriorly (8). The spine under consideration therefore occupies in this respect also a position intermediate between the normal spine and that of the giant.

Table XI.—Anterior and Posterior Heights of Body of Eleventh Thoracic Vertebra.

	A, Anterior Height.	B, Posterior Height.	Index: If A = 100, B =
	mm.	mm.	
11th thoracic vertebra of subject No. A 9 . .	19	29	152·6
Normal 11th thoracic vertebra, average of ten Irish specimens . .	22	24	109

One effect of the position of the column must have been to throw the centre of gravity of the body forward. This subject is referred to later in connection with the description of the femur.

The ossification of all the vertebræ is not complete.

Cervical Vertebræ— Ossification complete.

Thoracic Vertebræ— 1st and 2nd complete; 3rd, 4th, and 5th, epiphyses for tips of spines not yet joined.
6th and 7th complete.
8th, 9th, and 10th, epiphyses for spines not yet joined.
11th complete.
12th, epiphysis for spine not yet joined.

Lumbar— 1st and 2nd, epiphyses for spines not yet joined.
3rd and 4th, epiphysial plates for upper and lower surfaces of bodies not yet completely joined.
5th, epiphysis for spine not yet joined.

3. Thorax.

The sternum is a well-proportioned bone. Its total length is 163 mm., or 9·4 per cent. of the stature; the usual percentage is 9·5 (12).

The ribs are well formed and of normal shape. Their anterior extremities are not disproportionately large, as is the case in some giants. They are, in fact, rather slender in proportion to their length. This is well shown by the comparative figures in Table XII.

In this table the length of the rib, measured with the steel tape round its convex border, is compared with the antero-posterior breadth of its anterior extremity, measured with calipers.

Table XII.—Comparison of Length with Breadth of First Rib.

	A, Length of Rib.	B, Breadth of Rib.	Index: If A=100, B=
	mm.	mm.	
First rib of subject A 9 .	133	20	15
Normal first rib (average of ten Irish specimens)	120	19	15·8

The first rib is therefore slightly disproportionately narrow at its anterior end.

The ossification of the ribs is not quite complete.

In all the ribs from the first to the eleventh inclusive the epiphysis for the head is not completely joined. In the fifth, sixth, and seventh ribs the junction of the epiphysis for the articular part of the tubercle is extremely recent, the line of junction being still visible.

4. Pelvis.

The pelvis is large, but well-proportioned.

Before maceration a most striking feature was the great thickness of

Table XIII.—Dimensions of the Pelvis.

	Subject A 9.	Normal (13).
	mm.	mm.
Breadth	288	279*
Height	227	220*
Breadth-height index	*78*	*79*
Between anterior superior spines	233	231*
Between posterior superior spines	73	...
Between ischial tubera	126	...
Greatest diameter of cotyloid cavity	61	...
Transverse diameter of obturator foramen . . .	52	...
Vertical diameter of obturator foramen . . .	35	...
Obturator index	*67·3*	...
Transverse diameter of brim	132	130*
Conjugate diameter of brim	111	104*
Pelvic index	*84*	*80*
Depth of symphysis pubis	43	...
Depth of pelvic cavity	120	...
Length of sacrum	111	102†
Breadth of sacrum	128	113†
Sacral index	*115*	*111*†
Height of innominate bone	227	196†
Breadth of innominate	170	150†
Height-breadth index	*75·7*	*76*

* Average of sixty-three male pelves, Verneau, *Traité d'Anatomie*, Poirier and Charpy, tome i. p. 216. † Average of ten Irish specimens.

the cartilages at the symphysis and at the sacro-iliac articulations. The measurements given in the accompanying table were made after maceration.

The measurements show—

1. That the pelvis is large in every direction.

2. That the increase in breadth is slightly greater than the increase in length.

3. That the sacrum has increased slightly more in breadth than in length.

4. That the iliac bones are uniformly increased. To the eye they are slightly less convex outward than usual.

The characters of the pelvis of subject A 9, like those of his vertebral column, place him in a position midway between the normal and the giant types, for in giants the pelvis is very large in every direction, but especially in breadth.

In the giant Magrath, Professor Cunningham (8) found a greatly increased sacral breadth to be responsible for a truly enormous increase in the pelvic breadth. Langer (4), in his work on giants, lays particular stress upon their great breadth of hips. Pelvic breadth is indeed one of the most striking characteristics of living giants.

The ossification of the sacrum is incomplete. The lateral epiphysial plates have not yet joined. In the innominate bone the epiphyses for the crest of the ilium, the tuberosity of the ischium, and the symphysis, crest and spine of the pubis have not yet joined.

5. Appendicular Skeleton.

A. *Skeleton of the Upper Limb.*

1. *The Clavicle.*—The clavicle is of unusual shape; its acromial end is very broad. The measurements of the bone and comparative figures are shown in Tables XIV. and XV.

Table XIV.—Length of Clavicle compared with Antero-Posterior Breadth of Acromial End.

Length = 100.

	Length.	Acromial Breadth.	Index.
	mm.	mm.	
Subject A 9 . . .	153	33	21·6
Average of ten Irish clavicles . . .	134	22	16·5

Table XV.—The Antero-Posterior Thickness of the Sternal End compared with the Antero-Posterior Breadth of the Acromial End of the Clavicle.

Antero-Posterior thickness of Sternal End = 100.

	Sternal End.	Acromial End.	Index.
	mm.	mm.	
Subject A 9 . . .	15	33	220
Average of ten Irish clavicles . . .	20·4	22	108

These tables indicate the great development of the acromial end of the bone (fig. 3).

The ossification of the clavicle is not complete. In neither bone, indeed, is the normal epiphysial plate fully formed. In the right bone there is a small (mm. 7×9) epiphysial plate at the acromial extremity. In the left bone there is a smaller acromial epiphysis.

2. *The Scapula.*—The measurements of the scapula are shown in Table XVI.

Table XVI.—Measurements and Indices of the Scapula.

Length	167 mm.
Breadth	106 mm.
Scapular index	*63·4*
Infra-spinous length	135 mm.
Infra-spinous index	*78·5*

Sir Wm. Turner gives the figures 65·3 and 87·8 as the mean scapular and infra-scapular indices in the bones of Europeans. Both the indices of subject A 9's scapula are smaller in value; the macerated scapula is therefore unusually narrow, but the ossification of the bone is incomplete. Prior to maceration the vertebral border was extended by a broad strip of cartilage in which no bone centre had appeared. The epiphysis for the inferior angle is present, but is not joined. It is certain, therefore, that the bone had not reached its full development, and that it had not attained its definitive breadth.

3. *Humerus.*—The humerus is a normal, smooth bone, which is only remarkable for the fact that its upper epiphysis has not joined the shaft. The nutrient foramina are of considerable size. The length of the bone is 335 mm.

4. *Radius and Ulna.*—The radius and ulna are normal, smooth bones, which are only remarkable for the fact that their lower epiphyses have not yet joined the shaft. The length of the radius is 259 mm.; of the ulna, 275 mm.

The proportions which the lengths of the bones bear to the stature of subject A 9 and to each other are of interest. They are given in Tables XVII., XVIII., XIX., and XX.

TABLE XVII.—LENGTH OF HUMERUS AS COMPARED WITH THE STATURE.
Stature = 100.
(Average Index as given by Topinard (15) . . 19·8.)

	Length of Right Humerus.	Humerus-Stature Index.
	mm.	
Subject A 9 . . .	335	19·3
Magrath (8) . . .	431	19 7
Byrne (8) . . .	450	19

From Table XVII. it is evident that, in relation to stature, the humerus of A 9 is short, and that in this respect it resembles the humeri of the giants.

TABLE XVIII.—LENGTH OF RADIUS AS COMPARED WITH THE STATURE.
Stature = 100.
(Average Index as given by Topinard (15) . . 14·4).

	Length of Right Radius.	Radius-Stature Index.
	mm.	
Subject A 9 . . .	259	14·95
Magrath (8) . . .	331	15·2
Byrne (8) . . .	334	14·5

From Table XVIII. it is evident that, in relation to stature, the radius of A 9 is longer than normal, and that in this respect it resembles the radii of the giants.

TABLE XIX.—RADIO-HUMERAL INDEX.
Length of Humerus = 100.
(Average Index as given by Topinard (15) . . 73·4.)

	Length of Right Humerus.	Length of Right Radius.	Index.
	mm.		
Subject A 9 . . .	335	259	77·3
Magrath (8) . . .	431	331	76·8
Byrne (8) . . .	450	334	74·2

From Table XIX. it is evident that the radius of A 9 is disproportionately long, and that in this respect it resembles the radii of the giants.

It may be objected that Topinard's average indices, while useful for the comparison of averages, are of little value as standards of comparison in individual cases. To obviate this objection a method of comparison has been devised. In this the length of the bone is first taken, and from it the minimum and maximum possible statures of a normally-proportioned individual possessing such a bone are calculated. The calculation is based upon the known range of variations in the proportions of individuals, and the factors employed are those given in Professor Cunningham's *Text-book of Anatomy*, viz. humerus, 4·93–5·25; ulna, 6·26–6·66; radius, 6·70–7·11,—(17).

Thereafter, utilising the same factors, the probable lengths of the different bones in an individual of stature 1732 mm. are calculated. The minimum and maximum percentage of excess in the length of the bones of A 9 are then readily obtainable.

Table XX.

Name of Bone.	Length, mm.	Corresponding Stature Range in normally proportioned Individuals.	Normal Range of Bone Lengths in Individuals of Stature 1732.	Minimum and Maximum Percentages of Excess of Length in A 9.
Humerus . .	335	1651·5–1758·7	359·5–329·9	−7·3 . . +1·5
Ulna . . .	275	1721·5–1831·5	276·6–260	− ·5 . . +5·4
Radius . .	259	1735·3–1841·4	258·5–243·5	+ ·6 . . +6

This table shows that the humerus was relatively the shortest bone of the trio, the radius relatively the longest.

The minus signs do not necessarily mean that the humerus and ulna were absolutely short. The tibial and femoral lengths are included in stature, and the possibility is that it means that the humerus grew less than the tibia. It is shown later that this was the case.

5. *Hand.*—The aggregate length of the bones of the middle column of the hand, viz. the semilunar, the os magnum, the middle metacarpal, and the phalanges of the middle digit, is 187 mm. This is only 10·8 per cent. of the total stature in place of the normal 11·7 per cent. This, however, does not necessarily mean that the hand is absolutely short, only that it is short relatively to a stature which includes variables of unknown values. The most reliable standard for comparison, because it contains fewest variables, is furnished by the basi-nasal length. Using this as the basis of comparison, the relative size of the hand is shown by the measurements in Table XXI.

Table XXI.—Comparison of Hand Length with Basi-Nasal Length.
Basi-Nasal Length = 100.

	Hand Length.	Basi-Nasal Length.	Index.
	mm.	mm.	
Subject A 9 . . .	187	97	192·7
Average Irish measurements . . .	200	103	194

This shows that the relative hand length is slightly below the Irish average.

B. *Skeleton of the Lower Limb.*

(*Note.*—The innominate bone has already been considered in connection with the pelvis.)

1. *Femur.*—The measurements of the femur are given in Table XXII.

Table XXII.—Measurements and Indices of the Femur.

		mm.
Length		444
Diameters—		
Just below level of lesser trochanter	sagittal	33·5
	transverse	35
Index		*95·7*
Mid shaft	sagittal	31·5
	transverse	31·5
Index		*100*
Junction of mid and lower thirds	sagittal	33
	transverse	32
Index		*97*

The ossification of the femur is complete. The very slight degree of flattening of the upper part of the shaft of the femur is interesting in connection with the abnormal posture of the vertebral column. It is clear that an individual who has a forwardly-directed vertebral column must, when he walks, keep the thighs slightly flexed—*la marche en flexion.* With this Manouvrier (16) has associated the platymeric flattening of the femur.

2. *Tibia and Fibula.*—The measurements of the tibia and fibula are given in Table XXIII..

Table XXIII.—Measurements of Tibia and Fibula.

		mm.
Tibia—		
Length		398
Diameters—		
At mid shaft	sagittal	29·3
	transverse	24
Index		*81·8*
Fibula—		
Length		396

The ossification of the tibia is not quite complete, the superior epiphysis being not quite completely joined.

The ossification of the fibula is complete.

The most striking fact about the bones of the lower limb is the very great length of the tibia in comparison with the femur. This characteristic is universal among the giants. The point is well brought out in Table XXIV.

Table XXIV.—Tibio-Femoral Index.

Femur = 100.

Subject A 9	89·6
Magrath (8)	83·9
Byrne (8)	86·5
Normal (European)	81

As in the case of the bones of the upper limb, the proportions of the bones relative to stature are best worked out in minimum and maximum percentages of excess of length. This is done in Table XXV. The factors here employed are femur, 3·53–3·92; tibia, 4·32–4·80; fibula, 4·37–4·82,—(17).

Table XXV.

Name of Bone.	Length, mm.	Corresponding Stature Range in normal Individuals.	Range of normal Bone Lengths for Stature 1732.	Minimum and Maximum Percentages of Excess of Length of Bones of Subject A 9.
Femur	444	1567–1740	490–441·8	– 10·3 . . – ·4
Tibia	398	1719–1910	400–360·8	– ·5 . . – 9·3
Fibula	396	1730–1910	396–359	0 . . – 9·3

3. *Foot.*—The length of the foot is 254 mm. To obtain a satisfactory knowledge of the real proportionate size of the foot, the method employed in the case of the hand is again used in constructing Table XXVI.

Table XXVI.—Comparison of Foot Length with Basi-Nasal Length.

Basi-Nasal Length = 100.

	Foot Length.	Basi-Nasal Length.	Index.
	mm.	mm.	
Subject A 9	254	97	261·8
Average Irish measurements	263	103	255

This table shows that the foot of subject A 9 is longer than might have been expected. The difference is, however, very slight.

An interesting developmental abnormality was present in the right metatarsus of subject A 9, viz. a well-developed os tibiale (18).

Summary of Report upon Subject A 9.

A male subject aged 28 with functionless testicles, but no other visible lesions of the glands which control metabolism, has been found, in spite of his age, to possess an incompletely ossified skeleton.

This incomplete ossification affects certain of the vertebræ, the ribs, the bones of the limb girdles, and the bones of the limbs.

On examination, the skeleton is found to show many disturbances in its interproportions. These are similar to the disturbances in the interproportions of the skeletons of giants. In the skull, the great growth of the alveolar portion of the superior maxilla, the great depth of the mandibular symphysis, and the unusual height of the coronoid process, characters which have been associated with giantism, are found to be well marked. In the vertebral column the proportions of the cervical, thoracic, and lumbar segments are found to approximate to those of giant vertebral columns. Similarly, certain characteristics of the limbs resemble the corresponding peculiarities found in giants, more particularly the great relative and absolute length of the radius and ulna, and of the tibia and fibula.

The cause of the prolongation of the period of bone formation and of the disturbances in the size inter-relationships of the skeleton is hypothesised as being the failure of the testicles properly to develop.

In support of the hypothesis, the following observations made by the observers noted are adduced.

The first group of supporting observations deals with individuals whose testicles were functionless as a result of developmental error or disease (Group A), the second group with the effects of castration in animals (Group B), the third group with the structural peculiarities of eunuchs (Group C).

GROUP A.

Observations on the Structural Peculiarities of Individuals whose Testicles as a result of Developmental Error or Disease were Functionless. (Four Observations.)

Note.—The measurements of the limbs of the first three individuals of this group are considered together after Observation 3.

Observation A 1 (1).

X, aged 30 years, is unusually tall. His height is 1780 mm. The lower limbs are remarkable for their disproportionate length; the face and body are completely glabrous; the voice is small, and surprisingly infantile. The breasts are large, the pelvis is very wide. Examination of the genital organs show them to be extremely rudimentary; the skin is slightly pigmented and hairless. The penis is well formed, but is hardly as big as the little finger. The testicles are of the size of sparrows' eggs, and devoid of normal sensation. The patient has never experienced sexual desire.

Observation A 2 (1).

Y, aged 53, is thin and tall. His height is 1777 mm. The lower limbs are disproportionately long. The pelvis is broad. The genital organs are abnormal; the scrotum is small; the penis is very small. The right testicle feels like a small bean; the left is quite impalpable, owing to the presence of a scrotal hernia. At the operation for the hernia it was seen to be small, about the size of a pea, and fibrous.

Observation A 3 (1).

Z, aged 38, is unusually tall. His height is 1782 mm. His lower limbs are very long relatively to his height. The pelvis is very broad. The penis is smaller than usual; palpation of the scrotum reveals the presence of two very small and completely insensible testicles. There is no sexual desire.

Table XXVII.—Measurements of the Long Bones of the Limbs of X, Y, and Z.

	X.	Y.	Z.
Stature	mm. 1780	mm. 1777	mm. 1782
Humerus	350	364	370
Ulna	270	295	282
Radius	260	275	297
Femur	525	522	535
Tibia	440	432	455

From these measurements tables showing minimum and maximum percentages of excess of length can readily be constructed in the manner employed in Tables XX. and XXV.

These tables show that individuals X, Y, and Z, who possessed functionless testicles, were the subjects of an abnormal development of the long bones of the appendicular skeleton. The excess of growth was more marked in the tibia than it was in the femur, and more marked in the radius than it was in the humerus. It is clear that the method of comparison in utilising as its standard a figure whose magnitude is dependent upon the sum of a series of variables obscures the true value of the figures. Unfortunately, the observer (1) from whose writings the absolute measurements are taken has not provided any measurements more suitable.

Table XXVIII.—Limb Bones of X.

Name of Bone.	Length, mm.	Corresponding Stature Range in normal Individuals.	Range of normal Bone Length for Stature 1780.	Minimum and Maximum Percentages of Excess of Length of Bones of X.
Humerus . .	350	1695·5–1837·5	361 –339	– 3·1 . . 3·1
Ulna . . .	270	1690 –1798	284 –267	– 5·1 . . 1·1
Radius . .	260	1742 –1848·5	265·6–250	– 2·1 . . 3·8
Femur . . .	525	1853 –2058	504 –454	4 . . 13·5
Tibia . . .	440	1900 –2112	412 –370·8	6·3 . . 15·7

Table XXIX.—Limb Bones of Y.

Name of Bone.	Length, mm.	Corresponding Stature Range in normal Individuals.	Range of normal Bone Lengths for Stature 1777.	Minimum and Maximum Percentages of Excess of Length of Bones of Y.
Humerus . .	364	1794·5–1911	368·5–338·5	– 1·2 . . 7
Ulna . . .	295	1846·7–1974·7	283·8–266·8	3·7 . . 9·5
Radius . .	275	1842·5–1955	265 –250	3·6 . . 9
Femur. . .	522	1842 –2046·2	503·4–453	3·5 . . 13·2
Tibia . . .	432	1866 –2073·6	411·3–370	4·7 . . 14·3

Table XXX.—Limb Bones of Z.

Name of Bone.	Length, mm.	Corresponding Stature Range in normal Individuals.	Range of normal Bone Lengths for Stature 1782.	Minimum and Maximum Percentages of Excess of Length of Bones of Z.
Humerus . .	370	1824·1–1942·5	361·4–339·4	2·3 . . 7
Ulna . . .	282	1765·3–1878·1	284·6–267·5	– ·9 . . 9·5
Radius . .	297	1989·9–2111·6	266 –250·6	10·4 . . 15·6
Femur. . .	535	1888·5–2097·2	504·7–454·5	5·6 . . 15
Tibia . . .	455	1965·6–2184	412·5–371·2	9·3 . . 18·4

When compared with Tables XX. and XXV., the results of these three observations support in a most remarkable manner the findings with regard to the interproportions of the limb segments of subject A 9. Unfortunately they can give no further help.

The fourth observation in this group was made by MM. Launois and Roy (2).

Note.—It is not quite clear that this case belongs to the class under consideration. The observers record the case under the title "Anorchide hypermacroskele de 27 ans," and as such it has been accepted by Eberth (19). The difficulty is this: There is a class of pathological giants who remain sexually immature, but who owe their giantism and their infantilism to an underlying disturbance of their metabolism. These differ altogether from those who owe their slightly increased stature and their disproportionate bodies to a simple uncomplicated testicular failure. MM. Launois and Roy seem, to the writer, to have failed to recognise this, with the result that the interpretation they put on some of their cases appears to him to be incorrect (2). The same remark would apply to some other recorded cases. Consideration of these has been excluded from this communication, except in the comparisons of body form. With the reservation here expressed, the facts of Observation A 4 may be considered.

Observation A 4.

W., age 27, height 1860 mm., looks taller than he really is on account of his slender build and thin legs. His body is ill-proportioned. His height seated (870 mm.) is very short in comparison with his standing height (1860 mm.) and with his span (1900 mm.). This shows that his excessive stature is chiefly due to the length of his lower limbs. The height of his great trochanter is 1050 mm. According to Quetelet, the measurement in a normally-proportioned man 25 years of age, and 1682 mm. in height, should not be more than 873 mm. The same proportion for a stature of 1860 mm. is 960 mm., so that there is in this case an excess in the length of the lower limb of 90 mm.

An X-ray photograph shows that the epiphysial cartilages are not obliterated.

Further, the face is glabrous, the chest small, the pelvis large, the penis and scrotum rudimentary, and the testicles undiscoverable.

Summary of Observations. Group A 1–4.

Males whose testicles are functionless are found to possess unduly long limbs. This undue length (Observations 1–3) affects the radius and tibia more than the humerus and femur. The process of ossification is unduly prolonged (Observation 4).

This group of observations yields evidence in support of the hypothesis that the disturbances in the normal structural proportions found in the body of subject A 9 were the direct result of his testicular failure. Their evidence is strengthened by a study of the effects of the castration of healthy young animals.

The observations dealing with this matter constitute Group B.

GROUP B.

Observations upon the Effects of Castration in Animals.

Observation B 1 (1).

A. Two cocks of the same brood and of the same strain (Ardennes) were selected.

B. Two cocks of the same brood and of the same strain (Italian) were selected.

One animal in each pair was castrated, the other kept entire. They are distinguished as A. Capon, A. Cock, and B. Capon, B. Cock.

Before the operation was performed the weights of the animals were—

A. Cock	980 grammes.
A. Capon . . .	845 „
B. Cock	785 „
B. Capon . . .	805 „

The age of all the animals was 3 months and 12 days.

Eight months after operation the animals were killed, when their weights were—

A. Cock	1955 grammes.
A. Capon . . .	2800 „
B. Cock	2105 „
B. Capon . . .	2650 „

On examination, the bodies of the capons were found to be much fatter than those of the cocks.

Examination of the Skeleton.

The bones of the capons were longer, thicker, and denser than those of the cocks. The actual measurements of the long bones and the percentage increase of the bones of the capons are shown in Table XXXI.

Table XXXI.

	A. Cock.	A. Capon.	Percentage Increase.	B. Cock.	B. Capon.	Percentage Increase.
Femur .	9	9·8	9%	7·4	8·3	12%
Tibia . .	13·4	15·1	12·6%	12·2	13·2	8·2%

In all these birds the epiphysis and diaphysis had joined, but Sellheim (20) has shown that in the capon there is delay in the completion of the process of endochondral ossification.

Observation B 2 (1).

The Effect of Castration upon Guinea-pigs.

Three guinea-pigs of the same litter, age 45 days, were taken. One was left entire, two were castrated.

The weight of the animals was—

No. 1, castrated . . .	325 grammes.
No. 2, castrated . . .	400 „
No. 3, entire	355 „

After the operation the three animals were placed in one cage, and were treated absolutely alike. They were weighed regularly, and an interesting series of observations was recorded. At first the castrated animals got rapidly heavier. No. 1, castrated, which to start with weighed 30 grammes less than No. 3 entire, surpassed him in weight, and at the end of four months was the heavier by 150 grammes.

No. 1, castrated . . .	770 grammes.
No. 3, entire	625 „

For the next four months the weight of these animals remained stationary, at the end of which time they were killed.

No. 2, castrated, which to start with weighed 45 grammes more than No. 3 entire, gained weight up to three months, at which time he weighed 655 grammes. No. 3, entire, at the same time weighed 595 grammes. From the third month on, the weight of No. 2, castrated, did not materially alter until the end of the eighth month, when he was killed. As stated above, No. 3, entire, increased in weight to a total of 625 grammes.

No. 2, castrated . . .	655 grammes.
No. 3, entire . . .	625 „

In each of the castrated animals the penis remained undeveloped.

On examination of the bodies, naked-eye, there was no marked increase in the amount of fat in the castrated animals, in spite of their slightly greater weight. The bones of the skeletons were somewhat bigger than those of the entire male, being not only longer but thicker.

The measurements of the long bones of the limbs, and their percentage increase, as compared with those of the control, are given in Table XXXII.

Table XXXII.

	Control.		No. 1, Castrated.				No. 2, Castrated.			
	Length.		R.		L.		R.		L.	
	R.	L.	Length.	Increase per cent.	Length.	Increase per cent.	Length.	Increase per cent.	Length.	Increase per cent.
Femur .	44	44	46½	5·6	47	6·8	47½	7·9	47	6·8
Tibia .	46½	46½	49½	6·4	50	7·5	50½	8·6	50¼	8
Humerus .	38	38	39	2·6	39	2·6	40¾	7	41	9
Ulna .	42	42¼	44	4·7	43	1·7	43	2	43	1·7

The paws of the castrated animals were exceedingly large. This enlargement was so striking that it suggested acromegaly, and the pituitaries were examined, with negative results.

Ossification was complete in the control; none of the epiphysial cartilages of the castrated animals had disappeared.

Observation B 3 (3).

Poncet carried out experiments upon dogs and upon bitches.

A litter of five pups—two dogs and three bitches—was taken. One dog and two bitches were castrated, the remaining dog and bitch being kept as controls. Some months later the five animals were killed and their bones examined. The measurements and percentage increases in the dog are given in Table XXXIII.; in the bitches, in Table XXXIV.

Table XXXIII.

Name of Bone.	Length in mm. of Bone of		Percentage of Excess of Length.
	A, Control.	B, Castrated Animal.	
Femur . . .	14	17	21·4
Tibia . . .	50	56	12

Table XXXIV.

Name of Bone.	Length in mm. of Bone of			Average Percentage of Excess of Length.
	A, Control.	B, Castrated Bitch No. 1.	C, Castrated Bitch No. 2.	
Femur . . .	15	15·5	15·5	3·3
Tibia . . .	50	54	54	8

Another observation upon the effects of castration upon the skeleton of dogs is recorded by Brian (21). He compared the weight of the skeletons of castrated animals with the weight of the skeletons of controls, with result shown in Table XXXV.

Table XXXV.—Comparison of the Weight of the Skeleton of Castrated Animals with the Weight of the Skeleton of Controls.

Average weight of skeleton of castrated animals . .	240	grammes.
Average weight of skeleton of control animals . . .	175	"
Percentage of excess of weight	18·7	"

Summary of the Observations. Group B, 1–3.

In animals which have been castrated there is—

1. An increase in the length and weight of the bones.

This increase tends to affect especially the more distal segments of the limb. Exceptionally, it affects the proximal segment more than the distal. This occurred in the hind limb of one capon and one castrated dog, and in three fore limbs of two castrated guinea-pigs. In each of the two exceptional hind limbs the absolute growth was greater in the more distal segment. In the case of the three guinea-pig fore limbs the absolute growth was less in the more distal segment.

2. There is a delay in the obliteration of the epiphysial cartilages.

The second of these results is important, and it is desirable to emphasise it. As the result of observations (22), (23), (24) in the slaughter-house, it has been found that a bull's epiphyses have joined the bone shafts by the time the animal is 2 years old, whereas in the bullock the process of endochondral ossification is not completed until the animal is 3 or even 4 years old.

These facts show that the disturbance in the proportions of the limbs, and the undue prolongation of the process of ossification which takes place in the absence of functioning sexual glands, may fairly be ascribed to the metabolic effects of an uncomplicated testicular failure. That this is really so, and that not only may the changes in the limbs, but also the changes in the pelvis and skull of subject A 9 be justly regarded as effects of the same cause, is shown by the observations in Group C.

GROUP C.

The Structural Peculiarities of Eunuchs.

Observation C 1 (4).

Skeleton of a Eunuch (a negro from the Sudan).

Age at death, 24–25 years.

Total stature of skeleton . . .	1790 mm.
Allow for soft parts	30 „
Estimated stature of individual in life .	1820 mm.

Relatively to the length of the bones of the appendicular skeleton, the axial skeleton is short.

The long bones are exceedingly slender, of feminine type, and show no projecting crests for muscular attachment.

The humerus is long; the radius and ulna are definitely longer in proportion; the metacarpals and phalanges are extremely elongated, so that the hand is almost simian in appearance.

The femur is very weak, is devoid of its normal curve, and is unduly long; the tibia and fibula are even longer in proportion, and are very slender; the metatarsals and phalanges are very long and slender.

The measurements of the longer bones of the limbs are recorded in Table XXXVI., column 2. The remainder of the columns of the table were worked out in the manner already indicated in connection with Tables XX. and XXV.

Table XXXVI.

Name of Bone.	Length, mm.	Corresponding Stature Range in normal Individuals.	Range of normal Bone Lengths for Stature 1820.	Minimum and Maximum Percentages of Excess of Length of Bones of Eunuch.
Humerus . .	372	1834–1953	370–346	·5 – 7
Ulna . . .	324	2028–2158	290–273	10·5 – 18·6
Radius . .	305	2043–2169	272–256	10·8 – 16
Femur . .	530	1871–2078	513–464	3·2 – 12·4
Tibia . . .	464	2004–2227	421–379	9·2 – 18·2
Fibula . .	445	1945–2145	416–377	6·5 – 15·2

From this table it is obvious that, relatively to the total stature of the skeleton, all the bones of the limbs are unduly long; those of the proximal segments, however, to a markedly less extent than those of the distal.

Observation No. C 2 (1).

Abridged Description of the Skeleton of a Eunuch (negro from the Sudan).

Age at death 22 years.

A.—*Report upon the progress of ossification.*

1. *Upper Limb.*—The epiphysial cartilages of the scapula persist, as do those of the head of the humerus, of the lower end of the ulna, of the radius, and of the metacarpals.

2. *Lower Limb.*—None of the epiphysial cartilages show any sign of ossifying except those of the two first metatarsals, which have almost disappeared.

3. *Pelvis.*—The triradiate cartilage is still visible.

4. *Vertebral Column.*—Ossification is almost complete throughout; the junction of the epiphyses is obviously very recent. The pieces of the sacrum are still distinctly separate.

In normal individuals the triradiate cartilage disappears at about the age of 16; the upper epiphysis of the ulna and the lower of the humerus join the diaphysis during the 17th year. All the other epiphysial plates of the long bones normally disappear before the age of 22, except those situated at the upper ends of the humerus, tibia, and fibula, and at the lower ends of the radius and ulna. It therefore appears that in this eunuch there was delay in the completion of the process of endochondral ossification in the bones of the appendicular skeleton.

In the vertebral column, on the other hand, it is not usual to find fusion of the epiphysis until about the 25th year; yet in this eunuch, at the age of 22, the process of ossification of the vertebra was almost complete. It is therefore clear that there was no delay in the ossification of the bones of the axial skeleton.

B.—*Report upon the relative lengths of the long bones.*

The lengths of the longer bones and their various comparisons are given in Table XXXVII. The methods of calculation and comparison are exactly similar to those formerly adopted.

Total stature of skeleton	1800 mm.
Allow for soft parts	30 „
Estimated stature of individual in life .	1830 mm.

TABLE XXXVII.

Name of Bone.	Length, mm.	Corresponding Stature Range in normal Individuals.	Range of normal Bone Lengths for Stature 1830.	Minimum and Maximum Percentages of Excess of Length of Bones of Eunuch.
Humerus . .	372	1834–1953	371–348	·2 – 6·4
Ulna . . .	325	2034–2164	292–274	7 – 15·6
Radius . .	306	2050–2175	273–257	10·7 – 16
Femur. . .	535	1888–2097	518–466	3·1 – 12·9
Tibia . . .	463	2000–2222	423–379	8·6 – 18·1

From this table it is obvious that, relatively to the total stature of the skeleton, all the bones of the limbs are unduly long; those of the proximal segments, however, to a markedly less extent than those of the distal.

Observation No. C 3 (25).

ABRIDGED DESCRIPTION OF THE SKELETON OF A NEGRO EUNUCH.

Age at death 25 years.

A.—*Report upon the progress of ossification.*

Upper Limb.—The head of the humerus is separate from the shaft of the bone. The epiphysial cartilages at the lower end of the radius, ulna, and four inner metacarpals, and that at the upper extremity of the first metacarpal, still persist.

Pelvis.—The triradiate cartilage is still present.

Lower Limb. — None of the epiphysial cartilages have become obliterated.

In this eunuch, therefore, there was considerable delay in the completion of the process of endochondral ossification in the bones of the appendicular skeleton.

B.—*Report upon the relative lengths of the long bones.*

The lengths of the longer bones and their various comparisons are given in Table XXXVIII.

Total stature of the skeleton	1840 mm.
Allow for soft parts	30 „
Estimated stature of individual in life . .	1870 mm.

Table XXXVIII.

Name of Bone.	Length, mm.	Corresponding Stature Range in normal Individuals.	Range of normal Bone Lengths for Stature 1870.	Minimum and Maximum Percentages of Excess of Length of Bones of Eunuch.
Humerus . .	365	1799–1916	379–356	−3·8 − + 2·4
Ulna . . .	322	2015–2144	298–280	+7·4 − 13
Radius . . .	300	2010–2133	279–263	7 − 12·3
Femur . . .	556	1962–2179	529–477	4·8 − 14·2
Tibia . . .	470	2030–2256	432–389	8 − 17·2

From this table it is obvious that, relatively to the axial skeleton, all the bones of the limbs are unduly long; those of the proximal segments, however, to a markedly less extent than those of the distal.

Summary of Observations. Group C 1–3.

The Skeletons of Eunuchs.

From these three observations it is permissible to say that in eunuchs there is delay in the completion of the process of endochondral ossification. (In this connection an observation recorded by Gruber (39) in 1847 is of interest. He dissected the body of a eunuch whose age at death was 65 years, and he found the hyoid bone cartilaginous.)

Further, the long bones of the appendicular skeleton are unduly long. This excess of length is particularly remarkable in the more distal segments of the limbs. The bones are thin, smooth, and slender. As yet, however, there is no incontrovertible evidence to show whether the disproportion arises from excessive development of the limb bones or from deficient development of the axial bones, and unfortunately the fact that it is unknown to what tribe these negro eunuchs belonged prevents the determination of this point by the application of anthropological data.

From isolated cases, it is possible to turn to an anthropological survey, for there exists in Russia and Roumania a sect of religious fanatics, the Skoptzys (26, 27) or "White Doves," who believe that only those who have known no sexual desire are worthy to mount the "White Horse of the Apocalypse."

To secure this privilege for their sons, parents have them castrated when they are from 3 to 5 years old.

Observation No. C 4.

Physical Peculiarities of the Skoptzy Eunuchs.

The Skoptzy eunuchs are taller than the men of the same race. Pelikan (28) gives figures based upon an examination of seventeen Skoptzy skeletons, and of twenty-six skeletons of normal men of the same race.

The average of his measurements is—

	Skoptzy.	Normal.
Stature	1690 mm.	1656 mm.
Arm and forearm . .	550 „	
Tibia	476 „	

The proportionate length of the tibia to body height is as 1 to 4·32 4·80,—(17). This gives a tibia length of 345–378·7 for a stature of 1656. The tibia of the Skoptzy is therefore actually longer than that of normal men, but it is also relatively far longer, for a tibia length of 476 should be associated with a stature of between 2051 and 2285. But not only is the tibia relatively and absolutely longer than normal, the trunk is relatively and absolutely shorter. Pittard (29) gives confirmatory figures. He examined thirty individuals; these he divided into three groups, each of ten individuals, and obtained the following averages:—

Table XXXIX.

	Stature.	Height seated.	Length of Lower Limbs.
	mm.	mm.	mm.
1st Group, 10 Eunuchs, without beard and sexual hair . .	1751·8	1278·6	875·4
2nd Group, 10 similar individuals . .	1718·8	1277·6	844·2
3rd Group, 10 Eunuchs, possessing beard and sexual hair . .	1704·2	1287	818·2

The third group is composed of Skoptzys who have been converted after adult life has been attained, or of Skoptzys of a feeble faith who have had only one testicle removed. These furnish a control, and their measurements provide a useful standard of comparison.

Pittard (29), in a communication to L'Academie des Sciences, introduced by Laveran, 8th June 1903, gives further particulars arrived at as the result of an anthropometric survey of thirty of these eunuchs. His conclusions are—

1st. In the case of individuals castrated before puberty, the ablation of the testicles determincs an increase of stature. This increase is considerable; the average stature of the eunuchs easily surpasses the average for normal men of the same ethnic group.

2nd. The increase in length of the lower limb is the important factor in the general increase, the trunk developing relatively little. The proportion between the length of the lower limb and height is greater in eunuchs than in men of the same stature.

3rd. The length of the upper limb of eunuchs is relatively greater, and absolutely much greater, than that of men.

4th. On the other hand, the height of the head is relatively less in eunuchs than in men.

5th. The maximum antero-posterior diameter of the cranium, which normally increases proportionately with increased stature, in the Skoptzys shows an inverse proportion.

6th. The maximum transverse diameter also bears an inverse proportion to the increase of stature.

7th. Castration retards the growth of the skull; it must also affect adversely the growth of the brain.

8th. It retards or restrains equally the transverse growth of the forehead. The development of the frontal minimum, which in normal men is proportional to the increase of the stature, becomes inversely proportional.

9th. The effects of castration are also seen in the development of the height of the skull (auriculo-bregmatic diameter). This development is normally inversely proportional to the total stature. This order of growth persists, but it is retarded.

10th. Castration retards, and probably ultimately arrests, the lateral development of the face (represented by the bizygomatic diameter), and also, probably, its vertical height. It is especially the lower part of the face which is affected, and probably more especially the height of the inferior maxilla.

11th. In the Skoptzy eunuchs the region of the ophryon appears less developed.

12th. Castration causes an arrest in the growth of the nose, especially in height.

13th. It appears, on the other hand, to produce an increase in the depth of the orbit.

Summary and Conclusions of Section I.

Anatomical and Anthropological Investigation.

The results of this anatomical and anthropological investigation may be summarised thus:—

1. In the absence of functioning testicles, the process of endochondral ossification is stimulated and prolonged.

2. The stimulation and prolongation do not affect the cartilages equally; the distribution of the resulting growth is therefore not equal.

Although the distribution of the growth is unequal, it is not fortuitous, but follows a uniform scheme of distribution in men and animals. The bones most markedly affected are the bones of the leg, next those of the forearm, then that of the thigh and the arm, then the limb girdles, and, last and least, the vertebral column, within the limits of which the distribution is again unequal, the thoracic regional segment being most affected. The skull is also affected. The cranial capacity develops less completely than is normal. The expansion of the antrum of Highmore, alone of the nasal system of air cells, is retarded. As a direct result of this, the face remains narrow and the nasal part of its height short. The alveolar process of the superior maxilla grows out of proportion to the rest of the bone, as also do the mandibular symphysis and coronoid process.

4. These growth changes are clearly the result of an uncomplicated testicular failure, for they occur in otherwise healthy males if the testicles are removed by operation before the cartilages of the synchondroses are obliterated.

5. In their general incidence, though not in their intensity, the growth changes resemble those found in giants, who may or may not be sterile, though such abnormal beings usually are sexually deficient.

These are the mediate results of the absence of functioning sexual glands.

It is clear that the immediate results of removal of the testicles must be twofold:—

1. The constituents of the blood which normally provide for the nutriment of the testicles, and meet the demands of spermatogenesis, will no longer be required, and will become available for the use of the somatic cells.

2. The waste products (internal secretion ?) of the testicles will no longer be thrown into the blood, and the nervous stimuli arising from the collection of semen in the seminal vesticles will no longer affect the organism.

The somatic cells, under these circumstances, must experience a changed environment. An attempt to form some conception of the nature of that change forms the subject of the next section.

(It is worthy of note that in one case, Observation B 3, the removal of the ovaries from two young bitches was attended by generally similar results.)

SECTION II.—HISTOLOGICAL AND PHYSIOLOGICAL.

The histological facts of the process of endochondral ossification are universally known, but their metabolic interpretation is not. A short statement of the writer's views on the subject is therefore necessary as a preliminary to the complete discussion of the nature of the changes in the environment of the somatic cells induced by ablation of the testicles.

1. The Histology of Endochondral Ossification in terms of Metabolism.

In an embryo of the second month the mesoblastic cells which are placed in the axis of each limb bud become crowded together to form the cartilage which maps out the developing bone. After a time the cells situated near the centre of the mass gradually but progressively hypertrophy, and by their hypertrophy mark histologically the centre of ossification.

It is a commonplace that the surface of a sphere increases as the square, the content as the cube of its diameter. It follows that, as the hypertrophy of a cell continues, its surface area must become progressively smaller in proportion to its mass, and eventually must become too small to support that mass. The cell must then divide or die.

These simple facts are expressed in the cytological law: "The early growth of a cell, the increasing bulk of contained protoplasm, the accumulation of nutritive material, correspond to a predominance of protoplasmic processes which are constructive or anabolic. The growing disproportion between mass and surface must, however, imply a relative decrease of anabolism. Yet the life or general metabolism continues, and this entails a gradually increasing preponderance of destructive processes, or katabolism. So long as growth continues, the algebraic sum of the protoplasmic processes must be plus on the side of anabolism. The limit of growth, when waste was overtaken and is beginning to exceed the income or repair, corresponds in the same way to the maximum of katabolic preponderance consistent with life" (30).

In other words, the growth of a cell varies directly as the surplus of nutrition over expenditure, and directly as the rate at which this surplus decreases. The moment the surplus falls below zero, the cell must degenerate and die. This is the metabolic explanation of the histological fact, that soon after the cartilage cells attain to their maximum size they begin to exhibit indications of impaired vitality. Their protoplasm shrinks, their nuclei degenerate, and in a little while they are dead.

As the cartilage cells die out, the neighbouring matrix, no longer kept healthy by its guardian cells, has lime salts deposited in it. This interpretation of the meaning of the calcification of the cartilaginous matrix brings it into line with the pathological calcifications which occur in degenerating, dying, or dead tissue.

So soon as these changes are established, the cells of the osteogenetic layer of the primitive periosteum begin to multiply, to absorb the debris of the cartilage, and to eat out a space, the primitive marrow cavity, which they occupy, and in which they proliferate to form a tissue, the primitive marrow. From the primitive marrow the osteoblasts arise.

Around the primitive marrow cavity the surfeit and death of the cartilage cells continue, but before the cartilage cells undergo their fatal hypertrophy they provide for the safety of their race, temporarily at least, by dividing. One of the daughter cells apparently always escapes to the side furthest from the area of the greatest cell hypertrophy.

The deductions rendered inevitable by a study of these phenomena of endochondral ossification are—

1. That high states of nutrition compel the cartilage cells to proliferate rapidly. This determines growth in length of long bones, for such growth necessarily waits upon and follows cartilaginous proliferation.

2. That, assuming the composition of the blood to be uniform and the amount of growth in different parts of the body unequal, the inequality must be directly dependent upon an inequality in the absorptive power of the cartilage cells.

2. The Physiological Meaning of the ascertained Facts with Regard to Castrated Men and Animals.

As a result of the anatomical and anthropological survey undertaken in section 1 of this paper, it was found that—

1. The effect of castration is to stimulate and to prolong the period of endochondral ossification.

2. This stimulation and prolongation tends to be especially marked in the more distal segments of the limbs.

These generalisations can now be replaced by the statements—

1. The effect of castration is to determine a markedly anabolic type of nutrition which increases the liability of the cartilage cells to death by surfeit.

2. The cells forming the more distal segments of the limbs possess in the majority of cases a greater absorptive power than those forming the more proximal, and consequently experience the greatest liability to death by surfeit.

It is well to examine in detail the collateral evidence in support of the first of these statements. If it be established, the second, in the light of the definitely ascertained anatomical facts, is inevitably true.

Scattered through the records of the experiments upon castrated animals, there is evidence of the favourable type of their somatic nutrition.

Thus, in the experiment upon fowls (Observation B 1) the capons were found to be very fat, and weighed 2800 and 2650 grammes, compared with weights of 1955 and 2105 grammes attained by the cocks used as controls. But indeed, with regard to capons, it is unnecessary to go to the laboratory for such collateral evidence, for it is because of this very fact that castration does determine a greater body growth that poultry breeders capon their cockerels. A rather instructive point in this connection is, that many of the so-called capons sold by poultry dealers are in reality young cocks and hens subjected to forced feeding with the aid of cramming machines.

Similar evidence exists with regard to other castrated animals, and this condition of what may be called hyperanabolism persists for some time, but as the animals age it passes off. The reason for this is not difficult to understand, but its demonstration would involve a long digression. It is merely mentioned here to point out that it is not to be expected that old castrated animals will be heavier than old entire animals.

But though this collateral evidence is satisfactory so far as it goes, it does not equal in conclusive force the results that might be obtained from direct experiment. Of course the fact that the forced feeding of poultry produces capon-like birds is in one sense a direct experiment, and artificial experiments on any other lines than that of forced feeding are exceedingly difficult to devise. But nature herself sometimes carries out what are in essence perfect physiological experiments, and there is one which conclusively shows that rapid growth of cartilage and its attendant rapid bone formation are directly stimulated by the existence of a favourable environment.

The experiment is this: The human embryo is carried for nine months within the uterus of the mother, and during this period, as in after-life, different human beings are subjected to different conditions of nutritive supply. Sometimes the fœtus is over-nourished and grows to such an extent that it cannot be born alive. One such fœtus that was examined was enormous, and weighed close on 20 llb. in place of the normal 7–10 llb. Its long bones were most remarkable in shape. The shafts about their middle were of an approximately normal diameter, and then rapidly increased in size as they approached the extremities. The meaning of such an appearance is clear. It is that endochondral ossification has been responsible for the bulk of the bone, and that the rapid diametric increase of the bone shaft is due to the forced proliferation of the cartilage cells.

When to this observation we add the fact that in a poorly nourished fœtus the bones tend to be of a more uniform diameter throughout their length, there can be little doubt that a predominantly anabolic condition of metabolism is the direct stimulant of chondrogenesis, and this can only be explained on the lines of the metabolic interpretation of the histology of endochondral bone formation.

These facts are readily demonstrated by means of measurements. By measuring a young and growing femur at its centre and at its ends it is easy to construct an index of cartilaginous expansion, for periosteal bone formation is in the fœtus very slight, and any difference there may be in the diameter of the diaphysis of any long bone at its central point and at its end is due almost entirely to lateral expansion of the cartilage. The expansion is readily demonstrable microscopically as due to the great proliferation and hypertrophy of the cartilage cells immediately in front of the line of bone formation.

The result of a series of measurements undertaken to demonstrate this point in connection with the differing shapes of the shafts of bones in thin and fat fœtuses are shown in Table XL.

Table XL.—Index of Cartilaginous Expansion in developing Femora. Comparison of Diameter of the Central Part of the Diaphysis with that of its Extremity.

Central Diameter = 100.

	Diameter at Extremity.	Diameter Mid-point.	Index.
	mm.	mm.	
Normal (average of ten Irish femora)	73·1	29·3	249·4
Normal (average of ten Irish fœtal femora)	12	4	300
Fat Fœtus—			
No. 1	14·5	4·5	322·2
No. 2	30	9	333·3
No. 3	8	2·2	363·6
No. 4	26	6	433·3
Fat Baby	47	12	391·6

These measurements show conclusively that the shape of the bones in fat, well-nourished babies differs from that of the bones in the thinner, less well-nourished, in a way that is directly due to the greater relative and absolute growth of cartilage in the well-nourished. This supports the hypothesis that the stimulus to cartilaginous proliferation is food supply.

The reasonable assumption therefore is, that the changes in the growth of the bones of the skeleton, which follow on the removal of the sexual glands, and are due to stimulation of the cartilage cells, are occasioned by the setting free for general use of food-stuffs which would otherwise have been used to provide for the drain of spermatogenesis. In other words, the activity of the sexual glands is opposed to body growth.

SECTION III.—ANATOMICAL AND ANTHROPOLOGICAL (*continued*).

The acceptance of the hypothesis that the effects of castration are, in part at least, to be accounted for by the diminution in the demand for nutriment, and the consequent rendering of more of it available for the use of the somatic cells, is undoubtedly rendered difficult by the known facts in regard to the nutrition of the body during pregnancy. It is well known that young human mothers become fat during the early stages of pregnancy. Minot (6) has shown that the state of pregnancy acts as a stimulant to growth in the young guinea-pig. These facts have been interpreted as meaning that the animal is not growing at the maximum of its assimilative power, and the interpretation seems inevitable. But to use this as conclusive evidence that the effect of castration is not to set free for the use of the somatic cells nutriment that would otherwise be employed in providing for the cytogenic activities of the reproductive glands is unwarranted. To do so is tacitly to assume that the whole of the nutrition that can be absorbed by the intestine is immediately made available for the use of the tissues. There is evidence to show that this is not the case, and that the liver normally prepares a considerable proportion of the nitrogen-containing absorpta for immediate excretion, and only passes on for the use of the body a small percentage of the total nitrogenous nutriment delivered to it from the intestine (31). It may well be that in pregnancy the proportion of nutriment accepted and passed on by the liver is increased in amount. There is some evidence which suggests that this is so, and there is strong evidence to show that this is what happens in some cases of giant growth and in acromegaly. This paper is not concerned with such cases of giantism, but the writer hopes to be able to publish shortly, in a paper which is in course of preparation, the evidence upon which this suggestion is based (9). All that is desirable at the present time is to point out that the known effects of pregnancy on body growth are not necessarily opposed to the hypothesis that the stimulant effect of castration upon endochondral ossification is direct, and is due to the increase in the supply of nutrition delivered to the cartilage cells when the demands of the sexual glands are withdrawn. It is desirable to repeat that this hypothesis in no way excludes the possibility of some of the changes in body form being due to the absence of an internal secretion of the testicle, or to the absence of certain nervous stimulations normally consequent upon its activity. All that is advanced is that the increase of endochondral bone growth is a direct result of the increased nutrition of the somatic cells. It certainly seems highly improbable that the restraint in the growth of the penis and scrotum, the arrest of the development of the anterior part of the antrum of Highmore, and the possible arrest in the growth of the brain, are to be attributed to the same cause. These would seem more probably to be due to the absence of some internal secretion, or to the absence of some nervous stimulation.

Table XLI.—Height and Growth—Rate of Industrial School Children (32).

No. of Observations.	Age.	Sex.	Height.		Increase, Inches.
			Feet.	Inches.	
11	2	M.	...	...	...
		F.	2	7·95	...
15	3	M.	2	10·50	...
		F.	2	11·50	3·4
28	4	M.	3	0·27	1·8
		F.	3	1·07	1·5
80	5	M.	3	2·63	2·4
		F.	3	3.22	2·2
92	6	M.	3	5·14	2·5
		F.	3	5·15	1·9
109	7	M.	3	7·54	2·4
		F.	3	6·38	1·2
135	8	M.	3	8·61	1·1
		F.	3	8·68	2·3

It is clear that for this division of the effects of castration into two categories to be proved correct, it is necessary for those which are hypothesised as being the result of simple nutritional excess to be clearly demonstrated as occurring, to some extent at least, in normal, healthy, sexually active individuals who are the subjects of a favourable environment. In other words, members of the well-conditioned, prosperous classes of society should possess a more rapid rate and a more prolonged period of growth than the unfavourably situated, badly nourished classes. And further, the definitive body forms of the two classes should be different. That the children of the well-to-do grow more than the children of the poor, and that they are taller at corresponding ages, is shown by the figures in Tables XLI. and XLII.

Table XLII.—Height and Growth-rate of the Children of Prosperous Middle-class Parents (33).

No. of Observations.	Age.	Height.		Increase, Inches.
		Feet.	Inches.	
8	2	2	10·6	...
9	3	3	2·3	4·7
7	4	3	4·8	2·5
12	5	3	6·9	2·1
14	6	3	8·2	1·3
6	7	3	11·3	3·1
6	8	4	1·5	2·2

These measurements show that the Industrial School children, who are exposed to unfavourable, but not to the most unfavourable conditions of environment, increase in stature in six years from 2 feet 7·95 inches to 3 feet 8·68 inches, or 39·8 per cent.; whereas the children with a favourable environment increase in the same time from a greater height, viz. 2 feet 10·6 inches to 4 feet 1·95 inches, or 43 per cent. of their original stature. They therefore grow not only more quickly, but also both absolutely and relatively more. From this evidence the hypothesis gains support. It remains to be shown that the definitive body form of tall normal individuals is different from that of short normal individuals. This has been done by Papillault (34) and Manouvrier (35).

These observers' interesting results may be summarised as follows:—

1. The lower limb and the pelvic girdle show the most marked increase in tall men.

2. The upper limb also increases markedly, the increase being greater in the forearm than in the arm.

3. The foot grows less rapidly than the lower limb as a whole.

4. The hand does not increase very markedly.

5. The trunk diminishes in relative size as the stature increases. The cervical segment of the vertebral column increases most, the thoracic segment least.

6. The brain does not increase proportionately to the increase of stature. In result it is relatively small in tall men, and relatively large in short men, in women, and in children.

7. The antero-posterior diameter of the cranium increases with the increase of stature, as does the thickness of the frontal bone.

8. The face increases slightly in diameter.

9. The nose increases in height with the increase in stature.

10. In the mandible, the height of the mandibular symphysis is markedly increased, as is also the gonio-mental length.

11. The teeth are little affected.

Rollet (36) gives measurements which confirm these observations.

The body form of tall, normal men is therefore in many respects similar to that of the victims of an uncomplicated testicular failure. The only essential in which they differ is that in the tall normal man the measurements dependent upon development of the antrum of Highmore are increased, whereas in the anorchid they are diminished. The relatively small growth of the cervical segment of the vertebral column in anorchids and its relatively great growth in tall men is not improbably associated with the mass of the head. This point requires further investigation. There can, however, be no doubt that in essential points, such as the small increase in the size of the trunk and the marked increase in the size of the pelvis, the great increase in the length of the limbs, with the greater increase in the middle segment and the least increase in the hand and foot, the normal tall man is somewhat similar in body form to the anorchid.

Had this not been so, the hypothesis that the absence of the sexual glands determines an increase in the amount of nutriment available for the rest of the body, and thus stimulates the cartilage cells to proliferate, would have been practically untenable. As it is so, the hypothesis gains stability.

If the effects of removing the testicles are in large measure to be ascribed to the simple process of liberating food-stuffs for the use of the somatic cells; if, in other words, the effect of castration is to establish an environment for the cartilage cells, abnormal only in its favourability, the ultimate body shape of the individual affected will depend upon the degree of completeness of the process of endochondral ossification at the time when the condition is established.

For example, if all the synchondroses are open when the sexual demand is removed, the ensuing excessive growth will affect most markedly those segments of the limbs which contain the most absorptive cells, and then, in descending order of amplitude, those which contain the less and less absorptive cells; whereas if the early closing synchondroses in the neighbourhood of the ankle, hip, and elbow have disappeared, the growth changes can no longer affect them, and will concentrate upon the late closing epiphyses, namely, those at the knee, wrist, and shoulder, and in the limb girdles and vertebral column.

In addition, therefore, to the central type of anorchid body form, of which A 9 is an example, there must be a series of body forms leading, on the one hand, to a group of practically normally-shaped individuals, who may be typified by the adult converts to the Skoptzy faith, and, on the other, to a type of early development which will be found as a transient condition in anorchid adolescents who have not yet reached their limit of growth in stature. This early type, if our hypothesis be correct, should be characterised by a relatively long tibia, a femur of fair length, and a rather diminutive pelvis; a long radius, a moderate humerus, and a relatively small scapula.

Intermediate between the full late adult Skoptzy type and the central A 9 type there will be a group of individuals with long bodies, broad pelves, and short limbs, the reverse of the central type. The genesis of this intermediate late body form is described below.

In Observations C 5 and A 10 examples of the "**early**" and "**intermediate late**" types of anorchid body form are described.

Observation C 5 (37).

Abridged Description of the Skeleton of a Negro Eunuch.

Age at death 23 years. (Fig. 4a.)

A. Report upon the progress of ossification.

All the synchondroses, except those at the proximal end of the radius and ulna, are open.

B. Report upon the proportions of the bones of the skeleton.

Total stature of skeleton	1830 mm.
Allow for soft parts	30 „
Estimated stature of individual in life . .	1860 mm.

The lengths of the long bones and their various comparisons are given in Table XLIII. The methods of calculation and comparison are similar to those formerly employed.

Table XLIII.

Name of Bone.	Length, mm.	Corresponding Stature Range in normal Individuals.	Range of normal Bone Lengths for Stature 1860.	Minimum and Maximum Percentages of Excess of Length of Bones of Eunuch.
Humerus	364	1794·5–1911	377·2–354·2	−3·6 . . 2·6
Ulna *	322	2015·7–2144·5	297·1–279·3	7·7 . . 13·2
Femur	556	1962·6–2179·5	526·9–474·4	5·2 . . 14·6
Tibia	470	2030·4–2256	430·5–387·5	8·4 . . 17·5

* The measurement of the radius is not given by this observer.

These measurements show that, so far as limb lengths are concerned, this eunuch, whose bones had not attained to their full growth, conformed generally to what was hypothesised as the transient type of body form to be met with in cases of early removal of the functioning sexual glands. The figures in Table XLIV. show that in the small size of his pelvis he also conformed to the early type.

(*Note.*—The figures taken as the standard of comparison were obtained from the measurements of two negro skeletons.)

Table XLIV.—Absolute and Relative Dimensions of the Pelvis of C 5.

	Absolute Measurements, mm.			Indices : Stature—100.		
	C 5.	Negro 1.	Negro 2.	C 5.	Negro 1.	Negro 2.
Maximum breadth	242·5	217	279·5	13·25	14·32	17·47
Between anterior sup. spines	232	206	268·5	12·68	13·6	16·78
Conj. diam. of brim	106	90	105·2	5·79	5·94	6·58
Trans. diam. of brim	122·5	97	128·2	8·01	6·41	6·69
Breadth of sacrum	104·2	90	113·6	5·69	5·94	7·1

This table, though the series of measurements is incomplete, shows conclusively that the pelvis of C 5 at the time of his death was absolutely and relatively small. This observation is a most important support to the hypothesis in connection with the unequal distribution of absorptive power to the cartilage cells in the different parts of the body, as well as to the hypothesis that the general growth is due to increased nutrition of the somatic cells.

In Observation A 10 a skeleton belonging to the late type of development is described.

Observation A 10.

Abridged Description of the Body of an Anorchid dissected at the Royal College of Surgeons in Ireland.

Subject No. A 10. Age uncertain (between 20 and 30 ?), Male. Cause of death, Septic Peritonitis. Stature, 1675

General Description (see fig. 4B).

The subject was thin and ill-proportioned, the most striking peculiarity of his form being the great breadth of the perineum, and the long slope outwards between the crest of the ilium, and the prominence of the great trochanter. The cheeks were hollow and the face small; the skin freckled and glabrous. The penis and testicles were small.

The first step in the dissection was to remove the testicles. They measured 28 mm. in height, 19 mm. in antero-posterior diameter, 16 mm. in transverse diameter, in place of the normal $45 \times 25 \times 20$ mm. (38) of the average adult testicle.

As in A 9, the examination of the body was divided into three parts: (1) Microscopic examination of the testicles and ductless glands; (2) Dissection of the body; and (3) Examination of the macerated skeleton.

The testicles were functionless organs. No material abnormalities were discovered in the course of dissection.

The Skeleton.—The skull closely resembled the skull of A 9 (see fig. 5). There was a similar failure of the development of the anterior part of the antrum of Highmore, with its consequent diminution in the breadth of the face and nasal height. A similar increase in the depth of the alveolar process of the superior maxilla and in the symphysial height of the mandible.

The vertebral column was of interest. The ossification of the cervical and lumbar vertebræ was complete, whereas that of the dorsal vertebræ was incomplete, none of the dorsal epiphysial plates being joined.

Table XLV.—Proportions of the Different Segments of the Vertebral Column of A 10.

	A 10.	A 9.	Average (8).
Cervical Segment . .	21·3	20·1	21·8
Thoracic Segment . .	47·2	48·8	46·5
Lumbar Segment . .	31·5	31·1	31·7

The total length of the column was 530 mm., distributed as follows:—cervical segment 113 mm., thoracic segment 252 mm., lumbar segment

165 mm. The percentage proportions of the different segments are given in Table XLV. For comparison, the proportionate lengths of the spinal segments of A 9 and the average of six Irish spines are included (see Table VIII.).

These percentages show that the spine of A 10 approached to the type of A 9; and as the process of ossification in the thoracic segment was incomplete, the reasonable assumption is that had A 10 lived longer the approach would have been closer than it was.

An unusual feature of the atlas and axis vertebræ was the misfit between them. In no position could the inferior articular surfaces of the atlas be coapted with any accuracy to the superior articular surfaces of the axis (see fig. 6).

The ossification of the lateral plates of the sacrum was practically complete.

In the thorax the ossification of the ribs was incomplete. The first rib, like the first rib of A 9, was rather narrower than normal at its anterior end.

The more important measurements of the pelvis are given in Table XLVI.

Table XLVI.—Dimensions of the Pelvis.

	mm.
Breadth	263
Height	210
Breadth-height index	*79*
Transverse diameter	135
Conjugate diameter	91
Pelvic index	*67·3*
Length of sacrum	103
Breadth of sacrum	121
Sacral index	*117*
Height of innominate bone	210
Breadth of innominate bone	145
Height-breadth index	*69*

A comparison of the measurements in Table XIII. shows the pelvis as a whole to be small. The low pelvic index and the low height-breadth index of the innominate bones are evidence of the abnormality in pelvic form which is due to an increase in sacral and pubic breadth, and to unusual growth at the iliac crest and ischial tuberosity without corresponding growth at the triradiate cartilage.

The sacral index shows the sacrum to be a broad bone, and the incompleteness of its ossification that it had not attained its maximum breadth. Neither had the innominate bone attained to its maximum size, for the region of the pubic symphysis was still cartilaginous. The reasonable assumption therefore is that the pelvis had not acquired its definitive breadth.

On the other hand, it is difficult to believe that the length of the conjugate diameter was not approximately definitive.

In other words, the pelvis, as a whole, had nearly attained its definitive form when growth was reaccelerated. Because the triradiate cartilage was obliterated, this late growth could not affect the pelvis equally, but could only affect the lateral sacral, the pubic, the ischial, and the iliac epiphysial cartilages. Growth of these leads directly to increase in the breadth and height of the pelvis. The antero-posterior or conjugate diameter is unaffected by their activity. The evidence is definite that in this case the onset of anorchidism was late.

In the shoulder-girdle the clavicle of A 10 is almost a facsimile of the clavicle of A 9. It is somewhat shorter, but the antero-posterior diameter of its sternal end is identical with the corresponding diameter in A 9's clavicle, and the antero-posterior diameter of its acromial end is only one millimetre less than in the case of A 9. (See Table XV.)

The clavicular epiphysis is almost entirely cartilaginous.

The scapula of A 9, not including 7 mm. of cartilage along its vertebral border, is relatively much broader than the scapula of A 10 (see Table XVI.). Its measurements, including and not including the cartilage of the vertebral border, are given in Table XLVII.

Table XLVII.—Measurements and Indices of the Scapula of A 10.

	Bone.	Bone and Cartilage of Vertebral Border.
Length	115	151
Breadth	105	112
Scapular index	*69·6*	*74·1*
Infra-spinous length	114	114
Infra-scapular index	*92·1*	*98·2*

Including the cartilage, which would have become bone had life been prolonged, the relative breadth of the bone is phenomenal.

In the limb bones none of the epiphysial lines have quite disappeared, but they have all joined. In the region of the late closing epiphyses, more especially in that of the lower end of the femur, there is a marked broadening of the bone, with a peculiar splay upon the lower end of the diaphysis, which suggests that there was an unusual amount of endochondral bone formation after the central part of the intermediate cartilage was obliterated.

The measurements and comparisons of the bone lengths are given in Table XLVIII.

TABLE XLVIII.

Name of Bone.	Length, mm.	Corresponding Stature Range in normal Individuals.	Range of normal Bone Lengths for Stature 1675.	Minimum and Maximum Percentages of Excess of Length of Bones of A 10.
Humerus	324	1597·3–1701	339·7–319	– 4·8 . . –1·5
Ulna	238	1489·8–1585	267·5–251·5	–12·3 . . –5·6
Radius	222	1487·4–1578·4	250 –235·5	–12·6 . . –6·9
Femur	433	1528·4–1697·3	474·5–427·2	– 9·5 . . –1·3
Tibia	345	1490·4–1656	387·7–348·9	–12·6 . . –1·1
Fibula	334	1459·5–1609·8	383·2–348·7	–14·7 . . –4·4

This table shows most clearly that the relative deficiency in the length of the limb bones is most marked in the bones which have the early closing epiphyses, whereas the least deficiency is shown in the bones which have the latest closing epiphyses, namely, the femur and humerus.

This is exactly what is demanded by the hypothesis of the nature of the effect of functional removal of the testicles when growth is nearly complete. In cases of early removal, when all the epiphyses are open, the growth is greater in the more distal segments because of the greater absorptive power of the cells of the more distal cartilages, whereas in case of late removal growth is greatest centrally and least peripherally because of the order of closure of the late closing epiphyses.

As is well known, each of the principal long bones has at least two epiphyses, one of which closes considerably earlier than the other. The early closing epiphyses are situated in the neighbourhood of osseously strong joints (elbow, hip, and ankle). These usually close soon after puberty. The late closing epiphyses are situated in the neighbourhood of muscularly and ligamentously strong joints (shoulder, wrist, and knee). In the neighbourhood of the knee, the tibial epiphysis usually closes before the femoral; in the arm, the wrist epiphyses of the radius and ulna usually close before the shoulder epiphyses of the humerus.

In A 10 the functional failure of his testicles evidently occurred just before the last of his limb epiphyses (the proximal, humeral, and distal femoral) joined. Hence the reversal of his proportionate limb lengths. Further, as the central epiphyses of the pubis, the lateral epiphyses of the sacrum, the epiphyses of the vertebral border and inferior angle of the scapula, the epiphyses for the surface plates of the thoracic vertebral centra and for the heads of the ribs were all open, the growth that could take place was limited to them. Hence the peculiar shape of his pelvis and scapula and the unusual proportions of his vertebral column.

The knowledge that the form of which A 10 is a type can be produced in the absence of functioning testicles is strong evidence in favour of our hypothesis.

This completes the review of the effects of the removal of functioning sexual glands upon endochondral bone growth. The hypothesis that the absence of the testicles, whatever the cause of that absence be, sets free for the use of the somatic cells nutriment which would otherwise be required to meet the demands of spermatogenesis, accounts, without strain, for every test put upon it by the facts observed. It reduces to intelligible order the varying manifestations of the effects of testicular failure, which at first sight seem so complicated. With its help, it is easy to understand how it is that the body forms of anorchids constitute a series which ranges from the normal of the newly castrated boy to the normal of the castrated adult, and includes within its range (1) cases in which the limbs are long, the pelvis and scapula narrow, and the trunk short; (2) cases in which the limbs are long, the pelvis and scapula broad, and the thoracic segment of the trunk long; and (3) cases in which the limbs are short, the pelvis and scapula broad, and the whole trunk long; for all these variations can clearly be accounted for by the varying times of the establishment of a condition of somatic nutrition abnormal only in its favourability.

SUMMARY CONCLUSIONS AND THE END.

A. The immediate effects of the removal of functioning sexual glands are hypothesised as twofold:—

1. The demands upon the internal food-supply of the body are lessened.
2. The body is deprived of some internal secretion or nervous stimulus.

B. The mediate effects are certainly twofold:—

1. The cells of the epiphysial cartilages are stimulated more rapidly to proliferate.
2. There is an arrest in the development of the penis, the scrotum, the prostate, the antrum of Highmore, and possibly the brain.

C. There is no evidence to show why the arrest in the growth of these parts takes place. It may be supposed to be due to the absence of a stimulus to growth conveyed by an internal secretion or through the nervous system. However it be effected, it appears to differ in origin altogether from the stimulus to growth experienced by the epiphysial cartilages.

D. The normal mode of death of the cartilage cells in the process of endochondral ossification is by hypertrophy and surfeit. Increased rate of growth of the epiphysial cartilages is due to an increased rate of death by surfeit of the cartilage cells. This increased rate of surfeit is preceded by an increased rate of proliferation.

E. Since the composition of the arterial blood is uniform, departures from the normal inter-relations of the amount of growth can only be due to differences in the absorptive power of the cells of the different epiphysial cartilages.

F. When the sexual glands are removed or destroyed, there are departures from the normal inter-relations of the amount and rate of growth of the cartilages.

G. These departures are not fortuitous, but follow the same regular plan in men and animals.

H. So long as all the epiphysial cartilages are open, the stimulus to growth affects most markedly the forearm and leg, less markedly the femur and humerus, and, at first, still less markedly the limb girdles and the vertebral column. This leads to the evolution of the early type of anorchid body form which is exemplified by subject C 5. (See also Observations B 2, B 3, C 1, C 2, C 3.)

I. In cases of later onset of the state of anorchidism, and at a late stage in cases of early onset, the relative excess in length of the leg and forearm is less, whereas the pelvis is both relatively and absolutely larger. (Observations A 1, A 2, and A 3.)

Thus is evolved the central type of anorchid body form which is exemplified by subject A 9. To this type also belong the eunuchs whose growth is nearly or quite complete. (Observations C 1, C 2, C 3, C 4.)

This type may fittingly be called "eunuchoid."

K. In cases of late onset of the state of anorchidism, *i.e.* onset after the closure of all the epiphyses, there is no change in the proportions of the skeleton.

This is exemplified by the body form of the Skoptzys converted late in life (C 4).

L. Intermediate between the late, skeletally unaffected anorchids and the eunuchoids there is a type in which the body form is almost the reverse of the eunuchoid type. This is exemplified by A 10. The forearm and leg are relatively short, the femur and humerus of slightly increased size, the pelvis wide from side to side but shallow from front to back, the scapula broad, and the vertebral column, more especially the thoracic segment, long.

M. In cases of early testicular failure, the duration of the period of endochondral ossification is prolonged, it may be for years.

N. The bones of anorchids are smooth and graceful. (Observations A 9, B 1, C 1, C 2, C 3, (40)).

O. The skin of anorchids is glabrous, except in cases of post-pubertial onset.

P. In many, if not in all cases, of anorchidism the skin is freckled and pigmented.

In conclusion, I desire to express my indebtedness to Mr W. Gill for the great care with which he cleaned the bones of the skeletons of subjects A 9 and A 10, for his excellent microscopical preparations of the tissues specially examined, and for taking the photographs which illustrate this paper.

REFERENCES.

(1) Pirsche, E., *De l'influence de la castration sur le developpement du squellette*, Thèse de Lyon, 1902, A. Storck et Cie.

(2) Launois, P. E., and Roy, Pierre, *Études biologiques sur les géants*, Paris, 1904, Masson et Cie.

(3) Poncet, *Congrès de l'Association française pour l'avancement des Sciences* (Havres, 1877), quoted by Pirsche, *op. cit.*, and Eberth (reference 19).

(4) Lortet, "Allongement des membres inférieurs chez un eunuque," *Arch. d'Anthrop. criminelle, Lyon*, 1896, and *Soc. de Méd. de Lyon*, March 16, 1896 (latter reference from Pirsche, *op. cit.*).

(5) Hunt Morgan, T., *Experimental Zoology*, 1907, The Macmillan Company, New York.

(6) Minot, C. S., "Senescence and Rejuvenation," *Jour. of Physiol.*, xii., 1891; also "The Problem of Age, Growth, and Death," *Popular Science Monthly*, 1907, vol. lxx. No. 6, vol. lxxi. Nos. 1, 2, and 3; also *Age, Growth, and Death*, Popular Science Series.

(7) Geddes, A. C., "An Abnormal Nasal Duct," *Anat. Anzeiger*, vol. xxxvi., 1910.

(8) Cunningham, D. J., "The Skeleton of the Irish Giant, Cornelius Magrath," *Transactions of the Roy. Irish Acad.*, vol. xxix. part xvi., 1891.

(9) Geddes, A. C., "Acromegalic Giantism," Edinburgh University Thesis, 1908.

(10) Cunningham, D. J., *Cunningham Memoir*, No. II., "The Lumbar Curve in Man and the Apes, etc.," Roy. Irish Academy.

(11) Disse, Dr J., "Skeletlehre," *Handbuch der Anatomie*, von Prof. Dr Karl von Bardeleben, Bd. i. Abt. i., 1896, Jena, Gustav Fischer.

(12) Thane, G. D., *Quain's Elements of Anatomy*, 10th ed., vol. ii. part i.

(13) Verneau, M., "Le bassin dans les sexes et les races," Thèse de Paris, 1875, quoted by Poirier, livre 2me, tome 1er, *Traité d'anatomie humaine*, Paris, 1899, Masson et Cie.

(14) Langer, "Wachstum des menschlichen Skeletes mit Bezug auf den Riesen," *Denkschriften der kaiserlichen Akademie des Wissenschaften, Wien* (Math. Natur. Klasse), Band xxi., 1872.

(15) Topinard, P., *L'anthropologie.*

(16) Manouvrier, (1) *Bull. de la Soc. d'Anthrop. de Paris*, Séance du 23 Fév. 1893; (2) *Bull. de la Soc. d'Anthrop. de Paris*, tome vi. série iv., 1895.

(17) Thomson, A., *Text-book of Anatomy*, edited by D. J. Cunningham, 3rd edition, 1909, Hodder & Stoughton: article "Osteology."

(18) Krause, W., "Skelet der Oberen- und Unterenextremität," *Handbuch der Anatomie*, von Prof. Dr Karl von Bardeleben, Jena, 1909, Gustav Fischer.

(19) Eberth, C. J., "Die mannlichen Geschlechtsorgane," *Handbuch der Anatomie*, Prof. Dr Karl von Bardeleben, Jena, 1904, Gustav Fischer.

(20) Selheim, H., "Kastration und Knochenwachstum," *Hegars Beitr. zur Geburtsh. und Gynäk.*, Bd. ii., 1899.

(21) Brian, *Gazette hebdomadaire de Médicine et Chirurgerie*, Août, 1901.

(22) Bonnet, quoted by Launois and Roy, *op. cit.*, p. 87.

(23) Becker, *v. i.*, reference 37.

(24) Koudelka, "Das Verhältnis der Ossa longa zur Skelethöhe bei den Säugethieren," *Verhandl. des naturf. Vereins. Brünn*, 1885, Bd. xxiv.

(25) Ecker, K., "Zur Kenntniss des Körperbaues schwarzer Eunuchen," *Verhand. der Senken. Ges. in Frankfurt a.M.*, Bd. v.

(26) "Skoptzy Eunuchs," Launois and Roy, *op. cit.*, p. 141.

(27) Teinturier, E., "Les Skoptzys," *Progrès medical*, 1876–1877.

(28) Pelikan, E., *Gerichtlich-medizinische Untersuchungen über das Skopzentum in Russland*, übersetzt von Ivanoff Giessen, 1876.

(*Note.*—The figures in the text were taken from Launois and Roy, *op. cit.*, p. 142.)

(29) Pittard, E., (1) "La castration chez l'homme et les modifications qu'elle apporte," *Comptes rendus de l'Academie des Sciences*, 8 Juin 1903, p. 1411; (2) "Anthropologie de la Roumanie. Les Skoptzys: Modification anthropométriques apportées par la castration," *Bulletin de la Société des Sciences de Bucarest*, 1903.

(30) Geddes, Patrick, and J. A. Thomson, *The Evolution of Sex*, Walter Scott.

(31) Noël Paton, "On Folin's Theory of Proteid Metabolism," *Journal of Physiology*, vol. xxxiii., No. i., p. 1. (See also Folin, *Amer. Journ. of Physiol.*, vol. xiii. p. 117, 1905.)

(32) *British Association: Report of Anthropometric Committee*, 1883, quoted from Pye, reference 33.

(33) Pye, W., "On the Growth Rate of the Bones of the Lower Extremities, etc.," *Journ. of Anat. and Phys.*, vol. xxiii., 1889, pp. 116–123.

(34) Papillault, "L'homme moyen à Paris. Variations suivant le sexe et suivant la taille. Recherches anthropométriques sur 200 cadavres," *Bull. et Mém. de la Soc. d'Anthropologie*, 1902.

(35) Manouvrier, "Étude sur les rapports anthropométriques en général et sur les principales proportions du corps," *Bull. et Mém. de la Soc. d'Anthropologie*, 1902.

(36) Rollet, E., *De la mensuration des os longs des membres*, Lyon, 1889, quoted by Launois and Roy, *op. cit.*

(37) Becker, Dr Ph. F., "Ueber das Knochensystem eines Castraten," *Archiv für Anatomie und Physiologie*, Anatomische Abteilung, 1899.

(38) Piersol, G. A., *Human Anatomy*, Lippincott.

(39) Gruber, W., "Untersuchung eineger Organe eines Castraten," *Mullers Archiv für Anatomie, Physiologie, und wissenschaftliche Medicin*, Berlin, 1847.

(40) Geddes, A. C., "The Surface Details of the Bones of the Skeleton as an Index of Nutrition," *Proceedings of the Royal Physical Society of Edinburgh*, vol. xvii., No. 6.

EXPLANATION OF PLATES I.–III.

Fig. 1. Vertebral column of anorchid subject A 9. To show the unusual curvatures of the column. The upper part of the lumbar curve is flattened out. The thoracic portion of the column is nearly straight below; slightly convex forwards above. The cervical portion is straight and vertical.

Fig. 2. Sagittal section of the 11th thoracic vertebra of A 9. To show the wedge-shaped body, with its markedly greater depth behind.

Fig. 3. The clavicles of A 9, to show their great breadth at the acromial end.

Fig. 4. A. The skeleton of C 5 (after Becker). B. Photograph of subject A 10.

These are placed side by side to show the difference between the "**early**" and "**intermediate late**" types of anorchid body form. It is interesting to note that though the shoulders of the two subjects are on approximately the same level, the centre of the pelvic inlet of C 5 is on the same level as the umbilicus of A 10.

Fig. 5. Skull of A 10. To show the small size of the bodies of the superior maxillaries.

Fig. 6. Atlas and axis vertebræ of A 10. To show the misfit of their articulating surfaces.

(*Issued separately December* 26, 1910.)

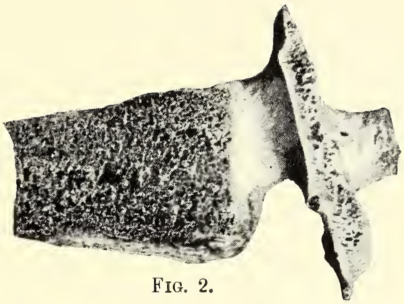
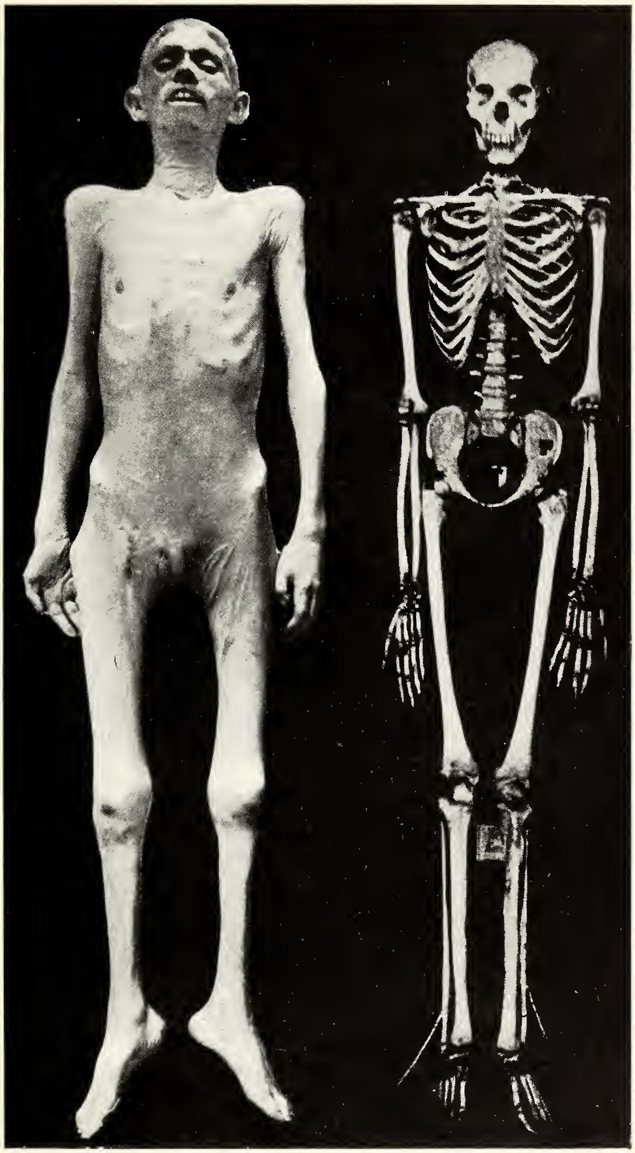


Fig. 2.

Fig. 3.

Fig. 1.



B. A.

Fig. 4.

Fig. 5.

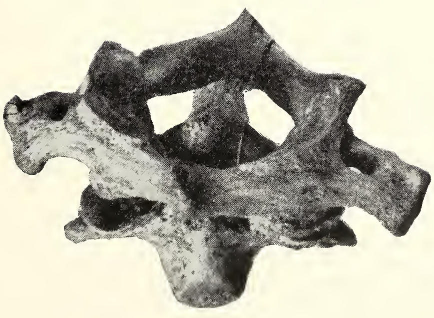


Fig. 6.

VII.—The Development of the Germ Cells in the Mammalian Ovary, with special reference to the Early Phases of Maturation. By A. Louise McIlroy, M.D., D.Sc. (From the Physiological Laboratory, University of Glasgow.) *Communicated by* Professor Noël Paton. (With Six Plates.) *

(MS. received June 1, 1910. Read July 4, 1910.)

Part I.—Introduction.

Although much literature has been published on the minute anatomy of the ovary, there are many controversial points remaining, and therefore the publication of the results of the examination of various mammalian types adds additional proof to the facts already stated by embryologists. Three years ago this work was taken up by me as a preliminary to an investigation into the distribution of the chromatic substance in cells which show malignant degenerative changes. The work has occupied so much time that the publication of the results has been necessarily delayed. In a paper on "Primary Cancer of the Ovary," published in 1906, I stated, as the result of a somewhat limited observation on the later stages of the development of the human ovary, that the cells of the stratum granulosum were mesoblastic in origin. This view I have modified by the more thorough and extensive investigation of the present paper.

Part II.—Historical.

The chief points of interest in the development of the ovary arise from the examination of the sex cells after their appearance on the genital ridge. Most observers are in agreement with the work of Waldeyer (28), who traced their early development from two thickened ridges of epithelium on the mesial aspects of the Wolffian bodies. The mode of implantation of these cells upon the peritoneum is still in doubt, some authors having traced the evolution of the sex cell from the peritoneal cell (Pflüger (19) and others), thus giving it a somatic origin. This theory, however, is not held by the majority of observers: the generally expressed opinion among recent writers is that the sex cells are differentiated at a very early stage, and are laid down while the segmentation of the ovum is

* The expenses of the research were defrayed by grants from the British Medical Association and Carnegie Trust.

taking place. The sex cells are therefore cells "sui generis." On a review of the literature, it is seen that while most observers are in agreement as to the origin of the sex cells from the genital ridge, the divergence of opinion lies in the origin of the cells of the stratum granulosum or follicle cells, the majority expressing the view that the ova and follicle cells have a common origin. Among these are Pflüger (19), Waldeyer (28), Balfour (1), Kölliker (12), Nagel (18), Bühler (5), Cœrt (7), von Velits (25), Whitridge Williams (30), v. Winiwarter (26), and Lane-Claypon (13), and their statements are based upon the proof of the persistence of small cells of the original epithelial type throughout all the stages of development of the cortical zone of the ovary, and the distribution of these smaller cells round the differentiated nuclei, both within the cell nests and separately. A minority of observers, including Foulis (9), Wendeler (31), Clark (6), Stevens (21), Griffiths and Williamson (11), take the view that the follicle cell is mesoblastic in its origin, on account of its spindle-shaped appearance, and also on the fact that many ova are seen to be lying in close contact with the stroma cells. It was not until the appearance of von Winiwarter's (26) paper in 1900 that the whole subject of the development of the ovary was put upon a clearer basis for the understanding of the growth of the germinal epithelium and the nuclear changes which take place in the germ cells. The material chosen by v. Winiwarter for examination was the ovary of the rabbit just before and after birth, and he describes the ovary in its early stages as consisting of two layers—a primitive cortical layer, made up of rows of epithelial cells, and a medullary, containing vascular stroma. The superficial layer of cells he calls "l'assise epitheliale." The growth of the stroma and epithelial cells is mutual, and nests or groups of cells are cut off by means of the stroma cells forming "les noyaux germinatifs." These nests become smaller and eventually contain one germ cell or ovum, with its surrounding follicle cells. The first formation in the ovary is that of the medullary cords, then the germinal epithelium proper or cell nests, and lastly the epithelial invaginations from the surface, all of these structures being in continuity, and each corresponding to a definite period of development of the ovary. In a recent series of papers by v. Winiwarter and Sainmont (27), (which have come into my hands since finishing this work), there is a full account of the development of the ovary from the examination of kitten embryos and the young animal after birth, and the conclusions arrived at differ somewhat in detail as to the distribution of the chromatin elements in the sex cells, but agree as to the main development of the ovary as a whole.

In Bryce's (4) *Embryology*, published in 1908, an outline of the development of the ovary is given. The main facts arrived at by an examination of the literature on the development of the ovary as a whole are that the growth of the germ and stroma cells is a mutual one; also that the medullary cords are the first structures to be differentiated after the formation of the genital ridge. These cords have in their substance small epithelial cells, which, according to v. Winiwarter and Sainmont, have an atrophic influence on the surrounding cells.

The confusion of terms between the tubes of Pflüger, cortical cords, and cell nests is easily explained when it is remembered that the observations of the different workers were drawn from a study of very varied mammalian types. In the kitten, these germinal cell clusters have the appearance of radiating tubules, in the puppy they are in cell masses, while they are not clearly differentiated in the human fœtal ovary, owing to the small amount of stroma present in that organ in its early stages. Although the different stages of development of the ovary occur at varying periods among the mammals, the human fœtal ovary in its early stages showing the same growth as some other mammalian ovaries after birth, the plan of development as a whole can be accepted from a comparison of the various types.

In the earlier accounts of the development of the ovary no mention of any length is made of the changes which occur in the nuclei of the germ cells, although Balfour has recognised some points of difference in these nuclei. The account of these changes must be sought for in the literature on the reproductive cells of plants and the lower animals, as the whole process of maturation can be followed from the earliest development of the sex cell up to fertilisation with almost mathematical precision. The favourite material for this observation is that of Ascaris, as in the mammalian types the early phases of maturation can only be followed, the ovum not being recognisable after its extrusion from the ovary until fertilisation takes place.

In mitotic division of somatic cells there is no reduction in the number of chromosomes, but in the reproductive cells this reduction takes place in the course of the last two cell divisions, four cells being formed, and having half the usual number of chromosomes. The germ cell divides into four, but only one of these cells forms an ovum; the other three form small cells which cannot develop, *i.e.* the polar bodies or "abortive eggs" (Mark (15)). The process of the formation of the polar bodies is given by Wilson (32) in detail. Von Winiwarter in 1900 describes the changes which take place in the germ cells up to the formation of the resting nucleus or germinal vesicle. The earliest cells or oogonia are small, oval, and granular, with masses of chromatin, and without a nucleolus. These cells proliferate rapidly by mitotic division, as in somatic cells. The superficial row of cells are larger and more spherical, "protobroque (*a*)," the remaining cells being smaller "protobroque (*b*)." The protobroque (*a*) cells divide and form similar cells which go to form the superficial layer on the ovary with its subsequent invaginations. The protobroque (*b*) cells form new oogonia, some of which become oocytes, others follicle cells. The author traces the changes which take place in the nucleus of the oocyte when it becomes larger, spherical, and clear, the chromatin spreading out as a fine reticulum, with a nucleolus and well-defined nuclear membrane "*deutobroque*" cells. Mitosis occurs in the protobroque cells, but ceases in the deutobroque variety. The chromatin filaments become twisted and form a fine spireme—*noyaux leptotenes*—and later this thread is gathered into a tangle at one portion of the nucleus, leaving a clear space with faintly staining nuclear membrane —*noyaux synaptenes.* The chromatin filaments emerge from the skein or *grumeau* and form a thick spireme filling the nucleus—*noyaux pachytenes* —the thickening of the cord being due to an apparent union of two parallel filaments by means of their achromatic processes. The final stages are where the cord becomes double, forming loops and rings—*noyaux diplotenes*—and later broken up and scattered throughout the nucleus forming the resting stages of the cell—*noyaux dictyés.* The development of the oocytes takes place from within outwards in the ovary, the further advanced stages being near the medullary portion. The cytoplasm of the cells increases in quantity with the increase in size of the nucleus, ultimately forming the deutoplasm. The more developed nuclei have rings of follicle cells surrounding them, and in the cytoplasm is seen a body of Balbiani. V. Winiwarter lays much stress upon the karyolytic changes which occur in the cells, but in his subsequent paper these degenerative changes are not given an important place. As nuclear changes increase, the process of mitosis becomes diminished, so that there are two stages in the growth of the ovary with regard to its cells—first the period of multiplication or mitosis, and second the period of growth or formation of oocytes of the first order. In the paper published by v. Winiwarter and Sainmont the protobroque cells are described, not as having granular nuclei, but as having some chromatin filaments in their protoplasm. An intermediate stage between them and the deutobroque cells is shown, where the nuclei contain filaments in the form of short rods and masses with a nucleolus—*noyaux transitoires* or *poussieroides.* The stage of the leptotene cells is described as being very transitory. The synaptic stage is further subdivided into *noyaux synaptenes abrégés.* Mitosis is occasionally seen in the deutobroque cells, but as an exception, and doubtfully also in the *noyaux* *poussieroides.* The nucleolus is more marked in the kitten's germ cells than in those of the rabbit.

Miss Lane-Claypon agrees with the work of v. Winiwarter in the main points, although she finds little evidence of a differentiation between the cells of the protobroque *a* and *b* varieties. Mitosis is found in the deutobroque cells for the most part. The oogonia form the follicle cells around the oocyte, their numbers being increased by the streaming in of others from the periphery. Some of the oogonia do not form follicle cells, but go to the formation of *interstitial* cells, which are metamorphosed deutobroque cells. The stroma cells only function as supporting and vascular. Bryce describes the nuclear changes in three phases,—a *pre-reduction* phase, which includes all generations up to that of spermatogonia or oogonia, the nuclei having the same number of chromosomes as the somatic nuclei. A *reduction* phase, which involves the generations, however, as spermatocytes, and oocytes of the first and second orders, and which have two divisions, during which the number of chromosomes is reduced to one-half that of the somatic number. The first division is called heterotypical, the second homotypical division. The *post-reduction* phase is the mature sex cell, and ends at its fertilisation, when the somatic number of chromosomes is restored in the segmentation nucleus formed by the union of the germ and sperm cells. The prophase of the first or heterotypical division is very prolonged.

A parallel process in the microsporcytes of plants is given by Jules Berghs (2), who shows that the chromatin filaments are in pairs in the synaptic stage. In his figures the nucleus is shown in different stages, the apparent reduction taking place in the synaptic stage; the true reduction only takes place by the separation of the daughter chromosomes towards the two poles of the first mitosis. Lerat (14) also gives a detailed account of the behaviour of the chromosomes in *Cyclops strenuus*, where are seen large cells with a nucleus four or five times greater than the others, being cells in all probability in repose. The skein in the synaptic stage is formed of numerous filaments, no centrosome being found which would influence the position of the chromatin mass. Two types of the synaptic stage are seen—the fine and thick filaments. The thick filaments split longitudinally, reduction during synapsis being only a pseudo-reduction. The nucleolus is present throughout all the nuclear changes, but stains faintly in some stages. At the stage of maturation the nucleolus exchanges chromation with the chromosomes, but not relationships; there is a "distinction parfaite entre le nucleole et les chromosomes." In the zone of multiplication mitosis takes place with the normal number of chromosomes, and therefore numerical reduction does not precede the first mitosis of maturation. During the period of synapsis the chromation element of the nucleus consists of a number of fine filaments; these assemble in a skein, out of which they emerge as a thick spireme which becomes reticulated. This thickened filament undergoes longitudinal division, which is only apparent as it is the separation of the fine filaments which were parallel in the previous stage. These fine filaments are probably the somatic chromosomes, and are in reality "chromosomes bivalents." The mitosis of maturation is identical in both spermatocytes and oocytes. The first mitosis is heterotypical and the second homotypical, and it is the latter which affects the numerical reduction (only apparent at the prophase), and consequently Cyclops verifies the pre-reduction type.

Farmer and Moore (8) point out that each heterotype chromosome is a bivalent structure, and their reduced number (half that of the somatic chromosomes) is due to the approximation and more or less intimate though temporary union of the equivalents of pairs of somatic chromosomes. The later stages of maturation in the oocyte can be studied from the papers of Victor Gregoire (10), Bolles Lee (3), Bryce and Van der Stricht (24).

The theories as to the duality of the filaments in the maturation phases are as follows:—(*a*) There is first longitudinal division of the filaments, separation then union of pairs, forming a thick cord, which undergoes a second longitudinal division. (*b*) The filaments of the first division remain parallel by means of their achromatic connections, having a thick appearance as of a single cord, then reappear as separate cords when these connections become lost. (*c*) In the synaptic stage two filaments unite and form a thick cord, then separate into parallel but distinct filaments. The third theory has the support of v. Winiwarter, owing to the observation that the concentration of the filaments in the synaptic stage is for the purpose of bringing different parts of the cord into contact, and by their union to form thick cords. When the chromatic segments are formed, the structure of the cord is modified, and their double constitution becomes visible, because the achromatic connections are relaxed. The synaptic stage is found in all animals, and the cords enter the skein singly. If the theory is true, the diminution of half the number of chromosomes at the time of the appearance of the first polar body is explained by union of the chromatin filament itself. The total number of segments will be equal to the typical number of chromosomes of the kind, but being united longitudinally two by two, their number is evidently diminished by half. Tetrads are formed by longitudinal division of each filament. The change which occurs in the nucleoli of the cell are referred to by most writers. Wilson has the view that there is one principal nucleolus and several accessory nucleoli.

The physiological function of the nucleoli is still involved in doubt, but there is strong evidence to prove that they do not contribute to the formation of the chromosomes. Moore and Robinson (17) have investigated the behaviour of the nucleolus, and find that it has much the same function as the nucleolus of the somatic cell.

Part III.—Present Investigation.

A. *Material and Methods.*

As the work of v. Winiwarter and Miss Lane-Claypon had been done on the rabbit, it was on the lines of their work that I first began my investigation. After working for some months on the rabbit embryo I abandoned this material, as the differentiation of the various structures in the ovary was not marked, the germ cells being small in size and their nuclear changes difficult to follow. For this reason, and also with the hope that fresh observations on parallel mammalian types would prove of more value, I carried out the investigation on the pig and human embryos, and the kitten and puppy for post-natal changes. In this way a plan of development has been formed by a comparison of the growth of the ovary in all these types, and conclusions have been arrived at which are in agreement with those of the majority of previous investigators. Some points in the development of the germ cells of the kitten and puppy differed from those recorded by v. Winiwarter in 1900. These points I demonstrated before the Physiological Section of the British Medical Association, July 1909. I find in the recent papers of v. Winiwarter and Sainmont (to which I have only had access within the last few months) that they have found the same changes in the kitten's ovary as I have found. The stages of the pig embryo were taken according to the measurements of the length of the embryo, as no approximation of the gestation period could be arrived at with accuracy; and as the embryo nearest the mesial portion of the uterus is more advanced in growth than that further away, the length measurement is the only accurate one to be used.

Embryos of 6, 8, 10, 12, 13, 14 cm. were examined, and finally 16–18 cm., the stage just before birth. The human fœtal ovaries were taken at the 4th, 5th, 6th, 7–7½ months, and the new-born. The puppies obtainable were new-born, 10 days after birth, 3 and 4 months. The kittens were 17 days, 4, 6, and 10 weeks. The examination of the rabbit's ovary was made on embryos of 21 and 26 days, at birth, 4 days after, and 4 months. The ovary of a new-born lamb was also investigated.

Methods.—For the young embryo the abdominal cavity was opened and the ovary fixed *in situ*, then removed later for dehydration and embedding. This method was found most satisfactory, as in the fresh state the organs are difficult to isolate and remove. Some very early pig embryos were sectioned *in toto* to show the genital ridge. I have found, in common with many other investigators, that the ovary, more especially that from the human fœtus, is an extremely difficult tissue to fix satisfactorily, and many methods were tried before a fixing solution was found which gave good results. The fluids used were Flemming's solution, Gilson's fluid, formalin 10 per cent., but the best results were obtained by using picro-formalin.* The length of time of immersion of the tissue varied according to the age of the embryo, very young tissue requiring only half to one hour; for older tissue six to twelve hours. The only difficulty in the method is the washing out of the picric acid by 50 per cent. alcohol. The sections were stained by Heidenhain's iron-hæmatoxylin method, but latterly I have used the Weigert rapid method. By the Weigert modification there is no time lost in mordanting the sections. The hæmatoxylin is made up in two stock solutions:—

Solution A.	Hæmatoxylin	1 grm.
	Alcohol 96 per cent.	100 c.c.
Solution B.	Ferri sesquichlorate	4 c.c.
	Official hydrochloric acid . . .	1 „
	Water	95 „

When required for use mix equal parts of A and B; precipitation ensues, and the mixture becomes quite black. The mixture must be freshly prepared, but can be used as long as no strong odour of ether is present (one to more days). Stain the sections for a few seconds, wash in water, counterstain in acid fuchsin picric acid † solution for a few seconds, wash rapidly in water, dehydrate in 90 per cent. alcohol, and clear in carbol-xylol, mounting in canada balsam.

B. *Ovary of Pig Embryo.*

Embryo 6 cm. in length. The ovary consists of an elongated narrow body with a pedicle or stalk. Under the microscope there is not much differentiation of structure of medullary or cortical zones. Through the

* Picro-formalin solution (Bouin)—

Sat. sol. Picric acid	75 parts
Formol	25 „
Acetic acid	5 „

† To 100 parts of an aq. sol. of picric acid, add 10 parts of 10 per cent. aq. sol. of acid fuchsin.

central portion are lacunæ filled with blood corpuscles, the walls of these spaces containing sparsely distributed oval stroma cells. In many places the blood corpuscles appear to be lying between the germ cells with a limiting membrane. The whole tissue is made up of small oval cells, granular, with a definite nuclear outline—the primitive sex-cells or oogonia. These cells appear to be scattered irregularly, but in places near the central zone they appear to be in rows of three or four cells: these rows correspond to the situation of the medullary cords. On the surface of the ovary the cells are already differentiated to form a row of protective cells or surface epithelium. For a clearer understanding of the description of the cells of the ovary this layer of cells will be referred to as the *capsular epithelium of the ovary.* The terms "germinal epithelium," "superficial epithelium," give rise to confusion among the various writers, and I feel that this term, "capsular," is the most appropriate, as it can be made use of when referring to the epithelium on the surface of the adult ovary. The cells of the capsular epithelium at this stage show a differentiation from the other oogonia by their slightly larger size, deeper staining, and elongated nuclei. They lie somewhat obliquely to the rest of the underlying cells, or have parallel axes. The nuclei appear granular, and contain several small masses of chromatin, one of which is probably a nucleolus. Very fine chromatin threads are seen in irregular distribution within the nuclear membrane. These correspond to the protobroque (a) cells of v. Winiwarter. The rest of the ovary is composed of smaller pale cells with rounded and oval nuclei, one pole appearing broader than the other. The chromatin threads are coarser than those of the capsular cells. The chromatin masses are not so large, and lie closer to the nuclear membrane. The nucleolus is present, but not well defined. Mitotic figures are numerous. Near the central zone are small elongated cells, *i.e.* stroma cells, the nuclei of which stain deeply. The oogonia form all the future cells of the ovary with the exception of the mesenchymatous tissue. Near the hilum of the ovary are a few gland-like structures, the rete tubules.

Embryo 8 *cm.*—Some differentiation is observed between the cortex and medulla; the oogonia are increased in number. The central vascular zone contains more stroma cells and blood spaces. There is a slight increase in the transverse diameter of the organ, and its whole volume is increased. The capsular cells are becoming placed more at right angles to the underlying cells. The cells in the deeper layers are larger, more spherical, with fine chromatin filaments in the granular protoplasm. The nuclear membrane is more distinct, with chromatin granules adherent to it. A nucleolus is seen in some cells. In the most central portion the cells are clear, with a well-defined nucleolus and marked chromatin filaments scattered irregularly through the nucleus. These are the oocytes of the first order, or deutobroque cells. Mitosis is seen among these cells. The stroma cells are beginning to push up through the germ cells, which in their turn burrow downwards, and thus egg clusters or nests are formed.

Embryo 10 *cm.*—The layer of oogonia is diminishing in breadth at the periphery, owing to the advance of the egg clusters. The stroma cells are pushing upwards and the central ends of the egg columns are rounded in outline. In the deeper portions of the nests are numerous oocytes of the first order, with a definite nucleolus and well-marked chromatin filaments. In addition to these are larger cells, where the filaments become lengthened and spread out over the nucleus; these filaments are in the form of very fine threads, and in the larger cells they are seen to be spread over the whole nucleus in the form of a spireme (noyaux leptotenes). The first stage of the oocyte may be called "the resting stage," and there are several transitional stages between it and the fine spireme stage, the latter being of extremely short duration. Like other observers, I cannot make out whether the filament is a single long cord cut in several lengths by the microtome, or numerous filaments from the beginning.

Embryo 12 *cm.* (fig. 1).—The cortical and medullary zones are well marked. The ovary is kidney-shape in outline. The cell nests are nearer the surface, very large nests being seen in the outer layers and smaller in the more central portions. In the medullary zone are large blood spaces. In the cell nests are masses of cytoplasm with irregularly distributed nuclei—the "syncytial masses"; some of the nuclei in these masses show signs of degeneration, and vacuoles are seen in the cytoplasm itself. The resting oocytes are in great numbers in the peripheral cell nests, and in the deeper are the transitional cells and spireme stage cells. In addition are cells with nuclei in synapsis, where the chromatin filaments are drawn to one side of the nucleus, and also transitional stages between these nuclei. The rest of the ovarian structure is made up of stroma cells and oogonia which have not become differentiated into oocytes. These cells are numerous under the capsular layer and at the medullary portion of the cell nests. Some of these cells are beginning to form round the larger cells—the early appearance of the primitive or primordial follicle. The medullary cords are not well marked, but some rows of small cells can be made out in the neighbourhood of the hilum; these show degenerative changes in their substance.

Embryo 14 *cm.*—The stroma cells have approached nearer to the periphery, and the layer of oogonia are therefore reduced in number.

Mitosis is seen among these cells and among the first stage of the oocytes. The cell nests are smaller near the centre and larger at the periphery. This is explained by the ingrowths of the stroma cells being more numerous and of greater density in the central portion. In the deeper layers are seen numerous cells in the stage of synapsis and in the thick spireme stage (pachytenes). The chromatin filaments seem to be thicker in the cells of the pig than those figured by v. Winiwarter in the rabbit. In the deepest layers of the cortical zone are seen primitive follicles with their enclosed oocyte, the chromatin filaments of which are arranged in the double-thread form (diplotenes). The oogonia of future follicle cells are not so numerous in the medullary portion as they were in the 12 cm. embryo. On the surface of the ovary are seen some invaginations of the capsular epithelium. The cells of this layer are placed with their axes at right angles to the underlying cells. In this embryo the nuclei in the synaptic stage are not numerous; the thick spireme stage nuclei are more numerous. (Fig. 2.)

Embryo 16 *cm.*—The germ cell nests are only in the superficial layers of the cortical zone. The capsular epithelium is isolated in places by the stroma cells, which reach the surface by means of a radiating distribution. In the medullary zone the stroma cells are loosely arranged, and there are large blood lacunæ and vessels with thick walls. Some glandular spaces are seen lined with cubical epithelium—the rete tubules. The primitive follicles in the deeper cortical layers are very numerous. The layer of oogonia has disappeared from the neighbourhood of the medullary zones. This disappearance is accounted for by the formation of the follicles in this region. Mitotic figures are not numerous, and are only seen in the most superficial layers of the ovary. The masses of cells forming a syncytium are well marked in the cell nests. Some of the nuclei in these masses have a much more definite outline than others, and it is probable that this syncytial mass may be used up as a pabulum for the developing oocytes. The blood-vessels are nearing the periphery of the ovary. The cells in synapsis are very numerous. The preceding stages of resting oocytes and their spireme cells are few in number. Thick spireme and double thread stage cells are seen, the latter being enclosed in follicles. In the medullary zone are seen groups of cells somewhat larger than oogonia. These cells are probably those which have been through one stage of development and there become reduced in size to form cells similar to the original oogonia, but with a layer and more spherical nucleus. These correspond to those described by Miss Lane-Claypon as "interstitial cells." The nuclei of the follicle cells are oval in shape, and have their axes at right angles to the cell membrane of the oocyte. In some the axes of the cells are parallel to the membrane, and these are the cells which have given rise to the statement that the follicle and stroma cell have a common origin. In some sections the nucleus is viewed from the side and has a fusiform appearance. Much stress has been laid by v. Winiwarter on karyolysis following the metaphase of mitosis. On observation of my specimens these nuclei do not call for comment, as they appear to me to represent a nucleus which has become degenerated during the process of mitosis. In the last paper on the kitten's ovary, v. Winiwarter does not appear to give to them the same degree of importance as in his previous publication.

It is seen on examination of the various stages of the pig embryo that the growth of the ovary corresponds with that of other mammalian types, but that it resembles the human fœtus more closely in its advanced stage of development before birth, primitive follicles being well marked in an embryo of 14 cm., corresponding to a rabbit of the third or fourth week after birth. The capsular epithelium is differentiated at a very early stage, and although mitosis occurs among its cells, it does not seem probable that these cells take much part in the formation of future oocytes of a permanent type. Mitotic figures are also seen in the epithelial invaginations, but I have found no evidence of the formation of differentiated nuclei from these cells. In the early formation of the follicle only a few cells arrange themselves round the oocyte; these increase in number, and the follicle takes on its typical appearance later. (Fig. 3.)

In the ovary of the pig the stages of development of the poles seem to vary, that pole nearest the abdominal end of the oviduct appearing in some sections to be somewhat further advanced than the opposite pole.

C. *Ovary of Puppy—new-born.*

The transitional stages in the germ cells are better seen in this ovary than in any other I have examined, therefore the description will be given in greater detail. The puppy's ovary is rounded in outline, being much thicker than that of the pig or human embryo, which is elongated; it is enclosed in a tough sac of peritoneum. The differentiation between cortex and medulla is not marked, as the stroma cells are so dense in this and in older stages. In the hilum are seen remains of the Wolffian body and gland-like spaces—the rete tubules. Numerous blood-vessels are seen, with a well-defined endothelial lining. The cortical zone is composed of nests of cells arranged in rows and columns; the convex borders of these columns are pressed into the medullary stroma. These nests are irregular in outline and branch in all directions, and are of various shapes and sizes. The smaller nests appear to be nearer the medullary zone. The capsular epithelium consists of a row of cubical cells arranged at right angles to the underlying cells. These cells are larger, and contain oval nuclei, which stain more deeply than the cells underneath. The chromatin is arranged in masses, one mass being larger than the others—the nucleolus. There are a few chromatin filaments scattered through the nucleus. The underlying cells are of the type of the oogonia and contain pale nuclei, with chromatin masses, short filaments, and a nucleolus. The chromatin granules are in contact in places with the nuclear membrane, giving it a more marked appearance. (Fig. 4.) The cytoplasm of the superficial layer is granular, and at the cell junctions there is a deeper stained portion, resembling the "glia" substance or intercellular cement described by Reid (20) in the epithelial cells of the cornea. The stroma cells have reached the capsular epithelium in places, and are separating it from the underlying tissue. (Fig. 5.) Underneath the capsular layer are masses of oogonia filling up the interstices between the cell nests, and having stroma cells scattered through them. Among the oogonia are numerous more spherical cells, which seem to be in a stage transitory between the oogonia and the oocytes of the first order, and which correspond to the "noyaux poussieroides" of v. Winiwarter and Sainmont. The larger spherical clear cells are well seen in these sections; some are just underneath the capsular layer, but the majority are within the cell nests. (Fig. 6.) The transition from the oogonium to the first oocyte stage (deutobroque) seems a prolonged process. The nucleus becomes spread over with fine filaments intersecting each other; these filaments increase in length and thickness until all the chromatin of the granules is taken up by the filaments, when we have the typical chromatin reticulum in the clear nucleus. In some nuclei the chromatin adheres to the membrane and to the filaments as granules, giving them a beaded appearance. The nucleolus is well marked, and several accessory nuclei are seen. Towards the medullary portion, but also just underneath the capsular layer, are seen enormous clear nuclei about three to four times the size of the first-stage oocyte. (Fig. 7.) These giant nuclei are usually solitary, but sometimes appear in clusters of three of varying size; they contain a large, well-defined nucleolus, and irregular masses of granular chromatin arranged around the nuclear membrane and near the nucleolus. These large nuclei seem to correspond to the resting stage of the cell, as described by Berghs in plants. After the first or somatic mitosis of the oogonium the cell remains in a resting stage, and then begins to take on the characters of the cell undergoing heterotypical division (prophase). These large cells may be a storehouse for chromatin for future chromosome formation. On the other hand, it may be a cell undergoing degenerative changes, becoming swollen and enlarged before atrophy and absorption take place. The cell membrane is indistinct, and the large nucleus appears in many places to be lying in a mass of granular cytoplasm or syncytium. As the ovary is examined from the periphery inwards, some cells in the fine spireme stage are seen, but this is a very transitory stage. The stage of the chromatin of the nucleus forming a reticulum—when the oocyte is first differentiated—is such a prolonged one that the chromatin filaments, becoming lengthened and gradually forming a spireme, can hardly be called a definite stage at any given moment. The chromatin spreads all over the nucleus as a fine filament, which becomes twisted into loops, as if wound round the nucleus like a spiral thread. At one side the threads seem to be more concentrated, and are gradually gathered into the skein close to the nuclear membrane. These fine filaments enter the skein in pairs, parallel rows, and loops; they become enmeshed in the chromatin mass, and at a further stage the whole nucleus is clear, with the exception of one side where the chromatin threads are closely intertwined. The nuclear membrane is present, but is faintly stained, giving the cells the appearance of a granular mass of cytoplasm (syncytium) with small rounded cavities in its substance; at the sides of the cavities are dark masses of chromatin. These are nuclei in the stage of *synapsis.* Fig. 8 shows an oocyte (*a*) with the chromatin distributed in the form of a reticulum and a well-marked nucleolus; (*b*) shows the thin spireme stage, with knots of chromatin at the intersection of the filaments. No nucleolus is shown, but it is present in many cells, although faintly stained. In (*c*) the filaments are gathered to one side of the nucleus and the membrane is not well marked. The filaments are entering the skein in pairs. In (*d*) a further stage is seen, with greater complexity of the filaments; (*e*) shows all the filaments in the mass, the loops appearing to be single and much thicker. The final stage of synapsis is seen in (*f*), where the loops of thick threads are becoming disentangled and spreading over the nucleus. These thick filaments have chromatin granules on their surface. These granules are also seen on the linin or achromatic threads, giving them a beaded appearance.

This last figure shows the first stage of the thick spireme nucleus with single filaments (pachytenes). There is no evidence that the skein in the synaptic stage is always formed at a particular pole of the nucleus; this is impossible to prove, owing to the varying directions in which the sections are cut. I could not find any evidence at this stage of an idiosome in the cytoplasm which might influence the position of the skein of filaments. Cells in the synaptic stage are seen in groups of four and five, and are deeper in the tissue of the ovary than the previous stages of oocyte development.

(Fig. 9.) Groups of cells are seen in synapsis. To left of field are cells in reticulum stage. To left of upper cell in synapsis is a nucleus showing a fine spireme. The cytoplasm is granular, and the cell membranes are not well seen. Surrounding the group of oocytes are dense bands of stroma cells, and in close contact with the group are oval cells, deeply stained; these are the oogonia or future follicle cells. The threads entering the synaptic skein are fine, those emerging are thicker. It would therefore appear as if a reduction of the filaments took place by means of this stage. On counting the filaments, the number which emerge are less numerous than those which enter, but no accurate conclusion can be arrived at, owing to the difficulty in making a count. It is most probable that two fine filaments become united by their achromatic threads to form a thick filament, which again separates, the union being a temporary one, and the succeeding longitudinal fission being only a pseudo-fission, as the filaments have never really united. The threads lying near one another are always fine, and there is no evidence of a thin and a thick filament lying side by side. Only a few nuclei in the thick spireme stage are seen in the ovary at birth. The most numerous cells are those of the first stage of the oocyte—deutobroque. In the deeper layers of the cortical zones are masses of oval cells lying in columns and tongues, divided up by the stroma cells. These cells have deeply-stained nuclei, oval in outline, granular, and with fine filaments a nucleolus is seen, and several masses of chromatin. Some of these cells have a more spherical and clear nucleus, with a reticulum of chromatin filaments. The cell outlines are well defined, and the cytoplasm is not granular. These cells have the appearance of oogonia and early oocyte stage cells, and are the future follicle and interstitial cells of the ovary. There may be a retrograde process taking place in these cells, causing them to assume an oval outline, but I cannot find any evidence of this. (Fig. 10.)

In the superficial layers of the cortical zone numerous cells in mitosis are seen in the forms of asters, diasters, and equatorial plates. These changes are occurring among the oocytes of the first stage as well as among the oogonia, being more numerous among the former. This is contrary to the view as expressed by v. Winiwarter. The filaments in the nucleus become thickened and applied against the nuclear membrane, the nucleolus increases in volume, and filaments are found in its vicinity. The filaments form chromosomes, which split and separate, and are scattered over the nucleus whose membrane is beginning to disappear. These chromosomes are formed into the equatorial plate, and lie on the spindle as thick rods, some being curved or horseshoe-shaped. In some cells the spindle threads are well marked; running into each attraction sphere with the row of chromosomes at the equator. In other cells the chromosomes form a diaster, but the majority of cells show the aster, where the chromosomes are distributed as a crown in the centre of the cytoplasm. No accurate count of these chromosomes can be made, as the individual rods are difficult to differentiate. Other cells show the chromosomes scattered in irregular masses or rods over the centre of the cell, and the nuclear membrane reforming round them. Numerous cells showing the latter formation are seen in these sections. Mitosis is not much in evidence among the cells of the capsular epithelium. (Fig. 11.) The chromatin spreads out in the form of filaments, a nucleolus appears and the cell takes on the form of the mother cell.

Some of these cells have their chromosomes broken up and degenerated. These are the cells in karyolysis, but they are not numerous, and are of little importance. Some of the cells, with the chromatin rods round their nuclear membrane, appear to have a resting stage before further changes take place. Mitotic figures are seen among the stroma cells, but there is no direct evidence of mitosis among the follicle cells at this stage. As the cell becomes surrounded by the stroma tissue, it develops so that the cells nearer the medulla are at an advanced stage. The masses of small cells at the periphery may develop as oocytes, or may become follicle or interstitial cells, or may act as nutrient cells to the growing oocyte. (Fig. 12.)

In the ovary, *ten days* after birth, there is not much change from the new-born. The layer of cell nests is smaller, the stroma cells are filling up the whole ovary, forming with the interstitial cells a dense mass of tissue. In the deeper parts cells in the thick spireme stage are seen. The interstitial cells in the central portions look like cords in outline, but no evidence is found of the so-called medullary cords further than that of the presence of some atrophied cells in the hilum.

Ovary of three-months-old Puppy.—In this specimen the changes from the previous stage are well marked. The capsular epithelium is entirely separated from the underlying cells by a band of stroma cells—the tunica albuginea. Invaginations into the deeper tissue are seen. The cell nests have disappeared, and in their place is seen groups of follicles arranged round the periphery of the ovary and coming close to the surface in places. (Fig. 13.) There is no distinction between cortex and medulla except that the blood-vessels are more numerous towards the centre. The whole centre is filled with masses of interstitial cells, divided up by stroma cells and blood-vessels. The cells surrounding the oocyte in the follicles consist of a coronet of cells with large oval nuclei, whose axes are perpendicular to the oocyte cell membrane, and in contact with it. These nuclei have a well-defined membrane, are granular, and contain masses of chromatin with a nucleolus. In some follicles in the deeper portions of the ovary the axes of the cells are parallel to the oocyte membrane. The nuclei are oval and have the appearance of stroma cells, being fusiform in outline. There is a considerable amount of cytoplasm in the follicle cells of the puppy. In many of the more superficial follicles the oocyte has fallen out, or has not been included in the sections examined. These empty follicles have the appearance of tubules lined with epithelium, almost columnar in type, and give rise to controversy as to their follicular origin. On examination of serial sections the oocyte is invariably found to be present, showing their origin as a primitive follicle, and not glandular, as some writers suppose. (Fig. 14.) In v. Winiwarter and Sainmont's work (*Arch. de Biol.*, Avril 1909) a drawing is given (fig. 9, pl. xi.) of a gland-like body very similar to these found in the cortex of the ovary. This is the figure of a tube of the epoophoron, but the structures in the ovary of the puppy cannot have any connection with these tubes as they are seen only in the periphery of the organ. For the deeper layers the follicle cells have the spindle-like forms. Fig. 15 shows follicles in the deeper layers.

In the ovary at *four months* the conditions are almost the same as at three months. In places large follicles are seen with several rows of cells, some containing one oocyte, others two or three, becoming divided up by the stroma ingrowths and follicle cells. Fig. 16 shows a mass of oocytes becoming separated up by ingrowths of follicle cells. Some of these oocytes are degenerating. To one side is seen a single oocyte, with smaller cells arranged round it. The nuclei at this stage show double filaments (diplotenes) and short paired rods (dictyés). The nucleus is now the germinal vesicle, and the principal nucleolus the germinal spot. The deutoplasm is granular. The follicle cells are in close contact with the cell membrane, but there is no appearance of processes entering this membrane for the purpose of nutriment. To all appearance this membrane is secreted by the cell cytoplasm itself. An organ of Balbiani is seen in some cells, but is not evident in the majority.

D. *Ovary of Kitten—seventeen days after birth.*

The ovary of the kitten is chiefly remarkable for the size of the germ cells, which stand out very clearly from the surrounding stroma cells. There is a distinct differentiation between the cortical and medullary zones. The latter consists of fairly dense strands of stroma cells, with numerous vessels and masses of small cells, which are cut off into groups by the stroma cells. In the cortical zone are large numbers of germ cells forming columns, "Pfluger's egg tubes." These columns are surrounded by dense strands of stroma cells. These stroma cells reach almost to the capsular epithelium, and in places are isolating it from the underlying tissue. The strands radiate out from the centre of the ovary. In the deeper layers of the cell nests there are solitary oocytes surrounded by stroma cells and follicle cells. The capsular layer consists of a single row of cubical cells, with deeply-staining nuclei arranged at right angles to the underlying cells. Just under this layer is a narrow zone of the primitive oogonial type, but the stroma cells have invaded this layer in places and cut off the cells into groups. The remaining portion of the cortical zone is made up of cell nests and columns. The larger aggregations are nearer the periphery, the masses becoming smaller as the centre is approached. In the groups near the surface are seen cells in mitosis. Some of the cells are in a state of karyolysis. Cells of the first oocyte stage are seen near the surface, and cells in synapsis and with thick spireme, but the majority of cells have the longitudinal fission of their filaments. These cells are surrounded by follicle cells. Fig. 17 shows cells from the surface of kitten's ovary; the stroma cells are isolating the capsular epithelium. In the medullary zone are masses of small cells, separated into groups by the stroma cells; these are the future follicle and interstitial cells, and may be called "reserve cells" on account of their function. They appear to be more numerous at this stage than at any other. A few oocytes with double filaments surrounded by follicle cells are seen in the deeper cortical layers.

Four weeks after birth.—There is not a marked difference in the structure. The cortical zone is broader, the cell nests filling the whole cortex, and the stroma cells forming a band under the capsular epithelium. Numerous cells in the first oocyte stage are seen and mitotic figures are also present. Cells in the synaptic stage are very numerous, more especially those with thick filaments. Thick spireme stage cells are also seen in numbers. When the cell is cut in transverse section the nucleus appears as if studded with dots of chromatin, thereby causing some confusion in the differentiation between these cells and those in the somatic anaphase of the oogonia. Nuclei with double filaments are seen in the deeper layers.

Four months after birth (fig. 18).—Groups of cells are seen in a narrow peripheral layer; there are numerous follicles in different stages of development, and the stroma cells are increased in number. The capsular layer is clearly differentiated from the underlying cells by a band of stroma cells. Epithelial invaginations are taking place from the surface. The cells are mainly in the double filament stage. There is no evidence of mitosis being present. Numerous masses of reserve cells are seen in the deeper portions of the ovary, some having large clear nuclei with nucleoli. The follicle cells near the deeper portions are spindle-shaped in outline, those nearer the surface being columnar or cubical. The deeper follicle cells proliferate rapidly and form several rows, and the follicles have the appearance of fully developed graafian follicles. These rows of cells become separated, an inner row adhering to the cell membrane of the oocyte; vacuoles form between the cells, and they appear irregular in their distribution. In the younger follicles the nuclei of the oocyte contain long double filaments, but in the more advanced stages the filaments are short and in the form of rings, figures of eight, and even tetrads. In some cells an organ of Balbiani can be made out.

E. *Human Fœtal Ovary—fourth month.*

The ovary at this stage is a long narrow body, showing on transverse section an oval or triangular outline. One pole is broader than the other, the cortex and medulla are differentiated, the medulla consisting of loose stroma and blood-vessels. In the deeper cortical layers are numerous follicles, some solitary, others in groups. The cortical zone is broad, and near its periphery are numerous cell nests, but these groups are not so easily differentiated as in other mammalian types, owing to the stroma being in very fine strands. At the periphery is a dense layer of cells, not divided into groups. The axes of the cells of the capsular epithelium are perpendicular to the underlying cells. The nuclei are oval or columnar in shape and are granular. Early invaginations of the epithelium are seen. Although the stroma cells have not reached the capsular epithelium, it can be differentiated easily from the underlying cells by a band of protoplasm, somewhat hyaline in appearance. There are numerous oocytes of the first stage in the peripheral layer, but the main number of cells are in the stage of synapsis, and also in the thick spireme stage. A few are seen undergoing transformation from the reticular or first-stage oocyte to the fine spireme. The chromatin filaments appear thick in the synaptic stages, and they have a beaded appearance. In the early synaptic stages the filaments appear fine and double, in the later they are thick and single. In the deeper layers of the cortical zone are long twisted double filament nuclei, with nodes of chromatin. The nucleolus is present during the whole development of the oocyte, but is faint in outline in the synaptic stage, where it is often lost to all appearance in the skein of filaments. It is well defined in the double-thread stage. Sometimes more than one nucleolus is seen, but a principal nucleolus is always to be made out in an eccentric position. The chromatin masses are irregular in size, but I have not seen the same outlines as those figured by v. Winiwarter and Sainmont. The cells with double filaments have a large amount of cytoplasm and a distinct cell membrane, which lies in contact with the follicle cells. Reserve cells are numerous near the early follicles. In the human fœtal ovary the cells are mixed up, and later stages of cells may be found near the periphery, while early stages are found near the central portion. The cells at the periphery are dense in appearance. The chromatin filaments are thicker than in other types, their distribution being easily made out in the nucleus. Syncytial masses are not so evident, and the cell membranes are well defined. Cells in mitosis are numerous in the peripheral layers. Irregular rows of cells in the hilum may be the remains of medullary cord formation (fig. 19). Nuclei containing short rods of chromatin are numerous, these being filaments cut in transverse section.

Fifth-month Fœtus.—The volume of the ovary is greater, the cortical zone is thicker, and the stroma dense and nearer the periphery. Mitotic figures are not so marked. Cells in synapsis are numerous, and those cells in the deeper layers show double filaments.

Sixth-month Fœtus.—The ovary is somewhat trilobed, with deep furrows on its surface. The differentiation between cortex and medulla is not well marked. The stroma cells are nearer the periphery. Cells of the first-stage oocyte are scanty and near the surface. In the deeper layers are synaptic, thick spireme and double filament stages. Follicles are more numerous, and contain a single row of follicle cells. Reserve cells are not so numerous.

Seven to seven-and-a-half months Fœtus.—The ovary is more flattened in appearance, with a distinct trilobed outline. The medulla is narrow. The capsular epithelium is isolated in places by the stroma cells. Follicles are seen all over the cortex, with oogonia in between. Reserve cells are not numerous. No oocytes of the first stage are seen. A few in synapsis and thick spireme are found, but the majority of the cells are in the double filament stage and surrounded with follicle cells. In some follicles the axes of the cells are perpendicular to the oocyte cell membrane, in others they are parallel to it. In the deeper cortical layers are several rows of follicle cells.

New-born.—The volume is greater and flattened antero-posteriorly. The stroma cells form a band under the capsular epithelium, whose nuclei are cubical in shape, their axes lying parallel or obliquely to the underlying cells. The oocytes are in the double-filament stage, those cells containing a large amount of deutoplasm which is granular and vacuolated in places. Some of the follicles contain two oocytes (fig. 20). The beaded appearance of the filaments is seen. These filaments are broken up into rods and loops in some places (dictyés). The stroma cells lie close to the follicles, but can be distinguished from them by their fusiform appearance. In the human fœtal ovary the plan of development is the same as in other mammalian types, although the distribution of the cells is somewhat irregular. Mitosis has ceased long before birth, and all the oocytes are in the double-thread stage before or just at birth.

Part IV.—Summary and Conclusions.

It is seen on examination of these four types of mammaliam ovaries that development takes place according to a fixed rule which is constant in all types, although some are further advanced in embryonic life than others. From the early formation of the female sex organ the germ cells develop through a given number of stages until they reach the resting nucleus or final stage (of the oocytes of the first order), the chromatin filaments being in the form of short double threads. The medullary cords are the first structures to appear in the genital ridge ; these atrophy early and disappear. Their function is not clearly proved, although some of their cells may form oocytes, whether permanent or not is not clear. All the remaining cells in the ovary are oogonia; these are small oval cells, with granular nuclei and masses of chromatin with a nucleolus faintly stained. The first differentiation which takes place is the formation of the *capsular epithelium.* The cells of this layer are larger in outline, and the axes of their nuclei in the early stages lie parallel to those of the oogonia (fig. 21). These nuclei afterwards become placed in a perpendicular direction, thus forming a cubical or columnar row of epithelium. Their function is mainly protective, and there is no evidence in my specimens that they form oocytes.

The epithelial invaginations are formed from a piling up of the cells like rows of coins, which become pushed down among the underlying tissue, forming gland-like processes, but without a circular lumen. The next change occurring is the mutual growth of the stroma upwards and germ cells downwards, the former penetrating between the latter and forming groups or cell nests.

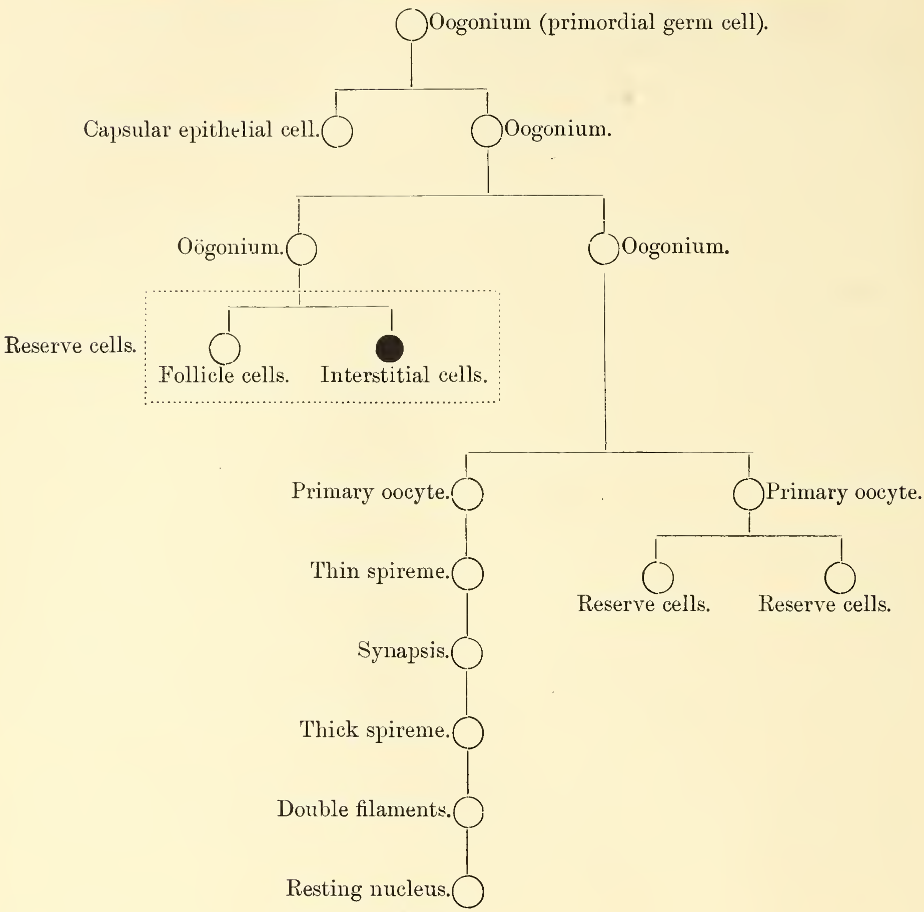
In the kitten's ovary these columns resemble tubules (Pflüger's tubules). As the ovary increases in volume the stroma cells come nearer the periphery, until finally the capsular epithelium is isolated by a band of stroma—the tunica albuginea. Fig. 22 shows the surface of a rabbit's ovary four days after birth, with the capsular epithelium at the surface. The stroma cells are fusiform in shape with deeply-staining granular nuclei; remains of the rete and Wolffian tubules are seen in the hilum of the ovary. The central portion of the cortical zone contains masses of cells which become follicle or interstitial cells, *i.e. reserve cells,* which are derived from the oogonia near the periphery. These cells differ in character, showing stages of progressive and regressive growth. The follicle cells are derived from the oogonia only, and have no relationship with the stroma cells. No evidence is found of processes extending from these cells to the oocytes which might be taken for food carriers, although it is most probable that the growing oocyte derives its nutriment from these cells. Mitotic figures are not seen in the follicle cells, their proliferation taking place in all probability by direct division. The germ cells in their early stages are oogonia when the nucleus is small, oval, and granular, with a nucleolus, and a few chromatin filaments and masses. These masses become broken up and scattered over the protoplasm and the filaments are more distinct. The nucleus becomes spherical, with a well-defined nucleolus; the chromatin is in the form of a reticulum, being beaded in appearance, the centre is clearer, and the nuclear membrane has chromatin granules attached to it by means of their linin processes. Mitosis takes place among the oogonia, and among these larger cells which may be called oocytes of the first stage (deutobroque). Some go on to form further advanced stages in the life-history of the oocyte. Others remain in a resting stage, or become smaller and function as follicle or interstitial cells. Those oocytes which are destined for fertilisation have the following stages.

The chromatin is spread over the nucleus as a fine reticulum like a thin spireme, the nucleolus is faint, *stage of fine spireme,* and is very transitory. The filaments become aggregated at one side of the nucleus near the membrane into the form of a skein of looped threads, some of which appear to be in pairs. This skein becomes more dense, until finally the whole of the filaments are gathered into a mass of chromatin substance. This mass becomes disentangled, and the filaments emerge as thick threads, and are spread out over the nucleus. The formation of the skein is called the *synaptic* stage, and is a prolonged one, and may be divided into three divisions—(*a*) where the fine paired filaments are entering the skein, (*b*) where all the filaments are in the skein, (*c*) where the thick single filaments are emerging from the skein. Synapsis is the most important stage of all, as it is during it that the filaments become united temporarily by their linin processes. This skein is not due to the fixing and staining processes employed in the preparation of the sections, as it has been seen without variation in all the types examined, and when fixed and stained by different reagents. The nuclear membrane is faint in outline; the nucleolus persists, but is lost sight of in some nuclei owing to the density of the skein. After emergence from the skein the filaments become spread over the nucleus and form a spiral—*the thick spireme stage.* This is also prolonged. During all these stages the nucleus is increasing in size and the cytoplasm becoming more abundant. In the next stage, that of the *double filament stage,* the thick rods are divided longitudinally and have the appearance of parallel pairs. This division is only apparent, as complete fusion has not been present (pseudo-reduction of the chromosomes). These filaments separate, some forming loops and rings. Transverse segmentation takes place, and the short filaments form figures of eight rings and short rods. This is the *resting stage* or germinal vesicle. It may last for years, and in the human ovary is very prolonged. The chromatin becomes broken up and the cytoplasm vacuolated. These chromatin filaments become more definite when the oocyte is ready for fertilisation at the time of the formation of the polar bodies. The nucleolus in the resting stage is well defined—the germinal vesicle; and the large amount of deutoplasm is the food for the future growing embryo. The cells of the ovary develop from the periphery inwards, the younger cells being near the surface. The stages of the cell development here described belong to the prophase of the heterotypical division. Reduction of the filaments is a temporary and only apparent process, as true reduction to half the number of somatic chromosomes does not take place until a late stage, *i.e.* after extrusion of the oocyte from the ovary, when it is termed an oocyte of the second order.

Multiplication by mitosis ceases at a given time, and in the human ovary the formation of the final stages of the oocytes of the first order is completed before or just after birth.

[DIAGRAM.

Diagram illustrating Oogenesis (Early Maturation Phases).



Conclusions.

1. My observations show that the same plan of development is followed among varied types of the mammalian ovary, the variation depending on the rate of development ante- and post-natal, the cells maturing from the periphery inwards.

2. The *capsular epithelium* is derived from the oogonia, and is differentiated at a very early stage, and has no function other than protective.

3. I find that mitosis occurs among the oogonia, and also among the primary oocytes of the reticular stage. In this latter point I differ from v. Winiwarter. Mitosis ceases at a given stage in the development of the ovary.

4. Large cells—*giant or resting cells*—are found by me in the reticular stage of the primary oocyte.

5. "Pflüger's tubules" are columns or clusters of cells derived from the oogonia, separated off by means of the stroma cells.

6. The growth of the stroma and germ cells is mutually correlated throughout the development of the ovary.

7. The follicle cells—which form the cells of the stratum granulosum—are derived from the oogonia, and not from the stroma cells. The latter are only supporting and vascular.

8. The *reserve cells* (oogonia and regressive oocytes scattered throughout the ovarian tissue) may function as follicle or as intersitial cells; they may become absorbed as pabulum for the developing oocyte.

9. The primary oocytes undergo transformations in their early maturation stages, which correspond to the prophase of the heterotypical divisions occurring in plants and animals.

10. The stage of *synapsis* I find is the most prolonged and of most importance. This agrees with Berghs and others. The chromatin filaments undergo a pseudo-reduction during this stage, and they undergo apparent longitudinal division in the stage following the formation of the thick spireme.

11. The nucleolus persists during all the stages of transition of the nucleus, although its staining capacity may vary.

12. I have found no evidence in the medullary cords of the presence of cells which may lead to the formation of primary oocytes. The glandular structures in the hilum are the remains of the Wolffian and rete tubules.

I gratefully acknowledge much kind help and encouragement in the carrying out of this research from Professor Nöel Paton; and also Dr G. Herbert Clark, who kindly procured specimens for the work. I am indebted for the photographs to Mr Richard Muir, Edinburgh University.

LITERATURE.

(1) Balfour, F. M., *Quart. Journ. of Microscop. Soc.*, vol. xviii., 1878: "On the Structure and Development of the Vertebrate Ovary."

(2) Berghs, Jules, *La Cellule*, t. xxii., 1905: "La formation des chromosomes hétérotypiques dans la sporogénèse végétale."

(3) Bolles Lee, A., *La Cellule*, t. xiii., "Les cinèses spermatogénétiques chez l'Helix pomatia."

(4) Bryce, T. H., *Quain's Elements of Anatomy*, vol. of Embryology, 1908. *Quart. Journ. of Micros. Soc.*, vol. xxiv., 1903: "Maturation of the Ovum in Echinus esculentus."

(5) Bühler, A , *Ziet. f. wiss. Zool.*, Bd. lviii., 1894: "Beiträge zur kenntniss der Eibildung beim kaninshen und der Markstränge des Eierstockes beim Fuchs und Menschen."

(6) Clark, *Johns Hopkins Hosp. Rep.*, vii., 1898: "The Origin, Growth, and Fate of the Corpus luteum."

(7) Cœrt, H. F., *Proefschrift*, Leiden, 1898, quoted by v. Winiwarter and Bryce.

(8) Farmer, F. B., and F. E. S. Moore, *Quart. Journ. of Micros. Soc.*, vol. xlviii., 1905: "On the Maiotic Phase (Reduction Divisions) in Animals and Plants."

(9) Foulis, *Quart. Journ. Micros. Soc.*, vol. xvi., 1876: "On the Development of the Ova and Structure of the Ovary in Man and other Mammalia."

(10) Grégoire, *La Cellule*, t. xxii., 1905 : "Les résultats acquis sur les cinèses de maturation dans les deux règnes."

(11) Griffiths and Williamson, *System of Gynæcology*, 1906: chapter on "The Ovary."

(12) Kölliker, A. V., *Entwicklungsgeschichte*, II., Aufl. 1879, Leipzig.

(13) Lane-Claypon, F., *Proc. Roy. Soc.*, 1905: "On the Origin and Life History of the Interstitial Cells of the Ovary of the Rabbit."

(14) Lerat, P., *La Cellule*, t. xxii., 1905: "Les phénomènes de maturation dans l'ovogénèse et la spermatogénèse."

(15) Mark, E. L., *Bull. Mus. Comp. Zoöl. Har. Coll.*, vi., 1881, quoted by Wilson.

(16) McIlroy, A. L., *Journ. of Obst. and Gyn. of Brit. Emp.*, 1906: "Primary Carcinoma of the Ovary."

(17) Moore, F. E. S., and L. E. Robinson, *Quart. Journ. Micros. Soc.*, vol. xlviii., 1905: "On the Behaviour of the Nucleolus in the Spermatogenesis of Periplaneta Americana."

(18) Nagel, W., *Arch. f. Mikr. Anat.*, Bd. xxxi., 1888: "Das menschliche Ei."

(19) Pflüger, W., *Ueber die Eierstöcke der Säuger u. des Menschen*, Leipzig, 1863.

(20) Reid, T., *Giornale della Reale Accad. di Med. di Torino*, 1904: "Riserche microscopiche fatte dal professore Thomas Reid di Glasgow della presenza fra gl' epiteli di elementi cellulori connettivali."

(21) Stevens, G., *Trans. Obst. Soc. Lond.*, vol. xlv.: "The Fate of the Ovum and Graafian Follicle in Præ-menstrual Life."

(22) Valentin, *Handbuch der Entwickelungsgeschichte der Menschen*, Berlin, 1835.

(23) Van Beneden, Ed., *Arch. de Biol.*, t. i., 1880: "Contribution à la connaissance de l'ovaire des mammifères."

(24) Van der Stricht, O., *Bull. de l'Acad. Roy. de Féd. de Belgique*, xix., 1905: "La structure de l'œuf des mammifères; seconde partie: structure de l'œuf ovarique de la femme."

(25) Von Velits, *Zeit. f. Geb. u. Gyn.*, xvii., 1889: "Beiträge zur Histologie u. Genese der Flimmerpapillärkystomen der Eierstocks."

(26) Von Winiwarter, H., *Arch. de Biol.*, 1900: "Recherches sur l'ovogénèse et l'organogénèse de l'ovaire des mammifères (lapin et homme)."

(27) Von Winiwarter, H., et Sainmont, G., *Arch. f. Biol.*, t. xxiv., Août 1908, Avril et Septembre 1909.

(28) Waldeyer, W., *Eierstock u. Ei*, Leipzig, 1870.

(29) Weigert, K., *Zeit. f. Mikros.*, Bd. xxi., 1904: "Eine kleine Verbesserung der Hämatoxylin—Van Gieson Methode."

(30) Whitridge Williams, F., *Johns Hopkins Hosp. Rep.*, iii., 1892: "Papillomatous Tumours of the Ovary"

(31) Wendeler, *Martin's Krankheit der Eierstöcke, etc.*, 1899.

(32) Wilson, E. B., *The Cell in Development and Inheritance*, 1906.

DESCRIPTION OF FIGURES.

Fig. 1. *Pig embryo*, 12 cm., photo, × 20 d., showing cortical and medullary zones.

Fig. 2. *Pig embryo*, 14 cm., photo, × 90 d., showing primitive follicles in cortex and blood sinuses in medulla.

Fig. 3. *Pig embryo*, 14 cm., drawing oc. 6, $\frac{1}{12}$ ol. i. Z., primitive follicle, with oocyte and follicle cells surrounding it.

Fig. 4. *Pig embryo*, 14 cm., drawing oc. 6, $\frac{1}{12}$ ol. i. Z., cells from capsular epithelium, darker in staining than oogonia underlying them.

Fig. 5. *New-born puppy*, photo, × 300 d., showing capsular epithelium on surface, nests of cells throughout cortex.

Fig. 6. *New-born puppy*, drawing, 6 oc. $\frac{1}{6}$. Capsular epithelium seen, cells underneath large and clear, first stage of oocyte, one cell seen in mitosis.

Fig. 7. *New-born puppy*, photo, × 500 d. Cell nests in cortex. Large solitary cell seen.

Fig. 8. *New-born puppy*, drawing, oc. 6, ol. im. $\frac{1}{12}$ Z., oocyte in the various stages of synapsis.

Fig. 9. *New-born puppy*, photo, × 1·000 d. Group of oocytes in stage of synapsis. To left of field are cells in reticulum stage.

Fig. 10. *New-born puppy*, photo, × 600 d. "Reserve" and interstitial cells in central portion of cortical zone.

Fig. 11. *New-born puppy*, photo, × 1000 d. Mitosis occurring in oocytes of the first stage.

Fig. 12. *New-born puppy*, schematic drawing of cortex, showing capsular epithelium, oocytes in various stages. Blood sinus. Masses of reserve cells near the centre.

Fig. 13. *Three months old puppy*, photo, × 300 d. Surface of ovary, capsular epithelium cut obliquely. Glandular-like spaces seen.

Fig. 14. *Three months old puppy*, follicle with oocyte in centre.

Fig. 15. *Three months old puppy*, photo, × 600 d. Follicles in deeper layers of ovary, oocytes showing double threads in a reticulum, with well-marked nucleolus and a great quantity of cell deutoplasm.

Fig. 16. *Three months old puppy*, drawing, $\frac{1}{12}$ ol. im. oc. 6. Follicle showing resting nucleus of oocyte (a); nucleus fallen out (b); follicle cells dividing up oocytes (c); stroma cells (d); isolated follicle (e).

Fig. 17. *Kitten, seventeen days old*, drawing, 6 oc. $\frac{1}{6}$, cortex stroma cells isolating capsular epithelium. Large oocytes seen.

Fig. 18. *Kitten, three months old*, photo, $\times$ 250 d., showing nests of large oocytes. Two follicles seen, one containing an oocyte and one without.

Fig. 19. *Human fœtus, four months old*, schematic drawing. Cells aggregated near the surface. Oocytes in various stages are seen. Some follicles near the centre; (*c*) cortex; (*bl.*) blood sinus.

Fig. 20. *New-born human ovary*, photo, $\times$ 1000 d., showing follicle containing two oocytes. The chromatin filaments have a beaded appearance.

Fig. 21. *Rabbit embryo, twenty-six days'*, $\times$ 600 d., showing differentiation of capsular epithelium on surface. The larger nuclei underneath are oocytes in reticular stage.

Fig. 22. *Rabbit, four days old*, photo, $\times$ 400 d., cortex showing capsular epithelium and cell nests.

(*Issued separately December* 29, 1910.)

× 20 Diam.

Fig. 1.

Pig Embryo. 12 cm. Showing cortical and medullary zones.

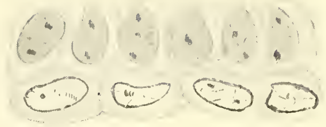


Fig. 4.

Capsular Epithelium from Surface of Ovary of New-born Puppy. Oogonia seen in deeper layer.

Fig. 2. (Diagram.)

Pig Embryo. 14 cm. Showing nuclei in synaptic stages. A few primitive follicles are seen in medullary portion. Blood sinuses are also seen.

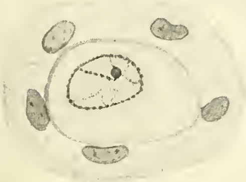
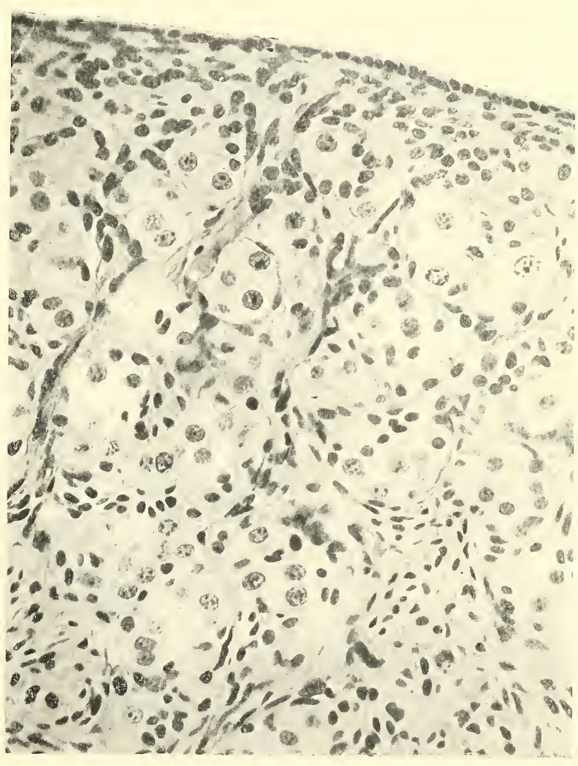


Fig. 3.

Oocyte from Pig Embryo. 14 cm. With follicle cells surrounding it.



× 300 Diam.

Fig. 5.

Surface of Ovary of New-born Puppy, showing capsular epithelium and cell nests.

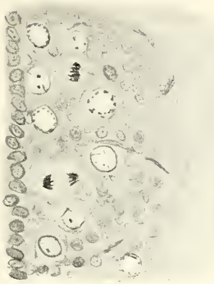


Diagram. Ol. Im. $\frac{1}{12}$.

Fig. 6.

Surface of Ovary of New-born Puppy, showing capsular epithelium, cells underneath in mitosis and in *deutobroque* stage.

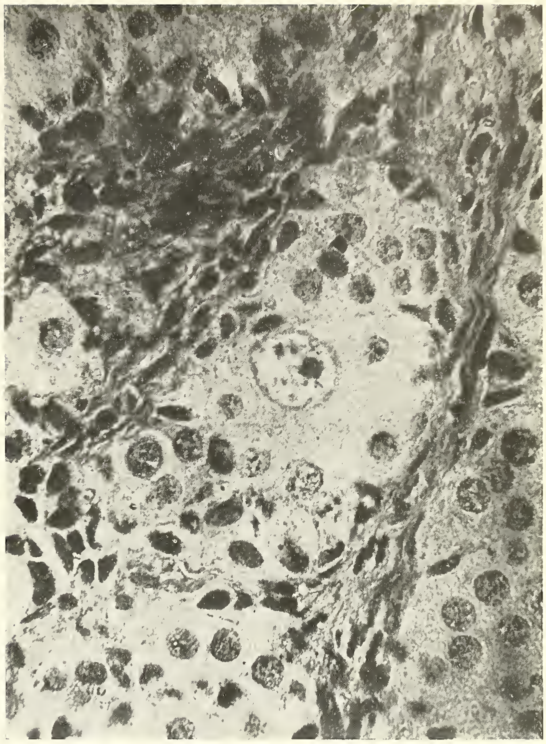


Fig. 7.

Ovary of New-born Puppy. Cell nests with large solitary (giant) cell from near the surface of the ovary.

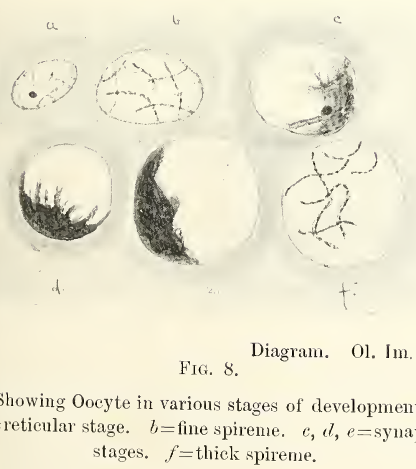
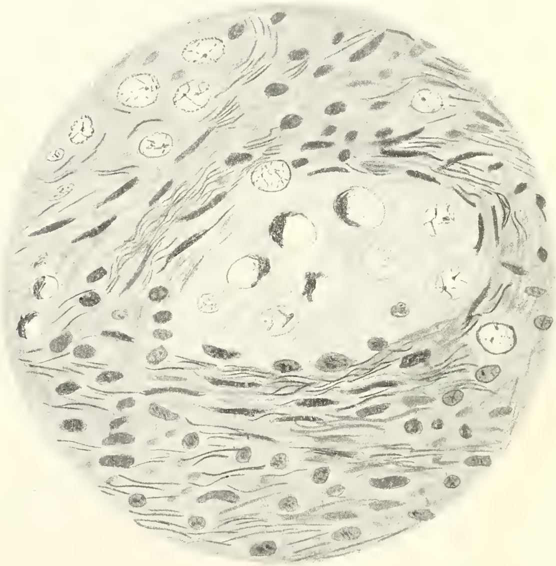


Diagram. Ol. Im. $\frac{1}{12}$.

Fig. 8.

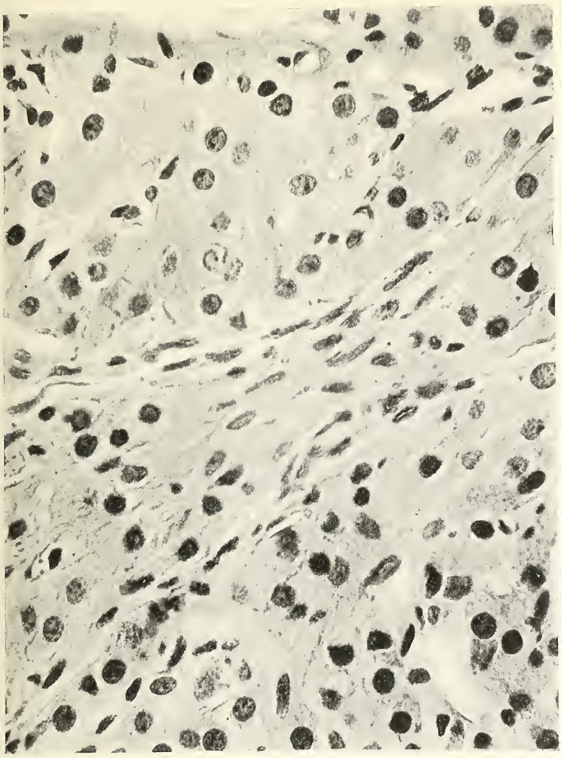
Showing Oocyte in various stages of development. a=reticular stage. b=fine spireme. c, d, e=synaptic stages. f=thick spireme.



× 1000 Diam.

Fig. 9.

Groups of Cells in Synapsis. To left of field are cells in reticulum stage.



× 500 Diam.

FIG. 10.
Interstitial Cells in Central Portion of Cortical Zone of New-born Puppy's Ovary.

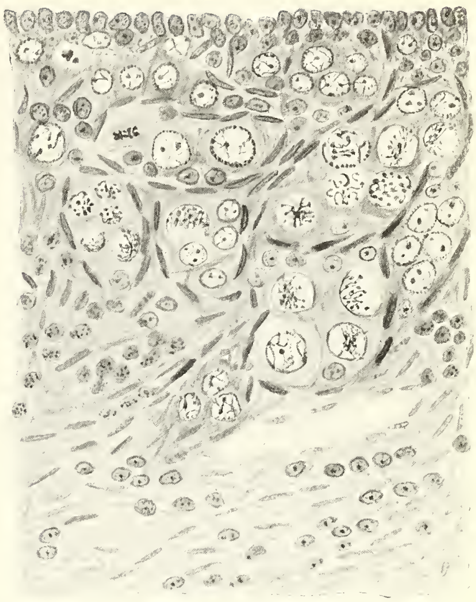
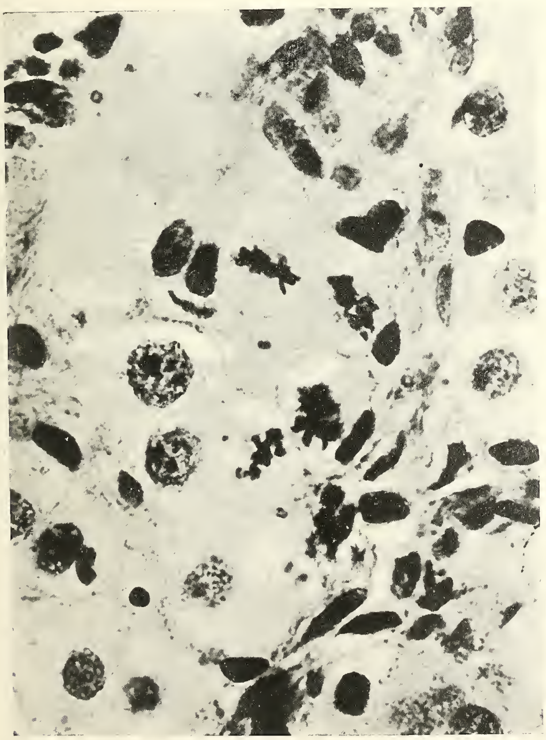
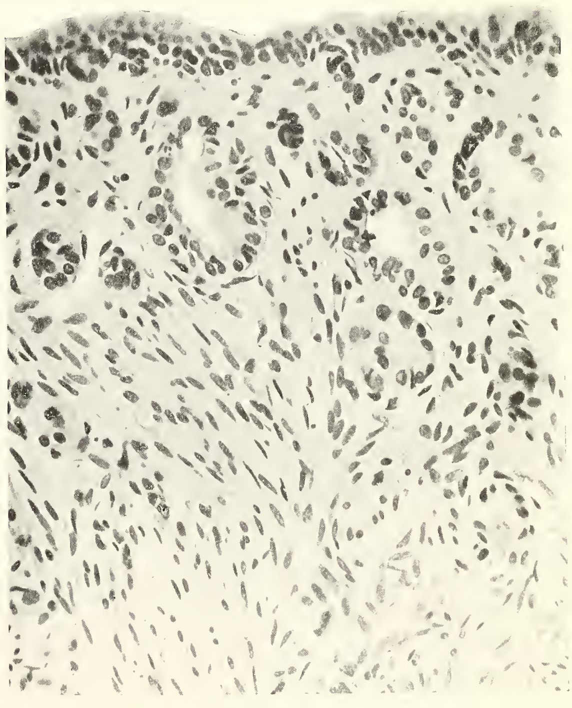


FIG. 12.
Schematic Drawing of Cortex of Ovary of New-born Puppy.



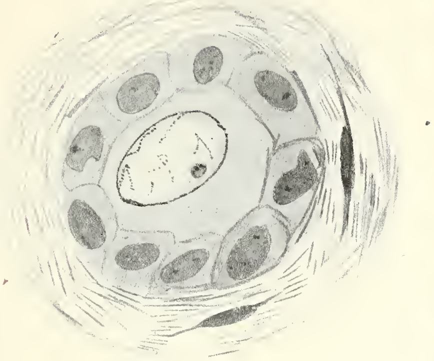
× 1000 Diam.

FIG. 11.
Mitosis occurring in Oocytes of first stage.



× 300 Diam.

FIG. 13.
From Surface of Ovary of three months' old Puppy. The capsular epithelium is cut obliquely. The gland-like spaces are seen.



Ol. I. $\frac{1}{12}$ Z.

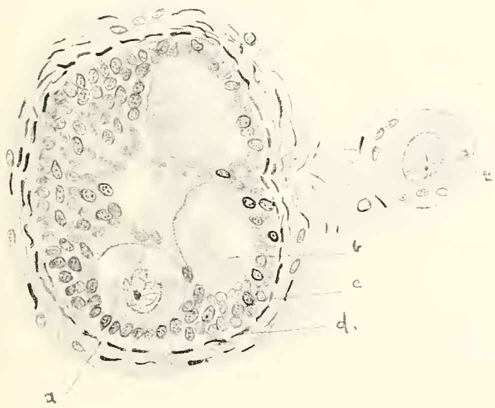
Fig. 14.

Follicle with Oocyte in its Interior.
Two stroma cells are seen.

× 600 Diam.

Fig. 15.

Follicles in deeper layers of Ovary of three months' old Puppy. Note spindle shape of follicle cells.

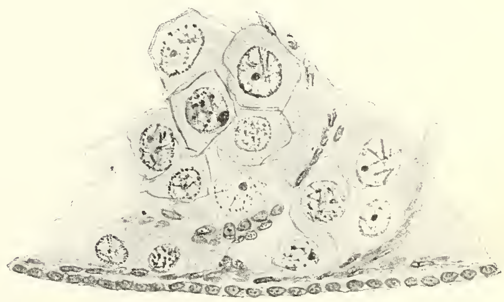


$\frac{1}{12}$ ol. Im. Zeiss.

Fig. 16.

Mass of Oocytes becoming separated up by ingrowths of follicle cells.

a=nucleus. b=cell with nucleus fallen out.
c=follicle cells. d=stroma cells. e=single Oocyte.



$\frac{1}{12}$ ol. Im. Zeiss.

Fig. 17.

Surface of Kitten's Ovary 17 days after birth.
Stroma cells are isolating the capsular epithelium.

× 250 Diam.

Fig. 18.

Surface of Ovary of three months' Kitten.

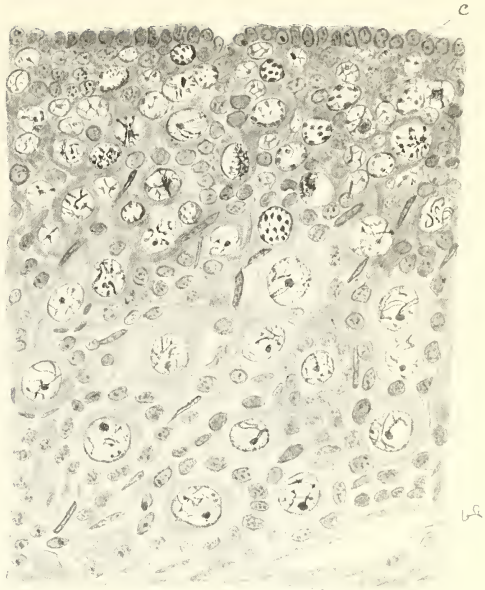
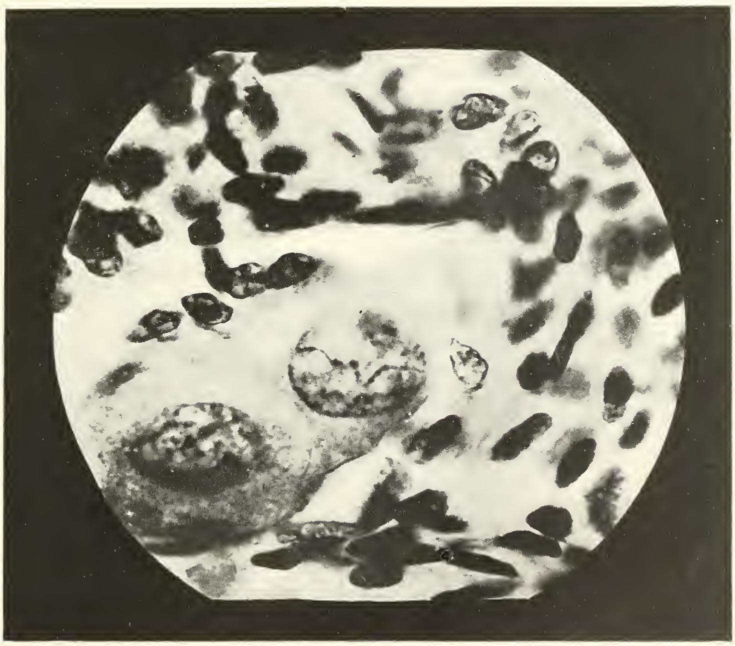


Fig. 19.

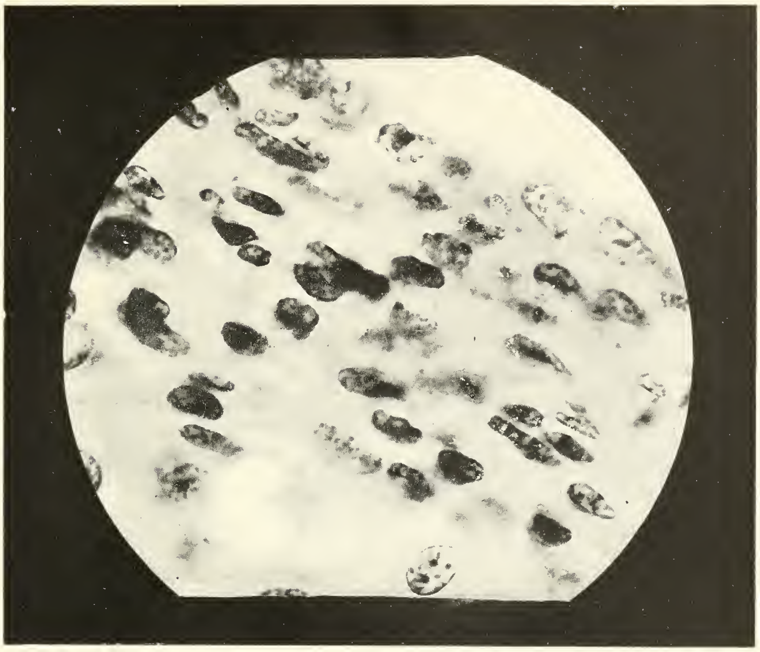
Schematic drawing of Cortex from an Ovary of a fourth month human Fœtus.



× 1000 Diam.

Fig 20.

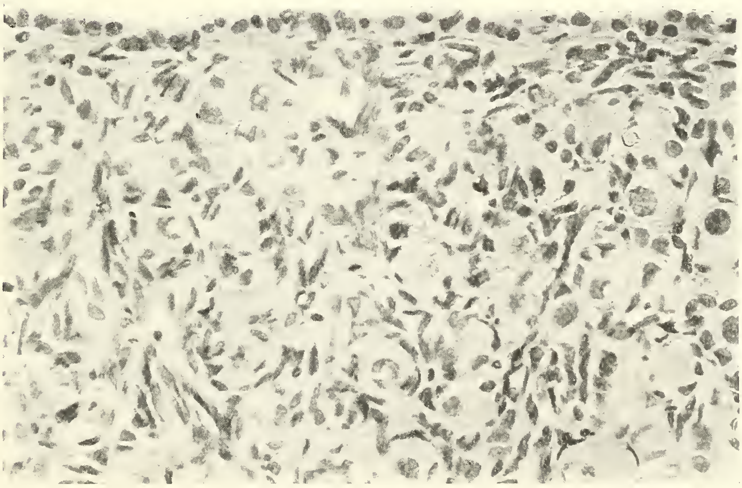
Ovary of New-born (human) Follicle containing two Oocytes.



× 800 Diam.

Fig. 21.

Ovary of Rabbit Embryo, 26 days, showing differentiation of capsular epithelium. The clear cell in lower part of photograph is in the reticular stage.



× 250 Diam.

Fig. 22.

Surface of Rabbit's Ovary 4 days after birth, showing capsular epithelium and cell nests.

VIII.—A Dynamic Method for measuring Vapour Pressures, with its Application to Benzene and Ammonium Chloride. By Alexander Smith and Alan W. C. Menzies.

(MS. received July 25, 1910. Read November 7, 1910.)

The simple "submerged bulblet" apparatus for determining vapour pressures, previously described in these *Proceedings* (vol. xxx. p. 437), may be modified to assume the form shown in fig. 1. The substance,

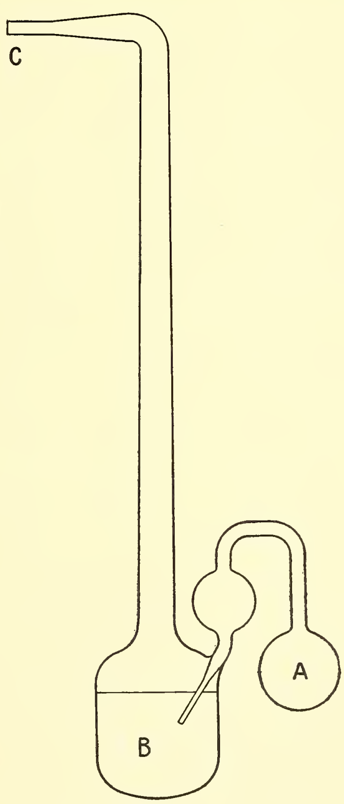


Fig. 1.

solid or liquid, is placed in the bulb A of this dynamic isoteniscope. The advantages gained are, that the new apparatus may be used for high pressures, that the confining fluid (bulb B) is reduced in amount, and that the thermometer is inserted directly in the bath and its bulb is, therefore, no longer in a region of varying pressures (*Proc.*, p. 436). The bath liquids, the mode of operation, and the corrections for submerged depth and capillary ascension are the same as before. Most of the advantages of this method of determining vapour pressures have already been enumerated (*l.c.* and *Proc.*, 434). The chief of these is that the vapour pressures of solids and of substances which interact with mercury may be determined, since any suitable material may be used as confining fluid. The apparatus has the advantage over the static isoteniscope (*Proc.*, 523) that here the confining fluid may have a considerable vapour pressure of its own, and the substance may be soluble in the confining fluid. Even interaction of the substance with the confining fluid does not necessarily interfere with the success of the measurements.

In the determinations given below, the bath, stirrer, gauge, and platinum resistance thermometer used with the static isoteniscope were employed. The temperatures, on the thermodynamic scale (S.B.P. assumed 445°), are accurate to ±0·01° below 120° (benzene), and to ±0·1° above 120° (NH_4Cl).

Vapour Pressures of Benzene.—In previous determinations, which are numerous, guarantees, either of the purity of the benzene or of the exactness of the experimental method, or both, are lacking. Sulphuric acid was used as confining fluid, and the values found were:—

t.	*p.*	*t.*	*p.*	*t.*	*p.*
65°	463	85°	879	105°	1542
70	551	90	1018	110	1751
75	650	95	1180	115	1983
80	757·5	100	1348	120	2240

Vapour Pressures of Ammonium Chloride.—In the previous determinations of Horstmann the temperatures are uncertain ±5°. In those of Ramsay and Young great experimental difficulties were encountered, and considerable divergences exist amongst the data, so that one may doubt whether the errors of ±1° at the lower temperatures and ±2° at the higher were not considerably exceeded. F. M. G. Johnson's measurements afford no assurance whatever of accuracy. At 335°, an error of ±1° corresponds to an error of ±15 mm. in the pressure. The existing values appear, therefore, to stand in need of confirmation or revision.

In our measurements, the mixture of potassium and sodium nitrates was used in the bath and as confining fluid. Since the vapour of the salt gave, by interaction with the confining fluid, a continuous stream of nitrous oxide, the method had to be modified, and the pressure at which the gas-evolution reached a minimum was taken. The error on

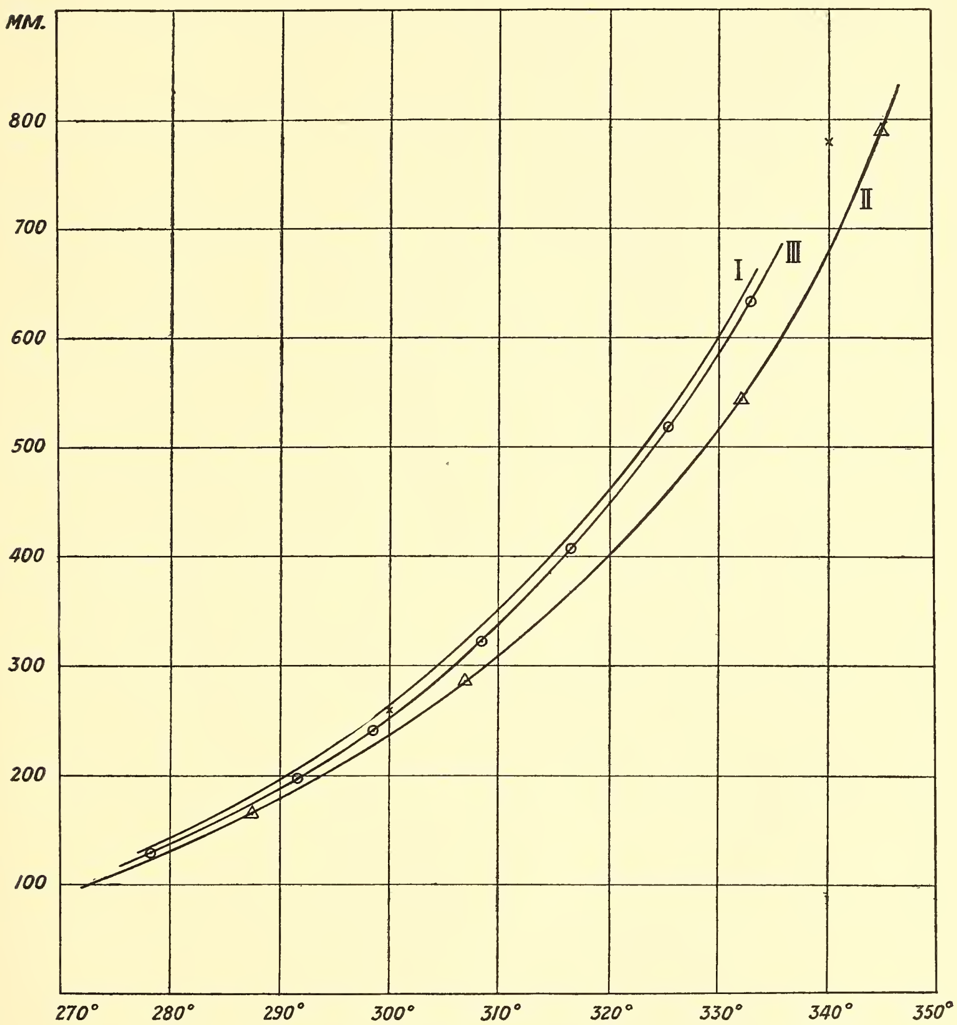


Fig. 2.

this account, which was not over 5 mm., was the greatest involved. The results of Horstmann (H.), Ramsay and Young (R. and Y.), including their final curve as well as the separate series by their static and dynamic methods, Johnson (J.), and ourselves (S. and M.) are given in the following table:—

t.	H.	R. and Y.			J.	S. and M.
		Curve.	Observations.			
			Static.	Dynamic.		
280°	...	143	140·5	147	132	138
290	...	195·5	189	200	178	189
300	259·5	264·5	251	275	237	252
310	...	350	333·5	367	309	336
320	...	460·5	443·5	500	401	447
330	...	599·5	596	675	518	587
333·5	...	661	653	750	566	642
340	778·1	777	759·5	...	678	...

The curves (fig. 2) show the results graphically: I. is R. and Y.'s curve, II. gives Johnson's results, III. gives our own. Two points (×) are from Horstmann's data.

The University of Chicago,
April 1910.

(*Issued separately December* 29, 1910.)

IX.—A Quantitative Study of the Constitution of Calomel Vapour. By Alexander Smith and Alan W. C. Menzies.

(MS. received July 25, 1910. Read November 7, 1910.)

THE vapour density of calomel vapour corresponds to the formula $HgCl$ or $Hg + HgCl_2$. Odling* showed, by amalgamation of gold-leaf, the presence of free mercury. Harris and V. Meyer† separated mercury from the vapour by diffusion, and also showed that solid potassium hydroxide turned yellow (HgO) when plunged into the vapour. Diffusion might naturally increase the dissociation, and even a pure mercurous salt would certainly give mercuric oxide at 240–260°, since at that temperature mercurous oxide is unstable. The quantitative conclusion of these observers, namely, that the vapour is completely dissociated, therefore, was not in any way justified by their qualitative experiments, although it has been accepted by many chemists. Hence the proportion of unchanged calomel ($HgCl$) and its dissociation products ($Hg + HgCl_2$) still remains to be ascertained.

The vapour contains, at most, three substances related by the following equilibria:—

$$\text{Calomel (solid)} \rightleftarrows 2HgCl \leftrightarrows Hg + HgCl_2.$$

It was proposed to make three sets of measurements in the same temperature-region: (1) the vapour pressures of calomel ($P_{calom.}$); (2) the vapour pressures of mercury ($P_{merc.}$); (3) the vapour pressures of a mixture of calomel and mercury ($P_{mixt.}$). In case the dissociation is negligible,

$$P_{mixt.} = P_{calom.} + P_{merc.} \qquad (1).$$

In case the dissociation is complete,

$$P_{mixt.} = P_{merc.} + p'_{corros.\,subl.} = P_{merc.} + \frac{(\frac{1}{2}P_{calom.})^2}{P_{merc.}} \qquad (2).$$

Thus, if either of these extreme cases represented the fact, the vapour pressures of the mixture could be obtained, both by calculation from the vapour pressures of mercury and of calomel, and by direct observation. In case undissociated calomel ($HgCl$) was present, its partial pressure ($p_{calom.}$) could be calculated from the observations:

$$p_{calom.} = P_{calom.} - 2P_{merc.} \pm 2\sqrt{(P_{mixt.} - P_{calom.})P_{merc.}} \qquad (3).$$

* *J. Chem. Soc.*, iii. 211. † *Berichte*, xxvii., 1842.

The first assumption gives the greatest value for the pressures of the mixture, the second the least. When the measurements proposed had been made, however, the pressure of the mixture was found to be even lower than the second assumption permitted. This could be explained provided the density of saturated mercury vapour, of saturated calomel vapour, or of the mixed vapours was abnormally high. Or it could be explained by depression in the vapour tension of the mercury in the mixture, due to dissolved calomel. The vapour densities were measured and found to be normal. The calomel vapour was then found to be very slowly dissolved by mercury, and the solubilities were determined. When the formula of the dissolved calomel is taken to be HgCl (not Hg_2Cl_2), the final calculated results demonstrate, with surprisingly exact quantitative correspondence, that the vapour of calomel is wholly dissociated.

The vapour pressures were all measured with the static isoteniscope and other apparatus used in the work on water * and on mercury.† In the following table the data are given as follows: (1) temperature; (2) solubility of calomel in mercury stated as mols. Hg: mols. HgCl; (3) $P_{merc.}$; (4) vapour pressure of Hg as reduced by dissolved calomel (calc.); (5) $P_{calom.}$; (6) $P_{mixt.}$ calc., assuming complete dissociation; (7) $P_{mixt.}$ observed; (8) divergence of (6) and (7).

1. t.	2. Solubility.	3. $P_{merc.}$	4. $P_{merc.}$ depressed.	5. $P_{calom.}$	6. $P_{mixt.}$, calc.	7. $P_{mixt.}$, observed.	8. Δ
360°	39·6	802·6	782·8	434	843·0	843·6	+0·6
365	32·6	876·7	850·6	491	921·5	924·9	+3·4
370	29·3	956·2	924·7	556	1008·3	1013·0	+4·7
375	27·3	1041·6	1004·8	630	1103·6	1108·4	+4·9
380	26·2	1133·0	1091·4	712	1207·5	1211·1	+3·6
385	25·6	1230·9	1184·6	805	1321·3	1321·7	+0·4
390	25·4	1335·4	1284·8	906	1444·5	1441·4	−3·4
395	25·2	1447·0	1391·8	1017	1577·6	1570·8	−6·8
400	25·1	1566·1	1506·1	1135	1719·9	1713·4	−6·5

The divergences from the theory of complete dissociation vary from −6·8 to +0·4 mm.: the sum of the divergences at nine temperatures is −0·9 mm. The average is thus only −0·1 mm., or about 1 part in 12,000, which is much less than the error involved in such difficult measurements.

An independent preliminary series of measurements of the vapour pressures, made with a mercury thermometer and a simple submerged

* *Proceedings*, xxx. 523. † *Proceedings*, xxx. 521.

bulblet apparatus,* gave results in which the agreement was almost as good (average divergence $-3{\cdot}3$ mm.).

The results show, therefore, (1) that the vapour is composed wholly of mercury and corrosive sublimate, HgCl and Hg_2Cl_2 being alike absent; (2) that calomel dissolves in mercury with a molecular weight corresponding to the formula HgCl; (3) that the isoteniscope may be used in the quantitative investigation of chemical problems.

The University of Chicago,
June 1910.

* *Proceedings*, vol. xxx. p. 437.

(*Issued separately December* 29, 1910.)

X.—The Variation of Young's Modulus under an Electric Current.

By Henry Walker, M.A., D.Sc. (*Communicated by* Professor J. G. MacGregor, F.R.S.)

(MS. received May 16, 1910. Read June 6, 1910.)

Part III.

The results of the investigations on four metals, viz. steel, iron, copper, and platinum, form the subject of my two first papers. In Parts I. and II. the effects on the modulus when the wire was stretched with a small load, and also with a much greater load, were examined. In this, my third paper, the investigation of these metals has been extended in several directions. The scope of the whole work has also been widened by subjecting nickel and cobalt to examination.

As the question of temperature still seemed doubtful, and as the justification given near the beginning of the second paper might not be altogether convincing, I thought it better to put the matter beyond all question. I therefore adopted the following method. Using the double-walled tube already described, the wires were passed through it and over the wheel in the same way as in the main experiments. The resistance of each wire was then determined at the temperature of the room, and at those of boiling sulphuric ether, ethyl-alcohol, amyl-alcohol, and steam. Also, at each of these temperatures each wire was subjected to the same loads as in the experiments in which the variations of the modulus were investigated. In all cases the difference between the resistance of the wire unstretched and stretched was small, even with the greatest load. In some cases the difference between the stretched and unstretched wire did not exceed what would have been produced by an increase in the temperature of 0·5° C. On the scale on which the curves are drawn, this difference in the temperature is inappreciable, so that there was no necessity to redraw the graphs in the two preceding papers. In this paper the temperature has been determined from the graph showing the resistance of the wire when subjected to the same load as in the experiment.

The investigation of the behaviour of soft iron, steel, copper, and platinum has been continued by subjecting each wire to various loads intermediate to the two loads employed in the experiments described in the second paper. As stated there, it was found that when the load was considerably increased the modulus diminished uniformly in value, a behaviour quite different from what was found when the load was small. It became necessary, therefore, to find out how each metal behaved as the load was increased step by step to the maximum.

As the curves for iron and steel are somewhat similar in nature to those obtained when these bodies are subjected to tensile strain in a magnetic field, the further necessity was felt of examining nickel and cobalt. These two metals give results different from each other, and also from iron, when under tensile strain in a magnetic field. Now, if there is a similarity between the variation of the modulus with magnetic field and that of intensity with magnetic field, it might be concluded that there was some relation between the modulus and magnetic induction.

In all my preceding experiments a definite weight was put into the pan

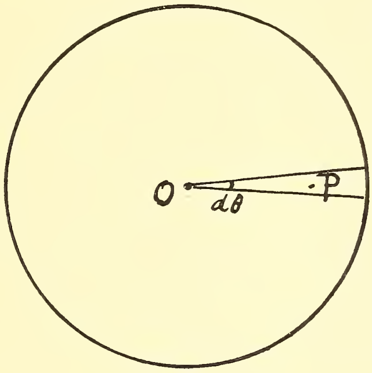


Fig. 1.

before any current was passed through the wire, then the current was gradually increased up to the maximum, and finally diminished to zero. Throughout each cycle of heating and cooling, therefore, the only change in the stress to which the wire was subjected was putting on and taking off the weight employed to produce the change of length. To vary the conditions of the experiments, another method was tried, viz. to alter the load while the magnetising force was kept constant. This method of investigating the effects of stress on magnetisation is described by Ewing,* and the results are quite different from those obtained by keeping the load constant and varying the magnetic field. The idea was to find out whether, under these conditions, there would be any change in the modulus as compared with the previous conditions of experiment, and, if so, whether the variation in the modulus was at all similar to that of the magnetic intensity. I thought that this comparison would be more accurate and complete if the strength of the magnetic field were known, and I determined

* *Magnetic Induction in Iron*, 3rd ed., p. 216.

the mean value of the field throughout the cross-section of the wire in the following manner:—

Let the circle of radius a represent a section of the wire, P any point at distance r from the centre O. Then the strength of the field at P, say H, is given by

$$H = \frac{2cr}{a^2},$$

where c is the current in C.G.S. units. Denote the area of the cross-section by S.

$$\therefore \text{ Mean value of } H = \frac{1}{S}\int H dS$$

$$= \frac{1}{\pi a^2}\int_0^a \int_0^{2\pi} \frac{2cr^2}{a^2} dr d\theta$$

$$= \frac{4c}{a^4}\Big[\frac{r^3}{3}\Big]_0^a$$

$$= \frac{4c}{3a}$$

$$= \frac{2}{15a} \times \text{current}$$

when the current is in amperes.

For the investigation of cobalt I employed a rectangular strip, and determined the mean value of the field as follows:—

Let the length and breadth of the cross-section be $2a$ and $2b$ respectively, the centre being the origin. Take any point P (x, y) and draw a rectangle

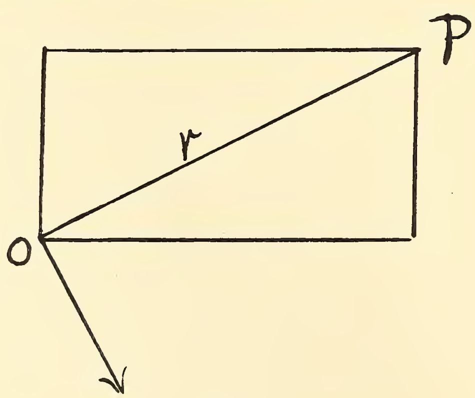


Fig. 2.

through P similar to the strip with sides $2x$ and $2y$. Also let the elementary rectangle $dxdy$ be drawn, having P for its centre. Suppose i to be the current density, and that the current is flowing upwards from the paper. Then the current passing through the elementary rectangle is $idxdy$.

∴ x-component of magnetic force at O due to upward current $idxdy$

$$= \frac{2idxdy}{r} \cdot \frac{y}{r}.$$

∴ Total x-component due to current through area xy is

$$\mathrm{I}^x = 2i \int_0^x \int_0^y \frac{y}{x^2+y^2} dxdy$$

$$= i \int_0^x dx \int_0^y \frac{d(y^2)}{x^2+y^2}$$

$$= \int_0^x dx \log \frac{x^2+y^2}{x^2}$$

Put $x = y \tan \phi$.

∴ $dx = y \sec^2 \phi d\phi$.

$$\therefore \mathrm{I}_x = iy \int (\log \operatorname{cosec}^2 \phi) \sec^2 \phi d\phi$$

$$= -iy \int \log \sin^2 \phi d (\tan \phi)$$

$$= -iy \Big[\tan \phi \log \sin^2 \phi \Big] + iy \int 2d\phi.$$

The limits are $\phi = \mathrm{O}$ and $\phi = \tan^{-1} \dfrac{x}{y}$.

$$\therefore \mathrm{I}_x = \left[-ix \log \frac{x^2}{r^2} + 2iy \tan^{-1} \frac{x}{y} \right]$$

$$= + \left[ix \log \left(1 + \frac{y^2}{x^2}\right) + 2iy \tan^{-1} \frac{x}{y} \right].$$

Similarly,

$$\mathrm{I}_y = -\left[iy \log \left(1 + \frac{x^2}{y^2}\right) + 2ix \tan^{-1} \frac{y}{x} \right],$$

where I_y is positive outwards from O.

To obtain the force at P due to the current in the strip, find the components along PD and PB.

To get the components along PD—

For rectangle	(1)	let	PC	be	x	and	PA	be	y and use	$-\mathrm{I}_x$.
„ „	(2)	„	PA	„	x	„	PD	„	y „ „	$+\mathrm{I}_y$.
„ „	(3)	„	PD	„	x	„	PB	„	y „ „	$+\mathrm{I}_x$.
„ „	(4)	„	PB	„	x	„	PC	„	y „ „	$-\mathrm{I}_y$.

Add these, and we get the resultant along PD. Let us call it F_1.

Similarly, to get the components along PB—

For rectangle	(1)	use	$-I_y$.
„ „	(2)	„	$-I_x$.
„ „	(3)	„	$+I_y$.
„ „	(4)	„	$+I_x$.

Add these, and we get the resultant along PB. Let us call it F_2, and also let the resultant field at P be H.

$$\therefore H = \sqrt{F_1^2 + F_2^2}.$$

We have then to evaluate $\iint H dx dy$.

This integral, however, is unmanageable, and I have made an approximate calculation as follows:—Suppose a quarter of the cross-section of the strip to be divided up into sixteen rectangles similar to the strip. Find the values of F_1 and F_2 at the centre of each of the rectangles, and obtain

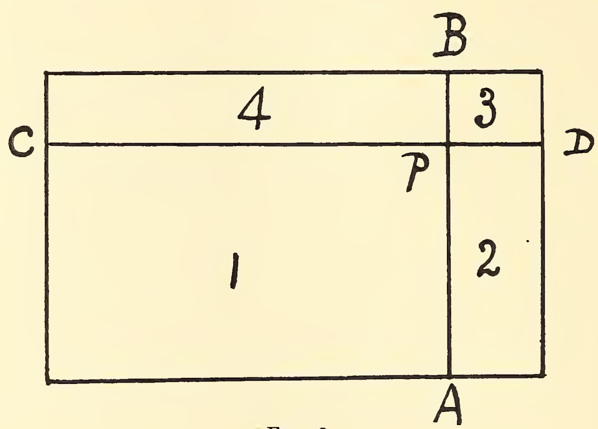


Fig. 3.

the value of H at each of these points, then finally calculate the arithmetical mean of the values of H. In this way I obtained an approximation from sixty-four points for the whole strip.

To eliminate the residual effects of stress, the wire was, in all the experiments described in this paper, demagnetised after the load had been put on. It was also placed at right angles to the earth's field.

Iron.

A wire from the same coil as was used in the experiment in the second paper was taken. First, the modulus was determined at the temperature of the room, and was found to be the same as in that experiment, the total load being 1 kilo. The current was next gradually increased up to

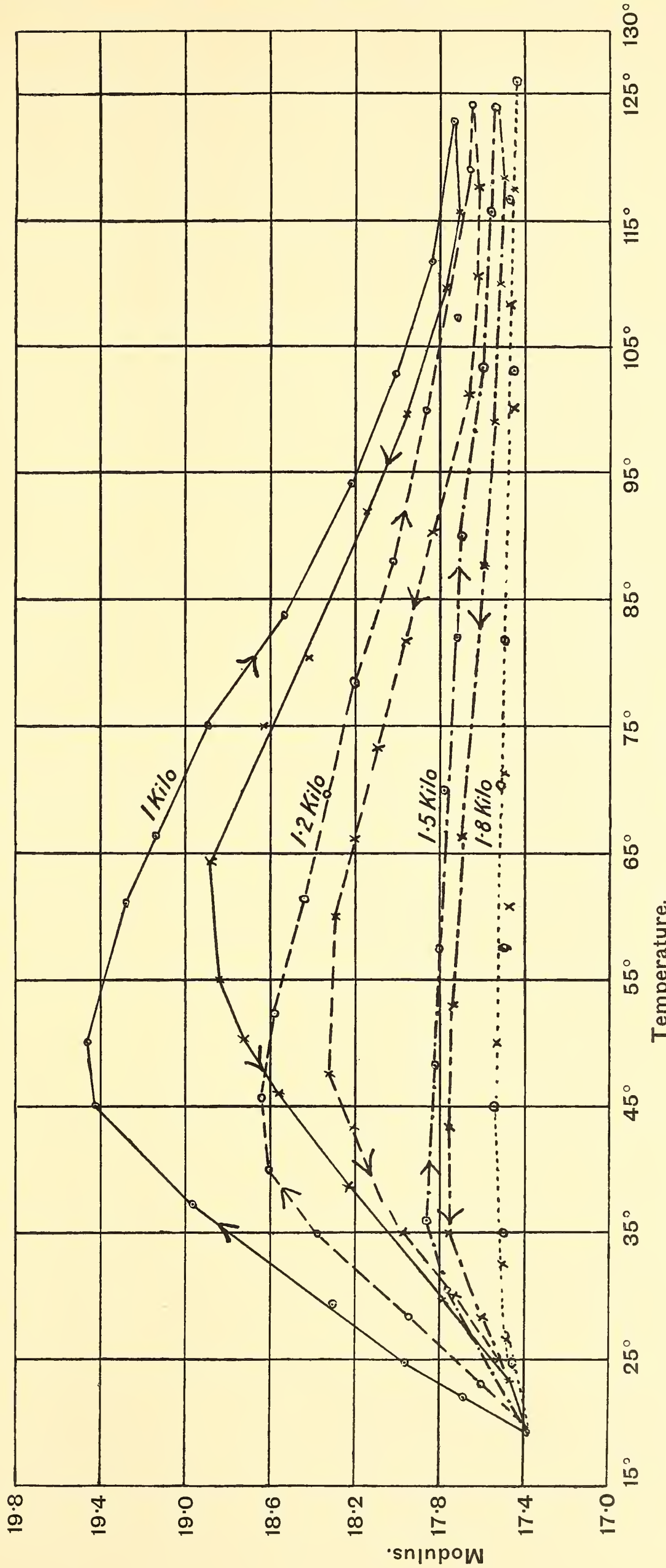

1 Kilo
1·2 Kilo
1·5 Kilo
1·8 Kilo
Temperature.
15°
25°
35°
45°
55°
65°
75°
85°
95°
105°
115°
125°
130°
Modulus.
19·8
19·4
19·0
18·6
18·2
17·8
17·4
17·0

Fig. 4.—Iron.

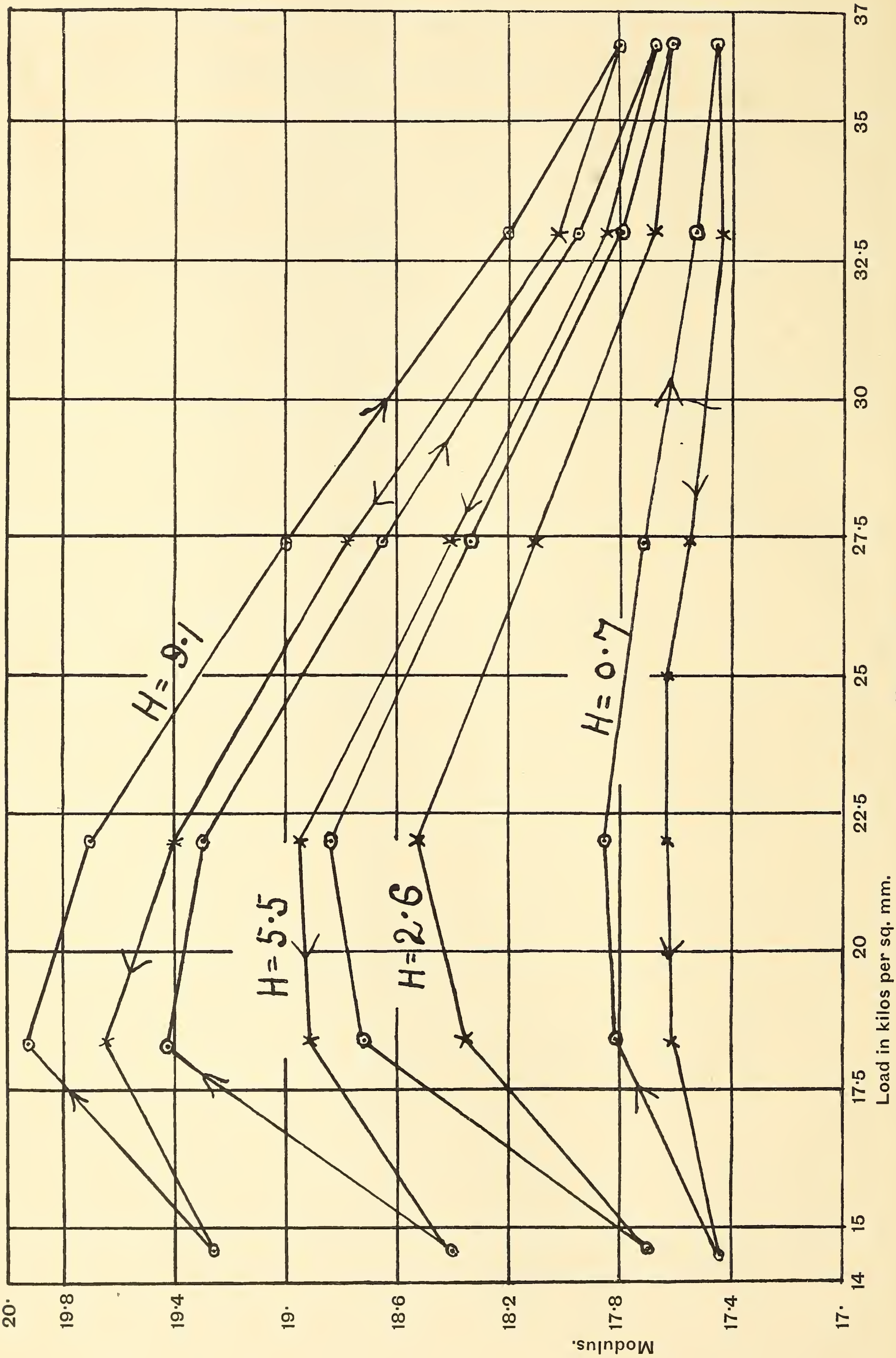

H = 9·1
H = 5·5
H = 2·6
H = 0·7
14
15
17·5
20
22·5
25
27·5
30
32·5
35
37
Load in kilos per sq. mm.
20·
19·8
19·4
19·
18·6
18·2
17·8
17·4
17·
Modulus.

Fig. 5.—Iron.

INSTRUCTIONS TO AUTHORS.

The 'copy' should be written on large sheets of paper, on one side only and the pages should be clearly numbered. The MS. must be easily legible, preferably typewritten, and must be absolutely in its final form for printing; so that corrections in proof shall be as few as possible, and shall not cause overrunning in the lines or pages of the proof. All tables of contents, references to plates, or illustrations in the text, etc., must be in their proper places, with the page numbers left blank; and spaces must be indicated for the insertion of illustrations that are to appear in the text.

Illustrations.—All illustrations must be drawn in a form immediately suitable for reproduction; and such illustrations as can be reproduced by photographic processes should, so far as possible, be preferred. Drawings to be reproduced as line blocks should be made with Indian ink (deadened with yellow if of bluish tone), preferably on fine white bristol board, free from folds or creases; smooth, clean lines, or sharp dots, but no washes or colours, should be used. If the drawings are done on a large scale, to be afterwards reduced by photography, any lettering or other legend must be on a corresponding scale.

If an author finds it inconvenient to furnish such drawings, the Society will have the figures re-drawn at his expense; but this will cause delay.

When the illustrations are to form plates, a scheme for the arrangement of the figures (in quarto plates for the Transactions, in octavo for the Proceedings) must be given, and numbering and lettering indicated.

Proofs.—In general, a first proof and a revise of each paper will be sent to the author, whose address should be indicated on the MS. If further proofs are required, owing to corrections or alterations for which the printer is not responsible, the expense of such proofs and corrections will be charged against the author.

All proofs must, if possible, be returned within one week, addressed to *The Secretary, Royal Society,* 22–24 *George Street, Edinburgh,* and *not to the printer.* To prevent delay, authors residing abroad should appoint some one residing in this country to correct their proofs.

The author receives fifty of the reprints free, and may have any reasonable number of additional copies at a fixed scale of prices which will be furnished by the printer, who will charge him with the cost. To prevent disappointment, especially if the paper contains plates, the author should, immediately after receiving his first proof, notify *to the printer* the number of additional copies required.

Index Slips.—In order to facilitate the compilation of Subject Indices, and to secure that due attention to the important points in a paper shall be given in General Catalogues of Scientific Literature and in Abstracts by Periodicals, every author is requested to return to the Secretary, along with his final proof, a brief index (on the model given below), of the points in it which he considers new or important. These indices will be edited by the Secretary, and incorporated in Separate Index Slips, to be issued with each part of the Proceedings and Transactions.

MODEL INDEX.

Schäfer, E. A.—On the Existence within the Liver Cells of Channels which can be directly injected from the Blood-vessels. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

Cells, Liver,—Intra-cellular Canaliculi in.
E. A. Schäfer. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

Liver,—Injection within Cells of.
E. A. Schäfer. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

The Papers published in this Part of the PROCEEDINGS *may be had separately, on application to the Publishers, at the following prices:—*

No. I.,	Price 1s.
No. II.,	,, 1s. 4d.
No. III.,	,, 3s.
No. IV.,	,, 10d.
No. V.,	,, 10d.
No. VI.,	,, 3s. 3d.
No. VII.,	,, 2s. 8d.
No. VIII.,	,, 6d.
No. IX.,	,, 6d.

PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

SESSION 1910–11.

Part II.] VOL. XXXI. [Pp. 193–288.

CONTENTS.

EDINBURGH:

PUBLISHED BY ROBERT GRANT & SON, 107 PRINCES STREET, AND WILLIAMS & NORGATE, 14 HENRIETTA STREET, COVENT GARDEN, LONDON.

MDCCCCXI.

Price Six Shillings and Tenpence.

REGULATIONS REGARDING THE PUBLICATION OF PAPERS IN THE PROCEEDINGS AND TRANSACTIONS OF THE SOCIETY.

THE Council beg to direct the attention of authors of communications to the Society to the following Regulations, which have been drawn up in order to accelerate the publication of the Proceedings and Transactions, and to utilise as widely and as fairly as possible the funds which the Society devotes to the publication of Scientific and Literary Researches.

1. MANUSCRIPT OF PAPERS.—As soon as any paper has been passed for publication, either in its original or in any altered form, and has been made ready for publication by the author, it is sent to the printer.

2. METHOD OF PUBLICATION.—As soon as the final revise of a Transactions paper has been returned, or as soon as the sheet in which the last part of a Proceedings paper appears is ready for press, a certain number of separate copies or reprints, in covers bearing the title of the paper and the name of the author, are printed off and placed on sale. The date of such separate publication will be printed on each paper.

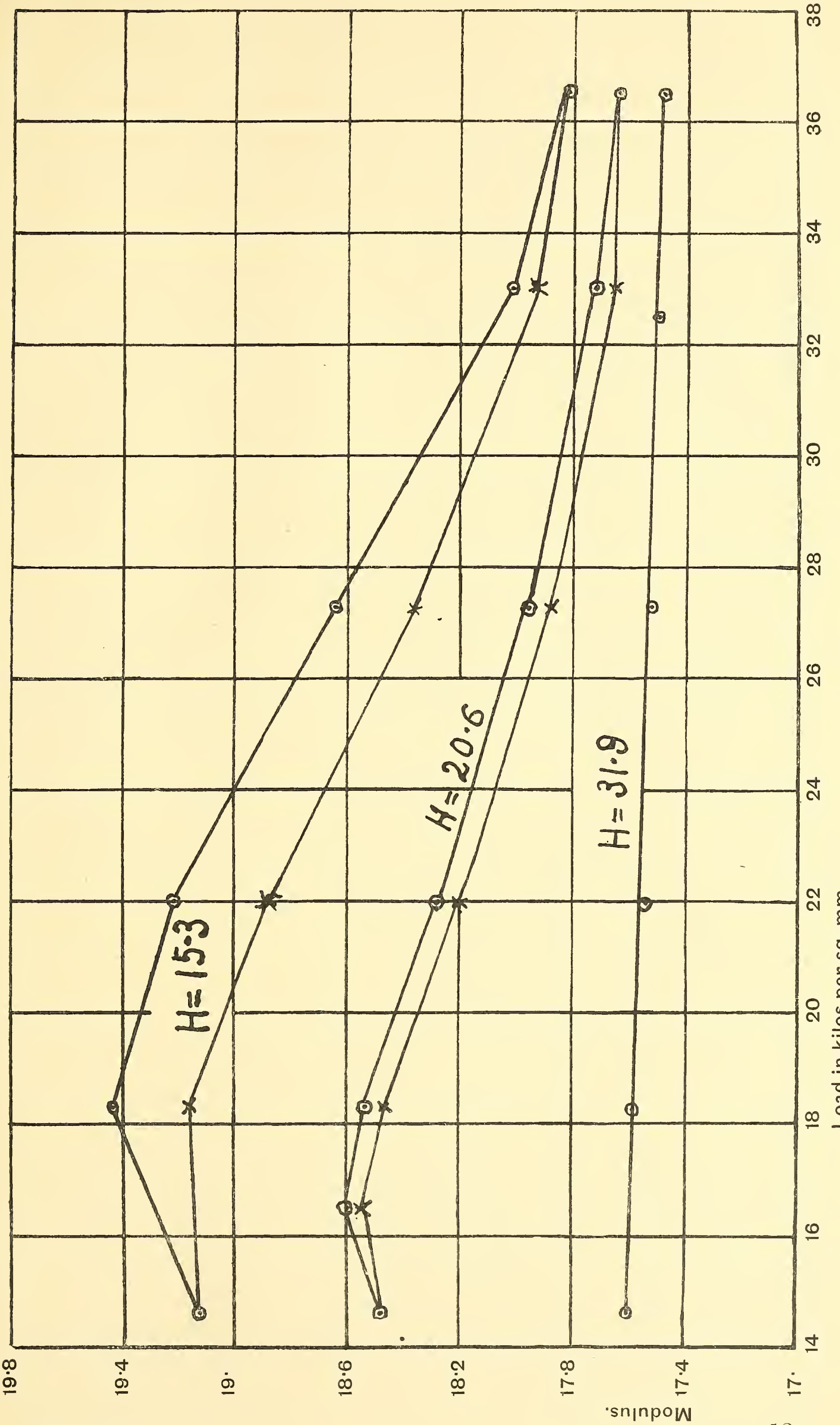
3. ADDITIONS TO A PAPER after it has been finally handed in for publication, if accepted by the Council, will be treated and dated as separate communications, and may, or may not, be printed immediately after the original paper.

4. BRIEF ABSTRACTS OF TRANSACTIONS PAPERS will be published in the Proceedings, provided they are sent along with the original paper.

5. SPECIAL DISCUSSION OF PAPERS ACCEPTED FOR PUBLICATION.—Where a paper has been accepted for publication, the Council may, with the consent of the author, select this paper for Special Discussion. In the case of such papers advanced proofs will be sent to the members of the Society desiring copies, and copies will be supplied to the author for distribution. A paper selected for Special Discussion will be marked with an asterisk (*) and placed first on the Billet for the day of reading. Any following papers for that day may be adjourned or held as read if the discussion prevents their being read.

6. COMMUNICATIONS NOT SUBMITTED FOR PUBLICATION, such as Demonstrations of Experiments, Statement of Scientific Problems, etc., may be received by the Council, and may also be selected for Special Discussion. The Council does not undertake to publish any notice of such communications in the Proceedings or Transactions of the Society.

[*Continued on page* iii *of Cover.*



H=15·3
H=20·6
H=31·9
Load in kilos per sq. mm.
14
16
18
20
22
24
26
28
30
32
34
36
38
Modulus.
19·8
19·4
19·
18·6
18·2
17·8
17·4
17·

Fig. 6.—Iron.

the maximum, and then diminished down to zero, the same procedure being followed as in the preceding experiments. This was repeated with weights of 1·2, 1·5, and 1·8 kilos., the results being shown in the graphs. These results are, in all cases, what was obtained after the cyclically steady state had been reached.

On examining the graphs, the first thing to be noticed is the diminution in the modulus as the weight is increased, and also the fact that the hysteresis at the same time becomes gradually smaller and smaller.

In figs. 5 and 6 are shown the results of those experiments in which the current was kept constant while the load was varied. An examination of them shows that as the current is increased the modulus at first increases and then diminishes. We also notice that when the field is weak the effect on the modulus is small. This is seen in the graph for a field whose strength is 0·7. In the next curve for field 2·6 the effect on the modulus is much more marked, and, both when the modulus is increasing and decreasing, the rate of change is greater than for the smaller field. The other curves are for increasing fields, and show the progressive change that takes place in the value of the modulus. There is a close correspondence between the results of the other method of experiment and this. In both we see that at first an increase of current is accompanied by an increase of the modulus, but that, when the current reaches a certain value, any further increase produces a diminution of the modulus. It is also to be noticed that there is a steady shift of the maximum value towards the zero of load, and that in the highest field, the value of which was 31·9, the decrease in the modulus was uniform.

[Table.

Iron.

Length = 98·34 cms.
Area of cross-section = ·0005474 sq. cm.
Elongation weight = 500 grams.
Total load on wire = 1 kilo.
Load per sq. mm. = 18·3 kilos.

Table V.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	19°·4 C.	·05072 cm.	8	$17·38 \times 10^{11}$
2	24·8	·04908	8	17·96
3	29·3	·04817	9	18·30
4	37·1	·04647	9	18·97
5	45·2	·04537	8	19·43
6	50·5	·04532	8	19·45
7	60·9	·04573	7	19·28
8	66·3	·04606	8	19·14
9	75·0	·04662	9	18·89
10	83·9	·04757	10	18·53
11	94·1	·04838	10	18·22
12	102·8	·04895	8	18·01
13	111·7	·04938	9	17·85
14	122·5	·04969	8	17·74
15	116·1	·04975	7	17·72
16	109·6	·04960	7	17·77
17	100·2	·04908	8	17·96
18	91·8	·04857	8	18·15
19	80·4	·04788	9	18·41
20	75·1	·04732	9	18·63
21	64·3	·04669	8	18·88
22	55·0	·04676	10	18·85
23	50·6	·04709	9	18·72
24	46·1	·04747	9	18·57
25	38·7	·04836	7	18·23
26	29·8	·04958	8	17·78
27	23·2	·05046	8	17·46
28	19·4	·05072	9	17·38

Total load = 1·2 kilos.
= 21·9 kilos. per sq. mm.

Table VI.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	19°·5 C.	·05074 cm.	8	$17·37 \times 10^{11}$
2	22·6	·05009	8	17·60
3	28·3	·04916	9	17·93
4	34·9	·04799	9	18·37
5	40·1	·04739	10	18·60
6	45·8	·04732	10	18·63
7	52·2	·04741	8	18·59
8	61·5	·04786	9	18·42
9	69·7	·04807	7	18·34

TABLE VI.—*continued.*

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
10	78°·4 C.	·04847 cm.	7	18·19 × 10^{11}
11	88·0	·04897	8	18·02
12	99·5	·04935	8	17·86
13	107·3	·04975	7	17·72
14	118·9	·04988	7	17·67
15	124·1	·04995	8	17·65
16	117·6	·05000	7	17·63
17	110·5	·04995	6	17·65
18	101·8	·04989	6	17·68
19	90·2	·04944	6	17·83
20	82·1	·04908	8	17·97
21	73·5	·04870	8	18·10
22	66·4	·04837	8	18·22
23	59·9	·04820	9	18·29
24	47·7	·04812	9	18·32
25	43·2	·04844	8	18·20
26	35·0	·04905	9	17·97
27	30·1	·04972	9	17·73
28	24·9	·05034	10	17·51
29	21·5	·05066	9	17·40
30	19·6	·05072	7	17·38

Total load = 1·5 kilos.
= 27·4 kilos. per sq. mm.

TABLE VII.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	19°·3 C.	·05075 cm.	7	17·37 × 10^{11}
2	25·4	·05011	7	17·59
3	30·7	·04958	9	17·78
4	36·1	·04938	8	17·85
5	48·2	·04944	9	17·83
6	57·5	·04952	8	17·80
7	69·4	·04961	8	17·79
8	81·9	·04972	6	17·73
9	90·6	·04980	6	17·70
10	103·3	·05003	8	17·62
11	115·7	·05014	8	17·58
12	124·0	·05026	10	17·54
13	118·2	·05032	10	17·52
14	109·8	·05029	9	17·53
15	99·1	·05023	9	17·55
16	87·5	·05009	8	17·60
17	74·8	·04983	7	17·69
18	66·2	·04980	7	17·70
19	52·9	·04969	8	17·74
20	43·3	·04963	8	17·76
21	35·0	·04960	8	17·77
22	28·4	·05006	7	17·61
23	24·3	·05049	6	17·46
24	19·3	·05075	8	17·37

Total load = 1·8 kilos.
= 32·9 kilos. per sq. mm.

TABLE VIII.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	19°·5 C.	·05078 cm.	7	$17·36 \times 10^{11}$
2	22·8	·05052	8	17·45
3	27·9	·05034	8	17·51
4	34·7	·05031	5	17·52
5	45·1	·05025	8	17·54
6	57·3	·05040	7	17·49
7	70·2	·05022	7	17·53
8	81·6	·05043	6	17·48
9	94·4	·05046	6	17·47
10	103·1	·05049	8	17·46
11	116·5	·05043	8	17·48
12	125·9	·05055	8	17·44
13	117·2	·05049	7	17·46
14	108·5	·05043	7	17·48
15	100·1	·05052	7	17·45
16	86·8	·05034	8	17·51
17	71·3	·05037	8	17·50
18	60·7	·05043	7	17·48
19	50·0	·05025	7	17·54
20	37·9	·05040	8	17·49
21	32·4	·05031	7	17·52
22	26·5	·05043	6	17·48
23	21·8	·05060	6	17·42
24	19·5	·05078	6	17·36

TABLE IX.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
0·7	14·6	·05052 cm.	8	$17·45 \times 0^{11}$
	18·3	·04949	9	17·81
	22·0	·04930	10	17·86
	27·4	·04972	9	17·73
	33·0	·05023	9	17·55
	36·5	·05049	10	17·46
	33·0	·05058	7	17·43
	27·4	·05020	8	17·56
	25·0	·05003	9	17·62
	22·0	·04991	9	17·66
	18·3	·05003	8	17·62
	14·6	·05052	7	17·45

Table X.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
2·6	14·6	·04983 cm.	8	$17·69 \times 10^{11}$
	18·3	·04707	9	18·73
	22·0	·04679	7	18·84
	27·4	·04801	8	18·36
	33·0	·04942	9	17·80
	36·5	·05000	9	17·63
	33·0	·04986	7	17·68
	27·4	·04865	7	18·12
	22·0	·04757	6	18·53
	18·3	·04801	8	18·36
	14·6	·04983	8	17·69

Table XI.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
5·5	14·6	·04788 cm.	8	$18·41 \times 10^{11}$
	18·3	·04539	9	19·42
	22·0	·04570	9	19·29
	27·4	·04832	10	18·67
	33·0	·04899	7	17·96
	36·5	·04991	7	17·66
	33·0	·04938	8	17·85
	27·4	·04783	10	18·43
	22·0	·04642	8	18·95
	18·3	·04648	8	18·93
	14·6	·04788	9	18·41

Table XII.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
9·1	14·6	·04577 cm.	8	$19·26 \times 10^{11}$
	18·3	·04425	7	19·92
	22·0	·04477	8	19·70
	27·4	·04637	5	19·01
	33·0	·04838	6	18·22
	36·5	·04955	6	17·79
	33·0	·04884	7	18·05
	27·4	·04694	7	18·78
	22·0	·04546	9	19·39
	18·3	·04487	8	19·65
	14·6	·04577	8	19·26

Table XIII.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
15·3	14·6	·04608 cm.	8	$19{\cdot}13 \times 10^{11}$
	18·3	·04535	7	19·44
	22·0	·04587	8	19·22
	27·4	·04729	7	19·64
	33·0	·04895	7	18·01
	36·5	·04947	9	17·82
	33·0	·04914	9	17·94
	27·4	·04801	8	18·36
	22·0	·04666	8	18·89
	18·3	·04599	6	19·17
	14·6	·04608	6	19·13

Table XIV.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
20·6	14·6	·04770 cm.	9	$18{\cdot}48 \times 10^{11}$
	16·5	·04742	7	18·59
	18·3	·04757	6	18·53
	22·0	·04823	6	18·28
	27·4	·04911	8	17·95
	33·0	·04975	8	17·72
	36·5	·05000	7	17·63
	33·0	·04994	7	17·65
	27·4	·04932	8	17·87
	22·0	·04838	6	18·22
	18·3	·04770	6	18·48
	16·5	·04752	8	18·55
	14·6	·04770	5	18·48

Table XV.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
31·9	14·6	·05009 cm.	8	$17{\cdot}60 \times 10^{11}$
	18·3	·05014	9	17·58
	22·0	·05029	5	17·53
	27·4	·05032	6	17·52
	33·0	·05037	6	17·50
	36·5	·05040	8	17·49
	33·0	·05040	9	17·49
	27·4	·05026	8	17·54
	22·0	·05026	8	17·54
	18·3	·05020	10	17·56
	14·6	·05006	8	17·61

Steel.

A wire of the same gauge as in the preceding experiments was used, and was put through successive cycles of heating and cooling, the loads being 1·1, 1·4, 1·7, and 2·0 kilos. The resulting curves are shown in fig. 7, each one being that obtained after the cyclic state had been reached. It will be seen that the modulus diminishes as the weight is increased, that is, at the same stage in any two cycles the modulus has a greater value in that cycle in which the load is smaller.

Speaking generally, we are able to say, from a comparison of the curves obtained under similar conditions—*e.g.* figs. 4 and 7, 5 and 8, 6 and 9—that effects of an electric current on Young's Modulus are the same in steel as in iron. At the same time, there are certain differences, for it will be seen that, when the current is diminishing, the curves at first lie above those at the same temperature for increasing current. Again, as the load is increased, the value of the modulus falls, and the maximum value is also reached in a weaker field, a result similar to what was found for iron.

An examination of the curves when the current was kept constant, and the load varied, shows that the curve for unloading is at first higher than that for loading, a point in which steel differs from iron. It is also seen that increase of field produces at first increase in the modulus, but that, when it rises above a certain value, a further increase causes the modulus to diminish. In the highest field there is a uniform decrease, and the curve becomes a straight line.

Steel.

Length	=	98·42 cms.
Area of cross-section	=	·0004714 sq. cm.
Elongation weight	=	500 grams.
Total load on wire	=	1·1 kilos.
Load per sq. mm.	=	23·3 kilos. per sq. mm.

Table V.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	14°·8 C.	·04835 cm.	8	$21·19 \times 10^{11}$
2	20·2	·04812	9	21·29
3	26·3	·04792	9	21·38
4	30·0	·04781	10	21·43
5	33·4	·04776	8	21·45
6	36·9	·04776	8	21·45

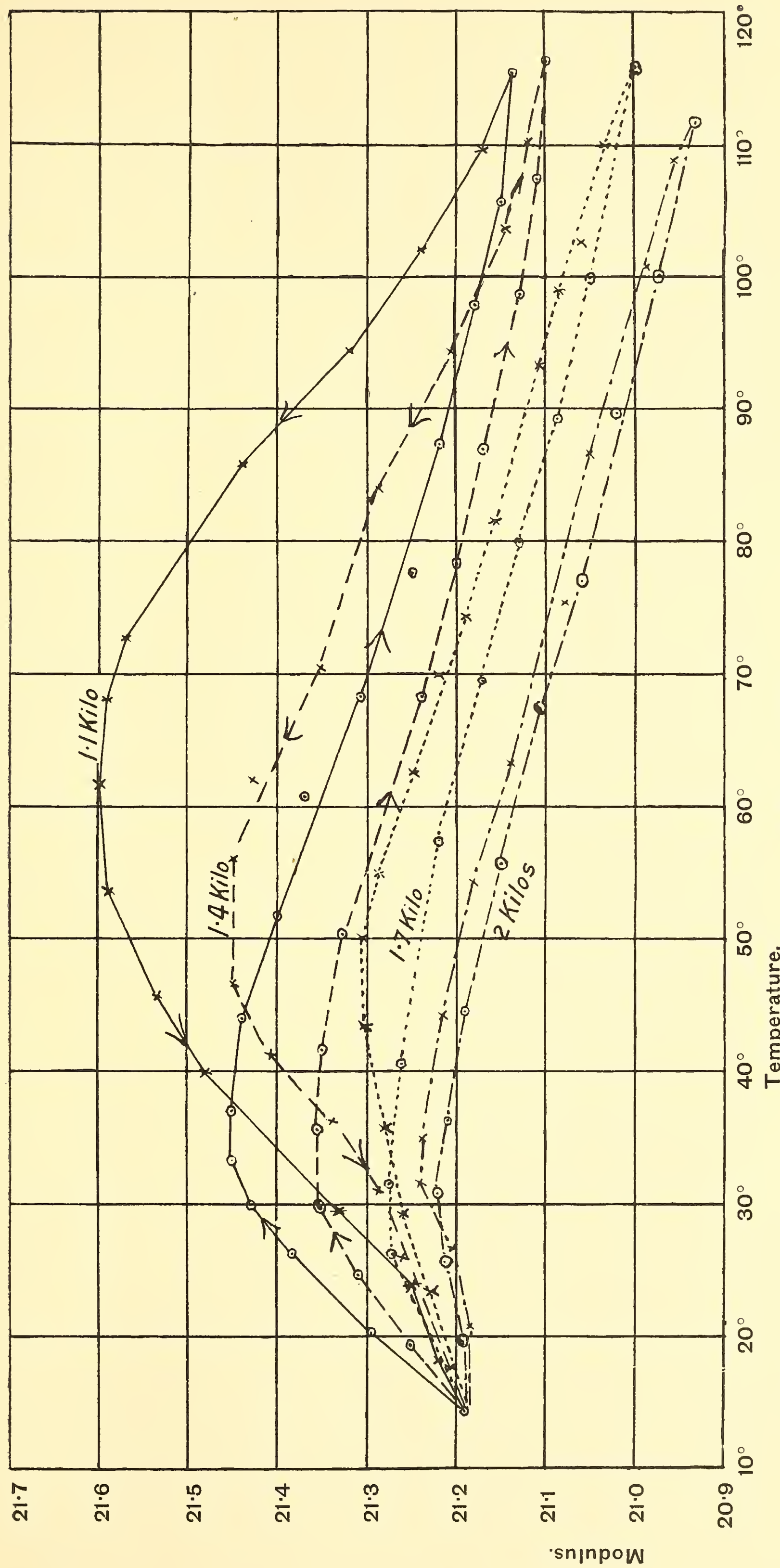

1·1 Kilo
1·4 Kilo
1·7 Kilo
2 Kilos
10° 20° 30° 40° 50° 60° 70° 80° 90° 100° 110° 120°
Temperature.
21·7 21·6 21·5 21·4 21·3 21·2 21·1 21·0 20·9
Modulus.

Fig. 7.—Steel.

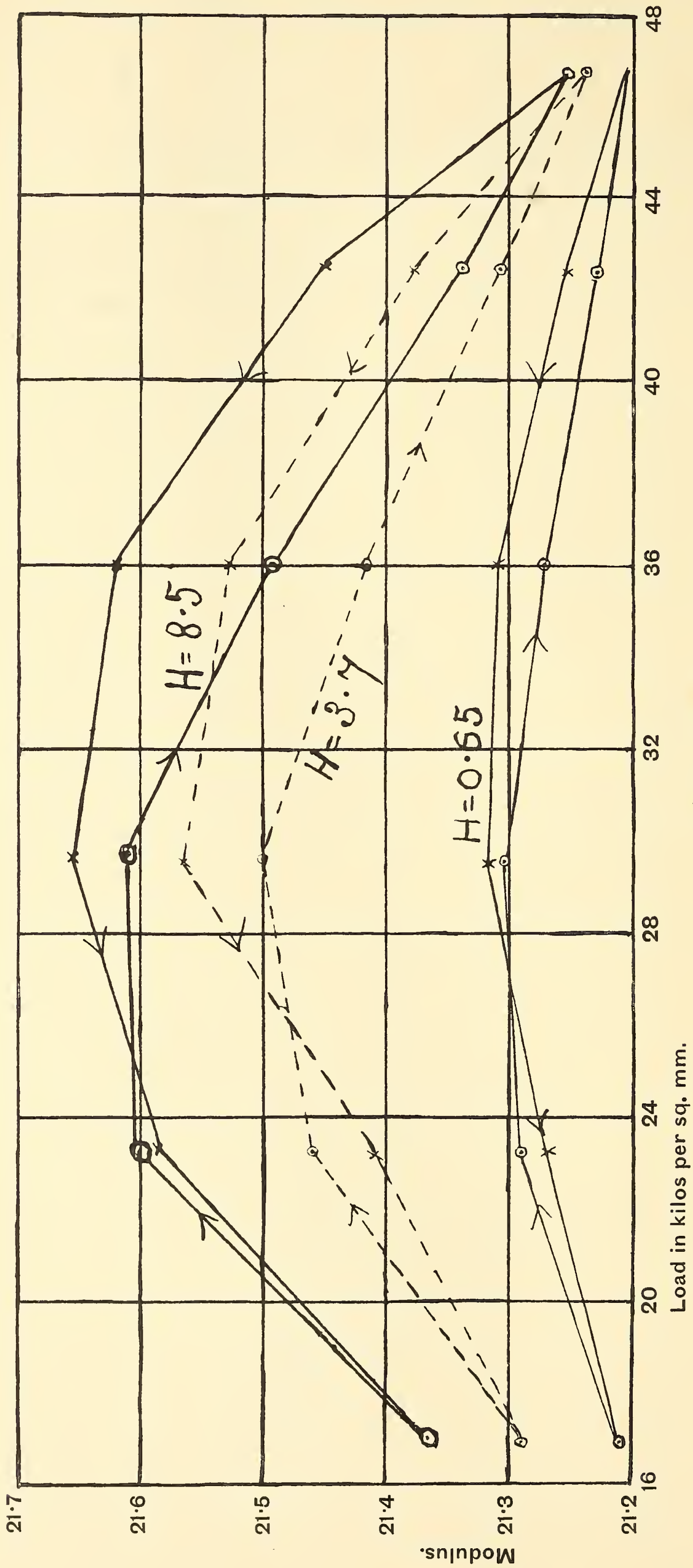

H=8·5
H=3·7
H=0·65
16
20
24
28
32
36
40
44
48
Load in kilos per sq. mm.
21·7
21·6
21·5
21·4
21·3
21·2
Modulus.

Fig. 8.—Steel.

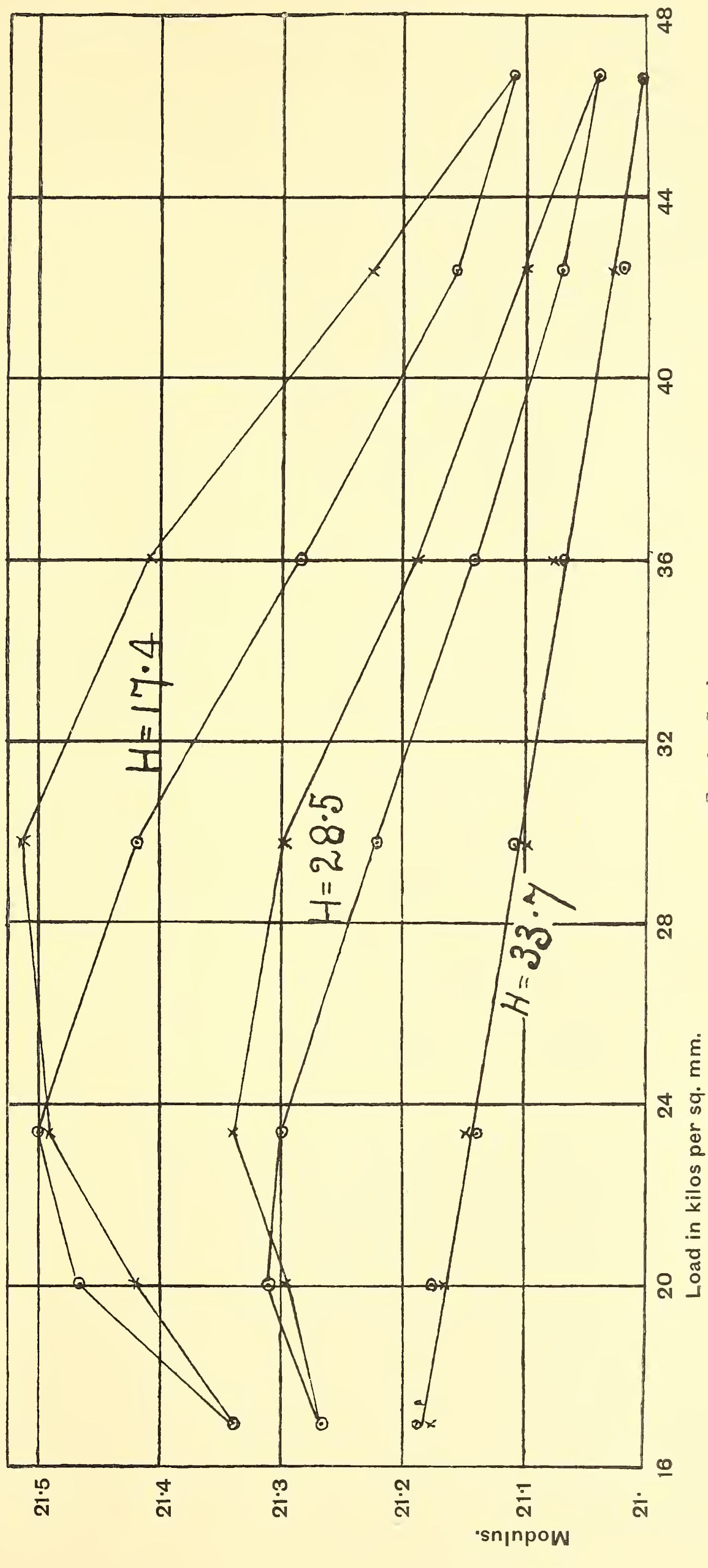

21·5
21·4
21·3
21·2
21·1
21·
Modulus.
16
20
24
28
32
36
40
44
48
Load in kilos per sq. mm.
H=17·4
H=28·5
H=33·7

Fig. 9.—Steel.

Table V.—*continued.*

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
7	44°·1 C.	·04779 cm.	8	$21{\cdot}44 \times 10^{11}$
8	51·7	·04788	7	21·40
9	60·6	·04794	7	21·37
10	68·3	·04806	9	21·32
11	77·8	·04825	9	21·25
12	87·5	·04828	8	21·22
13	98·0	·04838	8	21·18
14	105·7	·04844	8	21·15
15	115·4	·04846	7	21·14
16	109·2	·04840	7	21·17
17	101·9	·04824	9	21·24
18	94·3	·04806	9	21·32
19	85·7	·04779	8	21·44
20	72·8	·04750	8	21·57
21	68·1	·04745	8	21·59
22	61·5	·04743	7	21·60
23	53·6	·04745	7	21·59
24	45·2	·04759	8	21·53
25	40·0	·04770	6	21·48
26	34·7	·04788	6	21·40
27	29·8	·04804	6	21·33
28	22·3	·04821	8	21·25
29	18·4	·04828	7	21·22
30	14·8	·04835	7	21·19

Total load = 1·4 kilos.
= 29·7 kilos. per sq. mm.

Table VI.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	14°·7 C.	·04835 cm.	7	$21{\cdot}19 \times 10^{11}$
2	19·5	·04823	7	21·26
3	24·6	·04808	8	21·31
4	30·1	·04799	6	21·35
5	35·8	·04797	6	21·36
6	41·4	·04799	8	21·35
7	49·9	·04804	8	21·33
8	57·7	·04815	7	21·28
9	68·3	·04823	7	21·24
10	78·0	·04833	7	21·20
11	86·6	·04840	7	21·17
12	98·5	·04849	7	21·13
13	107·4	·04856	8	21·11
14	116·0	·04858	8	21·10
15	110·1	·04854	7	21·12

Table VI.—*continued.*

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
16	103°·8 C.	·04844 cm.	7	$21·15 \times 10^{11}$
17	94·2	·04831	9	21·21
18	82·5	·04813	9	21·29
19	70·7	·04797	8	21·36
20	61·3	·04781	7	21·43
21	55·9	·04776	7	21·45
22	46·7	·04776	8	21·45
23	41·2	·04786	8	21·41
24	36·4	·04802	9	21·34
25	31·1	·04813	8	21·29
26	25·6	·04819	9	21·26
27	20·5	·04826	9	21·23
28	17·8	·04831	7	21·21
29	14·7	·04835	6	21·19

Total load = 1·7 kilos.
= 36·0 kilos. per sq. mm.

Table VII.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	15°·0 C.	·04838 cm.	8	$21·18 \times 10^{11}$
2	20·3	·04825	7	21·23
3	25·9	·04817	7	21·27
4	31·4	·04815	6	21·28
5	40·8	·04819	6	21·26
6	48·7	·04819	5	21·26
7	57·6	·04828	5	21·22
8	69·5	·04840	8	21·17
9	80·1	·04849	9	21·13
10	89·3	·04861	9	21·08
11	99·8	·04867	9	21·05
12	111·6	·04878	9	21·01
13	115·9	·04880	8	21·00
14	109·5	·04872	8	21·03
15	102·7	·04865	10	21·06
16	93·2	·04856	10	21·11
17	81·8	·04842	8	21·16
18	70·0	·04828	8	21·22
19	62·5	·04821	7	21·25
20	55·1	·04813	7	21·29
21	50·2	·04808	8	21·31
22	43·4	·04810	8	21·30
23	35·7	·04815	7	21·28
24	28·6	·04819	7	21·26
25	23·3	·04826	7	21·23
26	18·8	·04835	9	21·19
27	15·1	·04838	9	21·18

Total load = 2 kilos.
= 42·4 kilos. per sq. mm.

TABLE VIII.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	15°1 C.	·04838 cm.	7	$21{\cdot}18 \times 10^{11}$
2	19·8	·04835	9	21·19
3	25·6	·04828	8	21·22
4	30·7	·04828	8	21·22
5	36·2	·04831	6	21·21
6	44·5	·04835	6	21·19
7	55·7	·04844	8	21·15
8	67·3	·04856	8	21·11
9	76·9	·04865	9	21·06
10	89·2	·04875	9	21·02
11	100·1	·04883	8	20·98
12	111·8	·04895	9	20·93
13	117·0	·04895	9	20·93
14	109·1	·04888	8	20·96
15	100·7	·04882	7	20·99
16	88·6	·04867	7	21·05
17	75·4	·04861	8	21·08
18	63·3	·04847	7	21·14
19	54·2	·04838	7	21·18
20	44·4	·04828	9	21·22
21	35·1	·04823	8	21·24
22	31·3	·04823	6	21·24
23	26·8	·04831	7	21·21
24	20·5	·04835	7	21·19
25	17·9	·04835	5	21·19
26	15·1	·04838	8	21·18

TABLE IX.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
0·65	16·9	·04830 cm.	8	$21{\cdot}21 \times 10^{11}$
	23·3	·04812	9	21·29
	29·7	·04808	9	21·31
	36·0	·04817	8	21·27
	42·4	·04826	6	21·23
	46·7	·04833	6	21·20
	42·4	·04820	8	21·26
	36·0	·04808	7	21·31
	29·7	·04806	7	21·32
	23·3	·04817	6	21·27
	16·9	·04830	8	21·21

Table X.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
3·7	16·9	·04812 cm.	8	$21{\cdot}29 \times 10^{11}$
	23·3	·04774	6	21·46
	29·7	·04765	6	21·50
	36·0	·04783	7	21·42
	42·4	·04808	7	21·31
	46·7	·04824	9	21·24
	42·4	·04792	8	21·38
	36·0	·04759	7	21·53
	29·7	·04752	6	21·56
	23·3	·04785	6	21·41
	16·9	·04812	6	21·29

Table XI.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
8·5	16·9	·04794 cm.	8	$21{\cdot}37 \times 10^{11}$
	23·3	·04743	6	21·60
	29·7	·04741	7	21·61
	36·0	·04767	7	21·49
	42·4	·04801	7	21·34
	46·7	·04825	9	21·25
	42·4	·04776	8	21·45
	36·0	·04739	8	21·62
	29·7	·04730	7	21·66
	23·3	·04745	7	21·59
	16·9	·04794	5	21·37

Table XII.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
17·4	16·9	·04801 cm.	7	$21{\cdot}34 \times 10^{11}$
	20·0	·04774	8	21·46
	23·3	·04767	5	21·49
	29·7	·04783	8	21·62
	36·0	·04812	8	21·29
	42·4	·04842	6	21·16
	46·7	·04853	7	21·11
	42·4	·04826	8	21·23
	36·0	·04785	9	21·41
	29·7	·04761	9	21·52
	23·3	·04767	8	21·49
	20·0	·04783	10	21·42
	16·9	·04801	10	21·34

Table XIII.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
28·5	16·9	·04815 cm.	7	$21{\cdot}28 \times 10^{11}$
	20·0	·04808	8	21·31
	23·3	·04810	9	21·30
	29·7	·04828	8	21·22
	36·0	·04847	10	21·14
	42·4	·04863	9	21·07
	46·7	·04870	9	21·04
	42·4	·04855	8	21·10
	36·0	·04835	7	21·19
	29·7	·04810	7	21·30
	23·3	·04801	8	21·34
	20·0	·04810	8	21·30
	16·9	·04815	6	21·28

Table XIV.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
33·7	16·9	·04835 cm.	8	$21{\cdot}19 \times 10^{11}$
	20·0	·04837	8	21·18
	23·3	·04847	8	21·14
	29·7	·04853	10	21·11
	36·0	·04863	10	21·07
	42·4	·04874	8	21·02
	46·7	·04879	8	21·00
	42·4	·04872	9	21·03
	36·0	·04861	8	21·08
	29·7	·04855	9	21·10
	23·3	·04844	9	21·15
	20·0	·04840	8	21·17
	16·9	·04370	6	21·18

Nickel.

As already stated in the introductory part of this paper, nickel was subjected to experiment in the same way as iron and steel. The wire was first stretched for twenty-four hours with the greatest load that was to be put on it, and then the modulus determined at the temperature of the room before any current was passed. The wire was demagnetised by reversals to eliminate the residual effects of the stress, the current increased step by step until the temperature was above 150° C., and at each step the modulus was determined. It will be seen that the modulus with the current was higher than before any current was passed through the wire, and also that the value when the load was 1·5 kilos. was greater after the cyclic state had been reached than during the first cycle. The other curves give the results that were obtained after the cyclic state had in each case been reached.

Before commencing each of the other cycles the wire was demagnetised, and the loads were 2·2, 2·9, 3·5, and 4 kilos. It will be seen from fig. 11 that as the weight is increased the modulus falls in value, that the hysteresis diminishes with increase of load, and that with the greatest load the modulus diminishes uniformly, the curve being a straight line.

At the conclusion of these cycles the wire was again demagnetised, and the modulus determined at the temperature of the room without any current. It was found to be higher than the first determination, so that the current had produced a permanent increase in the modulus.

Another piece of wire from the same coil was next passed through the double-walled tube, and the modulus determined for ordinary heating at the same temperatures as before by this method of heating—viz. that of the air, 35° C., 78° C., 100° C., and 130° C. As was to be expected, the graph was a straight line, showing the modulus diminishes uniformly when the wire is heated in this manner.

Finally, as stated in the general outline of the course at the beginning of this paper, the current was kept constant while the load was varied. The curves are shown in figs. 12 and 13, and it is at once evident that they are very similar to those for iron.

In order to eliminate any effect of torsion which might be produced by the wire hanging over the pulley, I attached the scale-pan to the wire by a string which was passed over the wheel.

On looking at the curves of fig. 11, we see that the effect of the current is first to produce an increase in the modulus, a maximum is reached, after which it begins to diminish. Next, as the load is increased the value of the modulus diminishes until, with the greatest load used, the decrease is uniform. It will also be seen that as the load is increased the maximum is reached in weaker fields, and that the curve for diminishing current always lies below the one for increasing current.

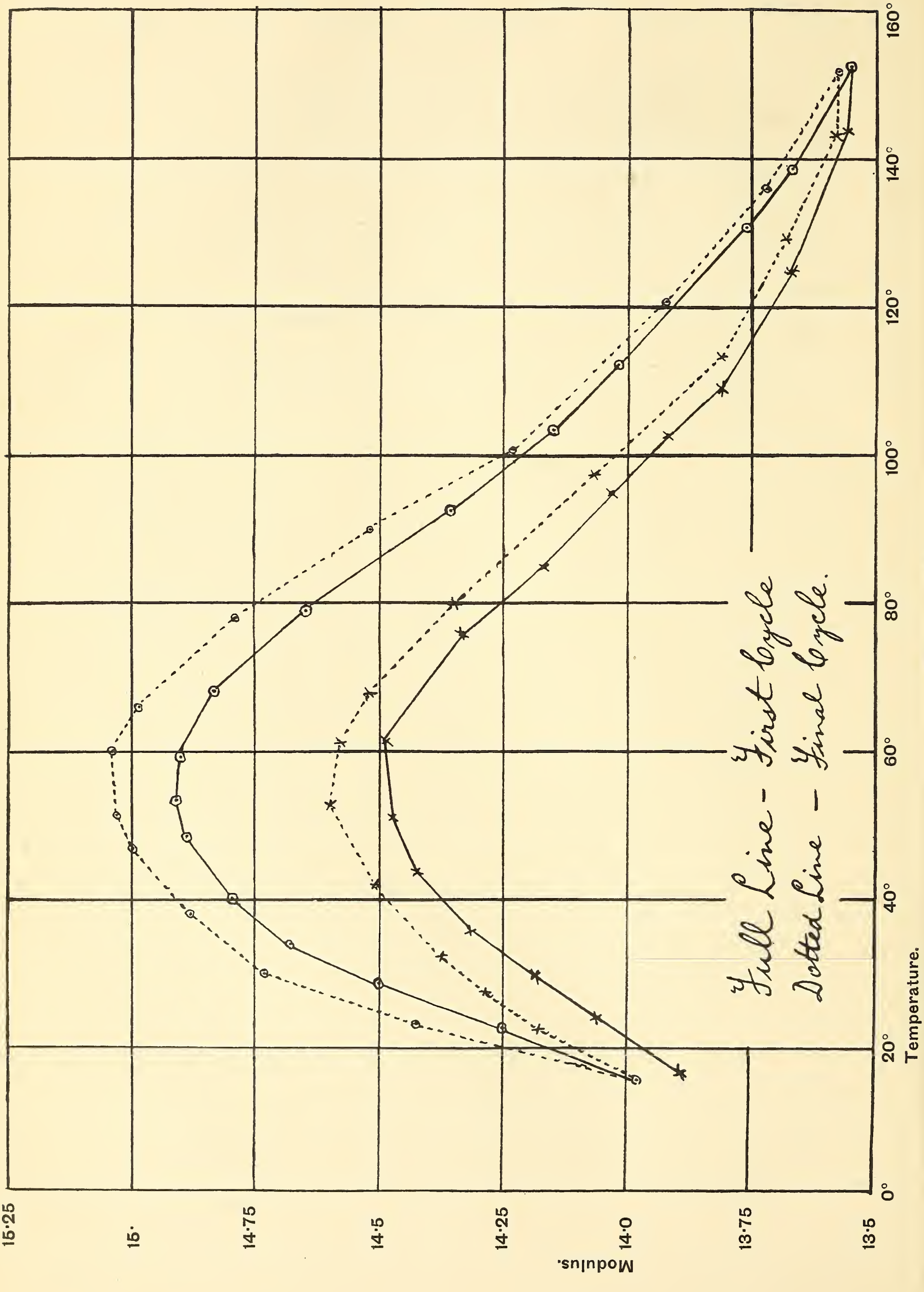


Fig. 10.—Nickel.

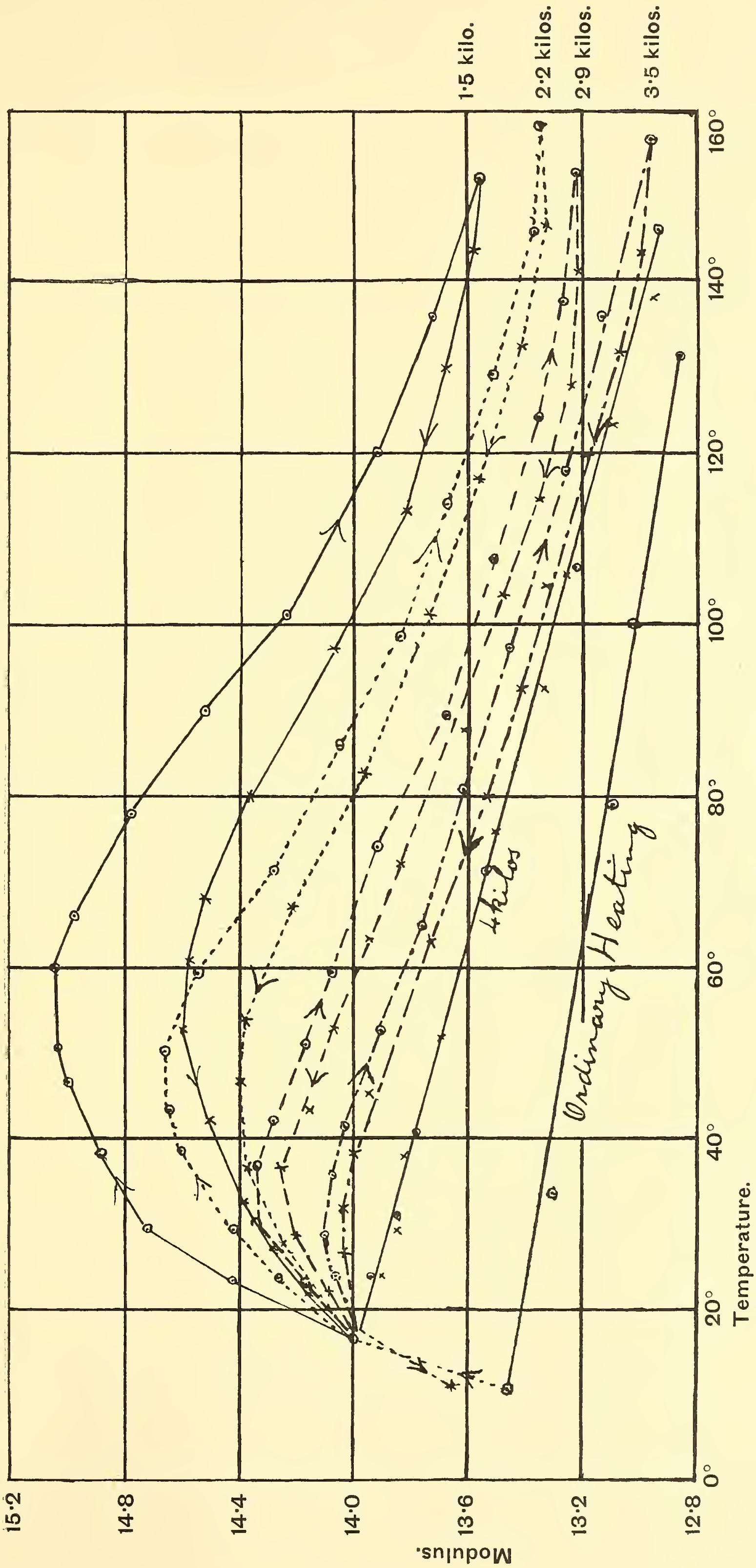


Fig. 11.—Nickel.

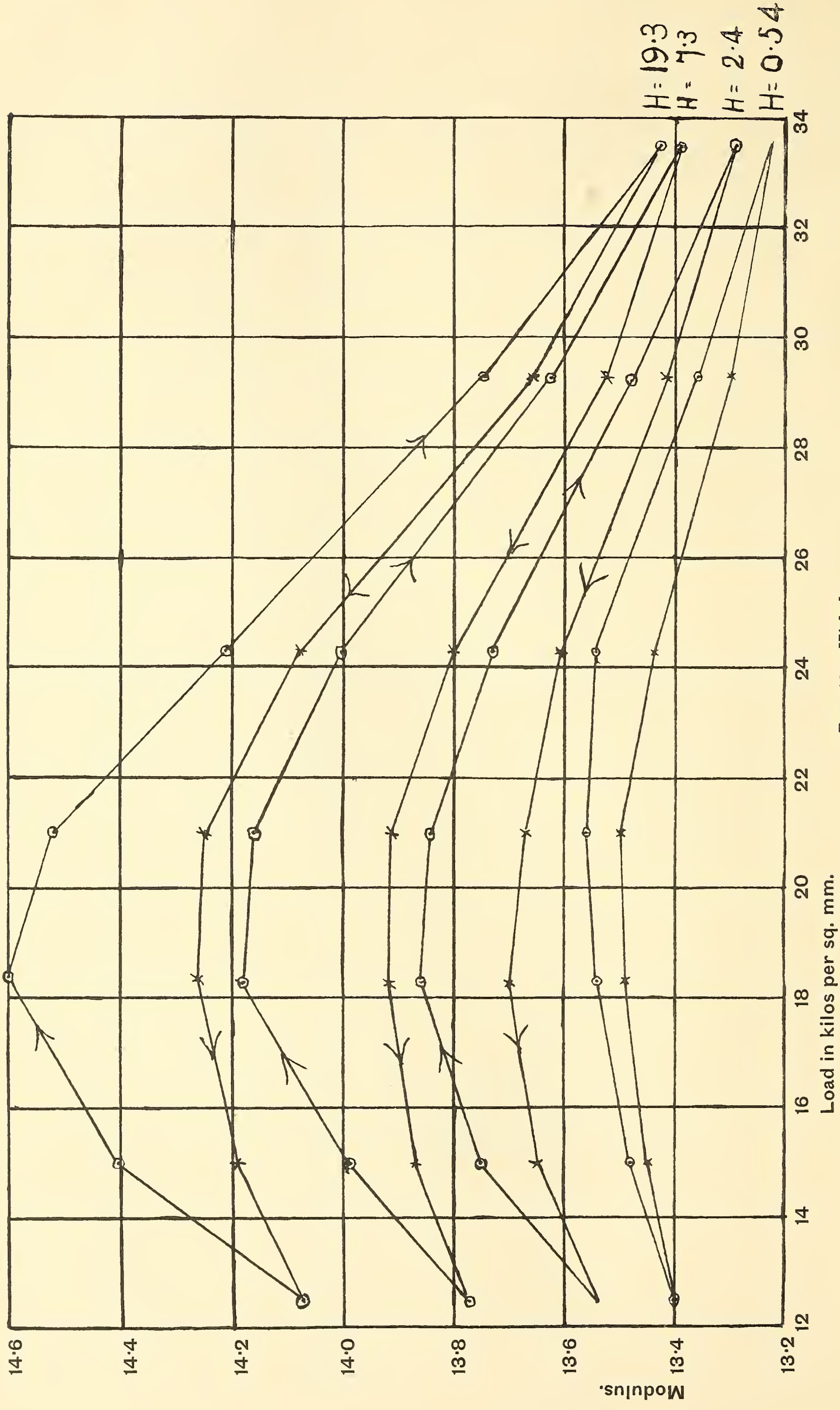


Fig. 12.—Nickel.

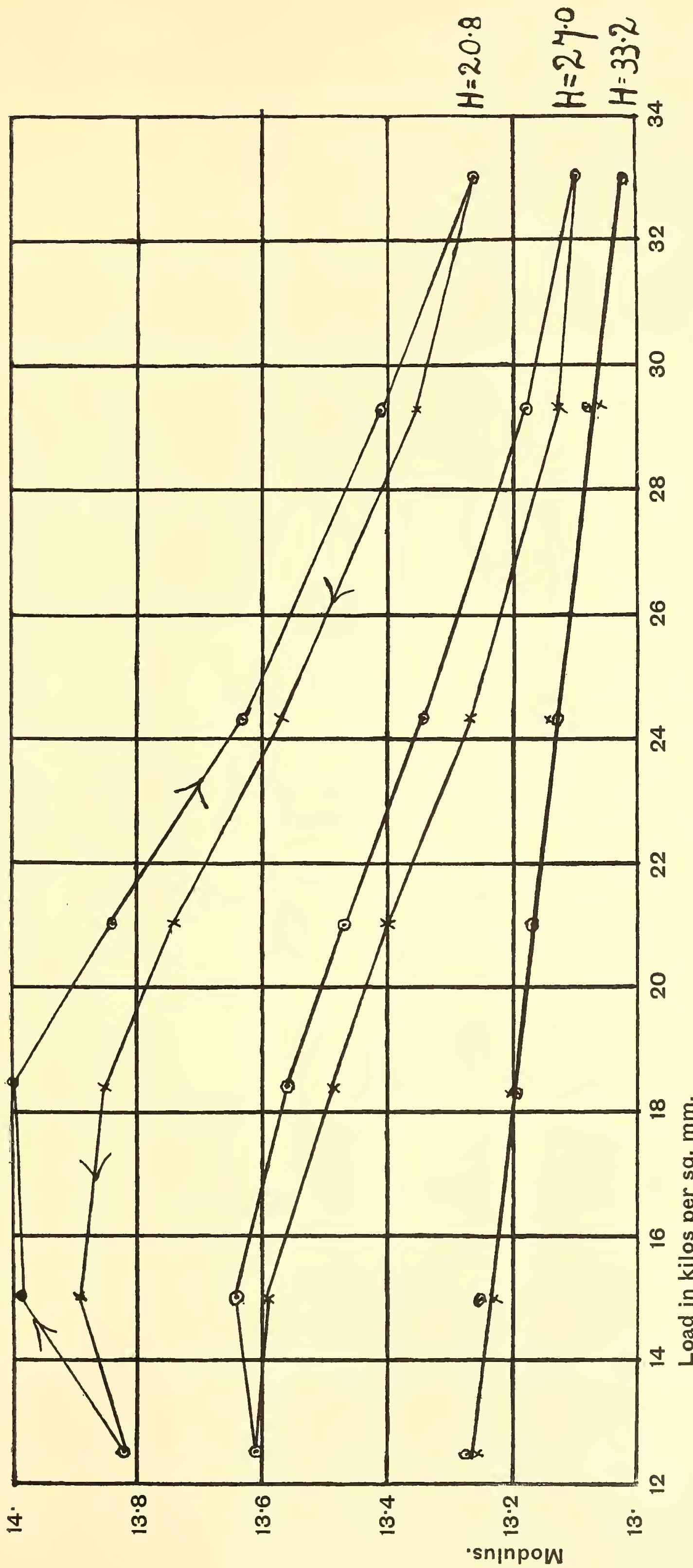


Fig. 13.—Nickel.

NICKEL.

Length = 97·82 cms.
Area of cross-section = 0·001195 sq. cm.
Elongation weight = 700 grams.
Total load on wire = 1·5 kilos.
Load per sq. mm. = 12·5 kilos.

TABLE I.—FIRST CYCLE.

No.	Temp.	Elongation for 700 grams.	No. of Observations.	M.
1	16°·0 C.	·04022 cm.	8	$13·98 \times 10^{11}$
2	22·3	·03969	9	14·17
3	28·6	·03878	9	14·50
4	33·7	·03830	8	14·68
5	40·1	·03799	10	14·80
6	48·4	·03776	8	14·89
7	53·2	·03771	9	14·91
8	59·8	·03774	9	14·90
9	68·5	·03792	8	14·83
10	79·1	·03838	8	14·65
11	92·9	·03916	10	14·36
12	103·6	·03965	9	14·15
13	117·4	·04011	8	14·02
14	130·7	·04086	7	13·76
15	139·0	·03771	7	13·67
16	152·6	·03774	6	13·55
17	143·5	·03792	6	13·56
18	132·7	·03838	6	13·63
19	125·4	·03916	6	13·68
20	114·3	·03965	6	13·82
21	102·9	·04039	8	13·92
22	94·6	·04005	6	14·04
23	83·0	·03966	6	14·18
24	75·8	·03921	9	14·34
25	61·3	·03882	9	14·49
26	52·7	·03884	8	14·48
27	44·2	·03897	5	14·43
28	38·1	·03930	5	14·32
29	29·6	·03963	7	14·19
30	23·5	·04000	6	14·06
31	16·2	·04048	9	13·89

TABLE II.—FINAL CYCLE FOR 1·5 KILOS.

No.	Temp.	Elongation for 700 grams.	No. of Observations.	M.
1	10°·6 C.	·04178 cm.	7	$13·46 \times 10^{11}$
2	16·7	·04014	7	14·01
3	23·6	·03900	9	14·42
4	30·1	·03817	9	14·73

TABLE II.—*continued.*

No.	Temp.	Elongation for 700 grams.	No. of Observations.	M.
5	38°·3 C.	·03779 cm.	8	$14·88 \times 10^{11}$
6	46·9	·03749	6	15·00
7	51·2	·03741	6	15·03
8	59·5	·03739	5	15·04
9	65·8	·03750	7	14·99
10	78·4	·03801	7	14·79
11	89·6	·03873	7	14·52
12	101·0	·03952	8	14·23
13	120·3	·04039	8	13·92
14	136·7	·04098	9	13·72
15	153·8	·04144	8	13·57
16	142·5	·04141	7	13·58
17	129·6	·04107	7	13·69
18	114·2	·04069	8	13·82
19	97·1	·03996	5	14·07
20	80·3	·03919	6	14·35
21	68·5	·03870	6	14·53
22	60·4	·03857	7	14·58
23	53·1	·03851	8	14·60
24	41·9	·03875	8	14·51
25	33·0	·03911	6	14·38
26	27·1	·03935	7	14·29
27	22·6	·03969	8	14·17
28	16·8	·04010	8	14·02

Total load = 2·2 kilos.
= 18·4 kilos. per sq. mm.

TABLE III.

No.	Temp.	Elongation for 700 grams.	No. of Observations.	M.
1	17°·0 C.	·04018 cm.	8	$13·99 \times 10^{11}$
2	24·1	·03941	7	14·27
3	29·4	·03900	8	14·42
4	38·6	·03851	7	14·60
5	43·5	·03838	7	14·65
6	50·2	·03836	9	14·66
7	59·7	·03857	9	14·58
8	71·9	·03938	8	14·28
9	85·4	·04000	8	14·06
10	98·7	·04059	9	13·85
11	114·5	·04110	10	13·68
12	129·1	·04162	8	13·51
13	145·8	·04203	7	13·38
14	158·7	·04212	7	13·35

Table III.—*continued.*

No.	Temp.	Elongation for 700 grams.	No. of Observations.	M.
15	146°·0 C.	·04206 cm.	7	$13·37 \times 10^{11}$
16	132·3	·04187	6	13·43
17	117·2	·04141	8	13·58
18	101·5	·04092	8	13·74
19	82·6	·04028	7	13·96
20	67·4	·03954	7	14·22
21	53·9	·03908	6	14·39
22	45·8	·03902	6	14·41
23	36·3	·03913	6	14·37
24	28·2	·03944	7	14·26
25	21·0	·03994	8	14·08
26	17·0	·04018	8	13·99

Total load = 2·9 kilos.
= 24·3 kilos. per sq. mm.

Table IV.

No.	Temp.	Elongation for 700 grams.	No. of Observations.	M.
1	17°·9 C.	·04021 cm.	7	$13·98 \times 10^{11}$
2	23·4	·03966	8	14·18
3	30·7	·03927	9	14·32
4	36·8	·03921	9	14·34
5	42·1	·03932	6	14·30
6	51·2	·03966	6	14·18
7	59·4	·03992	8	14·09
8	74·3	·04043	7	13·91
9	90·5	04107	7	13·69
10	107·6	·04162	7	13·51
11	121·9	·04209	7	13·36
12	138·1	·04238	7	13·27
13	152·7	·04250	6	13·23
14	141·5	·04253	6	13·22
15	128·4	·04244	8	13·25
16	115·0	·04209	8	13·36
17	103·2	·04168	7	13·49
18	88·3	·04125	9	13·63
19	71·6	·04059	9	13·85
20	62·9	·04028	10	13·96
21	52·6	·03994	10	14·08
22	44·1	·03969	8	14·17
23	37·0	·03941	8	14·27
24	28·5	·03954	8	14·22
25	22·9	·03997	8	14·07
26	18·0	·04021	7	13·98

Total load = 3·5 kilos.
= 29·3 kilos. per sq. mm.

TABLE V.

No.	Temp.	Elongation for 700 grams.	No. of Observations.	M.
1	18°·1 C.	·04024 cm.	10	$13·97 \times 10^{11}$
2	23·9	·03994	9	14·08
3	28·6	·03991	9	14·09
4	35·4	·03994	8	14·08
5	41·2	·04005	8	14·04
6	52·7	·04042	7	13·91
7	64·8	·04080	7	13·78
8	80·5	·04132	8	13·61
9	97·3	·04175	8	13·47
10	118·0	·04238	8	13·27
11	135·2	·04283	8	13·13
12	156·4	·04338	6	12·96
13	143·6	·04322	6	13·01
14	131·8	·04300	7	13·08
15	120·3	·04253	7	13·22
16	104·7	·04216	6	13·34
17	92·6	·04194	6	13·41
18	77·9	·04150	9	13·55
19	63·1	·04092	9	13·74
20	56·5	·04083	10	13·77
21	45·2	·04028	8	13·96
22	38·6	·04014	8	14·01
23	31·8	·04005	7	14·04
24	26·7	·04005	8	14·04
25	21·3	·04011	8	14·02
26	18·0	·04024	8	13·97

Total load = 4 kilos.
= 33·5 kilos. per sq. mm.

TABLE VI.

No.	Temp.	Elongation for 700 grams.	No. of Observations.	M.
1	17°·9 C.	·04028 cm.	8	$13·96 \times 10^{11}$
2	22·3	·04035	8	13·94
3	31·5	·04059	9	13·85
4	40·7	·04080	10	13·78
5	55·8	·04107	9	13·69
6	71·2	·04140	7	13·55
7	88·6	·04187	9	13·43
8	107·1	·04253	7	13·22
9	129·4	·04297	8	13·09
10	146·0	·04341	8	12·95
11	137·9	·04334	8	12·97
12	123·6	·04288	9	13·11

Table VI.—*continued.*

No.	Temp.	Elongation for 700 grams.	No. of Observations.	M.
13	105°·7 C.	·04235 cm.	8	$13·28 \times 10^{11}$
14	92·8	·04219	9	13·33
15	76·3	·04165	10	13·50
16	62·4	·04122	10	13·64
17	51·6	·04104	9	13·70
18	38·0	·04066	9	13·83
19	29·5	·04056	8	13·86
20	24·3	·04043	8	13·91
21	17·9	·04028	10	13·96
22	10·8	·03973	10	14·15

Table VII.

Field.	Load in kilos. per sq. mm.	Elongation for 700 grams.	No. of Observations.	M.
0·54	12·5	·04196 cm.	8	$13·40 \times 10^{11}$
	15·0	·04170	8	13·48
	18·4	·04153	9	13·54
	21·0	·04147	10	13·56
	24·3	·04153	9	13·54
	29·3	·04223	9	13·36
	33·5	·04254	8	13·22
	29·3	·04228	7	13·30
	24·3	·04183	7	13·44
	21·0	·04165	6	13·50
	18·4	·04168	6	13·49
	15·0	·04181	8	13·45
	12·5	·04196	8	13·40

Table VIII.

Field.	Load in kilos. per sq. mm.	Elongation for 700 grams.	No. of Observations.	M.
2·4	12·5	·04153 cm.	7	$13·54 \times 10^{11}$
	15·0	·04089	7	13·75
	18·4	·04057	8	13·86
	21·0	·04063	6	13·84
	24·3	·04095	6	13·73
	29·3	·04170	5	13·48
	33·5	·04232	7	13·29
	29·3	·04190	10	13·42
	24·3	·04132	8	13·61
	21·0	·04112	9	13·67
	18·4	·04104	7	13·70
	15·0	·04115	6	13·66
	12·5	·04153	9	13·54

Table IX.

Field.	Load in kilos. per sq. mm.	Elongation for 700 grams.	No. of Observations.	M.
7·3	12·5	·04083 cm.	8	$13·77 \times 10^{11}$
	15·0	·04018	8	13·99
	18·4	·03957	9	14·18
	21·0	·03960	9	14·17
	24·3	·04015	10	14·00
	29·3	·04125	8	13·63
	33·5	·04199	7	13·39
	29·3	·04156	6	13·53
	24·3	·04075	6	13·80
	21·0	·04042	8	13·91
	18·4	·04039	8	13·92
	15·0	·04054	8	13·87
	12·5	·04083	6	13·77

Table X.

Field.	Load in kilos. per sq. mm.	Elongation for 700 grams.	No. of Observations.	M.
13·9	12·5	·03997 cm.	10	$14·07 \times 10^{11}$
	15·0	·03908	8	14·39
	18·4	·03851	8	14·60
	21·0	·03873	7	14·52
	24·3	·03957	5	14·21
	29·3	·04092	7	13·74
	33·5	·04187	9	13·43
	29·3	·04115	10	13·66
	24·3	·03994	8	14·08
	21·0	·03946	8	14·25
	18·4	·03944	8	14·26
	15·0	·03963	9	14·19
	12·5	·03997	9	14·07

Table XI.

Field.	Load in kilos. per sq. mm.	Elongation for 700 grams.	No. of Observations.	M.
20·8	12·5	·04069 cm.	7	$13·82 \times 10^{11}$
	15·0	·04018	8	13·99
	18·4	·04012	8	14·01
	21·0	·04063	9	13·84
	24·3	·04125	5	13·63
	29·3	·04193	7	13·41
	33·5	·04241	7	13·26
	29·3	·04226	6	13·35
	24·3	·04144	6	13·57
	21·0	·04092	8	13·74
	18·4	·04060	8	13·85
	15·0	·04048	10	13·89
	12·5	·04069	8	13·82

TABLE XII.

Field.	Load in kilos. per sq. mm.	Elongation for 700 grams.	No. of Observations.	M.
27·0	12·5	·04132 cm.	8	$13·61 \times 10^{11}$
	15·0	·04122	7	13·64
	18·4	·04147	6	13·56
	21·0	·04173	9	13·47
	24·3	·04229	6	13·34
	29·3	·04267	8	13·18
	33·5	·04297	7	13·09
	29·3	·04283	7	13·13
	24·3	·04238	8	13·27
	21·0	·04196	9	13·40
	18·4	·04167	9	13·49
	15·0	·04137	8	13·59
	12·5	·04131	6	13·61

TABLE XIII.

Field.	Load in kilos. per sq. mm.	Elongation for 700 grams.	No. of Observations.	M.
32·2	12·5	·04238 cm.	7	$13·27 \times 10^{11}$
	15·0	·04244	6	13·25
	18·4	·04264	8	13·19
	21·0	·04270	9	13·17
	24·3	·04283	8	13·13
	29·3	·04300	9	13·08
	33·5	·04319	7	13·02
	29·3	·04303	7	13·07
	24·3	·04280	6	13·14
	21·0	·04270	6	13·17
	18·4	·04260	6	13·20
	15·0	·04250	8	13·23
	12·5	·04241	8	13·26

Load = 2·5 kilos.

TABLE XIV.—ORDINARY HEATING.

No.	Temp.	Elongation for 700 grams.	No. of Observations.	M.
1	10°·7 C.	·04178 cm.	8	$13·46 \times 10^{11}$
2	35·0	·04214	8	13·34
3	78·0	·04283	6	13·13
4	100·0	·04319	6	13·02
5	130·0	·04358	6	12·87

Cobalt.

The specimen of cobalt I used was a rectangular strip, as I had found it impossible to obtain a rod of a suitable size.

In the first place, the cobalt was heated in the ordinary way in the double-walled tube, the temperatures being the same as in the previous cases. The graph was a straight line, and the results are represented in fig. 15.

It was then passed through the glass tube preparatory to being heated by the current, and the modulus again determined at the temperature of the room. It was found to be higher, so that heating it to 130° C. had produced an increase in the modulus, a result similar to what was obtained by Shakespear and Gray, Blyth and Dunlop, in their experiments on Young's Modulus.

A weak current was then passed, with the result that the modulus was still further increased. On continuing to increase the current, the modulus also increased, until a maximum was reached at about 57° C., after which increase of current caused the modulus to diminish. The heating was continued until the temperature was nearly 160° C. When the current was diminished the modulus rose, and had at first a higher value than at the corresponding stage with increasing current. A maximum was reached at about 80° C., and it was lower than that with the increasing current. This was the first cycle, and the curve did not return into itself, so that the cyclic state had not been reached. The cycle was repeated, with the result that all through the value was a little higher than in the first; and it was not until the fourth cycle had been completed that the metal attained the cyclic condition. The first and last of these experiments are given in fig. 14, the weight being 15 kilos., or 13 kilos. per sq. mm.

The cycles were repeated with increasing loads, these being 20, 25, 30, 35, and 40 kilos., that is, 17·3, 21·6, 26·0, 30·3, and 34·6 kilos. per sq. mm. respectively. The results are shown in fig. 15, and the curves show in each case the values for the final cycle only.

Finally, the current was kept constant while the load was varied, and the results of this series of experiments are shown in figs. 16 and 17. These curves are also found to be similar in form to those for iron and nickel. As they display no marked peculiarity, it is needless to enter into any detailed discussion of them. We find at first the same increase in the value as the current is raised, that a maximum is reached, and thereafter increase of current causes the modulus to diminish.

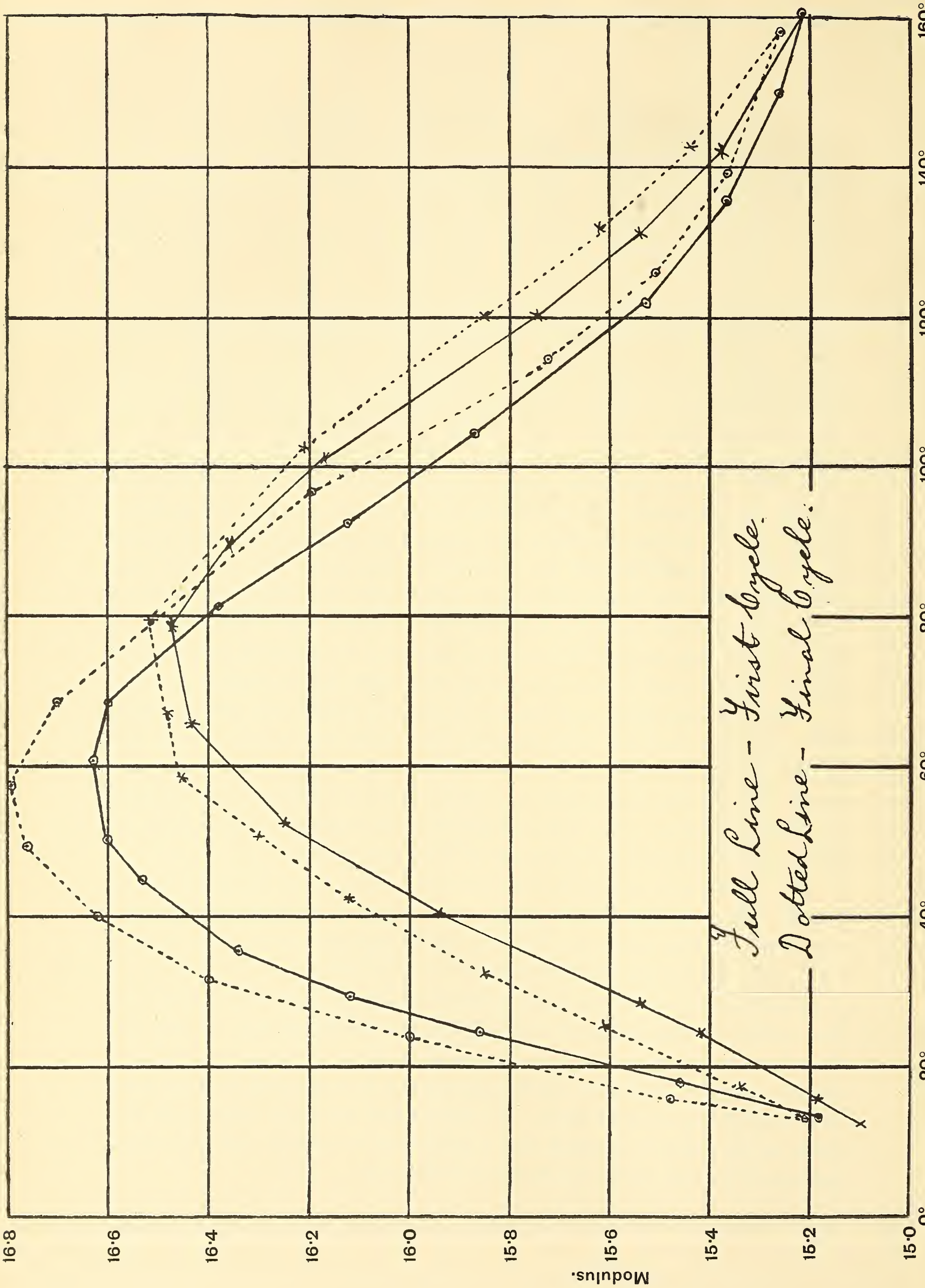


Fig. 14.—Cobalt.

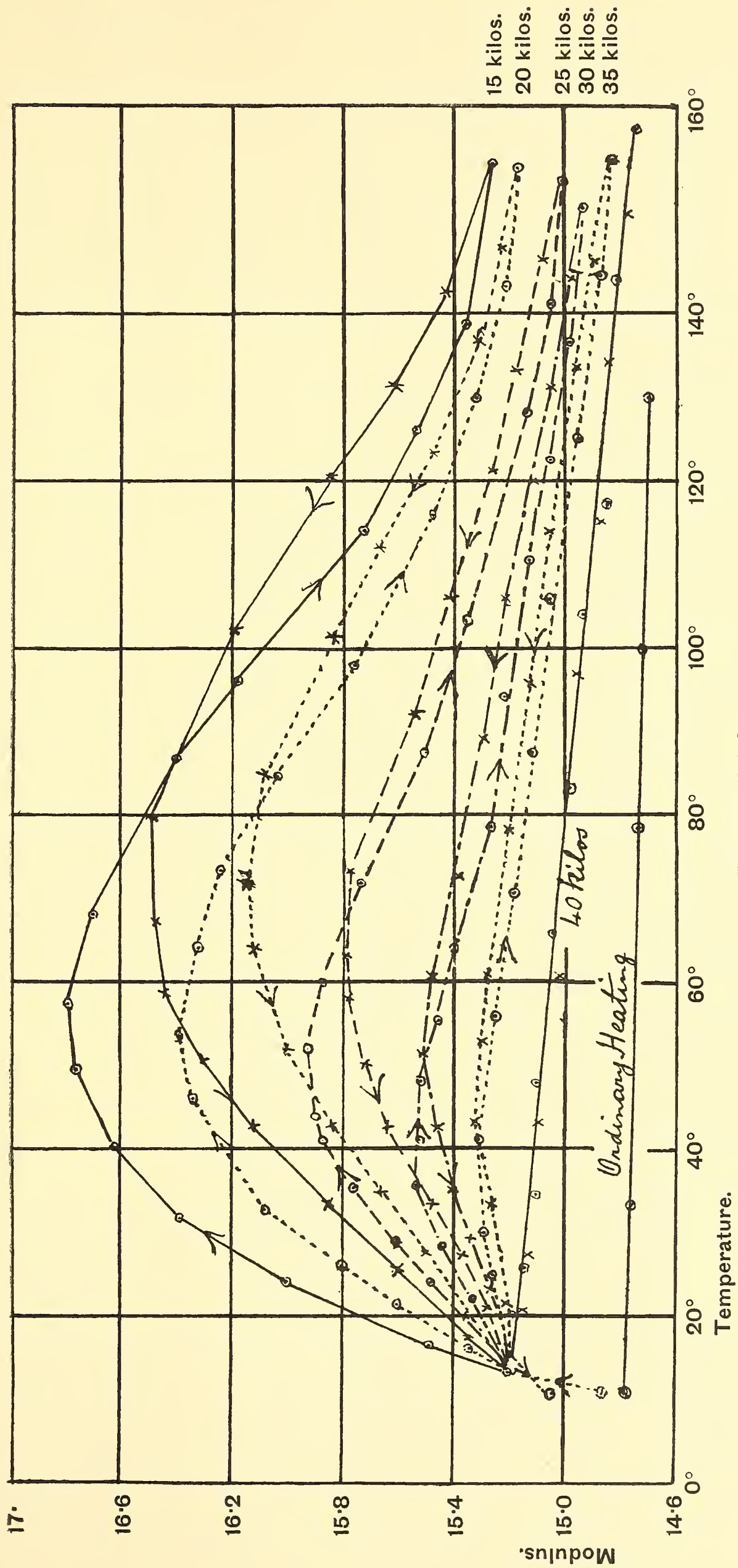


Fig. 15.—Cobalt.

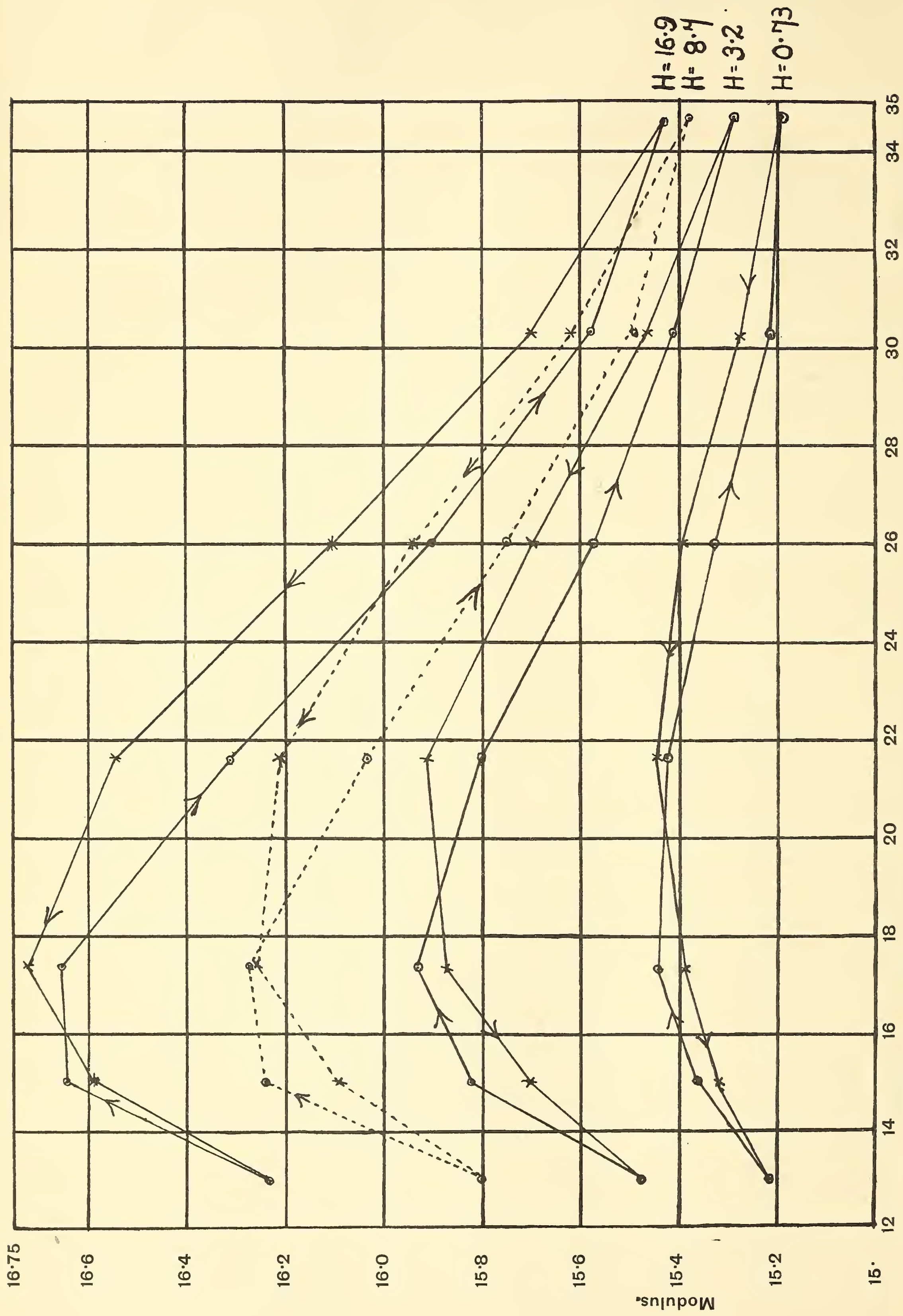


Fig. 16.—Cobalt.

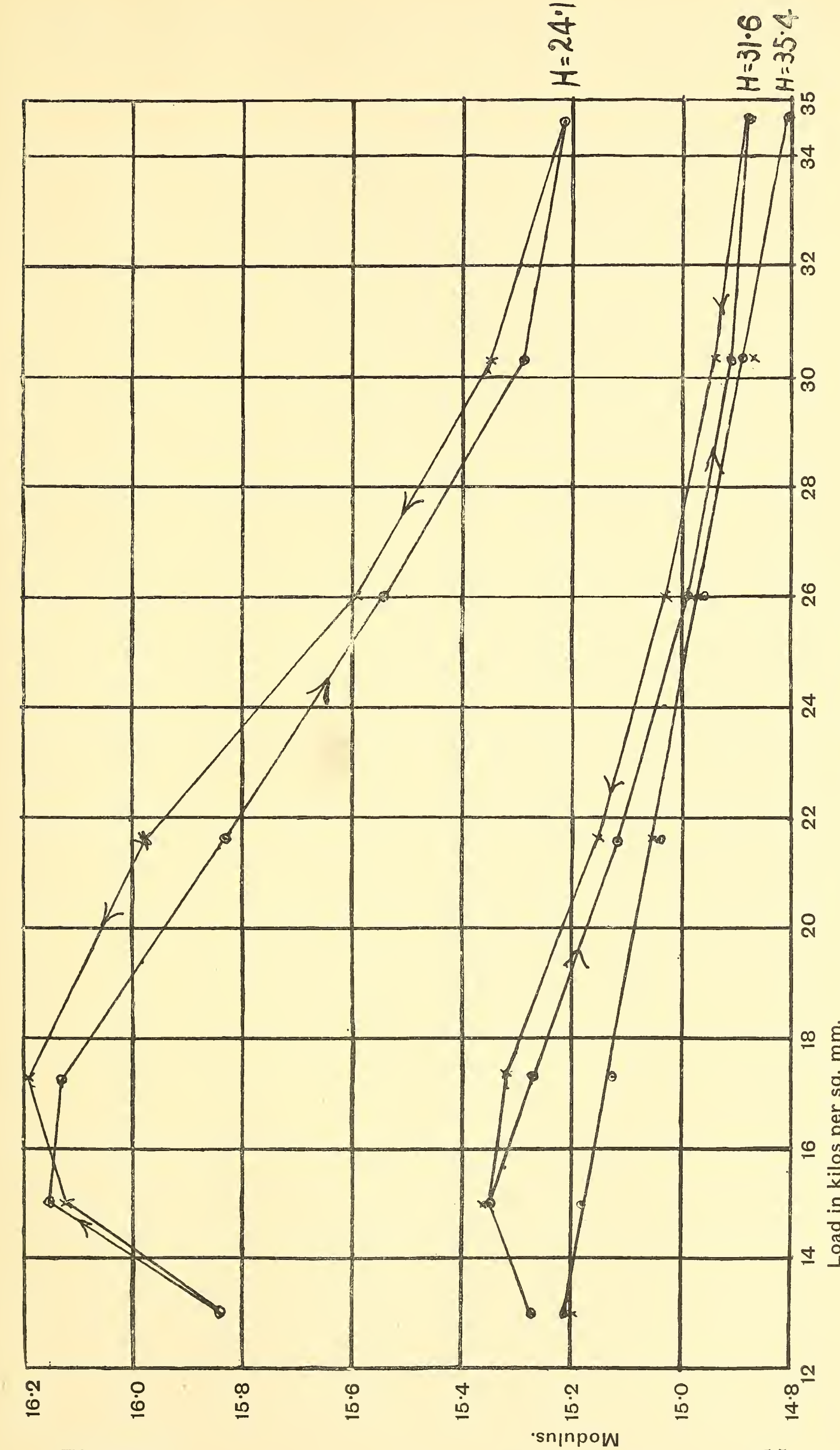


Fig. 17.—Cobalt.

Cobalt.

Length	=	73·1 cms.
Breadth	=	0·335 cm.
Thickness	=	0·0350 cm.
Area of cross-section	=	0·01155 sq. cm.
Elongation weight	=	5 kilos.
Total load	=	15 kilos.
Load per sq. mm.	=	13 kilos.

Table I.—First Cycle.

No.	Temp.	Elongation for 5 kilos.	No. of Observations.	M.
1	13°·5 C.	·02045 cm.	8	$15·19 \times 10^{11}$
2	17·2	·02009	8	15·46
3	24·9	·01970	10	15·76
4	29·6	·01925	10	16·13
5	35·3	·01901	9	16·34
6	44·7	·01879	8	16·53
7	50·1	·01871	10	16·60
8	60·8	·01867	10	16·63
9	68·4	·01871	9	16·60
10	81·0	·01896	9	16·38
11	92·6	·01925	7	16·13
12	104·5	·01957	7	15·87
13	122·3	·02001	6	15·53
14	135·1	·02021	8	15·37
15	149·7	·02035	6	15·26
16	160·4	·02040	7	15·22
17	152·1	·02033	9	15·28
18	142·2	·02019	10	15·38
19	131·5	·01998	8	15·54
20	120·7	·01972	8	15·75
21	101·3	·01921	10	16·17
22	89·4	·01898	8	16·36
23	78·9	·01886	7	16·47
24	65·8	·01890	7	16·43
25	52·3	·01911	9	16·25
26	40·5	·01948	0	15·94
27	31·6	·01998	8	15·54
28	24·3	·02014	8	15·42
29	16·2	·02045	8	15·19
30	12·6	·02057	8	15·10

TABLE II.—FINAL CYCLE FOR 15 KILOS.

No.	Temp.	Elongation for 5 kilos.	No. of Observations.	M.
1	10°·3 C.	·02090 cm.	8	$14·86 \times 10^{11}$
2	13·3	·02042	10	15·21
3	16·5	·02006	8	15·48
4	23·8	·01941	7	16·00
5	31·7	·01895	5	16·39
6	40·1	·01869	6	16·62
7	49·4	·01853	6	16·76
8	57·3	·01849	7	16·79
9	68·2	·01860	8	16·70
10	82·9	·01891	6	16·42
11	96·4	·01918	9	16·19
12	113·5	·01974	8	15·73
13	125·6	·02002	8	15·51
14	139·1	·02021	10	15·37
15	157·8	·02035	8	15·26
16	142·5	·02011	7	15·44
17	131·7	·01988	5	15·62
18	120·6	·01958	8	15·86
19	102·4	·01916	10	15·21
20	79·3	·01880	9	15·52
21	67·1	·01884	9	15·48
22	58·9	·01888	8	15·45
23	50·6	·01905	8	16·30
24	42·7	·01927	7	16·12
25	33·2	·01959	6	15·85
26	25·6	·01989	5	15·61
27	17·1	·02024	9	15·34
28	14·4	·02037	8	15·25
29	13·3	·02042	6	15·21

Total load = 20 kilos.
= 17·3 kilos. per sq. mm.

TABLE III.

No.	Temp.	Elongation for 5 kilos.	No. of Observations.	M.
1	12°·8 C.	·02041 cm.	8	$15·22 \times 10^{11}$
2	15·7	·02023	8	15·35
3	21·4	·01990	10	15·60
4	26·3	·01964	10	15·81
5	32·9	·01930	6	16·09
6	39·5	·01915	8	16·22
7	46·2	·01901	8	16·34
8	53·1	·01899	8	16·35
9	61·8	·01903	9	16·32
10	72·6	·01913	7	16·24
11	84·9	·01935	8	16·05
12	98·5	·01969	8	15·77
13	116·3	·02006	9	15·48
14	130·0	·02026	9	15·33
15	143·7	·02039	7	15·23
16	158·2	·02046	6	15·18

Table III.—*continued.*

No.	Temp.	Elongation for 5 kilos.	No. of Observations.	M.
17	147°·5 C.	·02037 cm.	9	$15·25 \times 10^{11}$
18	136·8	·02026	8	15·33
19	123·0	·02005	8	15·49
20	111·4	·01981	8	15·68
21	101·3	·01959	5	15·85
22	85·1	·01930	7	16·09
23	72·2	·01924	9	16·14
24	63·6	·01927	8	16·12
25	51·9	·01940	8	16·01
26	42·8	·01960	8	15·84
27	34·7	·01983	9	15·66
28	27·5	·02004	9	15·50
29	20·6	·02021	8	15·37
30	16·4	·02028	7	15·31
31	12·8	·02041	7	15·22

Total load = 25 kilos.
= 21·65 kilos. per sq. mm.

Table IV.

No.	Temp.	Elongation for 5 kilos.	No. of Observations.	M.
1	12°·7 C.	·02042 cm.	10	$15·21 \times 10^{11}$
2	16·1	·02033	8	15·28
3	22·5	·02005	9	15·49
4	29·3	·01988	7	15·62
5	35·6	·01970	6	15·76
6	40·7	·01957	6	15·87
7	44·2	·01952	9	15·91
8	57·9	·01951	8	15·92
9	59·6	·01956	8	15·88
10	72·1	·01973	6	15·74
11	87·8	·02000	7	15·50
12	103·4	·02023	8	15·36
13	124·9	·02046	8	15·18
14	141·2	·02061	8	15·07
15	155·8	·02069	8	15·01
16	146·5	·02057	9	15·10
17	133·1	·02046	6	15·18
18	121·7	·02034	7	15·27
19	105·6	·02013	6	15·43
20	92·3	·01997	8	15·55
21	74·2	·01971	9	15·76
22	64·7	·01966	9	15·80
23	57·9	·01968	8	15·78
24	50·4	·01974	8	15·73
25	41·8	·01984	5	15·60
26	33·3	·02006	7	15·48
27	26·9	·02021	6	15·37
28	21·0	·02033	8	15·28
29	15·7	·02040	7	15·23
30	12·7	·02042	7	15·21

Total load = 30 kilos.
= 26 kilos. per sq. mm.

Table V.

No.	Temp.	Elongation for 5 kilos.	No. of Observations.	M.
1	12°·9 C.	·02043 cm.	8	$15·20 \times 10^{11}$
2	16·1	·02037	10	15·25
3	21·8	·02026	10	15·33
4	27·6	·02011	8	15·44
5	35·7	·02001	8	15·52
6	40·5	·02001	8	15·52
7	48·3	·02002	7	15·51
8	55·6	·02009	9	15·46
9	63·8	·02017	6	15·40
10	79·2	·02033	6	15·28
11	94·5	·02042	8	15·21
12	110·4	·02053	10	15·13
13	122·6	·02061	9	15·07
14	137·1	·02071	7	14·99
15	154·0	·02077	6	14·95
16	143·7	·02071	6	14·99
17	131·2	·02062	7	15·06
18	119·8	·02053	8	15·13
19	105·9	·02043	6	15·20
20	88·6	·02032	7	15·29
21	72·5	·02020	7	15·38
22	60·7	·02009	6	15·46
23	51·3	·02009	6	15·50
24	43·0	·02010	9	15·45
25	35·1	·02017	10	15·40
26	29·2	·02026	8	15·33
27	22·7	·02033	8	15·28
28	16·8	·02039	8	15·24
29	12·9	·02043	8	15·20

Total load = 35 kilos.
= 30·3 kilos. per sq. mm.

Table VI.

No.	Temp.	Elongation for 5 kilos.	No. of Observations.	M.
1	13°·1 C.	·02044 cm.	5	$15·19 \times 10^{11}$
2	17·8	·02040	7	15·23
3	24·6	·02035	7	15·26
4	30·2	·02031	9	15·29
5	41·3	·02028	9	15·31
6	55·9	·02035	6	15·26

Table VI.—*continued.*

No.	Temp.	Elongation for 5 kilos.	No. of Observations.	M.
7	70°·7 C.	·02045 cm.	8	$15·19 \times 10^{11}$
8	87·4	·02053	6	15·13
9	105·8	·02061	8	15·07
10	125·4	·02070	9	14·98
11	142·5	·02085	9	14·89
12	159·0	·02093	8	14·84
13	141·6	·02084	8	14·90
14	131·8	·02076	8	14·96
15	114·0	·02062	9	15·06
16	97·2	·02054	9	15·12
17	78·5	·02043	7	15·20
18	60·4	·02033	6	15·28
19	53·1	·02028	6	15·31
20	42·7	·02029	7	15·32
21	33·2	·02034	8	15·27
22	21·5	·02041	8	15·22
23	17·4	·02042	9	15·21
24	13·1	·02044	9	15·19

Total load = 40 kilos.
= 34·6 kilos. per sq. mm.

Table VII.

No.	Temp.	Elongation for 5 kilos.	No. of Observations.	M.
1	13°·0	·02045 cm.	8	$15·19 \times 10^{11}$
2	19·6	·02046	9	15·18
3	25·9	·02052	9	15·14
4	34·7	·02055	8	15·11
5	48·2	·02056	10	15·10
6	65·8	·02063	9	15·05
7	83·1	·02071	9	14·99
8	102·5	·02078	6	14·94
9	121·4	·02091	6	14·85
10	143·6	·02095	7	14·82
11	162·4	·02105	5	14·75
12	151·9	·02101	7	14·78
13	134·2	·02090	8	14·86
14	115·7	·02088	8	14·87
15	97·3	·02077	9	14·95
16	76·1	·02070	9	14·98
17	60·8	·02063	10	15·02
18	42·0	·02057	10	15·10
19	27·6	·02053	8	15·13
20	20·2	·02050	8	15·15
21	13·1	·02045	6	15·19
22	10·7	·02066	5	15·03

TABLE VIII.

Field.	Load in kilos. per sq. mm.	Elongation for 5 kilos.	No. of Observations.	M.
0·73	13·0	·02040 cm.	8	$15·22 \times 10^{11}$
	15·0	·02023	8	15·36
	17·3	·02011	6	15·44
	21·6	·02012	6	15·43
	26·0	02026	7	15·33
	30·3	·02040	7	15 22
	34·6	·02045	9	15·19
	30·3	·02033	9	15·28
	26·0	·02017	10	15·40
	21·6	·02011	8	15·44
	17·3	·02018	8	15·39
	15·0	·02027	6	15·32
	13·0	·02040	6	15·22

TABLE IX.

Field.	Load in kilos. per sq. mm.	Elongation for 5 kilos.	No. of Observations.	M.
3·2	13·0	·02006 cm.	7	$15·48 \times 10^{11}$
	15·0	·01963	9	15·82
	17·3	·01950	9	15·93
	21·6	·01966	6	15·80
	26·0	·01993	8	15·58
	30·3	·02014	8	15·41
	34·6	·02032	8	15·29
	30·3	·02007	8	15·47
	26·0	·01978	9	15·70
	21·6	·01951	9	15·92
	17·3	·01955	8	15·88
	15·0	·01977	8	15·71
	13·0	·02006	8	15·48

TABLE X.

Field.	Load in kilos. per sq. mm.	Elongation for 5 kilos.	No. of Observations.	M.
8·7	13·0	·01966 cm.	9	$15·80 \times 10^{11}$
	15·0	·01912	9	16·24
	17·3	·01909	8	16·27
	21·6	·01937	8	16·03
	26·0	·01972	6	15·75
	30·3	·02005	6	15·49
	34·6	·02019	7	15 38
	30·3	·01988	7	15·62
	26·0	·01948	8	15·94
	21·6	·01937	5	16·21
	17·3	·01910	8	16·26
	15·0	·01930	6	16·09
	13·0	·01966	8	15·80

Table XI.

Field.	Load in kilos. per sq. mm.	Elongation for 5 kilos.	No. of Observations.	M.
16·9	13·0	·01914 cm.	5	$16·23 \times 10^{11}$
	15·0	·01866	8	16·64
	17·3	·01865	7	16·65
	21·6	·01904	9	16·31
	26·0	·01954	7	15·90
	30·3	·01993	6	15·58
	34·6	·02012	6	15·43
	30·3	·01978	8	15·70
	26·0	·01928	8	16·11
	21·6	·01878	9	16·54
	17·3	·01857	9	16·72
	15·0	·01872	7	16·59
	13·0	·01914	6	16·23

Table XII.

Field.	Load in kilos. per sq. mm.	Elongation for 5 kilos.	No. of Observations.	M.
24·1	13·0	·01960 cm.	7	$15·84 \times 10^{11}$
	15·0	·01922	8	16·16
	17·3	·01926	8	16·13
	21·6	·01962	9	15·83
	26·0	·01998	9	15·54
	30·3	·02032	10	15·29
	34·6	·02041	10	15·21
	30 3	·02024	9	15 35
	26 0	·01992	8	15·59
	21·6	·01943	7	15·98
	17·3	·01918	8	16·19
	15·0	·01927	8	16·12
	13·0	·01960	8	15·84

Table XIII.

Field.	Load in kilos. per sq. mm.	Elongation for 5 kilos.	No. of Observations.	M.
31·6	13·0	·02034 cm.	8	$15\ 27 \times 10^{11}$
	15·0	·02023	5	15·36
	17·3	·02034	7	15·27
	21·6	·02054	8	15·12
	26·0	·02071	6	14·99
	30·3	·02083	6	14·91
	34·6	·02087	7	14·88
	30·3	·02078	7	14·94
	26·0	02066	8	15·03
	21·6	·02050	8	15·15
	17·3	·02027	6	15·32
	15·0	·02023	6	15·36
	13·0	·02034	7	15·27

TABLE XIV.

Field.	Load in kilos. per sq. mm.	Elongation for 5 kilos.	No. of Observations.	M.
35·4	13·0	·02042 cm.	7	$15·21 \times 10^{11}$
	15·0	·02046	8	15·18
	17·3	·02053	7	15·13
	21·6	·02065	9	15·04
	26·0	·02077	6	14·95
	30·3	·02085	6	14·89
	34·6	·02098	5	14·80
	30·3	·02086	5	14·88
	26·0	·02074	6	14·97
	21·6	·02064	8	15·05
	17·3	·02053	8	15·13
	15·0	·02047	7	15·17
	13·0	·02043	9	15·20

Load = 25 kilos.

TABLE XV.—ORDINARY HEATING.

No.	Temp.	Elongation for 5 kilos.	No. of Observations.	M.
1	11°·2 C.	·02100 cm.	8	$14·79 \times 10^{11}$
2	35·0	·02103	7	14·76
3	78·0	·02105	9	14·75
4	100·0	·02108	6	14·73
5	130·0	·02111	8	14·71

COPPER.

The course of the experiments on copper was exactly the same as that described for the other metals.

The results are represented in fig. 18. The curves show how the modulus varies as the current is increased, also how the maximum values get less as the load is increased, and how with increase of load the maximum is reached at an earlier stage in the cyclic process.

The curves resemble those for steel and cobalt, as the curve for diminishing current lies at first above that for increasing current, attains a maximum for the diminishing current, which is lower than that for the increasing current, and finally, has a value less than what it was for the increasing current.

The results for those experiments in which the current was constant and the load altered are shown in figs. 19 and 20; but a minute description of them would merely be a repetition of statements already made in connection with the other metals.

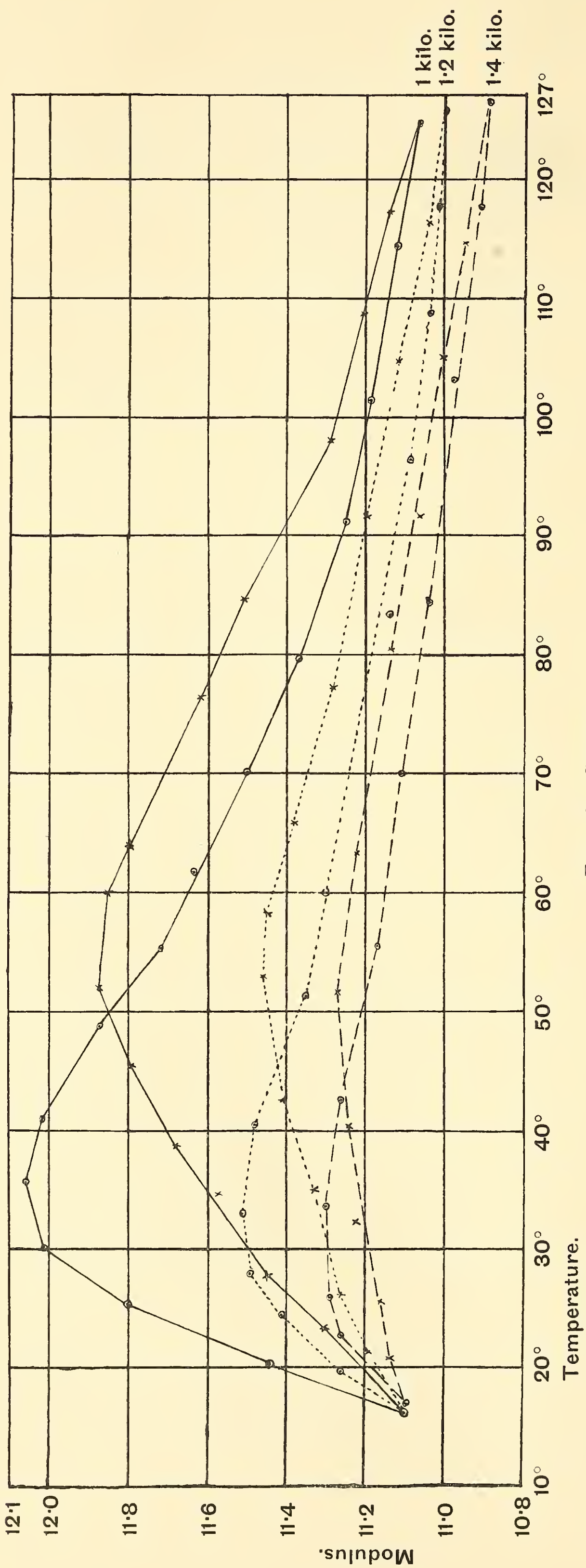


Fig. 18.—Copper.

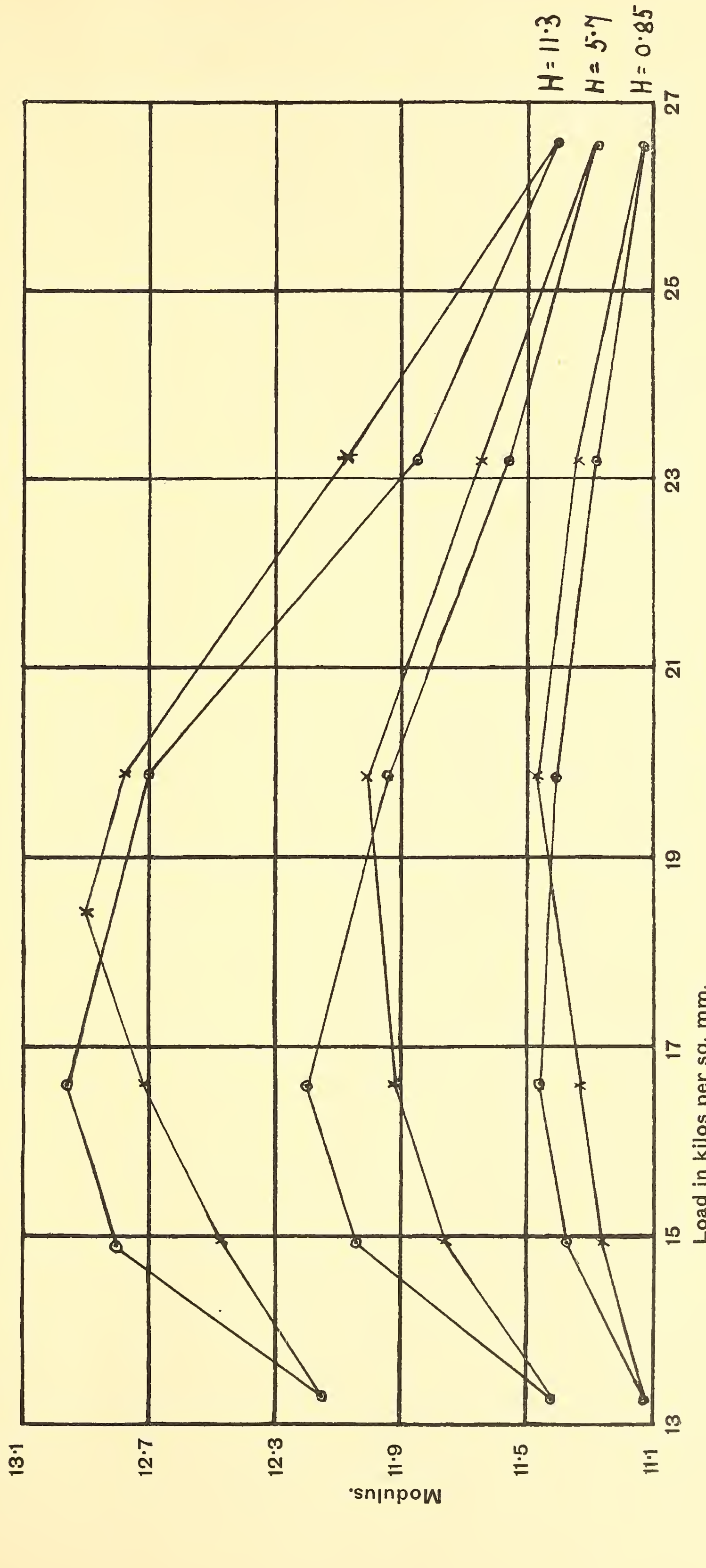


FIG. 19.—Copper.

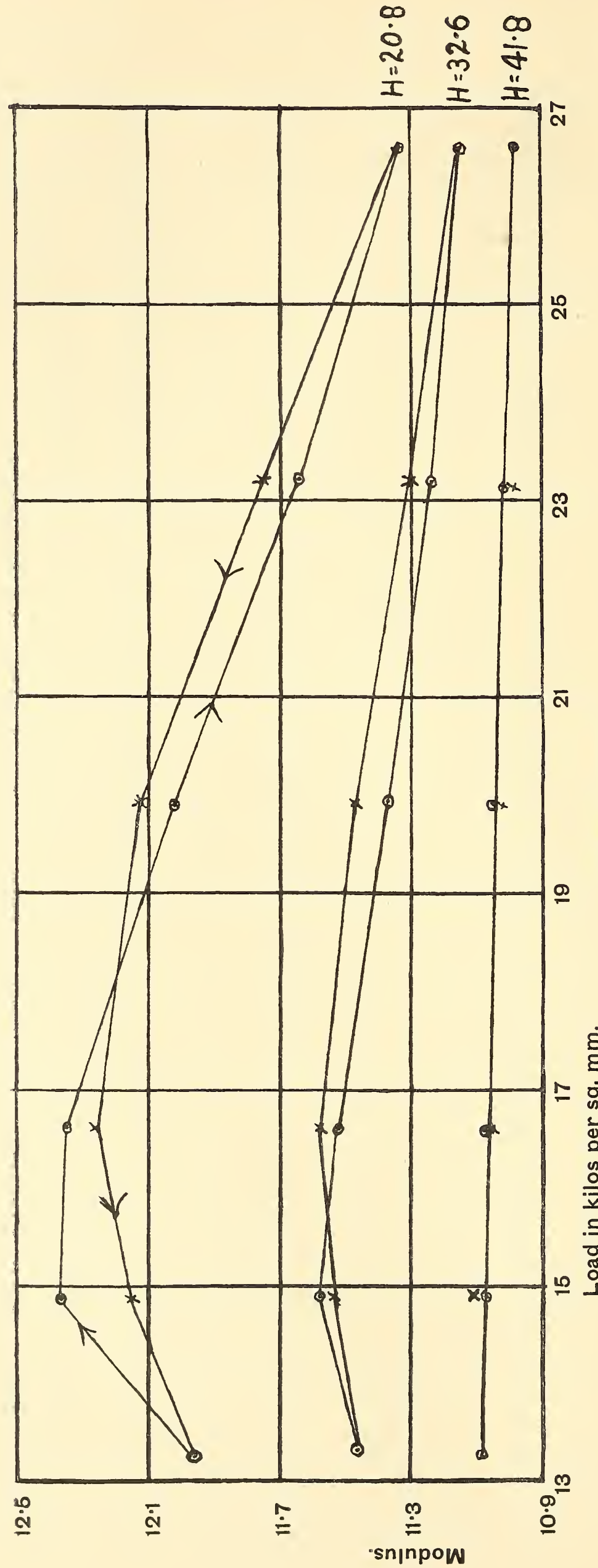


Fig. 20.—Copper.

Copper.

Length	=	98·15 cms.
Area of cross-section	=	0·0006026 sq. cm.
Elongation weight	=	300 grams.
Total load on wire	=	1 kilo.
Load per sq. mm.	=	16·6 kilos.

Table V.

No.	Temp.	Elongation for 300 grams.	No. of Observations.	M.
1	16°·9 C.	·04320 cm.	7	$11{\cdot}10 \times 10^{11}$
2	20·2	·04192	8	11·44
3	25·4	·04064	8	11·80
4	29·8	·03995	9	12·01
5	33·0	·03963	9	12·10
6	35·3	·03960	8	12·11
7	40·5	·03990	9	12·02
8	48·7	·04039	7	11·87
9	55·3	·04092	7	11·72
10	61·6	·04120	6	11·64
11	70·1	·04174	9	11·49
12	79·8	·04218	9	11·37
13	91·2	·04263	8	11·25
14	101·5	·04287	8	11·19
15	114·6	·04313	9	11·12
16	124·7	·04337	7	11·06
17	117·4	·04305	7	11·14
18	108·8	·04278	6	11·21
19	97·9	·04248	9	11·29
20	85·4	·04166	8	11·51
21	76·3	·04126	9	11·62
22	64·0	·04064	9	11·80
23	59·8	·04046	10	11·85
24	52·1	·04039	10	11·87
25	45·2	·04067	7	11·79
26	38·6	·04105	6	11·68
27	34·5	·04145	8	11·57
28	27·7	·04188	8	11·45
29	23·4	·04240	8	11·31
30	16·9	·04320	8	11·10

Total load = 1·2 kilos.
= 19·9 kilos. per sq. mm.

Table V.

No.	Temp.	Elongation for 300 grams.	No. of Observations.	M.
1	17°·2 C.	·04317 cm.	6	$11{\cdot}11 \times 10^{11}$
2	19·6	·04260	8	11·26
3	24·3	·04203	8	11·41
4	28·0	·04174	6	11·49

Table V.—*continued.*

No.	Temp.	Elongation for 300 grams.	No. of Observations.	M.
5	33°·1 C.	·04166 cm.	7	$11·51 \times 10^{11}$
6	40·5	·04177	8	11·48
7	51·7	·04225	8	11·35
8	59·9	·04244	10	11·30
9	71·6	·04287	8	11·19
10	83·4	·04305	8	11·14
11	96·3	·04325	7	11·09
12	108·7	·04344	6	11·04
13	117·2	·04352	6	11·02
14	125·8	·04359	8	11·00
15	116·5	·04344	8	11·04
16	104·6	·04313	8	11·12
17	91·7	·04281	9	11·20
18	77·2	·04248	9	11·29
19	65·9	·04214	6	11·38
20	58·3	·04188	7	11·45
21	53·0	·04184	6	11·46
22	42·8	·04203	8	11·41
23	35·1	·04233	9	11·33
24	26·4	·04260	10	11·26
25	21·5	·04287	8	11·19
26	17·3	·04317	8	11·11

Total load = 1·4 kilos.
= 23·2 kilos. per sq. mm.

Table VI.

No.	Temp.	Elongation for 300 grams.	No. of Observations.	M.
1	17°·2 C.	·04317 cm.	8	$11·11 \times 10^{11}$
2	22·8	·04260	8	11·26
3	26·1	·04248	10	11·29
4	33·4	·04244	9	11·30
5	42·6	·04260	7	11·26
6	55·7	·04294	8	11·17
7	69·9	·04317	8	11·11
8	84·2	·04344	6	11·04
9	103·0	·04366	7	10·98
10	117·5	·04395	7	10·91
11	126·3	·04402	8	10·89
12	114·8	·04378	8	10·95
13	105·1	·04356	9	11·01
14	91·7	·04337	9	11·06
15	80·6	·04305	10	11·14
16	63·3	·04270	8	11·23
17	51·7	·04255	7	11·27
18	40·2	·04269	6	11·24
19	32·5	·04275	6	11·22
20	25·8	·04298	6	11·16
21	20·9	·04305	7	11·14
22	17·2	·04317	7	11·11

Table VII.

Field.	Load in kilos. per sq. mm.	Elongation for 300 grams.	No. of Observations.	M.
0·85	13·3	·04313 cm.	10	$11{\cdot}12 \times 10^{11}$
	14·9	·04218	8	11·37
	16·6	·04185	9	11·46
	19·9	·04199	9	11·42
	23·2	·04248	8	11·29
	26·5	·04313	7	11·12
	23·2	·04229	6	11·34
	19·9	·04218	6	11·37
	16 6	·04225	8	11·35
	14·9	·04259	8	11·26
	13·3	·04313	6	11·12

Table VIII.

Field.	Load in kilos. per sq. mm.	Elongation for 300 grams.	No. of Observations.	M.
5·7	13·3	·04203 cm.	6	$11{\cdot}41 \times 10^{11}$
	14·9	·03983	6	12·04
	16·6	·03930	8	12·20
	19·9	·04012	8	11·95
	23·2	·04148	8	11·56
	26·5	·04248	7	11 29
	23·2	·04116	7	12·15
	19·9	·03993	9	11·01
	16·6	·04016	10	11·94
	14·9	·04074	8	11·77
	13·3	·04203	7	11·41

Table IX.

Field.	Load in kilos. per sq. mm.	Elongation for 300 grams.	No. of Observations.	M.
11·3	13·3	·03974 cm.	6	$12{\cdot}13 \times 10^{11}$
	14·9	·03749	8	12·79
	16·6	·03700	8	12·96
	19·9	·03776	10	12·70
	23·2	·04046	10	11·85
	26·5	·04207	8	11·40
	23·2	·03974	6	12·07
	19·9	·03752	7	12·78
	18·2	·03714	7	12·91
	16·6	·03769	9	12·72
	14·9	·03833	8	12·48
	13·3	·03947	9	12·15

Table X.

Field.	Load in kilos. per sq. mm.	Elongation for 300 grams.	No. of Observations.	M.
20·8	13·3	·04005 cm.	8	$11·97 \times 10^{11}$
	14·9	·03880	10	12·36
	16·6	·03886	8	12·34
	19·9	·03993	7	12·01
	23·2	·04119	6	11·64
	26·5	·04232	5	11·33
	23·2	·04081	8	11·75
	19·9	·03957	8	12·12
	16·6	·03909	7	12·27
	14·9	·03944	7	12·16
	13·3	·04005	8	11·97

Table XI.

Field.	Load in kilos. per sq. mm.	Elongation for 300 grams.	No. of Observations.	M.
32·6	13·3	·04181 cm.	8	$11·47 \times 10^{11}$
	14·9	·04141	8	11·58
	16·6	·04159	6	11·53
	19·9	·04214	6	11·38
	23·2	·04267	9	11·24
	26·5	·04305	9	11·14
	23·2	·04244	7	11·30
	19·9	·04181	7	11·47
	16·6	·04141	6	11·58
	14·9	·04155	4	11·54
	13·3	·04181	8	11·47

Table XII.

Field.	Load in kilos. per sq. mm.	Elongation for 300 grams.	No. of Observations.	M.
41·8	13·3	·04325 cm.	9	$11·09 \times 10^{11}$
	14·9	·04333	8	11·07
	16·6	·04329	8	11·08
	19·9	·04344	10	11·04
	23·2	·04352	8	11·02
	26·5	·04362	6	10·99
	23·2	·04356	8	11·01
	19·9	·04352	7	11·02
	16·6	·04337	7	11·06
	14·9	·04320	9	11·10
	13·3	·04320	9	11·10

Platinum.

The results for platinum are shown in fig. 21. These curves have marked differences from the others; there is a smaller rate of increase as the current rises, and, after the maximum has been attained, the rate of decrease is more rapid than the rate of increase. With diminishing current there is at first a fall, then a rapid increase to a maximum, and, finally, a diminution at about the same rate. The peculiarity platinum exhibits in the rapid rise with diminishing current is found to be greatly influenced by increase of load, so that it would appear as if the molecular arrangements, which give the wire its power to resist extension, were not formed when the load was increased.

The curves obtained from the experiments with constant field and varying load are shown in figs. 22 and 23. They show the same characteristic rise and fall, but not to the same degree as in the other method of experiment.

Platinum.

Length	=	62·12 cms.
Area of cross-section	=	0·0007548 sq. cm.
Elongation weight	=	500 grams.
Total load on wire	=	1·2 kilos.
Load per sq. mm.	=	15·9 kilos.

Table V.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	21°·9 C.	·03185 cm.	8	$12·68 \times 10^{11}$
2	29·2	·03142	8	12·85
3	40·6	·03090	8	13·07
4	51·3	·03034	9	13·31
5	65·0	·02974	8	13·59
6	82·1	·02871	8	14·08
7	97·8	·02755	8	14·67
8	104·5	·02711	7	14·91
9	113·3	·02716	6	14·88
10	120·7	·02763	8	14·63
11	131·4	·02871	8	14·06
12	145·0	·03034	10	13·31
13	157·3	·03185	9	12·68
14	148·1	·03257	9	12·41
15	137·4	·03234	8	12·50

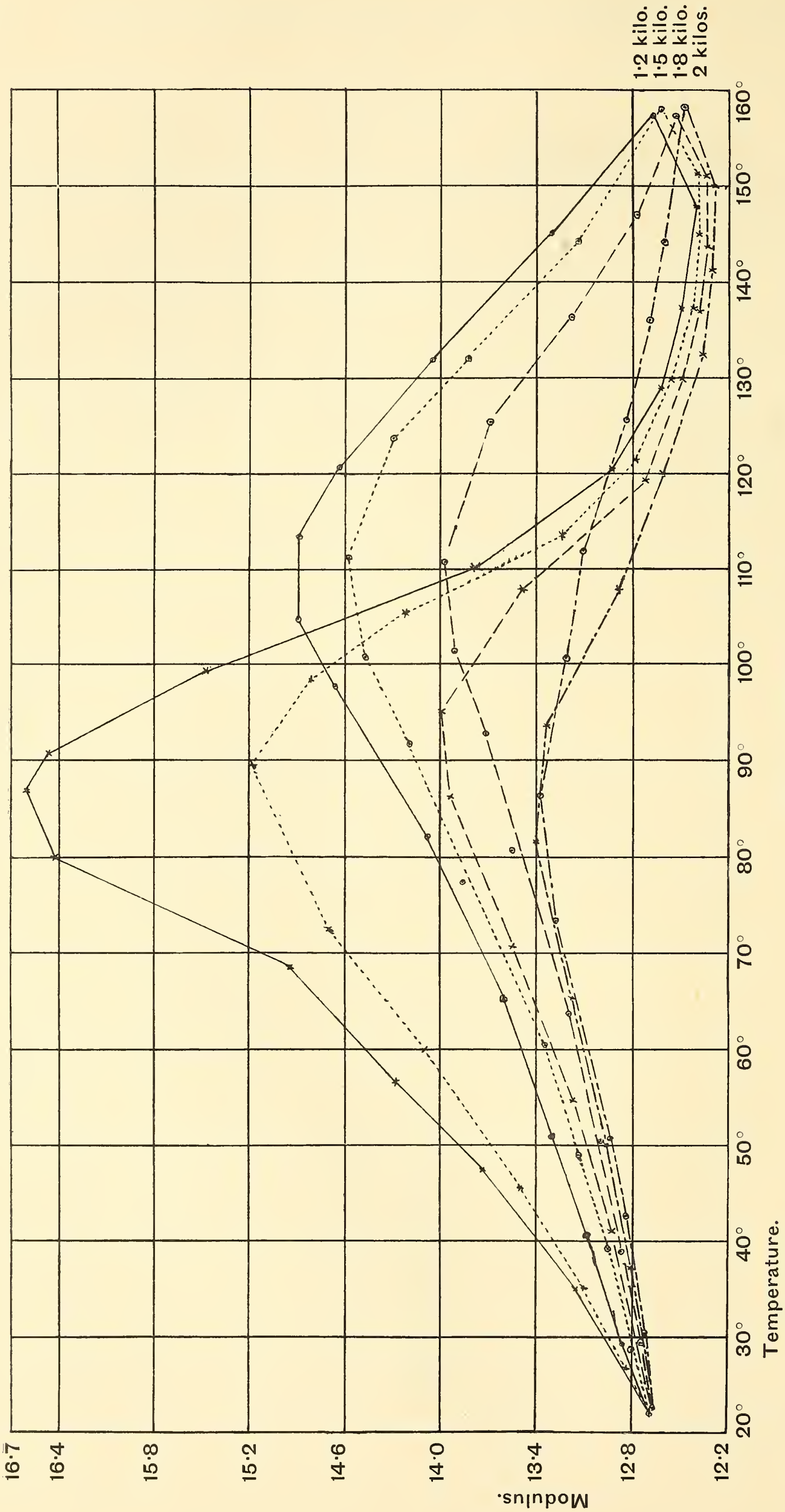


Fig. 21.—Platinum.

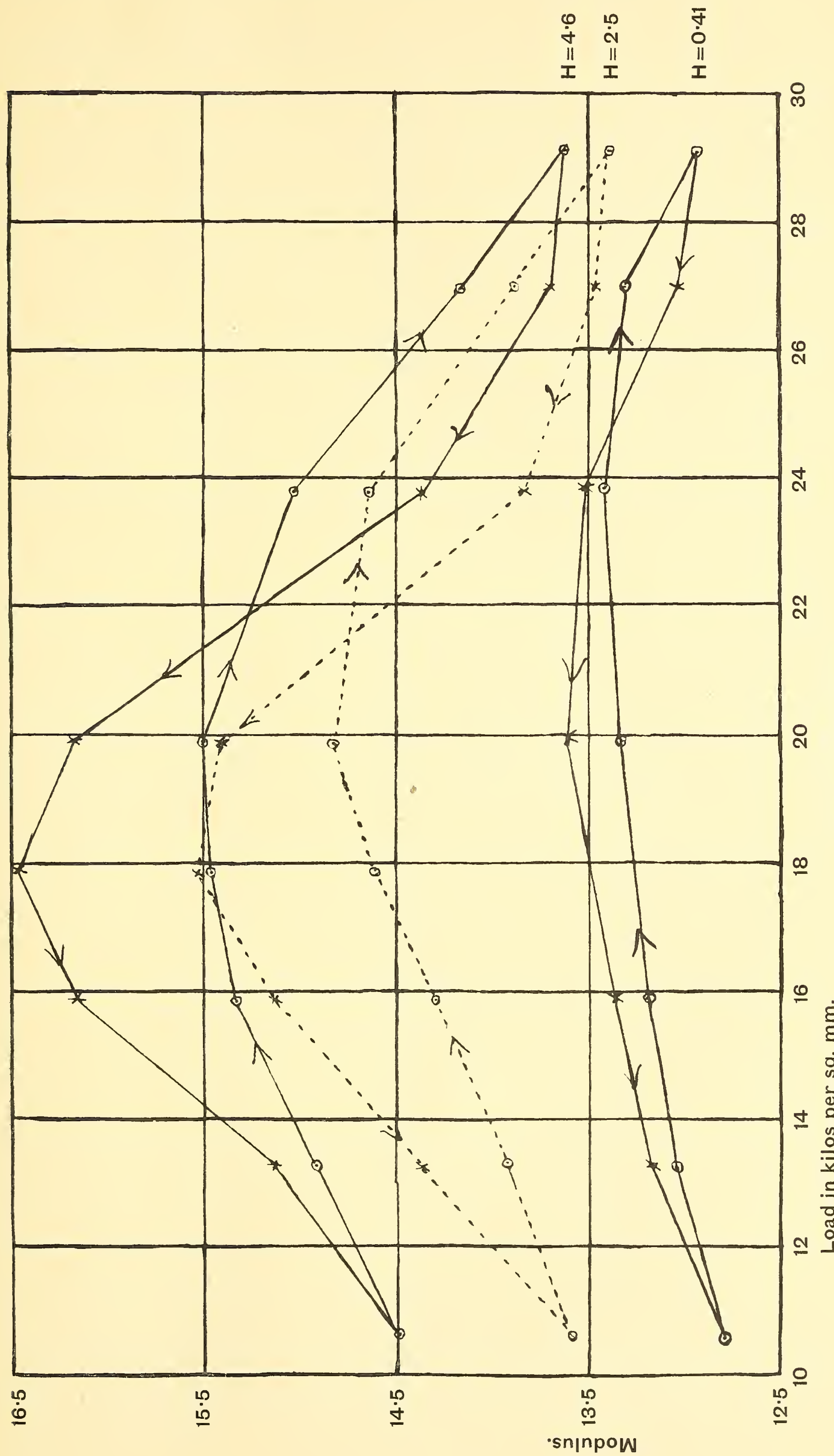


Fig. 22.—Platinum.

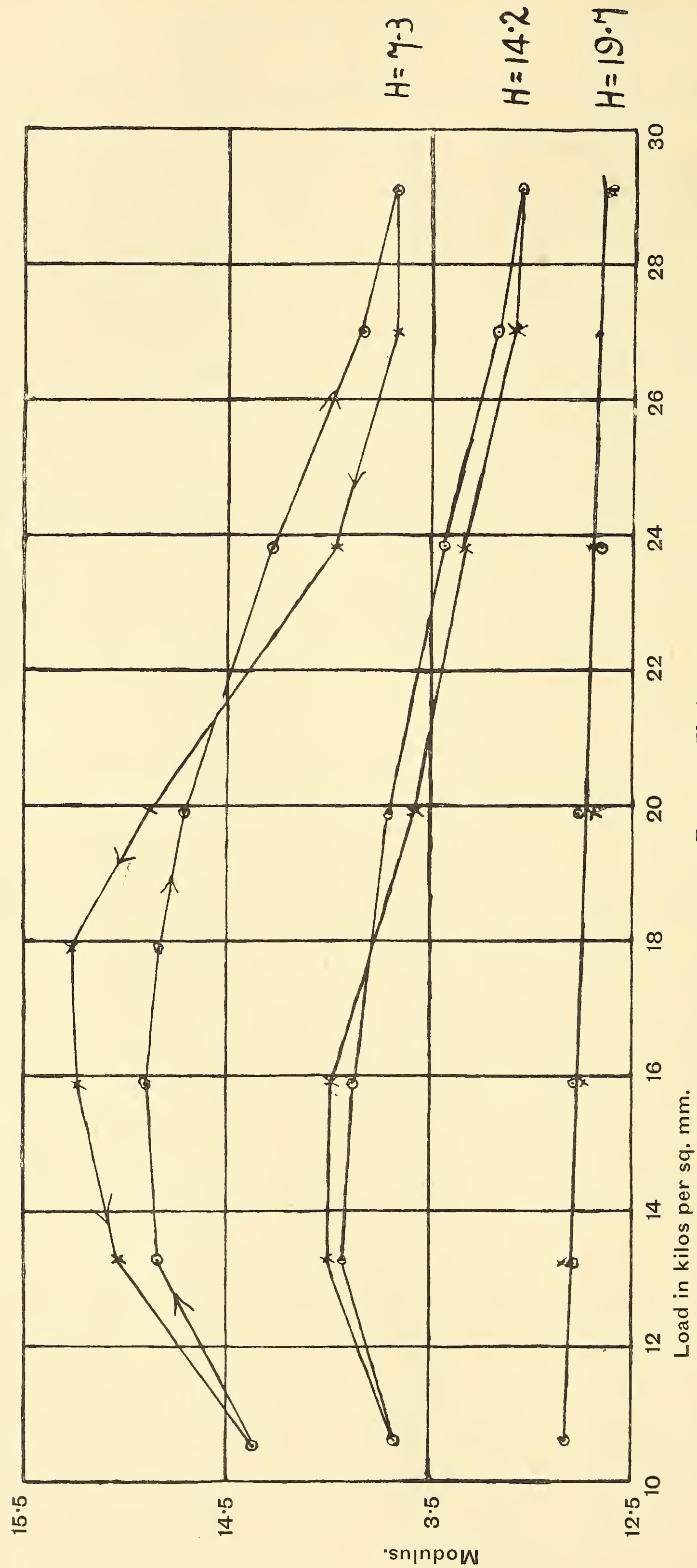


Fig. 23.—Platinum.

Table V.—*continued.*

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
16	129°·2 C.	·03203 cm.	8	$12{\cdot}62 \times 10^{11}$
17	120·7	·03126	8	12·93
18	110·5	·02929	8	13·80
19	99·6	·02611	8	15·48
20	90·8	·02454	8	16·47
21	87·0	·02431	8	16·63
22	79·9	·02463	7	16·41
23	68·7	·02703	6	14·95
24	56·6	·02829	6	14·29
25	47·4	·02942	8	13·74
26	35·1	·03072	9	13·16
27	27·5	·03128	8	12·92
28	21·9	·03185	8	12·68

Total load = 1·5 kilos.
= 19·9 kilos. per sq. mm.

Table VI.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	22°·0 C.	·03185 cm.	8	$12{\cdot}68 \times 10^{11}$
2	28·6	·03155	8	12·80
3	39·5	·03121	7	12·95
4	48·9	·03076	6	13·14
5	60·3	·03028	8	13·35
6	77·4	·02916	9	13·86
7	91·7	·02849	8	14·19
8	100·6	·02792	8	14·48
9	111·2	·02770	8	14·59
10	123·8	·02837	9	14·25
11	132·1	·02920	10	13·84
12	144·7	·03070	8	13·17
13	158·0	·03203	8	12·62
14	151·2	·03263	8	12·39
15	144·9	·03260	8	12·40
16	137·3	·03249	10	12·44
17	130·1	·03241	10	12·47
18	121·5	·03160	9	12·79
19	113·6	·03023	9	13·35
20	105·2	·02841	8	14·23
21	98·3	·02724	6	14·84
22	89·4	·02663	8	15·18
23	72·7	·02750	8	14·70
24	59·9	·02871	8	14·08
25	45·8	·02992	8	13·51
26	35·3	·03081	9	13·12
27	26·2	·03148	7	12·84
28	22·0	·03185	8	12·68

Total load = 1·8 kilos.
= 23·8 kilos. per sq. mm.

Table VII.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	22°·3 C.	·03193 cm.	8	$12·66 \times 10^{11}$
2	29·5	·03168	8	12·76
3	39·1	·03140	9	12·87
4	50·6	·03105	9	13·02
5	63·8	·03065	8	13·19
6	80·2	·03026	8	13·36
7	92·9	·02944	6	13·73
8	101·5	·02904	7	13·92
9	110·7	·02891	8	13·98
10	125·3	·02950	8	13·70
11	136·4	·03065	9	13·19
12	147·1	·03163	8	12·78
13	157·6	·03223	8	12·54
14	150·7	·03268	7	12·37
15	143·8	·03271	6	12·36
16	136·9	·03263	8	12·39
17	129·5	·03231	9	12·51
18	119·4	·03170	7	12·75
19	108·1	·02992	8	13·51
20	95·0	·02887	8	14·00
21	86·3	·02896	9	13·96
22	70·2	·02982	9	13·56
23	54·8	·03067	10	13·18
24	41·1	·03142	8	12·85
25	29·7	·03160	8	12·79
26	22·3	·03193	8	12·66

Total load = 2 kilos.
= 27 kilos. per sq. mm.

Table VIII.

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
1	22°·5 C.	·03195 cm.	8	$12·65 \times 10^{11}$
2	30·1	·03175	8	12·72
3	42·6	·03142	9	12·85
4	55·7	·03100	9	13·04
5	69·9	·03044	8	13·28
6	86·3	·03024	8	13·37
7	100·8	·03053	9	13·24
8	112·2	·03076	9	13·14
9	125·7	·03142	10	12·85
10	136·1	·03180	10	12·71
11	144·3	·03208	8	12·60
12	158·2	·03236	8	12·49
13	149·9	·03281	8	12·32
14	141·4	·03278	9	12·33

Table VIII.—*continued.*

No.	Temp.	Elongation for 500 grams.	No. of Observations.	M.
15	132°·5 C.	·03265 cm.	9	$12·38 \times 10^{11}$
16	120·0	·03200	7	12·63
17	109·1	·03130	7	12·92
18	93·6	·03033	6	13·33
19	81·7	·03016	8	13·40
20	65·3	·03067	8	13·18
21	48·9	·03116	6	12·97
22	37·0	·03115	6	12·81
23	28·5	·03178	8	12·72
24	22·6	·03195	8	12·65

Table IX.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
0·41	10·6	·03143 cm.	8	$12·85 \times 10^{11}$
	13·25	·03102	10	13·02
	15·9	·03046	8	13·26
	19·9	·03025	8	13·35
	23·8	·03007	9	13·43
	27·0	·03039	9	13·29
	29·1	·03131	7	12·90
	27·0	·03100	6	13·03
	23·8	·02989	6	13·51
	19·9	·02969	7	13·60
	15·9	·03021	9	13·37
	13·25	·03048	8	13·25
	10·6	·03143	8	12·85

Table X.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
2·5	10·6	·02974 cm.	7	$13·58 \times 10^{11}$
	13·25	·02895	6	13·95
	15·9	·02822	9	14·31
	17·9	·02762	8	14·62
	19·9	·02723	10	14·83
	23·8	·02760	8	14·63
	27·0	·02905	8	13·90
	29·1	·03011	8	13·41
	27·0	·02996	8	13·48
	23·8	·02911	6	13·87
	19·9	·02617	7	15·43
	17·9	·02605	9	15·50
	15·9	·02669	10	15·13
	13·25	·02814	8	14·35
	10·6	·02974	8	13·58

Table XI.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
4·6	10·6	·02785 cm.	9	$14{\cdot}50 \times 10^{11}$
	13·25	·02707	8	14·92
	15·9	·02638	6	15·31
	17·9	·02611	8	15·47
	19·9	·02607	7	15·49
	23·8	·02690	9	15·01
	27·0	·02848	10	14·18
	29·1	·02958	10	13·65
	27·0	·02948	8	13·70
	23·8	·02809	8	14·38
	19·9	·02495	8	16·19
	17·9	·02451	8	16·48
	15·9	·02495	8	16·19
	13·25	·02667	7	15·14
	10·6	·02785	8	14·50

Table XII.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
7·3	10·6	·02807 cm.	8	$14{\cdot}38 \times 10^{11}$
	13 25	·02725	10	14·82
	15·9	·02711	8	14·90
	17·9	·02723	7	14·83
	19·9	·02749	6	14·69
	23·8	·02828	6	14·28
	27·0	·02918	8	13·84
	29·1	·02956	8	13·66
	27·0	·02958	9	13·65
	23·8	·02885	10	13·97
	19·9	·02714	10	14·88
	17·9	·02648	9	15·25
	15·9	·02650	8	15·24
	13·25	·02690	8	15·01
	10·6	·02807	7	14·38

Table XIII.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
14·2	10·6	·02954 cm.	8	$13·67 \times 10^{11}$
	13·25	·02899	7	13 93
	15·9	02907	8	13·89
	19·9	·02952	8	13·68
	23·8	·03009	6	13·42
	27·0	·03067	6	13·17
	29·1	·03096	5	13 05
	27·0	·03083	6	13·10
	23·8	·03027	7	13·34
	19·9	·02971	7	13·59
	15·9	·02889	8	13·98
	13·25	·02885	9	14·00
	10·6	·02954	9	13·67

Table XIV.

Field.	Load in kilos. per sq. mm.	Elongation for 500 grams.	No. of Observations.	M.
19·7	10·6	·03143 cm.	8	$12·85 \times 10^{11}$
	13 25	·03151	6	12·81
	15·9	·03159	7	12·78
	19·9	·03161	8	12·77
	23·8	·03185	9	12·68
	27·0	·03197	10	12·63
	29·1	·03205	9	12·60
	27·0	·03195	8	12·64
	23·8	·03177	7	12·71
	19·9	·03177	6	12·71
	15·9	·03166	8	12·75
	13·25	·03145	8	12·84
	10·6	·03143	8	12·85

In concluding this paper, let me pass in rapid review the outstanding facts established by the different experiments. In all of them there are striking similarities. When the metals are heated by the ordinary method the graphs are uniformly straight lines, and there are no irreversible effects with rise of temperature. If, however, the rise of temperature be caused by current flow the results are more complex, for, when the heating is produced in this way, the variation of the modulus depends on the load. When this is fairly large the results are similar to those obtained with ordinary heating, that is, there are no irreversible effects. But when the load is moderate there are great and most important differences. Both the magnetic and non-magnetic metals exhibit marked hysteresis, which diminishes in amount as the load is increased, and which ultimately vanishes. Again, as the load is gradually increased, the maximum value is reached in continually diminishing fields; and when the field is kept constant and the load varied, the maximum value is reached with continually diminishing load. Thus we are led to the two important and fundamental facts brought out by the experiments as a whole—first, when rise of temperature is produced by an electric current combined with moderate longitudinal stress, that there are irreversible variations of Young's Modulus in both magnetic and non-magnetic metals; and second, that the curves exhibiting these variations display a marked similarity in both.

Finally, I desire to place on record my great indebtedness to Professor MacGregor for the loan of books, for access to the libraries of the University and the Royal Society, for references to the literature of the subject, and also for providing me with two strips of cobalt, as I had found it impossible to obtain rods of a size suitable for these experiments. I desire also to thank Professor Knott, Professor Peddie, and Mr. James Russell, Edinburgh, for many helpful suggestions and advice readily and kindly given at all times.

(*Issued separately, January* 13, 1911.)

XI.—The Inheritance of Complex Growth Forms, such as Stature, on Mendel's Theory. By John Brownlee, M.D., D.Sc.

(MS. received May 16, 1910. Read same date.)

THE inheritance of complexes is of great interest. The manner in which this arises on a Mendelian basis can be most easily seen by the consideration of a simple case. Let two races of different stature mix in equal numbers. Let for simplicity stature depend on two elements (a, a), (c, c) in one race, and on (b, b), (d, d) corresponding elements in the other. Then the permanent race obtained by free mating without any special selection of one parent or another will consist of the following proportions:—

$$1\left|\begin{matrix} a, a \\ \\ c, c \end{matrix}\right| + 2\left|\begin{matrix} a, a \\ \\ c, a \end{matrix}\right| + 1\left|\begin{matrix} a, a \\ \\ d, d \end{matrix}\right| + 2\left|\begin{matrix} a, b \\ \\ c, c \end{matrix}\right| + 4\left|\begin{matrix} a, b \\ \\ c, d \end{matrix}\right| + 2\left|\begin{matrix} a, b \\ \\ d, d \end{matrix}\right|$$

$$+1\left|\begin{matrix} b, b \\ \\ c, c \end{matrix}\right| + 2\left|\begin{matrix} b, b \\ \\ c, d \end{matrix}\right| + 1\left|\begin{matrix} b, b \\ \\ d, d \end{matrix}\right|$$

Now two factors may come into play: either dominance may not exist and the hybrid be a blend, or, on the other hand, dominance may determine the result of the mating. If dominance does not exist we may rearrange the elements according as they contain one or more element from each original race. On this hypothesis we have the following groups:—

$$\begin{array}{lccccc}
 & 1\left|\begin{matrix} a, a \\ \\ c, c \end{matrix}\right| & 2\left|\begin{matrix} a, a \\ \\ c, d \end{matrix}\right| & 4\left|\begin{matrix} a, b \\ \\ c, d \end{matrix}\right| & 2\left|\begin{matrix} b, b \\ \\ c, d \end{matrix}\right| & 1\left|\begin{matrix} b, b \\ \\ d, d \end{matrix}\right| \\
 & & 2\left|\begin{matrix} a, b \\ \\ c, c \end{matrix}\right| & 1\left|\begin{matrix} a, a \\ \\ d, d \end{matrix}\right| & 2\left|\begin{matrix} a, b \\ \\ d, d \end{matrix}\right| & \\
 & & & 1\left|\begin{matrix} b, b \\ \\ c, c \end{matrix}\right| & & \\
\text{Totals,} & 1 & 4 & 6 & 4 & 1
\end{array}$$

That is, stature tends to be graded according to the ordinary point binomial, it being granted probable that each group as above placed has essentially the same stature.

If dominance, however, exist it is a reasonable assumption that each race may supply an equal number of dominant elements (in this case one from each parent). Let a be dominant over b, and d over c, and the grouping becomes—

$$
\begin{array}{lcccc}
 & 1\left|\begin{matrix} a, a \\ c, c \end{matrix}\right|, & 2\left|\begin{matrix} a, a \\ c, d \end{matrix}\right| + 4\left|\begin{matrix} a, b \\ c, d \end{matrix}\right| + 2\left|\begin{matrix} b, b \\ c, d \end{matrix}\right|, & 1\left|\begin{matrix} b, b \\ a, d \end{matrix}\right|, \\
 & 2\left|\begin{matrix} a, b \\ c, c \end{matrix}\right|, & + \left|\begin{matrix} a, a \\ d, d \end{matrix}\right| + \left|\begin{matrix} b, b \\ c, c \end{matrix}\right| & 2\left|\begin{matrix} b, b \\ c, d \end{matrix}\right| \\
\text{Totals,} & 3 & 10 & 3
\end{array}
$$

It can be at once deduced from these results that when there are $2p$ pairs of elements in each parent determining stature, the subsequent grouping of the population is given by

$$(1+4+6+4+1)^p,$$

where there is no dominance,
and

$$(3+10+3)^p$$

where dominance applies to all the pairs and is equally divided between the pairs for each race. The mathematics of the first is easy and has been considered in my previous paper. It leads at once, when mating is random, to a correlation coefficient between parent and offspring of $r = \cdot 5$.

With regard to the second, there are several things to note. For convenience we may write the point binomial as $(1+n+1)^p$.

If we expand by the multinomial theorem, arrange the terms, and then determine the moments, we have:—

the second moment $\mu_2 = \dfrac{2p}{n+2}$,

the third moment $\mu_3 = 0$ as the multinomial is symmetrical,

the fourth moment $\mu_4 = \dfrac{2pn + 4p(3p-2)}{(n+2)^2}$.

The relationship of these moments is such that for all values of $n > 4$, $\dfrac{\mu_4}{\mu_2}$ is > 3, or the resulting curve partakes of the qualities of Type IV. rather than of the normal curve. But $n = 4$ is not the dividing point. When $n = 2$ the point binomial obviously is the square of $(1+1)$ and therefore is of the type which gives rise to the normal curve. In this case $\mu_2 = \dfrac{p}{2}$ and $\mu_4 = \dfrac{3p^2 - p}{4}$. We may therefore take it that this curve represents the dividing line, and that when n is greater than 2, a curve with moment relationships more nearly those of the symmetrical form of Type IV.

should describe the result and not the normal curve. In the same way if $n < 2$ we have a curve with the moment relationships of Type II. In the case considered $n = 3{\cdot}3$, or the point binomial is represented by $(3+10+3)^p$. To illustrate the manner in which this curve approximates to the normal we may take $p=4$, *i.e.* there are eight pairs of allomorphs originally present in each parent. The distribution of this and the corresponding normal distribution are then :—

Point binomial—81, 1080, 5724, 15240, 21286, 15240, 5724, 1080, 81.
Normal curve—115·2, 1136, 5764, 15226, 21042, 15226, 5764, 1136, 115·2.

It is to be noticed that the point binomial gives a "leptokurtic" distribution, that is, one in which the radius of curvature at the apex is less than that of the normal curve; in other words, one corresponding to that of Type IV., but not identical with it. On the numbers given by the total, namely, 66536, the difference between the two is most marked; but on one-tenth of the numbers, namely, 6654, a number much in excess of that obtained in ordinary observations, the normal curve is an exceedingly good fit, giving by the test $\chi^2 = 3{\cdot}1$ or $P = {\cdot}91$. It is interesting to observe that with only eight pairs of allomorphs a normal distribution of stature within the limits of error of observation is at once derived.

Such a condition of dominance is, however, hardly likely to occur; it is much more probable that there will be some blended inheritance as well, so that the resulting curve should be something between

$$(1+4+6+4+1)^p$$

and

$$(3+10+3)^p.$$

With free mating and equal fertility the proportion of the mixed population is $(3+10+3)^p$. The meaning of the value of n in the point binomial $(1+n+1)^p$ is therefore as follows:—If n is equal to $3{\cdot}3$ all matings are equally fertile, if n is $>3{\cdot}3$ then the matings of hybrid with hybrid are most fertile and if n is $<3{\cdot}3$, the matings of the purer races.

The manner in which asymmetry arises may be seen by going back to the simple case.

$$\text{Let} \qquad x \left|\begin{matrix} a,\, a \\ \\ c,\, c \end{matrix}\right| \quad \text{and} \quad y \left|\begin{matrix} b,\, b \\ \\ d,\, d \end{matrix}\right| \quad \text{mate at random;}$$

after one generation we have a stable population composed of

$$x^4 \left|\begin{matrix} a,\, a \\ \\ c,\, c \end{matrix}\right| \; 2x^2y \left|\begin{matrix} a,\, b \\ \\ c,\, c \end{matrix}\right| + \ldots .$$

Arranging these as before, we find for blended inheritance the following groups:—

$$x^4, \quad 4x^3y, \quad 6x^2y^2, \quad 4xy^3, \quad y^4,$$

and for mixed dominant inheritance,

$$x^4+2x^3y, \quad 2x^3y+6x^2y^2+2xy^3, \quad 2xy^3+y^4.$$

If $y=2x$ the latter reduce to

$$5, \quad 44, \quad 32,$$

so that the ultimate population is given by

$$(5+44+32)^p$$

where p is the number of double pairs originally present in either parent. This case is obviously asymmetric, but with p large quickly approaches symmetry. It is not, however, necessary to assume that two races mix. In each race diverse pairs must necessarily exist. It is possible that these may appear in proportions sensibly obeying the normal law. This, however, makes no difference in the preceding theory. Variation of individuals will but tend to smooth the curve (see note). Asymmetry also arises if the number of dominant elements from each race is unequal.

If the hypotheses above discussed are granted, two points emerge demanding consideration:—

(1) How far does this grouping accord with the observed statistics?

(2) How far does the grouping permit of the correlation coefficients found by observation?

(1) Every large series of statistics of a population give groups to which Type IV. corresponds better than the normal curve. Those of Pearson,* Powys,† etc., may be taken as types. In the case of the diagrams given by the former, it is very noticeable that at the apex the statistics give a value in excess of that shown by the normal curve fitted to the statistics, and at the limits there is a defect on the part of the statistics resembling that just found theoretically. With the curves of the Scottish insane given by Tocher ‡ the same is also found.

(2) With regard to the values of the correlation coefficients we are on firm ground. As we have seen, the hypothesis of blended inheritance leads to correlation coefficients as follows:—

Parent and offspring	·5
,, ,, grand-offspring	·25
,, ,, great-grand-offspring	·125
,, ,, great-great-grand-offspring . .	·0625

* Pearson, "On the Laws of Inheritance in Men," *Biometrika*, vol. ii. p. 357.

† Powys, "Data for the Problem of Evolution in Man," *Ibid.*, vol. i. p. 30.

‡ Tocher, "Anthropometry of the Scottish Insane," *Ibid.*, vol. v. p. 298.

These, however, are not what are found. The progression derived by Pearson from direct observation for the same relationship, is*

$$\cdot5, \quad \cdot33, \quad \cdot22, \quad \cdot15.$$

As before observed, these values arise on a Mendelian basis with dominance if a coefficient of assortive mating of $r = \cdot25$ be assumed. In the case of stature in man this is even exceeded. Professor Pearson finds that for this quality $r = \cdot28$, so that although for purposes of calculation it is assumed that $r = \cdot25$, this value is in defect. With the grouping assumed in this paper, the coefficient of correlation between parent and child without assortive mating is found to be $r = \cdot\dot{3}$, exactly as in the case considered by Professor Pearson.† In this case, however, the increase produced by the degree of assortive mating considered is not so large as in that where dominance is assumed to come exclusively from one side. The coefficient of correlation between parent and offspring is raised from $r = \cdot\dot{3}$ to $r = \cdot46$, somewhat in defect of the value $r = \cdot5$ found from observation. It is, however, in the highest degree improbable that pure dominance regulates inheritance. Blending, as already remarked, must also occur, and as a like coefficient of assortive mating must be held to apply in this case, the increase in the value of the coefficient to the neighbourhood of $r = \cdot5$ is almost a necessity, since for pure blended inheritance the correlation of parent and offspring is in the neighbourhood of $r = \cdot6$ when the coefficient of assortive mating is given by $r = \cdot25$.

To settle this question, inquiries into the constitution of races will require to be made, but I think that I have shown that there is nothing necessarily antagonistic between the evidence advanced by the biometricians and the Mendelian theory.

Conclusions.

(1) If the inheritance of stature depends upon a Mendelian mechanism, then the distribution of the population as regards height will be that which is actually found, namely, a distribution closely represented by the normal curve.

(2) There is nothing in the values of the coefficients of inheritance found by Sir Francis Galton and Professor Pearson which cannot be explained on the basis of Mendelian inheritance.

Note I.

Let $\phi(x)$, assumed symmetrical, represent the crude distribution of x as indicated above, and let each portion of the population vary according to

* *Biometrika*, vol. ii. p. 373. † *Trans. Roy. Soc.*, 1903, p. 53.

the normal curve $y = ae^{\frac{-x^2}{2\sigma^2}}$. Let in addition the second and fourth moments of ϕx be denoted by μ_2 and μ_4. Then for any point x the final distribution of the population will be given by

$$y = a\int_{-\infty}^{\infty}\phi(x')e^{\frac{-(x-x')}{2\sigma^2}}dx'.$$

Let the moments of this be denoted by ν_2 and ν_4: then

$$\nu_2\int_{-\infty}^{\infty}ydx = a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x^2\phi(x')e^{\frac{-(x-x')^2}{2\sigma^2}}dxdx'$$

$$= a\sqrt{2\pi\sigma^3}\int_{-\infty}^{\infty}\phi x'dx' + a\sqrt{2\pi\sigma}\int_{-\infty}^{\infty}x'^2\phi(x')dx',$$

or

$$\nu_2 = \sigma^2 + \mu_2\,;$$

likewise

$$\nu_4 = 3\sigma^4 + 6\sigma^2\mu_2 + \mu_4,$$

so that

$$\frac{\nu_4}{3\nu_2{}^2} = \frac{3\sigma^4 + 6\sigma^2\mu_2 + \mu_4}{3\sigma^4 + 6\sigma^2\mu_2 + 3\mu_2{}^2},$$

and is $>$ or <1 according as $\mu_4 >$ or $<3\mu_2$.

This curve then tends to approach the normal curve of error in its moment relationships, but it is to be remembered that σ^2 must be small compared with μ_2, and also that if dominance exist perfect normality is impossible.

Note II.

In case of misunderstanding, it may be well to state that I have used the word *blend* in the sense that the quality resulting from the combination of two different elements lies between that of the separate elements, and not in the sense that either of the elements is modified by the combination, as is sometimes done.

(*Issued separately January* 13, 1911.)

XII.—The Temperature Seiche. Part I. Temperature Observations in the Madüsee, Pomerania. Part II. Hydrodynamical Theory of Temperature Oscillations in Lakes. Part III. Calculation of the Period of the Temperature Seiche in the Madüsee. By E. M. Wedderburn, W.S. Part IV. Experimental Verification of the Hydrodynamical Theory of Temperature Oscillations. By E. M. Wedderburn, W.S., and A. M. Williams, M.A., B.Sc.

(MS. received December 23, 1910. Read December 5, 1910.)

[*Abstract.*]

THIS paper is published at length in the *Transactions* of the Society, vol. xlvii., part iv., p. 619, and describes observations in the Madüsee, Pomerania, made by Professor Halbfass and Mr Wedderburn, with the view of testing the latter's theory of temperature seiches in lakes, arrived at from a consideration of the observations of the Scottish Lake Survey.

PART I.

During the three weeks from 25th July to 14th August 1910, about 3000 temperature observations were made in the Madüsee, and oscillations with a period of from 24·4 hours to 25·3 hours were observed during most of the time. Observations were made at each end of the lake and at the centre. The phase of the oscillations was opposite at the two ends of the lake, and the amplitude was very small at the centre of the lake, indicating the proximity of a node.

During the period of observation the temperature discontinuity or *Sprungschicht* was very abrupt.

PART II.

An approximation to the actual conditions in a lake is got by supposing that to a depth h' the density of the water is uniformly ρ', and thereafter uniformly ρ. On this assumption, and taking the origin in the surface of separation (*i.e.* at depth h'), it is found that the temperature oscillations in lakes depend on the equation:

$$\frac{d^2\mathrm{P}}{dv^2} + \frac{n^2}{g\Sigma(v)(\rho - \rho')}\mathrm{P} = 0,$$

where

$$\Sigma(v) = \frac{b(x)}{\dfrac{\rho}{A(x)} + \dfrac{\rho'}{A'(x)}},$$

$b(x)$ being the breadth of the surface of separation at a distance x from the origin, $A(x)$ the area of a cross-section of the portion of the lake below the surface of separation, and $A'(x)$ of a cross-section above the surface of separation. The other symbols have the meaning assigned by Professor Chrystal in his memoir on the Hydrodynamical Theory of Seiches, *Trans. Roy. Soc. Edin.*, vol. xli. (iii.), p. 599, and the above equation is of the same form as Professor Chrystal's equation for the ordinary seiche.

Part III.

A temperature normal curve was drawn for the Madüsee and approximated to by a single parabola. The period of the temperature seiche calculated in this way was 24·8 hours. A second calculation, deduced from the known period of the ordinary seiche by considering the analogy between the ordinary seiche and the temperature seiche, gave 24 hours.

Part IV.

Laboratory experiments were carried out by Mr Wedderburn and Mr Williams to test the foregoing theory, and gave conformable results. Tanks of various shapes were used to imitate different shapes of lake basins, and difference of density was obtained by using a layer of paraffin resting on a layer of water.

(*Issued separately February* 23, 1911.)

XIII.—A Method for Determining the Molecular Weights of Dissolved Substances by Measurement of Lowering of Vapour Pressure. By Alan W. C. Menzies.

(MS. received November 26, 1910. Read January 9, 1911.)

[*Abstract.*]

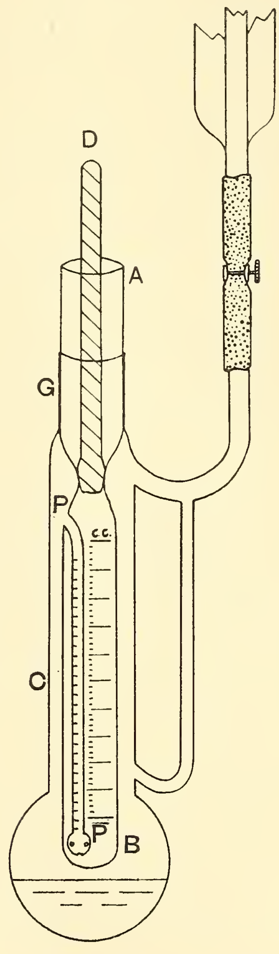
Although various methods have from time to time been described for determining the molecular weights of dissolved substances by measurement, not of boiling-point elevation, but of the reduction of vapour pressure of a solvent due to the presence of a dissolved substance, none of these methods has come into general use. The method here proposed, operating on this principle, offers the possibility of simply determining the molecular weights of non-volatile solutes in any of the ordinary solvents with an accuracy at least equal to that of the ebullioscopic methods, and in an apparatus at least as easily manageable.

As will be seen from the figure, the apparatus * consists of an inner test-tube AB, furnished with a pressure gauge tube PP and a glass stopper D, surrounded by a jacket C, in which the pure solvent is kept boiling. The jacket is connected to a small reflux condenser by a short length of wide rubber tubing furnished with a screw-clip. When the glass stopper is removed and the clip closed, the vapour of the liquid boiling in the jacket is obliged to escape by blowing through the gauge tube, which is open at both ends, into the test-tube. The latter fits into its jacket by a ground joint G, and is graduated in cubic centimetres. The gauge tube is graduated in millimetres of length. The purpose of the jacket is to maintain the test-tube and its contents at a constant temperature—the boiling-point of the solvent; while the purpose of the test-tube is to contain the solution whose vapour pressure is being measured. The pressure measurement is made in terms of the difference of level of the solution in the gauge tube and the test-tube—that is, in terms of millimetres of a liquid of low density compared to mercury. Differences of pressure that are small in terms of millimetres of mercury may therefore be measured with accuracy, and this permits of the use of very dilute solutions.

In making a determination, the bulb of the jacket is charged two-thirds full with the pure solvent, which is boiled ten minutes to expel dissolved

* This is made by Greiner & Friedrichs, Stützerbach, Thuringia.

gases. A portion of the boiled liquid is then poured into the test-tube so as to fill it two-thirds full. By partially closing the screw-clip, vapour from the boiling solvent is caused to "blow through" the liquid in the test-tube. When all air has been expelled, the stopper is inserted and the screw-clip simultaneously opened. After temperature equilibrium has become estab-



lished, the vapour pressure of the solvent within the test-tube is compared with the barometric pressure by observing the difference of level between the liquid in the test-tube and the gauge tube. If the solvent is pure, these two pressures are identical; in any case, the blowing-through process is repeated till constant results are obtained.

A known weight of solute is now added, the air expelled, the stopper inserted, and the reading of difference of level made, just as before, after temperature equilibrium is attained. To obtain the lowering of vapour pressure, the reading is corrected by the value of the zero reading already found. The stopper is removed, and the volume of the solution read off.

From the observed quantities,—weight of solute W, volume of solution V, and lowering of vapour pressure L—the molecular weight may be calculated from the formula:—

$$M = 1000 \frac{W \cdot K_{760} \cdot B}{L \cdot V \cdot 760}$$

where B is the barometric height, and K_{760} is a constant for the solvent, simply deducible from theoretical considerations.

The values of K_{760} for some of the common solvents are as follows:—

Solvent.	K_{760}.	Solvent.	K_{760}.
Benzene	1214	Ether	1577
Alcohol	871·5	Carbon bisulphide	526·6
Water	202·5	Ethyl acetate	1320
Chloroform	620·4	Ethylene dibromide	514·6
Acetone	1061		

The results tabulated below will serve to indicate the order of accuracy that may be attained by a beginner. For these determinations I am indebted to Mr Severin Gertken, who was without previous experience of molecular weight determination by any method.

Solvent.	Solute.	M. Wt. normal.	M. Wt. found.	Beckmann found.
Benzene	Benzil	210	203	216–256
	Ethyl Benzoate	150	164·5	163–172
Carbon bisulphide	Benzil	210	206·5	217
	Benzophenone	182	195·9–196·3	182–187
	Naphthalene	128	127·6	131–141
Water	Boric acid	62	61·4–62·1	64·6–66·9
	Potassium chloride	74·6	43·2–44·2	

University of Chicago,
May 1910.

(*Issued separately February* 23, 1911.)

XIV.—The Mathematical Theory of Random Migration and Epidemic Distribution. By John Brownlee, M.D., D.Sc.

(MS. received May 16, 1910. Read same date.)

The general theory of epidemic disease I have already considered in a communication to this Society.* In that communication I showed that the course of epidemics of all forms of infectious disease obeyed certain very definite laws. In the same paper it was also shown that the distribution of epidemic disease in a uniformly populated area obeyed a law essentially similar. Certain reasons were given why the normal curve of error $y = y_0 e^{-\frac{x^2}{2\sigma^2}}$ might be expected to give an approximate solution in both the cases considered, but why the distribution actually found (type iv.) should be the common form was not at all clear. I think, however, I have now arrived at the solution.

The distribution of an epidemic in space is evidently a problem in chance. If there is an infective group in the middle of a uniformly disposed population, then the distance from which friends come to visit a sick person or the distance a sick person travels while developing the disease determines the subsequent distribution of cases—a distribution, therefore, obeying some law on the average. This problem has since been attacked and solved by Professor Pearson under the title of "The Problem of Random Migration." The case which he considers refers specially to the prevention of malaria, which is now known to be spread through the agency of mosquitoes. The mathematical theory, which is very complex, leads to the determination that the normal surface of error gives a very close representation of this distribution. For epidemiological purposes the result is quite sufficiently close. To make the matter perfectly clear, the conditions of the problem solved are given in Professor Pearson's own words:—

"(1) Breeding grounds and food supply are supposed to have an average uniform distribution over the district under consideration. There is to be no special following of river beds or forest tracks.

"(2) The species scattering from a centre is supposed to distribute itself uniformly in all directions. The average distance through which an individual of the species moves from habitat to habitat

* *Proc. Roy. Soc. Edin.*, June 1906.

will be spoken of as a 'flight,' and there may be n such 'flights' from locus of origin to breeding ground, or again from breeding ground to breeding ground, if the species reproduces more than once. A flight is to be distinguished from a 'flitter,' a mere to-and-fro motion associated with the quest for food or mate in the neighbourhood of the habitat.

"(3) Now, taking a centre, reduced in the idealised system to a point, what would be the distribution after random flights of N individuals departing from this centre? This is the first problem. I will call it the 'Fundamental Problem of Random Migration.'

"(4) Supposing the first problem solved, we have now to distribute such points over an area bounded by any contour, and mark the distribution on both sides of the contour after any number of breeding seasons. The shape of the contour and the number of seasons dealt with will provide a series of problems which may be spoken of as 'Secondary Problems of Migration.'"

The proof of the theory given by Professor Pearson contains also implicitly the proof that if the normal surface of error describes the distribution at any moment, it will at all subsequent times. This can be seen, however, quite easily otherwise. Thus if $y = y_0 e^{-\frac{x^2+y^2}{2\sigma^2}}$ be the distribution of disease where y_0 is the number of cases per unit area at origin and $\sqrt{x^2+y^2}$ is the distance of any point from the origin, then the amount of disease at any point x', y' is $y_0 e^{-\frac{x'^2+y'^2}{2\sigma^2}}$. This element gives rise to a new normal surface $y_0 e^{-\frac{(x-x')^2+(y-y')^2}{2\sigma^2}-\frac{x'^2+y'^2}{2\sigma^2}}$ due to the infection at $x'y'$. If these several surfaces be integrated from $-\infty$ to $+\infty$ with respect to x' and y' successively, we get the new distribution

$$y = y_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{-(x-x')^2+(y-y')^2}{2\sigma^2}-\frac{x'^2+y'^2}{2\sigma^2}} dx'dy'$$

$$= y_0 \pi\sigma^2 e^{-\frac{x^2+y^2}{4\sigma^2}},$$

so that the standard deviation σ is multiplied by $\sqrt{2}$—that is, that the slope of the surface is flattened. In other words, the longer the disease is present in a town the more uniformly, other things being equal, it will tend to distribute itself.

At the present point it will perhaps be well to recapitulate the method by which it was considered likely that the normal curve of error should represent the course of an epidemic in time. If there be an amount of infectious disease a: if such element infect pa persons, and if q be the rate of loss of infectivity per unit time, as has already been shown to be probable, the number of each group of infected persons (supposing the supply of susceptible persons large) is $a : ap : ap \times pq : ap^2q \times pq^2 : ap^3q^3 \times pq^3$, etc., or, in general, if x denote the unit time, $y = ap^{x-1} q^{\frac{(x-1)(x-2)}{2}}$, which, as q is necessarily less than unity, is the normal curve of error. If, instead of finite, infinitesimal differences are employed, the result is expressed by the formula

$$y = ap^x q^{\frac{1}{2}x^2}.$$

That the normal curve is to be taken as that from which variation is to be expected both when the space and time distributions of epidemics are examined then seems clear, and it remains to discover in what manner the natural process differs from that so far developed theoretically.

It is, in the first place, to be noted that if the distribution is from a central area instead of a point, a disturbing influence on the shape of the curve comes into play. This can be allowed for at once. For purposes of convenience a two-dimensional solution is given, such a solution being easily extended to three dimensions in any particular case. If we consider the modification of the normal curve produced when the mosquitoes start from an area and not from a point, the moments of the resulting distribution will not be those of the normal curve, that is, not

$$\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2\sigma^2}} dx,$$

but

$$\int_{-a}^{a} da \int_{-\infty}^{\infty} (x+a)^n e^{-\frac{x^2}{2\sigma^2}} dx;$$

or, for the even moments, since the odd moments are zero,

$$\mu'_2 = \mu_2 + \frac{a^2}{3}$$

$$\mu'_4 = \mu_4 + 2a^2\mu_2 + \frac{a^4}{5},$$

so that

$$\frac{\mu_4}{\mu_2^2} = \frac{\mu_4 + 2a^2\mu_2 + \frac{a^4}{5}}{\left(\mu_2 + \frac{a^2}{3}\right)^2}$$

$$= \frac{3\mu_2^2 + 2a^2\mu_2 + \frac{a^4}{5}}{\mu_2^2 + \frac{2a^2\mu_2}{3} + \frac{a^4}{9}}$$

since for the normal curve $\mu_4 = 3\mu_2^2$.

The latter is always less than 3, so that if the centre of distribution is extended, that is, if the original mosquitoes are uniformly dispersed over a space bounded by two parallel lines, the subsequent distribution will resemble a curve of type ii. rather than the normal curve. But, as has been said, type iv. almost uniformly occurs. Some other modification is therefore necessary. This may be found in the fact that σ is not constant. In the simplified problem of Professor Pearson $2\sigma^2 = nl^2$ where n is the number of the flights and l the length of the mean flight. The curve is thus given by $y = \frac{\mathrm{N}}{\pi n l^2} e^{-\frac{x^2}{nl^2}}$. If l, however, vary, we do not get the normal curve at all, but a derivative. The frequency of any value of l or σ may be taken as given by $f(\sigma)$ when the limits of σ are a and β. The surface of distribution derived from this $y = \frac{\mathrm{N}}{\pi}\int_a^\beta \frac{f(\sigma)}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}} d\sigma$. Further, if the distribution take place from a definite area as above described, the final form of the distribution of the organism becomes

$$y = y_0 \int_{x-c}^{x+c} \phi x dx \int_a^\beta f(\sigma) e^{-\frac{x^2}{2\sigma^2}} \frac{d\sigma}{\sigma^2} \quad . \quad . \quad . \quad . \quad (1)$$

when ϕx denotes the mode of distribution in the area. This, if the forms of $\phi(x)$ and $(f\sigma)$ are known, I take to be the fundamental epidemic or random migration equation.

To return for a moment to the form found for the time distribution of an epidemic

$$y = ap^x q^{\frac{1}{2}x^2}.$$

This may be put in form

$$y = ae^{x \log p + \frac{1}{2}x^2 \log q}$$

$$= ae^{\frac{1}{2}\log q\left(x + \frac{\log p}{\log q}\right)^2 - \frac{1}{2}\frac{(\log p)^2}{\log q}}$$

$$= ae^{-\frac{(x+c)^2}{\sigma^2} + \frac{c}{\sigma^2}}$$

$$= ae^{-\frac{\xi^2}{\sigma^2} + \frac{c^2}{\sigma^2}} \quad \text{changing the origin of } x \quad . \quad . \quad (2)$$

$$\text{if } \log q = -\frac{2}{\sigma^2} \quad \text{and} \quad \frac{\log p}{\log q} = c.$$

From the symmetry of the epidemic c must in general be a constant, so that we have the relation

$$c \log q = \log p$$

or

$$p = q^c$$

as the relationship between the infectivity and the rate of loss of infectivity. As q is less than unity c will be negative in sign, so that we have as the general rule that as p increases q decreases, or the greater the infectivity the more rapidly it is lost.

In the case of the epidemic the frequency of each value of σ is quite unknown. As $\log q = -\frac{2}{\sigma^2}$ and as q must from the initial assumption lie between 0 and 1, the limits of σ are evidently $-\infty$ and 0. The situation of the commonest value of σ is therefore between these limits, and must in general lie nearer zero than infinity. In choosing an arbitrary form for σ so as to get an approximation, it must be of such a form that (2) will be integrable and expressible in a form suitable to calculation.

The equation obtained by integrating (2) is $y = \int_0^\infty e^{-\frac{\xi^2}{\sigma^2}+\frac{c^2}{\sigma^2}} f\sigma d\sigma$. It is obvious that if this be finite when $\sigma = 0$ that the term $e^{+\frac{c^2}{\sigma^2}}$ must disappear, so that part of $f(\sigma)$ must be $e^{-\frac{c^2}{\sigma^2}}$: the other part may be taken as $e^{-\frac{\sigma^2}{k^2}}$. The function $e^{-\frac{c^2}{\sigma^2}-\frac{\sigma^2}{k^2}}$ has a maximum when $\sigma = \sqrt{ck}$. The constant c is quite unknown, nor does it seem ascertainable from the method of analysis, but it disappears, and as k is at our disposal it is evident that the maximum value of σ can be placed where the statistics to be examined demand. We thus have as a possible form of the epidemic equation

$$y = \int_0^\infty e^{-\frac{\xi^2}{\sigma^2}-\frac{\sigma^2}{k^2}} dx.$$

To increase the variety of the distribution we might take $\sigma^n e^{-\frac{\xi^2}{\sigma^2}-\frac{\sigma^2}{k^2}}$ as representing the variation of σ: in this case the final integral becomes

$$y = \int_0^\infty \sigma^n e^{-\frac{\xi^2}{\sigma^2}-\frac{\sigma^2}{k^2}} d\sigma.$$

The epidemic due to an organism instantaneously becoming infective and thereafter losing its infectivity at a rate corresponding to the geometrical progression should at least approximately fit the above curve if the distribution of σ has been at all closely guessed. When we turn to Professor Pearson's approximate form of solution of the random migration problem we find that it also has a term with σ in the denominator. The distribution is given by

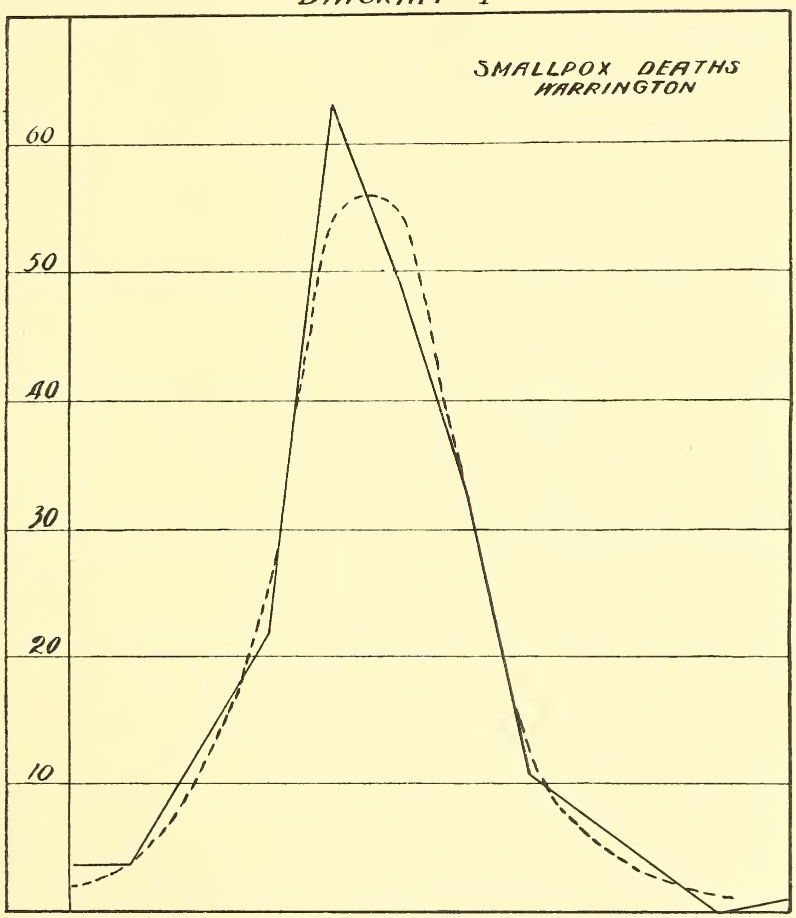
$$y = \frac{\mathrm{N}}{2\pi\sigma^2} e^{-\frac{x^2}{2\sigma^2}}$$

where N is the total number of mosquitoes starting from the point of origin and $\sigma = \frac{1}{2}nl^2$, n being equal to the number of flights and l to the length of the average flight. Now, if σ vary from 0 to ∞ for a first approximation, its frequency might be represented by type iii. or, say, $y=\sigma^n e^{-\frac{\sigma}{k}}$, in which case the migration form would be

$$\frac{\mathrm{AN}}{2\pi}\int_0^\infty \sigma^{n\cdot 2} e^{-\frac{r^2}{\sigma^2}-\frac{\sigma}{k}}.$$

This form is very intractable as a working integral, and as $\sigma^{2n}e^{-\frac{\sigma^2}{k^2}}$ gives

DIAGRAM I

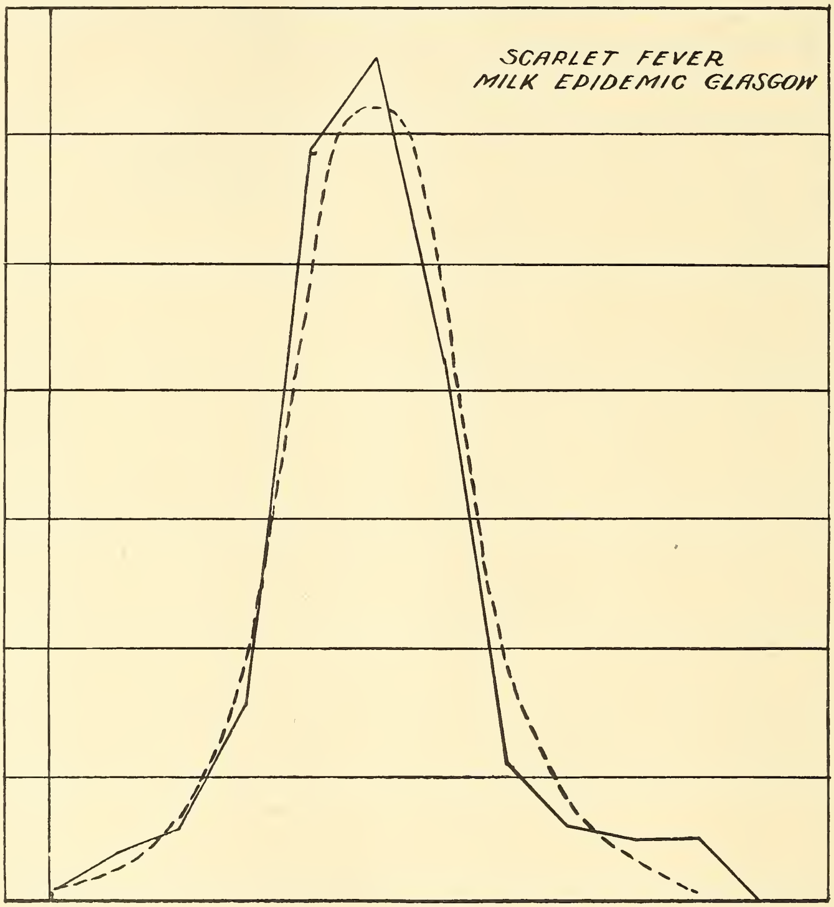


a somewhat similar series of curves, the latter may be substituted with much simplification of the calculation, so that the distribution of a species might be represented by

$$c\int_0^\infty \sigma^{2n+2} e^{-\frac{x^2}{\sigma^2}-\frac{\sigma^2}{k^2}} d\sigma,$$

a form easily reducible for any value of n. This is the same surface as was deduced as a possible one for the epidemic time wave. What has preceded is to a certain extent hypothetical in so far as the assumption of a form for the frequency of σ is connected; otherwise it is directly deduced in the case of the epidemic from the consideration of a large number of

DIAGRAM II.



typical epidemics, and in the second place by Professor Pearson by a rigid mathematical result from the assumptions already referred to.

The next assumption is the rate at which infective organisms are evolved. This may be taken as represented by $\phi(x)$, so that at any point x the ordinate will be represented by the integral of the curve taken between $x-c$ and $x+c$. We then, finally, obtain as the form of the epidemic

$$y = y_0 \int_{x-c}^{x+c} \phi(x) \int_0^\infty \sigma^{2n} e^{-\frac{x^2}{\sigma^2} - \frac{\sigma^2}{k^2}}.$$

It will be shown later that good agreement of theory with fact is obtained if $n=0$ and $\phi(x)=c$ (a constant), so that, in the first instance, the form

$$y=a\int_{x-c}^{x+c}dx\int_0^\infty e^{-\frac{x^2}{\sigma^2}-\frac{\sigma^2}{k^2}}d\sigma$$

will be considered.

As is well known

$$\int_0^\infty e^{-\frac{x^2}{\sigma^2}-\frac{\sigma^2}{k^2}}d\sigma=\sqrt{\pi}ke^{\pm\frac{2x}{k}},$$

the positive sign referring to the negative branch of x and the negative sign to the positive branch.

Hence the equation of the curve is

$$y=ak^2-ake^{-\frac{2c}{k}}\cosh\frac{2x}{k}\text{ from }x=0\text{ to }x=c,$$

$$y=ake^{-\frac{2x}{k}}\sinh\frac{2c}{k}\text{ from }x=c\text{ to }x=\infty.$$

The curve is symmetrical. Examples of this are given in Diagrams I. and II. When these are compared with Diagrams VII. and XXIII. respectively of my former paper,* it is seen how much better the fit now obtained is.

Mathematical Formulæ of Curves which represent possible Epidemic Forms.

Equations and formulæ which might describe more or less approximately epidemic or migration forms will now be considered in detail. The following symbols are used throughout:—

(1) Let $y=f(x)$ be the curve of distribution

$$\mathrm{A}=\int_{-\infty}^{\infty}ydx,\qquad\text{area of the curve}$$

$$\mu_2=\frac{1}{\mathrm{A}}\int_{-\infty}^{\infty}x^2ydx,\qquad\text{where the axis of } y \text{ is taken through the centre of gravity}$$

$$\mu_3=\frac{1}{\mathrm{A}}\int_{-\infty}^{\infty}x^3ydx,\text{ etc.};$$

and

$$\nu_1=\frac{1}{\mathrm{A}}\int_0^\infty xydx.$$

When the origin is not in a line through the centre of gravity the moments are denoted by

$$\mu'_1,\ \mu'_2,\ \mu'_3,\text{ etc.}$$

* *Proc. Roy. Soc. Edin.*, vol. xxvi. pp. 491 and 507.

To complete the symbols

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3},\ \beta_2 = \frac{\mu_4}{\mu_2^2}$$

The fundamental forms which have been chosen for investigation are

$$y = a\int_0^\infty \sigma^{2n} e^{-\frac{x^2}{\sigma^2} - \frac{\sigma^2}{k^2}} d\sigma \quad \text{and} \quad y = a\int_0^\infty \sigma^n e^{-\frac{x^2}{\sigma^2} - \sigma\gamma}.$$

These are symmetrical round the origin. On the hypothesis before mentioned they are integrated for each value of x from $x-c$ to $x+c$, multiplying each term by a suitable function $\phi(x)$, so that for the working equations we have,

$$y = a\int_{x-c}^{x+c} \phi x dx \int_0^\infty \sigma^{2n} e^{-\frac{x^2}{\sigma^2} - \frac{\sigma^2}{k^2}} d\sigma \quad . \quad . \quad . \quad . \quad (\text{M})$$

$$y = a\int_{x-c}^{x+c} \phi x dx \int_0^\infty \sigma^n e^{-\frac{x^2}{\sigma^2} - y\sigma} \quad . \quad . \quad . \quad . \quad (\text{N})$$

(2) The first of these forms is much easier to evaluate, and in addition gives the closest representation of the facts. Considering it in the first instance and assuming that $\phi(x) = a$ (a constant), we have the curve of distribution in time or space given by

$$y = a\int_{x-c}^{x+c} dx \int_0^\infty \sigma^{2n} e^{-\frac{x^2}{\sigma^2} - \frac{\sigma^2}{k^2}} d\sigma.$$

This form implies either that the rate at which infective organisms are given off is constant or that the distribution of organisms before migration begins is uniform.

In the application required n is in general zero. The areas and moments of the first three curves, however, are given in the following table:—

	$n=0.$	$n=1.$	$n=2.$
A	$ck^2\sqrt{\pi}$	$ck^4\sqrt{\pi}$	$2ck^6\sqrt{\pi}$
ν_1	$\frac{k}{2}$	$\frac{3k}{4}$	$\frac{15k}{16}$
μ_2	$\frac{k^2}{2} + \frac{c^2}{3}$	$k^2 + \frac{c^2}{3}$	$\frac{3}{2}k^2 + \frac{c^2}{3}$
μ_4	$\frac{3}{2}k^4 + c^2k^2 + \frac{c^4}{5}$	$\frac{9}{2}k^4 + 2c^2k^2 + \frac{c^4}{5}$	$9k^4 + 6c^2k^2 + \frac{c^4}{5}$

From the value of $\frac{\mu_4}{\mu_2^2}$ obtained from the statistics $\frac{k^2}{c^2}$ can be easily calculated, and thence k. The latter value may be compared with value of k deduced from the value of ν_1, and the nearest value of n ascertained. The equation of the curve corresponding to $n=0$ is as follows:—

$$y = ak - ake^{\frac{-2c}{k}} \cosh \frac{2x}{k} \text{ from } x=0 \text{ to } x=c$$

$$y = ake^{\frac{-2x}{k}} \sinh \frac{2c}{k} \text{ from } x=c \text{ to } x=\infty.$$

Those for higher values of n may be obtained by differentiating the above values with respect to $\frac{1}{k^2}$.

(3) To find k and c from the moments is easy, but for convenience a table is given by which the values of $\frac{k}{c}$ may be obtained when the value of $\frac{1}{6}\beta_2$, *i.e.* $\frac{1}{6}\frac{\mu_4}{\mu_2{}^2}$, has been calculated.

Table showing the values of $\frac{k}{c}$ for different values of

$$\frac{\frac{k^4}{c^4}+\frac{2}{3}\frac{k^2}{c^2}+\frac{2}{15}}{\frac{k^4}{c^4}+\frac{4}{3}\frac{k^2}{c^2}+\frac{4}{9}}=\frac{1}{6}\frac{\mu_4}{\mu_2{}^2}=\frac{1}{6}\beta_2\,.$$

$\frac{k}{c}$.	$\frac{1}{6}\beta_2$.	Differences.
·50	·4314	
		227
·55	·4541	
		230
·60	·4771	
		232
·65	·5003	
		230
·70	·5233	
		226
·75	·5459	
		222
·80	·5679	
		213
·85	·5892	
		205
·90	·6097	
		196
·95	·6293	
		185
1·00	·6478	
		346
1·10	·6826	
		310
1·20	·7136	
		275
1·30	·7411	
		244
1·40	·7655	
		216
1·50	·7871	

(4) A special case arises in random migration if the area to be invaded is bounded by a straight boundary on the one side of which there is an infinite field uniformly stocked with the organism which is migrating. In the case of animals actually migrating this form has been found by Pro-

fessor Pearson, but it is much simpler in the case of plant life. We have to deal with the integral

$$y = a\int_0^\infty e^{\frac{-2(x+\zeta)}{k}}d\zeta$$

$$y = \frac{ka}{2}e^{\frac{-2x}{k}}.$$

Application of this will be given.

(5) Secondly, the form

$$y = a\int_{x-c}^{x+c}dx\int_0^\infty \sigma^n e^{-\frac{x^2}{\sigma^2}-\gamma\sigma}d\sigma$$

may be considered. In this the distribution of frequency of σ is taken as $\sigma^{n+2}e^{-\gamma\sigma}$. The moments are easily calculated, and are given by

$$A = 2c\sqrt{\pi}\frac{\Gamma(n+2)}{\gamma^{n+2}}$$

$$\nu_1 = \frac{1}{2\sqrt{\pi}}\frac{n+2}{\gamma}$$

$$\mu_2 = \frac{c^2}{3} + \frac{(n+3)(n+2)}{2\gamma^2}$$

$$\mu_4 = \frac{c^4}{5} + \frac{c^2(n+3)(n+2)}{\gamma^2} + \frac{3(n+5)(n+4)(n+3)(n+2)}{\gamma^4}.$$

These equations are solved most easily if c and n be eliminated. This gives, if $2\sqrt{\pi}\nu_1$ is denoted by ζ,

$$\frac{9}{2}\zeta\frac{1}{\gamma^3} + \frac{36}{5}\zeta^2\frac{1}{\gamma^2} + \left(\frac{12}{5}\zeta^3 + \frac{6}{5}\mu_2\zeta\right)\frac{1}{\gamma} + \left(\frac{9}{5}\mu_2{}^2 + \frac{6}{5}\mu_2\zeta^2 - \frac{3}{10}\zeta^4 - \mu_4\right) = 0.$$

The rest of the solution is easy. I have not been able to arrive at the curve without mechanical quadrature. In the instance in which it has been fitted (Diagram III.) it proves exceedingly unsuitable, and it is therefore evident that the form discussed cannot represent the distribution of σ even approximately.

Asymmetry.

(6) Asymmetry in the epidemic curve deserves some notice. As we found before, $\frac{\log p}{\log q}$ must in general be equal to a constant, as the epidemic is symmetrical. If, however, $\frac{\log p}{\log q}$ is a function of σ it may, on account of the near symmetry of the epidemic, be assumed that it can be expanded in terms of σ of a rapidly convergent nature so that all the terms in the expansion except that of σ may be neglected.

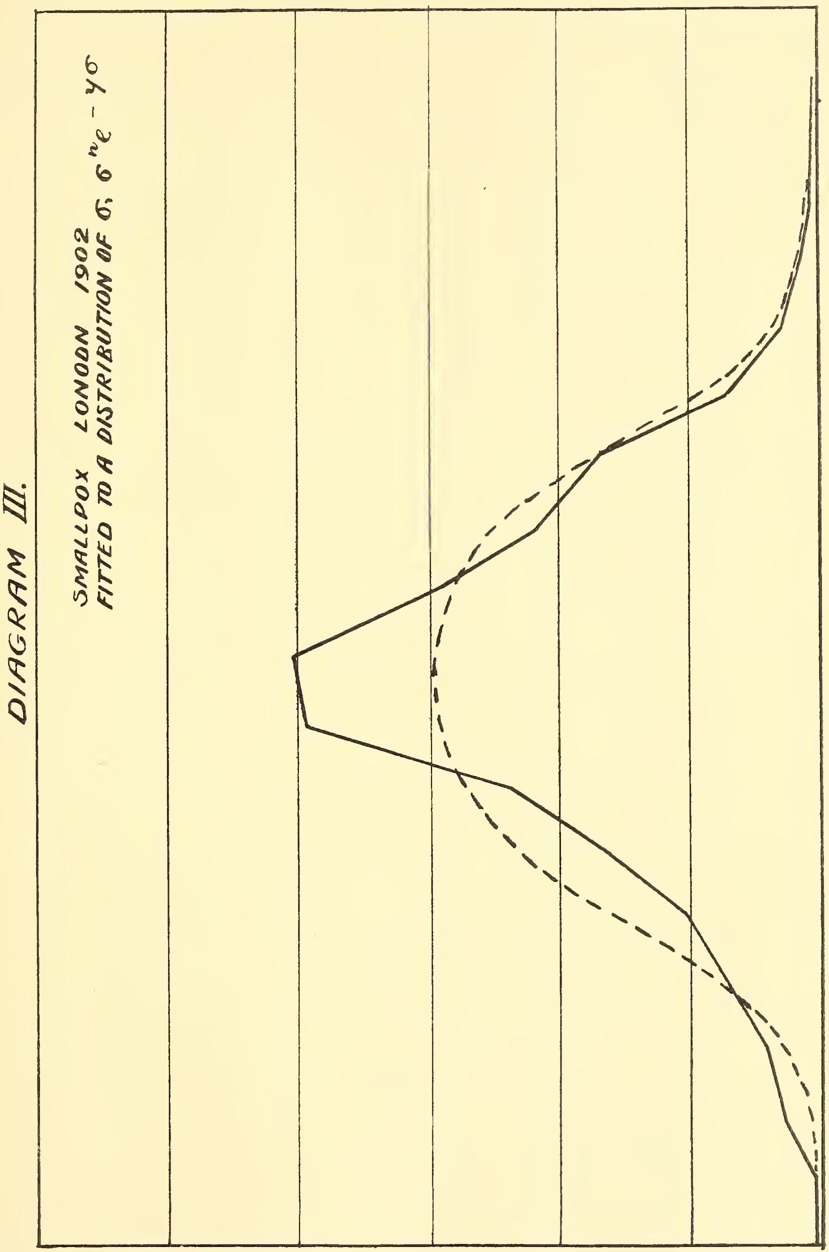


DIAGRAM III.
SMALLPOX LONDON 1902
FITTED TO A DISTRIBUTION OF σ, σ^n e^{-γσ}

The expression

$$a\int_{x-c}^{x+c}dx\int_0^\infty e^{-\frac{x^2}{\sigma^2}-\frac{\sigma^2}{k^2}}d\sigma$$

becomes on this hypothesis

$$a\int_{x-c}^{x+c}dx\int_0^\infty e^{\frac{-(x-a\sigma)^2}{\sigma^2}-\frac{\sigma^2}{k^2}}d\sigma.$$

In this case

$$\begin{aligned}
A &= c\sqrt{\pi}k^2\\
\mu'_1 &= \sqrt{\pi}ak\\
\mu'_2 &= \frac{c^2}{3}+\frac{k^2}{2}+a^2k^2\\
\mu'_3 &= \frac{9}{4}\sqrt{\pi}ak+\frac{3}{2}\sqrt{\pi}a^3k^3+a\sqrt{\pi}c^2k\\
\mu'_4 &= \frac{k^4}{2}(4a^4+12a^2+3)+k^2c^2(1+2a^2)+\frac{c^4}{5}
\end{aligned}$$

when μ'_1, μ'_2, μ'_3, etc., are the moments round the origin, which gives

$$\begin{aligned}
\mu_2 &= \frac{c_2}{3}+\frac{k^2}{2} \quad \text{if} \quad a^2 \doteq 0\\
\mu_3 &= \frac{9}{4}\sqrt{\pi}ak^3\\
\mu_4 &= \frac{3k^4}{2}+k^2c^2+\frac{c^4}{5}.
\end{aligned}$$

For the form of the curve

$$\begin{aligned}
y &= a\int_{x-c}^{x+c}dx\int_0^\infty e^{-\frac{(x-a\sigma)^2}{\sigma^2}-\frac{\sigma^2}{k^2}}d\sigma\\
&= ae^{-a^2}\int_{x-c}^{x+c}dx\int_0^\infty e^{-\frac{x^2}{\sigma^2}-\frac{\sigma^2}{k^2}}\left(1-\frac{a}{\sigma}\right)d\sigma\\
&= ae^{-a^2}\int_{x-c}^{x+c}dx\int_0^\infty e^{-\frac{x^2}{\sigma^2}-\frac{\sigma^2}{k^2}}d\sigma - aae^{-a^2}\int_{x-c}^{x+c}dx\int\frac{1}{\sigma}e^{-\frac{x^2}{\sigma^2}-\frac{\sigma^2}{k^2}}d\sigma
\end{aligned}$$

the solution of the first part is already given.

The second is equal to

$$-aae^{-a^2}\frac{2\pi}{2}\int_{x-c}^{x+c}H_0^{(1)}(ix)dx.*$$

This method for accounting for symmetry has likewise not been successful in representing the statistics.

(7) When the distribution is symmetrical around a centre it is evident that we have to deal with a similar integration of Professor Pearson's form and that a distribution of $y=ae^{-\frac{r}{k}}$ results.

* A table of this function is given in Jahnke und Emde, *Funktionentafeln*, p. 135.

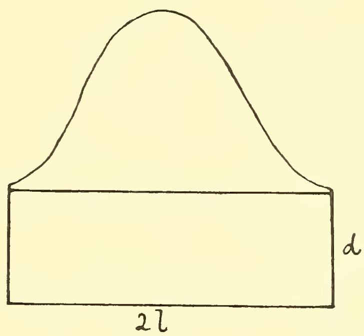
This can be fitted by radial moments, or, in general, more easily by first reducing the statistics so that the numbers per unit zone is taken as a basis of calculation.

The distribution may also be summed parallel to one axis and compared with a table of the integral

$$a\int_0^\infty e^{-\frac{\sqrt{x^2+y^2}}{k}}dx.$$

In some cases, especially when an epidemic occurs in a locality where a disease is more or less uniformly endemic, it is useful to have the means of separating the epidemic portion of the disease from that which is endemic. Such a combination is very common in the case of such diseases as scarlet fever, enteric fever, diarrhœa, etc. Successive approximations are necessary to obtain a solution, but a first approximation can be obtained by using the normal curve to represent the statistics of the course of the epidemic.

Let the whole amount of the disease be as represented in the diagram



The endemic prevalence of the disease is represented by the rectangular base and the epidemic portion by the curve above. In general the parts of the curve beyond the limits may be neglected as only affecting the result to a small extent. Taking the middle of the rectangle as origin, denoting its height by d and its length $2l$, where l is known and d unknown, we have for the area and for the moments of the rectangle round an axis through the middle,

$$\mathrm{A} = 2ld$$

$$\mathrm{A}\mu_2 = \frac{2l^3d}{3}$$

$$\mathrm{A}\mu_4 = \frac{2l^5d}{5}.$$

For the normal curve the equation is

$$y = y_0 e^{\frac{-(x-c)^2}{\sigma^2}},$$

and its area and moments become

$$A' = y_0 \sqrt{\pi}\sigma$$
$$A\mu'_1 = y_0 \sqrt{\pi}\sigma c$$
$$A\mu'_2 = y_0 \sqrt{\pi}\frac{\sigma^3}{2} + y_0 \sqrt{\pi}\sigma c^2$$
$$A\mu'_3 = y_0 \sqrt{\pi}\frac{\sigma^3 c}{2} + y_0 \sqrt{\pi}\frac{\sigma c^3}{3}.$$

Adding the corresponding portions of the moments, we have the following equations for determining the four unknowns, d, σ, c, and y_0,

$$A + A_1 = 2ld + y \quad = B_1 \qquad \ldots \qquad (a)$$
$$A\mu'_1 = y_0 \sqrt{\pi}\sigma c = B_2 \qquad \ldots \qquad (b)$$
$$A'\mu'_2 + A\mu_2 = \frac{2l^3 d}{3} + y_0 \sqrt{\pi}\frac{\sigma^3}{2} + y_0 \sqrt{\pi}\sigma c^2 = B_3 \qquad \ldots \qquad (c)$$
$$A\mu'_3 = y_0 \sqrt{\pi}\frac{\sigma^3 c}{2} + y_0 \sqrt{\pi}\frac{\sigma c^3}{3} = B_4 \qquad \ldots \qquad (d)$$

where B_1, B_2, etc., denote the area and moments obtained from the statistics.

By using (b) the equations (a), (c), and (d) become

$$\frac{B_2}{c} + 2ld = B_1 \qquad \ldots \qquad (a)$$
$$\frac{\sigma^2 B_2}{2c} + B_2 c + \frac{2ld^3}{3} = B_3 \qquad \ldots \qquad (\beta)$$
$$\frac{\sigma^2}{2}B_2 + \frac{c^2}{3}B_2 = B_4 \qquad \ldots \qquad (\gamma)$$

From a and β

$$\frac{\sigma^2}{2c}B_2 - \frac{B_2 l^2}{3c} + B_2 c = B_3 - \frac{l^2}{3}B_1,$$

or

$$2B_2 c^2 - (3B_3 - l^2 B_1)c + 3B_4 - B_2 l^2 = 0,$$

which gives c.

Hence immediately we have

$$\sigma^2 = 2\frac{B_4}{B_2} - \frac{2}{3}c^2$$
$$y_0 = \frac{B_2}{\sqrt{\pi}\sigma c}$$
$$2ld = B_1 - \frac{B_2}{c},$$

which complete the solution.

When the theory just described is applied to the elucidation of epidemic processes it is found to be fairly satisfactory. It gives a very good fit to many epidemics, as will be shown. It is also capable of expressing the moment relationships of epidemics satisfactorily. Thus it is generally found that $\mu_4 > 3\mu_2$: this is explained, but the form also allows of $\mu_4 < 3\mu_2$ without a change of theory.

Where this theory fails, however, is in its application to seasonable epidemics such as enteric fever when averaged over a number of years. This might be expected. Though the hypothesis that the infectious organism is given out for a definite time at a constant rate may be sufficiently accurate when applied to a single epidemic, it can hardly be expected to hold when an average of years is considered. An early or a late epidemic of the disease is the product of special weather conditions which will occur very seldom. So that though the epidemics occurring at the usual time will tend to have their irregularities smoothed, the beginning and the end of the compound curve will be framed on a different law from the middle. An endeavour has been made to frame an approximate formula to meet this, but none has been found fitting the facts.

To test the likelihood of a curve fitting, Professor Pearson's method is used throughout. To do this the actual and theoretical numbers are differenced, the difference squared and divided by the corresponding theoretical number. These are summed. The sum is denoted by χ^2. The table for testing curve fitting * is then consulted, and the value of P obtained.

Thus if $P = \cdot 8$ it signifies that in testing the matter over again a worse fit might be expected eight times out of ten, so the fit is good. The opposite interpretation is given when P is small, say $= \cdot 2$, when a worse fit might be expected only in two out of ten trials.

Plague in Hong-Kong.

The epidemics of plague in Hong-Kong are typical of the great majority in all the East. They are nearly bilaterally symmetrical; they spring from

Table showing the Number of Plague Cases, Actual and Theoretical, in the Epidemics of Plague in Hong-Kong in 1902 and 1904.

	1902. Number of Cases.		1904. Number of Cases.		
	Actual.	Theoretical.	Actual.	Actual Figures corrected so as to correspond with the Mean.	Theoretical.
March .	2	3·8	4	3·6	7·0
April . .	27	38·5	40	34·4	35
May . .	157	147·0	135	122·0	126·4
June . .	194	178·7	194	186·0	180·5
July . .	131	147·0	96	109·2	126·4
August .	50	38·5	19	29·3	35·1
September .	2	3·8	9	11·3	7·0

* *Biometrika*, vol. i. p. 155.

a zero line where the disease is nearly completely absent. The only point of difficulty in treating them mathematically is in settling how far the calculation of the moments is to be carried into the interepidemic period. In general good fits are obtained if the interepidemic cases are neglected.

Two epidemics are shown in the figures given in the table.

The fit of the first of these is not good as it stands; but when it is noted that the symmetry with regard to the first moment is produced from an unequal distribution with respect to the mean of a kind that the sums of the equidistant terms on both sides of the mean are nearly equal, we may put the distribution as follows:—

	Actual.	Theoretical.
2+2	4	7·6
27+50	77	77·0
157+131	288	294·0
194	194	178·7

This gives $\chi^2 = 3{\cdot}83$ or $P = {\cdot}28$.

Thus the variations in the rise cancel those of the fall. This method is subject to criticism, but the fit of any individual epidemic can hardly be expected to be good. With regard to the epidemic of 1904 the fit may be said to be good. Here $\chi^2 = 5{\cdot}96$ or $P = {\cdot}42$.

It is not necessary to give a large number of examples of the way in which the present theory fits the facts. In many instances better fits are obtained than those shown in my previous paper, where type iv. was found to closely represent the epidemic form. Two examples, however, may be given, that for the smallpox deaths in Warrington, and that for the milk epidemic of scarlet fever in Glasgow. These are shown in Diagrams I. and II. In both the correspondence of the facts with theory is very close, much closer than with type iv. Before leaving this part of the subject an example of the use of the distribution of $\sigma^n e^{-\gamma\sigma}$ for σ may be given. This is the only example I have thoroughly worked out, but in a number of others I have obtained the medium value. In none is there any evidence that the curve obtained has any resemblance to the facts. It is therefore very improbable that $\sigma^n e^{-\gamma\sigma}$ can represent the variation of σ. The example illustrated is that of smallpox in London in 1902. As will be seen by referring to my previous paper, type iv. gives a quite different representation from the curve shown in this case (Diagram III.).

Examples of the application to random migration will now be given; those with animal forms will be considered, and those with plant forms thereafter. With the former it is difficult to secure suitable examples. *Daphnia pulex* was used in many experiments. This crustacean does not move so consistently in one direction as others. It moves by jerks, and, moving more or less vertically, fulfils more nearly than any other Professor Pearson's criterion that motion in any one direction is as likely as in any other. It has also the advantages of being not specially attracted by light and of being more or less opaque, so that it is easily photographed. *Cyclops*, *Littorina rudis*, etc., were also used as is described. Of plants only two species were investigated, an Oscillatorian and *Aspargia hispida*.

Daphnia pulex.

Experiments were made in two ways with *Daphnia pulex*. In some experiments a large number of the crustaceans were placed inside a cylindrical tube in a large flat dish of white porcelain and then liberated; in others the water flea was allowed to take up its position as it liked in the dish. It usually chose to distribute itself from one corner along one side of the dish. Examples of the manner in which this happened are given below. The corner being an impenetrable boundary may be taken as representing a centre of diffusion, so that the simple fundamental integral should apply and the grouping should conform to the exponential. This is what takes place (Diagram IV.).

Table showing the Number of Daphnia per Unit of Length along the Margin of the Plate from one Corner.

Unit of Length.	Actual (*a*).	Theoretical.	Actual (*b*).	Theoretical.
0 – 1	18	18·1	23	20·7
1 – 2	11	10·1	11	13·6
2 – 3	3	5·6	7	9·1
3 – 4	4	3·9	6	6·0
4 – 5	2	2·5	5	3·9
5 – 6	1	1·5	1	2·6
6 – 7	...	1·1	4	1·7
7 – 8	1	·7	2	1·1
8 – 9	...	...	1	·75

The curve is given,

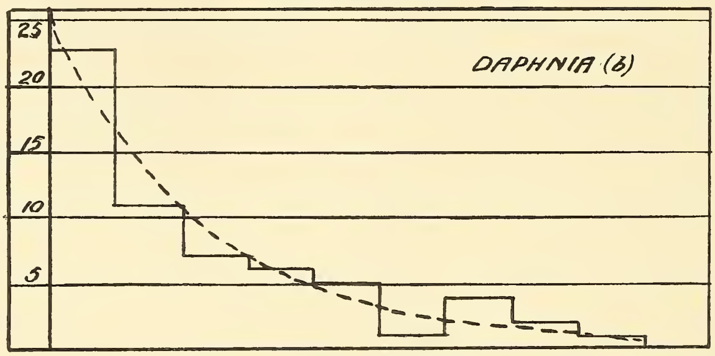
$$\text{for} \quad (a) \text{ by } y = 18 \cdot e^{-\frac{x}{2\cdot 225}}$$

$$\text{for} \quad (b) \text{ by } y = 25 \cdot e^{-\frac{x}{2\cdot 4}}.$$

In the first example the fit is excellent ($\chi^2 = 2\cdot 8$ P $= \cdot 9$), in the second not so good ($\chi^2 = 6\cdot 34$ P $= \cdot 6$). The want of fit, however, is largely due to the group of four near the tail end of the distribution, which contributes half of the divergence. Except for this group a further divergence might be expected nine times in each ten trials.

In several instances the Daphniæ, instead of grouping themselves along one edge of the dish, arranged themselves as if one corner of the dish were the centre of attraction. From this point outwards in all directions of the quadrant their numbers diminished. This, however, was observed only once, when the light was good enough for instantaneous photography. To

DIAGRAM IV.



count the numbers of the organism, concentric circles were drawn on the plate and the organisms between each pair of concentric circles in the quadrant counted. Some difficulty was experienced, however, at the greater distances, as in places it seemed that the outlyers of other groupings were even invading the periphery of this. All the organisms were counted, however, and none rejected. The figures are given in the following table :—

	i.	ii.	iii.	iv.
	Total Numbers.	Number per Unit Area.		Calculated Total.
		Actual.	Calculated.	
Centre quadrant . .	19	19	18·4	18·4
1st ,, zone .	33	11	11·1	33·3
2nd ,, ,, .	37	7·4	6·78	33·9
3rd ,, ,, .	18	2·67	4·16	29·12
4th ,, ,, .	24	2·67	2·55	22·95
5th ,, ,, .	19	1·72	1·57	16·27
6th ,, ,, .	17	1·31	·96	12·5
7th ,, ,, .	11	·73	·59	8·9
	178	46·50	46·11	177·14

It is easier to examine this grouping when the numbers are reduced to the population per unit area. On the hypothesis the grouping should be given by $y = ae^{-\frac{r}{k}}$ (par. 7) and the section of this by the plane of $y = 0$ gives $y = ae^{\pm\frac{x}{k}}$. The numbers reduced to the population per unit area are given in column ii. of the table. If we take the area and first moment and integrate from $x = 0$ to $x = 8$, the limit to which a count was made, we get $y = 23{\cdot}04e^{-\frac{x}{2{\cdot}056}}$. This gives $\chi^2 = 6{\cdot}13$ or $P = {\cdot}53$, a fit as good as could be expected, when the uncertainty regarding the numbers in the outer zone is remembered. A large part of the value of χ^2 is, as before noted, due to one zone alone. In all these groupings there seem to be secondary centres which interfere with results when the numbers are small.

The methods of dispersal from a centre were also investigated. For this many Daphniæ (from 100 to 200) were placed in the centre of the dish with a depth of water of about $\frac{3}{16}$ of an inch. They were contained in a cylindrical tube about $\frac{1}{2}$ inch in diameter. When the level of the water was the same on both sides of the tube and when the light was good and the camera ready, the tube was removed, the dispersal watched, and at a suitable moment instantaneously photographed. The photographs of course show no detail, the organisms being simply marked by a paler spot on the negative. In all cases a few Daphniæ were found greatly more energetic than the rest. These were generally above the mean size and probably represented an older generation. They were so exceptional that they possibly should be rejected from the statistics; all, however, have been included.

The experiments took much time. It was very difficult to manipulate the organism without damaging a number and thus introducing a new factor.

In the case of the negative from which the following table is made the centre of the group was found by counting the number of organisms in each half-inch square of the plate and calculating the mean. This being found, circles were drawn round this of diameter $\frac{1}{2}$ inch, 1 inch, $1\frac{1}{2}$ inches, etc., and the organisms in each zone counted.

The numbers in each zone are given in the table, column i.

The curve is given by $y = 39{\cdot}2e^{-\frac{x}{1{\cdot}309}}$, hence $\chi^2 = 9{\cdot}7$, which gives $P = {\cdot}4$.

In this case, again, one zone gives a large part of the value of χ^2, and it is also to be noted that again it is the third zone. Another fact of interest in all the experiments of this class is that the centre of the migration is not the original centre of dispersal; the whole mass has moved towards the light.

	i.	ii.	iii.	iv.
	Total Numbers.	Number per Unit Area.		Total Numbers calculated.
		Actual.	Calculated.	
Centre circle	28	28	27·8	27·8
1st zone	42	14	12·5	37·5
2nd „	28	5·6	5·8	29·0
3rd „	11	1·57	2·71	19·0
4th „	7	·78	1·26	11·3
5th „	7	·63	·58	6·3
6th „	5	·40	·274	3·6
7th „	4	·26	·128	1·9
8th „	2	·12	·06	1·0
9th „	1	·05	·028	·53

The dispersion of Daphnia may also be compared with the formula $y = ae^{-\frac{r}{k}}$ by summing parallel to one axis. A comparison is given below when the Daphniæ were so counted. The numbers are as follows:—

Actual.	Theoretical.
2	1·9
7	9·4
46	39·5
35	39·5
11	9·4
1	1·9

giving $\chi^2 = 2{\cdot}34$ or $P = {\cdot}75$.

Cyclops.

Cyclops were used in a few experiments but found very unsatisfactory. Not only did they photograph with great difficulty on account of their transparency, but they moved on dispersal strongly towards the light.

A sectional count of one instance is given (Diagram V.). It is not treated as a whole, but it is to be noticed that the advancing edge is very closely given by the exponential curve which has been fitted to it.

The values are:—

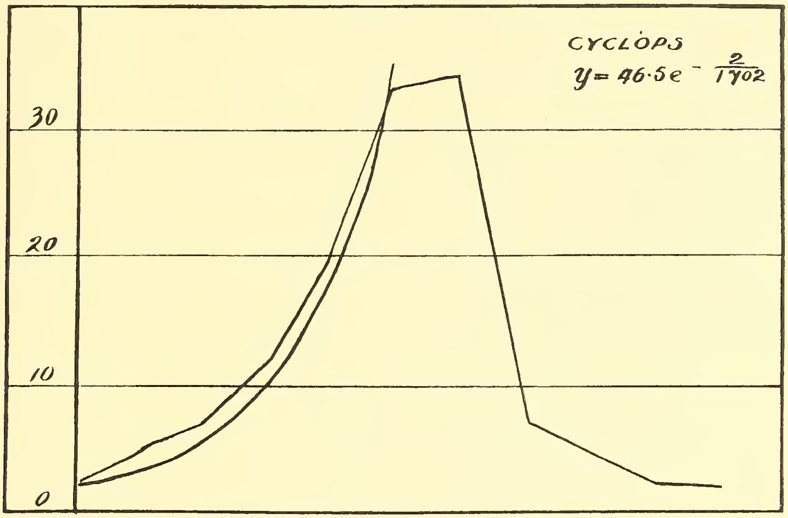
Actual.	Calculated.
33	35·3
20	19·2
12	10·7
7	5·9
5	3·3
2	1·8

which gives $\chi^2 = 1{\cdot}61$ or $P = {\cdot}9$, so that the fit is very close.

Other Crustacea.

Some copepods were experimented with, but their transparency and rate of motion unfitted them for this purpose; they were also too strongly attracted to light. In addition an attempt was made with a fresh-water

DIAGRAM V.



isopod, but a sufficiently large dish could not be obtained to make the experiment of any value.

Littorina rudis.

Several experiments were made with this mollusc. Numbers were placed in various vessels and their movements observed. It was found best to put them at the bottom of a deep glass jar with a little salt water

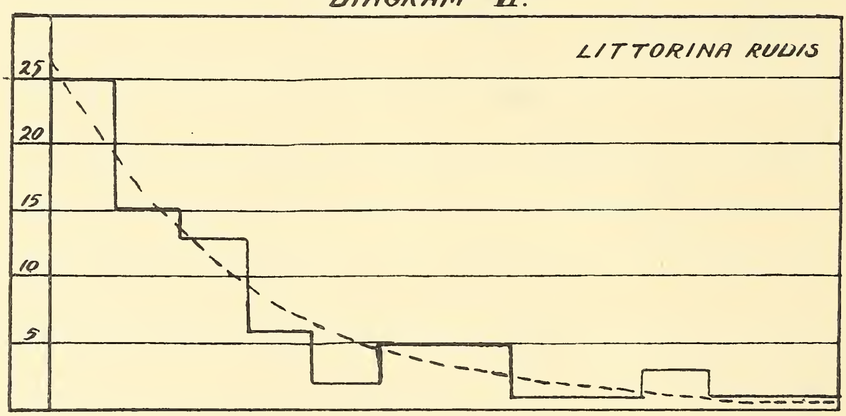
Distance in inches.	Numbers actual.	Numbers theoretical.
0 – ·5	25	23·2
·5 – 1	15	16·3
1 – 1·5	13	11·5
1·5 – 2·0	6	8·2
2·0 – 2·5	2	5·8
2·5 – 3·0	5	4·1
3·0 – 3·5	5	3·0
3·5 – 4·0	1	2·2
4·0 – 4·5	1	1·4
4·5 – 5·0	3	1·0
5·0 – 5·5	1	·7
5·5 – 6·0	1	·5

in the bottom and leave them to climb out; this they did, passing the edge and falling on the floor. Latterly a large number were placed in the jar one day and next day those on the side of the jar counted. It seems evident that we are dealing with the tail of a moving mass, and that this tail should obey the law of the exponential. In one experiment the number of Littorina in each half inch of the jar, measuring from the top, were as shown in table on page 283.

This gives a distribution $y = 2{\cdot}72e^{-\frac{x}{2{\cdot}572}}$.

As it stands $\chi^2 = 10{\cdot}2$, so that $P = {\cdot}53$, but, as before with Daphnia, 40 per cent. of the value of χ^2 is accounted for by the third last group. If this be subtracted $P = {\cdot}87$. In either case the fit is admissible.

DIAGRAM VI.



Other Marine Animals.

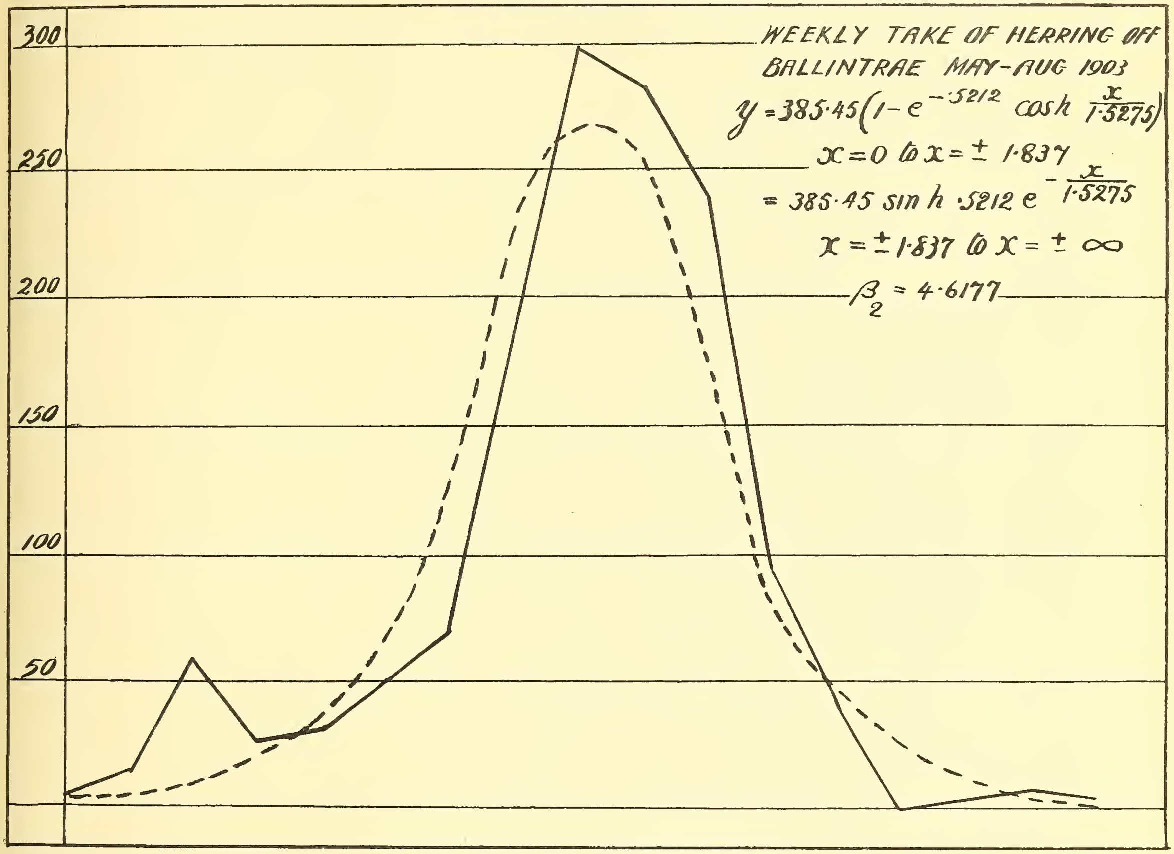
With regard to large animals few experiments can be recorded. Some observations were made on fish. In fish migration the form of the shoal should be capable of measurement. In this case the tanks in the Millport Biological Station seemed to afford some chance of success. One of the large tanks contained a small shoal of saithe (*Gadus virens*). These were photographed several times, but to get sufficient light for an instantaneous plate was difficult. In one such photograph the shoal is on the point of turning. The symmetry is remarkable, the numbers from the left to the right in each unit of length being as follows:—

1, 2, 2·5, 6, 6, 2·5, 1, 1. Total 22.

The numbers are too small to allow of differentiation of the type of curve, as they can be fitted either to the exponential curve described or to the normal curve of error.

Data for the other fish are lacking with the exception of herring. The weekly takes at given stations afford some guide, but not much. One selected from the Fishery Board Report of the take of herring off Ballantrae is illustrated in the diagram (Diagram VII.). Here the elevation at the beginning of the ascent prevents accurate curve fitting, but the general

DIAGRAM VII.



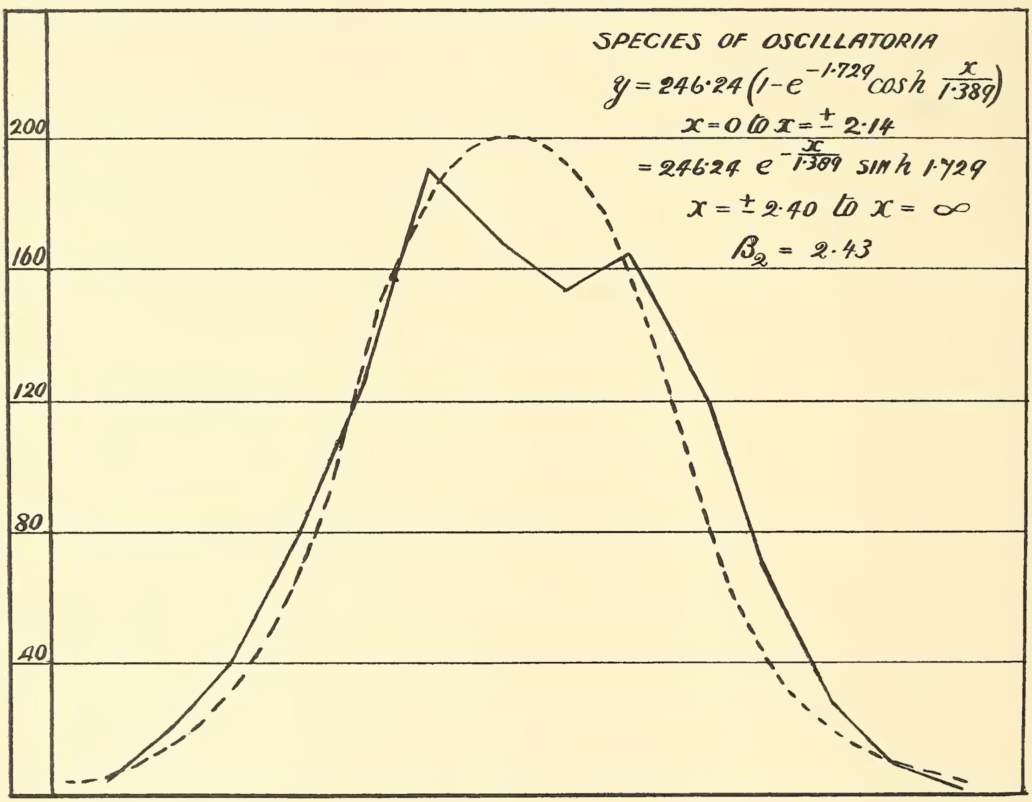
correspondence is good. It was hoped that with the figures at a fixed point some better results might be obtained, and the Fishery Board very kindly furnished me with the figures of the weekly takes in the trammel nets at Ballantrae. But these figures proved insufficient to determine the shoal form; in no case were the curves continuous. Factors such as the end of the open time, storms, etc., interfered with the returns to such an extent that it was not possible to use the figures.*

* Herring catch, Ballantrae, 1903, May–August, *Twenty-fifth Ann. Rep. Fish. Board Sc.*, p. 174.

Oscillatoria.

A species of Oscillatorian (not identified) was also experimented with. This, when put in a mass in the bottom of a test tube, started to climb the wall to the surface. After the migration had taken place so that the beginning and end of the mass was fairly clear, the test tube was carefully

DIAGRAM VIII.



emptied and filled with melted gelatine. When this was set the number of filaments per millimetre of length on the test tube was counted under the microscope. The numbers in each space of four millimetres were as follows:—

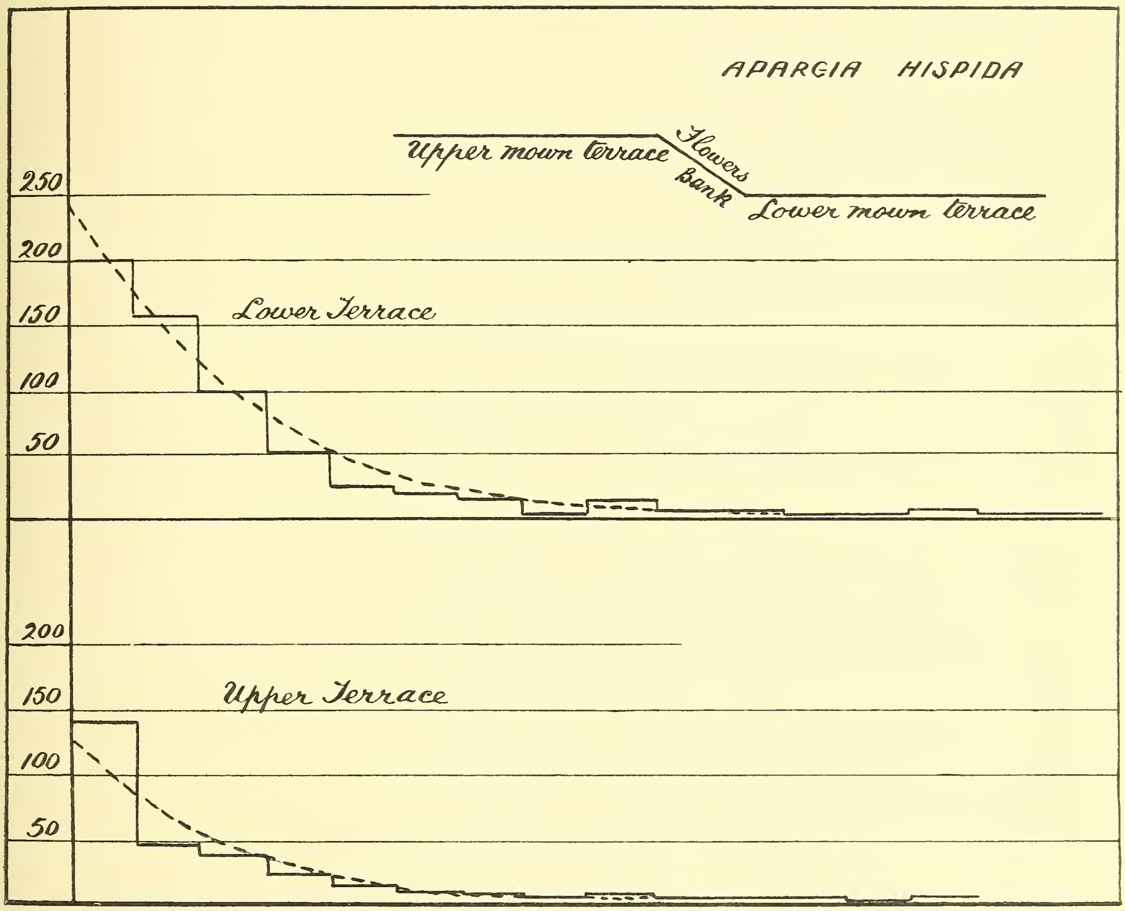
2, 20, 42, 80, 125, 190, 168, 154, 164, 128, 70, 28, 10, 2.

When fitted to a migration curve the correspondence is as illustrated in the accompanying diagram (Diagram VIII.). The fit is not good, but the shape of the curve suggests that it is made up of two and not one migration system.

Apargia hispida.

In the Belvidere Hospital there are long straight terraces adjacent to the Clyde constructed as in the diagram (Diagram IX.). It is the custom to mow the level parts of the terraces with a lawn mower.

DIAGRAM IX.



No plant, therefore, of any high habit can ever seed there. It can only grow herbaceously. On the bank between the terraces, however, such plants can develop, and a considerable stretch of the bank is thickly overgrown with Apargia. The seeds of this scatter over the lower and higher terraces, when they germinate and form plants in the grass.

This case can easily be considered as one where there is an infinite uniform source on one side of a straight line, and therefore may be represented by $y = ae^{-\frac{x}{k}}$

To obtain the form of distribution the ground on both terraces was lined with parallel lengths of string two feet apart for a distance of fifteen yards, and the number of plants in each rectangular space counted. The numbers are given in this table:—

Number of Plants.

Distance.	Upper Terrace.	Lower Terrace.
0– 2 feet	140	201
2– 4 „	47	157
4– 8 „	39	99
8–10 „	24	50
10–12 „	17	22
12–14 „	10	20
14–16 „	9	16
16–18 „	5	4
18–20 „	7	12
20–22 „	3	6
22–24 „	4	3
24–26 „	3	3
26–28 „	1	3
28–30 „	3	4
30–32 „	...	1
32–34 „	...	1
Total	312	605

The equation of the theoretical curves are:—

Higher terrace $y = 127e^{-\frac{x}{2\cdot458}}$

Lower „ $y = 248e^{-\frac{x}{2\cdot445}}$

The two areas show a practically identical distribution, with the exception that the seed spreads to twice the extent on the lower than on the higher terrace. The nature of the fit is shown on the diagram, and as the soil of the locality is all forced, clay coming to the surface in patches, and drainage being very irregular, it is as good as might be expected.

Conclusions.

(1) The general principles which underlie both epidemic distribution in space and time and random migration are identical.

(2) Both can be deduced almost directly from the laws of chance through assumptions which have considerable *a priori* probability.

INSTRUCTIONS TO AUTHORS.

The 'copy' should be written on large sheets of paper, on one side only and the pages should be clearly numbered. The MS. must be easily legible, preferably typewritten, and must be absolutely in its final form for printing; so that corrections in proof shall be as few as possible, and shall not cause overrunning in the lines or pages of the proof. All tables of contents, references to plates, or illustrations in the text, etc., must be in their proper places, with the page numbers left blank; and spaces must be indicated for the insertion of illustrations that are to appear in the text.

ILLUSTRATIONS.—All illustrations must be drawn in a form immediately suitable for reproduction; and such illustrations as can be reproduced by photographic processes should, so far as possible, be preferred. Drawings to be reproduced as line blocks should be made with Indian ink (deadened with yellow if of bluish tone), preferably on fine white bristol board, free from folds or creases; smooth, clean lines, or sharp dots, but no washes or colours, should be used. If the drawings are done on a large scale, to be afterwards reduced by photography, any lettering or other legend must be on a corresponding scale.

If an author finds it inconvenient to furnish such drawings, the Society will have the figures re-drawn at his expense; but this will cause delay.

When the illustrations are to form plates, a scheme for the arrangement of the figures (in quarto plates for the Transactions, in octavo for the Proceedings) must be given, and numbering and lettering indicated.

PROOFS.—In general, a first proof and a revise of each paper will be sent to the author, whose address should be indicated on the MS. If further proofs are required, owing to corrections or alterations for which the printer is not responsible, the expense of such proofs and corrections will be charged against the author.

All proofs must, if possible, be returned within one week, addressed to *The Secretary, Royal Society,* 22–24 *George Street, Edinburgh,* and *not to the printer.* To prevent delay, authors residing abroad should appoint some one residing in this country to correct their proofs.

The author receives fifty of the reprints free, and may have any reasonable number of additional copies at a fixed scale of prices which will be furnished by the printer, who will charge him with the cost. To prevent disappointment, especially if the paper contains plates, the author should, immediately after receiving his first proof, notify *to the printer* the number of additional copies required.

INDEX SLIPS.—In order to facilitate the compilation of Subject Indices, and to secure that due attention to the important points in a paper shall be given in General Catalogues of Scientific Literature and in Abstracts by Periodicals, every author is requested to return to the Secretary, along with his final proof, a brief index (on the model given below), of the points in it which he considers new or important. These indices will be edited by the Secretary, and incorporated in Separate Index Slips, to be issued with each part of the Proceedings and Transactions.

MODEL INDEX.

Schäfer, E. A.—On the Existence within the Liver Cells of Channels which can be directly injected from the Blood-vessels. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

Cells, Liver,—Intra-cellular Canaliculi in.
E. A. Schäfer. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

Liver,—Injection within Cells of.
E. A. Schäfer. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

The Papers published in this Part of the PROCEEDINGS *may be had separately, on application to the Publishers, at the following prices:—*

No. X.,	. . .	Price 3s. 8d.
No. XI.,	. . .	,, 6d.
No. XII.,	. . .	,, 6d.
No. XIII.,	. . .	,, 6d.
No. XIV.,	. . .	,, 1s. 8d.

PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

SESSION 1910–11.

Part III.] VOL. XXXI. [Pp. 289–416.

CONTENTS.

EDINBURGH:

PUBLISHED BY ROBERT GRANT & SON, 107 PRINCES STREET, AND
WILLIAMS & NORGATE, 14 HENRIETTA STREET, COVENT GARDEN, LONDON.

MDCCCCXI.

Price Twelve Shillings and Tenpence.

REGULATIONS REGARDING THE PUBLICATION OF PAPERS IN THE PROCEEDINGS AND TRANSACTIONS OF THE SOCIETY.

THE Council beg to direct the attention of authors of communications to the Society to the following Regulations, which have been drawn up in order to accelerate the publication of the Proceedings and Transactions, and to utilise as widely and as fairly as possible the funds which the Society devotes to the publication of Scientific and Literary Researches.

1. MANUSCRIPT OF PAPERS.—As soon as any paper has been passed for publication, either in its original or in any altered form, and has been made ready for publication by the author, it is sent to the printer.

2. METHOD OF PUBLICATION.—As soon as the final revise of a Transactions paper has been returned, or as soon as the sheet in which the last part of a Proceedings paper appears is ready for press, a certain number of separate copies or reprints, in covers bearing the title of the paper and the name of the author, are printed off and placed on sale. The date of such separate publication will be printed on each paper.

3. ADDITIONS TO A PAPER after it has been finally handed in for publication, if accepted by the Council, will be treated and dated as separate communications, and may, or may not, be printed immediately after the original paper.

4. BRIEF ABSTRACTS OF TRANSACTIONS PAPERS will be published in the Proceedings, provided they are sent along with the original paper.

5. SPECIAL DISCUSSION OF PAPERS ACCEPTED FOR PUBLICATION.—Where a paper has been accepted for publication, the Council may, with the consent of the author, select this paper for Special Discussion. In the case of such papers advanced proofs will be sent to the members of the Society desiring copies, and copies will be supplied to the author for distribution. A paper selected for Special Discussion will be marked with an asterisk (*) and placed first on the Billet for the day of reading. Any following papers for that day may be adjourned or held as read if the discussion prevents their being read.

6. COMMUNICATIONS NOT SUBMITTED FOR PUBLICATION, such as Demonstrations of Experiments, Statement of Scientific Problems, etc., may be received by the Council, and may also be selected for Special Discussion. The Council does not undertake to publish any notice of such communications in the Proceedings or Transactions of the Society.

[*Continued on page* iii *of Cover.*

(3) It is found in both cases that the exponential curve can be taken as giving a fair representation of most of the facts. The manner in which this curve, however, appears is somewhat different in the two instances. It is doubtful why this formula should express both results.

NOTE.—The experimental work recorded in this paper was done chiefly at the laboratory of the Millport Marine Biological Station, and I desire to thank the Superintendent for much assistance.

(*Issued separately February* 25, 1911.)

XV.—Illustration of the Modus Operandi of the Prism. By George Green, D.Sc., Assistant to the Professor of Natural Philosophy in the University of Glasgow. *Communicated by* Professor A. Gray, F.R.S.

(MS. received December 1, 1910. Read January 9, 1911.)

§ 1. It was proved by Gouy * and Lord Rayleigh † that interference may be observed by the aid of the spectroscope even if the incident light were entirely irregular, consisting, for instance, of an irregular succession of impulses. This means that the "regularity" of the emergent light arising from an impulse is due to the spectroscope itself, and the extent to which interference may be observed is limited only by the resolving power of the instrument used. The greater the resolving power, the more homogeneous is the light in any part of the spectrum. The question regarding the *modus operandi* of the prism has been further dealt with by Professor Schuster in his paper "On Interference Phenomena" (*Phil. Mag.*, vol. xxxvii., 1894) and, later, in his explanation of Talbot's bands by consideration of group-velocities (*Phil. Mag.*, vol. vii., 1904). In a paper by J. S. Ames, entitled "An Elementary Discussion of the Action of a Prism on White Light" (*Astrophysical Journal*, vol. xxii., 1905), a view similar to that of Professor Schuster is adopted. The theory given in these papers is now regarded as complete, but it is difficult to form from it a definite general impression regarding the wave-system produced by a prism from a given impulse; and as this may be useful, the object of this short note is to illustrate the action of the prism by associating with it a definite wave-pattern.

§ 2. Consider the case of an incident light pulse in a plane perpendicular to the paper, whose trace is ac (fig. 1), when first it meets the side of the prism at a. The pulse travels along the face of the prism from a to b with uniform velocity, and the problem is to determine the wave-system which it originates, and to follow the configuration of each group of waves characterised by a given wave-length after emergence. The wave-system within the prism can be determined directly by means of Fourier's theorem, provided the wave-velocity of all the wave-trains, into which the original disturbance can be analysed, is known; or more conveniently perhaps by the principle of stationary phase as applied by Dr T. H. Havelock in his

* *Journal de Physique* (2), v., 1886.

† *Phil. Mag.*, vol. xxvii. p. 463, 1889.

paper on "The Propagation of Groups of Waves in Dispersive Media" (*Proc. Roy. Soc. Lon.*, lxxxi., 1908). As it is only the general features of the wave-system after emergence that are of interest to us, it is unnecessary

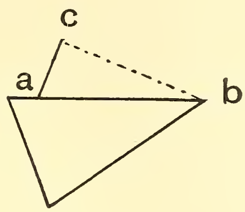


Fig. 1.

to consider effects due to the plane boundaries of the prism not touched by the incident light pulse.

§ 3. The problem in hydrodynamics analogous to that of § 2 is to determine the wave-system arising from the application of a point-pressure which moves with constant velocity over the water surface. In each case the disturbance can be determined by the consideration of wave-trains whose velocity varies with the wave-length, and we can therefore take the ship-waves pattern to illustrate the main features of light-propagation through the prism. The analogy here referred to was also remarked upon by Lord Rayleigh with respect to the origin of the prismatic colours (*Phil. Mag.*, Oct. 1905). By means of it the expression for the number of waves having a certain length λ arising from the action of a given pulse is easily obtained, provided the greatest thickness of the prism traversed by waves of the corresponding length in glass is known.

The ship-waves pattern indicated by the curved lines of constant phase in the accompanying diagram, fig. 2, which is taken from Lord Kelvin's paper on "Deep-sea Ship-waves" (*Proc. R.S.E.*, vol. xxv., 1904–1905), gives, of course, the wave-system built up after a long application of the moving point-pressure. When the pulse is applied for only a short time, few of the transverse waves have had time to form, and there is, in addition, wave-disturbance beyond the line of cusps OC on either side. But for the application of the diagram which we intend to make, the argument is the same whether the pulse be of long or short application, as it involves only the principle of stationary phase in relation to each group of diverging waves of definite wave-length chosen for observation.

§ 4. In the diagram (fig. 2), AO is the line of motion of the ship, and the straight line OC is the line of cusps beyond which there is no disturbance. The curved lines OC, OD, OE, are lines of equal phase in the diverging wave-system. For convenience they may be regarded as three successive

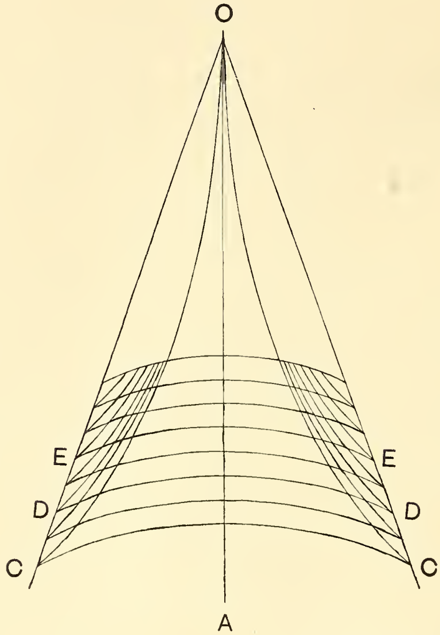


Fig. 2.

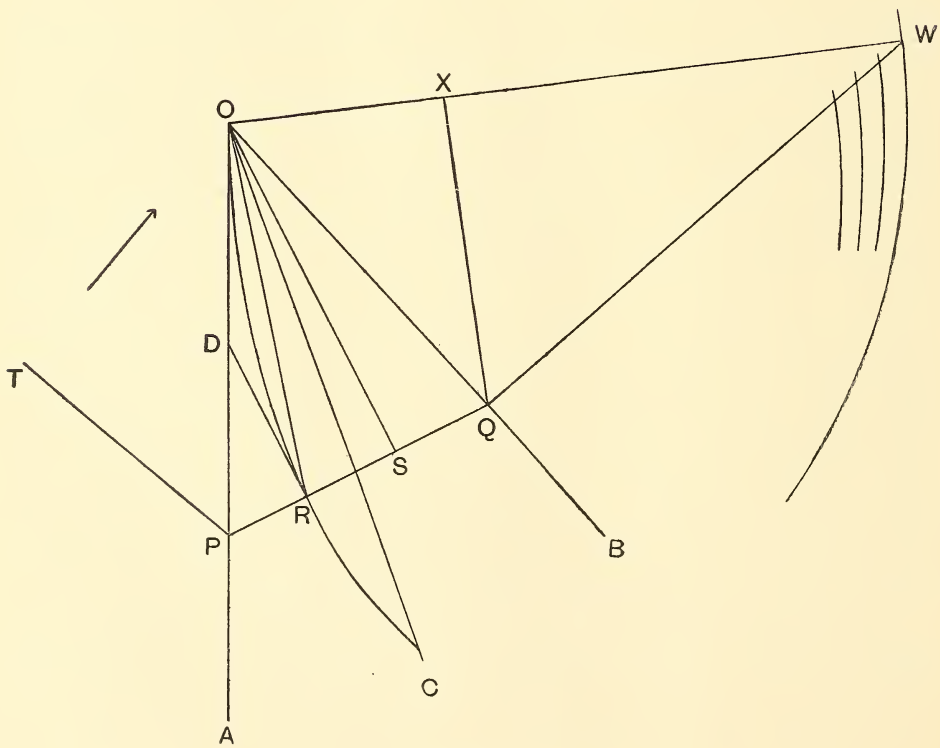


Fig. 3.

wave-crests. The tangent to an isophasal line at any point is parallel to the wave-crests and perpendicular to the line of advance of the waves in that neighbourhood. Taking v as the velocity of the ship along AO, the condition that the wave-pattern maintains the same position relative to the ship requires that the wave-crests moving forward in a direction inclined at angle ψ to the line of motion of the ship should advance with velocity $v \cos \psi$. Thus the velocity of the waves, and therefore the wave-length, varies according to the direction of the crest, being constant at all points where the tangents to the isophasal curves are parallel. As, according to the known results for ship-waves, the tangents to the isophasal curves are parallel at all points along any straight line through O such as OR, in fig. 3, the line OR is the locus of points where a constant wave-length is to be observed, and the crests of the whole group of waves having this wave-length are parallel to the tangent RD to the isophasal OC. The line AO, and the two lines CO in fig. 3 are reproduced from fig. 2. The group moves perpendicularly to RD. If we suppose the moving point-pressure to be first applied at P, then R marks the extreme rear of the group of the observed wave-length at the instant the pressure has reached O, which then marks the front of the same group.

§ 5. For the application to the prism, OA in fig. 3 represents the side of the prism affected from P to O by the incident light pulse whose trace is TP when first it meets the prism at P: OB is the side at which the light emerges. PQ, being perpendicular to the tangent to the isophasal OC at R, indicates the direction of motion of the group whose crests lie along OR, and which we shall observe until it leaves the prism at Q. OS is perpendicular to PR.

Let t be the time taken by the pulse to pass from P to O; then $PS = v \cos \psi \,.\, t$, and $PR = Ut$ according to the principle of stationary phase, where U is the group-velocity corresponding to the wave-velocity $v \cos \psi$. For convenience we shall write V instead of $v \cos \psi$ to denote the wave-velocity in glass corresponding to wave-length λ *in vacuo*: and we shall take V_0 to denote the velocity of the waves *in vacuo*, and T as the greatest thickness of the prism traversed by the group under observation, *i.e.* $T = PQ$. The length of the group at time t is represented by RS, which is $(V - U)t$. Immediately after time t the group begins to emerge from the prism at O.

§ 6. To follow the condition of the disturbance after complete emergence, remark that the rear of the group advances at the group-velocity U, and that it will reach Q in a time t', reckoned from the time of the diagram, where

$$t' = \frac{RQ}{U} = \frac{T - PR}{U} = \frac{T - Ut}{U}.$$

In the same time the front of the group advances *in vacuo* a distance $V_0 t'$, represented by OW. Thus

$$V_0 t' = OW = V_0 \frac{T - Ut}{U}.$$

The direction of OW is determined by the ordinary law of refraction, as we are in reality now dealing with the emergence of a regular group of waves, although their crests are short, and are arranged in echelon pattern. Hence, after the rear of the group has reached Q, the wave-crests are all parallel to QX and lie along a line QW similar to their former arrangement along OR. In other words, QW is the locus of points where a constant wave-length λ is to be observed. By the law of refraction

$$OX = \frac{V_0}{V} SQ = \frac{V_0}{V}(T - Vt).$$

The length of the group on emergence is therefore given by

$$XW = OW - OX = V_0 \left\{ \frac{T - Ut}{U} - \frac{T - Vt}{V} \right\} = T \left\{ \frac{V_0}{U} - \frac{V_0}{V} \right\}.$$

Using in this the following equations, $\frac{V_0}{U} = \mu - \lambda \frac{d\mu}{d\lambda}$ and $\frac{V_0}{V} = \mu$, we have finally

$$XW = - T\lambda \frac{d\mu}{d\lambda},$$

and the number of wave-lengths in the group after emergence is therefore $-T\frac{d\mu}{d\lambda}$, which is in agreement with Lord Rayleigh's expression.

§ 7. It may be of interest to show that this value for the number of waves of length λ in the emergent wave-system can be arrived at without making use of the law of refraction. As stated in § 5, at time t the length of the group in the prism is $(V - U)t$, and the number of waves it contains is therefore $(V - U)t/\lambda'$, where λ' is the wave-length in glass. During the passage of the group through the prism fresh waves are continually entering it, owing to the fact that the individual waves move with greater velocity than the group. Thus the number of waves which enter the group per second is $(V - U)/\lambda'$, and the total addition to the group by the time the rear reaches Q is $(V - U)t'/\lambda'$ where t' is given in § 6. The total number of waves in the group on emergence is therefore given by

$$\frac{(V - U)t}{\lambda'} + \frac{(V - U)}{\lambda'} \cdot \frac{(T - Ut)}{U},$$

which reduces to $\frac{(V - U)T}{\lambda' U}$; or with $\lambda' = \frac{V}{V_0}\lambda$ to $\frac{TV_0}{\lambda}\left\{\frac{V - U}{UV}\right\}$, which is easily transformed into $-T\frac{d\mu}{d\lambda}$ by means of the optical equations used in § 6.

§ 8. A general idea can now be obtained from the diagram of the whole wave-system outside the prism at the instant the rear of our group has reached Q. For clearly, if we repeat what we have done with the group of wave-length λ, with groups of greater and less wave-lengths we shall find that the circle whose centre is O and radius OW is a line of constant phase; and if we draw the isophasal lines through the successive crests of our group lying along QW, they must clearly become closer for the shorter wave-lengths and open out for the greater, as indicated by the three short isophasal lines in the diagram.

It is evident, also, from the echelon formation of the crests of the wave-group which lie along QW in the diagram, why Talbot's and Powell's bands are obtained only when the retarding plate is inserted from the thin side of the prism. The amplitudes of the successive crests of each group within the prism could be determined as in the case of ship-waves, though it is probable that the number of crests formed by any actual light impulse is very small. But, as we have seen in § 7, the group in emerging is largely extended by the well-known process of the successive wave-crests passing through the group, so that we may assume that the amplitudes of the successive crests vary slowly as we pass from W to Q in the emergent group.

(*Issued separately March* 30, 1911.)

XVI.—The Theory of Wronskians in the Historical Order of Development up to 1860. By Thomas Muir, LL.D.

(MS. received June 13, 1910. Read July 4, 1910.)

The previous history of Wronskians being not at all lengthy, was included in the chapter on "Miscellaneous Special Forms" (*History*, i., chap. xvi.), and is to be found there under Wronski 1812, Wronski 1815, Wronski 1816–17, and Schweins 1825 (pp. 472–478, 482–485).

The name dates only from 1882, being first suggested on p. 224 of my Text-book on Determinants.*

Liouville, J. (1838).

[Note sur la théorie de la variation des constantes arbitraires. *Journ. (de Liouville) de Math.*, iii. pp. 342–347.]

The Wronskian which incidentally appears here is of a special kind, namely, that in which the originating functions are in the so-called relation of being first differential-quotients of one and the same function, for example, in later notation,

$$\begin{vmatrix} \dfrac{\partial x}{\partial a} & \dfrac{\partial x}{\partial b} & \dfrac{\partial x}{\partial c} \\ \dfrac{\partial^2 x}{\partial a\partial t} & \dfrac{\partial^2 x}{\partial b\partial t} & \dfrac{\partial^2 x}{\partial c\partial t} \\ \dfrac{\partial^3 x}{\partial a\partial t^2} & \dfrac{\partial^3 x}{\partial b\partial t^2} & \dfrac{\partial^3 x}{\partial c\partial t^2} \end{vmatrix} .$$

It is worthy of note also that the expression for the differential-quotient of this with respect to t is obtained in the form which accords with the case of Schweins' theorem of 1825 (*Hist.*, i. p. 484).

$$Zd \,\Big\| (Zd)^a A_1 \,.\, (Zd)^{a+1} A_2 \,.\, \ldots\, (Zd)^{a+n-1} A_n \Big)$$
$$= \Big\| (Zd)^a A_1 \,.\, (Zd)^{a+1} A_2 \,.\, \ldots\, (Zd)^{a+n-2} A_{n-1} \,.\, (Zd)^{a+n} A_n \Big)$$

where $Z=1$ and $a=0$.

* Muir, Th. A Treatise on the Theory of Determinants, . . . viii + 240 pp., London.

Malmsten, C. J. (1849).

[Moyens pour trouver l'expression de la n-ième intégrale particulière de l'équation $y^{(n)}+\mathrm{P}y^{(n-1)}+\ldots+\mathrm{S}y^{(1)}+\mathrm{T}y=0$ à l'aide des $n-1$ valeurs $y_1, y_2, \ldots, y_{n-1}$ qui satisfont à celle équation. *Crelle's Journ.*, xxxix. pp. 91–98: or abstract in *Cambridge and Dub. Math. Journ.*, iv. pp. 286–288.]

The result obtained, after an introductory note on Determinants, is

$$y_n = z_1 y_1 + z_2 y_2 + \ldots + z_{n-1} y_{n-1},$$

where

$$z_r = (-1)^{n-1}\int \frac{d\mathrm{R}}{dy_r^{(n-2)}} \cdot e^{-\int \mathrm{P}dx} dx, \quad \text{and} \quad \mathrm{R} = 1 \Big/ \sum \pm y_1 y_2' y_3'' \ldots y_{n-1}^{(n-2)}.$$

Only one special property of the Wronskian is used, namely, that regarding its differential-quotient.

Puiseux, V. (1851).

[Sur la ligne dont les deux courbures ont entre elles un rapport constant. *Journ.* (*de Liouville*) *de Math.*, vi. pp. 208–211.]

At the close of his paper Puiseux remarks that his proof would have been shortened by using the theorem, "*Les lettres* t, u, v, . . . , w *désignant* n *variables, si le déterminant du système de* n *quantités.*

$$\begin{array}{cccc} t & u & \ldots & w \\ dt & du & \ldots & dw \\ d^2t & d^2u & \ldots & d^2w \\ \cdot & \cdot & \cdot & \cdot \\ d^{n-1}t & d^{n-1}u & \ldots & d^{n-1}w \end{array}$$

est égal à zéro, on a necessairement l'équation

$$at + bu + cv + \ldots + gw = 0$$

où a, b, c, . . . , g *sont des constantes*" The theorem is spoken of as known, but no reference is given.

Tissot, A. (1852).

[Sur un déterminant d'intégrales définies. *Journ.* (*de Liouville*) *de Math.*, xvii. pp. 177–185.]

Tissot incidentally asserts that if $y, y_1, y_2, \ldots, y_n$ all satisfy a linear differential equation similar to that dealt with by Malmsten above, then

$$\sum(\pm y y_1' y_2'' \ldots y_n^{(n)}) = \gamma \epsilon^{-\int \mathrm{P}dx}$$

where γ is independent of x. It is further stated that Liouville proved this in vol. x. of his journal, but such is not the case, the subject being not even referred to in that volume.

Prouhet, E. (1852).

[Mémoire sur quelques formules générales d'analyse. *Journ.* (*de Liouville*) *de Math.*, (2) i. pp. 321–344.]

The third and last section (§§ 40–45) of Prouhet's memoir is headed "Théorèmes sur quelques déterminants de fonctions." The first theorem proved is that just mentioned as having been used by Malmsten. He then takes the set of $m+1$ equations,

$$\begin{array}{l} x_0 . d^0(u\phi^0) + x_1 . d^1(u\phi^0) + \ldots + x_m . d^m(u\phi^0) = d^{m+1}(u\phi^0) \\ x_0 . d^0(u\phi^1) + x_1 . d^1(u\phi^1) + \ldots + x_m . d^m(u\phi^1) = d^{m+1}(u\phi^1) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_0 . d^0(u\phi^m) + x_1 . d^1(u\phi^m) + \ldots + x_m . d^m(u\phi^m) = d^{m+1}(u\phi^m) , \end{array}$$

and with the help of the said theorem obtains at once

$$x_m = \frac{d\Delta}{\Delta} = d(\log \Delta) ,$$

where Δ stands for

$$\left| d^0(u\phi^0) . d^1(u\phi^1) . \ldots . d^m(u\phi^m) \right| .$$

Next, by using in connection with the same sets of equations the multipliers

$$\pm\phi^m , \quad \mp m\phi^{m-1} \quad , \ldots , \quad \tfrac{1}{2}m(m+1)\phi^2 , \quad -m\phi^1, \quad \phi^0$$

and performing addition, the coefficients of $x_0, x_1, \ldots, x_{m-1}$ are found to vanish, with the result

$$\{x_m m! u(d\phi)^m\} = \tfrac{1}{2}m(m+1).m!u(d\phi)^{m-1}d^2\phi + (m-1)!du.(d\phi^m) ,$$

this being due to the theorem in differentiation * that the expression of $m+1$ terms

$$d^r(u\phi^m) - m\phi . d^r(u\phi^{m-1}) + \tfrac{1}{2}m(m+1)\phi^2 . d^r(u\phi^{m-2}) - \ldots .$$

has the values

$$0 , \quad m!u(d\phi)^m , \quad \tfrac{1}{2}m(m+1).m!u(d\phi^{m-1})d^2\phi + (m+1)!du.(d\phi^m)$$

according as $r<m$, $=m$, or $=m+1$. An alternative value for x_m is thus found, namely,

$$\tfrac{1}{2}m(m+1)\frac{d^2\phi}{d\phi} + (m+1)\frac{du}{u} \quad \text{i.e.} \quad d\log\left[u^{m+1}(d\phi)^{\frac{1}{2}m(m+)}\right] ;$$

and from the two values it follows that

$$\Delta = u^{m+1}(d\phi)^{\frac{1}{2}m(m+1)} \times \text{a constant,}$$

the constant being determined to be $\Sigma[\pm 1^0 2^1 3^2 \ldots (m+1)^m]$, or $1!2!3!\ldots n!$

* Attributed in part to Lexell (1772) and to Arbogast (1800).

by considering the particular case where $u=\phi=e^x$. The final result thus is

$$\begin{vmatrix} u\phi^0 & d(u\phi^0) & \dots\dots & d^m(u\phi_0) \\ u\phi^1 & d(u\phi^1) & \dots & d^m(u\phi^1) \\ \dots & \dots & \dots & \dots \\ u\phi^m & d(u\phi^m) & \dots & d^m(u\phi^m) \end{vmatrix} = 1\,!\,2\,!\,3\,!\dots m\,!\,u^{m+1}(d\phi)^{\frac{1}{2}m(m+1)},$$

which on putting $u=1$ becomes Wronski's result of the year 1816, and therefore a case of Schweins' generalisation of 1825.

A[BADIE, T.] (1852).

[Sur la différentiation des fonctions de fonctions: séries de Burmann, de Lagrange, de Wronski. *Nouv. Annales de Math.*, xi. pp. 376–383: or French translation of Brioschi's *Teorica dei Determinanti*, pp. 182–193.]

By a method similar to Prouhet's, namely, by solving a set of equations in two different ways, Abadie obtains

$$\frac{\dfrac{d^{n-1}}{dh^{n-1}}\left[\theta^{-n}\mathrm{F}'\left(n+h\right)\right]_{h=0}}{1\,.\,2\,.\,3\,.\,.\,.\,n} = \frac{\sum[\pm \mathrm{D}^1\phi\,.\,\mathrm{D}^2\phi^2\,.\,\dots\,.\,\mathrm{D}^{n-1}\phi^{n-1}\,.\,\mathrm{D}^n\mathrm{F}]}{\sum[\pm \mathrm{D}^1\phi\,.\,\mathrm{D}^2\phi^2\,.\,\dots\,.\,\mathrm{D}^{n-1}\phi^{n-1}\,.\,\mathrm{D}^n\phi^n]}$$

where θ and $\mathrm{D}^r\phi^s$ stand for

$$\frac{\phi(x+h)-\phi(x)}{h} \quad\text{and}\quad \frac{d^r}{dx^r}\left\{\phi(x)\right\}^s$$

respectively. Thence, by equating coefficients of $\mathrm{D}^n\mathrm{F}$, Wronski's result

$$\sum[\pm \mathrm{D}^1\phi\,.\,\mathrm{D}^2\phi^2\,.\,\dots\,.\,\mathrm{D}^n\phi^n] = 1\,!\,2\,!\,3\,!\dots n\,!\{\phi'(x)\}^{\frac{1}{2}n(n+1)}$$

is reached; and this in its turn is then used to simplify the result which has just originated it, another of Wronski's formulæ being thus arrived at.

BRIOSCHI, F. (1855).

[Sur une propriété d'un déterminant fonctionnel. *Quart. Journ. of Math.*, i. pp. 365–367: or *Opere Mat.*, v. pp. 389–392].

In later phraseology the property in question is that *if the Wronskian, W say, of* $y_1, y_2, \dots y_n$ *be a known function of the independent variable x, then any one of the y's,* y_r *say,* can be *expressed in terms of the others and* W. Denoting the s^{th} differential-quotient of y_r by $y_r^{(s)}$ we have of course

$$\mathrm{W} \equiv \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1^{(1)} & y_2^{(1)} & \dots & y_n^{(1)} \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

and using "cof" for "cofactor of" we readily see that since

$$\frac{d}{dx}(\operatorname{cof} y_s^{(n-1)}) = -\operatorname{cof} y_s^{(n-2)}$$

we have

$$\frac{d}{dx}\left(\frac{\operatorname{cof} y_s^{(n-1)}}{\operatorname{cof} y_r^{(n-1)}}\right) = \frac{-\operatorname{cof} y_r^{(n-1)} \operatorname{cof} y_s^{(n-2)} + \operatorname{cof} y_s^{(n-1)} \operatorname{cof} y_r^{(n-2)}}{(\operatorname{cof} y_r^{(n-2)})^2}$$

$$= \frac{\mathrm{W}.\operatorname{cof}(y_s^{(n-1)} y_r^{(n-2)})}{(\operatorname{cof} y_r^{(n-1)})^2}.$$

Integration of both sides with respect to x then gives

$$\operatorname{cof} y_s^{(n-1)} = \operatorname{cof} y_r^{(n-1)} \int \frac{\mathrm{W}.\operatorname{cof}(y_s^{(n-1)} y_r^{(n-2)})}{(\operatorname{cof} y_r^{(n-1)})^2} dx\,;$$

and it is seen that for the cofactor of any element in the last row of W except the r^{th} there is an expression in which the function y_r never explicitly occurs. It only remains then to take the well-known identity

$$0 = y_1 \operatorname{cof} y_1^{(n-1)} + y_2 \operatorname{cof} y_2^{(n-1)} + \ \ldots \ + y_r \operatorname{cof} y_r^{(n-1)} + \ \ldots \ + y_n \operatorname{cof} y_n^{(n-1)},$$

write therein for $\operatorname{cof} y_s^{(n-1)}$ the substitute thus provided, and divide both sides by $\operatorname{cof} y_r^{(n-1)}$; for, this being done there results

$$0 = y_1 \mathrm{I}_1 + y_2 \mathrm{I}_2 + \ \ldots \ + y_r + \ \ldots \ + y_n \mathrm{I}_n$$

where I_s stands for the integral above written.

Brioschi then proceeds to establish Malmsten's theorem of 1849.

Christoffel, E. B. (1857).

[Ueber die lineare Abhängigkeit von Functionen einer einzigen Veränderlichen. *Crelle's Journ.*, lv. pp. 281–299.]

The results directly bearing on the subject specified in the title of the paper are summed up in five propositions (pp. 293–294), the one comparable with Puiseux's of 1851 being that *If* $\Sigma \pm \mathrm{f}(\mathrm{x}) \cdot \mathrm{f}'_1(\mathrm{x}) \ldots \mathrm{f}_{\mathrm{n}}^{(\mathrm{n})}(\mathrm{x})$ *vanishes for all values of* x *from* $\mathrm{x} = \mathrm{x}_0$ *to* $\mathrm{x} = \mathrm{x}_1$, *where* $\mathrm{x}_0 < \mathrm{x}_1$, *then the functions* $\mathrm{f}(\mathrm{x}), \mathrm{f}_1(\mathrm{x}), \ldots, \mathrm{f}_{\mathrm{n}}(\mathrm{x})$ *are for those values linearly dependent.* In a concluding section are brought together (pp. 297–299) the properties of the determinants which had been used in reaching the said results. The first theorem is

$$\sum \pm u u_1' u_2'' \ldots u_n^{(n)} = \sum \pm u . \Delta u' . \Delta^2 u'' \ldots \Delta^n u^{(n)} \qquad (1)$$

where

$$\Delta u_\nu^{(\mu)} = u_{\nu+1}^{(\mu)} - u_\nu^{(\mu)}, \quad \Delta^2 u_\nu^{(\mu)} = \Delta u_{\nu+1}^{(\mu)} - \Delta u_\nu^{(\mu)}, \quad \ldots .$$

This, it is stated, can be proved in the same way as the theorem

$$\sum \pm f(m) . f_1(m+1) \ldots f_n(m+n) = \sum \pm f(m) . \Delta f_1(m) \ldots . \Delta^n f_n(m) \qquad (1')$$

given earlier in the paper (p. 293), namely, by evolving the left-hand side from the right-hand side after substituting for the differences in the latter their equivalents obtainable from the identity

$$\Delta^i f_k(m) = f_k(m+i) - if_k(m+i-1) + \frac{i(i-1)}{1\,.\,2}f_k(m+i-2) - \dots .$$

There is next derived the theorem

$$\begin{aligned}&\sum \pm vu.\Delta(vu').\Delta^2(vu'')\dots\Delta^n(vu^{(n)})\\ &\qquad = vv_1v_2\dots v_n\,.\,\sum \pm u.\Delta u'.\Delta^2 u''\dots.\Delta^n u^{(n)}\end{aligned} \qquad (2)$$

and from this again by putting $v_r = 1/u_r$ there is obtained

$$\sum \pm u.\Delta u'.\Delta^2 u''\dots\Delta^n u^{(n)} = uu_1u_2\dots u_n\,.\,\sum \pm \Delta\left(\frac{u'}{u}\right).\Delta^2\left(\frac{u''}{u}\right)\dots\Delta^n\left(\frac{u^{(n)}}{u}\right).$$

Then, by using this last theorem on itself, and continuing in like manner, there is finally reached the result

$$\begin{aligned}&\sum \pm u.\Delta u'.\Delta^2 u''\dots\Delta^n u^{(n)}\\ &= (uu_1\dots u_n)(u^{10}u_1^{10}\dots u_{n-1}^{10})(u^{21}u_1^{21}\dots u_{n-2}^{21})\dots.(u^{n-1,\,n-2}u_1^{n-1,\,n-2})u^{n,\,n-1}\end{aligned} \qquad (3)$$

where

$$u^{\mu 0} = \Delta\frac{u^{(\mu)}}{u}, \quad u^{\mu,1} = \Delta\frac{u^{\mu,0}}{u^{1,0}}, \quad u^{\mu,2} = \Delta\frac{u^{\mu,1}}{u^{2,1}}, \quad \dots .$$

In order to pass from differences to differentials Christoffel then puts

$$u_\mu^{(\nu)} = f_\nu(m\epsilon+\mu\epsilon), \quad v_\mu = \phi(m\epsilon+\mu\epsilon), \quad m\epsilon = x, \quad \epsilon = \partial x,$$

the result obtained from (2) being

$$\sum \pm \phi f\,.\,\frac{\partial\phi f_1}{\partial x}\,.\,\frac{\partial^2\phi f_2}{\partial x^2}\dots\frac{\partial^n\phi f_n}{\partial x^n} = \phi^{n+1}\sum \pm f\,.\,\frac{\partial f_1}{\partial x}\,.\,\frac{\partial^2 f_2}{\partial x^2}\dots\frac{\partial^n f_n}{\partial x^n} \qquad (4)$$

and from (3) being

$$\sum \pm f\,.\,\frac{\partial f_1}{\partial x}\,.\,\frac{\partial^2 f_2}{\partial x^2}\dots\frac{\partial^n f_n}{\partial x^n} = f^{n+1}\,.\,f_{1,0}^n\,.\,f_{2,1}^{n-1}\dots f_{n,n-1} \qquad (5)$$

where

$$f_{\mu,0} = \frac{\partial}{\partial x}\left(\frac{f_\mu}{f}\right), \quad f_{\mu,1} = \frac{\partial}{\partial x}\left(\frac{f_{\mu,0}}{f_{1,0}}\right), \quad f_{\mu,2} = \frac{\partial}{\partial x}\left(\frac{f_{\mu,1}}{f_{2,1}}\right), \quad \dots .$$

There is next given a result dealing with change of variable, namely,

$$\sum \pm f\,.\,\frac{\partial f_1}{\partial x}\,.\,\frac{\partial^2 f_2}{\partial x^2}\dots\frac{\partial^n f_n}{\partial x^n} = \left(\frac{\partial t}{\partial x}\right)^{\frac{1}{2}n(n+1)}\sum \pm f\,.\,\frac{\partial f_1}{\partial t}\,.\,\frac{\partial^2 f_2}{\partial t^2}\dots\frac{\partial^n f_n}{\partial t^n}$$

which is proved like (1) and (1′), the identity used for substitution purposes being now

$$\frac{\partial^r f}{\partial x^r} = \frac{\partial^r f}{\partial t^r}\left(\frac{\partial t}{\partial x}\right)^r + a_1^{(r)}\frac{\partial^{r-1} f}{\partial t^{r-1}} + a_2^{(r)}\frac{\partial^{r-2} f}{\partial t^{r-2}} + \dots .$$

where $a^{(r)}$, $a_2^{(r)}$, . . . do not involve differential-quotients of f.

Lastly, denoting the cofactors of $f^{(n)}, f_1^{(n)}, \ldots, f_n^{(n)}$ in

$$\sum \pm f \cdot \frac{\partial f_1}{\partial x} \cdot \frac{\partial^2 f_2}{\partial x^2} \cdots \frac{\partial^n f_n}{\partial x^n}$$

by $W_0, W_1, \ldots, W_n$ he announces the set of results

$$\begin{aligned}
\sum \pm W_0 W_1' W_2'' \ldots W_n^{(n)} &= W^n, \\
\sum \pm W_0 W_1' W_2'' \ldots W_{n-1}^{(n-1)} &= (-1)^n W^{n-1} f_n, \\
\sum \pm W_0 W_1' W_2'' \ldots W_{n-2}^{(n-2)} &= (-1)^{2n} W^{n-2} . \sum \pm f^{n-1} f'^{n'}, \\
\sum \pm W_0 W_1' W_2'' \ldots W_{n-3}^{(n-3)} &= (-1)^{3n} W^{n-3} . \sum \pm f_{n-2} f'_{n-1} f''_n, \\
&\ldots\ldots\ldots\ldots \\
\sum \pm W_0 W_1' &= (-1)^{(n-1)n} W . \sum \pm f_2 f_3' \ldots f^{(n-2)}, \\
W_0 &= (-1)^{nn} \sum \pm f_1 f_2' f_3'' \ldots f_n^{(n-1)}.
\end{aligned}$$

In regard to these we may note in passing that as $W_0, W_1, \ldots, W_n$ are themselves Wronskians, the name "Compound Wronskian" would not be inappropriate for the determinants on the left. Also, that in form the results bear a resemblance to those included in Jacobi's theorem regarding any minor of the adjugate of a general determinant.

Hesse, O. (1857).

[Ueber die Criterien des Maximums und Minimums der einfachen Integrale. *Crelle's Journ.*, liv. pp. 227–273: or *Werke*, pp. 413–467.]

In the course of his investigation Hesse pauses (p. 249) to enunciate the result which we have numbered (4) in dealing with Christoffel's paper. He also gives Christoffel's fifth result, having arrived at it, however, by a different method, namely, by the repeated use of the immediately preceding result. Thus, in the case of the 3rd order, the first part of his procedure would be

$$\begin{vmatrix} u & u' & u'' \\ v & v' & v'' \\ w & w' & w'' \end{vmatrix} = \begin{vmatrix} u.1 & (u.1)' & (u.1)'' \\ u.\frac{v}{u} & \left(u.\frac{v}{u}\right)' & \left(u.\frac{v}{u}\right)'' \\ u.\frac{w}{u} & \left(u.\frac{w}{u}\right)' & \left(u.\frac{w}{u}\right)'' \end{vmatrix} = u^3 \begin{vmatrix} 1 & (1)' & (1)'' \\ \frac{v}{u} & \left(\frac{v}{u}\right)' & \left(\frac{v}{u}\right)'' \\ \frac{w}{u} & \left(\frac{w}{u}\right)' & \left(\frac{w}{u}\right)'' \end{vmatrix}$$

$$= u^3 \begin{vmatrix} \left(\frac{v}{u}\right)' & \left(\frac{v}{u}\right)'' \\ \left(\frac{w}{u}\right)' & \left(\frac{w}{u}\right)'' \end{vmatrix};$$

and he would next treat this two-line determinant in similar fashion, the result being

$$\sum \pm uv'w'' = u^3 . \left\{ \left(\frac{v}{u} \right)' \right\}^2 . \left\{ \frac{\left(\frac{w}{u} \right)'}{\left(\frac{v}{u} \right)'} \right\}' .$$

MONTFERRIER, A. S. DE (1858).

[ENCYCLOPÉDIE MATHÉMATIQUE, ou exposition complète de toutes les branches des mathématiques d'après les principes . . . . de Hoëne Wronski. Première Partie : Mathématiques Pures. Tomes i.–iv., Paris.*]

Although, from the nature of this work, it cannot be expected to contain fresh results, it would be a mistake to undervalue it, as in the matter of exposition the disciple had more skill than his master. Whether, therefore, as a substitute for, or a commentary on, the original, it deserves attention. It is only in the *third* volume that the "Schin" functions appear, a short general account being given in §§ 1041, 1042 (pp. 423–428), and special instances dealt with under the headings "Les Séries" (§§ 953–960, pp. 267–276) and "La Loi Suprême" (§§ 1006–1023, pp. 358–391).

LIST OF AUTHORS

whose writings are herein dealt with.

* None of the four volumes is dated, but they appeared in 1856–59 : they do not complete the First Part.

(*Issued separately March* 30, 1911.)

XVII.—The Theory of Recurrent Determinants in the Historical Order of Development up to 1860. By Thomas Muir, LL.D.

(MS. received June 13, 1910. Read July 4, 1910.)

Like Wronskians, and for the same reason, *recurrents* were at first dealt with among "Miscellaneous Special Forms": their previous history is thus to be found under Wronski 1812, Scherk 1825, and Schweins 1825 in the chapter so entitled. (*History*, i. pp. 472–474, 478–481.)

The name is quite recent, having been first proposed by E. Pascal in 1907 in a paper published in the *Rendiconti* . . . . *Ist. Lombardo*, (2) xl. pp. 293–305.

Spottiswoode, W. (1853, August).

[Elementary theorems relating to determinants. Second edition, . . . *Crelle's Journ.*, li. pp. 209–271, 328–381.]

In the last chapter or section (§ xi.), which is headed "Miscellaneous Instances of Determinants," Spottiswoode gives (pp. 373–374) an expression for the n^{th} differential-quotient of u/v in terms of the n^{th} and lower differential-quotients of u and of v. The first four cases are

$$\left(\frac{u}{v}\right)' = \begin{vmatrix} v & v' \\ u & u' \end{vmatrix} v^{-2}, \qquad \left(\frac{u}{v}\right)'' = - \begin{vmatrix} . & v & 2v' \\ v & v' & v'' \\ u & u' & u'' \end{vmatrix} v^{-3},$$

$$\left(\frac{u}{v}\right)''' = \begin{vmatrix} . & . & v & 3v' \\ . & v & 2v' & 3v'' \\ v & v' & v'' & v''' \\ u & u & u'' & u''' \end{vmatrix} v^{-4}, \qquad \left(\frac{u}{v}\right)^{\text{iv}} = - \begin{vmatrix} . & . & . & v & 4v' \\ . & . & v & 3v' & 6v'' \\ . & v & 2v' & 3v'' & 4v'' \\ v & v' & v'' & v''' & v^{\text{iv}} \\ u & u' & u'' & u''' & u^{\text{iv}} \end{vmatrix} v^{-5},$$

where the arithmetical coefficients appearing in the elements of the determinants are those of the binomial theorem.

He also notes that any binary quantic may be expressed as a determinant: thus $a_0x^n + a_1x^{n-1}y + x_2a^{n-2}y^2 +$ . . . . is written by him in the form

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_n \\ y & x & . & . & \ldots & . & . \\ . & y & x & . & \ldots & . & . \\ . & . & y & x & \ldots & . & . \\ . & . & . & . & \ldots & . & . \\ . & . & . & . & \ldots & y & x \end{vmatrix}.$$

BRIOSCHI, FR. (1854, 1855, February).

[Sur deux formules relatives à la théorie de la décomposition des fractions rationnelles. *Crelle's Journ.*, l. pp. 239–242: or *Opere Math.*, vol. v. pp. 267–276. See also *Nouv. Annales de Math.*, xiii. p. 352.]

In a review of the second edition of Serret's *Cours d'Algèbre Supérieure*, Terquem, the editor of the *Nouvelles Annales*, takes occasion to enunciate the theorem that if s_r denote the sum of the r^{th} powers of the roots of the equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n = 0\,,$$

then

$$s_r = (-1)^r \begin{vmatrix} a_1 & 1 & . & \dots & . \\ 2a_2 & a_1 & 1 & \dots & . \\ 3a_3 & a_2 & a_1 & \dots & . \\ \dots & \dots & \dots & \dots & \dots \\ ra_r & a_{r-1} & a_{r-2} & \dots & a_1 \end{vmatrix},$$

attributing it to Brioschi, and indicating that it had been arrived at by the solution of a "suite indéfinie d'équations périodiques du premier degré." * The origin of the determinant is thus exactly similar to that of the first determinant of like kind, namely, that occurring in the statement of Wronski's "loi générale des séries."

A few months later we find on p. 240 of vol. l. of *Crelle's Journal* Brioschi himself enunciating and proving with some trouble the theorem that if

$$\phi(x) = c_0 x^n + c_1 x^{n-1} + \dots + c_n\,,$$

and

$$\begin{aligned} f(x) &= a_0 x^n + a_1 x^{n-1} + \dots + a_n\,, \\ &= a_0(x-x_1)(x-x_2)\dots(x-x_n)\,, \end{aligned}$$

then

$$\frac{x_1{}^r\phi(x_1)}{f'(x_1)} + \frac{x_2{}^r\phi(x_2)}{f'(x_2)} + \dots + \frac{x_n\phi(x_n)}{f'(x_n)}$$

$$= (-1)^{r+1}\frac{1}{a_0^{r+2}} \begin{vmatrix} c_0 & a_0 & . & \dots & . \\ c_1 & a_1 & a_0 & \dots & . \\ \dots & \dots & \dots & \dots & \dots \\ c_r & a_r & a_{r-} & \dots & a_0 \\ c_{r+1} & a_{r+1} & a_r & \dots & a_1 \end{vmatrix}.$$

The subject, however, is not pursued further.

* By this, of course, is meant the set of identities known as "Newton's formulæ,"—

$$\begin{aligned} s_1 + a_1 &= 0 \\ s_2 + a_1 s_1 + 2a_2 &= 0 \\ \dots & \dots \end{aligned}$$

(See NEWTON, *Arith. Univ.*, Tom. ii., cap. iii., § 8.)

FAURE (1855, March).

[Théorème sur la somme des puissances semblables des racines. *Nouv. Annales de Math.*, (1) xiv. pp. 94–97: or pp. 172–175 of Combescure's translation of Brioschi's *Teorica dei Determinanti.*]

Having seen Brioschi's result regarding s_r, Faure takes up the subject and succeeds in throwing on it fresh light. His fundamental proposition is not connected with the roots of equations at all, being to the effect that if

$$\begin{aligned}\phi(x) &= c_0x^m + c_1x^{m-1} + \dots + c_m,\\ f(x) &= a_0x^n + a_1x^{n-1} + \dots + a_n\end{aligned}$$

and

$$\phi(x) \div f(x) = \mathrm{A}_0x^{m-n} + \mathrm{A}_1x^{m-n-1} + \mathrm{A}_2x^{m-n-2} + \dots$$

then

$$\mathrm{A}_r = (-1)^r\frac{1}{a_0^{r+1}}\begin{vmatrix} c_0 & a_0 & . & \dots & . \\ c_1 & a_1 & a_0 & \dots & . \\ c_2 & a_2 & a_1 & \dots & . \\ . & . & . & \dots & . \\ c_{r-1} & a_{r-1} & a_{r-2} & \dots & a_0 \\ c_r & a_r & a_{r-1} & \dots & a_1 \end{vmatrix}.$$

Having stated this he recalls the theorem * that if

$$f(x) = a_0(x-x_1)(x-x_2)\dots(x-x_n)$$

we have

$$\psi(x_1) + \psi(x_2) + \dots + \psi(x_n) = \text{coeff. of } x^{-1} \text{ in } \frac{f'(x)\psi(x)}{f(x)}$$

and therefore as a special case

$$x_1{}^r + x_2{}^r + \dots + x_n{}^r = \text{coeff. of } x^{-r-1} \text{ in } \frac{f'(x)}{f(x)}.$$

It is thus seen that to obtain a determinant expression for s_r we have only to make $\phi(x)$ identical with $f'(x)$,—in other words, put $m=n-1$, $c_0=na_0$, $c_1=(n-1)a_1$, $c_2=(n-2)a_2$, ..., and find the coefficient of x^{-r-}. Doing this we obtain

$$(-1)^r\frac{1}{a_0^{r+1}}\begin{vmatrix} na_0 & a_0 & . & \dots & . \\ (n-1)a_1 & a_1 & a_0 & \dots & . \\ (n-2)a_2 & a_2 & a_1 & \dots & . \\ . & . & . & \dots & . \\ (n-r)a_r & a_r & a_{r-1} & \dots & a_1 \end{vmatrix}$$

* Said to be first given by Cauchy in his *Exercices de Math.* for 1826.

which is readily reduced to

$$(-1)^r\frac{1}{a_0{}^r}\begin{vmatrix} a_1 & a_0 & \cdot & \cdots & \cdot \\ 2a_2 & a_1 & a_0 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ ra & a_{r-1} & a_{r-2} & \cdots & a_1 \end{vmatrix},$$

and so agrees with the result obtained from Newton's relations between the a's and s's.

Here Faure leaves the subject, but he might equally easily have established Brioschi's more general result. Instead of specialising by putting $\psi(x)=x^r$ he might have made $\psi(x)=x^r\phi(x)f'(x)$ and so have got

$$\frac{x_1{}^r\phi(x_1)}{f'(x_1)}+\frac{x_2{}^r\phi(x_2)}{f'(x_2)}+\ \cdots\ +\frac{x_n{}^r\phi(x_n)}{f'(x_n)} = \text{coeff. of } x^{-r-1} \text{ in } \frac{\phi(x)}{f(x)}.$$

Bearing in mind that $\phi(x)$ as used by Brioschi was of the n^{th} degree, we have from Faure's fundamental theorem the said coefficient

$$=\mathrm{A}_{r+1}=(-1)^{r+1}\frac{1}{a_0^{r+1}}\begin{vmatrix} c_0 & a_0 & \cdot & \cdots & \cdot \\ c_1 & a_1 & a_0 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ c_r & a_r & a_{r-1} & \cdots & a_0 \\ c_{r+1} & a_{r+1} & a_r & \cdots & a_1 \end{vmatrix}$$

as it ought to be.

BRUNO, FAÀ DI (1855, December).

[Note sur une nouvelle formule du calcul différentiel. *Quart. Journ. of Math.*, i. pp. 359–360: or, with a different title, *Annali di Sci. Mat. e Fis.*, vi. pp. 479–480.

The formula referred to is

$$\frac{\partial^{n+1}}{\partial x^{n+1}}\phi\{\psi(x)\} = \begin{vmatrix} \psi'\phi & n\psi''\phi & \tfrac{1}{2}n(n-1)\psi'''\phi & \cdots & \psi^{(n+1)}\phi \\ -1 & \psi'\phi & (n-1)\psi''\phi & \cdots & \psi^{(n)}\phi \\ \cdot & -1 & \psi'\phi & \cdots & \psi^{(n-1)}\phi \\ \cdot & \cdot & -1 & \cdots & \psi^{(n-2)}\psi \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \psi'\phi \end{vmatrix}$$

where the coefficients in the r^{th} row are those of the expansion of $(a+b)^{n-r+1}$, and where after development of the determinant ϕ^r is to be taken as meaning the r^{th} differential-quotient of ϕ with respect to ψ.*

* An opportunity was here lost by Bruno of noting that a recurrent with the elements in its zero-bordered diagonal all negative has all its terms positive.

BRUNO, FAÀ DI (1856, February).

[Sulle funzioni isobariche. *Annali di Sci. Mat. e Fis.*, vii. pp. 76–89.]

On p. 81 Bruno enunciates, as having been recently ("ultimamente") discovered by him, a theorem which is essentially Faure's, the A_r of Faure being given by Bruno as the coefficient of x^r in the expansion of

$$\frac{c_0 + c_1x + c_2x^2 + \ldots\ldots}{a_0 + a_1x + a_2x^2 + \ldots\ldots}.$$

The reason for A_r being the same in both is evident on putting $m = n = 0$ in Faure's.

ALLEGRET, A. (1857).

[Solutions de quelques problèmes curieux d'arithmétique. *Nouv. Annales de Math.*, xvi. pp. 136–139.]

In the course of Allegret's work, the determinant

$$\begin{vmatrix} 1 & . & . & . & 1 \\ 1 & a_1 & . & . & . \\ 1 & 1 & a_2 & . & . \\ 1 & . & 1 & a_3 & . \\ 1 & . & . & 1 & a_4 \end{vmatrix}$$

appears, which he says

$$= a_1a_2a_3a_4 + \begin{vmatrix} 1 & a_1 & . & . \\ 1 & 1 & a_2 & . \\ 1 & . & 1 & a_3 \\ 1 & . & . & 1 \end{vmatrix} = a_1a_2a_3a_4 - \begin{vmatrix} 1 & . & . & 1 \\ 1 & a_1 & . & . \\ 1 & 1 & a_2 & . \\ 1 & . & 1 & a_3 \end{vmatrix},$$

and, the four-line determinant now reached being similar in form to the original, he concludes that the final expansion of the latter must be

$$a_1a_2a_3a_4 - a_1a_2a_3 + a_1a_2 - a_1 + 1.$$

By passing the first row over the others to occupy the last place the determinant is recognisable as a special case of recurrent, and it is seen that the expansion in terms of the elements of the last row and their cofactors leads at once to Allegret's result.

BRIOSCHI, FR. (1857).

[Solution de la question 350 (Wronski). *Nouv. Annales de Math.*, xvi. pp. 248–249.]

The problem having been set to find what Wronski called the "Aleph" functions* of the roots $x_1, x_2, \ldots, x_n$ of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$$

* WRONSKI, H. Introduction à la Philosophie des Mathématiques . . . (pp. 65, . . .) vi + 270 pp., Paris, 1811.

in terms of the coefficients, Brioschi begins by saying that the $r^{t}{}_{h}$ of the said functions, being the complete homogeneous function of degree r, is the coefficient of z^r in the product

$$(1+x_1z+x_1{}^2z^2+\ldots)(1+x_2z+x_2{}^2z^2+\ldots)\ldots(1+x_nz+x_n{}^2z^2+\ldots)$$

i.e.

$$\frac{1}{(1-x_1z)(1-x_2z)\ldots(1-x_nz)}, \quad \text{or} \quad \frac{1}{\phi(z)} \text{ say.}$$

He thus has

$$\frac{1}{\phi(z)} = 1+\aleph_1z+\aleph_2z^2+\ldots.$$

and therefore by differentiation

$$-\frac{\phi'(z)}{\phi(z)} = \frac{\aleph_1+2\aleph_2z+3\aleph_3z^2+\ldots}{1+\aleph_1z+\aleph_2z^2+\ldots}.$$

But having also by a well-known theorem

$$\begin{aligned}-\frac{\phi'(z)}{\phi(z)} &= \frac{x_1}{1-x_1z}+\frac{x_2}{1-x_2z}+\ldots+\frac{x_n}{1-x_nz}, \\ &= \left.\begin{array}{l} x_1+x_1{}^2z+x_1{}^3z^2+\ldots. \\ +x_2+x_2{}^2z+x_2{}^3z^2+\ldots. \\ +\ldots\ldots\ldots\ldots \\ +x_n+x_n{}^2z+x_n{}^3z^2+\ldots. \end{array}\right\} \\ &= s_1+s_2z+s_3z^2+\ldots.\end{aligned}$$

he deduces

$$s_1+s_2z+s_3z^2+\ldots = \frac{\aleph_1+2\aleph_2z+3\aleph_3z^2+\ldots.}{1+\aleph_1z+\aleph_2z^2+\ldots.}$$

whence by equating like coefficients of z there results

$$\begin{aligned}\aleph_1 &= s_1, \\ 2\aleph_2 &= s_2+\aleph_1s_1, \\ 3\aleph_3 &= s_3+\aleph_1s_2+\aleph_2s_1, \\ &\ldots\ldots\ldots\ldots \\ r\aleph_r &= s_r+\aleph_1s_{r-1}+\ldots.+\aleph_{r-1}s_1.\end{aligned}$$

Multiplying now by $a_{r-1}, a_{r-2}, \ldots, a_0$ and adding, he has, on using Newton's relations between the a's and s's,

$$a_{r-1}\aleph_1+2a_{r-2}\aleph_2+\ldots.+ra_0\aleph_r = -\{ra_r+(r-1)a_{r-1}\aleph_1+\ldots.+a_1\aleph_{r-1}\},$$

whence comes Wronski's relation

$$a_r+a_{r-1}\aleph_1+\ldots+a_0\aleph_r=0;$$

and, on solution of the set of relations obtained from this by putting $r=1, 2, \ldots, r$,

$$\aleph_r=\frac{(-1)^r}{a_0{}^r}\begin{vmatrix} a_1 & a_0 & . & \ldots & . \\ a_2 & a_1 & a_0 & \ldots & . \\ . & . & . & \ldots & . \\ a_{r-1} & a_{r-2} & a_{r-3} & \ldots & a_0 \\ a & a_{r-1} & a_{r-2} & \ldots & a_1 \end{vmatrix}.$$

CATALAN, E. (1857).

[Note sur la question 350 (Wronski). *Nouv. Annales de Math.*, xvi. pp. 416–417.]

Catalan, under the anagrammatic signature of "M. Ange le Tauneac," points out a simplification. Having got as far as

$$\frac{1}{(1-x_1z)(1-x_2z)\ \ldots\ (1-x_nz)} = 1 + \aleph_1 z + \aleph_2 z^2 + \ \ldots$$

he merely draws attention to the fact that the denominator on the left being

$$= (a_0 + a_1 z + a_2 z^2 + \ \ldots\ldots\ + a_n z^n)\frac{1}{a_0}\,,$$

there results

$$a_0 = (1 + \aleph_1 z + \aleph_2 z^2 + \ \ldots\ldots\ )(a_0 + a_1 z + a_2 z^2 + \ \ldots\ + a_n z^n)\,,$$

whence Wronski's set of relations follow at once.

This, of course, is much preferable to Brioschi's procedure. It has to be noted, however, that by taking a roundabout way Brioschi came across the equations connecting the $\aleph$'s and the s's.*

LIST OF AUTHORS

whose writings are herein dealt with.

* After all, it is this set of equations and the two other similar sets that are worth knowing, namely,

Newton's, $a_0 s_r + a_1 s_{r-1} + a_2 s_{r-2} + \ \ldots\ + a_{r-1} s_1 + r a_r = 0\,.$

Wronski's, $a_r + \aleph_1 a_{r-1} + \aleph_2 a_{r-2} + \ \ldots\ + \aleph_r a_0 = 0\,.$

Brioschi's, $s^r + \aleph_1 s_{r-1} + \aleph_2 s_{r-2} + \ \ldots\ \aleph_{r-1} s_1 = r\aleph_r\,.$

(*Issued separately March* 27, 1911.)

XVIII.—The Less Common Special Forms of Determinants up to 1860. By Thomas Muir, LL.D.

(MS. received June 13, 1910. Read July 4, 1910.)

There now only remain for consideration those special forms which, prior to 1860, had not received any noteworthy attention. These will be found to include: (α) permanents, which are touched on by three authors; (β) determinants with the typical element $a_{rs}+b_{rs}i$, which are referred to in four memoirs; (γ) two other forms, which are each dealt with in two papers; and (δ) nine others, which make their appearance only once.

(α) PERMANENTS.

As we have already seen, Cauchy, in his memoir of 1812, widened the ordinary meaning of the term "symmetric function," and was consequently led to call such expressions as

$$a_1b_2+a_2b_1\,,\qquad a_1b_2+a_2b_3+a_3b_1+a_1b_3+a_2b_1+a_3b_2\,,\quad \ldots$$

"fonctions symétriques permanentes," denoting them by $S^2(a_1b_2)$, $S^3(a_1b_2)$, . . .

In the same year, as we have also noted, Binet gave the identities

$$\begin{aligned}\Sigma ab' &= \Sigma a\Sigma b - ab,\\ \Sigma ab'c'' &= \Sigma a\Sigma b\Sigma c + 2\Sigma abc - \Sigma a\Sigma bc - \Sigma b\Sigma ca - \Sigma c\Sigma ab\\ &\ldots\ldots\ldots\ldots\end{aligned}$$

which in Cauchy's notation would have been written

$$\begin{aligned}S^n(a_1b_2) &= S^n(a_1)S^n(b_1) - S^n(a_1b_1),\\ S^n(a_1b_2c_3) &= S^n(a_1)S^n(b_1)S^n(c_1) + 2S^n(a_1b_1c_1) - \ldots\end{aligned}$$

but which, in reality, are due to Waring, who, denoting the sum of the p^{th} powers of $\alpha, \beta, \gamma, \ldots$ by s_p asserted in his *Miscellanea Analytica* of the year 1762 that

$$\begin{aligned}\sum\alpha^p\beta^q &= s_p.s_q - s_{p+q}\,,\\ \sum\alpha^p\beta^q\gamma^r &= s_q.s_q.s_r + 2s_{p+q+r} - \ldots\\ &\ldots\ldots\ldots\ldots\end{aligned}$$

Proofs of Waring's identities were given by Paoli in his *Supplemento agli Elementi di Algebra*, published in 1804 (see Op., ii. § 28), and by Meier Hirsch in his *Sammlung von Aufgaben aus der Theorie der algebraischen Gleichungen*, published in 1809 (see pp. 34–41).

It is only symmetric functions like $S^2(a_1b_2)$, $S^3(a_1b_2c_3)$, $S^4(a_1b_2c_3d_4)$, . . . , whose every term involves the full number of letters, that at the present day are spoken of as *permanents*.

BORCHARDT, C. W. (1855, 5/8).

[Bestimmung der symmetrischen Verbindungen vermittelst ihrer erzeugenden Function. *Monatsb. . . . Akad. d. Wiss. zu Berlin*, 1855, pp. 165–171: or *Crelle's Journ.*, liii. pp. 193–198: or *Gesammelte Werke*, pp. 97–105.]

Having already fully dealt with this paper under the heading *Alternants*, it suffices merely to recall the identity therein given, namely,

$$\sum\left(\frac{1}{t-a}\cdot\frac{1}{t_1-a_1}\cdots\frac{1}{t_n-a_n}\right)\times\sum\left(\pm\frac{1}{t-a}\cdot\frac{1}{t_1-a_1}\cdots\frac{1}{t_n-a_n}\right)$$
$$=\sum\left(\pm\frac{1}{(t-a)^2}\cdot\frac{1}{(t_1-a_1)^2}\cdots\frac{1}{(t_n-a_n)^2}\right),$$

where the first factor on the left differs from the determinant which is its cofactor merely in having the signs of all its terms positive.

JOACHIMSTHAL, F. (1856, September).

[De æquationibus quarti et sexti gradus quæ in theoria linearum et superficierum secundi gradus occurrunt. *Crelle's Journ.*, liii. pp. 149–172.]

Joachimsthal, requiring the use of the so-called "Binet's" identities, devotes section iii. of his paper to them, combining them in one proposition, and showing more or less satisfactorily, after the manner of Meier Hirsch, how the proof of each case can be made dependent on the previous case. His proposition is—*There being* m *rows each of* z *quantities*

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \cdots\cdots & \alpha_z \\ \beta_1 & \beta_2 & \cdots\cdots & \beta_z \\ \cdot & \cdot & \cdots\cdots & \cdot \\ \lambda_1 & \lambda_2 & \cdots\cdots & \lambda_z \\ \mu_1 & \mu_2 & \cdots\cdots & \mu_z \end{array}$$

and z *being not less than* m, *the sum*

$$\sum \alpha_1\beta_2\ldots\mu_m$$

consisting of z(z−1)(z−2)...(z−m+1) *terms can be expressed as an integral function of the sums*

$$\begin{array}{llll} \Sigma\alpha_1, & \Sigma\beta_1, & \ldots\ldots, & \Sigma\mu_1 \\ \Sigma\alpha_1\beta_1, & \Sigma\alpha_1\gamma_1, & \ldots\ldots, & \Sigma\lambda_1\mu_1 \\ \Sigma\alpha_1\beta_1\gamma_1, & \Sigma\alpha_1\beta_1\delta_1, & \ldots\ldots, & \Sigma\kappa_1\lambda_1\mu_1 \\ \cdot & \cdot & \cdots\cdots & \cdot \\ \Sigma\alpha_1\beta_1\gamma_1\ldots\mu_1, & & & \end{array}$$

each consisting of z *terms: further, the said function when* z < m *vanishes identically.*

To prove the proposition when m = 3 he takes the previous case

$$\Sigma\alpha_1\beta_2 = \Sigma\alpha_1\Sigma\beta_1 - \Sigma\alpha_1\beta_1 \qquad (x_2)$$

and multiplies both sides by $\Sigma\gamma_1$, thus obtaining

$$\Sigma\alpha_1\beta_2\gamma_3 + \Sigma\alpha_1\gamma_1\beta_2 + \Sigma\beta_1\gamma_1\alpha_2 = \Sigma\alpha_1\Sigma\beta_1\Sigma\gamma_1 - \Sigma\alpha_1\beta_1.\Sigma\gamma_1,$$

in which the previous case enables him to replace

$$\Sigma\alpha_1\gamma_1\beta_2 \quad \text{by} \quad \Sigma\alpha_1\gamma_1.\Sigma\beta_1 - \Sigma\alpha_1\beta_1\gamma_1,$$

and

$$\Sigma\beta_1\gamma_1\alpha_2 \quad \text{by} \quad \Sigma\beta_1\gamma_1.\Sigma\alpha_1 - \Sigma\alpha_1\beta_1\gamma_1,$$

with the result that

$$\Sigma\alpha_1\beta_2\gamma_3 = \Sigma\alpha_1\Sigma\beta_1\Sigma\gamma_1 - \Sigma\alpha_1\beta_1.\Sigma\gamma_1 - \Sigma\alpha_1\gamma_1.\Sigma\beta_1 - \Sigma\beta_1\gamma_1.\Sigma\alpha_1 + 2\Sigma\alpha_1\beta_1\gamma_1 \qquad (x_3)$$

as desired. Similarly, on multiplying both sides of this by $\Sigma\delta_1$ there is obtained on the left

$$\Sigma\alpha_1\beta_2\gamma_3\delta_4 + \Sigma\alpha_1\delta_1\beta_2\gamma_3 + \Sigma\beta_1\delta_1\alpha_2\gamma_3 + \Sigma\gamma_1\delta_1\alpha_2\beta_3,$$

the last three terms of which have only to be replaced by expressions warranted from (x_3) in order to give the desired equivalent * for $\Sigma\alpha_1\beta_2\gamma_3\delta_4$.

It is then pointed out that when $z = m$ the sum $\Sigma\alpha_1\beta_2\ldots\mu_m$ "tantum a determinante differt, quod omnes ejus termini sunt positivi," and that therefore when $z < m$ the sum vanishes.

The three sections following (iv, v, vi) are occupied, as has been noted elsewhere, with the generalisation of Borchardt's theorem of the previous year.

Cayley, A. (1857, 10/3).

[Note sur les normals d'une conique. *Crelle's Journ.*, lvi. pp. 182–185: or *Collected Math. Papers*, iv. pp. 74–77.]

In dealing with essentially the same geometrical subject as Joachimsthal, Cayley gives, in support of part of his demonstration, the identity

$$\left\{\begin{matrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{matrix}\right\} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} x_1^2 & y_1^2 & z_1^2 \\ x_2^2 & y_2^2 & z_2^2 \\ x_3^2 & y_3^2 & z_3^2 \end{vmatrix} + \begin{vmatrix} y_1z_1 & z_1x_1 & x_1y_1 \\ y_2z_2 & z_2x_2 & x_2y_2 \\ y_3z_3 & z_3x_3 & x_3y_3 \end{vmatrix}$$

where the first factor on the left is what Cauchy denoted by $S^3(x_1y_2z_3)$.

* By an oversight three terms of this are left out by Joachimsthal.

(β) DETERMINANTS WITH COMPLEX ELEMENTS.

Hermite, Ch. (1854).

[Extrait d'une lettre . . . . sur le nombre des racines d'une équation algébrique comprises entre des limites données. *Crelle's Journ.*, lii. pp. 39–51: or *Œuvres,* T. i. pp. 397–414.]

On p. 40 it is pointed out that any determinant whose conjugate elements are of the form $a_{rs}+b_{rs}\sqrt{-1}$, $a_{rs}-b_{rs}\sqrt{-1}$, and whose diagonal elements are therefore of the form a_{rr}, must be real, for the reason that it is not altered in value by changing $\sqrt{-1}$ into $-\sqrt{-1}$.

Hermite, Ch. (1855, August).

[Remarque sur un théorème de M. Cauchy. *Comptes rendus . . . . Acad. des Sci.* (Paris), xli. pp. 181–183: or *Œuvres,* T. i. pp. 479–481.]

The remark concerns the determinant just referred to, and is to the effect that the equation

$$\begin{vmatrix} a_{11}-x & a_{12}+b_{12}i & \dots & a_{1n}+b_{1n}i \\ a_{21}+b_{21}i & a_{22}-x & \dots & a_{2n}+b_{2n}i \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1}+b_{n1}i & a_{n2}+b_{n2}i & \dots & a_{nn}-x \end{vmatrix} = 0,$$

where $a_{rs}=a_{sr}$, $b_{rs}=-b_{sr}$, $i=\sqrt{-1}$, has all its roots real if the a's and b's be real,—a result which degenerates into one previously known (Lagrange, 1773, Cauchy, 1829) when all the b's vanish. No proof is given, but it is stated that one is obtainable by transforming "le déterminant en un autre à éléments réels, d'un nombre double de colonnes et symétrique par rapport à la diagonale." A rule is formulated for determining the number of roots of the equation which lie between two limits. Lastly, it is remarked that the equation arises in connection with the study of forms of the type

$$\begin{array}{cc|c} x+x'i & y+y'i & \\ \hline \alpha_{11} & \alpha_{12}-\beta_{12}i & x-x'i \\ \alpha_{12}+\beta_{12}i & \alpha_{22} & y-y'i, \end{array}$$

that is to say,

$$\left.\begin{array}{r} \alpha_{11}x^2+2\alpha_{12}xy+\alpha_{22}y^2 \\ +\alpha_{11}x'^2+2\alpha_{12}x'y'+\alpha_{22}y'^2 \end{array}\right\} + 2\beta_{12}\,|\,xy' \;.$$

Rubini, R. (1857, May).

[Applicazione della teorica dei determinanti. *Annali di Sc. Mat. e Fis.*, viii. pp. 179–200.]

In treating of determinants with binomial elements Rubini's most interesting example is that in which the element in the $(r,s)^{\text{th}}$ place is $a_{rs}+b_{rs}\sqrt{-1}$. By substitution in his general result he readily obtains the expansion of the determinant in the form $\mathrm{C}+\mathrm{D}\sqrt{-1}$, which is seen to alter into $\mathrm{C}-\mathrm{D}\sqrt{-1}$ on changing the signs of the b's. The product of $|\, a_{1n}+b_{1n}\sqrt{-1}\,|$ and $|\, a_{1n}-b_{1n}\sqrt{-1}\,|$ is consequently expressible as the sum of two squares. His next point is that on using the ordinary multiplication-theorem the same product is got in the form

$$\begin{vmatrix} \alpha_{11} & \alpha_{12}-\beta_{12}\sqrt{-1} & \dots & \alpha_{1n}-\beta_{1n}\sqrt{-1} \\ \alpha_{12}+\beta_{12}\sqrt{-1} & \alpha_{22} & \dots & \alpha_{2n}-\beta_{2n}\sqrt{-1} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{1n}+\beta_{1n}\sqrt{-1} & \alpha_{2n}+\beta_{2n}\sqrt{-1} & \dots & \alpha_{nn} \end{vmatrix},$$

and that a comparison of the two forms may be fruitful of results. When $n=2$, the identity resulting from such comparison is

$$\begin{aligned}&(ad-bc-\alpha\delta+\beta\gamma)^2+(a\delta-b\gamma+\alpha d-\beta c)^2\\ &=(a^2+\alpha^2+b^2+\beta^2)(c^2+\gamma^2+d^2+\delta^2)-(ac+\alpha\gamma+bd+\beta\delta)^2-(a\gamma-\alpha c+b\delta-\beta d)^2,\end{aligned}$$

—a result which gives the product of two sums of four squares as a like sum.

In connection with this special example, however, note should be taken that Hermite in a letter to Jacobi published in 1850 (see *Crelle's Journ.*, xl. p. 297), had pointed out that it followed from the row-by-row multiplication of

$$\begin{vmatrix} a+\alpha\sqrt{-1} & b+\beta\sqrt{-1} \\ -\beta+\beta\sqrt{-1} & a-\alpha\sqrt{-1} \end{vmatrix} \quad \text{by} \quad \begin{vmatrix} -c+\gamma\sqrt{-1} & -d+\delta\sqrt{-1} \\ d+\delta\sqrt{-1} & -c-\gamma\sqrt{-1} \end{vmatrix}$$

Clebsch, A. (1859).

[Theorie der circularpolarisirenden Medien. *Crelle's Journ.*, lvii. pp. 319–358.]

In § 3 (pp. 324–330) Clebsch is led to consider the nature of the roots of the equation dealt with by Hermite in 1855, not knowing, apparently, what the latter had done. Unfortunately the proof given of the reality of the roots is not effected without the use of a set of unessential equations of which the determinant is the eliminant.

The interesting fact is noted that when $n=3$ the equation can be changed into

$$\begin{vmatrix} a_{11}-x & a_{12} & a_{13} \\ a_{12} & a_{22}-x & a_{23} \\ a_{13} & a_{23} & a_{33}-x \end{vmatrix} - \begin{array}{ccc|c} b_{23} & -b_{13} & b_{12} & \\ \hline a_{11}-x & a_{12} & a_{13} & b_{23} \\ a_{12} & a_{22}-x & a_{23} & -b_{13} \\ a_{13} & a_{23} & a_{33}-x & b_{12} \end{array} = 0 .$$

(γ) DETERMINANTS CONNECTED WITH ANHARMONIC RATIOS.

Cayley, A. (1854, February).

[On some integral transformations. *Quart. Journ. of Math.*, i. pp. 4–6: or *Collected Math. Papers,* iii. pp. 1–4.]

This paper opens with two statements in reference to the determinant

$$\begin{vmatrix} 1 & \alpha & \alpha' & \alpha\alpha' \\ 1 & \beta & \beta' & \beta\beta' \\ 1 & \gamma & \gamma & \gamma\gamma' \\ 1 & \delta & \delta & \delta\delta' \end{vmatrix} , \quad \text{or } \Psi \text{ say} .$$

The first is to the effect that the equation

$$\Psi = 0$$

asserts the equality of the anharmonic ratios of $\alpha, \beta, \gamma, \delta$ and $\alpha', \beta', \gamma', \delta'$: and the second that the said equation may also be expressed in the forms *

$$\begin{aligned} \mathrm{K}\alpha &= -\left\{ \gamma\delta(\gamma'-\delta')(\alpha'-\beta') + \delta\beta(\delta'-\beta')(\alpha'-\gamma') + \beta\gamma(\beta'-\gamma')(\alpha'-\delta') \right\}, \\ \mathrm{K}(\alpha-\beta) &= (\delta-\beta)(\beta-\gamma)(\gamma'-\delta')(\alpha'-\beta'), \\ \mathrm{K}(\alpha-\gamma) &= (\beta-\gamma)(\gamma-\delta)(\delta'-\beta')(\alpha'-\gamma'), \\ \mathrm{K}(\alpha-\delta) &= (\gamma-\delta)(\delta-\beta)(\beta'-\gamma')(\alpha'-\delta'), \end{aligned}$$

---

* These may be established as follows. By separating the terms of K which involve α from those which do not, we see that

$$\mathrm{K} = -\begin{vmatrix} . & 1 & . & \alpha' \\ 1 & \beta & \beta' & \beta\beta' \\ 1 & \gamma & \gamma' & \gamma\gamma' \\ 1 & \delta & \delta' & \delta\delta' \end{vmatrix} ,$$

a determinant differing from Ψ in the first row only, and consequently on multiplying by α and adding we obtain

$$\Psi + \mathrm{K}\alpha = \begin{vmatrix} 1 & . & \alpha' & . \\ 1 & \beta & \beta' & \beta\beta' \\ 1 & \gamma & \gamma' & \gamma\gamma' \\ 1 & \delta & \delta' & \delta\delta' \end{vmatrix} = \begin{vmatrix} \beta & \beta'-\alpha' & \beta\beta' \\ \gamma & \gamma'-\alpha' & \gamma\gamma' \\ \delta & \delta'-\alpha' & \delta\delta' \end{vmatrix} = \ldots .$$

if we use K to stand for

$$\beta(\gamma'-\delta')(\alpha'-\beta') + \gamma(\delta'-\beta')(\alpha'-\gamma') + \delta(\beta'-\gamma')(\alpha'-\delta').$$

Accepting the first statement, and knowing that the equality referred to is

$$\frac{(\gamma-\alpha)(\beta-\delta)}{(\alpha-\beta)(\gamma-\delta)} = \frac{(\gamma'-\alpha')(\beta'-\delta')}{(\alpha'-\beta')(\gamma'-\delta')},$$

we readily make the deduction that

$$\Psi = (\gamma-\alpha)(\beta-\delta)(\alpha'-\beta')(\gamma'-\delta') - (\alpha-\beta)(\gamma-\delta)(\gamma'-\alpha')(\beta'-\delta').$$

By accepting the second statement, like conclusions may be drawn; for then the elimination of K from any two of the equations involving it must of course lead us back to some form or other of the equation with which we started. Thus, multiplying K by a and using the first of the four derived equations we obtain by subtraction

$$0 = (\alpha\beta+\gamma\delta)(\gamma'-\delta')(\alpha'-\beta') - (\alpha\gamma+\beta\delta)(\beta'-\delta')(\alpha'-\gamma') + (\alpha\delta+\beta\gamma)(\beta'-\gamma')(\alpha'-\delta'),$$

whence we deduce in the same manner as before that the expression * on the right when changed in sign is equal to Ψ; and using any pair of the remaining equations we reach either the form of Ψ previously obtained or one of the two forms derivable from it by means of the simultaneous circular substitutions

$$\begin{aligned}\beta,\gamma,\delta &= \gamma,\delta,\beta,\\ \beta',\gamma',\delta' &= \gamma',\delta',\beta'.\end{aligned}$$

Cayley, A. (1858, February).

[A fifth memoir on quantics. *Philos. Trans. R. Soc. London*, cxlviii. pp. 429–460: or *Collected Math. Papers*, ii. pp. 527–557.]

The second part (§§ 96–114) of the memoir deals with two or more quadrics, and forming part of it is a digression (§§ 105–114) on involution

Similarly,

$$\Psi+\mathrm{K}(\alpha-\beta) = \begin{vmatrix} 1 & \beta & \alpha' & \beta\alpha' \\ 1 & \beta & \beta' & \beta\beta' \\ 1 & \gamma & \gamma' & \gamma\gamma' \\ 1 & \delta & \delta' & \delta\delta' \end{vmatrix} = \begin{vmatrix} 1 & . & \alpha' & . \\ 1 & . & \beta' & . \\ 1 & \gamma-\beta & \gamma' & (\gamma-\beta)\gamma' \\ 1 & \delta-\beta & \delta' & (\delta-\beta)\delta' \end{vmatrix} = \dots .$$

and so of the others.

In doing this we learn, too, that

$$\Psi+\mathrm{K}\alpha=\Psi_{\alpha=0}, \quad \Psi+\mathrm{K}(\alpha-\beta)=\Psi_{\alpha=\beta}, \quad \dots .$$

* This second form of Ψ may be got directly from the determinant by expanding in terms of the two-line minors formable from the first and third columns, and the minors complementary to these. Of course we also have

$$\Psi = (\alpha'\beta'+\gamma'\delta')(\alpha-\beta)(\gamma-\delta) - (\alpha'\gamma'+\beta'\delta')(\alpha-\gamma)(\beta-\delta) + (\alpha'\delta'+\beta'\gamma')(\alpha-\delta)(\beta-\gamma).$$

and the anharmonic relation. The determinant Ψ thus again makes its appearance, and associated with it is the determinant

$$\begin{vmatrix} 1 & \alpha+\alpha' & \alpha\alpha' \\ 1 & \beta+\beta' & \beta\beta' \\ 1 & \gamma+\gamma' & \gamma\gamma' \end{vmatrix} \quad \text{or } \Upsilon \text{ say},$$

for the reason that, when $\delta=\alpha'$ and $\delta'=\alpha$, Ψ is readily shown to be equal to

$$(\alpha'-\alpha)\Upsilon.$$

To obtain the required non-determinant forms of the two the multiplication-theorem is used with pleasing effect. In the first place Υ is multiplied row-wise by

$$\begin{vmatrix} u^2 & -u & 1 \\ v^2 & -v & 1 \\ w^2 & -w & 1 \end{vmatrix},$$

the result, being of course,

$$\Upsilon.(w-v)(w-u)(v-u) = \begin{vmatrix} (u-\alpha)(u-\alpha') & (v-\alpha)(v-\alpha') & (w-\alpha)(w-\alpha') \\ (u-\beta)(u-\beta') & (v-\beta)(v-\beta') & (w-\beta)(w-\beta') \\ (u-\gamma)(u-\gamma') & (v-\gamma)(v-\gamma') & (w-\gamma)(w-\gamma') \end{vmatrix}.$$

In this Cayley then puts $u=\alpha$, $v=\alpha'$, obtaining

$$\Upsilon.(\alpha'-\alpha) = (\alpha-\beta)(\alpha-\beta')(\alpha'-\gamma)(\alpha'-\gamma') - (\alpha'-\beta)(\alpha'-\beta')(\alpha-\gamma)(\alpha-\gamma'),$$

and putting $u, v, w=\alpha, \beta, \gamma$ obtains

$$\Upsilon = (\alpha-\beta')(\beta-\gamma')(\gamma-\alpha') - (\alpha-\gamma')(\beta-\alpha')(\gamma-\beta'),$$

—a result known to Hesse in 1849 (see *Crelle's Journ.*, l. p. 265).

In the next place (§ 114) Ψ is multiplied by the similar determinant

$$\begin{vmatrix} ss' & -s' & -s & 1 \\ tt' & -t' & -t & 1 \\ uu' & -u' & -u & 1 \\ vv' & -v' & -v & 1 \end{vmatrix}, \quad \text{or } \Psi' \text{ say},$$

the result being

$$\Psi\Psi' = \begin{vmatrix} (s-\alpha)(s'-\alpha') & (t-\alpha)(t'-\alpha') & (u-\alpha)(u'-\alpha') & (v-\alpha)(v'-\alpha') \\ (s-\beta)(s'-\beta') & (t-\beta)(t'-\beta') & (u-\beta)(u'-\beta') & (v-\beta)(v'-\beta') \\ (s-\gamma)(s'-\gamma') & (t-\gamma)(t'-\gamma') & (u-\gamma)(u'-\gamma') & (u-\gamma)(v'-\gamma') \\ (s-\delta)(s'-\delta') & (t-\delta)(t'-\delta') & (u-\delta)(u'-\delta') & (v-\delta)(v'-\delta') \end{vmatrix},$$

so that on putting

$$\left.\begin{matrix} s, t, u, v \\ s', t', u', v' \end{matrix}\right\} = \left\{\begin{matrix} \alpha, \beta, \gamma, \delta \\ \beta', \alpha', \delta', \gamma' \end{matrix}\right.$$

the product becomes

$$\begin{vmatrix} \cdot & \cdot & (\gamma-\alpha)(\delta'-\alpha') & (\delta-\alpha)(\gamma'-\alpha') \\ \cdot & \cdot & (\gamma-\beta)(\delta'-\beta') & (\delta-\beta)(\gamma'-\beta') \\ (\alpha-\gamma)(\beta'-\gamma') & (\beta-\gamma)(\alpha'-\gamma') & \cdot & \cdot \\ (\alpha-\delta)(\beta'-\delta') & (\beta-\delta)(\alpha'-\delta') & \cdot & \cdot \end{vmatrix}$$

and there is obtained

$$\begin{vmatrix} 1 & \alpha & \alpha' & \alpha\alpha' \\ 1 & \beta & \beta' & \beta\beta' \\ 1 & \gamma & \gamma' & \gamma\gamma' \\ 1 & \delta & \delta' & \delta\delta' \end{vmatrix} \cdot \begin{vmatrix} 1 & \alpha & \beta' & \alpha\beta' \\ 1 & \beta & \alpha' & \beta\alpha' \\ 1 & \gamma & \delta' & \gamma\delta' \\ & \delta & \gamma' & \delta\gamma' \end{vmatrix} = \left\{ \begin{array}{c} (\alpha-\gamma)(\beta-\delta)(\alpha'-\delta')(\beta'-\gamma') \\ -(\alpha-\delta)(\beta-\gamma)(\alpha'-\gamma')(\beta'-\delta') \end{array} \right\}^2 .$$

From this Cayley concludes (1) that Ψ is not changed by the transposition

$$\begin{pmatrix} \alpha' & \gamma' \\ \beta' & \delta' \end{pmatrix},$$

and (2) that either form equals

$$(\alpha-\gamma)(\beta-\delta)(\alpha'-\delta')(\beta'-\gamma') - (\alpha-\delta)(\beta-\gamma)(\alpha'-\gamma')(\beta'-\delta') .$$

Sardi, C. (1864).

[Quistione 39. *Giornale di Mat.*, ii. p. 256, pp. 315–316.]

On the determinant Ψ Sardi performs the operation which we may indicate by

$$\mathrm{col}_4 - \beta\,\mathrm{col}_3 - \delta'\,\mathrm{col}_2 + \beta\delta'\,\mathrm{col}_1 ,$$

thus obtaining

$$\begin{vmatrix} 1 & \alpha & \alpha' & (\alpha-\beta)(\alpha'-\delta') \\ 1 & \beta & \beta' & \cdot \\ 1 & \gamma & \gamma' & (\gamma-\beta)(\gamma'-\delta') \\ 1 & \delta & \delta' & \cdot \end{vmatrix}$$

in which the cofactor of $(\alpha-\beta)(\alpha'-\delta')$ is

$$-\begin{vmatrix} 1 & \beta & \beta' \\ 1 & \gamma & \gamma' \\ 1 & \delta & \delta' \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} \beta-\gamma & \beta'-\gamma' \\ \delta-\gamma & \delta'-\gamma' \end{vmatrix}$$

and the cofactor of $(\gamma-\beta)(\gamma'-\delta')$ is

$$-\begin{vmatrix} 1 & \alpha & \alpha' \\ 1 & \beta & \beta' \\ 1 & \delta & \delta' \end{vmatrix} \quad \text{or} \quad -\begin{vmatrix} \beta-\alpha & \beta'-\alpha' \\ \delta-\alpha & \delta'-\alpha' \end{vmatrix}$$

where, be it observed, it is the rows $1, \beta, \beta'$ and $1, \delta, \delta'$ that are diminished on both occasions.

There is thus obtained

$$\begin{aligned}\Psi &= (\alpha-\beta)(\alpha'-\delta')\left\{(\beta-\gamma)(\delta'-\gamma') - (\delta-\gamma)(\beta'-\gamma')\right\}\\ &\quad -(\gamma-\beta)(\gamma'-\delta')\left\{(\beta-\alpha)(\delta'-\alpha') - (\beta'-\alpha')(\delta-\alpha)\right\},\\ &= (\alpha-\beta)(\alpha'-\delta')(\gamma-\delta)(\beta'-\gamma') + (\gamma-\beta)(\gamma'-\delta')(\beta'-\alpha')(\delta-\alpha),\\ &= -(\alpha-\beta)(\gamma-\delta)(\alpha'-\delta')(\gamma'-\beta') + (\alpha'-\beta')(\gamma'-\delta')(\alpha-\delta)(\gamma-\beta).^*\end{aligned}$$

(γ_2) SYLVESTER'S UNISIGNANT.

SYLVESTER, J. J. (1855, April).

[On the change of systems of independent variables. *Quarterly Journ. of Math.*, i. pp. 42–56 : or *Collected Math. Papers*, ii. pp. 65–85.]

In the course of Sylvester's investigations a peculiar three-line determinant turns up, which he considers deserving of attention on its own account, namely, the determinant

$$\begin{vmatrix} a_1+a_2+a_3 & -a_2 & -a_3 \\ -b_1 & b_1+b_2+b_3 & -b_3 \\ -c_1 & -c_2 & c_1+c_2+c_3 \end{vmatrix},$$

the final expansion of which consists of 16 terms, all positive. To obtain this expansion a "simple rule" is laid down, namely, to substitute

$$\left.\begin{matrix} a & a_b & a_c \\ b_a & b & b_c \\ c_a & c_b & c \end{matrix}\right\} \text{ for } \left\{\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}\right.$$

and then multiply together the elements of the diagonal, rejecting every term such as $a_b b_a$, $a_b b_c c_a$, ... in which the letters form a cycle. Two examples are given, but no justification of the "rule" is vouchsafed. The examples are—

$$\begin{vmatrix} a+a_b+a_c & -a_b & -a_c \\ -b_a & b+b_c+b_a & -b_c \\ -c_a & -c_b & c+c_a+c_b \end{vmatrix} = \begin{aligned}[t] &abc + (c_a+c_b)ab + (a_b+a_c)bc + (b_c+b_a)ca \\ &+ a(b_a c_a + b_a c_b + c_a b_c) + b(c_b a_b + c_b a_c + a_b c_a) \\ &+ c(a_c b_c + a_c b_a + b_c a_b),\end{aligned}$$

---

* It will be seen that merely by accident the three ways in which Ψ can be expressed as the difference of two products have turned up in succession, and that they may be written

$$|PQ'|, |QR'|, |PR'|$$

if we put

$$\begin{aligned}(\alpha-\beta)(\gamma-\delta) &= P,\\ (\alpha-\gamma)(\beta-\delta) &= Q,\\ (\alpha-\delta)(\beta-\gamma) &= R.\end{aligned}$$

$$\begin{vmatrix} a+a_b+a_c+a_d & -b_a & -c_a & -d_a \\ -a_b & b+b_c+b_d+b_a & -c_b & -d_b \\ -a_c & -b_c & c+c_d+c_a+c_b & -d_c \\ -a_d & -b_d & -c_d & d+d_a+d_b+d_c \end{vmatrix}$$
$$= abcd + \sum abc(d_a+d_b+d_c) + \sum ab(c_d d_a + \ldots\ldots) + \sum a(b_c c_d d_a + \ldots\ldots).$$

The arrangement of the two developments almost raises doubts as to whether the "rule" had been utilised, suggesting indeed that in the latter instance, for example, the cofactor of ab was first obtained in the form

$$\begin{vmatrix} c_d+c_a+c_b & -d_c \\ -c_d & d_a+d_b+d_c \end{vmatrix},$$

and the cofactor of a in the form of a similar determinant of the third order. The "rule," however, is noted by Cayley in *Crelle's Journal*, lii. (1855), p. 279.

The number of terms is $(n+1)^{n-1}$, n being the order-number of the determinant. This Sylvester obtains by putting $a, a_b, a_c, \ldots$ all equal to 1. It will be observed that from the form of the development we have

$$\begin{aligned} 1+3\cdot2+3\cdot3 &= 4^2 \\ 1+4\cdot3+6\cdot8+4\cdot16 &= 5^3 \\ 1+5\cdot4+10\cdot15+10\cdot50+5\cdot125 &= 6^4 \\ \ldots\ldots\ldots\ldots & \end{aligned}$$

BORCHARDT, C. W. (1859, May).

[Ueber eine der Interpolation entsprechende Darstellung der Eliminations-Resultante. *Crelle's Journ.*, lvii. pp. 111–121: or *Monatsb. d. Akad. d. Wiss.* (Berlin), pp. 376–388: also abstract in *Annali di Mat.*, ii. pp. 262–264.]

The representation in question is in terms of the values which the two functions $\phi(x)$ and $\psi(x)$, both of the n^{th} degree, assume for the values $a_0, a_1, a_2, \ldots, a_n$ of x. It emerges as a special determinant of the form

$$\begin{vmatrix} \sigma_1-(11) & -(12) & \ldots\ldots & -(1n) \\ -(21) & \sigma_2-(22) & & -(2n) \\ \ldots & \ldots & \ldots & \ldots \\ -(n1) & -(n2) & \ldots\ldots & \sigma_n-(nn) \end{vmatrix}$$

where $\sigma_r=(r0)+(r1)+\ldots\ldots+(rn)$ and $(rs)=(sr)$,—a form which we readily recognise to be the axisymmetric case of Sylvester's determinant of the year 1855. To the consideration of it Borchardt, supposing it to be new, devotes the last six pages of his paper.

Denoting it by $(0, 1, 2, \ldots, n)$, since it is a function of the $\frac{1}{2}n(n+1)$ quantities,

$$\begin{array}{cccc} (01) & (02) & \ldots & (0n) \\ & (12) & \ldots & (1n) \\ & & \cdots\cdots & \\ & & & (n-1, n), \end{array}$$

he first shows with some prolixity that the cofactor of (01) in it is $(\overline{0+1}, 2, 3, \ldots, n)$, next that the cofactor of $(01)(02)\ldots(0i)$ is

$$(\overline{0+1+\ldots+i}, i+1, i+2, \ldots, n),$$

and finally that

$$\begin{aligned} (0, 1, 2, \ldots, n) = & \sum(01)(1, 2, \ldots, n) \\ & + \sum(01)(02)(\overline{1+2}, 3, \ldots, n) \\ & + \ldots\ldots\ldots\ldots \\ & + \sum(01)(02)\ldots(0k)(\overline{1+2+\ldots+k}, k+1, \ldots, n) \\ & + \ldots\ldots\ldots\ldots \\ & + (01(02)\ldots(0n). \end{aligned}$$

Resuming consideration, but proceeding on a different tack, he arrives at Sylvester's "rule," namely, that $(0, 1, 2, \ldots, n)$ is "gleich der Summe aller nicht-cyclischen Producte, die aus je n jener $\frac{1}{2}n(n+1)$ Elemente $(i\ k)$ gebildet werden können." Unlike Sylvester, however, he is careful to give a justification of it based on four observed facts, namely, (1) that $(0, 1, 2, \ldots, n)$ is unaltered by interchanging any two of the umbræ; (2) that the coefficient of the term $(01)(02)\ldots(0n)$ is 1; (3) that none of the terms is free of the umbra 0; (4) that, as already mentioned, the cofactor of (01) is $(\overline{0+1}, 2, \ldots, n)$.* As the proof, which extends to two pages (pp. 119–120), applies only to the case of axisymmetry, it need not be given.

Lastly, the number of terms in the development of $(0, 1, 2, \ldots, n)$ is investigated, the result obtained agreeing with Sylvester's.

We may note for ourselves in passing that the first three of the basic facts of the proof are, like the last, most readily appreciated by observing the determinant form, the case where $n=3$, namely,

$$\begin{vmatrix} 10+12+13 & -12 & -13 \\ -21 & 20+21+23 & -23 \\ -31 & -32 & 30+31+32 \end{vmatrix}$$

being amply sufficient. Thus, increasing any column by all the others, and thereafter increasing the corresponding row by all the other rows, we

* As (01) occurs only in the element $\sigma_1 - (11)$, its cofactor is the primary minor obtained by deleting the first row and the first column, and this is seen to be $(\overline{0+1}, 2, \ldots, n)$ by definition.

obtain the first result, learning at the same time that it only holds when axisymmetry exists; the second is self-evident; and the third follows from the fact that the aggregate of the terms which are free of 0, being got by deleting 10, 20, 30, is expressible as a vanishing determinant.

(δ) MISCELLANEOUS SPECIAL FORMS.

Cayley, A. (1845).

[On certain results relating to quaternions. *Philos. Magazine*, xxvi. pp. 141–145: or *Collected Math. Papers*, i. pp. 123–126.]

Assuming that in each term of the development of a determinant the elements are arranged in the order of the columns from which they are taken, Cayley points out that if the elements be quaternions

$$\begin{vmatrix} \pi & \pi' \\ \pi & \pi' \end{vmatrix} = \pi\pi' - \pi\pi' = 0,$$

but

$$\begin{vmatrix} \pi & \pi \\ \pi' & \pi' \end{vmatrix} = \pi\pi' - \pi'\pi \neq 0.$$

He is thus led to inquire what the non-zero value is in this latter case and in other similar cases. Taking

$$\begin{aligned} \pi &= x + iy + jz + kw, \\ \pi' &= x' + iy' + jz' + kw', \\ \pi'' &= x'' + iy'' + jz'' + kw'', \end{aligned}$$

he says it is easy to show * that

$$\begin{vmatrix} \pi & \pi \\ \pi' & \pi' \end{vmatrix} = -2\begin{vmatrix} i & j & k \\ y & z & w \\ y' & z' & w' \end{vmatrix}, \qquad \begin{vmatrix} \pi & \pi & \pi \\ \pi' & \pi' & \pi' \\ \pi'' & \pi'' & \pi'' \end{vmatrix} = -2\begin{vmatrix} 3 & i & j & k \\ x & y & z & w \\ x' & y' & z' & w' \\ x'' & y'' & z'' & w'' \end{vmatrix},$$

* Probably the easiest way is to express the determinant as a sum of determinants with monomial elements. In the case of the third order the number of such determinants is 64, of which 40 vanish, the sum remaining being

$$\begin{aligned} &123+132+213+231+312+321 \\ &+124+142+\ldots. \\ &+134+143+\ldots. \\ &+234+243+\ldots. \end{aligned}$$

where rst stands for the determinant whose columns are in order, the r^{th}, s^{th}, t^{th} columns of the array

$$\begin{matrix} x & iy & jz & kw \\ x' & iy' & jz' & kw' \\ x'' & iy'' & jz'' & kw'' \end{matrix}$$

but that for higher orders the result is 0. He next notes the identity

$$\begin{vmatrix} \phi & \chi \\ \phi' & \chi' \end{vmatrix} + \begin{vmatrix} \chi & \phi \\ \chi' & \phi' \end{vmatrix} = \begin{vmatrix} \phi & \phi \\ \chi' & \chi' \end{vmatrix} - \begin{vmatrix} \phi' & \phi' \\ \chi & \chi \end{vmatrix},$$

adding "etc. for determinants of any order";* and then from this set of identities and the previous set he concludes that *if any four adjacent columns of a quaternion determinant be transposed in every possible manner, the sum of the determinants thus obtained vanishes*—a property which, he says, is much less simple than the analogous one for the rows, this last being the same that holds in the case of determinants with ordinary elements. Lastly, he gives the important warning that the eliminant of

$$\left.\begin{aligned} \pi\Pi + \phi\Phi = 0 \\ \pi'\Pi + \phi'\Phi = 0 \end{aligned}\right\}$$

is neither $\pi\phi' - \pi'\phi$ nor $\pi\phi' - \phi\pi'$, but

$$\pi^{-1}\phi - \pi'^{-1}\phi'.$$

TISSOT, A. (1852, May).

[Sur un déterminant d'intégrales définies. *Journ.* (*de Liouville*) *de Math.*, xvii. pp. 177–185.]

The subject here is the evaluation of the determinant of the $(n+1)^{\text{th}}$ order whose $(r, s)^{\text{th}}$ element is

$$\int_{a_{s-1}}^{a_s} e^{-x} \frac{x^r dx}{(\phi)_{s-1}(x)},$$

where

$$\phi_i(x) = (x-a_0)^{m_0}(x-a_1)^{m_1} \ldots . (x-a_i)^{m_i}(a_{i+1}-x)^{m_{i+1}} \ldots (a_n - x)^{m_n},$$

---

and where therefore

$$123 + 132 + \ldots = |xy'z''| . (ij + ij + ij - ij - ij + ij) = 2k\,|xy'z''|,$$

and so on. The multiplication table of i, j, k, it may be recalled, is

$$\begin{pmatrix} ii & ij & ik \\ ji & jj & jk \\ ki & kj & kk \end{pmatrix} = \begin{pmatrix} -l & k & -j \\ -k & -l & i \\ j & -i & -l \end{pmatrix}.$$

* Very probably the next case is the identity

$$\begin{vmatrix} \phi & \chi & \psi \\ \phi' & \chi' & \psi' \\ \phi'' & \chi'' & \psi'' \end{vmatrix} + \begin{vmatrix} \phi & \psi & \chi \\ \phi' & \psi' & \chi' \\ \phi'' & \psi'' & \chi'' \end{vmatrix} + \begin{vmatrix} \chi & \phi & \psi \\ \chi' & \phi' & \psi' \\ \chi'' & \phi'' & \psi'' \end{vmatrix} + \ldots + \begin{vmatrix} \psi & \chi & \phi \\ \psi' & \chi' & \phi' \\ \psi'' & \chi'' & \phi'' \end{vmatrix}$$

$$= \begin{vmatrix} \phi & \phi & \phi \\ \chi' & \chi' & \chi' \\ \psi'' & \psi'' & \psi'' \end{vmatrix} - \begin{vmatrix} \phi & \phi & \phi \\ \chi'' & \chi'' & \chi'' \\ \psi' & \psi' & \psi' \end{vmatrix} - \begin{vmatrix} \phi' & \phi' & \phi' \\ \chi & \chi & \chi \\ \psi'' & \psi'' & \psi'' \end{vmatrix} + \ldots - \begin{vmatrix} \phi'' & \phi'' & \phi'' \\ \chi' & \chi' & \chi' \\ \psi & \psi & \psi \end{vmatrix},$$

where, as in the other cases, the r^{th} determinant on the left is equal to the aggregate of the r^{th} terms of all the determinants on the right.

the m's are all less than 1, and $a_{n+1}=\infty$. The simplest example is

$$\begin{vmatrix} \int_{a_0}^{a_1} e^{-x}\frac{dx}{\phi_0(x)} & \int_{a_1}^{\infty} e^{-x}\frac{dx}{\phi_1(x)} \\ \int_{a_0}^{a_1} e^{-x}\frac{xdx}{\phi_0(x)} & \int_{a_1}^{\infty} e^{-x}\frac{xdx}{\phi_1(x)} \end{vmatrix} = \Gamma(1-m_0)\,.\,\Gamma(1-m_1)\,.\,(a_1-a_0)^{1-m_0-m_1}\,.\,e^{-a_0-a_1}.$$

In establishing the result, use is made of the fact that the determinant is expressible also as a multiple integral: for example, the two-line determinant just written is equal to

$$\int_{a_0}^{a_1}\int_{a_1}^{\infty} \frac{e^{-x-x_1}(x_1-x)dxdx_1}{(a-x_0)^{m_0}(a_1-x)^{m_1}(x_1-a_0)^{m_0}(x_1-a_1)^{m_1}}.$$

Bazin, [H.] (1854 July).

[Démonstration d'un théorème sur les déterminants. *Journ.* (*de Liouville*) *de Math.*, xix. pp. 209–214.]

The theorem in question is to the effect that if there be two n-by-m arrays R, R′ with integral elements, and such that the ratio of any n-line minor of R to the corresponding minor of R′ is constant and integral, and if the n-line minors of R have 1 for their highest common factor, then it is possible to find a determinant S of the n^{th} order with integral elements so that the product of S by any n-line minor of R′ shall equal the corresponding minor of R. For example, it being given that

$$k\begin{Vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{Vmatrix} = \begin{Vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{Vmatrix}$$

where all the letters denote integers, and that the highest common factor of $|b_1c_2|$, $|b_1c_3|$, $|b_2c_3|$ is 1, four integers α, β, γ, δ can be found such that

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} . \begin{Vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{Vmatrix} = \begin{Vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{Vmatrix}.$$

Brioschi, F. (1855).

[Additions à l'article No. 15, page 239 de ce tome. *Crelle's Journ.*, L. pp. 318–321: or *Opere mat.*, v. pp. 271–276.]

The determinant here (pp. 320–321) dealt with is, for shortness' sake, taken to be of the 4th order, namely, $|m_1\ \delta_2\ b_3\ c_4|$, in which δ_1, δ_2, δ_3, δ_4 stand for

$$\begin{gathered} ax_1+ex_2+fx_3+gx_4, \\ ex_1+bx_2+hx_3+kx_4, \\ fx_1+hx_2+cx_3+lx_4, \\ gx_1+kx_2+lx_3+dx_4, \end{gathered}$$

and where the δ's, b's, c's are such that

$$\begin{aligned} \delta_1x_1+\delta_2x_2+\delta_3x_3+\delta_4x_4 &= 0\,, \\ b_1x_1+b_2x_2+b_3x_3+b_4x_4 &= 0\,, \\ c_1x_1+c_2x_2+c_3x_3+c_4x_4 &= 0\,. \end{aligned}$$

The cofactor of m_r in $|m_1\ \delta_2\ b_3\ c_4|$ being denoted by M_r, we see that, as an example,

$$\mathrm{M}_4{}^2 = - \begin{vmatrix} a & e & f & \delta_1 & b_1 & c_1 \\ e & b & h & \delta_2 & b_2 & c_2 \\ f & h & c & \delta_3 & b_3 & c_3 \\ \delta_1 & \delta_2 & \delta_3 & \cdot & \cdot & \cdot \\ b_1 & b_2 & b_3 & \cdot & \cdot & \cdot \\ c_1 & c_2 & c_3 & \cdot & \cdot & \cdot \end{vmatrix}$$

which after performance of the operations

$$\begin{aligned} &\mathrm{col}_4 - x_1\mathrm{col}_1 - x_2\mathrm{col}_2 - x_3\mathrm{col}_3\,, \\ &\mathrm{row}_4 - x_1\mathrm{row}_1 - x_2\mathrm{row}_2 - x_3\mathrm{row}_3\,, \end{aligned}$$

becomes

$$\mathrm{M}_4{}^2 = - \begin{vmatrix} a & e & f & g & b_1 & c_1 \\ e & b & h & k & b_2 & c_2 \\ f & h & c & l & b_3 & c_3 \\ g & k & l & d & b_4 & c_4 \\ b_1 & b_2 & b_3 & b_4 & \cdot & \cdot \\ c_1 & c_2 & c_3 & c_4 & \cdot & \cdot \end{vmatrix} x_4{}^2\,, \quad \text{or say } -x_4{}^2\Delta\,.$$

As it can be shown similarly that

$$\mathrm{M}_3{}^2 = -x_3{}^2\Delta\,, \quad \mathrm{M}_2{}^2 = -x_2{}^2\Delta\,, \quad \mathrm{M}_1{}^2 = -x_1{}^2\Delta\,,$$

Brioschi obtains *

$$|\,m_1\delta_2b_3c_4\,| = (m_1x_1+m_2x_2+m_3x_3+m_4x_4)\sqrt{-\Delta}\,,$$

nothing being said as to the sign to be taken in extracting the square root of x_r^2.

We have only to add for ourselves that the first of the conditioning equations is the vanishing of the quaternary quadric

$$\begin{array}{cccc|l} x_1 & x_2 & x_3 & x_4 & \\ \hline a & e & f & g & x_1 \\ e & b & h & k & x_2 \\ f & h & c & l & x_3 \\ g & k & l & d & x_4\,, \end{array}$$

and that the δ's are the halved differential quotients of this with respect to x_1, x_2, x_3, x_4.

* The minus sign is omitted by him throughout. If the number of x's had been odd, the sign would have been $+$.

HERMITE, CH. (1855, January).

[Sur la théorie de la transformation des fonctions abéliennes. *Comptes rendus . . . . Acad. des Sci.* (Paris), xl. pp. 249–254: or *Œuvres*, t. i. pp. 444–478.]

The special determinant here considered, as being auxiliary to Hermite's main purpose, is $|a_1b_2c_3d_4|$ with its elements subject to the conditions

$$\left.\begin{array}{l} |a_1d_2| + |b_1c_2| = 0 = |a_1d_3| + |b_1c_3|, \\ |a_1d_4| + |b_1c_4| = k = |a_2d_3| + |b_2c_3|, \\ |a_2d_4| + |b_2c_4| = 0 = |a_3d_4| + |b_3c_4|, \end{array}\right\}$$

and the results in regard to it are:—(1) that it is equal to k^2; (2) that the row-by-row product of two such determinants is a determinant of the same type. No proof is given, but from the way in which Hermite writes the conditions, it would appear that the first was obtained by multiplying the given determinant columnwise by itself in the form

$$\begin{vmatrix} d_1 & d_2 & d_3 & d_4 \\ c_1 & c_2 & c_3 & c_4 \\ -b_1 & -b_2 & -b_3 & -b_4 \\ -a_1 & -a_2 & -a_3 & -a_4 \end{vmatrix}.$$

A generalisation by Brioschi (1855) has already been dealt with under Skew Determinants.

ZEHFUSS, G. (1858).

[Uebungsaufgaben für Schüler. *Archiv d. Math. u. Phys.*, xxxi. p. 246: or *Nouv. Annales de Math.*, xviii. p. 171: (2) ii. pp. 60–61.]

The proposition offered for proof by Zehfuss is in modern phraseology to the effect that the determinant of the difference of the two square matrices

$$\begin{array}{cccc} a_1 & a_1 & \dots & a_1 \\ a_2 & a_2 & \dots & a_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_n & a_n & \dots & a_n , \end{array} \qquad \begin{array}{cccc} b_1 & b_2 & \dots & b_n \\ b_1 & b_2 & \dots & b_n \\ \cdot & \cdot & \cdot & \cdot \\ b_1 & b_2 & \dots & b_n \end{array}$$

vanishes for all orders higher than the second. The proof given by Gustave Harang in the *Nouvelles Annales* rests on the operations

$$\mathrm{col}_1 - \mathrm{col}_2 , \quad \mathrm{col}_2 - \mathrm{col}_3 , \quad \dots\dots$$

When $n = 2$ we have

$$\begin{vmatrix} a_1 - b_1 & a_1 - b_2 \\ a_2 - b_1 & a_2 - b_2 \end{vmatrix} = (a_1 - a_2)(b_1 - b_2) .$$

CAYLEY, A. (1859, March).

[On the double tangents of a plane curve. *Philos. Trans. R. Soc. London,* cxlix. pp. 193–212: or *Collected Math. Papers,* iv. pp. 186–206.]

The theorem on which an important part of this investigation rests is enunciated by its author as follows: *If the* 2n–1 *columns of the matrix*

$$\begin{array}{ccccc|cccc} a_0 & a_1 & a_2 & \ldots & a_{n-1} ; & a'_0 & a'_1 & \ldots & a'_{n-2} \\ a_1 & a_2 & a_3 & \ldots & a_n ; & a'_1 & a'_2 & \ldots & a'_{n-1} \\ a'_0 & a'_1 & a'_2 & \ldots & a'_{n-1} ; & a''_0 & a''_1 & \ldots & a''_{n-2} \end{array}$$

be represented by

$$1 \quad 2 \quad 3 \quad \ldots \quad n \quad ; \quad (1) \quad (2) \quad \ldots \quad (n-1)$$

respectively; *the determinant whose columns are those thus represented by* r, s, (t) *be denoted by* {r, s, (t)}; *and the determinant aggregates*

$$\begin{aligned} &\{n, n-1, (2)\} + \{n, n-2, (3)\} + \ldots + \{n, 2, (n-1)\}, \\ -&\{n, n-1, (1)\} - \{n, n-2, (2)\} - \ldots - \{n, 2, (n-2)\} - \{n, 1, (n-1)\}, \\ -&\{1, 2, (n-1)\} - \{1, 3, (n-2)\} - \ldots - \{1, n-1, (2)\} - \{1, n, (1)\}, \\ &\{1, 2, (n-2)\} + \{1, 3, (n-1)\} + \ldots + \{1, n-1, (1)\}, \end{aligned}$$

by I, II, III, IV; *then*

$$a_0\mathrm{I} + a_1\mathrm{II} + a_{n-1}\mathrm{III} + a_n\mathrm{IV} = 0.$$

The mode of verification suggested consists in showing that there exist six quantities (12), (13), (14), (23), (24), (34), say, such that

$$\begin{aligned} \mathrm{I} &= a_0.0 + a_1(12) + a_{n-1}(13) + a_n(14), \\ \mathrm{II} &= -a_0(12) + a_1.0 + a_{n-1}(23) + a_n(24), \\ \mathrm{III} &= -a_0(13) - a_1(23) + a_{n-1}.0 + a_n(34), \\ \mathrm{IV} &= -a_0(14) - a_1(24) - a_{n-1}(34) + a_n.0; \end{aligned}$$

and then taking the sum of the requisite multiples. The six quantities in question are actually found for the cases where $n = 3, 4, 6$. In the last case, the matrix being

$$\begin{array}{cccccc|ccccc} a & b & c & d & e & f & a' & b' & c' & d' & e' \\ b & c & d & e & f & g & b' & c' & d' & e' & f' \\ a' & b' & c' & d' & e' & f' & a'' & b'' & c'' & d'' & e'', \end{array}$$

their values are written by Cayley in the form

$$\begin{aligned} (12) &= -e''g && + f'f', \\ (13) &= b''f + c''e + d''d + e''c && - b'f' - c'e' - d'd' - e'c' - f'b', \\ (14) &= -b''e - c''d - d''c && + b'e' + c'd' + d'c' + e'b', \\ (23) &= -a''f - b''e - c''d - d''c - e''b && + a'f' + b'e' + c'd' + d'c' + e'b' + f'a', \\ (24) &= a''e + b''d + c''c + d''b && - a'e' - b'd' - c'c' - d'b' - e'a', \\ (34) &= -a''a && + a'a'. \end{aligned}$$

The final lemma used in the verification may be formulated thus: *If from the* n *quantities* $x_1, x_2, \ldots, x_n$ *and the* $\frac{1}{2}n(n-1)$ *others*

$$\begin{array}{r} 12\,,\,13\,,\,\ldots\,,\,1n \\ 23\,,\,\ldots\,,\,2n \\ \cdot\ \ \cdot\ \ \cdot\ \ \cdot \\ nn \end{array}$$

there be formed n *lineo-linear functions, namely,*

$$\left.\begin{array}{lcl} f_1 & = & x_1.0 \quad + x_2(12) + x_3(13) + \ldots + x_n(1n)\,, \\ f_2 & = & -x_1.(12) + x_2.0 \quad + x_3(23) + \ldots + x_n(2n)\,, \\ f_3 & = & -x_1(13) - x_2(23) + x_3.0 \quad + \ldots + x_n(3n)\,, \\ \cdot & \cdot & \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot \\ f_n & = & -x_1(1n) - x_2(2n) - x_3(3n) - \ldots + x_n.0 \end{array}\right\},$$

then $x_1f_1 + x_2f_2 + \ldots + x_nf_n = 0$.* It may be viewed as included in the identity

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & \ldots & x_n & \\ \hline . & 12 & 13 & \ldots & 1n & x_1 \\ -12 & . & 23 & \ldots & 2n & x_2 \\ -13 & -23 & . & \ldots & 3n & x_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \ldots \\ -1n & -2n & -3n & \ldots & . & x_n \end{array} = 0\,;$$

or in the statement that *Any quadric whose discriminant is a zero-axial skew determinant vanishes identically.*

Hirst, T. A. (1859, 1/9).

[Question 489. (A determinant which vanishes for every order higher than the fourth.) *Nouv. Annales de Math.*, xviii. p. 358: (2) ix. pp. 561–563.]

Hirst's theorem is that *if*

$$a_{rs} = (\alpha_r + \beta_r s)\cos s\phi + (\gamma_r + \delta_r s)\sin s\phi$$

then the determinant

$$\left|\begin{array}{cccc} a_{1,s} & a_{1,s+1} & \ldots & a_{1,s+n-1} \\ a_{2,s} & a_{2,s+1} & \ldots & a_{2,s+n-1} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n,s} & a_{n,s+1} & \ldots & a_{n,s+n-1} \end{array}\right|$$

* When the coefficient of x_r in f_r is not 0 but (rr), the result of course is

$$x_1f_1 + x_2f_2 + \ldots + x_nf_n = x_1^2(11) + x_2^2(22) + \ldots + x_n^2(nn)\,;$$

and in this connection it may be well to recall a step in Hermite's mode of effecting the automorphic transformation of a quadric (see under *Orthogonants*).

vanishes when $n>4$, and has a non-zero value independent of s *when* n$=4$; and the real significance of it is best grasped by noting—as is not done in the *Annales*—that the determinant is the product

$$\begin{Vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_n & \beta_n & \gamma_n & \delta_n \end{Vmatrix} \cdot \begin{Vmatrix} \cos s\phi & s\cos s\phi & \sin s\phi & s\sin s\phi \\ \cos(s+1)\phi & (s+1)\cos(s+1)\phi & \sin(s+1)\phi & (s+1)\sin(s+1)\phi \\ \cdots & \cdots & \cdots & \cdots \\ \cos(s+n-1)\phi & (s+n-1)\cos(s+n-1)\phi & \sin(s+n-1)\phi & (s+n-1)\sin(s+n-1)\phi \end{Vmatrix}$$

The vanishing of it when $n>4$ is then self-evident, and its value when $n=4$ being

$$|\alpha_1\beta_2\gamma_3\delta_4| \cdot \begin{vmatrix} \cos s\phi & s\cos s\phi & \sin s\phi & s\sin\phi \\ \cos(s+1)\phi & (s+1)\cos(s+1)\phi & \sin(s+1)\phi & (s+1)\sin(s+1)\phi \\ \cos(s+2)\phi & (s+2)\cos(s+2)\phi & \sin(s+2)\phi & (s+2)\sin(s+2)\phi \\ \cos(s+3)\phi & (s+3)\cos(s+3)\phi & \sin(s+3)\phi & (s+3)\sin(s+3)\phi \end{vmatrix},$$

we have only to show that the second determinant here is independent of s. The solver (Lucien Bignon) does this by multiplying the determinant by itself in the form

$$\begin{vmatrix} s\cos s\phi & -\cos s\phi & s\sin s\phi & -\sin s\phi \\ (s+1)\cos(s+1)\phi & -\cos(s+1)\phi & (s+1)\sin(s+1)\phi & -\sin(s+1)\phi \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

and so finding for its square a determinant whose every element is independent of s, the element in the place i,j being in fact

$$(j-i)\cos(j-i)\phi .$$

He does not note, however, that such a determinant is zero-axial and skew, and that its value is thus readily seen, by a theorem of Cayley's, to be

$$(\cos^2\phi - 4\cos^2 2\phi + 3\cos\phi\cos 3\phi)^2 ,$$

i.e.

$$(-4\sin^4\phi)^2 .$$

Cayley, A. (1859).

[Note on the value of certain determinants, the terms of which are the squared distances of points in a plane or in space. *Quart. Journ. of Math.*, iii. pp. 275–277: or *Collected Math. Papers*, iv. pp. 460–462.]

The five results given in the paper are more important than the title would imply, being true when instead of Cayley's elements $\overline{12}^2, \overline{13}^2, \ldots$

we write any elements whatever, namely, 12, 13, . . . This change being made, the fourth and fifth are

$$\begin{vmatrix} . & 12 & 13 & 14 \\ 21 & . & 23 & 24 \\ 31 & 32 & . & 33 \\ 41 & 42 & 43 & . \end{vmatrix} = \sum 12\ 21\ .\ 34\ 43\ - \sum 12\ 23\ 24\ 41\,,$$

$$\begin{vmatrix} . & 12 & 13 & 14 & 15 \\ 21 & . & 23 & 24 & 25 \\ 31 & 32 & . & 34 & 35 \\ 41 & 42 & 43 & . & 45 \\ 51 & 52 & 53 & 54 & . \end{vmatrix} = \left\{ \begin{array}{l} \sum 12\ 23\ 34\ 45\ 51 \\ -\sum 12\ 23\ 31\ .\ 45\ 54\,, \end{array} \right.$$

where the Σ's cover 3, 6, 24, 20 terms respectively. No commentary is added, nor any indication of a law including both results. The three of the other set are less general, namely,

$$\begin{vmatrix} . & 12 & 12 & 1 \\ 21 & . & 23 & 1 \\ 31 & 32 & . & 1 \\ 1 & 1 & 1 & . \end{vmatrix} = \sum 12\ 21\ - \sum 12\ 23\,,$$

where the Σ's cover 3 and 6 terms respectively;

$$\begin{vmatrix} \cdot & 12 & 13 & 14 & 1 \\ 21 & . & 23 & 24 & 1 \\ 31 & 32 & . & 34 & 1 \\ 41 & 42 & 43 & . & 1 \\ 1 & 1 & 1 & 1 & . \end{vmatrix} = \sum 12\ 23\ 34\ - \sum 12\ \ 34\ \ 43\ - \sum 12\ 23\ 31\,,$$

where the Σ's cover 24, 12, 8, terms respectively;

$$\begin{vmatrix} . & 12 & 13 & 14 & 15 & 1 \\ 21 & . & 23 & 24 & 25 & 1 \\ 31 & 32 & . & 34 & 35 & 1 \\ 41 & 42 & 43 & . & 45 & 1 \\ 51 & 52 & 53 & 54 & . & 1 \\ 1 & 1 & 1 & 1 & 1 & . \end{vmatrix} = \left\{ \begin{array}{l} -\sum 12\ 23\ 34\ 45\ \ -\sum 12\ 21\ .\ 34\ 43 \\ +\sum 12\ 23\ .\ 45\ 54\ \ +\sum 12\ 23\ 34\ 41 \\ +\sum 12\ 23\ 31\ .\ 45\,, \end{array} \right.$$

where the Σ's cover 120, 15, 60, 30, 40 terms respectively. Here again no generalisation is attempted.

(ϵ) CENSUS OF TERMS IN SPECIAL DETERMINANTS.

The first instance of the finding of the number of terms in the final development of a determinant of special form has already been drawn attention to, the investigator being Scherk, and the date 1825.

During the period now occupying us, the earliest suggestion on the subject occurs in 1844 in *Crelle's Journ.*, xxviii. pp. 191–192, the determinant being of the 8th order, and the specialisation consisting in having a zero in the places

$$\begin{array}{c} 16, 17, 18, 27, 28, 31, 38, 41, 42 \\ 83, 82, 81, 72, 71, 68, 61, 58, 57. \end{array}$$

The proposer of the problem, so far as it appears, received no satisfaction. The next instance occurred to Sylvester, who in 1855 having hit upon a peculiar determinant whose terms were all positive, ascertained the number of them by evaluating a special circulant. (See *Quart. Journ. of Math.*, i. pp. 42–56, or our notice of it given above.) The third instance, like the first, arose as a problem and remained long unsolved. It appeared in 1858 in the *Nouv. Annales de Math.*, xvii. p. 262, under the heading "Question 445," the requirement being to find the number of terms remaining in the case of a determinant of the n^{th} order when all those terms have been deleted which contain two or more diagonal elements. The fourth instance —which is more closely connected with the third than might at first appear —was the actual but incidental determination by Cayley in 1859 of the number of terms in a zero-axial determinant whose order is not greater than the 7^{th}. The numbers found were 9, 44, 265, for the 4^{th}, 5^{th}, 6^{th} orders respectively. (See *Quart. Journ. of Math.*, iii. pp. 275–277, or our notice of it given above.)

LIST OF AUTHORS

whose writings are herein dealt with.

(*Issued separately March* 27, 1911.)

XIX.—A Study of Artificial Pyrexia produced by Tetra-hydro-β-Naphthylamine Hydrochloride. By Adam Black. (From the Physiological Department, Glasgow University.) *Communicated by* Professor D. Noël Paton.

(MS. received February 6, 1911. Read January 23, 1911.)

The object of the present investigation is to throw further light upon the differences between the disturbances in the chemical processes in the body in fever caused, on the one hand, by the mere rise of body temperature, and on the other by the action of the micro-organismal poisons on the tissues of the body.

The influence of such micro-organismal poisons has been already studied by several investigators—among others, by Noël Paton, Dunlop, and Macadam in the case of diphtheria (*Journ. of Phys.*, vol. xxiv. p. 331, 1899; see also chapter "Fever and Infection," by Kraus, Von Noorden's *Metabolism and Practical Medicine*, vol. ii. p. 90); but so far few detailed studies of the metabolism in high temperature due to non-infectious processes have been recorded.

It has been maintained that, while bacteria and their toxins cause an increased disintegration of proteins, simple pyrexia is entirely caused by an increased combustion of glycogen.

The peculiar action of tetra-hydro-β-naphthylamine hydrochloride in causing a rise of temperature seems to afford a means of differentiating the effects of mere high temperature from the effects of ordinary infective fevers.

In 1889 Stern (1 and 2) showed that after the injection of a small dose of this substance there is a distinct rise of temperature. He investigated the effects of the drug with considerable care, but came to no definite conclusions as to its mode of action. Fawcett and Hale-White (3) also made observations on the action of the drug, and concluded that the rise of temperature is not due to diminution in the loss of heat, and that it is not due to muscular movement. They also attempted to study the effect of the drug when the spinal cord is cut, and when the rabbit is under the influence of curare; but they came to the conclusion that the influence of the fixing down of the animal renders all conclusions from such experiments valueless. They also record some experiments on the combined effect of fixing down the rabbit and performing artificial respiration with the administration of ether alone and combined with the drug. The result seems to have been negative. Later, some work was done by Hale-White (4) and Pembrey, who concluded that the action is due to increased tone of the muscles and increased muscular action. Ott (5) stated that tetra-hydro-β-naphthylamine hydrochloride causes a rise of temperature after the animal has been freed from its stored glycogen, and he argues that the stimulation of the thermogenic mechanism acts upon proteins to increase their combustion and to cause increased heat-production; but as his observations are not substantiated by determinations of the nitrogen excreted, his conclusions must be accepted with some reserve.

The evidence upon which he supports his contention that the rise of temperature is not due to increased heat elimination through contraction of the peripheral vessels does not appear to be absolutely satisfactory.

His early experiments to show that the drug no longer causes a rise of temperature after section below the tuber cinerium, and his later experiments to prove that it does not act if prevented by the injection of paraffin (5A) into the carotids from reaching his hypothetical thermogenic centres, do not fully dispose of the possibility of the rise of temperature being due to decreased heat-elimination. His further experiments on the effect of other vaso-constrictions upon the temperature are open to criticism, since the marked action of the drug upon the heart may have been sufficient to mask the effects of constriction of the arterioles upon the blood-pressure.

A. Mode of Action of the Drug.

The well-known action of such anæsthetics as ether and chloroform in causing a marked fall in temperature, with increased loss of heat from the skin, seemed to indicate that some light might be thrown upon the mode of action of tetra-hydro-β-naphthylamine hydrochloride by its administration along with ether or chloroform, and, before proceeding to investigate its action on the metabolism, an investigation of this point was made.

As a preliminary, (*a*) the normal variation in the rectal temperature of the rabbit was first investigated. Its limits were found to be between 36·9° C. and 38·3° C.

In these and in all my experiments the animal was not fixed down in any way.

(*b*) On administering the drug the following results were obtained:—*

0·025 grm. steadied the temperature at 38·3° C. for two hours.

0·03 gave an almost similar result.

0·10 gave a rise of temperature occurring in three-quarters of an hour, the maximum temperature of 39·1° C. being reached in two and a half hours.

* The injections were made by Dr E. P. Cathcart and by Professor Noël Paton.

0·13 grm. gave a rise of temperature which began in half an hour, and reached a maximum of 40·3° C. in one and a half hours. On a second trial with the same dose on the same rabbit, the rise came in half an hour

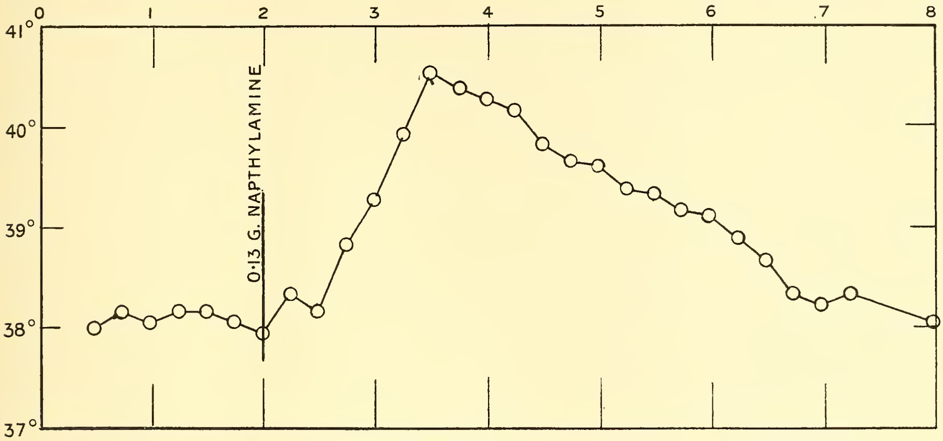


Chart I.

as before, but the maximum did not come until two and three-quarter hours and was 39·4° C. (Charts I. and II.).

For the effect on dogs, see p. 339.

It was next necessary to follow the course of the loss of heat under ether, and its effect upon the temperature.

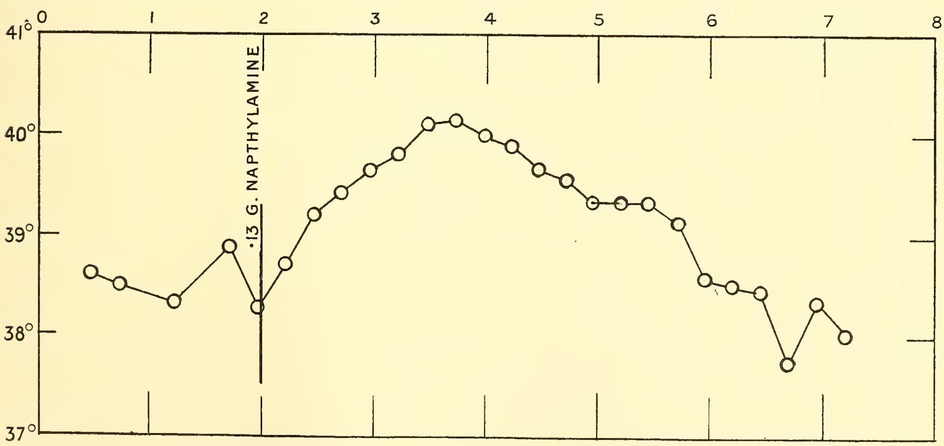


Chart II.

The following results were obtained in two experiments with the anæsthetic. In each the temperature fell in a curve similar to that of a mass cooling by radiation and conduction unaffected by physiological action. The rate of cooling was at first 3° in the hour (Charts III. and IV.).

The drug was next given concurrently with the ether, and in every experiment the fall of temperature characteristic of the anæsthetic was observed. The rate of the fall of the temperature was not retarded by the drug being injected. In one experiment, it may be noted, in which the

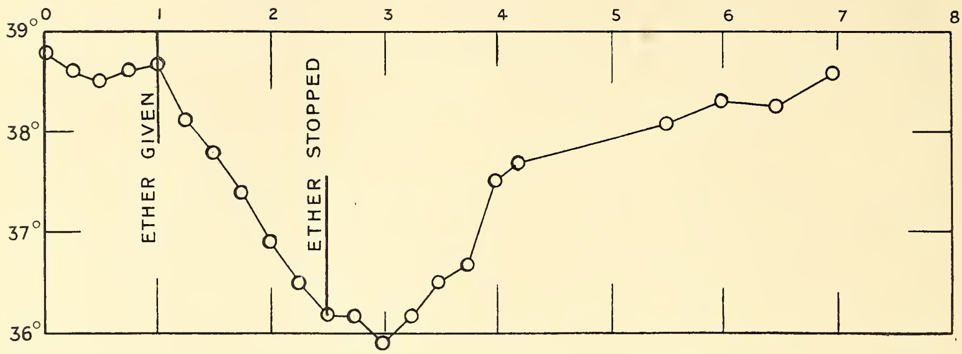


Chart III.

ether was not given in sufficient amount to keep the animal fully paralysed, and where muscular action as of running and chewing continued, the rate of the fall of temperature was retarded, being only about ·5° C. in the first hour. The only difference between those experiments when the drug was

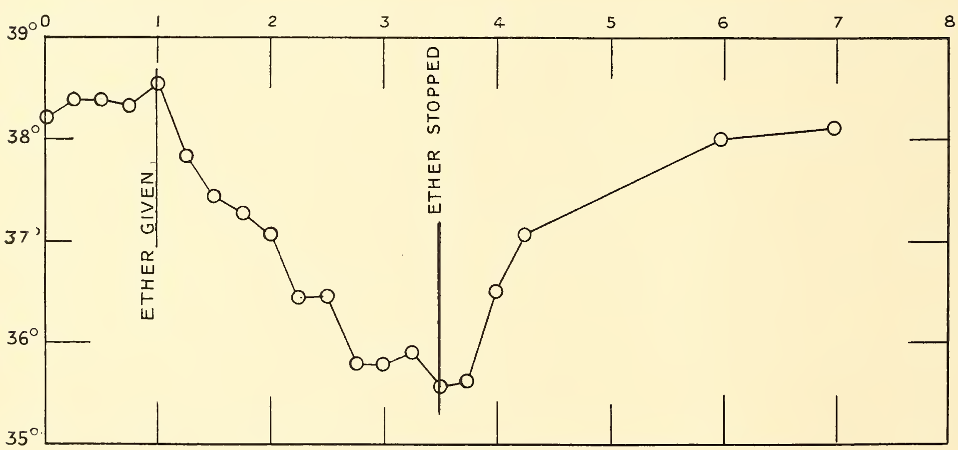


Chart IV.

present and those when it was absent was that the anæsthetic plus the drug did not so markedly lower the temperature as when the anæsthetic was given alone. Further, there was a tendency on the part of the temperature, after the animal recovered from the influence of the anæsthetic plus the drug, to reach a higher level than the normal temperature of the animal before the experiment commenced (see Chart V.).

A further set of experiments were carried out in which the drug was first injected and the temperature of the animal allowed to rise definitely before the administration of the ether was commenced. In these experiments a fall of temperature began at once, and at about the

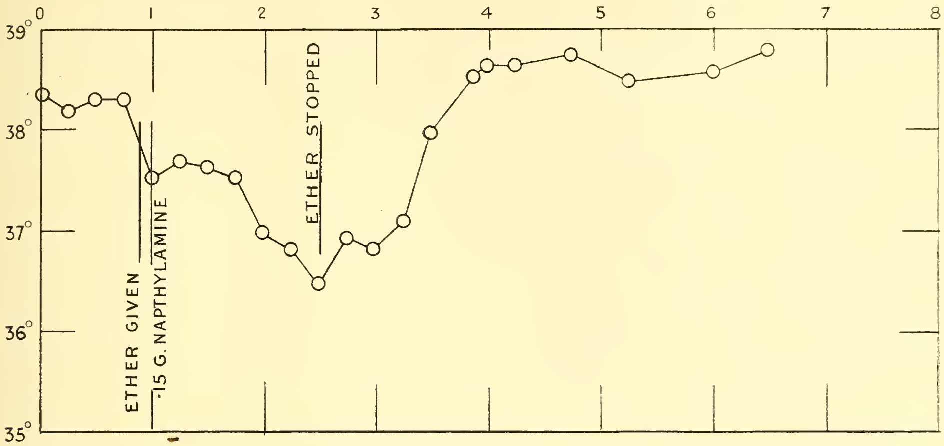


Chart V.

rate of 1·7° C. in the first hour. When the ether was stopped there was not the same tendency as in the previous set of experiments for the temperature to rise above the normal, the recovery, in fact, being rather slow (see Chart VI.).

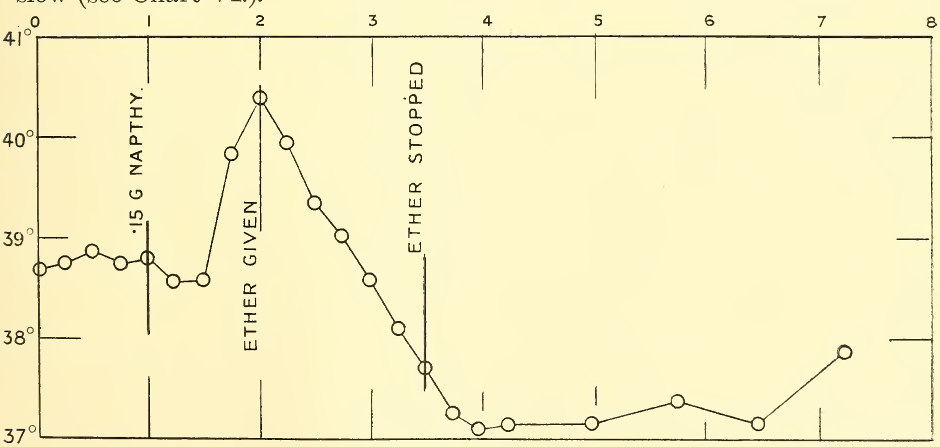


Chart VI.

The administration of an anæsthetic (ether) prevents or checks the rise of temperature produced by the drug. This action may be due to the anæsthetic acting in one of three ways:—

(1) By poisoning a hypothetical heat-producing centre upon which the drug may act.

(2) By causing a dilatation of the cutaneous blood-vessels and an increased loss of heat, thus leading to a lowering of the temperature.

(3) By preventing the nervous system from acting upon the muscles, *i.e.* by paralysing the muscular system and thus checking the liberation of heat in the body.

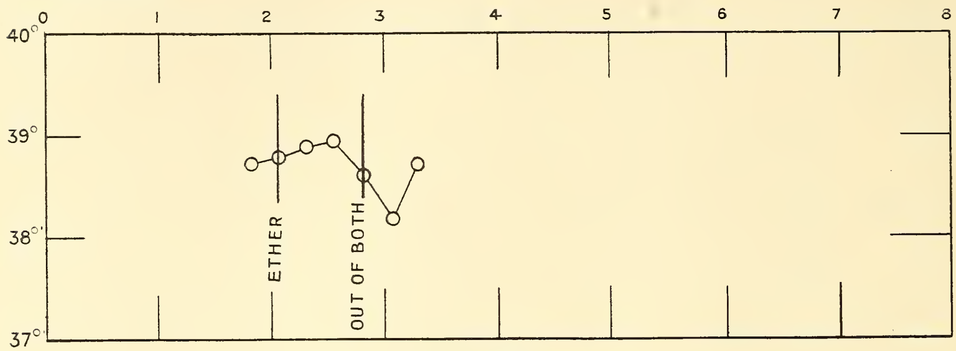


Chart VII.

Against the third hypothesis may be set the fact, which Harnack (6) and Marshall (7) have so clearly demonstrated, that various convulsants which cause powerful muscular contractions do not raise the temperature of the animal.

In order to test the second hypothesis several experiments were carried out in which the animal, during the period of experiment, was well wrapped

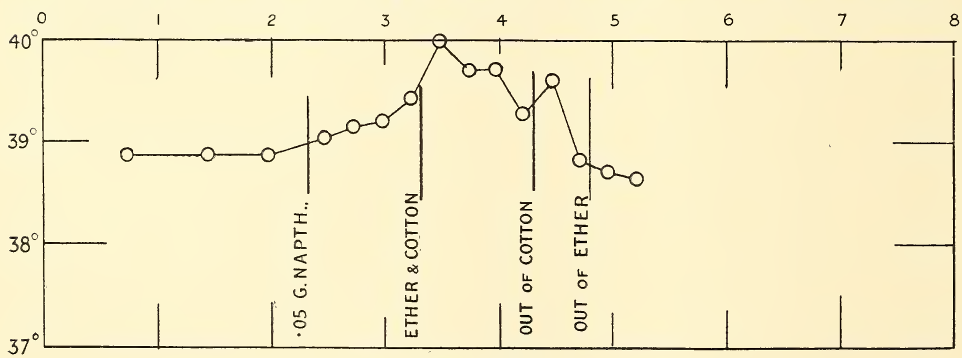


Chart VIII.

up in cotton-wool. With ether alone under these conditions the fall of temperature did not take place (see Chart VII.), and in the case of the drug plus ether and cotton-wool there was likewise no fall, indeed the rise of temperature continued (see Chart VIII.).

These last experiments point very directly to the fact that the fall of temperature due to the anæsthetic is to a large extent the result of heat loss by vaso-dilatation. The observation that the ears of the animals were always cold during the rise of temperature under the influence of the drug, and continued cool while the temperature was falling, suggests that vaso-contraction with decreased loss of heat is an important factor in bringing about the rise of temperature.

As regards the first hypothesis, that the anæsthetic paralyses a thermogenic centre on which the drug acts, no definite experiments are at present put forward. But the evidence given above seems to indicate that this antagonistic action is not of primary importance.

B. Influence of the Drug on Nitrogenous Metabolism.

Two experiments were done on dogs.

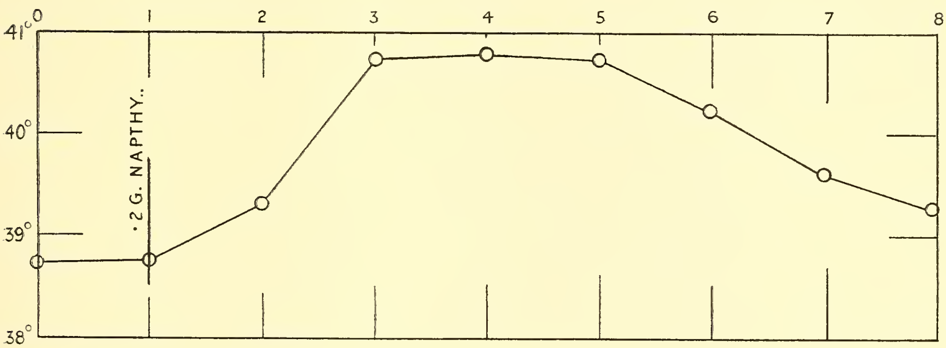


Chart IX.

Exp. 1. The results of this experiment are given in detail (see Temperature Chart IX.).

Exp. 2. The results of this experiment confirm 1 in all respects, and are not given (see Temperature Chart X.).

Experiment 1.—*Setter, Female.*

Weight, Nov. 13: 18·200 kilos.
" " 22: 16·700 "
Diet: 150 grams oatmeal.
25 " plasmon.
500 c.c. milk.

Given daily in one meal at 10 a.m.

Dates given opposite urinary analysis are those of the days of collecting the urine passed in the preceding twenty-four hours: the urine was collected at 10 a.m.

At 12 noon on the 17th November 0·2 gram of drug was given hypodermically.

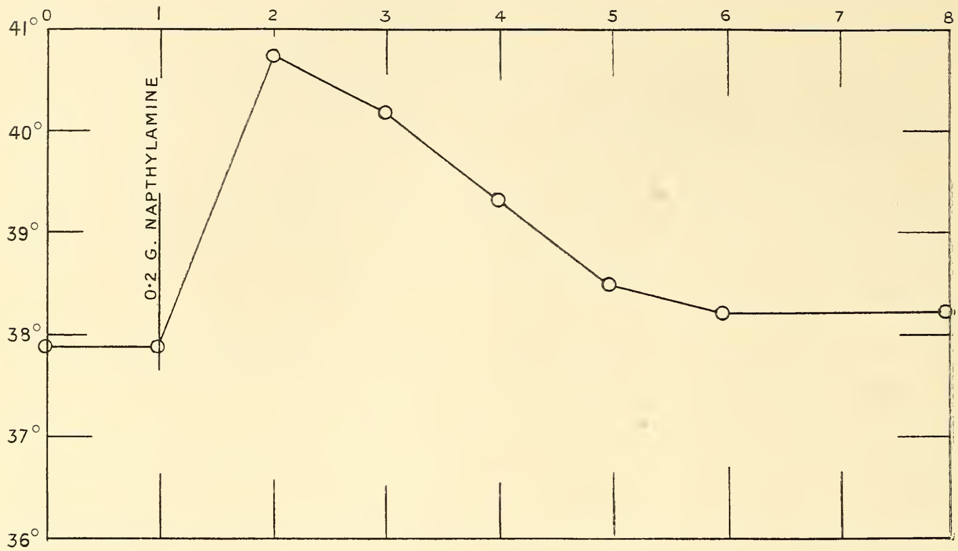


Chart X.

Temperatures.

Daily at 10 a.m.	Hourly on 17th.
16th. 101·8° F.	12 noon. 101·4° F.
17th. 101·7° „	1 p.m. 102·4° „
18th. 102·0° „	2 „ 105·0° „
19th. 102·0° „	3 „ 105·1° „
20th. 101·8° „	4 „ 105·0° „
21st. 101·6° „	5 „ 104·1° „
	6 „ 103·0° „
	...
	8 „ 102·4° „

Composition of Urines.

	Amount.	Specific Gravity.	Total Nitrogen.	Urea Nitrogen.	Ammonia Nitrogen.	Total Purin Nitrogen.	Creatinine Nitrogen.	Creatine Nitrogen.	Sodium Chloride.	Phosphoric Acid.	% Urea Nitrogen in Total.	% Ammonia Nitrogen in Total.	% Purin Nitrogen in Total.	% Creatinine Nitrogen in Total.
Nov. 15	1100	1012	7·62	6·76	·44	·06	·09	·11	...	...	88·6	5·8	·73	1·13
„ 16	1095	1012	7·24	5·97	·37	·07	·09	·11	10·4	...	82·4	5·2	·91	1·24
„ 17	930	1010	4·99	4·34	34	·04	·15	·00	7·9	·230	87·1	6·7	·84	3·01
„ 18	1150	1009	6·36	5·32	·32	·05	·18	·01	12·7	·142	83·6	5·0	·76	2·96
„ 19	730	1015	7·46	6·11	·44	·07	·21	·01	7·3	·323	81·9	5·8	1·00	2·78
„ 20	720	1010	5·65	4·53	·35	·05	·11	·00	3·6	·251	80·3	6·3	·96	1·99
„ 21	870	1009	5·42	4·56	·27	·05	·13	·00	8·3	·210	84·1	5·0	1·01	2·30
„ 22	900	1010	6·30	5·36	·36	·06	·16	·01	6·3	·239	85·0	5·8	·90	2·59

From the table it will be noted that as the result of the injection there is a very slight rise in the output of total nitrogen. This agrees with the result of a similar experiment carried out by Stern (2). There is a slight fall in the percentage of urea nitrogen and a slight rise in the total purins and in the creatinine, although the dog's food was practically purin and creatine free. Leathes (9) found that the output of both of these was increased during fever. Van Hoogenhuyze and Verploegh (8) also found an increased output of creatinine in the same condition. The retention of chlorides so characteristic of all pyrexias is quite marked.

Conclusions.

The investigation thus goes to show (1) that tetra-hydro-β-naphthylamine hydrochloride is a drug which induces a high temperature by acting upon the nervous system, and that its action may be antagonised by ether, and that in all probability the rise of temperature under the drug is due to decreased heat-elimination.

(2) That the change produced in the nitrogenous metabolism by so marked a pyrexia as that induced by the drug is very small when compared with the change produced by such agents as diphtheria toxin. This tends to show that it is the action of toxins on the tissues rather than the pyrexia which modifies the protein metabolism.

My thanks are due to Dr Cathcart for suggesting to me the subject of research, and to him and to Professor Noël Paton for assistance in its prosecution.

(Towards the expenses of the investigation a grant was made by Mr Francis Mason.)

BIBLIOGRAPHY.

1. Stern, *Virchow's Archiv*, cxv. p. 13, 1889.
2. ,, ,, ,, cxxi. p. 376, 1890.
3. Fawcett and Hale White, *Journal of Physiology*, xxi. p. 435, 1897.
4. Hale-White and Pembrey, "Proc. of the Phys. Soc.," *Journ. of Phys.*, 1903.
5. Ott, *Journ. of Exp. Medicine*, vol. ix. No. 6, 1907; *Contributions to Physiology*, part xvii., 1907; *ibid.*, part xviii., 1909.
6. Harnack, *Arch. f. exp. Pathol.*, xlix. p. 157, 1903.
7. Marshall (communication to Dr Cathcart).
8. Van Hoogenhuyze and Verploegh, *Ztsch. f. physiol. Chemie*, Bd. lvii. p. 161, 1908.
9. Leathes, *Journal of Physiology*, vol. xxxv. p. 205, 1907.

(*Issued separately April* 12, 1911.)

XX.—On the Determination of Small Degrees of Enzymatic Peptolysis. By **Dorothy Court**, B.Sc., Carnegie Research Scholar, Mycology Department, Heriot-Watt College. *Communicated by* Dr E. Westergaard.

(MS. received December 31, 1910. Read February 20, 1911.)

In studying the conditions under which enzymes are produced in seeds at the beginning of germination, two main difficulties have been encountered. The one has been in devising means of maintaining sterility in the material without at the same time interfering with the formation and action of enzymes. Some experiments have been carried out with the object of determining the relative efficiency of some of the more commonly used antiseptics, but, at present, these are not sufficiently far advanced to admit of any general conclusions being drawn. They have, however, shown that saturation with chloroform ensures sterility under the conditions of the experiments which form the subject of this paper. The other difficulty has been in demonstrating small degrees of enzymatic activity. It is with this latter question I wish to deal here.

In the present research, attention has been mostly confined to the activation of the proteolytic and peptolytic enzymes of the seed. The presence of traces of such in resting seeds of several plants has been shown by Vines (*Annals of Botany*, vol. xxii.), Miss J. White (*Proc. Roy. Soc.*, B 550), and others, and it has been further demonstrated in a large number of cases that the enzymatic activity rapidly increases during germination. For the purposes of this investigation it was accordingly necessary to devise a method whereby even slight differences in the enzymatic activity might be accurately determined. The usual methods of detecting and measuring proteolysis and peptolysis are only of service where the degree of activity is relatively high. The method of Kjeldahl and Weis, in which the enzymatic preparation is digested with a protein substrate and the increase in non-precipitable nitrogen estimated after digestion, is unsuitable for this reason, and also since it is too long and complicated to allow of many estimations being carried out simultaneously. Sörensen's method of estimating peptolysis, in which the material is treated with formol and then titrated against standard alkali, is also unsuitable, since in dealing with substances of entirely unknown chemical constitution such as ground-up seeds or animal tissue there is always the possibility of subsidiary processes taking place which may give rise to change in the reaction.

The synthetic production, by Emil Fischer and his school, of numerous di- and polypeptids made it possible to investigate peptolysis by studying the action of enzymes on such substances. The general employment of these ideal methods is, however, prevented by the difficulties attaching to the preparation of polypeptids. In order to overcome these a new method has been devised by Abderhalden (*Zeitschrift für physiologisches Chemie*, vol. lxi.) which involves the use of a silk pepton—Pepton Roche. This, on hydrolysis, gives a large yield of tyrosin, which, being sparingly soluble, is precipitated, and may be filtered off, dried, and weighed, the weight indicating the degree of peptolysis and hence the enzymatic activity. Two difficulties arise here, however. The activity may be so slight that the amount of tyrosin thrown down is too small to determine accurately by weighing. Secondly, it is not always possible to use clear solutions, and even where such are used the possibility remains of insoluble substances other than tyrosin being produced. The formation of such substances has actually been observed in various instances during my experiments. Consequently, a method of estimating tyrosin by chemical means was sought.

It was suggested to me at this point by Dr Westergaard that a colorimetric method might be applicable, and to this I turned my attention. A suitable colour reaction is described by Mörner in the *Zeitschrift für physiologisches Chemie*, xxxvii. Mörner's reagent, which consists of sulphuric acid and formol, gives with a small amount of tyrosin an intense green coloration easily determinable by tintometer. It now remained to prove, firstly, that the intensity of colour is in direct relation to the concentration of tyrosin, and, secondly, that the weight of tyrosin produced under standard conditions from the same quantities of enzyme and substrate is constant. In order to demonstrate the first of these relations, a series of estimations was carried out, using quantities of tyrosin varying from ·004 gm. to ·012 gm. The tyrosin used for this purpose was Merck's preparation, which, although possibly not perfectly pure, is nevertheless sufficiently so for the purpose of showing the proportionality existing between the degree of colour and the quantity of tyrosin. The preparation gave on ignition a very small trace of reddish ash, the quantity obtained from ·14 gm. tyrosin being too small to ascertain by weight. The solubility determined at 20° C. was slightly above that stated in the handbooks, being 1 in 2090, while Beilstein gives it as 1 in 2454. The melting point is stated variously by different authors. Beilstein gives it as 235°, while Oppenheimer in the *Handbuch der Biologie* gives it as between 314° and 318°. The present preparation was so far decomposed at 270° that I was unable to ascertain the actual melting point. In each case the estimation was made in 20 c.c. reagent, using the $\frac{1}{4}$-inch cell for the reading.

The following readings were obtained:—

				Average.
·012	gm. tyrosin	gave readings	8·0, 7·8, 7·6	7·8
·010	,,	,,	6·6, 6·4, 6·6	6·53
·008	,,	,,	5·2, 5·1, 5·0	5·1
·007	,,	,,	4·5, 4·4, 4·4	4·43
·006	,,	,,	3·8, 3·8, 3·7	3·76
·005	,,	,,	3·0, 3·0, 3·1	3·03
·004	,,	,,	2·4, 2·3, 2·4	2·36

From these results a curve may be obtained which, within these limits, gives a straight line. This, although coming very near to the starting

point, does not actually pass through it. Whether this is due to inaccuracies in the readings, or whether, as was suggested by some readings taken with lower concentrations using a wider cell, the curve is really of hyperbolic nature, could not be determined. For practical purposes, however, the curve may be considered a straight line and the reading directly proportional to the concentration of tyrosin.

A series of digestions were now carried out in order to discover if the quantity of tyrosin produced by the same proportions of enzyme and substrate under standard conditions were constant. For this purpose it seemed advisable to use an enzymatic preparation whose properties were fairly well known, and which could be obtained in reasonably large quantities. It was therefore decided to use the preparation known as Armour's Pancreatin.

Seven series of digestions in all were carried out. In each series the digestions were carried out simultaneously, using the same amount of enzyme and substrate. The quantity of pancreatin used was either ·05 gm. or ·1 gm., the time of digestion being varied in the different series.

The following results were obtained:—

(1) 8·0, 8·0, 8·0, 8·4, 8·2, 8·0, 8·0.
(2) 5·8, 5·6, 5·6, 5·6, 5·4, 5·4, 5·4, 5·4, 5·6.
(3) 7·4, 7·2, 7·3, 7·2, 7·6, 7·4, 7·4, 7·4, 7·6.
(4) 8·0, 8·0, 7·8, 8·0, 8·2, 8·2.
(5) 7·2, 7·0, 7·4, 7·4, 7·4, 7·0, 7·2.
(6) 8·0, 8·0, 8·4, 8·2, 8·4, 8·2, 8·4.
(7) 4·0, 4·2, 4·4, 4·0, 4·0, 4·0.

In carrying out the method the following materials are required:—

1. *Pepton.*—The solution used in these experiments has been of 20 per cent. strength. As this solution usually gives an acid reaction, sodium bicarbonate should be added till neutral. The pepton solution was kept over chloroform, so that it was always saturated with that antiseptic when used.

2. *Sulphuric acid.*—Kahlbaum's pure acid has been used. As solvent for the tyrosin, a mixture of 1 volume of the acid with 10 volumes of distilled water was used. Mörner's reagent consists of 55 volumes of concentrated acid, 45 volumes of distilled water, and 1 volume of formol. The concentration used for dissolving the tyrosin was fixed upon as being of suitable strength for solvent purposes, and, at the same time, as admitting of being brought up to the concentration required by Mörner by the subsequent addition of an easily measurable volume of concentrated acid.

Sulphuric acid of the same concentration as the reagent is also required. This is made up by mixing 55 volumes of concentrated acid with 45 volumes of distilled water.

3. *Formol.*—In these experiments Schering's preparation has been used. The method of procedure was as follows:—

The pepton solution was measured out into dry flasks with the material to be tested. If an alkaline medium is desired, a definite quantity of sodium carbonate or bicarbonate may be added to the pepton solution beforehand, or the alkali may be used in extracting the enzyme. In these experiments a concentration of sodium bicarbonate equal to 1 per cent. has been employed.

In the preliminary experiments an enzymatic solution was used, which was obtained by extracting pancreatin with water in proportions varying from $\frac{1}{10}$ to $\frac{1}{1000}$. Apart from the difficulties attending filtration, it was found that only a very small amount of the enzyme went into solution, even when relatively large quantities of water were used. It was, moreover, repeatedly observed that it was impossible, by filtration through paper, to obtain a solution which was quite homogeneous as regards activity. It was therefore decided to weigh out the desired quantity of pancreatin directly into each flask used in the experiment.

During digestion the development of micro-organisms has been prevented by the presence of chloroform. Originally, a few drops were added in each case before digestion. This method, however, was discarded owing to the possible retarding influence of excess chloroform. It was found quite sufficient to saturate the pepton beforehand by shaking with excess of chloroform, while in this way the presence of excess during digestion was avoided. In those cases where a liquid extract of the enzyme was used, it was also saturated before digestion.

The digestions have been performed in stoppered flasks which were kept at 40° C. in an incubator or a water-bath. The time of digestion has been regulated according to the activity of the enzyme. Abderhalden has, in his experiments, found an incubation of a few hours sufficient to produce a weighable quantity of tyrosin, but, when working with smaller degrees of enzymatic activity, a more prolonged digestion is necessary. From observation of the reaction it would appear that the separation of the tyrosin is by no means an immediate process. Apparently, during the first period of decomposition, soluble products only are produced, no deposit being visible for some time. Eventually, tyrosin begins to crystallise out, and, once commenced, the process goes on rapidly. In one series of digestions the material remained clear after twenty hours' incubation. At the end of twenty-three hours, however, a considerable deposit was found in all the flasks.

After digestion the flasks were cooled in water for a few minutes and the contents filtered in the ordinary way. The precipitates were well washed with cold water and finally dried in an oven. It is necessary to remove by washing all trace of the pepton solution, as this itself gives a slight green colour with the reagent. This does not, however, present any difficulty. The dried precipitates were now dissolved in measured quantities of the solvent acid.

The quantity of reagent employed was in each case determined by the approximate quantity of tyrosin, but the same volume of reagent will give suitable readings within a fairly wide range of tyrosin. Therefore, if an apparently large precipitate is obtained, a large quantity of reagent is employed in preference to diluting a possibly too dark liquid after the reaction. This latter method has been found to give inaccurate results.

The acid was allowed to run on to the filter paper from a burette, the solution being received in a flask graduated for the final amount of reagent to be used. For 20 c.c. reagent, 9·8 c.c. of the solvent acid was used and was received in a 20 c.c. flask. To this was added ·2 c.c. of formol. Finally, 10 c.c. concentrated sulphuric acid was added, the contents of the flask well mixed by stirring or shaking, and the flask immersed in a bath of boiling water. In these experiments the flasks have been allowed to remain in the bath for twenty minutes, after which they have been cooled in water. The period of boiling—twenty minutes—was simply fixed upon as giving ample time for the completion of the reaction, and also in order to minimise the possible experimental error due to slight variations in the time. Since some shrinkage always occurred, the contents of the flask, when cool, were made up to the graduation mark with sulphuric acid of the same concentration as the reagent. The colour was now estimated by means of the tintometer.

The usual method of estimating colour is by matching it as nearly as possible. In a paper read before the Society of Chemical Industry, Dr Westergaard describes a method of estimating colour by neutralisation. This method has been adopted in preference to the former, in these experiments, partly because it appeared easier to decide when a colour was neutralised than when it was exactly matched, and also because the neutralisation of the green colour may be effected by a single red slide, while matching would at least demand two. The estimation becomes considerably simplified when only a single slide is required for each determination. It is now a comparatively simple matter to discover a red slide of such intensity that, on superposition, the green colour of the solution is entirely neutralised.

As the method is based upon the formation of tyrosin by the hydrolysis of certain polypeptids, it follows that it may be used for measuring the activity of all enzymatic preparations which are capable of catalysing this reaction. Abderhalden has demonstrated the presence of such enzymes in numerous cases in both animal and plant tissues, such as kidney, lung, intestine, various fungi, etc. (*Zeitschrift für physiologisches Chemie,* lxv. p. 180; lxvi. pp. 137 and 276; lxviii. p. 312). I have myself tried the method for testing the activity of the peptolytic enzyme in yeast with good results, and am now testing the applicability in the case of seed enzymes.

It further follows that peptolytic enzymes which are incapable of catalysing this reaction, and for which the method can accordingly not be used, must be of an entirely different nature from that produced by the pancreas.

(*Issued separately April* 12, 1911.)

XXI.—The Independence of Peripheral Sensory Neurons in view of the Results of Experimental Section of the Optic Nerve in the Rabbit.

By **Janie Hamilton M'Ilroy**, M.A., B.Sc., M.B., Ch.B., Barbour Research Scholar in Physiology. (From the Physiological Laboratory, University of Glasgow.) *Communicated by* Professor Noël Paton.

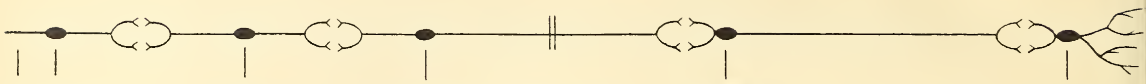
(MS. received February 24, 1911. Read January 23, 1911.)

[*Short abstract of a paper which will appear in extenso in "Brain."*]

The problem of the amount of independence possessed by the units which go to make up the nervous system is one which has long occupied the attention of neuro-pathologists, and is still far from solved. That degenerative changes, as the result of peripheral injury, may extend across several intermediate neurons is indicated by the work of many investigators, notably v. Monakow, Campbell, Bolton, etc. Thus after enucleation of the eyeball changes have been met with in the visual cortex, showing that degeneration has travelled over the neurons situated at the basal ganglia of the cerebrum to the neurons in the visual cortex itself. Interruption of the sensory conduction path affects the integrity of the more centrally placed neurons. The question of the stability of the peripheral neurons under similar conditions has not received attention. Birch-Hirschfeld in 1900 published the results of a large series of experiments upon the retina, including a few cases of the effects of experimental section of the optic nerve. He examined the retina, by the Nissl method, 55 hours, 5, 10, and 15 days after section of the nerve, and found definite changes both in the ganglion cells and in the nuclear layers. The number of these experiments, however, is too small to be convincing. As no other definite evidence could be found, it was decided that more extensive investigation would be profitable.

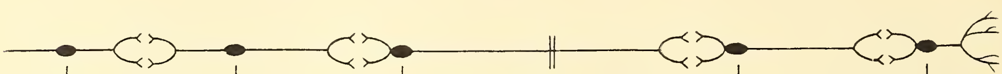
The peripheral sensory neuron as it exists in the eye, ear, nose, and skin is well adapted for experimental research, and the eye especially offers likelihood of satisfactory results. The optic nerve is accessible for operation, and the retina is an organ which can be quickly and easily preserved for histological purposes.

The visual sensory conduction path consists of the following stations:—



The rods and cones and outer nuclei may be looked upon as true peripheral neurons. The inner nuclei may be looked upon as the homologues of the sensory nerve fibres. The ganglion cells are central and are the homologues of the cells of the medullary nucleus of the sensory chains of other special sense mechanisms.

Compare the gustatory sensory path as given below:—



Mode of Experiment.—The experiments were performed on rabbits. The duration of the experiment varied from 24 hours to 38 weeks. Histological investigation was by the Held-Nissl method.

The results of the investigation may be tabulated as follows:—

Tabulated Results.

Exp. i. 24 hours. No change in retina.

Exp. ii. 48 hours. Slight blurring of Nissl's granules in the ganglion cells. Outer neurons normal.

Exp. iii. 48 hours. Same as ii.

Exp. iv. 72 hours. Slightly increased blurring of Nissl's granules. Outer neurons normal.

Exp. v. 4 days. Much the same as iv.

Exp. vi. 7 days. Some shrinking of ganglion cells. Nissl's granules often dust-like. Slightly diffuse staining of protoplasm. Outer neurons normal.

Exp. vii. 8 days. Diffuse staining more marked. Granules often dust-like. Outer neurons normal.

Exp. viii. 3 weeks. Ganglion cells much shrunken. Nissl's granules in bead-like rows, or in form of very fine dust. Outer neurons normal.

Exp. ix. 3 weeks. Ganglion cells much shrunken, pale or densely staining. Granules dust-like. Outer neurons normal.

Exp. x. 6 weeks. Ganglion cells much shrivelled, the nucleus often alone remaining. Occasionally dust-like granules are seen, but the staining is mostly diffuse. Outer neurons normal.

Exp. xi. 15 weeks. Ganglion cells very much shrivelled, and diffusely stained. Outer neurons normal.

Exp. xii. $16\frac{1}{2}$ weeks. Ganglion cells mostly atrophic. Outer neurons practically normal.

Exp. xiii. 38 weeks. Ganglion cells completely atrophic. Outer neurons practically normal.

Discussion of Results.—Many more experiments were performed, but were rejected on account of complications, *e.g.* hæmorrhage, injury to the ciliary nerves, etc. The thirteen cases given above may be taken as uncomplicated examples of the effect of section of the optic nerve upon the retina.

These results show that as early as 48 hours after operation slight degenerative changes have appeared in the ganglion cells. The degeneration, however, does not proceed very rapidly, no pronounced change being observed before the lapse of 3 weeks. At 15 weeks the changes in the ganglion cells are very marked, and at 38 weeks atrophy is complete. The outer neurons, on the other hand, remain practically unchanged throughout.

The question of *inherent* stability on the part of the outer neurons had to be considered, and a series of autolytic experiments was carried out. Fresh eyeballs from the ox, cat, dog, and rabbit were dipped in melted paraffin (in order to seal from the atmosphere and obviate putrefactive conditions) and left for varying lengths of time. On histological examination it was found that the outer neurons, far from exhibiting any marked power of resistance to autolysis, were of all the retinal elements the most perishable. The ganglion cells, on the other hand, were comparatively well preserved, even after a considerable lapse of time.

The conclusion is arrived at that the changes, which in the case of the ganglion cells after section of the optic nerve were so pronounced, were degenerative and the result of the abnormal anatomical conditions. The preservation of the easily perishable outer neurons indicated that these latter elements are unaffected by such conditions; in other words, that the peripheral sensory neuron of the retina is anatomically an independent organism.

(*Issued separately April* 12, 1911.)

XXII.—The Sex and Age Incidence of Mortality from Pulmonary Tuberculosis in Scotland and in its Groups of Registration Districts since 1861. By C. Hunter Stewart, M.B., D.Sc., Professor of Public Health, University of Edinburgh. (With Sixteen Tables, nine of which in text, and Eight Plates.)

(MS. received June 23, 1910. Read November 21, 1910.)

No subject in the domain of preventive medicine is attracting greater attention at present than that of the prevention of tuberculosis, more especially that of tuberculosis of the lungs. The fact that this disease is one of the largest contributors to the death-rate makes the question pressing, and the great diminution of the mortality from it that has already taken place in most civilised countries makes the prospect of still greater diminution most hopeful.

The statistics of the mortality from this disease in England and Wales contained in the Annual Report of the Registrar General have been carefully investigated by Dr Tatham, late Superintendent of Statistics, and the results obtained have been tabulated and discussed in his Supplements to the 55th and 65th Reports of the Registrar General for England and Wales, etc. These results are of great epidemiological value, showing as they do, not only the course of the mortality and its sex and age incidence, but also the effect of locality, urban and rural, on this mortality.

In the following paper the results are given of an investigation on somewhat similar lines of the statistics of the mortality from tuberculosis of the lungs in Scotland during the period 1861–1907. The yearly deaths and the estimated population for that year are taken from the Annual Reports of the Registrar General for Scotland. The death-rates are per 10,000 living, and are calculated on the mean estimated population. The sex and age constitution of the population, *i.e.* the number of each sex at each of the age periods 0–5 years, etc., is calculated in each case from that of the census at the beginning of the corresponding decade.

Table I. shows the course of the mortality from tuberculosis of the lungs in Scotland since 1861, and, for purposes of comparison, that in England and Wales during the same period and in Ireland since 1881, the rates for the latter being calculated from the statistics contained in the Annual Reports of the Registrar General for Ireland. In columns 2, 3, and 4 are given the rates which actually occurred in Scotland in the respective quinquennia. For a proper comparison of the series of rates, however, account must be taken of the change in sex and age constitution of the population during the period under review, seeing the mortality varies greatly according to age and, though in a lesser degree, according to sex. From the censuses of Scotland 1861 and 1901 the following table has been prepared:—

Age and Sex Constitution of Population in Scotland per 10,000.

	Census 1861.		Census 1901.	
	Males.	Females.	Males.	Females.
0–5	692	670	600	592
5–10	602	584	557	544
10–15	539	515	533	517
15–20	491	513	515	504
20-25	416	499	470	498
25–30	331	430	405	442
30–35	296	364	337	373
35–40	256	312	297	325
40–45	241	290	265	283
45–50	204	235	224	242
50–55	184	222	187	208
55–60	140	168	147	172
60–65	133	174	124	157
65–70	80	107	82	111
70–75	58	84	59	84
75+	66	95	55	91

This change in the sex and age constitution of the population is due chiefly to the decrease in the birth-rate, modified to some degree by changes in the death-rate. In Table I. the figures given in columns 5, 6 and 7 are all comparable, as they show the death-rates which would have resulted from the rates prevailing in Scotland at the several age periods if the sex and age constitution of the population in all of the quinquennia had been the same as that of Scotland at census 1901. Columns 8, 9, and 10 show the ratio (as whole numbers) of the rates thus corrected to that of 1866–70 taken as 100. It will be observed that this correction raises the death-rate in each case, showing that the sex and age constitution of the population from 1861–1900 was favourable to a low death-rate from tuberculosis of the lungs as compared to that at census 1901. The rates for Ireland have been similarly corrected to make them comparable with those of Scotland. The rates for England and Wales are uncorrected.

TABLE I.—MORTALITY FROM TUBERCULOSIS OF THE LUNGS PER 10,000 LIVING AT ALL AGES OF MALES, OF FEMALES, AND OF BOTH SEXES IN SCOTLAND, ENGLAND AND WALES, AND IRELAND.

	England and Wales.	Scotland.			Scotland (corrected to Sex and Age Constitution of Scotland, Census 1901).						Ireland.			Ireland (corrected to Sex and Age Constitution of Scotland, Census 1901).		
	Death-rate.	Death-rate.			Death-rate.			Ratio to Death-rate of 1866–70 as 100.			Death-rate.			Death-rate.		
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	Both Sexes.	Both Sexes.	M.	F.	Both Sexes.	M.	F.	Both Sexes.	M.	F.	Both Sexes.	M.	F.	Both Sexes.	M.	F.
1861–65	25·3	25·51	25·0	26·0	26·22	26·14	26·30	...	...	...	...	...	...	...	...	...
1866–70	24·48	26·22	25·54	26·83	27·11	26·94	27·28	100	100	100	...	...	...	...	...	...
1871–75	22·18	24·98	24·10	25·79	26·00	25·49	26·58	96	94	97	...	...	...	...	...	...
1876–80	20·4	23·08	22·43	23·66	24·00	23·63	24·37	89	88	89	...	...	...	...	...	...
1881–85	18·3	20·90	19·88	21·84	21·74	20·80	22·63	80	77	83	20·79	19·08	22·03	21·96	20·96	22·91
1886–90	16·35	18·63	18·25	18·97	19·40	19·16	19·63	72	71	72	21·19	20·24	22·07	22·32	21·84	22·76
1891–95	14·62	17·50	17·53	17·48	18·05	18·19	17·92	67	67	66	21·39	20·39	22·36	22·27	21·44	23·05
1896–1900	13·23	16·64	17·14	16·18	17·20	17·84	16·59	64	66	61	21·29	20·73	21·83	22·13	21·84	22·41
1901–05	12·1	14·45	14·72	14·19	14·45	14·72	14·19	53	55	52	21·53	21·37	21·69	21·46	21·30	21·59
1906–07	11·45	13·58	13·87	13·30	13·58	13·87	13·30	50	51	49	20·26	20·31	20·21	20·25	20·34	20·17

The Decrease in Mortality from Tuberculosis of the Lungs.—From a study of this table it is seen that in Scotland a decrease in mortality from tuberculosis of the lungs began in quinquennium 1871–75 and has continued since. The rate of the decrease has varied from quinquennium to quinquennium, and has been more marked since 1901. During the thirty-seven years 1871–1907 the reduction has been 50 per cent. in both sexes taken together. Two-thirds of this reduction took place in the twenty-five years 1871–95, and one-third in the twelve years 1896–1907. In England and Wales the decrease began earlier than in Scotland, and has been more marked. This will be more fully discussed later. In Ireland there has been a very small decrease.

Change in Sex Incidence.—In 1855 the first Detailed Annual Report of the Registrar-General for Scotland appeared. From 1861–65 to 1886–90 the rate of mortality was higher among females than among males, not only as an average, but in every individual year of that period. In 1891–95 the mortality among males was higher, but in three of the years the female rate was in excess. From 1896 to 1907 the female rate was lower in each quinquennium, but was slightly higher than the male rate in two of the years. In England and Wales the mortality among females was higher than among males till 1865, but from 1866 onwards *the excess has been invariably and increasingly among males.*

In Scotland this change in sex incidence began twenty-six years later than in England, and has been much more gradual. While in England and Wales in 1907 the difference of mortality between the two sexes was nearly 4 per 10,000 in favour of females, in Scotland it was practically only ·5 per 10,000.

A similar change in sex incidence of mortality has occurred in other countries. The following table is condensed from one given by Frederick L. Hoffman * :—

Mortality from Tuberculosis of the Lungs among White Population in the United States (five Eastern States and three other Cities).

	Per 10,000 living at all ages. Males.	Females.
1871–75	34·50	35·30
1876–80	31·40	33·33
1881–85	30·17	32·03
1886–90	28·66	27·23
1891–95	25·10	22·63
1896–1900	23·70	19·24

* *Transactions British Congress on Tuberculosis*, 1901, vol. ii. p. 350.

Table II.—Comparison between Mortality Rates from Tuberculosis of the Lungs per 10,000 Living at All Ages of Males, of Females, and of Both Sexes, in Scotland, and in England and Wales.

	Scotland.						England and Wales.					
	Death-rate.			Ratio to Death-rate of 1861–70 as 100.			Death-rate.			Ratio to Death-rate of 1861–70 as 100.		
	Both Sexes.	M.	F.	Both Sexes.	M.	F.	Both Sexes.	M.	F.	Both Sexes.	M.	F.
1861–70	26·66	26·54	26·79	100	100	100	25·45	25·38	25·52	100	100	100
1871–80	25·00	24·56	25·47	94	92	95	21·90	22·88	20·98	87	90	82
1881–90	20·57	19·98	21·13	77	75	79	17·75	19·04	16·55	70	75	65
1891–1900	17·62	18·01	17·25	66	68	64	13·91	15·80	12·14	55	62	48
1901–5	14·45	14·72	14·19	54	55	53	12·14	14·34	10·09	48	56	39
1906–7	13·58	13·87	13·30	51	52	50	11·45	13·48	9·55	45	53	37

In Table II. the statistics for Scotland and for England and Wales are more fully compared. The rates for Scotland are corrected to sex and age constitution of census 1901. In the first three decades the rates for England and Wales are corrected to the sex and age constitution of that country in 1891–1900. The ratio of the corrected rates to that occurring in decade 1861–70 in Scotland and in England and Wales respectively are also given. It shows that in decade 1861–70 the mortality from pulmonary tuberculosis in Scotland was higher than that in England and Wales, and that since that period the decrease in the mortality among both sexes in Scotland has been only 49 per cent., while it has been 55 per cent. in England and Wales. The decrease according to sex has been very different in the two countries. While in both the decrease among males has been practically the same, indeed has been rather greater in Scotland, the decrease in mortality among females has been only 50 per cent. in Scotland compared to 63 per cent. in England and Wales. *At the present time the greater mortality from tuberculosis of the lungs among the entire population of Scotland as compared with that of England and Wales is due to the greater mortality among females, and this latter is due to the fact that the decrease in the rate of mortality among females has been much less than in England and Wales.*

Mortality from Tuberculosis of the Lungs according to Sex and Age.—In Table III. the statistics for Scotland and England are compared. As the age periods in the English statistics are taken as ten years after the age 25, the Scottish statistics have been tabulated in the same way. In both countries there has been a decrease in the mortality at each age period.

Mortality among Males.—In Scotland the maximum mortality was at age 20–25 years in the four decades 1861–1900. In 1901–5 it was at age 25–35, and in 1906–7 at 45–55. In England and Wales the maximum rate has always been later than in Scotland. In the three decades 1861–90 it was at age 35–45, and from 1891–1907 it has been at 45–55 years. As in Scotland this postponement has been gradual.

Mortality among Females. — In Scotland the maximum mortality coincided with that among males in 1871–80 at 20–25 years. In 1861–70, and from 1881 to 1907, it has been at age 25–35 years, and, as a rule, at age 25–30 years. In England and Wales the maximum rate was at 25–35 years from 1861 to 1890, and from 1891 to 1907 it has been at 35–45 years. The curves of these mortalities, which are plotted out on Plate I., show graphically the comparison between the sex and age incidence of the mortality in the two countries.

Percentage Reduction of the Mortality in each Sex at Age Periods.—Taking the mortality rate in each sex and at each of the age periods during 1861–70 as 100, the amount of reduction at each period has been calculated as a percentage. For example:—The mortality in Scotland among males 0–5 years was in 1861–70 15, and in 1906–7 it was 4·38. The reduction is 10·62 per 10,000, *i.e.* 70·8 per cent. These percentage reductions in Scotland and in England and Wales are plotted out as curves on Plate II.

Reduction of Mortality among Males.—The reduction at ages 10 to 45 years has been smaller in Scotland than in England and Wales. Above 45 years it has been greater.

Reduction of Mortality among Females.—In both countries the reduction among females 0–5 years has been greater than among males. In England and Wales the reduction in female mortality at age periods 5–10 up to and including 45–55 is greater than in Scotland, with the result that while in the former country in 1861–1870 this mortality was lower than that in Scotland only from 0–5 to 20–25, in 1906–7 it was lower, from 0–5 to 45–55 years.

But there is a still more striking difference between the two countries. In England and Wales the reduction is greater in *female* than in male mortality from 15–20 years onwards, while in Scotland this is the case only from period 35–45 years. In 1871–80, indeed, there was an increase in the female mortality from 15–25 years in Scotland as compared with 1861–70.* The curves themselves bring out these points more clearly.

Table II. shows that while in 1861–70 the mortality at all ages was *higher* among females than among males in both countries, and the difference between the two sexes was practically the same in each, in 1906–7 the female mortality in England and Wales was nearly 4 per 10,000, and in Scotland only ·53 per 10,000 *lower* than the male mortality. This is due to the greater reduction in the female mortality. If the rate of the reduction of this mortality had been the same in Scotland as in England during the entire period under review, there would have been a greater saving of life among females at their most productive and reproductive period. During 1906–7 there would have been an annual saving of 660 lives, and the death-rate per 10,000 living, which was 13·58 in both sexes and 13·30 in females, would have been 12·2 and 10·2 respectively, rates not very different from those of England and Wales.

Discussing the experience of England and Wales, Dr Tatham states:† "The age of maximum phthisis mortality has been postponed in both sexes. In other words, either the saving of life has been greater at the ages which were formerly most liable to phthisis than at the ages immediately following, or persons specially liable to phthisis have lived longer than they would have done under the earlier conditions." Had the latter condition obtained to any great extent the curves would have shown a greater *diminution* of the rate of reduction at more advanced age periods, which is not the case.

Mortality from Tuberculosis of the Lungs in the different groups of Registration Districts in Scotland.—In 1871 Scotland was divided for statistical purposes into five groups of registration districts constituted as follows:—(1) Principal towns with population above 25,000; (2) large towns between 10,000 and 25,000; (3) small towns between 2000 and 10,000; (4) mainland rural; and (5) insular rural districts. The principal towns were Glasgow, Edinburgh, Dundee, Aberdeen, Paisley, Greenock, Leith, and Perth. Owing to increase of population Kilmarnock was added to the principal towns, and in 1892 Coatbridge was also added, and

* There is now a tendency to a relatively greater increase in the reduction of female mortality at earlier ages in Scotland, shown by the percentage reductions at age 25–35 years being the same in both sexes during period 1901–7. From 55 years onwards the reduction among females in Scotland has been always greater than in England and Wales.

† "Memorandum on Mortality from Tubercular Phthisis in England and Wales during the last Forty Years," by John Tatham, M.D., F.R.C.P., *Transactions of British Congress on Tuberculosis*, vol. ii., 1902, p. 497.

certain small towns were transferred to the large town districts. In 1901 a further change was made, the minimum population of a principal town being raised to 30,000. There are now fifteen of these, viz. those above enumerated, and Govan, Partick, Kirkcaldy, Hamilton, and Motherwell. An effort has thus been made to retain the distinctive feature of the population of each of these districts as regards aggregation on area, which was adopted in 1871. The population of Scotland was distributed at census periods as follows:—

	Principal Towns.	Large Towns.	Small Towns.	Mainland Rural.	Insular Rural.
1881.	1,411,536	388,797	790,796	1,014,056	130,388
1901.	1,950,297	578,769	897,326	929,042	116,505

The population of the insular rural district is so small in comparison with that of the others that its statistics are not fully comparable, and, besides, a large number of the deaths in this district are not medically certified as to cause. For these reasons the statistics of the first four districts only are given. These are set out, corrected to sex and age constitution of Scotland 1901, in Table IV. The rate of diminution among both sexes, taken together and in each sex, is greatest in the principal towns, and least in the mainland rural. In all the districts the diminution among females is greater than among males. In the principal town districts the female death-rate exceeded that of males till quinquennium 1886–90,* and in the mainland rural districts till 1896–1900, while in the large town and small town districts it has exceeded that of the males during the entire period 1871–1907.

Effect of Urban and Rural Conditions.—In order to study this question in sufficiently large populations, the death-rates of the principal town and large town districts combined, and of the small town and mainland rural districts combined, have been calculated and are set out in Table V. For purposes of comparison the rates for Scotland, excluding the insular rural districts, are also given. The urban area so constituted had an average population in 1891–1900 of 2,239,487, and the rural area of 1,832,601.

A study of Table V. shows that the reduction in the mortality has been greater in the urban than in the rural area (1) among both sexes taken together, and (2) among females both absolutely and in comparison with that among males.

* The female rate did not exceed the male rate in all the principal towns. In Edinburgh, *e.g.*, it has been always much less than the male rate. This is discussed later.

Table VI. shows the mortality according to sex and age in these two areas, and in Plate III. this is represented graphically. Up to ten years of age the mortality among males and females is lower (even in proportion to the mortality at all ages) in the rural than in the urban area. During the whole period under review the maximum mortality among *males* has been at an earlier age in the rural area, where it has ranged from 20-25 to 30–35 years, than in the urban area, where it has ranged from 35–40 to 50–55. In both areas there is a distinct and progressively increasing retardation of the age period of maximum mortality in both sexes, though this is more marked among males. The percentage reduction of mortality at age periods is shown graphically in Plate IV. At all age periods up to 45–50 it has been less in the rural area among both males and females. The reduction of the female rate has been less than that of the male rate up to 20–25 years in the rural and up to 15–20 years in the urban area. In order to contrast the influence of pronounced urban and rural conditions, the sex and age mortality in the principal town and in the mainland rural districts are set out in Table VII.

Comparison with Urban and Rural Phthisis Mortality in England and Wales.—Dr Tatham investigated* the phthisis mortality in two selected areas representing urban and rural England respectively. The urban area, with a population 1891–1900 of 16,465,427, consists of those counties which at the census 1901 were mainly urban in character, containing the chief centres of industry, and the rural area, with a population of 4,265,578, consists of those counties which, though containing some considerable urban communities, were nevertheless mainly rural in character.

Mortality at all Ages.—In Table VIII. the statistics of mortality at all ages for these areas in England and Wales are given, as also for comparison those of the two areas in Scotland. From this Table it is seen (1) that the mortality among males in the urban area of Scotland, which was much higher than that in urban England in 1891–1900, fell to a lower figure in 1901–7; (2) that the male rate exceeded the female rate in both areas of England and Wales and in the urban area of Scotland during 1891–1907, but only since 1901 in the rural area of Scotland; (3) that the difference between them is much less in Scotland, and that taking the period 1906–7, while in Scotland the male exceeded the female rate by 4 per cent. in the urban, and by less than 3 per cent. in the rural area, in England and Wales the male rate exceeded the female rate by 33 per cent. in the urban and by 13 per cent. in the rural; (4) that (1906–7) in

* *Supplement to 65th Annual Report*, p. xcv, and *Detailed Annual Reports for* 1901–7.

Table VIII.—Mortality from Tuberculosis of the Lungs per 10,000 living at All Ages of Males, of Females, and of Both Sexes in the Urban and Rural Areas of Scotland and of England and Wales.

	England and Wales.†						Scotland.‡					
	Urban.			Rural.			Urban. §			Rural. §		
	Both Sexes.	Male.	Fem.	Both Sexes.	Male.	Fem.	Both Sexes.	Male.	Fem.	Both Sexes.	Male.	Fem.
1891–1900	14·93	17·53	12·50	12·92	13·47	12·40	18·75	19·35	18·14	15·10	15·05	15·25
1901–5	13·03	15·81	10·43	11·40	12·56	10·31	15·23	15·56	14·98	13·30	13·25	13·34
1901–5*	,,	,,	,,	,,	,,	,,	15·59	15·87	15·32	13·49	13·54	13·44
1906–7	12·16	14·73	9·76	10·90	11·65	10·21	13·81	14·12	13·52	13·15	13·33	12·95

Scotland the female mortality was higher in the urban than in the rural area, while in England it was lower. The rural female rate in Scotland is higher than that in England and Wales, but lower than the urban female rate in Scotland. In England and Wales the rural and urban female rates are nearly the same, with a tendency recently to an excess in the former.

Mortality at different Age Periods.—Plate V. shows the mortality at age periods in rural and urban areas in Scotland and in England and Wales for 1891 to 1907.

Mortality among Females.—The mortality up to 35–45 years is greater in Scotland than in England and Wales, both in the urban and rural areas. While it was greater in Scotland from 10–15 to 25–35 in the urban than in the rural area during 1891–1900, and was practically the same in both during 1901–7, it was smaller in England and Wales between these age periods in the urban than in the rural area from 1891–1907. Further, the maximum mortality was always at an earlier age period in the rural area in England and Wales and in Scotland till 1906–7, when it was the same as in the urban.

Mortality among Males.—In both countries the age period of maximum

* Corrected to census England 1901. This correction makes the male rates in the Scottish and English areas strictly comparable. In both countries the urban male rates are practically the same.

† 1891–1900 corrected to census England 1891.
1901–7 ,, ,, ,, ,, 1901.

‡ 1891–1900 corrected to census Scotland 1891.
1901–7 ,, ,, ,, ,, 1901.

§ For constitution of these areas see Table VI.

mortality coincides with or slightly precedes that among females in the rural area, while it is always later in the urban.

Mortality among Females in certain Towns and in groups of Towns in Scotland.—While in the principal town districts the mortality among females exceeds that among males till 1886–90, the individual towns differ among themselves in this respect.

TABLE IX.—MORTALITY AT ALL AGES* FROM TUBERCULOSIS OF THE LUNGS PER 10,000 MALES AND FEMALES IN CERTAIN TOWNS AND GROUPS OF TOWNS IN SCOTLAND.

	EDINBURGH.		ABERDEEN.		DUNDEE.	
	M.	F.	M.	F.	M.	F.
1871–80	25·9	20·0	24·46	24·20	27·60	28·30
1881–90	23·4	16·8	20·10	20·42	23·65	24·74
1891–1900	21·6	14·56	18·80	17·80	21·30	21·50
1901–7	16·4	11·92	13·52	12·97	17·50	17·05

1901–7.	M.	F.
Glasgow . . .	16·84	15·23
Paisley . . .	13·50	14·22
Four towns (Paisley, Kilmarnock, Kirkcaldy, and Perth).	13·43	13·60
Four towns (Coatbridge, Govan, Hamilton, and Motherwell).	10·70	13·07

NOTE.—It was not possible to make a correction for deaths occurring in institutions in these towns. But the value of the statistics, as showing the *relation* between the male and female rate and the *reduction* that has taken place in each, is probably not appreciably affected thereby.

Plate VI. shows the age and sex incidence of mortality in these towns. The male rate at all ages in Edinburgh is not very different from that in Dundee, but the female rate is very markedly less. In 1901–7 the female rate was less than the male rate in all these towns except Paisley. The percentage reduction of the mortality between 1871–80 and 1901–7 has been practically the same in Edinburgh and Dundee.

EDINBURGH.		DUNDEE.	
M.	F.	M.	F.
36·6 per cent.	40·4 per cent.	36·6 per cent.	40 per cent.

The mortality at age period 15–35 and 35–55 years was as follows:—

	1871–80.				1901–7.			
	15–35.		35–55.		15–35.		35–55.	
	M.	F.	M.	F.	M.	F.	M.	F.
Edinburgh .	36·0	28·8	45·0	28·0	19·5	15·0	29·8	18·7
Dundee .	38·0	41·5	42·0	39·0	23·6	22·0	27·0	25·0;

* Corrected to sex and age constitution of Scotland, census 1901.

showing the following percentage reduction at each of these age periods:—

	15–35.		35–55.	
	M.	F.	M.	F.
Edinburgh . .	45·8 per cent.	47·6 per cent.	33·8 per cent.	33·0 per cent.
Dundee . .	37·7 ,,	47·0 ,,	35·7 ,,	36·0 ,,

The reduction in female mortality has been nearly the same in both cities, leaving the female mortality in Dundee much higher than that in Edinburgh.

Though in the entire population the number of females exceeds that of males, individual towns differ in this respect. Among the principal towns there are four in each of which *the males* are in excess, viz. Govan, Coatbridge, Motherwell, and Hamilton (Group 1), and four in each of which *the females* are in excess, viz. Paisley, Kilmarnock, Kirkcaldy, and Perth (Group 2). The mean population in 1901–7 was 199,910 in Group 1 and 191,060 in Group 2, so that they are comparable as regards size. The sex constitution of the population at census 1901, and the corrected phthisis mortalities for 1901–7, were as follows:—

Group 1.		Group 2.	
M.	F.	M.	F.
5300	4700	4740	5260
Phthisis mortality per 10,000 at all ages.*			
10·7	13·07	13·43	13·60

Both groups of towns are industrial, but in Group 1 there is much less employment for females (Table XII.). In Group 1 the mortality among females is greater than that among males, but is less than among females in Group 2, in Scotland as a whole (Table I.) or in its mainland rural districts (Table IV.). The mortality among males in Group 1 is less than that in Scotland as a whole, or any of its divisions. Plate VII. shows the sex and age incidence of the mortality. At the early age periods especially the male rate is markedly less than the female. The rates for age periods 15–35 and 35–55 years are as follows:—

	15–35.		35–55.	
	M.	F.	M.	F.
Group 1 . .	13·0	20·0	17·0	17·68
Group 2 . .	19·5	22·5	22·2	16·35

The conditions in the towns of Group 1, while most favourable to males, are more favourable to females than those obtaining in the towns of Group 2 or of Scotland at the same time.

* Corrected to sex and age constitution, Scotland, 1901.

Summarising the results of the foregoing statistical inquiry, it is to be noted:—

1. That in the quinquennium 1861–65 the mortality from tuberculosis of the lungs at all ages was practically the same in Scotland and in England and Wales in *both* sexes, the mortality among females being greater than among males (Table I.).

2. That between 1861–70 and 1906–7 there has been a smaller reduction of the mortality in Scotland than in England and Wales, and this is almost entirely due to the much smaller reduction in female mortality, the reduction in male mortality being practically the same in both countries (Table II.).

3. That while in England and Wales the female rate fell below the male rate in 1866–70 and has continued so to an increasing degree, this did not occur in Scotland till quinquennium 1891–95 (Table I.).

4. That while in England and Wales the reduction has been greater among females than among males from 15–20 years onwards, in Scotland this has been the case only from 35–45 years onwards (Plate II.).

5. That in the rural area of Scotland the percentage reduction of mortality at age periods 0 to 45–50 years has been less in both sexes than in the urban area, and that up to 25 years of age there was a smaller reduction in female relatively to male mortality in the rural than in the urban area (Plate IV.).

6. That contrasting the urban and rural areas of Scotland with the somewhat similar areas in England and Wales since 1891–1900, the female rate in the urban area is much higher than that in England and Wales, while the male rate is now lower. The rural female rate is also higher, and, as in England and Wales, it is lower than the male rate. The ratio of the female rate in the urban and rural areas of Scotland to that in England and Wales is as follows:—

	Urban Area.		Rural Area.	
	England.	Scotland.	England.	Scotland.
1891–1900 . . .	100	150	100	119
1901–5	100	146	100	124
1906–7	100	132	100	123

In both urban and rural areas in Scotland the male and female rates from 0–5 to 35–45 years are greater than those in the corresponding areas of England and Wales.

7. That in certain towns and groups of towns in Scotland the female rate at all ages exceeds, and in others is the same or less than, the male rate.

The cause or causes of the greater mortality from tuberculosis of the lungs among females in Scotland as compared to that in England and Wales must have operated during more than forty years (Table II.) both in urban and in rural areas (Plates III. and V., Tables V. and VI.), the effect being most marked in the age periods up to 35–45 years (Table III.). The reduction in Scotland at all ages has been practically the same among males and females, but between age periods 5–10 and 25–35 years it was less among females until 1901–5, since when it has been the same in both sexes. In England and Wales, while the reduction in female mortality at all ages has been much greater than in male mortality, the reduction among females has been the same as or greater than among males from age period 15–20 years onwards (Plate II.). Between the ages 5 and 20 years the phthisis mortality both in England and in Scotland, alike in urban and rural districts, has always been greater than among males. Leaving this point for later discussion, it remains that some influence has been at work in Scotland which has retarded the reduction of female mortality at age period 15–20 to 30–35 years as compared to that among females in England and Wales, and among males in both countries.

In the etiology of tuberculosis of the lungs two factors are generally recognised, the bacillus and the condition of the body into which it enters, the relative importance of these being estimated differently by different authorities. The cause of the greater mortality among females in Scotland than in England and Wales may be either a greater exposure to infection or diminished resisting power among the female population of Scotland, especially at the early age periods, or to both of these. Any light on this point is most important in connection with the success of preventive measures. Fortunately many measures, such as improved housing, improved conditions in factories and workshops, school hygiene, etc., tend both to lessen risk of infection and to increase the bodily resistance. If the chief cause be greater exposure to infection, measures must be directed specially against the bacillus; if diminished resistance of the body be also of great importance, then success will lie to a large extent in measures for securing general conditions of healthy living.

Greater Risk of Infection in Scotland.—If this be the explanation, then it does not apply in the case of males, as their mortality has decreased rather more in Scotland than in England, and markedly so in the urban area since 1891–1900 (Tables I. and VIII.). Women admittedly spend a larger part of their lives in the dwelling than men, but probably this is not more the case in Scotland than in England. It is difficult to compare the housing conditions of the two countries. In the insular rural districts of Scotland, where the housing is as a rule far from satisfactory, and where the women will largely be the attendants on the cases of the disease, the female mortality has always been markedly lower than the male at all ages, and at age periods above 20 years. That infection in the dwelling is an important cause of spread of tuberculosis of the lungs is undoubted, but that this is a greater factor in Scotland than in England, to such an extent, at all events, as to explain the great difference in female mortality, is very difficult of acceptance.

Conditions affecting Bodily Resistance.—Lowered resistance of the tissues, produced by exhausting work and insufficient nutrition, is recognised as a factor in tuberculosis of the lungs. Dr Burton Fanning (quoted by Dr Newsholme), speaking of the effect of over-exertion, says: "To my mind, there are few causes more powerful to determine the outbreak of

Table X.—Number of Females above Ten Years of Age engaged in "All Occupations" and in certain Occupations in Scotland and in England and Wales per 10,000 Females at Age Periods.

		Scotland.			England and Wales.		
		Per 10,000 Females at Age Periods					
		Above 10 Years.	10–20 Years.	Above 20 Years.	Above 10 Years.	10–20 Years.	Above 20 Years.
1871	All occupations	3314	3983	3076	3709	4478	3436
1881	,,	3409	4392	3054	3406	4038	3175
1891	,,	3479	4567	3079	3504	4347	3191
1901	,,	3305	4208	2995	3163	3870	2926
	Average of 40 years	3376	4298	3048	3419	4162	3155
1871	Indoor domestic service	916	1220	808	1396	2096	1147
1881	,,	919	1265	795	1259	2006	986
1891	,,	1050	1436	940	1250	1836	1034
1901	,,	803	1033	724	1009	1400	878
	Average of 40 years	919	1229	806	1208	1802	997
1871	Agricultural labourer	321	424	285	66	77	62
1881	,,	302	380	274	40	40	40
1891	,,	138	147	139	21	19	21
1901	,,	116	137	101	9	8	9
	Average of 40 years	209	259	190	31	33	30
1871	Textile industries	1028	1487	866	642	973	524
1881	,,	897	1273	761	590	832	503
1891	,,	833	1205	720	548	825	445
1901	,,	734	1059	623	516	755	418
	Average of 40 years	862	1241	727	563	836	466
1871	Engaged in "dress"	405	329	432	831	582	920
1881	,,	430	405	438	617	575	632
1891	,,	479	576	459	603	713	562
1901	,,	456	555	422	524	686	490
	Average of 40 years	445	476	434	633	646	628

pulmonary tuberculosis than physical over-exertion. . . . It is important to recognise that although such exercise be taken in the open air it is conducive to the development of consumption if it entails exhaustion or fatigue." Setting on one side the dangers to health in special industries, the exhaustion produced in young females by hard physical exertion will tend to accentuate that predisposition to tuberculosis of the lungs observed in them between 5 and 20 years of age, as compared with males. The comparative conditions obtaining in Scotland and in England and Wales in this respect among females may be deduced from the tables calculated from the census returns (Tables X. and XII.).

Indoor Domestic Service.—There is and has been for the last forty years a much smaller relative number of the female population in Scotland engaged in indoor domestic service than in England and Wales. If "the average number of domestic servants be, as it probably is, a good standard by which to measure the average degree of comfort in which a community is living,"* then in this respect the inhabitants of Scotland are not so well off as those of England and Wales. Greater comfort of living involves better average feeding and probably better housing, and a less degree of care and nervous strain. On an average during the forty years, in England and Wales more than 35 per cent. and in Scotland 27 per cent. of the occupied females above 10 years were engaged in indoor domestic service. But as regards the female domestic servants themselves, at least to many of them, it certainly ensures better feeding and housing than they would have had in their own homes. About 46 per cent. more females between 10 and 20 years of age are engaged in domestic service in England and Wales than in Scotland, and hence have to this extent less severe labour and the better conditions as regards food and housing above referred to, at the time of greater predisposition to pulmonary tuberculosis as compared with males.

Agricultural Labour.—Since 1871, on an average 6·2 per cent. of the occupied females above 10 years were engaged as agricultural labourers in Scotland, while in England and Wales only 0·9 per cent. were thus employed. There was 87 per cent. more females thus employed at age period 10–20 than in England and Wales. Though this is outdoor work, it is often hard, and, in the past at all events, the sleeping accommodation was insufficient both as regards space and ventilation. In the rural districts of Scotland, therefore, the female population above 10 years of age have been much more engaged in the hard work of agricultural labour.

Textile Industries.—The chief textile industry in Scotland is jute manufacture; in England and Wales it is cotton manufacture. There is a great

* Census England and Wales 1891, vol. iv. p. 40.

difference between the two countries in the relative number of females employed in these industries. In Scotland since 1871 on an average nearly 26 per cent. of the occupied females above 10 years of age were engaged in textile manufacture, while in England and Wales less than 17 per cent. were so employed; 33 per cent. fewer females from 10–20 years of age were engaged in it in England and Wales than in Scotland.

Dress.—This includes dressmaking, tailoring, staymaking, millinery, etc. The work is on an average of a lighter nature than either agricultural or textile labour. In Scotland 13 per cent. and in England and Wales above 18 per cent. of the occupied females above 10 years of age were so employed. Nearly 40 per cent. more females from 10–20 years of age were employed in this industry in England and Wales than in Scotland.

Thus there is and has been in Scotland since 1871 a markedly smaller number of the female population between 10 and 20 years engaged in the relatively light occupations of indoor domestic service and dress manufacture, and a markedly larger number in the harder work of agricultural and textile industries, than in England. Any diminution in the bodily resistance produced by these conditions would probably be chiefly operative on the mortality rates between 15 and 35 years, and it is between these age periods that the reduction in female mortality is so markedly smaller in Scotland (Plate II.). In the following Table, where the rate at each age period is compared with that at all ages taken as 100, the great diminution since 1861–70 at age period 15–35 in England as compared with Scotland is well seen.

Table XI.—Comparative Mortality from Tuberculosis of the Lungs among Females in Scotland and England and Wales.

	1861–70.		1901–5.	
	Scotland.	England and Wales.	Scotland.	England and Wales.
Females all ages . . .	100	100	100	100
„ 0–5 years . . .	56	38	31	31
„ 5–10 „ . . .	33	19	28	18
„ 10–15 „ . . .	58	41	59	40
„ 15–20 „ . . .	130	126	130	96
„ 20–25 „ . . .	150	156	149	121
„ 25–35 „ . . .	153	172	165	152
„ 35–45 „ . . .	137	153	154	172
„ 45–55 „ . . .	104	112	108	142
„ 55–65 „ . . .	80	81	71	110
„ 65–75 „ . . .	49	49	43	79

MORTALITY FROM SCOTLAND AND IN ENGLAND AND WALES.

SCOTLAND	0	55–65		65–75		75 +	
	M.	M.	F.	M.	F.	M.	F.
1861–70	15·00	27·00	21·35	19·00	13·00	7·90	5·70
1871–80	12·30	23·26	16·40	16·54	10·27	6·90	6·45
1881–90	7·00	20·06	13·05	14·50	8·14	5·82	4·34
1891–1900	5·26	22·00	11·40	13·86	7·49	5·60	3·70
1901–5	4·73	19·03	10·05	13·72	6·05	5·47	2·88
1906–7	4·38	18·82	9·84	14·05	7·66	4·60	2·43
ENGLAND AND WALES	0	55–65		65–75		75 +	
	M.	M.	F.	M.	F.	M.	F.
1861–70	9·94	33·12	20·75	20·37	12·46	6·63	4·48
1871–80	7·87	32·06	17·86	19·28	10·97	6·04	4·07
1881–90	5·53	29·20	15·15	18·23	9·80	6·90	3·98
1891–1900	4·41	26·18	12·39	15·84	8·07	5·56	3·52
1901–5	3·63	25·06	11·11	15·85	8·01	5·51	3·69
1906–7	3·42	24·59	10·49	16·46	9·49	5·15	3·55

TABLE III.

Mortality from Tuberculosis of the Lungs among Males and Females per 10,000 of each Sex Living at Age Periods in Scotland and in England and Wales.

Scotland	0–5		5–10		10–15		15–20		20–25		25–35		35–45		45–55		55–65		65–75		75 +	
	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.
1861–70	15·00	15·06	8·21	8·82	10·10	15·60	28·40	34·90	47·36	40·07	42·60	41·00	33·10	36·70	30·00	27·85	27·00	21·35	19·00	13·00	7·90	5·70
1871–80	12·30	12·20	7·22	8·29	9·72	15·64	27·70	36·56	42·80	42·70	39·47	39·70	32·32	33·48	28·36	23·57	23·26	16·40	16·54	10·27	6·90	6·45
1881–90	7·00	6·60	4·68	6·46	6·88	13·04	23·00	30·84	33·93	34·55	32·72	36·79	28·77	23·04	25·55	18·43	20·06	13·05	14·50	8·14	5·82	4·34
1891–1900	5·26	5·00	2·80	4·60	4·52	10·12	18·70	24·03	30·31	26·29	29·19	30·08	29·00	26·09	25·37	15·31	22·00	11·40	13·86	7·49	5·60	3·70
1901–5	4·73	4·33	2·39	4·00	3·66	8·36	13·65	18·43	22·03	21·17	23·98	23·47	23·64	21·80	23·14	15·33	19·03	10·05	13·72	6·05	5·47	2·88
1906–7	4·38	3·93	2·36	3·19	2·92	8·22	11·75	17·12	19·81	19·37	22·82	22·00	22·54	20·24	23·30	14·32	18·82	9·84	14·05	7·66	4 60	2·43
England and Wales	0–5		5–10		10–15		15–20		20–25		25–35		35–45		45–55		55–65		65–75		75 +	
	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.
1861–70	9·94	9·51	4·33	4·79	6·08	10·50	21·96	31·21	38·94	39·72	41·11	43·95	41·70	39·09	38·80	28·67	33·12	20·75	20·37	12·46	6·63	4·48
1871–80	7·87	7·53	3·42	3·77	4·83	8·51	16·85	24·09	31·09	31·54	37·13	35·56	41·37	34·12	38·65	24·68	32·06	17·86	19·28	10·97	6·04	4·07
1881–90	5·53	5·18	2·54	3·28	3·44	7·02	12·93	18·09	23·41	23·26	30·37	28·01	35·77	27·40	35·05	20·62	29·20	15·15	18·23	9·80	6·90	3·98
1891–1900	4·41	3·85	1·74	2·39	2·34	5·02	9·95	12·90	18·87	15·91	23·69	19·23	30·95	21·21	31·44	16·42	26·18	12·39	15·84	8·07	5·56	3·52
1901–5	3·63	3·14	1·49	1·96	1·73	4·01	7·65	9·74	15·72	12·26	21·00	15·41	26·87	17·34	30·41	14·37	25·06	11·11	15·85	8·01	5·51	3·69
1906–7	3·42	2·98	1·26	1·98	1·52	3·67	7·33	9·51	13·90	11·86	19·54	14·59	25·34	15·96	28·65	13·41	24·59	10·49	16 46	9·49	5·15	3·55

ND OF BOTH SEXES,

Period.	S	Mainland Rural.					
		Death-rate.*			Ratio.		
		Both Sexes.	M.	F.	Both Sexes.	M.	F.
1871–75		19·00	19·15	18·85	100	100	100
1876–80		18·13	18·16	18·10	95	95	96
1881–85		16·10	15·29	16·88	85	80	90
1886–90		15·40	15·06	15·73	81	79	83
1891–95		14·34	14·21	14·45	75	74	77
1896–1900		14·90	15·44	14·37	78	81	76
1901–5		13·49	13·70	13·28	71	72	70
1906–7		13·84	14·30	13·39	73	75	71

BOTH SEXES, IN SCOTLAND
OTLAND.

Period.	Rural Area.†				
	th-rate.‡		Ratio.		
	M.	F.	Both Sexes.	M.	F.
1871–75	21·80	22·26	100	100	100
1876–80	20·36	20·97	94	93	94
1881–85	17·06	19·19	82	78	86
1886–90	16·38	17·36	77	75	78
1891–95	15·52	15·79	71	71	71
1896–1900	15·75	15·36	70	72	69
1901–5	13·37	13·23	60	61	59
1906–7	13·34	12·97	60	61	58

* Consis x and age constitution of Scotland, census 1901.

PROF. HUNT

TABLE IV.

Mortality from Tuberculosis of the Lungs per 10,000 Living at all Ages, of Males, of Females, and of both Sexes, in Four Registration Areas of Scotland.

Period.	Principal Towns.						Large Towns.						Small Towns.						Mainland Rural.					
	Death-rate.*			Ratio.			Death-rate.*			Ratio.			Death-rate.*			Ratio.			Death-rate.*			Ratio.		
	Both Sexes.	M.	F.	Both Sexes.	M.	F.	Both Sexes.	M.	F.	Both Sexes.	M.	F.	Both Sexes.	M.	F.	Both Sexes.	M.	F.	Both Sexes.	M.	F.	Both Sexes.	M.	F.
1871–75	31·40	30·72	32·07	100	100	100	31·25	28·85	33·66	100	100	100	26·06	25·31	26·78	100	100	100	19·00	19·15	18·85	100	100	100
1876–80	28·19	28·04	28·34	90	91	88	28·85	27·03	30·63	92	94	91	23·93	23·18	24·64	92	92	92	18·13	18·16	18·10	95	95	96
1881–85	25·97	25·51	26·42	83	83	82	22·74	21·14	24·30	73	73	72	20·70	19·27	22·05	79	76	82	16·10	15·29	16·88	85	80	90
1886–90	22·06	22·38	21·76	70	73	68	20·77	19·50	21·98	66	68	65	18·66	18·00	19·31	72	71	72	15·40	15·06	15·73	81	79	83
1891–95	20·54	21·33	19·73	65	70	62	19·10	18·21	19·98	61	63	59	17·10	16·93	17·26	66	67	64	14·34	14·21	14·45	75	74	77
1896–1900	18·81	20·24	17·47	60	66	54	17·80	17·62	17·98	57	61	53	16·23	16·08	16·37	62	63	61	14·90	15·44	14·37	78	81	76
1901–5	15·20	15·80	14·62	48	51	46	15·33	14·63	16·00	49	51	48	13·11	13·04	13·17	50	51	49	13·49	13·70	13·28	71	72	70
1906–7	13·45	13·93	12·99	43	45	40	15·14	14·86	15·42	48	51	46	12·49	12·42	12·55	48	49	47	13·84	14·30	13·39	73	75	71

* Corrected to sex and age constitution of Scotland, census 1901.

TABLE V.

Mortality from Tuberculosis of the Lungs per 10,000 Living at all Ages, of Males, of Females, and of both Sexes, in Scotland (excluding the Insular Rural Districts) and in the Urban* and Rural† Areas of Scotland.

Period.	Scotland (excluding Insular Rural Districts).						Urban Area.*						Rural Area.†					
	Death-rate.‡			Ratio.			Death-rate.‡			Ratio.			Death-rate.‡			Ratio.		
	Both Sexes.	M.	F.	Both Sexes.	M.	F.	Both Sexes.	M.	F.	Both Sexes.	M.	F.	Both Sexes.	M.	F.	Both Sexes.	M.	F.
1871–75	26·18	25·56	26·77	100	100	100	31·39	30·28	32·44	100	100	100	22·04	21·80	22·26	100	100	100
1876–80	24·19	23·76	24·59	92	93	92	28·36	27·80	28·90	90	92	89	20·67	20·36	20·97	94	93	94
1881–85	21·75	20·85	22·61	83	82	84	25·28	24·57	25·97	81	81	80	18·15	17·06	19·19	82	78	86
1886–90	19·45	19·23	19·67	74	75	73	21·77	21·74	21·81	69	72	67	16·88	16·38	17·36	77	75	78
1891–95	18·10	18·25	17·95	69	71	67	20·17	20·57	19·79	64	68	61	15·66	15·52	15·79	71	71	71
1896–1900	17·24	17·90	16·61	66	70	62	18·57	19·60	17·60	59	65	54	15·55	15·75	15·36	70	72	69
1901–5	14·43	14·64	14·23	55	57	53	15·22	15·55	14·93	48	51	46	13·29	13·37	13·23	60	61	59
1906–7	13·54	13·81	13·29	52	54	50	13·82	14·12	13·52	44	47	42	13·15	13·34	12·97	60	61	58

* Consisting of principal town and large town districts. † Consisting of small town and mainland rural districts. ‡ Corrected to sex and age constitution of Scotland, census 1901.

Prof. Hunter Stewart.

Mortality fr n Urban and Rural Areas of Scotland.

	0-	55–60		60–65		65–70		70–75		75+	
	M.	I.	F.	M.	F.	M.	F.	M.	F.	M.	F.
Urban Area (consisting of Principal Town and Large Town Districts).											
1871–80 . . .	18·79	·53	23·02	23·80	15·29	23·24	11·66	14·02	7·59	9·0	6·67
1881–90 . . .	9·69	·51	15·22	21·84	12·44	18·15	9·14	8·66	5·92	6·45	3·97
1891–1900 . . .	6·84	·52	15·21	23·76	9·43	19·45	10·36	11·90	4·89	7·53	3·70
1901–5 . . .	6·14	·47	11·32	18·97	9·09	17·93	9·0	11·50	4·40	6·29	3·40
1906–7 . . .	5·60	·95	11·33	19·82	8·51	20·63	7·30	10·96	5·63	4·82	2·51
Rural Area (consisting of Small Town and Mainland Rural Districts).											
1871–80 . . .	7·47	·36	15·86	18·81	13·33	18·10	12·29	12·39	8·53	6·03	6·62
1881–90 . . .	4·33	·51	13·48	17·37	11·36	17·0	9·58	11·80	5·97	5·28	4·55
1891–1900 . . .	3·53	·15	11·44	15·85	9·27	14·18	8·27	10·90	5·54	4·80	3·72
1901–5 . . .	2·97	·00	10·34	13·81	9·19	13·97	7·26	10·22	4·74	4·94	2·68
1906–7 . . .	2·74	·34	11·75	13·39	7·84	17·45	10·17	7·58	8·03	4·06	2·05

Mortality s in the Principal Town Districts

	0-5	55–60		60–65		65–70		70–75		75+	
	M.	.	F.	M.	F.	M.	F.	M.	F.	M.	F.
Principal Towns.											
1871–75 . . .	20·10	90	24·30	26·00	17·50	21·30	11·40	15·70	8·90	8·60	7·00
1876–80 . . .	17·83	96	21·52	20·45	11·65	25·43	12·17	14·97	6·00	8·60	5·33
1881–85 . . .	12·53	00	16·40	20·00	12·33	18·52	8·34	8·75	6·30	6·83	3·53
1886–90 . . .	8·77	00	15·10	23·75	11·71	20·14	8·23	6·88	5·89	4·26	4·62
1891–95 . . .	7·69	27	14·98	25·61	8·83	19·77	10·79	10·16	4·32	6·32	2·77
1896–1900 . . .	6·53	35	15·46	23·51	8·97	23·90	10·59	11·91	4·84	9·16	3·48
1901–5 . . .	6·75	30	10 52	20·26	9·38	20·00	9·00	11·53	4·33	6·04	3·18
1906–7 . . .	6·05	82	11·85	20·20	8·80	20·66	7·71	10·16	5·96	5·98	3·36
Mainland Rural.											
1871–75 . . .	5·30	20	15·30	18·80	13·20	17·00	12·90	13·20	6·50	5·50	5·60
1876–80 . . .	5·02	44	14·19	14·31	12·17	17·68	10·54	13·63	7·43	4·25	5·80
1881–85 . . .	3·59	77	14·03	15·64	12 54	16·06	8·54	11·54	5·32	5·11	4·40
1886–90 . . .	2·84	16	13·80	17·05	10·33	14·96	10·50	11·63	5·18	4·36	4·29
1891–95 . . .	2·98	72	11·55	14·16	9·23	13·08	8·62	12·14	5·78	4·84	3·30
1896–1900 . . .	2·36	73	10·18	14·98	8·95	15·72	6·97	9·34	5·90	5·55	4·20
1901–5 . . .	2 68	01	11·35	14·88	8·74	13·72	7·38	9·33	4·08	5·04	2·46
1906–7 . . .	2·05	26	10·82	11·72	8·11	18·25	12·34	8·18	6·50	4·39	1·93

Prof. Hunter Stewart.

TABLE VI.

Mortality from Tuberculosis of the Lungs among Males and Females per 10,000 of each Sex Living at Age Periods in Urban and Rural Areas of Scotland.

	0–5		5–10		10–15		15–20		20–25		25–30		30–35		35–40		40–45		45–50		50–55		55–60		60–65		65–70		70–75		75+	
	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.
Urban Area (consisting of Principal Town and Large Town Districts).																																
1871–80 . . .	18·79	18·23	11·27	12·0	13·63	19·83	34·37	42·88	41·66	46·34	39·69	45·69	40·15	45·21	43·94	45·24	37·84	39·25	39·67	32·88	33·92	24·61	30·53	23·02	23·80	15·29	23·24	11·66	14·02	7·59	9·0	6·67
1881–90 . . .	9·69	8·92	6·54	8·36	8·60	14·97	28·19	34·01	34·76	37·78	34·80	40·59	35·02	41·84	35·49	35·61	33·15	29·83	35·86	24·74	28·58	18·06	25·51	15·22	21·84	12·44	18·15	9·14	8·66	5·92	6·45	3·97
1891–1900 . . .	6·84	6·41	3·68	5·56	5·04	10·25	20·84	24·71	30·35	27·51	29·98	32·02	29·98	32·04	32·65	31·56	34·14	27·11	31·60	19·59	29·47	14·32	30·52	15·21	23·76	9·43	19·45	10·36	11·90	4·89	7·53	3·70
1901–5 . . .	6·14	5·46	2·87	5·01	3·83	8·34	14·27	18·13	19·71	20·88	22·05	23·00	24·31	24·67	25·16	25·77	25·42	22·14	27·47	19·29	27·30	15·05	25·47	11·32	18·97	9 09	17·93	9·0	11·50	4·40	6·29	3·40
1906–7 . . .	5·60	4·79	3·11	3·98	2·80	8·07	11·13	16·86	18·00	18·51	18·85	23·24	22·24	21·28	23·27	21·64	22·89	20·14	24·81	16·85	26·22	13·62	23·95	11·33	19·82	8·51	20·63	7·30	10·96	5·63	4·82	2·51
Rural Area (consisting of Small Town and Mainland Rural Districts).																																
1871–80 . . .	7·47	7·78	4·41	5·91	7·39	13·08	23·01	31·97	46·03	39·62	41·23	37·95	33·13	31·88	24·75	28·58	22·50	24·13	22·29	21·64	22·38	18·05	23·36	15·86	18·81	13·33	18·10	12·29	12·39	8·53	6·03	6·62
1881–90 . . .	4·33	4·45	3·02	4·70	5·55	11·68	18·44	27·58	32·48	31·0	30·41	33·98	27·10	30·90	23·93	24·60	20·75	21·30	19·54	17·04	19·03	13·90	17·51	13·48	17·37	11·36	17·0	9·58	11·80	5·97	5·28	4·55
1891–1900 . . .	3·53	3·44	1·98	3·77	4·13	10·08	16·13	23·42	29·22	25·13	29·10	27·82	25·82	26·81	24·40	23·84	22·03	19·95	21·39	11·14	19·15	11·77	19·15	11·44	15·85	9·27	14·18	8·27	10·90	5·54	4·80	3·72
1901–5 . . .	2·97	2·77	1·85	2·79	3·55	8·34	12·44	18·52	23·79	21·05	24·51	22·53	23·73	23·71	21·21	21·50	18·91	15·84	18·71	13·96	16·95	11·50	17·00	10·34	13·81	9·19	13·97	7·26	10·22	4·74	4·94	2·68
1906–7 . . .	2·74	2·78	1·42	2·12	2·98	8·26	11·07	17·62	20·71	20·20	24·63	21·06	25·35	21·94	21·86	20·23	20·96	18·07	19·93	14·90	21·65	12·84	18·34	11·75	13·39	7·84	17·45	10·17	7·58	8·03	4·06	2·05

TABLE VII.

Mortality from Tuberculosis of the Lungs among Males and Females per 10,000 of each Sex Living at Age Periods in the Principal Town Districts and Mainland Rural Districts of Scotland.

	0–5		5–10		10–15		15–20		20–25		25–30		30–35		35–40		40–45		45–50		50–55		55–60		60–65		65–70		70–75		75+	
	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.
Principal Towns.																																
1871–75 . . .	20·10	19·00	12·70	13·00	13·90	19·10	34·80	42·20	43·20	49·00	39·40	47·10	44·40	46·00	46·00	49·60	43·20	43·90	43·50	38·30	37·90	27·50	34 90	24·30	26·00	17·50	21·30	11·40	15·70	8·90	8·60	7·00
1876–80 . . .	17·83	17·77	11·18	11·92	13·80	19·58	33·18	42·07	38·74	41·63	37·83	42·97	41·41	39·59	44·68	41·37	35·60	35·86	39·50	29·20	31·29	21·04	27·96	21·52	20·45	11·65	25·43	12·17	14·97	6·00	8·60	5·33
1881–85 . . .	12·53	11·16	8·53	10·26	10·20	16·12	32·19	38·26	38·57	41·83	37·91	45·32	36·56	44·18	37·72	37·84	37·05	34·82	36·89	24·96	29·79	18·44	28·00	16·40	20·00	12·33	18·52	8·34	8·75	6·30	6·83	3·53
1886–90 . . .	8·77	7·88	5·30	6 79	7·56	13·26	26·26	29·56	31·85	33·38	33·30	36·52	35·00	38·84	36·10	34·40	33·28	27·17	39·04	23·19	30·05	18·17	25·00	15·10	23·75	11·71	20·14	8·23	6·88	5·89	4·26	4·62
1891–95 . . .	7·69	7·71	4·41	6·30	5·64	11·14	23·70	26·35	32·38	27·56	30·97	33·94	34·14	34·63	34·16	34·16	35·22	28·98	32·07	19·31	31·03	14·61	30·27	14·98	25·61	8·83	19·77	10·79	10·16	4·32	6·32	2·77
1896–1900 . . .	6·53	6·05	3·23	4·89	4·84	8·82	19·34	20·48	27·69	25·86	29·87	29·63	29·65	30·63	33·51	31·13	36·22	26·62	35·70	20·35	31·99	14·37	34·85	15·46	23·51	8·97	23·90	10·59	11·91	4·84	9·16	3·48
1901–5 . . .	6·75	5·82	3·09	5·22	3·97	8·21	13·89	16·79	19·33	20·11	22·16	21·96	23·99	24·18	24·86	25·40	27·61	22·40	29·17	19·28	28·31	15·19	26·30	10 52	20·26	9·38	20·00	9·00	11·53	4·33	6·04	3·18
1906–7 . . .	6·05	4·87	3·24	4·31	2·78	8·10	11·48	15·82	17·45	17·50	18·09	21·64	21·80	18·67	22·47	20·62	23·31	20·44	24·07	16·18	26 00	13 04	24·82	11·85	20·20	8·80	20·66	7·71	10·16	5·96	5·98	3·36
Mainland Rural.																																
1871–75 . . .	5·30	5·10	3·10	4·20	5·30	10·20	19·20	27·00	42·40	33·60	40·90	33·40	31·50	28·40	25·40	26·20	19·10	22·50	20·10	21·30	19·70	16·50	18 20	15·30	18·80	13·20	17·00	12·90	13·20	6·50	5·50	5·60
1876–80 . . .	5·02	5·54	2·87	4·69	5·75	10·85	18·21	24·95	37·28	33·99	39·39	31·53	29·26	29·59	25·48	24·43	20·05	18·71	17·66	18·52	18·16	14·34	22·44	14·19	14·31	12·17	17·68	10·54	13·63	7·43	4·25	5·80
1881–85 . . .	3·59	3·32	2·56	4·24	4·99	9·68	16·55	22·33	30·57	29·00	31·44	32·34	23·52	30·36	22·02	24·36	17·39	19·90	15·63	15·97	17·96	12·80	15·77	14·03	15·64	12 54	16·06	8·54	11·54	5·32	5·11	4·40
1886–90 . . .	2·84	3·14	2·11	3·34	4·59	10·40	15·31	22·37	28·32	25·91	27·70	30·00	26·89	26·31	23·65	21·31	22·23	19·83	17·64	14·73	16 86	12 81	14·16	13·80	17·05	10·33	14·96	10·50	11·63	5·18	4·36	4·29
1891–95 . . .	2·98	3·73	1·87	3·33	3·80	8·81	14·29	21·28	26·67	23·31	27·62	27·09	22·77	24·09	21·37	22·21	19·72	17·91	18 61	11·88	18·10	10·34	17·72	11·55	14·16	9·23	13·08	8·62	12·14	5·78	4·84	3·30
1896–1900 . . .	2·36	2·46	1·39	2·85	3·56	8·04	14·34	21·20	29·28	22·60	29·90	26·26	28·32	25·84	25·31	20·94	23·30	19·86	20·88	15·41	16·17	12·07	18·73	10·18	11 98	8·95	15·72	6·97	9·34	5·90	5·55	4·20
1901–5 . . .	2 68	2·17	1·21	2·36	3·46	8·53	11·90	17·86	24·42	22·46	27·26	21·19	25·88	23·14	22·07	21·83	18·49	16·52	19·25	13·92	17·46	10·84	18·01	11·35	14·88	8·74	13·72	7·38	9·33	4·08	5·04	2·46
1906–7 . . .	2·05	2·48	0·95	1·97	3·38	8·39	10·63	18·36	23·04	22·00	29·18	21·72	26·92	22·47	23·82	19·59	25·03	19·63	19·06	14·77	22·80	15·10	22·26	10·82	11·72	8·11	18·25	12·34	8·18	6·50	4·39	1·93

STAT S OF TOWNS IN SCOTLAND, AND OF

	d in Dress.		Engaged in Indoor Domestic Service.		
	Females at Ages		Per 10,000 Females at Ages		
	0–20	Above 20	Above 10	10–20	Above 20
SCOTLAND	476	434	919	1235	806
ENGLAND	646	628	1208	1802	998

		Engaged in Indoor Domestic Service.		
	Ages	Per 10,000 Females at Ages		
	Above 20	Above 10	10–20	Above 20
EDINBURG	592	1519	1632	1484
ABERDEEN	496	909	1146	826
FOUR To Kilmarı and Pei	562	560	680	515
DUNDEE	322	355	396	342
FOUR Tow Govan, Mother	497	408	664	305

PROF. I

TABLE XII.

Statistics (Census 1901) of Occupation of Females in England and Scotland and in certain Towns and Groups of Towns in Scotland, and of their Mortality from Tuberculosis of the Lungs at all Ages (1901–7).

	Death-rate per 10,000 Females 1901–7.	Occupied. Per 10,000 Females at Ages			Engaged as Agricultural Labourers. Per 10,000 Females at Ages			Engaged in Textiles. Per 10,000 Females at Ages			Engaged in Dress. Per 10,000 Females at Ages			Engaged in Indoor Domestic Service. Per 10,000 Females at Ages		
		Above 10	10–20	Above 20	Above 10	10–20	Above 20	Above 10	10–20	Above 20	Above 10	10–20	Above 20	Above 10	10–20	Above 20
Scotland . . .	13·93	3376	4298	3048	208	260	190	861	1241	727	445	476	434	919	1235	806
England and Wales .	9·94	3419	4162	3155	31	33	30	563	836	466	633	646	628	1208	1802	998

	Death-rate per 10,000 Females 1901–7.	Occupied. Per 10,000 Females at Ages			Engaged in Textiles. Per 10,000 Females at Ages			Engaged in Dress. Per 10,000 Females at Ages			Engaged in Indoor Domestic Service. Per 10,000 Females at Ages		
		Above 10	10–20	Above 20	Above 10	10–20	Above 20	Above 10	10–20	Above 20	Above 10	10–20	Above 20
Edinburgh . . .	11·92	3950	4638	3735	76	96	70	626	736	592	1519	1632	1484
Aberdeen . . .	12·97	3674	4803	3274	687	1203	506	500	508	496	909	1146	826
Four Towns (Paisley, Kilmarnock, Kirkcaldy and Perth), Group 2 .	13·60	4119	5560	3587	1952	3172	1502	547	507	562	560	680	515
Dundee . . .	17·05	5285	6617	4826	3898	5270	3426	317	305	322	355	396	342
Four Towns (Coatbridge, Govan, Hamilton, Motherwell), Group 1	13·07	2324	3626	1804	222	384	158	614	909	497	408	664	305

Fever, per 10,000

	) and Wales. *				
		10–15		15–20	
		M.	F.	M.	F.
	5 †	1·15 †	1·28 †	0·55 †	0·61 †
Tuberculo	9	1·10	1·20	0·60	0·63
	2	1·14	1·22	0·62	0·64
	2 †	0·67 †	0·84 †	0·47 †	0·64 †
Tabes mes culous p	4	0·62	0·74	0·49	0·66
	7	0·65	0·75	0·47	0·65
	1	0·44	0·52	0·57	0·62
Bronchitis	8	0·38	0·43	0·42	0·46
	3	0·25	0·29	0·30	0·33
		...	...	...	...
	7	1·02	1·10	1·95	1·51
Pneumonia	6	1·18	1·22	2·37	1·68
	1	1·10	1·13	2·56	1·57
		...	...	...	...
	6	2·74	3·52	3·77	4·38
Enteric fev	9	1·92	2·26	3·00	2·81
	4	1·52	1·72	2·79	2·33
		...	...	...	...

the period 1881–90.

Prof. H

TABLE XIV.

Mortality from Tuberculous Meningitis, Tuberculous Peritonitis, Bronchitis, Pneumonia, and Enteric Fever, per 10,000 at Age Periods in Scotland and in England and Wales.

		Scotland.								England and Wales.*							
		0–5		5–10		10–15		15–20		0–5		5–10		10–15		15–20	
		M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.
Tuberculous meningitis	1886–90	16·72	14·30	4·03	4·58	1·86	1·76	0·75	0·58	16·88 †	12·25 †	3·19 †	2·95 †	1·15 †	1·28 †	0·55 †	0·61 †
	1891–1900	17·62	15·72	3·62	3·93	1·38	1·75	0·82	0·85	14·81	11·61	2·83	2·69	1·10	1·20	0·60	0·63
	1901–5	16·56	13·63	3·37	3·73	1·45	1·58	0·60	0·80	11·76	10·24	2·75	2·62	1·14	1·22	0·62	0·64
Tabes mesenterica (tuberculous peritonitis)	1886–90	11·98	10·0	2·76	2·69	1·68	1·81	0·73	1·07	20·05 †	16·12 †	1·21 †	1·22 †	0·67 †	0·84 †	0·47 †	0·64 †
	1891–1900	12·06	9·83	2·26	2·23	1·19	1·78	0·84	1·14	16·13	13·04	1·02	1·04	0·62	0·74	0·49	0·66
	1901–5	11·86	9·38	2·51	2·84	1·57	1·66	0·76	1·27	11·76	9·35	1·02	1·07	0·65	0·75	0·47	0·65
Bronchitis	1871–80	96·70	83·7	3·76	4·06	1·21	1·28	1·16	1·30	76·86	64·21	1·78	1·91	0·44	0·52	0·57	0·62
	1881–90	68·90	60·0	2·26	2·65	0·68	1·04	0·80	1·0	81·62	68·19	1·75	1·88	0·38	0·43	0·42	0·46
	1891–1900	54·28	45·34	1·26	1·31	0·46	0·52	0·54	0·64	68·96	57·20	1·06	1·13	0·25	0·29	0·30	0·33
	1901–5	39·40	31·87	0 69	0·83	0·26	0·43	0·36	0·39	...	...	...	..	...	...	...	...
Pneumonia	1871–80	27·70	22·80	2·60	2·69	1·29	1·31	2·63	1·89	43·70	36·28	2·61	2·57	1·02	1·10	1·95	1·51
	1881–90	34·69	29·50	3·06	3·09	1·57	1·37	3·0	2·20	40·0	33·29	3·03	2·96	1·18	1·22	2·37	1·68
	1891–1900	51·60	44·0	3·04	2·87	1·46	1·41	3·11	1·84	54·25	44·68	3·03	2·91	1·10	1·13	2·56	1·57
	1901–5	60·50	51·0	3·14	2·96	1·59	1·38	3·21	2·04	...	...	...	...	...	...	...	...
Enteric fever	1871–80	4·47	4·80	4·38	5·35	3·60	4·48	4·95	5·08	3·98	4·05	3·09	3·66	2·74	3·52	3·77	4·38
	1881–90	1·97	2·02	2·59	2·81	2·53	3·20	3·24	3·35	1·31	1·28	1·70	1·89	1·92	2·26	3·00	2·81
	1891–1900	0·86	0·77	1·43	1·70	1·74	1·82	3·00	2·76	0·85	0·80	1·20	1·34	1·52	1·72	2·79	2·33
	1901–5	0·35	0·27	0·85	0·83	0·95	0·94	1·89	1·90	...	...	...	...	...	...	...	...

* Supplement to 65th Annual Report Registrar General for England, and Annual Report for 1906.

† These figures are for the period 1881–90.

Prof. Hunter Stewart.

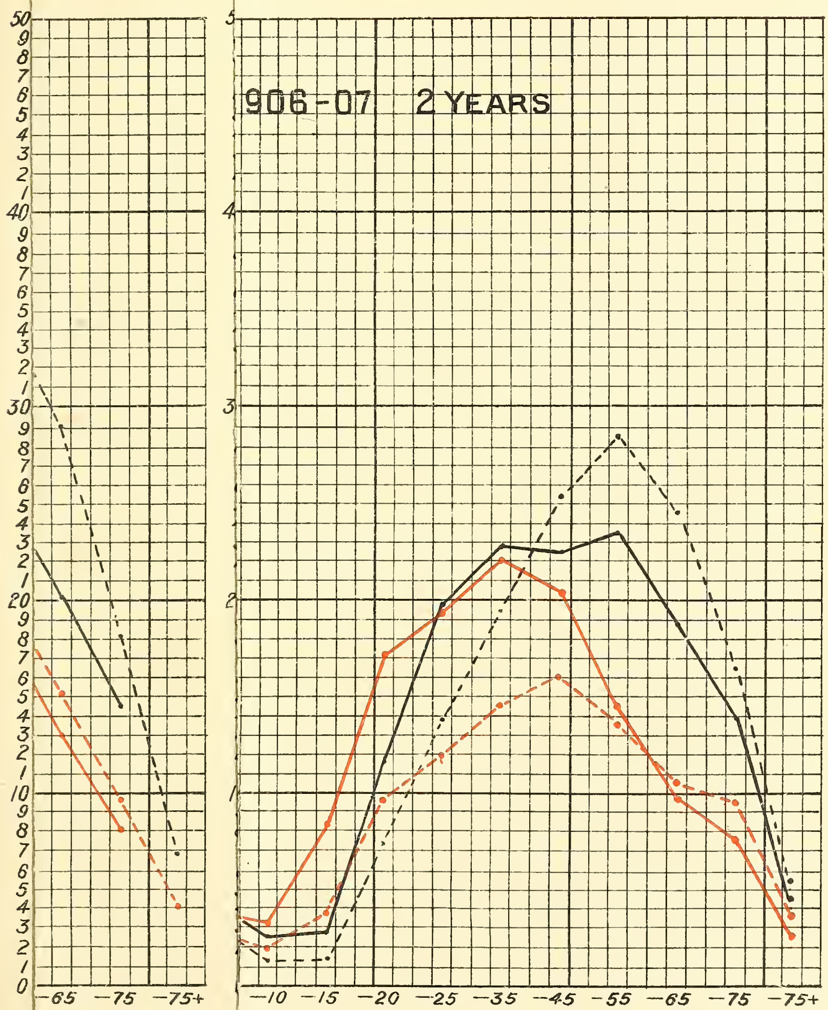
PLAT

CULOSIS OF

F EACH SEX

MALES

LES: MALES



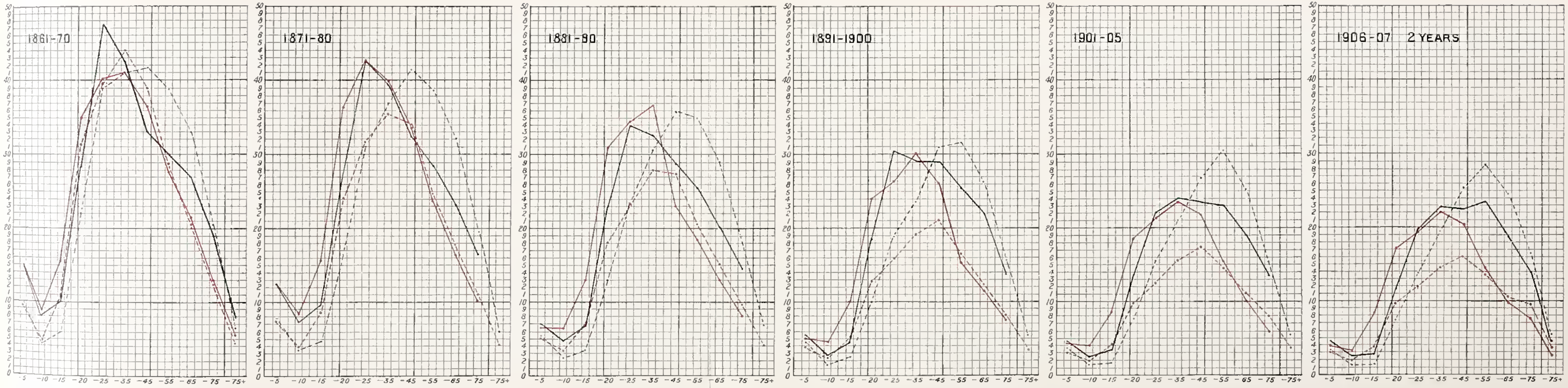
Prof. C.H. Stewart.

Plate I.

DEATH RATE FROM TUBERCULOSIS OF THE LUNGS AMONG MALES & FEMALES
PER 10,000 OF EACH SEX LIVING AT AGE PERIODS

SCOTLAND: MALES ——— FEMALES ———

ENGLAND & WALES: MALES - - - - - FEMALES - - - - -



Prof C. H. Stewart

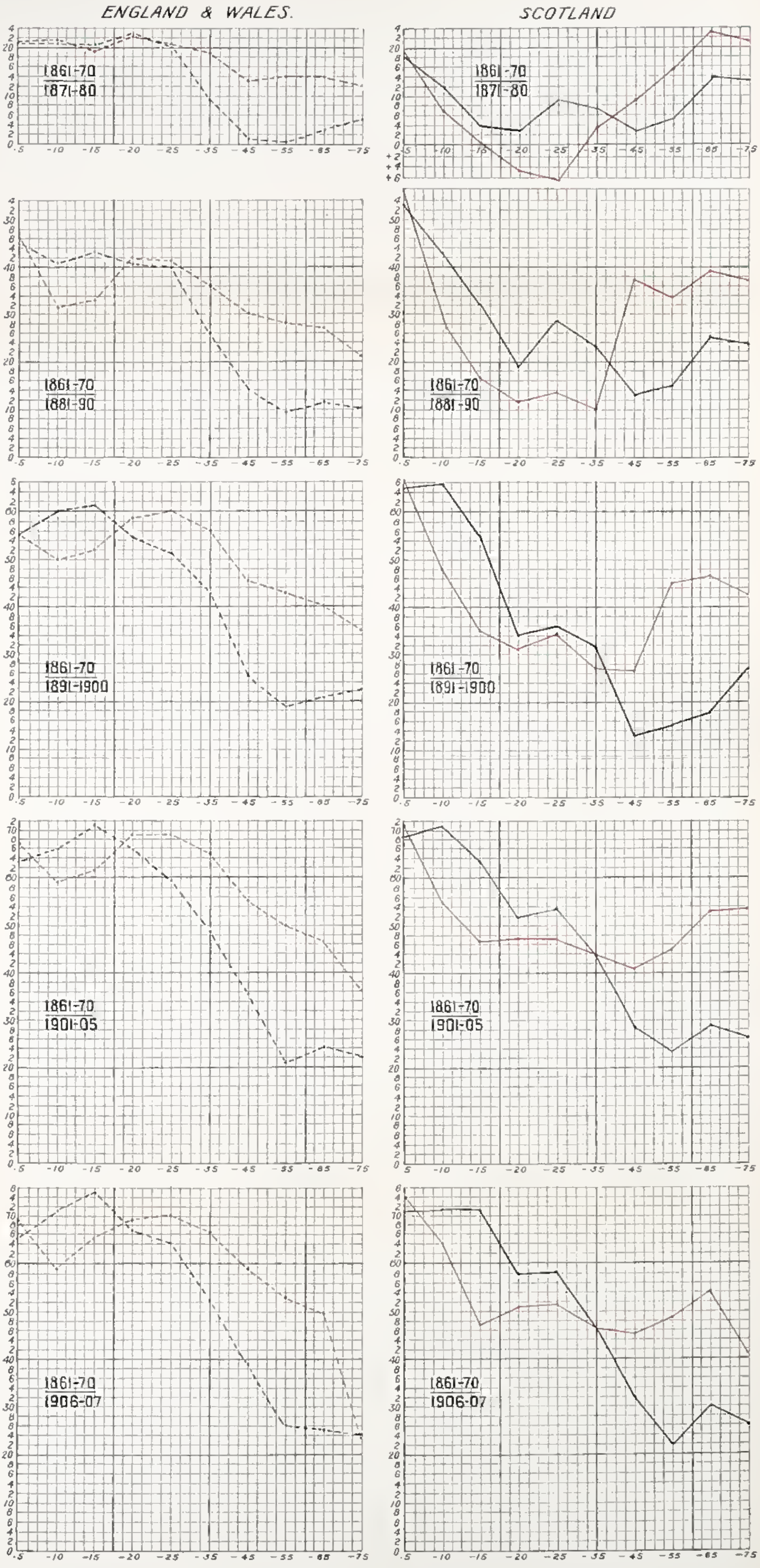
PLATE II.

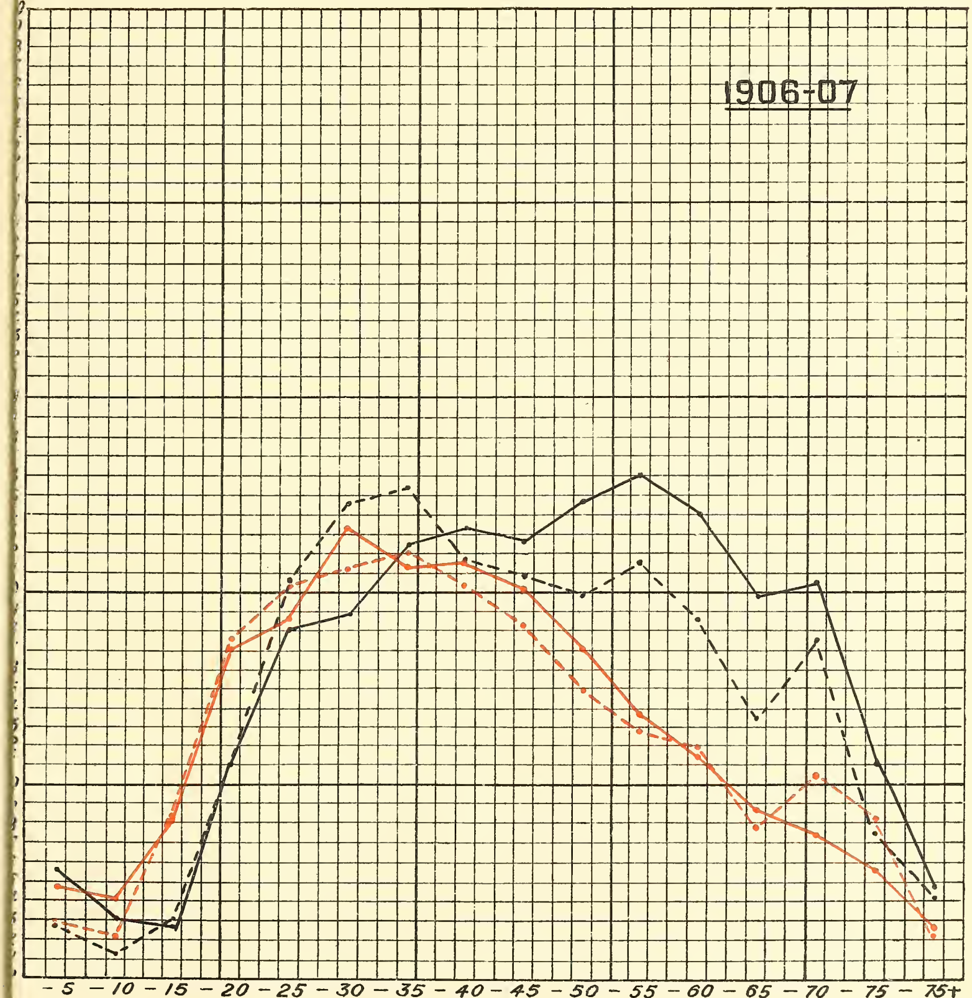
PLATE II.

PERCENTAGE REDUCTION IN THE MALE & FEMALE MORTALITY FROM TUBERCULOSIS OF THE LUNGS AT AGE PERIODS SINCE 1861-70.

MALES BLACK.

FEMALES. RED

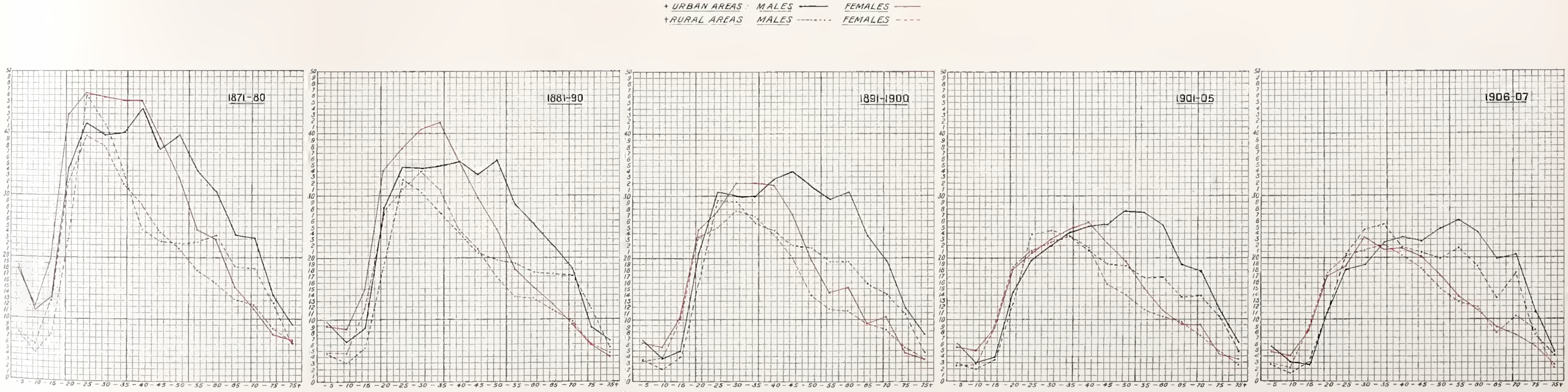




Prof. C. H. Stewart.

PLATE III.

MORTALITY FROM TUBERCULOSIS OF THE LUNGS AMONG MALES & FEMALES PER 10,000 OF EACH SEX LIVING AT AGE PERIODS IN URBAN AND RURAL AREAS OF SCOTLAND.



+ FOR CONSTITUTION OF THESE AREAS SEE TABLE VI

Prof C.H. Stewart

PLATE IV.

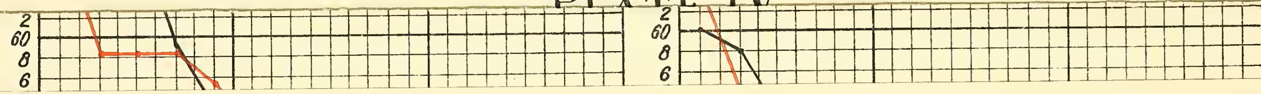
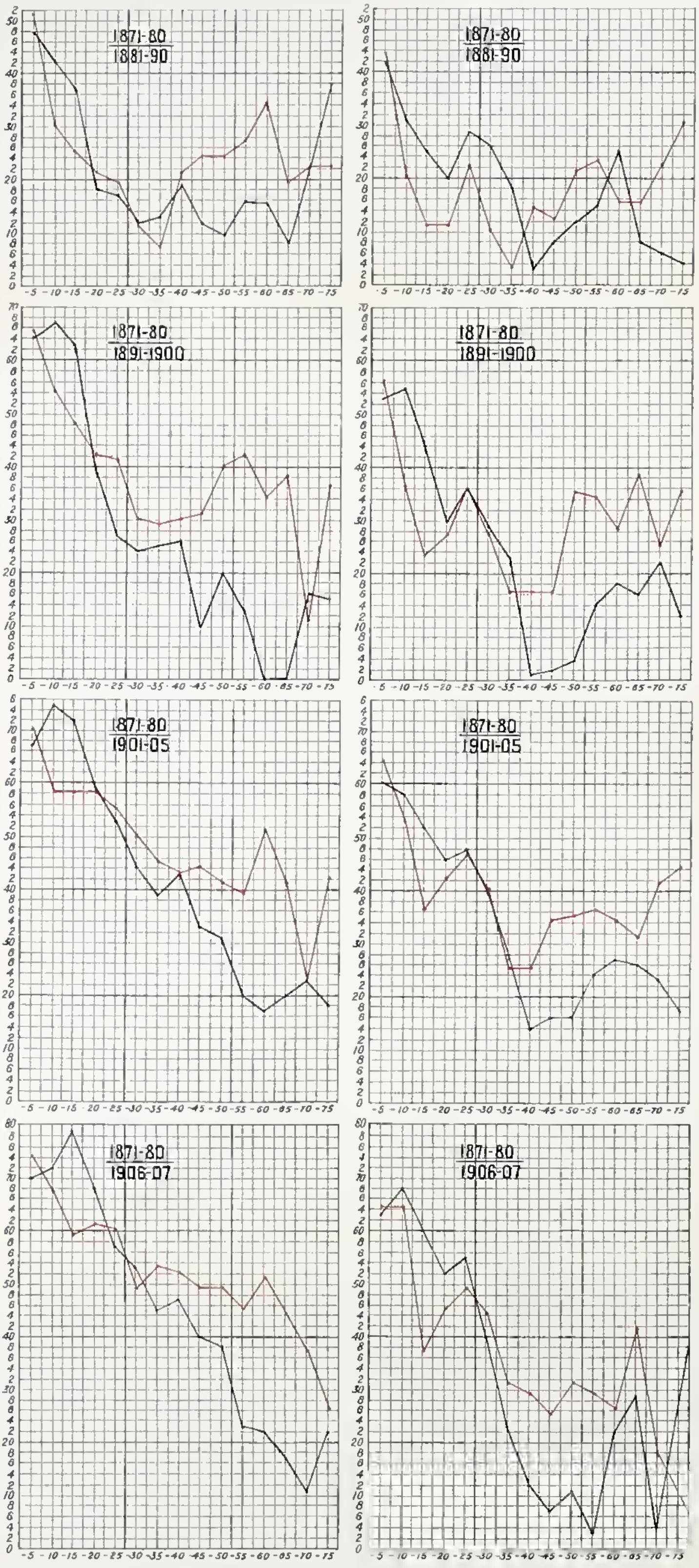


Plate IV.

PERCENTAGE REDUCTION IN THE MALE & FEMALE MORTALITY FROM TUBERCULOSIS OF THE LUNGS AT AGE PERIODS SINCE 1871-80 IN THE URBAN & RURAL AREAS OF SCOTLAND.

MALES - BLACK
FEMALES - RED

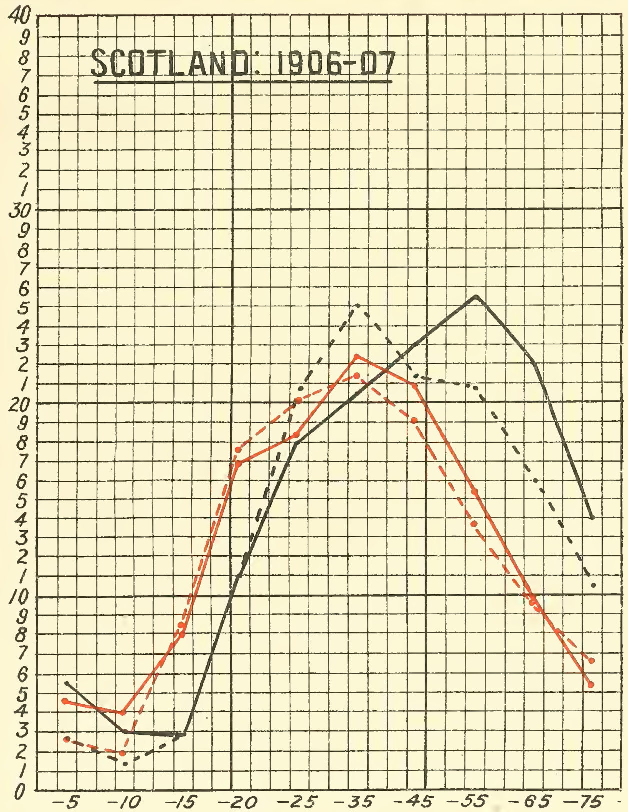
+ URBAN AREA + RURAL AREA

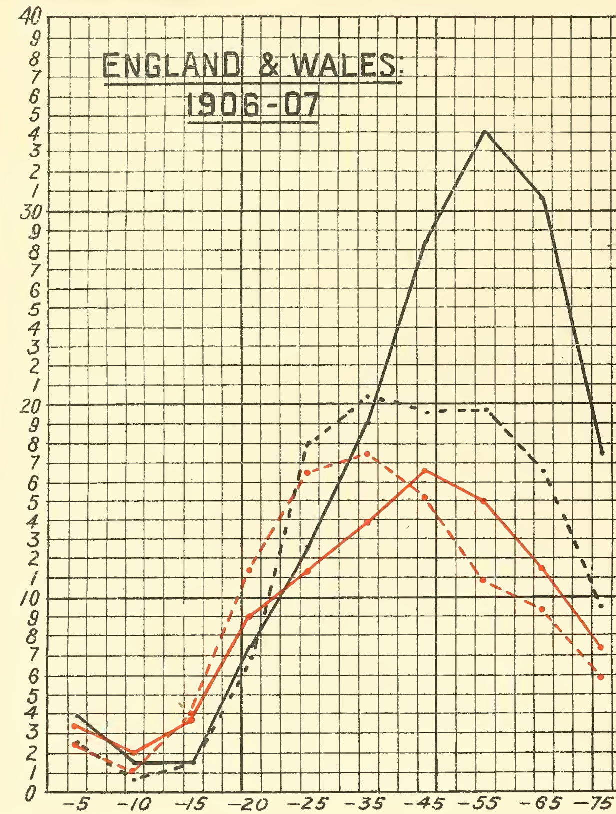


+ FOR CONSTITUTION OF THESE AREAS SEE TABLE VI.

Prof. C.H. Stewart

PLATE V.





† FOR CONSTITUTION OF THESE AREAS IN SCOTLAND SEE TABLE VI.

McFarlane & Erskine, Edinr

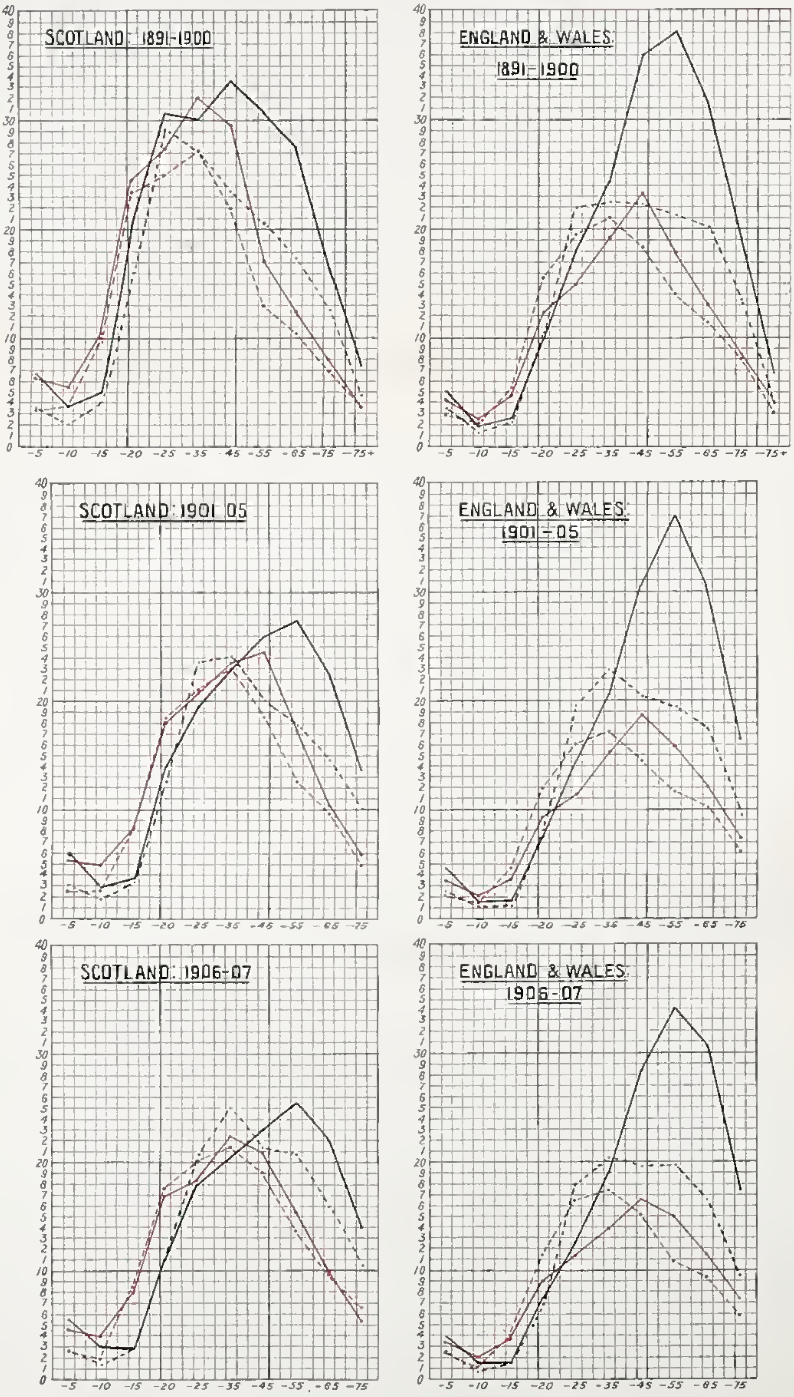
Prof. C. H. Stewart.

Plate V.

MORTALITY FROM TUBERCULOSIS OF THE LUNGS IN THE URBAN & RURAL AREAS OF SCOTLAND AND OF ENGLAND & WALES, AMONG MALES & FEMALES PER 10,000 OF EACH SEX LIVING AT AGE PERIODS.

† URBAN AREAS: MALE —— FEMALE ——

† RURAL AREAS: MALE - - - - - FEMALE - - - - -



† FOR CONSTITUTION OF THESE AREAS IN SCOTLAND SEE TABLE VI.

Prof. C. H. Stewart

ALES

TLAND

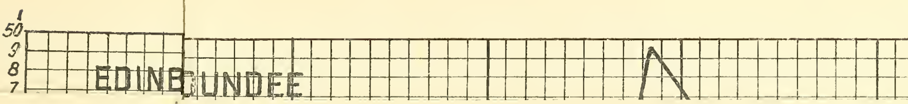
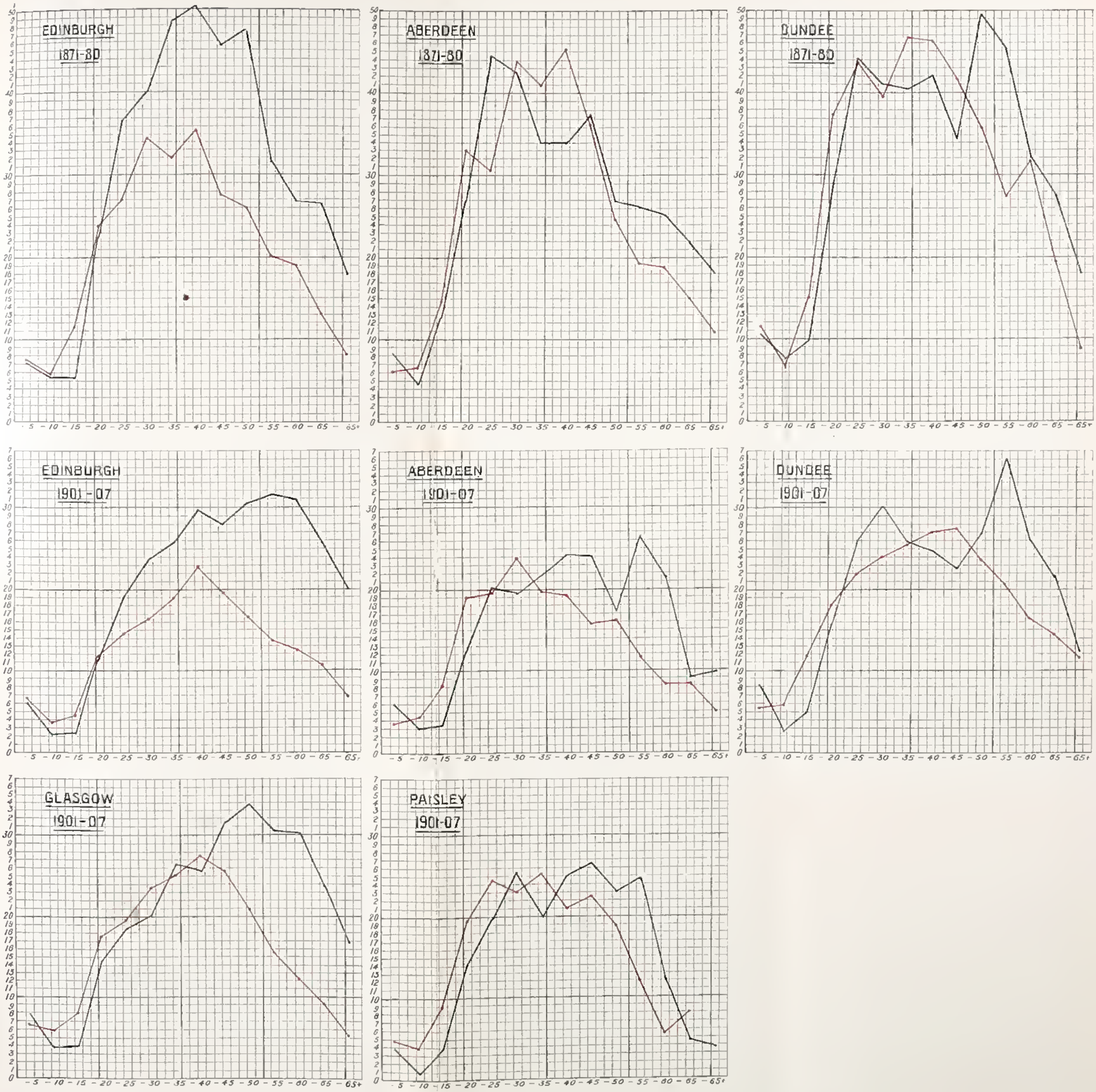


Plate VI.

MORTALITY FROM TUBERCULOSIS OF THE LUNGS AMONG MALES & FEMALES PER 10,000 OF EACH SEX LIVING AT AGE PERIODS IN CERTAIN TOWNS OF SCOTLAND

MALES = BLACK
FEMALES = RED

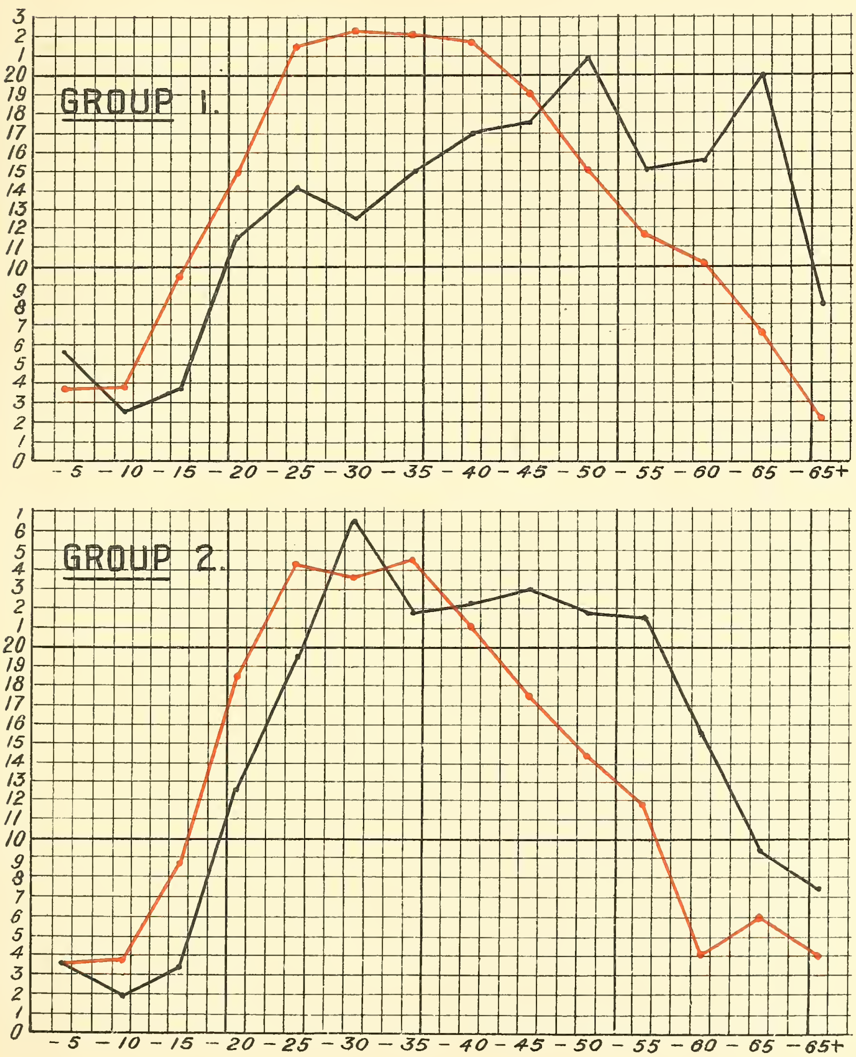


Prof C H Stewart

PLATE VII.

MORTALITY FROM TUBERCULOSIS OF THE LUNGS AMONG MALES & FEMALES PER 10,000 OF EACH SEX AT AGE PERIODS IN TWO GROUPS OF TOWNS IN SCOTLAND 1901_07.

MALES = BLACK.
FEMALES = RED.



GROUP 1 = (COATBRIDGE, GOVAN. HAMILTON & MOTHERWELL)

GROUP 2 = PAISLEY, KILMARNOCK, KIRKCALDY & PERTH)

McFarlane & Erskine, Edin.

Prof C.H. Stewart.

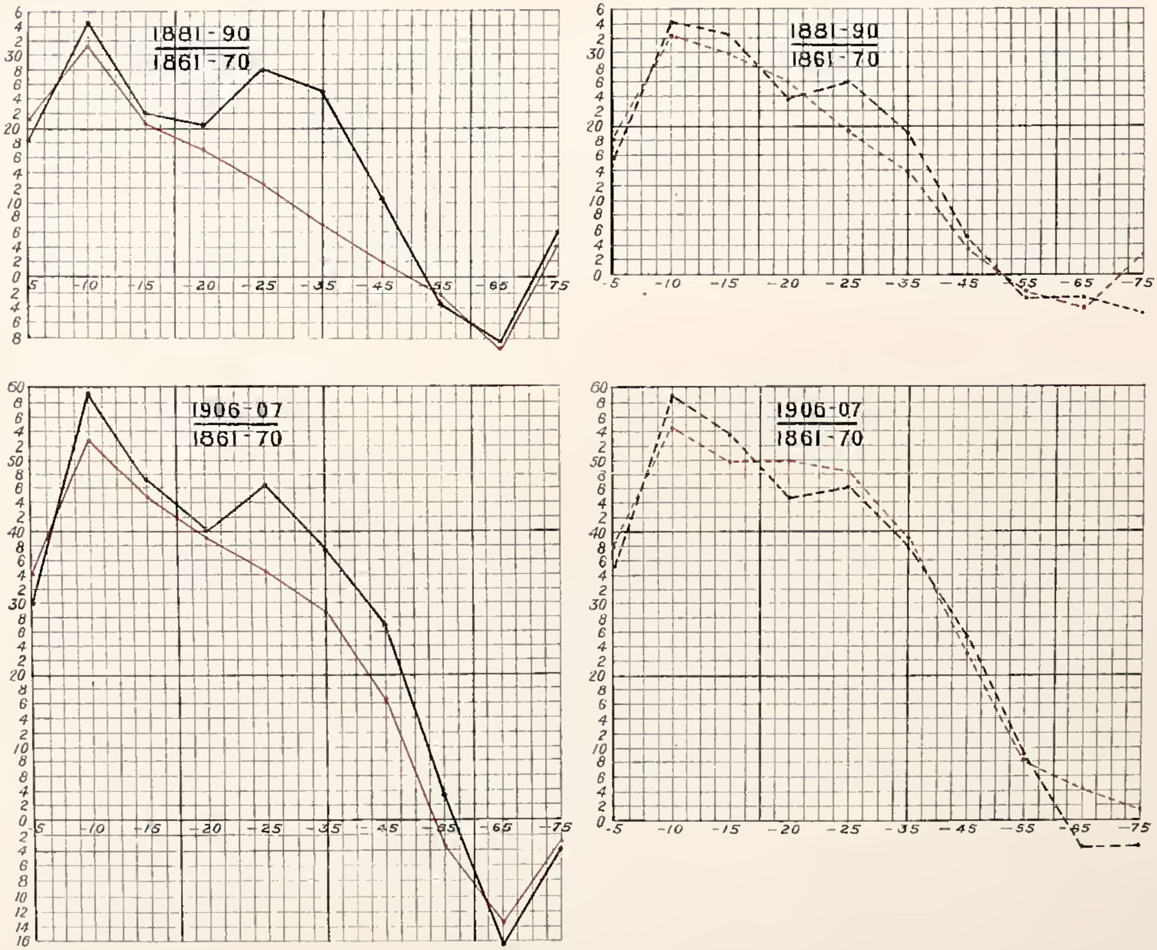
Plate VIII.

Plate VIII.

PERCENTAGE REDUCTION IN THE MALE & FEMALE MORTALITY FROM ALL CAUSES EXCEPT TUBERCULOSIS OF THE LUNGS AT AGE PERIODS SINCE 1861–70.

MALES BLACK.
FEMALES RED.

SCOTLAND. ENGLAND & WALES.



In Table XII. the female rates (1901–7) for Scotland and England and for certain Scottish towns and groups of towns are set out, along with their statistics of occupation among females, from the census 1901. Without suggesting direct numerical relation between these, there seems sufficient accordance to indicate that the occupation is exerting an influence on the female mortality in Scotland as compared with England, and in the Scottish towns of greater as compared with those of lesser industrial occupation among females.

In England textile industries *per se* are not necessarily associated with a high female mortality from tuberculosis of the lungs. In the following Table the statistics of 1901 for Dundee, and for three of the most important centres of cotton manufacture, are set out, together with the mortality rates from this disease in the latter.

Table XIII.

	Death-rate per 10,000 1898–1907.*		Engaged in Textiles. Per 10,000 Females at Ages			Engaged in Dress. Per 10,000 Females at Ages			Engaged in Indoor Domestic Service. Per 10,000 Females at Age
	Male.	Female.	Above 10	10–15.	10–25.	Above 10	10–15.	10–25.	Above 10
Dundee . .	...	...	3800	2435	5079	287	81	361	280
Blackburn. .	13·30	8·83	4311	3650	5896	372	172	515	363
Bolton . .	17·48	10·94	2797	2782	4845	290	137	398	435
Oldham . .	19·87	12·02	3012	2465	4962	280	131	400	374

Mortality Rates from 0–15 years.—During this age period the rate in Scotland has always been greater both among males and females than in England and Wales. From 0–5 the female has been less than the male rate in both countries, while from 5–15 it has been greater. The relation between the male and female rates in the two countries and in their urban and rural areas during the period 1891–1907 is shown in the following Table, the male rate in each case being taken as 1. For comparison the relation at 15–20 is also given.

* Article "The Local Incidence of Tuberculosis," by Dr Scurfield, *Public Health*, vol. xxii., No. 10.

	0–5		5–10		10–15		15–20	
	M.	F.	M.	F.	M.	F.	M.	F.
Scotland	1	0·94	1	1·62	1	2·30	1	1·32
England and Wales	1	0·87	1	1·37	1	2·21	1	1·29
Scotland—								
Principal towns	1	0·92	1	1·50	1	2·02	1	1·14
Mainland rural	1	1·06	1	1·92	1	2·36	1	1·51
Scotland—								
Urban area	1	0·92	1	1·54	1	2·13	1	1·23
Rural area	1	0·97	1	1·75	1	2·44	1	1·48
England and Wales—								
Urban area	1	0·86	1	1·33	1	2·06	1	1·22
Rural area	1	0·90	1	1·41	1	2·66	1	1·50

The excess of mortality among females over males at 0–15 is greater in Scotland than in England and Wales, especially at 5–10. In both countries the excess in rural areas is greater than in the urban. It is seen to be still more marked in rural conditions when the principal town and mainland rural districts of Scotland are compared. It will be observed that in the latter also the female rate at 0–5 exceeds the male rate.

At age period 0–15 occupation can have no influence in causing either the greater mortality among both sexes in Scotland or the excess of female over male mortality in both countries, which, as has been seen, is more marked in Scotland and in the rural than in the urban areas of both countries. The mortality from tuberculous meningitis is markedly greater in Scotland, and also that of tuberculous peritonitis, except at the age period 0–5, when it is greater in England and Wales (Table XIV.). The relation between the female and the male mortality (taken as 1) is as follows:—

	0–5		5–10		10–15		15–20	
	M.	F.	M.	F.	M.	F.	M.	F.
Scotland—								
Tuberculous meningitis	1	0·87	1	1·10	1	1·13	1	1·03
Tuberculous peritonitis	1	0·81	1	1·02	1	1·25	1	1·46
England and Wales—								
Tuberculous meningitis	1	0·79	1	0·94	1	1·09	1	1·06
Tuberculous peritonitis	1	0·80	1	1·02	1	1·20	1	1·36

In both these diseases the male exceeds the female mortality at 0–5 in both countries. In tuberculous peritonitis the female rate at 5–15 is in excess of the male to about the same extent in both countries. In tuberculous meningitis the excess of the female over the male rate is greater in Scotland. Thus at age period 5–15 the mortality in both sexes from tuberculosis of the lungs tuberculous meningitis, and tuberculous peritonitis, is greater in Scotland than in England and Wales, the relation between the male and female rates in the two latter diseases, especially in tuberculous peritonitis, being not very dissimilar.

Certain other diseases are recognised as predisposing to pulmonary tuberculosis. Table XIV. shows the rates for age periods 0–20 from bronchitis, pneumonia, and enteric fever. The rate from bronchitis is greater in Scotland at 5–15 both among males and females, and in pneumonia at 10–15. The mortality from enteric fever is greater in both sexes in Scotland. The relation of the female to the male mortality is similar in both countries.

The excess of female over male mortality from tuberculosis of the lungs at age period 5–15, which is observed in other countries also, is difficult to explain, unless the developmental changes associated with puberty predispose females to attack from this disease.

Addendum.

(MS. received March 16, 1911.)

Comparison between the Mortality from Tuberculosis of the Lungs and that from All Causes except Tuberculosis of the Lungs.—In Table XV. the rates of mortality from all causes except tuberculosis of the lungs at all ages and in each sex which occurred in Scotland and in England and Wales during 1861–70, 1901–05, and 1906–07, are set out, and in Table XVI. the rates at the different age periods are given. Table XV. shows that the reduction of mortality among females between 1906–07 and 1861–70 has been less in Scotland than in England and Wales, being 20 per cent. in the former and 26 per cent. in the latter, and that in Scotland the reduction in the female has been rather less than in the male mortality, while in England the reverse is the case.

Mortality at Age Periods.—At age period 0–5 this has been lower in Scotland than in England and Wales since 1861–70 in both sexes owing to the lower infantile mortality in Scotland. During the period 1861–70 to 1881–90 the mortality among females in Scotland was higher than among females in England and Wales from age period 5–10 to 45–55 years, and since 1901–05 from 5–10 to 65–75 years.

Percentage Reduction in Mortality from All Causes except Tuberculosis of the Lungs.—The percentage reduction during period 1861–70 to 1906–07 at age periods is graphically shown on Plate VIII., and is calculated in the same way as for pulmonary tuberculosis (Plate II.). Since 1871–80 the reduction among females has been smaller in Scotland than in England and Wales during age periods 10–15 to 25–35 years, and since 1881–90 from 10–15 to 35–45 years of age.

Reduction in Female as compared with Male Mortality.—Though the difference between the rate of reduction among females as compared with males during age periods 15–20 to 25–35 years in the respective countries is not so marked as in the case of pulmonary tuberculosis, it is yet striking. In Scotland, during 1881–90 and since that time, the rate of reduction among females has been markedly less than among males from 10–15 to 35–45 years, and in England and Wales from 1881–90 to 1891–1900, it has also been less than among males, but not nearly to the same amount, and from 1901–05 it has been greater than, or the same as, among males from 15–20 to 25–35 years. The curves on Plate VIII. bring this out more clearly.

Some influence has operated in Scotland since 1861–70 which has kept the mortality among females, from all causes other than tuberculosis of the lungs, higher in that country than among females in England and Wales at age periods beyond 5–10 years, and has retarded the rate of reduction of mortality among females in Scotland as compared with that among females in England and Wales between the ages 10–15 and 35–45 years, and as compared with males in Scotland during the same age periods.

A comparison of the curves on Plate II. and Plate VIII. shows that the rate of reduction among females from tuberculosis of the lungs follows a somewhat similar course to that from all causes except this disease, and the question naturally arises as to whether in both cases the same cause is operative, and whether this consists in a lowered bodily resistance to disease in general among females in Scotland as compared with females in England and Wales, brought about, it may be, by the different conditions detailed on page 366. This I hope to discuss in a second paper.

[Table.

Table XV.—Comparison between Mortality Rates from All Causes except Tuberculosis of the Lungs per 1000 Living at All Ages of Males and of Females.

	Scotland.*				England and Wales.			
	Death-rate.		Ratio to Death-rate of 1861–70 as 100.		Death-rate.		Ratio to Death-rate of 1861–70 as 100.	
	M.	F.	M.	F.	M.	F.	M.	F.
1861–70	19·23	17·99	100	100	†19·93	18·04	100	100
1901–05	15·93	14·97	83	83	‡15·66	13·96	79	77
1906–07	15·01	14·47	78	80	‡14·82	13·31	75	74

Table XVI.—Mortality from all Causes except Tuberculosis of the Lungs among Males and Females per 1000 of Each Sex living at Age Periods in Scotland and in England and Wales.

	Scotland.						England and Wales.					
	1861–70.		1901–05.		1906–07.		1861–70.		1901–05.		1906–7.	
	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.
0–5	63·0	56·89	48·6	41·10	44·06	37·43	72·48	62·76	53·42	44·56	46·69	38·85
5–10	9·03	8·56	3·76	4·00	3·70	4·00	7·76	7·32	3·49	3·54	3·22	3·31
10–15	4·31	3·99	2·36	2·24	2·27	2·19	3·87	3·45	1·93	1·80	1·80	1·72
15–20	4·64	3·71	2·90	2·59	2·76	2·24	3·98	3·53	2·37	1·99	2·21	1·77
20–25	5·68	4·21	3·26	3·03	2·94	2·75	4·59	4·00	2·72	2·34	2·47	2·06
25–35	6·92	5·74	4·62	4·51	4·32	4·09	5·83	5·32	3·82	3·50	3·63	3·21
35–45	10·17	8·55	8·40	7·28	7·45	7·13	9·31	8·15	7·23	6·47	6·94	6·21
45–55	16·12	12·98	16·19	13·35	15·59	13·35	15·38	12·78	14·28	11·88	13·98	11·68
55–65	28·85	23·56	34·10	27·87	33·82	27·16	29·83	25·84	30·42	24·59	30·97	24·69
65–75	66·62	56·44	67·97	56·72	69·24	57·73	65·08	57·87	65·61	55·65	67·86	56·96
75 +	154·18	153·85			151·04	135·51	164·80	154·48			147·48	145·29

* Corrected to sex and age constitution of census 1901 (Scotland).

† Corrected to sex and age constitution of population of England and Wales 1891–1900.

‡ Corrected to sex and age constitution of population at census 1901 (England and Wales).

(Issued separately April 15, 1911.)

XXIII.—An Investigation into the Effects of Errors in Surveying. By **Henry Briggs**, B.Sc., A.R.S.M. *Communicated by* Principal A. P. Laurie.

(MS. received January 7, 1911. Read February 6, 1911.)

(Abstract.)

(Paper appears in *Transactions R.S.E.*, 1911.)

The paper discusses the effects of errors in linear and angular measurements on the accuracy of surveys, and is divided into six sections, thus:—

Section I. The Sum of Vector Errors.

Section II. The Average Error due to Imperfect Centring.

Section III. The Relative Effects of Errors in Centring and those of Sighting and Reading.

Section IV. The Propagation of Errors in Traversing.

Section V. The Propagation of Errors in Minor Triangulation.

Section VI. A Summary of Results.

Under the first section a simple mode of summing vector errors is established, which is made use of throughout the rest of the paper, and particularly in studying the accuracy of triangulation in transmitting distance. Special stress is laid on the practical conclusions to be drawn from the inquiry, and curves and diagrams are given illustrating the effects of error in different branches of surveying.

The last section is divided into two parts. In the first of these are placed the results, eight in number, which may be considered as already recognised by surveyors, and in the second are given twelve results which the author believes to be new.

XXIV.—On the Temperature Coefficient of Concentration Cells, in which the same salt is dissolved in two different solvents. By A. P. Laurie, M.A., D.Sc.

(MS. received February 3, 1911. Read February 20, 1911.)

In a paper read before the Royal Society of Edinburgh (*Proc. Roy. Soc. Edin.*, xxviii., part v., p. 382 (1908); *Zeit. phys. Chem.*, lxiv. 5) I described a new type of concentration cell, in which the one platinum electrode was surrounded by a solution of ·025 molecules of KI containing ·001 molecules of iodine dissolved in absolute alcohol, and the other electrode was surrounded by ·025 molecules of KI and ·001 molecules of iodine, dissolved in water. This cell developed a considerable E.M.F. of ·198 volts at 25° C. in the direction which would transfer the iodine from water to alcohol and potassium iodide from alcohol to water.

The temperature coefficient was positive and amounted to ·00037 pr. deg., the E.M.F. rising with rise of temperature and the cell being therefore endothermic.

Cells of this type were originally investigated by Luther (*Zeit. phys. Chem.*, xix. p. 567, 1896), who experimented with a cell n which zinc electrodes were surrounded by zinc sulphate dissolved in ether and water respectively; and he showed that the conditions for equilibrium, and therefore for the E.M.F., were determined by the partition coefficient of the salt between the two solvents.

To take for instance Luther's cell: if a solution of zinc sulphate in water is shaken up with ether, the zinc sulphate will share itself between the two solvents in a certain ratio. If two zinc electrodes are now introduced and the cell connected to an electrometer, no E.M.F. will be detected, as is obvious on thermodynamical principles. If there was an E.M.F. the zinc sulphate would be transferred during the passage of the current from one solvent to the other, again returning by diffusion, and a source of energy would be so obtained under conditions which are, as Luther has shown, thermodynamically impossible.

If a solvent is chosen which mixes with water, the conditions are complicated by the fact that the two solvents diffuse into each other. But there is still a difference of solubility of the salt in the two solvents, which means that there is a partition coefficient equilibrium, though it cannot be determined in this case by shaking up the two solvents with the salt.

It is also obvious in general that if both solvents are saturated with the salt, they will be in equilibrium with each other, though this is not necessarily the case if the solvents themselves act on each other, as in the case of alcohol and water, where admixture results in contraction and evolution of heat and precipitation of dissolved salt.

If the solvents do not mix, the saturated solutions will be in equilibrium.

In the case of the cell already described, ignoring for the present the disturbing factor due to the interdiffusion and chemical action on each other of alcohol and water, it is evident that we have here at least two partition coefficients to deal with, namely, the partition coefficient between iodine in water and iodine in alcohol, and between potassium iodide in water and potassium iodide in alcohol. A consideration of the solubilities of these substances in the respective solvents will at once show that we should expect to find, when the solutions on both sides are of equal strength, an E.M.F. tending to transfer iodine from water to alcohol and potassium iodide from alcohol to water, until a ratio of strength of solution for each was established approximately equal to the ratio of their solubilities.

Unfortunately, however, the necessary data for exact calculations in the case of this cell are wanting, as, while we know the ratio between KI, I_2, and KI_3 in the water solution, we have not similar data for alcohol and so cannot determine the amount of free iodine present.

The next paper of importance for our present purpose is by Abel, dealing with the mathematical theory of such cells.*

The following is Abel's treatment of the problem :—

To take the case of a simple salt like potassium iodide, distributed between two solvents and partially ionised in each, it is evident that there is a partition coefficient between the unionised solvent and a partition coefficient between the ions in the two solvents respectively.

The ionic partition coefficient will here be the same for both ions.

Professor Abel investigated this case as follows :—

In the first place, let us suppose the solutions to be in partition equilibrium, then

M	Phase I. Solvent L_I M-ion concentration $\bar{c}_I$	Phase II. Solvent L_{II} M-ion concentration $\bar{c}_{II}$	M

M, M are the two metal electrodes, and $\bar{c}_I$, $\bar{c}_{II}$ the ionic concentrations of the metallic ions in the two solvents respectively.

* *Zeit. phys. Chem.*, lvi. p. 612 (1906). For the researches of other investigators on cells with organic solvents, along with fresh results, see the Dissertation by Joseph Neustadt, Breslau University, 4th Aug. 1909.

Let $c_{\mathrm{I}}, c_{\mathrm{II}}$ be the solution pressure of M in the two solvents, n_{I}, n_{II} the transfer numbers of the kations in I. and II., K_{K}, K_{A} the specific partition coefficients of the ions between the two phases, then

$$\mathrm{E} = \frac{\mathrm{RT}}{m}\,ln\,\frac{\mathrm{C_I}}{\bar{c}_{\mathrm{I}}} + x\frac{\mathrm{RT}}{m}\,.\,ln\,\frac{k_{\mathrm{K}}\bar{c}_{\mathrm{I}}}{\bar{c}_{\mathrm{II}}} + (1-x)\frac{\mathrm{RT}}{m}\,ln\,\frac{\bar{c}_{\mathrm{II}}}{k_{\mathrm{A}}\bar{c}_{\mathrm{I}}} + \frac{\mathrm{RT}}{m}\,ln\,\frac{\bar{c}_{\mathrm{II}}}{\mathrm{C_{II}}} = 0.$$

If now k be taken as the partition coefficient between the undissociated solvents, and $\mathrm{K_I}$ $\mathrm{K_{II}}$ the ionisation constant for each solvent, then

$$\mathrm{K_{II}} = \mathrm{K_I}\frac{k_{\mathrm{K}}k_{\mathrm{A}}}{k};$$

therefore

$$\frac{1}{k_{\mathrm{A}}} = \frac{\mathrm{K_I}}{\mathrm{K_{II}}k}k_{\mathrm{K}} = \frac{(\bar{c}_{\mathrm{I}})^2}{(\bar{c}_{\mathrm{II}})^2}k_{\mathrm{K}},$$

and therefore

$$\mathrm{E} = \frac{\mathrm{RT}}{m}\,ln\,\frac{\mathrm{C_I}}{\bar{c}_{\mathrm{I}}} + \frac{\mathrm{RT}}{m}\frac{\mathrm{K_K}\bar{c}_{\mathrm{I}}}{\mathrm{K}\bar{c}_{\mathrm{II}}} + \frac{\mathrm{KT}}{m}\,ln\,\frac{\bar{c}_{\mathrm{II}}}{\mathrm{C_{II}}} = 0:$$

consequently the relation follows,

$$\frac{\mathrm{C_I}}{\mathrm{C_{II}}}k_{\mathrm{K}} = 1 \text{ or } \mathrm{C_{II}} = \mathrm{C_I}k_{\mathrm{K}},$$

Having thus established the general conditions for a cell in which the concentrations are in partition equilibrium, Abel proceeds to consider the case where the concentrations in the two solvents are not in such equilibrium and there is consequently an actual E.M.F. developed.

It is evident that when the two solutions are brought into contact the partition equilibrium will be at once established at the surface of contact, and that therefore each cell can be supposed to be divided into four parts, namely a concentration c_{I} and c_{II} round the two electrodes, and a concentration $\bar{c}_{\mathrm{I}}$ and $\bar{c}_{\mathrm{II}}$ on each side of the boundary between the two liquids, and the E.M.F. therefore is made up of two parts due to the concentration slope in each liquid, from c_{I} to $\bar{c}_{\mathrm{I}}$ and from c_{II} to $\bar{c}_{\mathrm{II}}$, and therefore if v_{I} u_{I} and v_{II} u_{II} be the ionic velocities in the two solvents respectively, the E.M.F. is given by the equation

$$\mathrm{E} = -\frac{2v_{\mathrm{I}}}{u_{\mathrm{I}}+v_{\mathrm{I}}}\frac{\mathrm{RT}}{m}\,ln\,\frac{c_{\mathrm{I}}}{\bar{c}_{\mathrm{I}}} - \frac{2v_{\mathrm{II}}}{u_{\mathrm{II}}+v_{\mathrm{II}}}\frac{\mathrm{RT}}{m}\,ln\,\frac{\bar{c}_{\mathrm{II}}}{c_{\mathrm{II}}}.$$

The experimental testing of this equation involves, unfortunately, an amount of knowledge as to the conditions pertaining in organic solutions, which is seldom available, and then usually for such solvents like ethyl and methyl alcohol, which are mixable with water.

The above equations are in complete agreement with what has already been stated as to the conditions pertaining to such cells.

When the two solvents can be mixed a fresh factor is introduced owing to the fact that a series of layers will be formed containing various proportions of the two solvents. Professor Abel investigated these conditions and arrived at the following equation:—

$$E = 2\,(1 - n_{II})\frac{RT}{m} ln\ c_{II} + 2\,(1 - n_{I})\frac{RT}{m} ln\frac{\chi^{I-II}}{c_I} + 2\frac{RT}{m}\int_{n^{II}}^{n^{I}} ln\frac{\chi}{c}\,.\,dn,$$

where the expression within the integral sign is the summation of the effects due to the interdiffusion of the two solvents, and χ is the ionic partition coefficient.

Abel comes to the conclusion that in such cells the E.M.F. will not be constant for a given concentration, owing to the uncertain element introduced by the diffusion layers of the two electrolytes into each other. He therefore proceeds to investigate the case when a third liquid is introduced between the other two and mixes with neither but contains the salt in solution.

It is unnecessary to state here the steps of this investigation.

His equation is then further simplified by defining the unit of electric current as the current which will transfer a gramme molecule of salt, thus avoiding the factors, for the time being, depending on the ionic velocities in the various solvents, and defining χ^{II-I} as the ionic partition coefficient simplifies the equation to

$$A = 2\frac{RT}{m} ln\frac{c_I}{c_{II}}\chi II - I.$$

This is in a slightly different form from the equation I have used in this paper for pot. iodide cells, namely:—

$$E = RT\,(\log c \times x - \log c'),$$

when x is a constant for high dilutions, depending on the partition coefficient and capable of expression as a definite experimentally determined ratio of concentration.

I shall also throughout the paper take the above definition of unit quantity of electricity in order to lighten the equations.

The equations which I shall give in developing the theory of the change of E.M.F. with temperature will all be capable of expanding into the form of Abel's equation if required.

Abel takes in his equations the ionic partition coefficient, while I have taken throughout the partition coefficient for the salt as a whole, which includes the partition for the unionised solvents and for the ions, and is the experimental value with which we meet in practice. Evidently, if this is known and the ionisations in both solutions, Abel's partition coefficient can be calculated. When the solutions are in mechanical equilibrium they are also in electrical equilibrium (the electrical equilibrium depending on the ions alone), and the relationship is given by Abel's equation,

$$K_{II} = K_I \frac{k_A k_K}{k}.$$

With reference to experimental determinations of the E.M.F.s of cells of this type, Abel comes to the conclusion that when the liquids do not mix it is necessary to have the cell in four divisions for reliable readings, the two central divisions being in partition equilibrium, and that in the case of interdiffusing liquids the measurements will be seriously disturbed by this interdiffusion. The magnitude of the corrections due to these conditions would require special researches to decide.

In the case of the alcohol water cells with silver silver iodide electrodes there are changes of E.M.F. which are evidently taking place at the electrode which is immersed in the alcohol solution, the cause of which is very obscure.

While I have found such large changes in the liquid contact conditions of alcohol water cells, as introducing a solution of potassium iodide in glycerine, produce little or no change in the E.M.F. (thus going far to demonstrate that the errors at the liquid contacts due to interdiffusion are not serious), I have on the other hand found sudden capricious changes of E.M.F. take place to as much as 20 per cent. or 30 per cent. in the silver silver iodide, potassium iodide, alcohol, water cell, with which I have principally experimented—changes the cause of which defied investigation in spite of hundreds of measurements.

Probably the best plan with cells of this type is, for each concentration, to determine the ratios for zero E.M.F., thus eliminating at any rate some sources of error.

All such measurements must, however, be received with great caution until more is known about the conditions of reliable readings.

As far as this paper is concerned, however, qualitative results are all that are needed to demonstrate the correctness of the theory developed as to the behaviour of these cells with changes of temperature; while in the latter part of the paper, where the actual quantitative results are necessary, they have been confirmed by readings taken under various conditions, and have been shown to be in harmony with other experimental facts, and therefore justify the conclusions arrived at.

The first experiments, then, were made with the cell already described.

Round one platinum electrode was placed a solution of ·025 molecules of

KI and an excess of solid iodine. Round the other electrode was placed a series of solutions of iodine in alcohol containing ·025 KI.

The following table contains the approximate strengths of these iodine solutions and E.M.F.s of the cells at 25° C. and 14° C.:—

Current in the direction of transferring KI from alcohol to water defined as +ve:—

Table I.

Water.	Alcohol.	E.M.F. at 25° C.	E.M.F. at 14° C.
·025 KI + ·0138 I_2	·025 KI ·043 I_2	·069 +	·033 +
·025 KI + ·0138 I_2	·025 KI ·086 I_2	·009 +	·014 −
·025 KI + ·0138 I_2	·025 KI ·172 I_2	·028 −	·070 −
·025 KI + ·0138 I_2	·025 KI ·345 I_2	·071 −	·099 −

Result obtained before—

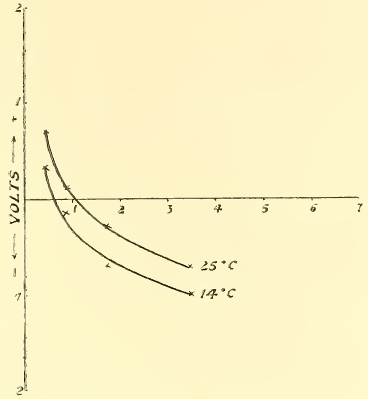
·025 KI + ·001 I_2	·025 KI ·001 I_2	·198 +	T.C. ·00037 + ve.

If the table of E.M.F.s at 25° C. is examined, it will be noticed in the first place that as the iodine in alcohol is increased in strength the E.M.F. is finally reversed, the current flowing the other way, and the neutral point of zero E.M.F. being somewhere about a concentration of ·086 molecules of I_2.

This is all in accord with the theory of such cells as developed by Luther and Abel. If now the column of E.M.F.s at 14° C. is examined, it will be noted that while the current was in the direction necessary to transfer potassium iodide from alcohol to water (which for convenience in this paper will be defined as the positive direction) the temperature coefficient was positive, but when the current was reversed so as to convey potassium iodide from water to alcohol the temperature coefficient became negative and the cell therefore exothermic, the cell doing work and also setting free heat.

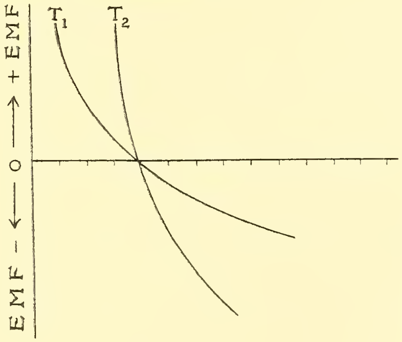
The result given at the beginning of the paper, although the concentration dealt with was different, evidently harmonises with and fits into this table.

If the results are plotted, two roughly logarithmic curves are obtained with approximate temperature coefficients:—



Iodine concentrations.

This is evidently quite different from the curves for an ordinary concentration cell, which are of this form, the curves for two different temperatures crossing at the zero E.M.F. concentration:—



This result seemed so interesting that it was worth testing in a cell made up with quite different materials, and therefore nitrobenzene and water were selected for the purpose.

Potassium iodide is almost insoluble in nitrobenzene, but dissolves freely in presence of iodine to give a complex series of poly-iodides (Dawson).

A solution of ·025 molecules KI and ·025 molecules I_2 in nitrobenzene was placed in one electrode, and ·025 KI molecules and ·01 I_2 molecules in water was placed round the other electrode.

The E.M.F. was positive (that is, in the direction of transferring potassium iodide from nitrobenzene to water, and iodine from water to nitrobenzene), and amounted to ·43 volts.

The two solutions were next shaken up together for some hours in the thermostat at 25° C. and allowed to separate, and introduced into their respective electrodes.

No E.M.F. could now be detected in the electrometer in accord with Luther and Abel's theory.

This cell was now raised to 30°, and at once a positive E.M.F. was developed, and in cooling to 12·5 a negative E.M.F. was developed.

In each case these E.M.F.s amounted to some one or two hundredths of a volt.

This cell is evidently therefore the same in its temperature reactions as the alcohol cell already described. That is, when the E.M.F. is reversed from positive to negative the cell has a negative temperature coefficient and becomes exothermic.

Experiments made with alcohol water cells with potassium iodide in solution and silver silver iodide electrodes gave the same results. When the solutions were of equal strength, a considerable E.M.F. was obtained in the direction of transferring potassium iodide from alcohol to water, and the cell was endothermic. On increasing the concentrations of the water solution until the E.M.F. was reversed, the temperature coefficient became negative and the cell exothermic.

The experiments with these potassium iodide cells will presently be more particularly described, as the attempt was made to get certain quantitative measurements with them; but in the meantime the above result is sufficient for our purpose.

In the following mathematical investigation, although the alcohol cell is being used as an illustration, the nitrobenzene cell is a more strictly typical cell for our purpose, as in the alcohol cell we have the complication of the interdiffusing alcohol, setting free heat and precipitating salt from solution. The effect of this on the results will ultimately have to be considered.

In these cells, then, we have a very neat form of heat engine. If the salt is in partition equilibrium at a particular temperature, then on raising the temperature of the cell above this amount an E.M.F. is developed and heat is absorbed. On now cooling below the neutral temperature, the current is reversed, the chemical changes are replaced, and heat is set free.

Mr Ernest Gibson, in a letter to me, has developed the theory of these cells directly from Abel's equation in the following manner:—

Taking the Abel equation in the following simple form,

$$E = RT\left\{\log\frac{P_w}{\bar{c}_w} - \log\frac{P_s}{\bar{c}_s}\right\} = 0 \quad . \quad . \quad . \quad (1)$$

when $P_w P_s$ are the solution pressures of the metal in the two solvents, then if the partition ratio varies with temperature we must have a temperature coefficient as observed, because $\bar{c}_w$ and $\bar{c}_s$ at a higher or lower temperature are no longer the partition equilibrium concentrations.

The temperature coefficient is therefore

$$\frac{de}{dt} = R\left(\log\frac{P_w}{\bar{c}_w} - \log\frac{P_s}{\bar{c}_s}\right) + \frac{RT}{P_w}\frac{dP_w}{dt} - \frac{RT}{P_s}\frac{dP_s}{dt} = RT\left(\frac{1}{P_w}\frac{dP_w}{dt} - \frac{1}{P_s}\frac{dP_s}{dt}\right) \quad (2),$$

since

$$R\left(\log\frac{P_w}{\bar{c}_w} - \log\frac{P_s}{\bar{c}_s}\right) = 0 \text{ by } (1).$$

The signs of the T.C. will therefore be positive or negative according as $\frac{1}{P_w}\frac{dP_w}{dt}$ is greater or less than $\frac{1}{P_s}\frac{dP_s}{dt}$.

A simple concentration cell which is at zero E.M.F. at t will have no temperature coefficient, for in this case $P_w = P_s$, and therefore

$$\frac{de}{dt} = RT\left(\frac{1}{R_w}\frac{dP_w}{dt} - \frac{1}{P_w}\frac{dP_w}{dt}\right) = 0.$$

I prefer, however, to approach the matter in another way, which seems to me to lead us further into an understanding of these cells and their behaviour.

The actual and only physical result of the passage of a current through the alcohol water cell with its silver silver iodide electrodes is the transfer of a certain quantity of potassium iodide from the one solvent to the other. Now we know that the solution of potassium iodide in water results in a large absorption of heat, and this absorption is not entirely due to the conversion of the potassium iodide from the solid to the fluid condition, as there is a further absorption of heat on diluting the solution. We do not know the latent heat of solution of KI in alcohol. It may be positive or negative, though doubtless small in amount.

We are considering, here, not the latent heat involved in passing from the solid salt to the solvent, but the difference in latent heats involved in passing a molecule of the salt from the one solvent to the other solvent. This difference of latent heats is the only apparent source of energy available to give us a voltaic cell from these materials which is both doing external work and producing heat.

Let, then, λ' be the latent heat due to the solution of one gramme molecule of potassium iodide in water (calling heat absorbed positive) and λ the latent heat of solution of an equal quantity of salt in alcohol, then applying the Helmholtz equation (our unit quantity of electricity being still defined as that necessary to transfer a gramme molecule of the salt), then when the E.M.F. is in the direction of transferring salt from alcohol to water the equation will be

$$E = \lambda' - \lambda + T\frac{de}{dt}.$$

If λ' is greater than λ, then, remembering that λ and λ' mean heat absorbed, this process is endothermic.

If the concentrations, however, are so arranged as to transfer the salt from water to alcohol, then, λ' being assumed, as above, to be the greater, the equation becomes

$$E = \lambda - \lambda' + T\frac{de}{dt},$$

and the cell is exothermic.

If the cell is so arranged as to concentrations that at temperature T, E equals 0, that is, that the solutions are in partition equilibrium, then

$$\lambda - \lambda' = T\frac{de}{dt}$$

from equation (1).

That is, still assuming λ' to be greater than λ, $\lambda - \lambda'$, and therefore $T\frac{de}{dt}$ has a positive value, and therefore the E.M.F. is rising with increase of temperature; and similarly from equation (2)

$$\lambda' - \lambda = T\frac{de}{dt},$$

and therefore $T\frac{de}{dt}$ has a negative value, and the E.M.F. is falling with rise of temperature.

As has already been pointed out, in order to determine the difference between the latent heats of solution in dilute solutions by this method, we require to know as well the relative transport numbers of the salt in the two solutions, so as to know the quantity of salt transferred by unit current in each case.

If we next take the particular case of two solutions in contact which are both saturated at the particular temperatures selected, we can obtain the same result from a different point of view, and thus confirm the conclusion that the latent heats are the only available source of energy.

Let us suppose the same salt dissolved in two different solvents which do not mix to any appreciable extent, and let the two solvents be called A and B.

If both solvents are saturated with the salt, and the solvents have no action on each other, then the saturated solutions will be in partition equilibrium and the cell will have no E.M.F.

With a view to fixing our ideas, let us suppose that solution A is much weaker than solution B when equilibrium is established, so that if solution A and B are of equal concentration there will be an E.M.F. transferring the salt from A to B.

Let us now proceed to consider the effect of temperature on such a cell. Let us suppose a cell in which both solutions are saturated and are therefore in equilibrium, and let us suppose the cell to be raised from temperature T_1 to T_2. The electromotive force of such a cell may be stated as follows:—

$$E = RT_1 \left(\log C \frac{S'_1}{S_1} - \log C'\right) = 0,$$

when S_1 is the solubility of the salt in A, S'_1 is the solubility in B, and C and C′ the concentration in A and B respectively at temp. T_1.

Let the cell be now raised in temperature to T_2, then

$$E + de = RT_2 \left(\log C \frac{S'_2}{S_2} - \log C'\right),$$

when S'_2, S_2 are the solubilities of the salt at the new temperature T_2.

Evidently if the rate of increase of solubility in both solutions is the same, and $E_1 = 0$, then $E_1 + de = 0$.

For in an ordinary concentration cell, if

$$E_1 = RT_1 (\log C - \log C') = 0,$$

then

$$E_1 + de = RT_2 (\log C - \log C') = 0.$$

Let us suppose the solubility coefficients with temperature are not the same: then

$$(E_1 + de) - E_1 = RT_2 \left(\log C\frac{S'_2}{S_2} - \log C'\right) - RT_1 \left(\log C\frac{S'_1}{S_1} - \log C'\right).$$

But $E_1 = 0$, therefore

$$de = RT_2 \left(\log \frac{S'_2}{S'_1} - \log \frac{S_2}{S_1}\right) \quad . \quad . \quad . \quad (1)$$

Now according to the Van 't Hoff equation, if λ is the latent heat of solution (heat absorbed being reckoned positive) and S_1, S_2 the solubilities of the salt, at T_1, T_2 the

$$\log \frac{S_2}{S_1} = \frac{1}{R} \lambda \left(\frac{1}{T_1} - \frac{1}{T_2}\right).$$

Let λ be the latent heat of solution A and λ' of solution B, then

$$\log \frac{S'_2}{S'_1} = \frac{1}{R} \lambda' \left(\frac{1}{T_1} - \frac{1}{T_2}\right).$$

Therefore

$$\log \frac{S'_2}{S'_1} - \log \frac{S_2}{S_1} = \frac{1}{R}\lambda'\left(\frac{1}{T_1} - \frac{1}{T_2}\right) - \frac{1}{R} \lambda \left(\frac{1}{T_1} - \frac{1}{T_2}\right).$$

Substituting this value in equation (1),

$$de = (\lambda' - \lambda)\frac{T_2 - T_1}{T_1} \qquad \lambda' - \lambda = T\frac{de}{dt}.$$

Thus we find that the source of energy in this cell when the temperature is changed is equal to the difference between the latent heats of the salt in the two solutions. We can now consider in detail the effect of temperature on such a cell.

Let us suppose that in raising the temperature of the cell the solubility of S' increases faster than S. Then it will be necessary to transfer salt from S to S' to restore the equilibrium, and there will be an electromotive force in the direction necessary to transfer salt from A to B. But if the temperature coefficient of S' is greater than that of S, then λ' is greater than λ, and therefore during the running of the cell heat is absorbed $= \lambda' - \lambda$, and the cell is endothermic.

If the temperature, however, is lowered below T_1, then, since S' has the higher temperature coefficient, its solubility will now diminish quicker than the solubility of S, and the current will be reversed and flow the other way, so that now heat will be set free instead of absorbed, and $\frac{de}{dt}$ has a negative value; that is, the E.M.F. diminishes with rise and increases with fall of temperature.

I shall now proceed to describe in more detail the experiments made with the alcohol, water, potassium iodide cells, with silver silver iodide electrodes.

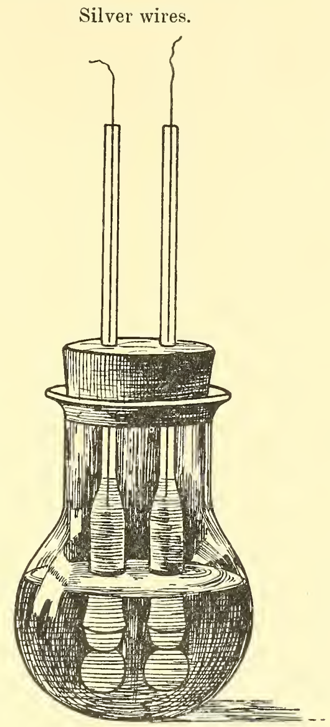
It will be remembered that the general investigation given above will probably only apply with a modification, owing to the diffusion of alcohol into water and salt precipitation.

The type of cell used was a modification of the type used in my experiments on iodine concentration cells, little vessels about two cms. long and half a cm. broad, stoppered at one end and with a fine open tube at the other for the introduction of the silver wire, being used. These fine tubes were passed through a cork, the stoppered ends dipping into a bottle, as shown in the diagram on p. 387 :—

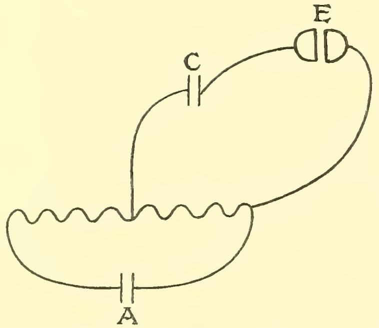
The iodised silver wires were then introduced and cemented in with a little paraffin wax.

As in my former experiments, a Dolezalek electrometer was used to measure the E.M.F., the E.M.F. of the cell to be measured being balanced against the E.M.F. of a storage cell through a resistance, and the storage cell being standardised from time to time against a standard cadmium cell.

The Dolezalek electrometer is very convenient for experiments with high resistance solutions, and enables the stoppered vessels to be used with



impunity, thus simplifying all the manipulation. The following is a diagram of the connections :—



C = cell being measured ; E = electrometer ; A = accumulator.

It will be noted that no current beyond that necessary to charge the quadrants is ever drawn from the cell.

The silver wires were pure silver obtained from Heralus. Cumming*

* *Trans. Faraday Society*, 11th Dec. 1906.

has shown that this silver is quite reliable. The wires were iodised by being dipped into a strong solution of iodine in alcohol, washed with absolute alcohol and allowed to dry by waving them about in the air.

The alcohol used was a fresh supply of Kahlbaum's absolute alcohol. The solubility of pure potassium iodide in this alcohol at 25° C. was determined as ·0955 molecules. For another purpose, Dr Denison had, after drying this alcohol over lime and distilling off calcium, determined the solubility of KI at 25° C. to be ·0957. This alcohol of Kahlbaum's may therefore be regarded as sufficiently dry for the purposes of these experiments.

Changing the wires about and reiodising them made no perceptible change in the E.M.F.

Every solution before being used was mixed with pure dry silver iodide and kept in the thermostat for some hours, with occasional shaking so as to saturate it with silver iodide.

In order to test the method, the following experiments were made:—

A concentration cell of ·01 molecules KI against ·001 molecules KI with the silver iodide electrodes gave an E.M.F. at 25° C. of ·0552 volts.

According to Kohlrausch's latest results, ·01 KI is ionised 94 per cent. Plotting his results and continuing the curve graphically, ·001 KI would be ionised about 96 per cent. Therefore, the E.M.F. of this cell should be ·0591, $=$RT (log ·0094 $-$ log ·00096). But a small correction has to be made in this for the diffusion E.M.F.

Taking the values of U and V from Whetham's *Theory of Solution*, we get for the theoretical E.M.F. the value ·0577.

Similarly an experiment was made with a ·1, ·01 concentration cell. The measured E.M.F. was ·0536 and the calculated was ·0551. It will be noted that in both cases the measured E.M.F.s are about ·002 volts below the calculated. This agrees with the results obtained by Jahn * for silver chloride, potassium chloride cells. It is not necessary here to discuss the conclusions at which Jahn arrives from these discrepancies. It is sufficient for our purpose that the readings obtained agree so closely with his, and that therefore the whole method is reliable.

It is now necessary to describe the experiments made with the alcohol, water, potassium iodide cells with silver silver iodide electrodes.

After the preliminary experiments, two solutions were compared containing each ·001 molecules of KI and saturated with silver iodide. The result of my measurements with this cell was to establish ·08 volts as the most probable E.M.F. Occasionally this would capriciously change to a

* *Zeit. phys. Chem.*, xxxiii. p. 370.

lower value, but a higher reading for this dilution was never obtained, though a slightly higher value was indicated for a still greater dilution.

The results for higher dilutions were, however, very unstable, and readings for greater concentrations gave lower E.M.F.s.

These results are to be expected when we remember that potassium iodide ionises far more rapidly on dilution in water than in alcohol, the ionising power of alcohol being supposed to be about one-third that of water.

On the assumption that a ·0001 solution in alcohol might be regarded as completely ionised, I calculated back the ionisation of the alcoholic solution and found the results in good general agreement with the observations of others, giving for ·01 molecules 34%; ·005, 39%; ·001, 67%.

If the experimental difficulties could be overcome, these cells would be of value as giving another method of determining the ionisation of organic solvents.

Starting with a cell consisting of ·001 alcohol solution and ·001 water solution, I altered the water concentrations, and the results are given in the following table:—

Table II.

Molecules K.I. in 1000 cc. Alcohol.		Molecules K.I. in 1000 cc. Water.		E.M.F. found.	E.M.F. calculated.	
·001	–	·001	=	·080		
·001	–	·0025	=	·049	·057	+ ve
·001	–	·005	=	·034	·039	+ ve
·001	–	·0075	=	·025	·029	+ ve
·001	–	·01	=	·017	·021	+ ve
·001	–	·02	=	·0	·005	+ ve
·001	–	·04	=	·016	·013	– ve
·001	–	·06	=	·027	·023	– ve
·001	–	·08	=	·033	·030	– ve
·001	–	·1	=	·041	·041	– ve

T.C. for ·001 al. – ·001 water = ·00092 + ve

T.C. for ·001 al. – ·04 water = ·00072 – ve

The calculations were made from the equation

$$E = RT\ (\log C \times x - \log C'),$$

where x is a constant depending on the partition coefficient, and treated as a concentration.

A correction is made for the ionisation of the water solution of the potassium iodide.

The ionisation of the KI is determined by interpolating on Kohlrausch's latest results. It will be noted that the agreement between the found and calculated is very fair.

I also tried the effects of introducing in the cell between the stoppered vessels a solution of potassium iodide in glycerine. This very slightly affected the E.M.F. readings, small changes being caused by altering the concentration of the potassium iodide in the glycerine.

It is evident from these results that the concentrations which are in equilibrium are about 1 to 20. Now the solubility of potassium iodide in alcohol is ·096 molecules, and in water it is 6·2 molecules, so that there is here a big departure from the partition coefficient which might have been expected.

To throw further light on this discrepancy, experiments were next made with concentrated solutions.

The saturated solution of potassium iodide in alcohol was prepared by dissolving potassium iodide in excess in alcohol at a higher temperature, letting it cool in the thermostat, and shaking the solution up with the deposited crystals at intervals for several days, the solution being kept in the thermostat. This solution poured off the crystals was saturated with silver iodide at the same temperature.

Another alcohol electrode was prepared by taking this solution and packing the stoppered electrode with dried potassium iodide and silver iodide and then introducing the same solution.

This electrode usually gave against the first an E.M.F. of ·015 volts. This E.M.F. is worthy of further investigation.

The preparation of the water electrode was more difficult.

If crystals of potassium iodide and silver iodide are introduced into a saturated potassium iodide solution, both dissolve, a liquid being obtained on which potassium iodide crystals float, and further addition of both substances finally produces a stiff paste.

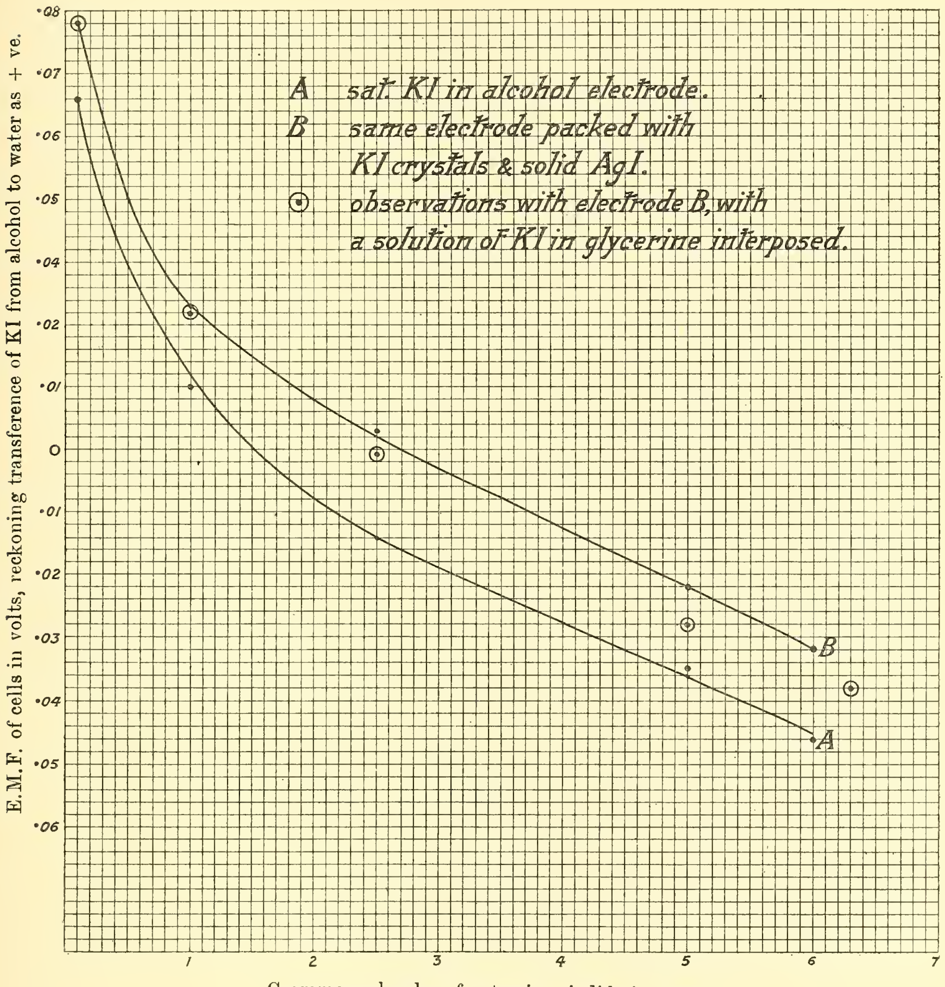
It is therefore difficult to define what is meant by a saturated potassium iodide solution under these circumstances.

I determined, therefore, to prepare a saturated solution by repeated shaking in the thermostat with a large excess of potassium iodide crystals, then pouring off this solution and introducing an excess of silver iodide into it. This is the solution used in the following experiments.

The paste above described gave a slightly higher E.M.F. than the above solution. The following curves represent the E.M.F. obtained by reading a series of water solutions against the two alcohol electrodes, described above, and also with the interposition of glycerine.

The result is to show that instead of the saturated solutions giving no E.M.F., the equilibrium of no E.M.F. is not reached till the water has been diluted to about two molecules per 1000 cc. This represents very nearly the mean of the results obtained.

As has already been pointed out, in the case of alcohol, the diffusion



results in evolution of heat and precipitation of the dissolved salt. It seemed therefore advisable to investigate the conditions of this precipitation, and determine whether there was a particular strength of water solution which would not be precipitated by the addition of a large excess of alcohol already saturated with KI.

To test this a series of glass tubes were filled with alcohol saturated with

KI, and two or three drops introduced of water containing 3 molecules, 2·75 molecules, 2·25 molecules, and 2 molecules of potassium iodide respectively.

The tubes were then sealed and hung in the thermostat, with repeated shaking. Crystals quickly separated from the solution containing a few drops of 3 molecules and 2·75 molcules, but no crystals could be detected, after several days, in the 2 molecule or 2·25 molecule solution. Slight formation of crystals was detected after some time in a 2·5 molecule solution in another experiment.

These results, then, show that there is an equilibrium between alcohol saturated with KI and water containing about 2 molecules of KI, no further precipitation taking place. It seemed important to confirm this in another way.

A series of solutions was therefore prepared of KI in water, and to equal quantities of water increasing volumes of alcohol saturated with KI were added. The water solutions were chosen so that the amount of crystals separated should be small.

The various mixtures were stirred for several days in a thermostat at 25° C., and then known portions of the total volume drawn off and evaporated. On adding alcohol to water there is of course a contraction; but by adding a known volume of alcohol to a known volume of water and evaporating a known portion of the total volume, accurate results are obviously obtainable.

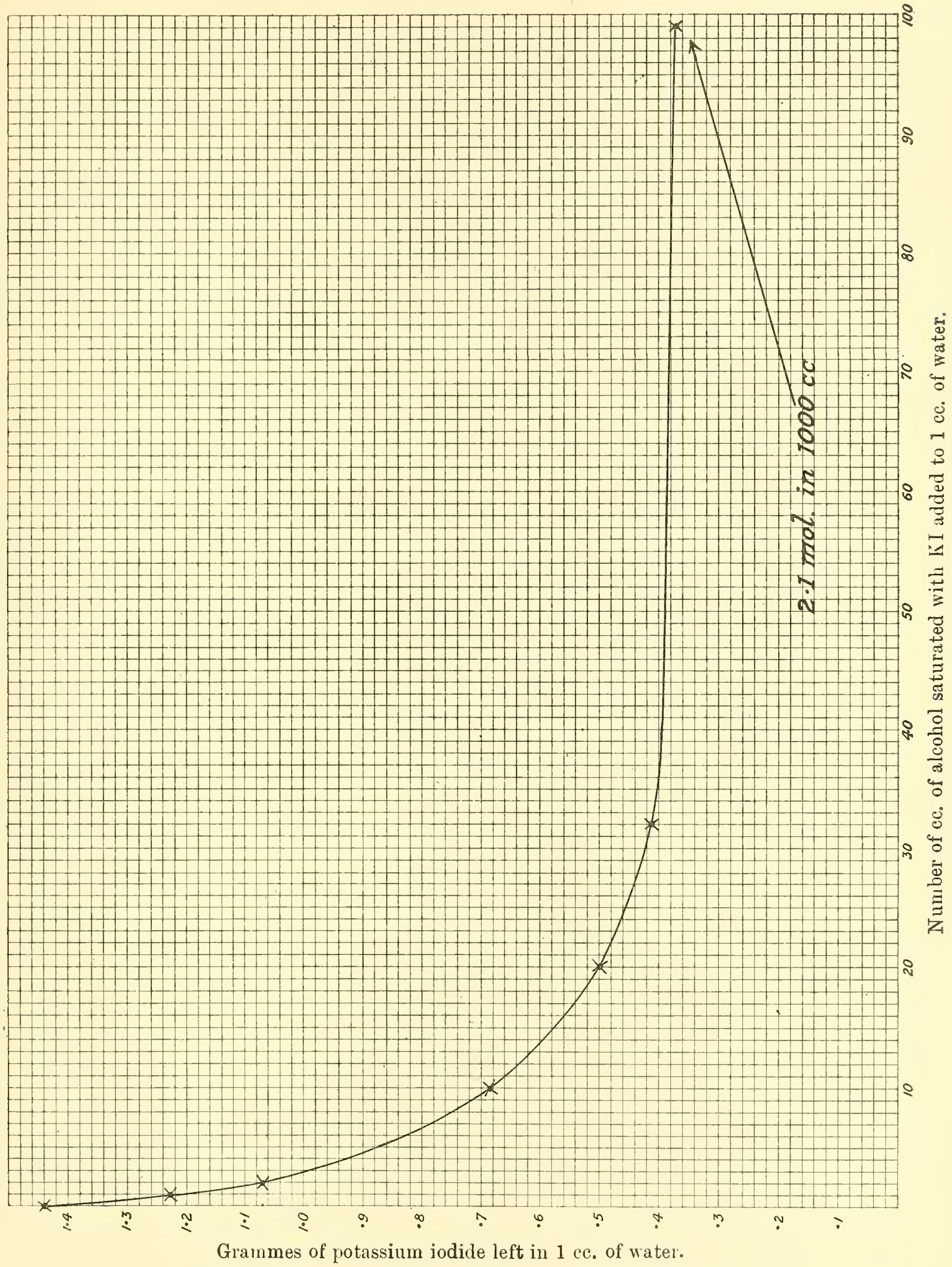
The method adopted was therefore to take a given volume of water, then dissolve in it a known weight, of potassium iodide, then make up this solution to 100″ with the KI saturated alcohol, measuring the volume of alcohol used for this purpose independently.

Then after treatment in the thermostat 10″ was withdrawn, evaporated, and weighed, and from the total weight, the weight of salt due to the alcohol content (which is known from the volumes of water and of alcohol originally taken) was subtracted.

The following curve contains the results, in which the additions of saturated KI alcohol to the potassium iodide solutions in water are plotted against the KI left in solution in the water content of the mixture.

It will be seen that in approximating to 2 molecules of water, precipitation has practically ceased.

We find, therefore, as the result of these experiments, that if a solution of alcohol saturated with KI is brought in contact with a solution of water containing about 2 molecules, the diffusion of the alcohol into the water will produce no further precipitation of KI, and that such an arrangement, if silver silver iodide elements are introduced, is in equilibrium and gives no E.M.F. Moreover, if both solutions are diluted, the conditions of zero E.M.F. are a ratio nearly corresponding to this, so



Curve showing the precipitation of potassium iodide from water by alcohol saturated with the salt.

that the partition coefficient ratio must be defined, when the liquids interdiffuse and the one precipitates the salt in the other, as the ratio between the saturated solution of the first solvent and the strongest solution of the second solvent, which is not precipitated by the first.

If we examine Table II. it is evident that if the condition of equilibrium was the ratio of the saturated solutions, that ratio being 1 to 60, the E.M.F. zero would be shifted about $-·027$ volts.

Taking curve (1), the shifting would be about $-·04$ volts, to give equilibrium at the saturation point of each solution.

We may therefore express these results by saying that there is apparently a positive quantity of heat involved in the passage of KI from alcohol to water, which is subtracted from the latent heats of solution and which shifts the E.M.F. somewhere about ·027 to ·04 volts, and which is not due apparently to the latent heats of solution of the salt but to some other cause. I discuss this more fully further on.

Probably, of these figures the one obtained from the dilute solution is the more reliable.

With reference to the small alteration made by the introduction of the glycerine between the electrodes, it must be remembered that ordinary glycerine was used, which doubtless contained water, and that therefore the alcohol was linked up through this contained water with the water electrode. The object of the introduction of the glycerine was to produce a big change in the surface contact conditions of the two solvents in order to test how far the slopes of concentration were affecting the results, according to the suggestions made by Abel.

The E.M.F. which exists in the cell in which both the alcohol and water are saturated with KI seems to be only capable of explanation on the assumption that the ions are combined with water and also possibly combined with alcohol. For if the action of the cell be carefully considered, it is evident that as far as the two ends of the cell are concerned there is here no source of E.M.F., as at the two ends we have solid KI. Our only source, therefore, of electrical energy is in some way derived from the mixing of water and alcohol. Now, the mere diffusion of these two liquids into each other can hardly enter into the mechanism of the electrical process. By suitable arrangements the diffusion could be reduced to a minimum, and yet the electrical process would go on and must involve the using up of some energy within the cell. This difficulty is entirely explained if we suppose that the ions are combined with water and alcohol respectively, and carry their combined solvent into the other solvent, thus causing electrolytic mixing, the current flowing from the cell until this electrolytic mixing is complete, the interdiffusion of the two liquids simply resulting in a waste of some of the energy otherwise available. Calorimetric experiments might therefore enable us, along with the E.M.F. measurements, and ionic velocities to calculate the total amount of water plus alcohol carried by the ions.

In the case of two liquids that do not mix, the ion has to part with its combined liquid at the margin of contact and take up the new liquid into combination. If we imagine the case of a cell with liquids that do not mix and in which the dilutions are sufficient to give complete ionisation on both sides, then the only phenomenon with which we are apparently dealing is the heat absorbed or set free by the combinations of the ions with the solvents. In the case of water and alcohol and KI, when iodine ions are travelling from water to alcohol and the K ions from alcohol to water, the summation of these combinations results in the absorption of heat. An approximate calculation of this heat absorption can be made from the measurement of the E.M.F. and the temperature coefficient of the cell in which ·001 molecules of KI in alcohol was measured against ·001 molecules of KI in water (with the assumption that the velocity of the potassium and iodine ions in alcohol is practically the same, just as it is practically the same in water). The result is to show total heat absorption of 9000 calories. Thomsen gives the latent heat of solution of KI in water at 5100 calories, but it must be remembered that he is here measuring the nett result, from which the heat of ionisation has to be subtracted. While I do not wish to lay much weight on this calculation, it certainly suggests that the heats involved in the very dilute solutions are much higher than those obtained in the ordinary way in the calorimeter. When both solutions are so dilute as to be completely ionised, then the partition coefficients are ionic partition coefficients; and if K_1 and K_2 are the ionic partition coefficients at temperatures T_1 and T_2, the general Van 't Hoff equation, of which the one already used is a particular case, would surely apply to these solutions; and if λ is the nett result of the heats absorbed or set free by the combination of the respective ions with the two solvents, then the value of λ can be obtained from the general Van 't Hoff equation,

$$\log \frac{K_2}{K_1} = \frac{\lambda}{R}\left(\frac{1}{T_1} - \frac{1}{T_2}\right).$$

To sum up, the following, as far as I am aware, new results have been obtained:—

First, the demonstration of the effects of change of temperature on these cells.

Second, the demonstration of the connection between the latent heats of solution and the observed results.

Third, the demonstration that the condition of electrical equilibrium, in the potassium iodide-water-alcohol cell, is not when both solutions are saturated with the salt, but is, when the strength of the water solution is such, that, when in contact with alcohol saturated with potassium iodide, diffusion produces no salt precipitation. This equilibrium is reached when the strength of the water solution is about 2 molecules of potassium iodide per 1000 cc.

In conclusion, I must thank Mr Andrew King and Mr Ernest Gibson for their assistance, and the Carnegie Trust for the grant which enabled me to carry out these experiments.

(*Issued separately April* 25, 1911.)

XXV.—The Topography of the Cerebral Cortex of the Guinea-pig.

By **Williamina Abel**, M.D., McCunn Scholar. (From the Physiology Department, Glasgow University.) *Communicated by* Professor D. Noël Paton.

(Read January 23, 1911. MS. received February 6, 1911.)

CONTENTS.

1. Introduction.

This investigation was begun as a result of certain observations made by Dr T. Graham Brown, on reflexes in the guinea-pig (45). In various experiments he removed parts of the cerebral cortex and found that little if any disturbance, either motor or sensory, ensued. It was thereupon suggested that an examination should be made of the cortex histologically and by electrical stimulation in order to locate the position of the various areas.

2. Literature.

Space only allows of the briefest references to some of the numerous papers on the question of cerebral structure and localisation. Among the earlier workers may be mentioned Fritz and Hitzig (1, 2), Goltz (3, 4), Munk (5), Fürstner (6), and Ferrier (7). These all describe numerous investigations made by them on the cerebral cortex of the lower animals, and considerable divergence of opinion is noticeable. The writings of Munk show that in his opinion the so-called motor area is also a sensory zone. Ferrier's work is extremely interesting, as he describes definite cortical centres associated with definite movements in the brains of the rabbit, guinea-pig, and rat. In the rabbit the centres for fore and hind limb movement, as well as for movements of ear, nose, and facial muscles, are described as lying in the middle and posterior thirds of the cerebrum, while the centres for eye and mouth movements are placed in the anterior third. In the guinea-pig the centres are not so numerous or so definitely localised, but two centres are described in the posterior third of the cerebrum associated with movements of fore and hind limbs. These centres are placed near the middle line, while immediately below them is the centre for eye movements. The centre for movements of the ear is described as lying more posteriorly than the others, while the centre for the mouth is placed in the anterior third. In the rat the centres described correspond fairly exactly with those of the guinea-pig.

Goltz describes the results of removal of large portions of the anterior portion of the cerebral hemispheres, and calls attention to the fact that the symptoms presented by dogs so operated on are not those which would naturally be expected were the generally accepted view of the significance of the motor area correct. In this connection an interesting report is given by Klein, Langley, and Schäfer (9) on a dog, a large portion of whose cerebral hemispheres had been removed by Goltz, but which showed freedom of movement and marvellously normal sight and hearing.

Ferrier and Yeo describe a monkey from which a large portion of the so-called motor area had been removed and which showed definite paralytic symptoms. The brains of those animals were subsequently removed and reported on by Klein (11), Langley (10), and Schäfer (12).

An interesting account is given by Langley and Grünbaum (13) of the brain of a dog which Goltz had submitted to an operation somewhat similar to that already described.

Fürstner (6) describes some interesting experiments on the cerebrum of various animals, among which is the rabbit.

Fritz and Hitzig (1) demonstrate the excitability of definite areas of the cortex, and the presence of definite changes in the function of a limb whose centre has been removed.

The work of Nothnagel (14) is here only referred to.

Moeli (15) gives an interesting description of changes occurring after lesion of the cerebrum in rabbits. He describes disturbance of vision as a sequence to injury of the posterior part of the opposite cerebral hemisphere.

The investigations of Bevan Lewis (16), Ferrier and Yeo (8), Schäfer (12), Beevor and Horsley (17, 18, 19), are here only referred to, as, although of the greatest importance with regard to the question of cerebral localisation as a whole, they bear but very indirectly on the work in hand.

Professor Sir W. Turner (21, 23) gives a full review of the arrangement and type of the cerebral convolutions in various orders of mammalia. This he has amplified by a paper on the cerebral hemisphere of the ornithoryhnchus paradoxus.

Ziehen (22), in a study of the comparative anatomy of the cerebral convolutions, criticises several of the views expressed by Turner in his paper (21). Special attention is given in this article to the convolutions in the brains of the bear and walrus.

Baginsky (24) contributes an interesting paper on cerebral localisation with special regard to the centre for hearing.

Bechterew (25) discusses the question of the so-called motor zone, being in reality sensori-motor. Work on the ape is quoted in support of this latter view, and attention is called to clinical cases which strongly suggest that in man the motor and sensory functions are closely united.

Bechterew (26) describes the localisation in the occipital lobe of the ape of two centres associated with dilatation and contraction of the pupil.

Gustav Mann (27) describes in detail the homoplasty of the brain in rodents, insectivores, and carnivores. In the brain of the rabbit he is able, by means of electrical stimulation, to locate centres for movement of limbs, head, neck, eye, and facial muscles. These centres he finds lie in the posterior half of the brain; the centre for the hind limbs lying close to the middle line, and posterior to the other centres.

Wesley Mills (28) gives an account of an investigation made by him on cerebral localisation in the rabbit, cavy, rat, and bird. As a result of his work on the cavy, he decides that the motor areas are not well defined although he is able to describe centres for all movements except those of the hind limbs.

Cunningham (29) describes the motor areas in the cerebrum of the opossum, which he finds lie in the anterior half of the cerebral hemisphere. He fails to find a satisfactory centre for the hind limb.

Martin (30), who examined the cerebrum of the ornithoryhnchus, finds the motor areas in the anterior half of the brain, but describes no centre for the hind limbs.

Elliot Smith (31) gives a description of the brain in monotremata, and draws comparisons between the cerebral cortex in monotremata, marsupialia, insectivora, and other orders.

Bechterew (32) discusses the question of the localisation of the auditory centre. A considerable part of the paper is devoted to a résumé of the most recent work on this point. The work of Larinoff is described in considerable detail. He worked on dogs, and localised the centre in the first and third temporal convolutions with subsidiary centres associated with the appreciation of different tones.

Yonakow (33) gives an interesting résumé of work on the question of cerebral localisation.

Bolton (34, 35), in two papers—one on the histological basis of amentia and dementia, the other on the function of the frontal lobes—gives the conclusions arrived at by him on the functions of the various portions of the cortex cerebri. He concludes that the prefrontal region is the centre for the highest co-ordinated and associated processes of the mind. From examination made by him of human fœtuses of four and six months, he finds that lamination begins in the cerebral cortex at the sixth month of fœtal life. At this stage he finds the polymorphic layer almost fully formed, the granular layer about half its full size, while the pyramidal layer is barely a fourth of its full depth. He finds that the pyramidal layer develops enormously up to and after birth. From this, and from the fact that in mental diseases the pyramidal layer invariably degenerates, he concludes that this layer subserves the psychic or associative functions. The polymorphic layer is never found to have undergone change in mental disease, and, as already stated, it is practically fully developed at birth; it evidently subserves the voluntary or lower animal functions. The granular layer apparently subserves the transference of afferent impulses from sense organs to other parts of the brain.

Miss Allan (37) describes a series of experiments made to test the psychical processes peculiar to the guinea-pig, and the changes occurring in those processes from birth to maturity. As a result of these experiments, Miss Allan finds that there is no evidence of an increase in the complexity of psychical processes after the third day of life. It also seemed evident that the memory of a particular set of movements—such as would be required to retrace a labyrinth to find food—is registered by the kinæsthetic sense. At birth nerve medullation is found to be far advanced, its completion occurring in a few weeks. An interesting comparison is made between the conditions found in the guinea-pig and in the white rat, as described by Watson (36). In the latter animal none of the nerve fibres are medullated at birth, while the animal is quite helpless. This is in sharp contrast to the newly born guinea-pig, who starts life with its psychical powers almost mature. The psychical maturity of the white rat is reached about the twenty-seventh day after birth, but is far in advance of that found in the guinea-pig.

Edinger (38) contributes an interesting description of the brain of the lamprey. The prominence of the olfactory bulb and its structure are minutely described.

Watson (39), in a paper on the mammalian cerebral cortex, describes minutely the brains of the mole, shrew, and hedgehog, and defines the position of the motor, sensory, and undifferentiated areas, while the probable sites of the visual and fifth nerve centres are defined on the mesial surface. As a result of this work he draws the following conclusions :—

1. The relative depth of the layers is of more value from the point of view of drawing conclusions than a consideration of the cortex as a whole.

2. Regarded developmentally, ontogenetically, and phylogenetically, the facts adduced point to the mammalian cortex being built on an infra-granular basis.

3. The supra-granular layer, corresponding to the pyramidal layer of Bolton, is practically absent in lower mammals.

In this paper Watson points out certain peculiarities of the neopallium of the guinea-pig, such as the well-marked supra-granular layer. The cells in this layer, though numerous, are apparently mostly functionless, since they undergo little development after birth.

Haller (40) gives an interesting paper on the phylogenetic relationship existing between the brains of various mammals. Three types—Chiroptera, Erinaceus, and Mustelidae—are fully described and compared.

Mott and Halliburton (41) describe the cerebral localisation in the brain of the lemur. They localise the motor area in the central region of the cortex.

Mott and Kelly (42) follow with a detailed description of the histological characteristics of the brain of the lemur. They describe the motor area as easily recognisable from histological appearances.

Mott, Schuster, and Halliburton (43) give a detailed account of the cortical lamination and motor centres in the brain of the marmoset. They describe the motor areas as lying in the anterior half of the brain, and find, further, that a large portion of this area is associated with the movements of face, tongue, and mouth. This point is of interest, since the marmoset is dependent on the delicate play of those muscles for its insect prey.

Bolton (44), in his chapter in *Further Advances in Physiology*, recapitulates much of what has already been published in his papers previously quoted (34, 35), and goes on to discuss the evolution of cerebral function, the work of Flechsig, Smith, Brodmann and others being referred to.

Schuster (47) describes the cell lamination in the cerebrum of the Echidna in considerable detail, mapping out the whole cortex into some five different types of cellular arrangement.

Schäfer (46) describes a series of experiments made on monkeys, which show that the volitional impulses may pass from the cortex to the cord by other paths than the pyramidal tract, an alternative path being furnished by the ventral columns and ventral parts of the lateral columns of the cord.

3. Methods.

The cerebral hemispheres of the guinea-pig were first hardened in graduated baths of alcohol, then cleared with cedar oil and embedded in paraffin. Sections were then made 10 μ in thickness and stained by Nissel's method.

One hemisphere was cut in a complete series of transverse sections, from the posterior to the anterior pole, great care was taken to preserve the full sequence of these sections. From this series a general idea was got of the lamination of the cerebrum, and the extent and distribution of the various types of cortex. It was, however, impossible to make any estimations of the depth of the cortex, owing to the obliquity which is unavoidable in a continuous series of sections through a cerebral hemisphere. A second hemisphere was therefore mapped out into suitably sized blocks, cut and stained. The position, size, shape, and relationship of these blocks was recorded in a diagram drawn to scale. From these sections measurements were made and the areas occupied by the different types of cortex definitely mapped out.

4. Histology of the Cortex.

A. Neopallium.—The cerebrum of the guinea-pig is triangular in outline and of very smooth contour. There are three fissures, which occupy much the same position as the fissures described in the brain of the rabbit by Mann (27). From the fact that they coincide so exactly in shape and position, the same nomenclature as that used by Mann has been adopted. The fissures are.

1. The rhinal fissure.
2. The Sylvian fissure.
3. The pre- and post-lateral fissures of the rabbit form one continuous fissure—the lateral fissure.

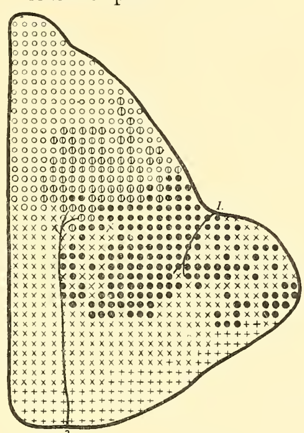
The histological examination of the cortex of the cerebrum led to the differentiation of five different types of cell lamination. The areas occupied by the different types are shown on the accompanying diagrams 1 and 2. The admirable method used by Mott has been adopted in demarcating these areas. Drawings of the cell lamination in the different areas which are shown in figures 3 to 8 were made by a camera lucida method devised by Dr G. H. Clark.

Type I. (+ + +) *lying at the posterior pole.*

Depth of cortex 1·5 mm.

1. On the surface is a zone of very small, faintly stained cells, lying in a non-stained fibrous network. Since this zone is common to all the types, it will merely be referred to as Zone I. in the descriptions of the remaining types.

Anterior pole.



Posterior pole.

Fig. 1.—Upper surface of cerebral hemisphere, showing the distribution of the various types of cortex described in the text. 1, Sylvian fissure; 2, lateral fissure; + +, cortex, Type I.; × ×, cortex, Type II.; ●●, cortex, Type III.; ⦶ ⦶, cortex, Type IV.; ○○, cortex, Type V.

2. A thick zone of irregular round or polygonal cells distinctly larger than the cells of the first zone.

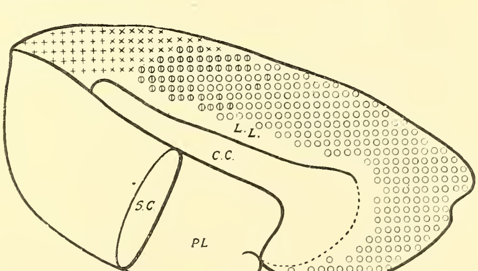


Fig. 2.—Mesial surface of cerebrum. *c.c.*, corpus callosum; *P.L.*, pyriform lobe; *s.c.*, section through crus cerebri; *L.L.*, position of limbic lobe; + +, cortex, Type I.; × ×, cortex, Type II.; ⦶ ⦶, cortex, Type IV.; ○○, cortex, Type V.

2′. A narrow irregular zone of very small pyramidal cells.

3. A narrow granular zone; this zone varies much in depth but is nowhere prominent. It corresponds to the granular layer of Bolton.

4. A broad zone composed of cells more or less pyramidal in shape; among them lie some larger and more prominent cells, distinctly pyramidal in shape and measuring some 15 to 25 μ in length by 10 to 15 μ broad.

Fig. 3.—Type I. × 100.

From comparisons made between them and the cells forming a prominent feature in the fourth and fifth zones of the second type of cortex, it is evident that they are similar. Their significance will be discussed in the description of the second type.

5. A broad polymorphic zone.

(See fig. 3.)

Type II. (× × ×) *in front of Type I.*

Depth of cortex varies from 1·5 mm. to 1·6 mm.

1. Zone I.
2. A narrow zone of angular and pyramidal cells.

Fig. 4.—Type II. ×100.

3. A very indefinite granular zone—the granular layer of Bolton.

4 and 5. These two zones may be considered as one broad zone of pyramidal-shaped cells. The cells, more especially in the deeper parts of the zones, are large and clearly defined, varying in length from 30 to 40 μ and 12 to 20 μ in breadth. In a cursory examination of this cortical area

they form the most prominent feature in the whole field, and, as they are most abundant in this area, they evidently have some significance. The

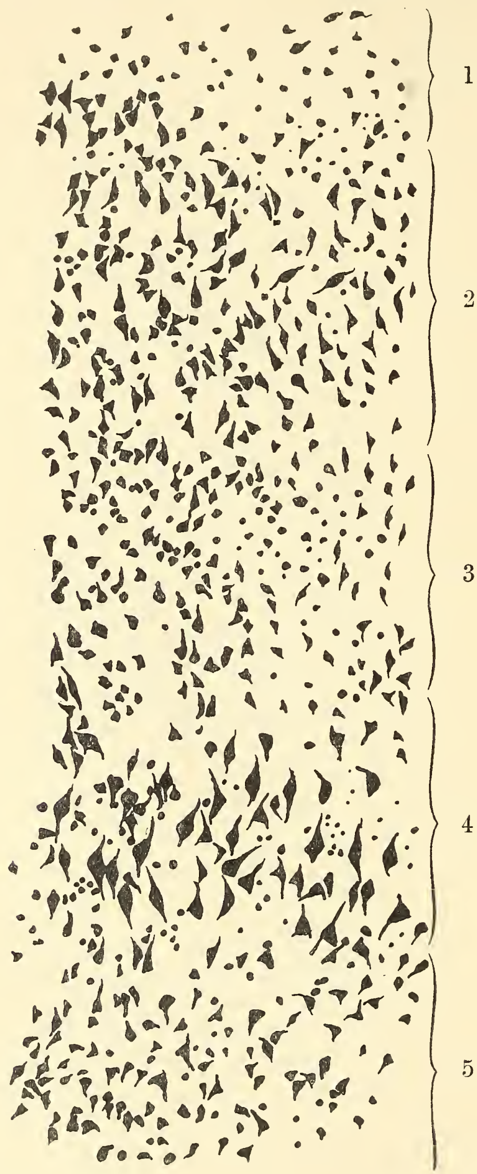


Fig. 5.—Type III. ×100.

results of electrical stimulation of the cortex will be described later, but it is sufficient here to point out that motor reaction was obtained from the areas showing this particular type of cell. Further, the fact that these areas are comparatively poor as regards their granular zones would imply—

if Watson's view be accepted—that they are not at least wholly sensory. Arguing from these facts, it would seem that these large pyramidal cells are associated with outgoing volitional impulses. They merge into the next zone.

6. A wide zone of polymorphic cells. These three zones correspond to Bolton's sub-granular zone.

(See fig. 4.)

Type III. (● ● ●) *below and in front of Type II.*

Depth of cortex 1·5 mm. varying to 1·4 mm.

1. Zone I.
2. A zone of small angular and pyramidal cells.
3. A broad granular zone.
4. A zone of pyramidal shaped cells, some of which are 30 μ long and 12 to 15 μ broad, but these larger cells are not nearly so abundant as in Type II.
5. A broad zone of polymorphic cells.

(See fig. 5.)

Type IV. (⦶ ⦶ ⦶) *in front of Type III.*

Depth of cortex 1·3 to 1·2 mm.

1. Zone I.
2. A zone of small angular and pyramidal cells.
3. A broad, well-marked granular zone.
4. A zone composed of granular cells interspersed with pyramidal cells.
5. A broad polymorphic zone, the basis of which seems to be largely composed of small, almost granular, cells.

(See fig. 6.)

Type V. (○ ○ ○) *at the anterior pole.*

Depth of cortex 1·2 to 1 mm.

1. Zone I.
2. A broad zone of polygonal cells interspersed with a few pyramidal cells.
3. A well-defined granular zone.
4. A broad, clearly defined zone of angular cells, among which are a very few small pyramidal cells.
5. A broad zone of very small polymorphic cells.

(See fig. 7.)

On the mesial surface (fig. 2) four of the five types of cortex appear, and, in addition, extending round the upper border of the corpus callosum is the limbic lobe. The upper boundary of this lobe has only been

Fig. 6.—Type IV. × 100.

recognised microscopically, as examination of several cerebri has failed to identify a definite fissure corresponding to the calloso-marginal. The cortex cannot be divided into layers, and is composed of granular and polymorphic cells. The larger cells lie in the deepest parts of the cortex, and in some parts, more especially at the posterior pole of the lobe, there are faint indications of the formation of a zone, but the limbic lobe is extremely rudimental in structure.

B. Archipallium.—Sections taken through the pyriform lobe show a type of cortex absolutely distinct from any of the types described, but

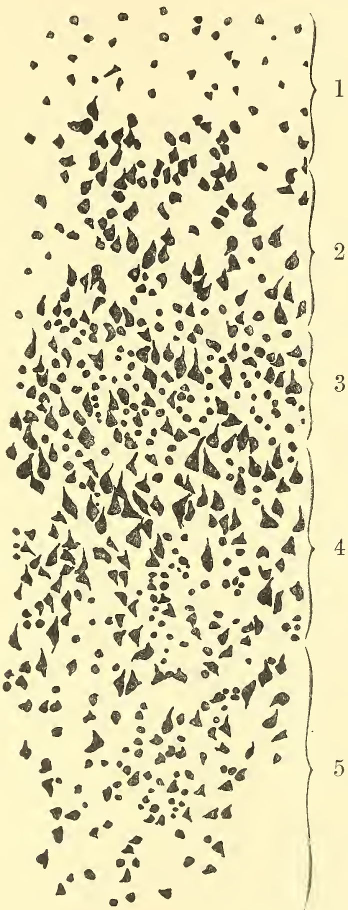


Fig. 7.—Type V. ×100.

corresponding to a certain extent with the typical description of archipallial cortex given by Bolton.

The cortex is divisible into two zones, an upper and a lower. The upper zone is granular in structure, with the exception of a slight band of larger cells lying near the surface. The lower zone is composed almost wholly of polymorphic cells, among which are interspersed a fair number of granular cells.

(See fig. 8.)

In the anterior portion of the archipallium the laminar arrangement is all but lost, the only sign of lamination which persists is an irregular and broken band of polymorphic cells in the same position as the band

Fig. 8.—Type VI. ×100.

of larger cells described in the granular zone of the pyriform lobe. The remainder of the archipallium is composed of a homogeneous arrangement of granular and polymorphic cells.

5. The Results of Electrical Stimulation of the Cortex.]

For the investigation of the cerebral cortex unipolar stimulation was employed with the Berne induction coil. The strengths of current vary from ·008 to ·0104 and ·0123 milliamperes as determined for me by Dr James Gray.

As a result of electrical stimulation of the cerebral cortex of the guinea-pig, the motor area was located in the posterior half of the cerebrum. It was, however, found impossible to mark off different centres for the various movements. In practically all the experiments performed it was found that a stronger current was required to cause movements of the hind limbs as contrasted with the fore limbs. Movements of the eyelids were generally most easily elicited, but twitching movements of the nose and face were frequently well marked and elicited with almost equal ease.

The accompanying diagram (fig. 9) shows the position and approximate extent of the motor area, while the letters correspond to the position where certain movements were most frequently elicited.

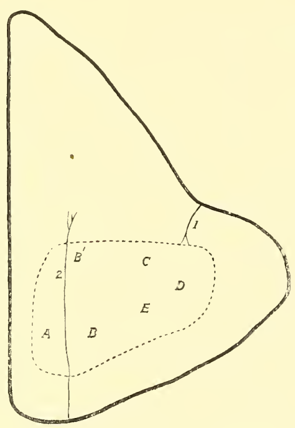


Fig. 9.—Upper surface of cerebrum, showing the excitable area (marked by dotted line) as determined by electrical stimulation.

1. Sylvian fissure. 2. Lateral fissure.

A. Movements of the hind and fore limbs were elicited by stimulation of this part of the cortex, an increase of current being necessary to cause hind limb movements.

B and B′. This represents the cortical area over which stimulation was found to cause fore limb movements. B represents the area which gave an active reaction, B′ the area associated with a comparatively weak reaction.

C. Over this area stimulation is associated with movements of the eyelid. This is probably one of the most definite areas, as the movements are very easily elicited.

D. Movements of the nostrils, and occasionally twitching of the facial muscles, were produced in some of the experiments by stimulation of this area.

E. A twisting movement of the neck was in some cases associated with stimulation of this area.

A comparison of figs. 1 and 3 will show that the distribution of the motor cortex in fig. 3 coincides to a large extent with the distribution of Type II. of the histological classification; it also overlaps into the areas occupied by Type I. and Type III. As the significance of this coincidence between the motor area as delimited by electrical stimulation, and the area showing large pyramidal cells in the infra-granular zone has already been referred to, it is unnecessary to do more than point out the similarity in the areas.

6. Conclusions.

The results of this investigation are of special interest when compared with the work done on the brain of insectivora by Watson (36). If the interesting conclusions he came to be accepted, they throw considerable light upon the significance of the cerebral lamination in the guinea-pig.

In the cerebrum of the mole, shrew, and hedgehog, Watson found that the motor area occupied an extensive area relative to the size of the brain. This is also seen in the guinea-pig, as both histological examination and electrical stimulation locate the motor area over the greater part of the posterior half of the cerebrum. It is seen from the histological description that the lamination in the so-called motor area is not characteristic of exclusive motor function, as both supra-granular, or pyramidal and granular, layers are present. This type of cortex was described in the motor area of the mole by Watson; and Bolton suggested that the area, besides regulating the passage of efferent impulses, was also a centre for the reception of sensory impulses projected to it from the various sensory areas.

This certainly seems to be the case in the "motor" cortex of the guinea-pig, for the large pyramidal cells in the sub-granular zone are practically an addition to a cortex which is in other characteristics sensory. The probable function of the large pyramidal sub-granular cells has already been discussed; they are evidently associated with movement, as they occur in the area over which electrical stimulation produced movement. The fact that they extend beyond the area figured as "motor" in fig. 9 is not unfavourable to this view, since this area really coincides with their richest distribution. In no part of the cerebrum is there anything like a brusque line between the various types of cortex; the distribution of the large pyramidal sub-granular cells is therefore limited by no definite line.

Bolton's view with regard to the sensori-motor nature of the area associated with movement in the lower mammals is similar to that of Horsley and others regarding the sensori-motor nature of the Rolandic area in the higher apes and in man.

With the exception of the "sensori-motor" area just described, almost the whole of the guinea-pig cerebrum shows a sensory type of cortex. The cortex at the anterior pole, or Type V., is probably scarcely sensory, and might be classified as undifferentiated.

No definite conclusion was come to as regards the position of sensory centres from an examination of the cortex, but a consideration of the work done on the guinea-pig by Allen and Watson led to the suggestion that a large part of this sensory zone must correspond to the kinæsthetic sense. From Miss Allen's work particularly it is clearly shown that the kinæsthetic sense is the most important factor in the associative or psychical processes in the guinea-pig.

In the mole, shrew, and hedgehog, Watson mapped out on the mesial surface a probable visual and fifth nerve centre. Careful examination of the mesial surface in the guinea-pig failed to discover any parallel formation. Round the posterior pole the cerebrum shows a "sensori motor" type of cortex, but with fewer large sub-granular pyramidal cells than in Type II. Since this is homologous to the visual centre as demonstrated in rabbits and other animals, the modified sensory type of cortex found here seems to suggest the probability of this being a visual centre. The somewhat atypical sensory cortex does not seem to contradict such a suggestion, since sight is not the most important sense in the guinea-pig.

Briefly summarised, the results are:—

1. Histologically, the whole cerebral cortex may be divided into five distinct types of cell lamination.

2. Electrical stimulation demonstrated the presence of a motor area in the posterior half of the cerebrum.

3. This motor area was characterised by large sub-granular pyramidal cells.

4. No true motor area was shown to exist, the so-called motor area being evidently sensori-motor.

5. The sensory area is extremely diffuse, and, from a consideration of the work of Allen and Watson, the suggestion is made that a great part of the area is associated with the kinæsthetic sense.

6. The suggestion is made—based on the homology of the area—that the visual centre is located at the posterior pole.

I wish to acknowledge my deep indebtedness to Professor Noël Paton for much help and direction in the work, also to Dr T. Graham Brown in association with whom this investigation was begun.

REFERENCES.

(1) Fritz and Hitzig, *Archiv f. Anatomie*, 1870.

(2) Hitzig, *Untersuchungen ueber das Gehirn*, 1874.

(3) Goltz, *Beiträge z. Lehre v. a. Functionen der Nervencentren des Frosches*, 1869.

(4) Goltz, *Pflüger's Archiv*, 1879.

(5) Munk, *Ueber die Functionen der Grosshirnvinde*, 1890.

(6) Fürstner, *Archiv f. Psychiatr.*, vol. vi.

(7) Ferrier, *The Functions of the Brain*, 1876 and 1886.

(8) Ferrier and Yeo, *Philosophical Transactions*, p. 479, 1884.

(9) Klein, Langley, and Schäfer, *Jour. of Physiology*, p. 231, 1883.

(10) Langley, *Jour. of Physiology*, p. 286, 1883.

(11) Klein, *Jour. of Physiology*, p. 310, 1883.

(12) Schäfer, *Jour. of Physiology*, p. 316, 1883.

(13) Langley and Grünbaum, *Jour. of Physiology*, p. 606, 1890.

(14) Nothnagel, *Virchow's Archiv*, vol. lvii.

(15) Moeli, *Archiv für Pathol. Anatomie*, p. 475, 1879.

(16) Lewis, Bevan, *Philosophical Transactions*, p. 35, 1880.

(17) Beevor and Horsley, *Philosophical Transactions*, p. 153, 1887.

(18) Beevor and Horsley, *Philosophical Transactions*, p. 49, B, 1890.

(19) Beevor and Horsley, *Philosophical Transactions*, p. 129, B, 1890.

(20) Schäfer, *Brain*, p. 1, 1888.

(21) Turner, Professor Sir W., *Journal of Anatomy and Physiology*, p. 357, vol. xxvi.

(22) Ziehen, *Anat. Anzeig.*, p. 692, 1890.

(23) Turner, Professor Sir W., *Journal of Anatomy and Physiology*, p. 357, vol. xxvi.

(24) Baginsky, *Archiv für Physiologie*, 1891.

(25) Bechterew, *Archiv für Physiologie*, p. 22, 1900.

(26) Bechterew, *Archiv für Physiologie*, p. 25, 1900.

(27) Mann, Gustav, *Journal of Anatomy and Physiology*, p. 1, vol. xxx.

(28) Mills, Wesley, *British Medical Journal*, p. 1485, 20th Nov. 1897; *Transactions of the Royal Society of Canada*, 1896.

(29) Cunningham, *Jour. of Physiology*, p. 264, vol. xxii.

(30) Martin, *Jour. of Physiology*, p. 383, vol. xxiii.

(31) Smith, Elliot, *Jour. of Anatomy and Physiology*, p. 309, vol. xxxiii.

(32) Bechterew, *Archiv für Physiologie*, p. 391, 1899.

(33) Yonakow, Von, *Ergebnisse der Physiologie*, p. 534, 1902.

(34) Bolton, *Archive of Neurology*, p. 424, vol. ii.

(35) Bolton, *Brain*, p. 215, 1903.

(36) Watson, *The University of Chicago Press*, p. 122, 1903.

(37) Allen, *Jour. of Comp. Neurology and Psychology*, 1904.

(38) Edinger, *Anat. Anzeig.*, p. 633, 1905.

(39) Watson, *Archive of Neurology*, p. 49, 1907.

(40) Haller, *Archiv für mik. Anatomie*, p. 117, 1907.

(41) Mott and Halliburton, *Proc. of the Royal Society*, p. 136, 1908.

(42) Mott and Kelly, *Proc. of the Royal Society*, p. 488, 1908.

(43) Mott, Schuster, and Halliburton, *Proc. of the Royal Society*, p. 124, 1910.

(44) Bolton, *Further Advances in Physiology*, p. 284, 1909.

(45) Graham Brown, T., *Quart. Jour. of Exper. Phys.*, vols. ii. and iii.

(46) Schäfer, *Quart. Jour. of Exper. Phys.*, p. 355, 1910.

(47) Schuster, *Proc. of the Royal Society*, 1910.

(*Issued separately May* 12, 1911)

XXVI.—On the Accuracy attainable with a Modified Form of Atwood's Machine.* By John P. Dalton, M.A., B.Sc., Carnegie Research Fellow. *Communicated by* Professor W. Peddie.

(MS. received March 6, 1911. Read same date.)

1. Introduction.

A careful determination of g by means of the ordinary type of Atwood's machine does not, as a rule, lead the average student in a Physical Laboratory to a better result than 930 or 940 cm/sec^2. From the point of view of successful teaching, it is somewhat unfortunate that, after bestowing reasonable care and attention to his work, a student should be unable to obtain a result approximating satisfactorily to what he knows to be the correct figure. Not unnaturally he takes it for granted that the actual numerical result obtained from his experimental labours is quite immaterial as long as the processes involved are clearly comprehended, and to him Experimental Physics is anything but an exact science. On the other hand, to set before the ordinary student a complicated apparatus specially designed for reaching an accuracy of 0·1 per cent. would be proceeding to the other extreme, and one could hardly expect much benefit to be derived from its use. But even the student who has already had some experimental training, and who has realised that quantitative relationship is just as important as qualitative, could not do any better in this case, for the defects are inherent to the method usually followed of timing the fall through a distance of 150 or 200 cm. with a metronome or stop watch: in his interests, at any rate, a more accurate procedure should be adopted.

The purpose of the present paper is to show how the usual type of Atwood machine may very readily be modified so as to give synchronous chronographical records of both time and distance at various points of the fall. The apparatus was, in fact, devised in the course of a research upon the wind pressure law and the efficiency of air-drags, for the purpose of obtaining accurate time-distance curves for the fall of a parachute. Some results of this research will soon appear; but as the calibration of the apparatus showed that it could be used with tolerable accuracy for the determination of g, it seemed desirable to publish a description of it, along with a few examples of the degree of accuracy that may be reached with it.

* The expenses of this research were met by a grant made by the Carnegie Trust.

INSTRUCTIONS TO AUTHORS.

The 'copy' should be written on large sheets of paper, on one side only and the pages should be clearly numbered. The MS. must be easily legible, preferably typewritten, and must be absolutely in its final form for printing; so that corrections in proof shall be as few as possible, and shall not cause overrunning in the lines or pages of the proof. All tables of contents, references to plates, or illustrations in the text, etc., must be in their proper places, with the page numbers left blank; and spaces must be indicated for the insertion of illustrations that are to appear in the text.

Illustrations.—All illustrations must be drawn in a form immediately suitable for reproduction; and such illustrations as can be reproduced by photographic processes should, so far as possible, be preferred. Drawings to be reproduced as line blocks should be made with Indian ink (deadened with yellow if of bluish tone), preferably on fine white bristol board, free from folds or creases; smooth, clean lines, or sharp dots, but no washes or colours, should be used. If the drawings are done on a large scale, to be afterwards reduced by photography, any lettering or other legend must be on a corresponding scale.

If an author finds it inconvenient to furnish such drawings, the Society will have the figures re-drawn at his expense; but this will cause delay.

When the illustrations are to form plates, a scheme for the arrangement of the figures (in quarto plates for the Transactions, in octavo for the Proceedings) must be given, and numbering and lettering indicated.

Proofs.—In general, a first proof and a revise of each paper will be sent to the author, whose address should be indicated on the MS. If further proofs are required, owing to corrections or alterations for which the printer is not responsible, the expense of such proofs and corrections will be charged against the author.

All proofs must, if possible, be returned within one week, addressed to *The Secretary, Royal Society,* 22–24 *George Street, Edinburgh,* and *not to the printer.* To prevent delay, authors residing abroad should appoint some one residing in this country to correct their proofs.

The author receives fifty of the reprints free, and may have any reasonable number of additional copies at a fixed scale of prices which will be furnished by the printer, who will charge him with the cost. To prevent disappointment, especially if the paper contains plates, the author should, immediately after receiving his first proof, notify *to the printer* the number of additional copies required.

Index Slips.—In order to facilitate the compilation of Subject Indices, and to secure that due attention to the important points in a paper shall be given in General Catalogues of Scientific Literature and in Abstracts by Periodicals, every author is requested to return to the Secretary, along with his final proof, a brief index (on the model given below), of the points in it which he considers new or important. These indices will be edited by the Secretary, and incorporated in Separate Index Slips, to be issued with each part of the Proceedings and Transactions.

MODEL INDEX.

Schäfer, E. A.—On the Existence within the Liver Cells of Channels which can be directly injected from the Blood-vessels. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

Cells, Liver,—Intra-cellular Canaliculi in.
E. A. Schäfer. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

Liver,—Injection within Cells of.
E. A. Schäfer. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

The Papers published in this Part of the PROCEEDINGS *may be had separately, on application to the Publishers, at the following prices:—*

No. XV.,	Price	6d.	No. XXI.,	Price	6d.
No. XVI.,	,,	6d.	No. XXII.,	,,	6s.
No. XVII.,	,,	6d.	No. XXIII. (*Abstract.*)		
No. XVIII.,	,,	1s. 4d.	No. XXIV.,	,,	1s. 4d.
No. XIX.,	,,	6d.	No. XXV.,	,,	1s. 2d.
No. XX.,	,,	6d.			

PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

SESSION 1910–11.

Part IV.] VOL. XXXI. [Pp. 417–576.

CONTENTS.

EDINBURGH:

PUBLISHED BY ROBERT GRANT & SON, 107 PRINCES STREET, AND
WILLIAMS & NORGATE, 14 HENRIETTA STREET, COVENT GARDEN, LONDON.

MDCCCCXI.

Price Eleven Shillings and Threepence.

REGULATIONS REGARDING THE PUBLICATION OF PAPERS IN THE PROCEEDINGS AND TRANSACTIONS OF THE SOCIETY.

THE Council beg to direct the attention of authors of communications to the Society to the following Regulations, which have been drawn up in order to accelerate the publication of the Proceedings and Transactions, and to utilise as widely and as fairly as possible the funds which the Society devotes to the publication of Scientific and Literary Researches.

1. MANUSCRIPT OF PAPERS.—As soon as any paper has been passed for publication, either in its original or in any altered form, and has been made ready for publication by the author, it is sent to the printer.

2. METHOD OF PUBLICATION.—As soon as the final revise of a Transactions paper has been returned, or as soon as the sheet in which the last part of a Proceedings paper appears is ready for press, a certain number of separate copies or reprints, in covers bearing the title of the paper and the name of the author, are printed off and placed on sale. The date of such separate publication will be printed on each paper.

3. ADDITIONS TO A PAPER after it has been finally handed in for publication, if accepted by the Council, will be treated and dated as separate communications, and may, or may not, be printed immediately after the original paper.

4. BRIEF ABSTRACTS OF TRANSACTIONS PAPERS will be published in the Proceedings, provided they are sent along with the original paper.

5. SPECIAL DISCUSSION OF PAPERS ACCEPTED FOR PUBLICATION.—Where a paper has been accepted for publication, the Council may, with the consent of the author, select this paper for Special Discussion. In the case of such papers advanced proofs will be sent to the members of the Society desiring copies, and copies will be supplied to the author for distribution. A paper selected for Special Discussion will be marked with an asterisk (*) and placed first on the Billet for the day of reading. Any following papers for that day may be adjourned or held as read if the discussion prevents their being read.

6. COMMUNICATIONS NOT SUBMITTED FOR PUBLICATION, such as Demonstrations of Experiments, Statement of Scientific Problems, etc., may be received by the Council, and may also be selected for Special Discussion. The Council does not undertake to publish any notice of such communications in the Proceedings or Transactions of the Society.

[*Continued on page* iii *of Cover.*

The results here appended are not to be taken as giving the limits of accuracy of the method, for, unfortunately, the construction of the friction rollers in the apparatus used is not quite satisfactory, with the result that friction is somewhat variable and in need of constant evaluation: with a more carefully constructed apparatus friction would be much smaller, and its constancy more assured.

2. Description of Apparatus. (See fig.)

A six-spoked aluminium wheel, cut with a V-groove in the rim, and mounted on a steel spindle, runs on four aluminium friction rollers. These rollers have conical sockets and are mounted on points screwed into a brass plate, so that if the steel spindle were completely homogeneous and were laid on the rollers, electrical contact would be complete from one side of the apparatus to the other during one whole revolution of the wheel. At one end of the spindle, however, a semi-cylindrical portion of the steel is removed and replaced by an identical piece of hard ebonite, the whole being then turned true in the lathe; it follows that at a certain point during each revolution the ebonite is in contact with both friction rollers on which it rests, and at that moment electrical contact is no longer possible between the two sides of the apparatus. Hence, by connecting with a chronograph, a single record is obtained of the distance fallen through during each revolution of the wheel (being the distance equal to the effective circumference of the wheel and string), and of the time taken to describe that distance. A sensitive relay is interposed between the revolving wheel and the chronograph, so that a very small current may be used, and no sparking occurs during motion. In the experiments given in this paper a three-pen motor-driven chronograph was used in conjunction with a clock beating half seconds, and times were easily obtained to $\frac{1}{100}$th of a second. The third pen served to record the actual moment at which motion began. Its magnet is connected in parallel with a small electro-magnet held in an adjustable stand. When an experiment is about to be made, the pan with the smaller load, to the bottom of which is soldered a small iron disc, is brought down to this release magnet, and, if necessary, its height is adjusted until the point of no contact in the wheel revolution is just reached. Paper is inserted between the magnet and the pan to ensure immediate release; the current is then broken, setting the system in motion and recording the initial point of the experiment on pen No. 3; every complete revolution of the wheel is recorded on pen No. 2, while the half seconds are marked by pen No. 1.

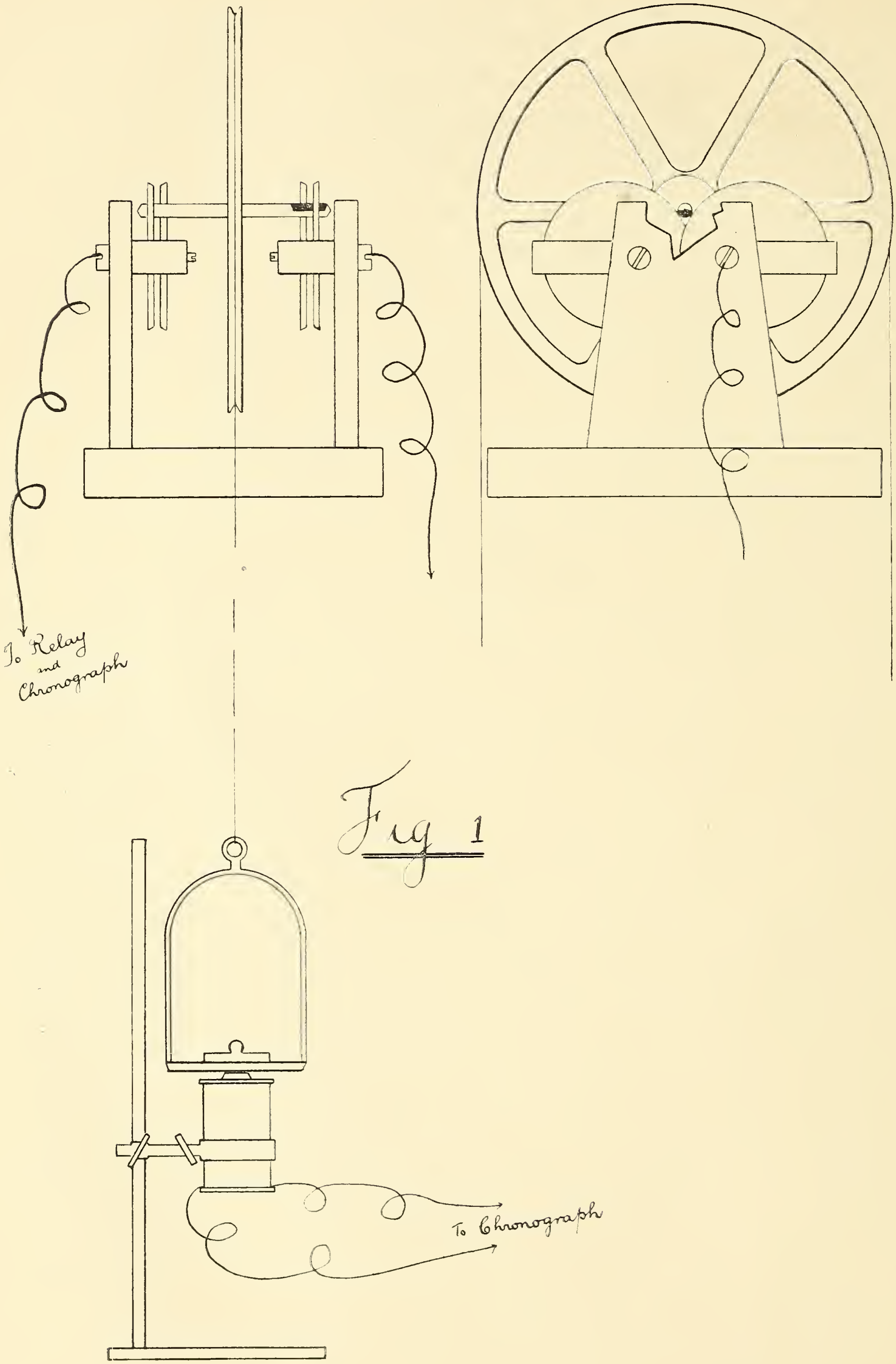


Fig 1

The string used was a strong silk fishing line, fitting well into the groove and continued beneath the pans to form an endless loop. The effective fall for one pulley revolution was determined by attaching a 10-metre tape to one of the pans and reading against a fixed point the distances covered between successive markings of the chronograph while the pulley was slowly rotated. The mean of fourteen separate measurements gave an effective fall of 38·92 cm., and consequently a mean pulley radius $p = 6{\cdot}194$ cm. The constancy of these individual measurements showed that the string did not slip appreciably in the V-groove.

3. Dynamical Equations and Data.

Putting

L = load on each side, including pans and string,
w = driving weight,
P = weight of revolving pulley,
p = effective radius of pulley,
k = radius of gyration of pulley,
a = observed acceleration,
a = radius of spindle,
and $a \sin \lambda$ = effective friction radius,

the friction moment becomes $2L + P + w\left(1 - \frac{a}{g}\right) a \sin \lambda$, and we readily obtain the well-known result

$$g = \frac{a\left[2L + w\left(1 - \frac{a}{p}\sin \lambda\right) + P\frac{k^2}{p^2}\right] + g\frac{a}{p}\sin \lambda[2L + w + P]}{w} \quad . \quad . \quad (1)$$

Frictional retardation, a', is determined by observing the time taken to come to rest after communicating a certain speed to the system symmetrically loaded. This is also done on the chronograph, it being now necessary to observe and record on pen No. 3 the moment at which motion *ceases*. To get as near as possible to the same conditions of load as those obtaining in the actual a experiment, it is well to observe a' with a load $L' = L + \frac{w}{2}$ on each side, and in that case a' is given by

$$\frac{g\frac{a}{p}\sin \lambda[2L + w + P]}{2L + w + P\frac{k^2}{p^2}}$$

Hence equation (1) reduces to the very simple form

$$g = \frac{a + a'}{w}\left[2L + w + P\frac{k^2}{p^2}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In this form, viz. Driving Force *minus* Frictional Force *equal to* Effective Force, the meaning of the equation can be grasped even by a student whose dynamical knowledge is small.

With the present apparatus it was found necessary to determine a' several times before and after each determination of a; with an apparatus of more satisfactory construction this would probably be unnecessary.

To get k, the radius of gyration, the pulley wheel was removed from its position on the friction rollers and was attached to bi-filar suspensions. Three separate determinations gave $k = 4{\cdot}221$, $4{\cdot}205$, $4{\cdot}230$, giving a mean $k = 4{\cdot}218$. This, with the weight $P = 44{\cdot}0$ g, and the radius $p = 6{\cdot}194$ cm., gives for the equivalent mass of the pulley $P\frac{k^2}{p^2} = 20{\cdot}5$ g.

The inertia of the four friction rollers was found from their dimensions and their weights to be one-tenth of that of the pulley wheel itself. As their angular speed is less than one-tenth that of the pulley, their total kinetic energy is less than one-thousandth of the kinetic energy of the pulley, and has therefore been left out of account in the subsequent calculation of g.

A graphical evaluation of $P\frac{k^2}{p^2}$ made in the usual way from the results appended, by plotting $\frac{w}{a+a'}$ against $2L+w$ and reading off the intercept on the load axis, led to a value $21{\cdot}2$ g.

A graphical method may also be adopted for ascertaining the fraction of a revolution at the beginning in finding a, and at the conclusion in finding a'. If R is the number of revolutions from and to rest respectively, we have $\frac{R}{T^2}$ in each case a constant, and so x, the unknown fraction of a revolution, can be at once obtained by plotting (0, 1, 2, 3, etc.) R against T_0^2, T_1^2, T_2^2, T_3^2, etc., where T_0, T_1, etc., are the times of x, $1+x$, $2+x$, etc., revolutions. In the a measurements this fraction is reduced to the smallest possible value by initial adjustment of the level of the release magnet, but one has no control over its value in the a' determinations. As a rule, however, it is not necessary to evaluate x, the incomplete part of a revolution, in the determination of a and a'; it is sufficient to plot squares of times from beginning and end respectively against number of complete revolutions, and the products of the slopes of the resulting straight lines into twice the effective distance of one revolution at once give the acceleration and retardation required. This was the procedure adopted in obtaining the results communicated in this paper: the experimental points, as long as the speed did not become excessive, lay exactly on a straight line, whose slope could easily be found to 1 in 1000.

The weight of the two pans and string used in the following experiments was 87·5 g, while additional loads of 50, 100, 150, and 200 g were added to each side. In each case two driving weights were tried, viz. 10 g and 20 g. The chronograph clock was carefully calibrated, giving a mean nominal second equivalent to 0·995 true seconds.

The maximum fall available was about 700 cm., but although records were obtained for the complete fall in every case, the $s-t^2$ curves showed some curvature for the last few metres of fall, and in the case of large acceleration this curvature was quite pronounced, the acceleration in every case diminishing as the speed increased. This is obviously due to the resistance of the pans to motion through the air, and, in fact, the apparatus is used chiefly for the determination of these resistances with larger surfaces. The resistance was allowed for in the present experiments by using for the determination of a only the first part of the fall, where the speed was low and the graph was accurately straight; should it be desired to use it over greater distances, it would be well to dispense entirely with pans, and to use weights made in the form of rods, so that the area presented normally to the direction of motion would be a minimum.

4. Results.

The following are eight different values of g obtained by the method indicated above. The experiments are grouped in pairs, at the beginning and end of which a' was determined a few times, the mean of each set being taken as holding for the actual experiments.

Values of g determined with Modified Form of Atwood's Machine.

2L (gram).	a' (cm/sec²).	w (gram).	$2L+w$ (gram).	$2L+w+P\frac{k^2}{r^2}$ (gram).	a (cm/sec²).	g (cm/sec²).
187·5	3·49	10	197·5	218	41·37	978
		20	207·5	228	82·00	975
						...976
287·5	3·64	10	297·5	318	27·16	979
		20	307·5	328	56·19	981
						...980
387·5	3·56	10	397·5	418	19·96	983
		20	407·5	428	42·35	983
						...983
487·5	3·31	10	497·5	518	15·65	982
		20	507·5	528	33·75	978
						...980
				Mean value of g	....	980

5. Discussion of Results.

Considering the magnitude of the frictional correction for the particular apparatus used in these experiments and the slight uncertainty in its numerical value, the remarkably good mean value obtained for g must be regarded as somewhat fortuitous. That there are irregularities is evident from the above values of a', which, as a' varies as $\left[1+\mathrm{P}\dfrac{1-\dfrac{k^2}{p^2}}{2\mathrm{L}'+\mathrm{P}\dfrac{k^2}{p^2}}\right]$, should decrease asymptotically as the load increases; but it would also seem that the effect of these disturbances can be determined by a proper evaluation of friction for each individual experiment. No doubt, more carefully constructed friction rollers would prove more regular in action; but as the accuracy here attained is more than sufficient for the author's immediate purpose, he did not think it necessary to have another wheel constructed in order to be able to test this point further.

It is, perhaps, of interest to examine under what conditions greatest accuracy may be attained. For a given absolute possibility of time measurement a should be as small as possible in order that it should be known with the highest relative accuracy, but then the difficulty arises that a' is a large fraction of the total, and any slight uncertainty in its value affects the result accordingly. On the other hand, to increase a so as to make a' relatively small would entail less accuracy in the time measurement, the square of which is involved, and, moreover, air resistance at such comparatively high speeds would become appreciable even during the earlier stages of the fall. It is, perhaps, significant that the worst value amongst the foregoing results is that obtained from the greatest acceleration. What should be attempted, therefore, is the reduction of the absolute value of a' to a minimum. Since $a'=g\dfrac{a}{p}\sin\lambda\left[1+\mathrm{P}\dfrac{1-\dfrac{k^2}{p^2}}{2\mathrm{L}'+\mathrm{P}\dfrac{k^2}{p^2}}\right]$, for a given load L′, with P fixed by considerations of stability, the desired result will be attained by making a and λ a minimum, and k as nearly equal to p as possible. This means that the supporting pivots must be thin and well lubricated, and that the mass of the pulley must be concentrated in the rim, the spokes being as light as possible, consistent with the load they have to bear.

In conclusion, it might be well to draw attention to the fact that the only modification in the usual type of Atwood pulley necessary to adapt it to the foregoing method is the very simple one of forming the wheel spindle partly of metal and partly of non-conducting material. Before the plan described above was adopted, the effect was tried of coating half of one end of the spindle with a very thin layer of hard varnish. This answered the purpose sufficiently well for one or two experiments, but the varnish soon cracked in places, giving rise to confused records on the chronograph. Something more permanent is required.

When the apparatus is to be used as an ATWOOD machine for determining the acceleration due to gravity, an inking chronograph is not at all essential; in fact, the accuracy of the time measurement obtained with the simpler forms, in which a smoked plate travels or a smoked drum revolves in front of a vibrating tuning fork, would reach the order of 1/500th second. Most laboratories now possess such a chronograph in their equipment, and most students are called upon to use it at some stage of their laboratory experience; and as equation (2), containing the dynamics of the method, is extremely simple, there is no reason why any junior student should be unable to apply it: the extra knowledge of experimental work required is but small, while the resulting gain in accuracy is great.

UNIVERSITY COLLEGE, DUNDEE,
March 1, 1911.

(*Issued separately May* 15, 1911.)

XXVII.—The Dissipation of Energy in Torsionally Oscillating Wires of Brass and other Materials, with the Effects produced on the Law of Torsional Oscillation by Change of Temperature, etc.

By J. B. Ritchie, B.Sc., Carnegie Research Scholar in Physics, University College, Dundee. *Communicated by* Professor W. Peddie.

(MS. received March 6, 1911. Read same date.)

In the determination of the law of decrease of torsional oscillations of an iron wire, when the range of oscillation is large in comparison with the palpable limits of elasticity, an equation of the form

$$y^n (x+a)=b$$

has been shown by Dr Peddie (*Phil. Mag.*, July 1894) to give close representation of results where—

y = the range of oscillation.

x = the number of oscillations since the commencement of observations.

n, a, b = quantities, constant for any one experiment, depending on the initial conditions of the experiment and the previous treatment of the wire.

The present work has been undertaken to find if this equation can with equal accuracy be applied in the case of wires of brass and other materials, and to find the effect produced on the constants of this equation by altering the initial conditions of the wire by change of temperature and by fatigue induced in the wire by repeated extensional or torsional strains.

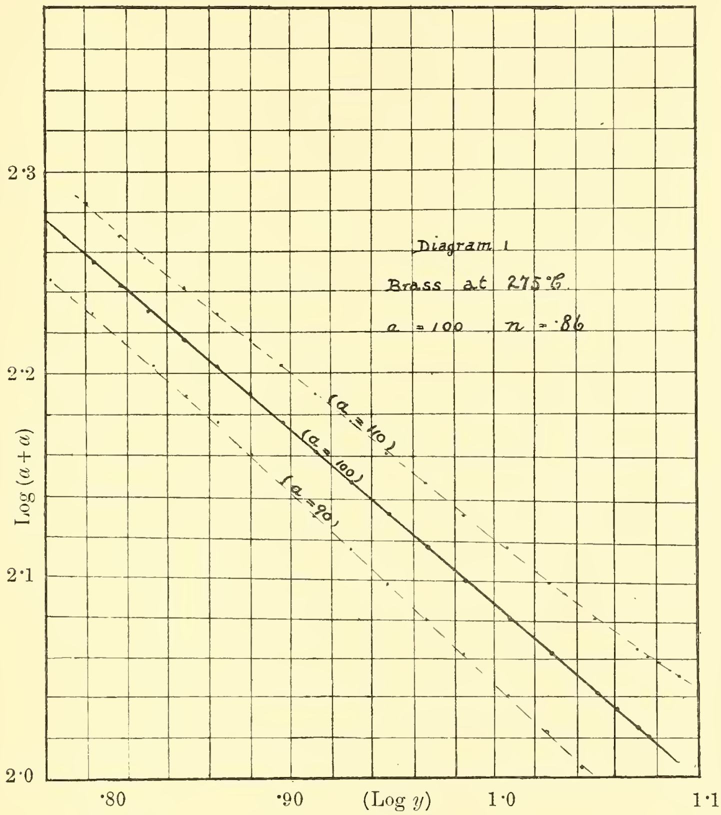
Method of Calculating the Constants.

The method described by Dr Peddie in a second paper on the same subject (*Trans. R.S.E.*, 1896) was employed for the determination of the quantities n, a, and b.

Since

$$n \log y + \log (x+a) = \log b,$$

then if $\log (x+a)$ be plotted against $\log y$, the corresponding points will lie on a straight line which intersects the axis along which $\log y$ is measured at an angle whose tangent is n, provided that the proper value of a be inserted. The actual value of this constant to be added to x depends upon the interval which elapses after starting the experiment until the first reading is taken. A rough idea of the value of a to be taken is got from the curve with scale readings as ordinates and number of oscillations as abscissæ, when the distance from the y axis of the line which the curve approaches asymptotically gives the value of a. If a wrong value of a be taken, the points in the curve of log y against log $(x+a)$ will not lie in a



straight line, a curve convex to the origin being obtained if the value of a be too large, and a curve concave to the origin if the value of a be too small. This is seen* to be the case in diagram 1, when with $a=90$ the curve is concave, and with $a=110$, convex. The value of a which gives the straightest line is taken, and from the tangent of the angle included by the line and the axis along which log y is measured n is found, and b can then be got by substitution.

* The scale readings y on the diagrams correspond to a rotation through 1° per ·175 cm. of scale.

Method of Conducting the Experiment.

The wire under consideration was suspended from a clamp attached to a torsion head, and at the other end was clamped, symmetrically and horizontally, a heavy lead ring of large moment of inertia. To the outer surface of this ring was fastened a scale divided into millimetres. The vibrations of the apparatus were damped out, and the torsion head then carefully turned so that no pendulum oscillation should be set up in the wire. Exterior disturbances were also, as far as possible, avoided. Readings of successive maxima ranges of oscillation were taken by means of a telescope with cross wires inserted, the crossing point being fixed in the same horizontal plane as the lead ring, at a distance of about 6 feet from the scale. It was found convenient to miss the first reading, and to take readings at the end of every oscillation after the first until ten oscillations had been completed, and thereafter to take readings after every fifth oscillation. Except in the case of tin wire, in which case the oscillations died down with extreme rapidity, the readings were extended over a hundred oscillations. The zero of the scale was found by taking successive readings to right and left at intervals, and the average of these values was then taken. A curve drawn with the scale readings as ordinates and the number of swings as abscissæ showed by means of the waviness of the curve if ordinary pendulum oscillations had been appreciably started in the apparatus. The values of log y were then plotted against those of log $(x+a)$, and when the proper value of a had been found, so that the points lay practically on a straight line, the constants were obtained.

Confirmation of the Empirical Law.

Wires of nine different metals were tested, brass, copper, aluminium, tin, zinc, silver, german silver, platinum, and nickel. Of these, brass, tin, zinc, silver, german silver, and nickel were found at the ordinary temperature to give close agreement, over the very large range of oscillations taken, to the general law, in each case a suitable value of a being found which caused all the points to lie on a straight line. It was found, however, that in the case of the remaining metals, and especially in the cases of aluminium and copper, no one value of a could be found to bring all the points into one line, an s-shaped curve being obtained in general. When this was first observed, it was thought that the law did not hold in such cases, or at least that it did not hold over the range taken. In attempting to straighten, in this case, one part of the curve, however, it was found that, with a certain value of a, the points could be brought to lie on *two* straight lines inclined at an angle not differing much from 180°. It was further found that this could not be done in every case with the same value of a for the two portions, but, by choosing a slightly different value of a, in every case the points could be brought to lie on two straight lines. The doubling of the line, as will be seen when the metals are considered separately, was found to depend upon controllable conditions, *e.g.* in brass it occurred when the metal had been brought to a certain temperature in the neighbourhood of 375° C. In most cases it was found that the value of n was greater in the line drawn through the points corresponding to the smaller oscillations.

Experiments on Brass Wire.

In the present series of experiments brass was the material most studied in detail, and, for the purpose of experiment, lengths of brass wire, approximately one millimetre in diameter,* were used. The length was in each case chosen so that, from clamp to clamp on the torsion apparatus, there should be exactly one foot of wire. It was found in a subsequent experiment, however, that change of length had no effect on the constants a and n, although b might differ considerably. In an experiment on 6 inches of brass wire, the values of a and n were found to be equal to those got with 12 inches of the same wire. The reason can readily be seen, as follows.

If we postulate that the loss of potential energy in a breaking down of molecular groups is proportional to a power of the angle of torsion, we can approximately write (Peddie, *Phil. Mag.*, July 1894) the loss of energy per swing in the form

$$-kydy = py^m dx.$$

Now, in a wire of half length, k is doubled for the same value of y; and the loss of energy, with the same y at half length, is half of what it would have been in the wire of whole length at $2y$. But in the wire of whole length at $2y$ the loss is

$$p2^m y^m dx.$$

Thus

$$-2kydy = p2^{m-1}y^m dx$$

$$-kydy = p2^{m-2}y^m dx,$$

so

$$y^{m-2}(x+a) = \frac{k}{2^{m-2}p(m-2)},$$

i.e.

$$y^n(x+a) = \frac{k}{pn}\cdot\frac{1}{2^n}$$

$$= b' \equiv b\frac{1}{2^n}.$$

The empirical law was found to hold over a very long range at the ordinary temperature. In all cases the points lay, with the proper value of a, in

* ·0975 cm.

straight lines. The following table gives some of the results got with wires each one foot long:—

Best Value of a.	Value of n.	Value of b.
140	·79	947
130	·76	765
120	·78	535
160	·74	783
150	·79	1035
180	·78	1242

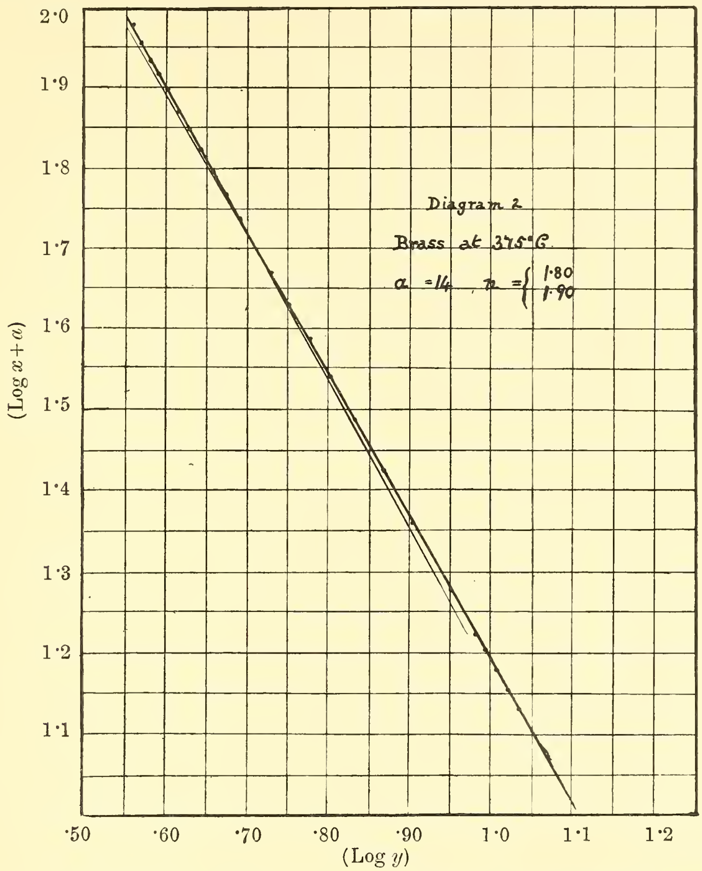
From these it is seen that the best value of a in each case lies between one and two hundred. Of course, the variations here are explained to some extent by the fact that when a reaches this high value, a difference of ten or twenty in its value may make very little difference in the closeness with which the points tend to lie on one line. In each case the value chosen is the mean of these two values which respectively tend to just render the curve visibly concave and convex to the origin. The value of n is seen to remain practically constant, but the values of b obtained do not seem to follow simply any corresponding change in the other constants.

Effect of Heating.

After the wire had been heated to redness, a complete change in the behaviour when oscillated was experienced. The oscillations died down much more rapidly, and it was found that the value of a required fell to the neighbourhood of zero, whilst the value of n had greatly increased. It was observed, however, as already stated, that the points could not now be fitted in to one straight line, but that a value of a could in general be found to cause the points to lie along two portions of straight lines.* In a few cases, however, two slightly differing values of a were required for the two parts. The heating was carried out in an electric furnace, and a series of temperature observations carried out to find at what temperature the change from a single to a double line occurred, and in what manner the value of n changed with increasing temperature. The temperature was measured by means of a platinum thermometer and bridge ratio box. It was found that the value of n gradually increased with increase of temperature until in the region of 300°, when the change became more apparent; n then increased rapidly until 375° C., after which two values of n were always obtained, which, owing to the value of a being so small, could not be got so accurately, the least change in a causing an appreciable difference in the

* Diagram 3.

slope of the line. Diagram 4 shows the results graphically, whilst the following table gives them in detail. It is seen that after this point of



change for one line, the values of n increase to a practically constant value within the range of experimental error.

Temperature.	Value of a.	Value of n.
275° C.	100	·86
339	120	1·02
358	60	1·26
368	40	1·60
372	20	1·80
375	14	1·80 and 1·90
377	6 and 5	2·15 and 1·90
400	5 and 0	2·10 and 2·30
413	2	2·40 and 2·60
466	4	2·30 and 2·80
485	3 and 2	2·15 and 2·70
535	4	2·30 and 2·55
625	1 and 0	2·60 and 3·20
675	2	2·40 and 2·90

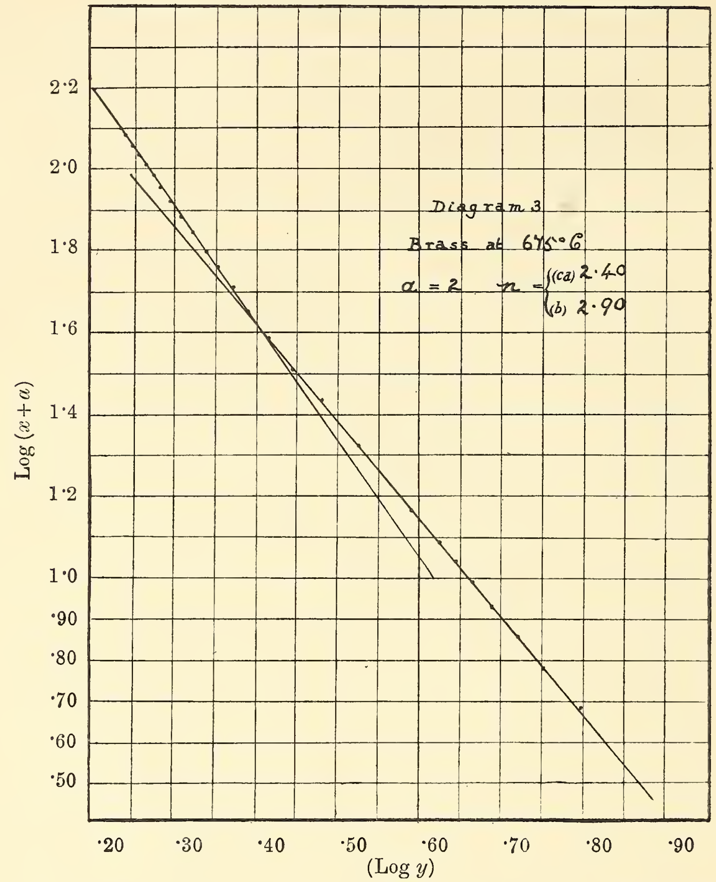


Diagram 3
Brass at 675°·6
a = 2 n = (ca) 2·40
(b) 2·90
Log (x+a)
2·2
2·0
1·8
1·6
1·4
1·2
1·0
·90
·80
·70
·60
·50
·20
·30
·40
·50
·60
·70
·80
·90
(Log y)

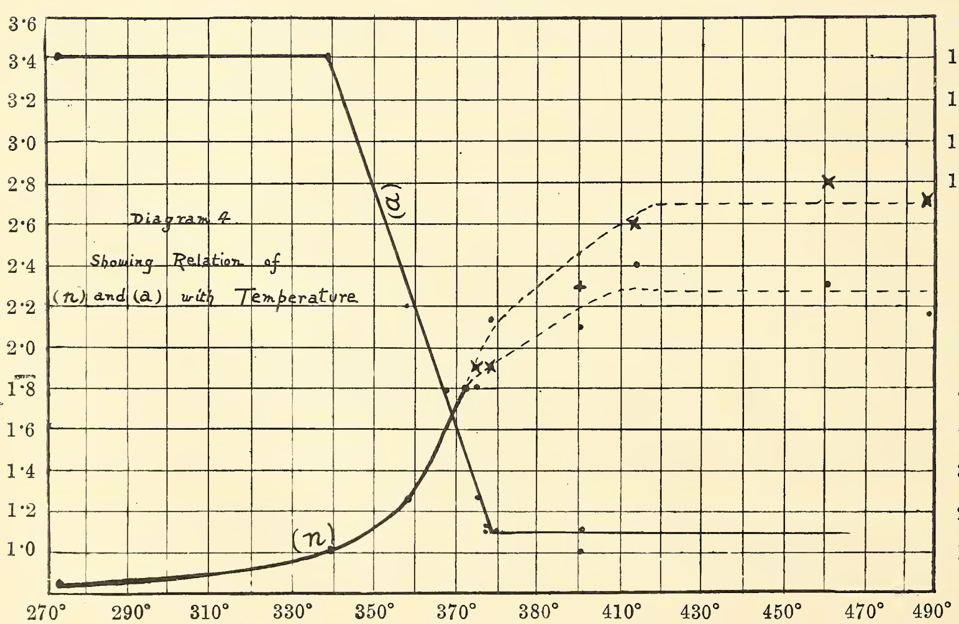


Diagram 4.
Showing Relation of
(n) and (a) with Temperature
(a)
(n)
3·6
3·4
3·2
3·0
2·8
2·6
2·4
2·2
2·0
1·8
1·6
1·4
1·2
1·0
40
130
120
110
100
90
80
70
60
50
40
30
20
10
270°
290°
310°
330°
350°
370°
380°
410°
430°
450°
470°
490°

Diagram 2 shows the condition at 375° C., *i.e.* the first temperature noted at which the double line begins to be necessary, whilst diagram 3 shows a decided angle between the two lines.

Time Effect.

In these temperature experiments, the plan adopted was to raise the brass wire up to the temperature indicated, and to remove it from the furnace at once, as it was found by previous tests that the change was a sudden one. Wires were introduced for $\frac{1}{2}$ hour, $\frac{1}{4}$ hour, 5 minutes, and 1 minute, respectively, at a temperature of 400° C., and in each case the same values of n got.

Lowering of Temperature.

A length of original * wire was allowed to remain in liquid air for over half an hour and subsequently tested, but this seemed to have no effect on the value of n; and this result was also got on treating in the same manner a portion of wire already heated past 375° C.

Effect of Extensional Strain.

A length of the original wire was hung with a 14-lb. weight attached in a long vertical shaft, and portions cut from it were tested at intervals of days or weeks, but no change was observed. The weight was then replaced by one of 28 lbs., and latterly of 56 lbs., but still the same values of n were observed. The wire after heating to 400° C., however, could be easily elongated by a pull; and on subsequent testing it was found to give results tending to approach those obtained with an unheated length of wire. A wire 12 inches in length was raised to 400° C. and then stretched till rupture occurred. The value of a was found to have changed to 100, whilst n was found to be 1·34,—the points again all falling into one straight line. The diameter was now 0·90 mm., and the change of length 3 inches. Thus stretching is followed by a reversion to the original conditions of the unheated wire; and it is quite probable that, with more careful stretching, the value of n could be reduced to its original amount. The effects of repeated extensional and torsional strains on brass are treated in another paper.

Wires of Varying Diameter.

It was found that a wire of diameter 1·2 mms. gave results similar to those got for that of diameter 1·0 mm. The exact values got were

$$a = 95, \quad n = \cdot 70, \quad b = 439.$$

* This term will be used in subsequent pages to denote a length of wire cut from a coil as supplied by the dealer.

A thicker wire, of diameter 1·65 mms., gave the results

$$a=85, \quad n=\cdot 66, \quad b=360.$$

In this case the readings had to be taken very quickly, since the oscillations died down with extreme rapidity, and thus the accuracy could not be so great as in the former experiments. This was found to be even more apparent in the next experiment, with a wire of diameter 2·0 mms. It was found impossible to take readings with any degree of accuracy with 12 inches of this wire, and so a double length was taken. Distinct pendulum oscillations were also of more frequent occurrence when the thick wire was used, and the curve obtained showed a wavy appearance. By drawing a straight line through the observed points, so as to eliminate the disturbing effect, the following results were obtained:—

$$a=90, \quad n=\cdot 84, \quad b=427.$$

This value of b, as shown on page 427, can be compared with the values got with the wires already discussed. For it was shown that in a wire of half length

$$b' \equiv b\frac{1}{2^n}.$$

Thus the value of b will be got for that of double length by

$$\begin{aligned} b &= b' \cdot 2^n \\ &= 238. \end{aligned}$$

The value of b, then, although not exactly under control, is seen to decline steadily with increases of diameter, thus:—

Diameter.	a.	n.	b.
1 mm.	About 100	·70–·80	700–1000
1·2 mms.	95	·70	439
1·65 „	80	·66	360
2·0 „	90	·84	238

The values of a and n, then, are independent of change of sectional area, just as they were of change of length taken. This is to be expected, in accordance with the original assumption that the loss of energy for a given material depends solely on a power of the angle of distortion.

Annealing from High Temperatures.

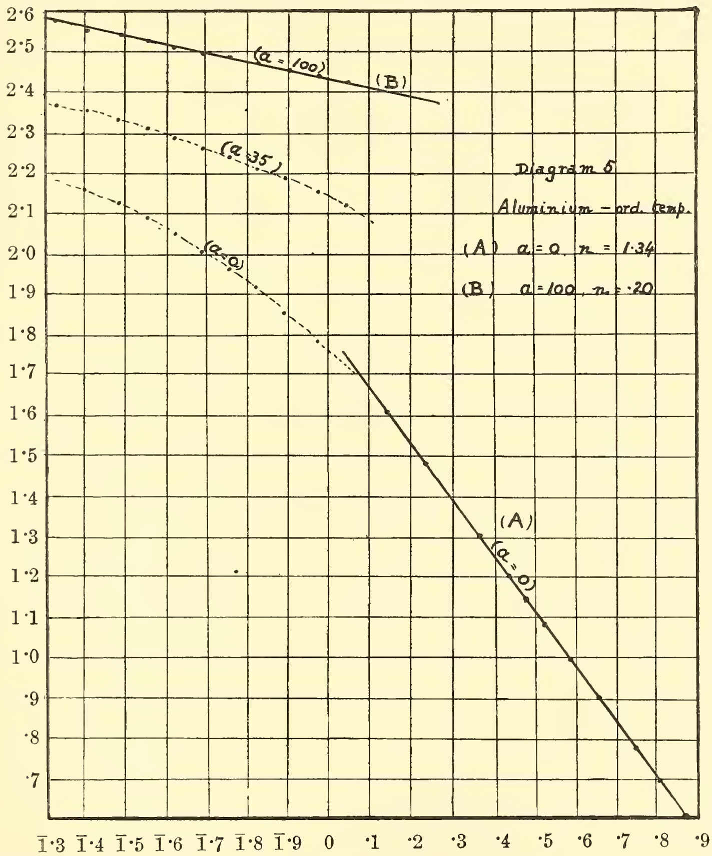
A length of brass wire heated to 535° C. was annealed and then tested. The values got were

$$\begin{aligned} &\text{A}: \quad a=1, \quad n=2\cdot 30, \\ &\text{B}: \quad a=1, \quad n=2\cdot 70, \end{aligned}$$

show no change from those for wire heated to 535° C., and quickly withdrawn from the furnace. On that occasion the values got were $a=4$, $n=2\cdot30$ and $2\cdot60$. An experiment when the wire was annealed from 485° C. also showed no change.

Aluminium Wire.

In the experiments, wire of diameter ·95 mm. and of length 12 inches was used. In observing the readings on oscillating the original wire, it



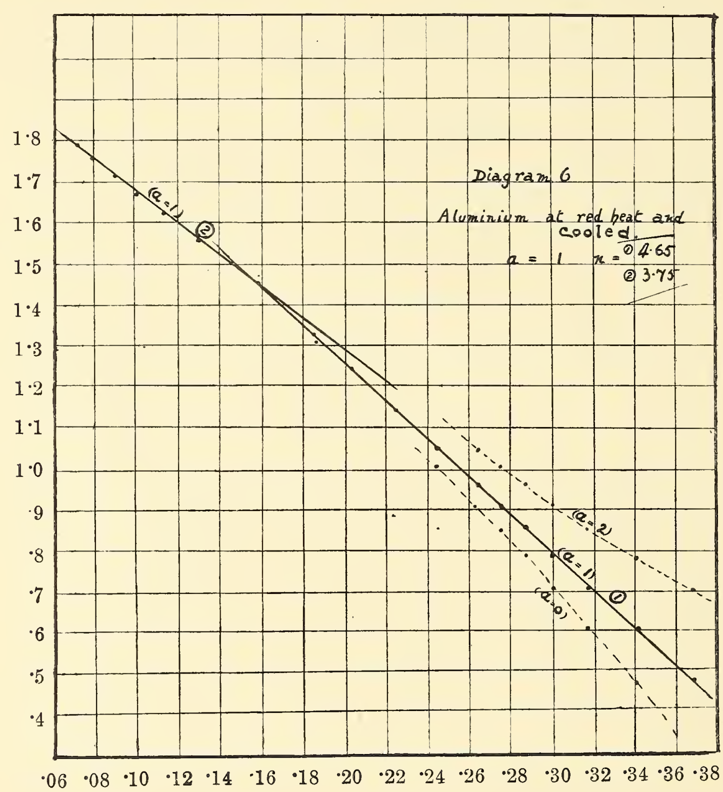
was noticed that after about twenty oscillations the relative decrease per swing in scale reading became larger, and after this point the oscillations seemed to die down in a different manner. This was seen to be the case on plotting log $(x+a)$ against log y. No value of a could be got to cause all the points plotted to lie on a single straight line; with $a=0$ a straight line was got for the first ten points (see diagram 5), but the others lay along a curve concave to the origin. The break seemed to be quite sudden. Higher and higher values of a were taken, but it was not until a

had reached the value of 100 that the remaining points could be fitted into one straight line. Of course, as a is increased, the curve tends to flatten in any case, but in the present case the slope of the line is still large, so that the linearity is real.

The values got here are

$$\text{Line A}: \quad a = 0, \quad n = 1{\cdot}34, \quad b = 31{\cdot}1.$$
$$\text{,, B}: \quad a = 100, \quad n = {\cdot}20, \quad b = 212.$$

This wire, then, offers a marked contrast to brass, as not only is it found



impossible to group the points along one line, but very different values of a have to be taken to cause them to lie along two lines.

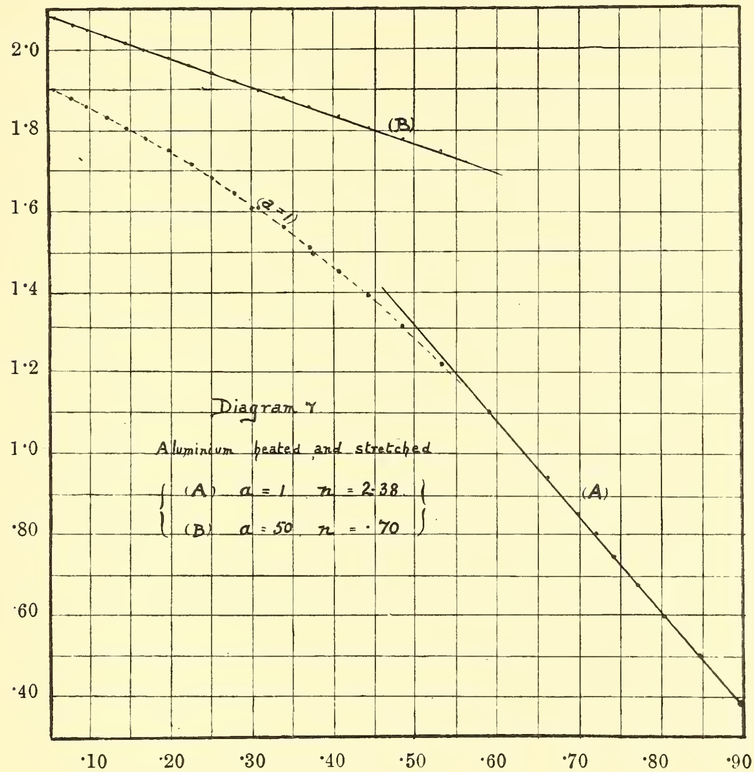
The effect of heating the wire to redness was next studied, and, whilst a double line was still the result, the curve showed a marked tendency to assume the condition of one straight line, as the two values of a required were identical, and n did not have greatly differing values.

The values got were

$$\text{1st line}, \quad a = 1, \quad n = 4{\cdot}65, \quad b = 150.$$
$$\text{2nd ,,} \quad a = 1, \quad n = 3{\cdot}75, \quad b = 109.$$

The values of n are seen to have increased greatly, whilst the average value of b has remained nearly the same. The value of a required for the first line has not altered much, but considerably for the second.

The effect of stretching the wires was then observed. The original wire broke in each case before any lengthening was obtained, but the heated wire pulled easily, and a length of 6 inches was got, whose diameter had been reduced from ·95 mm. to ·88 mm. This stretching was



found to have the same effect as in the case of the brass in the sense of a reversion to the non-heated condition, but, as is seen in the diagram, the actual type of effect is quite different.

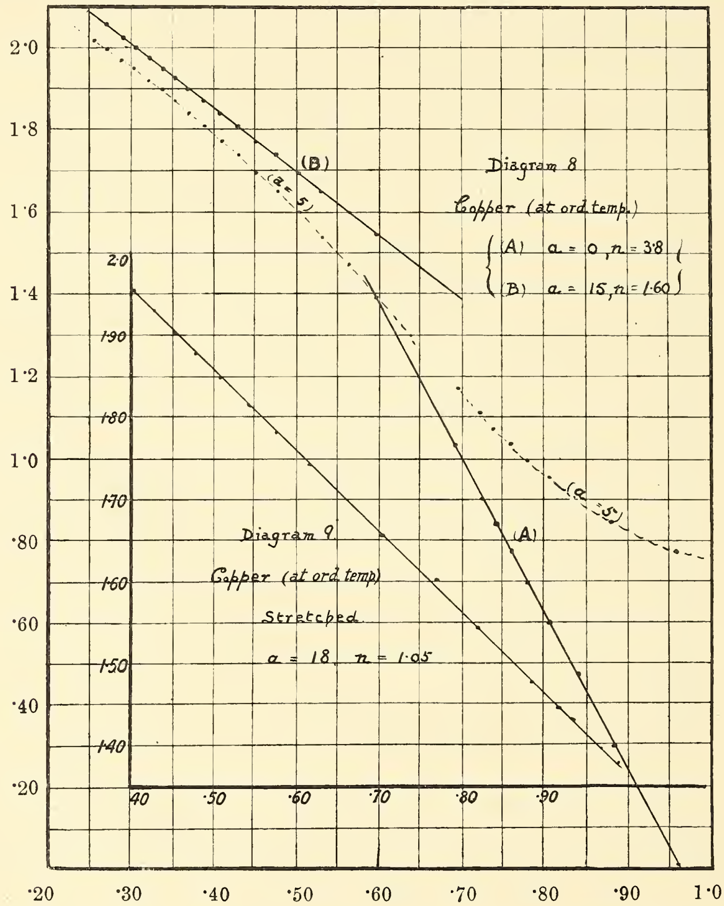
The second line again breaks away from the first, and a totally different value of a is required. The condition here is seen to be intermediate between those of the original unheated and the heated conditions. The actual values were

$$\text{A}: \quad a = 1, \quad n = 2{\cdot}38, \quad b' = 336.$$
$$\text{B}: \quad a = 50, \quad n = {\cdot}70, \quad b' = 130.$$

In no condition, then, has it been found possible to obtain one straight line including all the points over the period of 100 oscillations, and the tendency towards a straight line *after* heating to redness is quite an opposite result to that noted for brass.

Copper Wire.

Length of wire used = 12 inches. Diameter = ·80 mm. This wire, tested at the ordinary temperature, with values of log y plotted against those of log $(x+a)$, gave with $a=5$ an s-shaped curve (see diagram 8).



It was found possible to obtain two straight lines here by taking different values of a for the two parts, the actual figures being

$$\text{A}: \quad a=0, \quad n=3{\cdot}80, \quad b=446.$$
$$\text{B}: \quad a=15, \quad n=1{\cdot}60, \quad b=317.$$

Another specimen cut from a different part of the wire gave

$$\text{A}: \quad a=0, \quad n=3{\cdot}20, \quad b=1030.$$
$$\text{B}: \quad a=10, \quad n=1{\cdot}70, \quad b=216.$$

After heating to redness, there seemed to be in this case no tendency to change the form of the curve, and again a pair of straight lines was got, with values not far different from those got for the original wire. The values found were

$$\text{A}: \quad a = 0, \quad n = 4{\cdot}00, \quad b = 753.$$
$$\text{B}: \quad a = 25, \quad n = 1{\cdot}25, \quad b = 120.$$

Heating to the limit of temperature reached has thus only a slight effect. It tends to increase the angle between the pair of lines by a small amount.

On stretching the wire at the ordinary temperature, the form of curve is totally altered. Now all the points fall on one straight line (diagram 9), the values of the constants being

$$a = 18, \quad n = 1{\cdot}05, \quad b = 252.$$

Another set of values obtained were

$$a = 20, \quad n = 1{\cdot}05, \quad b = 270.$$

Stretching has lowered the value of n below either of the values got with the original or the heated wires.

German Silver Wire.

Similar experiments were carried out with German silver wire, of which a length of 1 foot was used, the diameter being 1 mm. Tested at the ordinary temperature, results were got which resembled those got for brass in the same condition. With $a = 110$ a perfectly straight line was obtained, the values being

$$a = 110, \quad n = 1{\cdot}05, \quad b = 1080.$$

On heating to redness and testing, however, the resemblance to brass ceases, as one straight line is still obtained, the values being

$$a = 20, \quad n = 1{\cdot}70, \quad b = 1092.$$

Whilst a has decreased considerably and n increased, b has remained practically stationary.

This heated wire was now pulled till rupture occurred, and half the original length, of diameter ·85 mm., oscillated. A rather unusual result was here obtained, as two lines were got with the values

$$\text{A}: \quad a = 50, \quad n = {\cdot}66, \quad b = 223.$$
$$\text{B}: \quad a = 60, \quad n = {\cdot}94, \quad b = 202.$$

These values of n are lower than either of the former values. The supposition might be made that the original unheated wire, if pulled, would assume values approaching these latter. Attempts were made to test this point, but no appreciable effect could be got, the wire resisting effective extension under the pulls that could be given it.

Zinc Wire.

This wire, of diameter 1·6 mms., was tested in its original condition, and found to obey the law over a range about one-sixth of that of the wires already mentioned. The oscillations in this case decreased very rapidly, and it was only with great difficulty and after many attempts that a satisfactory series of readings was obtained. The readings had to be taken consecutively, and after a dozen of these had been got, the oscillations had so far decreased as to render further readings of remaining values of little accuracy. On graphing the values in the usual way, it was found that the first twelve points lay on a straight line, with the constants

$$a=0, \quad n=\cdot60, \quad b=419.$$

After this point there was extreme inaccuracy, although there was a marked tendency to a decrease in the value of n.

Silver Wire.

Diameter of Wire used = 1·10 *mms.*

In this case the law was found to hold over the usual range, and one straight line included all the points when a was taken equal to 65, the result being

$$a=65, \quad n=\cdot45, \quad b=166.$$

Tin Wire.

Diameter of Wire used = 1·07 *mms.*

The lead ring used with the preceding wires elongated the tin wire to breaking point, and a light brass ring, to the bottom of which a thin brass tube was attached, was substituted. Even with this oscillation elongation occurred, a wire of 1 foot increasing by ·5 cm. in 24 hours. As in the case of zinc, the oscillations died down with extreme rapidity, and great difficulty was experienced in getting readings. The readings had to be taken at the end of every oscillation, and, as with zinc, only a dozen readings could be got with any degree of accuracy. Here, again, the points are found to lie on a single line until the oscillations had become so small as to prohibit exact readings. Again the tendency was noted for n to diminish greatly when the oscillations were small. Values got in various experiments were similar, the mean result being

$$a = 3, \quad n = \cdot 32, \quad b = 5\cdot 6.$$

Platinum Wire.

Diameter of Wire used = ·75 *mm.*

This wire was kindly lent, from the Chemical Department, by Professor Hugh Marshall. The points plotted over the range of a hundred oscillations could not be brought to lie on one line, and values were got as follows:—

$$\text{A}: \quad a = 28, \quad n = \cdot 95, \quad b = 341.$$
$$\text{B}: \quad a = 30, \quad n = 1\cdot 05, \quad b = 429.$$

Nickel Wire.

Diameter of Wire = ·5 *mm.*

Mr Butchart, one of the members of the Physics Research Section of the Scientific Society in this College, has made several experiments on nickel wire, and has found that over a large range it also conforms to the empirical equation.

Further work will be directed to the obtaining of a relation between the values of a and n and temperature in the case of aluminium and some of the other metals showing change on heating, and to the testing of alloys of varying proportions of constituents.

(*Issued separately May* 16, 1911.)

XXVIII.—An Apparatus for inducing Fatigue in Wires by means of repeated Extensional and Rotational Strains, with the Effects produced by such Fatigue in the Laws of Torsional Oscillation.

By **J. B. Ritchie**, B.Sc., Carnegie Research Scholar in Physics, University College, Dundee. *Communicated by* Prof. W. Peddie.

(MS. received March 6, 1911. Read same date.)

In the preceding paper the law of torsional oscillation

$$y^n(x+a)=b,$$

where y = range of oscillation, x = number of oscillations since start of experiment, and n, a, b constants for any one experiment, is considered with reference to wires of various materials, together with the effects produced on the constants of that empirical equation, and in particular on the constant n, by altering the initial conditions of the experiment. The wires were subjected to changes of length, diameter, temperature, and rigidity, and the resultant changes in the values of n, a, and b were given in detail. It was noted that the application of a large extensional force had a great effect in some cases, notably in the case of several of the metals after having been raised to a red heat in the electric furnace. The present work was undertaken to find the effect produced by a repeated application of an extensional force and by a repeated application of a twist to one end of the wire, the other end being held firm, thus tending to induce fatigue in the wires, the supposition being that such treatment would have an effect in the mode of oscillation when tested immediately afterwards. The latter step is necessary since it has been shown (Peddie, *Trans. Roy. Soc. Edin.*, vol. xxxix., 1897–1898) that rapid partial recovery from fatigue is possible.

The apparatus used for imparting these continued strains is a modification of an earlier machine used by Dr Peddie in these investigations, and was designed by him, the cost of construction being defrayed by means of a Royal Society grant given for the purpose.

The accompanying photograph shows the complete apparatus, with the battery and rheostat introduced into the circuit, but unfortunately the wires, which are in position for being fatigued, are not evident. The various parts of the apparatus will be seen to better advantage in the following diagram, and the description will be facilitated by the use of the various letters.

The whole apparatus is supported by a three-legged table, in height $4\frac{1}{4}$

feet, which has a lower stance at a distance of one foot from the ground. The brass rod T, which is six feet in length, is terminated by a soft iron

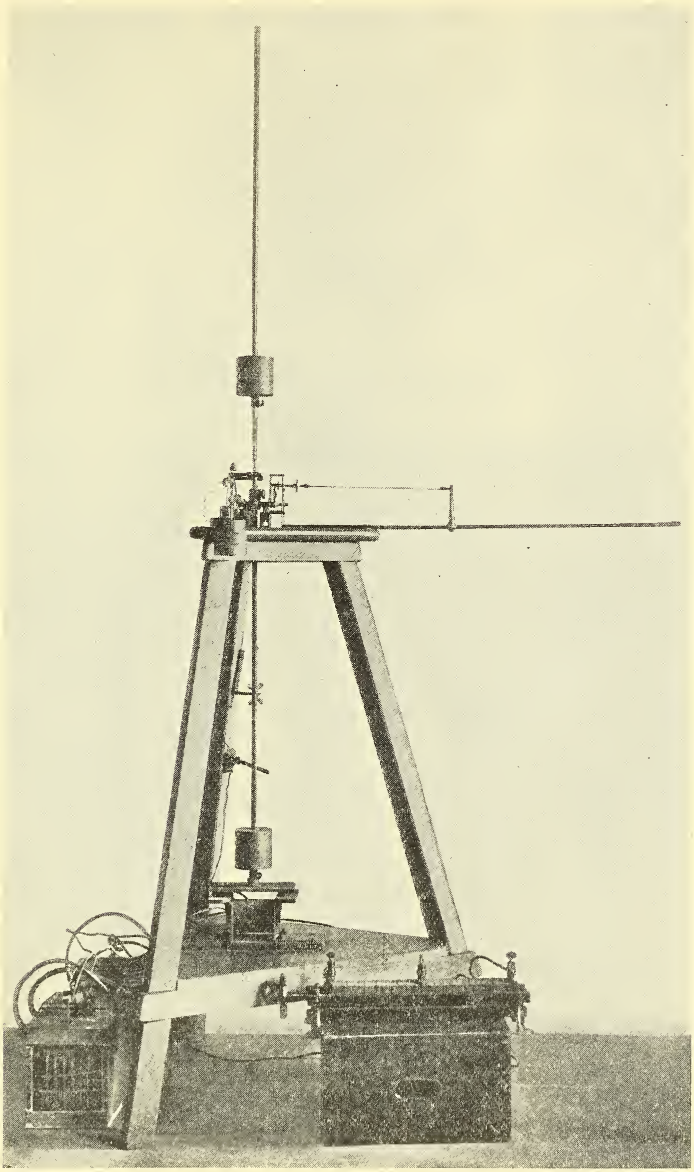
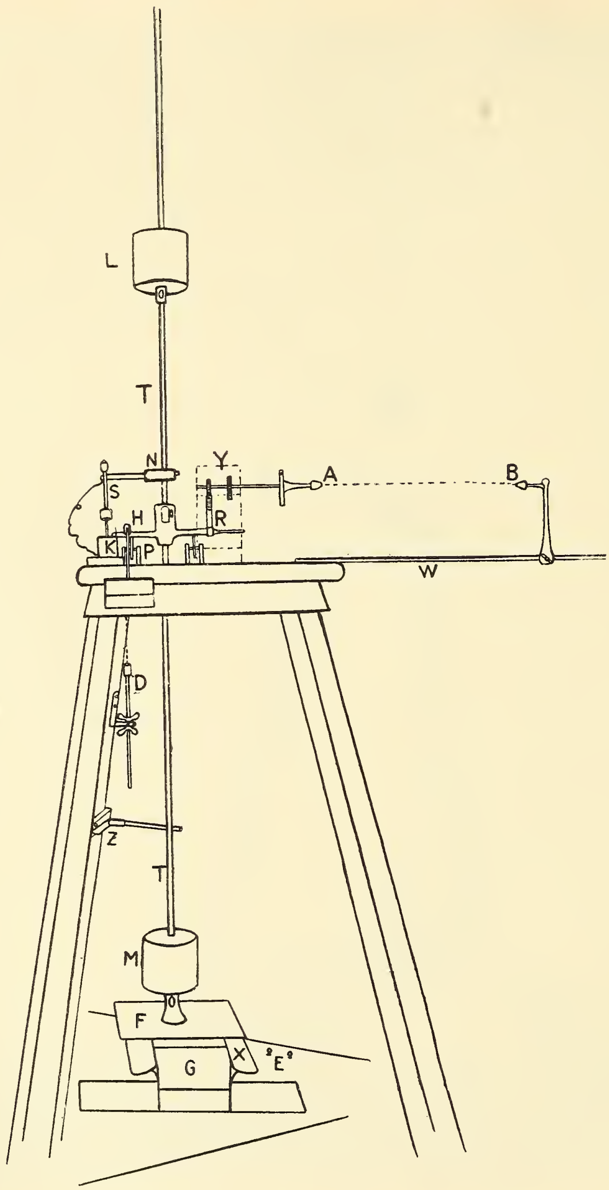
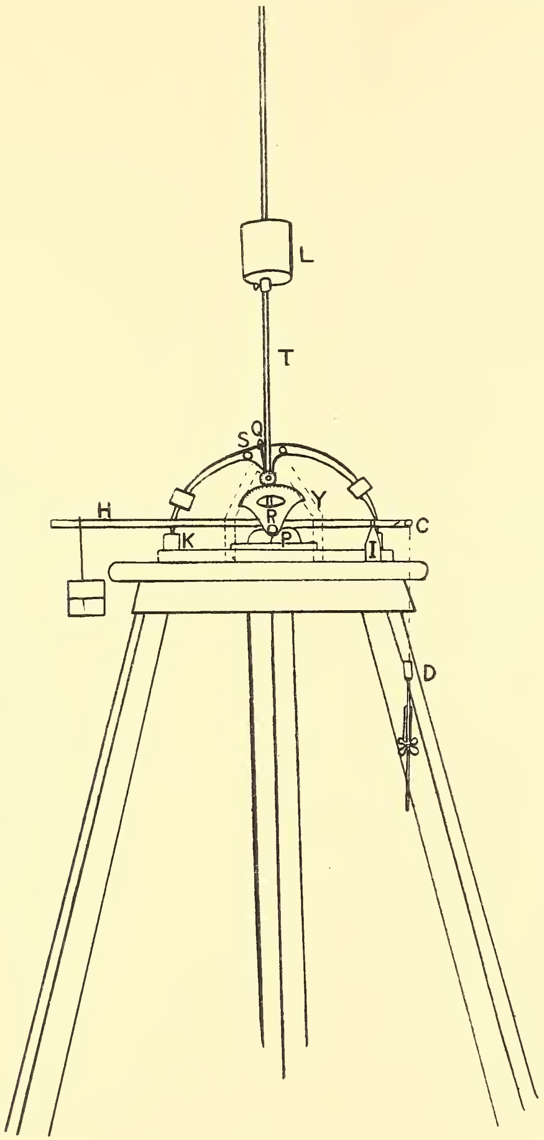


plate. Above this plate a heavy mass, M, of the same material is fastened, whilst another such mass, L, is fixed above the point of suspension. This pendulum is supported by a short steel rod which rests between two pairs of friction wheels, and oscillates freely between them, passing through a rectangular aperture cut in the table. The masses M and L are movable, and can be clamped in any desired position, so that any required moment of



inertia may be imparted to the pendulum. By raising L this can be increased, and the positions actually occupied were: M close to the lower plate, and L at a distance of $3\frac{1}{2}$ feet from it. Towards the end of the short steel rod, to which the pendulum is fastened by being passed through its middle point and clamped there, is fixed a V-shaped brass-toothed wheel R, which works into either of two smaller circular wheels of the same material attached to the brass rod terminating in the clamp A. One of these wheels has a



circumference double that of the other, so that the pointer on A can be given a single or double revolution. B is another clamp which is movable along a brass rod W fixed at one end on the table. The rod to which A is fixed can be shifted, by means of a catch, so that the V-shaped toothed wheel can fit into another smaller wheel on it, and so produce larger oscillations when the apparatus is set in motion. These toothed wheels are contained in an iron framework Y, fixed to the table. Fitting into the clamps A and B is the wire to which torsional fatigue is to be given, the length of wire taken being variable. In the farther side of the supporting steel rod is cut a little trench, and in this rests horizontally a long iron rod H C. At one end of this rod weights are attached, whilst at C is a clamp in a line above another one, D, fixed to one leg of the table. Between C and D is the wire to be given extensional strains. S is a semicircular strip of brass supported about a movable axis Q. The arms of this arc are loaded, and dip respectively into two iron cups, K being the one evident in the diagram. These cups are partially filled with mercury, the depth of which is so arranged as to ensure neither end of the brass arc being in contact when the apparatus is at its position of rest, and the arc being thus vertical. Projecting from, and clamped behind, slits on either side of the vertical support of the arc are two short brass rods N, which cause the oscillating pendulum to force the ends of the arc alternately into the mercury pools. Pieces of rubber tubing are fixed on these rods at the points of contact with the rod T. The leads from the battery enter at E, and a rheostat is introduced into the circuit so that the current is under control. From E one wire passes under the table and again emerges to form the coil G, which is surmounted by the two iron plates X, thence passing up the leg of the table and making contact by means of an iron bar fixed below the upper stance with the two mercury pools. The other wire from E passes to the key Z and thence upwards to the vertical rod at Q, a piece of wire inserted between Q and S ensuring the completing of the circuit. The mercury cups are fixed by screws to the table, and can be raised or lowered to any desired level. The pendulum is raised by hand, and allowed to oscillate. When M is at its maximum outward swing, the brass arc S is forced over until its end dips into the mercury at K. The magnetic action of G is then called into play, and M swings back, this time forcing over the arc to the other side by means of the projecting rod N, the end of the arc formerly dipping into the mercury thereupon breaking contact with the mercury and producing a spark. The pendulum now oscillates freely about the spindle P, alternately making and breaking contact with the two mercury pools. The current is regulated so that the pointer on A describes one complete revolution at each swing. If the current becomes too strong and the swings of the pendulum increase, the rod finally knocks against the contact maker at Z, causing it to be forced out of position, and thus breaking contact, so that the apparatus automatically prevents an increase in current causing the oscillations to exceed the normal maximum. If a wire be now clamped between A and B, and the pendulum set in motion, the end of the wire at A is twisted through an angle of 360° at each half swing, and thus fatigue is imparted to any desired degree in the wire by noting the time of oscillation. In the steel spindle on which the pendulum rests, and at the other side from the toothed wheels R, is a small groove into which the thin iron bar H rests, and this bar is movable about a fulcrum I. As the oscillator moves to its maximum outward swing from the vertical position of rest, the end H is raised, and consequently C lowered, and in the inward swing of the pendulum this is repeated, so that a wire clamped between C and D receives two pulls during a complete oscillation. To measure the power applied to the end of the wire per pull the weight hung on the end H and the ratio of the arms of the lever are noted, whilst the total number of pulls is found by doubling the number of oscillations in the given time. The wire between C and D can be kept fully stretched by means of screws above and below the projecting arm to which D is clamped.

The pendulum having been set oscillating, and the necessary resistance introduced by means of the rheostat, it was generally found that the range of the oscillation kept quite uniform, and the apparatus could be left for any desired time. On one occasion it was left going overnight, but this had worn down the teeth of the brass wheels so far that unsteady oscillation of A was being produced.

Results to Date.

Brass wire, similar to that used in the temperature experiments, is the only wire yet tested, and experiments included the finding of the effects on the constants n, a, and b of the equation

$$y^n (x+a) = b,$$

after subjecting the wire (1) to rotational strain in the original, unheated state; (2) to rotational strain after the wire had been raised to a red heat and cooled, *i.e.*, after the points obtained by plotting $\log(x+a)$ against $\log y$ could not be fitted into one straight line. It was seen, however, that these points could, with the proper value of a, lie along two straight lines; (3) to extensional strain in the unheated state; and (4) to extensional strain after raising to a red heat and then allowing to cool. It was shown in a paper accompanying this, that, since $n \log y + \log(x+a) = \log b$, then if $\log y$ be plotted against $\log(x+a)$, the points will, if the empirical law hold, fall into a straight line, provided the proper value of a be inserted, and the tangent of the angle this line makes with the $\log y$ axis will then be $= n$; b can then be calculated.

1. Wires were fatigued in this manner for periods varying from five minutes to three and a half hours, and subsequently tested. The values got were as follows:—

Time of Fatigue.	Best Value of a.	Value of n.	Calculated Value of b.
5 mins.	100	·78	1000
1 hour	180	·78	834
$2\frac{1}{2}$ hours	70	·78	560
$3\frac{1}{2}$ "	100	·76	484

The value of n is seen to remain constant over this range of fatigue, whilst that of a oscillates about 100, the values of a when large being more variable than when small, since in trying to find the best value of a, values of the latter differing by ten or twenty units may make very little difference in the slope of the line, or the ability of the points to lie along it. b seems to be the only constant affected, although it was seen in the previous paper that the values of b varied greatly. There appears to be a gradual fall in value here, however. Rotational fatigue, then, has little or no effect on the constants of the equation. Further work will be directed to ascertaining if this fall in the value of b with increase of fatigue is real.

2. Rotational strain on heated wires. It was shown also in the previous paper that after brass wire had been raised to a temperature above 375° C., then two values of n were obtained, one value extending for one range of oscillation, and another for the remaining part. It was shown that these values of n increased to a constant value. Fatigue, again, is found to have no effect on the constants a and n, as the following results will show:—

Time of Fatigue.	Best Value of a.	Values of n.	
		1st Part of Curve.	2nd Part of Curve.
0 mins.	0	2·52	3·00
5 "	2	2·50	3·20
10 "	1	2·51	3·10
15 "	0	2·53	3·02
30 "	0	2·81	3·00
60 "	0	*3·36	3·03
120 "	0	2·56	3·03
180 "	2	2·59	3·07

The behaviour of b after the wire has been heated and fatigued will be the object of further work.

* *This* value, 3·36, got for the line lying along the points of the first oscillations, is quite abnormal, and the wire used in this experiment was examined to discover, if possible, the

3. A length of brass wire, clamped vertically between C and D, and fatigued for two and a half hours, gave the values

$$a = 90, \quad n = \cdot 76, \quad b = 841.$$

These are the values got for the original wire, and thus extensional fatigue has no effect on the constants when the wire is unheated. This was confirmed by further experiments.

4. After heating to a red heat and then cooling, extensional fatigue was found to have the same tendency as stretching had—*i.e.* to tend to straighten the curve so as to allow of one straight line covering all the points. A typical experiment gave the results

$$a = 30, \quad n = 1\cdot 24, \quad b = 524.$$

This wire was fatigued for one hour after having been raised to 485° C. It was found that the pendulum made twenty-five complete oscillations per minute; therefore, since the wire between C and D receives two pulls during one oscillation, the total number of pulls given was 3000. The weight attached to the end H of the lever was 3000 gms., and the ratio of the arms 3 to 1, the fulcrum I being half way between the spindle P and the end C further from H. The leverage exerted per pull is, therefore, in this case 9000 gms. The values got on testing this wire after heating to 485° C., and before fatigue was applied, were

$$a = 2, \quad n = 2\cdot 15 \text{ and } 2\cdot 70\,;$$

a then is seen to be raised to a value intermediate to those obtained from the fresh and heated wires. Similarly n has now an intermediate value.

The effects of both forms of fatigue on the empirical equation as applied to other materials will be the object of further research.

cause of the unusual value of n. A repeated experiment gave the same result. The wire was then halved, and each half tested separately. Change of length has been shown to have no effect on the values of n or a.

Each half tested gave values as follows :—

$$(1)\ a = \cdot 5, \quad n = 2\cdot 00 \text{ and } 2\cdot 80$$
$$(2)\ a = \cdot 5, \quad n = 2\cdot 04$$

for first oscillations, and curve for rest, which could not be fitted with one straight line.

This latter half was then again halved, and the two small parts tested. Again the two parts were found to give different results, and the abnormality was traced to one of the quarters.

(*Issued separately May* 16, 1911.)

XXIX.—Boole's Unisignant. By Thomas Muir, LL.D.

(MS. received January 17, 1911. Read February 20, 1911.)

1. In two memoirs on the "Theory of Probabilities," * Boole was led to the consideration of a peculiarly interesting form of determinant which has only positive terms in its final form of development, and which in the case of the fourth order may be written

$$\begin{vmatrix} V & x\frac{\partial V}{\partial x} & y\frac{\partial V}{\partial y} & z\frac{\partial V}{\partial z} \\ x\frac{\partial V}{\partial x} & x\frac{\partial V}{\partial x} & xy\frac{\partial^2 V}{\partial x \partial y} & xz\frac{\partial^2 V}{\partial x \partial z} \\ y\frac{\partial V}{\partial y} & xy\frac{\partial^2 V}{\partial x \partial y} & y\frac{\partial V}{\partial y} & yz\frac{\partial^2 V}{\partial y \partial z} \\ z\frac{\partial V}{\partial z} & xz\frac{\partial^2 V}{\partial x \partial z} & yz\frac{\partial^2 V}{\partial y \partial z} & z\frac{\partial V}{\partial z} \end{vmatrix},$$

V standing therein for

$$axyz + byz + czx + dxy + ex + fy + gz + h\,.$$

In the first of the two memoirs the determinant is not explicitly referred to, but in the second it receives considerable attention, Boole, indeed, there saying that the memoir "involves discussions relating to the properties of a certain functional determinant, and to the possible solutions of a system of algebraical equations of peculiar form, discussions which will, I trust, be thought to possess a value as contributions to mathematical analysis, independent of their present application." The results, so far as determinants are concerned, occupy pages 235–240, and are summed up in a lemma and two propositions, the first proposition being of a general character, and the second concerning the special form just referred to.

The object of the present note is to present these two propositions in an entirely fresh light, and to co-ordinate therewith some recent investigations on similar functions.†

2. Boole's general proposition may be formulated as follows: *If each element of an axisymmetric determinant be an aggregate of multiples of*

* Boole, G., "On the Application of the Theory of Probabilities to the Question of the Combination of Testimonies or Judgments," *Trans. Roy. Soc. Edin.*, xxi. (1857), pp. 597–652.

Boole, G., "On the Theory of Probabilities," *Philos. Trans. Roy. Soc. Lond.*, clii. (1862), pp. 225–252.

† Muir, T., "A New Unisignant," *Messenger of Math.*, xl. pp. 177–192.

one and the same series of variables, the multipliers in the diagonal elements being all positive, and if the multipliers of any one of the variables in any row be in order proportional to the multipliers of the same variable in any other row, then the final development of the determinant contains nothing but positive terms.

The variables being denoted by $a, b, c, \ldots$ and the multipliers by Greek letters, the first column of the determinant must be of the form

$$\begin{array}{l} \alpha_1 a + \alpha_2 b + \alpha_3 c + \ \ldots \\ \beta_1 a + \beta_2 b + \beta_3 c + \ \ldots \\ \gamma_1 a + \gamma_2 b + \gamma_3 c + \ \ldots \\ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \end{array}$$

where we have only to bear in mind that $\alpha_1, \alpha_2, \alpha_3, \ldots$ are positive. This being the case, the axisymmetry and proportionality posited in the enunciation entail that the whole determinant must be

$$\begin{vmatrix} \alpha_1 a + \alpha_2 b + \alpha_3 c + \ \ldots & \beta_1 a + \beta_2 b + \beta_3 c + \ \ldots & \gamma_1 a + \gamma_2 b + \gamma_3 c + \ \ldots \\ \beta_1 a + \beta_2 b + \beta_3 c + \ \ldots & \frac{\beta_1^2}{\alpha_1} a + \frac{\beta_2^2}{\alpha_2} b + \frac{\beta_3^2}{\alpha_3} c + \ \ldots & \frac{\beta_1\gamma_1}{\alpha_1} a + \frac{\beta_2\gamma_2}{\alpha_2} b + \frac{\beta_3\gamma_3}{\alpha_3} c + \ \ldots \\ \gamma_1 a + \gamma_2 b + \gamma_3 c + \ \ldots & \frac{\beta_1\gamma_1}{\alpha_1} a + \frac{\beta_2\gamma_2}{\alpha_2} b + \frac{\beta_3\gamma_3}{\alpha_3} c + \ \ldots & \frac{\gamma_1^2}{\alpha_1} a + \frac{\gamma_2^2}{\alpha_2} b + \frac{\gamma_3^2}{\alpha_3} c + \ \ldots \end{vmatrix}$$

where the order is restricted to the third merely for shortness' sake in writing. This, however, is seen to be the product,

$$\left\| \begin{matrix} \alpha_1 a & \alpha_2 b & \alpha_3 c & \alpha_4 d & \ldots \\ \beta_1 a & \beta_2 b & \beta_3 c & \beta_4 d & \ldots \\ \gamma_1 a & \gamma_2 b & \gamma_3 c & \gamma_4 d & \ldots \end{matrix} \right\| \cdot \left\| \begin{matrix} 1 & 1 & 1 & 1 & \ldots \\ \frac{\beta_1}{\alpha_1} & \frac{\beta_2}{\alpha_2} & \frac{\beta_3}{\alpha_3} & \frac{\beta_4}{\alpha_4} & \ldots \\ \frac{\gamma_1}{\alpha_1} & \frac{\gamma_2}{\alpha_2} & \frac{\gamma_3}{\alpha_3} & \frac{\gamma_4}{\alpha_4} & \ldots \end{matrix} \right\|$$

and therefore by Binet's theorem

$$= \begin{vmatrix} \alpha_1 a & \alpha_2 b & \alpha_3 c \\ \beta_1 a & \beta_2 b & \beta_3 c \\ \gamma_1 a & \gamma_2 b & \gamma_3 c \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ \frac{\beta_1}{\alpha_1} & \frac{\beta_2}{\alpha_2} & \frac{\beta_3}{\alpha_3} \\ \frac{\gamma_1}{\alpha_1} & \frac{\gamma_2}{\alpha_2} & \frac{\gamma_3}{\alpha_3} \end{vmatrix} + \begin{vmatrix} \alpha_1 a & \alpha_2 b & \alpha_4 d \\ \beta_1 a & \beta_2 b & \beta_4 d \\ \gamma_1 a & \gamma_2 b & \gamma_4 d \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ \frac{\beta_1}{\alpha_1} & \frac{\beta_2}{\alpha_2} & \frac{\beta_4}{\alpha_4} \\ \frac{\gamma_1}{\alpha_1} & \frac{\gamma_2}{\alpha_2} & \frac{\gamma_4}{\alpha_4} \end{vmatrix} + \ldots.$$

$$= abc \frac{|\alpha_1\beta_2\gamma_3|^2}{\alpha_1\alpha_2\alpha_3} + abd \frac{|\alpha_1\beta_2\gamma_4|^2}{\alpha_1\alpha_2\alpha_4} + \ldots. \qquad \text{(I.)}$$

3. The advantage of this proof does not lie merely in its brevity; for we have to note that whereas Boole's three pages of the *Philosophical Transactions* establish only the fact that the terms of the final development must be all positive, what is here obtained is the actual positive terms themselves.

We also see from it that the proposition as above stated on the lines of the original is over-conditioned, it being unnecessary to stipulate that the multipliers in *all* the diagonal elements shall be positive. What is required is merely that this shall hold in regard to the *first* of these elements. (II.)

Further, we learn that the number of terms in the development cannot exceed $C_{n+m,\,n}$ where n is the order-number of the determinant and $n+m$ is the number of variables. (III.)

4. Boole's second proposition is in effect nothing more than the statement that a determinant of the special form exemplified in § 1, and which, without loss of generality, we may write

$$\begin{vmatrix} a+b+c+d+e+f+g+h & a+c+d+e & a+b+d+f & a+b+c+g \\ a+c+d+e & a+c+d+e & a+d & a+c \\ a+b+d+f & a+d & a+b+d+f & a+b \\ a+b+c+g & a+c & a+b & a+b+c+g \end{vmatrix},$$

is included as a special case of the determinant dealt with in his first proposition. That this statement is justified is at once seen by making it more definite, namely, by saying that the special case is that in which the values of the α's, β's, ... are given by the matrical equation

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_8 \\ \beta_1 & \beta_2 & \ldots & \beta_8 \\ \gamma_1 & \gamma_2 & \ldots & \gamma_8 \\ \delta_1 & \delta_2 & \ldots & \delta_8 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & . & 1 & . & . & . \\ 1 & 1 & . & 1 & . & 1 & . & . \\ 1 & . & 1 & 1 & . & . & 1 & . \end{pmatrix},$$

and in which, therefore, the final development is

$$abcd \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & . \\ 1 & 1 & . & 1 \\ 1 & . & 1 & 1 \end{vmatrix}^2 + \ldots + efgh \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \end{vmatrix}^2.$$

5. The special determinant of § 4 being denoted by

$$(\mathrm{B}(a\,;\,b,c,d\,;\,e,f,g\,;\,h) \quad \text{or} \quad \mathrm{B}(a\,,\,bcd\,,\,efg\,,\,h)$$

it is readily shown by interchanging pairs of the last three rows, and subsequently corresponding pairs of the last three columns, that it is the same as any one of the five

$$\mathrm{B}(a,\ b\,d\,c,\ e\,g\,f,\ h),$$
$$\mathrm{B}(a,\ c\,b\,d,\ f\,e\,g,\ h),$$
$$\mathrm{B}(a,\ c\,d\,b,\ f\,g\,e,\ h),$$
$$\mathrm{B}(a,\ d\,b\,c,\ g\,e\,f,\ h),$$
$$\mathrm{B}(a,\ d\,c\,b,\ g\,f\,e,\ h).$$

In other words, *the first and last variables being retained, it is allowable to take any permutation of the first triad, provided the same permutation of the second triad be taken at the same time.* (IV.)

6. Again, by subtracting the first row from one or more of the other rows, and subsequently treating the columns in the same way, we obtain

$$\left.\begin{array}{l} B(b,\ agf,\ hdc,\ e), \\ B(c,\ gae,\ dhb,\ f), \\ B(d,\ fea,\ cbh,\ g), \end{array}\right\}$$
$$\left.\begin{array}{l} B(e,\ hdc,\ agf,\ b), \\ B(f,\ dhb,\ gae,\ c), \\ B(g,\ cbh,\ fea,\ d), \end{array}\right\}$$
$$B(h,\ efg,\ bcd,\ a).$$

In other words, we learn that—

The first letter may be interchanged with any one of the letters of the first triad, provided that the last letter be interchanged with the corresponding letter of the second triad, and the remaining letters of the first triad be interchanged with the corresponding letters of the second triad. (V.)

The first letter may be interchanged with any one of the letters of the second triad, provided the last letter be interchanged with the corresponding letter of the first triad. (VI.)

The first and last letters may be interchanged, provided all the letters of the one triad be interchanged in order with the corresponding letters of the other. (VII.)

7. From a combination of the results of §§ 5, 6 it is seen that there are in all forty-eight different orders in which the variables can be taken. If we note that from (IV.) and (VII.) *the order of the variables may be reversed*, we may obtain the forty-eight forms most readily by using (V.) or (VI.), then performing reversal, and finally using (IV.).

As one of the results of § 5 is to the effect that the function is invariant to the simultaneous performance of the cyclical substitutions

$$\left.\begin{array}{l} b, c, d = c, d, b \\ e, f, g = f, g, e \end{array}\right\}$$

we may, in general, on knowing one term of the development, obtain in this way two others. Thus *ecdh* gives rise to *fdbh* and *gbch*, and we may also, therefore, conveniently denote the sum of the three by

$$\sum^{\circ} ecdh.$$

8. As a first step towards arriving at the final expansion of the function, we take note from the original determinant form that the aggregate of the terms containing h

$$= h \begin{vmatrix} a+c+d+e & a+d & a+c \\ a+d & a+b+d+f & a+b \\ a+c & a+b & a+b+c+g \end{vmatrix}$$

$$= h \begin{vmatrix} a+c+d+e & c+e & d+e \\ c+e & b+c+e+f & b+e \\ d+e & b+e & b+d+e+g \end{vmatrix},$$

and consequently that the aggregate of the terms containing ha is

$$ha\left\{(b+e)(c+f) + (c+f)(d+g) + (d+g)(b+e)\right\}.$$

We then, from the same source, obtain the terms containing h without a, namely,

$$\begin{aligned} &4bcd + (cd+db+bc)(e+f+g) \\ &\quad + (c+d)fg + (d+b)eg + (b+c)ef + efg\,; \end{aligned}$$

thence, by means of the interchange (VII.), the terms containing a without h; and finally, by recurring to the original determinant, the terms independent of both h and a, namely,

$$be(c+f)(d+g) + cf(d+g)(b+e) + dg(b+e)(c+f).$$

The result thus reached,

$$\begin{aligned} &\sum^{0}(ha+be)(c+f)(b+g) + \sum^{0}(a+h)be(\overline{c+f}+\overline{d+g}) \\ &\quad + a(bcd+4efg+\textstyle\sum^{3} bfg) + h(efg+4bcd+\textstyle\sum^{3} ecd), \end{aligned} \qquad \text{(VIII.)}$$

though not elegant, is useful for purposes of verification.

9. Having, however, obtained the terms in ha, we are led to use the first three forms of B in § 6 to arrive at three other aggregates of like constitution; and a little additional trouble results in the complete expression

$$\begin{aligned} &ah\left\{(b+e)(c+f) + (c+f)(d+g) + (d+g)(b+e)\right\} + (a+h)(efg+bcd) \\ &+be\left\{(c+f)(d+g) + (d+g)(a+h) + (a+h)(c+f)\right\} + (b+e)(agf+hdc) \\ &+cf\left\{(d+g)(a+h) + (a+h)(b+e) + (b+e)(d+g)\right\} + (c+f)(gae+dhb) \\ &+dg\left\{(a+h)(b+e) + (b+e)(c+f) + (c+f)(a+h)\right\} + (d+g)(fea+cbh) \end{aligned} \qquad \text{(IX.)}$$

which, until we reach the last binomial in each line, is a function of

$$ah\,,\, be\,,\, cf\,,\, dg \quad\text{and}\quad a+h\,,\, b+e\,,\, c+f\,,\, d+g\,.$$

10. Again, by performing on the original determinant the operations

$$\text{row}_1 - \text{row}_4\,, \quad \text{col}_1 - \text{col}_4$$

we obtain

$$\begin{vmatrix} d+e+f+h & d+e & d+f & . \\ d+e & a+c+d+e & a+d & a+c \\ d+f & a+d & a+b+d+f & a+b \\ . & a+c & a+b & a+b+c+g \end{vmatrix},$$

and are thus led to elaborate the result

$$\left.\begin{array}{l} (a+b+c+g)\,.\,(def+deh+dfh+efh) \\ +(ab+cg)\,.\,(d+f)(e+h) \\ +(ac+bg)\,.\,(d+e)(f+h) \\ +(ag+bc)\,.\,(d+h)(e+f) \\ +(abc+abg+acg+bcg)\,.\,(d+e+f+h) \end{array} \begin{array}{l} \\ \\ \left.\begin{array}{l} +\ \ 4ag\,.\,ef \\ +\ \ 4bc\,.\,dh \end{array}\right\} \\ \\ \\ \end{array}\right. \qquad \text{(X.)}$$

where the factors on the left of the multiplication dot are functions of a, b, c, g, and those on the right are functions of d, e, f, h. As this result cannot be altered by the interchange

$$a, b, c, g \;=\; d, e, f, h \;=\; a, b, c, g$$

we learn from making this interchange that

$$\left.\begin{array}{l} (ab+cg)\,.\,(d+f)(e+h) \\ +(ac+bg)\,.\,(d+e)(f+h) \\ +(ag+bc)\,.\,(d+h)(e+f) \end{array}\right\} = \left\{\begin{array}{l} (de+fh)\,.\,(a+c)(b+g) \\ +(df+eh)\,.\,(a+b)(c+g) \\ +(dh+ef)\,.\,(a+g)(b+c) \end{array}\right. \qquad \text{(XI.)}$$

11. By subtracting the first row of the original determinant from each of the other rows, thereafter increasing the first row by the sum of the others, and finally diminishing the first column by the sum of the others, we obtain the interesting alternative form

$$\begin{vmatrix} 2(a+h) & a-e & a-f & a-g \\ b-h & . & b+f & b+g \\ c-h & c+e & . & c+g \\ d-h & d+e & d+f & . \end{vmatrix}. \qquad \text{(XII.)}$$

This, by monomialising the elements of the first row and first column, becomes

$$-ha\begin{vmatrix} . & 1 & 1 & 1 \\ 1 & . & b+f & b+g \\ 1 & c+e & . & c+g \\ 1 & d+e & d+f & . \end{vmatrix} + h\begin{vmatrix} 2 & e & f & g \\ 1 & . & b+f & b+g \\ 1 & c+e & . & c+g \\ 1 & d+e & d+f & . \end{vmatrix}$$

$$+ a\begin{vmatrix} 2 & 1 & 1 & 1 \\ b & . & b+f & b+g \\ c & c+e & . & c+g \\ d & d+e & d+f & . \end{vmatrix} - \begin{vmatrix} . & e & f & g \\ b & . & b+f & b+g \\ c & c+e & . & c+g \\ d & d+e & d+f & . \end{vmatrix}.$$

which may profitably be compared with the result of § 8. Of the new forms of unisignant which make their appearance in it, the first and the last are of little importance, being really both included in the simple form

$$-\begin{vmatrix} \cdot & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & x & \cdot & \cdot \\ \beta_2 & \cdot & y & \cdot \\ \beta_3 & \cdot & \cdot & z \end{vmatrix}$$

which equals $\alpha_1\beta_1yz + \alpha_2\beta_2zx + \alpha_3\beta_3xy$: but the form which includes the two others is not at all so trivial, the proposition regarding it being that *any positive elements whatever may be inserted in the vacant places of the determinant*

$$\begin{vmatrix} 2 & 1 & 1 & 1 \\ -\alpha & -\alpha & & \\ -\beta & & -\beta & \\ -\gamma & & & -\gamma \end{vmatrix}, \qquad \text{(XIII.)}$$

and yet all the terms of the final development remain positive. Further, *if the elements so inserted be* α_3, α_2, β_3, β_1, γ_2, γ_1, *the determinant*

$$\begin{vmatrix} \alpha+\alpha_2+\alpha_3 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta+\beta_3+\beta_1 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma+\gamma_1+\gamma_2 \end{vmatrix} \qquad \text{(XIV.)}$$

has the same development, namely,

$$\begin{aligned} &\alpha\beta\gamma + \alpha\beta(\gamma_1+\gamma_2) + \beta\gamma(\alpha_2+\alpha_3) + \gamma\alpha(\beta_3+\beta_1) \\ &+ \alpha(\beta_3\gamma_1+\beta_1\gamma_1+\beta_1\gamma_2) + \beta(\alpha_2\gamma_1+\alpha_2\gamma_2+\alpha_3\gamma_2) + \gamma(\alpha_2\beta_3+\alpha_3\beta_1+\alpha_3\beta_3) \\ &+ 2(\alpha_2\beta_3\gamma_1+\alpha_3\beta_1\gamma_2), \end{aligned}$$

a result which ought to be classed along with Sylvester's of the year 1855, as his determinant (which is also unisignant) differs from it only in having the non-diagonal elements all negative.

12. From any one of the forms (VIII.), (IX.), (X.), we can readily verify the fact that the number of terms in the final development is sixty-four. This agrees with the result of putting each variable equal to 1 in either of the determinant forms,—a substitution which is warranted as soon as the unisignancy becomes established. *When the determinant is of the* n^{th} *order, the number of variables is* 2^{n-1}, *and the number of terms in the final development is* $2^{(n-1)(n-2)}$. (XV.)

13. We may note, in conclusion, that when in the determinant of § 1 the function V is different in form from Boole's, the result is different in

character. Indeed there are cases where the signs of the terms in the final development are all *negative*. Thus, when

$$\mathrm{V} = ax^2 + 2bxy + cy^2 ,$$

the determinant is equal to

$$-4xy(a^2bx^4 + a^2cx^3y + ac^2xy^3 + bc^2y^4) ;$$

and when

$$\mathrm{V} = ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy ,$$

the determinant is equal to

$$-8xyz \left\{ \overset{0}{\sum} a^2efx^5 + \overset{0}{\sum} a^2(be+df)x^4y + \overset{0}{\sum} a^2(cf+de)x^4z + \overset{0}{\sum} a^2bdx^3y^2 \right.$$

$$\left. + \overset{0}{\sum} a(abc+2def)x^3yz + \overset{0}{\sum} a^2cdx^3z^2 + 2abdex^2y^2z \right\} \quad \text{(XVI.)}$$

where the symbol of summation implies the performance of the cyclic substitutions

$$x, y, z = y, z, x, \qquad a, b, c = b, c, a, \qquad d, e, f = e, f, d .$$

(*Issued separately May* 17, 1911.)

XXX.—Isopiestic Expansibility of Water at High Pressures and Temperatures. By W. Watson, M.A., B.Sc.; 1851 Exhibition Research Scholar, 1909–1910. (*Communicated by* Professor J. G. MacGregor.)

(Read February 6, 1911. MS. received February 13, 1911.)

These experiments were made in the Physical Institute of the University of Leipzig at the suggestion of Professor Des Coudres, who very kindly placed at my disposal for the purpose a compression-cylinder suitable for high temperature work, which I shall describe later. The original description by Professor Des Coudres appeared recently in the *Berichte der mathematisch-physischen Klasse der Königlichen Sächsischen Gesellschaft der Wissenschaften zu Leipzig*, vol. lxii., 18th July 1910.

Previous Work on the Subject.

To investigate the volume of water above 100° C., we must submit it to pressure in order to avoid vaporisation. Hirn * kept the water under the pressure of about 10 metres of mercury and worked up to temperatures of 200° C. He calculated out a formula of the form $V_T = V_0 (1 + at + bt^2 + ct^3 + dt^4)$ and found the values of the constants a, b, c, d for temperatures ranging from 100° to 200° C.

Waterston † investigated the water in a closed graduated glass tube with thick walls. Correcting for the mass of vapour which was always unavoidably present, he found the volumes of water up to 320° C.

Amagat's ‡ great work on the compressibility of water, ranging to very high pressures, gives us data from which, at various pressures, we can calculate the specific volume of water at temperatures up to about 200° C., the observations being specially exhaustive between 0° and 100° C.

In the experiments of Hirn and Waterston the pressure is comparatively small and the temperatures never exceed 320° C.; in Amagat's, the pressures are in general high and the temperatures have an upper limit at 200° C.

The experiments with which this paper deals bear a certain resemblance to Amagat's in this respect, that the pressures are invariably high, having,

* *Ann. Chim. Phys.* (4), 10, 32, 1867.

† *Phil. Mag.* (4), 21, 401, 1861, and 26, 116, 1863.

‡ *Ann. Chim. Phys.* (6), 29, 551, 1893, etc.

however, a maximum limit of 1300 atmospheres; they differ from all three in the sense that with very few exceptions the temperatures observed were also high, having a range from 200° C. to 1000° C.

The dilatometer method used was such that while the whole dilatometer and contents were subjected to the high pressure, only that part of it containing the liquid under investigation was subjected to the high temperature. How this was done will be clear from the following. I shall first describe the compression-cylinder with its connections. I shall then discuss the various forms of dilatometer tried, and describe the general procedure with the form ultimately adopted. The last part of the paper will contain an account of the observations taken and the results obtained.

Before the general description is begun, however, it should here be mentioned that most of the experiments were not made with pure water, but with a solution, slightly acid, formed by diluting 80 parts of a normal HCl solution with approximately 1000 parts of pure distilled water. The main reason for this is that pure water attacks quartz-glass strongly, at high temperatures two or three experiments at most being possible with the same dilatometer bulb if pure water be employed. The HCl solution, while differing very little from unity in density at room temperature, does not attack the quartz-glass so strongly. To justify the procedure several observations were made with pure water, and the results easily fell within the limits of error as determined by the other observations. They are included in the table on page 470 and are marked with an asterisk. When water is spoken of in the following, it is invariably this weak solution of HCl that is meant.

The Compression-Cylinder.

As mentioned on page 456, I had at my disposal the compression-cylinder devised for high-temperature work by Professor Des Coudres. By means of it one is enabled to subject a body to high pressures and high temperatures simultaneously. The general arrangement will be best explained by means of figures taken from the above-mentioned paper. B is a steel cylinder 60 cm. long and 18 cm. diameter. It has an inner bore of 6 cm. width. It is closed above and below by plugs K_1 and K_2, which are of the bayonet-joint pattern and rendered pressure-tight by vulcanite rings *d,d.* The steel tube *z* connects the compression-cylinder with a pump, compression-screw, and gauge as supplied by Schaeffer and Budenberg. During the course of the experiments the plug K_1 does not require to be removed. It carries the cooling spiral S, whose ends *rr* are hard soldered into it, and in which the water from the city mains is allowed to circulate. Through K_1 also passes the insulated copper lead E_1 which forms one terminal of the heating circuit. Leak is prevented by the steel cone arrangement C, which is surrounded by an ivory collar. From C the current passes through copper rods to the iron vessel *n* containing mercury, into which the lower end of the oven O dips. The upper end of the oven O is rigidly attached to the plug K_2, through the centre of which, similarly, a thick copper rod passes and makes connection

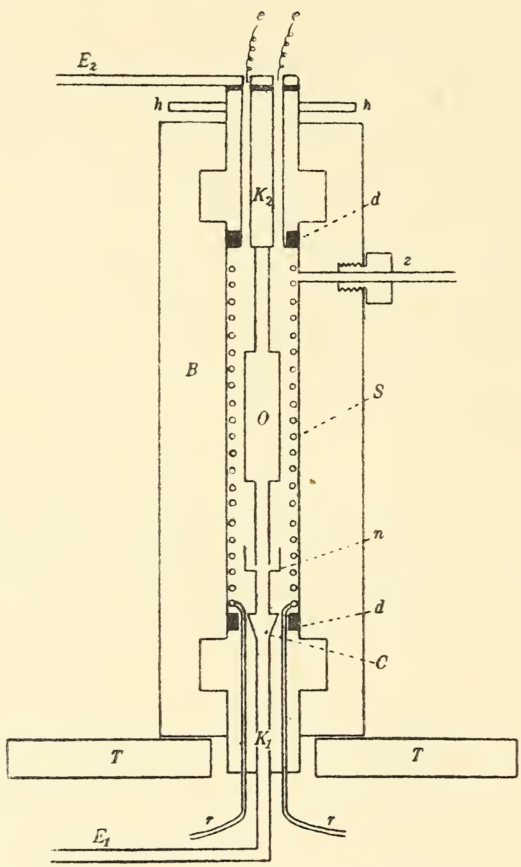


Fig. 1.—High-temperature compression-cylinder.

with E_2. E_1 and E_2 are connected to the terminals of an alternating current apparatus with a transformer which gives 1200 amperes. In addition, after the pattern given by Amagat, four insulated copper wires *e* of 2 mm. diameter (two of which are represented in the figure) are led through the plug K_2. A vertical section of the oven itself is shown on a larger scale in fig. 2. Copper rings F_1 and F_2, to which the current is led by means of the copper pieces A and B, are connected by the platinum tube H, 10 cm. long and about 15 mm. in diameter. Its resistance was approximately

0·85 ohm. The plug S, which fitted tightly into the upper end, was made of nickel, but was coated with platinum on its under surface. The concentric protecting cylinders N are soldered to the upper copper plate. They are insulated from the lower plate by a vulcanite ring G. The cylindrical spaces thus separated off from one another by these nickel tubes have a width of 2 mm. These narrow chambers prevent the water which fills them from circulating too strongly, though the pressure of the water

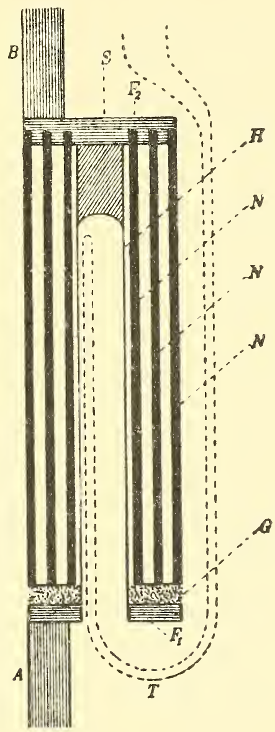


Fig. 2.

is the same as in the rest of the compression-cylinder. This water acts as a heat-insulator. The temperature in the heating chamber is measured by a Pt.—Pt. Rd. thermoelement, which is enclosed in a thin-walled quartz-glass capillary tube and has the form shown by the dotted lines T in fig. 2.

The Dilatometer.

As regards the material of the dilatometer several conditions have to be satisfied in my experiments. The material must have a high fusing-point and otherwise stand temperatures up to 1000° C. Ordinary glass cannot therefore be used; but both platinum and quartz-glass are suitable in this respect, the devitrification and loss of strength in the case of the latter being very small below 1200° C. In quantitative volume experiments it is practically essential that the dilatometer be made of some transparent material, and, if not essential, it is desirable that the coefficient of expansion of the material of the dilatometer should be small. Platinum is not transparent, and its coefficient of expansion is seventeen times that of quartz-glass. The latter is therefore more suitable, and the dilatometers used were all made from quartz-glass, obtained in tubes of various sizes from Sieverts and Kühn, Cassel, and made to the desired shape by a local glass-blower. Two properties of quartz-glass which caused difficulties in the course of my work were: (1) it was found impracticable to fuse metal electrodes into the walls of a quartz-glass tube, and (2), as Berthelot and others have found, quartz-glass is at high temperatures not impervious to gases. How these difficulties affected the work will appear later.

As regards the form and dimensions of the dilatometers, both were limited to a large extent by the space available in the compression-cylinder, and details of both appear in the course of the following discussion.

In experiments involving the measurement of volume changes under pressure an electrical method of detecting these changes has often been found the most suitable. To take two examples: Tait, in his experiments on compressibility, and later Richards, both used such a method. Tait's method, which was successfully employed by Amagat, can best be exemplified by a figure.

The piezometer bulb is filled to a definite point C * with the liquid under investigation. The bend CD is filled with mercury. The part BC has a number of platinum electrodes fused into its walls at given distances apart. These are all connected with one another on the outside by wires wound round BC, and the highest and lowest are connected by wires with a circuit containing a battery and a galvanometer. When the liquid in AB is compressed, the mercury rises in CB, and, as it passes each contact, there is a change in the deflection of the galvanometer.

Richards' method consists in having only two platinum contacts at A and at B,† which are connected direct with the galvanometer and battery circuit. Mercury again fills the part CBA. The liquid in CD is compressed until the mercury level, originally above A, falls below A, causing a break in the galvanometer circuit. A known weight of mercury is added at A and the experiment repeated.

In my experiments a similar volume change had to be measured, so I

* Fig. 3*b*. † Fig. 3*a*.

endeavoured to apply a similar electrical method. A great advantage of Tait's method is that for a single immersion of the piezometer in the compressor we can get a series of readings for various pressures; while with Richards' only one reading is obtained for a given pressure. Owing to the evident advantage to be obtained from a series of readings, I first tried to apply Tait's method. A dilatometer was made of the form shown in the figure.* The part AE is made of quartz-glass, while the tube EF

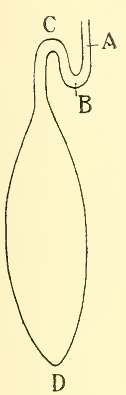


Fig. 3*a*.

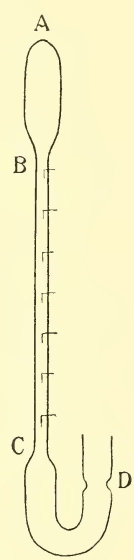


Fig. 3*b*.

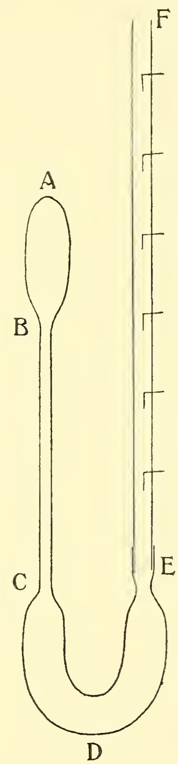


Fig. 3*c*.

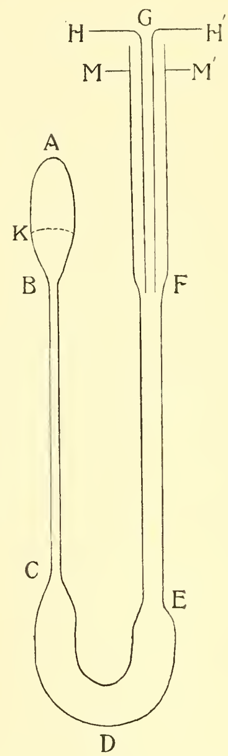


Fig. 3*d*.

with the contacts had to be of ordinary glass, since the high fusing-point and low coefficient of expansion of quartz-glass make it unsuitable for the purpose of having metal electrodes fused into it. This means that we have a junction of two different materials at E. The junction was made by wrapping asbestos paper round the lower end of the glass tube till it fitted as tightly as possible, and then coating the whole joint with sealing-wax. This is quite an effective junction at ordinary atmospheric pressure, but it frequently ceased to be mercury-tight at high pressures, fully 50 per cent. of the observations being spoiled in this way. Owing to the

* Fig. 3*c*.

junction, too, the determination of the volumes is rendered much more difficult than with a form in which no such junction occurs. To avoid the junction, and at the same time have a series of observations for one immersion of the dilatometer, two other forms were tried. In both forms the dilatometer was made entirely of quartz-glass, and had the open limb comparatively long and wide. In one of them, contacts connected as before were fused into a thin glass rod, which was inserted in the long limb of the dilatometer, and the mercury, as it rose between the rod and the walls of the dilatometer, gave galvanometer deflections. This proved unsatisfactory, owing mainly to capillary effects. In the other case, a fine wire was stretched in the long limb of the dilatometer, and the change in its resistance with the rise of the mercury was measured. Here, however, the temperature variations made the resistance measurements practically useless for this purpose.

Finally, then, I had to have recourse to a form which had early suggested itself, but which did not lend itself to more than single measurements for a given pressure. The figure * shows the form with which the majority of the observations were made.

The dimensions were approximately as follows:—

Part.	Length.	External Diameter.	Internal Diameter.
	cm.	cm.	cm.
AB	2·6	0·8	>0·50
BC	5·5	0·4	<0·10
CD	1·9	0·6	0·34
AD	10·0	...	...
DF	9·0	0·5	0·25
FG	9·5	0·6	0·34

BC is made with small diameter, to prevent too great conduction of heat through the mercury. H and H′ are two stout platinum wires insulated from one another in FG by quartz-glass tubing except at F, at their lowest points, where GF becomes narrower, and connected with a battery and galvanometer circuit. They are held in position by being bound with thread to the excrescences M and M′. At the commencement of an experiment we have water from A to K and mercury from K to E, the volume EF being taken approximately equal to the volume KB. When the volume AK of the water has increased to AB, the level of the mercury is at F, and this is shown by a deflection on the galvanometer. The method therefore bears a certain resemblance to that used by Richards for compressibility.

* Fig. 3*d*.

General Procedure.

The following account refers more particularly to the procedure in the case of a dilatometer of the form last described; but the variations necessary in the other cases will be clear.

The dilatometer is carefully cleaned and weighed empty. Let its weight be W_1.

By means of exhaustion under the air-pump, or by gently heating, it is filled with mercury approximately to the point F. To bring it to a uniform temperature, it is immersed in a beaker of water at 15° C. and allowed to stand for some minutes. After removal, the wires HH′ are introduced and connected with the galvanometer circuit, the dilatometer being clamped so as to stand vertically. Mercury is added or removed so that contact is just made at F. The dilatometer and contents are then weighed. Let this weight be W_2.

By means of a long thin glass tube some water is introduced round to C, and, by gently tapping with the finger, it is made to rise into the bulb AB. A gentle heat was sometimes also necessary. After bringing the whole once more to a uniform temperature, we remove mercury from GF until contact is again just made at F. The dilatometer and contents are again weighed. Let this weight be W_3.

Finally, an amount of mercury, depending on the volume of the liquid in the bulb and on the temperature to be worked at, is removed from EF and weighed. Let its weight be W_4.

Assuming the density of mercury at the said temperature to be 13·56, and the density of the liquid to be 1·00, and neglecting corrections which will be discussed later, we have: (1) Original volume of liquid at 15° C. is equal to

$$\frac{W_2 - W_3}{12{\cdot}56}.$$

(2) Change in volume necessary before contact is again made at F is given by

$$\frac{W_4}{13{\cdot}56}.$$

After the removal of the mercury (W_4), the remaining space EG is filled with pure distilled water to prevent impurities coming in from the compression-cylinder in the course of the experiment and spoiling the contacts. The wire-system HH′ is then tied by means of thread to MM′ in the same definite position as it was when used to determine the volumes above.

The plug K_2 with the oven O is removed from the compression-cylinder, and the dilatometer is introduced in such a manner that the upper end of the bulb AB lies about 5 mm. from the lower end of the plug S (fig. 2), the whole dilatometer thus being roughly in the same position as shown for the thermoelement T in the figure. The dilatometer is held in a fixed vertical position by means of a horizontal copper platform screwed into A (fig. 2), and also by a number of wires which bind it tightly to the outside of N. The quartz-glass-covered thermoelement is similarly introduced and fastened to N. The thermoelement terminals and the wires HH′ are connected to binding screws which connect with the lower ends of the conductors *e*. The plug is then inserted in the compression-cylinder and all the outer connections made. By a few strokes of the hand-pump the pressure is raised to approximately 300 atmospheres. The heating current is then gradually switched in. The heating gradually raises the pressure inside the cylinder; and after a little practice it is possible to arrange the initial pressure such that the final pressure desired is just attained at the desired temperature, or at least attainable within the range of action of the regulating screw. By means of this regulating screw the pressure could be kept constant to within 10 atmospheres. At this stage two experimenters were usually required—one to regulate the pressure and temperature, and the other to read the temperature and mark the galvanometer deflections. The temperature at which contact was made, and the temperature at which contact was broken, were read on an average three times each. The normal duration of such a series for one value of the pressure was about 10 minutes, as it was not advisable to continue heating longer, lest damage should be done to the insulations. The cylinder was accordingly left to cool for about half an hour, the pressure being meanwhile kept high. After that time the pressure was adjusted at a different value and the observations repeated. At the end the plug is again removed and the dilatometer taken out. If everything is in order, the mercury in the bulb is collected and weighed, as a check on the original volume. The dilatometer is then ready to be refilled. Each separate dilatometer bulb could be used only a few times, owing both to the material being attacked at the high temperature and to the fact that some gas—probably hydrogen—seems gradually to occlude itself in the walls of the bulb. After a few experiments, even at comparatively low pressures and temperatures, this gas forces its way into the interior of the bulb and reduces the value of the observations. Whenever it began to be troublesome, the old bulb was cut off and a new one fused on without interfering with the other parts of the dilatometer.

To give an idea of the various quantities involved, I will quote the details of a single series.

Before the experiment the following weights were taken:—

Wt. of empty dilatometer	=10·018 g. . .	(1)
Wt. of mercury entirely filling bulb AB .	= 6·856 g. . .	(2)
Wt. of dilatometer with mercury A to F .	=32·612 g. . .	(3)
Wt. of dilatometer with liquid A to K, and mercury K to F	=28·994 g. . .	(4)
Wt. of mercury removed from EF . .	= 2·785 g. . .	(5)

Calculations:—

Neglecting meanwhile all corrections, $\frac{(3)-(4)}{12{\cdot}56}$ gives original volume;

∴ original volume is equal to ·288 c.c. . . . . (a)

Again $\frac{(5)}{13{\cdot}56}$ gives change of volume;

∴ change of volume is equal to ·205 c.c. . . . . (b)

Finally, (4) − (5) gives weight of dilatometer with contents before experiment. This weight is equal to 26·209 g. . . . . . . (c)

While the dilatometer was in the compression-cylinder the observations given in the following Table were made:—

Observed Pressure in kgs./cm.^{2}=	1000.		700.		400.	
Observed Temperature °C. when contact is—	Made.	Broken.	Made.	Broken.	Made.	Broken.
	420	410	375	368	338	325
	413	410	373	365	328	325
	413	410	373	360	328	323
Calculated Average Temperature in °C.	413		369		328	

After the experiment the following weights were taken:—

Wt. of dilatometer and contents after experiment	=26·198 g. .	(6)
Wt. of mercury in bulb K to B	= 2·922 g. .	(7)
Wt. of empty dilatometer	= 9·995 g. .	(8)

From these and the previous weights we calculate again the original volume from $\frac{(2)-(7)}{13{\cdot}56}$;

∴ original volume is equal to ·290 c.c. . . . . (d)

Remarks:—

(1) (*a*) and (*d*) give the value of the original volume determined in two ways.

(2) Comparing (1) with (8) we find a decrease of ·023 g. in the weight of the dilatometer. The average decrease in weight after four experiments was ·125 g.

(3) Weight (6) is a rough check that no mercury has escaped in the course of the manipulation.

(4) A glance at the table of temperatures shows a gradual decrease in the temperature with time. This can be explained on two grounds. Firstly, in all the observations but the first, the observer has the advantage of knowing the approximate position of contact and can regulate the change of temperature accordingly. In the first case, the temperature is invariably rising more or less rapidly when contact is made. Secondly, the gradual conduction of heat through the mercury from KB to EF will evidently tend to lower the temperature of contact.

In the table of temperatures given above, the average of the six observations for a definite pressure is taken as the temperature of observation. This is justifiable on the grounds that it counterbalances the effects of the two factors just mentioned, and, besides, gives fairly consistent results. That the latter is true can be seen from the following four experiments, in which the dilatometer after one experiment was allowed to stand under pressure till the following day, when the observations were repeated.

Experiment	1.	2.	3.	4.
Average temperature for 1st day ...	383	508	476	583
„ „ „ 2nd „ ...	380	516	478	586

We come now to the question of treatment of observations obtained. The first question which arises is—What temperature are we to ascribe to the liquid in the bulb when the thermoelement shows a given temperature? To aid in estimating this temperature, I made a series of observations with two similar thermoelements placed at varying heights in the heating oven. The results are given in the subjoined Table, the distances being measured from the lower surface of the plug S.

Distances from top in cm. =	1·0 Normal Position.	2·2	4·0	5·0	6·5	Near C. of Dilatometer.	Near F. of Dilatometer.
Temperature in degrees C.	150	150	90	...	...	...	...
	210	210	140	...	...	...	...
	300	300	220	230	240	...	...
	350	350	270	...	...	...	...
	400	400	320	...	300	...	...
	450	450	380	...	...	...	...
	500	500	420	...	325	...	15
	550	550	470	...	...	...	...
	600	600	520	450	...	200	...
	700	...	570	...	...	...	...
	800	...	660	...	350	...	30
	900	...	740	...	...	...	...
	1000	...	860	750	450	...	15
	1040	920	...	...	...	...	...
	1100	...	990	...	...	100	...
	1150	...	...	...	400	...	...
	1200	...	1100	...	...	...	...

In this Table, the first column is to be compared directly with each of the remaining columns. It must be remembered that only two thermoelements could be inserted at once, and, as the important question was a comparison of the other temperatures with the temperature in the normal position, one thermoelement was kept fixed there and the other varied. In the calculation we are concerned with the temperature of the contents of the bulb AB when the temperature given by the thermoelement in the normal position was T_1 (say). Now the normal position of the bulb was such that the lower end B was always about 4 cm. from top of oven. The bulb was slightly wider towards end B. I proceeded, therefore, as follows: I ascribed to the upper two-thirds of the bulb the temperature T_1, and to the remaining third the temperature T_2 (say), 4 cm. from the top.

Let T_3 be the average temperature of whole bulb; then $T_3 = \frac{2}{3}T_1 + \frac{1}{3}T_2$, where, from table of temperatures,

$$T_2 \text{ may be taken as } T_1 - 80\text{; from } T_1 = 0^\circ \text{ to } T_1 = 600^\circ,$$
$$\text{and } T_2 \text{ may be taken as } T_1 - 150\text{; from } T_1 = 600^\circ \text{ to } T_1 = 1000^\circ;$$
$$\textit{i.e. } T_3 = T_1 - 25 \text{ (approx.) between } 0^\circ \text{ and } 600^\circ,$$
$$\text{and } T_3 = T_1 - 50 \text{ (approx.) between } 600^\circ \text{ and } 1000^\circ.$$

This full correction had to be applied only in those cases in which the lower surface of the liquid was at B when the temperature T_1 was registered.

In several cases, however, the liquid level was below B, in other cases above B, for the temperature T_1.

The general procedure was as follows:—

(a) Cases in which the excess of the final liquid volume (registered temperature $=T_1$) over the volume AB was greater than ·025 c.c. (approximately one-third of the volume of the capillary part BC) were discarded.

(b) Cases in which the excess of the volume AB over the final liquid volume was greater than ·150 c.c. were taken as having the registered temperature T_1.

(c) For all other cases:—

Temperature Range =	0° – 600°.	600° – 1000°.
Excess of Vol. AB over Final Liquid Volume.	T_3.	T_3.
– ·025 to + ·050	$T_1 - 25$	$T_1 - 50$
+ ·050 to + ·100	$T_1 - 15$	$T_1 - 30$
+ ·100 to + ·150	$T_1 - 5$	$T_1 - 10$

Volume Corrections.

The volumes $\frac{W_2 - W_3}{12{\cdot}56}$ and $\frac{W_4}{13{\cdot}56}$, given on p. 463, have to be corrected for change of pressure.

Let α, β, γ be the mean compressibility coefficients of quartz-glass, mercury, and water respectively.

Also let ${}_tV_p$ denote water-volume at temperature t and pressure p.

and ${}_tV'_p$ denote mercury-volume at temperature t and pressure p.

The uncorrected initial volume of the water is $\frac{W_2 - W_3}{12{\cdot}56}$ or ${}_{15}V_1$.

The corrected initial volume of the water is ${}_{15}V_p$,

$$\text{where } {}_{15}V_p = {}_{15}V_1(1 - \gamma \,.\, p) = \frac{W_2 - W_3}{12{\cdot}56}(1 - \gamma \,.\, p) \qquad (1)$$

Similarly, the original mercury volume is ${}_{15}V'_1$.

This becomes, as a result of the compression of the mercury, ${}_{15}V'_p$ at pressure p,

$$\text{where } {}_{15}V'_p = {}_{15}V'_1(1 - \beta \,.\, p).$$

Finally, if v be the internal volume of dilatometer from A to F at atmospheric pressure, this becomes at p atmospheres $v\,(1 - \alpha \cdot p)$.

Hence the corrected change in volume is

$$\frac{W_4}{13{\cdot}56} + \frac{W_2 - W_3}{12{\cdot}56} \,.\, \gamma \,.\, p + {}_{15}V'_1 \,.\, \beta \,.\, p - v \,.\, \alpha \,.\, p \qquad (2)$$

In this formula sufficient approximations are given by taking

$$v = 2{\cdot}0 \text{ c.c.} \quad \alpha = 2{\cdot}10^{-6} \quad \gamma = 41{\cdot}10^{-6}.$$
$$_{15}V'_1 = 1{\cdot}5 \text{ c.c.} \quad \beta = 4{\cdot}10^{-6}$$

In our example given above—

$$\frac{W_4}{13{\cdot}56} = {\cdot}205 \text{ c.c.}$$

and

$$\frac{W_2 - W_3}{12{\cdot}56} = {\cdot}288 \text{ c.c.}$$

Substituting in (1) and (2), and calculating the corrections for various values of p, we get the following Table:—

Pressure in kgs./cm.^{2}	Uncorrected Initial Vol.	Corrected Initial Vol.	Uncorrected Change of Vol.	Corrected Change of Vol.	Corrected Final Vol.	Ratio. $\frac{\text{Corrected Final Vol.}}{\text{Corrected Initial Vol.}}$
400	·288	·282	·205	·211	·493	1·75
700	·288	·279	·205	·215	·494	1·77
1000	·288	·276	·205	·219	·495	1·80

Correcting also the temperatures to which these volumes correspond, we get—

Pressure.	$\frac{\text{Final Vol.}}{\text{Initial Vol.}}$	Uncorrected Temperature T.	Corrected Temperature (T − 25).
400	1·75	328°	303°
700	1·77	369°	344°
1000	1·80	413°	388°

As explained above, the correction is 25° in this case, because here—

Vol. of bulb = ·506 c.c. and final volume of liquid = ·494 c.c.

All the observations were treated as explained above, and the following table gives the values of V_T and $\frac{V_T - V_0}{T{\cdot}V_0}$ for various values of T, where V_T represents the volume at temperature T degrees centigrade, and V_0 is the initial volume, taken throughout as 100 c.c., *i.e.*—

$$V_T = 100 \times \frac{\text{initial volume} + \text{change in volume}}{\text{initial volume}}.$$

Pressure in kgs./cm.^{2} =	400.			700.			1000.			1300.		
	V_T.	T.	$\frac{V_T - V_0}{V_0 \cdot T}$.	V_T.	T.	$\frac{V_T - V_0}{V_0 \cdot T}$.	V_T.	T.	$\frac{V_T - V_0}{V_0 \cdot T}$.	V_T.	T.	$\frac{V_T - V_0}{V_0 \cdot T}$.
	135	245	·00143	137	266	·00139	139	295	·00132	146	305	·00151
	140	244	164	141	275	149	144	305	144	151	324	158
	159	286	206	161	325	188	163	358	176	165	382	170
	175	303	247	177	344	224	180	388	206	193	438	212
	186	325	264	188	365	241	189	410	217	194	441	213
	186	331	260	188	369	238	190	411	219	203	476	216
	196	312	308	198	362	271	*191	407	224	256	546	286
	203	321	321	205	380	276	200	425	235	257	568	276
	208	330	327	210	387	284	209	431	253	299	587	339
	209	333	328	210	392	281	214	433	264			
	228	351	365	231	395	332	235	445	304			
	232	345	383	235	410	329	252	475	320			
	246	357	409	248	394	376	253	466	328			
	246	337	433	249	419	356	253	487	314			
	246	344	425	255	415	373	258	489	323			
	259	352	452	262	430	377	263	476	342			
	286	361	516	291	435	439	265	497	332			
	*509	383	·01068	*514	550	753	294	503	386			
	*509	407	1005	*514	547	757	327	571	397			
	527	395	1081	583	607	796	369	573	469			
	575	423	1123	635	597	896	*394	610	584			
	*678	441	1311	*690	642	919	*519	703	596			
	784	472	1449	758	690	954	*519	723	580			
				794	715	971	562	798	579			
					731	949	572	794	594			
							591	765	642			
							611	889	575			
							643	757	717			
							735	1002	634			
							745	960	672			
							750	936	694			
							769	888	754			
							812	924	771			

* See p. 457.

The values of V_T and T given in the preceding Table were plotted on square paper, and smooth curves were drawn as shown in fig. 4. Amagat's values for the volume of water at high pressures give us the volumes for 200° C. The scale of the diagram is too small to allow the curves to be accurately drawn in below that temperature. It is seen that, with the exception of a few points above 700° on the 1000 isopiestic, all the points lie fairly well on the curves drawn. Over 80 per cent., in fact, of the points lie on the curves within an error limit of 2 units in V and of 12 units in T. The real possible error in the case of T is difficult to determine, but we should expect it to be greater than 10° at least, because the temperature could be observed at only one point, introducing a considerable error in spite of the correction, and, besides, the smallest division on the registering

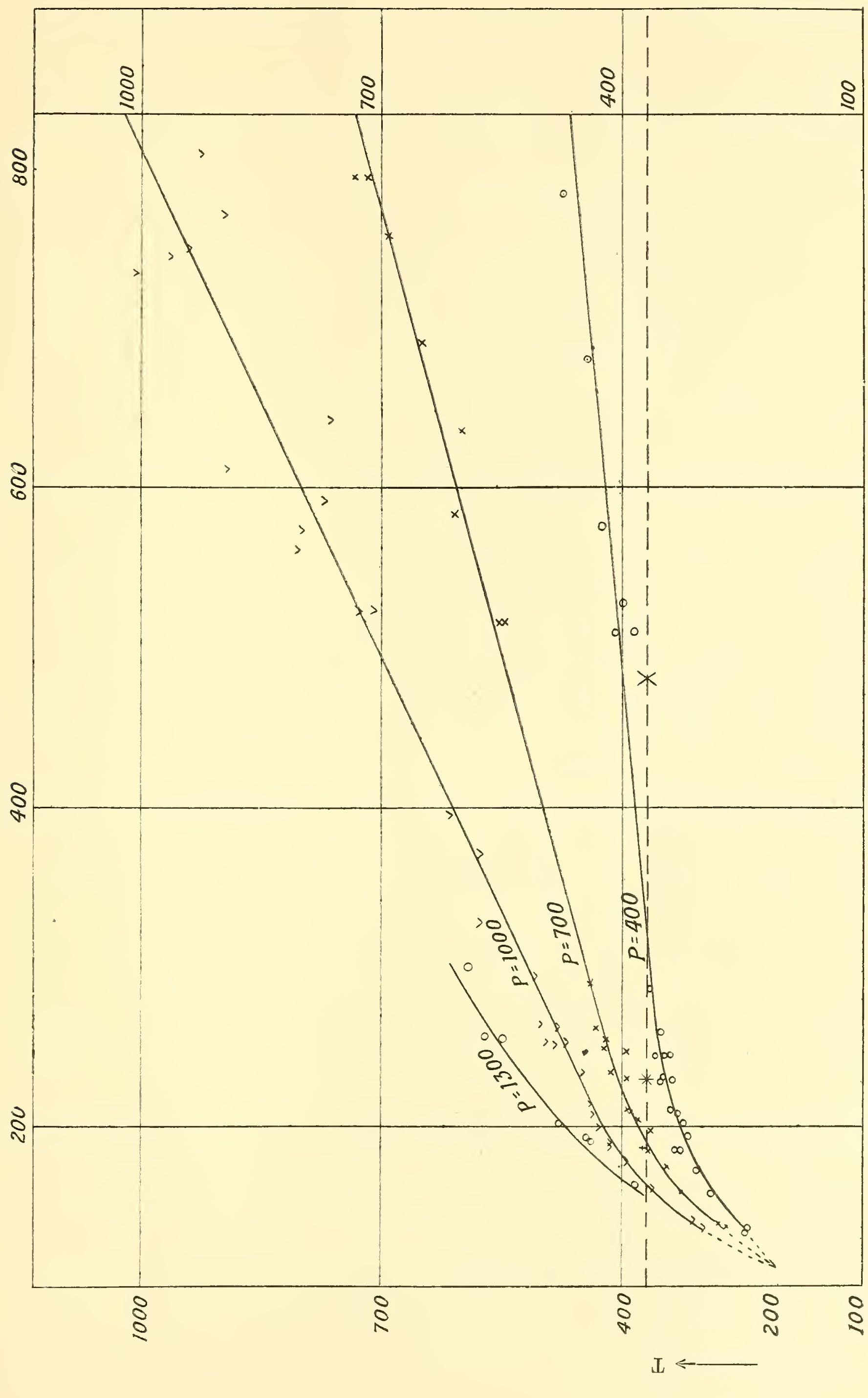


Fig. 4.—Isopiestic curves for water.

pyrometer scale was 10°, though quarters of a division could be estimated. The curves show that up to a certain point, which varies with the pressure, the volume increases more rapidly than the temperature; but that above that point the isopiestic becomes a straight line. The greater divergences above 700° on the 1000 isopiestic can be explained thus:—a small error in the determination of the initial volume has a relatively large effect on the value of V_T, and, at these temperatures also, a small change in the pressure causes large deviations in the temperature and vice versa. As regards the 1300 isopiestic, sufficient observations were not obtained to enable so complete curves to be drawn in. All the curves drawn show that for a given value of V the temperature is higher the greater the pressure. This means that above 200° water continues to behave as between 125° and 200°, it being remembered that for water the isopiestics have a crossing point between 100° and 200°, and that, in fact, below 125° for a given value of V, the temperature is lower the higher the pressure.

The third column for each pressure in the preceding table gives the value of the mean coefficient of expansion of the liquid between 0° and T° if we neglect the change in V_0 between 0° and room temperature. The change is approximately 4 in 1000, and therefore does not appreciably affect the results. It is seen that in general the mean coefficient of expansion increases with the temperature, and that the increase is relatively greater the lower the pressure. For a given temperature, also, the mean coefficient of expansion decreases with increasing pressure, the decrease being relatively greater the higher the temperature; as can also be seen directly from the curves in fig. 4. These conclusions agree entirely with Amagat's results for the coefficients of expansion of liquids at lower temperatures.

To obtain values for the true coefficient of expansion $\frac{1}{V_0}\frac{dV}{dT}$ we can proceed in two ways. We can either read off from the curves the tangents of the angles of inclination to the T-axis and find $\frac{dV}{dT}$ from these tangents by multiplying by a constant factor which depends on the scales used for V and T in the figure, or we can try to obtain equations for the curves, expressing V in terms of T and calculating $\frac{dV}{dT}$ from them. Considering the comparatively large error limit involved in these experiments, and the complex formulæ which previous observers have found necessary to express V_T in terms of V_0 and T even for very small ranges of temperature, I adopted the first method of finding the true coefficient of expansion. The curves show that $\frac{dV}{dT}$ constantly increases with increasing temperature to a maximum at the point where the isopiestic becomes a straight line. The limiting values of $\frac{1}{V_0}\frac{dV}{dT}$ are given in the following Table:—

Pressure.	Range of T.	Range of $\frac{1}{V_0}\frac{dV}{dT}$.
400	240–370	506×10^{-5}–430×10^{-4}
700	260–430	409×10^{-5}–188×10^{-4}
1000	290–450	319×10^{-5}–106×10^{-4}

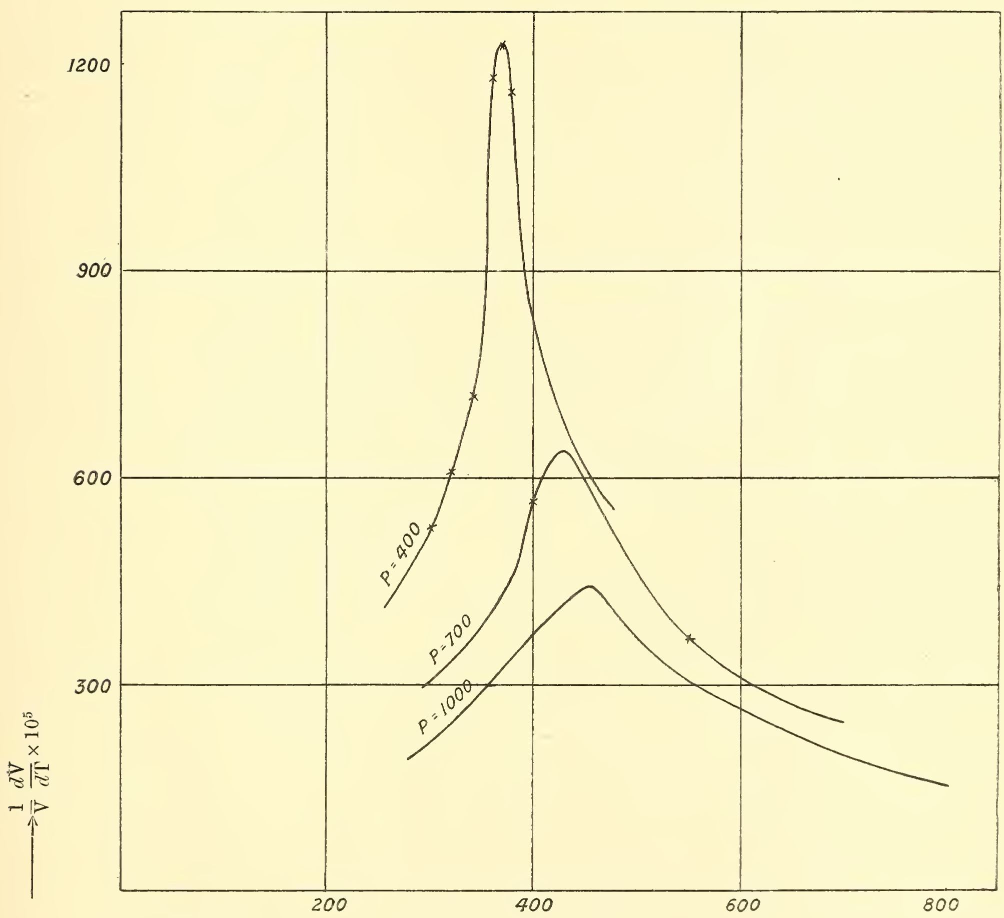


Fig. 5.—Expansibility curves.

Similarly, if we calculate $\frac{1}{V_0}\frac{V_{T_1}-V_T}{T_1-T}$ for various intervals, we get the following values, the figures for the range 100–200 being taken from Amagat's results:—

Pressure.	Mean Coefficient of Expansion $\times 10^5$ between							
	100–200.	200–300.	300–400.	400–500.	500–600.	600–700.	700–800.	800–900.
400	99	600	3280	4300	...	...	...	...
700	91	380	880	1760	1880	1880	...	...
1000	84	280	520	1000	1060	1060	1060	1060

The Table shows that, while for a pressure of 1000 the maximum value of a_1 is about thirteen times as great as its value between 100° and 200°, for a pressure of 400 the maximum value of a_1 is forty-three times its value for the same range, the maximum value thus rapidly increasing with decrease of pressure.

If we finally calculate $\frac{1}{V}\frac{dV}{dT}$, or the rate of change of volume with temperature per unit volume at that temperature, we find that this quantity constantly increases with temperature to a maximum which again corresponds to the point where the straight part of the isopiestic begins and thereafter decreases as shown by the curves in fig. 5. The maximum values of $\frac{1}{V}\frac{dV}{dT}$ with the corresponding values of V and T are approximately as follows:—

Pressure.	Maximum $\frac{1}{V}\frac{dV}{dT}$.	V.	T.
400	·0124	350	370
700	·0065	280	428
1000	·0046	235	454

The values of V and T here given were obtained from curves similar to fig. 4, but drawn on a very much larger scale.

Looking at the curves in fig. 4 now from another point of view, we see that, if a series of such isopiestics for still lower pressures could have been obtained, the critical constants might have been graphically determined. These critical constants, as tabulated in Winkelmann's *Handbuch der Physik*, are for water as follows:—

Critical Temperature.	Critical Pressure.	Critical Volume.	Observer.
358·1	...	2·3	Nadejdine.
364·3	194·61	4·8	Battelli.
365·0	200·5	...	Cailletet and Colardeau.
359·0	...	...	Knipp.
374·0	...	...	Traube and Teichner.

The temperatures are here expressed in degrees centigrade, the pressures in atmospheres, and the volume is that of 1 gramme of the substance in the critical state expressed in c.c. Approximately, then, these units are the same as we have been dealing with, except that our unit of volume is a hundred times greater. The values for the critical temperature and the critical pressure show very little deviation from one another as compared with the two values for the critical volume, one of which is more than twice as great as the other. Of the four isopiestics drawn in fig. 4 we see, therefore, that the value of the pressure for the lowest is well over the critical. Attempts to obtain values at much lower pressures than 400 were attended by the risk of serious damage to the apparatus through the possibility of the mercury being suddenly expelled from the dilatometer and coming in contact with the metal work, such as the cooling spiral. Two such attempts, one to gradually lower the pressure when the temperatures were about the critical temperature, and another to gradually raise the temperature while the pressure was below the critical pressure, gave so unsatisfactory results that I decided to devote the time at my disposal to the investigation of the higher isopiestics. Besides, at these lower temperatures and pressures the method employed is not sufficiently accurate, as may be seen from the following results. The same change of volume * was obtained for the quoted values of T and P:—

$P = 50$	100	250	500	750	1000 kgs./cm.^{2}
$T_1 = 225$	232	242	260	280	290° C.
$T_2 = 150$	145	155	172	185	196° C.

* In the Table, T_1 gives the temperatures at which the liquid has a definite volume V_1 (say), and T_2 gives the corresponding temperatures at which the volume is V_2 (say), where V_2 is different from V_1.

e.g. A change in pressure of 150 gives a change of temperature which is of the same order as the possible error in the measurement of the temperature. The results given in above Table were obtained with a dilatometer of the shape in fig. 3*c*, and I quote here a few more results got by the same method:—

Pressure =	300.	400.	500.	600.	700.	750.	800.	1000.
Corresponding Temperatures.	320	...	352	...	...	...	...	430
	318	335	358	378	...	...	...	...
	...	...	355	...	...	...	...	430
	...	...	348	...	...	...	395	426
	...	...	426	...	...	...	549	658
	...	...	505	...	...	710	...	905
	...	...	335	340	357	...	360	375
	...	...	385	410	445	...	460	500

These corresponding temperatures mean that the water had the same volume for the various pressures at the temperatures given in the horizontal rows. The values of the volumes for the several rows could not be accurately determined; but by taking the values for pressure 1000 we could get the corresponding points on the curves in fig. 4, and might thus draw in with fair accuracy the isopiestic for 500 at any rate up to about 400°.

Referring back to the table of critical values on page 475 I should like to add the following remark. The method employed by Nadejdine in his determinations, using as he did a differential densimeter, was such that the critical temperature was observed directly, while Battelli's values were indirectly obtained in the sense that his method was a graphical one. That the latter method, however, is a reliable one has been shown by Amagat's results for the isotherms of CO_2 and also by Ramsay and Young's results for other substances. If in fig. 4 we draw in the isotherm for 365° (shown by a broken line in the figure), we see that it cuts the 400 isopiestic at a point whose abscissa is 320. It must, therefore, cut the isopiestic for 200, or the critical isopiestic at a point for which V is greater than 320. It is clear, then, that Nadejdine's value for the crit. vol.—viz. 2·3, marked ✻ in figure—is inconsistent with our results; while the value 4·8, given by Battelli, marked ×, seems quite reasonable. The two graphical methods, therefore, agree in placing the critical volume nearer 4·8, than 2·3.

The results obtained in this paper, then, refer to water under pressures above the critical pressure. The range of temperature for all pressures investigated includes the critical temperature. It is clear from the figure that there is no sudden change in the isopiestics on crossing the critothermal. It is a common assumption, however, to consider a substance under pressures above the critical as gaseous or liquid according as it is above or below the critical temperature. This assumption is made in the following general statement.

The volume and coefficients of expansion of water under pressures above the critical pressure have been determined for temperatures reaching in some cases to 1000° C. The results show :—

1. The rate of change of volume with temperature increases continuously with the temperature to a maximum the value of which decreases with increase of pressure.

2. While in the liquid state water under high pressures and at temperatures above 240° obeys the ordinary laws of dilatation of liquids.

3. Almost immediately after entering into the gaseous state, water at high pressures behaves as a substance the coefficient of expansion of which is independent of the temperature and decreases with increase of pressure. It behaves, therefore, like such gases as oxygen, nitrogen, and hydrogen at the same pressures but at lower temperatures.

I think these results justify the statement that wonderfully consistent results can be obtained by using a compression cylinder of the type described, and I feel deeply indebted to Professor Des Coudres for the facilities and guidance given me in carrying out the experiments.

(*Issued separately May* 18, 1911.)

XXXI.—On some Nuclei of Cloudy Condensation. By John Aitken, F.R.S.

(MS. received March 21, 1911.)

Part II.

There was some difficulty in deciding what the title of this paper should be. It ought to have been "The Abnormal Dust Readings at Kingairloch"; but as that would convey no suggestion of its contents, it was thought better to make the paper a continuation of a previous one under the above title,* as the Kingairloch abnormal readings are a particular case of the same subject.

Those who were interested in the Kingairloch observations may remember that during the series of tests of atmospheric dust at that place the number of particles per c.c. rose occasionally very high. When the wind was north-west and the sky clouded, the number of particles was generally very low and the air very clear; but if the north-west wind was accompanied by sunshine, then the numbers became very high. In the former case, the number was generally about 200 to 300 per c.c. and sometimes under 100; in the latter, the number increased to thousands per c.c. and sometimes was over 10,000 per c.c. The interesting point connected with the observations made under these conditions was that though the number of particles became very high the air did not lose its transparency, but retained much the same appearance as when the number was only a few hundreds; if the wind, however, was from any other direction, the amount of haze was directly proportional to the number of particles present, and with 10,000 per c.c. there would have been produced a thick haze.

The abnormal dust readings at Kingairloch have long remained a mystery, though investigations were made at Kingairloch at the time. To the north-west of Kingairloch there are some sulphur springs, and the baneful influence of sulphur compounds in forming nuclei of cloudy condensation being known, these were suspected; but on investigation the number of particles was found to be the same to the windward as to the lee of the springs, which therefore secured a verdict of "Not guilty."

* *Trans. Roy. Soc. Edin.*, vol. xxxix. part i., No. 3.

The matter remained a mystery until last summer, when I was observing at Appin in connection with the work communicated to this Society on 18th July last. On the last day of my visit to Appin I was fortunate in stumbling on a clue to the cause of these abnormally high readings. There are a number of dwellings at Appin, and it was sometimes difficult to get clear of the products of their fires, as shown by the highness and the want of uniformity in the numbers of particles in each of the ten tests usually made to get a fair mean value. Generally it was possible to get out of these pollutions by moving cross-wind ways, but on this occasion it was not found possible; showing that the nuclei were coming from some other source. And further, from the direction of the wind, it was evident there were no houses near which could have polluted the air; nor were there any steamers within sight on the loch to account for the high numbers.

An investigation was at once begun as to the source of these nuclei. There did not seem to be any possible source of artificial pollution, and the question was, where and how was nature producing them? The answer to this question was likely to explain the high readings at Kingairloch, as they were produced under similar conditions, namely, bright sunshine and clear air.

In this investigation the dust-counter was used just as a dog uses its nose when quartering its ground in searching for game. The dog, after finding the scent, runs up the wind in the line of the scent towards the quarry; with the dust-counter we usually go cross wind to get out of the scent, but in this case it was used to run up the scent and point to the source of the sun-made nuclei.

The first thing to be done was to test the air cross-wind ways to try and strike the main stream of nuclei, but this gave no satisfactory result. Cross wind, the numbers varied but gave no clue, as they were everywhere high. On descending to the meadow near the shore, the numbers were found to be much higher, but a search over the meadow gave no result; so, selecting the place where the numbers were highest, I walked up wind to seek the source of the nuclei. This led me to the shore, where the numbers continued to be very high. Walking still up wind over the foreshore, I at last arrived at the water's edge. On testing the air at this point I was at last rewarded with something definite to work on. At the water's edge the numbers were much lower than back above high-water mark, showing that I had overrun the scent and had arrived at the purer air to the windward, and that therefore the source of the sun-made nuclei was something taking place on the foreshore behind me. The next two hours were spent in walking backward and forward over the foreshores, from high-water mark to the water's edge, at different parts of the beach, and always directly up or down wind; that is, testing the air as it arrived from off the water and the same stream of air after it had passed on to the shore. The result of the observations was that much lower numbers were observed in the air at the water's edge than after it had passed over the foreshore. At the water's edge the numbers varied from 15,000 to 20,000 per c.c., while back on shore they were from 50,000 to 150,000 per c.c. These observations show that the sun-born nuclei are produced by some changes taking place on the foreshore.

It will be remarked that the lowest numbers here are very high, far above anything previously observed in the air of that district, but a slight account of the history of the air arriving at the water's edge explains these high numbers. The wind was southerly at the time, and the air arriving at the point of observation, though coming off the water, had a short time previously travelled over the foreshore at Port Appin or Lismore to the south, and had been exposed to the same conditions as on the foreshore at Appin; its sun-formed nuclei had thus been greatly added to before it arrived at the place of observation.

After my return home I began an investigation of the physics and chemistry of these sun-formed nuclei. At first the explanation seemed simple enough, but on further study it was found to be very complicated. Though satisfactory results were sometimes obtained, they could not be repeated with certainty, showing a lack of knowledge of some of the conditions. As the discovery of the source of the sunshine nuclei had only been made on the last day of my visit, I determined to revisit Appin and see if further light could be obtained by a study of the subject on the spot. October 8th was selected for this visit, and I was extremely fortunate in getting a day in every way suited for the purpose. The morning was cloudy, and when I arrived at Appin about mid-day I found the wind was light and from W.N.W. The air was very clear, and while the tests were being made the tide was out, thus exposing a good deal of the foreshore to the action of the sun. When the test began the sky was still clouded, but showed signs of clearing; the number of particles observed on shore was low, being only 105 per c.c. Shortly afterwards there were bright gleams of sunshine, when the numbers rose at times to from 2100 to 4000 per c.c. On going down to the water's edge to test the air before it passed over the foreshore, I found that the numbers fell to 103 per c.c. On the sky again clouding, the numbers were 2240 on shore, while at the water's edge they fell to 70 per c.c. When work was again started at 2.10, the sky was only half clouded, and the conditions had considerably changed. On the shore there were now 21,000, while at the water's edge there were only 154 per c.c. On walking back above high-water mark the numbers varied from 1700 to 10,000 per c.c. Down at the water's edge the figure was 140. At 3 p.m. the readings on shore were from 20,000 to 8000, and at the water's edge they were now up to 427. Later, when the sky was cloudless, the numbers on shore varied from 10,000 to 40,000, and at the water's edge they had increased to 1120 per c.c.

Here we have a repetition of the July observations; only in this case we have the history of an increase in the numbers with increase in the sunshine. In Table I. are given the number of particles per c.c. observed at water's edge and on shore at the different hours, and the amount of cloud at

Table I.—Dust Observations made at Appin on 8th October 1910.

Hour.	Number of Particles in the Air when it left the Surface of the Water.	Number of Particles in the Air after it had passed over the Foreshore.	Amount of Cloud. 10 = totally covered.
p.m. 12.30	...	105	10
	...	2,100	9
	...	4,000	
	...	460	10
	103	...	10
	...	420	10
	...	2,240	9
	280	...	
	84	...	10
1.30	70	...	10
2.10	...	1,825	5
	...	21,000	
	154	...	
	...	1,700	
	...	10,000	2
	140	...	
	...	490	
	...	4,200	
3	...	20,000	1
	...	8,000	
	427	...	
	...	40,000	0
	1,120	...	
	...	40,000	
3.45	...	10,000	0

the time, 10 being wholly covered. At the beginning of the tests the sky was clouded all over, and both above high-water mark and at water's edge the numbers were low, about 100 per c.c. With the coming of gleams of sunshine came spasmodic increases in the numbers on shore, up to as high as 20,000 per c.c., while the numbers remained low at the water's edge. At the end of the observations, when the sky was cloudless, the numbers at the water's edge had risen to over 1000 per c.c., or ten times their original number. This increase was probably due to sun-formed nuclei brought from the shore on the other side of Loch Linnhe by the inshore wind. These high numbers at the water's edge would probably have come sooner, but the high hills across the loch kept that side clouded to a much later hour than the Appin side. It will be noticed that the numbers did not rise so high on this occasion as they did on the previous visit. The highest reading in the October visit was 40,000, while in July it was 150,000 per c.c., a result easily accounted for by the weaker sunshine of October compared with that of July, and by the air arriving pure at the foreshore on the former date.

It cannot be said that this second visit to Appin gave any suggestion as to how these nuclei were produced on the foreshore, while it added difficulties, since it showed that they were not steadily produced, as one might expect if they were formed by sunshine out of something which was being evaporated by the heat. The tests clearly indicated that their production, instead of being a steadily increasing and decreasing quantity, was a very spasmodic one. The shore tests, at times when the sun was shining occasionally, only gave low numbers, one or two hundred per c.c.; then these would suddenly increase and rise to thousands for a short time, and then again disappear, to return again in a little while, but the high figures never remained steady for long. This rapid variation in the numbers may be explained in two ways: either the production is spasmodic, or it is constant but very local and the variation in the numbers is to be explained by a slight shifting of the wind, causing the nuclei to be carried direct to the observer or made to pass to one side of him.

In the Kingairloch observations the abnormally high readings were generally observed in N.W. winds; and perhaps some explanation is necessary of why N.W. winds are so frequent at that station, while winds from W. and N. are rare. The cause of this is the geographical situation of Kingairloch, which is near the opening of a deep valley between high mountains. This valley has a trend to the N.W., so that all winds from N. to W. are deflected by the mountains and blow down the valley, arriving at Kingairloch as north-westerlys. An examination of the map of the country to the west and north of Kingairloch shows it to be full of sea lochs and islands; so that there is a large area of foreshores, over which the air passes before arriving at Kingairloch, on which the sun may act and produce nuclei in the same way as has been found at Appin.

The Physics and Chemistry of the Sun-formed Nuclei.

Turning now to the physics and chemistry of these sun-formed nuclei, one naturally wishes to know something about their origin—what is the material out of which they are made, and what is their nature. There are three substances which suggest themselves as the possible origin of these nuclei. These are, first, the salt formed in the drying sea-weed; second, iodine, which is supposed to be present in the air on the sea-shore; and third, hydrochloric acid, another substance generally admitted to be present. Experimental work was begun to see if any of these substances produced nuclei under the influence of sunshine.

The method of testing is shown in fig. 1. S is a flask in which the substance to be tested is placed and exposed to sunshine. This flask is connected by a tube with a stopcock on it with the flask T, in which is placed some water and its sides kept wet. T is connected with the air-pump P. For convenience we will call the flask T the test flask, as the air to be tested for nuclei is drawn into it and saturated with water vapour, and afterwards expanded by the pump P, and the density of the condensation noted. These experiments should be made in a darkened room, only as much light being admitted as is necessary for sunning the flask. A black background to the test flask also helps to make the observations accurate and easy. For illuminating the test flask a gas jet, and sometimes an incandescent burner L, has been used, the light being placed in the dark lantern D, which is provided with a vertical opening so arranged that it can be made of the most suitable width; and between this opening and the test flask T is placed a condenser C in the form of a globular glass flask filled with water. This condenser is a great convenience, as it absorbs the radiant heat and prevents the test flask from getting heated on the side next the light. This unequal heating often gives rise to what looks like spontaneous condensation, though it is only due to the hot moist air rising from the heated glass and mixing with the colder air. Further, by preventing the heating of the side of the test flask it is not necessary to turn it upside down so frequently in order to wet the sides.

At first it was proposed to use unfiltered air so as to interfere as little as possible with the conditions in nature. It was hoped that, by noting the difference in the density of the condensation with the sunning flask S empty and with the substance under test in it and sunned, definite conclusions might be obtained. This, however, was not found practicable, as the purity of the air was not constant, and it was impossible to separate the alterations in density from this cause from those due to any-

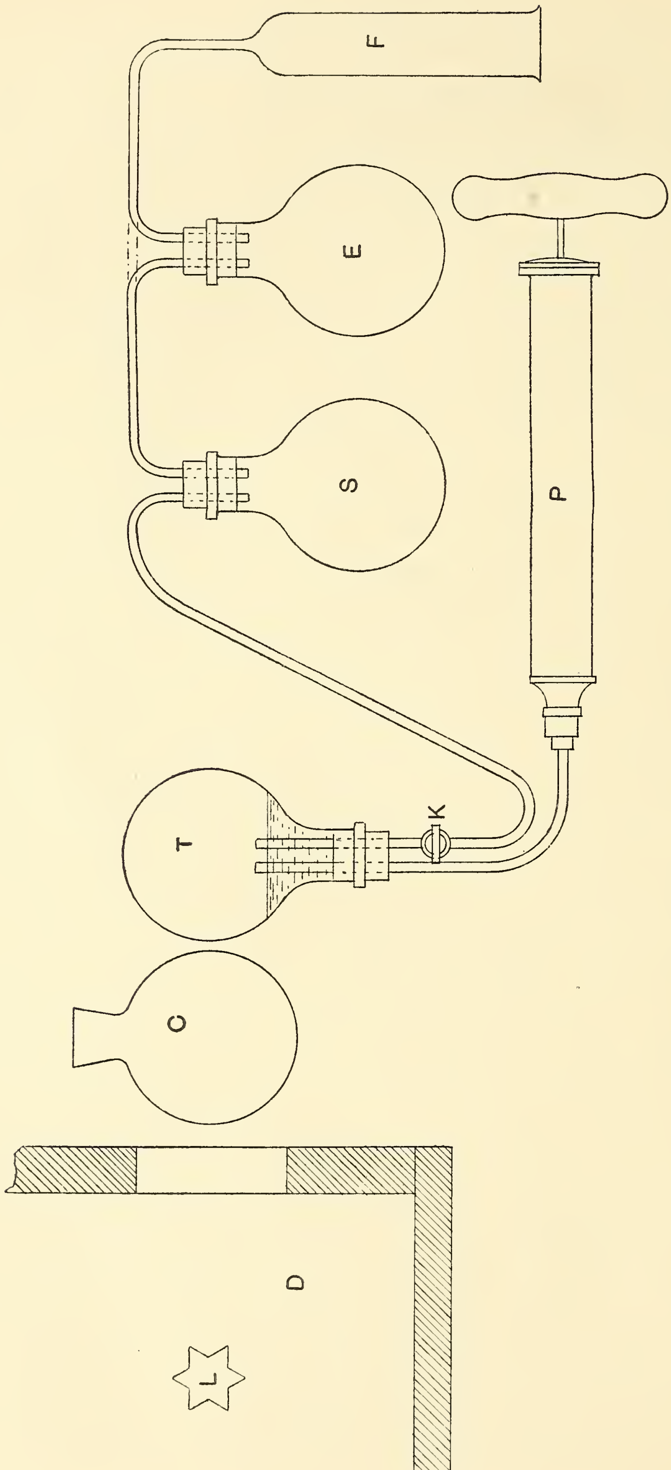


Fig. 1.

thing taking place in the sunning flask. The filter F was therefore added. As this filter stopped all dust, the air when it arrived at the test flask gave no condensation, so that any condensation which might now occur would be entirely due to what was taking place in the sunning flask.

Salt and iodine, separately and together, were the first substances submitted to the test. The apparatus was first fitted together, the flask E being omitted and the filter connected direct to S. The pump was now worked for a time to fill the apparatus with dustless air. The stopcock K was then closed and a stroke of the pump made to test the air in T. If any condensation took place, the pump was worked till it ceased. If all condensation ceased, then the joints would be all tight and the filter right. The flask S was then sunned to see that there was nothing due to anything taking place in the flask. If the air kept free from all condensation, then a little of the substance to be experimented with was put in S. The pump was again worked till all condensation ceased. The sunning flask with the substance in it was next sunned for one minute or more. A stroke of the pump was now made to bring some of the air from the sunning flask into the test flask, when the stopcock K was closed and the air in T expanded by means of the pump. While this is being done the air can be examined through a lens if necessary and the density of the condensation, if any, noted. The experiments with salt and iodine seemed at first quite satisfactory. It looked as if the explanation of these sunshine nuclei had been obtained when salt or iodine, or both, were in the sunning flask and it was exposed to sunshine; the air when passed into the test flask gave a dense, foggy condensation, while, if the sunning flask were kept in the dark, no condensation took place in the test flask. This result seemed highly satisfactory. It was soon found, however, that the formation of this fogging was far from a certainty. Sometimes it would appear a number of times and then cease, and sometimes it did not appear at all. One might imagine that the salt had given off all of the something which was the fog-producer, and soon ceased, and certainly on adding new salt the action started again; but this explanation will not hold for the iodine, which would continue to evaporate till it was all gone.

This occasional activity of the sodium chloride may be explained by the fact that this salt is decomposed into soda and hydrochloric acid by heat in the presence of water vapour. This supposition is supported by tests, as it was found that on heating the salt slightly it recovered its activity.

The great difficulty in investigations of this kind is the extremely minute quantities of matter which produce surprising results and make the work full of pitfalls for the hasty. As an example of this may be mentioned a part of this investigation. I wanted to introduce some electrical conditions into the sunning flask, and to avoid making another hole in the stopper a brass tube was substituted for one of the glass ones, the brass tube forming one connection and a wire the other. The results were most successful, and it looked as if the electrical conditions had solved the problem, as the salt and iodine were steadily fog-producing under the action of sunshine. But now the cross-examination of the apparatus began. First, the electrical connections were broken, but still the salt and iodine steadily responded to the sunshine. The brass tube was now taken out and a glass one put in. All action ceased. An examination of the part of the brass tube that had been inside the flask showed that the vapours had slightly discoloured it; there had evidently been some chemical action on the metal, and it seemed possible that the products of this action might have made the gases active. An answer to this question was obtained by retaining the glass tube and putting a piece of brass in the sunning flask, to see if it would make the gases active: but there was no condensation with it, nor with zinc or copper, showing that the activity with the brass tube was not due to any action on the metal. Attention was again directed to the brass tube. The end which entered the flask was now highly heated in a bunsen flame to cleanse it. The other end of the tube could not be highly heated, as there were some attachments put on with solder. The tube was then returned to the flask. On again sunning for one minute, the gases gave no condensation; but if some minutes' sunning were given, a dense fogging appeared in the test flask as before. The only explanation I could think of was that the upper part of the tube had not been purified by the heating, and that when plenty of time was given the gases diffused up to the impure part of the tube. To test this, the air in the flask was drawn up to the upper part of the tube and allowed to go back. When this was done, one minute of sunshine was ample time to give a dense condensation; thus showing that the first hopeful results obtained with this apparatus were due to some impurity in the inside of the brass tube, which made the salt and iodine active. This conclusion was confirmed in other ways.

Hydrochloric Acid and newly prepared Gases.

So far the investigation had thrown but little light on the occasional activity of salt and iodine. Attention was next directed to the other constituent of sea-shore air, namely, hydrochloric acid. Some weak acid was put in the sunning flask and its vapour sunned, but when proper care was taken there was no response; the air and acid vapour gave no condensation in the test-flask, nor did it do so when salt and iodine were added. The investigation of hydrochloric acid was, however, continued; and, though not directly bearing on our subject, the results seem to be of sufficient interest to be recorded. With this acid the well-known condensing powers of newly prepared gases were investigated. For this purpose the same apparatus was used as before, with the addition of two metal wires which passed through the stopper of the sunning flask; one of these wires terminated in a thick copper wire, the other in a strip of amalgamated zinc about 3 mm. broad. The copper wire and zinc plate dipped about 3 mm. into the weak solution of hydrochloric acid in the flask. If the wires outside the flask were not connected, so that no gases were generated, then as before there was no condensation after sunning; but if the wires were brought into contact for a few seconds so as to liberate a little hydrogen, and the gases sunned for one minute, then very dense fogging took place when tested. If the gases were kept in the dark there was no condensation on testing, only a few drops appearing such as might be expected from the bubbling of the gases at the pole of the battery; but if these same gases formed in the dark were afterwards sunned they gave dense fogging. It may be stated that the zinc and copper couple may be outside the sunning flask and the gases drawn in through the cotton-wool filter F. The density of the condensation so produced shows that the gases have passed freely through the filter.

These experiments seemed to indicate that the hydrogen escaping from the small battery had in a very high degree the clouding tendencies usually ascribed to newly prepared gases. It was therefore necessary to investigate further, as in other experiments I had found very little tendency in newly prepared gases to cause condensation. Great precautions were taken in the next experiments. New flasks, etc., were used and everything done to ensure purity. With the new apparatus the hydrochloric acid and newly prepared hydrogen no longer responded to sunshine, so experiments were made to find out what had caused them to be sensitive in the previous tests. As iodine had been in the flasks previously used, it was suspected. A minute trace of that substance added to the hydrochloric acid, or admitted to the flask as a vapour, at once made the gases active and caused dense fogging. Further, one of the old flasks in which iodine had been experimented with was first washed with alcohol and afterwards carefully with water, yet on using this flask in place of the new one it gave dense condensation with hydrochloric acid and hydrogen, showing that enough iodine had remained on the washed walls of the flask to make the gases respond to the action of sunshine. This shows that though the newly prepared hydrogen was not an active fog-producer, yet it exerted some influence on the impurity which made the gases sensitive to sunshine. Further, even with the aid of the impurity it had no condensing effect by itself, and it was inactive so long as it was kept in the dark.

A few other experiments were made with newly prepared gases to see if they deserved the character usually given them of being powerful fog-producers. The gases were prepared by means of a small battery, the gases being given off at platinum electrodes, dipped either in tap water or very weak acid; but neither hydrogen nor oxygen, nor both combined, showed any tendency to cause fogging. In all the tests only a few drops were seen falling in the test flask, such as one expects from the bubbling of the gases. Other methods were used for preparing the gases, but so far I have not found them either fog-producers or even nucleus-producers, so long as they are not prepared from strong acids nor dried in sulphuric acid—a common practice. Now, in passing gases through that acid they become contaminated with one of the worst fog-producers we have, and the stronger the acid the worse it is. As I showed long ago, if we wet a glass rod with sulphuric acid and heat it we get a stream of as fine a fog-producing vapour as any experimenter could desire; and if we bubble ordinary air through sulphuric acid into moist air it soon makes a fog. The density of this fog seems to depend on the acid used. Some samples give denser fogs than others. The strength of the acid also affects the density of the fog produced by it; but all the samples tested were fog-producers, and some gave very bad fogs even with filtered air, though the fogs are always less with filtered air than with unfiltered.

Another test of the condensing power of newly prepared gases was made by placing some iron wire and sulphuric acid in the flask S and drawing in filtered air. When the newly prepared gases entered the test flask they only gave a slight condensation, due to the bubbling gases; and this was stopped if the gases were passed through a glass-wool filter before they entered the test flask, and the gases showed no sign of spontaneous condensation, which constitutes a true fog-producer.

Oxidising Gases.

The experiments on salt and iodine showed that, though these are occasionally active nucleus-producers, their activity is not constant; and as this activity is always at the beginning of a test, it is not therefore likely to be due to the composition of the air entering the flask, but the composition of the air may have some effect and may make them active. The method of experimenting was therefore expanded and attempts were made to see if there was anything in nature which would keep these substances active. An answer to this question has been found to be extremely difficult, as the quantities of matter involved are extremely minute, of the ultra-balance order, as has already been found in these experiments. One naturally turns to the oxidising gases in the atmosphere to see if any of them will make the salt and iodine active with sunshine. In all pure air on the sea-shore there is generally some peroxide of hydrogen and ozone, and it seemed possible that their presence might bring about the change. Experiments were therefore made with them.

Peroxide of Hydrogen.

To test the effect of peroxide of hydrogen some of the ordinary solution was put in the flask E, which was introduced between the filter and the sunning flask S, so that the peroxide might be carried by the air into the sunning flask and there get mixed with the vapours or gases from the salt, iodine, or hydrochloric acid. Another method of introducing the peroxide was to saturate some cotton-wool with the solution and pass the air through it. No satisfactory results were obtained. If a little condensation did appear, as it sometimes did, without the peroxide, the peroxide did not ensure its always forming. In these tests only weak peroxide was used, as it has been shown in Part I. that strong peroxide is active after sunning. The strength used, however, was much greater than is likely to be met with in nature.

Ozone.

Ozone is another powerful oxidising agent in our atmosphere, but unfortunately our knowledge of it is somewhat vague. Its origin and even the tests for it are under constant revision, most of the early tests not being trustworthy, as the indications supposed to be due to ozone may be produced either by that or by some other gas, leaving the observer in doubt as to what is the agent which produces the results. In experimenting with ozone another difficulty presents itself. All methods of preparing that gas for experimental purposes give us not only ozone but other active gases, with properties in some ways similar to those of ozone; so that when experimenting with ozone we are left in doubt as to whether the results are produced by the ozone or by the other gases. If we prepare ozone by the electrical methods, we get not only ozone but nitrogen compounds; if we use turpentine, we get, along with the ozone, turpentine vapour; and so on. The most common way of preparing this gas is by the electrical methods, the point discharge and the Siemens ozone tube.

Point Discharge.

In a previous part of this investigation it was found that the newly prepared hydrogen escaping from the small battery made it an active nucleus-producer when it met with salt and iodine. It was therefore thought that an electric discharge through the gases might have a similar effect. The first thing to be done was to ascertain the effect of the point discharge on normal air, and to see if the gases and vapours under test

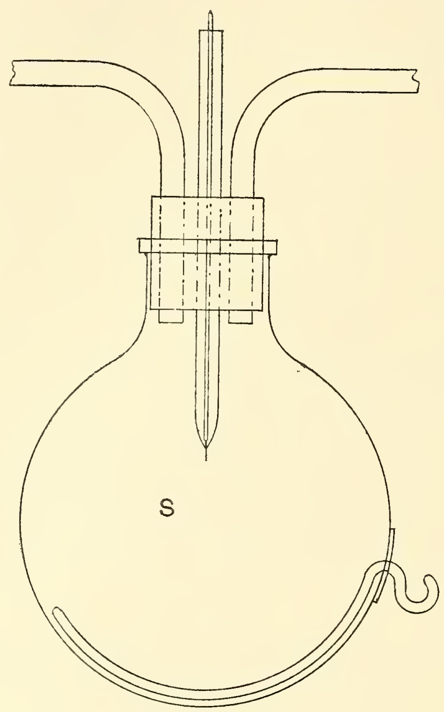


Fig. 2.

in any way altered it. The apparatus used for this purpose is shown in fig. 2. S is an ordinary glass flask about four inches in diameter. The flask is fitted with a stopper, through which pass two tubes for letting the air in from the filter and out to the test flask; when in use this flask takes the place of the flask S in fig. 1. Through the stopper also passes a glass tube inside of which is cemented a copper wire terminating inside the flask in a fine platinum point. Through the side of the flask passes, as shown, a metal wire for connecting the lower part of the flask with earth. An electrophorus or a small induction electrical machine was used for electrifying the point.

It is long since it was shown by many investigators that the point discharge in dusty air rapidly clears the dust out of the air and deposits it on the surrounding surfaces. But there is a particular I do not remember to have seen previously noticed, and that is, that though the electricity can clear away the dust it does not always leave the air nucleus-free; only under certain conditions will the electricity leave the air free from nuclei of cloudy condensation when it is supersaturated. It is only when the potential of the electricity is kept below a certain value that the air is made dustless and unable to give any condensation with slight supersaturation. If the potential rises above a very small value, then, though the dust be deposited, something is produced by the discharge—probably ozone and oxides of nitrogen—which when tested in moist air is found to give a dense fog with but little supersaturation. This product of the electric discharge seems to be in the gaseous state, because after it has been formed it cannot be removed by a discharge at a low potential, which it would be if the something were in a solid or liquid condition, such as might be produced by the disintegration of the discharging point. As the potential which gives this gaseous nucleus-producer is very low, all attempts at clearing out dust and fog on a large scale, where high potentials must be used, only result in getting rid of the solid and liquid particles and probably moisture-absorbing ones, and replacing them with gaseous matter, which forms nuclei whenever the air is cooled to the dew-point. Further, the point discharge not only acts on pure air, producing nuclei, but if there are any impurities, such as sulphurous acid, which is always present in town air, the density of the condensation is greatly increased; water vapour has also a similar effect, and it is always in abundance when fogs are being cleared with electricity. From this it would appear that air that has been cleared from fog by electric discharge is still in a condition to give a dense condensation when cooled to the dew-point.

Having ascertained the effect of the point discharge in ordinary air, namely, that only very low potentials can be used without producing nuclei, and that the potential must be lower for damp air; further, that it was necessary to cleanse the platinum point by keeping it in a flame for a time, as an unclean point gave dense condensation with low potentials;—a little salt or iodine, or both, was put in the flask and the point electrified. It was now found that a much lower potential caused the formation of nuclei, which gave a very dense form of fogging. Even a glass rod rubbed with silk gave sufficient electricity to cause a fog in the test flask, showing that only a small quantity of electricity is required, though the potential requires to be high. It is not necessary that the salt and iodine should be put into the electrifying flask, as the products of the electrifying flask can be drawn into the sunning flask with like results. If an almost invisible crystal of salt, or iodine, be put on the platinum point, a very dense condensation is produced with very slight electrification.

Ozone Tube.

Experiments were also made with ozone prepared by means of a Siemens ozone tube. A small coil and one-cell bichromate battery, or a small induction electrical machine with two ebonite plates, was used for exciting the tube. The results were similar to those obtained with the point discharge, which probably owes most of its action to the production of ozone. With the electrical machine sparks of 3 mm. could be given without producing any nuclei if the sunning flask was clean; but whenever salt or iodine was in the flask, dense condensation took place in the test flask if the gases were sunned, but none without sunning.

In making these experiments with ozone it was soon found that the cleanness of the sunning flask was of great importance. Indeed, it is not certain that the condensation which takes place with the higher potentials is not due to some impurity in the flask, as it has been found that the more care taken in cleaning this flask the higher the potential from the point discharge, and for working the ozone tube, that may be given without producing nuclei. In the experiments with the ozone tube it was found that damp air gave nuclei with a potential discharge that gave none when the air was dry. This result was also obtained in the point discharge experiments.

The product of the Siemens tube seems to vary with the potential and density of the charge, which is what one might expect, as its product is of a mixed kind and the proportion of ozone to the other products does not seem to be constant.

Tests were made to see if the products of the electric discharge in the Siemens tube were particles or gases. For this purpose the air from the tube was passed through a filter, then through a tube in which were placed some ozone test papers, the air being then led into the test flask. On exciting the ozone tube somewhat strongly and drawing the air through the apparatus into the test flask, it was noticed that no effect was at first produced on the test papers, nor did any condensation make its appearance in the test flask; but after two or three strokes of the pump the test papers got rapidly discoloured and dense condensation took place in the test flask. The test papers used were the iodized starch and the red litmus moistened with potassium iodide, which seems to be the best test paper for ozone, as it turns blue on the liberation of the alkali and is unaffected by acids. Different kinds of filters were tried in these experiments, but all of them passed the ozone. Asbestos seemed to pass it most freely, cotton-wool was next, and curiously glass-wool seemed to check more than the others; but all three held back the active gases for a time, probably absorbed by the substance of the filter or condensed on the surface of its fibres. But though the filters absorbed the active gases they gave them out again, as they required considerable washing with air before they were free of them. These three filters did not seem to act equally on the different constituents of the ozone tube. For instance, the two test papers seemed to be acted on in the same proportion and to the same amount when the asbestos filter was used as when no filter intervened; but with the cotton-wool filter, while the iodized starch paper was well darkened, the red litmus paper was not changed in the same proportion as it would have been if no filter had been used. A filter made of ground charcoal was found to hold back all the active gases of the ozone tube; test papers were not discoloured, and even all condensation ceased in the test flask. Wet cotton-wool acts like carbon, and destroys all the products of the ozone tube.

In these tests with ozone and other substances glass flasks were generally used, though some tests were made with a silica flask at an open window to get the maximum light effect. It was found, however, that glass flasks sunned through a window gave working results. Glass connecting tubes were generally used, the ends being brought close and joined by a short length of india-rubber tube. India-rubber stoppers were generally used in the flasks. Some tests were made to see if the rubber had any effect on the gases and vapours used in the experiments. Pieces of rubber were put into the sunning flask and subjected to the conditions of the experiment, but no effect was observed. Another test was made to see if the action of ozone on rubber had any tendency to produce nuclei. The ozone tube was first connected direct with the test flask by means of a new piece of glass tube, all stopcocks, etc., being removed, as they might influence the results; there was only one joint between the ozone tube and the test flask, and that was made by entering the one tube some distance inside the other and sealing the outside. The density was then noted of the condensation produced by a certain number of sparks of a fixed length given to excite the ozone tube. After a number of trials, and density of condensation noted, the joint between the ozone tube and the test flask was broken and the ends of the two tubes were connected by means of a rubber tube $1\frac{1}{2}$ metres long. Though the ozone had to travel through all that length of rubber tubing, no difference was noted in the density of the condensation in the test flask. It would thus appear that, though ozone is known to act on rubber, this action does not result in the production of nuclei; we may therefore conclude that the rubber used in these experiments has played no part in the effects under observation.

From the experiment on the effect of ozone it is evident that the different substances experimented on were made active by it and sunshine, or at least they were made active by something produced by the discharge of the electricity. But as the products of both methods of using the electricity are but little known, it is difficult to say whether the action was due to ozone or some nitrogen compounds or to the combination of them. We cannot therefore draw from the experiments any satisfactory conclusion as to what takes place in nature. It is probable that some of the ozone in nature has an electrical origin; it may therefore be presumed that some of the nitrogen compounds exist along with the ozone in the air. There is, accordingly, a presumption that ozone and the other gases may be the something required to make the salt and iodine on the foreshores active as nucleus-producers. Difficulties, however, meet us in this explanation. In nature the point discharge is very feeble under all but exceptional conditions, and is probably too feeble to produce any effect.

Other methods of producing ozone which did not give at the same time other active gases were now tried. There are two substances which are generally supposed to produce ozone under certain conditions. One is phosphorus while undergoing slow oxidation, the other turpentine while evaporating. The former of these was found to be unsuitable for tests of this kind, as it gave off products which keep up a constant dense condensation; but turpentine only gives off a vapour. Turpentine vapour was found to give the usual reactions with the two ozone test papers, so experiments were made with it. A small flask was introduced at E between the filter and the sunning flask S. Into this flask was put a piece of paper wetted with turpentine. After all the dusty air had been pumped out, the apparatus was sunned and the air tested and found to be free of nuclei, showing that the vapour of turpentine gave no nuclei. Salt, iodine, and hydrochloric acid were now tested with the ozone made in this way. Both salt and iodine were found to be dense fog-producers after being sunned, but gave no condensation if not sunned, while hydrochloric acid gave no condensation with it. It would thus appear that salt and iodine are active fog-producers with ozone produced by the oxidation of turpentine vapour, just as they are with the products of the ozone tube. Kingsett, however, says that this product of the oxidation of turpentine is not ozone, nor peroxide of hydrogen, because it is destroyed at a temperature of 160°, while these gases require a higher temperature for their destruction; another complication is thus added to the so-called ozone effects.

Hazing Effect of Sun-formed Nuclei.

Some experiments were made to ascertain what was the nature of these sunshine nuclei. The observations made in nature show that they have but little hazing effect. Air was observed at Kingairloch which was very clear yet had a great number of these nuclei. To explain this non-hazing effect of these nuclei three suppositions may be advanced. First, that they are not in the form of solid or liquid nuclei while in the atmosphere, but are in the gaseous state, and only form nuclei when mixed with air saturated with water vapour; second, that they may be extremely small, so small as not to affect the transparency of the air; or, third, that the number produced by the sun may not be sufficient to affect the transparency. It is true the number per c.c. was occasionally very high, but the depth of the stream of air charged with them can only be a few feet, and this stream when mixed with the higher air will not have much effect, and it will take long to get to the upper air in which the haze observations are made. That this stream of nuclei-laden air is very thin is probable from the fact that the nuclei are formed on the ground, and the wind cannot mix them up with the higher air till they have passed some distance over irregular surfaces. But that this mixing quickly takes place after the air has travelled over the rough ground on shore is shown by the observations taken at Appin on 10th July, the day referred to at the beginning of this communication, when the number of particles was from 50,000 to 150,000 per c.c. on the shore, while the ordinary records for that day already given* show that the number observed a few hundred yards back from the shore was only about 3000. The number was thus reduced to about one-thirtieth of what it was, showing that the nuclei-laden stream near the shore must have been thin to undergo so great a dilution in so short a flow.

Experiments were made to test the nature of some of these sun-made nuclei. The method of testing was to see if they would pass through a filter. If they did, then we might conclude they were gases. Unfortunately, the converse is not true, namely, that if they do not pass through they are not gases, because all filters stop gases more or less, and it is only if there is enough of the gas to saturate the filter that any gets through. Tried by this test, only slight indications of the passage of some of them were obtained, the greater part of the condensation being stopped. The product of the ozone tube passed freely through after a time, but in that case we were dealing with a large quantity, and the first part of it was even then entirely stopped by the filter.

* *Proc. Roy. Soc. Edin.*, vol. xxx., part vii., No. 39.

These experiments help us but little to an understanding of the conditions in nature, as we are not certain whether the nuclei formed by the sun on the foreshores are the product of any of the substances experimented with, and the study of the salt, iodine, and hydrochloric acid only suggests certain ways in which the sun may act as a nucleus-producer. The Kingairloch observations show that these sun-formed nuclei do not appear in the early morning though the sun be shining. It is generally about ten o'clock before they come in any numbers, but they remain till near sunset. This would seem to indicate that the foreshores require to be dried to a certain extent and heated by the sun, and this seems to support the supposition that the nuclei may be due to the decomposition of the salt on the heated seaweed over which is passing damp air.

Kingairloch Observations.

Before concluding, I should like to call attention to the bearing of the results given in this paper on the conclusions previously arrived at. In a paper communicated to this Society on 19th February 1894 there is given a summary of the dust observations previous to that date. It is there shown that the haze in the atmosphere depends on the number of particles in the air and on the relative humidity. The higher the humidity, the smaller the number of particles required to produce a certain thickness of haze. It is shown that for the same humidity the transparency is inversely proportional to the number of dust particles in the air at the time. So that if the number of particles per c.c. be doubled, the limit of visibility is halved. The result of this is that if we multiply the number of particles by the limit of visibility in miles we get a constant for the wet-bulb depression at the time. By means of this constant, called C in the paper referred to, we can compare the different observations at the same and at different wet-bulb depressions.

In the paper above referred to are given tables in which are shown for the different places of observation the number of particles and the limit of visibility in miles at the different wet-bulb depressions; also the value of the constant C for each observation, as well as the mean value of all the observations taken at the same wet-bulb depression. In Table II. is here reproduced Table XXII. from the above paper. In this table are entered the names of the places where the observations were made, and in the three columns for the different wet-bulb depressions at the time the observations were made are entered the mean values of the constants of all the observations. As these numbers are obtained by multiplying the limit of visibility in miles by the number of particles per c.c., it follows that if we multiply any of these numbers by 160932—that is, by the number of centimetres in a mile—we get the number of particles required to produce a complete haze of view in a column of air 1 centimetre square, whatever the length of the column may be; it may be 1 mile or 250 miles. A complete haze is one that completely hazes out of view a distant hill or other object.

TABLE II.

Place.	Value of C at different Wet-bulb Depressions.		
	2° to 4°.	4° to 7°.	7° and over.
Kingairloch, 1893 . .	77,525	105,923	140,628
„ 1892 . .	No observations	116,677	174,832
Alford	75,474	95,153	124,921
Rigi Kulm . . .	75,176	104,430	124,211
Mean . .	76,058	105,545	141,148

Now, the point to which I wish to direct attention is: Do these sun-formed nuclei, which were found in such numbers on the foreshore at Appin, affect the number which was found to be required at Kingairloch to form a complete haze? The abnormal numbers were observed at Kingairloch in N.W. winds—that is, in pure air, where their presence could be easily detected; but, one naturally asks, are not all the observations at that place vitiated by these sun-formed nuclei, because it is surrounded, more or less, on all sides by sea lochs and islands, and from the foreshores of these the sun-formed nuclei may be brought with winds from every direction?

Let us see how Table II. helps to answer this question, namely, Does it require more particles at Kingairloch to make a complete haze than at an inland place? In Table II. the values of C for Kingairloch are evidently higher than at the Rigi Kulm in Switzerland, or at Alford, situated about 25 miles from the coast in Aberdeenshire. The figures in Table II. are put in a more convenient form for our purpose in Table III. The two years' observations at Kingairloch are averaged, and also the observations at the two inland stations. Underneath these figures are given the differences; and it will be seen that the values for C at Kingairloch are higher for all conditions of humidity of the air than at the inland places, showing that these sun-formed nuclei have entered into all the observations. In making up the original tables for calculating the relation between the amount of haze and the number of particles at Kingairloch, all the observations made in N.W. winds while the sun was shining were rejected, as they contained an unknown quantity, since the amount of haze at the time bore no relation to the number of particles. The increase in the Kingairloch numbers thus represents only the sun-formed nuclei brought by other than N.W. winds.

It will be noticed that the Kingairloch number is only about 2 per cent. higher than the inland number when the air is fairly humid, and that the difference increases with the dryness, being 11 per cent. at wet-bulb depressions between 4° and 7°, and is 26 per cent. higher when the air is very dry,

Table III.

Place.	Values of C at the different Wet-bulb Depressions.		
	2° to 4°.	4° to 7°.	7° and over.
Kingairloch, 1892–93 . . .	77,525	111,300	157,730
Rigi Kulm and Alford . . .	75,325	99,791	124,566
Difference . .	2,200	11,509	33,164
Per cent. . . .	3	11	26

a result easily explained by the state of the weather under the different conditions. High humidities will generally be accompanied by much cloud, while with low humidities the sunshine will generally be plentiful and so produce more nuclei. It is evident from Table III. that the Kingairloch observations give too high a figure for the number of particles required to produce a complete haze, especially when the air is dry, being in error to as much as 26 per cent.

I feel that an apology is due for presenting this paper. The subject is an uninteresting one to most people, and I have to admit that its presentation is badly focussed and gives but a hazy impression. If it had not been for the discovery of the origin of the sun-formed nuclei at Kingairloch, the work would not have been done. Perhaps the difficulty of experimenting on a subject in which such extremely small quantities of matter produce such marked effects, and the varied nature of the contents of the paper, may be some excuse for these defects.

(*Issued separately July* 6, 1911.)

XXXII.—On the New Genus of Iron-bacteria, Spirophyllum ferrugineum (Ellis). (A reply to criticism.) By Dr David Ellis, Lecturer in Botany and Bacteriology, Glasgow, and West of Scotland Technical College, Glasgow. (With Two Plates.)

(MS. received March 9, 1911. Read February 20, 1911.)

In 1907 I had the honour of reading a paper before this Society in which a new genus of iron-bacteria was described.* This organism was first discovered in Renfrew, near Glasgow, and subsequently observed in various samples of iron-water from different parts of the country. At Renfrew this organism was at the time of its discovery the sole bacterial occupant of the iron-waters of this neighbourhood. In most samples of these waters—and I have examined hundreds—only the remains of iron-bacteria can be obtained, but occasionally the samples are found to contain iron-bacteria in process of growth and multiplication. After a little experience, it is possible to ascertain which streams contain iron-bacteria in this favourable condition. At Renfrew such streams were then common, and so gave me an opportunity of tracing the life-history of this species. The results of this investigation seemed to me to justify the conclusion that I had obtained a new genus of iron-bacteria. In 1910 Professor Molisch's book on the iron-bacteria was published,† and in it he expresses the opinion that *Spirophyllum ferrugineum*, the name which I had given to this new species, was none other than a form of *Gallionella ferruginea*. The object of the present paper is to support the conclusions of the former paper, and chiefly to supplement them by the publication of some photomicrographs which were taken at the time, though not published. I am indebted to Mr Robert Garry for the excellent negatives which he took for me.

In the first place, I should like to note that my experience of the iron-bacteria has shown that their multiplication is spasmodic and short-lived. In Great Britain, at any rate, the iron-streams are usually devoid of living iron-bacteria, though there are plenty of remains in the sediments. Occasionally, however, as I have proved by my own observations and those of a band of assistants, here and there a zone of growth, local in its extent,

* Ellis, "On the Discovery of a New Genus of Thread Bacteria (*Spirophyllum ferrugineum*, Ellis)," *Proc. Roy. Soc. Edin.*, vol. xxvii., part i., No. 6.

† Molisch, *Die Eisen-bakterien*; Gustav Fischer, Jena, 1910.

is often encountered. In the case of the genera Leptothrix and Gallionella the periods of growth are limited both in space and time. The bulk of the deposit of the iron-waters seems to have been made up of the remains of organisms, the growth of which, though limited as regards space and time, yet took place at frequent intervals. This admits of an easy physiological explanation, into which I need not at present enter. All my observations of Spirophyllum were made from material in which the bacteria were in a state of growth. Diagrammatic representations of the two organisms in question are given in figs. 1 and 2.

I. The appearance of Gallionella is that of a thread spirally twisted

Fig. 1.

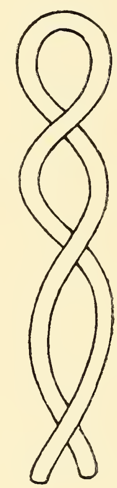


Fig. 2.

round itself, whereas Spirophyllum is a band-shaped structure. The primary forms of the two organisms are thus fundamentally different.

II. As will be seen by a perusal of my former paper,* my observations were taken from such favourable material that I was able to follow the whole of the life-history, and, by comparison of the individuals of various ages, to trace the method of growth. Small, spirally twisted *bands* could be seen that were no wider than the conidia from which they had germinated, so that it was evident that the peculiar spiral band shape had been assumed *immediately on emerging from, or elongating from, the conidium.* Again, the comparison of individuals in various stages of development showed that no departure was made from this spiral band form during the whole period of its existence. I regard this fact as of paramount importance in connection with this point, because it is evident

* *Ibid.*

that Molisch regards Spirophyllum as a Gallionella which *has grown up in the Gallionella form, and which has subsequently developed into the form characteristic of Spirophyllum.*

III. During the whole time that the investigation lasted (about six months) no traces were observed of any transitions between Gallionella and Spirophyllum.

IV. Gallionella has never been known to be motile; but in the case of Spirophyllum, the young forms, for a short time after emergence from the conidium, possessed an independent movement. This motility was confined to young organisms, the stoppage being evidently expedited by the deposition of iron on the surface. The young organisms are quite colourless, but as they get older their colour gradually changes into a reddish-brown tint, which becomes deeper with age, until the tint characteristic of the sediment is observed on the individual organism.

V. I may say that I am well acquainted with the life-history of Gallionella, which I found growing in several places near Glasgow, and have published the results of my investigations,* so that it cannot be reasonably stated that I am not acquainted with the organism with which Spirophyllum is supposed to be identical. In my observations of Gallionella I noticed certain forms which seem to me to form the basis of Molisch's criticism. As is well known, part of the membranes of the iron-bacteria after a certain age become changed into a mucilaginous substance, which has a great influence in determining the amount of iron which will later be deposited on the organism. When, therefore, a Gallionella thread becomes so mucilaginous that the mucilage fills the spaces between the windings of the thread, an appearance somewhat similar to a Spirophyllum band is obtained. In such cases, however, as can be seen by reference to figs. 3, 4, 5, and 6, the two organisms are easily distinguished by their ends, which are essentially different. Spirophyllum has square-cut ends, which can be observed in all stages of growth, whilst as Gallionella forms convolutions the ends are rounded. Whatever forms these organisms possess later, the appearance of the ends is always a distinguishing mark.

Another approach to a similarity in form is presented by the behaviour of Spirophyllum in some of its later stages, viz., during decomposition, when there is a tendency, though only occasionally, for the thinner middle part of the band to be detached, leaving behind the thicker edge, so that a superficial resemblance to Gallionella is presented. (See fig. 3; also explanation on p. 504.) If I had not followed the phases of the life-history

* Ellis, "A Contribution to our knowledge of the Thread Bacteria (I.)," *Centralblatt für Bakteriologie*, Bd. xix., 1907.

of this organism from start to finish, and if I had not the appearance of the square-cut ends to assure me, and likewise my knowledge of the life-history of Gallionella, I should also have regarded Spirophyllum, if presented for observation only at this stage, as belonging to Gallionella.

VI. Another remarkable attitude taken by Molisch is the position of doubt which he takes up with regard to my discovery of conidia-formation in Spirophyllum and Gallionella, and likewise to the previous discovery of the same mode of reproduction discovered by others in the case of Leptothrix. He regards the conidia as foreign particles which have accidentally been deposited on the organisms from the surrounding water. His reason for this attitude lies in the fact that in his pure cultures of *Leptothrix ochracea*, upon the success of which I should like to take this opportunity of congratulating Professor Molisch, no signs of conidia-formation were apparent. In the first place, so far as *Leptothrix ochracea* is concerned, conidia-formation has been described both by Migula * and myself.† In this organism, however, the retaining power of the mucilage surrounding the cells is so great that the conidia sometimes germinate *in situ*, with the result that in some cases outgrowths radiating in all directions from the organism can be seen, so that the latter looks like a porcupine, except that the ends of the "bristles" are rounded instead of being pointed. It is very difficult to understand how the presence of these outgrowths is to be explained except by the germination of conidia, which have not been able to free themselves from the organism which gave rise to them. Surely the outgrowths cannot be ascribed to foreign particles which have adhered to the organism. Further, I have seen threads of Spirophyllum and also of Leptothrix so beset with these conidia that the outline of the organism itself could only be seen as a dim outline barely discernible through the mass of conidia. In such masses the conidia were distinctly visible, each as a sharply defined oval body: the uniformity in size, the definiteness of the outline, allowed no possibility of confusion with foreign particles. Again, these oval bodies could not possibly belong to other organisms, for there were very few, if any, other bacteria in the water, and in any case they were too large to belong to any of the lower bacteria, and there was no trace of any other organisms. In a case of this kind, the final proof is afforded by an observation of the development of the structures in question. I was able to accomplish this by observing individuals on which the number of conidia was small. In such cases some were sure to be seen in the first stages of development. Small nodules could be seen in process of develop-

* Migula, *System der Bakterien.* † *Ibid.*

ment. A comparison of the different stages showed that the conidium grew to a certain length, and then was pinched off from the mother-organism. In fig. 6 an individual is shown at this stage. Over the whole surface nodular projections are seen, which are conidia in process of formation. Of course one cannot see these with the same distinctness in a photograph as when looking through the microscope. On the same plate, a band of Spirophyllum is seen completely covered with conidia, all of the same size and shape; and though the individual outlines of the conidia cannot be shown on the plate, yet through the microscope there was no more difficulty in seeing them than in seeing the conidia of a mould such as Penicillium. Molisch's remarks on the observations of other investigators on the morphology of the iron-bacteria are of a most remarkable nature. In effect it amounts to this, that any recorded phenomena which he cannot verify in his own cultures are to be put aside as being due to mistaken observations. Although he worked with pure test-tube cultures in his investigations of Leptothrix, whilst I worked from samples—almost pure cultures—taken from nature, it does not follow that all the morphological capabilities of an organism are always demonstrated in test-tube cultures. Every bacteriologist knows that such is very seldom, if ever, the case. Artificial cultures of an organism are notoriously different in character, both physiologically and morphologically, from cultures of the same organism growing free in nature. This applies particularly to the methods of reproduction; and one is not surprised to learn that conidia-formation does not take place in test-tube cultures of Leptothrix. But to assume that because conidia-formation is not found in artificial cultures, therefore no conidia-formation can take place under any circumstances, in spite of the researches of other investigators—which is in effect the standpoint taken by Molisch—is to take up a very remarkable attitude. As Molisch denies the existence of conidia-formation in Leptothrix, which he has seen and investigated, so also the same incredulity is expressed with regard to my discovery of conidia-formation in Spirophyllum, which he has not seen. From this standpoint we could, with equal justification, assert that the morphological and physiological peculiarities of growth of any pathogenic bacillus inside the host cannot be true because the same phenomena had not been observed in test-tube cultures of the organism. I should like to state here, that the objections which Molisch has brought forward are objections as to facts, and not as to the interpretation of facts; and such being the case, I can do no more than leave the matter in the hands of later researchers on this branch of bacteriology. The chief object of the paper is the publication of the photomicrographs which were taken at the time of the investigation, and which will, I think, convince all who are familiar with Gallionella, and have no preconceived opinions on the subject, of the generic value of the organism which I have named *Spirophyllum ferrugineum.*

EXPLANATION OF FIGURES.

Fig. 1. Diagrammatic representation of *Spirophyllum ferrugineum.*

Fig. 2. Diagrammatic representation of *Gallionella ferruginea.*

Fig. 3. Photomicrograph of *Spirophyllum ferrugineum.* At × two individuals in a stage of disintegration are shown, giving a superficial resemblance to Gallionella. Note, however, that in all cases the square-cut ends characteristic of this species are well shown.

Fig. 4. Photomicrograph of *Spirophyllum ferrugineum*: some of the individuals are in a state of incipient conidia-formation.

Fig. 5. Central figure is composed of two intertwining bands of *Spirophyllum ferrugineum.*

Fig. 6. Central figure is composed of two intertwining bands of *Spirophyllum ferrugineum*, both of which are in a stage of incipient conidia-formation. The individual to right of central figure is covered with conidia, so that its outline has become somewhat vague.

(*Issued separately July* 7, 1911.)

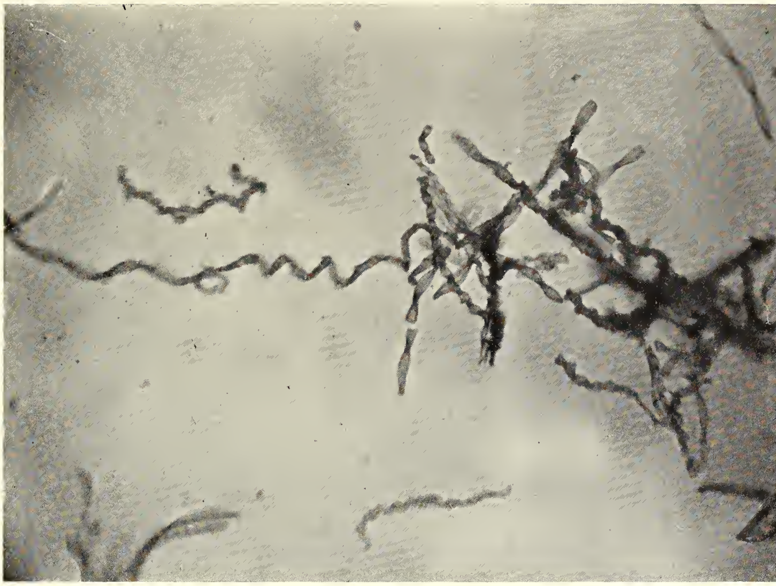


Fig. 4.

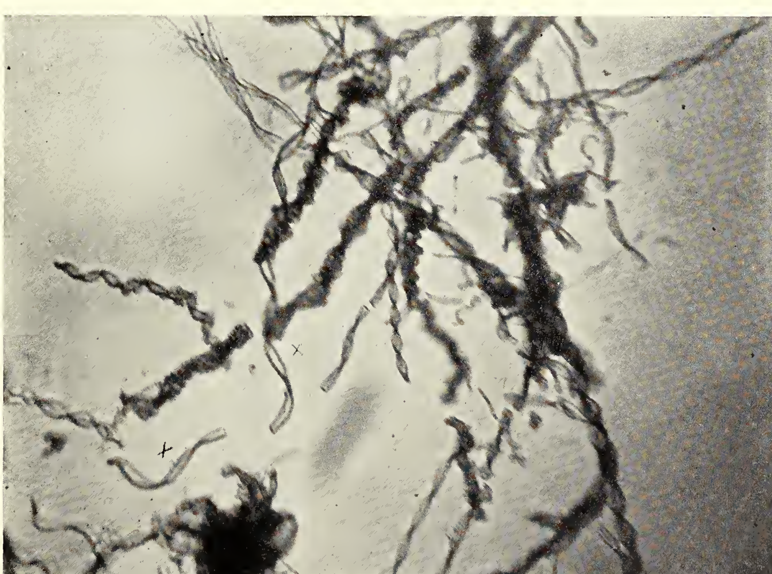


Fig. 3.

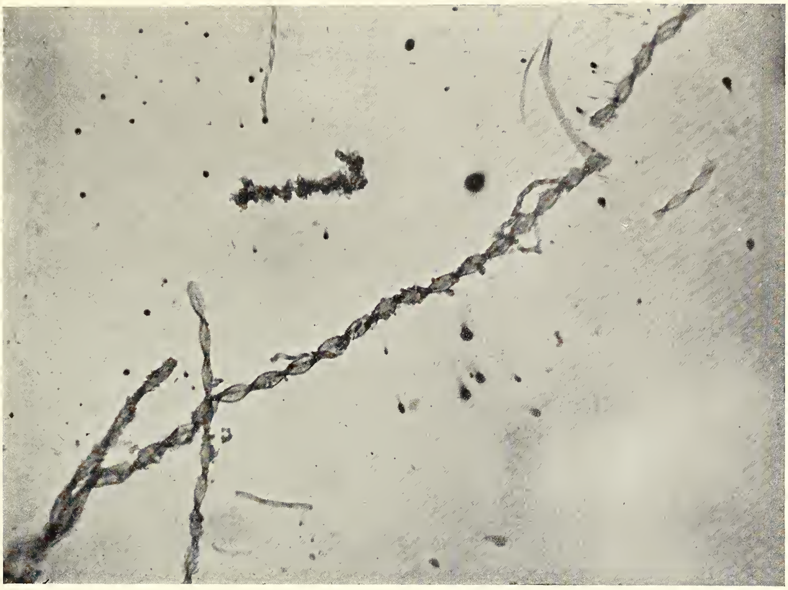


Fig. 6.

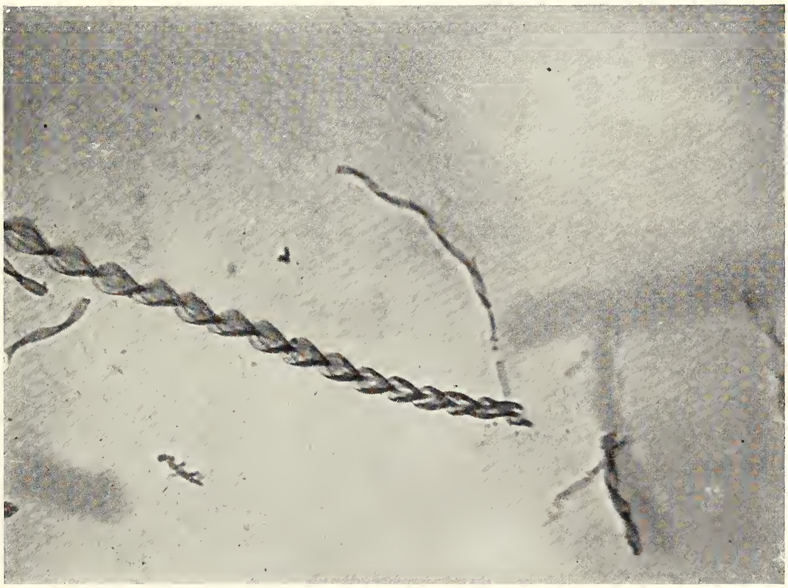


Fig. 5.

XXXIII.—On the Influence of Temperature upon the Magnetic Properties of a Graded Series of Carbon Steels.

By **Margaret B. Moir**, M.A., B.Sc., Carnegie Research Scholar in the University of Glasgow. *Communicated by* Professor A. Gray, F.R.S.

(MS. received March 20, 1911. Read same date.)

The magnetic properties of iron, steel, nickel, and cobalt at moderate and high temperatures have been investigated by numerous experimenters, perhaps the most notable work being that carried out by Hopkinson,* M. Curie,† and D. K. Morris.‡ Hopkinson employed the ring method of Rowland, insulating the windings from each other by means of asbestos, and deducing the temperature from the electrical resistance of a platinum wire wound upon the specimen. He obtained magnetisation curves, at temperatures lying between ordinary room temperature and the critical temperature of the material, for soft iron, mild steel, hard steel, nickel, and cobalt. In the case of soft iron he found that for low values of the magnetising force the effect of increasing the temperature was to bring about an increase in the permeability; as the temperature approached the critical temperature of the material an enormous increase took place in the magnetic quality. For a value of the magnetising force of 0·3 c.g.s. units the permeability was about 400 at room temperature; as the temperature increased the permeability steadily increased, and at 600° C. had attained a value of 900. From this point on, the increase in permeability with temperature became more rapid; at 700° C. the permeability was about 1900, at 750° C. it had the value 4000, and at 775° C. it reached the maximum value of 11,000. Further heating brought about a very rapid loss of magnetic quality, and at the temperature of 790° C. the iron had become practically non-magnetic. At this temperature its permeability was about unity. For large fields the permeability remained practically constant until a temperature of 600° C. was reached; there was then a steady falling off in magnetic quality up to 790° C., at which temperature the iron became practically non-magnetic.

Specimens of mild and hard steels exhibited much the same general behaviour; the critical temperature was found to be 740° C. for the former and 690° C. for the latter material.

In 1895 an important paper on the magnetic properties of iron, nickel,

* *Phil. Trans. Roy. Soc.*, 1889, A, p. 443. † *Journ. de Phys.*, vol. v. p. 289, 1895.
‡ *Phil. Mag.*, vol. xliv. p. 213, 1897.

and cobalt at different temperatures was published by M. Curie,* who employed fields up to 1350 c.g.s. units and temperatures extending from room temperature to 1400° C. He found the critical temperature of iron to be 760° C., which is somewhat lower than Hopkinson's value; he showed also that as the temperature is further increased the permeability remains practically zero up to 1300° C., when there is an abrupt increase in magnetic quality, indicating the existence of a further transformation point.

In 1897 D. K. Morris † repeated Hopkinson's work for soft iron, special attention being paid to the influence of temperature upon the magnetic hysteresis of the materials tested. It was found that for small and moderate magnetising forces heating the specimen resulted in a general improvement of the magnetic quality, as had already been shown by Hopkinson. The experiments showed, however, that for small field strengths the permeability-temperature curve passes through several maximum and minimum points before the critical temperature is reached. Morris verified the existence of a transformation point for iron at a temperature considerably above the critical temperature. The fact that turning points occur in the permeability-temperature curves indicates that as the temperature is increased the crystalline structure undergoes modifications; and Morris' work showed that for iron, in addition to the main critical point, there are at least three further such points lying between room temperature and the critical temperature.

It is interesting to notice that the changes which the magnetic qualities of iron and steel undergo on being heated are accompanied by changes in the other physical properties of the materials. Gore, in 1869, found that an iron wire cooling from a bright red heat undergoes a momentary elongation at a dull red heat, and then goes on contracting as before. Barrett,‡ in 1874, showed that at the temperature at which Gore's phenomenon takes place the material passes from the non-magnetic to the magnetic condition. Tait § found that at certain temperatures the thermo-electric powers of iron and nickel undergo very remarkable alterations.

In Hopkinson's work, and in that of many of his successors, the procedure adopted was as follows. The ring specimen was first heated up above the critical temperature and allowed to cool to the temperature at which it was desired to carry out a test; the cooling was then arrested and the test carried out by the method of reversals. The specimen was next submitted to the action of an alternating magnetic field of gradually diminishing strength; the initial value of this field being great and the

* *Journ. de Phys.*, vol. v. p. 289, 1895.

† *Phil. Mag.*, vol. xliv. p. 213, 1897.

‡ *Phil. Mag.*, Jan. 1874.

§ *Trans. Roy. Soc. Edin.*, 1873.

final value zero, the specimen was left devoid of residual magnetism and of magnetic history. The specimen was now allowed to cool by a suitable amount, when the cooling was again arrested and a further test carried out; and so on. In this way a series of magnetisation curves was obtained from which permeability-temperature curves corresponding to various field strengths were deduced.

It is to be carefully noticed in connection with what follows that the specimen was submitted to the action of the alternating magnetic field (in what follows I shall refer to this process, by means of which the specimen is rendered neutral, as a "process of reversals") *after* and *not before* each test, *with its preliminary thermal change.* Thus a change in temperature intervened between each test and the previous application of the process of reversals.

Now, it is not generally known that purely thermal treatment, no matter what temperature is reached in the process, develops in the specimen a peculiar state which renders additional precautions necessary. The bearing of this fact upon magnetic testing has been investigated by Dr J. G. Gray * and Mr A. D. Ross, who have shown that in order that a magnetisation curve yielded by a specimen at a particular temperature (following upon a temperature change) should be characteristic of the material at the new temperature, it is necessary that the specimen should be submitted to a process of reversals *at the new temperature.*

The importance of attending to this point will be evident from the following results, which were obtained on testing a specimen of hard steel by the magnetometer method. The specimen was first tested at room temperature after having been rendered neutral, when the following readings were obtained:—

Magnetising current in amperes .	0·2	0·4	0·6	0·8	1·0	1·5
Magnetometer deflection . .	32	73	117	167	221	365

The specimen was now submitted to a process of reversals at 15° C.; the temperature was then raised to 105° C., and a further test carried out with the following results:—

Magnetising current in amperes .	0·2	0·4	0·6	0·8	1·0	1·5
Magnetometer deflection . .	40	88	139	195	253	410

* "On Magnetic Testing," *Phil. Mag.*, Jan. 1911.

Finally the specimen was submitted to a process of reversals at 105° C. and retested at that temperature, when the following results were obtained:—

Magnetising current in amperes .	0·2	0·4	0·6	0·8	1·0	1·5
Magnetometer deflection . .	38	79	127	183	240	397

It will be seen that a magnetisation curve yielded by the specimen at 105° C. following upon the application of the process of reversals at 105° C. lies everywhere below the curve yielded by the specimen at 105° C. following upon the application of the process of reversals at 15° C. The former curve is the true magnetisation curve of the material corresponding to 105° C. If the latter curve is taken, the errors introduced into the results are very considerable.

If the temperature of a test-specimen is changed and a test carried out without first applying the process of reversals at the new temperature, the magnetisation curve obtained depends not only on the new temperature but on the thermal change effected. The magnetisation curve yielded by a test following upon a process of reversals carried out at the new temperature depends only on the temperature of the specimen.

In view of the above facts it seemed of importance to repeat Hopkinson's work on steels. Accordingly a series of specimens containing varying amounts of carbon were obtained from Messrs Armstrong, Whitworth & Co. Ltd. These steels form a suitably graded series, extending from mild steel, through medium carbon steel, to high carbon steel and cast iron. The following table gives their compositions:—

Table I.

Compositions of Steels tested.

Description of Material.	Percentage of Composition.				
	C.	Mu.	Si.	P.	S.
Cast iron . . .	3·15	0·15	0·13	...	...
High carbon steel .	1·64	0·13	0·85	0·02	0·02
Medium carbon steel .	0·80	0·20	0·075	0·012	0·02
Low carbon steel. .	0·30	0·60	Trace	0·025	0·03
Soft iron . . .	0·06	0·10	...	...	...

The specimens were supplied in the form of cylinders 20 cms. in length and 0·9 cms. in diameter. Previous to being tested each specimen was carefully annealed from 900° C. in a Fletcher gas furnace, care being taken to exclude atmospheric air from contact with the specimen. For the magnetic tests a Gray-Ross magnetometer,* provided with the newest type of electric furnace,† was employed. The furnace is contained within the magnetising coil of the magnetometer, and by its means the specimen can be brought to any desired temperature while under the influence of the magnetising helix. The furnace is air-tight, and consequently there is no possibility of the carbon being burnt out of the specimen. The procedure followed in carrying out the magnetic tests was as follows. The test-bar, having been placed within the furnace, was submitted to a process of reversals and tested, all the operations being carried out at the temperature of the room. The temperature was next raised by a suitable amount, the heating arrested, the specimen submitted to a process of reversals, and a test carried out; and so on. In this way a sufficient number of magnetisation curves at different temperatures was obtained to admit of the variation in magnetic quality with temperature being deduced.

In deducing the effective field strengths from the applied field strengths the demagnetising factors investigated by Du Bois were employed.

Discussion of Results obtained.

Cast Iron.—The magnetisation curves obtained on testing the specimen of cast iron are shown in fig. 1, and a series of susceptibility-temperature curves deduced from them is exhibited in fig. 2. As the temperature increases from 15° C. to 180° C. the susceptibility of the material, for field strengths extending from 2 to 12 c.g.s. units, steadily increases. At the latter temperature the slope of the curve diminishes rapidly; the curve exhibits a maximum point at about 190° C., and the effect of further heating is to bring about an abrupt diminution in magnetic equality. The susceptibility arrives at a minimum value at about 260° C., after which it again slopes steadily upwards until the temperature reaches 600° C.; from 600° C. to 790° C. the magnetic quality falls off abruptly, becoming practically zero at 790° C. It will thus be seen that in cast iron there is a transformation point in the neighbourhood of 200° C. Further, an inspection of fig. 2 shows that while the maximum point on the susceptibility curve in the neighbourhood of the critical temperature is more marked for low fields than for high ones, the maximum point in the neighbourhood of 200° C. is more marked for high fields than for low ones.

High Carbon Steel.—The results yielded by the specimen of high carbon steel are shown in figs. 3 and 4. An examination of fig. 4 shows

* *Proc. Roy. Soc. Edin.*, vol. xxix. p. 182, 1909.

† *Proc. Roy. Phil. Soc. Glasg.*, vol. xii., 1910.

that for a value of the magnetising force of 5 c.g.s. units the magnetic quality at first steadily increases with the temperature to a maximum at about 230° C.; from 230° C. to 270° C. it diminishes, and from this point it continually increases, reaching a second maximum at about 730° C.; further increase of the temperature brings about a rapid diminution in suscepti-

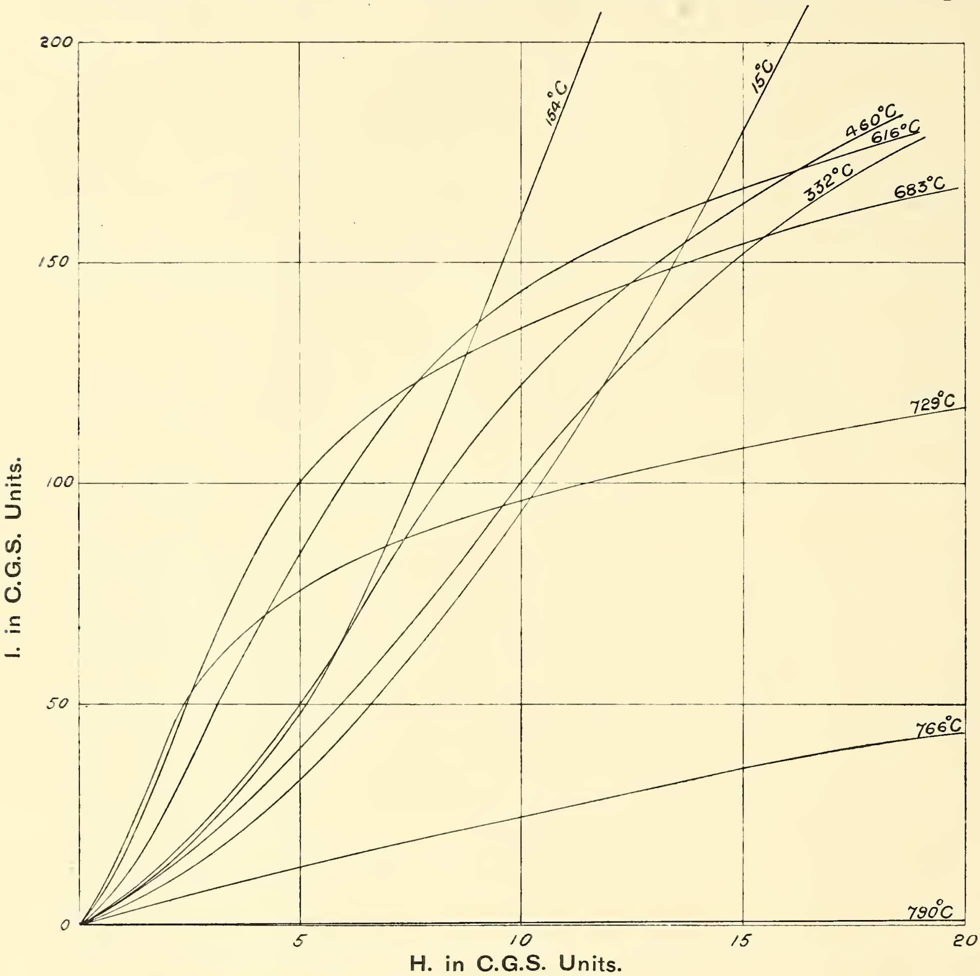


Fig. 1.—Specimen of cast iron.

bility, and the material becomes non-magnetic at about 770° C. The corresponding curves for higher values of the magnetising force present the same general features. For H = 30 c.g.s. units the curve slopes upward until the temperature becomes 220° C., after which there is an abrupt falling off in magnetic quality, succeeded by an abrupt increase, and a second maximum is arrived at when the temperature reaches 300° C. From 300° C. to 500° C. the susceptibility remains practically constant; from

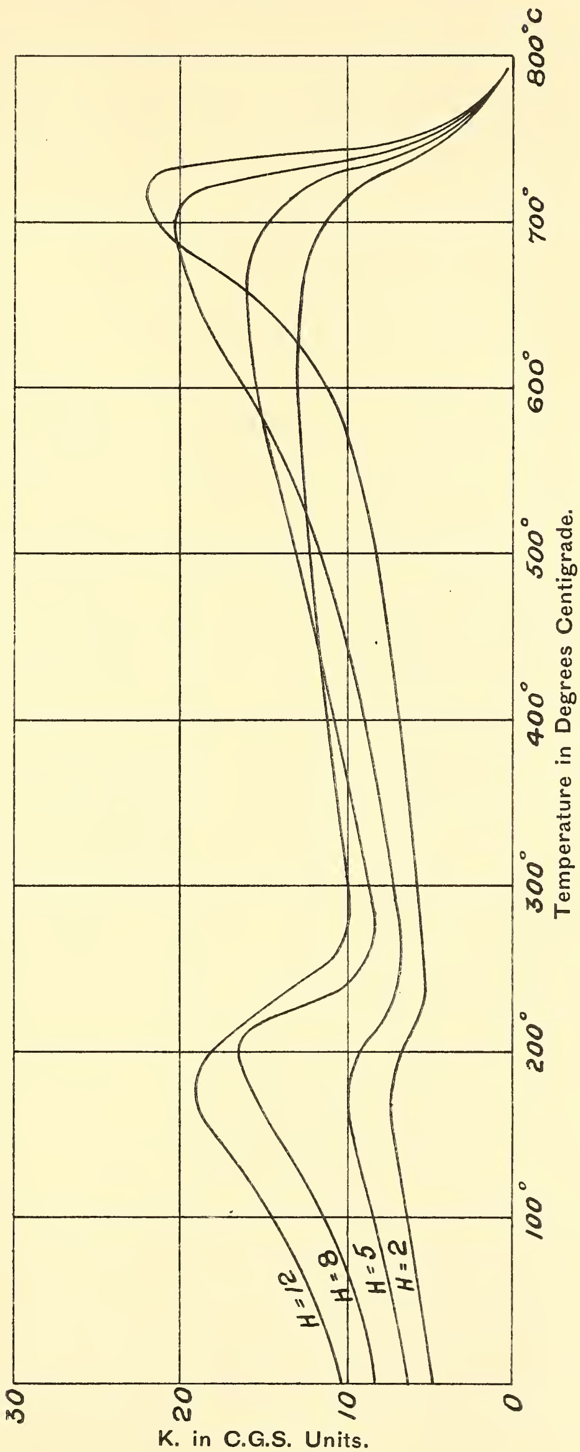


Fig. 2.—Specimen of cast iron.

500° C. to 700° C. there is a gradual falling off in magnetic quality; and finally the susceptibility becomes zero at 770° C.

It will be seen that the susceptibility-temperature curves for this high carbon steel present precisely the same general features as those which characterise the specimen of cast iron.

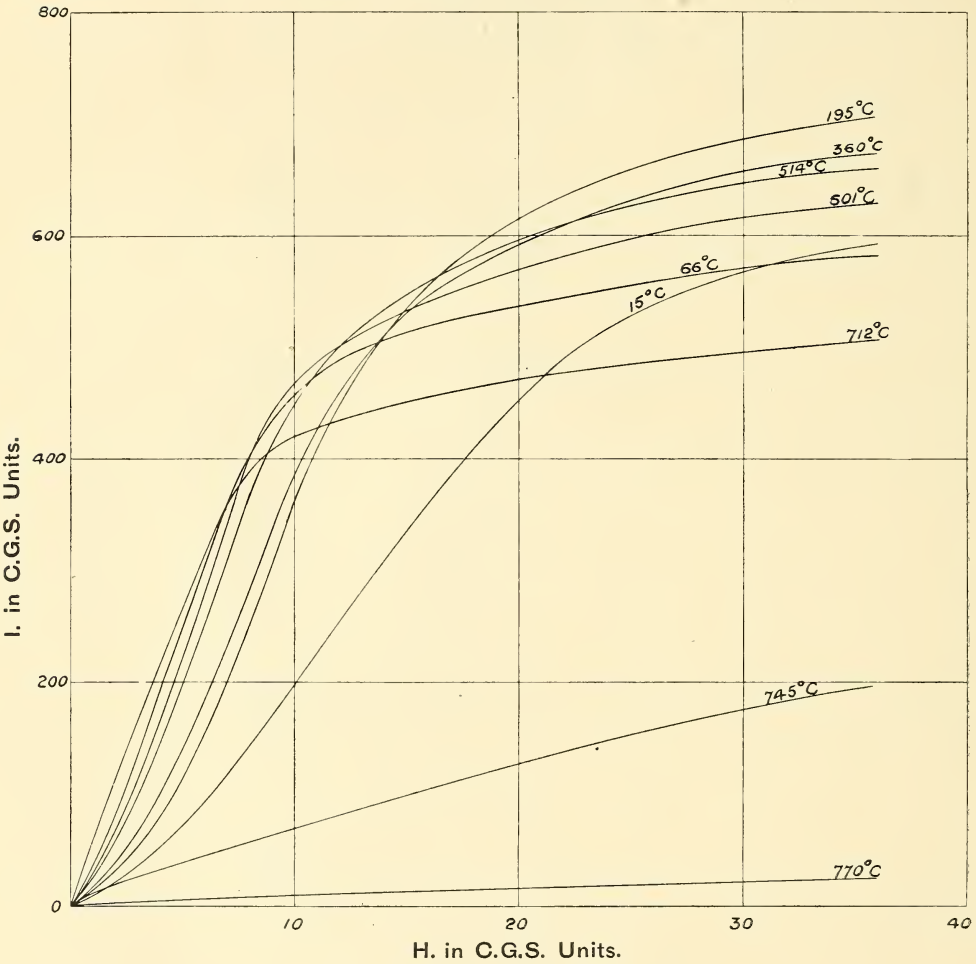


Fig. 3.—Specimen of high carbon steel.

Medium Carbon Steel.—On testing the specimen of medium carbon steel very similar results were obtained. An improvement in magnetic quality takes place as the temperature rises from room temperature to 150° C.; this is followed by a falling off, and then a further improvement, in magnetic quality. This improvement continues until the critical temperature is approached, when a very rapid falling off in magnetic susceptibility takes place.

In the case of this variety of steel, the first maximum occurs at about 180° C., and the first minimum at about 220° C., the bend in the curve

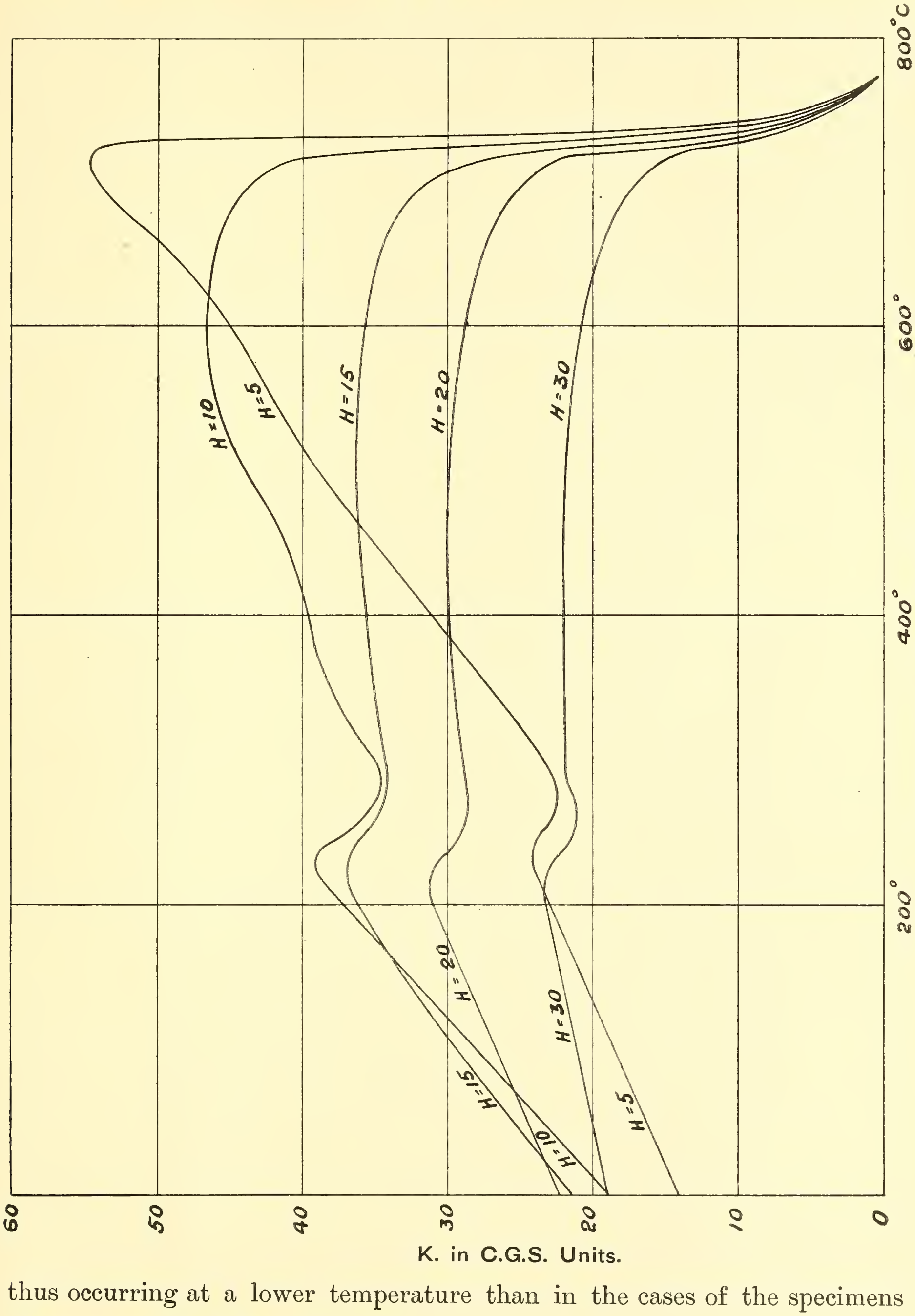


FIG. 4.—Specimen of high carbon steel.

thus occurring at a lower temperature than in the cases of the specimens of cast iron and high carbon steel.

Low Carbon Steel.—A similar examination of a specimen of the low carbon steel showed that similar turning points occur in the susceptibility-temperature curve. They are, however, much less marked than those contained in the corresponding curves for the steels containing higher percentages of carbon.

Soft Iron.—The susceptibility-temperature curves yielded by a specimen of soft iron were entirely normal. There is no indication of a set-back in the susceptibility until the temperature approaches the critical temperature.

In the following table the most important of the preceding results are collected together for the sake of comparison, and it may be observed at this point that these results refer only to values of the magnetising force exceeding 2 c.g.s. units, and are therefore not necessarily true for very low fields. As a matter of fact, the lower the value of the magnetising force the less marked is the first transformation point, as has already been pointed out. In the table, T_1 is the temperature at which the first maximum appears in the susceptibility-temperature curve, and T_2 is the temperature at which the first minimum appears.

Table II.

Variety of Steel.	Percentage of Carbon.	T_1.	T_2.
Cast iron . . .	3·15	180° C.–200° C.	240° C.–270° C.
High carbon . . .	1·64	210° C.–230° C.	250° C.–270° C.
Medium carbon . .	0·8	180° C.	220° C.
Low carbon . . .	0·3	110° C.	140° C.
Soft iron . . .	0 06	...	...

Comparing first the temperatures of minimum susceptibility we see that while that for cast iron and that for high carbon steel are almost the same, that for medium carbon steel is lower than for either of the specimens containing a high percentage of carbon; that for low carbon steel is lower still. Also the range of temperature within which the bend takes place tends to become smaller as the percentage of carbon diminishes. Again, the magnitude of the set-back in magnetic quality in the neighbourhood of 200° C. becomes less marked with diminishing quantity of carbon. When the percentage of carbon is as low as 0·06 per cent. the set-back is no longer perceptible.

In conclusion, it may be remarked that the writer has failed to detect any indication of the presence of a transformation point for steel in the neighbourhood of 200° C. in the curves obtained by previous workers in magnetism. Hopkinson's curves give no indication of the presence of such a transformation point. For hard steel the permeability-temperature curve obtained by this investigator is, for a field strength of 1·5 c.g.s. units, and for temperatures lying between 15° C. and 550° C., practically a straight line. Morris demonstrated the existence of several minor transformation points for soft iron lying between room temperature and the critical temperature. And in this connection it should be stated that Morris submitted the specimen under test to a process of reversals immediately before carrying out a test: the magnetisation curves yielded were therefore the true magnetisation curves corresponding to the various temperatures. It is not pointed out by Morris, however, in his paper, that it is essential that the process of reversals should be carried out at the particular temperature at which it is desired to test the specimen. As a matter of fact, had the process of reversals not been so applied the small waves in the curves would have been completely masked by the want of neutrality brought about by the thermal changes.

Summary.

1. Experiments of Hopkinson, Curie, and D. K. Morris.

2. Precautions necessary in magnetic testing. If the temperature of a test-specimen is changed, the specimen must be rendered neutral at the new temperature previous to carrying out the test.

3. The effect of increasing the temperature of a specimen of cast iron from 15° C. to 190° C. is to improve its magnetic quality for moderate values of the magnetising force; from 190° C. to 260° C. the magnetic quality falls off, after which there is a further improvement. Finally the specimen becomes non-magnetic at the critical temperature. These results indicate the presence of a transformation point for carbon alloys of iron in the neighbourhood of 200° C.

4. Steel containing 1·64 per cent. of carbon resembles cast iron in its behaviour. The set-back in magnetic quality takes place at much the same temperature as that which occurs in cast iron, the magnitude of the set-back being much the same for both.

5. Steel containing 0·8 per cent. of carbon exhibits the transformation point, but in a less marked degree. The first maximum point in the susceptibility curve occurs at 180° C., and the first minimum point at 220° C.

6. Low carbon steel containing 0·3 per cent. of carbon shows the transformation point; it is, however, very much less marked than in the case of the steels containing the higher percentages of carbon.

7. The susceptibility-temperature curve of a specimen of soft iron containing 0·06 per cent. of carbon exhibits no turning points in the neighbourhood of 200° C. for fields lying between 2 and 15 c.g.s. units.

The work described in the present paper was carried out in the Natural Philosophy Institute of the University of Glasgow. The author desires to express her thanks to Professor Gray and to Dr J. G. Gray for the interest they have taken in the progress of the experiments.

(*Issued separately July* 7, 1911.)

XXXIV.—Laboratory Note: A Simple Method of finding the Radius of Gyration of a Body. By W. G. Robson, A.R.C.S. *Communicated by* Professor W. Peddie.

(MS. received April 29, 1911. Read July 3, 1911.)

In the course of some work in this laboratory a determination of the radius of gyration of an Atwood's machine pulley was necessary. The radius was found by using a bifilar suspension. Having been unable to find any published record of the method, which is capable of giving fairly accurate results, and as the measurement of moment of inertia or of radius of gyration is a frequently occurring and ill-understood exercise in a physical laboratory, it seemed to me that the following details might be of interest.

If a body be suspended by two threads of equal length, l, the distances between the threads being d_1 and d_2 at the upper and lower ends respectively, the time of a small oscillation about a vertical axis through the centre of gravity is given by

$$T^2 = \frac{16\pi^2 l k^2}{g d_1 d_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

k being the radius of gyration.

Now let the body be made to swing as a simple pendulum (using the same suspension threads). Then

$$T_1{}^2 = \frac{4\pi^2 l}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2) it follows that

$$k = \frac{T\sqrt{d_1 d_2}}{2T_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

—which reduces in the case of parallel threads to the simple form

$$\frac{Td}{2T_1}.$$

In (3) T and T_1 may, of course, be the times of any number of swings, and evidently the result is independent of the rate of the timepiece.

A few typical results are given in the following table. The suspension threads were of thin copper wire about two metres in length; d_1 and d_2

varied from 8 to 25 cms. Allowance was made for the fact that l in (1) and (2) are nearly, but not quite, equal, the centre of gravity being a small distance below the line joining the lower points of suspension.

	Axis.	Values of k in cms. from	
		Dimensions.	Experiment.
Cylinder (1)	Perp. to length	6·56	6·58
" (2)	" "	3·64	3·64
Circular Disc	Perp. to diar.	2·83	2·82
" "	Diar.	2·02	2·03
Ring	Perp. to diar.	3·54	3·54
"	Diar.	2·52	2·53
Rectangular Plate	Perp. to plane	5·32	5·32
" "	" "	5·32	5·32
" "	Length	2·09	2·10
" "	Width	4·90	4·91
Fly-wheel	Axle	19·10	18·83

Physical Laboratory,
University College, Dundee.

(*Issued separately, July* 7, 1911.)

XXXV.—The Vapour Pressure of Dry Calomel. By Alexander Smith and Alan W. C. Menzies.

(MS. received April 4, 1911. Read June 5, 1911.)

(Abstract.)

We have shown * that calomel vapour contains no measurable amount of Hg_2Cl_2 or of HgCl, and consists wholly of the dissociation products, mercury and mercuric chloride. According to chemical theory, therefore, when, by the removal of all moisture, the dissociation is prevented, the vapour pressure of the dry substance should be equal only to the partial pressure of the undissociated molecules of the undried vapour; that is to say, it should be negligibly small. Brereton Baker, however, was able to measure the vapour density of such dry calomel, finding it to be almost double the ordinary density. Since the measurement was made by the V. Meyer method, at 445°, it follows that this dry calomel must have possessed a vapour pressure approximating, if not exceeding, one atmosphere. The same anomaly occurs in the case of ammonium chloride, and its existence has been confirmed in that case by Abegg and Johnson, who found the vapour pressure of the ordinary and the dry salts to be exactly equal. The explanation offered in this case by Wegscheider is that salammoniak is dimorphous, and the form proper to low temperatures persists, when dry, in a temperature region in which it is unstable, and there gives, as the theory would predict, a high vapour pressure. The exact equality between the pressures is due, according to him, to a coincidence. Since Baker's result seems to point to a similar dimorphism in calomel, it was of the highest interest to ascertain the actual value of the vapour pressure in the case of the dried calomel.

The measurement was made by placing a known weight of calomel in a small bulb, and fusing the latter on to the elongated and down-turned capillary neck of a large bulb. The drying was performed by phosphorus pentoxide contained in another bulb (subsequently fused off) much as in Baker's experiments, except that the apparatus was evacuated and was kept in an oven at 115°, while the pentoxide bulb hung in running cold water. After the period of drying was over, the apparatus was immersed in a bath of potassium and sodium nitrates, kept constant at 352°, and after

* *Proceedings*, vol. xxxi. p. 183.

fifteen minutes the top of the arched capillary neck was raised out of the bath, and was instantly sealed by encountering a small blast-lamp flame. Two such experiments were made.

At 352° the vapour pressure of ordinary calomel is 347 mm. The weight of 1 c.c. of the vapour at this temperature is therefore 0·002096 g.

Experiment A.—The drying at room temperature lasted sixty-three days, and in the oven eight days. The weight of calomel vaporised was 0·1320 g., and the volume of the bulb (at 352°) 65·53 c.c. The weight of 1 c.c. found is 0·002015. The density is therefore normal, and the vapour pressure also normal (347 mm.) The drying was presumably insufficient.

Experiment B.—The drying in the oven lasted five and a half months. After the experiment, no calomel was to be seen in the large bulb, but a few drops of the nitrates had entered accidentally, through a small perforation. The contents of the bulb, however, gave evidence of containing the slightest trace of mercury only. The vapour pressure of this dried calomel was therefore negligible, as the theory predicts.

The fact that Baker was able to vaporise the dried substance in the form of Hg_2Cl_2 at 445° remains unexplained. It may be that we reached a profounder stage of dryness, at which the volatility disappears.

The University of Chicago,
December 1910.

(*Issued separately July* 7, 1911.)

XXXVI.—The Absorption of Light by Inorganic Salts. No. I.: Aqueous Solutions of Cobalt Salts in the Infra-Red. By R. A. Houstoun, M.A., Ph.D., D.Sc., Lecturer on Physical Optics in the University of Glasgow. *Communicated by* Prof. A. Gray, F.R.S.

(MS. received March 8, 1911. Read June 5, 1911.)

The present article is intended to be the first of a series on the absorption of light by solutions of inorganic salts of different elements. With a few scattered exceptions all the work hitherto done on the absorption spectra of inorganic salts has been merely qualitative and has been confined to the visible spectrum. Kayser in his *Spectroscopie*, vol. iii. p. 45, states that in this field there is work for years and for numerous observers. E. C. C. Baly states in his *Spectroscopy*, p. 407, that not much is known about the absorption of light in inorganic salts. Merely for its own sake, then, an accurate determination of the molecular extinction coefficient for as many salts under as many different conditions of temperature and concentration, and for as many wave-lengths as possible, would be very valuable.

But the molecular extinction coefficient is also of great theoretical interest. Clerk Maxwell said that the constitution of the atom was accessible to the student who was armed with the spectroscope and the calculus. The emission spectrum, could we only read it, was a key to the structure of the atom. But although we are now able to measure a wave-length to an infinitesimal fraction of a centimetre, comparatively little progress has been made since Maxwell's time in reading emission spectra. Some of the spectra, too, have so many thousand lines that the prospects of constructing a dynamical system with degrees of freedom to correspond to the lines appears hopelessly remote. Emission spectra vary considerably with the means of production, and with the coming of radioactivity there has come also the idea that the emission spectrum is not so much a key to the structure of the atom as a key to the manner in which the atom is being destroyed. In order to get at the structure of the atom we must study its spectrum under normal circumstances at ordinary temperatures and under specified conditions that we can easily reproduce. That is, we must take the absorption spectrum of a solution. The molecular extinction coefficient may thus help to a knowledge of the structure of the atom.

The molecular extinction coefficient is defined by the following equation, which also gives the law according to which light is absorbed by a solution:

$$I = I_0 10^{-Acd}.$$

The original intensity is I_0, the final intensity is I, c is the concentration of the solution in gramme-molecules per litre, d the length of path traversed in centimetres, and A the molecular extinction coefficient. A is, of course, a function of λ, the wave-length of the light in question. It may also vary with c, the concentration of the solution.

In these *Proceedings* * I have investigated two alleged phenomena connected with absorption, and have found no evidence in favour of the existence of either of them. In another recent paper,† using all data at present obtainable, I showed that the vibrating particles which correspond to the absorption bands of anilin colouring matters have masses of the same order of magnitude as the mass of an electron, whereas in the case of inorganic salts we are possibly dealing with ions. In order to go on with that work it is necessary to get more data, hence the present series of investigations.

After considering the literature of the subject, I concluded that the most promising way of measuring A in the infra-red was by means of a Rubens linear thermopile and Du Bois Rubens ironclad galvanometer, and accordingly a galvanometer and thermopiles were obtained with the aid of a grant from the Carnegie Trust for the Universities of Scotland. Each thermopile has twenty iron-constantan junctions on a length of two centimetres, and has a resistance of about four ohms. The solder at the junctions is hammered into the form of discs of about one millimetre diameter. By placing a screen with a slit in it in front of the thermopile, its effective width can be reduced and readings taken closer together in the spectrum. The galvanometer has a total resistance of 10·5 ohms, and when the light suspension system was used the sensitiveness was 3×10^{-10} amps./half mm. at one metre distance for a period of five seconds.

A Boys radio-micrometer was borrowed from the Glasgow and West of Scotland Technical College and tested against the thermopile and galvanometer. The readings agreed very well. The surface for receiving radiation in the radio-micrometer is, however, rectangular, measuring 2×3 sq. mm.; hence it is not so suitable for use in the spectrum. Apart from that, it is not so sensitive. Boys' own instrument ‡ gave a deflection of 0·9 cm. per sq. mm. of sensitive surface for one candle, candle and scale being at a distance of one metre, and was about three times as sensitive as the radio-micrometer I used. For a period of five seconds with candle

* "On a Question in Absorption Spectroscopy," *Proc. Roy. Soc. Edin.*, xxix. p. 68 (1908). "A Negative Attempt to detect Fluorescence Absorption," *Proc. Roy. Soc. Edin.*, xxix. p. 401 (1909).

† "On the Mechanism of the Absorption Spectra of Solutions," *Proc. Roy. Soc.*, 82 A, p. 606 (1909).

‡ *Phil. Trans.*, 180 A, p. 159 (1889).

and scale at one metre my thermopile and galvanometer gave a throw of 0·6 cm. per sq. mm. of sensitive surface. When the area of the surface that can be utilised is taken into consideration, this is six times as sensitive as Boys' radio-micrometer.

The spectroscope, thermopile, and galvanometer were the same as I used for determining the efficiency of metallic filament lamps,* only the spectroscope was fitted with a new and better prism. It was calibrated in the same way as formerly, by bringing the thermopile into coincidence with various spectral lines in the visible spectrum and by using the absorption maxima of water in the infra-red at $\cdot 966\mu$, $1 \cdot 500\mu$, and $1 \cdot 956\mu$, and the minimum at $1 \cdot 708\mu$ as determined by Aschkinass.† The spectroscope had glass lenses and a glass prism, but as only aqueous solutions were used and as water absorbs before glass does, glass was quite satisfactory.

A straight Nernst filament was used as source of light, and at first it was focussed directly on the slit of the spectroscope by a lens. Two similar glass cells were taken, and one was filled with water and the other with the solution to be investigated. The cell filled with water was placed in front of the slit, a wooden screen placed before the cell, and the zero noted. The screen was then removed, the light allowed to fall on the slit, and the reading noted. The screen was then replaced and the zero again noted. The zero was always drifting, and the deflection was calculated from the mean of the readings before and after the deflection. The cell with the water was then replaced by the cell with the solution and the deflection obtained for the latter. The second reading, divided by the first, then gave 10^{-Acd}.

This method was, however, unsatisfactory on account of stray heat. Radiant heat, of course, is to a certain extent reflected from both lenses and scattered from the surfaces of the prism, and consequently the spectrum is not quite pure. As the deflections at one part of the spectrum used are fully twenty times the deflections at another part, diffuse heat from the first part may appreciably alter the reading at the second part. Attempts were made to eliminate the effects of this stray heat with water filters, ferrous ammonium sulphate filters, and six different kinds of coloured glass. Also screens were tried at different places inside the instrument. But all these attempts were unsuccessful.

I eventually removed this cause of error in a satisfactory way by resolving the light spectrally before it reached the slit, and this method

* "The Efficiency of Metallic Filament Lamps," *Proc. Roy. Soc. Edin.*, xxx. p. 555 (1910).

† E. Aschkinass, *Wied. Ann.*, lv. p. 401 (1895).

was used for obtaining all the results given in this paper. A single straight Nernst filament without a heater, of the kind known in Germany as Nernststiften an Brücken, was used as source. It took a current of one ampere, which was supplied by a battery of 113 storage cells. The current through the filament, and consequently its brightness, was thus very steady. The filament was shunted by a current balance and resistance for the purpose of checking the voltage on it. The light from the filament was made parallel by an achromatic lens; it then passed through a prism set approximately at minimum deviation, and was focussed by a second lens on the slit. By rotating the prism the various colours could be passed across the slit. The arrangement thus practically consisted of two spectroscopes in series, the filament itself being the slit of the first spectroscope. The cells and wooden screen were placed in front of the slit of the second spectroscope as before.

Another cause of error arose from the fact that the cells were usually not quite plane parallel but slightly prismatic, or perhaps they were not placed in position accurately. With the first arrangement this moved the image of the filament slightly off the slit and altered the deflection appreciably; but with the final arrangement the error was not so serious, and with a little care it could be avoided.

The salts experimented on were the fluoride, chloride, bromide, iodide, nitrate, and sulphate. They were obtained from Kahlbaum.

The curves on the next page, which give the results of three independent determinations of A for the nitrate, give an idea of the accuracy obtained. The solutions were made up independently, and different cells were employed for each determination.

The following table gives the results:—

Values of A for Cobalt Salts in the Infra-Red.

Scale Reading.	λ.	$d=3$ cm.	$d=1{\cdot}00$ cm.				
		$c=\cdot077$. Fluoride.	$c=\cdot67$. Chloride.	$c=\cdot65$. Bromide.	$c=\cdot417$. Iodide.	$c=\cdot67$. Nitrate.	$c=\cdot67$. Sulphate.
50° 25′	·645μ	·21	·41	·27	·53	26	·25
50 10	·684	·25	·24	·15	·25	·11	·14
49 58	·720	·14	·31	·15	·12	·10	·05
49 49	·750	·05	·28	·09	·11	·10	·03
49 40	·794	·12	·28	·12	·14	·12	·07
49 31	·850	·12	·28	·11	·12	·13	·12
49 22	·910	·15	·28	·14	·16	·18	·12
49 13	·980	·16	·40	·23	·30	·33	·22
49 4	1·070	·46	·70	·60	·61	·66	·59
48 55	1·170	·68	1·14	1·05	1·06	1·15	1·04
48 46	1·270	1·02	1·40	1·40	1·43	1·42	1·54

The figures for the iodide are the mean of two independent determinations, for the chloride the mean of four independent determinations, and for the others the mean of three independent determinations, taken in the same way as for the case of the nitrate. The value of c gives the concentration of the solution for which the determination was made, in gramme-molecules per litre. The thickness of the cell used is given by d.

When the chloride, bromide, and particularly the iodide solutions were

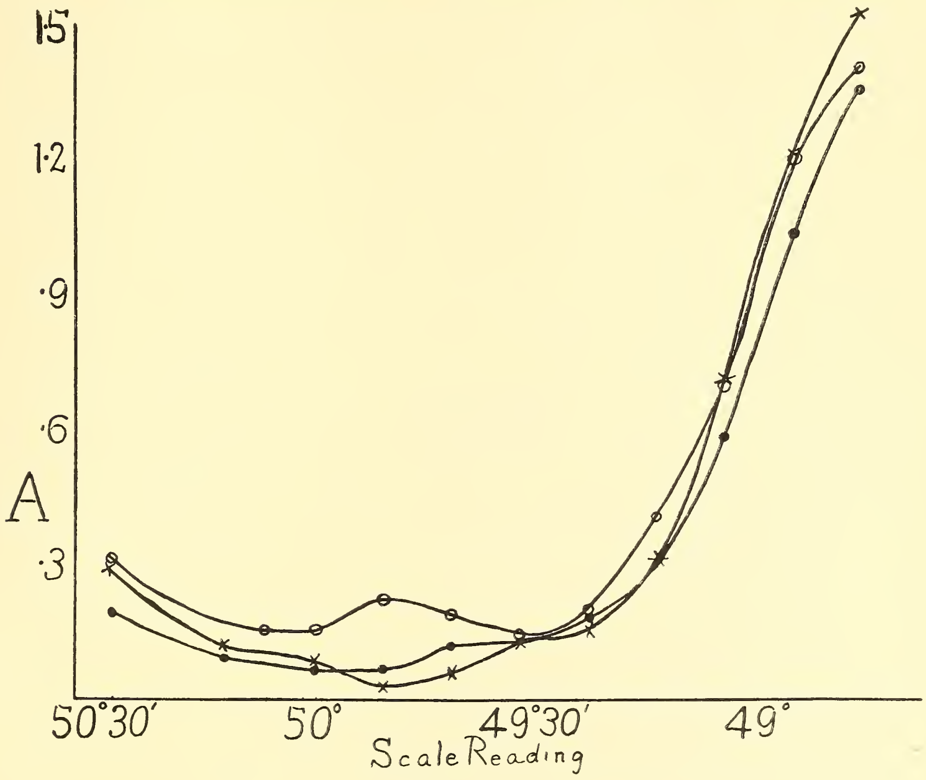


Fig. 1.

left standing some time a slight precipitate formed, which in the case of the first two salts was yellowish, and in the case of the iodide was chocolate-coloured. It sometimes made the solutions slightly turbid, and caused a noticeable increase in the value for A for those points for which it is small. On filtering the solution, however, A always dropped to its former low value. The solution of the fluoride used was a saturated one. The effective width of the thermopile was ·4 mm., the width of the slit was ·3 mm., except for the first two points, for which it was 1·0 mm. With a thickness of solution equal to 1 cm. it is not possible to go further into the infra-red owing to the absorption of the water itself.

After the results given in the above table were obtained, a determination of A was made for more concentrated solutions. It is, of course, well known that the absorption spectra of some cobalt salts undergo changes when the solutions become very concentrated. For more concentrated solutions thinner layers of solution must be used, 1 mm. and thereabouts. There is difficulty in measuring this thickness, and I thought at first that it might be possible to obtain A without knowing the thickness by using the arrangement illustrated in the diagram. M and M′ are mirrors. The solution is contained in a glass box, and the light passes vertically down through the bottom of the box. At first the box is filled with concentrated solution of strength c', depth d', and molecular extinction coefficient A′, and

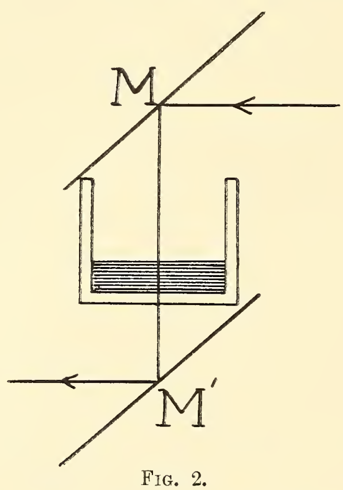


Fig. 2.

the deflection is noted. Some water is then added, the strength, depth, and molecular extinction coefficient change to c, d, and A, and the deflection is again taken. Any change in the deflection must be due to a change in A for $cd = c'd'$. It is easy to measure d, as it is much larger than d'; c can be obtained by comparing the tint of the solution spectrophotometrically with solutions of known strength; hence A′. This method failed, however, owing to a curvature of the surface of the liquid which was produced by capillary forces. The liquid thus acted as a concave lens of changing power, the curvature altering when more water was added.

I therefore used cells of 1 cm. inside thickness, putting parallel-sided pieces of plate glass inside them to diminish the thickness of the liquid, and determined the thickness of the layer with a reading microscope. The value of the smallest thickness could not be relied on to less than 4 per cent. by this method; consequently, the values of A obtained with it are uncertain to a constant factor—that is, they may have to be multiplied by something between 1·04 and 0·96 to make them correct. The changes produced by concentration are, however, in many cases much greater than 4 per cent. The following table gives the results:—

Values of A for more Concentrated Solutions.

Scale Reading.	λ.	$d=1$ mm.			$d=5$ mm.	
		$c=3·10$. Chloride.	$c=2·78$. Bromide.	$c=2·96$. Iodide.	$c=3·66$. Nitrate.	$c=2·00$. Sulphate.
50° 10′	·684μ	2·00	·67	·41	·20	·117
49 58	·720	0·41	·37	·10	·10	·078
49 49	·750	·37	·23	·12	·09	·057
49 40	·794	·16	·16	...	·12	·073
49 31	·850	·18	·23	·20	·16	·136
49 22	·910	·29	·15	...	·17	·114
49 13	·980	·38	·26	·23	·31	·211
49 4	1·07	·74	·70	·74	·66	·60
48 55	1·17	1·43	1·52	1·37	1·20	1·11
48 46	1·27	1·92	2·00	1·64	1·54	1·47
48 37	1·38	1·67	1·64	1·26	1·19	...

These results are the mean of two independent determinations in the case of the chloride, nitrate, and sulphate. One determination only was made in the case of the bromide and iodide. One cell with a piece of glass in it held the solution, another, with a similar piece of glass in it, the water; for the second determination, if two were made, the solution was put in the other cell. The fluoride is not given here, as the solution used for the last table was a saturated one. The solutions of the chloride and sulphate were saturated solutions prepared by leaving the water in contact with an excess of salt for days. For each wave-length the effective width of the thermopile and the width of the slit were the same as for the former concentration. The thinner layer of water has enabled us to gain an additional point in the infra-red, and it is seen that the end absorption has reached a maximum and is beginning to decrease. This conclusion is supported by results for the weaker solutions for the same wave-lengths, which were not given in the table on account of their uncertainty. It may be stated that two cobalt glasses examined gave an absorption band in the infra-red with a maximum at about 1·15μ. Abney and Festing* also find one band in the infra-red for cobalt glass, but they place it at 1·32μ.

It will be noticed that increased concentration always produces the same effect except in the case of the sulphate, namely, A increases at both ends of

* "Absorption-spectra Thermograms," *Proc. Roy. Soc.*, xxxviii. p. 77 (1885).

the region investigated and is practically unaltered for intermediate values of λ. In the red the increase is much more marked in the case of the chloride, then come in order of magnitude the bromide iodide and nitrate, and for the sulphate we have a very slight decrease.

After finishing the effect of concentration, I commenced to investigate the effect of temperature. The experimental difficulties usually met with here will be more fully discussed in one of the later papers of this series. It was resolved to keep the method as simple as possible. One cell was taken, of the same type as before, $3 \times 2 \times 1$ cm., inside measure; it was filled with solution, and if the strength of the latter made it necessary a glass block was also placed inside the cell. The cell was then placed in a thin strip of aluminium, as illustrated in the diagram. AB is a rubber band

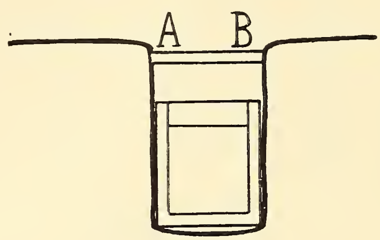


Fig. 3.

which keeps the strip tight. A copper vessel was filled with water, the cell placed in it with the horizontal parts of the strip resting on the edge of the vessel, and the upper surface of the water almost level with the top of the cell. The cell was heated by placing a bunsen below the copper vessel.

Before heating, the cell was placed before the slit and the deflection noted twice. The cell was then heated until the temperature of the solution as read by a thermometer was 80° C. It was then placed before the slit and the deflection noted twice, during which time the temperature usually fell to about 60° C. The cell was then cooled down to room temperature by placing it back in the heater and gradually cooling the water; two readings were then taken when it was cold. It was then heated a second time, other two readings taken, then again cooled to room temperature and a final two readings taken. The set of readings took an hour. From all these observations s and s', the mean deflections respectively for room temperature and for 70° C., were calculated. If A denote the molecular extinction coefficient at room temperature and A′ at 70° C.,

$$\frac{s}{s'} = \frac{10^{-\mathrm{A}cd}}{10^{-\mathrm{A}'cd}} = 10^{cd(\mathrm{A}'-\mathrm{A})}.$$

Hence

$$\mathrm{A}' - \mathrm{A} = \frac{1}{cd} \log s/s'.$$

As d was known, and as solutions were used for which A and c were known, A′ could thus be calculated.

The alternate heating and cooling was necessary. During a single heating evaporation takes place and there is a consequent increase of concentration. Bubbles form sometimes. Also, particularly in the case of the bromide and iodide, there is a formation of a precipitate. These changes take place in the one direction and have the same effect, no matter whether the solution is being heated or cooled. Hence, by taking the mean of the effects of heating and cooling, these disturbing influences cut out and we get the reversible change due solely to the difference of temperature of the solution.

Only four wave-lengths were taken, but they are sufficient to show the nature of the change. The fluoride was not experimented on, as it is decomposed by heating. Two different strengths of the chloride were used. The following table gives some results :—

Increase in A produced by Heating.

λ.	Chloride. c=·67.	Chloride. c=3·10.	Bromide. c=2·78.	Iodide. c=2·96.	Nitrate. c=3·66.	Sulphate. c=2·00.
·684	·34	7·9	3·7	·48	·12	08
·794	·00	·14	·07	·14	·02	·01
·980	·07	·07	·07	·28	·03	·01
1·27	·05	·40	·20	·19	0·12	·06

The concentrated solutions were taken in the case of the bromide, iodide, nitrate, and sulphate, as they gave more uniform results.

It is evident that heating has the same effect as increasing the concentration. It increases the absorption at both ends of the region investigated, the change in the red being most marked in the case of the chloride and the other salts following in the same order as before.

The experiments described in this paper were carried out in the Physical Laboratory of the University of Glasgow.

(*Issued separately July* 8, 1911.)

XXXVII.—The Absorption of Light by Inorganic Salts. No. II.: Aqueous Solutions of Cobalt Salts in the Visible Spectrum.

By **R. A. Houstoun**, M.A., Ph.D., D.Sc., Lecturer on Physical Optics, and **Alex. R. Brown**, M.A., Thomson Experimental Scholar in the University of Glasgow. *Communicated by* Prof. A. Gray, F.R.S.

(MS. received March 8, 1911. Read June 5, 1911.)

As in the case of the work in the infra-red, the salts experimented on were the nitrate, sulphate, chloride, bromide, fluoride, and iodide, and they were obtained from Kahlbaum. Very little trouble was experienced with the sulphate, nitrate, and chloride. The salts were supplied as $CoSo_4 7H_2O$, $Co(NO_3)_2 6H_2O$, $CoCl_2 6H_2O$. The fluoride was a pinkish powder sparingly soluble in water. It dissolved with formation of insoluble oxyfluoride which seemed to increase on standing. By heating the salt and driving off the water of crystallisation, it was found to be $CoF_2 2H_2O$. The iodide was an extremely deliquescent green powder. A slight insoluble chocolate-coloured residue remained in the solution and was filtered off. By heating the salt it was found to have the composition $CoI_2 2H_2O$. The bromide consisted of very deliquescent crimson lumps, and was found to have the composition $CoBr_2 6H_2O$.

The apparatus used was a spectrophotometer belonging to the Carnegie Trust for the Universities of Scotland, of a new type, which has already been fully described,* and which has been used for several photometric researches. In this instrument there are two slits illuminated by the same piece of the same incandescent mantle. Corresponding to the slits, there are formed two spectra, the one directly above the other, the intensities of which are matched by the rotation of a nicol in the eyepiece.

The absorption was specified as before by the molecular extinction coefficient which is defined by the equation

$$I = I_0 10^{-Acd},$$

where A is the molecular extinction coefficient, c the concentration of the solution in gramme-molecules per litre, d the thickness of the layer of solution through which the light passes, I_0 and I the intensities of the light before and after passing through the layer.

The dilute solutions were examined first. The usual method was to take two glass cells the internal thickness of which was 1 cm.; one was filled with water and the other with the solution. They were then placed one before each slit of the instrument, and the intensities of the light, which they transmitted, matched in two adjacent quadrants, the difference

* "A New Spectrophotometer of the Hüfner Type," *Phil. Mag.* (6) xv., p. 282, 1908.

of the nicol readings being $2\theta_1$. The cells were then interchanged and the readings repeated, the difference of the two settings being now $2\theta_2$. Then the value of Acd for the wave-length in question is given by

$$Acd = \log \frac{\tan \theta_1}{\tan \theta_2} \text{ or } \log \frac{\cot \theta_1}{\cot \theta_2}$$

according as θ_1 or θ_2 is larger.

Readings were taken in this manner for different wave-lengths throughout the spectrum, care of course being taken that all the readings were independent. That is, the range of wave-lengths compared never overlapped for two adjacent points in the spectrum.

In order to prevent error, solutions for each salt were made up at least twice, and for each salt three independent determinations of Acd were made throughout the spectrum. To give an idea of the accuracy usually obtained, all the determinations for the solution of cobalt nitrate are tabulated below:—

λ	*Acd.* 1st Set.	2nd Set.	3rd Set.	Mean.	λ	*Acd.* 1st Set.	2nd Set.	3rd Set.	Mean.
434 μμ	·24	·30	·28	·28	547	·61	·74	·69	·68
444	·41	·42	·45	·42	563	·36	·30	·35	·34
453	·54	·63	·66	·61	582	·20	·17	·12	·16
463	·83	·93	·88	·88	602	·13	·10	·10	·11
475	1·06	1·13	1·02	1·07	625	·11	·08	·07	·09
486	1·29	1·34	1·28	1·30	653	·11	·04	·05	·07
499	1·55	1·55	1·43	1·52	687	·08	·03	·03	·05
514	1·60	1·62	1·56	1·60	717	·05	·00	·01	·02
529	1·20	1·31	1·21	1·24					

The following table gives the results for all six salts:—

Values of A for Cobalt Salts in the Visible Spectrum.

λ.	$c=$ ·075. Fluoride.	$c=$ ·361. Chloride.	$c=$ ·303. Bromide.	$c=$ ·077. Iodide.	$c=$ ·305. Nitrate.	$c=$ ·306. Sulphate.
424 μμ	...	...	·68	...	...	...
434	·82	1·19	1·19	...	·91	1·06
444	1·10	1·54	1·80	27·6	1·39	1·82
453	1·55	1·99	2·32	20·2	2·00	2·55
463	2·07	2·69	3·26	15·6	2·91	3·43
475	2·65	3·29	3·80	11·0	3·51	4·02
486	3·08	3·77	3·89	9·20	4·29	4·69
499	3·44	4·60	4·86	7·97	4·97	5·53
505	...	5·16	...	...	...	...
514	3·57	4·87	5·18	7·08	5·23	5·85
529	3·00	3·82	4·03	5·86	4·64	4·70
547	1·75	2·17	1·94	4·06	2·24	2·72
563	·80	1·13	1·08	2·72	1·12	1·38
582	·43	·64	·45	1·20	·54	·72
602	·35	·44	·33	·95	·37	·56
625	·27	·38	·32	·70	·29	·45
653	·31	·35	·25	·60	·22	·35
687	·23	·25	·17	·51	·16	·24
717	·15	...	·11	...	·07	·25

They are represented by the curves, which, in order to facilitate comparison with the nickel salts, are printed in the third article of the series. To make the curves complete, the values of A for the infra-red and ultra-violet are added from the first and fourth articles of the series. The full consideration of the curves is postponed until the fourth article of the series. The values of A obtained with the spectrophotometer are represented by o s, the values obtained otherwise are represented by x s.

As in the infra-red, the bromide and iodide tended to become turbid on standing. The following three curves for the bromide illustrate this very nicely. The heavy curve was taken within twenty-six hours of making up the solution, and in this case the solution was filtered before taking

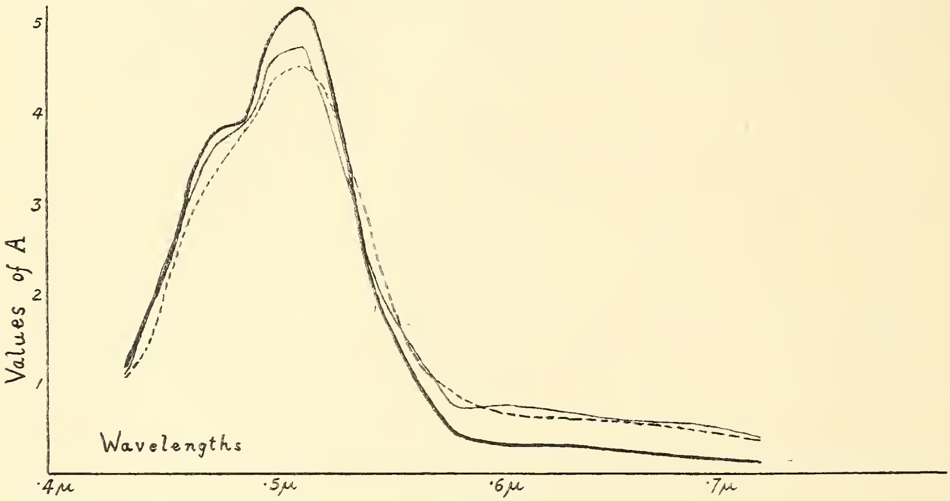


Fig. 1.

the readings. The light curve was made by diluting a stronger solution which had been previously filtered, and the readings were taken forty-eight hours after the solution was made. The dotted curve was obtained from the same diluted solutions four days later, the solution not being filtered during the interval. It will be noticed that the heavy curve is both highest at the maximum and lowest in the red, while the dotted curve is lowest at the maximum and higher in the red. This is easily explained. When the bromide decomposes, the solution absorbs less all over; A decreases everywhere in the same ratio; hence the decrease in the maximum value of A. But, at the same time, there is a turbidity produced which increases A by approximately the same amount throughout the visible spectrum. Hence the increase in the red. Instead of the mean, in this case the values represented by the highest curve were entered in the table.

Afterwards more concentrated solutions of the salts were experimented on. On account of the greater concentration it was necessary to use a thinner layer of solution. At first an attempt was made to make cells of a suitable thickness by taking two rectangular plates of mirror-glass 4 cm. by 3 cm., placing a U-shaped sheet of mica between them and cementing the glasses and mica together with seccotine. By using different sheets of mica the thickness of the cell could thus be varied at will. But the solutions seemed to make the seccotine swell, the solutions were sometimes affected by the seccotine and the cells sometimes leaked. So this method was abandoned.

More satisfactory results were obtained by using the same method as in the infra-red and placing rectangular blocks of glass in the cells so as to diminish the thickness of the layer of solution. The thickness of the block was obtained by the screw-gauge, the thickness of the cell with a microscope, and hence the thickness of the layer was determined.

The results tabulated below are, in the case of the sulphate and nitrate, the mean of three sets of readings. In the other cases only one set of readings was made:—

Values of A for more Concentrated Solutions.

λ.	$c=3{\cdot}10$. Chloride.	$c=2{\cdot}79$. Bromide.	$c=2{\cdot}96$. Iodide.	$c=1{\cdot}78$. Nitrate.	$c=1{\cdot}83$. Sulphate.
434 $\mu\mu$	2·89	1·56	...	1·09	·73
444	...	...	...	1·69	1·17
453	3·76	3·22	...	2·45	1·75
463	...	...	...	3·45	2·45
475	6·22	5·23	...	4·26	2·86
486	...	...	...	5·02	3·22
499	8·95	7·08	13·0	5·62	3·78
514	...	...	...	6·06	4·02
529	8·63	6·50	9·53	4·94	3·11
547	...	...	...	2·64	1·72
563	3·47	2·02	3·83	1·41	·87
582	...	...	...	·67	·39
602	1·38	·61	1·44	·44	·26
625	...	...	...	·39	·21
653	1·59	·45	·57	·29	·18
687	...	...	...	·23	·11
717	1·28	·40	·13	·15	·11

An attempt was then made to study the effect of heating. The difficulties here are evaporation, slow chemical changes, loosening of the cement, and cracking of the glass of the cells. We at first used a method which was a development of that formerly used by one of us.* The figure

* "Untersuchungen über den Einfluss der Temperatur auf die Absorption des Lichtes in isotropen Körpern," *Ann. d. Phys.* (4), xxi. p. 535, 1906.

represents the arrangement. A copper casting was made somewhat like a saucepan in shape, the inside being shaped cylindrically, of depth 6 cm. and of internal diameter 8 cm. There was a circular hole in the bottom of the hollow. The one beam of light was reflected by the mirror M_1,

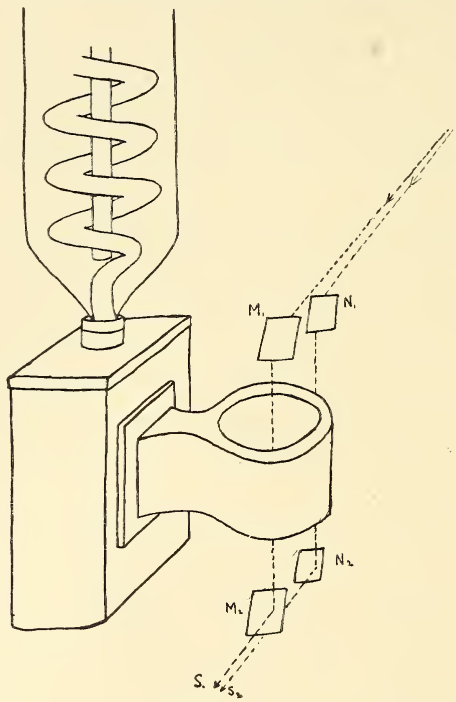


Fig. 2.

passed vertically downwards through the hole, was reflected by the mirror M_2 and arrived at the upper slit S_1; the other beam was reflected by the mirrors N_1, N_2, and entered the lower slit S_2. Two moulded glass cells with plane bottoms were fixed one inside the other, as shown in the figure, with

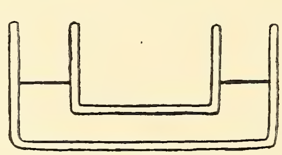


Fig. 3.

their bases parallel and so that the distance between them could be varied. The liquid investigated was contained between the two cells. There was a lid for the cylinder, not shown in the diagram, with a circular hole for admitting the light from the mirror M_1. Water or amyl alcohol was boiled in the tin and condensed in the condenser above, the heat flowed along the copper and thus the liquid was maintained at a constant temperature.

One advantage of this method is its simplicity and the very steady temperatures obtained by it. The bottoms of the moulded cells were, however, not plane enough for photometry, and these cells had to be given up. Square glass boxes were then tried, one inside the other, and some results were obtained. These boxes were, however, not very satisfactory, and the figures recorded in this paper were obtained by the simple method used in the infra-red. The apparatus illustrated in the diagram is being altered at present, and will be employed in a more accurate investigation of the effect of temperature and density which Mr Brown is at present undertaking.

In the simple method, then, which was chiefly employed here, the lower slit was kept free. The cell with the solution was placed before the upper slit and the difference of the nicol readings in two adjacent quadrants taken (2θ). The cell was then heated, placed again before the upper slit,

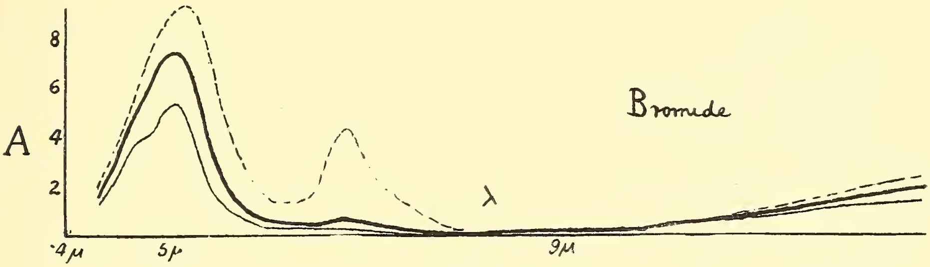


Fig. 4.

and the difference of the nicol readings in two adjacent quadrants again taken ($2\theta'$). Let A′ denote the molecular extinction coefficient at the higher temperature. Assume that 2θ, $2\theta'$ are the angles less than 90°. Let s be the fraction of the incident light transmitted by the glass cell when filled with water only, and let σ be the ratio of the intensity of the light falling upon the upper slit to the intensity of the light falling upon the lower slit when both slits are free. Then

$$s\sigma 10^{-\mathrm{A}cd} = \tan^2\theta\,;\ s\sigma 10^{-\mathrm{A}'cd} = \tan^2\theta'.$$

Hence

$$10^{cd(\mathrm{A}'-\mathrm{A})} = \frac{\tan^2\theta}{\tan^2\theta'},$$

$$\text{or } \mathrm{A}' - \mathrm{A} = \frac{2}{cd}\,(\log\tan\theta - \log\tan\theta').$$

As c and d are known, the increase in A is thus easily determined.

In order to eliminate irreversible changes, the mean of repeated heatings and coolings was taken. The average temperature of the hot solution was

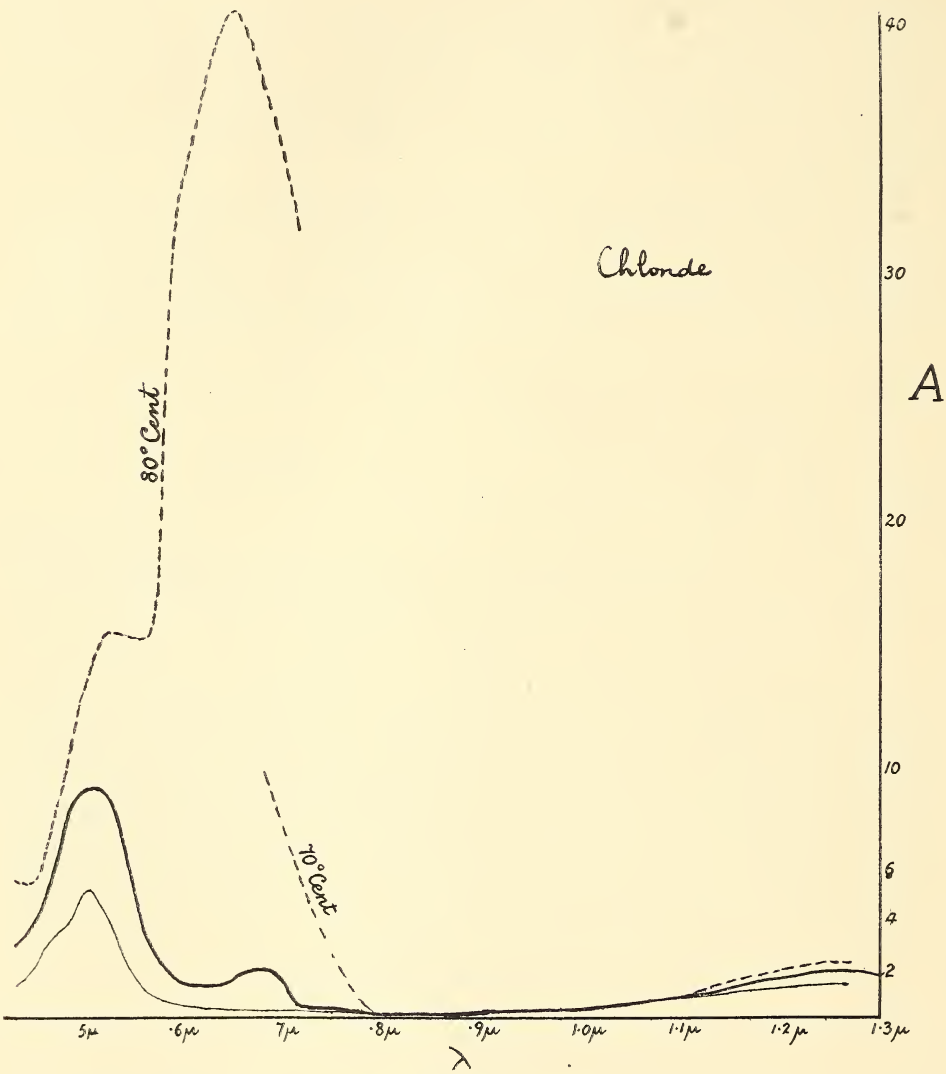

Chloride
80° Cent
70° Cent
A
40
30
20
10
6
4
2
·5μ
·6μ
·7μ
·8μ
·9μ
1·0μ
1·1μ
1·2μ
1·3μ
λ

Fig. 5.

about 80° C., as against 70° C. in the case of the work in the infra-red. The results are given below :—

Increase in A produced by Heating (A′ − A).

λ.	$c=3·10$. Chloride.	$c=2·79$. Bromide.	$c=2·96$. Iodide.	$c=3·66$. Nitrate.	$c=2·00$. Sulphate.
434μμ	2·68	·33	...	·93	·04
453	1·72	·43	...	− ·17	·40
475	2·36	·90	...	+ ·50	·61
499	3·63	1·30	...	2·15	·59
529	6·87	2·58	...	2·92	1·42
563	11·8	2·44	1·97	4·28	·94
602	32·6	·97	·74	1·31	·18
653	38·9	1·42	·42	·12	·09
717	30·4	1·69	·35	·73	− ·13

The change is very striking in the case of the chloride, the bromide, and the nitrate. For the chloride and bromide the values of A for the cold dilute, the cold concentrated and the hot concentrated solutions are represented in figs. 4 and 5, the values from the infra-red being added from the first paper of the series. In both diagrams the values for the cold dilute solution are shown by a light line, the values for the cold concentrated solution by a heavy line, and the values for the hot concentrated by a dotted line.

(*Issued separately July* 8, 1911.)

XXXVIII.—The Absorption of Light by Inorganic Salts. No. III.: Aqueous Solutions of Nickel Salts in the Visible Spectrum and the Infra-Red.

By **R. A. Houstoun**, M.A., Ph.D., D.Sc., Lecturer on Physical Optics in the University of Glasgow. *Communicated by* Prof. A. Gray, F.R.S.

(MS. received March 8, 1911. Read June 5, 1911.)

The methods and the apparatus were the same as in the two previous articles of the series and hence need not be described, the only change being in connection with the galvanometer. Previously it had rested on the table, but when half the results in the infra-red recorded in this paper were obtained, the arrangement was changed; it was suspended by three iron wires each about one and a half metres long from a bracket in the wall and hung with its three levelling screws clearing the table by about one centimetre. Between the table and levelling screws were placed loose wads of cotton wool for the purpose of damping any vibrations that might arise. With this arrangement the zero is very much less sensitive to vibration. With the lamp and scale at one and a half metres—the distance at which they are ordinarily used now—vibrations of the laboratory rarely cause the zero to move $\frac{1}{4}$ mm. either way.

Some trouble was caused by the iron wires of the thermopiles rusting through.

The salts were obtained from Kahlbaum. The nitrate, sulphate, and chloride, which were supplied as $Ni(NO_3)6H_2O$, $NiSO_47H_2O$, and $NiCl_26H_2O$, gave no trouble. The bromide was a yellowish-brown powder with a very slight greenish tint in some parts. By heating so as to drive off the water of crystallisation it was found to be $NiBr_23H_2O$. It dissolved very readily with froth, gave a turbid green solution, and on filtering left some yellow precipitate on the paper in analogy with cobalt bromide. Dilute solutions also became slightly turbid on standing some time. The iodide was of an iron-black colour. By driving off the water of crystallisation it was found to be NiI_26H_2O. It dissolved frothily, and the solution looked turbid. On filtering, a brown solution was obtained and a chocolate-coloured precipitate similar to the precipitate in the case of cobalt iodide. The weight of the precipitate could be neglected in comparison with the weight of the dissolved salt. The solution did not keep well. For example, in three months in a normal solution exactly half the dissolved salt decomposed, giving the chocolate-brown precipitate referred to. The fluoride was a light green powder sparingly soluble in cold water. It was taken as $NiF_2 3H_2O$.

The following table gives the results for the visible spectrum:—

Values of A for Nickel Salts in the Visible Spectrum.

λ.	$d = 2$ cm.	$d = 1$ cm.				
	$c = \cdot 140$.	$c = 2{\cdot}27$; first five and last two pts. $\frac{1}{3}$ strength.	$c = 1{\cdot}40$; first four and last pt. $\frac{1}{2}$ strength.	First ten pts. $c = \cdot 488$, next three $\frac{1}{3}$ strength, last four $\frac{1}{12}$ strength.	$c = 1{\cdot}56$; first five and last pt. $\frac{1}{3}$ strength.	$c = 1{\cdot}398$; first three and last pt. $\frac{1}{2}$ strength.
	Fluoride.	Chloride.	Bromide.	Iodide.	Nitrate.	Sulphate.
717$\mu\mu$	1·83	2·07	2·06	2·17	1·87	2·03
687	1·59	1·88	1·91	1·99	1·65	2·04
653	1·57	1·78	1·76	2·00	1·57	1·93
625	1·24	1·20	1·22	1·66	1·16	1·27
602	·71	·68	·72	1·41	·64	·70
582	·39	·353	·40	1·48	·363	·41
563	·27	·204	·237	1·80	·206	·21
547	·30	·137	·172	2·10	·139	·15
529	·22	·100	·141	2·58	·089	·12
514	·24	·064	·108	3·10	·063	·07
499	·19	·049	·082	4·5	·046	·057
486	·26	·108	·120	7·0	·083	·102
475	·32	·222	·237	11·6	·174	·174
463	·45	·390	·39	13·2	·299	·341
453	·61	·580	·52	18·8	·427	·460
444	·68	·62	·73	29·0	·58	·63
434	...	1·12	1·12	33·7	1·04	·88

In every case except the sulphate the values in the table are the mean of two independent determinations. For the sulphate only one determination was made. The values are not for the same value of c throughout the spectrum. Towards the ends where the absorption is greater the solutions were diluted. It would have been less open to objection to keep c constant and alter d, but this would have entailed more trouble. The variation of A with c for these points was investigated afterwards. Owing to the solutions being so weak, the values for the fluoride did not agree nearly so well with one another as in the case of the other salts. The values of A for adjacent wave-lengths are, of course, fully independent; the successive strips of spectrum taken do not overlap.

Two of these solutions have been investigated with a spectrophotometer

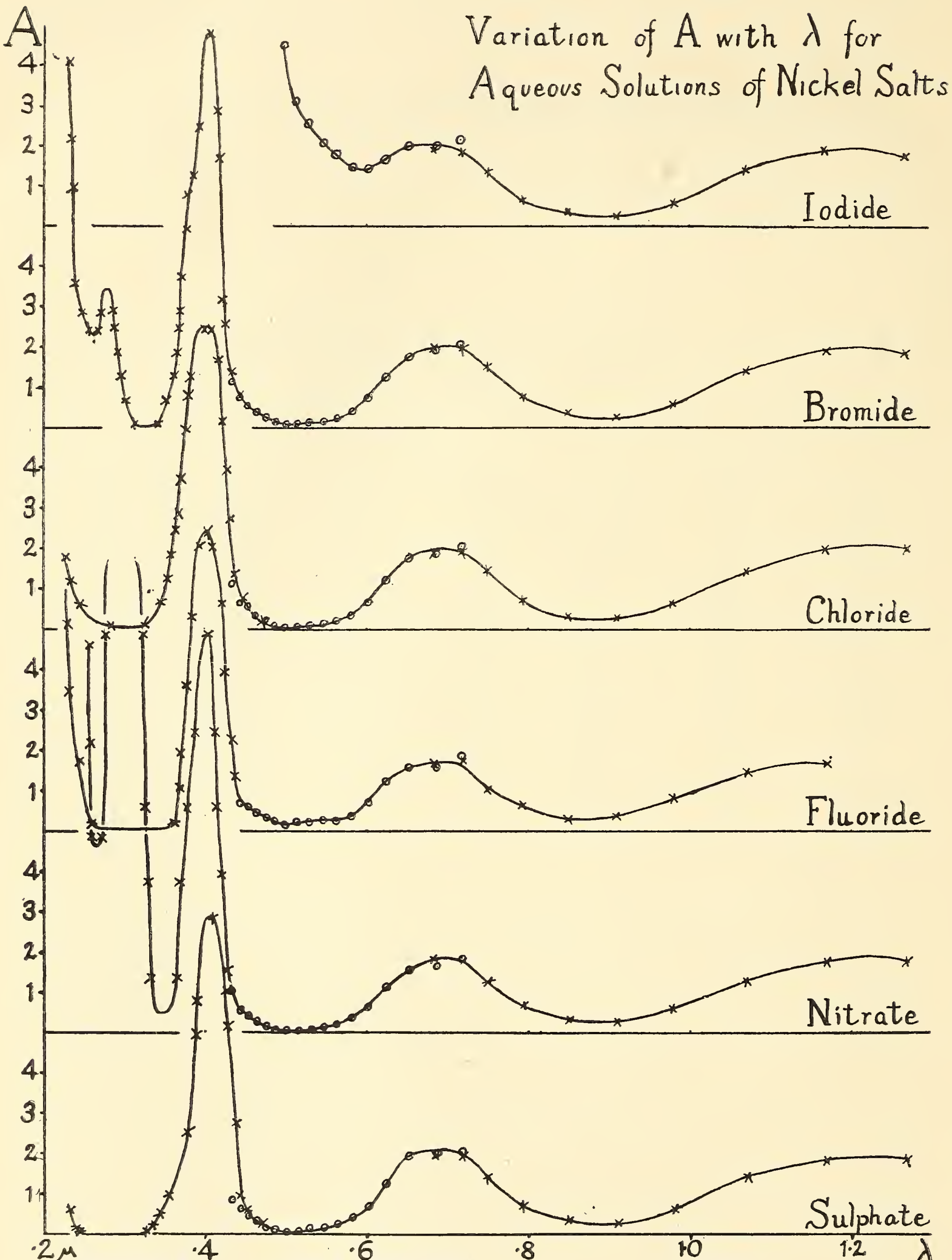

Variation of A with λ for
Aqueous Solutions of Nickel Salts
A
Iodide
Bromide
Chloride
Fluoride
Nitrate
Sulphate
·2 μ
·4
·6
·8
1·0
1·2
λ
ULTRA-VIOLET
VISIBLE SPECTRUM
INFRA-RED

Fig. 1.

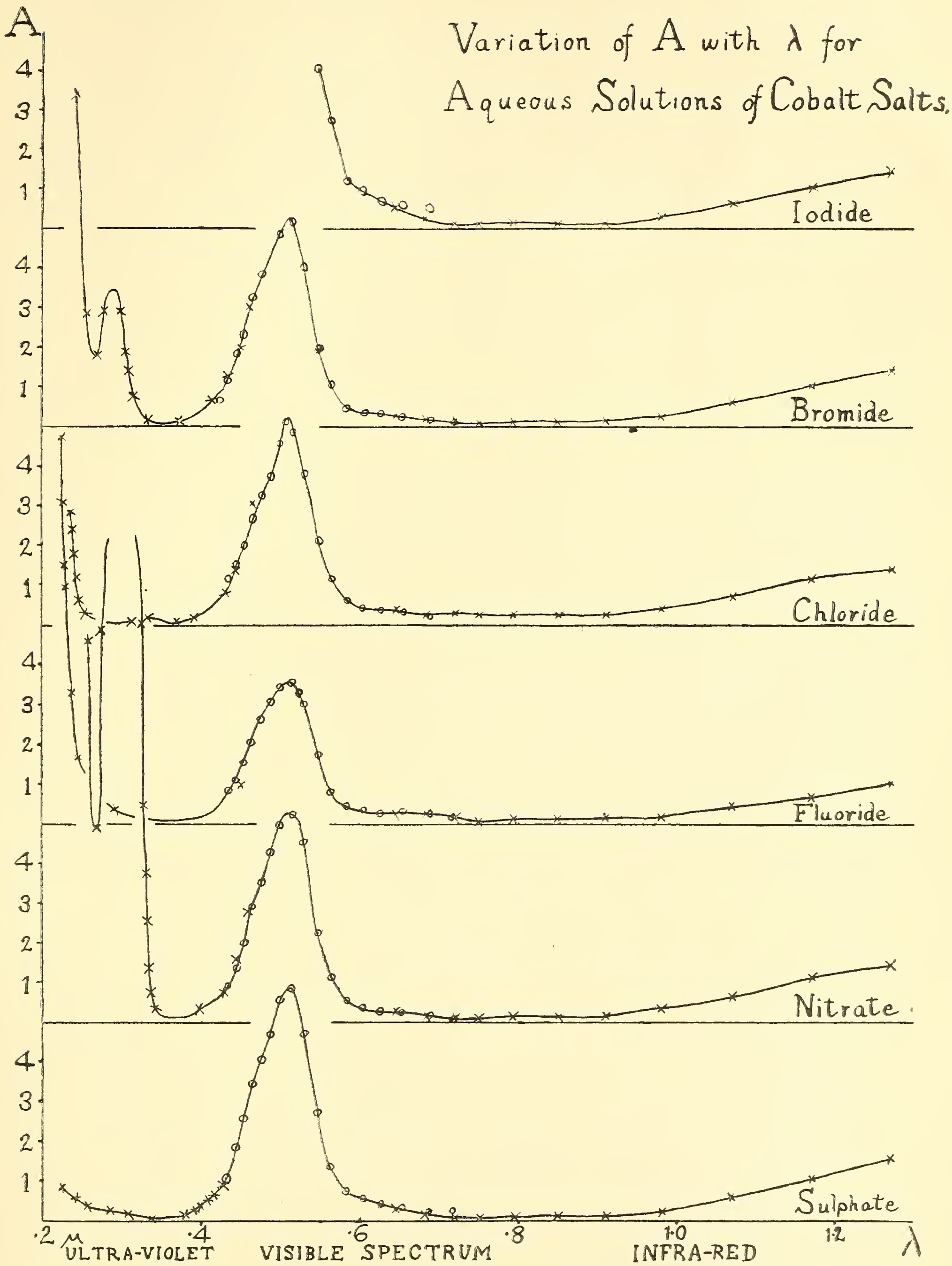

Variation of A with λ for
Aqueous Solutions of Cobalt Salts.
A
λ
Iodide
Bromide
Chloride
Fluoride
Nitrate
Sulphate
·2 μ
·4
·6
·8
1·0
1·2
ULTRA-VIOLET
VISIBLE SPECTRUM
INFRA-RED

Fig. 2.

before, the nitrate by E. Müller * for $c = \cdot 053$ but only for the middle of the spectrum, and the sulphate by Nichols and Spencer,† E. Müller ‡ and A. S. Russell and myself.§ As this formed a means of testing the accuracy of my work, I plotted the figures for the sulphate in the following diagram. Nichols and Spencer give only four points, and the accuracy of their work is probably not intended to be great; I have therefore not used it. Müller's figures are for $c = \cdot 211$, and are shown as black discs. Our former results, which are for $c = \cdot 408$, are shown as x *s*, and the figures recorded in this

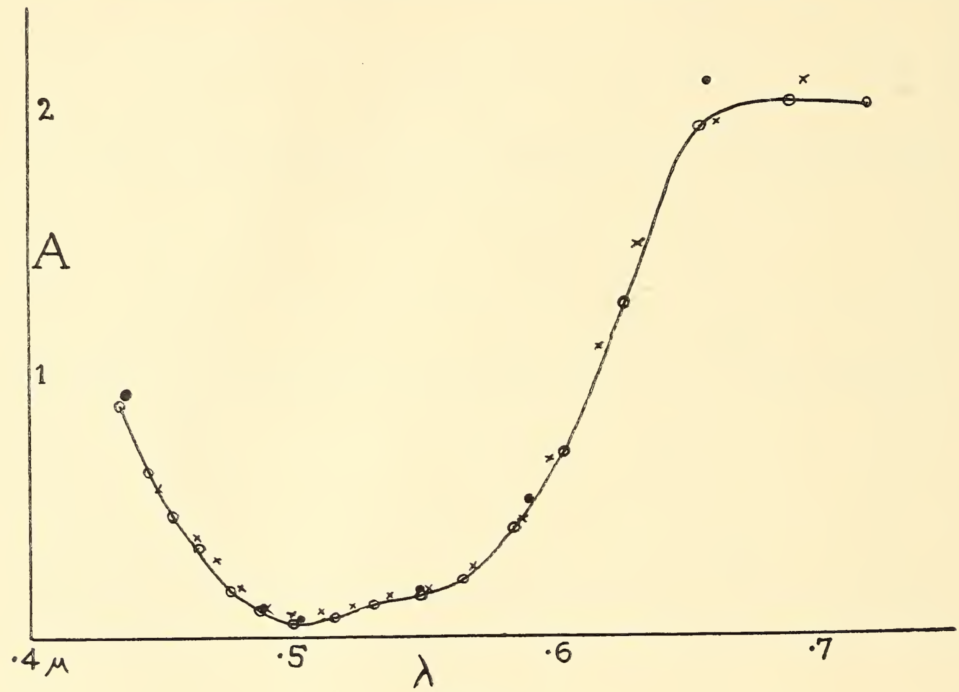


FIG. 3.

paper are plotted as o *s*. || The agreement is satisfactory. Our former results were, of course, made with the same instrument, but during the interval it has been taken to pieces once or twice.

* "Ueber die Lichtabsorption wässriger Lösungen von Kupfer und Nickelsalzen," *Ann. d. Phys.* (4), xii. p. 767, 1903.

† E. L. Nichols and Mary C. Spencer, "The Influence of Temperature upon the Transparency of Solutions," *Phys. Rev.*, ii. p. 344 (1895).

‡ "Untersuchungen über die Absorption des Lichts in Lösungen," *Ann. d. Phys.* (4), xxi. p. 575, 1906.

§ "On a Question in Absorption Spectroscopy," *Proc. Roy. Soc. Edin.*, xxix. p. 68 (1908).

|| In the former paper the values of c, for which the different values of A are determined, were not given. They are, for copper sulphate ·226, cobalt chloride $c = \cdot 452$, potassium dichromate ·039, and uranyl nitrate ·9906 (last eight points $\frac{1}{5}$ this strength).

The following table gives the results for the infra-red :—

Values of A for Nickel Salts in the Infra-Red.

λ.	$d=3$ cm.	$d=1$ cm.				
	$c=\cdot140$. Fluoride.	$c=\cdot757$. Chloride.	$c=\cdot303$. Bromide.	$c=\cdot488$. Iodide.	$c=\cdot52$. Nitrate.	$c=\cdot35$. Sulphate.
·684μ	1·68	1·9	1·97	1·88	1·82	1·96
·720	1·74	1·9	1·96	1·82	1·73	1·97
·750	1·01	1·46	1·50	1·37	1·26	1·41
·794	·63	·71	·78	·65	·65	·70
·850	·28	·33	·39	·33	·32	·34
·910	·37	·30	·27	·285	·24	·28
·980	·84	·62	·59	·59	·58	·61
1·07	1·48	1·46	1·40	1·40	1·28	1·42
1·17	1·67	2·00	1·90	1·89	1·76	1·83
1·27	...	1·99	1·88	1·75	1·75	1·81

The values of A given in this table and the preceding one are represented by fig. 1. To make the curves complete, the values for the ultra-violet are added from the fourth article of this series. To facilitate comparison the similar set of cobalt curves is printed in this article in fig. 2.

The effects of increasing the concentration and the temperature were also studied. The methods used were the same as for the cobalt salts and hence need not be described. The results are given in the following tables and represented by the following curves, the heavy line giving the increase in A due to concentration and the dotted line the increase due to heating.

Values of A for more Concentrated Solutions. Visible Spectrum.

λ.	$d=1$ mm.	
	$c=4\cdot09$. Chloride.	$c=3\cdot19$. Bromide.
717$\mu\mu$	2·16	2·03
653	1·68	1·73
602	·58	·84
563	·18	·38
529	·05	·43
499	·06	·44
475	·33	·49
453	·94	·86
434	2·65	1·78

λ.	$d=1$ mm.
	$c=4\cdot15$. Nitrate.
687$\mu\mu$	2·12
625	1·36
582	·35
547	·09
514	·00
486	·04
463	·35
444	·73
...	...

Values of A for more Concentrated Solutions. Infra-Red.

λ	$d=1$ mm.	
	$c=4{\cdot}09$. Chloride.	$c=3{\cdot}19$. Bromide.
·684μ	2·29	2·20
·720	2·58	2·38
·750	1·68	1·32
·794	1·01	·73
·850	·45	·40
·910	·39	·51
·980	·89	1·02
1·07	1·70	2·07
1·17	2·46	2·53
1·27	2·03	2·03
1·38	1·33	1·11

Increase in A produced by Heating.

λ	$d=1$ mm.	
	$c=4{\cdot}09$. Chloride.	$c=3{\cdot}19$. Bromide.
·717μ	·17	·24
·602	− ·07	·07
·529	+ ·00	·14
·475	·49	·35
·434	2·00	1·32
·684μ	·21 ?	·49 ?
·794	·81	·32
·980	− ·02	− ·04
1·270	+ ·28	+ ·40

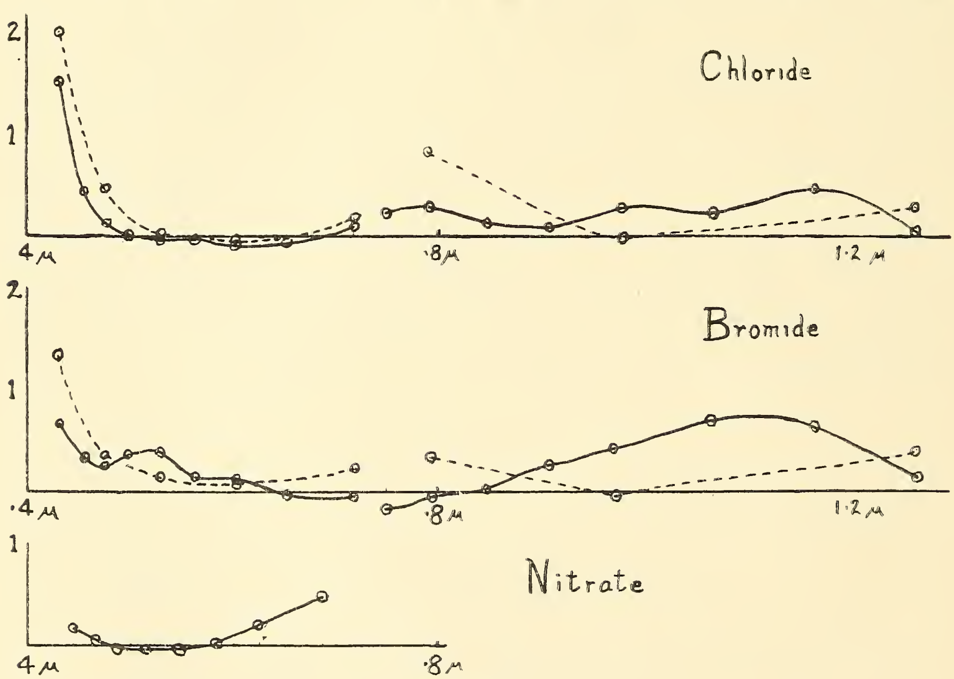


Fig. 4.

The concentrations used above are near saturation. An almost saturated solution of the sulphate was prepared, for which c was equal to 2·10, but when it was examined with the spectrophotometer no change in A could be detected.

The values obtained with the thermopile for $\cdot 684\mu$ and $\cdot 720\mu$ are not given by the curves, as, owing to the smallness of the deflections, together with the great absorption in that region, they are not accurate.

The striking feature brought out by the above curves is the great increase in absorption in the violet in the case of the chloride and bromide produced by increase of concentration and by heating. Yet we have never been accustomed to associate nickel chloride and bromide solutions with colour changes as we do the corresponding cobalt salts. This is principally

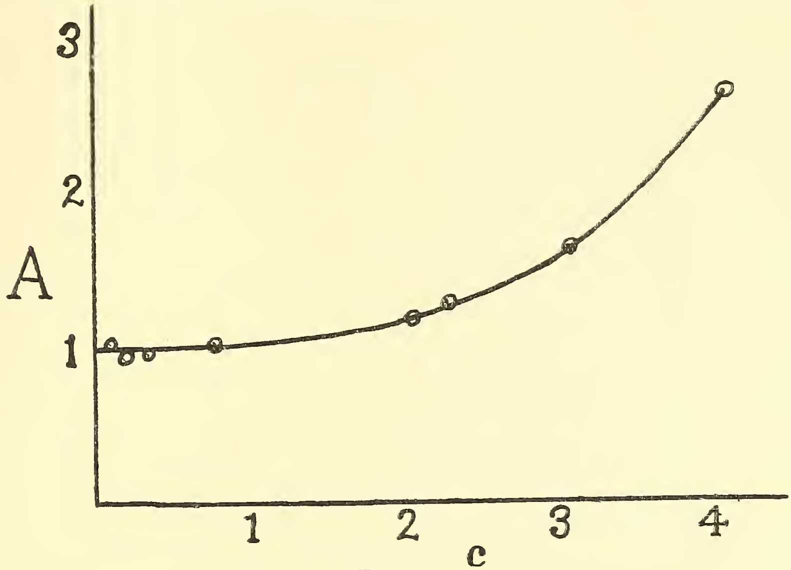


Fig. 5.

due to the changes in the cobalt salts being greater, but partly to our eyes not being sensitive to violet light.

In order to study this change further, I made up a set of solutions of nickel chloride of gradually increasing concentration and determined A as a function of c for the wave-length $\lambda = 434\mu\mu$. The results are given in the following table, and illustrated by fig. 5.

c.	A.
4·09	2·65
3·07	1·64
2·27	1·27
2·05	1·17
·76	1·03
·34	·96
·17	·94
·085	1·04 ?

The difference between the last two values of A in the table is less than the error of observation, which in this case is greater than usual owing to the solution being so dilute and the wave-length being so far in the violet. A is therefore constant for small values of c, but increases rapidly as the solution approaches saturation. The change has thus apparently nothing to do with ionisation. The question of the cause of the change, together with the similar colour change of the cobalt salts, is being held over for more exhaustive investigation.

The effect of temperature upon very dilute solutions of the sulphate and chloride has already been investigated with a spectrophotometer by Nichols and Spencer.* No change was found in the case of the sulphate, but in the case of the chloride in the middle of the spectrum a permanent increase of absorption was obtained. Their original value of A is, however, too high; possibly the solution was slightly turbid to begin with, and this turbidity was increased permanently by the heating.

* E. L. Nichols and Mary C. Spencer, "The Influence of Temperature upon the Transparency of Solutions," *Phys. Rev.*, ii. p. 344 (1895).

(*Issued separately July* 8, 1911.)

XXXIX.—The Absorption of Light by Inorganic Salts. No. IV.: Aqueous Solutions of Cobalt and Nickel Salts in the Ultra-Violet.

By **R. A. Houstoun**, M.A., Ph.D., D.Sc., Lecturer in Physical Optics, and **John S. Anderson**, M.A., B.Sc., Carnegie Scholar in the University of Glasgow. *Communicated by* Prof. A. Gray, F.R.S.

(MS. received March 8, 1911. Read June 5, 1911.)

In Kayser's *Spectroscopie*, vol. iii. pp. 45–49, an account is given of all the different methods that have hitherto been employed for photometry in the ultra-violet part of the spectrum. Photography, phosphorescent plates, selenium cells, and the ionising effect of ultra-violet light have all been used, but with limited success, and Professor Kayser considers that the only really practical method is that which has been recently introduced by Pflüger. Pflüger has discovered that there is relatively an enormous amount of energy in the ultra-violet spectrum of the electric spark produced by the discharge of a condenser between metal electrodes. He therefore uses the electric spark as a source, and takes the deflections with a thermopile and galvanometer, just as in the infra-red. He finds the deflections to be wonderfully steady, considering the inconstant nature of the spark.

Pflüger's method requires a powerful induction coil and a very sensitive galvanometer. The universal method in the ultra-violet is photography. The latter involves much less expense. We felt, therefore, that it would be of great advantage if a simple and practical spectrophotometer, suitable for use throughout the ultra-violet, could be designed and built. We set ourselves this task, and after some trouble have been completely successful.

The density of the image on a photographic plate is not proportional to the time of exposure, even within the range of the plate. Nor, when two beams of light produce the same density on the same plate in different times, are their intensities inversely proportional to the time of exposure. Work of a kind can be done in such cases by making auxiliary experiments, but it is unsatisfactory. Only when two beams of light of the same wavelength fall on adjacent parts of the same photographic plate and produce equal blackening in the same time can we say that their intensity is equal. This principle is essential. It was adopted from the first, and made the basis of all our experiments.

The first step was to find a means of diminishing the intensity of a beam of light. The one beam under comparison must, of course, go through the cell with the solution, and its intensity is diminished; we must have a means of diminishing the intensity of the other beam to the same value, and at the same time of letting us know in what ratio it is diminished. Nicol's prisms are of no use for this purpose, for Iceland spar and Canada balsam absorb all wave-lengths below $\cdot 300\mu$. Neither can a micrometer slit be used as in Vierordt's spectrophotometer. The brightness of a continuous spectrum can, it is true, be diminished by diminishing the width of a slit, but there are no continuous spectra below $\cdot 300\mu$. In this region the only possible source of light is the iron arc or the electric spark, and these sources give line spectra. And if we have two line spectra, one above the other, and gradually close the one slit, we only diminish the width of the lines, not their brightness. A rotating sector cannot be used, because it does not diminish the intensity of the beam, but only the time during which it acts. The effect is the same as far as the eye is concerned, but it is different with a photographic plate. Twice the intensity for half the time does not produce the same density.

We were therefore compelled to fall back upon the inverse square law, and in order to study its possibilities a preliminary experiment was made with a spectrograph with glass prism and lenses. A ground glass plate was fixed permanently 14 cm. in front of the slit. This ground glass plate was large enough for the rays from it to fill the full aperture of the collimator. A small inverted incandescent mantle was mounted so that it could be moved back and forward in the line of the collimator, but always at such a distance in front of the ground glass that the intensity of illumination on the latter could be regarded as inversely proportional to the square of their relative distance. Draw-slides were placed across the upper and lower halves of the slit. A glass cell with the solution to be investigated was placed before the draw-slides, between the latter and the ground glass. The incandescent mantle was placed at a standard position near the ground glass, one-half of the slit was opened, and an exposure made. The cell was then replaced by a similar cell filled with water, the relative distance of incandescent mantle and ground glass increased—say doubled—and another exposure made for exactly the same time, the other half of the slit being used. If we assume, now, that the intensity of the incandescent mantle remains unaltered, the intensity of illumination in the second case is only one-quarter what it was in the first case. If, therefore, we seek out those wave-lengths for which the density on the adjacent spectra is equal, we obtain the wave-lengths for which one-quarter of the incident light is transmitted through the solution, *i.e.* for which $10^{-Acd} = \frac{1}{4}$. Hence Acd.

This procedure was repeated then for other ratios of the relative distances of mantle and ground glass, and thus A was determined for different wave-lengths.

This method was tested for cobalt fluoride by comparison with the spectrophotometer for the blue and green regions of the spectrum, and although perhaps not quite so accurate as the latter, it proved very satisfactory. It might be thought that it would be slow and laborious. But this is not the case; the exposures only took three minutes. Twelve could be taken on one plate, and if we were dealing with an absorption band, one exposure could give A for two points, one on each side of the band. This apparatus seems eminently suited for doing the spectrophotometry of the salts of the rare earths, which has as yet not been attempted. The

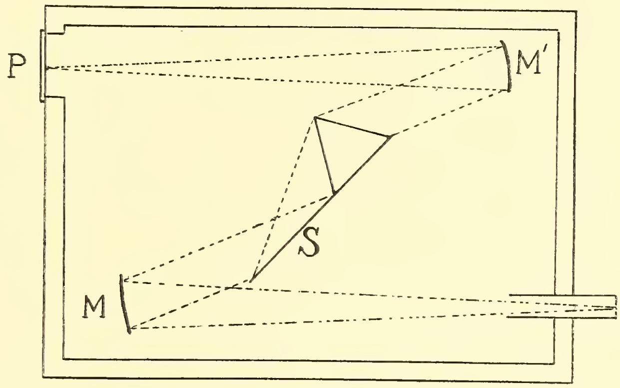


Fig. 1.

bands of the rare earths are so narrow that the ordinary spectrophotometers cannot very well be applied to them.

When we were convinced of the suitability of the inverse square method we turned our attention towards the necessary quartz spectrograph. This we built ourselves, partly in the interests of economy and partly because we thought we could improve on the usual type. The diagram shows our first spectrograph; the lenses are replaced by concave mirrors M, M′. The light from the slit is made parallel by M, falls on a plane mirror S, passes through the prism at minimum deviation, and is focussed by M′ on the photographic plate P. This spectrograph will be described when it is fully successful, as we are convinced of its advantages. But until now we have had trouble with the mirrors; silver does not reflect the ultra-violet well. Considerable time was wasted in waiting for nickel mirrors; when they came, we found they were silver mirrors with nickel deposited electrolytically on them and then all polished off again. As we could not afford to wait longer, we built a spectrograph of the usual type with quartz lenses, and it is with it that the results recorded in the present paper are obtained.

The incandescent mantle is not suitable as a source of light in the ultra-violet. When we attempted to replace it with an electric spark we could not get the latter bright enough with the means at our disposal. The iron arc, though brighter, was not found steady enough. So we came to the conclusion that it was not advisable to make the comparison exposures one after the other. They must be made simultaneously. We had therefore to recast our whole photometric arrangement. The diagram shows how this was done.

The arrangement is easier to understand if we suppose the rays to be going in the reverse direction, from the camera to the slit. SS′ is the slit of the collimator. PQRT is a quartz prism of special design in which PQ = PT and QR = TR. The sides PQ and PT are each 2 cm., angle P = 70°,

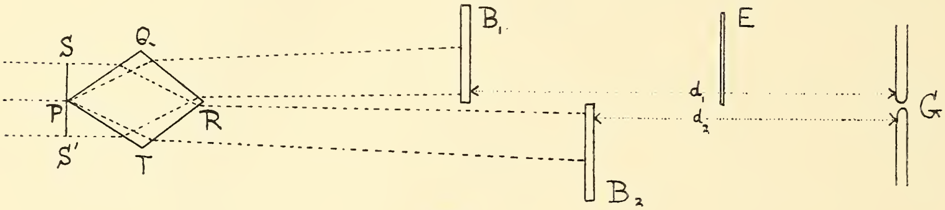


Fig. 2.

angle R = 75°, and the axis of the quartz is perpendicular to the plane of the diagram. B_1 and B_2 are two square plates of ground quartz with their planes perpendicular to the plane of the figure; they can be moved backwards and forwards in the line of the axis of the collimator. G is the iron arc; it is shown vertical in the diagram, although in reality it was horizontal. Now consider all the rays that can be drawn from the collimator lens to the slit. They are divided into two beams by the sharp edge P of the prism, and every ray of each of these beams after refraction through the prism PQRT meets one of the ground quartz plates, no matter at what point of its range the latter happens to be. Hence, when the light travels in the proper direction, from the arc to the slit and thence to the camera, the brightness of either of the two spectra formed by the two halves of the slit does not vary with the distance of the quartz plate from the slit, but only with the state of illumination of the latter. Hence it varies as the inverse square of the distance of the quartz plate from the arc.

The illumination of B_1 is proportional to $1/d_1^2$ and that of B_2 to $1/d_2^2$. The amount of light falling from B_2 on B_1 is inappreciable in comparison with the amount received by B_1 direct from the arc. If B_1 and B_2 are at their extreme positions, namely, 20 and 10 cm. respectively, the illumination of B_1 is $\frac{1}{4}$ that of B_2. In order to increase the effective range of the photometer a piece of wire mesh E is placed between the arc G and the quartz plate B_1. Several pieces of mesh were treated with nitric acid for different lengths of time, and that one was chosen which allowed about one-quarter of the incident light to pass through it. Thus the intensities of illumination of B_1 and B_2 are the same when $d_1 = 10$ cm. and $d_2 = 20$ cm., and the illumination of B_2 is sixteen times that of B_1 when $d_1 = 20$ cm. and $d_2 = 10$ cm.

At first, instead of using the mesh, a screen was pushed into the upper beam from one side at a point close up to the prism, in order to diminish the intensity. It displaced the lines in the spectra, however, and thus did not prove satisfactory.

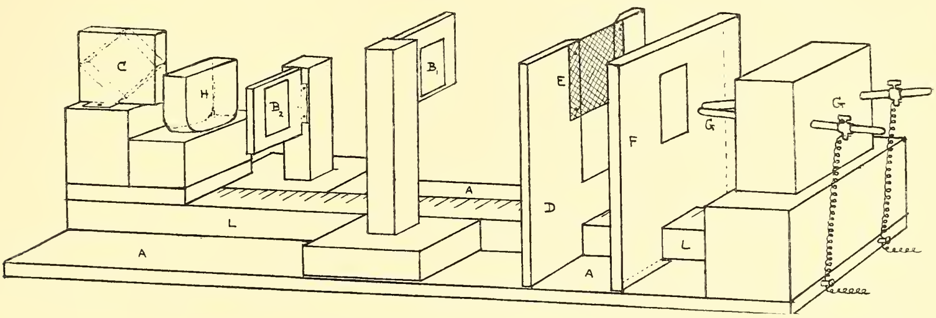


Fig. 3.—Quartz photometer.

The essential parts of the photometer are shown in position in the above diagram. B_1 and B_2 are the two square plates of ground quartz; they rest in metal supports fixed to upright wooden stands, which can be moved backwards and forwards alongside the horizontal scale L. The metal supports are so made as to serve the additional purpose of screening light off all but the necessary portion of each of the quartz plates. C is the quartz prism of special design, enclosed in a metal frame. D is an upright wooden screen with a rectangular slot in it, and E is the wire mesh nailed across the upper half of the slot. A zinc screen F is placed in front of the iron arc G which is formed by two horizontal converging rods passing through a plaster of Paris mould. H is a quartz cell for holding the solution to be examined. All the woodwork connected with the photometer and all the metal supports were painted a dead black.

Imperial Special Rapid plates were used. With exposures of about $\frac{1}{2}$ min. and a slit width of about 1 mm. a fairly continuous spectrum extending to $\lambda = \cdot 233\mu$ was obtained with the iron arc as source, and with slightly longer exposures $\lambda = \cdot 223\mu$ was reached. Spark lines were obtained down to $\cdot 202\mu$ (Zn 29). The length of the iron spectrum on the plate from $\cdot 233\mu$ to about $\cdot 530\mu$ was about 4 cm. Eleven exposures could be obtained on one plate.

The method of working was as follows:—The iron-arc spectrum was first of all calibrated by means of the cadmium-zinc spark and a table drawn up by which the wave-length of any portion of it could be read. The ratio of the light transmitted by the mesh to the light transmitted by the quartz cell filled with water and placed before the lower face of the prism was next determined from the positions of the quartz plates for equal density on the photographic plate. This ratio varied slightly throughout the spectrum, but the latter could be conveniently divided into two regions, for each of which the ratio could be taken as constant. Then certain standard positions of the quartz plates B_1 and B_2 were determined which gave equally spaced values of Acd and therefore of A, Acd being of course defined by the equation $I = I_0 10^{-Acd}$. The quartz cell was next filled with the solution, the absorption of which was to be measured, placed in position, and exposures were made for the different standard positions of the quartz plates. The plate was then developed and examined under a microscope of low power. The points at which the two comparison spectra matched were thus determined. The table then gave the wave-lengths, and the positions of the quartz plates gave the corresponding values of Acd. Owing to the sharp edge P of the quartz prism PQRT (fig. 2) the comparison spectra touched one another very sharply, and it was thus possible to determine very accurately the positions at which the intensities matched. One exposure sometimes gave A for as many as five points, thus indicating at least two maxima and two minima in the curve representing A as a function of λ.

Only one quartz cell was used, its inside thickness being 1 cm., and for most of the salts more than one strength of solution had to be used in order to bring A within the range of the instrument for all wave-lengths. The results obtained are tabulated below. In each case the first column gives the concentration of the salt used in gramme-molecules per litre, the second gives the wave-lengths, and the third gives the corresponding values of A.

$CoF_2 2H_2O$.			$CoCl_2 6H_2O$.			$CoBr_2 6H_2O$.			$CoI_2 2H_2O$.		
c.	λ.	A.	*c*.	λ.	A.	*c*.	λ.	A.	*c*.	λ.	A.
·074	·224	9·72	·40	·235	2·83	·10	·231	11·32	·0001	·237	11,320
	·226	8·09		·237	2·40		·244	8·40	·001	·250	1,132
	·228	6·50		·240	1·79	·40	·256	2·83		·251	959
	·239	3·30		·243	1·20		·269	1·79	·002	·255	566
	·244	1·65		·245	·61		·276	2·90		·258	239
	·290	·38	·80	·254	·30		·296	2·90	·005	·259	192
	? ·450	·98		·312	·09		·305	1·86		·260	143
	·521	3·30	·40	·332	·18		·309	1·38	·010	·263	113
			·80	·368	·09		·315	·79		·265	96
			·40	·390	·18		·332	·18		·269	96
				·430	·79		·373	·18		·271	113
				·441	1·38		·414	·68	·005	·278	149
				·463	3·01		·432	1·27		·293	197
							·450	1·97	·002	·303	158
							·460	3·01	·010	·311	116
										·317	99
										·334	99
										·337	116
										·355	149
										·376	116
										·382	99
										·397	74
										·404	51
										·410	27

$Co(NO_3)_2 6H_2O$.			$CoSO_4 7H_2O$.			$Ni(NO_3)_2 6H_2O$.			$NiSO_4 7H_2O$.		
c.	λ.	A.	*c*.	λ.	A.	*c*.	λ.	A.	*c*.	λ.	A.
·010	·243	113	1·0	·225	·84	·040	·250	27·1	·20	·233	·61
	·245	60		·244	·60	·050	·251	22·6	1·0	·239	·12
	·248	24·4	2·0	·257	·36		·252	19·2	2·0	·244	·06
·050	·250	22·6		·285	·24		·255	14·3		·326	·07
	·251	19·2		·307	·16	·10	·256	9·59	1·0	·334	·15
	·255	14·3		·336	·03		·258	7·16		·343	·51
	·257	9·58		·380	·16	·050	·259	4·87		·353	·99
	·266	4·87		·392	·27		·270	4·87	·20	·376	2·53
	·274	9·87		399	·39	·10	·276	9·87		? ·389	4·94
	·282	14·9		·409	·51		·278	11·6		·390	5·80
	·292	19·7	1·0	·416	·67	·050	·279	14·9	·10	·410	7·89
	·296	23·2		·426	·91		·289	19·7	·20	·426	6·02
	·308	23·2					·292	23·2		? ·427	5·16
	·312	19·7				·040	? ·300	27·1		·437	2·76
	·316	14·9				·050	·314	23·2		·441	·97
	·324	10·1					·316	19·7	2·0	·452	·60
	·327	5·44					·319	14·9		·470	·27
·20	·330	3·72				·10	·321	9·87			
	·332	2·53				·20	·325	5·60			
	·334	1·36					·329	3·72			
·10	·336	·72					·332	1·36			
·20	·341	·36					·365	1·36			
	·396	·36					·367	3·72			
·10	·426	·72					·376	5·60			
·20	·441	1·58				·10	·387	7·44			
	·456	2·76					? ·404	9·87			
							·411	7·44			
						·20	·414	5·60			
							·420	3·94			
							·427	1·58			

$NiF_2 3H_2O$.			$NiCl_2 6H_2O$.			$NiBr_2 3H_2O$.			$NiI_2 6H_2O$.		
c.	λ.	A.	*c*.	λ.	A.	*c*.	λ.	A.	*c*.	λ.	A.
·10	?·226	7·16	·40	?·226	1·79	·040	?·233	9·10	·00004	?·227	27000
·14	?·228	5·12		·233	1·20	·10	·235	7·16		?·233	18000
	?·230	3·42		·244	·61		·237	5·96	·0001	·239	10000
	·244	1·74		·284	·07	·20	·240	3·58	·0002	·242	5413
	·284	·20		·325	·07	·40	·248	2·83	·0001	·243	4790
	·363	·20		·344	·69		·257	2·40	·001	·250	1100
	·367	1·07		·353	1·27		·267	2·40		·252	959
	·369	1·94		·357	1·86		·271	2·83		·253	716
	·377	3·62		·362	2·47		·287	2·90	·002	·254	566
	·383	5·32		·366	2·90		·289	2·47	·001	·256	364
	·393	7·05	·20	·370	3·72		·292	1·86	·002	·259	239
·10	·403	7·44		·378	4·94		·296	1·27		·265	239
·14	·410	7·05		·380	5·80		·304	·67	·001	·268	364
	·420	5·63	·10	·382	6·24		·314	·07	·002	·271	566
	·423	3·94		·399	7·44		·343	·07	·001	·278	744
	·433	2·26		·407	7·44		·352	·69		?·290	987
	·436	1·38		·418	6·69		·364	1·27		·298	744
			·20	·421	5·16		·366	1·86		·305	392
				·429	3·94		·368	2·47		·312	272
				·432	2·76		·369	2·90		?·315	149
			·40	·437	1·38	·20	·370	3·72		?·335	149
				·448	·79		·378	4·94		·336	392
				?·470	·18		·379	5·80	·002	·337	580
						·10	·387	6·24	·001	·349	744
							·393	7·44	·002	·370	580
						·04	?·407	9·80	·001	·374	592
						·10	·416	7·89	·002	·395	253
							·418	6·69	·005	·410	197
							·421	3·16		·420	158
						·40	·425	2·58	·01	·429	120
							·434	1·38		·445	103
							·443	·79		·454	79
										·472	55
										·503	32

The values of A in the above tables have already been plotted as functions of λ in the third paper of this series. In these curves the results obtained by the thermopile, spectrophotometer, and photographic photometer all join on quite well. It is, of course, to be remembered that where the values obtained by the different instruments overlap, each of these instruments is getting out of its proper region and is hence working with less than its usual accuracy. The sets of the cobalt and nickel curves are drawn to the same scale.

It is seen by examining these curves that the molecular extinction coefficient is, to a first approximation, apparently an additive property, at least for the salts examined. In each of the cobalt salts the same bands occur at $\lambda = {\cdot}510\mu$ and at the infra-red end of the spectrum. These bands can, therefore, only be due to something vibrating in the cobalt atom. The height of the bands varies somewhat in each case, this being probably due to the acid radicles with which the metal is associated. Again, in the nickel salts we have in each case the same three maxima at $\lambda = \cdot 405\mu$, $\lambda = \cdot 690\mu$, and $\lambda = 1\cdot 210\mu$. These bands can, therefore, be due only to something in the nickel atom. Of course in cobalt iodide the band at $\lambda = \cdot 510\mu$ and in nickel iodide the band at $\lambda = \cdot 405\mu$ are swamped by the bands due to iodine.

If now we look for an effect due to the acid, we can trace none in the case of the sulphates. In the sulphates there is no band we can attribute to the SO_4 radicle, though it probably influences the height of the bands due to the metal. The chlorides and fluorides each show indications of a band in the extreme ultra-violet just off our range; we can trace the presence of this band in the rapid increase of the value of A. This increase is more marked in the fluoride than in the chloride. In the chloride there is also a weak band at $\lambda = \cdot 330\mu$, which has already been noticed by Jones and Anderson.* Both the bromides indicate a band off our range and a decidedly interesting one at $\lambda = \cdot 285\mu$, which attains the same height in each case. It has apparently not been observed before.

The case of the nitrates and iodides is, however, by far the most interesting. Owing to the large values of A, the parts of the curves in the ultra-violet could not be shown well on the same scale as the rest. They are, therefore, plotted separately in the two following diagrams, the values for the cobalt salts being shown by o *s* and for the nickel salts by x *s*. The two nitrates are almost identical. The band at $\cdot 300\mu$ has, of course, been observed by Hartley and by Jones and Anderson, but no measurements of A have previously been made on it. The iodides both show three bands, situated respectively at the end of the range, at $\lambda = \cdot 292\mu$ and at $\lambda = \cdot 355\mu$. The heights for nickel iodide are about five times the values for cobalt iodide. It should be stated that, owing to their unstable nature, the values of A for the iodides are not so reliable as for the other salts, especially at the low concentrations employed in this part of the spectrum. The values given are, it is hoped, free from serious error, as they are each the result of three or four careful independent determinations. For purposes of comparison the values of A for a saturated aqueous solution of iodine are shown by means of □'s on the same diagram. In determining the absolute value of A for iodine some trouble was experienced owing to the latter vaporising from the solution. To show the effect on A of the decomposition of cobalt iodide a dotted curve is shown which represents the values

* Jones and Anderson, Publication of the Carnegie Institution, No. 110.

obtained with a solution two months old. The bands at $\lambda = \cdot 292\mu$ and at $\lambda = \cdot 355\mu$ have disappeared.

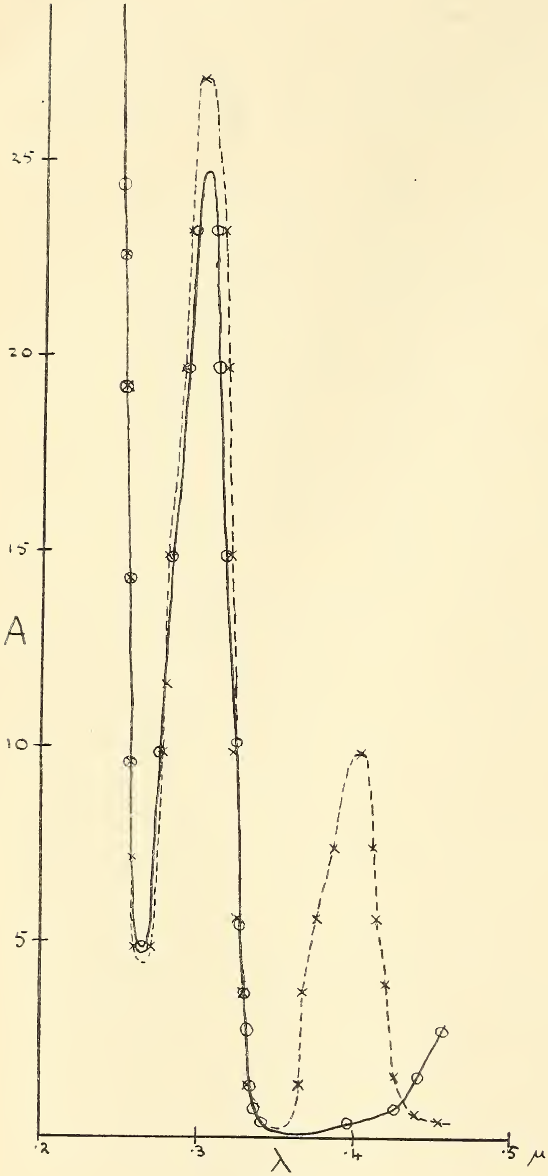


Fig. 4.

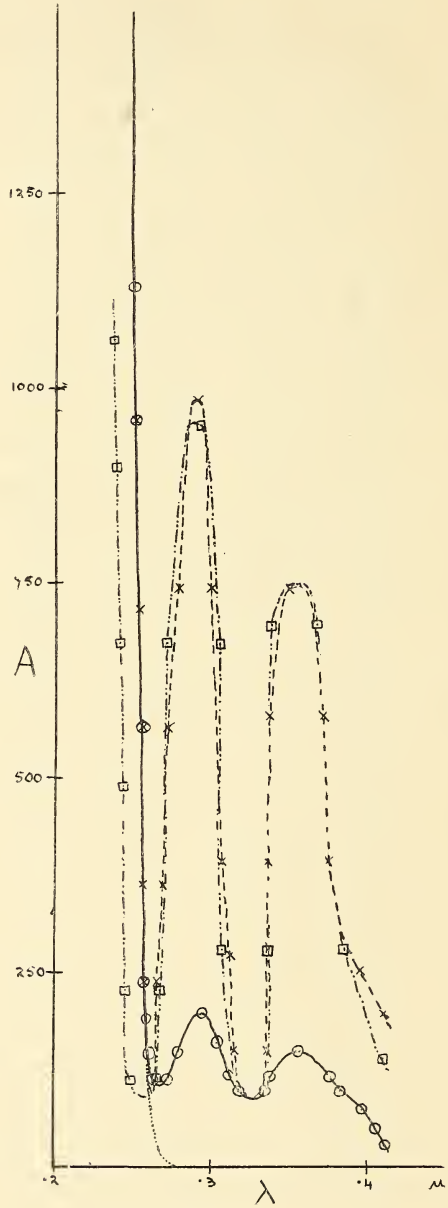


Fig. 5.

Owing to the strengths of the solutions employed the regularities obtained cannot be attributed to the "colour of the ions." This hypothesis has now lost its meaning owing to the new conception of the atom introduced by recent developments of the theory of electrons, and in addition it has, of course, not been borne out by experimental work. Some irregularities in the curves may be explained by the fact that we have not used the same concentration throughout the spectrum, though, as a rule, A does not vary much with c. But, owing to the wide limits within which A varies, it would be impossible to use the same value of c throughout the spectrum except at a very great cost of time and money. Owing to the want of suitable quartz cells we could not investigate the variation of A with c in the ultra-violet. It was noticed, however, that some of the bands in this region narrowed slightly with dilution.

It was, of course, impossible for us to go further into the infra-red owing to the absorption of water, and into the ultra-violet owing to the absorption of quartz.

The only quantitative work hitherto done on the absorption of light by aqueous solutions of cobalt and nickel salts has been noticed in one of the previous papers. It is that of E. Müller on nickel sulphate and nitrate in the visible spectrum. Much qualitative work has been done, some of it of great value. We reserve comparison of our results with those of other workers until we have data for more chemical elements. Meanwhile, for the complete bibliography of the subject we refer to Kayser's *Spectroscopie*, iii. pp. 409 and 416, and to *The Absorption Spectra of Solutions*, by Harry C. Jones and John A. Anderson, Carnegie Institution, Washington, Publication No. 110, 1909. We are continuing our work on other elements.

In conclusion, a few values of pe/m, calculated by the formula *

$$\frac{pe}{m} = \frac{\cdot 3170\, A_0(\lambda_1 - \lambda_0)}{\lambda_0^2},$$

may be of interest. p is the number of electrons or ions per molecule of salt, causing the absorption band under consideration, e being the charge on, and m the mass of, the electron or ion in question. A_0 is the value of A at the maximum, λ_0 the position of the maximum, and λ_1 the wave-length for which A has half its maximum value. The formula, of course, holds only on the special assumption that the electrons or ions in question are not much influenced by the rest of the vibrating system, but in other cases the order of magnitude of pe/m probably gives us a clue as to whether electrons or ions are in question. In any case, we cannot theorise more about the structure of the atom until we have more data.

* R. A. Houstoun, "On the Mechanism of the Absorption Spectra of Solutions," *Proc. Roy. Soc.*, 82 A, p. 606 (1909).

Band.	A_0.	λ_0.	λ_1.	pe/m.
Cobalt visible . . .	5	·510μ	·545μ	2·13 10^3
,, infra-red . .	1·5	1·3	1·1	5·6 10^2
Nickel ultra-violet . .	8	·405	·425	3·1 10^3
,, visible . .	2	·690	·770	1·1 10^3
,, infra-red . .	2	1·21μ	1·02μ	6·5 10^2
Bromide ultra-violet .	3·5	·285	·300	2 1 10^3
Nitrates	26	·302	·320	1·6 10^4
Iodides				Varying from 10^5 to 5 10^5.

If m is the mass of the cobalt or nickel atom e/m should be about 163.

(*Issued separately July* 8, 1911.)

XL.—On Fourier's Repeated Integral. By W. H. Young, Sc.D., F.R.S.

Communicated by Professor G. A. Gibson.

(MS. received December 28, 1910. Read February 6, 1911.)

Introduction.

§ 1. Pringsheim has recently reopened the question as to the circumstances under which Fourier's repeated integral exists and represents the function to which it corresponds.* In its simplest form the theorem concerning this integral asserts that, with provisos to be specified—

$$\int_0^\infty dv \int_0^\infty f(u) \cos uv \, du = \frac{\pi}{2} f(+0),$$

$f(+0)$ being the unique limit, supposed to exist, of $f(u)$ as u approaches the value zero. From this equation, indeed, the remainder of the theory follows immediately. If it is to be true, certain conditions must be satisfied at the origin, at infinity, and in the finite part of the range of values of the independent variable. Till Pringsheim's paper appeared, the only condition known to be sufficient at infinity was of a very special character, requiring no less than the absolute integrability of the function at infinity. Thus the theorem did not, for instance, apply to the function $(1+u)^{-\frac{1}{2}}$. The step taken by Pringsheim is an important one. He has shown that it is sufficient if the function is a monotone decreasing one, with zero as unique limit at infinity. What is even more noteworthy, it appears that it is sufficient for the function to be expressible as the product of two factors, one, ϕ, having the character above described, while the other is a cosine, or a sine, or, more generally, is expressible in the form of a trigonometrical series whose coefficients form an absolutely convergent series, provided that in the last two cases the former of the two functions when divided by the independent variable is absolutely integrable in the whole infinite interval.

Pringsheim requires certain other conditions to be satisfied by the function expressed by the trigonometrical series; these appear, however, only to have been introduced to secure that the product of the two functions also satisfies sufficient conditions near the origin. It is more convenient to consider the behaviour of the function in the neighbourhood of the origin,

* A Pringsheim, "Ueber neue Gültigkeitsbedingungen für die Fouriersche Integralformel," (1909), *Math. Ann.*, lxviii. pp. 367–408.

in the neighbourhood of the point infinity, and in the intermediate portions of the range of the independent variable separately. It then at once appears that the function which in Pringsheim's work is expressible in the form of a trigonometrical series, need not be differentiable, whereas Pringsheim requires it to possess very restricted properties as regards differentiability. A very slight re-arrangement, therefore, of Pringsheim's reasoning enables us to increase the scope of his results. Again, throughout Pringsheim's paper the reference is to ordinary integration, or to the various extensions of this concept known before Lebesgue had so materially increased its generality. Here a slight change of wording, and a reference to theorems recently discovered, allows us to interpret Pringsheim's theorems, so that they may be applicable to the more generalised concept in question.

The desirability of reconsidering the subject from this point of view is responsible in part for the length of the present paper, whose original object, however, was that of extending in an important particular Pringsheim's result, as to the condition to be satisfied by the function in the neighbourhood of infinity, and so, *en passant*, to confirm its validity.

Occasion is also taken to expose corresponding results for the repeated integral

$$\int_0^\infty dv \int_0^\infty f(u) \sin uv \, du.$$

The interest of this integral lies in the fact that it is the analogue of the very important but little-studied trigonometrical series obtained from a Fourier series by interchanging the coefficient of cos nx and that of sin nx, and changing the sign of one of them, for all values of n.

I now give for convenience of reference a table of the results proved in this paper, at the same time indicating which of them are to be regarded as new, and to what extent.

Sine Integral.

Result.

$$\int_0^\infty dv \int_{-\infty}^\infty f(u) \sin uv \, du = \int_{-\infty}^\infty \frac{f(u) - f(-u)}{2u} du.$$

Condition in any Finite Interval (p, q).

$\int_p^q \frac{f(u) - f(-u)}{2u} du$ *exists as a Lebesgue integral.*

Cosine Integral.

$$\int_0^\infty dv \int_{-\infty}^\infty f(u) \cos uv \, du = \frac{\pi}{2}(f(+0) + f(-0)).$$

Condition in any Finite Interval (p, q), not containing the Origin.

$\int_p^q f(u)du$ *exists as a Lebesgue integral.*

Alternative Conditions at the Origin.

(i.) $f(u)$ *has bounded variation in some interval containing the origin as internal point: or* (ii.) *for some values of the constants* C_+ *and* C_-,

$\int_0^p \frac{f(u) - C_+}{u} du$ *and* $\int_{-q}^0 \frac{f(u) - C_-}{u} du$ *both exist as Lebesgue integrals; the right-hand side of the above equation then becomes*

$$\frac{\pi}{2}(C_+ + C_-);$$

or

(iii.) f(u) *is expressible in an interval containing the origin as internal point, as the product* g(u)h(u) *of two functions, one of which,* g(u), *satisfies* (i.), *and the other,* h(u), *satisfies* (ii.). *The right-hand side of the above equation must then be suitably altered in accordance with* (ii.).

(iv.) *the repeated integral exists*; f(u) *has a Lebesgue integral in some interval* $(-p, p)$, *and the origin is a point at which* f(u) *is the differential coefficient of its indefinite integral.*

Alternative Conditions at Infinity.

(I.) $\int_q^\infty |f(u)| du$ *and* $\int_{-\infty}^{-q} |f(u)| du$ *exist, or, more generally,* $\int_q^\infty |f(u) + f(-u)| du$ *exists, for some value of the positive quantity* q ; *or*

(II.) *in some interval* (q, ∞), $f(u) = g(u)h(u)$, *where* g(u) *is monotone decreasing with zero as limit at infinity, while* h(u) *is any summable function whose Lebesgue integral can be expanded in a series of the form*

$$\int^u h(u)du = A_0 u + \sum_{n=1}^{\infty}(A_n \cos k_n u + B_n \sin k_n u),$$

the positive quantities k_n *increasing without limit as* n *increases, and the series whose general term is* $|A_n| + |B_n|$ *converging, while the integrals* $\int_q^\infty \frac{g(u)}{u} du$ *and* $\int_q^\infty \left|\frac{f(u)}{u}\right| du$ *both exist.*

It is hardly necessary here to point out that $f(u)$ may, in any one of the parts into which we thus divide the straight line, be the sum of any finite number of functions, satisfying the above conditions, each multiplied by a suitable constant. In particular, we may take $g(u)$ in (II.) to be a function of bounded variation in the whole infinite interval, instead of monotone decreasing, provided each of the monotone descending functions of which it is the difference satisfies the further conditions given.

There are various special cases of the condition (II.) which are of greater interest than the general case.

(II*a*.) *The function* h(u) *is a constant*; in this case the condition that $\int_q^\infty \left|\frac{f(u)}{u}\right| du$ should exist may be omitted, and the condition that $\int_q^\infty \frac{g(u)}{u} du$ exists is only necessary in the case of the sine-integral.

(II*b*.) *The function* h(u) *is of the form* cos ku *or* sin ku; in this case the condition that $\int_q^\infty \left|\frac{f(u)}{u}\right| du$ should exist may be omitted, and the condition that $\int_q^\infty \frac{g(u)}{u} du$ should exist is unnecessary in the former case for the cosine-integral, and in the latter case for the sine-integral.

(II*c*.) *The expansion of* $\int^u$h(u)du *contains no sine terms, or no cosine terms*; in the former case we may, in considering the cosine-integral, omit the condition that $\int_q^\infty \frac{g(u)}{u} du$ should exist, and in the latter case we may omit this condition in considering the sine-integral.

(II*d*.) f(u) *is expressible as the product of two factors, one of which,* g(u), *is a monotone decreasing function with zero as limit, while the second,* h(u), *is any periodic function whose square is summable, provided further that* $\int_q^\infty \frac{g(u)}{u} du$ *and* $\int_q^\infty \left|\frac{f(u)}{u}\right| du$ *both exist.*

These conditions differ, as already remarked, from those hitherto given in the fact that the generalised integration has been freely used. It has moreover been usual to consider only the cosine-integral. With regard to the conditions when we are considering the latter integral, it may be noticed that there is a slight advance in the statement of condition (ii.) at the origin, as previous writers have omitted to remark that $f(u)$ need not have an unique limit. The condition (iii.) at the origin includes as a particular case that virtually given by Pringsheim. The condition (iv.) is here stated for the first time.

As regards the conditions at infinity, (II*a*, *b*, and *c*) are due to Pringsheim. The general condition (II.) and the special one (II*d*.) appear to constitute a material advance on the corresponding condition given by Pringsheim.

In obtaining these extensions all turns on the use of a theorem in the theory of Fourier's series, and of a more general theorem in the integration of series, recently stated and proved by myself. The reasoning adopted is moreover differently arranged from that of Pringsheim, though the general line of argument is the same.

Section 1.

On the Fourier Sine-integral $\int_0^\infty dv \int_p^q \sin uv\, du$, *when the Limits* p *and* q *of Integration with respect to* u *are finite.*

§ 2. Theorem 1.—*The repeated integral*

$$\int_0^\infty dv \int_p^q f(u) \sin uv\, du,$$

where p *and* q *are finite quantities, positive, negative, or zero, necessarily exists, and is equal to*

$$\int_p^q \frac{f(u)}{u} du,$$

provided only the latter integral exists.

Since $\frac{f(u)}{u}$ is summable, so is $f(u)$, and therefore also $|f(u)|$. Hence the repeated integrals of $|f(u)|$, taken over a finite rectangle, exist and are equal. Also $\sin uv$ is bounded. Hence, by a known theorem,* the repeated integrals of $|f(u)| \sin uv$, and therefore of $f(u) \sin uv$, exist and are equal over the same finite rectangle. We may therefore write

$$\int_0^B dv \int_p^q f(u) \sin uv\, du = \int_p^q du \int_0^B f(u) \sin uv\, dv = \int_p^q \frac{f(u)}{u}(1 - \cos Bu) du.$$

But, by a theorem of Riemann-Lebesgue,† $\int_p^q \frac{f(u)}{u} \cos Bu\, du$ has, as B increases, the unique limit zero, whence the required result at once follows.

§ 3. If, as throughout the present paper, $f(u)$ is assumed *a priori* to be summable in every finite interval, it is clear that the only doubt as to the existence of the integral in the theorem of the preceding article, relates to the behaviour of the function $f(u)$ in any interval, however small, containing the origin. But, further, it is not necessary for the existence of the repeated integral that the function $f(u)/u$ should be summable; it is sufficient that $(f(u) - f(-u))/u$ should be summable. This is of some importance if we reflect that, otherwise, the addition of a constant to $f(u)$ would invalidate the existence of the repeated integral. The corresponding theorem is stated most conveniently when the limits of integration with respect to u differ only in sign; but evidently the theorem may be suitably modified so as to embrace any case.

* W. H. Young, "On Change of Order of Integration in a Repeated Integral," 1909, *Camb. Phil. Trans.*, vol. xxi. p. 364.

† B. Riemann, "Ueber die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe," § 10 (1854), *Ges. Werke*, 2nd ed., p. 254; H. Lebesgue, *Leçons sur les séries trigonométriques* (1905); see also Hobson's *Theory of Functions of a Real Variable*, p. 674.

THEOREM 1, bis.—*The repeated integral*

$$\int_0^\infty dv \int_{-p}^{p} f(u) \sin uv \, du,$$

where p *is any finite positive quantity, necessarily exists, and is equal to*

$$\int_{-p}^{p} \frac{f_1(u)}{u} du,$$

where $f_1(u) = \frac{1}{2}(f(u) - f(-u))$, *provided only the latter integral exists.*

The proof is immediate, since the substitution of $f_1(u)$ for $f(u)$ does not alter the repeated integral. In fact,

$$\int_{-p}^{p} f(u) \sin uv \, du = -\int_{-p}^{p} f(-u) \sin uv \, du,$$

which proves the truth of the statement.

SECTION 2.

On the Fourier Cosine-integral, $\int_0^\infty dv \int_p^q f(u) \cos uv \, du$, *in which the Limits of Integration* p *and* q *with respect to* u *are finite.*

§ 4. THEOREM 2.—*The repeated integral*

$$\int_0^\infty dv \int_p^q f(u) \cos uv \, du,$$

where p *and* q *are finite and have the same sign, necessarily exists and is zero, provided only that* f(u) *is summable.*

To prove this theorem, we have merely to repeat the argument used in Section 1 to prove that change of order of integration between finite limits is allowable. We hence get for the value of our repeated integral

$$\underset{B=\infty}{\mathrm{Lt}} \int_p^q \frac{f(u)}{u} \sin Bu \, du.$$

But, since p and q have the same sign, $f(u)/u$ is summable in the interval (p, q). Hence, by the theorem of Riemann-Lebesgue, the required result follows.

COR.—*If* p *and* q *have opposite signs, the value or values of the limits represented by the repeated integral in question are independent of the particular fixed values we may choose to assign to* p *and* q.

§ 5. THEOREM 3.—*If* f(u) *is a function of bounded variation in the interval* $(-p, p)$, *then*

$$\int_0^\infty dv \int_{-p}^{p} f(u) \cos uv \, du = \frac{\pi}{2} \{f(+0) + f(-0)\} = \pi f_2(+0),$$

where $f_2(u)$ *is defined to be* $\frac{1}{2}(f(u) + f(-u))$.

It is clearly sufficient to prove the theorem on the hypothesis that $f(u)$

is zero when u is negative, and when u is positive it is a monotone decreasing function of u.

Now

$$\int_0^B dv \int_0^p f(u) \cos uv \, du = \int_0^p du \int_0^B f(u) \cos uv \, dv = \int_0^p \frac{f(u)}{u} \sin Bu \, du$$

$$= \int_0^{Bp} f\left(\frac{u}{B}\right) \frac{\sin u}{u} du \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But $f(u/B)$ is a monotone function of u, and converges boundedly to $f(+0)$, as B increases indefinitely, and $\sin u/u$ has a Harnack integral in the interval $(0, \infty)$.

Hence *

$$\int_0^Q f\left(\frac{u}{B}\right) \frac{\sin u}{u} du$$

has, as Q and B approach infinity in any manner whatever, an unique double limit, which is, therefore, that got by letting first B and afterwards Q increase without limit. Thus

$$\underset{B=\infty}{Lt} \int_0^{Bp} f\left(\frac{u}{B}\right) \frac{\sin u}{u} du = \int_0^\infty f(+0) \frac{\sin u}{u} du = \frac{\pi}{2} f(+0).$$

This is therefore also the limit of the integral on the extreme left of (1), which proves the theorem.

§ 6. Theorem 4.—*If for some value of the constant* C

$$\frac{f_2(u) - C}{u}, \quad \text{where} \quad f_2(u) = \tfrac{1}{2}\{f(u) + f(-u)\},$$

has a Lebesgue integral in the interval $(-p, p)$, *then*

$$\int_0^\infty dv \int_{-p}^p f(u) \cos uv \, du = \pi C.$$

Evidently we may replace $f(u)$ by $f_2(u)$ without altering the value of the integral. Also, since $(f_2(u) - C)/u$ is summable, so is $f_2(u)$, in the interval considered, and therefore since $\cos uv$ is bounded, we may, as usual, change the order of integration over a finite rectangle. Thus, as before, we have

$$\tfrac{1}{2}\int_0^\infty dv \int_{-p}^p f_2(u) \cos uv \, du = \underset{B=\infty}{Lt} \int_0^p f_2(u) \frac{\sin Bu}{u} du$$

$$= C \underset{B=\infty}{Lt} \int_0^p \frac{\sin Bu}{u} du + \underset{B=\infty}{Lt} \int_0^p \frac{f_2(u) - C}{u} \sin Bu \, du.$$

Now, the first of the two integrals on the right is equal to $\int_0^{Bp} \frac{\sin u}{u} du$, and has therefore $\frac{\pi}{2}$ for unique limit when B increases indefinitely. Also,

* For the general form of the theorem here used see W. H. Young, "On a Theorem in the Harnack Integration of Series" (1910), § 3, *Mess. Math.*, p. 105.

by the theorem of Riemann-Lebesgue, since $\dfrac{f_2(u)-C}{u}$ is a summable function, by hypothesis, the second integral on the right has zero for unique limit, which proves the theorem.

NOTE.—If as u approaches zero, $f_2(u)$ has an unique limit—in other words, if $f_2(+0)$ exists,—it is evident that the only possible value for C is $f_2(+0)$.

That the expression may still have a Lebesgue integral without this occurring is evident, if we reflect that the integrability of $(f_2(u)-C)/u$, when $f_2(u)$ has C as unique limit, will not be affected, if, for example, at a countable set of points having $u=0$ as limiting point, we ascribe any value whatever to $f_2(u)$.

§ 7. THEOREM 5.—*If in the interval* (0, p), f(u) *can be expressed as the product of two functions, one of which,* g(u), *has bounded variation, and the other,* h(u), *is such that*

$$\frac{h(u)-C}{u}$$

has a Lebesgue integral in that interval, then

$$\int_0^\infty dv \int_0^p f(u)\cos uv\, du = \frac{\pi}{2}C\{g(+0)\}.$$

For, as before, we may without loss of generality suppose $g(u)$ to be monotone decreasing. Also, since $f(u)$ is summable, and therefore change of order of integration between finite limits is allowable,

$$\int_0^\infty dv \int_0^p g(u)h(u)\cos uv\, du = \underset{B=\infty}{\mathrm{Lt}} \int_0^p h(u)\frac{g(u)\sin Bu}{u}du$$

$$= C\underset{B=\infty}{\mathrm{Lt}} \int_0^p \frac{g(u)\sin Bu}{u}du + \underset{B=\infty}{\mathrm{Lt}} \int_0^p \frac{h(u)-C}{u}g(u)\sin Bu\, du.$$

But since $(h(u)-C)/u$ is summable and $g(u)$ is bounded, their product is summable, whence, by the theorem of Riemann-Lebesgue, the last integral has the unique limit zero. Further, since $g(u)$ is monotone decreasing, it follows, as in the proof of Theorem 3, by the Second Theorem of the Mean, that the first of the two integrals on the right of the preceding equality has the unique limit $\frac{\pi}{2}g(+0)$.

COR.—*If a similar statement is also true for the interval* $(-p, 0)$, *then,* C′ *being the corresponding constant,*

$$\int_0^\infty dv \int_{-p}^p f(u)\cos uv\, du = \frac{\pi}{2}\{Cg(+0) - C'g(-0)\}.$$

§ 8. Theorem 6.—*If* f(u) *is a summable function in the interval* (−p, p), *and* $\int^u$f(u)du *has at the origin a differential coefficient whose value is* f(0), *then*

$$\int_0^\infty dv \int_{-p}^{p} f(u)\cos uv\, du = \pi f(0),$$

provided only this repeated integral exists.

Let

$$g(u) = f(u) \sin \tfrac{1}{2}u/u,$$

then

$$\frac{1}{h}\int_0^h g(u)du = \frac{1}{h}\left(\frac{\sin\frac{1}{2}h}{h}\right)\int_0^h f(u)du - \int_0^h \frac{\frac{1}{2}u\cos\frac{1}{2}u - \sin\frac{1}{2}u}{u^2}\left(\int_0^u f(u)du\right)du.$$

But, since the differential coefficient of $\int^u f(u)\,du$ at $u=0$ is $f(0)$, the first term on the right has, when h approaches zero, the unique limit $\frac{1}{2}f(0)$. Hence *

$$\underset{h=0}{\mathrm{L}lt}\frac{1}{h}\int_0^h g(u)du = \tfrac{1}{2}f(0) + \underset{h=0}{\mathrm{L}lt}\frac{1}{h}\int_0^h \frac{\sin\frac{1}{2}u - \frac{1}{2}u\cos\frac{1}{2}u}{u^2}\left(\int_0^u f(u)du\right)du.$$

But

$$\frac{\sin\frac{1}{2}u - \frac{1}{2}u\cos\frac{1}{2}u}{u^2} \quad\text{and}\quad \int_0^u f(u)du$$

are both continuous functions of u; hence the integrand of the integral on the right, being the product of these two functions, is itself a continuous function of u, and consequently the limits on the right are the same as

$$\underset{h=0}{\mathrm{L}lt}\frac{\sin\frac{1}{2}h - \frac{1}{2}h\cos\frac{1}{2}h}{h^2}\int_0^h f(u)du,$$

which are uniquely zero.

Hence, finally,

$$\underset{h=0}{\mathrm{L}t}\frac{1}{h}\int_0^h g(u)du = \tfrac{1}{2}f(0) = g(0),$$

that is, at the point $u=0$, the function $g(u)$ is the differential coefficient of its integral.

Denote the repeated integral, whose existence was postulated, by I. Then

$$\mathrm{I} = \underset{\mathrm{B}=\infty}{\mathrm{L}t}\int_{-p}^{p} f(u)\frac{\sin \mathrm{B}u}{u}du.$$

Since this limit is unique, we may replace B by $\frac{1}{2}(2m+1)$, so that

$$\mathrm{I} = \underset{m=\infty}{\mathrm{L}t}\int_{-p}^{p} g(u)\frac{\sin\frac{1}{2}(2m+1)u}{\sin\frac{1}{2}u}du = \underset{m=\infty}{\mathrm{L}t}\int_{-\pi}^{\pi} g(u)\frac{\sin\frac{1}{2}(2m+1)u}{\sin\frac{1}{2}u}du,$$

where, if need be, we regard $g(u)$ as being zero outside the interval $(-p, p)$. This last integral is 2π times the sum of the first $(2m+1)$ terms of the

* I use the symbol $\underset{h=0}{\mathrm{L}lt}$ to denote a possible plurality of limits, and $\underset{h=0}{\mathrm{L}t}$ to denote an unique limit, known to exist.

Fourier series of $g(u)$, at the point $u=0$. Hence, since the limit in question is supposed to exist, the Fourier series converges at the point $u=0$. But, as we have seen, $g(u)$ is, at the point $u=0$, the differential coefficient of its indefinite integral. From these two facts we have, by a known theorem,* the result that the Fourier series of $g(u)$ converges at the point $u=0$ to $g(0)$. Hence

$$I=2\pi g(0)=\pi f(0).$$

Cor.—*If, except for a set of values of* x *of content zero in the interval* (−p, p),

$$\int_0^\infty dv\int_{-p}^{p} f(u)\cos v(u-x)du$$

exists, it is throughout that interval equal to πf(x), *except for a set of values of* x *of content zero.*

If x be an internal point of the interval $(-p, p)$ at which the repeated integral exists, the latter is equal to

$$\int_0^\infty dv\int_{-p-x}^{p-x} f(u+x)\cos uv\,du=\int_0^\infty dv\int_{-p}^{p} f(u+x)\cos uv\,du.$$

Hence, since $f(x)$ is necessarily the differential coefficient of its integral except for a set of values of x of content zero,† the required result follows.

Section 3.

On the Fourier Sine-integral $\int_0^\infty dv\int_{-\infty}^{\infty} f(u)\sin uv\,du$.

§ 9. Here the limits of integration with respect to u are both infinite. Hence, if we adopt the usual interpretation, we assume the existence of the repeated integral when these infinite limits are replaced by finite limits. We accordingly assume further that one of the conditions in Section 2 is fulfilled.

Breaking the repeated integral, then, up into three parts, in one of which the limits of integration with respect to u are of opposite signs, we reduce the discussion to that of $\int_0^\infty dv\int_q^\infty f(u)\sin uv\,du$, where q is positive and not zero, and a precisely similar integral in which the limits of integration with respect to u are the same in magnitude, but of opposite sign.

§ 10. In the sections that follow, *we shall require the theorem of Riemann-Lebesgue,* quoted in § 2, *in the extended form in which the*

* P. Fatou, "Séries trigonométriques et Séries de Taylor," 1905, *Acta Math.*, xxx. pp. 335–400; also H. Lebesgue, "Sur les intégrales singulières," 1910, *Ann. de Toulouse*, p. 90.

† For this known result see, for instance, my paper on "Functions of Bounded Variation," 1910, *Quart. Journ.*, p. 82.

interval of integration is infinite. It will be at once seen that this extension is allowable, provided the conditions which Lebesgue postulates in a finite interval hold in the infinite interval—that is, *provided the absolute value of the integrand may be integrated over the infinite interval.*

In fact, if $|g(x)|$ has an integral from q to infinity, so has $g(x)\sin Bx$, since it lies between $+|g(x)|$ and $-|g(x)|$; thus we may write

$$\left|\int_q^\infty g(x)\sin Bx\,dx\right| = \left|\int_q^Q g(x)\sin Bx\,dx + \int_Q^\infty g(x)\sin B(x)dx\right|$$
$$\leqq \left|\int_q^Q g(x)\sin Bx\,dx\right| + \int_Q^\infty |g(x)|\,dx.$$

Therefore, by the theorem of Riemann-Lebesgue, the interval (q, Q) being finite,

$$\underset{B=\infty}{\mathrm{L}lt}\left|\int_q^\infty g(x)\sin Bx\,dx\right| \leqq \int_Q^\infty |g(x)|\,dx,$$

which, by choosing Q sufficiently large, is smaller than any assigned positive quantity. Therefore

$$\underset{B=\infty}{\mathrm{L}t}\int_q^\infty g(x)\sin Bx\,dx = 0.$$

Similarly,

$$\underset{B=\infty}{\mathrm{L}t}\int_q^\infty g(x)\cos Bx\,dx = 0.$$

§ 11. Theorem 7.—*If* $\int_q^\infty |f(u)|\,du$ *exists, then*

$$\int_0^\infty dv\int_q^\infty f(u)\sin uv\,du = \int_q^\infty \frac{f(u)}{u}du.$$

Since $f(u)\sin uv$ lies between $+|f(u)|$ and $-|f(u)|$, it may be integrated between the limits q and infinity. Also

$$\int_q^Q f(u)\sin uv\,du \leqq \int_q^Q |f(u)|\,du \leqq \int_q^\infty |f(u)|\,du,$$

which shows that the integral on the left is a bounded function of the ensemble (v, Q) in the whole region

$$q \leqq Q = \infty, \quad 0 \leqq v \leqq \infty.$$

Hence

$$\int_0^B dv\,\underset{Q=\infty}{\mathrm{L}t}\int_q^Q f(u)\sin uv\,du = \underset{Q=\infty}{\mathrm{L}t}\int_0^B dv\int_q^Q f(u)\sin uv\,du = \underset{Q=\infty}{\mathrm{L}t}\int_q^Q du\int_0^B f(u)\sin uv\,dv$$
$$= \int_q^\infty \frac{f(u)}{u}du - \int_q^\infty \frac{f(u)}{u}\cos Bu\,du,$$

change of order of integration being allowable by the theorem already quoted in § 2. But, since $\dfrac{1}{u} \leqq \dfrac{1}{q}$, $|f(u)/u|$ has an integral from q to ∞. Hence, by the theorem of Riemann-Lebesgue, as pointed out in § 10, this last integral has zero as unique limit when B is indefinitely increased, whence the required result at once follows.

Cor.—*If* f(u) = g(u)h(u), *where* $\int_q^\infty |g(u)|\,du$ *exists,* and h(u) *oscillates finitely at infinity, the same result holds.*

§ 12. Theorem 8.—*If in the interval* (q, ∞) *the function* f(u) *is monotone decreasing, and approaches the unique limit zero at infinity, then*

$$\int_0^\infty dv \int_q^\infty f(u)\sin uv\,du = \int_q^\infty \frac{f(u)}{u}du$$

provided the latter integral exists.

Since $\int_a^b \sin uv\,du = (\cos va - \cos vb)/v$, it follows, by the Second Theorem of the Mean, that if $a<b$ and $v>e>0$,

$$\left|\int_a^b f(u)\sin uv\,du\right| \angle\, 2f(a)/e.$$

Hence, firstly, $\int_Q^\infty f(u)\cos uv\,du$ exists for each positive value of v (since, by hypothesis, $f(u)$ tends to zero at infinity); secondly, it is, for all values of v greater than e, a bounded function of v; and, thirdly, $\int_Q^A f(u)\sin uv\,du$ converges boundedly to its limit when A becomes infinite, for all such values of v.

Hence we may integrate with respect to v between the finite positive limits e and B, and we get, denoting for the moment the integrand $f(u)\sin uv$ by $F(u, v)$,

$$\int_e^B dv\int_Q^\infty F(u,v)du = \int_e^B dv \operatorname*{Lt}_{A=\infty}\int_Q^A F(u,v)du = \operatorname*{Lt}_{A=\infty}\int_e^B dv\int_Q^A F(u,v)du = \operatorname*{Lt}_{A=\infty}\int_Q^A du\int_e^B F(u,v)dv$$
$$= \int_Q^\infty \frac{f(u)}{u}(\cos eu - \cos Bu)du.$$

Now, in the interval (Q, ∞), the function $f(u)/u$ is positive, and the function $(\cos eu - \cos Bu)$ is numerically $\angle$ 2. Therefore the right-hand side of the preceding equation is numerically $\angle\, 2\int_Q^\infty \frac{f(u)}{u}du$, which, since by hypothesis $f(u)/u$ can be integrated over the infinite interval, is a function of Q which has the unique limit zero at infinity. Denoting this function by $g(Q)$, we have, therefore,

$$\int_e^B dv\int_q^Q F(u,v)du - g(Q) \angle \int_e^B dv\int_q^\infty F(u,v)du \angle \int_e^B dv\int_q^Q F(u,v)du + g(Q).$$

Let e diminish to zero in such a manner that the repeated integral in the middle of this inequality has an unique limit, say I. We thus get

$$\int_0^B dv\int_q^Q F(u,v)du - g(Q) \angle\ \mathrm{I} \angle \int_0^B dv\int_q^Q (Fu, v)du + g(Q),$$

or, which is the same thing,

$$\int_q^Q du\int_0^B F(u,v)dv - g(Q) \angle\ \mathrm{I} \angle \int_q^Q du\int_0^B F(u,v)dv + g(Q).$$

Now let Q move off to infinity in such a way that the repeated integral which occurs in both the extreme members of this inequality has an unique limit, say J. We thus get, since $g(Q)$ has the unique limit zero,

$$I = J.$$

But this is true whichever of the possible limits J is. Therefore there is only one such limit—that is, $\int_q^\infty du \int_0^B F(u, v)dv$ exists, and is equal to I. Since this is true whichever of the possible limits I is, this proves that there is only one such limit—that is, $\int_0^B dv \int_q^\infty F(u, v)du$ exists, and is equal to $\int_q^\infty du \int_0^B F(u, v)dv$.

Thus, as a preliminary step, we have proved that we can reverse the order of integration in $\int_0^B dv \int_q^\infty F(u, v)du$, *both the repeated integrals in question in fact existing and being equal.*

Now

$$\int_q^\infty du \int_0^B F(u, v)dv = \int_q^\infty f(u) \frac{1 - \cos Bu}{u} du = \int_q^\infty \frac{f(u)}{u} du - \int_q^\infty \frac{f(u)}{u} \cos Bu\, du,$$

and by the Second Theorem of the Mean, the last integral is numerically $\leqq 2f(q)/Bq$. Hence, letting B increase without limit, the required result follows.

§ 13. Theorem 9.—*If in the interval* (q, ∞) *the function* f(u) *can be expressed as the product of two functions, one of which,* g(u), *is monotone decreasing with zero as limit at infinity, and the other,* h(u), *is of the form* cos ku, *then, provided* $\int_q^\infty \frac{g(u)}{u} du$ *exists,*

$$\int_0^\infty dv \int_q^\infty f(u) \sin uv\, du = \int_q^\infty \frac{f(u)}{u} du,$$

both integrals necessarily existing.

For, by the preceding theorem,

$$\int_0^\infty dv \int_q^\infty g(u) \sin u(v+k) du = \int_k^\infty dv \int_q^\infty g(u) \sin uv\, du$$

$$= \int_q^\infty \frac{g(u)}{u} du - \int_0^k dv \int_q g(u) \sin uv\, du,$$

all these integrals existing. Changing the order of integration in the last integral (which we may do, as shown in the course of the preceding proof), and then performing the integration with respect to v, the last integral becomes

$$\int_q^\infty g(u) \frac{1 - \cos ku}{u} du.$$

Hence we have

$$\int_0^\infty dv \int_q^\infty g(u) \sin u(v+k) du = \int_q^\infty \frac{g(u) \cos ku}{u} du.$$

Changing k into $-k$, the right-hand side remains unaltered. Hence also

$$\int_0^\infty dv \int_q^\infty g(u) \sin u(v-k)du = \int_q^\infty \frac{g(u) \cos ku}{u} du.$$

Adding these two results, we get the above theorem; subtracting them we get a result to be stated in its proper place (Section 4, § 24).

Cor.—*When the upper limit of integration with respect to* v *is finite, not infinite, we may reverse the order of integration.*

For, as above,

$$\int_0^B dv \int_q^\infty g(u) \sin u(v+k)du = \int_0^{B+k} dv \int_q^\infty g(u) \sin uv\, du - \int_0^k dv \int_q^\infty g(u) \sin uv\, du,$$

and in each of the integrals on the right we may change the order of integration, as shown in the course of the proof of Theorem 8. Similarly we may change the order of integration, in $\int_0^B dv \int_q^\infty g(u) \sin u(v-k)du$, and therefore also in $\int_0^B dv \int_q^\infty g(u) \cos ku \sin uv\, du$, which is half the sum of these two integrals.

§ 14. Theorem 10.—*If in the interval* (q, ∞) *the function* f(u) *can be expressed as the product of two functions, one of which,* g(u), *is monotone decreasing with zero as limit at infinity, and the other,* h(u), *is of the form* sin ku, *then,*

$$\int_0^\infty dv \int_q^\infty f(u) \sin uv\, du = \int_q^\infty \frac{f(u)}{u} du,$$

both integrals necessarily existing.

For, by a theorem proved in the next section (Section 4, § 23),

$$\int_0^\infty dv \int_q^\infty g(u) \cos uv\, du = 0,$$

and we may change the order of integration, writing

$$\int_0^B dv \int_q^\infty g(u) \cos uv\, du = \int_q^\infty du \int_0^B g(u) \cos uv\, du,$$

these integrals existing. Hence, proceeding as in the preceding proof, we have

$$\int_0^\infty dv \int_q^\infty g(u) \cos u(v+k)du = \int_k^\infty dv \int_q^\infty g(u) \cos uv\, du = -\int_0^k dv \int_q^\infty g(u) \cos uv\, du$$

$$= -\int_q^\infty du \int_0^k g(u) \cos uv\, dv = -\int_q^\infty \frac{g(u) \sin ku}{u} du.$$

Changing k into $-k$, we have,

$$\int_0^\infty dv \int_q^\infty g(u) \cos u(v-k)du = \int_q^\infty \frac{g(u) \sin ku}{u} du.$$

Subtracting these two results, and dividing by 2, our theorem follows; adding them, we get the corresponding result to be stated later (Section 4, § 24).

Cor.—*When the upper limit of integration with respect to* v *is finite, we may reverse the order of integration.*

For, as in the proof of the preceding corollary,

$$\int_0^B dv\int_q^\infty g(u)\cos u(v+k)du = \int_0^{B+k} dv\int_q^\infty g(u)\sin uv\,du - \int_0^k dv\int_q^\infty g(u)\cos uv\,du,$$

and, by the theorem quoted in the proof of the present theorem, we may change the order of integration in each of the integrals on the right, and therefore we may do so in the integral on the left. Similarly we may do so in $\int_0^B dv\int_q^\infty g(u)\cos u(v-k)du$, and therefore in $\int_0^B\int_q^\infty g(u)\sin ku\sin uv\,du$.

§ 15. Lemma.—*If the series of proper or improper Lebesgue or Harnack-Lebesgue integrals,*

$$\int_q^z f_1(x)dx + \int_q^z f_2(x)dx + \dots + \int_q^z f_n(x)dx + \dots$$

converge to the proper or improper Lebesgue or Harnack-Lebesgue integral

$$\int_q^z f(x)dx,$$

for all values of q *and* z *in the completely open interval* (p<q<z<B), *in such a manner that a convergent series of positive quantities,*

$$c_1 + c_2 + \dots + c_n + \dots,$$

can be found, each term of which is not less than the absolute value of the corresponding term of the series of integrals, whatever be the values of q *and* z *in the interval in question, then, provided only in addition*

$$\int_p^B f_n(x)dx$$

exists, for each value of n, *we can assert that*

(i.) $\int_p^B f(x)dx$ *exists*;

(ii.) *the series*

$$\int_p^B f_1(x)dx + \int_p^B f_2(x)dx + \dots + \int_p^B f_n(x)dx + \dots$$

converges;

(iii.) *the sum of the latter series is equal to the former integral. Here the limits of* p *and* B *may be either finite or infinite.*

We first prove the lemma when the lower limit of integration is q instead of p.

For since, by hypothesis, $\left|\int_q^z f_n(x)dx\right| \leqq c_n$, and $\int_q^B f_n(x)dx$ exists, so that it is the unique limit of the former integral when z approaches B

$$\left|\int_q^B f_n(x)dx\right| \leqq c_n.$$

Hence, by Weierstrass's test for convergency, the result (ii.) follows when for p we write q.

Again, it immediately follows from the premisses that

$$\left|\int_q^z f(x)dx - \int_q^z f_1(x)dx - \int_q^z f_2(x)dx - \dots - \int_q^z f_n(x)dx\right| \leqq c_{n+1} + c_{n+2} + \dots$$

Therefore, taking the repeated limit of both sides, first with respect to z, and then with respect to n, we get

$$\underset{n=\infty}{\mathrm{Lt}}\ \underset{z=\mathrm{B}}{\mathrm{Lt}}\left[\int_q^z f(x)dx - \int_q^z f_1(x)dx - \int_q^z f_2(x)dx - \ .\ .\ .\ .\ - \int_q^z f_n(x)dx\right] = 0.$$

But, by (ii.), already proved, $\underset{n=\infty}{\mathrm{Lt}}\ \underset{z=\mathrm{B}}{\mathrm{Lt}}\left[\int_q^z f_1(x)dx + \int_q^z f_2(x)dx + \ .\ .\ .\ .\ + \int_q^z f_n(x)dx\right]$ exists, therefore, by the last equation, (i.) and (ii.) follow simultaneously.

Similarly the lemma is true when the upper limit of integration is q and the lower limit is p. Hence by addition the required result follows.

Example.—If the series whose general term is $n^{\frac{1}{2}}a_n$ converges, and

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx, \quad g(x) = \sum_{n=1}^{\infty} a_n(1 - \cos nx),$$

then

$$\int_0^\infty x^{-\frac{3}{2}} f(x)dx \quad \text{and} \quad \int_0^\infty x^{-\frac{3}{2}} g(x)dx$$

exist, and we have their values, since, by a known result,*

$$\int_0^\infty x^{-\frac{3}{2}} \sin nx\, dx = \sqrt{2\pi n} = \int_0^\infty x^{-\frac{3}{2}}(1 - \cos nx)dx,$$

given as follows,

$$\int_0^\infty x^{-\frac{3}{2}} f(x)dx = \sqrt{2\pi}\sum_{n-1}^{\infty} n^{\frac{1}{2}}a_n = \int_0^\infty x^{-\frac{3}{2}} g(x)dx.$$

§ 16. We shall also require the following lemma, which is an immediate consequence of a theorem of Harnack's, recently generalised by Fatou †:—

Lemma.—*If* a_n, b_n *are the Fourier constants of a function* f(x) *whose square is summable, then the series* $\Sigma a_n/n^{\frac{1}{2}+\epsilon}$, $\Sigma b_n/n^{\frac{1}{2}+\epsilon}$ *converge absolutely,* $\epsilon > 0$.

For, by Harnack's theorem, $\Sigma(a_n^2 + b_n^2)$ *converges, since it is equal to*

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2 dx.$$

Hence each of the series Σa_n^2, Σb_n^2 converges, and therefore so do also the series whose general terms are respectively

$$a_n^2 + \frac{1}{n^{1+2\epsilon}} \quad \text{and} \quad b_n^2 + \frac{1}{n^{1+2\epsilon}}.$$

But

$$\frac{|a_n|}{n^{\frac{1}{2}+\epsilon}} \leqq \frac{1}{2}\left(a_n^2 + \frac{1}{n^{1+2\epsilon}}\right),$$

therefore the series $\Sigma|a_n|/n^{\frac{1}{2}+\epsilon}$ converges. Similarly the series $\Sigma|b_n|/n^{\frac{1}{2}+\epsilon}$ converges. This proves the lemma.‡

* See Pringsheim, *loc. cit.*, pp. 375–6.
† Fatou, *loc. cit.* Harnack, *Math. Ann.*, xix. (1882), p. 225.
‡ See A. Pringsheim, *Münch. Ber.*, 30 (1900), p. 63.

§ 17. THEOREM 11.—*If in the interval* (q, ∞) *the function* f(u) *can be expressed as the product of two functions, one of which,* g(u), *is monotone decreasing with zero as limit at infinity, and the other,* h(u), *is any periodic function whose square is summable, then, provided*

$$\int_q^\infty \frac{g(u)}{u}du \quad \text{and} \quad \int_q^\infty \left|\frac{f(u)}{u}\right| du$$

both exist, we have

$$\int_0^\infty dv \int_q^\infty f(u)\sin uv\, du = \int_q^\infty \frac{f(u)}{u}du.$$

We shall, in the first instance, assume that the Fourier series of $h(u)$ is free of constant term.

Since the square of $h(u)$ is summable, the Fourier cœfficients a_n and b_n of the function $h(u)$ are such that the series, whose general terms are respectively $|a_n|/n$ and $|b_n|/n$, converge absolutely, by the lemma of § 16. Now, in any finite interval (q, Q) of values of u, the function $g(u)\sin uv$ has bounded variation, and $h(u)$ is summable, since $(h(u))^2$ is summable. Therefore* we may integrate the product term-by-term, using the not necessarily convergent Fourier series of $h(u)$. Thus

$$\int_q^{\mathrm{Q}} g(u)h(u)\sin uv\, du = \sum_{n=1}^{\infty}\int_q^{\mathrm{Q}} g(u)\sin uv(a_n\cos nu + b_n \sin nu)du \quad . \quad . \quad (1)$$

Let m be the first integer greater than 2B, so that, for all integers $n \geqq m$,

$$n - \mathrm{B} > \tfrac{1}{2}n,$$

and therefore, for all values of v in the closed interval (0, B),

$$n + v \geqq n - v \geqq n - \mathrm{B} > \tfrac{1}{2}n.$$

Hence, using the Second Theorem of the Mean to bring $g(q)$ outside the sign of integration, and expressing the two products of sines and cosines which appear in the integrand as the sum or difference of sines and cosines in the usual way, we get for the absolute value of the integral under the summation sign in (1) the following inequality, supposing $n \geqq m$:—

$$g(q)\left|\int_q^{\mathrm{Q}}(a_n \sin uv \cos nu + b_n \sin uv \sin nu)du\right| \leqq g(q)\left(\frac{1}{n+v}+\frac{1}{n-v}\right)(|a_n|+|b_n|)$$

$$= g(q)\frac{4}{n}(|a_n|+|b_n|) \quad . \quad . \quad . \quad . \quad (2)$$

Now, denoting the sum of the first $2(m-1)$ terms of the Fourier series of $h(u)$ by s_m, so that

$$s_m = \sum_{n=1}^{m-1}(a_n\cos nu + b_n \sin nu),$$

* W. H. Young, "On the Integration of Fourier Series" (1910), Theorem 2, presented to the London Mathematical Society.

we may write the equation (1) in the following form :—

$$\int_q^Q g(u) \sin uv\{h(u) - s_m\}du = \sum_{n=m}^{\infty} \int_q^Q g(u) \sin uv\{a_n \cos nu + b_n \sin nu\}du.$$

The inequality (2) shows that we may apply the lemma of § 15 to the infinite series on the right-hand side of this equation, bearing in mind that the individual integrals in this series continue to exist when we write infinity for Q. Thus we can assert that both sides of the last equation continue to exist when we write infinity for Q, and that the equation still holds when we put $Q = \infty$. Now, when this has been done, the right-hand side becomes a series of functions of v, which converges boundedly for all values of B in the closed interval (0, B). Hence we may integrate term-by-term with respect to v between the limits 0 and B, and assert that

$$\int_0^B dv\int_q^\infty g(u) \sin uv\{h(u) - s_m\}du = \sum_{n=m}^{\infty} \int_0^B dv\int_q^\infty g(u) \sin uv\{a_n \cos nu + b_n \sin nu\}du \quad (3)$$

so that both sides exist and are equal.

Again, since $g(u)u/u$ is monotone decreasing in the interval (q, Q), and $\cos Bu$ may evidently take the place of $\sin uv$ in the preceding argument,

$$\int_q^Q \frac{g(u)}{u} \cos Bu\{h(u) - s_m\}du = \sum_{n=m}^{\infty} \int_q^Q \frac{g(u)}{u} \cos Bu\{a_n \cos nu + b_n \sin nu\}du,$$

and by the same argument as that used above, this equation has a definite meaning, and remains true when we write infinity for Q.

But by a direct application of the extended form of the theorem quoted which holds for an infinite interval, the result just obtained is equally true if the factor $\cos Bu$ is omitted. Subtracting these two equations, we get

$$\int_q^\infty \frac{g(u)}{u}(1 - \cos Bu)\{h(u) - s_m\}du = \sum_{n=m}^{\infty} \int_q^\infty \frac{g(u)}{u}(1 - \cos Bu)(a_n \cos nu + b_n \sin nu)du.$$

But this equation may obviously be written in the form

$$\int_q^\infty du[g(u)\{h(u) - s_m\}\int_0^B \sin uv\, dv] = \sum_{n=m}^{\infty} \int_q^\infty du[g(u)\{a_n \cos nu + b_n \sin nu\}\int_0^B \sin uv\, dv] \quad (4)$$

both sides accordingly existing and having equal values.

Now, since $\int_q^\infty \frac{g(u)}{u}du$ exists, the corresponding terms of the infinite series on the right-hand sides of (3) and (4) have been already (§§ 13, 14) proved to be equal. Hence the right-hand sides of (3) and (4) are equal, and therefore the same is true of the left-hand sides.

But, again, for any integral value of n less than m it is equally true, and has been shown in the same corollaries, that

$$\int_q^\infty du\int_0^B g(u) \sin uv \begin{matrix}\cos nu\\ \cdot\\ \sin nu\end{matrix} dv = \int_0^B dv\int_q^\infty g(u) \sin uv \begin{matrix}\cos nu\\ \cdot\\ \sin nu\end{matrix} du,$$

INSTRUCTIONS TO AUTHORS.

The 'copy' should be written on large sheets of paper, on one side only and the pages should be clearly numbered. The MS. must be easily legible, preferably typewritten, and must be absolutely in its final form for printing; so that corrections in proof shall be as few as possible, and shall not cause overrunning in the lines or pages of the proof. All tables of contents, references to plates, or illustrations in the text, etc., must be in their proper places, with the page numbers left blank; and spaces must be indicated for the insertion of illustrations that are to appear in the text.

Illustrations.—All illustrations must be drawn in a form immediately suitable for reproduction; and such illustrations as can be reproduced by photographic processes should, so far as possible, be preferred. Drawings to be reproduced as line blocks should be made with Indian ink (deadened with yellow if of bluish tone), preferably on fine white bristol board, free from folds or creases; smooth, clean lines, or sharp dots, but no washes or colours, should be used. If the drawings are done on a large scale, to be afterwards reduced by photography, any lettering or other legend must be on a corresponding scale.

If an author finds it inconvenient to furnish such drawings, the Society will have the figures re-drawn at his expense; but this will cause delay.

When the illustrations are to form plates, a scheme for the arrangement of the figures (in quarto plates for the Transactions, in octavo for the Proceedings) must be given, and numbering and lettering indicated.

Proofs.—In general, a first proof and a revise of each paper will be sent to the author, whose address should be indicated on the MS. If further proofs are required, owing to corrections or alterations for which the printer is not responsible, the expense of such proofs and corrections will be charged against the author.

All proofs must, if possible, be returned within one week, addressed to *The Secretary, Royal Society,* 22–24 *George Street, Edinburgh,* and *not to the printer.* To prevent delay, authors residing abroad should appoint some one residing in this country to correct their proofs.

The author receives fifty of the reprints free, and may have any reasonable number of additional copies at a fixed scale of prices which will be furnished by the printer, who will charge him with the cost. To prevent disappointment, especially if the paper contains plates, the author should, immediately after receiving his first proof, notify *to the printer* the number of additional copies required.

Index Slips.—In order to facilitate the compilation of Subject Indices, and to secure that due attention to the important points in a paper shall be given in General Catalogues of Scientific Literature and in Abstracts by Periodicals, every author is requested to return to the Secretary, along with his final proof, a brief index (on the model given below), of the points in it which he considers new or important. These indices will be edited by the Secretary, and incorporated in Separate Index Slips, to be issued with each part of the Proceedings and Transactions.

MODEL INDEX.

Schäfer, E. A.—On the Existence within the Liver Cells of Channels which can be directly injected from the Blood-vessels. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

Cells, Liver,—Intra-cellular Canaliculi in.
E. A. Schäfer. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

Liver,—Injection within Cells of.
E. A. Schäfer. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

The Papers published in this Part of the PROCEEDINGS *may be had separately, on application to the Publishers, at the following prices:—*

No. XXVI. (begins in Pt. III.), Price	6d.	No. XXXIII., Price	8d.
No. XXVII., „	1s.	No. XXXIV., „	6d.
No. XXVIII., „	6d.	No. XXXV. (*Abstract*), „	6d.
No. XXIX., „	6d.	No. XXXVI., „	6d.
No. XXX., „	1s. 4d.	No. XXXVII., „	6d.
No. XXXI., „	1s. 2d.	No. XXXVIII., „	6d.
No. XXXII., „	9d.	No. XXXIX., „	8d.
		No. XL., „	1s. 8d.

PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

SESSION 1910–11.

Part V.] VOL. XXXI. [Pp. 577–782.

CONTENTS.

[*Continued on page* iv. *of Cover*

EDINBURGH:

PUBLISHED BY ROBERT GRANT & SON, 107 PRINCES STREET, AND WILLIAMS & NORGATE, 14 HENRIETTA STREET, COVENT GARDEN, LONDON.

MDCCCCXII.

Price Seven Shillings and Sixpence.

REGULATIONS REGARDING THE PUBLICATION OF PAPERS IN THE PROCEEDINGS AND TRANSACTIONS OF THE SOCIETY.

THE Council beg to direct the attention of authors of communications to the Society to the following Regulations, which have been drawn up in order to accelerate the publication of the Proceedings and Transactions, and to utilise as widely and as fairly as possible the funds which the Society devotes to the publication of Scientific and Literary Researches.

1. MANUSCRIPT OF PAPERS.—As soon as any paper has been passed for publication, either in its original or in any altered form, and has been made ready for publication by the author, it is sent to the printer.

2. METHOD OF PUBLICATION.—As soon as the final revise of a Transactions paper has been returned, or as soon as the sheet in which the last part of a Proceedings paper appears is ready for press, a certain number of separate copies or reprints, in covers bearing the title of the paper and the name of the author, are printed off and placed on sale. The date of such separate publication will be printed on each paper.

3. ADDITIONS TO A PAPER after it has been finally handed in for publication, if accepted by the Council, will be treated and dated as separate communications, and may, or may not, be printed immediately after the original paper.

4. BRIEF ABSTRACTS OF TRANSACTIONS PAPERS will be published in the Proceedings, provided they are sent along with the original paper.

5. SPECIAL DISCUSSION OF PAPERS ACCEPTED FOR PUBLICATION.—Where a paper has been accepted for publication, the Council may, with the consent of the author, select this paper for Special Discussion. In the case of such papers advanced proofs will be sent to the members of the Society desiring copies, and copies will be supplied to the author for distribution. A paper selected for Special Discussion will be marked with an asterisk (*) and placed first on the Billet for the day of reading. Any following papers for that day may be adjourned or held as read if the discussion prevents their being read.

6. COMMUNICATIONS NOT SUBMITTED FOR PUBLICATION, such as Demonstrations of Experiments, Statement of Scientific Problems, etc., may be received by the Council, and may also be selected for Special Discussion. The Council does not undertake to publish any notice of such communications in the Proceedings or Transactions of the Society.

[*Continued on page* iii *of Cover.*

so that both sides exist and are equal. Hence, adding all such equations for integral values of n less than m, after multiplying by suitable coefficients, we have

$$\int_q^\infty du \int_0^B g(u) s_m \sin uv\, dv = \int_0^B dv \int_q^\infty g(u) s_m \sin uv\, du.$$

Whence, expressing the fact already proved, that the left-hand sides of (3) and (4) are equal, and adding the equation so obtained to that last written down, we have

$$\int_0^B dv \int_q^\infty f(u) \sin uv\, du = \int_q^\infty du \int_0^B f(u) \sin uv\, dv = \int_q^\infty \frac{f(u)}{u}(1-\cos Bu)\, du.$$

Proceeding to the limit with B, we get, in this case, the required result, using the theorem of Riemann-Lebesgue (§ 10).

Next, let $\frac{1}{2}a_0$ be the constant term in the Fourier series of $h(u)$. Then, by what has just been proved,

$$\int_0^\infty dv \int_q^\infty g(u)\{h(u)-\tfrac{1}{2}a_0\} \sin uv\, du = \int_q^\infty \frac{1}{u} g(u)\{h(u)-\tfrac{1}{2}a_0\}\, du.$$

But, by Theorem 10,

$$\tfrac{1}{2}a_0 \int_0^\infty dv \int_q^\infty g(u) \sin uv\, du = \tfrac{1}{2}a_0 \int_q^\infty \frac{g(u)}{u}\, du.$$

Adding these two equations, the required result follows.

COR. 1.—*If* h(u) *is an odd function, the condition that* $\int_q^\infty \frac{g(u)}{u} du$ *exists may be omitted.*

For, in this case, the coefficients a_n are identically zero; therefore Theorem 9, § 13, is not used, while Theorem 10 does not require the condition in question.

COR. 2.—*If* h(u) *is bounded in the interval* (q, ∞), *the condition that* $\int_q^\infty \left|\frac{f(u)}{u}\right| du$ *should exist need not be explicitly mentioned in the enunciation.*

For the condition is satisfied, since $h(u)$ is bounded and $\int_q^\infty \frac{g(u)}{u} du$ exists, $g(u)$ being ≥ 0.

§ 18. It is clearly unnecessary for the argument in the preceding article that the square of $h(u)$ should be summable, provided the series whose general term is $(|a_n|+|b_n|)/n$ is convergent. More generally, applying instead of Theorem 2 of my paper on the "Integration of Fourier Series," the theorem on which that theorem itself is based, namely Theorem 6 of the companion paper,* we may replace the integer n by any positive quantity

* W. H. Young, "On the Theory of the Application of Expansions to Definite Integrals," presented to the London Mathematical Society.

k_n, having, as n increases, the unique limit infinity; the argument will remain valid, with the sole change that the choice of the integer m, immediately after the equation (1), must now be such that, for all values of $n \geqq m$,

$$k_n - B > \tfrac{1}{2}k_n .$$

We thus have the following extended form of the theorem:—

Theorem 11, bis.—*If for all values of* u *in the open interval* (q, ∞),

$$f(u) = g(u)h(u),$$

where g(u) *is monotone decreasing with zero as limit at infinity, and* h(u) *is a summable function whose Lebesgue integral, proper or improper, can be expanded in a series of the form*

$$\int^u h(u)du = B_0 u + \sum_{n=1}^{\infty}(A_n \cos k_n u + B_n \sin k_n u),$$

where the positive quantities k_n *have infinity as unique limit as* n *increases, and the series* $\sum\limits_{n=1}^{\infty}(|A_n| + |B_n|)$ *converges, then, if* $\int_q^\infty \left|\frac{f(u)}{u}\right| du$ *exists,*

$$\int_0^\infty dv \int_q^\infty f(u) \sin uv\, du = \int_q^\infty \frac{f(u)}{u} du,$$

provided $\int_q^\infty \frac{g(u)}{u} du$ *exists, or the coefficients* A_n *are identically zero.*

§ 19. If we make the more stringent condition that the series $\sum\limits_{n=1}^{\infty}|a_n|$ and $\sum\limits_{n=1}^{\infty}|b_n|$ both converge, where

$$a_n/k_n = B_n , \quad \text{and} \quad b_n/k_n = A_n ,$$

the series whose general term is

$$a_n \cos k_n u + b_n \sin k_n u$$

will converge uniformly, and may therefore be integrated term-by-term; thus we may take $h(u)$ to be the sum of this series, and it will then be a bounded function of u, so that $\int_q^\infty \left|\frac{f(u)}{u}\right| du$ will exist, provided $\int_q^\infty \frac{g(u)}{u} du$ exists.

Thus we get the following corollary:—

Cor.—*If the series whose general terms are respectively* $|a_n|$ *and* $|b_n|$ *both converge, and* h(u) *denote the sum of the series whose general term is*

$$a_n \cos k_n u + b_n \sin k_n u,$$

$k_1, k_2 \ldots . k_n$ *being any succession of positive quantities increasing without limit as* n *increases, then*

$$\int_0^\infty dv \int_q^\infty g(u)h(u) \sin uv\, du = \int_q^\infty \frac{g(u)h(u)}{u} du,$$

g(u) *being a monotone decreasing function with zero as limit at infinity, provided* $\int_q^\infty \frac{g(u)}{u} du$ *exists; or provided the coefficients* a_n *are identically zero, and* $\int_q^\infty \left| \frac{g(u)h(u)}{u} \right| du$ *exists.*

§ 20. If $g(u)$ is a function of bounded variation in the whole infinite interval (q, ∞) with zero as unique limit at infinity, and $P(u)$ and $-N(u)$ are the positive and negative variations of $g(u)$,

$$g(u) = P(u) - N(u) = (W - N(u)) - (W - P(u)),$$

where W is the common limit of $P(u)$ and $N(u)$ when u increases indefinitely. $W - N(u)$ and $W - P(u)$ are then both monotone decreasing functions of u, with zero as unique limit at infinity, and if either of them, when divided by u, may be integrated from q to infinity, so can the other, always supposing that $g(u)/u$ may be integrated from q to infinity.*

The total variation of $g(u)$ is $P(u) + N(u)$, which has at infinity the limit $2W$; thus $2W - P(u) - N(u)$, or, say, $\gamma(u)$, may be called *the total variation of* g(u) *measured from infinity.* Under the above circumstances, therefore, this function is a monotone decreasing function with zero as limit at infinity, and, after division by u, may be integrated from q to ∞. Conversely, if this function $\gamma(u)$, when divided by u, may be integrated from q to infinity, so may $g(u)$, as well as its positive and negative variations measured from infinity.

In this way we have at once the following extension of the preceding theorems:—

Theorem 12.—*If throughout the interval* (q, ∞),

$$f(u) = g(u)h(u),$$

where g(u) *is a function of bounded variation in the whole infinite interval, with zero as limit at infinity, and* h(u) *is any periodic summable function whose Fourier coefficients* a_n *and* b_n *are such that the series whose general terms are respectively* $|a_n|/n$ *and* $|b_n|/n$ *both converge, then, provided* $|f(u)/u|$ *may be integrated from* q *to infinity,*

$$\int_0^\infty dv \int_q^\infty f(u) \sin uv \, du = \int_q^\infty \frac{f(u)}{u} du,$$

* There is a certain want of clearness on this point in Pringsheim's remarks on p. 397, as to his condition b_∞, referred to again on p. 400; if $g(u)$ is only of bounded variation, not monotone, the existence of the integrals $\int_q^\infty \frac{W - P(u)}{u} du$ and $\int_q^\infty \frac{W - N(u)}{u} du$ *entails* that of $\int_q^\infty \frac{g(u)}{u} du$, but it is not a *consequence* of it.

provided either h(u) *is an odd function, or* $\int_q^\infty \frac{\gamma(u)}{u}du$ *exists,* $\gamma(u)$ *denoting the total variation of* g(u) *measured from infinity.*

THEOREM 12, bis.—*If throughout the interval* (q, ∞)

$$f(u) = g(u)h(u),$$

where g(u) *has bounded variation in the whole infinite interval and approaches the limit zero at infinity, and* h(u) *is any summable function whose Lebesgue integral, proper or improper, can be expanded in a series of the form*

$$\int^u h(u)du = B_0 u + \sum_{n=1}^{\infty} (A_n \cos k_n u + B_n \sin k_n u),$$

where the positive quantities k_n *have infinity as unique limit as* n *increases, and the series whose general term is* $|A_n| + |B_n|$ *converges, then, if* $\int_q^\infty \left|\frac{f(u)}{u}\right| du$ *exists,*

$$\int_0^\infty dv \int_q^\infty f(u) \sin uv\, du = \int_q^\infty \frac{f(u)}{u} du,$$

provided the coefficients A_n *are identically zero, or* $\int_q^\infty \frac{\gamma(u)}{u}du$ *exists,* $\gamma(u)$ *denoting the total variation of* g(u) *measured from infinity.*

This last theorem includes all the Theorems 8–12 as special cases; moreover, we may, in the enunciations of those theorems, change the monotone function into a function of bounded variation in the whole infinite interval, provided we add the condition that $\int_q^\infty \frac{\gamma(u}{u} du$ should exist, $\gamma(u)$ denoting the total variation of function in question measured from infinity.

SECTION 4.

On the Fourier Cosine-integral $\int_0^\infty dv \int_{-\infty}^\infty f(u) \cos uv\, du$.

§ 21. As in Section 3, the discussion reduces itself to that of

$$\int_0^\infty dv \int_q^\infty f(u) \cos uv\, du.$$

The theorems which occur are in striking parallelism to those of the preceding section.

§ 22. THEOREM 13.—*If* $\int_q^\infty |f(u)| du$ *exists, then*

$$\int_0^\infty dv \int_q^\infty f(u) \cos uv\, du = 0.$$

By exactly the same argument as in the proof of the parallel theorem in Section 3 (§ 12), we arrive at the equation

$$\int_0^B dv \operatorname*{Lt}_{Q=\infty} \int_q^Q f(u) \cos uv\, du = \operatorname*{Lt}_{Q=\infty} \int_q^Q du \int_0^B f(u) \cos uv\, dv = \int_q^\infty \frac{f(u)}{u} \sin Bu\, du,$$

which, by the theorem of Riemann-Lebesgue (§ 10), has the unique limit zero when B is indefinitely increased. This proves the theorem.

Cor. 1.—*If* f(u) = g(u)h(u) *where* $\int_q^\infty |\,g(u)\,|\,du$ *exists, and* h(u) *oscillates finitely at infinity, the same result holds.**

Cor. 2.†—*If* $\int_0^\infty |\,f(u)\,du\,|$ *exists, then, supposing the integral*

$$\int_0^\infty dv \int_0^\infty f(u) \cos uv\, du$$

to exist, it is certainly equal to f(0), *provided only that the origin does not belong to the exceptional set of content zero at which* f(u) *is not the differential coefficient of its integral.*

We have, in fact, only to combine the present theorem with that of § 8.

§ 23. Theorem 14.—*If in the interval* (q, ∞) *the function* f(u) *is monotone decreasing, and approaches the unique limit zero at infinity, then*

$$\int_0^\infty dv \int_q^\infty f(u) \cos uv\, du = 0.$$

Since $\int_a^b \cos uv\, du = (\sin vb - \sin va)/v$, it follows by the Second Theorem of the Mean that, if $a < b$ and $v > e > 0$,

$$\int_a^b f(u) \cos uv\, du \leqq 2f(a)/e.$$

Hence, denoting for shortness the integrand $f(u) \cos uv$ by $F(u, v)$, it follows, firstly, that $\int_Q^\infty F(u, v) du$ exists for each positive value of v (since, by hypothesis, $f(u)$ has zero as unique limit at infinity); secondly, it is, for all values of v greater than e, a bounded function of v; and, thirdly, $\int_Q^A f(u) \cos uv\, du$ converges boundedly to its limit for all such values of v.

Hence we may integrate, and get

$$\int_e^B dv \int_Q^\infty F(u, v) du = \int_e^B dv \operatorname*{Lt}_{A=\infty} \int_Q^A F(u, v) du = \operatorname*{Lt}_{A=\infty} \int_e^B dv \int_Q^A F(u, v) du = \operatorname*{Lt}_{A=\infty} \int_Q^A du \int_e^B F(u, v) dv$$
$$= \int_Q^\infty f(u) \frac{\sin Bu - \sin eu}{u} du,$$

which is numerically $\leqq 2\pi f(Q+0)$, using the Second Theorem of the Mean, and noticing that

$$\left| \int_Q^{Q'} \frac{\sin ku}{u} du \right| = \left| \int_{kQ}^{kQ'} \frac{\sin u}{u} du \right| \leqq \pi.$$

Hence,

$$\int_e^B dv \int_q^Q F(u, v) du - 2\pi f(Q+0) \leqq \int_e^B dv \int_q^\infty F(u, v) du \leqq \int_e^B dv \int_q^Q F(u, v) du + 2\pi f(Q+0).$$

* This includes that case of Pringsheim's theorem, *loc. cit.*, p. 399, § 5, in which the first of Pringsheim's conditions at infinity (p. 391) is satisfied by one of the factors of the integrand. The other factor is, in consequence of Pringsheim's further assumptions, a bounded function of all the variables.

† This may be compared with a theorem given by Michel Plancherel on p. 42 of his "Contribution à l'étude de la représentation d'une fonction arbitraire par des intégrales définies," 1910, *Rend. di Palermo*, xxx. pp. 1–47. In Plancherel's theorem $\int_0^\infty \{f(u)\}^2 du$ exists, instead of, as here, $\int_0^\infty |\,f(u)\,|\,du$.

Hence, as in the proof of the corresponding theorem in Section 3 (§ 13), letting, first, e diminish to zero in such a manner as to get an unique limit for the central integral, and then letting Q move off to infinity in such a way as to get an unique limit for the integral which appears in both the extreme members of this inequality, we see that these limits are equal, and hence that, however e and Q approach their limits, the two integrals have the same unique limit—that is, $\int_0^B dv \int_q^\infty F(u, v)du$ and $\int_q^\infty du \int_0^B F(u, v)dv$ both exist and are equal.

Hence

$$\int_0^\infty dv \int_q^\infty f(u) \cos uv\, du = \int_q^\infty \frac{f(u)}{u} \sin Bu\, du,$$

which, by the Second Theorem of the Mean, is numerically $\leq 2f(q)/qB$. Hence, letting B increase without limit, the required result follows.

§ 24. Theorems 15 and 16.—*If in the interval* (q, ∞) *the function* f(u) *can be expressed as the product of two functions, one of which,* g(u), *is monotone decreasing with zero as unique limit at infinity, and the other,* h(u), *is of the form sin* ku, *or of the form cos* ku, *then*

$$\int_0^\infty dv \int_q^\infty f(u) \cos uv\, du = 0,$$

provided in the former case $\int_q^\infty \frac{g(u)}{u}$ *exists.*

These are, in fact, the results referred to at the end of the proofs of the corresponding theorems in Section 3 (§§ 13 and 14), as got by subtraction and addition respectively of the results there added and subtracted.

Cor.—*When the upper limit of integration with respect to* v *is finite, we may reverse the order of integration.*

§ 25. Theorem 17.—*If, in the interval* (q, ∞), *the function* f(u) *can be expressed as the product of two functions, one of which,* g(u), *is monotone decreasing with zero as limit at infinity, and the other,* h(u), *is any periodic function whose square is summable, then, provided*

$$\int_q^\infty \frac{g(u)}{u} du \quad \text{and} \quad \int_q^\infty \left| \frac{f(u)}{u} \right| du$$

both exist, we have

$$\int_0^\infty dv \int_q^\infty f(u) \cos uv\, du = 0.$$

We shall, in the first instance, assume that the Fourier series of $h(u)$ is free of constant term.

Since the square of $h(u)$ is summable, the Fourier coefficients of $h(u)$, say a_n and b_n, are such that the series $\Sigma a_n/n$ and $\Sigma b_n/n$ converge absolutely, by the lemma of § 16.

Now, in any finite interval (q, Q) of values of u, the function

$g(u)\cos uv$ has bounded variation, and $h(u)$ is summable, since $(h(u))^2$ is summable. Therefore * we may integrate the product term-by-term, using the not necessarily convergent Fourier series of $h(u)$. Thus

$$\int_q^Q g(u)h(u)\cos uv\,du = \sum_{n=1}^{\infty}\int_q^Q g(u)\cos uv(a_n\cos nu + b_n\sin nu)du \quad . \quad . \quad (1)$$

Let m be the first integer $\geqq 2B$, so that, for all integers $n \geqq m$,

$$n - B \geqslant \tfrac{1}{2}n.$$

For such values of n, provided $0 \leqq v \leqq B$,

$$n + v \geqq n - v \geqq \tfrac{1}{2}n,$$

Therefore, expressing the integrand of the integral under the summation sign in (1) in the usual way as a sum of sines and cosines, after removing the factor $g(u)$ outside the integral sign, by the Second Theorem of the Mean, we get, for $n \geqq m$,

$$\int_q^Q g(u)\cos uv(a_n\cos nu + b_n\sin nu)du \leqq g(q)\left(\frac{1}{n+v} + \frac{1}{n-v}\right)(\,|\,a_n\,| + |\,b_n\,|\,)$$
$$\leqq 4g(q)(\,|\,a_n\,| + |\,b_n\,|\,)/n \quad . \quad . \quad . \quad . \quad (2)$$

Now, denoting the sum of the first $2(m-1)$ terms of the Fourier series of $h(u)$ by s_m, we may write the equation (1) in the following form :—

$$\int_q^Q g(u)\cos uv\{h(u) - s_m\}du = \sum_{n=m}^{\infty}\int_q^Q g(u)\cos uv\{a_n\cos nu + b_n\sin nu\}du.$$

The inequality (2) shows that we may apply the lemma of § 15 to the infinite series on the right-hand side of this equation, bearing in mind that the individual integrals in this series continue to exist when we write infinity for Q. Thus we can assert that both sides of the last equation continue to exist when we write infinity for Q, and that the equation still holds when we do this. Now, when this has been done, the right-hand side converges boundedly for all values of v in the closed interval (0, B). Hence we may integrate term-by-term with respect to v, and assert that

$$\int_0^B dv\int_q^\infty g(u)\cos uv\{h(u) - s_m\}du = \sum_{n=m}^{\infty}\int_q^Q g(u)\cos uv\{a_n\cos nu + b_n\sin nu\}du, \quad (3)$$

so that both sides exist and are equal.

Again, since $g(u)/u$ is monotone decreasing, and $\sin Bu$ may evidently take the place of $\cos uv$ in the preceding argument,

$$\int_q^Q \frac{g(u)}{u}\sin Bu\{h(u) - s_m\}du = \sum_{n=m}^{\infty}\int_q^Q \frac{g(u)}{u}\sin Bu\{a_n\cos nu + b_n\sin nu\}du\,;$$

and, by the same argument as that used above, this equation has a definite

* *Loc. cit.*, *supra*, § 17.

meaning and remains true when we write infinity for Q. Replacing $\frac{\sin Bu}{u}$ by its integral form, this gives us

$$\int_q^Q du[g(u)\{h(u)-s_m\}\int_0^B \cos uv\, dv] = \sum_{n=m}^{\infty}\int_q^Q du[g(u)\{a_n \cos nu + b_n \sin nu\}\int_0^B \cos uv\, dv] \quad (4)$$

both sides accordingly existing and having equal values.

Now, since $\int_q^\infty \frac{g(u)}{u}du$ exists, the corresponding terms of the infinite series on the right-hand sides of (3) and (4) have been already proved (§ 24) to be equal, and therefore the same is true of the left-hand sides.

But, again, for any integral value of n less than m it is equally true, and has been shown in the same theorems, that

$$\int_q^\infty du\int_0^B g(u)\cos uv \begin{matrix}\cos nu\\ \cdot\\ \sin nu\end{matrix} dv = \int_0^B dv\int_q^\infty g(u)\cos uv \begin{matrix}\cos nu\\ \cdot\\ \sin nu\end{matrix} du,$$

so that both sides exist and are equal.

Adding all such equations for integral values of n less than m, after multiplying by suitable coefficients, we have

$$\int_q^\infty du\int_0^B g(u)s_m \cos uv\, dv = \int_0^B dv\int_q^\infty g(u)s_m \cos uv\, du.$$

Hence, expressing the fact, already proved, that the left-hand sides of (3) and (4) are equal, and adding the equation so obtained to that last written down, we have

$$\int_0^B dv\int_q^\infty f(u)\cos uv\, du = \int_q^\infty du\int_0^B f(u)\sin uv\, dv = \int_q^\infty \frac{f(u)}{u}\sin Bu\, du.$$

Proceeding to the limit with B, we get, in this case, the required result, using the theorem of Riemann-Lebesgue (§ 10).

Next, let $\frac{1}{2}a_0$ be the constant term in the Fourier series of $h(u)$. Then, by what has just been proved,

$$\int_0^\infty dv\int_q^\infty g(u)\{h(u)-\tfrac{1}{2}a_0\}\cos uv\, du = 0.$$

But, by Theorem 14,

$$\tfrac{1}{2}a_0\int_0^\infty dv\int_q^\infty g(u)\cos uv\, du = 0.$$

Adding these two equations, the required result follows.

Cor. 1.—*If* h(u) *is an even function, the condition that* $\int_q^\infty \frac{g(u)}{u}du$ *should exist, may be omitted.*

For the coefficients b_n are then identically zero, therefore Theorem 15 is then not used, while Theorem 16 does not require the above condition.

Cor. 2.—*If* h(u) *is a bounded function, the condition that* $\int_q^\infty \left|\frac{f(u)}{u}\right|du$ *exists need not be mentioned in the enunciation.*

For, in this case, $\int_q^\infty \left|\frac{f(u)}{u}\right| du$ necessarily exists, provided $\int_q^\infty \frac{g(u)}{u}$ does so.

§ 26. As in § 20, we can extend the preceding theorems to the case when $g(u)$ has bounded variation in the whole infinite interval; and $\gamma(u)$, the total variation of $g(u)$ measured from infinity, after division by u, can be integrated from q to infinity. We thus have the following theorem:—

Theorem 18.—*If throughout the interval* (q, ∞),

$$f(u) = g(u)h(u),$$

where g(u) *is a function of bounded variation in the whole infinite interval, with zero as limit at infinity, and* h(u) *is any periodic summable function whose Fourier coefficients* a_n *and* b_n *are such that the series whose general terms are respectively* $|a_n|/n$ *and* $|b_n|/n$ *both converge, then, provided* $|f(u)/u|$ *may be integrated from* q *to infinity*

$$\int_0^\infty dv \int_q^\infty f(u) \cos uv\, du = 0,$$

provided, in addition, when h(u) *is not an even function,* γ(u), *that the total variation of* g(u) *measured from infinity, after division by* u, *may be integrated from* q *to infinity.*

§ 27.—As in discussing the corresponding theorems for the sine-integral, it is evidently unnecessary for the argument of the preceding article that the square of $h(u)$ should be summable, provided the series whose general term is $(|a_n| + |b_n|)/n$ converges. Thus we have, as in Section 3, § 18, an extended theorem, as follows:—

Theorem 18, bis.—*If for all values of* u *in the open interval* (q, ∞),

$$f(u) = g(u)h(u),$$

where g(u) *has bounded variation in the whole infinite interval, and approaches the limit zero at infinity, while* h(u) *is any summable function whose Lebesgue integral, proper or improper, can be expanded in a series of the form*

$$\int^u h(u)du = A_0 u + \sum_{n=1}^{\infty} (A_n \cos k_n u + B_n \sin k_n u),$$

the positive quantities k_n *increasing without limit with* n, *and the series whose general term is* $|A_n| + |B_n|$ *converging, then, if* $\int_q^\infty \left|\frac{f(u)}{u}\right| du$ *exists,*

$$\int_0^\infty dv \int_q^\infty f(u) \sin uv\, du = 0,$$

provided either the coefficients B_n *are identically zero, or* $\int_q^\infty \frac{\gamma(u)}{u} du$ *exists,* γ(u) *denoting the total variation of* g(u) *measured from infinity.*

§ 28. As at the corresponding point in Section 3 (§ 19), we have now the following corollary:—

Cor. 1.—*If the series whose general terms are respectively* $|\mathrm{a_n}|$ *and* $|\mathrm{b_n}|$ *both converge, and* h(u) *denote the sum of the series whose general term is*

$$a_n \cos k_n u + b_n \sin k_n u,$$

the positive quantities $\mathrm{k_n}$ *increasing without limit with* n, *then if* $\int_q^\infty \left|\frac{g(u)}{u}\right| du$ *exists,*

$$\int_0^\infty dv \int_q^\infty g(u)h(u) \cos uv = 0,$$

g(u) *denoting a function of bounded variation in the whole infinite interval, having zero as unique limit at infinity, provided, in addition, either the coefficients* $\mathrm{b_n}$ *are identically zero, or* $\int_q^\infty \frac{\gamma(u)}{u} du$ *exists,* γ(u) *being the total variation of* g(u) *measured from infinity.*

If we write $(u+x)$ for u, $c_n \cos r_n$ for a_n, and $c_n \sin r_n$ for b_n, we get, as a second corollary, Pringsheim's result *:—

Cor. 2.—*If the series* $\sum_{n=1}^{\infty} \mathrm{c}_n$ *converges absolutely and* h(u+x) *denote the sum of the series whose general term is*

$$c_n \cos \{k_n(u+x) + r_n\},$$

then if g(u) *is a function of bounded variation in the whole infinite interval, with zero as limit at infinity, and* $\int_q^\infty \left|\frac{g(u)}{u}\right| du$ *exists,*

$$\int_0^\infty dv \int_q^\infty g(u)h(u+x) \cos uv \, du = 0,$$

provided, in addition, either $\mathrm{r_n} = 0$, (mod. π), *or* $\int_q^\infty \frac{\gamma(u)}{u} du$ *exists,* γ(u) *being the total variation of* g(u), *measured from infinity.*†

* *Loc. cit.*, pp. 399–403. Pringsheim's proof, as printed, appears to require further justification. At the top of p. 402 the author divides a certain infinite summation into two parts : (*a*) the sum of the first *n* terms, and (*b*) the remainder. He then draws the conclusion that the part (*a*), being the sum of a finite number of terms, behaves in the desired manner, when the quantity B, which also appears in the summation, increases indefinitely; *n* has, however, previously been chosen so as to satisfy two inequalities, viz. (25) and (26) at the bottom of p. 400, of which the former inequality makes *n* increase indefinitely with B. Again, the conclusion, drawn later, as to the remainder also depends implicitly on the inequality (25), since it involves explicitly (27) and (28); thus the condition (25) cannot be simply dispensed with at this stage of the proof.

† The condition that $\int_q^\infty \frac{g(u)}{u} du$ exists is stated by Pringsheim (p. 397, referred to on p. 400 as Condition *b*) to be sufficient; see above, footnote to § 20. The alternative condition, $r_n = 0$, is not mentioned by Pringsheim.

Added 2nd May 1911.

Professor Pringsheim asks me to say that, in consequence of my criticisms, he is amending his proof, and that the corrections needed will shortly appear in the form of a Note in the *Mathematische Annalen.*

(*Issued separately, July* 11, 1911.)

XLI.—On Sommerfeld's Form of Fourier's Repeated Integrals. By W. H. Young, Sc.D., F.R.S. *Communicated by* Professor G. A. Gibson.

(MS. received January 4, 1911. Read February 6, 1911.)

§ 1. In his treatise on *Fourier Series and Integrals* Carslaw quotes without proof Sommerfeld's theorem that

$$\underset{t=0}{\mathrm{Lt}}\int_0^\infty dv\int_0^p f(u)\cos uv\, e^{-kv^2t}du=\frac{\pi}{2}f(+0),$$

when the limit on the right-hand side exists.* In applied mathematics, he remarks, it is this limit, rather than the corresponding Fourier repeated integral $\int_0^\infty dv\int_0^p f(u)\cos uv\, du$, which occurs.

In the present paper I propose to extend this result in various ways. After proving Sommerfeld's result on the general hypothesis, not considered by him, that the integral is a Lebesgue integral, I show that the limit in question is $\frac{\pi}{2}f(0)$, whenever the origin is a point at which $f(u)$ is the differential coefficient of its integral, and I obtain the corresponding results for

$$\underset{t=0}{\mathrm{Lt}}\int_0^\infty dv\int_0^p f(u)\sin uv\, e^{-kv^2t}du.$$

In all their generality these statements are only true when the interval $(0, p)$ is a finite one. I then show how, under a variety of hypotheses with respect to the nature of $f(x)$ at infinity, they can be extended so as to be still true when $p=+\infty$. These hypotheses correspond precisely to those which have been proved † to be sufficient for the corresponding statements as to the Fourier sine and cosine repeated integrals in their usual forms,

$$\int_0^\infty dv\int_0^\infty f(u)\cos uv\, du \quad \text{and} \quad \int_0^\infty dv\int_0^\infty f(u)\sin uv\, du.$$

They are as follows:—

(i.) $\int_q^\infty |f(u)|\, du$ exists,

* *Cf.* Carslaw, *op. cit.*, p. 186. The theorem as there stated differs only in form from the above, from which it may be easily deduced. The proviso as to the existence of the limit $f(+0)$ is moreover, though tacitly assumed, not expressly stated in the enunciation.

† A. Pringsheim, "Ueber neue Gültigkeitsbedingungen für die Fouriersche Integralformel," 1909, *Math. Ann.*, lxviii. pp. 367–408. W. H. Young, "On Fourier's Repeated Integrals," 1910, presented to the Royal Society of Edinburgh.

or

(ii.) $f(u)$ is monotone decreasing with zero as limit at infinity,

or

(iii.) $f(u)$ is the product of two functions, one of which, $g(u)$, is monotone decreasing with zero as limit at infinity, and the other, $h(u)$, is a sine or a cosine or any periodic function whose square is summable in every finite interval, provided in the latter case $\int_q^\infty \frac{f(u)}{u} du$ exists.

Still more general forms may be given to the sufficient conditions (iii.). Thus the square of $h(u)$ need not be summable, provided $h(u)$ is summable, and the series whose general terms are $|A_n|$ and $|B_n|$ are convergent, where A_n and B_n are constants such that

$$\int^u h(u)du = A_0 u + \sum_{n=1}^{\infty} (A_n \cos k_n u + B_n \sin k_n u),$$

the quantities k_n being positive and increasing without limit as n increases. In particular, therefore, this is true if the series whose general terms are a_n/n and b_n/n are absolutely convergent, a_n and b_n being the Fourier constants of $h(u)$.

I have not in the present paper thought it worth while to call special attention to the case when $f(u)$ or $g(u)$ is a function of bounded variation. This case belongs to the more general case when $f(u)$ or $g(u)$ is the algebraic sum of a number of functions belonging to the types above specified, multiplied by $+1$ or -1.

In conclusion, I here give the statement of a theorem which, though now well known, is rarely stated in its most general form, and which is continually used throughout the present paper.*

"If $s_n(x)$ is a bounded function of (n, x), and, except at a possible set of zero content of values of x, converges to $f(x)$, and $g(x)$ is any function possessing a Lebesgue integral, proper or improper, in the finite or infinite interval (a, b), then the sequence of Lebesgue integrals $\int_a^b s_n(x)g(x)dx$ converges uniformly to $\int_a^b f(x)g(x)dx$."

§ 2. Theorem.—*If* f(u) *is any summable function in the interval* (0, p), *having at the origin the unique limit* f (+0), *then*

$$\underset{t=0}{\mathrm{Lt}} \int_0^\infty dv \int_0^p e^{-kv^2 t} f(u) \cos uv\, du = \tfrac{1}{2}\pi f(+0).$$

* The statement and indications of proof of this theorem are given in my recent paper on the "Theory of the Application of Expansions to Definite Integrals," § 4, presented to the London Mathematical Society.

For, since $f(u)$ is a summable function of u alone, it certainly possesses finite repeated integrals of equal value, therefore,* since $e^{-kv^2t}\cos uv$ is a bounded function of (u, v),

$$\int_0^B dv \int_0^p e^{-kv^2t} f(u) \cos uv\, du = \int_0^p du \int_0^B e^{-kv^2t} f(u) \cos uv\, dv \quad . \quad . \quad (1)$$

Again, since

$$\left| \int_0^B e^{-kv^2t} \cos uv\, dv \right| \leqq \int_0^\infty e^{-kv^2t}\, dv,$$

and is therefore a bounded function of (B, u) in the interval $0 \leqq \mathrm{B} \leqq \infty$, $0 \leqq u \leqq p$, and $f(u)$ is summable, it follows † that, if in (1) we let B approach infinity, we may, on the right, introduce the limit under the first sign of integration, since $\int_0^\infty e^{-kv^2t} \cos uv\, dv$ exists. We thus get

$$\begin{aligned}\int_0^\infty dv \int_0^p e^{-kv^2t} f(u) \cos uv\, du &= \int_0^p du\, f(u) \int_0^\infty e^{-kv^2t} \cos uv\, dv \\ &= \int_0^p f(u) \sqrt{\frac{\pi}{4kt}}\, e^{-\frac{u^2}{4kt}} du \\ &= \sqrt{\pi} \int_0^{p/\sqrt{4kt}} f(u\sqrt{4kt}) e^{-u^2} du \quad . \quad . \quad . \quad (2)\end{aligned}$$

Now, $\int_0^\infty e^{-u^2} du$ exists and $= \frac{1}{2}\sqrt{\pi}$, and $f(u\sqrt{4kt})$ converges boundedly to $f(+0)$. Hence ‡ the above expression has an unique limit, when the two t's which occur are regarded as approaching their limit zero independently. Thus we get the required equality.

Cor.
$$\underset{t=0}{\mathrm{Lt}} \int_0^\infty dv \int_p^q e^{-kv^2t} f(u) \cos uv\, du = 0,$$

provided p *and* q *have the same sign, and* f(u) *is summable in the interval* (p, q). This follows at once from the above theorem, putting $f(u) = 0$, when $u < p$.

§ 3. Theorem.—*If* f(u) *is a summable function, and the origin does not belong to the exceptional set of content zero at which* f(u) *is not the differential coefficient of its integral,*

$$\underset{t=0}{\mathrm{Lt}} \int_0^\infty dv \int_0^p e^{-kv^2t} f(u) \cos uv\, du = \frac{\pi}{2} f(0).$$

As in the preceding proof we obtain equation (2), in which for shortness

* W. H. Young, "On the Change of Order of Integration in an Improper Repeated Integral," 1910, *Trans. Camb. Phil. Soc.*, vol. xxi. p. 364, § 4.

† By the theorem enunciated at the end of § 1.

‡ W. H. Young, "On a Theorem of Tannery's and its Analogue in the Theory of Definite Integrals," 1910, *Mess. Math.*, p. 58, Cor. 2. See also Bromwich's *Introduction to the Theory of Infinite Series* (1908), p. 438, where the condition of uniform convergence may be changed to bounded convergence.

we write n for $1/\sqrt{4kt}$, and so get, omitting for the present the factor $\sqrt{\pi}$,

$$\int_0^{np} f(u/n)e^{-u^2}du,$$

as the expression whose limit when n increases indefinitely is sought.

Integrating by parts, and denoting $\int_0^u f(u)du$ by $\mathrm{F}(u)$, this becomes

$$\left[n\mathrm{F}(u/n)e^{-u^2}\right]_0^{pn} + 2\int_0^{pn}\left\{\frac{n}{u}(\mathrm{F}(u/n)\right\}u^2e^{-u^2}du \quad . \quad . \quad . \quad (3)$$

Now the quantity in square brackets is zero at the inferior limit of integration, and at the superior limit is $n\mathrm{F}(p)e^{-p^2n^2}$, which approaches zero as limit when n increases indefinitely. Thus we have only to retain the integral in (3), the first factor of the integrand of which has by hypothesis the limit $f(0)$, and, moreover, approaches this limit boundedly, since $\mathrm{F}(x)/x$ is a continuous function with an unique limit at $x=0$. Hence we can again use the last theorem referred to in the proof of the preceding theorem, since

$$\int_0^\infty 2u^2e^{-u^2}du = \int_0^\infty(-u)d(e^{-u^2}) = \left[ue^{-u^2}\right]_0^\infty + \int_0^\infty e^{-u^2}du = \frac{\sqrt{\pi}}{2}.$$

We thus get for (3) the limit $\frac{\sqrt{\pi}}{2}f(0)$, which, replacing the omitted factor, proves the required result.

§ 4. THEOREM.—*If* f(u)/u *is summable in the interval* (p, q), *where* $0 \leqq p < q$, *then*

$$\underset{t=0}{\mathrm{L}t}\int_0^\infty dv\int_p^q e^{-kv^2t}f(u)\sin uv\,du = \int_p^q\frac{f(u)}{u}du.$$

For since $e^{-kv^2t}u\sin uv$ is a bounded function of (u, v), and $f(u)/u$ is summable,

$$\int_0^{\mathrm{B}}dv\int_p^q e^{-kv^2t}f(u)\sin uv\,du = \int_p^q du\,f(u)\int_0^{\mathrm{B}}e^{-kv^2t}\sin uv\,dv.$$

Also, since $\left|\int_0^{\mathrm{B}}e^{-kv^2t}u\sin uv\,dv\right| \leqq q\int_0^\infty e^{-kv^2t}dv$ is a bounded function of (B, u) in the rectangle $0 \leqq \mathrm{B} \leqq \infty$, $p \leqq u \leqq q$, and $f(u)/u$ is a summable function, we have as in § 2, since $\int_0^\infty e^{-kv^2t}u\sin uv\,dv$ exists,

$$\int_0^\infty dv\int_p^q e^{-kv^2t}f(u)\sin uv\,du = \int_p^q du\frac{f(u)}{u}\int_0^\infty e^{-kv^2t}u\sin uv\,dv \quad . \quad . \quad (1)$$

Now

$$\begin{aligned}\int_0^\infty e^{-kv^2t}u\sin uv\,dv &= -\left[e^{-kv^2t}\cos uv\right]_0^\infty + \int_0^\infty 2ktv\,e^{-kv^2t}\cos uv\,dv\\ &= 1 + 2\int_0^\infty v\,e^{-v^2}\cos\frac{uv}{\sqrt{kt}}dv,\\ &= 1 + 2\int_0^\infty ve^{-v^2}\cos\mathrm{B}'v\,dv,\end{aligned}$$

where, as t approaches zero, B′ approaches infinity, provided u is not zero. Hence, by the extended theorem of Riemann-Lebesgue,* the integral last written down has, as t approaches zero, the unique limit zero when u is not zero. Moreover, since, whether u is zero or not, it is numerically $\leq \int_0^\infty ve^{-v^2}dv$, it is a bounded function of the ensemble (u, t). Hence we may write, by the preceding equality,

$$\underset{t=0}{\mathrm{Lt}}\int_p^q du\frac{f(u)}{u}\int_0^\infty e^{-kv^2t}u\sin uv\,dv = \int_p^q du\frac{f(u)}{u}\underset{t=0}{\mathrm{Lt}}\int_0^\infty e^{-kv^2t}u\sin uv\,dv$$
$$= \int_p^q \frac{f(u)}{u}du.$$

By (1) this proves the theorem, whether p is zero or not.

§ 5. THEOREM.—*If* $\int_q^\infty |\mathrm{f(u)}|\,\mathrm{du}$ *exists, then*

$$\underset{t=0}{\mathrm{Lt}}\int_0^\infty dv\int_q^\infty e^{-kv^2t}\cos uv\,du = 0.$$

For

$$\int_0^{\mathrm{B}} dv\int_{\mathrm{A}}^\infty e^{-kv^2t}f(u)\cos uv\,du$$

is numerically not greater than the result of changing $f(u)\cos uv$ into $|f(u)|$, that is, $\int_0^{\mathrm{B}} e^{-kv^2t}dv \times \int_{\mathrm{A}}^\infty |f(u)|\,du$, which, in consequence of the hypothesis made, has the unique limit zero when A is indefinitely increased. Therefore

$$\underset{\mathrm{A}=\infty}{\mathrm{Lt}}\int_0^{\mathrm{B}} dv\int_{\mathrm{A}}^\infty e^{-kv^2t}f(u)\cos uv\,du = 0.$$

But this suffices to justify us in reversing the order of integration,† and writing

$$\int_0^{\mathrm{B}} dv\int_q^\infty e^{-kv^2t}f(u)\cos uv\,du = \int_q^\infty du\int_0^{\mathrm{B}} e^{-kv^2t}f(u)\cos uv\,dv \quad . \quad . \quad (1)$$

Now the integral with respect to v on the right-hand side of (1) is a bounded function of (u, B), since it is numerically $\leq \int_0^\infty e^{-kv^2t}dv$, and converges, since $\int_0^\infty e^{-kv^2t}f(u)\cos uv\,du$ exists. Also $f(u)$ is summable in the infinite interval (q, ∞). Hence, proceeding to the limit infinity with B, and zero with t, by the theory of the integration of sequences, (1) gives us the following:—

* This extension to an infinite interval of integration may be found in my recent paper on "Fourier's Repeated Integrals," § 10, presented to the Royal Society of Edinburgh. For the original theorem see B. Riemann, *Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe*, § 10; and H. Lebesgue, *Leçons sur les séries trigonométriques*; also Hobson's *Theory of Functions of a Real Variable*, p. 674.

† See my paper on "Change of Order of Integration in an Improper Repeated Integral," already quoted, § 9 (β).

$$\underset{t=0}{\mathrm{L}lt}\int_0^\infty dv\int_q^\infty e^{-kv^2t}f(u)\cos uv\,du = \underset{t=0}{\mathrm{L}lt}\int_q^\infty du\int_0^\infty e^{-kv^2t}f(u)\cos uv\,dv$$
$$= \underset{t=0}{\mathrm{L}lt}\int_q^\infty f(u)\sqrt{\frac{\pi}{4kt}}e^{-\frac{u^2}{4kt}}du$$
$$= \underset{t=0}{\mathrm{L}lt}\sqrt{\frac{\pi}{4kt}}e^{-\frac{u^2}{4kt}}\int_q^{\mathrm{Q}} f(u)du = 0,$$

which proves the theorem.

§ 6. For the next theorems we require the following lemma:—

Lemma.—*If* f(u) *is monotone decreasing with zero as limit at infinity, and*

$$\mathrm{F}(u, v, t) \text{ is } e^{-kv^2t}f(u)\cos u(v-\mathrm{V}), \text{ or } e^{-kv^2t}f(u)\sin u(v-\mathrm{V}),$$

we may reverse the order of integration and write,

$$\int_0^{\mathrm{B}} dv\int_q^\infty \mathrm{F}(u, v, t)du = \int_q^\infty du\int_0^{\mathrm{B}}\mathrm{F}(u, v, t)\,dv,$$

both these repeated integrals existing, provided that when F(u, v, t) *has its second value, we know that* $\int_q^\infty \frac{f(u)}{u}du$ *exists.*

To prove this, we affirm that

$$\int_0^{\mathrm{V}} dv\int_q^\infty \mathrm{F}(u, v, t)du = \int_q^\infty du\int_0^{\mathrm{V}}\mathrm{F}(u, v, t)dv,$$
$$\text{and}\quad \int_{\mathrm{V}}^{\mathrm{B}} dv\int_q^\infty \mathrm{F}(u, v, t)du = \int_q^\infty du\int_{\mathrm{V}}^{\mathrm{B}}\mathrm{F}(u, v, t)dv,$$

from which, by addition, the required result at once follows.

The proofs of these two results are very similar; we shall therefore content ourselves with proving the latter.

Replacing the lower limit V by V + c, for the variation of v, so that

$$\mathrm{V}+c, \leqq v \leqq \mathrm{B},$$

we have, for the range of values of v in question, using the Second Theorem of the Mean,

$$\left|\int_q^{\mathrm{Q}}\mathrm{F}(u, v, t)du\right| = e^{-kv^2t}f(q+0)\left|\int_q^{\mathrm{Q}}\frac{\cos u(v-\mathrm{V})}{\sin u(v-\mathrm{V})}du\right| \leqq 2f(q)/c.$$

Hence, firstly, the left-hand side vanishes when q moves off to infinity, so that $\int_q^\infty \mathrm{F}(u, v, t)du$ certainly exists; secondly, $\int_q^{\mathrm{Q}}\mathrm{F}(u, v, t)du$ converges boundedly to its limit $\int_q^\infty \mathrm{F}(u, v, t)du$, and the latter integral is a bounded function of (v, t) for the range of values of v considered. Hence we may integrate and get

$$\int_{\mathrm{V}+c}^{\mathrm{B}} dv\int_q^\infty \mathrm{F}(u, v, t)du = \int_{\mathrm{V}+c}^{\mathrm{B}} dv\,\underset{\mathrm{Q}=\infty}{\mathrm{L}t}\int_q^{\mathrm{Q}}\mathrm{F}(u, v, t)du = \underset{\mathrm{Q}=\infty}{\mathrm{L}t}\int_{\mathrm{V}+c}^{\mathrm{B}} dv\int_q^{\mathrm{Q}}\mathrm{F}(u, v, t)du$$
$$= \underset{\mathrm{Q}=\infty}{\mathrm{L}t}\int_q^{\mathrm{Q}} du\int_{\mathrm{V}+c}^{\mathrm{B}}\mathrm{F}(u, v, t)dv = \underset{\mathrm{Q}=\infty}{\mathrm{L}t}\, e^{-k(\mathrm{V}+c)^2t}\int_q^{\mathrm{Q}} du\int_{\mathrm{V}+c}^{\mathrm{B}'} f(u)\frac{\cos u(v-\mathrm{V})}{\sin u(v-\mathrm{V})}dv.$$

using the Second Theorem of the Mean, B′ lying between V + c and B.

Now in the case when we have $\cos u(v-\mathrm{V})$, the last integral is numerically

$$\leqq f(q+0)2\pi \,.\, e^{-k(\mathrm{V}+c)^2 t},$$

using again the Second Theorem of the Mean, and remembering the fact that

$$\left|\int_q^{\mathrm{Q}'} \frac{\sin ku}{u}\right| du \leqq \pi.$$

Thus in this case we may, changing q to Q, write

$$\int_{\mathrm{V}+c}^{\mathrm{B}} dv \int_{\mathrm{Q}}^{\infty} \mathrm{F}(u,\, v,\, t) du = z,$$

where z has zero as unique limit when the ensemble $(c,\ \mathrm{Q})$ approaches $(0,\ \infty)$ in any manner.

But this is also true when we have $\sin u(v-\mathrm{V})$, since in this case $\displaystyle\int_q^{\infty} \frac{f(u)}{u} du$ exists by hypothesis, and we have

$$\left|\int_q^{\mathrm{Q}} du \int_{\mathrm{V}+c}^{\mathrm{B}'} f(u) \sin u(v-\mathrm{V}) dv\right| = \left|\int_q^{\mathrm{Q}} \frac{f(u)}{u} (\cos uc - \cos u(\mathrm{B}'-\mathrm{V}) du\right| \leqq 2\int_q^{\infty} \frac{f(u)}{u} du.$$

Thus we have only to take

$$z = e^{-k(\mathrm{V}+c)^2 t} 2 \int_q^{\infty} \frac{f(u)}{u} du.$$

Hence, in either case, we have the inequality

$$\int_{\mathrm{V}+c}^{\mathrm{B}} dv \int_q^{\mathrm{Q}} \mathrm{F}(u,\, v,\, t) du - z \leqq \int_{\mathrm{V}+c}^{\mathrm{B}} dv \int_q^{\infty} \mathrm{F}(u,\, v,\, t) du \leqq \int_{\mathrm{V}+c}^{\mathrm{B}} dv \int_q^{\mathrm{Q}} f(u,\, v,\, t) du + z.$$

Now let c approach zero in such a way as to give an unique limit I for the central integral, and subsequently let Q move off to infinity in such a way that $\int_{\mathrm{V}}^{\mathrm{B}} dv \int_q^{\mathrm{Q}} \mathrm{F}(u,\, v,\, t) du$, which has taken the place of the integral appearing in the extremes of the last inequality, has an unique limit J. Since at the end of this process z has vanished, we have

$$\mathrm{I} = \mathrm{J}.$$

But this is true whichever of the possible limits J was; therefore all such limits coincide, and we have

$$\int_{\mathrm{V}}^{\mathrm{B}} dv \int_q^{\infty} \mathrm{F}(u,\, v,\, t) du = \mathrm{I},$$

the integral on the left existing. This is, however, true whichever of the possible limits I was, so that all such limits coincide with I, and we have

$$\int_q^{\infty} du \int_{\mathrm{V}}^{\mathrm{B}} v\mathrm{F}(u,\, v,\, t) dv = \int_{\mathrm{V}}^{\mathrm{B}} dv \int_q^{\infty} \mathrm{F}(u,\, v,\, t) du,$$

both these integrals existing.

This proves the second of the subsidiary results. The first may be proved in the same way, replacing V by $V-c$.

Thus the lemma is proved.

It need hardly be added that the reversal of the order of integration is still allowable when the lower limit of integration with respect to v is altered from zero to, say, b, where $b<B$. This follows in fact at once by subtracting the equation in which the limits are zero and b, from that in which the limits are zero and B.

§ 7. Theorem.—*If* f(u) *is monotone decreasing with zero as limit at infinity, then*

$$\underset{t=0}{\mathrm{Lt}}\int_0^\infty dv\int_q^\infty e^{-kv^2t}f(u)\cos uv\,du=0.$$

For, by the preceding lemma, in which we put $V=0$,

$$\int_0^B dv\int_q^\infty F(u,\ v,\ t)du=\int_q^\infty du\int_0^B F(u,\ v,\ t)dv, \quad . \qquad . \qquad . \quad (1)$$

where

$$F(u,\ v,\ t)=e^{-kv^2t}f(u)\cos uv\,du.$$

Moreover, $\int_0^\infty dv\int_q^\infty F(u,\ v,\ t)du$ exists, for

$$\int_b^B dv\int_q^\infty F(u,\ v,\ t)du=f(q)\int_b^B e^{-kv^2t}\frac{\sin qv-\sin Qv}{v}dv,$$

using the Second Theorem of the Mean; therefore, using it a second time, the left-hand side is numerically

$$\frac{\leqq f(q)}{b}\int_0^\infty 2e^{-kv^2t}dv,$$

which vanishes when b moves off to infinity.

Hence, by (1), proceeding to the limit with B, and changing q into Q,

$$\int_0^\infty dv\int_Q^\infty F(u)du=\underset{B=\infty}{\mathrm{Lt}}\int_Q^\infty du\int_0^B F(u,\ v,\ t)dv$$

$$=\underset{B=\infty}{\mathrm{Lt}}\int_Q^\infty f(u)du\int_0^{B'}\cos uv\,dv=\underset{B=\infty}{\mathrm{Lt}}f(Q)\int_Q^{Q'}\frac{\sin B'u}{u}du,$$

using the Second Theorem of the Mean twice. Hence the left-hand side is numerically $\leqq \pi f(Q)$, and therefore the same is true of the limits of that integral, when $t=0$. But, in any finite interval $(q,\ Q)$, $f(u)$ is bounded and therefore summable. Hence by § 2,

$$\underset{t=0}{\mathrm{Lt}}\int_0^\infty dv\int_q^Q e^{-kv^2t}f(u)\cos uv\,du=0\ ;$$

therefore, by what was already proved,

$$\underset{t=0}{\mathrm{Llt}}\int_0^\infty dv\int_q^\infty F(u,\ v,\ t)du\leqq \pi f(Q).$$

But the left-hand side is independent of Q and, letting Q move off to infinity, the right-hand side vanishes. Therefore all the limits on the left coincide, and we get the required equation,

$$\operatorname*{Lt}_{t=0}\int_0^\infty dv\int_q^\infty \mathrm{F}(u,\ v,\ t)du = 0.$$

§ 8. Theorem.—*If in the interval* (q, ∞) *the function* f(u) *is monotone decreasing with zero as limit at infinity, then*

$$\operatorname*{Lt}_{t=0}\int_0^\infty dv\int_q^\infty e^{-kv^2t}f(u)\sin uv\,du = \int_q^\infty \frac{f(u)}{u}du,$$

provided $\int_q^\infty \frac{\mathrm{f(u)}}{\mathrm{u}}\mathrm{du}$ *exists.**

By the lemma of § 6, putting $\mathrm{V}=0$, we have

$$\int_0^{\mathrm{B}} dv\int_q^\infty e^{-kv^2t}f(u)\sin uv\,du = \int_q^\infty du\int_0^{\mathrm{B}} e^{-kv^2t}f(u)\sin uv\,dv$$
$$= \int_q^\infty du\frac{f(u)}{u}\int_0^{\mathrm{B}} e^{-kv^2t}u\sin uv\,dv.$$

Now

$$\int_0^{\mathrm{B}} e^{-kv^2t}u\sin uv\,dv = -\Big[e^{-kv^2t}\cos uv\Big]_0^{\mathrm{B}} + 2kt\int_0^{\mathrm{B}} v\cos uv\,e^{-kv^2t}dv$$
$$= 1 - e^{-k\mathrm{B}^2t}\cos \mathrm{B}u + 2\int_0^{\mathrm{B}\sqrt{kt}} v\cos\frac{uv}{\sqrt{kt}}e^{-v^2}dv \qquad . \quad (1)$$

and is therefore numerically less than $2+2\int_0^\infty ve^{-v^2}dv$, *i.e.* less than 3 always.

Similarly

$$\int_b^{\mathrm{B}} e^{-kv^2t}u\sin uv\,dv < 3e^{-kb^2t},$$

so that $\int_0^\infty e^{-kv^2t}u\sin uv\,dv$ exists, and $\int_0^{\mathrm{B}} e^{-kv^2t}u\sin uv\,dv$ converges boundedly to it as limit when B increases indefinitely.

Hence also $\int_0^\infty dv\int_q^\infty e^{-kv^2t}f(u)\sin uv\,du$ also exists, for by the same reasoning as the above $\int_b^{\mathrm{B}} dv\int_q^\infty e^{-kv^2t}f(u)\sin uv\,du$ is numerically less than $3e^{-kb^2t}\int_q^\infty \frac{f(u)}{u}du$, and therefore vanishes when B increases indefinitely.

Thus we may write

$$\operatorname*{Lt}_{t=0}\int_0^\infty dv\int_q^\infty e^{-kv^2t}f(u)\sin uv\,du = \operatorname*{Lt}_{t=0}\operatorname*{Lt}_{\mathrm{B}=\infty}\int_q^\infty du\frac{f(u)}{u}\int_0^{\mathrm{B}} e^{-kv^2t}u\sin uv\,dv$$
$$= \operatorname*{Lt}_{t=0}\int_q^\infty du\frac{f(u)}{u}\int_0^\infty e^{-kv^2t}u\sin uv\,dv,$$

since $f(u)/u$ has an integral from (q, ∞), and the integral multiplying it

* This theorem, and that of § 15, can also be proved even more simply by the method of § 14.

has been shown to be a bounded function of (B, u, t), for the range of variables considered.

By the same reasoning, we may introduce the limit with respect to t under the integral sign, whence using (1), in which we put $B = \infty$,

$$\underset{t=0}{\mathrm{L}t}\int_0^\infty dv\int_q^\infty e^{-kv^2t}f(u)\sin uv\,du = \int_q^\infty du\frac{f(u)}{u}\underset{=0}{\mathrm{L}t}\left(1 + 2\int_0^\infty e^{-v^2}v\cos\frac{uv}{\sqrt{kt}}dv\right).$$

But, by the extended Riemann-Lebesgue theorem, the integral with respect to v last written down has, provided u is not zero, the unique limit zero when t approaches zero. Hence, when multiplied by the summable function $f(u)/u$ and integrated from q to ∞ with respect to u, this integral disappears from our equation. Thus we get the required equation

$$\underset{t=0}{\mathrm{L}t}\int_0^\infty dv\int_q^\infty e^{-kv^2t}f(u)\sin uv\,du = \int_q^\infty\frac{f(u)}{u}du.$$

§ 9. Theorem.—*If* f(u) *is expressible as the product of two factors, one of which,* g(u), *is monotone decreasing with zero as limit at infinity, while the other is of the form cos* uV, *then*

$$\underset{t=0}{\mathrm{L}t}\int_0^\infty dv\int_q^\infty e^{-kv^2t}f(u)\cos uv\,du = 0.$$

By the lemma of § 6 we have

$$\int_0^B dv\int_Q^\infty e^{-kv^2t}g(u)\cos u(v-V)du = \int_Q^\infty du\int_0^B e^{-kv^2t}g(u)\cos u(v-V)dv$$
$$= g(Q+0)\int_Q^{Q'}du\int_0^{B'}\cos u(v-V)dv,$$

using the Second Theorem of the Mean twice. Hence the first of these repeated integrals is numerically

$$= g(Q+0)\left|\int_Q^{Q'}\frac{1}{u}(\sin u(B'-V) + \sin uV)du\right| \leq 2\pi g(Q+0),$$
$$\leq z, \text{ say,}$$

where z has the unique limit zero when Q moves off to infinity.

Hence

$$\left|\int_0^\infty dv\int_Q^\infty e^{-kv^2t}g(u)\cos u(v-V)du\right| \leq z, \quad . \quad . \quad . \quad . \quad (1)$$

the integral on the left certainly existing, since, by the same argument as that used above,

$$\left|\int_b^B dv\int_q^\infty e^{-kv^2t}g(u)\cos u(v-V)du\right| = e^{-kb^2t}2\pi g(q+0),$$

and therefore vanishes when b moves off to infinity.

Again, $g(u)$ is summable in the finite interval (q, Q); therefore the same is true of $g(u)\sin uv$ and $g(u)\cos uv$. Hence, considering, if we please.

$g(u)$ as being zero when u is less than the positive quantity q, we have by §§ 2 and 4,

$$\underset{t=0}{\mathrm{Lt}}\int_0^\infty dv\int_q^{\mathrm{Q}} e^{-kv^2t}g(u)\cos u(v-\mathrm{V})du=\int_q^{\mathrm{Q}}\frac{g(u)\sin u\mathrm{V}}{u}du \quad . \quad . \quad (2)$$

From (1) and (2) we have

$$\underset{t=0}{\mathrm{Lt}}\int_0^\infty dv\int_q^\infty e^{-kv^2t}g(u)\cos u(v-\mathrm{V})du=\int_q^{\mathrm{Q}}\frac{g(u)\sin u\mathrm{V}}{u}du+\theta z,$$

where $|\theta| \leqq 1$. But the left-hand side of this equation is independent of Q, and the second term on the right has the unique limit zero, when Q moves off to infinity. Therefore,

$$\underset{t=0}{\mathrm{Lt}}\int_0^\infty dv\int_q^\infty e^{-kv^2t}g(u)\cos u(v-\mathrm{V})du=\int_q^\infty\frac{g(u)\sin u\mathrm{V}}{u}du \quad . \quad . \quad (3)$$

Changing V into $-\mathrm{V}$, we have, similarly,

$$\underset{t=0}{\mathrm{Lt}}\int_0^\infty dv\int_q^\infty e^{-kv^2t}g(u)\cos u(v+\mathrm{V})du=-\int_q^\infty\frac{g(u)\sin u\mathrm{V}}{u}du \quad . \quad . \quad (4)$$

Adding the equations (3) and (4), we get the required result.

§ 10. Theorem.—*If* f(u) *is expressible as the product of two factors, one of which,* g(u), *is monotone decreasing with zero as limit at infinity, while the other is of the form sin* uV, *then*

$$\underset{t=0}{\mathrm{Lt}}\int_0^\infty dv\int_q^\infty e^{-kv^2t}f(u)\cos uv\,du=0,$$

provided $\int_q^\infty\frac{\mathrm{g(u)}}{\mathrm{u}}\mathrm{du}$ *exists.*

By the lemma of § 6,

$$\int_0^{\mathrm{B}}dv\int_q^\infty e^{-kv^2t}g(u)\sin u(v-\mathrm{V})\,du=\int_q^\infty du\int_0^{\mathrm{B}}e^{-kv^2t}g(u)\sin u(v-\mathrm{V})dv$$
$$=\int_q^\infty du\frac{g(u)}{u}\int_0^{\mathrm{B}}e^{-kv^2t}u\sin u(v-\mathrm{V})dv.$$

Now,

$$\int_0^{\mathrm{B}}e^{-kv^2t}u\sin u(v-\mathrm{V})dv=\left[-e^{-kv^2t}\cos u(v-\mathrm{V})\right]_0^{\mathrm{B}}+2kt\int_0^{\mathrm{B}}ve^{-kv^2t}\cos u(v-\mathrm{V})dv,$$

which, as in § 8, is numerically less than 3. Hence the integral on the left converges boundedly to its limit $\int_0^\infty e^{-kv^2t}u\sin u(v-\mathrm{V})dv$, whatever values, fixed or varying, be ascribed to u and t. Similarly

$$\left|\int_b^{\mathrm{B}}dv\int_q^\infty e^{-kv^2t}g(u)\sin u(v-\mathrm{V})du\right|<3e^{-kb^2t}\int_q^\infty\frac{g(u)}{u}\cdot du,$$

and therefore has, as b increases, the limit zero, so that

$$\int_0^\infty dv \int_q^\infty e^{-kv^2t} g(u) \sin u(v-\mathrm{V}) du$$

exists. Hence

$$\underset{t=0}{\mathrm{Lt}} \int_0^\infty dv \int_q^\infty e^{-kv^2t} g(u) \sin u(v-\mathrm{V}) du = \underset{t=0}{\mathrm{Lt}} \underset{\mathrm{B}=\infty}{\mathrm{Lt}} \int_q^\infty du \frac{g(u)}{u} \int_0^{\mathrm{B}} e^{-kv^2t} u \sin u(v-\mathrm{V}) dv$$
$$= \underset{t=0}{\mathrm{Lt}} \int_q^\infty du \frac{g(u)}{u} \int_0^\infty e^{-kv^2t} u \sin u(v-\mathrm{V}) dv,$$

$g(u)/u$ being positive and summable in (q, ∞). But this last expression may be written

$$\int_q^\infty du \frac{g(u)}{u} \underset{t=0}{\mathrm{Lt}} \left(\cos u\mathrm{V} + 2\int_0^\infty ve^{-v^2} \cos u\left(\frac{v}{\sqrt{kt}} - \mathrm{V}\right) dv\right),$$

by the same reasoning, using the value of the integral already found, and putting $\mathrm{B}=\infty$. In fact $\frac{g(u)}{u}$ is summable in (q, ∞), and the remaining factor is a bounded function of (u, t).

Now, the integral with respect to v on the right of the preceding equation is a bounded function of (u, t), and, except for the value zero of u, has the unique limit zero for each fixed value of u when t diminishes to zero, by the extended theorem of Riemann-Lebesgue. Hence, when we multiply this limit by the bounded function $g(u)/u$, and integrate from q to infinity, the result is zero. Thus the last equation becomes

$$\underset{t=0}{\mathrm{Lt}} \int_0^\infty dv \int_q^\infty e^{-kv^2t} g(u) \sin u(v-\mathrm{V}) du = \int_q^\infty \frac{g(u)}{u} \cos u\mathrm{V}\, du \quad . \quad . \quad (1)$$

Similarly, changing V into $-\mathrm{V}$,

$$\underset{t=0}{\mathrm{Lt}} \int_0^\infty dv \int_q^\infty e^{-kv^2t} g(u) \sin u(v+\mathrm{V}) du = \int_q^\infty \frac{g(u)}{u} \cos u\mathrm{V}\, du \quad . \quad . \quad (2)$$

Subtracting (1) and (2), the required result follows.

§ 11. Theorem.—*If* f(u) *is expressible as the product of two factors, one of which,* g(u), *is monotone decreasing with zero as limit at infinity, while the other is of the form sin* uV, *then*

$$\underset{t=0}{\mathrm{Lt}} \int_0^\infty dv \int_q^\infty e^{-kv^2t} f(u) \sin uv\, du = \int_q^\infty \frac{f(u)}{u} du,$$

both these integrals necessarily existing.

For, subtracting the equations (3) and (4) of § 9, this result follows.

§ 12. Theorem.—*If* f(u) *is expressible as the product of two factors, one of which,* g(u), *is monotone decreasing with zero as limit at infinity, while* *the other is of the form cos* uV, *then*

$$\underset{t=0}{\mathrm{Lt}}\int_0^\infty dv\int_q^\infty e^{-kv^2t}f(u)\sin uv\,du=\int_q^\infty\frac{f(u)}{u}du,$$

provided $\int_q^\infty \frac{\mathrm{g(u)}}{\mathrm{u}}\,\mathrm{du}$ *exists.*

For, adding the equations (1) and (2) of § 10, we have at once the required result.

§ 13. To prove the next two theorems we require the following lemma:—

Lemma.—*If* g(u) *is a monotone decreasing function of* u *with zero as limit at infinity, and* h(u) *has its square summable in every finite interval, then we may reverse the order of integration and write*

$$\int_0^{\mathrm{B}} dv\int_q^\infty \mathrm{G}(u,\ v,\ t)h(u)du=\int_q^\infty du\int_0^{\mathrm{B}}\mathrm{G}(u,\ v,\ t)h(u)dv,$$

where

$$\mathrm{G}(u,\ v,\ t)=e^{-kv^2t}g(u)\,\frac{\cos uv}{\sin uv}.$$

We shall, in the first instance, assume that the Fourier series of $h(u)$ is free of constant term.

Since the square of $h(u)$ is summable, the Fourier coefficients a_n and b_n of the function $h(u)$ are such that the series whose general terms are respectively $|a_n|/n$ and $|b_n|/n$ converge absolutely.*

Now, in any finite interval $(q,\ \mathrm{Q})$ of values of u, $h(u)$ is summable, since its square is summable, and $\mathrm{G}(u,\ v,\ t)$ has bounded variation; therefore † we may integrate the product term-by-term, using the not necessarily convergent Fourier series of $h(u)$. Thus

$$\int_q^{\mathrm{Q}}\mathrm{G}(u,\ v,\ t)h(u)du=\sum_{n=1}^{\infty}\int_q^{\mathrm{Q}}\mathrm{G}(u,\ v,\ t)(a_n\cos nu+b_n\sin nu)du \quad . \quad . \quad (1)$$

Let m be the first integer greater than 2B. Then, for all integers $n \geqq m$, and for all values of v in the closed interval (0, B),

$$n+v\geqq n-v\geqq n-\mathrm{B}>\tfrac{1}{2}n.$$

Hence, using the Second Theorem of the Mean to bring $g(q)$ outside the sign of integration, and breaking up the products cos nu cos uv, cos nu sin uv, sin nu cos uv, and sin nu sin uv into the sum or difference of two sines or cosines in the usual way, we get for the absolute value of the integral

* In fact, the series whose general terms are $n^{-\frac{1}{2}-e}a_n$ and $n^{-\frac{1}{2}-e}b_n$ where $0<e$, converge absolutely. See my recent paper on Fourier integrals already cited, § 16.

† W. H. Young, "On the Integration of Fourier Series" (1910), Theorem 2, presented to the London Mathematical Society.

under the summation sign in (1) the following inequality, for all integers $n \geqq m$.

$$\int_q^Q \mathrm{G}(u, v, t)(a_n \cos nu + b_n \sin nu)du \leqq g(q)e^{-kv^2t}\left(\frac{1}{n+v}+\frac{1}{n-v}\right)(|a_n|+|b_n|)$$

$$\leqq g(q)\frac{4}{n}(|a_n|+|b_n|) \quad . \quad . \quad . \quad . \quad (2)$$

Again, $\int_0^B e^{-kv^2t} \cos uv\, dv$ is a function of bounded variation of the variable u, since it has a bounded differential coefficient with respect to u, viz., $-\int_0^B e^{-kv^2t} v \sin uv\, dv$. Therefore, the product of two functions of bounded variation being a function of bounded variation, $g(u)\int_0^B e^{-kv^2t} \cos uv\, dv$ is a function of bounded variation of the variable u in the interval (q, Q), and $h(u)-s_m$, where s_m denotes the sum of the first $2m-2$ terms of the Fourier series of $h(u)$, is summable; therefore we may integrate term-by-term, as before, and write

$$\int_q^Q du\int_0^B e^{-kv^2t}g(u)(h(u)-s_m)\cos uv\, dv = \sum_{n=m}^{\infty}\int_q^Q du\int_0^B e^{-kv^2t}g(u)(a_n \cos nu + b_n \sin nu)\cos uv\, dv.$$

Similarly $g(u)\int_0^B e^{-kv^2t}\sin uv\, dv$ is a function of bounded variation, so that in the preceding equation we may change $\cos uv$ into $\sin uv$. Thus we may write, with either signification of $\mathrm{G}(u, v, t)$,

$$\int_q^Q du\int_0^B \mathrm{G}(u, v, t)\{h(u)-s_m\}dv = \sum_{n=m}^{\infty}\int_q^Q du\int_0^B \mathrm{G}(u, v, t)(a_n \cos nu + b_n \sin nu)dv \qquad (4)$$

Now we may write the equation (1) in the following form:—

$$\int_q^Q \mathrm{G}(u, v, t)\{h(u)-s_m\}du = \sum_{n=m}^{\infty}\int_q^Q \mathrm{G}(u, v, t)(a_n \cos nu + b_n \sin nu)du.$$

Bearing in mind that the individual integrals in this last summation continue to exist when we write infinity for Q, the inequality (2) shows that the equation last written down continues to hold when we put $\mathrm{Q}=\infty$.* When this is done, the right-hand side becomes a series of functions of (v, t) which converges boundedly for all values of v in the

* See § 15 of the paper already quoted on Fourier's integrals. The lemma here used is as follows:—*If the series of proper or improper Lebesgue, or Harnack-Lebesgue, integrals* $\sum_{n=1}\int_q^z f_n(x)dx$ *converge to the proper or improper Lebesgue, or Harnack-Lebesgue, integral* $\int_q^z f(x)dx$ *for all values of* q *and* z *in the completely open interval* $(p<q<z<B)$, *in such a manner that a convergent series of positive quantities* $\sum_{n=1} c_n$ *can be found, each term of which is not less than the absolute value of the corresponding term of the series of integrals, whatever be the values of* q *and* z *in the interval, then, provided only in addition* $\int_q^B f_n(x)dx$ *exists for each integer* n, *we can assert that* $\int_p^B f(x)dx$ *exists and* $=\sum\int_p^B f_n(x)dx$.

closed interval (0, B), and all values of $t \geqq 0$. Hence we may integrate term-by-term with respect to v between the limits 0 and B, and assert that

$$\int_0^{\mathrm{B}} dv \int_q^\infty \mathrm{G}(u,\, v,\, t)\{h(u) - s_m\}du = \sum_{n=m}^{\infty} \int_0^{\mathrm{B}} dv \int_q^\infty \mathrm{G}(u,\, v,\, t)(a_n \cos nu + b_n \sin nu)du \tag{3}$$

so that both sides exist and are equal.

Now, by the inequality (2),

$$\int_0^{\mathrm{B}} dv \int_q^{\mathrm{Q}} \mathrm{G}(u,\, v,\, t)(a_n \cos nu + b_n \sin nu)du \leqq \frac{4}{n} g(q)(\,|\,a_n\,| + |\,b_n\,|\,) \int_0^{\mathrm{B}} e^{-kv^2 t} dv\,;$$

therefore, reversing the order of integration,

$$\int_q^{\mathrm{Q}} du \int_0^{\mathrm{B}} \mathrm{G}(u,\, v,\, t)(a_n \cos nu + b_n \sin nu)dv \leqq \frac{4}{n} g(q)(\,|\,a_n\,| + |\,b_n\,|\,) \int_0^{\infty} e^{-kv^2 t} dv.$$

This shows that, when q recedes to infinity, the left-hand side vanishes, so that the left-hand side still exists when we replace Q by infinity. It also shows that the conditions of the lemma quoted are satisfied, and we may assert that the equation (4) still persists when we put $\mathrm{Q} = \infty$. Thus,

$$\int_q^\infty du \int_0^{\mathrm{B}} \mathrm{G}(u,\, v,\, t)\{h(u) - s_m\}dv = \sum_{n=m}^{\infty} \int_q^\infty du \int_0^{\mathrm{B}} \mathrm{G}(u,\, v,\, t)(a_n \cos nu + b_n \sin nu)dv \tag{5}$$

But, by the lemma of § 6, we may reverse the order of integration in the integral under the summation sign on the right, as well as in the integrals of the same form in which $n < m$, whose sum is $\int_q^\infty du \int_0^{\mathrm{B}} \mathrm{G}(u, v, t) s_m dv$. Hence

$$\int_q^\infty du \int_0^{\mathrm{B}} \mathrm{G}(u,\, v,\, t) s_m dv = \int_0^{\mathrm{B}} dv \int_q^\infty \mathrm{G}(u,\, v,\, t) s_m du,$$

and

$$\begin{aligned}\int_q^\infty du \int_0^{\mathrm{B}} \mathrm{G}(u,\, v,\, t)\{h(u) - s_m\}dv &= \sum_{n=m}^{\infty} \int_0^{\mathrm{B}} dv \int_q^\infty \mathrm{G}(u,\, v,\, t)(a_n \cos nu + bn \sin nu)du \\ &= \int_0^{\mathrm{B}} dv \int_q^\infty \mathrm{G}(u,\, v,\, t)\{h(u) - s_m\}du,\end{aligned}$$

using (3).

From these last two equations we get by addition,

$$\int_q^\infty du \int_0^{\mathrm{B}} \mathrm{G}(u,\, v,\, t) h(u) dv = \int_0^{\mathrm{B}} dv \int_q^\infty \mathrm{G}(u,\, v,\, t) h(u) du.$$

This is the required result, and has been proved under the assumption that the Fourier series of $h(u)$ is free of constant term. If this is not the case, and $\frac{1}{2}a$ be the constant term, the above shows that we may write

$$\int_q^\infty du \int_0^{\mathrm{B}} \mathrm{G}(u,\, v, t)\{h(u) - \tfrac{1}{2}a_0\}dv = \int_0^{\mathrm{B}} dv \int_q^\infty \mathrm{G}(u,\, v,\, t)\{h(u) - \tfrac{1}{2}a_0\}du.$$

Moreover, since $\mathrm{G}(u,\, v,\, t)$ is a bounded function of all the variables,

$$\tfrac{1}{2}a_0 \int_q^\infty du \int_0^{\mathrm{B}} \mathrm{G}(u,\, v,\, t)dv = \tfrac{1}{2}a_0 \int_0^{\mathrm{B}} dv \int_0^\infty \mathrm{G}(u,\, v,\, t)du.$$

Adding the last two equations, we get the required result, which proves the lemma.

§ 13. THEOREM.—*If in the interval* (q, ∞) *the function* f(u) *can be expressed as the product of two functions, one of which,* g(u), *is monotone decreasing with zero as limit at infinity, and the other,* h(u), *is any periodic function whose square is summable in every finite interval, then*

$$\operatorname*{Lt}_{t=0}\int_0^\infty dv\int_q^\infty e^{-kv^2t}f(u)\cos uv\,du=0,$$

provided $\int_q^\infty \frac{f(u)}{u}du$ *exists.*

For, by the lemma of § 12,

$$\int_0^B dv\int_Q^\infty e^{-kv^2t}f(u)\cos uv\,du=\int_Q^\infty du\int_0^B e^{-kv^2t}f(u)\cos uv\,dv$$
$$=\int_Q^\infty f(u)\frac{\sin B'u}{u}du,$$

using the Second Theorem of the Mean, B′ lying between 0 and B. Hence the integral on the left is numerically

$$\leqq\int_Q^\infty\frac{f(u)}{u}du,$$

which, by the hypothesis that $\int_q^\infty\frac{f(u)}{u}du$ exists, has zero as unique limit when Q recedes to infinity.

Moreover, by a similar argument,

$$\int_b^B dv\int_Q^\infty e^{-kv^2t}f(u)\cos uv\,du=e^{-kb^2t}\int_Q^\infty\frac{f(u)}{u}du,$$

which vanishes when b increases without limit.

Therefore $\int_0^\infty dv\int_0^\infty e^{-kv^2t}f(u)\cos uv\,du$ exists, and, by the above

$$\left|\int_0^\infty dv\int_Q^\infty e^{-kv^2t}f(u)\cos uv\,du\right|<\int_Q^\infty\frac{f(u)}{u}du \qquad . \quad . \quad . \quad (1)$$

But, in any finite interval (q, Q), $h(u)$ is summable and $g(u)$ is bounded; therefore $f(u)$ is summable. Hence, by § 2,

$$\operatorname*{Lt}_{t=0}\int_0^\infty dv\int_q^Q e^{-kv^2t}f(u)\cos uv\,du=0 \qquad . \quad . \quad . \quad . \quad (2)$$

Adding (1) and (2)

$$\left|\operatorname*{Lt}_{t=0}\int_0^\infty dv\int_q^\infty e^{-kv^2t}f(u)\cos uv\,du\right|\leqq\int_Q^\infty\frac{f(u)}{u}du.$$

But the left-hand side is independent of Q, and the right-hand side vanishes when Q moves off to infinity; hence the required result follows.

§ 15. Theorem.—*If in the interval* (q, ∞) *the function* f(u) *can be expressed as the product of two factors, one of which,* g(u), *is monotone decreasing with zero as limit at infinity, and the other,* h(u), *is any periodic function whose square is summable, then*

$$\underset{t=0}{\mathrm{L}t}\int_0^\infty dv\int_q^\infty e^{-kv^2t}f(u)\sin uv\,du=\int_q^\infty\frac{f(u)}{u}du,$$

provided the latter integral exists.

The proof of this theorem is word for word the same as that in § 8, using the lemma of § 12 instead of that of § 6 to reverse the order of integration at the beginning.

§ 16. It remains only to point out that the lemma of § 13, and, consequently the theorems of §§ 13 and 14, remain true when we remove the restriction that the square of $h(u)$ is summable, provided only $h(u)$ is summable and the series whose general terms are $|a_n|/n$ and $|b_n|/n$ are absolutely convergent. The reasoning of the text is, in fact, unaffected.

Still more generally, reverting to the general theorem on the integration of series * which forms the basis of the theorem on the integration of Fourier series quoted in § 12, we may replace the integer n by any positive quantity k_n, having, as n increases, the unique limit infinity, and $|a_n|/n$ and $|b_n|/n$ by $|A_n|$ and $|B_n|$, provided $h(u)$ is summable and its Lebesgue integral can be expanded in a series of the new form, viz.

$$\int^u h(u)du=A_0u+\sum_{n=1}^{\infty}(A_n\cos k_nu+B_n\sin k_nu).$$

* W. H. Young, "On the Theory of the Application of Expansions to Definite Integrals," 1910, presented to the London Mathematical Society.

(*Issued separately July* 26, 1911.)

XLII.—The Sectional Anatomy of the Head of the Australian Aboriginal: A Contribution to the Subject of Racial Anatomy. By Richard J. A. Berry, M.D. Edin. et Melb., F.R.C.S. Edin., F.R.S. Edin., Professor of Anatomy in the University of Melbourne. (With Fourteen Plates.)

(MS. received November 21, 1910. Read December 19, 1910.)

So far back as 1867, Flower and Murie (1), in giving an account of the dissection of a Bushwoman, stated that "observations upon the comparative anatomy of the different races of Man have hitherto been confined too exclusively to the external characters and to the skeleton. With very few exceptions the arrangement of the muscles, vessels, viscera, and even of the brain and nervous system, constitute at present an unexplored field; and numerous well-marked races of our species are passing away from the face of the earth without the slightest record being left on any one of these points. And yet in discussing questions, daily becoming of greater interest, relating to the unity or plurality of Mankind, and the amount of divergence of races, data such as these afford, whether their testimony be negative or positive, whether they tend to show absence or presence of variation from a given standard, cannot be neglected by the conscientious inquirer."

The above remarks make two points sufficiently clear, one being the vast importance of the matter, and the other the paucity of the material. Writing nearly forty years later than Flower and Murie, Duckworth (2) can only add that the "material is still very scanty, although the observations are daily increasing in number and accessibility."

As regards the amount of material available for the study of the racial anatomy of the softer parts, such as the brain, which can be obtained from the post-mortem room, there are far more data available than there are of other parts, such as muscles, vessels, or viscera, which of necessity imply the retention of the whole body.

Literature.

Of the soft parts in general we have most information of the Negro Race. Chudzinski in his various writings (3, 4, 5, 6, and 7) has given us much information as to the soft parts of twelve members of the *African or*

Western section of the Negro race, and of two of the *Oceanic or Eastern section* of the same race.

Turner (8) describes the muscular and vascular anomalies of a pure Negro, without racial admixture, from the British West Indies. In two subsequent communications (9 and 10) he describes the dissections of two more Negroes, but he had no data as to their birthplace.

Wood (11), so far back as 1865, had an opportunity of dissecting a Negro.

Bryce (12) gives a careful description of all the muscular variations met with in the dissection of a pure-bred Negro of unknown nationality of 34 years of age.

Giacomini (13) also deals with the muscular variations of a Negro. The same Author also describes the existence of Harder's gland in a Bushman, the duplicity of the cartilage of the plica semilunaris, the ciliary muscles in Negroes, and the distribution of pigment (14). Giacomini has also a third paper (15) dealing with the same subject—the anatomy of the Negro.

Testut (16 and 17) has two valuable papers on the comparative anatomy of the Negro, and on the muscular anomalies in Negroes and whites.

Michel (18) has described two cervical muscular anomalies in the Negro.

Duckworth (2) describes the dissection carried out by himself of a Kroo native, and also figures the larynx of the same.

Popovsky (19) has dealt pretty fully with the facial musculature of an Ashantee Negro, whilst Anthony and Hazard (20) give a short sketch of the individual muscles of a dissected arm and lower limb from a Negro of Obangui. They state that no specially striking or numerous variations occurred, but that the muscles appeared to be shorter and thicker than in the European.

Fallot and Alezais (21) had an opportunity of making an autopsy on the body of a Negro from Martinique.

Bartels (22) examined the larynges of eleven Negroes, whilst Livini (23) had an opportunity in Florence of dissecting the corpse of a mulatto, the father a Mozambique Negro and the mother a white woman, and also of a Cuban-born mulatto, the father a Negro and the mother a white woman. Livini describes a large number of variations which occurred in the muscles, arteries, and intestines.

Flower and Murie (1) in 1867 gave a very full account of their dissection of a Bushwoman aged about 22. They deal with the external characters, muscular system, arterial system, nervous system (the brain excepted, which was handed over to Mr Marshall), the thoracic and abdominal viscera, and the generative organs.

Of the *Oceanic or Eastern section of the Negro race* our knowledge is naturally not nearly so extensive as of the African or Western section, but there are, nevertheless, some data available.

Forster (24) has had the opportunity of examining the muscular system of a new-born Papuan infant, wherein he found numerous variations, the retrogressive characters of which were sufficiently obvious. The progressive formations were but few, but included the marked separation of the index finger extensor, and the rich differentiation of the thumb and little finger muscles. The facial muscles also showed a typically atavistic condition and closely resembled that of the anthropoids.

Fischer (25) studied the thickness of the soft parts in the heads of two Papuans—Baining people. He also states that their facial musculature was of a primitive character, with but little differentiation of the individual muscles.

Amongst the Melanesians Chudzinski is apparently the only observer, as he dissected the body of a New Caledonian (6).

In the Australian division of this section of the Negro race—the division with which this paper is particularly concerned—Cunningham (26) has published a study of the head of an Australian aboriginal, in which he deals almost exclusively with external features.

Duckworth (27) has also made some observations on the dissection of the head of an Australian aboriginal, and elsewhere (2) he has figured the facial musculature of an aboriginal native of South Australia presently in the Anatomy Museum, Cambridge.

Amongst the *Northern or Mongolo-Turki section* of the Mongolian Race, Adachi (28 to 35), both alone and in conjunction, is doing work of a high order on the anatomy of the soft parts of the Japanese. His work is far too important to be abstracted here, but some of it, bearing more directly upon the present paper, will be referred to hereafter.

Amongst the *Southern, Tibetan, Indo-Chinese, and Chinese section* of the Mongolian race, Adachi (33) has made observations on the facial musculature of three Chinese heads. Birkner (36 and 37) has examined, by Kollmann's method, the thickness of the soft parts of the face on six Chinese heads. Anderson Stuart (38) has published an account of the dissection of the body of a Chinese, and Chudzinski (6) has furnished us with data from the dissections of the bodies of two Annamese and two Cochin-Chinese.

Of the *Oceanic Section* of the Mongolian race there appear to be no records of any dissections, though several papers are extant treating of the external anatomy of the ear and the eye.

Amongst the primitive inhabitants of the American Continent the available data of dissections are but few. Chudzinski (6) has dissected the body of a Peruvian, and Fallot and Alezais (21) made an autopsy on the body of an American Indian.

Of the less frequently obtained members of the Caucasian Race Chudzinski (6) has dissected the body of an Arab (*Hamito-semitic section*); and Bryce, amongst the *Indo-European section*, has recorded (12) the dissection of a native of Dacca in Bengal, the subject being an adult male, a Lascar on board ship.

As there is also an active Anatomical Department in Cairo, it may safely be inferred that much has been done there on the bodies of Egyptian fellaheen, though as yet there are apparently no records except upon the brains.

Sources of the Material of the Present Work.

The material employed in the present investigation consisted of the entire heads of two Australian aboriginal natives from the Lower Murray in South Australia. Of these, one was the head of a male, aged 25 years, who had lived at Point M'Leay, near the mouth of the River Murray, and who died from pneumonia in 1907. The other was that of a female, aged 50 years, who resided at Murray Bridge, and who also died in 1907, from some unexplained cause. It is almost unnecessary to add that although both were pure-bred, both had lived under European conditions.

Sectional and Reconstructional Technique.

After death both corpses were injected with a strong solution of formaldehyde. In January 1909 the heads were removed from the bodies, frozen, and then sawn through into transverse sections, two centimetres thick, by means of an electrically driven band saw. Each section was washed and then photographed under water with a vertically directed camera. As it was desired to enlarge from the half-plate negative to life size the camera was screwed down to the floor and its position never changed during the whole of the photographic observations.

After the photography of the surfaces of the several sections the sections were reimposed upon each other and the heads again reconstructed. In this reconstruction, and in order to compensate for the tissue removed by the saw—the thickness of which was known—sheets of paper were introduced between each section, and after the final reconstruction the auricular height was measured and compared with the same height as measured prior to the sectionising, with the result that in neither head was there a difference of a millimetre.

The reconstructed head was next placed in a box in a vertical position, and one half of the head was completely embedded in plaster of Paris. In this way the future immobility of the replaced sections was absolutely guaranteed, whilst one lateral half of the head was free for further investigation.

The box containing the reconstructed and partially embedded head was next firmly screwed to the board of Martin's dioptrograph. A sheet of paper was pinned down to the drawing-board of the apparatus, and upon this was recorded an outline tracing of the head, with its various anatomical landmarks, such as the ear, etc., together with the lines of section. Upon this same sheet of paper were subsequently recorded the dioptrographic tracings of bone and brain, the latter for subsequent investigation.

The skin was next removed from one half of every section and the osseous outline dioptrographically recorded, together with the positions of the various suturæ cranii. The segments of bone were next similarly removed from the surface of every section, the meninges removed, and the lateral surfaces of the cerebral hemisphere fully exposed. Its various sulci and gyri were then drawn in by means of the dioptrograph, and lastly the various sections of cerebral hemisphere were removed from every section and the structures in the median plane of the head exposed and recorded, including, in conclusion, the medial surface of the brain itself.

Upon a single piece of paper there were thus dioptrographically recorded the outline of the head and its surface anatomy, the outline of the skull and its various sutures, the lateral surface of the brain with its sulci and gyri, the medial surface of the brain, the processes of dura mater, and the larger blood-vessels. This procedure necessarily produced an excessively complicated result, but it ensured absolute accuracy, displayed the relative relations of structures on the surface to those at deeper planes; and as each successive tracing was recorded in differently coloured waterproof inks, it was easy to obtain subsequent simplified tracings as required.

All the various sections of the brain were carefully removed, after the completion of the dioptrographic tracings, for subsequent examination.

Prior to the section cutting, a plaster mould of each head was taken, from which casts were made, which are now located in the Anatomy Museum of the University of Melbourne.

Anthropometric Measurements.

In dealing, in the first place, with the external anthropometric measurements it may be stated that these were recorded with the most modern and approved instruments upon the heads prior to their being cut, and that the measurements so recorded are those of the Monaco International Commission so far as they go, and of the British Association for the Advancement of Science. In connection with these measurements a third Australian Aboriginal head was available, but was not suitable for the examination of the brain, as the hardening process had not been completely successful. This third head was that of a male from New South Wales, and in the accompanying list of measurements is numbered two.

External Measurements of Three Australian Aboriginal Heads.

	1. Male.	2. Male.	3. Female.
A. *The Cranium.*			
1. Maximum length	195	183	190
2. Maximum breadth	152	126	135
3. Height of the head poised perpendicularly upon the condyles	115	125	129
4. Minimum frontal breadth	110	115	100
5. Maximum bimastoid diameter	137	120	120
6. Maximum circumference	574	520	560
7. Longitudinal arc	344	334	343
8. Transverse arc	326	295	320
9. Vertical radius or auricular height	122	118	125
10. Fronto-glabellar radius	109	105	103
11. Fronto-ophryon radius	113	103	106
12. Maximum frontal radius	125	116	118
13. Occipital radius	90	90	105
14. Inial radius	80	75	86 ?
15. Estimated cranial capacity	1592	1198	1446
16. Actual cranial capacity	...	1000	...
B. *The Face.*			
1. Upper face length or naso-alveolar diameter	73	65	62
2. Total face length or naso-mental diameter	130	113	109
3. Total face height	185	172	180
4. Naso-buccal diameter	80	71	70
5. Maximum bizygomatic diameter	143	135	125
6. Maximum bimalar diameter	139	121	112
7. External orbital width	122	111	110
8. External ocular or bipalpebral width	95	91	90
9. Internal ocular or bipalpebral width	35	29	38
10. Bigonial diameter	109	112	92
11. Orbito-nasal diameter	145	134	120
12. Upper nasal radius	103	99	94
13 Midnasal radius	103	103	95
14. Lower nasal radius	114	115	107
15. Alveolar radius	110	108	103
16. Mental radius	118	120	111

External Measurements of Three Australian Aboriginal Heads—*continued.*

	1. Male.	2. Male.	3. Female.
C. *The Nose.*			
1. Nasal height	48	50	44
2. Nasal breadth	46	41	43
3. Nasal depth	16	11	22
4. Nasal length	56	48	36
5. Nostril length	9	13	10
6. Nostril breadth	11	15	12
D. *The Mouth.*			
1. Width of the mouth	67	61	60
2. Bilabial height	21	15	15
E. *The Ear.*			
1. Length of ear basis	45, 45	54, 55	44, 46
2. Maximum length of ear	60, 63	66, 69	60, 69
3. Maximum breadth of ear	31, 30	34, 37	36, 38
4. Distance from the Darwinian tubercle to the upper border of the tragus . .	28, 24	29, 38	37, 42
5. Distance from the highest point of the ear to the bottom of the incisura intertragica	50, 50	52, 51	46, 47
6. Length of the lobule of ear . . .	12, 19	18, 18	14, 18
7. Length of cartilaginous ear . . .	48, 52	54, 60	49, 49

As the anthropometric form analysis of these aboriginal heads was only a subsidiary object of the investigation no extensive comments upon the preceding figures are necessary. They are given in detail for future comparison and for the benefit of fellow-workers in other parts of the world who have not access to Australian aboriginal material. This notwithstanding, there are one or two points to which attention may be directed.

It has already been stated that head number two turned out to be useless for any observations upon the brain, but this was not discovered until the entire head had been sawn into sections.

Estimated Cubic Capacity.

The estimated cubic capacity of this head had been previously calculated from the peripheral measurements of length, breadth, and auricular height, as recommended by Lee, who says that the substitution of the auricular height for the more customarily employed basal height gives results of much greater accuracy and value.

The formula employed for this calculation was therefore as follows:—

$$\frac{\text{Length} \times \text{breadth} \times \text{auricular height}}{2 \times 1{\cdot}135 \text{ for male}} = \text{estimated cubic capacity.}$$

$(2 \times 1{\cdot}108$ for female).

In the case of head number two the estimated cubic capacity worked out by this method yielded a result of 1198 c.c. After the discovery that the brain was useless it was removed and the sections built up again in order to reconstruct the head for storage purposes. Whilst doing this it suddenly occurred to me that by filling up the skull cavity with shot I should be enabled to submit the method of estimation to a somewhat rigorous test as to its accuracy or otherwise. The several sections were therefore displaced and again rebuilt, but this time with a thin layer of vaseline between every section, with the dual object of restoring the tissue lost by the saw cuts and at the same time of making the reconstructed head impervious to water. The skull was then cubed some six times with both water and shot, with the nett result that with both an actual cubic capacity of 1000 c.c. was obtained. There was thus a difference between the estimated and the actual cubic capacity of 198 c.c., which is more than the 4 per cent. difference to which Lee claims that her method reduces the error.

As it was impossible to carry out the like procedure on the remaining two heads, on account of the brain investigations, and struck as I was with the possibilities opened out by this chance experiment, I arranged with Dr J. H. Anderson, my senior assistant, to carry out an elaborate investigation into the whole subject, and his results have already been published. (50 and 51.)

Schwalbe's Fronto-biorbital Index.

In view of the well-known fact that in the crania of *Pithecanthropus erectus*, Homo primigenius, and Homo fossilis there is a well-marked constriction at the frontal end of the skull, due to the extreme narrowness of the minimum frontal diameter, an examination of Schwalbe's "fronto-biorbital index" (39), for these three Australian heads become of some importance, particularly as Schwalbe remarks, "besonders bedaure ich, dass mir für die Australierneger nur ein einziger Schädel vorliegt und dass ich diese Lucke auch aus Krause's Arbeit nicht ausfüllen kann, da auch dieser den ausseres biorbitalen Durchmesser nicht berücksichtigt."

Schwalbe's "fronto-biorbital index" seeks to compare the minimum frontal diameter with the external orbital diameter, the latter being taken as 100. It is expressed by the following formula:—

$$\frac{\text{Minimum frontal diameter} \times 100}{\text{External orbital diameter}}.$$

In *Pithecanthropus erectus* this index is 82. In Homo primigenius it ranges from 84·3 in Spy 1 to 88·8 in Neanderthal. In four Kalmucks it averaged 88·7. In fifteen Dschagga Negroes the index averaged 90·5. In 19 modern European males 91·8, and in 16 modern European females 94.

From his figures Schwalbe concludes that the majority of the skulls of recent races possess an index of over 90, but that in the lower races of mankind the value of the index is more frequently below 90 than in the higher races.

Schwalbe quotes one Australian skull with an index of 90·8. Cunningham (26) has described the head of an Australian aboriginal, named "Boco," whose head possessed a peculiarly low index, lower even than either Pithecanthropus or Neanderthal, and which worked out at 78·8.

In the present series of Australian aboriginal heads, number one gives an index of 90; number two 103; and number three 91. The range of variation over the only four recorded Australian heads, not skulls, is thus somewhat remarkable. It ranges from 78·8 to 103, but gives notwithstanding an average of 90·7, which agrees pretty closely with Schwalbe's one solitary Australian skull, with an index of 90·8. Personally I am not inclined to lay much stress on Cunningham's case, for the individual had been an inmate of a lunatic asylum for many years; and the present case, number two, with the high figure, I am also disposed to regard somewhat sceptically, inasmuch as the head is peculiarly shaped throughout. The nett result is, however, pretty much the same, as the Australian works out at an index of about 90·7 or 90·8, which brings him within the range of modern man, and seems once more to throw some doubt as to the pristine character of the race so strongly insisted upon by the modern German School of thought.

The Nose and its Index.

Turning next to the question of the size of the nose and its index I have thought it advisable to examine this feature in some detail, because Cunningham (26) makes the statement that the relative breadth of the nose in the Australians is not so great as in a single specimen of a Negro, and this "is in accordance with the general belief on this matter." The statement is based on an examination of only three Australian aboriginal heads and of one Negro.

In the comparison now to be instituted I shall avail myself of height and breadth measurements of the nose from the three Australian heads with which this work deals, with the three heads described by Cunningham, with thirty measurements recorded on the living subject by Spencer and Gillen (40), and with another series of forty recorded by the same authors in another work (41). There are, therefore, in all a series of 76 nasal measurements recorded on the Australian aboriginal native. Of these 48 are males and 28 females, and of the total number 70 were recorded on the living subject.

On analysing these measurements, the average heights, breadths, and indices for the Australian aboriginal are as follows:—

A. *Nasal Height.*

76 unsexed	48·2
48 males	49·6
28 females	45·8

B. *Nasal Breadths.*

76 unsexed	47·7
48 males	49·0
28 females	45·4

C. *Nasal Index.*

78 unsexed	98·9
48 males	98·8
28 females	99·1

From these figures there emerge three facts. First, that the Australian nose is nearly as broad as it is long; second, that there is a distinct sexual difference in the nasal form; and third, that the female nose is relatively broader in proportion to its height than is that of the male.

Since the present paper with its larger amount of material is free from errors due to the use of insufficient numbers, and since its results show that the nasal index in all three groups, unsexed, males, and females, is always greater than is the figure quoted by Cunningham for the Negro namely, 95·6, it necessarily follows that the statement that the Australian nose is not so broad as that of the Negro falls to the ground.

As the above results were obtained by the ordinary arithmetical method of averaging, I thought it advisable to test the point still further by Pearson's biometrical methods, and I asked my assistant, Dr A. W. D. Robertson, to work out for me the mean nasal index and its standard deviation with the probable errors.

From this biometrical analysis of the nasal index for both sexes combined there emerges the interesting fact that the mean nasal index is even higher than as worked out arithmetically, for it results in a mean of 99·53, with a probable error of ·92. This, therefore, seems finally to disprove the view that the Negro nose is broader than the Australian, for so far as figures are available to us the Negro nasal index is 95·6 and the Australian 99·53±·92.

The standard deviation of the nasal index, as worked out by Dr Robertson from my figures, is, for the Australian, extraordinarily high. It results in the figure 11·90, with a probable error of ·651. This figure is so high that it can only mean one or other of two things: either that the Australian nose is not that of a homogeneous type, or that the nasal index is a morphological feature of very little value.

With our present knowledge it is impossible to express any opinion as to which of the two possibilities just indicated by the standard deviation is the correct one. On the one hand we have the mathematical work of Berry, Robertson, and Cross (42) on the Tasmanian, Australian, and Papuan, tending to indicate the impurity of the Australian aboriginal; and, on the other hand, we have Reche (43), who is of opinion that the nasal index hitherto employed does not sufficiently indicate the difference between the lower and higher forms of nose, because it is influenced in an equal degree by an increasing nasal breadth and great length.

Whichever of these two possibilities may eventually prove to be correct, it is clear that in the Australian aboriginal there is an enormous diversity of nasal form, as indicated by the standard deviation of nasal index and by the great range of variation in the index itself. In my series this index ranges from a minimum of 79 to a maximum of 133. I can therefore confirm Cunningham's statement when he says "it is evident that there is a considerable amount of variation in Australian aboriginals" in respect to the shape of the nose, but I cannot agree with his statement that the "greatest number of examples are ranged in the immediate vicinity of the index 94," nor that the relative breadth of the nose in the Australian is not so great as in the Negro. Cunningham's errors, if errors they be, are due not to defective observation, but to insufficiency of numbers. Biometrical methods—the most rigidly exact which can be employed—show that the average nasal index of the Australian altogether exceeds that for the Negro, as has already been pointed out.

On the shape and form of the nose of the living subject there is therefore much still to be done, not only for the lower races, but for all other races as well.

The Ear.

In view of the large amount of attention which has been directed by criminologists, and to a lesser extent by anthropologists, to the ear, it will be advisable to say something of this feature.

The criminological literature of the ear is already very extensive, but as this work is not specially concerned with this aspect of the question it is unnecessary to say anything thereon.

Of the anthropological literature, Holl (44) treats of the position of the ear on the head. Before him, Henle, Hyrtl, Harless, Hasse, and Langer had already treated of the subject, and Langer has pointed out that the only correct determination of the position of the ear on the head is the position of the meatus acusticus externus and its positional relation to the vertical dimension. This usually falls in the middle of a line connecting the vertex of the head with the angulus mandibulæ, the projection being on a vertical plane. Higher or lower positions of the meatus indicate individual, racial, or age peculiarities in the height formation of the brain and of the facial skeleton. In very high brain skulls, or in low mandibular facial skeletons, the meatus acusticus externus is displaced downwards on this vertical.

In the three Australian heads employed in the present research the distance from vertex to angulus mandibulæ, the distance from vertex to meatus acusticus externus, and the average relation of the latter distance to the former, are as follows:—

Number.	Vertex to Angulus.	Vertex to Meatus.	Percentage.
1	167	134	71·6
2	188	129	68·6
3	178	125	70·3

From these figures two facts follow: first, that in the Australian aboriginal the position of the meatus acusticus externus is not in the middle of the vertex-angulus vertical, but is rather more than two-thirds of the distance downwards, that is, caudally; and second, that as the Australian is not characterised by great height of skull, the low position must be due, according to Holl's work, to a low mandibular facial skeleton.

Karutz (45) has devoted a large amount of attention to the anthropology of the ear. He states that of all ear measurements the one which has received most consideration is the ear length; but if people are simply grouped according to this measurement it equally simply results that tall races have long ears, and *vice versa*. The relative ear length to the bodily height is a more instructive comparison, and shows that Mongolians, Americans, Finns, Malays, and Micronesians are "long-eared"; Papuans, Australians, and Polynesians are moderately "long-eared"; whilst the genuinely "short-eared" include the Negroes, Bushman, and Cingalese.

As Karutz's work only came to my notice after the bulk of the present investigation was completed, I regret that I have not the available data of bodily stature of these three aboriginals necessary for the instructive comparison suggested by Karutz; the rest of his work, in which he deals with the relative proportions of ear breadth to ear length, will however be utilised.

Of Australian aboriginal ear measurements of length and breadth I have been enabled to utilise 49 examples, derived from 46 individuals. These 49 examples are composed of both ears of the three heads of this work, of three ears from the three individuals mentioned by Cunningham (26), and of the single ears of forty individuals measured on the living subject by Spencer and Gillen (41). Of these 49 ear measurements, 30 are from males and 19 from females.

On analysing these ear measurements, the average lengths, breadths, and indices for the Australian aboriginal ear are found to be as follows:—

A. *Greatest Ear Length.*	
49 unsexed	64·2
30 males	65·5
19 females	62·1
B. *Greatest Ear Breadth.*	
49 unsexed	34·4
30 males	34·1
19 females	34·9
C. *Ear Index.*	
49 unsexed	53·6
30 males	52·1
19 females	56·2

From these figures there emerge two facts: first, that there is a sexual difference in the Australian aboriginal ear; and, second, that the female ear is relatively broader and shorter than the male. It is, however, right to observe that Cunningham (26) says, in reference to this ear index, "I question very much if these indices possess any real importance in establishing racial distinctions, and a study of the figures given by other observers tends to confirm me in this opinion."

Karutz (45), on the other hand and in opposition to Cunningham, would seem to regard the relative proportions of ear breadth to ear length as of very distinct racial value, and gives the following proportions:—

The proportion of ear breadth to ear length is in:—

Germans, Semites, Hamites, and Mongolians as 1 is to 1·8.
Papuans as 1 is to 1·75.
Hottentots as 1 is to 1·7.
Negroes as 1 is to 1·6.

The Negro ear is not only short, but is also relatively broad; that of the Papuan broad and of middle length; that of the Hamite of middle length and small; whilst that of the Mongolian is the longest and smallest.

For the Australian aboriginal my figures work out as follows:—

For the males as 1 is to 1·9.
For the females as 1 is to 1·7.
For both sexes combined as 1 is to 1·8.

The Australian aboriginal male has therefore a relatively narrower and longer ear than any of the races mentioned by Karutz, whilst the Australian aboriginal female ear is of the same relative proportions as the Hottentot.

Schwalbe (46), writing in Bardeleben's *Handbuch der Anatomie des Menschen*, terms the ear index of Topinard, with which we have just dealt, the "physiognomischen Ohrindex," and, unlike Cunningham, regards it as of considerable importance, for he says the "physiognomischen Ohrindex ist ein anthropologisch wichtiger Index." He further tells us that in the adult male it ranges from 50 to 78, with an average of 60·5. In the female it ranges from 45 to 74, with an average of 59·0.

In addition to this physiognomical ear index, Schwalbe (48) has introduced what he calls the *morphological ear index*, in which the length of the ear base is compared with the true length of the ear, the latter being taken as 100.

As Spencer and Gillen did not record either of these measurements, my examination of the morphological ear index in the Australian aboriginal is restricted to 9 ears of six individuals, 6 ears from the three individuals of the present work, and the remainder from Cunningham's memoir. Of these 9 ears, 7 are from males and 2 from one female.

On analysing these measurements, the ear base, the true ear length, and the morphological ear index are, for the Australian aboriginal, as follows:—

A. *Length of Ear Base.*

9 unsexed	48·4
7 males	49·4
2 females	45·0

B. *True Ear Length.*

9 unsexed	31·7
7 males	29·5
2 females	39·5

C. *Morphological Ear Index.*

9 unsexed	152·4
7 males	167·1
2 females	113·9

Schwalbe tells us that the *length of the ear base* ranges in the male from 33 to 58 mm., with an average of 44·4, and in the female the range is from 30 to 61, with an average of 40·1.

The *true ear length* ranges in the male from 22 to 49, with an average of 33·9. In the female the range is from 24 to 41, with an average of 33·7. As regards this former measurement, the Australian aboriginal, both male and female, is well above the average; but for true ear length the male Australian is below the average, and the female above it.

The *morphological ear index* is of chief significance for the comparison of the human ear with that of the lower animals. It expresses the degree of reduction which the ear has undergone from the long-eared type of animal. The greater the numerical value of the index the more has the ear become reduced from the long-eared type. Hence a low index expresses an animal-like ear; and, as a general rule, the high indices occur most frequently in the female, because in the female the ear has progressed farthest from the animal type. In man the index ranges from 83 to 195.

In the following table of the morphological ear index I have availed myself of the figures given by Schwalbe in Bardeleben's *Handbuch* and of Wilhelm's figures which he there quotes, of Sakaki's (47) figures for 200 Ainus, and of my own observations on the Australian aboriginal.

Rabbit	21·3
Kangaroo	33·
Lemur	76·
Chimpanzee	105·
2 Australian females	113·9
Orang	122·
Gorilla	125·
Modern degenerates	138·7
Modern epileptics	159·7
Modern healthy criminals	163·
7 Australian males	167·1
Average of 200 Ainu females	168·9
Average of 200 Ainu males	169·5
Modern criminal lunatics	171·6
Modern healthy man	175·3

From the paucity of the Australian figures it would be unwise to draw any decided conclusions, especially as the female figures merely deal with the two ears of one individual. It is, however, of some interest to note that in both of the Australian examples, as in Sakaki's Ainus, the female ranks lower than the male. It would therefore appear that the tendency of the modern European female to possess a higher morphological ear index than the male is a sign of marked and high civilisation and of advancement, and is not found in the lower races.

Schwalbe (48), in one of his earlier writings, has classified ears into six types, according to the degree and size of the Darwinian tubercle. Cunningham devotes some attention to this and lays considerable stress upon Schwalbe's work. In the present series of three heads I should say that all, but particularly case 1, approximate to Schwalbe's type two or Cercopithecus form, where the ear apex is pressed very closely to the posterior (dorsal) border of the helix, but the latter is not inrolled. This is the form which Schwalbe states is characteristic of the human embryo from the fourth to the sixth month.

Form Analysis of the Skull.

Passing next to the consideration of what Schwalbe has so aptly termed the "form analysis" of the skulls of the heads dealt with in the present work, it is well known that Schwalbe, in his examination of the *Pithecanthropus* calvaria, as well as in his other similar works on the crania of primitive man, has introduced some novel and useful methods. His methods have been adopted with, or without, modification by Klaatsch in his work on the Australian aboriginal skull, by Berry and Robertson (49) in their work on the Tasmanian calvaria, and by most other investigators of the calvaria of primitive and fossil man. I have therefore thought it advisable to institute such a form analysis of the skulls of two of the heads with which this work deals, and to utilise, for purposes of comparison, the observations I have already employed, in conjunction with Dr Robertson (49), in the examination of the Tasmanian crania. The observations recorded on the Tasmanian crania were 27 in number, but to them I have now added five others employed by Klaatsch, in order that such data may be available for comparative purposes in future investigations.

The following table gives the results of the form analysis of the Australian aboriginal heads one and three of the present work, with the minimum, average, and maximum figures from the Tasmanian work just referred to for purposes of comparison:—

	Australian.		Tasmanian.		
	I.	III.	Minimum.	Average.	Maximum.
1. Glabella-inion length . . .	183	180	157	173	188
2. Calvarial height	101	100	87	97	108
3. Calvarial height index . . .	55	55	48	56	62
4. Maximum breadth	152	135	120	134	145
5. Calvarial height breadth index .	66	74	66	72	79
6. Half the sum of one and four . .	167	157	140	154	164
7. Calvarial height half-sum index .	60	63	55	63	69
8. Distance of calvarial height foot point from glabella . . .	106	101	85	102	115
9. Calvarial height foot point positional index	58	56	53	59	65
10. Frontal angle	81	84	72	86	96
11. Bregma angle	61	68	51	56	64
12. Distance of bregma foot point from glabella	48	56	43	58	71
13. Bregma foot point positional index .	26	31	26	34	41
14. Length of frontal arc . . .	127	127	113	126	143
15. Length of frontal chord . . .	109	115	97	109	120
16. Curvature index of os frontale. .	86	90	81	87	97
17. Angle of frontal curvature . .	149	142	131	139	149
18. Length of the chord of the pars glabellaris	36	38	18	24	29
19. Length of the chord of the pars cerebralis	82	85	73	94	106
20. Glabellar cerebral chord index .	44	45	18	25	36
21. Length of the parietal arc . .	146	141	112	126	145
22. Length of the parietal chord . .	122	119	98	113	127
23. Curvature index of os parietale .	83	84	84	90	98
24. Angle of parietal curvature . .	139	136	125	134	141
25. Parietal frontal arc index . .	115	111	86	100	114
26. Lambda angle	78	82	74	80	88
27. Opisthion angle	33	36	34	41	47
28. Nasio-inion length	173	174	...	...	...
29. Glabella-lambda length . . .	182	182	...	...	...
30. Lambda-glabella inion angle . .	22	20	...	...	...
31. Distance of bregma foot point from glabella on glabella-lambda line .	78	84	...	...	...
32. Bregma foot point glabella-lambda-index	43	46	...	...	...

It need hardly be said that it is not here intended to draw any conclusions from the comparison above instituted of two Australian aboriginal heads with the crania of fifty-two Tasmanians, for the insufficiency of the Australian material here employed would make it futile to do so. This notwithstanding, it is of some interest to note that the two Australians, one male the other female, rank, in the majority of the observations, within the Tasmanian range. Before pointing out wherein the Australians are not so within the Tasmanian range, it may be as well to state that all the figures are confined to whole numbers. Where decimal points occurred the nearest whole number is given, and if the decimal be a ·5 then the lower whole number is recorded.

As regards the *glabella-inion length*, both Australians are within the Tasmanian range of variation; but in both instances they are well above the Tasmanian average, which is in accordance with modern research on the subject of the relative lengths of the crania of these two peoples.

In *breadth* the Australian male is greater than the greatest breadth for any Tasmanian, whilst the female almost exactly coincides with the average Tasmanian breadth.

In view of the great morphological significance of Schwalbe's *bregmatic angle* it is of exceptional interest to note that, whilst both Australians are within the Tasmanian range, they are both well above the Tasmanian average; the same remarks applying to another equally important morphological observation, namely, the *angle of frontal curvature.*

The *length of the chord of the pars glabellaris* of the os frontale is, in the Australians, very striking. In both male and female this length is considerably greater than the corresponding maximum figure in the Tasmanian. Whilst the length of the chord of the pars glabellaris cannot express anything more than the linear area occupied by the glabellar part of the frontal bone on the total length of the os frontale, it is somewhat remarkable that this area should be so much greater in the Australian than in the Tasmanian, for the latter is as ancient a race as the Australian and probably more so. The question of the evolution of the glabellar and cerebral portions of the os frontale is, however, far too complicated and intricate a one to be entered upon here; but it must not be forgotten that Cunningham (52) has thrown much light upon this problem, and that Cross (53) has practically proved that of all Schwalbe's observational methods of examining the calvaria those which have to do with the glabellar portion of the os frontale are of least value.

For both the *length of the parietal arc* and the *parietal frontal arc index* the female Australian is within the Tasmanian range of variation; but the male is, in each instance, just beyond the Tasmanian maximum. This implies that in this individual the parietal segment of his skull has attained a relatively greater degree of development in proportion to the frontal arc than in any Tasmanian.

The Surfaces of the Sections.

In the illustrations which accompany this work there are displayed the surfaces of the sections of the male Australian aboriginal head number one.

An examination of these sections will show that the topographical anatomy of the Australian aboriginal head, as studied in section, does not differ in any important essential from that of other races.

This general statement does not imply, nor is it intended to imply, that there are no differences between the Australian and other races in minutiæ. For example, Adachi (29), working on the well-known fact that the eyes project farther forwards in Japanese than in Europeans, examined the topographical position of the bulbus oculi in the orbit by means of frozen sections on 9 Japanese males and 5 females. He treats of the differences between the bulbus oculi of the Japanese and the European, and he notes that the distance from the point of entrance of the N. opticus in the bulbus oculi to the foramen opticum is in the Japanese 25 mm., whereas in the European it is only 18·1 mm.

This same distance is, in the male Australian aboriginal number one, 23·5 mm., and in this respect the Australian more nearly approaches the Japanese type than the European. This fact illustrates the main contention, that in general the sectional anatomy of the Australian aboriginal does not differ from that of any other race, but that in details of minutiæ there are many minor differences of the character illustrated. The negative evidence adduced is, however, of value as supporting the monogonist theory of the origin of the several races of Mankind, whilst the differences of detail are not of sufficient importance as to warrant careful description. I can therefore merely refer those who are interested in such racial differences to the plates themselves, which, as they are recorded to scale, will enable any future observer to study such racial differences at his leisure.

Comparison of the Brain Surface of the Australian and the European.

Of much greater importance than the minutiæ of the sections is the comparison of the brain surface of the Australian and the European. For the purposes of establishing this comparison I have selected the head of the male Australian number one, and the head of a male European aged 75 years. The latter forms one of the series of heads figured in Cunningham's *Contribution to the Surface Anatomy of the Cerebral Hemisphere.* In that work it forms the subject of illustration number 16, and is further accessible in the form of a cast. This European head from Cunningham's work is a particularly favourable one for the comparison with the Australian, inasmuch as the skull measurements of glabella-inion length, calvarial height, and maximum breadth are as nearly as possible identical, as the following figures show :—

	European.	Australian.
Glabella-inion length . .	181	183
Maximum breadth . .	160	152
Calvarial height . . .	98	101

A dioptrographic tracing of the European skull and lateral surface of the cerebral hemisphere was then superimposed on a similar tracing of the Australian skull and brain, both heads being oriented on the glabella-inion plane. A study of these superimposed tracings showed that the European brain attained its maximum degree of development in the vicinity of the frontal pole and the Australian in the occipital pole. The European brain exceeded that of the Australian, because laterally it had extended farther caudally.

In order to obtain a mathematically correct idea of the relative amounts of surface occupied by the frontal, parietal, and occipital lobes on the lateral surfaces of the two brains respectively, each lobe was divided up into a series of triangles and the area in square millimetres calculated in the ordinary mathematical way. The results of, this procedure, though sufficiently striking in themselves, must be regarded with a certain amount of caution for two reasons: first, the curved area of the brain was worked out on a flat surface; and, second, the European was calculated from a cast upon which the original Author (Cunningham), for the purposes of his research, had left in a bar of bone over the region of one of the most important fissures for this calculation, namely, the fissura parieto-occipitalis. It was, therefore, impossible to say exactly where the fissure was really situated. Its determination was, however, done as accurately as possible both from the cast itself and also from the illustration of the cast in Cunningham's original memoir, where the different cerebral lobes are represented in colour.

Bearing in mind the caution just mentioned, the results in square millimetres of the frontal, parietal, and occipital lobes in the European and Australian were as follows:—

	European.	Australian.
Lobus frontalis . . .	4710	4857
Lobus parietalis . . .	4577	2534
Lobus occipitalis . . .	758	1204

It would thus appear that the higher and more civilised races of Mankind owe their more advanced evolutionary position to an increased development of the lobus parietalis rather than to the lobus frontalis; that this parietal increase is partly actual as the result of the increased brain, and partly relative, being gained at the expense of the occipital lobe, and that these important facts are not altogether indicated by a study of the parietal and frontal arcs. In other words, the infinitely greater "intellectuality" of the higher races is gained by the growth of the educational portion of the neopallium at the expense of the visual area of the lower races.

BIBLIOGRAPHY.

(1) Flower and Murie, "Account of the Dissection of a Bushwoman," *Journ. Anat. and Phys.*, vol. i., 1867, pp. 189–208.

(2) Duckworth, W. L. H., *Morphology and Anthropology*, Cambridge, 1904; pp. xxviii + 564.

(3) Chudzinski, T., "Quelques mots sur la splanchnologie des races humaines," *Revue d'Anthropologie*, Year 16, 1887, Series 3, vol. ii. pp. 275–291.

(4) Chudzinski, T., "Quelques observations sur le grand droit de l'abdomen dans les races humaines," *Bull. de la Soc. d'Anthropologie de Paris*, Series 4, vol. vi., 1895, pp. 522–540.

(5) Chudzinski, T., *Quelques observations sur les muscles peaussiers du crâne et de la face dans les races humaines*, Paris, Masson et Cie., 90 pp., with fig.

(6) Chudzinski, T., "Observations sur les variations musculaires dans les races humaines," *Mém. Soc. d'Anthropol. Paris*, vol. ii., Series 3, 1898.

(7) Chudzinski, T., "Quelques mots sur l'anatomie de deux nègres," *Revue d'Anthropol.*, 1884, pp. 603–616.

(8) Turner, Sir W., "Notes on the Dissection of a Negro," *Journ. Anat. and Phys.*, vol. xiii., 1878, pp. 382–386.

(9) Turner, Sir W., "Notes on the Dissection of a Second Negro," *Journ. Anat. and Phys.*, vol. xiv., 1879, pp. 244–248.

(10) Turner, Sir W., "Notes on the Dissection of a Third Negro," *Journ. Anat. and Phys.*, vol. xxxi., 1896–7, pp. 624–626.

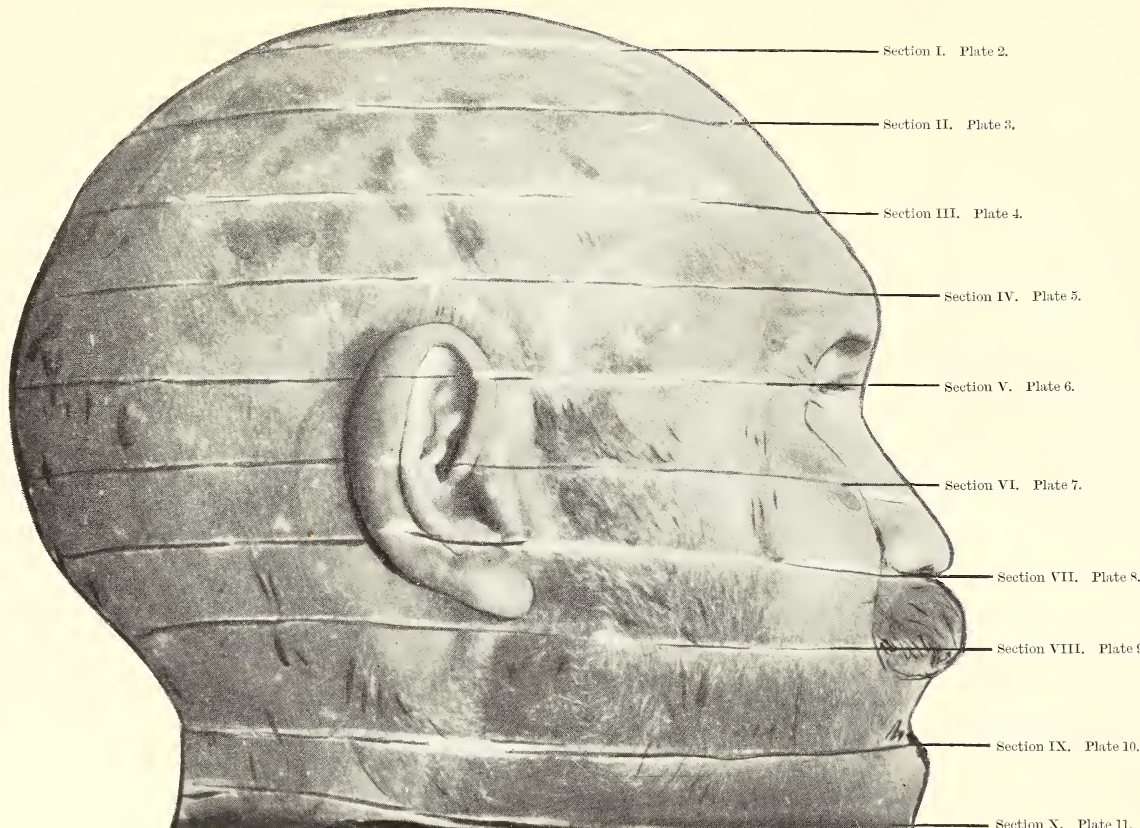
(11) Wood, J., Account of the Dissection of a Negro, published in the *Proceedings of the Royal Society of London*, June 1865.

(12) Bryce, T. H., "Notes on the Myology of a Negro," *Journ. Anat. and Phys.*, vol. xxxi., 1896–7, pp. 607–616.

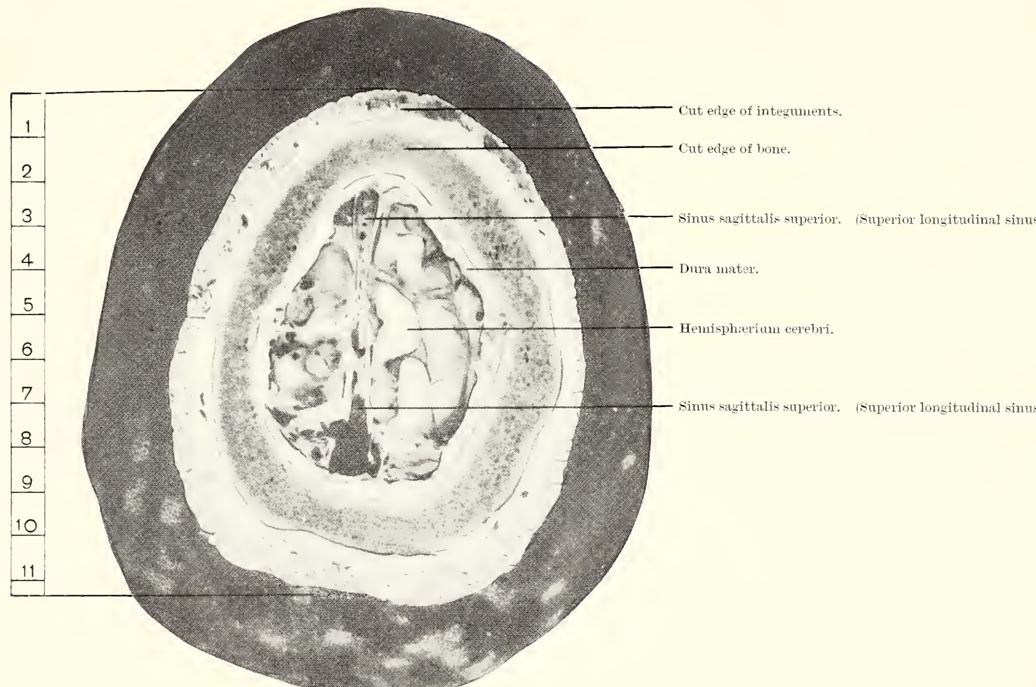
(13) Giacomini, C., *Sopra l' anat. del Negro IV varieta nel Sistema muscolare*, Torino, 1882.

(14) Giacomini, C., "Notes sur l'anatomie du nègre," *Archives italiennes de Biologie*, vol. ix. pp. 119–138.

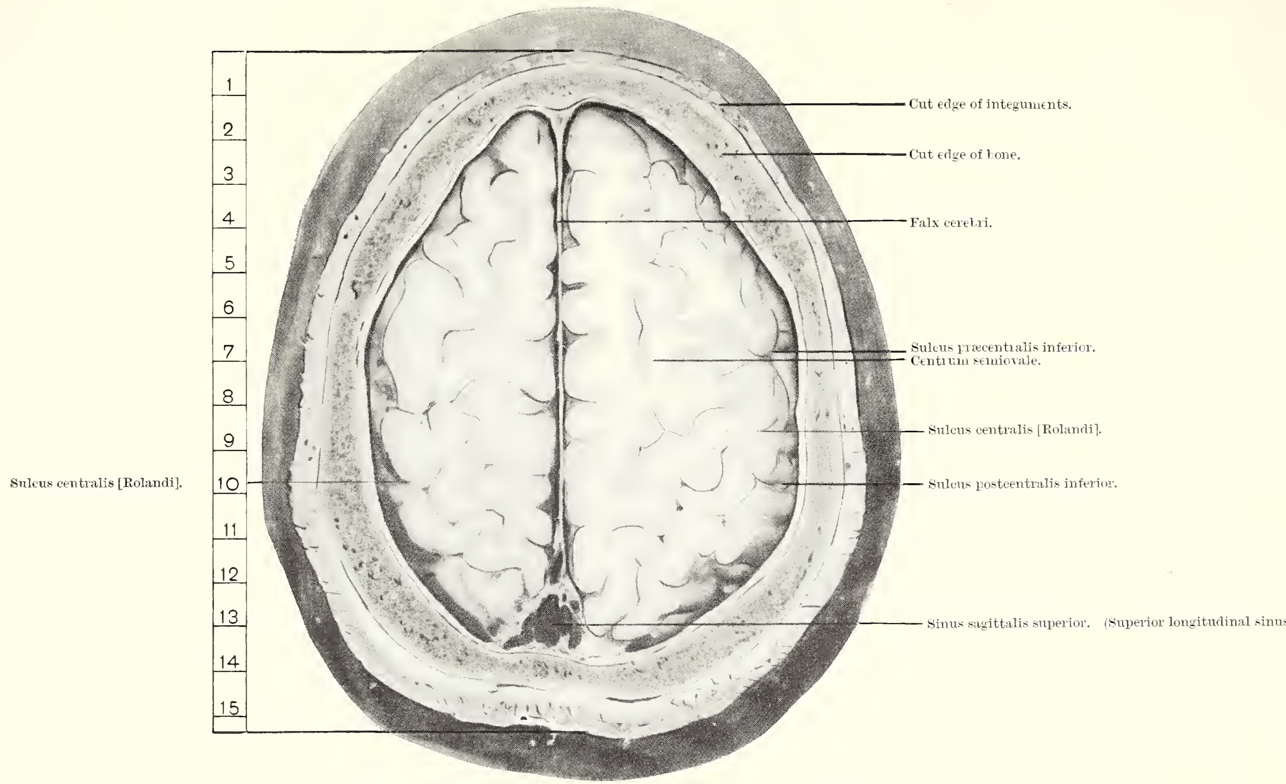
(15) Giacomini, C., "Annotations sur l'anatomie du nègre," *Archives italiennes de Biologie*, vol. xvii., 1892, pp. 337–371.



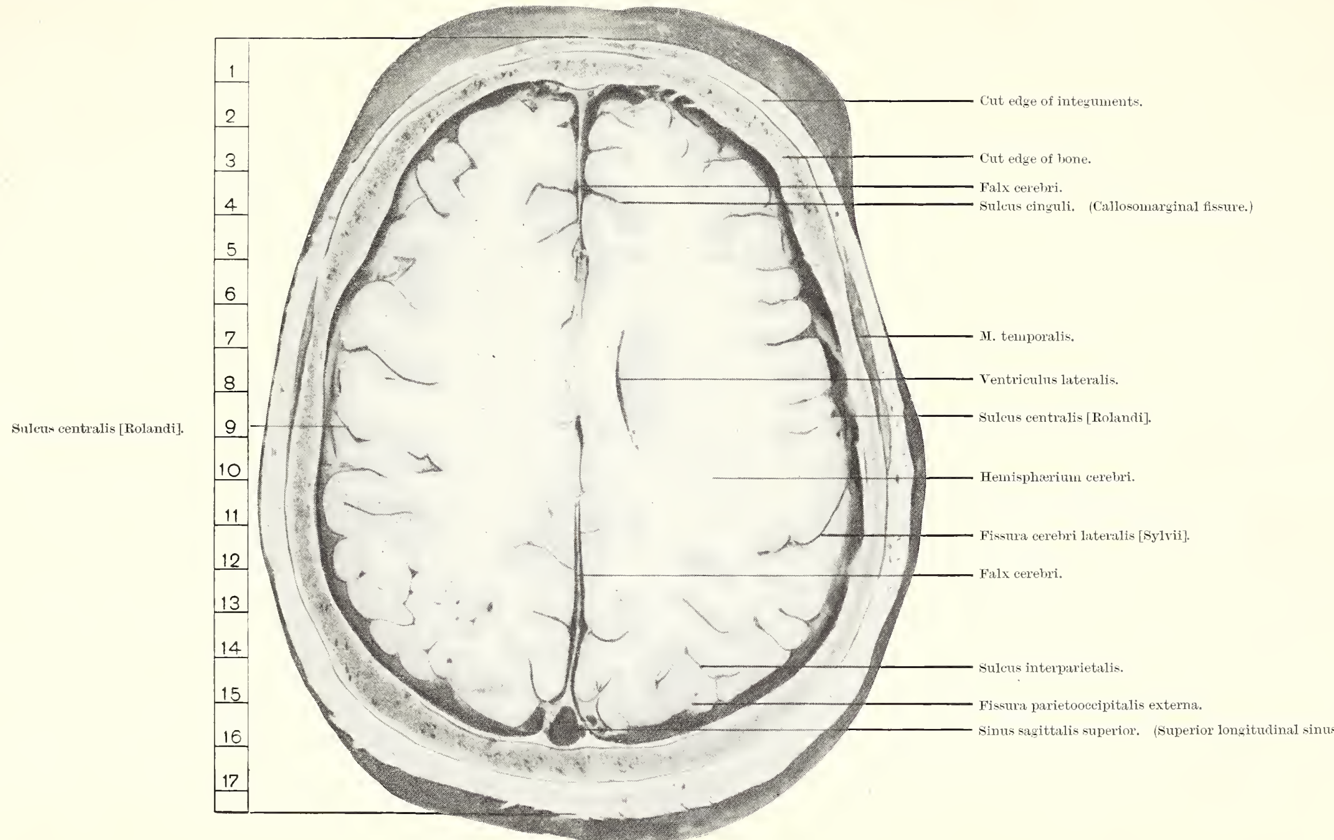
Right Lateral View of the Head, showing the Planes of the Sections.



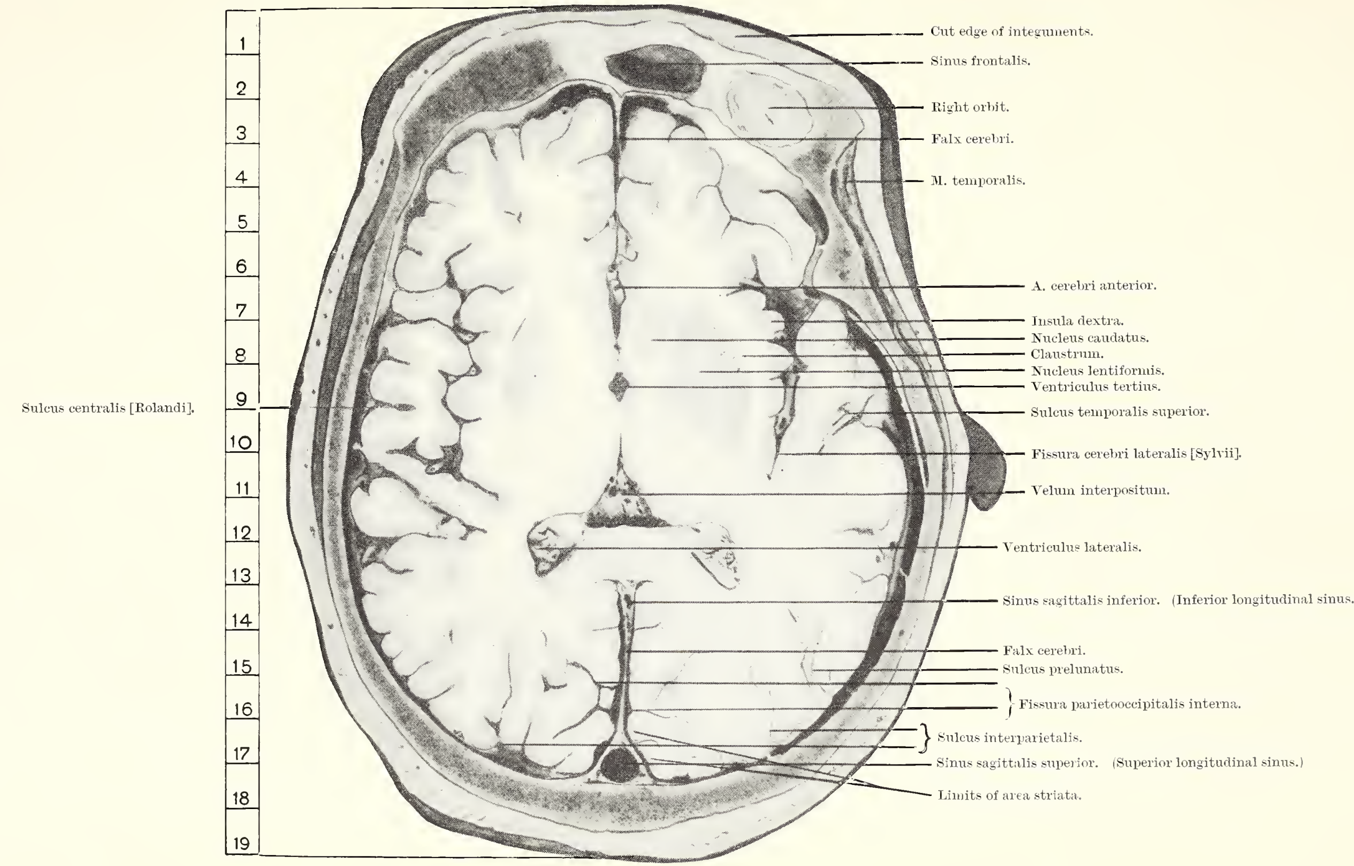
Section Number One, 2 cm. Caudal to the Vertex.



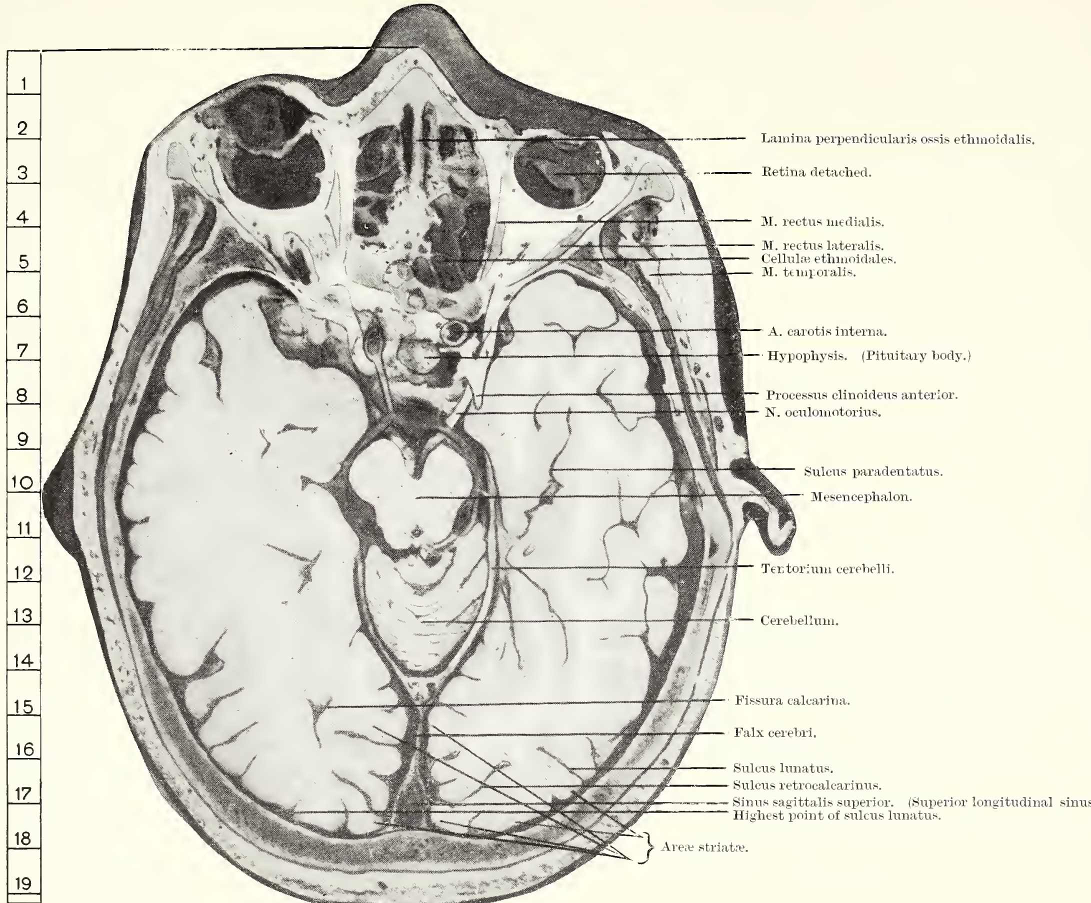
SECTION NUMBER TWO, 4 CM. CAUDAL TO THE VERTEX.



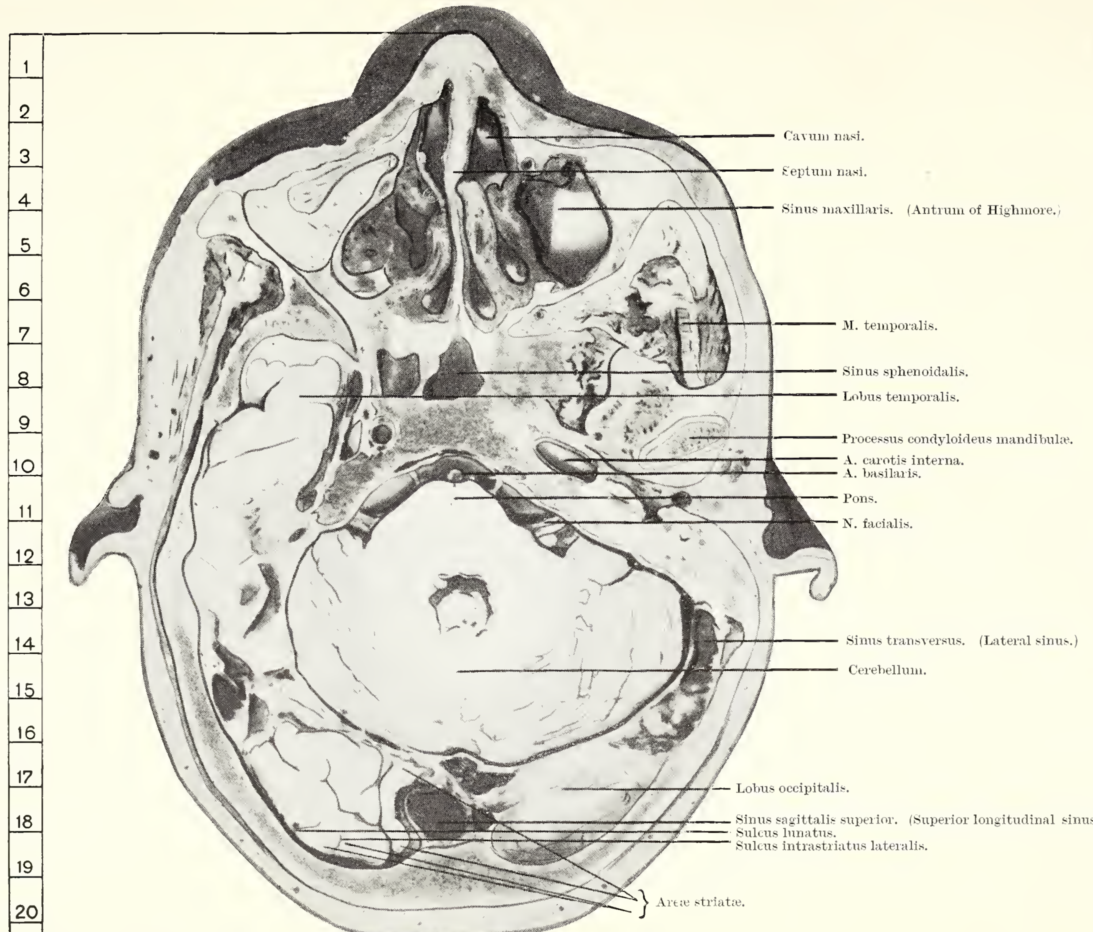
SECTION NUMBER THREE, 6 CM. CAUDAL TO THE VERTEX.



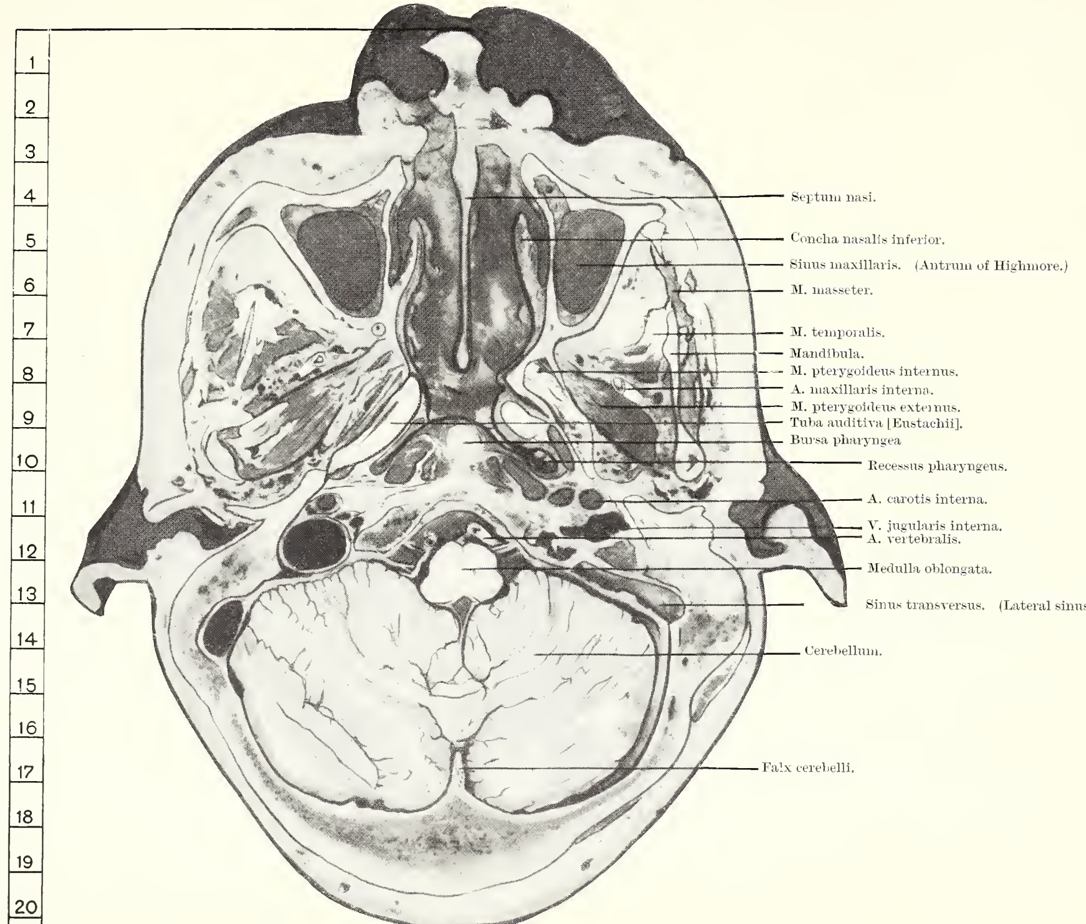
SECTION NUMBER FOUR, 8 CM. CAUDAL TO THE VERTEX.



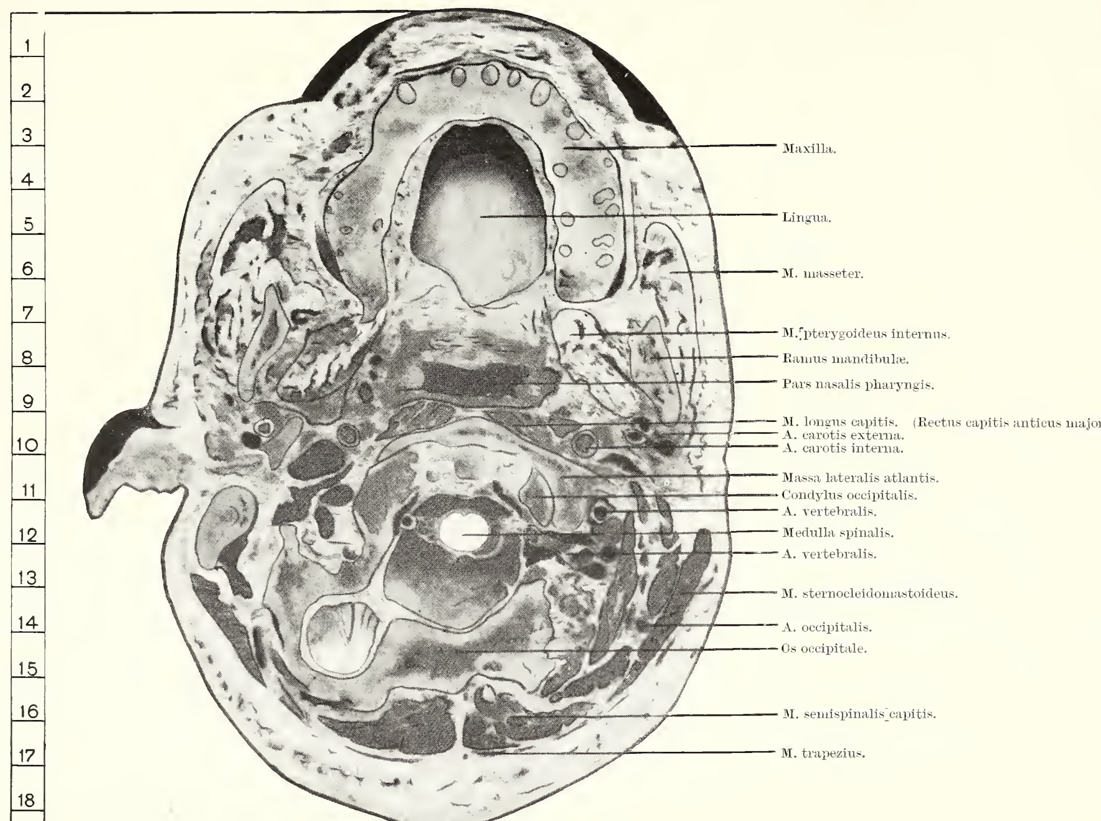
SECTION NUMBER FIVE, 10 CM. CAUDAL TO THE VERTEX.



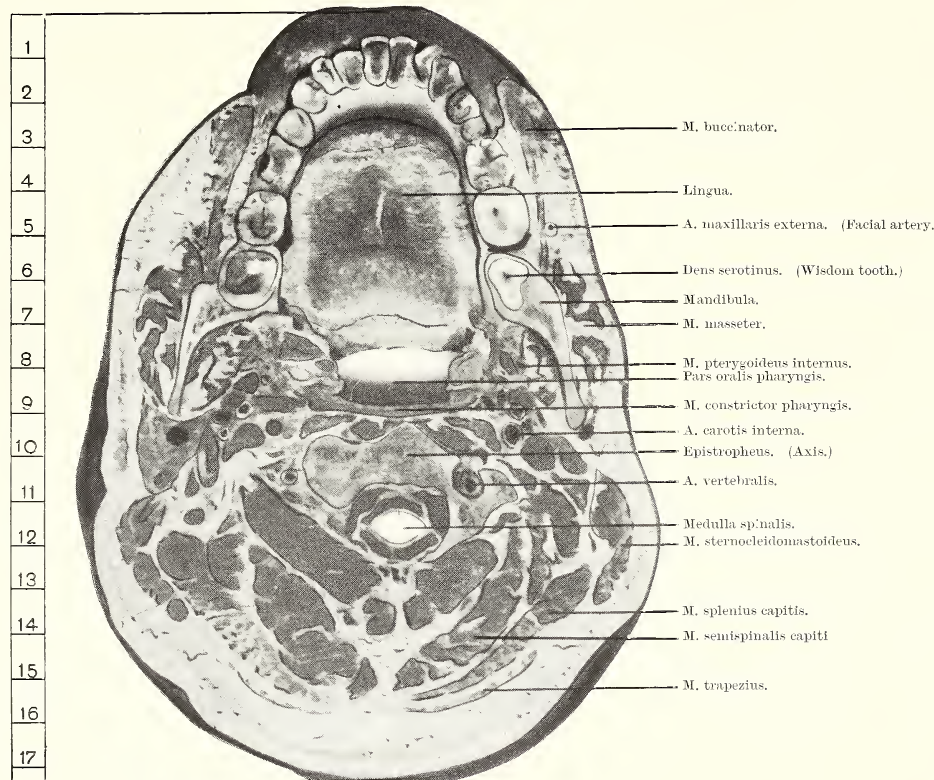
Section Number Six, 12 cm. Caudal to the Vertex.



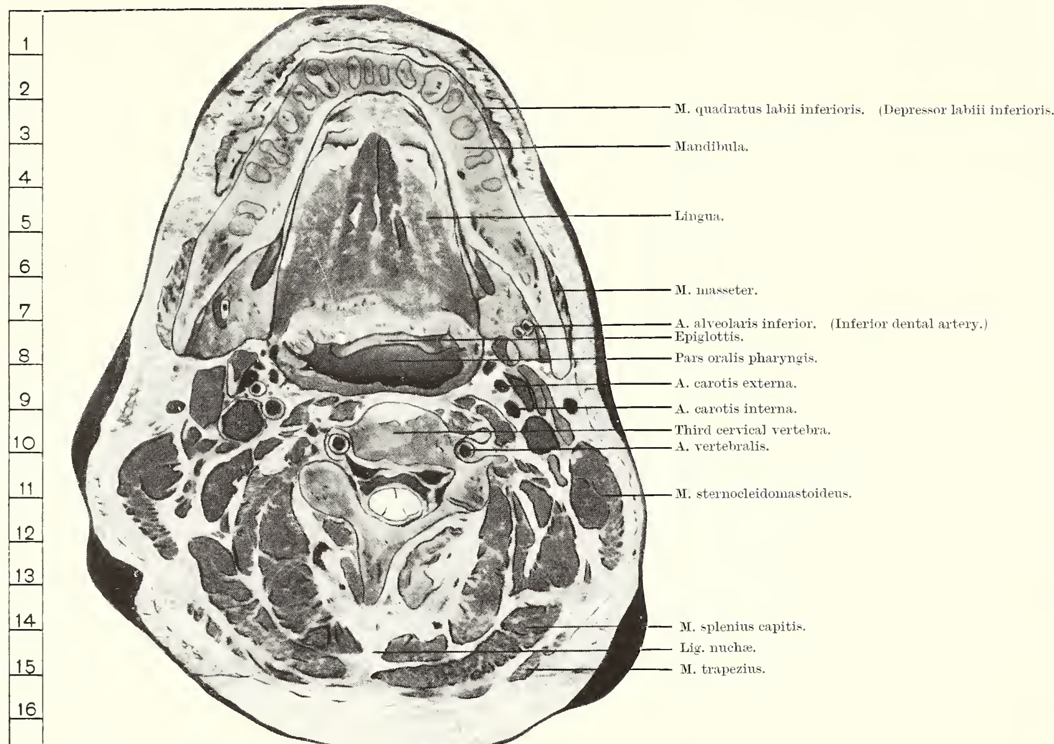
SECTION NUMBER SEVEN, 14 CM. CAUDAL TO THE VERTEX.



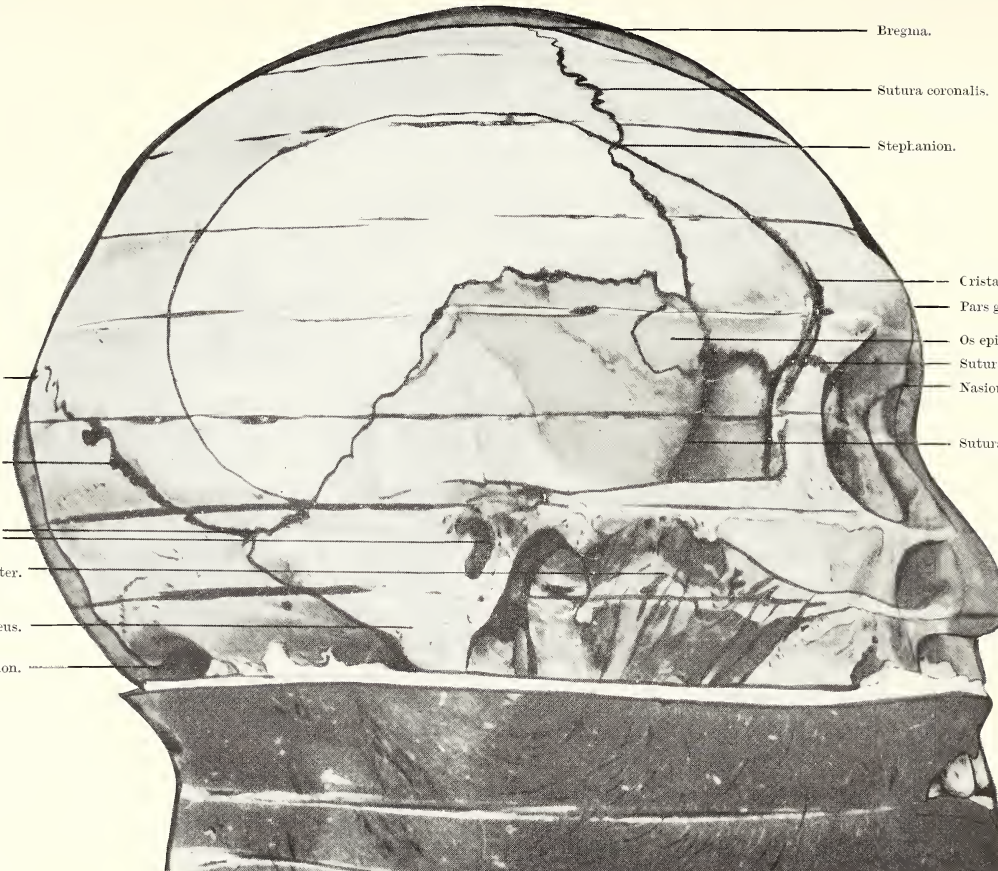
Section Number Eight, 16 cm. Caudal to the Vertex.



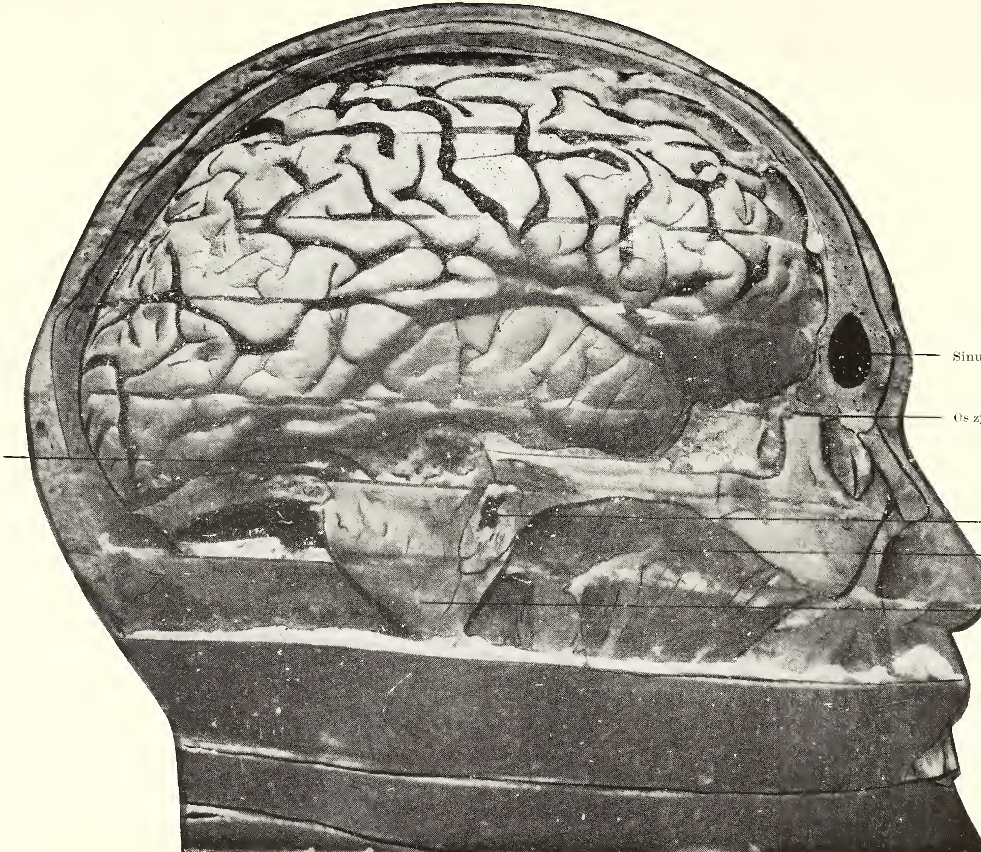
SECTION NUMBER NINE, 18 CM. CAUDAL TO THE VERTEX.



Section Number Ten, 20 cm. Caudal to the Vertex.

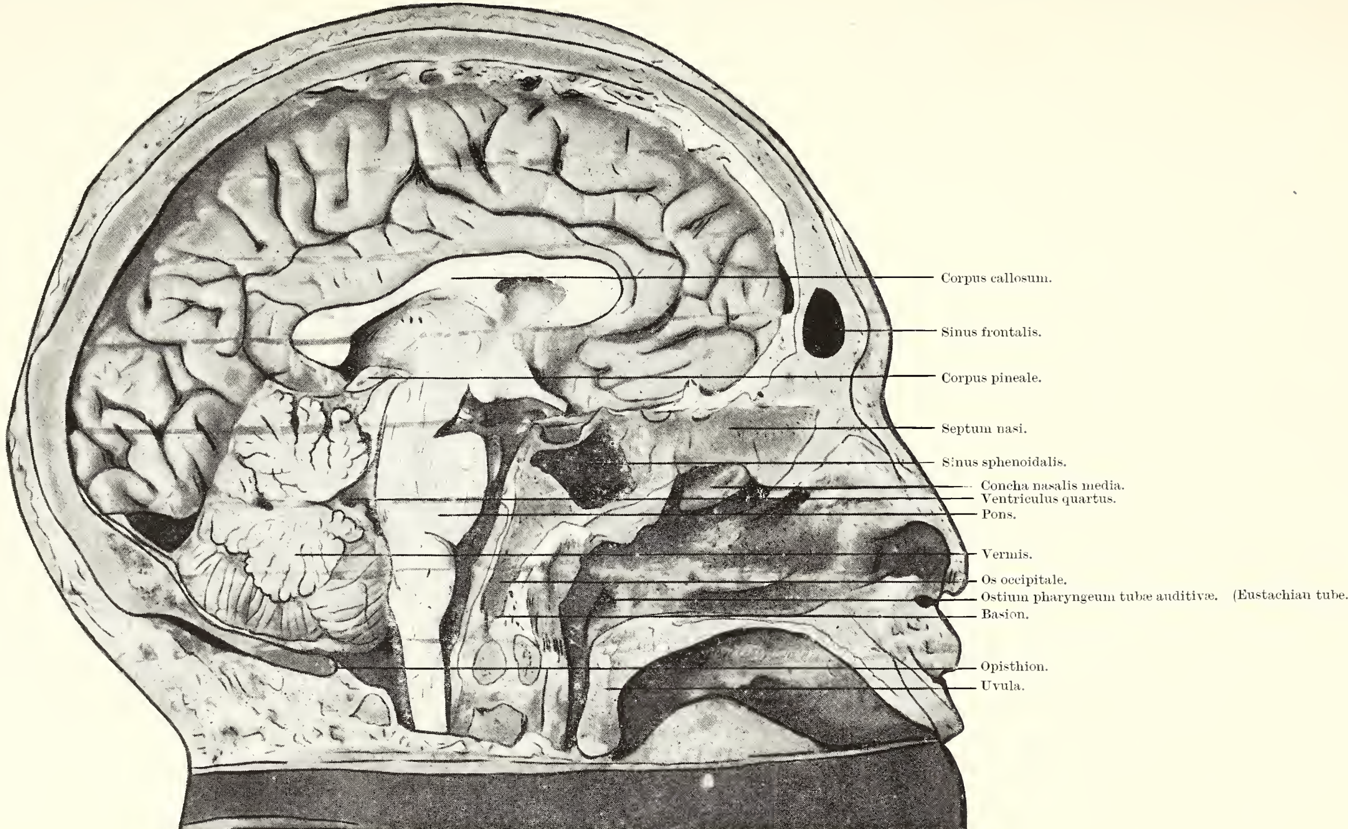


Reconstruction of the Head, showing the Norma Lateralis Dextra of the Skull.



RECONSTRUCTION OF THE HEAD, SHOWING THE FACIES LATERALIS OF THE RIGHT CEREBRAL HEMISPHERE.

PROF. R. J. A. BERRY.



Reconstruction of the Head, showing the Facies Medialis of the Left Cerebral Hemisphere.

Prof. R. J. A. Berry. [Plate XIV.

(16) Testut, L., *Contrib. à l'anatomie comparée des races nègres*, Bordeaux, 1884, Bellini et Cie.

(17) Testut, L., "Les anomalies musculaires chez les nègres et chez les blancs," *Internat. Monatsschrift*, Band i., 1884.

(18) Michel, "Two Cervical Muscular Anomalies in the Negro," *Med. Record New York*, vol. xli., 1892, pp. 125–128.

(19) Popovsky, J., "Les muscles de la face chez un nègre achanti: Matériaux pour l'histoire de l'homme," *Revue d'Anthropol. Revue d'Ethnographie réunis*, vol. i., 1890, pp. 413–422.

(20) Anthony, R., and Hazard, A., "Notes sur la myologie d'un nègre de l'Oubangui," *L'Anthrop.*, vol. xvi. pp. 445–456.

(21) Fallot and Alezais, "Notes sur l'autopsie d'un Indien d'Amérique et d'un nègre de la Martinique," *L'Anthrop.*, vol. i., 1890, pp. 655–678.

(22) Bartels, P., "Über die Nebenraume der Kehlkopfhole: Beiträge zur vergleichenden- und zur Rassenanatomie," *Zeitsch. Morphol. und Anthropol.*, vol. viii. pp. 11–61.

(23) Livini, F., "Contribuzioni alla anatomia del Negro," *Arch. l'Anthrop.*, vol. xxxix. p. 203.

(24) Forster, A., "Das Muskelsystem eines männlicher neugeborenen Papua," *Arch. K. Leop. Carol. deutsch. Akad. Naturf.*, vol. lxxxii. pp. 1–140.

(25) Fischer, E., "Anatomische Untersuchungen an den Kopfweichteilen zweier Papua," *Corrbl. deutsch. Anthropol. Ges. Verh. Salzburg*, Year 36, pp. 118–122.

(26) Cunningham, D. J., "The Head of an Australian Aboriginal," *Journ. Anthropol. Inst.*, vol. xxxvii., 1907, pp. 47–56.

(27) Duckworth, W. L. H., "Notes on the Dissection of the Head of an Aboriginal of Australia," *Anthropological Studies*, Cambridge, pp. 61–64.

(28) Adachi, B., "On the Prepuce of the Japanese," *The Chugai-Ljishimpo*, vol. xx. No. 17.

(29) Adachi, B., "Anatomische Untersuchungen an Japanen," *Zeitsch. Morph. und Anthrop.*, vol. ii. pp. 198–222.

(30) Adachi and Azaza, "Über die Lage des Augapfels bei der Japanern," *Mitteil. d. med. Ges. Tokio*, vol. xiv., 1900.

(31) Adachi, B., "Haufigeres Vorkommen des Musculus sternalis bei Japanern," Anat. Untersuchungen an Japanern, IV., *Zeits. Morphol. und Anthropol.*, vol. vii. pp. 133–144.

(32) Adachi, B., "Topographische Lage des Augapfels der Japaner," *Zeits. Morphol. und Anthropol.*, vol. vii. pp. 481–501.

(33) Adachi, B., "Preliminary Notes on the Facial Muscles of the Japanese and the Chinese," *Journ. Anthrop. Soc. Tokio*, vol. xx. pp. 499–500.

(34) Adachi, B., "Das Knorpelstuck in der Plica semilunaris conjunctivæ der Japaner," *Zeitsch. Morphol. und Anthropol.*, vol. ix. pp. 325–326.

(35) Adachi, B., "Mikroskopische Untersuchungen über die Augenlides der Affen und des Menschen (Insbesondere der Japaner)," *Mitteil. med. Fakult. Kaiserl. Japan. Univ. Tokio*, vol. vii. pp. 47–90.

(36) Birkner, F., "Beiträge der Rassenanatomie der Gesichtsweichenteile," *Corrbl. deutsch. Ges. Anthropol.*, Year 34, 1903, p. 163.

(37) Birkner, F., "Beiträge zur Rassenanatomie der Chinesen," *Corrbl. deutsch. Ges. Anthropol.*, Year 35, pp. 144–148.

(38) Anderson Stuart, "Notes from the Dissection of a Chinaman," *Journ. Anat. and Phys.*, vol. xix., 1885, pp. 227–228.

(39) Schwalbe, G., "Studien über *Pithecanthropus erectus* Dubois," *Zeitsch. Morph. und Anthropol.*, Band i., 1899.

(40) Spencer and Gillen, *The Native Tribes of Central Australia*, Macmillan & Co., London, 1899.

(41) Spencer and Gillen, *The Northern Tribes of Central Australia*, Macmillan & Co., London, 1904.

(42) Berry, R. J. A., Robertson, A. W. D., and Cross, K. S., "A Biometrical Study of the Relative Degree of Purity of Race of the Tasmanian, Australian, and Papuan," *Proc. Roy. Soc. Edin.*, vol. xxxi., 1910.

(43) Reche, C., "Über den Nasenindex," *Corrbl. deutsch. Ges. Anthropol.*, Year 38, pp. 49–52.

(44) Holl, M., "Über die Lage des Ohres," *Mitt. Anthropol. Ges. Wien*, Band xxxix., 1899, pp. 177.

(45) Karutz, "Ein Beitrag zur Anthropologie des Ohres," *Arch. Anthrop.*, Band xxviii., 1900, p. 733.

(46) Schwalbe, G., "Sinnesorgane," *Handbuch der Anatomie des Menschen*, herausgegeben von Prof. Dr Karl von Bardeleben: Gustav Fischer, Jena, 1897.

(47) Sakaki, "Über die Ohrmuschel der Ainu," *Mitteil. med. Fakult. Tokio*, vol. viii. pp. 25–60, 1903.

(48) Schwalbe, G., "Das Darwin'sche Spitzohr beim menschlichen Embryo," *Anat. Anz.*, vol. iv., 1889, pp. 176–189.

(49) Berry, R. J. A., and Robertson, A. W. D., "The Place in Nature of the Tasmanian Aboriginal, etc.," *Proc. Roy. Soc. Edin.*, vol. xxxi., 1910.

(50) Anderson, J. H., "An Investigation as to the most Accurate Method of estimating the Cubic Capacity of the Living Head, together with some remarks on the relative thickness of the cranial integuments," *Journ. Royal Anthrop. Instit.*, 1911.

(51) Anderson, J. H., "The Proportionate Contents of the Skull as demonstrated from an Examination of 40 Caucasian Crania," *Journ. Royal Anthrop. Instit.*, 1911.

(52) Cunningham, D. J., "The Evolution of the Eyebrow Region of the Forehead, with special reference to the excessive supraorbital development in the Neanderthal Race," *Trans. Roy. Soc. Edin.*, vol. xlvi., part ii., 1908, pp. 283–311.

(53) Cross, K. S., "On a Numerical Determination of the Relative Positions of certain Biological Types in the Evolutionary Scale and of the Relative Values of various Cranial Measurements and Indices as Criteria," *Proc. Roy. Soc. Edin.*, vol. xxxi., 1910.

(*Issued separately July* 27, 1911.)

XLIII.—The Relation of the Mono-Molecular Reaction to Life-Processes and to Immunity. By John Brownlee, M.D., D.Sc.

(MS. received January 9, 1911. Read January 9, 1911.)

Some of the relationships, such as those between toxins and antitoxins, discovered during the last twenty years have lately been investigated by Arrhenius. It has been found by this observer that many of these

Agglutinating Power of the Blood.

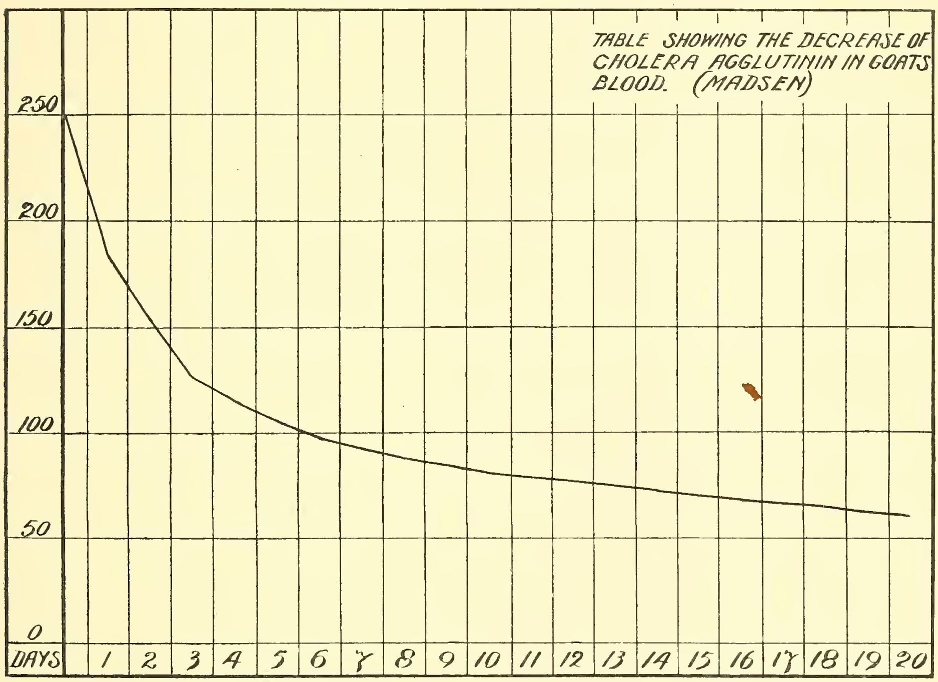


Diagram I.

processes can be adequately represented by the equations which govern homogeneous mass reactions; for instance, the rates of the disappearance of diphtheria antitoxin and of typhoid agglutinin from the circulation are described at least empirically by such a formula (Diagram I.) (1). Some years ago, in a communication to this Society on the laws of epidemics (2), I showed that the course of an epidemic could be accounted for if the organism, having a high infectivity at the beginning of the epidemic, lost its power of infecting thereafter at a rate corresponding to a geometrical progression. Miss Chick (3), in some experiments on disinfection, came to a like conclusion, finding that, in the application of a germicide to spore-bearing organisms, the number of survivors at the end of each period of time obeyed the same law. Miss Chick's problem is, however, fundamentally different from mine: their relationships will be discussed later. Such observations suggested a search through the records of disease for analogous examples. Five such in all have been discovered up to the present—

1. The decline of the death-rate in children's zymotic diseases from birth to the age of ten years.
2. The increase of mortality in zymotic diseases of adults with increase of age.
3. The general increase of mortality with old age (Gompertz' Law).
4. The course of epidemics.
5. The loss of protective power of vaccination against small-pox.

1. In this group three diseases are included—measles, whooping-cough, and scarlet fever (4). All are specially fatal to young children. During the first year of life the mortality varies in a manner which is not susceptible to any simple explanation; but by the time the age of one year is reached there is evidence that a certain chemical stability is attained. From this age to the age of ten years the population may be taken as homogeneous. The patients are mostly children born in the city. Above ten years immigration introduces a serious error, as the patients suffering from these diseases at high ages are largely country born, and therefore susceptible to an attack of greater severity. It is to be noted (Table I.) that in measles and scarlet fever the death-rate diminishes in a geometrical progression almost uniformly. Where the fitted geometrical progression deviates largely from the actual figures, it is only at one age that this deviation occurs. In the case of measles it is at the age of four years. In the case of scarlet fever it is at the age of six in Glasgow and at that of three in Manchester. When the fitting of these curves is tested by Pearson's method of estimating goodness of fit the value of P is fairly large, but in each case at least two-thirds of the value of χ^2 is due to one year, and in each case a different year. We may, then, hold it probable that the deviation is accidental. The case of whooping-cough is somewhat different, but the selection for admission in this case is much more stringent; specially ill children only are chosen, or those whose home circumstances are very bad, so that at the higher ages there will be a larger proportion of severe cases. Thus *a priori* a concordance of fact with theory should

TABLE I.

Table showing the Case Mortality at Ages from 1 Year to 10 Years in the City of Glasgow Fever Hospital, Belvidere, for Measles, Whooping-cough, and Scarlet Fever, with a like Table added for Comparison for Scarlet Fever in Manchester.

Age Period.	Measles.		Whooping-cough.		Scarlet Fever.			
					Glasgow.		Manchester.	
	Actual C. M. %.	Theor. C. M. %.	Actual C. M. %.	Theor. C. M. %.	Actual C. M. %.	Theor. C. M. %.	Actual C. M. %.	Theor. C. M. %.
1–2	26·1	25·0	38·0	38·0	24·3	22	19·3	19·3
2–3	15·5	16·0	25·6	25·1	16·5	16·5	14·7	14·2
3–4	10·4	10·2	15·3	16·1	12·6	12·4	12·2	10·6*
4–5	5·0	6·5*	11·7	11·7	9·1	9·3	8·7	7·8
5–6	4·3	4·2	9·1	7·2*	7·0	7·0	5·7	5·8
6–7	2·7	2·7	6·6	4·7*	4·1	5·2*	4·5	4·3
7–8	1·4	1·7	...	...	3·6	3·9	3·4	3·2
8–9	1·9 { 1·5	1·1 } 1·8	...	...	3·1	2·9	2·2	2·3
9–10	·4	·7	...	...	2·2	2·2		
	$\chi^2 = 8{\cdot}3$	$P = {\cdot}40$	$\chi^2 = {\cdot}83$	$P > {\cdot}80$	$\chi^2 = 7{\cdot}9$	$P \doteq {\cdot}43$	$\chi^2 = 7{\cdot}2$	$P \doteq {\cdot}53$

* Ages at which divergence of fact from theory is greatest.

not be expected. As a matter of fact, from one year to five years the concordance is excellent, but thereafter it falls away.

A certain check on the value of this comparison is seen in Table II.,

TABLE II.

Table showing the Factor which represents the Rate of Loss of Mortality in each of the above Cases.

	Scarlet Fever.		Measles.	Whooping-cough.
	Glasgow.	Manchester.	Glasgow.	Glasgow.
Factor . . .	·75	·74	·64	·65

where the factor which expresses the ratio of the decline from year to year is given. With regard to scarlet fever, it is to be noted that, though the disease is at each age more severe in Glasgow than in Manchester, yet the progressions from year to year have the same ratio. This is in strict accordance with what I have elsewhere stated, that the average illness only is due to the virulence of the germ, and that the relative illness depends wholly on the susceptibility of the persons affected. When measles and whooping-cough are compared, though the values of the ratios differ considerably from that of scarlet fever, they have a marked concordance, a fact to be expected, as the chief cause of death is in both instances the same, namely, broncho-pneumonia.

2. In this group the evidence is very scanty and not of the highest value. The essential condition, namely, the homogeneous population, can hardly be attained. For instance, in Glasgow, small-pox draws such a large number of its victims from the model lodging-house population, a population of inferior resistance and of lower mean ages, that the figures for the class mortality at middle ages cannot be compared with those at high ages. For this disease the epidemic in Gloucester (5) in 1895–6 has been chosen as affording more homogeneous material. With regard to typhus, on the other hand, Glasgow, with its large Irish population, among whom the disease chiefly spreads, affords more homogeneous material than London. When, however, data are selected in this manner their importance is much diminished. It is further to be noted that an epidemic attacking a practically virgin population may be expected to give better evidence than one where the susceptible persons have largely been protected by an attack of the disease before they arrived at old age.

TABLE III.

Table showing the Increase of Case Mortality in Typhus Fever at Ages above 25 Years in Glasgow and London.

Age Period.	Glasgow, 1865–1871.				London.			
	Cases.	Deaths.	Case Mortality.		Cases.	Deaths.	Case Mortality.	
			Actual.	Theor.			Actual.	Theor.
25–35	841	147	17·6	17·7	3245	574	17·6	17·7
35–45	648	183	28·3	26·0	2965	842	28·3	26·3
45–55	309	127	41·0	38·2	1829	834	45·4	38·2*
55–65	107	57	53·0	54·9	881	479	54·4	54·9
65–75	20	16	80·0	82·4	272	203	74·7	82·4

* Age when the law does not hold.

$\chi^2 = 2$ P = ·74 $\chi^2 = 25·5$ P = ·0001

In Table III. a comparison of the death-rate of typhus fever in Glasgow (4) and London (6) for the times when that disease was common is given. It is to be noted in the first place that at these age periods—25–35, 35–45, and 55–65—the case mortalities in both cities are practically identical. These were therefore taken as a basis, and a geometrical progression fitted by the method of least squares. The result for Glasgow is very good, but from London far from satisfactory. The error is again almost wholly at one age period, 45–55.

TABLE IV.

TABLE SHOWING THE CASE MORTALITY FOR SMALL-POX AMONG THE VACCINATED, GLOUCESTER, 1895–6.

Age Period.	Cases.	Deaths.	Case Mortality.	
			Actual.	Theor.
20–30	367	23	6·2	8·1*
30–40	272	29	10·6	10·6
40–50	142	21	14·8	13·8
50–60	63	12	19·8	18·0
60–70	34	8	23·5	23·6
70–80	13	4	30·8	30·8

* Age when the law does not hold.

For all ages $\chi^2 = 2{\cdot}4$ $P = {\cdot}62$.
For ages above 30 $\chi^2 = {\cdot}3$ $P = {\cdot}91$.

With regard to small-pox, Table IV., the' Gloucester figures show a very good agreement between fact and theory. From 30 upwards the concordance is exact. The theoretical figures were fitted to the mortality rates for the latter years, and that for the age period 20–30 thence calculated. When the whole series thus obtained is considered, the value of P is not specially high, but for the period above 30 years the fit is exceptionally good.

3. *Gompertz' Law.*—This law is the product of an earlier science, and has almost been forgotten. Briefly expressed, it is to the effect that the liability to death increases in a geometrical progression with age. Its truth is undoubted for all ages above 50 for both sexes. In this law Makeham made a change which rendered it much more convenient to actuaries, but which obscured the biological significance of the original statement. The following figures from the Registrar-General's tables for England sufficiently explain the facts:—

TABLE V.

SHOWING THE AVERAGE DEATH-RATE IN MALES AND FEMALES AT AGES 55 AND UPWARDS FOR THE PERIOD 1858–1901.

Age Period.	Males.	Females.
55–65	33·3	28 1
65–75	68·3	60·0
75–85	147·4	133·3
85–	308 6	281·9

The death-rate thus approximately doubles itself every ten years.

4. *The Law of the Epidemic.*—It is not necessary to discuss this again fully. In my two communications to this Society (2), I have shown that the decline of infectivity of an organism must follow very closely a geometrical progression.

5. *The Loss of Protective Power of Vaccination against Small-pox.*—This is the only example in which the data are in the least sufficient to permit of the calculation of the rate of loss of acquired immunity. The subject, however, is specially difficult. It is not sufficient to take the case rate for each age period of population and examine that, as has been done in previous instances. For, in the first place, both vaccinated and unvaccinated are specially susceptible to small-pox at the ages of 10–15, and, in the second place, old age confers a very real immunity against small-pox. In one instance alone do data exist which permit the elimination of these factors, namely, the epidemic in Sheffield in 1887–88 (7), discussed in a former paper. In this case a census of the vaccinated and unvaccinated was made, so that for each age period we have the numbers of both vaccinated and unvaccinated, and the numbers of cases of small-pox occurring among each.

This permits the calculation of the coefficients of correlation showing the degree of protection present (8). The values of these coefficients and of the geometrical progression fitted to them are given in the accompanying diagram (Diagram II.). As will be seen, the correspondence is very fairly close. Certain of the values are obviously too low, and certain others too high. When these are adjacent it will be noted that the curve almost exactly bisects the distance between them. The fit is as good as might be expected when the condition of spread of small-pox in a large town is considered, and there is sufficient correspondence of the facts with the theory to make the interpretation given here at least probable.

As a general commentary on what has been said, it may be remarked in the first case that young children are more easily killed by certain organisms than older children. This susceptibility to death is related quantitatively to age, and this relation requires explanation. It may be explained if we assume that the young child has some substance present which affords a means of foothold to the attacking organism. This substance disappears with age in the manner of the mono-molecular chemical reaction. We then have a case of athreptic * immunity, that is, an immunity due to lack of nourishing substance in the attacked organisms.

The second and third cases may be considered together. As the death-

* Ehrlich's "atreptic" in the barbarous German spelling.

rate increases in a geometrical progression, it may be expected that some substance essential to vitality disappears in the organisms, according to the assumed law. In this way protection against fevers becomes more difficult with each year of age. So that the loss of immunity is due to the inability of the organisms to furnish protective substances. It is therefore distinct from case 1. With case 4 something essentially similar takes

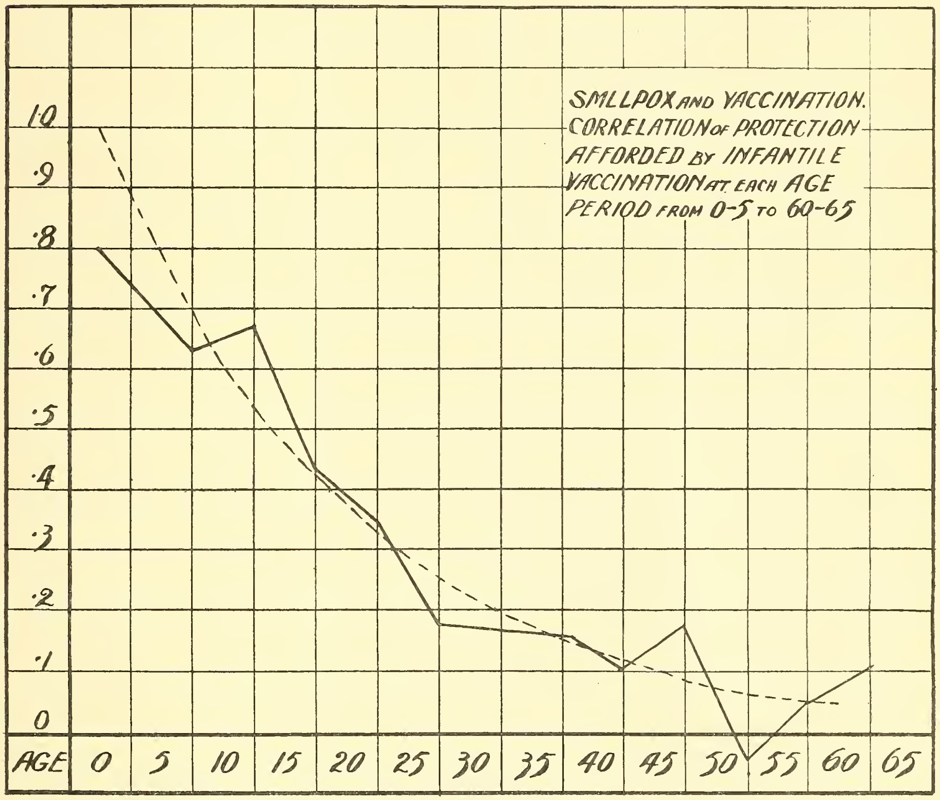


Diagram II.

place; the attacking organism in this case loses something necessary to the power of attack, and this lost, the epidemic terminates for the moment till by some means, possibly sexual, the vitality of the organism is regained. With some epidemic diseases this seems roughly cyclical. The fifth case is specially interesting as showing that acquired immunity passes in the same way, and that here also the substances in the body which control the reaction to disease behave essentially as if the loss of power were due to rearrangement of the various atoms of individual molecules.

Returning to Miss Chick's problem, the following remarks at once suggest themselves. In her paper the living organism is shown to act towards disinfectants as if it were itself a chemical substance. Thus the actual numbers of anthrax spores living after a specific period is in a geometrical progression. It is the same problem as if the whole population of a district were uniformly at the same instant exposed to an infective disease, and the uniform medium of infection kept constantly present in the same amount. In this case an epidemic curve would be, on her analogy, given by $\chi = ae^{-\beta t}$. In the cases here considered, however, it is some substance contained by the organism which is assumed to obey the mono-molecular law, and in each case a different kind of substance.

Her case is really in a different dimension from those given in this communication. She indeed touches on the general problem when she discusses the behaviour of *Bacillus paratyphosus*, and shows that the difference of age in the organism makes a great difference in the rate of the action. Thus the old organisms die more quickly than young, and thus the numbers living no longer obey the exponential law; but the data she gives are not sufficient to find the law of death even approximately.

I do not propose to carry this further at the present moment. The examples chosen are the chief obtainable. Those in case 2 are the most subject to criticism as selected. It is essential, however, that in this class we have equal freedom of infection, and that the disease be definitely infectious in a homogeneous population. Other diseases, such as enteric fever, do not seem to fulfil the conditions sufficiently. It can hardly be expected that the whole progress of immunity and mortality during life should be comprehended in one chemical law.

REFERENCES.

(1) Arrhenius, *Immuno-Chemistry*, pp. 5–6.

(2) Brownlee, "The Law of the Epidemic," *Proc. Roy. Soc. Ed.*, June 1906.

(2) Brownlee, "The Mathematical Theory of Epidemic Distribution and Random Migration," *Proc. Roy. Soc. Ed.*, May 16, 1910.

(3) Chick, "An Investigation of the Laws of Disinfection," *Journal of Hygiene*, 1908, vol. viii. p. 92.

(4) Brownlee, *Rep. of City of Glasgow Fever Hosp., Belvidere*, 1901–2, 1902–3.

(5) *Report of Royal Commission on Vaccination. Appendix on Gloucester.*

(6) Murchison, *Continued Fevers of Great Britain*, p. 237.

(7) Barry, *Report on Epidemic of Small-pox in Sheffield in* 1887–8.

(8) Brownlee, "Small-pox and Vaccination," *Biometrika*, vol. iv. p. 313.

(*Issued separately September* 28, 1911.)

XLIV.—Experimental Researches on the Specific Gravity and the Displacement of some Saline Solutions. By J. Y. Buchanan, F.R.S.

[*Abstract.—Extended paper in Transactions R.S.E.*]

(MS. received July 20, 1911. Read July 14, 1911.)

In this paper are collected and discussed the observations of the specific gravity of saline solutions of moderate and high dilution made during the course of the last ten years. The instrument employed was the hydrometer, of the type used by me during the cruise of the *Challenger* in the years 1872 to 1876. The results then obtained were such as to convince me that the instrument, in capable hands, and with strict attention to the obligatory conditions of experiment, gives more exact results than any other instrument in use for such work. In the course of the last forty years the method has been improved in the sense that more and more attention has been paid to the conditions, especially those of temperature, under which the experiment is made, because it was found that the method is so delicate that it responds to every increase of care in experimenting by greater exactness in the results which it furnishes. Almost the whole of the work of the *Challenger* was done between latitudes 40° N. and 40° S., where the temperature of the air is high and subject to very small diurnal variation. Besides this natural advantage, the conditions obtaining in a laboratory on the main deck of a wooden man-of-war of the ancient type were all that could be wished for the performance of physical experiments requiring invariability of temperature. For this reason the results obtained during the cruise were exceptionally good.

When I took the subject up again in the climate of the British Islands it was a long time before I was able to manage the climate of the laboratory so as to produce the desired temperature in the liquid, and to maintain it without variation during the time that the experiment lasted. In this I have had the good fortune to be aided by assistants of rare ability and perseverance, with the result that any variation of the temperature of the liquid during the experiment, which is sensible on a thermometer divided into tenths of a degree Centigrade, each such interval occupying a length of 1 millimetre, is considered an imperfection.

In the extended paper the principal solutions of moderate and high dilution considered are those of the chlorides, bromides, iodides, nitrates,

chlorates, bromates, and iodates of the alkalies and some salts of the alkaline earths. The concentration of the solutions is expressed in terms of gram-molecules salt dissolved in 1000 grams of water. We express by w the weight of m gram-molecules of the salt, and the total weight of the solution by $W=(w+1000)$ grams. The specific gravity of solution, S, is referred to that of distilled water at the same temperature (T), as unity. In the majority of cases $T=19{\cdot}5°$ C., but in some cases $T=15°$ C., and in others $T=23°$ C.

In the solutions of any particular salt, for which the general symbol used is MR, the values of m, representing the amount of salt expressed in gram-molecules dissolved in 1000 grams water, are in a descending geometrical series, beginning usually with $m=\frac{1}{2}$ and proceeding $m=\frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, down to, in most cases, $\frac{1}{512}$, and in some $\frac{1}{1024}$, of a gram-molecule.

The results obtained are set forth in various tabular forms. In the first twenty-four tables, class A, the specific gravity, S, of each solution is the mean of several series of observations made with two different hydrometers, generally Nos. 17 and 21. The facts of observation which are entered in these tables are:—under m the quantity of salt, in terms of the gram-molecule, dissolved in 1000 grams of water; under W the weight, in grams, of this solution; and under S the specific gravity of the solution at the temperature T referred to that of distilled water of the same temperature as unity. These are the observed data, and the first derived datum, namely, that of the *Displacement*, $\Delta=W/S$ of the solution, is given. It is expressed as the weight in grams of distilled water having the temperature T which is displaced by, or has the same volume as, the weight W of the solution, having also the temperature T. The advantage of using the term "displacement" in preference to "volume" is that a weight is never thought of as a function of temperature, whereas a volume is always so regarded. In our experiments the temperature is a constant.

On page 637 is given an example of a table in class A. It is Table No. 1 referring to solutions of m KCl + 1000 grams water at $19{\cdot}5°$ C. There are nine concentrations, $m=\frac{1}{2}$ to $\frac{1}{512}$, inclusive. The two columns of differences, $d\Delta$ and $d\log\Delta$, are omitted.

In this research all the weights given represent weights *in vacuo*. The values under m and W represent direct weighings on delicate balances with standard weights. The exactness of the values under S form the subject of a table of series B for each table of series A in which two hydrometers were used. An example is given on page 637.

CHLORIDES.

TABLE No. 1 A.

POTASSIUM CHLORIDE. MR = KCl = 74·6.

T = 19·5°.

	Weight of Solution.	Specific Gravity.	Displacement.
m.	W.	S.	Δ.
1/2	1037·3000	1·022977	1014·001
1/4	1018·6500	1·011670	1006·899
1/8	1009·3250	1·005889	1003·416
1/16	1004·6625	1·002973	1001·684
1/32	1002·3312	1·001489	1000·841
1/64	1001·1656	1·000741	1000·423
1/128	1000·5828	1·000365	1000·217
1/256	1000·2914	1·000193	1000·098
1/512	1000·1457	1·000082	1000·064

TABLE No. 1 B.

POTASSIUM CHLORIDE. MR = KCl = 74·6.

T = 19·5°.

m.	S_{21}.	s_{21}.	S_{17}.	s_{17}.	$s_{21}+s_{17}$.	r_0.	*d.*
1/2	1·022986	4	1·022969	4	8	2·9	18
1/4	1·011674	4	1·011665	4	8	3·0	20
1/8	1·005895	4	1·005883	4	8	2·1	16
1/16	1·002980	3	1·002967	3	6	3·6	18
1/32	1·001494	3	1·001485	4	7	2·2	14
1/64	1·000756	4	1·000730	4	8	3·1	27
1/128	1·000368	3	1·000361	3	6	1·5	10
1/256	1·000199	4	1·000188	4	8	2·2	14
1/512	1·000088	4	1·000076	4	8	2·2	14

It is Table No. 1, series B, and refers to the experiments made with solutions of chloride of potassium at 19·5° recorded in Table No. 1, class A. In it the value, $\bar{S}$, being the same as that recorded under S in Table No. 1 A, is not repeated. After the column *m*, which has the same signification in both tables, we have S_{21}; the numbers under it give for each value of *m* the mean specific gravity derived from s_{21} series of observations made with hydrometer No. 21, and under S_{17} the mean specific gravity derived from s_{17} series of observations made with hydrometer No. 17. The mean of the sum of the series $s_{21}+s_{17}$ is the value of S entered in Table No. 1 A. All the series consist of nine individual observations each, and they are to be taken to be of equal weight. We then obtain by the method of least squares the probable error ($\pm r_0$) of the value of S in Table No. 1 A, expressed in units of the sixth decimal place, and under d the greatest departure from it of the mean of any individual of the $(s_{21}+s_{17})$ series.

There are twenty-four tables of each class, referring in all to 185 different solutions. The total number of series of observations of specific gravity made with the two hydrometers is 1203. Each of these series consisted of nine separate and independent observations, so that the total number of individual observations is 10,827. Dividing the total number of series of observations made (1203) by the number of different solutions experimented on (185), we obtain 6·50 as the mean number of series of observations made per solution.

The sum of the probable errors (Σr_0) is 532·4, whence the mean probable error of the mean specific gravity per solution is $\pm 3{\cdot}04$ in the sixth decimal place. Amongst the 185 separate values of r_0, 171 are not above 5; and of the remainder, 13 lie between 5 and 10, and only 1 value is above 10 in the sixth decimal place. We see, then, that the mean specific gravities in these twenty-four tables have a probable error which hardly exceeds ± 3 in the sixth place. Consequently the exactness of units in the fifth place is abundantly guaranteed.

In the column under d we have the maximum departure from the mean (S) of any of the serial values which form the basis of the mean. The mean of the 185 values of d is 15·4. Of these, 33 are above 20 and 3 above 30.

Returning to the values of r_0, the probable error of the arithmetical mean S per solution, we have seen that it is $\pm 3{\cdot}04$ for a mean of 6·50 series per solution. Amongst observations, all having equal weight, the probable error of the arithmetical mean varies inversely as the square root of the number of observations used in arriving at it; consequently the probable error of one of the 6·50 serial means is $3{\cdot}04\sqrt{6{\cdot}50} = 7{\cdot}75$. As all the observations were made uniformly and with the same care and attention, they are to be taken as having equal weight; and we have in the following table the probable error r_0 of the mean specific gravity of a solution derived from any number s of serial means.

$s=$	1	2	3	4	5	6	7	8	9
$\pm r_0=$	7·75	5·48	4·47	3·88	3·46	3·16	2·92	2·74	2·55

This table is of use for showing the probable exactness of mean specific gravities depending on other numbers of serial means, whether made with one or more than one hydrometer. The specific gravities of the solutions of the Ennead (MRO_3) consisting of the salts of $\{K, Rb, Cs; ClO_3\ BrO_3, IO_3\}$ and those of the nitrates of the same metals, at 19·5° C., were all made with hydrometer No. 17 alone. This was done because there was not time to make the double determinations in the case of all the solutions, and the agreement between the results obtained in the case of the salts quoted in the first twenty-four tables was so good that a complete set of observations with one hydrometer seemed to be more valuable than an incomplete set with two hydrometers. The mean number of serial means on which the arithmetical mean specific gravity of each of these solutions depends is 2·97, so that the mean probable error of the mean specific gravity of any of these solutions may be taken as $\pm 4 \cdot 5$ in the sixth place.

We have seen that the probable error of the mean of a single series is $\pm 7 \cdot 75$ in the sixth decimal place, and the number of independent observations included in each series is 9; consequently the probable error of any one of these observations is $3 \times 7 \cdot 75 = \pm 23 \cdot 25$ in the sixth place, or $\pm 2 \cdot 3$ in the fifth place.

The standard temperatures of the solutions and of the distilled water at which our observations have been made are 15°, 19·5°, and 23° C. The majority of the determinations have been made at 19·5° because it has been found to be comparatively easy to keep this temperature constant in my laboratory at most seasons of the year. In cold winter weather it has sometimes been necessary to use 15° C. as the standard temperature, and in hot anticyclonic weather in summer it has been necessary at times to adopt 23° C. as the standard temperature.

In the experiments which I made before 1902 the variation of the temperature of the liquid during the experiment was allowed to reach an amplitude of 0·3° C. without being rejected, but the amplitude did not usually exceed 0·1° C. A series, consisting of nine separate observations, takes from 12 to 15 minutes to complete.

After 1902, when I began systematic work in a room of the Davy-Faraday laboratory, which had a single window to the north, and was otherwise very well suited to the maintenance of a constant temperature, the permissible amplitude of variation was rapidly reduced so as to be usually insensible. Indeed, in the work done during the last year by Mr S. M. Bosworth, he has been so successful in the management of the climate of the room that cases in which the temperature of the liquid showed any variation have been so few that it has been possible to reject them and use only observations made at absolutely constant temperature. It is true, however, that the climate of that year exhibited no extremes of either heat or cold.

In the following table the statistics of the variations of the temperature of the liquid, while a total of 1316 series of observations was made with hydrometers Nos. 17 and 21, namely, 837 with No. 17, and 479 with No. 21, are given. In 68 per cent. of the series made with No. 17 there was no sensible variation of temperature, and the same was the case in 55·2 per cent.

of the series made with No. 21. If we consider the series made with both hydrometers for which the variation of temperature was not greater than 0·05°, the percentages are almost identical, namely, 89·5 for No. 17 and 89·2 for No. 21.

In the paper, *a sensible variation of temperature* is defined to be one that is perceptible on a thermometer divided into tenths of a Centigrade degree, when each such interval of temperature on the scale occupies a length of one millimetre on the stem of the thermometer.

Table No. 2.—Giving Statistics of Constancy or Variability of Temperature during Experiments in the Earlier Years of Work in the Davy-Faraday Laboratory.

Amplitude of Variation of the Temperature of the Liquid while a Series of Observations was being made with the Hydrometer.	Number of Series of Observations so made with Hydrometer.			
	No. 17.		No. 21.	
	Absolute.	Per cent.	Absolute.	Per cent.
0·00°	568	68·0	265	55·2
·01°	48	5·7	51	10 6
·02°	68	8·1	54	11·3
·03°	19	2·3	23	5·0
·04°	12	1·4	16	3·3
·05°	34	4·0	18	3·8
·05° to ·10°	69	8·3	40	8·4
·10° to ·20°	16	1·9	11	2·2
·20° to ·30°	3	0·3	1	0·2
Totals . .	837	100·0	479	100·0

In order to obtain full advantage of the degree of precision of which the hydrometric method is capable, the experimenter must train himself in the management of the climate of his laboratory, so as to produce the constancy of temperature of which the numbers in this table are evidence.

The hydrometer itself is only an instrument like the balance or the thermometer. It is true that all of them are necessary; but what is indispensable is the competence and the earnestness of the investigator who handles them.

When the systematic research was begun in the year 1902 my first object was to study the specific gravity and displacement of equivalent solutions of the salts of the Ennead (K,Rb,Cs; Cl,Br,I). The most concentrated solution contained $\frac{1}{2}$ gram-molecule of the salt dissolved in 1000 grams of water. Using hydrometers Nos. 17 and 21, it was necessary to operate with at least 1000 grams of the solution. In view of the rarity and very high price of the salts of cæsium, it was impossible to work with solutions containing more than $\frac{1}{2}$ gram-molecule.

Table No. 1, class A, already quoted, is an example of one of the tables of the Ennead, and relates to chloride of potassium.

The results obtained with the hydrometer in dilute solutions of the Ennead having proved so interesting, I undertook the investigation of the more concentrated solutions with the specific gravity bottle or pyknometer, using generally the same common temperature $T = 19{\cdot}5°$. The specific gravity bottles used held 25 and 50 cubic centimetres respectively. The most soluble of the salts is cæsium chloride; 1000 grams of water saturated with it at 19·5° C. contains 12·1563 gram-molecules or 2048·34 grams of the salt. The specific gravities of the solutions saturated at temperatures between 21° and 24°, and those of the solid salts which crystallised out of them, formed the subject of a memoir read before the Chemical Society of London, and published in the *American Journal of Science*, vol. xxi., January 1906. It is entitled, "On a method of Determining the Specific Gravity of Soluble Salts by Displacement in their own mother-liquor; and its application in the case of the Alkaline Halides." The work then done with the saturated solution has now been joined on to that done with the hydrometer in dilute solutions, by observations made with the pyknometer in solutions prepared so as to contain 1, 2, 3 . . . in whole numbers of gram-molecules per 1000 grams of water. By these determinations the specific gravity of solutions of all concentrations of the salts of the Ennead has been well investigated.

While it was necessary to use the pyknometer for the concentrated solutions of such costly salts as those of rubidium and cæsium, it was found convenient to adapt the hydrometer, by a slight alteration in the distribution of its weight, for use in solutions of all concentrations, and so obtain results of as high precision in concentrated solutions as I had already obtained in dilute solutions.

In the extended paper many interesting observations on the specific gravity of solutions in the neighbourhood of saturation are recorded; and in the case of a supersaturated solution of chloride of calcium particularly interesting results were obtained, furnishing volumetric evidence of the *labour* which takes place in such a solution before it finally becomes a mother-liquor by giving birth to a crop of crystals. This highly interesting quantitative work could not be effected by any other instrumental means than the hydrometric method here described.

In the paper published in the *American Journal of Science* above referred to, the specific gravities of the crystallised salts were determined by displacement in their own mother-liquors; it appeared to be interesting to extend this work so as to include the salts of the Ennead of *oxyhalides*, having the general formula MRO_3, in which M may be K,Rb,Cs, and RO_3 may be ClO_3,BrO_3,IO_3.

In contrast with the salts of the Ennead of halides, MR, which are very soluble, the oxyhalides are only sparingly soluble. The determination of the specific gravity of the crystals in their mother-liquors is therefore much easier, and was effected quite successfully by my assistant, Mr G. Fermor. The results so obtained are given in the following table. In the first line, T, we have the temperature at which the crystals separated from their mother-liquor, and at which their specific gravity, when immersed in the same mother-liquor, was determined. In the second line, MRO_3, we have the formula of the salt In the third line, m, we have the concentration of the mother-liquor expressed in terms of the gram-molecule per 1000 grams of water. In the fourth line, S, we have the specific gravity of the mother-liquor at T referred to that of distilled water at the same temperature as unity. In the fifth line, D, we have the specific gravity of the crystals at T referred to that of distilled water also at T as unity, and in the sixth line $\frac{MRO_3}{D}$ we have the displacement of the crystal per gram-molecule at T expressed in grams of water having the temperature T. In the eighth line

Table No. 3.—Giving the Numerical Data relating to the Specific Gravity and the Molecular Displacement of the Crystals and the Mother-liquors of the Salts of the Ennead MRO_3 at the Temperature T.

Temperature . T	14·8° C.	19·2°.	18·6°.	16·2°.	16·0°.	15·6°.	16·0°.	16·0°.	15·4°.
Salt. . MRO_3	$KClO_3$.	$KBrO_3$.	KIO_3.	$RbClO_3$.	$RbBrO_3$.	$RbIO_3$.	$CsClO_3$.	$CsBrO_3$.	$CsIO_3$.
Concentration . m	0·5139	0·4446	0·4016	0·2983	0·1068	0·1113	0·2635	0·0989	0·0770
Specific gravity of mother-liquor . . S	1·0360	1·0475	1·0708	1·0339	1·0169	1·0233	1·0402	1·0203	1·0188
Specific gravity of crystals . D	2·319	3·221	3·921	3·170	3·680	4·336	3·582	4·109	4·849
Molecular displacement of crystals . . $\frac{MRO_3}{D}$	52·86	51·88	54·60	53·31	58·00	60·07	60·44	63·52	63·52
Molecular displacement of crystals for Ennead MR . $\frac{MR}{D}$	38·23	44·46	54·58	44·71	51·55	61·99	42·31	47·82	57·67
Difference $\frac{MRO_3}{D} - \frac{MR}{D}$	14·63	7·42	0·02	8·60	6·45	−1·92	18·13	15·70	5·85

of the table we have MR/D, the molecular displacement of the crystals of the salts of the Ennead MR; and in the ninth line we have the difference of the displacement of the crystals of corresponding salts of the two Enneads.

The difference in constitution between the salts of MRO_3 and the corresponding one of MR is O_3, so that the difference of molecular weight $MRO_3 - MR$ is constant, and equal to 48. Further, $ClO_3 = Br + 3{\cdot}5$ and $BrO_3 = I + 1$, so that the acid Radicals ClO_3 and BrO_3 may be looked on as isomers of Br and I respectively. By similar reasoning, IO_3 would fall to be the isomer of a possible *Ekaiodine*, in the spirit of Mendeléeff's nomenclature.

Only in one case is the displacement of the halide greater than that of the corresponding oxyhalide. The displacement of RbI is 61·99 grams, and that of $RbIO_3$ is 60·07. In the corresponding potassium salts there is practically equality in the displacements, namely that of KIO_3, 54·60 grams, and that of KI, 54·58. Amongst the chlorates the excess of displacement above that of the chloride is greatest, 18·13 grams, in the case of the cæsium salts; it is least, 8·59, in that of the rubidium salts, and 14·63 in that of the potassium salts. The excess of the displacement of the bromate over the bromide is greatest, 15·70 grams, in the cæsium salts, 7·42 in the potassium salts, and 6·45 in the rubidium salts. We see, then, that the increment of displacement of the crystals produced by an addition of O_3 to the molecular weight of the salt is negative for MR = RbI; it is sensibly zero for MR = KI, and rises in the following order of equivalents of MR,—CsI, RbBr, KBr, RbCl, KCl, CsBr, the maximum difference, 18·13, occurring when MR = CsCl.

One of the most remarkable results obtained in the research on the crystals and the mother-liquors of the salts of the Ennead, MR, was that the sum of the displacements of the crystalline salt and of the water required to form the saturated solution is greater than that of the resultant solution when MR is KCl, KBr, KI, RbCl, RbBr, or RbI; and less when MR is CsCl, CsBr, or CsI. The excess ($\pm$) of the sum of the displacements of the constituents over that of the solution is given for each salt of the two Enneads in the following table.

[Table.

TABLE No. 4.

Designation of Salt	KCl.	KBr.	KI.	RbCl.	RbBr.	RbI.	CsCl.	CsBr.	CsI.
Excess of sum of displacements of salt and water above that of saturated solution	7·010	5·422	5·074	6·641	7·422	3·411	-6·711	-0·317	-10·237
Designation of Salt	$KClO_3$.	$KBrO_3$.	KIO_3.	$RbClO_3$.	$RbBrO_3$.	$RbIO_3$.	$CsClO_3$.	$CsBrO_3$.	$CsIO_3$.
Excess of sum of displacements of salt and water above that of saturated solution	2·14	-5·65	19·26	-0·23	3·69	10·03	-0·12	8·78	0·79

In the Ennead MR the difference of the behaviour of the cæsium salts from those of potassium and rubidium is very marked. In the case of cæsium iodide the displacement of the saturated solution is fully 1 per cent. greater than the sum of those of the salt and water which it contains.

In the case of the salts of the Ennead MRO_3, given in the third and fourth lines of the table, it is only in the case of bromate of potassium that the displacement of the saturated solution shows a marked excess above that of the sum of the salt and water. The most remarkable feature is that of the iodates. The displacement of the saturated solution of cæsium iodate is only a trifle, 0·79, less than that of the sum of its constituents; in the case of rubidium iodate it is 10·03 grams, and in that of potassium iodate it is 19·26 grams less.

It was pointed out in the paper in the *American Journal of Science* above quoted that in the case of the salts of the Ennead MR the crystallisation of the potassium and rubidium salts must be hindered by increased pressure, while that of cæsium salts must be helped by the same agency. In the case of the salts of the Ennead MRO_3 it is only the solution of potassium bromate the crystallisation of which would be much helped by increase of pressure. Regarding the iodates, we see that the crystallisation of the saturated solution of cæsium iodate would be but little influenced by change of pressure, whereas the crystallisation of the rubidium salt, and still more that of the potassium salt, would be very seriously hindered by increase of pressure.

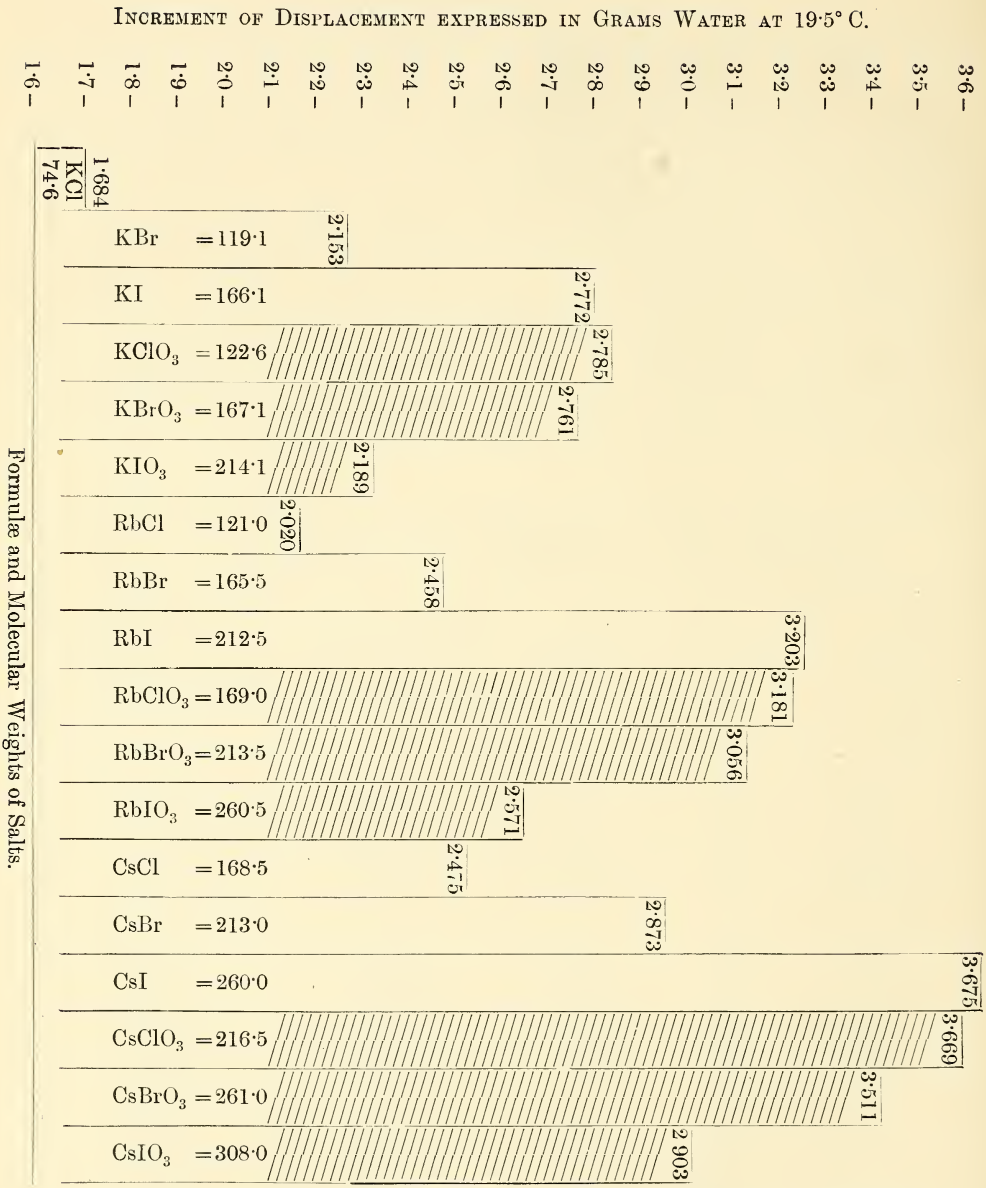
The two Enneads, or the double Ennead (MR, MRO_3), offer many points of great interest, of which an example is furnished in Table No. 5. It gives under w the weight of $\frac{1}{16}$ gram-molecule of the salt dissolved in 1000 grams of water; under S, the specific gravity of this solution at 19·5° C., referred to that of distilled water at the same temperature as unity; and under v, the increment of displacement caused by dissolving $\frac{1}{16}$ gram-molecule of the salt in 1000 grams of water, expressed in grams of distilled water having the temperature 19·5° C.

The solutions are arranged in three groups, each group containing six solutions of salts having the same metallic base (K, Rb, Cs). These six solutions fall into two groups of three, or triads, the first three being the salts having the general formula MR, and the second triad those having the general formula MRO_3. Each triad is entered in the ascending order of the molecular weights of the salts which compose it, and each group of three triads forms the Ennead of MR or MRO_3 respectively.

Table No. 5.—Giving the Specific Gravity, S, at 19°·5, of Solutions of Salts of the Double Ennead (MR, MRO_3). The Solutions have the General Formula 1/16 MR, or 1/16 MRO_3 + 1000 Grams of Water.

Salt in Solution.	Molecular Weight of Salt.	Weight of 1/16 grm.-mol. Salt. w.	Specific Gravity of Solution of 1/16 grm.-mol. Salt in 1000 grams Water. S.	Increment of Displacement produced by the Dissolution of w grams Salt in 1000 grams Water. v.
KCl	74·6	4·6625	1·002973	1·684
KBr	119·1	7·4437	1·005279	2·153
KI	166·1	10·3812	1·007588	2·772
$KClO_3$	122·6	7·6625	1·004863	2·785
$KBrO_3$	167·1	10·4443	1·007662	2·761
KIO_3	214·1	13·3812	1·011169	2·189
RbCl	121·0	7·5625	1·005531	2·020
RbBr	165·5	10·3437	1·007868	2·456
RbI	212·5	13·2812	1·010046	3·203
$RbClO_3$	169·0	10·5593	1·007354	3·183
$RbBrO_3$	213·5	13·3412	1·010253	3·057
$RbIO_3$	260·5	16·2812	1·013673	2·576
CsCl	168·5	10·5312	1·008036	2·475
CsBr	213·0	13·3125	1·010409	2·873
CsI	260·0	16·2500	1·012529	3·675
$CsClO_3$	216·5	13·5312	1·009825	3·669
$CsBrO_3$	261·0	16·3125	1·012756	3·511
$CsIO_3$	308·0	19·2500	1·016299	2·903

[Diagram.



The above diagram is illustrative of Table 5, and shows graphically, by the heights of the columns, the different increments of displacement produced by the dissolution in 1000 grams of water at 19·5° C. of $\frac{1}{16}$ gram-molecule of each salt of the double Ennead (MR, MRO_3). The columns representing the increment of displacement produced by salts of the Ennead MRO_3 are shaded.

$10^5(S-1)$ MR	$10^5(S-1)$ MR	$10^5(S-1)$ MR	MRO_3 (10^5S-1)	MRO_3 (10^5S-1)	MRO_3 (10^5S-1)
					$CsIO_3$ 1630
					308·0
1253 CsI				$RbIO_3$ 1367	$CsBrO_3$ 1276
260·0				260·5	261·0
1041 CsBr	1005 RbI		KIO_3 1117	$RbBrO_3$ 1025	$CsClO_3$ 983
213	212·5		214·1	213·5	216·5
804 CsCl	787 RbBr	759 KI	$KBrO_3$ 766	$RbClO_3$ 735	
168·5	165·5	166·1	167·1	169·0	
	553 RbCl	528 KBr	$KClO_3$ 486		
	121·0	119·1	122·6		
		297 KCl			
		74·6			

The above scheme shows the parallel periodic progression of molecular weight of the dissolved salt, and the excess above unity, multiplied by 10^5, of the specific gravity of the solution $\frac{1}{16}$ MR + 1000 grams water, or $\frac{1}{16}$ MRO_3 + 1000 grams water, at 19·5° C. respectively.

If we consider the specific gravity of the solutions, we see that in each triad it rises with the molecular weight of the salt dissolved. If we consider the increments of displacement (v) produced by the dissolution of $\frac{1}{16}$ gram-molecule of each of these salts in 1000 grams of water, we see that, for the salts of the same metal, the values increase from the chloride to the bromide, and from the bromide to the iodide; and that for the chlorates the values of v are almost identical with those for the iodides; they diminish from the chlorates to the bromates, and suffer a considerable fall from the bromates to the iodates. There is also a decided fall in the value of v from that of the iodate of potassium or rubidium to that of chlorate of rubidium or cæsium respectively. This is very clearly shown in the diagram (page 646), in which the heights of the columns are proportional to the values of v produced by the dissolution in 1000 grams water of $\frac{1}{16}$ gram-molecule of the salts inscribed at their bases. The columns referring to the oxyhalides are shaded.

One of the principal reasons for determining the specific gravity of solutions of the high dilutions used in this research was to ascertain whether the condensation which is commonly observed when a saline solution is diluted with water takes place when the concentration of the solution produced is continuously diminished. The difference between the sum of the displacements of the concentrated solution and the water used and that of the resultant solution produced is a question which has often been raised; but it has hitherto been impossible to decide it, in the case of dilute solutions, owing to the disabilities and imperfections of the methods used for determining their specific gravity.

One result of the observations made in this research is that in the case of the solutions of some salts there is a limit beyond which contraction does not take place on dilution, and some in which it is replaced by expansion. This is certainly the case with solutions of chloride of sodium which contain less than $\frac{1}{16}$ gram-molecule of salt per 1000 grams of water at 19·5° C.

A large amount of material bearing on this and other important points connected with the specific gravity and displacement of saline solutions has been collected in the course of the investigation, and much of it will be found to be discussed in the extended paper which is now in the press.

(*Issued separately October* 24, 1911.)

XLV.—The Rate of Multiplication of Micro-organisms: A Mathematical Study.

By A. G. M'Kendrick, Captain I.M.S., and M. Kesava Pai, M.D. (Pasteur Institute of Southern India). *Communicated by* Professor M'Kendrick.

(MS. received March 13, 1911. Read June 19, 1911.)

The problem of the rate of multiplication of micro-organisms is one which has often been attacked, but which has not, to our knowledge, been reduced to a simple law.

If there be an unlimited supply of nutriment, an organism reproduces itself by compound interest: in a geometrical progression—*i.e.* 1, 2, 4, 8, etc. That is to say, that the rate of growth under unimpeded conditions is proportional to the number present, at any moment, or

$$\frac{dy}{dt} = by. \tag{1}$$

In test-tube experiments, however, this simple state of affairs is complicated by the fact that the supply of nutriment is limited, and consequently, as time goes on, the rate of multiplication falls off.

Every living organism employs the nutriment which it has absorbed for two objects: first, the maintenance of the individual; and, second, its reproduction. As, however, in the case of those micro-organisms with which we shall deal, the rate of multiplication is very fast, we may, for all practical purposes, consider that the amount of food-stuff utilised for their upkeep is negligible, and assume that the whole of it is employed in reproduction.

If we accept this simplifying assumption we may say that organisms in a test-tube multiply, by a simple conversion of the available food-stuff, into other organisms, and that the rate of multiplication is proportional to the concentration of that food-stuff.

If a be the original concentration of food-stuff, the concentration at the time t will be $(a-y)$.

Introducing this factor into equation (1), we have

$$\frac{dy}{dt} = by(a-y), \tag{2}$$

which means that the *rate of increase of fast-growing organisms is proportional to the number of organisms present, and to the concentration of the food-stuff.*

It is the object of this paper to show that this simple law is in accordance with experimental facts.

Experimental Procedure.

B. coli was used throughout.

The broth employed was composed of—

Liebig's Extract of Meat . .	0·3 per cent.
Salt	0·5 ,,
Peptone	1·0 ,,

Enumerations were made by plating out various dilutions of samples, taken at definite intervals of time, and counting the colonies, after twelve to twenty-four hours' growth.

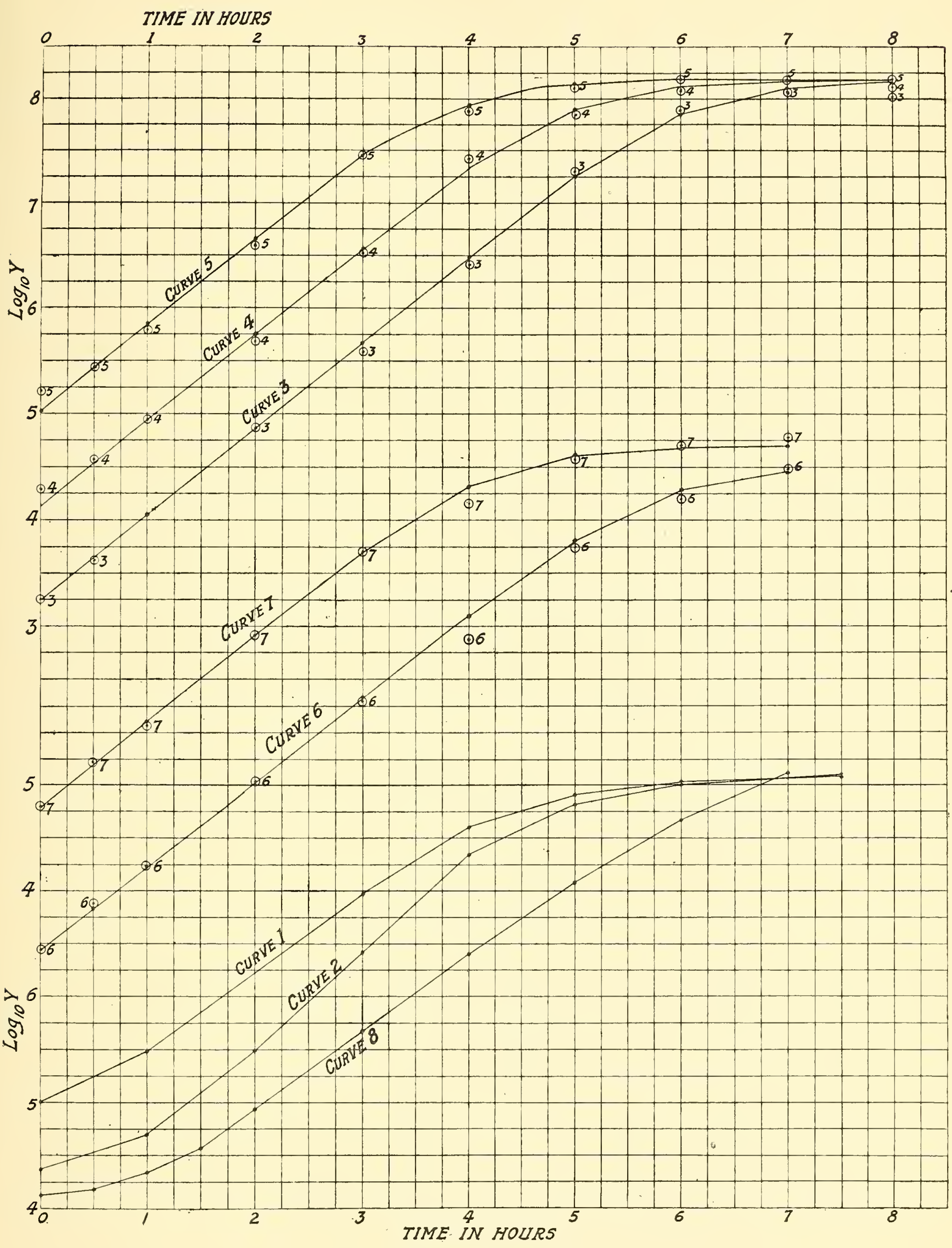
Similar experiments made by Myer Coplans (*Journal of Pathology and Bacteriology*, vol. xiv. p. 1) showed that in the early stages of multiplication a period of "latency" occurred, in which growth was slower than would be expected. We considered that this fact was not in accordance with the simplest experimental conditions, and attempted to eliminate it.

For numerical study it is essential that the physical state of the organisms should be identical throughout the experiment. It was necessary therefore to make certain that each organism inoculated was similar in character to its neighbour, and also that this character was that which would be maintained throughout the experiment. As this condition is one of *active growth*, we therefore used for inoculation only organisms obtained from cultures which were already actively growing, *i.e.* cultures which had only been growing for from one to three hours. The importance of this point is seen on a comparison of curve 8 with curves 6 and 7. In the former the culture used for inoculation had been incubated for fourteen days, in the latter for only one hour.

A second point of importance is that the temperature should be absolutely uniform throughout the whole procedure. As the taking of samples and their manipulation occupies at least three minutes, it is essential that during this period the tube should be maintained at the temperature of the incubator (37° C.).

Curves 1 to 8 show the effects of improvements in the technical methods which we employed.

(*a*) In curve 1. The tubes were incubated in the air of the incubator: no precautions were taken to keep the tube at 37° C. during manipulation.


TIME IN HOURS
0
1
2
3
4
5
6
7
8
Log10 Y
CURVE 5
CURVE 4
CURVE 3
CURVE 7
CURVE 6
CURVE 1
CURVE 2
CURVE 8
TIME IN HOURS

(*b*) In curve 2. The broth was heated to 37° C. prior to inoculation. Tube placed in the air of the incubator. No external precautions taken.

(*c*) In curves 3, 4, 5, as in (*b*), but in addition the tubes were placed in a water bath in the incubator, and during manipulation, bath and tubes were removed, the tubes being kept as much as possible in the water bath.

(*d*) In curves 6 and 7. As in (*c*), but in this instance two water baths were used, one in the incubator, the other outside of it. The tubes were manipulated in the second water bath, which was also kept at 37° C.

(*e*) In curve 8. The same precautions were taken as in (*d*): the culture used for inoculation had been incubated for fourteen days.

We found, then, that by adopting the procedure as stated in (*d*), and by using for inoculation only actively growing organisms from very young cultures, the latent period was entirely eliminated (as is shown in curves 6 and 7).

The figures of curves 3, 4, 5, 6, 7 are given on the opposite page.

[*Note.*—It will be noted that in curves 4 and 5 the latent period was still in evidence.]

We return to the mathematical theory.

Equation (2) $$\frac{dy}{dt} = by(a-y)$$

becomes on integration

(3) $$y = \frac{a}{1 + \frac{a-y_0}{y_0} e^{-abt}}$$

where y_0 is the value of y at the time $t=0$, *i.e.* the number of organisms originally inoculated, and a and b are constants to be determined.

When $\frac{dy}{dt} = 0$ (*i.e.* when multiplication has ceased)

$$y = a.$$

Enumerations beyond this point give irregular results; the samples taken are uneven, probably on account of clumping.

From a study of the manner in which the curves flatten after eight or nine hours' growth, a suitable value of a may be inferred in each case.

	Curve 3.		Curve 4.		Curve 5.		Curve 6.		Curve 7.	
Time in hours.	y	$\log_{10}y$	y	$\log_{10}y$	y	$\log_{10}y$	y	$\log_{10}y$	y	$\log_{10}y$
0	1,760 1,760	3·246 3·246	19,000 14,070	4·279 4·148	176,000 110,200	5·246 5·042	2,850 2,850	3·455 3·455	64,250 64,250	4·808 4·808
½	4,020 4,483	3·604 3·652	35,900 35,833	4·555 4·554	280,000 280,390	5·447 5·448	7,500 7,080	3·875 3·850	165,000 163,550	5·217 5·214
1	... 11,419	... 4·058	88,000 91,243	4·944 4·960	608,500 712,200	5·784 5·852	17,500 17,587	4·243 4·245	357,500 415,960	5·553 5·619
2	72,000 74,057	4·857 4·870	482,000 590,100	5·683 5·770	3,870,000 4,506,000	6·588 6·654	105,000 108,440	5·021 5·035	2,625,000 2,660,900	6·419 6·425
3	380,000 479,240	5·580 5·680	3,400,000 3,749,900	6·531 6·574	28,200,000 25,194,000	7·450 7·401	625,000 664,050	5·796 5·822	16,250,000 15,820,600	7·211 7·199
4	2,600,000 3,057,200	6·415 6·485	26,300,000 21,462,000	7·419 7·332	74,200,000 86,131,000	7·870 7·935	2,250,000 3,971,200	6·352 6·599	45,375,000 66,542,000	7·657 7·823
5	19,750,000 17,877,000	7·296 7·252	71,600,000 78,903,000	7·855 7·897	127,000,000 137,320,000	8·104 8·138	17,750,000 20,333,000	7·249 7·308	122,125,000 131,520,000	8·087 8·119
6	77,000,000 72,718,000	7·886 7·862	118,000,000 134,299,000	8·072 8·128	150,000,000 151,170,000	8·176 8·179	50,000,000 61,169,000	7·699 7·787	158,500,000 154,834,000	8·200 8·190
7	120,000,000 130,420,000	8·079 8·115	... 150,590,000	... 8·178	149,000,000 153,550,000	8·173 8·186	97,500,000 90,672,000	7·989 7·957	194,500,000 159,181,000	8·289 8·202
8	106,000,000 149,830,000	8·025 8·175	135,000,000 153,460,000	8·130 8·186	154,000,000 153,900,000	8·188 8·187	...	...	...	...
Quantity of broth.	25 c.c.	...	25 c.c.	...	25 c.c.	...	12 c.c.	...	12 c.c.	...
Age of culture inoculated.	$1\frac{1}{2}$ hours	...	$1\frac{1}{2}$ hours	...	$1\frac{1}{2}$ hours	...	1 hour	...	1 hour	...
Constants.	$a=154$	$ab=1{\cdot}87$	$a=154$	$ab=1{\cdot}87$	$a=154$	$ab=1{\cdot}87$	$a=100$	$ab=1{\cdot}82$	$a=160$	$ab=1{\cdot}87$

Upper figures are observed. Lower figures are calculated.

The following figures illustrate this maximum value:—

y_0.	a after fourteen hours.
22,800	147,000,000
32,150	151,000,000
100,550	167,000,000

From a practical point of view this is of extreme importance, as obviously error in the number inoculated has little or no effect on the total number attained to. Provided that the number planted be comparatively small, it is unnecessary to enumerate each flask, in the preparation of vaccines in large quantities. Indeed, with a suitable and accurately based system (concentration of broth, etc., being kept constant) vaccines could be prepared month after month at standard strength without any enumerations being made. B. coli, in the quantities of broth which we have used, reaches a maximum in from twelve to fifteen hours.

When $t=0$

$$y=y_0,$$

and $\frac{dy_0}{dt}$ (the initial rate of increase) $=by_0(a-y_0)$,

i.e.
$$\frac{d\log y_0}{dt}=b(a-y_0).$$

As in our experiments a is measured in hundred millions and y_0 rarely exceeds a hundred thousand, we may consider

$$\frac{d\log y_0}{dt}=ab.$$

That is to say, with log y as ordinate and t as abscissa the slope of the curve at the commencement is equal to ab. This value can be obtained from observed results, and its value substituted in equation (3). The values of ab so obtained are entered on the Table of Numbers given above.

[*Note.*—ab is measured in logarithms to base e, whereas in the accompanying tables logarithms to base 10 are employed. Log y to base 10 multiplied by $2{\cdot}3026=\log y$ to base e.]

Since the constant a denotes the original concentration of food-stuff, and b depends on the ability of the organism to acquire its food (modified by such accelerating or retarding influences as temperature, degree of alkalinity, presence of medicaments, etc.), it might have been hoped that in the relation

$\frac{d \log y_0}{dt} = ab$ lay a method of estimating nutritive values and possibly even of determining whether a particular substance acted as a food or as a mere accelerator. But it is at this point that the simplifying assumption breaks down, for obviously an extreme degree of concentration of food-stuff cannot cause an infinite rate of multiplication. The simplifying assumption ascribes to the organism the business of obtaining its food; it presupposes a lightning rapidity of assimilation after it has come in touch with its food, and a lightning rate of multiplication after sufficient nutriment has been assimilated. These conceptions are impossible, and it is only where the period of a generation is large in comparison with the times required for assimilation and division that the simplifying assumption holds good. We have, in fact, applied molecular physics to molar vital phenomena, and we meet with necessary limitations.

The rate of growth (ab) may, however, be applied to a comparison of bacteria as to their multiplying properties; and it may also be most advantageously employed in the comparative investigation of fluid media (bouillon, sugar, etc.) with a view to their standardisation and improvement.

The period of a generation can be deduced as follows:—One generation corresponds to a multiplication of the original number by 2; or to an addition of 0·301 to its logarithm to base 10. But ab is the rate of change of $\log_{10} y_0$, or change per unit time. Hence $\frac{ab}{0·301}$ = number of generations per unit time. In curves 3, 4, 5, $ab = 0·812$, and unit time is one hour; consequently the period of a generation is 22·27 minutes.

Conclusions.

1. The rate of multiplication of fast-growing micro-organisms is proportional to the number of organisms, and to the concentration of food-stuff.

2. The initial rate of multiplication affords a factor of comparison both of efficiency for media and of reproductive properties of organisms.

3. Vaccines may be prepared in large quantities on the basis that a maximum number of organisms is attained to, this maximum being dependent on the concentration of nutriment, and independent of the amount of culture inoculated.

(*Issued separately October* 24, 1911.)

XLVI.—On Photometric Paddle-Wheels. By James Robert Milne, D.Sc.

(Read March 21, 1910. MS. received August 2, 1911.)

CONTENTS.

§ 1. Introduction.

The purpose of the following paper is to describe a modification of the well-known rotating sector, so extensively used in photometric investigations. What led the writer to devise this new form was the need for a sector apparatus, or its equivalent, which would allow of varying the intensity of a beam of light without stopping the rotation of the sector, and which would at the same time give accurate readings. In the ordinary form of rotating sector, consisting of two sectored discs fixed to a common axle by a thumb-screw, there is of course no possibility of altering the angle of the sectors while the apparatus is in motion. This drawback is partially overcome in the "stepped sector" now so much employed in photographic researches, and in the ingenious modification of the stepped sector due to Mr J. de Graft Hunter:* but both these forms only permit of the use of a limited number of fixed light-reduction ratios. Neither of these objections applies to Sir W. de W. Abney's sector, with which the light-reduction ratio can be continuously modified, and its value also read at any time without stopping the rotation of the sectors. It is evident, however, that great accuracy cannot be looked for in any form of mechanism which has to perform the duty of imparting small relative displacements to two parts each of which is moving with a high velocity. The principle

* *Proc. Roy. Soc.*, Series A, vol. lxxxii. p. 307, 1909.

on which the writer's apparatus is based obviates the necessity for this, and permits of making the only moving part in *one* solid piece.

This rotating part, which is somewhat similar to a paddle-wheel in appearance, may take various forms, of which the simplest is shown in fig. 1. It consists of two flat triangular vanes V and V′ fixed on the shaft of an electric motor C, by means of which they are rotated. The motor is fixed to a circular base E, which turns on a vertical pivot D; and when in use the apparatus is placed so that the pivot is directly underneath the beam of light L. In spinning round, V and V′ interrupt the light, and the amount of interruption evidently depends on the azimuth of the base, which is

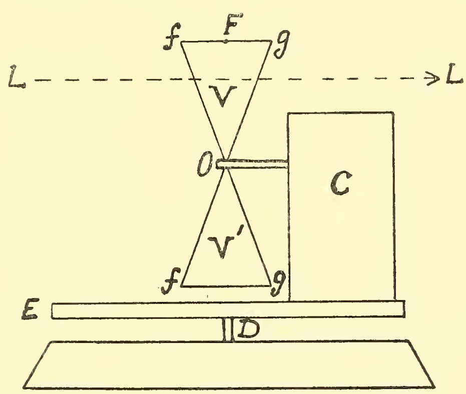


Fig. 1.

ascertained from a circular scale round its edge, or in any other convenient way. It is clear that this angular measurement, on which the light intensity depends, can be made with any required degree of accuracy.

The following is a List of Symbols, with their Meanings:—

a. The azimuth angle through which the motor is turned about its vertical pivot. The zero position is that in which the axle of the paddle-wheel is parallel to the beam of light; and rotation of the motor counter-clockwise about the pivot, looking down upon the latter from above, is considered to give a positive value to a, and *vice versa*.

a_m. That maximum value which it has been decided that a shall not exceed.

c. A constant for each paddle-wheel; which may be defined as half the distance between two imaginary planes of indefinitely great area, perpendicular to the axle of the paddle-wheel, and so placed, one in front of the wheel, and the other behind, as to just be in contact with the extreme tips of the vanes: *e.g.* for the paddle-wheel shown in fig. 1, $c = \frac{1}{2} fg$.

E. The "Eclipse-Angle," *i.e.* the angle through which the paddle-wheel turns while the particular ray of light considered is being intercepted by a vane (or pair of vanes, if there be two sets of vanes one behind the other).

e. Is restricted to indicate the eclipse-angle of the simplest type of vane (shown in fig. 1) and no other.

$\left.\begin{matrix} E_0. \\ e_0. \end{matrix}\right\}$ The values which these symbols have for the "medial" rays of the beam.

ϵ. Is equal to $e_0 - e$.

λ. The "Position-Ratio" of a ray, *i.e.* the horizontal distance of the ray from O, divided by its vertical distance.

λ'. The value of λ which is such that the rays having that particular position-ratio differ in their transmission coefficient by p per cent. from t_0, the value of the transmission coefficient of the medial rays.

λ_m. The smallest value that λ' assumes through the whole range of azimuth angles through which the motor will be swung.

n. The number of vanes possessed by a paddle-wheel (when, however, the latter is of a type having two sets of vanes, one behind the other, the vanes of only one set are to be counted).

O. The point at which the vanes are attached to the motor axle, *e.g.* see fig. 1.

p. The greatest percentage difference considered to be permissible between the values of the transmission coefficients of the medial and the extreme lateral rays respectively.

t. The "Transmission Coefficient" of any ray, *i.e.* the fraction of the light of that ray transmitted by the paddle-wheel.

t_0. The transmission coefficient of the medial rays.

u. See equation 14, page 670.

u_m. See the sentence which follows equation 16, page 670.

Definitions.

Eclipse-Angle—See E, above.

Medial Rays—Those rays which pass through an imaginary vertical line through O.

"Position-Ratio" of a Ray—See λ, above.

"Transmission Coefficient" of a Ray—See t above.

N.B.—In all cases it is supposed that the ends of the vanes remote from the axle of the motor lie on the surface of an imaginary circular cylinder, the axis of which coincides with the axle: and furthermore, the radius of this imaginary cylinder is taken as the unit of length. To put the matter more simply, but less exactly, the radius of the paddle-wheel is always taken to be unity.

§ 2. Different Types of Paddle-Wheel.

It is evident that an endless variety of paddle-wheels might be suggested, differing in the number, shape, and arrangement of their vanes, and perhaps also in other ways. In what follows, however, only a few of the simplest forms will be considered.

In the type of wheel depicted in fig. 1, it is clear that whether the motor be turned round its pivot to right, or to left, the same reduction of light intensity will ensue; and this repetition is undesirable since it shortens the total range of the values of the light intensity, because the amount of turning that can be done in either direction is limited, for reasons that will appear later on. Duplication of values may, however, be got rid of by inclining all the vanes of the wheel at an angle to the axle, instead of having them parallel to it. In fig. 2, AB represents the top edge of such an inclined vane, when at its highest position, looked at from above. Now it is evident that the effect of this vane will be the same as the joint effect of two others which we may imagine to take its place, namely, a vane BC parallel to the axle as before, and a triangular vane AC, having its outer edge AC perpendicular to the outer edge BC of the other, all the vanes, both the real and the imaginary, meeting the axle at a common point of attachment O.

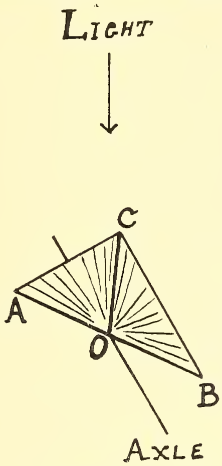


Fig. 2.

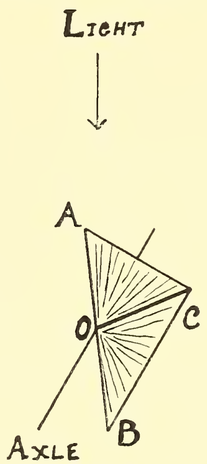


Fig. 3.

If now the *eclipse-angle* of a vane be defined as the angle through which the motor axle turns while the particular ray of light considered is being intercepted by that vane, then clearly the eclipse-angle of the vane AB is equal to that of the vane BC, plus that of the vane AC. Or in symbols,

$$E = \sigma + e \qquad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where E is the eclipse-angle of the vane AB, σ that of the vane AC (σ is evidently a constant), and e that of BC. As the figure is drawn σ is evidently positive; but in the cases where AB lies across the axle in the opposite direction, A being on what in the present figure is the B side of the axle, and *vice versa*—then σ is negative.

It will be convenient to continue this notation throughout, using E as the symbol for the eclipse-angles of all vanes whatever their shape or inclination, and employing the letter e only for the eclipse-angles of the sort of vane shown in fig. 1, which has its plane parallel to the axle. We shall further consider E to be always positive, but e to have the same sign as α the azimuth angle, the latter being positive when the motor is turned counter-clockwise about its pivot, looked at from above, and negative when turned clockwise.

With this convention it is evident that (1) will apply also to the case shown in fig. 3, when a and e are negative, provided the equation be written

$$E = |\sigma + e| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A')$$

Again, we write for a vane of the paddle-wheel shown in fig. 1,

$$E = |e| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

an equation which affords no new information in itself, but is required in connection with the general theory.

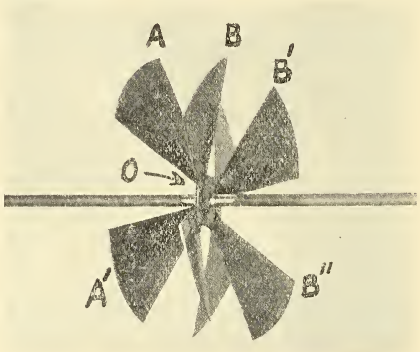


Fig. 4.

Another type of paddle-wheel closely analogous to the preceding is shown in fig. 4. If the reader will imagine the following changes to take place in this new type of wheel, he will see how it may be transformed back into the type of wheel just considered. Suppose that each edge of each vane is connected to the edge of the vane directly behind it by a flat triangular piece, *e.g.* OA connected to OB; and then suppose that all the vanes are subsequently removed, leaving only these triangular pieces. They will form a paddle-wheel only differing from that shown in fig. 1 by the possession of a greater number of vanes. Fig. 5 shows diagrammatically a pair of vanes such as are pictured in fig. 4, when looked at from above. In this case the eclipse-angle E caused by the action of these two vanes is clearly equal to the eclipse-angle of an imaginary vane BOD of the fig. 1 kind, plus the eclipse-angle of the actual vane AB. This latter angle of course is constant and essentially positive; calling it σ', and calling e the eclipse-angle of the (imaginary) vane BOD, we have

$$E = \sigma' + e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

When the axle is turned to the left, as in fig. 6, it is still the *sum* of the two quantities that must be taken to give E; hence, as in accordance with our convention e changes sign with a, (2) must be written

$$E = \sigma' + |e| \qquad \text{(B)}$$

if it is to apply in all cases.

If each rear vane B, fig. 4, is not directly behind the corresponding front vane A, all the rear vanes as we may suppose having been rotated as a whole slightly round the axle with respect to the front set, then a paddle-wheel is obtained which has for its formula

$$E = \sigma' + |\sigma + e| \qquad \text{(B')}$$

where σ is the angle through which the back vanes have been turned

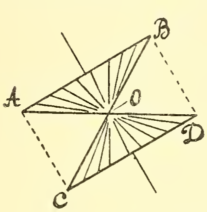


Fig. 5.

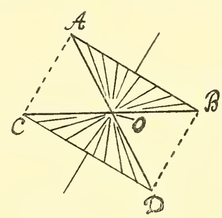


Fig. 6.

relative to the front, σ being considered positive if the angle be turned through counter-clockwise from zero looking with the light, and *vice versa*; and where σ' and e have their former meanings.

The four types of paddle-wheel described above, namely those which lead to equations (A), (A′), (B), and (B′), may be designated respectively types A, A′, B, B′.

Types A′ and B′ have the advantage that they avoid, or can be arranged so as to avoid, repetition in their values of E; an advantage which is not possessed by types A and B. Types A and A′ fan the air as they rotate;* this is avoided by the wheels of types B and B′. But then again these latter have the disadvantage that their E's cannot be reduced to zero, as in the case of the E's of the former pair. It will also appear from what follows that the different types of wheels may differ considerably in the restrictions they impose on the size of the cross section of the beam of light. Hence it cannot be said that there is any "best" type; it all depends on the particular use to which the wheel is to be put; the pattern which is the most suitable for one purpose will probably be less suitable for another. The cost of the actual wheel, however, is small, and the mounting to be described subsequently has

* On referring to the photograph of fig. 17 it will be seen that in practice the construction of the wheels of types A and A′ may be such as to reduce the air resistance considerably.

been so designed as to be equally serviceable with a variety of different wheels. There is therefore no reason why a number of such wheels should not be provided, and employed as occasion may require.

Fig. 4 suggests another type of wheel, shown in fig. 7, consisting of two *plane* circular discs fixed on a shaft at a small distance apart, parallel to each other, and having the same number of equal sectors cut out of each.

As in the analogous case of the paddle-wheel of fig. 4,

$$E = \sigma' + |\,e\,|\,,^*$$

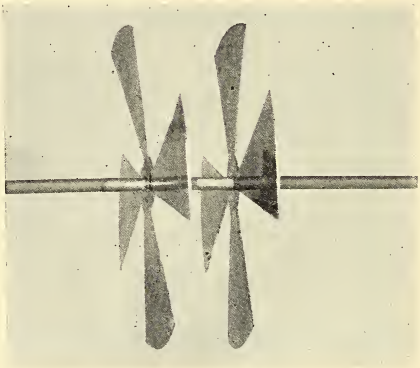


Fig. 7.

but here e is *not* given by the equation which will be deduced below for the e of the previous cases; but, as may readily be shown, by the equation,

$$\tan e = \frac{2yc \tan \alpha}{y^2 + x^2 \sec^2 \alpha - c^2 \tan^2 \alpha},$$

where x and y are the coordinates of the ray of light considered, referred respectively to a horizontal axis through O (which here is not the point of attachment of the vanes to the axle, but the point of the axle midway between the discs) perpendicular to the ray, and to a vertical axis through O; and where c is half the distance between the discs. Now, it will be found in this formula—and herein it differs from the other referred to—that a small change in the value of either x or y will cause a considerable change in the value of e; that is to say, in this case the values of the eclipse-angles for the different rays of the beam of light will differ too much among themselves for such a type of paddle-wheel to be useful in practice.

It may be well at this point to emphasise the fact that the mathematical investigation which follows has been worked out chiefly for the purpose of

* σ' being the eclipse-angle of one of the actual vanes, and e being the eclipse-angle of one of the imaginary vanes, each of which must in this case be considered as consisting of a rectangle connecting the edge of a front vane to the corresponding edge of the vane behind it.

discovering what are the best forms and dimensions for photometric paddle-wheels generally, and therefore of course it is not required in connection with the actual use of such wheels, any more than the elaborate calculations relating to the design of a ship's hull are required in connection with the actual navigation of the ship.

§ 3. The Relation between the Angle of Azimuth of the Axle of the Paddle-Wheel and the Corresponding Reduction of the Light Intensity.

We begin by finding an expression for e. It is clear that e may be regarded as the angle between two planes passing through the y axis of fig. 8 and through the two intersections respectively of the chosen ray of

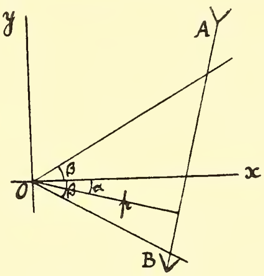


Fig. 8.

Here Oy coincides with the axle of the motor (supposed horizontal); Ox is horizontal and at right angles to Oy; while Oz (z not shown) is vertical.
The ray of light AB is parallel to the xy plane, and at a height q above it; and makes an angle α with the yz plane.

light AB with a cone whose axis is the y axis and whose semi-vertical angle is $90° - \beta$, 2β being equal to the angle fOg, fig. 1. It will be observed that we are considering the motor axle to be stationary, while the direction of the beam of light is movable, the opposite of what obtains in practice, but it is simpler to do this, and the results are of course identical.

The ray AB is given by its equations,

$$z = q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\cos\alpha \, . \, x - \sin\alpha y = p \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The equation of the cone is

$$x^2 - \cot^2\beta \, . \, y^2 = -z^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The equation of any plane through the y axis is, θ being the angle it makes with the z axis,

$$x = \tan\theta \, . \, z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If (7) passes through the intersection of (4), (5), and (6), this determines two values for θ, *i.e.* determines two planes (7), the angle e between them being given by the equation,

$$\cos e = \frac{1 + \lambda^2 \sec^2 a - 2c^2 \tan^2 a}{1 + \lambda^2 \sec^2 a},$$

λ being written for the ratio p/q, which may be called the *position-ratio* of the ray of light.

$$\therefore \quad 2 \sin^2 \frac{e}{2} = 1 - \cos e = \frac{2c^2 \tan^2 a}{1 + \lambda^2 \sec^2 a}$$

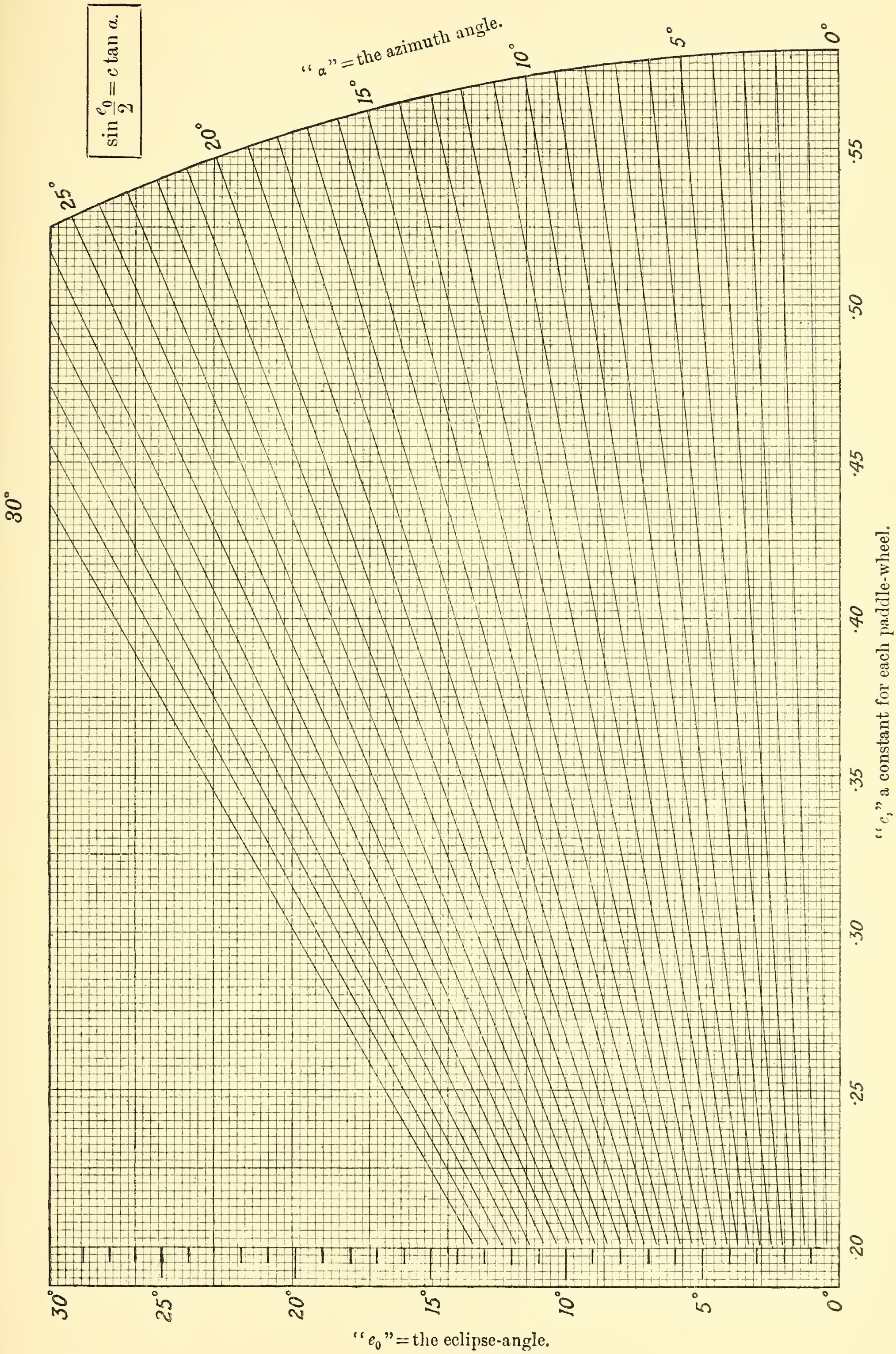
$$\sin \frac{e}{2} = \frac{c \tan a}{\sqrt{(1 + \lambda^2 \sec^2 a)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The presence of λ in this equation shows that the reduction of brightness caused by the vanes is different for different rays of the beam. It can only be sensibly the same for all the rays if the term involving λ^2 is in all cases negligibly small in comparison with unity. When this is the case, then each ray undergoes sensibly the same reduction of brightness as those rays of the beam which are vertically above O, fig. 1, and for all of which $\lambda = 0$. The question of the greatest value that can be allowed to λ (and therefore the greatest permissible width of the beam of light that can be used), without violating the condition that the term of (8) in which it appears is negligible, will be discussed later; meantime we shall simply assume that the condition is complied with. On such an understanding (8) becomes

$$\sin \frac{e_0}{2} = c \tan a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In order to facilitate the discovery of the properties and behaviour of any proposed design of paddle-wheel, the above equation, and all the subsequent equations of fundamental importance, have been "graphically tabulated." By this means the quantitative results for any paddle-wheel may be found at once without having to resort to calculation.

The principle on which Chart I. is constructed will be understood by reference to fig. 9. OA, the scale along which c is measured, is of unit length, and is linearly divided. The arc AG, along which the angles are numbered off, has its centre at O. If then AOB be the value of a, $\tan a = \text{AB}$; and if OD be the value of c, $c \tan a = \text{CD} = \text{EF}$. But $\text{EF} = \sin \text{EOF}$, therefore EOF must be the required value of $e_0/2$, corresponding to the given values of a and of c. The actual chart has been drawn in two parts, (a) and (b), for the sake of greater accuracy, and there are some slight modifications in detail; but these ought to present no difficulty: *e.g.* on the $\frac{e_0}{2}$ scale the numbers have all been doubled, so that it gives not $\frac{e_0}{2}$ but e_0, which of course is more convenient. *To use either part*



$\sin \frac{e_0}{2} = c \tan a.$
"a" = the azimuth angle.
0° 5° 10° 15° 20° 25° 30°
·20 ·25 ·30 ·35 ·40 ·45 ·50 ·55
"c," a constant for each paddle-wheel.
"e_0" = the eclipse-angle.

CHART I. (a).

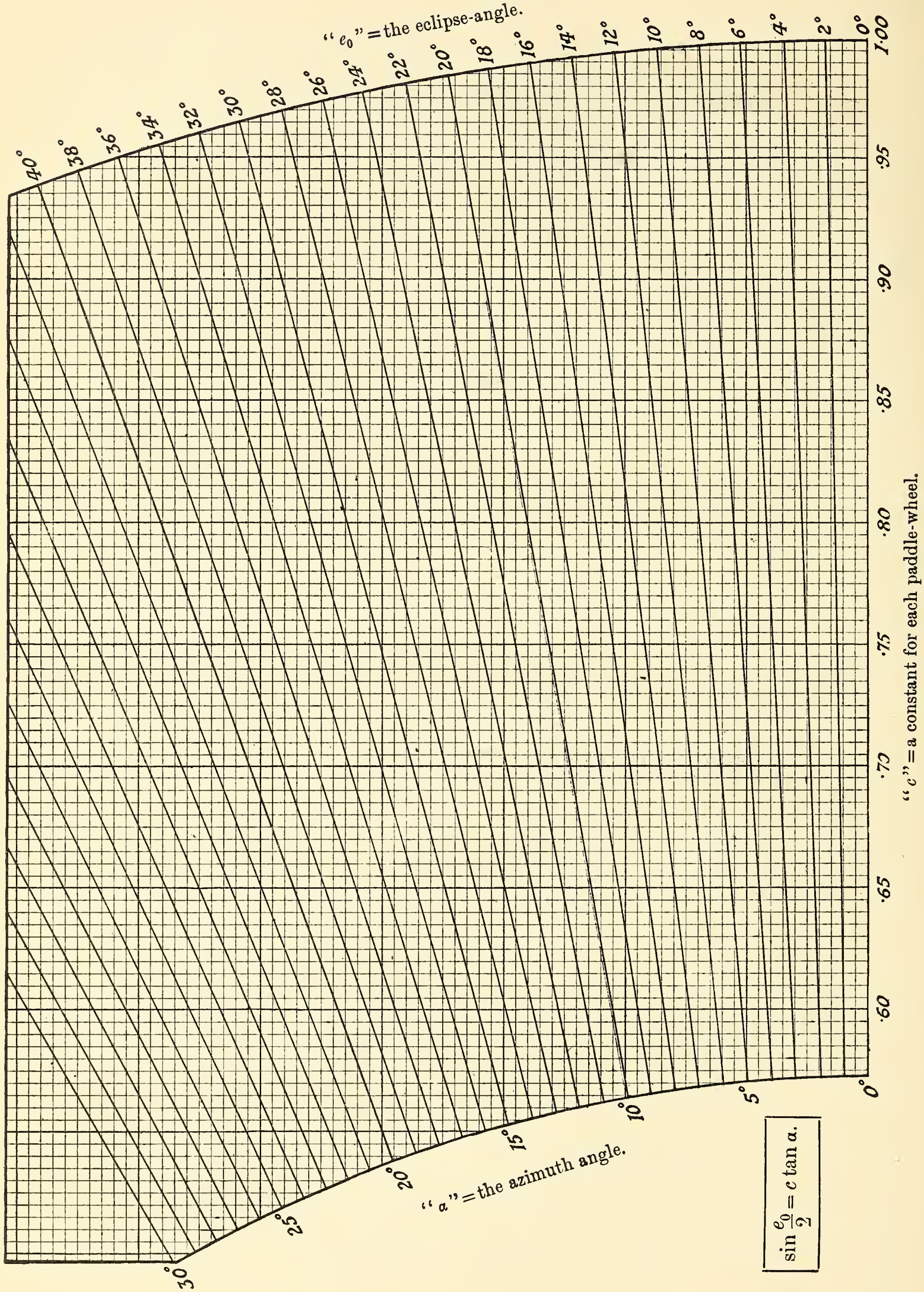

" e_0 " = the eclipse-angle.
40° 38° 36° 34° 32° 30° 28° 26° 24° 22° 20° 18° 16° 14° 12° 10° 8° 6° 4° 2° 0°
1·00 ·95 ·90 ·85 ·80 ·75 ·70 ·65 ·60
" c " = a constant for each paddle-wheel.
0° 5° 10° 15° 20° 25° 30°
" a " = the azimuth angle.
$\sin\frac{e_0}{2} = c \tan a.$

CHART I. (b).

of the chart enter the value of a *in the scale so marked; follow the nearest radial line until it meets the vertical line which passes through the given value of* c, *and the horizontal line through this point of meeting will cut the* e_0 *scale at the required value.*

In the above only the effect of one vane has been considered: we now go on to find the fraction of the light which is transmitted by the entire wheel, taking account of the whole set of vanes.

Evidently the transmitted fraction t is given by the equation

$$(1-t)=\frac{n\mathrm{E}}{360} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where, as before, E is the eclipse-angle, n being the total number of vanes of

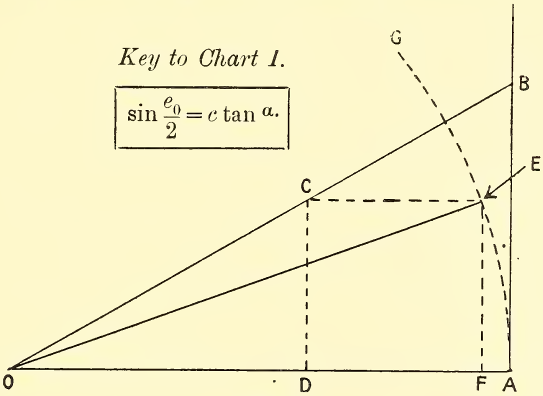


Fig. 9.

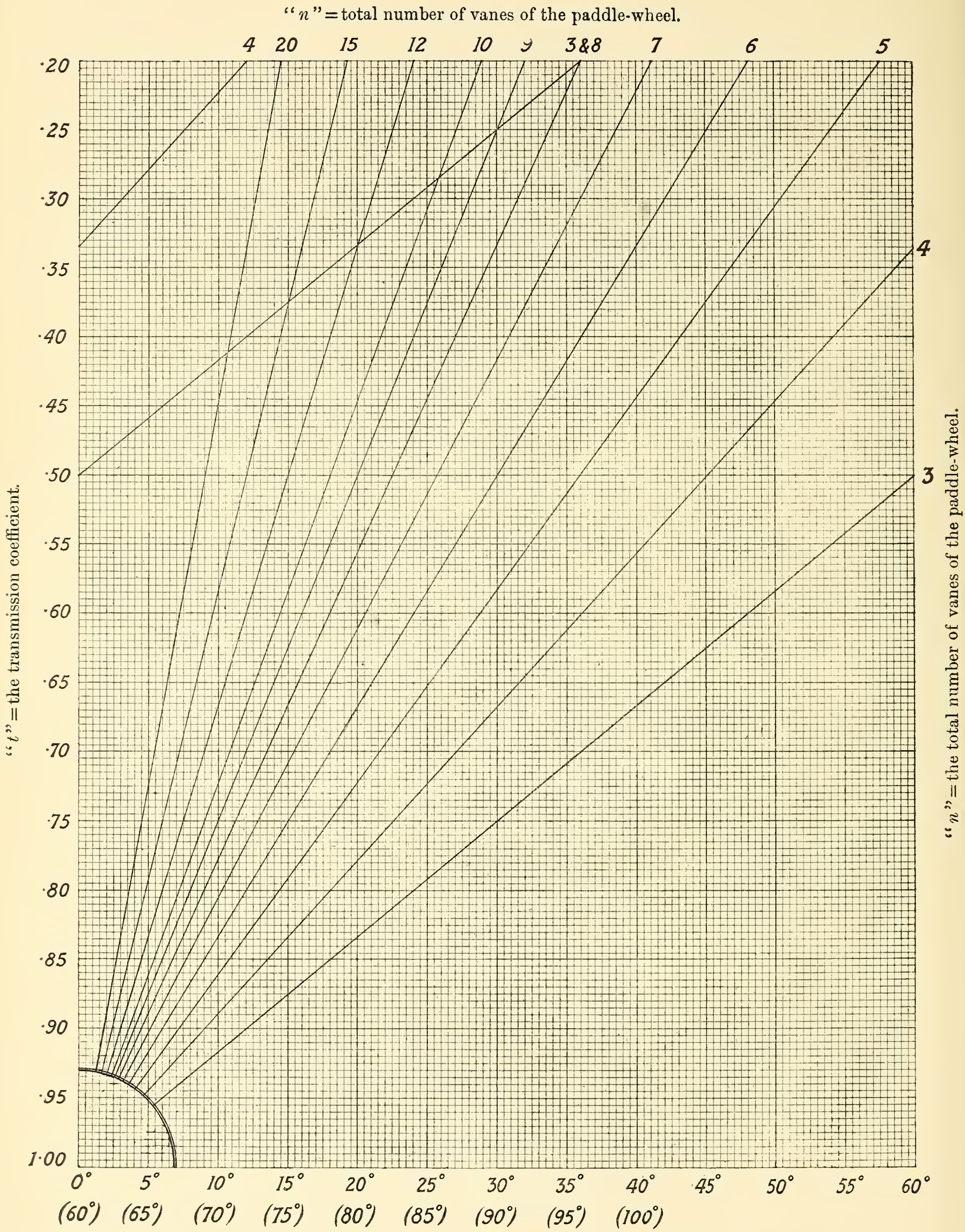
the wheel. This equation is graphically plotted in Chart II. If we consider t to be the dependent variable, E the independent, and n a variable parameter which takes only integral values, then the equation is linear, and is represented by a series of straight lines, corresponding to the different values of n. The value of E is obtained from e_0 (Chart I.) and equations A, A′, B, or B′ as the case may be; and the method of using the chart is obvious.

§ 4. The Condition necessary to ensure that all the Rays of the Beam of Light shall have sensibly the same Intensity after their Transmission by the Paddle-Wheel.

We shall consider first the case of wheels of types A and B, the formulas for which may be written as one,

$$\mathrm{E}=\mathrm{K}+|e| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where K is a constant not less than zero. As in the above equation the sign of e is indifferent, it will be sufficient to consider only positive values of e.



" E " = the eclipse-angle.

CHART II.

$$1 - t = \frac{nE}{360}.$$

In order that the chart should be on as large a scale as possible, it has been necessary to draw the lines for 3-vaned and for 4-vaned paddle-wheels in two parts. These lines, it will be noticed, run out at the right-hand side and reappear at the same height on the left-hand side. The lower row of figures (within brackets) on the E scale applies to the additional pieces of the lines.

Consider a beam of light transmitted by a paddle-wheel of type A or B. As we pass outwards from the middle of the beam towards either side of it, the position-ratio of the rays successively encountered goes on numerically increasing, and at the same time (8) shows that simultaneously the value of e will go on decreasing, and (11) and (10) show that the value of t will go on increasing. Therefore, after a beam of light has passed through the wheel, the greatest difference in the values of the transmission coefficients of the various rays will be found, between the coefficients of the medial rays, for which λ is zero, and the coefficients of the outermost rays for which λ has its greatest value; and consequently, if the difference between *these* coefficients be sufficiently small, the arrangement will be satisfactory. If then p per cent. be the largest variation that can be permitted in the value of the transmission coefficient, the condition for equality of illumination of the rays of the beam is simply that the cross section of the latter should be so restricted that no ray has a position-ratio greater than λ', where λ' is that value of λ which makes the value of t the transmission coefficient of the corresponding rays differ by p per cent. from t_0, the transmission coefficient of the "medial rays."

λ' is found as follows:—

From (10)

$$1 - t = \frac{nE}{2\pi}$$

where E is in radians; therefore, from (11)

$$1 - t_0 = \frac{n(K + e_0)}{2\pi}.$$

Also

$$1 - t = 1 - t_0\left(1 + \frac{p}{100}\right) = \frac{n(K + e)}{2\pi},$$

therefore

$$\frac{t_0 p}{100} = \frac{n}{2\pi}(e_0 - e) = \frac{n\epsilon}{2\pi} \quad \text{(say)};$$

i.e.

$$\epsilon = \cdot 0628 \frac{t_0 p}{n} \quad . \quad . \quad . \quad . \quad . \quad (11^*)$$

We have also

$$\sin\frac{e_0}{2} = c \tan \alpha,$$

and

$$\sin\frac{e}{2} = \sin\frac{e_0 - \epsilon}{2} = \frac{c \tan \alpha}{\sqrt{(1 + \lambda'^2 \sec^2 \alpha)}},$$

therefore

$$\sin\frac{e_0 - \epsilon}{2} = \frac{\sin\frac{e_0}{2}}{\sqrt{(1 + \lambda'^2 \sec^2 \alpha)}};$$

i.e.

$$\frac{\sin\frac{e_0-\epsilon}{2}}{\sin\frac{e_0}{2}}=\frac{1}{\sqrt{(1+\lambda'^2\sec^2\alpha)}}\,; \qquad (12)$$

and as it can easily be shown that in practice ϵ would never exceed about $4^\circ\!\cdot\!5$, therefore, expanding and keeping only the first power of ϵ, we have

$$1-\frac{\epsilon}{2}\cot\frac{e_0}{2}=\frac{1}{\sqrt{(1+\lambda'^2\sec^2\alpha)}}\,,$$

so that

$$\lambda'=\sqrt{\left\{\frac{1}{\left(1-\frac{\epsilon}{2}\cot\frac{e_0}{2}\right)^2}-1\right\}}\cos\alpha\,,$$

and from (11*)

$$\lambda'=\sqrt{\left\{\frac{1}{\left(1-\cdot 0314\frac{t_0p}{n}\cot\frac{e_0}{2}\right)^2}-1\right\}}\cos\alpha \qquad (13)$$

Let

$$u=\frac{t_0p}{n}\cot\frac{e_0}{2} \qquad (14)$$

then

$$\lambda'=\sqrt{\left\{\frac{1}{(1-\cdot 0314u)^2}-1\right\}}\cos\alpha \qquad (15)$$

It appears from the above that the connection between λ' and p in the case of any given paddle-wheel of the types A or B will depend on α, the azimuth angle through which the latter is turned. And the position-ratios of the rays of the beam are none of them to be greater than λ', if the beam is to be sensibly homogeneous in intensity. But it is necessary, of course, that this homogeneity should exist whatever be the value of the angle through which the paddle-wheel is turned. Therefore the position-ratios of the rays are none of them to be greater than λ_m, where λ_m is the smallest value which λ' assumes throughout the whole range of azimuth angles through which the motor will be swung (say $0 \not> \alpha \not> \alpha_m$). λ_m is found as follows. In (12) the left-hand member is always a positive proper fraction (assuming $e_0 < \pi$), hence its equivalent $(1-\epsilon/2\,.\cot e_0/2)$ equal to $(1-\cdot 0314t_0p/n\,.\cot e_0/2)$ is always a positive proper fraction, hence from (13) λ' is a minimum, *i.e.* $\lambda'=\lambda_m$, when simultaneously,

t_0 is a min. [*see* (11) & (10)] when e_0 is a max.
e_0 is a max. [*see* (9)] when α is a max.
α is a max. when $\alpha=\alpha_m$

Hence from (15)

$$\lambda_m=\sqrt{\left\{\frac{1}{(1-\cdot 0314u_m)^2}-1\right\}}\cos\alpha_m \qquad (16)$$

where u_m denotes the minimum value of u, obtained by taking $a = a_m$ in finding t_0 and e_0 the constituents of u.

The value of λ_m for all the cases likely to occur in practice may be found by an inspection of Charts III. and IV. The theory of Chart III. will be understood on reference to the key fig. 10. If AF be measured off equal in length to n, AE equal in length to p,* and OG equal in length to t_0;† and GH be drawn parallel to FE, then by similar triangles $OH = OG.AE/AF = t_0 p/n$. If HK be drawn parallel to ON and AB, and the angle NOM be drawn equal to $e_0/2$,‡ then $OL = LK \cot e_0/2 = OH \cot e_0/2 = t_0 p/n \,.\, \cot e_0/2$: *i.e.* the length of OL is equal to the value of u_m

Key to Chart III.

$$u = \frac{t_0 p}{n} \cot \frac{e_0}{2}.$$

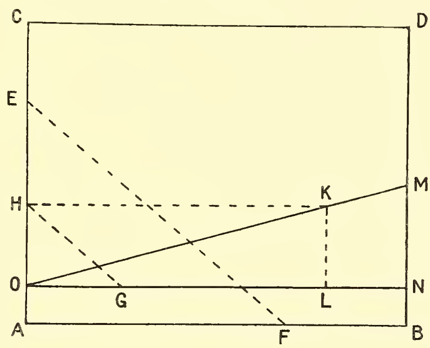


Fig. 10.

in (16). *To use Chart III. one places the edge of a parallel ruler at the given values of* n *and of* p *on their respective scales, and then moves the edge parallel to itself until it intersects the* t_0 *scale at the proper value of* t_0 *(as found from Chart II., putting* $a = a_m$*). One follows the horizontal line from the point at which the edge now cuts the* p *scale until it meets the radial line, which passes through the given value of* e_0 *(found from Chart I., putting* $a = a_m$*) on the* e_0 *scale: the vertical line through this meeting place cuts the* u *scale at the required value of* u_m. *If preferred, a set square and straight-edge may of course be employed instead of a parallel ruler:* in either case, despite the necessarily somewhat long description, the use of the chart will be found both quick and easy.

u_m having been thus ascertained, λ_m follows from an inspection of

* When dealing with small values of p greater accuracy will be obtained if AE is made equal to *twice* p, it being remembered that in that case the value of OL which will be obtained is twice the value of n.

† Found from Chart II., putting $a = a_m$. ‡ Found from Chart I., putting $a = a_m$.

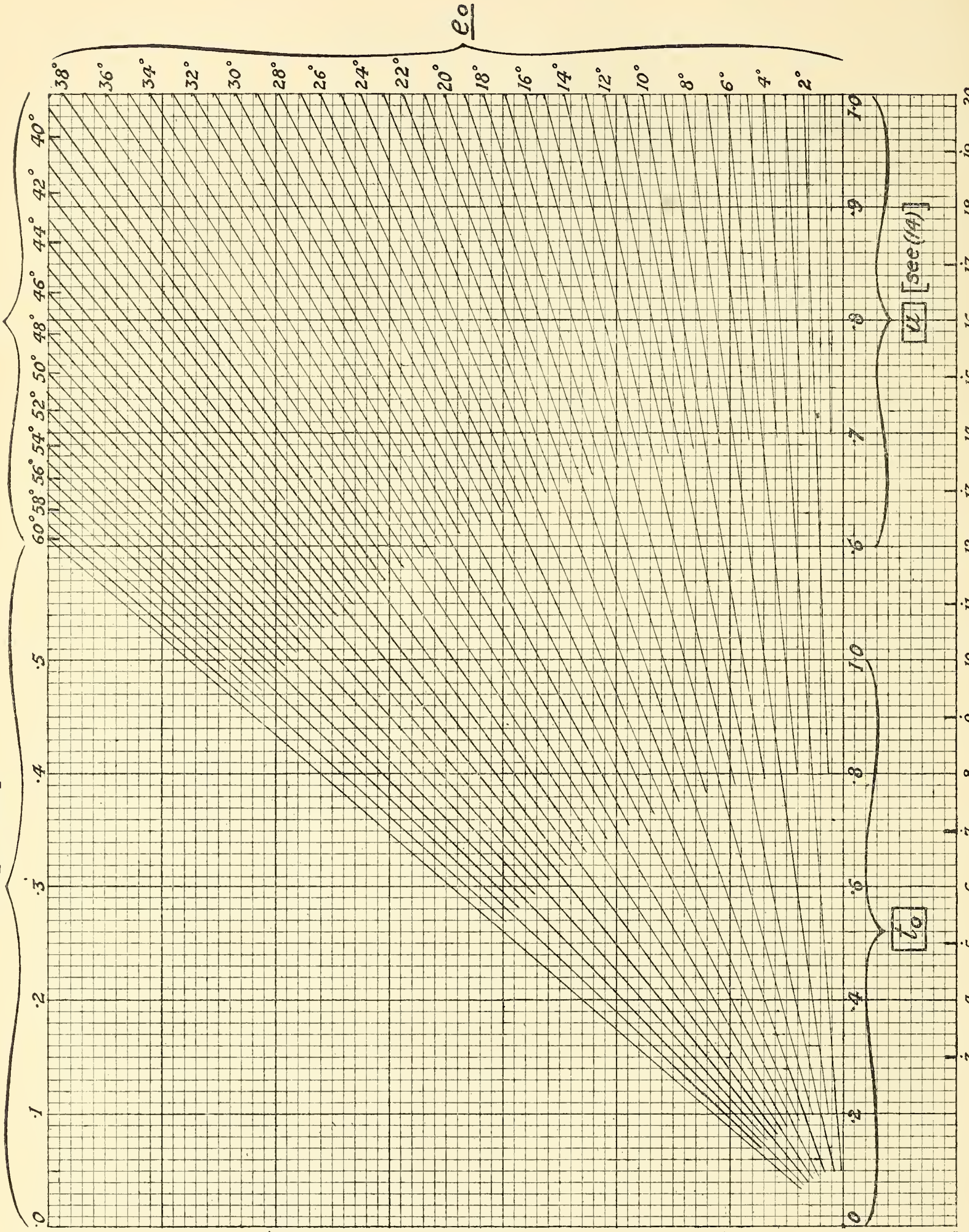


CHART III.

" n " = the number of vanes of the paddle-wheel.

" p " = the percentage difference of intensity.

$$u = \frac{t_0 p}{n} \cot \frac{e_0}{2}.$$

For greater convenience the values of e_0 are marked off instead of the values of $e_0/2$. Because of this, and because the vertical scale is twice the horizontal, the angles made with the horizontal by the respective radial lines, are not the angles marked at the ends of these lines, but are equal to $\tan^{-1}(2 \tan e_0/2)$.

Chart IV., as will be understood from the key fig. 11. Various values of u_m are marked off along AC, each being placed at a distance from A equal to $\sqrt{\left\{\frac{1}{(1-\cdot 0314u_m)^2}-1\right\}}$. Suppose, for instance, that the given value of u_m is found at D, then $AD=\sqrt{\left\{\frac{1}{(1-\cdot 0314u_m)^2}-1\right\}}$. But from the geometry of the figure, it follows, that $AF = AE\cos a = AD\cos a$, therefore, $AF=\sqrt{\left\{\frac{1}{(1-\cdot 0314u_m)^2}-1\right\}}\cos a_m = \lambda_m$ (*see* 16). Only the part (shown dotted) of fig. 11 that is needed in practice is drawn in

Key to Chart IV.

$$\boxed{\lambda_m=\sqrt{\left\{\frac{1}{(1-\cdot 0314u_m)^2}-1\right\}}\cos a_m.}$$

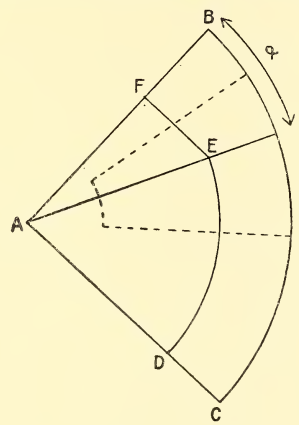
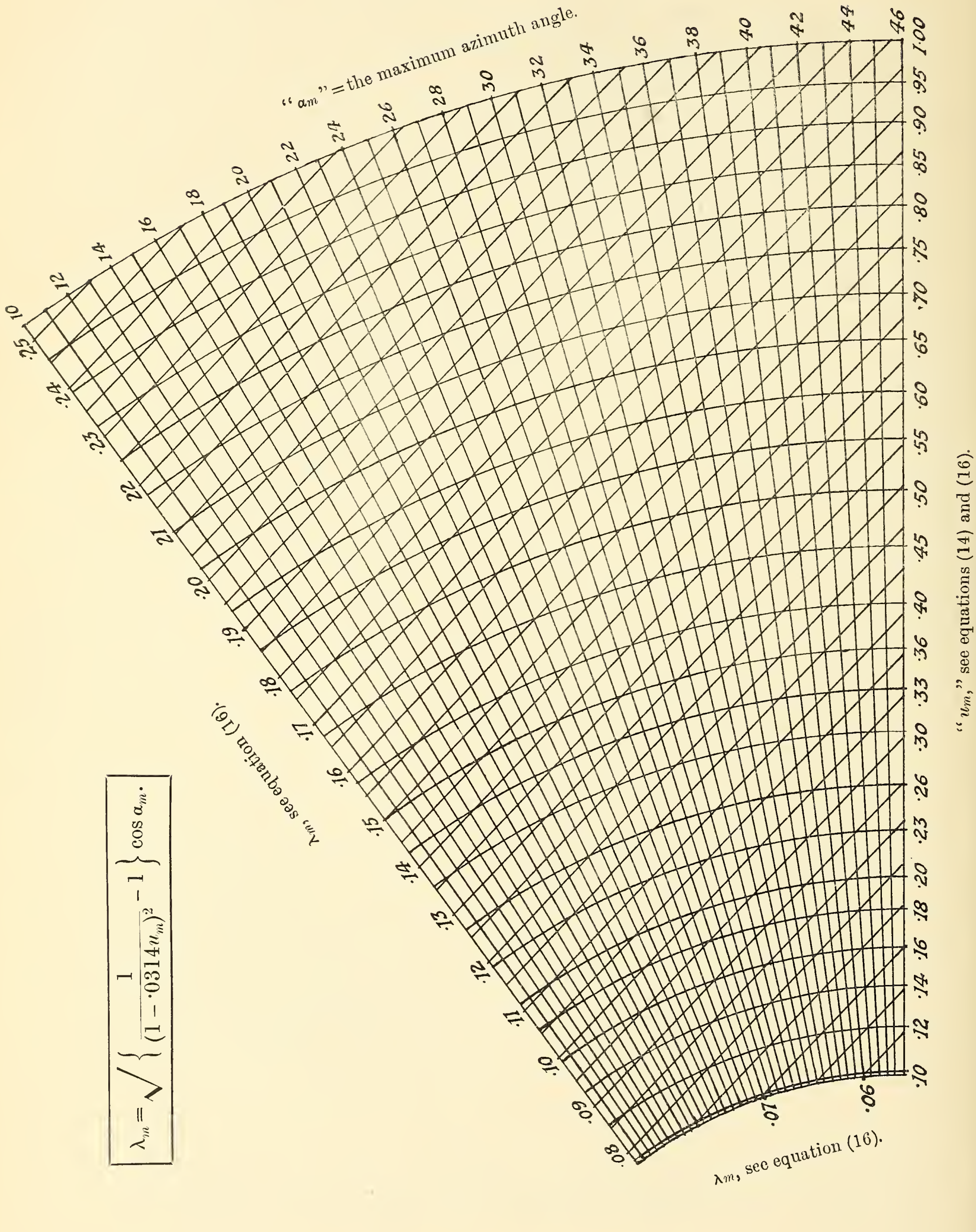


Fig. 11.

Chart IV. In accordance with the above explanation, in using the chart *one enters the value of* u_m *at the bottom, runs up the corresponding circular arc until one meets the radial line corresponding to the required value of* a_m, *and then runs along the nearest of the parallel lines, and this line will intersect the* λ_m *scale at the required value.* Now the beam of light is to be so restricted that none of its rays has a position-ratio greater than λ_m. To see what this means we may imagine two straight lines to be drawn upwards through the point O in fig. 1, in a plane perpendicular to the beam of light, one to the right of the vertical through O, the other to the left, and each making with the vertical an angle $\tan^{-1}\lambda_m$; then the necessary condition for equality of intensity among the various rays of the beam is that without exception they shall all pass between these two lines.



$$\boxed{\lambda_m = \sqrt{\left\{\frac{1}{(1 - \cdot 0314 u_m)^2} - 1\right\}} \cos \alpha_m .}$$

CHART IV.

In the above discussion only wheels of the A and B types have been considered, but before passing on to the others it is first of all necessary to treat the theory somewhat more generally than has yet been done. If t_0 and e be roughly plotted as functions of a, the graph shown in fig. 12 will be obtained, ϵ being here defined as the change in e_0 caused by altering λ from zero to some fixed maximum value. Now, for the medial rays of the beam $\lambda = 0$, and from (10) and (11),

$$1 - t_0 = n\mathrm{E}_0/360 = n(\mathrm{K} + \mid e_0 \mid)/360\ ;$$

while for the most lateral rays, λ is a maximum, e_0 becomes $(e_0 - \epsilon)$, and (from the above equation) t_0 becomes $(t_0 + n\epsilon/360)$. Hence the change in t_0 is simply proportional to ϵ. But unfortunately it appears from fig. 12 that when t_0 is least, ϵ, and therefore the change in t_0, is greatest; *i.e.* things so fall out that the greatest absolute variation occurs in the value of t_0 precisely when the value of t_0 is such that it will be most disturbed by any change.

This state of affairs reaffirms a conclusion already arrived at, namely, that λ' attains its minimum value λ_m when a is a maximum; and it also accounts for the fact—which a little experimenting with the charts will soon elicit—that the use of paddle-wheels of the A and B types necessitates a comparatively severe restriction on the width of the beam.

Much better results, however, can be obtained from the A′ and B′ types of wheel. In equations (A′) and (B′) the sign of σ depends only on the way in which the positive and negative directions are chosen for a, and, as this choice is quite arbitrary, σ can always be made positive. We shall assume that this has been done; and further that σ—to avoid repetition of values of E—is taken not less than the largest numerical value of e_0. With this understanding (A′) and (B′) may be written as one equation,

$$\mathrm{E} = \mathrm{K} + e \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where K is a positive constant always greater than e.

Hence from (10) and the above equation, when $\lambda = o$,

$$1 - t_0 = n\mathrm{E}_0/360 = n(\mathrm{K} + e_0)/360 \quad . \quad . \quad . \quad . \quad (18)$$

and when λ is a maximum, e_0 becomes $(e_0 - \epsilon)$, and from (18), t_0 becomes $(t_0 + n\epsilon/360)$; so that the change in t_0 is proportional to ϵ, as before. But now t_0 is a different function of a, and on reference to fig. 13 (which like fig. 12 is only diagrammatic) it will be seen that by restricting a to a range of negative values terminating at the origin, we have the power to so arrange matters that as t_0 decreases, so also (numerically) does ϵ, and with it the variation of t_0; thus bringing about the ideal result that, when t_0 is at its smallest, the absolute amount of the change to which it is liable is simultaneously a minimum.

For the purpose of finding quantitatively in such a case the necessary restriction on the width of the light beam the former discussion is still serviceable.

For positive values of e equation (17), the fundamental equation in the present case, is identical with equation (11), the fundamental equation in the last case; and on the other hand, when e is negative, so also are a, p, and ϵ, and the argument will be found to proceed *mutatis mutandis*

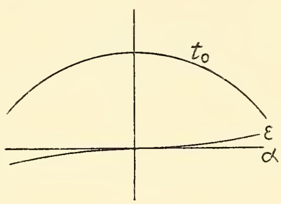


Fig. 12.

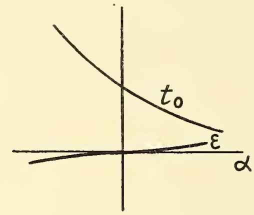


Fig. 13.

precisely as before, and to lead up to the use of Charts III. and IV. as explained above.

Another restriction on the permissible size of the cross section of the beam is due to the evident necessity for having all the rays near enough to the paddle-wheel to be duly intercepted by the vanes as they spin round; and this may be called the "contact condition."

§ 5. The "Contact Condition."

The following method of discussing the condition applies to all the types of paddle-wheel that have been described above. Imagine two plane discs of unit radius fastened to the axle of the paddle-wheel perpendicular to the axle; the front one having its periphery in contact with the front extremities of the vanes (*e.g.* with ff in fig. 1); and the rear one having its periphery in contact with the rear extremities of the vanes (*e.g.* with gg in fig. 1). Then clearly all rays of the beam, that are properly intercepted by the vanes as they rotate, will pass through both these imaginary discs, not outside either; or, to put the matter in another way, through the common part of the orthogonal projection of the two discs on an imaginary plane through O perpendicular to the beam. That is, the entire beam must be contained within the boundary ABC, fig. 14 (only the *upper* halves of the projections of the discs are shown, because of course the beam is always entirely above the axle).

The beams of light employed in photometry have as a rule either circular or rectangular sections. Hence, what concerns one in practice is the question of what is the largest size of circle, or of rectangle (with a given ratio of sides), that can be included within the fig. ABC. To facilitate the finding out of this, Chart V. has been constructed, in which for a number of different values of the determining variables c and a the restricting boundary ABC is drawn to scale. Or, rather, the essential parts of ABC are drawn to scale, for evidently it will be sufficient to give half of it—

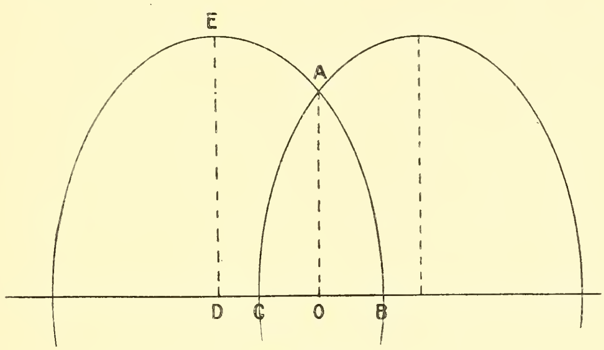


Fig. 14.

namely, one of the elliptic sides AB, and the centre line AO drawn in the proper position relative to AB.

The restriction on the size of the beam will be most severe when a is a maximum; hence, it will be enough to consider the case of $a = a_m$, where a_m is the largest azimuth angle through which the paddle-wheel will be turned.

On considering the way in which fig. 14 is obtained, it will be clear that the semi-major-axis DE must always be unity, because the radius of the paddle-wheel is taken as unity *; and that the distance DO of the centre line AO, from DE, must be equal to $c \sin a_m$. In Chart V. (where DE corresponds to the DE of fig. 14) this distance can be found as follows. The values of a_m marked off along the line DH are placed at distances from D equal to $\sin a_m$; and along ED are marked off values of c equal to their distances from E. It follows that the required centre line, at a distance $c \sin a_m$ from DE, must pass through the intersection of the horizontal line which cuts ED at the given value of c, with a radial line from E which

* See the final sentence of the "Definitions" of § 1.

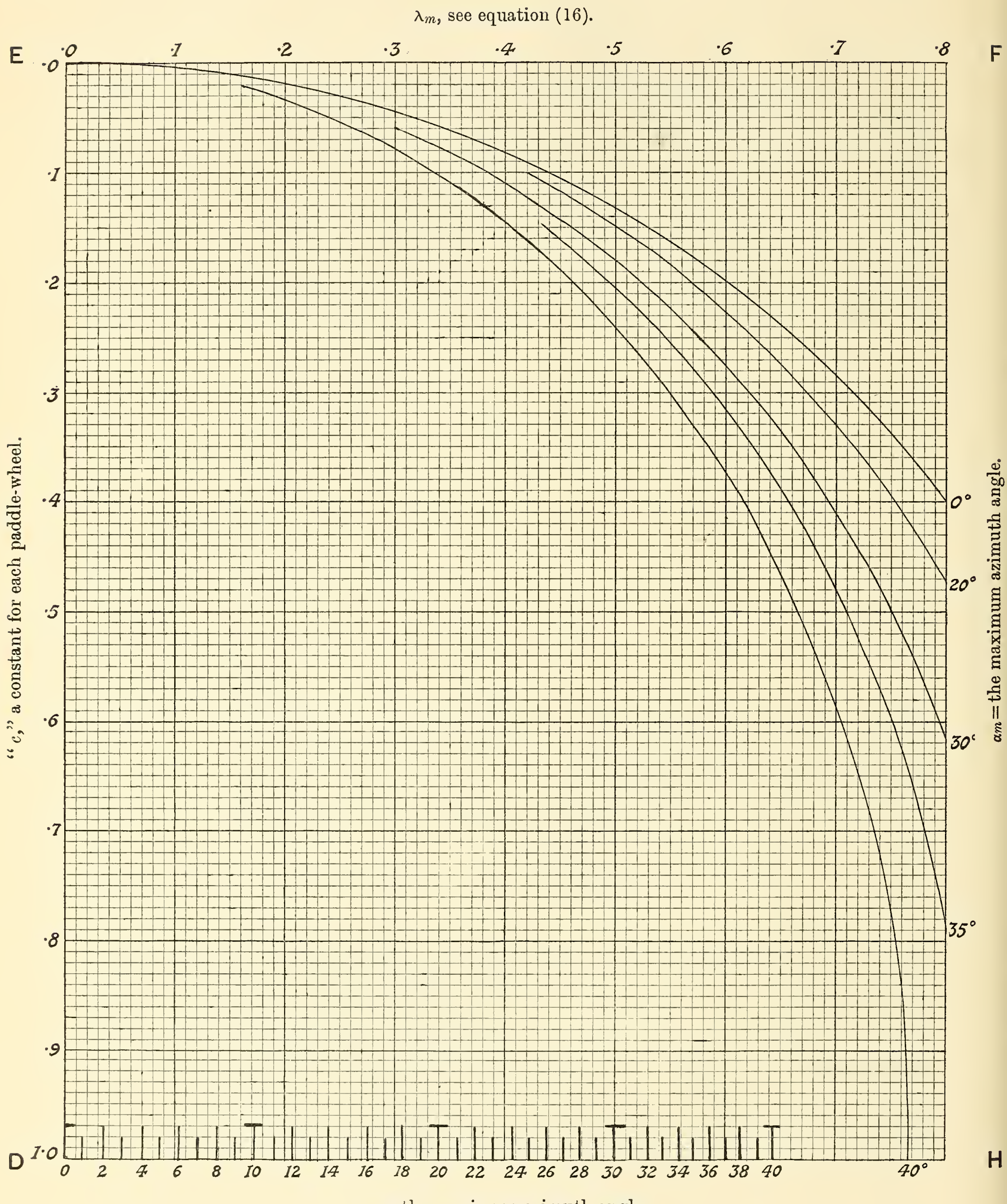

λ_m, see equation (16).
E
F
D
H
·0 ·1 ·2 ·3 ·4 ·5 ·6 ·7 ·8
·0 ·1 ·2 ·3 ·4 ·5 ·6 ·7 ·8 ·9 1·0
"c," a constant for each paddle-wheel.
0°
20°
30°
35°
a_m = the maximum azimuth angle.
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40
40°
a_m = the maximum azimuth angle.

CHART V.

cuts DH at the given value of a_m. The necessary vertical and horizontal lines appear on the chart; but the radial lines have not been drawn, as greater accuracy can be obtained by stretching a fine black thread in the required position. The corresponding elliptic part of the boundary (EB in fig. 14) is the curved line from E, which cuts FH at the given value of a_m—the reason why nothing to the right of FH need be considered will appear in the course of the next paragraph.

§ 6. On what is Necessary to Satisfy both the Foregoing Conditions.

It has just been shown that the beam of light must lie within a boundary of the shape ABC, fig. 14, and it was shown in § 4 that it must lie

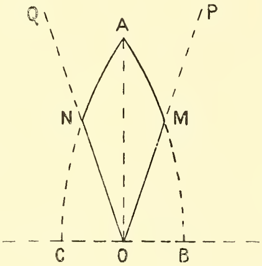


Fig. 15.

between two lines through O each making an angle with the vertical OA equal to $\tan^{-1}\lambda_m$. To comply with both conditions it is evidently necessary and sufficient that the beam should be confined within the boundary AMON, fig. 15. As before, owing to the symmetry of this boundary, it will be sufficient to provide a means of determining AMO, one half of it. The parts AM and AO can be found drawn to scale in Chart V., as just explained; and the position of the line OP relative to them, although not marked on the chart, can readily be ascertained as follows. DE is of unit length, and therefore a line joining D with a point on EF distant from E by a length equal to λ_m would make an angle $\tan^{-1}\lambda_m$ with the vertical; *i.e.* would have the same *slope* as the required line. If therefore a parallel ruler be laid on the chart with its edge on D and on the given value of λ_m on EF, and then be moved parallel to itself until its edge is at the point on DH corresponding to O, the required line is found. And this completes the quantitative determination of the restricting boundary.

§ 7. Description of an actual Paddle-Wheel Instrument: and of its Self-Recording Device for Registering the Readings.

The apparatus (figs. 16 and 17) was designed so as to be as rigid as possible. It is constructed chiefly of flat wrought-iron bars $1'' \times \frac{1}{4}''$ in section, welded or bolted together, lock-nuts being used on all the bolts. The base A is of wood strengthened by various iron bars, and it is elevated

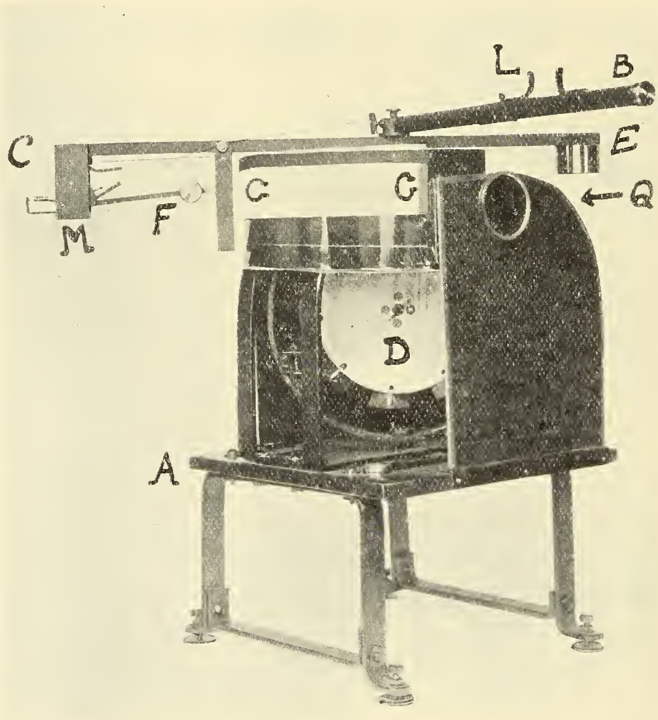


Fig. 16.

on legs 7″ high in order that it may be put astride the long base bar of a spectro-photometer with which it is to be used. The electric motor, which is out of sight in both figures owing to the paddle-wheel D fixed on the end of its axle, is bolted to a horizontal platform, attached to a vertical hoop H. This hoop turns about the pivots PP′, which revolve in brass plummer blocks connected to the support S. This is made of a single welded piece, in shape like the letter D laid on its side ⊍, and is provided with projecting feet, which are bolted to the base A. The paddle-wheel D consists of an aluminium disc $\frac{1}{8}''$ thick, having a brass hub by which to fix it to the motor axle, and also a brass rim $\frac{1}{16}''$ thick and $\frac{1}{2}''$ wide for the attachment of the vanes. The latter, nine in number, were cut out of sheet brass by the aid of a template, and are fixed in slotted studs screwed into the brass rim. A simple optical method was employed, by means of which each vane in turn was accurately set at the desired angle of 60° with the axle; and after each one was adjusted, it was finally soldered so as to guard against any possibility of its working loose and changing its position.

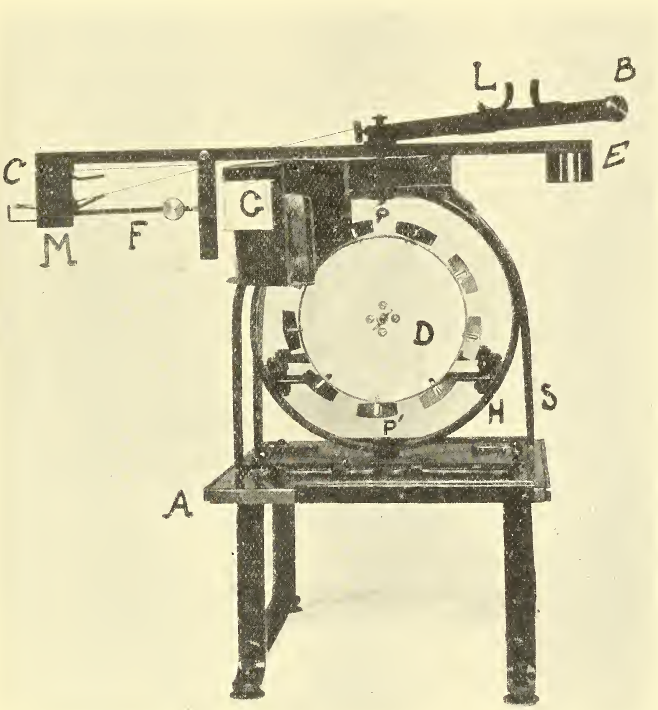


Fig. 17.

The whole apparatus is enclosed from light by means of two sheet metal covers, one of which is shown in position in fig. 16, the beam of light passes through the (very short) tube Q, across the top of the wheel, and out through a similar tube on the other side. Velvet bags attached by rubber bands to these tubes make a light-tight connection with corresponding short tubes on the neighbouring parts of the spectro-photometer. The instrument is provided with a handle B, fixed to the pivot P, on which the observer keeps his hand for the purpose of altering the azimuth of the hoop, and thus changing the intensity of the beam of light.

The self-recording device consists of an arm CE, rigidly fixed to the pivot P, and therefore swinging round with the hoop, which has a spear F, whose point can be made to puncture the strip of paper GG, by pulling the trigger L on the handle. This strip is 13″ long and $2\frac{1}{2}$″ high, and is kept in position by the wide rubber band seen at the top, on a cork-faced brass piece, bent into a circular arc having a radius of $6\frac{1}{2}$″, and which subtends an angle of 120° at its centre. This arrangement for holding the strip is satisfactory in practice, and makes the changing of it an easy matter. After use, the strip is a permanent record of the intensities of the light transmitted by the instrument at the moments when the observer pulled the trigger Z. [By marking off a suitable scale on a transparent strip of celluloid, it is possible to read off without any calculation opposite the punctures on each strip of paper the corresponding percentages of light transmitted.] In order to avoid the possibility that a second puncture might be made by the spear at a place already occupied by a first, the return backwards of the spear is made to produce each time a slight permanent elevation of its point, so that the successive punctures take place higher and higher up on the face of the strip of paper. The mechanism which effects this is as follows: The rear end of the spear is pivoted, and the point end is weighted and suspended from a cord which passes over a pulley above and is then led back to, and wound round, a small drum (not shown) in the part M. This drum is fixed on the same axle as a toothed wheel, which, by means of a ratchet attached to the spear, is made to revolve slightly at every retraction of the latter. Any lateral motion of the spear, which would of course impair the accuracy of its record, is prevented by means of a guide on each side of its point.

In a former paper * I indicated the advantages of using self-recording apparatus in connection with ocular photometric work. Briefly recapitulated, these are: (1) saving of labour and of eye-strain, (2) elimination of possible errors of reading, (3) great saving of time, (4) the obtaining of a permanent record, and (5) elimination of personal bias.

I am glad to take this opportunity of expressing my thanks to the Trustees of the Moray Fund for a grant to defray the cost of the apparatus, and also to the Carnegie Trust for a subsequent small grant which enabled me to carry out some necessary calibrations.

Summary.

This paper describes a new form of rotating photometric "sector," capable of very accurate adjustment to any desired light-transmission value

* *Proc. Roy. Soc. Edin.*, vol. xxix. p. 176, 1909.

while running, and possessed of the great advantage of having its rotating part in *one* piece only.

A mathematical discussion of this new type of apparatus follows, in which are deduced formulæ for the intensities of the light transmitted under different conditions, and for the greatest width of the beam of light that can be employed; and a graphical tabulation of the values of these formulæ in different cases is provided.

The mounting and details of an actual instrument are described; together with an additional mechanism for the purpose of automatically recording the photometric measurements obtained.

(*Issued separately October* 24, 1911.)

OBITUARY NOTICES.

George Barclay. By Dr George A. Berry.

(Read March 20, 1911.)

George Barclay was born in 1820 and died on November 24, 1910, having nearly completed the first half of his ninety-first year. For the last seventy years his associations had been entirely with Edinburgh, where he came to reside from his native Aberdeenshire.

In Aberdeenshire the name of Barclay is one of the oldest in the county. George Barclay's grandfather was the lineal representative of the Barclays of Tollie. His father, Dr George Barclay, attained great distinction in his profession. Though only twenty-seven at the time of his death, which occurred some months before his son George's birth, he was the recognised authority on surgery in the north of Scotland, and the first to occupy the post of Lecturer on Surgery in the Aberdeen University. William Macgillivray, the naturalist, published for private circulation a long elegiac poem on the death of his friend and teacher, the cutting short of whose brilliant career was looked upon in Aberdeen as a public calamity.

George Barclay retained a vivid recollection of the old customs and school fights at the ancient Grammar School of Aberdeen. There he was a pupil of the famous Melvin and a class-fellow of the late William Garden Blaikie. With Blaikie he not only divided the first Greek prize and won the coveted "silver pen," but shared the distinction of obtaining the first place on leaving Marischal College with its degree of M.A.

In 1848, after some years spent in Belgium and Germany, and before settling down as a partner in the business firm of his uncle in Leith, Barclay undertook a journey to the Holy Land—somewhat of an enterprise at that time, and, as his experience showed, attended with some risk. The homeward journey led him through Rome, where in those turbulent days—the year of the revolution—he narrowly escaped imprisonment, or worse, through the prank of a companion, who set the great bell of the Capitol tolling, a bell which was never heard except on occasion of great public emergency. He often subsequently visited Rome, in whose historical and artistic associations he, as an accomplished linguist and scholar, took the deepest interest. Equally at home in France and Germany, his sojourns abroad were enlivened by many interesting companionships among men of culture of his own and other nationalities, who were attracted by his broad-minded views and the charm of his conversation.

Barclay combined with his knowledge of languages a high critical appreciation of good literature. A great reader and endowed with a retentive memory, he enjoyed the friendship of many literary men. His interests, however, were not confined to *belles-lettres*. There were few sciences of which he had not some grasp of an order more intimate than what could be called merely popular, and amongst whose votaries he could not count friends—Janssen the distinguished French astronomer, Agassiz, Tait, Wyville Thomson, and many other fellows of the Royal Society. One great and constant interest to him was the work of the late Sir Wyville Thomson and Sir John Murray in connection with the *Challenger* expedition. Professor Tait dedicated to him, along with the late Thomas Stevenson, his book on *Recent Advances in Physical Science*, which was published in recognition of their wish to have the latest theories on that subject disseminated more widely by being presented in a popular form, and so brought within the reach of those who did not lay claim to be scientific men. Barclay's attitude to science may be gathered from a characteristic rejoinder of Tait's, who, when somebody asked him if Mr Barclay was a man of science, replied, "Mr Barclay is *not* a *man of science*; he *is* a man *intelligently interested in science*." His connection with the Royal Society arose partly from this "intelligent interest" in science and partly from his special taste for marine zoology, to which for some years he devoted much of his spare time. This pursuit, which entailed much work with the microscope and the making of very delicate drawings, he was reluctantly obliged to give up, as it led to an affection of sight which eventually caused the total loss of one eye.

Barclay enjoyed for over thirty years the closest friendship with Dr John Brown, an intimacy which brought him into contact with other famous writers. Foremost among these were Thackeray, whose outstanding genius he had recognised from the first, and Ruskin. In the note to Dr Brown's paper on "Thackeray's Death" it is George Barclay who is referred to as the friend with whose help the eighty half-crowns were collected to buy the silver Punch inkstand inscribed "Grati et gratæ Edinensis" which Thackeray said was his very first testimonial. He, too, it was who brought to his friend's notice the brief story of a child's life which Dr Brown amplified and embellished in the exquisite story of *Pet Marjory.*

During his business connection with Leith, George Barclay's name was associated with all that was most honourable and public-spirited.

He was chosen as a representative of the Treasury on the Harbour and Dock Commission under its old constitution, and was a governor of Watt's Hospital.

In Edinburgh he held, among other important directorships, that of the Bank of Scotland; and on the death of the manager, David Davidson—a Fellow of the Royal Society—he contributed an obituary notice of him to the Society's *Proceedings.*

As a citizen of Edinburgh he never spared himself time or trouble for the common good. Reference need only be made here to one of the many charities with which he was associated. In the minutes of our Sick Children's Hospital his part in its creation is thus recorded:

"When in 1859 a scheme was set on foot for the founding of a Sick Children's Hospital no one threw himself more earnestly into the proposal than Mr Barclay. He supported the claims of such an hospital to sympathy and support by admirable letters to the press, and by a very handsome donation he became its *first contributor.*"

In 1850 he married his cousin Elizabeth Berry, a woman of strong character and boundless hospitality, who predeceased him in 1896.

To the last day of his life George Barclay retained his balanced judgment, his warmth of feeling, and his keen sense of humour, even as he retained unaltered the virile ring of his sympathetic voice.

Those who knew him will remember him as a delightful companion, a wise counsellor, a loyal friend, a just man.

The late Dr Alexander Bruce, Edinburgh. By Dr J. H. Harvey Pirie.

(Read November 6, 1911.)

By the death of Alexander Bruce, M.A., M.D., LL.D., F.R.C.P.E., F.R.S.E., of 8 Ainslie Place, Edinburgh, on June 4th, at the comparatively early age of fifty-six, the Edinburgh Medical School has lost one of its most prominent teachers, and Great Britain one of her most brilliant investigators in the domain of diseases of the nervous system. Born in East Aberdeenshire in 1854, Dr Bruce received his early education in Aberdeen. Entering Aberdeen University as first bursar at the early age of sixteen, he had a distinguished career in the Faculty of Arts, graduating with first-class honours in classics, and gaining the Simpson Prize for Greek, the Seafield Medal in Latin, and the Town Council gold medal as the best student of his year. He then proceeded to Edinburgh University, where he had an equally brilliant undergraduate medical career, gaining, amongst other awards, the Leckie-MacTier Fellowship, and again carrying off the blue ribbon of his year, in the shape of the Ettles Scholarship, when he graduated M.B., C.M., with first-class honours, in 1879. Subsequent to graduation he held resident posts in the Royal Infirmary, Edinburgh, and at West Riding Asylum, Yorkshire, and studied abroad for a period at Paris, Frankfort, Heidelberg, and Vienna. Returning to Edinburgh, he acted for a time as assistant to the late Dr Argyll Robertson, but soon gave up this post to devote his energies to a growing general practice, and to original research in what was to become his life's work—the structure and diseases of the nervous system. A firm believer in a thorough knowledge of pathology as a basis for medical research, he taught pathology for a time in the extra-mural medical school, and acted as pathologist successively to the Royal Hospital for Sick Children, the Royal Infirmary, and the Longmore Hospital for Incurables. Later he relinquished the teaching of pathology for that of the practice of medicine, and at the time of his death was one of the physicians and lecturers on clinical medicine in the Royal Infirmary. Some ten years ago he gave up general practice and restricted himself to consulting work, more particularly in connection with nervous diseases, in which branch of medical science he was by this time a well-known authority, and soon acquired wide recognition as a consultant. An indefatigable worker, he found time, in addition to the cares of teaching and practice, for much original research, the results of which are embodied in many contributions to medical literature. Of his more important works may be mentioned the *Illustrations of the Mid and Hind Brain,* a work which was an elaboration of his M.D. thesis for which he received a University gold medal, and which was for many years a standard work of reference on the anatomy of that part of the brain; his *Topographical Atlas of the Spinal Cord*; and the paper entitled "Distribution of the Cells in the Intermedio-lateral Tract of the Spinal Cord," published in the *Transactions* of the Royal Society of Edinburgh, and for which he was awarded the Keith Prize by the Society in 1907. In 1903 he founded, and has since edited, the *Review of Neurology and Psychiatry,* a monthly journal devoted to nervous and mental diseases, which now has a wide circulation both in this country and abroad. Dr Bruce was an excellent linguist and a frequent attender at the meetings of various medical societies and congresses. He had the distinction of being a corresponding member of the Neurological Society of Paris, and he had translated from German Thoma's *Manual of Pathology,* and shortly before his death had just published in two large volumes a translation of Oppenheim's celebrated *Text-book of Nervous Diseases.* Amongst other honours it may be mentioned that he was an original member of the Neurological Society of the United Kingdom and was on the editorial staff of its magazine, *Brain*; that some years ago he was the Morrison Lecturer at the Royal College of Physicians, Edinburgh; and that in 1909 the University of Aberdeen conferred on him the honorary degree of LL.D. in recognition of his eminence as a teacher, physician, and scientist. To his patients, both in hospital and in private, he was ever the beloved physician; his services were much in demand, and to all he gave of his best ungrudgingly. Many of his patients were of the type to whom the personal element in the physician is everything, and there can be little doubt that the constant strain of his practice, combined with so much teaching and literary work, was responsible for his untimely break-down and death. As a colleague he enjoyed the highest respect of his professional brethren; and to those who were privileged to know him intimately there was revealed, in addition to his uprightness of character and general charm of manner, a great depth of quiet humour and a tenderness which greatly endeared him and make his loss deeply felt.

The Rev. Robert Flint (ob. Nov. 25, 1910, æt. 73).
By Rev. Bruce M'Ewen, D.Phil.

(Read November 6, 1911.)

Both for his pre-eminent merit as a theologian and for long and faithful services in this Society a record is here made of the life of Professor Flint. That life was so exclusively, so whole-heartedly, and so successfully devoted to laborious study that little time or energy was left for other interests. Nature had perhaps unfitted him for taking any active part in ecclesiastical or political affairs: he certainly shrank from prominence in academic government, and steadily avoided many public distractions that offered themselves to one in his position; and therefore the fact is all the more notable of his having been a diligent Councillor of the Society for twenty-two years in succession, and a Vice-President during sixteen of them.

The future Professor—almost forty years of his life were passed in University Chairs at St Andrews and Edinburgh—was born of humble parents, near Dumfries, in 1837. Very soon the family removed to Moffat, and later to Glasgow, it so happening that in each place only a wayside school was available for his early education. He entered the University of Glasgow in 1852, and completed his Divinity course with the highest distinction at the age of twenty. While awaiting licence to preach, he served as a lay missionary under the "Elders' Association" of Glasgow—a distracting winter; but out of it sprang a lifelong friendship with the late Dr James A. Campbell of Stracathro. On attaining his majority he became assistant to Norman Macleod, minister of the Barony, and was minister of the East Church of Aberdeen from 1859 to 1861. A call to the quiet country parish of Kilconquhar proved irresistible after his experiences of city life, and there, during the next three years, he laid the foundations of his great teaching fame by a strenuous course of omnivorous reading. Academic recognition of his talents came speedily, and in 1864 he succeeded Professor Ferrier in the Chair of Moral Philosophy and Political Economy at St Andrews, defeating so strong a candidate as the late T. H. Green of Oxford. He was at once recognised as a power in the University; he was wonderfully popular among his students, and a good standing in his class was specially prized. It is remembered of him that he never thought it necessary to barricade his house against the boisterous revelry of Kate Kennedy's Day, and in the troubles that followed the suppression of that noisy festival he had kept serenely apart from the assertion of authority, losing neither popularity nor dignity thereby. He is reported to have said that neither principle nor want of principle was involved.

The method of teaching adopted in his class was mainly historical, analytical, and critical. Each lecture began with the giving out of a series of "headings," usually about a dozen in number. These were dictated slowly, and formed a complete summary of the subject of the day: then followed the lecture proper, delivered in an even, unpretentious manner, with many naïve provincialisms, and now and then a sly pun or quaint conceit to enliven the presentation. Whether teaching Ethics or Political Economy or Theology, he retained this method to the end of his professorial career, attracting admirers by the bold yet carefully planned exposition of his reasonings, and also by the charm of his strong personality and transparent character, rather than by any arts of rhetoric.

None of Professor Flint's published works reproduce the matter of his St Andrews lectures. Those on Moral Philosophy were highly thought of, giving the judgment on centuries of human speculation by a capable, unprejudiced, and sagacious mind. A short course on Political Economy was considered at the time to be less weighty, and gave no indication of that deeper interest in Sociology which successfully occupied a much later phase of his intellectual activity.

In 1876 he was transferred to the Chair of Divinity in the University of Edinburgh, repeating the experience of Dr Thomas Chalmers, who was his predecessor in both Chairs half a century earlier. His connection with this Society dates from his election as a Fellow in 1880. He took his place at the Council table in November 1883, and was Vice-President of the Society for three periods—from 1886 to 1892, from 1894 to 1899, and finally from 1902 to 1906. On this last occasion he did not complete the normal six years of office. He had already resigned his active work as Professor of Divinity; his home was out of Edinburgh; old friends, and among them Professor Tait, had passed away; and with regret he expressed his desire to be relieved from any further duties on the Royal Society Council.

Professor Flint did his full share of presiding at the meetings, and on two occasions, in 1887 and in 1898, he delivered the opening address from the chair, giving a clear account of the work of the preceding session and a brief obituary of the Fellows whom the Society had lost during the year. In addition to his duty as Vice-President, he attended all the meetings most faithfully, taking part in discussions when these fell within the range of his own special knowledge. Flint and Tait had a great regard, indeed affection, for each other. Both were strong intellectual men, whose opinions were always clear-cut and reasonable, and both held in a remarkable degree the confidence of their colleagues on the Council Board. In common they possessed the same broad sympathy with all kinds of human knowledge, and each too sought, in a very special way of his own, to combine the theological and the scientific view of the universe.

At home Professor Flint was a student, almost a recluse, spending every available moment in his library. He composed and wrote slowly, and with such deliberate precision that it would be nigh impossible to detect any inconsistency or serious ambiguity in his published pages. His literary output was considerable, ranging over a wide field where accuracy and impartiality can be obtained only by independent and exhaustive inquiry, and yet one somehow never expected Flint to have anything to retract.

The little book, *Christ's Kingdom on Earth* (1865), was intended to form the portico to a much larger edifice (of Biblical Theology) still uncompleted. *The Philosophy of History in France and Germany* (1874) was a fine example of pioneer work in a department little cultivated in this country. Its accurate scholarship and power of independent criticism established the author's reputation on a firm basis. It was translated into French, and led to his election as a Corresponding Member of the Institute of France and as an Honorary Member of the Royal Society of Palermo. This vein of research was continued to some extent in the later volume, *Historical Philosophy in France* (1894); but the results of subsequent labours upon the same subject in Italy and England, though believed to exist in manuscript in a fairly complete form, have not yet been given to the public.

Professor Flint was Baird Lecturer in 1876–77, Stone Lecturer at Princeton, U.S.A., 1880, and Croall Lecturer, 1887–88. The substance of his Baird Lectures was incorporated in the two widely circulated books, *Theism* (1877) and *Anti-Theistic Theories* (1879); while the Croall Lectures, very much expanded, appeared as a systematic treatise on *Agnosticism* (1902). To Blackwood's "Philosophical Classics" he contributed a volume on *Vico* (1884), and to the ninth edition of the *Encyclopædia Britannica* two masterly articles on "Theism" and "Theology." A large work on *Socialism* (1894) was a breaking of new ground, as was also another, *Philosophy as Scientia Scientiarum* (1904); while a variety of occasional papers appeared as *Sermons and Addresses* (1899) and *On Theological, Biblical, and other Subjects* (1905). All are marked by the same thoroughness, clearness of expression, insight, and intellectual grasp. The two Baird Lectures, although of a *quasi*-popular nature, contain much that was characteristic of the lecturer's own system of theology, and by them he will probably be judged in the future.

As already mentioned, Professor Flint retired from the active duties of University teaching in 1903, and in announcing his intention he had said that his chief motive was to gain more time for writing; but neither health nor years were given him. The delivery of his Gifford Lectures for 1908–9 was eagerly awaited by many who believed that in them he would finally outline a Philosophy of Religion adumbrated for years, and find satisfaction at last for that passion for completeness which delayed this and many another of the Professor's literary projects. But long before the lectures were due bodily weakness suddenly intervened, and all the hopes thus formed were disappointed.

Both in Theology and Philosophy Professor Flint definitely adopted the traditional British or, we may say, Scottish position, and was what would have been called an orthodox man. In Ethics an Intuitionist, in Theology he started from a perfect trust in reason and its powers, and in every argument chose to be dominated by the necessity for purely intellectual conviction, insisting upon it with a vigour that distinguishes him from any contemporary thinker. In justifying this initial trust in reason, he criticised adversely and almost with passion every other avenue that professes to lead to truth. He rejected utterly the prevalent tendency of his own day to take refuge from the agnostic verdict against reason in philosophies of faith or feeling, will or conscience. And yet in all Professor Flint's published works the historical, critical, and controversial elements prevail over the positive establishment of his own conservative position. It was from the Chair of Divinity that he specially devoted himself to the task of vindicating the older methods, which were, in his opinion, being needlessly thrust aside; and it was only in his class lectures, unfortunately as yet unpublished, that his constructive genius found full scope. In them he often sounded the characteristic note of all-comprehensiveness. He conceived the task of Systematic Theology to be to collect materials from every possible source, from Natural Theology and from the other religions of the world as well as from the springs of Christian revelation; he regarded every movement of speculative and scientific thought as combining to clear the ground, and there is no existing treatise on Christian doctrine which takes so wide a view or contains so much supplementary matter. He stood for a strong and vigorous and exacting type of thought, and stood for it almost alone.

Professor Flint held the degrees of Doctor of Divinity and Doctor of Laws from more than one university, and the many honours bestowed upon him both at home and abroad testify to his international reputation. Those who knew himself can but add their testimony to the singular moral dignity, the kindly chivalry, and the absolute sincerity that marked his life.

The Very Rev. James MacGregor, D.D., one of His Majesty's Chaplains.

(Read November 6, 1911.)

Dr MacGregor was born near Scone in 1832. Entering the University of St Andrews when fifteen years of age, he studied there till 1855; and such was the distinction of his career, that in the same year he was appointed to the important benefice of the High Church of Paisley. Afterwards in Glasgow, and later in Edinburgh, he devoted himself with characteristic energy and eloquence to the work of a parish minister, and it is, perhaps, as Minister of the Parish Church of St Cuthbert in this city that he will be longest remembered. In 1891 the General Assembly of the Church of Scotland called him to the Moderator's chair, in recognition, not only of his work as a preacher and pastor, but also as a philanthropist. He was elected Fellow of the Society in 1886. He died on November 25, 1910. A scholarly man and a great preacher, whose eloquence never wearied in pulpit or on platform in the cause of the moral welfare of his country, his memory is at once a pride and an incentive.

Lord M'Laren. By Professor C. G. Knott.

(Read December 4, 1911.)

John M'Laren, Q.C., LL.D. (Edin., Glas., Aber.), Lord of Session from 1881, was born in Edinburgh on April 17, 1831. He was the eldest son of Duncan M'Laren, one of the most prominent figures in political and municipal life of Edinburgh in his day and generation. He was a delicate boy, and his life was in serious danger through a severe illness which followed a bad wetting he experienced when about the age of twelve. For years he lived much abroad, visiting such places as Madeira, Jamaica, and Algeria. He had a great thirst for useful information and possessed a retentive memory, so that his broken school life did not impede his career when he passed into college and began to study for the Law.

In 1856 he was admitted a member of the Faculty of Advocates, and, chiefly by his literary work, soon established a reputation as an able and erudite lawyer. His treatises on *Trusts* and *Wills* appeared respectively in 1863 and 1868; and the third edition of the combined work in 1894 is regarded as the leading authority on these subjects north of the Tweed. The seventh edition of Bell's *Commentaries* was edited by him in 1870, and holds a very high place in the estimate of the legal profession. It is a recognised piece of legal etiquette that the opinions or verdicts of a living judge cannot be quoted in court as being of any authority; and when an advocate had to refer to Lord M'Laren's writings on Trusts and Wills, as was not unusual before Lord M'Laren himself, it had to be done in a mysterious way, without explicit mention of the author, but yet so as to leave no doubt as to the "authority" who was being appealed to.

Lord M'Laren was married on December 14, 1868, to a daughter of H. L. Schwabe, of Glasgow. In 1869 he was appointed Sheriff of Chancery, resigning the appointment in 1880 when he entered public political life.

In politics Lord M'Laren was a Liberal, and came to the front in 1880, at the time of Gladstone's famous Midlothian campaign. He was elected Member of Parliament for the Wigtown Burghs, and on the formation of Gladstone's Government became Lord Advocate for Scotland. At the succeeding by-election, necessitated by his accepting a Crown appointment, he lost his seat, and it was not till January of 1881 that he regained his place in Parliament as one of the members for Edinburgh. In August of the same year, however, he retired from parliamentary life, and accepted the judgeship on the Scottish Bench, which he adorned to the end of his life. He made an admirable judge, his opinions and judgments being models of lucid exposition.

Throughout his life Lord M'Laren took a keen interest in science, especially in botany and astronomy. He became a Fellow of the Royal Society of Edinburgh as early as 1869, and in 1883 was elected a Member of Council. From that date to the year before his death he served faithfully on the Council, filling three terms of office as Vice-President, namely, from 1885 to 1891, from 1893 to 1899, and from 1901 to 1906. He took his full share in presiding at meetings both of the Council and of the Society, acting in this capacity on more occasions than have fallen to the lot of any other Fellow. He was also of invaluable service to the Society when delicate questions arose requiring careful deliberation and knowledge of affairs. His skill in drafting documents was in continual request, and several of the more recent modifications in the rules of the Society were framed by him. In the negotiations which preceded the transference of the Society from its original rooms in the Royal Institution in Princes Street to its present abode in George Street, Lord M'Laren's advice and support formed an important factor.

Outside the Royal Society, Lord M'Laren's scientific activities found expression in his interest in the Royal Observatory on Blackford Hill, the Scottish Meteorological Society, of which he was President for several years, and the Ben Nevis Observatory, of which he was a Director. In the organisation of the Royal Observatory he was closely associated with Lord Lindsay and Professor Copeland; and in helping to carry out the work of the Ben Nevis Observatory, he ably seconded the efforts of Dr Buchan, Sir John Murray, and others in keeping the great experiment before the minds of his countrymen.

Although never of a robust constitution, Lord M'Laren continued to do his work till within a year of his death, which took place at Brighton on April 6, 1910. To the end he remained in touch with the affairs of the Royal Society and of the Royal Society Club, of which he had been a member since 1883. He was a close friend of Lord Kelvin, who was frequently his guest when Royal Society business brought the great natural philosopher to Edinburgh.

Lord M'Laren was an amateur of science in the highest sense of the term, and devoted himself whole-heartedly to the advancement of the varied interests of the Royal Society.

As will be seen from the list of published papers given below, Lord M'Laren was an analyst of considerable skill, as well as a student of astronomical science on its observational side.

The following is the list of papers published in the *Transactions* and *Proceedings* of the Royal Society of Edinburgh:—

1. Tables for Facilitating the Computation of Differential Refraction in Position Angle and Distance. 1886. *Trans.*, vol. xxxiii.

The Tables contain the values of two quantities calculated for the parallel 55° 56′ and 57° 30′, and for each interval of two degrees of declination from 40° north to 90°.

2. On Systems of Solutions of Homogeneous and Central Equations of the nth Degree and of Two or More Variables; with a Discussion of the Loci of such Equations. 1888. *Trans.*, vol. xxxv.

The underlying idea of this elaborate paper of 55 pages is to find exact solutions in equations between variables, so that the precise form of plane curves and contours of surfaces may be determined.

3. On the Four Surfaces of an Aplanatic Objective. 1888. *Proc.*, vol. xv.

4. On the Solution of the Three-Term Numerical Equation of the nth Degree. 1890. *Proc.*, vol. xvii.

The solutions are obtained in an interesting manner by use of what are known as addition and subtraction logarithms.

5. On the Reflexion-Caustics of Symmetrical Curves. 1890. *Proc.*, vol. xvii.

6. Equation of the Glissette of the Two-Term Oval $x^n/a^n + y^n/b^n = 1$ and Cognate Curves. 1891. *Proc.*, vol. xviii.

7. On the Eliminant of the Glissette Equations of the Ellipse Glissette. 1892. *Proc.*, vol. xix.

These two papers are a following up of Professor Tait's glissette investigations.

8. A New Solution of Sylvester's Problem of the Ternary Equations. 1893. *Proc.*, vol. xix.

9. Elimination of Powers of Sines and Cosines between Two Equations. 1893. *Proc.*, vol. xx.

This is an extension of the process of No. 7 above.

10. Symmetrical Solution of the Ellipse-Glissette Elimination Problem. 1899. *Proc.*, vol. xxii.

In this last attack on a difficult problem of elimination the author gives the general eliminant as expressed in the form of a single symmetrical bordered determinant.

11. Opening Address, Session 1901–2. *Proc.*, vol. xxiv.

Contains an estimate of the work of Professor Tait.

APPENDIX.

CONTENTS.

PROCEEDINGS OF THE STATUTORY GENERAL MEETING.

The 128th Session, 1910–1911.

At the Annual Statutory Meeting of the Royal Society of Edinburgh, held in the Society's Lecture Hall, 24 George Street, on Monday, 24th October 1910, at 3 p.m.

Dr Horne, F.R.S., Vice-President, in the Chair,
the Minutes of last Statutory Meeting, 25th October 1909, were read, approved, and signed.

On the motion of Dr Knott, seconded by Dr Kidston, Prof. F. W. Dyson and Mr J. W. Inglis were appointed Scrutineers, and the ballot for the New Council commenced.

The Treasurer's Accounts for the past year were submitted. These, with the Auditor's Report, were read and approved.

The Scrutineers reported that the following New Council had been duly elected:—

Principal Sir Wm. Turner, K.C.B., D.C.L., F.R.S., President.
Professor Crum Brown, M.D., LL.D., F.R.S.,
Professor J. C. Ewart, M.D., F.R.S.,
John Horne, LL.D., F.R.S., F.G.S.,
James Burgess, C.I.E., LL.D., M.R.A.S.,
Professor T. Hudson Beare, M.Inst.C.E.,
Professor F. O. Bower, M.A., D.Sc., F.R.S., } Vice-Presidents.
Professor George Chrystal, LL.D., General Secretary.
Cargill G. Knott, D.Sc.,
Robert Kidston, LL.D., F.R.S., F.G.S., } Secretaries to Ordinary Meetings.
James Currie, M.A., Treasurer.
John S. Black, M.A., LL.D., Curator of Library and Museum.

ORDINARY MEMBERS OF COUNCIL.

Professor J. W. Gregory, D.Sc., F.R.S.
A. P. Laurie, M.A., D.Sc.
Professor Wm. Peddie, D.Sc.
Professor H. M. Macdonald, M.A., F.R.S.
Professor D. Noël Paton, M.D., B.Sc., F.R.C.P.E.
William S. Bruce, LL.D.
Professor F. G. Baily, M.A.
J. G. Bartholomew, LL.D., F.R.G.S.
Ramsay H. Traquair, M.D., LL.D., F.R.S.
Professor James Walker, D.Sc., Ph.D., LL.D., F.R.S.
Professor Arthur Robinson, M.D., M.R.C.S.
W. S. M'Cormick, M.A., LL.D.

On the motion of Professor Chrystal, thanks were voted to the Scrutineers.

On the motion of Professor Chrystal, thanks were voted to the Auditors, and they were reappointed.

On the motion of Professor Chrystal, thanks were voted to the Treasurer for his admirable management of the Finances of the Society during the past year, which has been an important one in the Society's history.

PROCEEDINGS OF THE ORDINARY MEETINGS,
Session 1910–1911.

FIRST ORDINARY MEETING.

Monday, 7th November 1910, *at* 8 *p.m.*

Professor Crum Brown, LL.D., F.R.S., Vice-President, in the Chair.

The following communications were read :—

1. Thermo-Electric Diagram from $-200°$ C. to $+100°$ C., deduced from the Observations of Professors Dewar and Fleming. By J. D. Hamilton Dickson, M.A. (*With Lantern Illustrations.*) *Trans.*, vol. xlvii. pp. 737–791.

2. The Efficiency of Metallic Filament Lamps. By Dr R. A. Houstoun. Communicated by Professor A. Gray, F.R.S. (*With Lantern Illustrations.*) *Proc.*, vol. xxx. pp. 555–561.

3. A Dynamic Method for Measuring Vapour Pressures, with its Application to Benzene and Ammonium Chloride. By Professor Alex. Smith and A. W. C. Menzies. *Proc.*, vol. xxxi. pp. 179–182.

4. A Quantitative Study of the Constitution of Calomel Vapour. By Professor Alex. Smith and A. W. C. Menzies. *Proc.*, vol. xxxi. pp. 183–185.

The following, nominated for Honorary Fellowship, were balloted for and duly declared elected :—

As British Honorary Fellows :—

1. James George Frazer, D.C.L., LL.D., Litt.D., F.B.A., Fellow of Trinity College, Cambridge, Professor of Social Anthropology in the University of Liverpool.
2. Sir Joseph Larmor, D.Sc., LL.D., D.C.L., F.R.S., Lucasian Professor of Mathematics in the University of Cambridge, Secretary of the Royal Society.
3. Alfred Russel Wallace, O.M., LL.D., D.C.L., F.R.S.

As Foreign Honorary Fellows :—

1. Hugo de Vries, Sc.D., LL.D., Professor of Anatomy and Physiology of Plants in the University of Amsterdam.
2. F. A. Forel, Naturalist and Limnographer, Morges, Switzerland.
3. Karl F. von Goebel, D.Sc., Professor of Botany in the University of Munich.
4. Jacobus Cornelius Kapteyn, Ph.D., Professor of Astronomy in the University of Groningen.
5. Elie Metchnikoff, Vice-Director of the Pasteur Institute, Member of the Academy of Medicine, Paris.
6. Albert Abraham Michelson, Ph.D., Sc.D., LL.D., F.R.S., Professor and Head of the Department of Physics in the University of Chicago.
7. Wilhelm Ostwald, D.M., D.Sc., LL.D., Emeritus Professor of Physical Chemistry in the University of Leipsic.
8. Frederick Ward Putnam, Professor of American Archæology and Ethnology in Harvard University.
9. August F. L. Weismann, D.M., Ph.D., D.Bot., D.C.L., LL.D., Professor of Zoology in the University of Freiburg (Baden).

The Rev. R. S. Calderwood signed the Roll and was duly admitted a Fellow of the Society.

SECOND ORDINARY MEETING.

Monday, 21st November 1910, *at* 4.30 *p.m.*

Professor J. C. Ewart, F.R.S., Vice-President, in the Chair.

The following communications were read :—

1. The Sex and Age Incidence of Mortality from Pulmonary Tuberculosis in Scotland, and in its Groups of Registration Districts since 1861. By Professor C. Hunter Stewart, D.Sc., on which a special discussion took place. *Proc.*, vol. xxxi. pp. 352–373.

2. The Nemertines of Millport and its Vicinity. By Dr J. STEPHENSON. Communicated by Professor D'ARCY W. THOMPSON, C.B. *Trans.*, vol. xlviii. pp. 1–29.

3. Some Littoral Oligochæta of the Clyde. By Dr J. STEPHENSON. Communicated by Professor D'ARCY W. THOMPSON, C.B. *Trans.*, vol. xlviii. pp. 31–65.

Dr G. W. LEE signed the Roll and was duly admitted a Fellow of the Society.

THIRD ORDINARY MEETING.

Monday, 5th December 1910, *at* 4.30 *p.m.*

Dr James Burgess, C.I.E., Vice-President, in the Chair.

The following communications were read :—

Temperature Seiche, The. Part I.—Observations in the Madüsee, Pomerania. Part II.—Hydrodynamical Theory of Temperature Oscillations in Lakes. Part III.—Calculation of the Period of the Temperature Seiche in the Madüsee, by E. M. WEDDERBURN. Part IV.—Experimental Verification of the Hydrodynamical Theory of Temperature Seiches, by E. M. WEDDERBURN and A. M. WILLIAMS. (*With Experimental Illustrations.*) (*Abstract.*) *Proc.*, vol. xxxi. pp. 257–258. *Trans.*, vol. xlvii. pp. 619–642.

Observations on the Body Temperature of the Domestic Fowl (*Gallus gallus*) during Incubation. By Dr SUTHERLAND SIMPSON. Communicated by Prof. E. A. SCHÄFER. *Trans.*, vol. xlvii. pp. 605–617.

Mr BRUCE M'GREGOR GRAY signed the Roll and was duly admitted a Fellow of the Society.

FOURTH ORDINARY MEETING.

Monday, 19th December 1910, *at* 4.30 *p.m.*

Professor F. O. Bower, F.R.S., Vice-President, in the Chair.

The following communications were read :—

1. Obituary Notice of the late Professor D. J. Cunningham. By Dr C. G. KNOTT. *Proc.*, vol. xxx. pp. 569–579.

2. The Jurassic Flora of Sutherland. By Professor A. C. SEWARD, F.R.S. Communicated by Dr R. KIDSTON, F.R.S. *Trans.*, vol. xlvii. pp. 643–709.

3. Phase of the Nucleus known as Synapsis. By Dr A. ANSTRUTHER LAWSON. *Trans.*, vol. xlvii. pp. 591–604.

4. The Sectional Anatomy of the Head of the Australian Aboriginal: A Contribution to the Subject of Racial Anatomy. By Professor R. J. A. BERRY. *Proc.*, vol. xxxi. pp. 604–626.

The following Candidates for Fellowship were balloted for and declared duly elected :—ANDREW WILSON, A.M.I.C.E., ALEXANDER C. COWAN, GILBERT MACINTYRE HUNTER, A.M.I.C.E., GREGORY MACALISTER MATHEWS, F.L.S., FRANK WYVILLE THOMSON, M.A., M.B., C.M., D.P.H., WM. INGLIS CLARK, D.Sc., JOHN WM. M'INTOSH, A.R.C.V.S., HENRY WALKER, M.A., D.Sc., ARCHIBALD CAMPBELL ADAMS, A.M.I.M.E., and RURIC WHITEHEAD WRIGLEY, B.A. (Cantab.).

FIFTH ORDINARY MEETING.

Monday, 9th January 1911, at 8 *p.m.*

Professor T. Hudson Beare, Memb.Inst.C.E., Vice-President, in the Chair.

The following Communications were read :—

1. A Method for Determining the Molecular Weights of Dissolved Substances by Measurement of Lowering of Vapour Pressure. By ALAN W. C. MENZIES. (*Abstract.*) *Proc.*, vol. xxxi. pp. 259–261.

2. Illustration of the *Modus Operandi* of the Prism. By Dr GEORGE GREEN. Communicated by Professor A. GRAY. *Proc.*, vol. xxxi. pp. 290–295.

3. The Relation of the Mono-Molecular Reaction to Life Processes and to Immunity. By Dr JOHN BROWNLEE. *Proc.*, vol. xxxi. pp. 627–634.

Mr ARCHIBALD CAMPBELL ADAMS signed the Roll, and was duly admitted a Fellow of the Society.

SIXTH ORDINARY MEETING.

Monday, 23rd January 1911, at 4.30 *p.m.*

Professor J. C. Ewart, F.R.S., Vice-President, in the Chair.

The following Communications were read :—

1. Note on an Entoproctan Polyzoon (*Barentsia benedeni*) new to the British Fauna, with remarks on Related Species. By JAMES RITCHIE, M.A., B.Sc. Communicated by Dr R. H. TRAQUAIR, F.R.S. *Trans.*, vol. xlvii. pp. 835–848.
2. The Topography of the Cortex Cerebri of the Guinea-pig. By WILLIAMINA ABEL, M.D. Communicated by Professor D. NOËL PATON. *Proc.*, vol. xxxi. pp. 397–415.
3. A Study of Artificial Pyrexia. By Mr ADAM BLACK. Communicated by Professor D. NOËL PATON. *Proc.*, vol. xxxi. pp. 333–341.
4. The Independence of the Peripheral Sensory Neurone in view of the Results of Experimental Section of the Optic Nerve in the Rabbit. By JANIE HAMILTON M'ILROY, M.A., B.Sc., M.B. Communicated by Professor D. NOËL PATON. (*Lantern Demonstration.*) *Proc.*, vol. xxxi. pp. 349–351.

The following Gentlemen signed the Roll and were duly admitted Fellows of the Society :— Professor JAMES MACKINNON, WILLIAM INGLIS CLARK, D.Sc., and RURIC WHITEHEAD WRIGLEY B.A.

SEVENTH ORDINARY MEETING.

Monday, 6th February 1911, at 4.30 *p.m.*

Dr James Burgess, C.I.E., Vice-President, in the Chair.

The following Communications were read :—

1. On Fourier's Repeated Integral. By Dr W. H. YOUNG. Communicated by Professor G. A. GIBSON. *Proc.*, vol. xxxi. pp. 559–586.
2. On Sommerfeld's Form of Fourier's Repeated Integral. By Dr W. H. YOUNG. Communicated by Professor G. A. GIBSON. *Proc.*, vol. xxxi. pp. 587–603.
3. The Isopiestic Expansibility of Water at High Temperatures and Pressures. By W. WATSON, M.A., B.Sc. Communicated by Professor J. G. MACGREGOR. (*Lantern.*) *Proc.*, vol. xxxi. pp. 456–477.
4. An Investigation into the Effects of Errors in Surveying. By HENRY BRIGGS, B.Sc. Communicated by Principal A. P. LAURIE. (*Lantern.*) *Trans.*, vol. xlvii. pp. 849–877.

EIGHTH ORDINARY MEETING.

Monday, 20th February 1911, at 4.30 *p.m.*

Dr John Horne, F.R.S., Vice-President, in the Chair.

The following Communications were read :—

1. On the Determination of Small Degrees of Enzymatic Peptolysis. By DOROTHY COURT, B.Sc. Communicated by Dr E. WESTERGAARD. *Proc.*, vol. xxxi. pp. 342–348.
2. Concerning the New Genus of Iron Bacteria, "Spirophyllum ferrugineum" (Ellis). By Dr DAVID ELLIS. *Proc.*, vol. xxxi. pp. 499–504.
3. On the Temperature Coefficient of Concentration Cells, in which the same Salt is dissolved in two Different Solvents. By Principal A. P. LAURIE. *Proc.*, vol. xxxi. pp. 375–396.

The following Candidates for Fellowship were Balloted for and declared duly elected :—JAMES WATT, W.S., F.F.A., JAMES HARTLEY ASHWORTH, D.Sc., REGINALD JOHN GLADSTONE, M.D., JAMES ANDREW GUNN, M.A., M.D., JOHN ARNOLD FLEMING, F.C.S., STEPHEN SMITH, B.Sc., SUTHERLAND SIMPSON, M.D., Rev. LAUCHLAN M'LEAN WATT, B.D., and EDALJI MANEKJI MODI, F.C.S.

NINTH ORDINARY MEETING.

Monday, 6th March 1911, *at* 8 *p.m.*

Professor A. Crum Brown, LL.D., F.R.S., Vice-President, in the Chair.

The following Communications were read :—

1. On the Accuracy attainable with a Modified Form of Atwood's Machine. By J. P. DALTON, M.A., B.Sc. *Proc.*, vol. xxxi. pp. 416–423.

2. The Dissipation of Energy in Torsionally Oscillating Wires of Brass and other Materials, with the Effects produced by Change of Temperature, etc. By J. B. Ritchie, B.Sc. (*With Lantern Illustrations.*) *Proc.*, vol. xxxi. pp. 424–439.

3. An Apparatus for Inducing Fatigue in Wires by means of Repeated Extensional and Rotational Strains, with the Effects of Fatigue on the Laws of Torsional Oscillation. By J. B. Ritchie, B.Sc. (*With Lantern Illustrations.*) Above three papers communicated by Professor W. Peddie, Dundee. *Proc.*, vol. xxxi. pp. 440–447.

4. Boole's Unisignant. By Dr Thos. Muir. *Proc.*, vol. xxxi. pp. 448–455.

The following gentlemen signed the Roll, and were duly admitted Fellows of the Society:— J. Campbell Dewar, C.A., and James Watt, F.F.A.

TENTH ORDINARY MEETING.

Monday, 20th March 1911, *at* 4.30 *p.m.*

Professor T. Hudson Beare, M.Inst.C.E., Vice-President, in the Chair.

The following Communications were read:—

1. Obituary Notice of Mr George Barclay. By G. A. Berry, P.R.C.S. (Ed.). *Proc.*, vol. xxxi. p. 684.

2. Measurements on the Scattering of Light by "Ground" Glass. By Dr J. R. Milne. (*With Lantern Illustrations.*)

3. On the Influence of Temperature upon the Magnetic Properties of a Grated Series of Carbon Steels. By Margaret B. Moir, M.A., B.Sc. Communicated by James G. Gray, D.Sc. (*With Lantern Illustrations.*) *Proc.*, vol. xxxi. pp. 505–516.

4. The Pharmacological Action of Harmine. By Dr J. A. Gunn. (*With Lantern Illustrations.*) *Trans.* vol. xlviii. pp. 83–96.

The following Candidates for Fellowship were balloted for, and declared duly elected:— Charles Richard Whittaker, F.R.C.S. (Edin.).

Mr Stephen Smith and Dr James A. Gunn signed the Roll, and were duly admitted Fellows of the Society.

ELEVENTH ORDINARY MEETING.

Monday, 1st May 1911, *at* 4.30 *p.m.*

Emeritus-Professor Crum Brown, F.R.S., Vice-President, in the Chair.

The following Address was delivered:—

On Ice and its Natural History. By John Young Buchanan, M.A., F.R.S., Christ's College, Cambridge. (*With Lantern Illustrations.*) *Trans.*

Dr J. R. Ratcliffe and Mr J. A. Fleming signed the Roll, and were duly admitted Fellows of the Society.

TWELFTH ORDINARY MEETING.

Monday, 15th May 1911, *at* 4.30 *p.m.*

Sir William Turner, K.C.B., President, in the Chair.

The following Address was delivered:—

On Sand Dunes and Golf Links. By Professor F. O. Bower, F.R.S., University, Glasgow. (*With Lantern Illustrations.*)

Mr Peter Pinkerton signed the Roll, and was duly admitted a Fellow of the Society.

The following Candidates for Fellowship were balloted for, and declared duly elected:— William Guy, F.R.C.S., William Jacob Holland, LL.D. (St. And.), James A. S. Watson, B.Sc., Henry A. J. Gidney, L.M., S. Socts. Ap. London, John Smith Purdy, M.D., Adam A. Rankin, and James Ramsay Tosh, M.A.

THIRTEENTH ORDINARY MEETING.

Monday, 5th June 1911, *at* 8 *p.m.*

Dr James Burgess, C.I.E., Vice-President, in the Chair.

The following Communications were read :—

The Absorption of Light by Inorganic Salts—

1. Aqueous Solutions of Cobalt Salts in the Infra-red. By Dr R. A. Houstoun. Communicated by Professor A. Gray. *Proc.*, vol. xxxi. pp. 521–529.
2. Aqueous Solutions of Cobalt Salts in the Visible Spectrum. By Dr R. A. Houstoun and Alex. R. Brown, M.A. Communicated by Professor A. Gray. *Proc.*, vol. xxxi. pp. 530–537.
3. Aqueous Solutions of Nickel Salts in the Visible Spectrum and the Infra-red. By Dr R. A. Houstoun. Communicated by Professor A. Gray. *Proc.*, vol. xxxi. pp. 538-546.
4. Aqueous Solutions of Cobalt and Nickel Salts in the Ultra-Violet. By Dr R. A. Houstoun and John S. Anderson, M.A. Communicated by Professor A. Gray. *Proc.*, vol. xxxi. pp. 547–558.
5. The Vapour Pressure of Dry Calomel. By Professor Alexander Smith and Alan W. C. Menzies, M.A., B.Sc. (*Abstract.*) *Proc.*, vol. xxxi. pp. 519–520.

FOURTEENTH ORDINARY MEETING.

Monday, 19th June 1911, *at* 4.30 *p.m.*

Professor F. O. Bower, F.R.S., Vice-President, in the Chair.

The following Communications were read :—

1. Les Mousses de l'Expédition nationale antarctique écossaise. Par M. Jules Cardot. Présentée par Professeur Bayley Balfour, F.R.S., etc. *Trans.*, vol. xlviii. pp. 67–82.
2. On some Nuclei of Cloudy Condensation. Part II. By Dr John Aitken, F.R.S. (*With Lantern Illustrations.*) *Proc.*, vol. xxxi. pp. 478–498.
3. Nuclear Osmosis as a Factor in the Mechanism of Mitosis. By Dr A. Anstruther Lawson. *Trans.*, vol. xlviii. pp. 137–161.
4. The Rate of Multiplication of Micro-organisms: A Mathematical Study. By Captain A. G. M'Kendrick, I.M.S., and Dr M. Kesava Pai. Communicated by Dr C. G. Knott. *Proc.*, vol. xxxi. pp. 649–655.

Mr A. A. Rankin and Dr J. R. Tosh signed the Roll, and were duly admitted Fellows of the Society.

The following Candidates for Fellowship were balloted for, and declared duly elected :— Duncan M'Laren Young Sommerville, M.A., D.Sc., and David Rainy Brown.

FIFTEENTH ORDINARY MEETING.

Monday, 3rd July 1911, *at* 4.30 *p.m.*

Dr Horne, F.R.S., Vice-President, in the Chair.

The Chairman read the following Address to the King :—

TO HIS MOST EXCELLENT MAJESTY

GEORGE V.

KING AND EMPEROR.

May it please Your Majesty :—

We, your Majesty's loyal subjects, the President and Fellows of the Royal Society of Edinburgh, desire most cordially to welcome Your Majesty on Your visit to the Capital of your Ancient Kingdom of Scotland.

As a Society mainly though not wholly devoted to the advancement of Science, we thankfully recall the great services rendered by Your Majesty's illustrious Grandfather, the Prince Consort, in striving to secure adequate public recognition of the value to the Nation of scientific thought and work.

The reigns of Your Majesty's August Predecessors were characterised by remarkable progress in the various branches of human knowledge and activity; and we are confident that under Your Majesty's Rule still greater advance will be witnessed in all that makes for the welfare of the British Empire and of the many Races that own Your sway.

We would also express our cordial welcome to Her Most Gracious Majesty, Queen Mary; and would place on record our humble and earnest prayer that Your Majesties may long be spared to rule over a prosperous and contented people.

(Signed by the President and General Secretary.)

PRIZE.

The Council having awarded—

The MAKDOUGALL-BRISBANE PRIZE for the biennial period 1908-09, 1909-10 to ERNEST MACLAGAN WEDDERBURN, M.A., LL.B., for his series of papers bearing upon the Temperature Distribution in Fresh-water Lochs, and upon Seiche Phenomena which occur at the interface of two layers of different density, whether that difference be due to difference of temperature or difference of salinity, published in the *Proceedings* of the Society within the prescribed period, and also in the *Transactions* and *Proceedings* before and after that period:—this Prize was presented.

On Presenting the above prize the CHAIRMAN read the following statement:—

Having been appointed physicist to the Scottish Lake Survey under the direction of Sir John Murray and Mr Laurence Pullar, Mr E. M. Wedderburn, first in association with Mr E. R. Watson, and subsequently by himself, investigated the temperature distribution in Loch Ness in a more thorough manner than had ever before been attempted in a fresh-water loch. The results are published in the *Transactions* of the Society (vol. xlv.). The presence of a "sprungschicht," or layer through which the temperature of the lake changes at a rapid rate with depth, had already been recognised as occurring at certain periods of the year and at depths which varied with each lake and with the meteorological conditions of each year. By his observations at Loch Ness, Mr Wedderburn established the existence of a temperature or density seiche which occurred at this layer of rapid change of temperature, and showed that the period of the seiche could be roughly calculated for Loch Ness by means of an approximate formula. The temperature seiche had been discussed by Mr E. R. Watson in 1904; and ten years earlier by Professor Thoulet of Nancy; but the latter did not pursue the speculations which he had made.

In the same paper Mr Wedderburn also discussed the effect of winds on the temperature distribution in lakes, an effect which had generally been disregarded by limnologists, although Sir John Murray had called attention to it. As a verification of the theory and a demonstration of the fact of a temperature seiche, experiments were made in a long glass tank containing two liquids of different density. Over the surface of the upper and lighter liquid a current of wind from a rotary fan was blown, with a consequent production of a seiche at the interface of the two liquids. Thus it was shown that winds are a *vera causa* in the production of the temperature seiche; and fresh light was thrown upon the whole question of the influence of wind upon lake temperatures. See *Proceedings* of the Society, vol. xxviii.

Following up these researches along a new line, Mr Wedderburn demonstrated for the first time by direct observation the existence of a return current in deep water, which compensated for the surface current due to the action of the wind. The depth at which this return took place was shown to depend on the temperature distribution for the time being.

Analogies between the temperature seiche and long-period oscillations observed in the sea were also suggested, from which it appears probable that temperature observations in lakes may be of importance in solving oceanographical problems.

Doubt having been expressed by limnologists in Germany and the United States as to the true nature of the temperature seiche, Mr Wedderburn arranged an expedition, jointly with Professor Halbfass, to study the temperature distribution in the Madüsee, Pomerania. The observations and conclusions, which are given in a paper published in the *Transactions* of the Society (vol. xlvii.), have proved to the satisfaction of Mr Wedderburn's critics that a temperature seiche exists in fresh-water lakes whenever the layer of rapid temperature change is well marked. By an extension of the method employed by Professor Chrystal in his discussion of ordinary seiche phenomena, Mr Wedderburn in this latest paper gives a mathematical discussion of the temperature seiche, and deduces a hydrodynamical theory applicable to lakes of variable depth and breadth. The theory, which gives a close numerical accord with the data of observation, was also tested by experiments in tanks of various shape. So close is the analogy between the temperature seiche and the ordinary seiche that the period of the one may be calculated from the period of the other.

For this interesting and important series of investigations into problems closely connected with the phenomena of the temperature seiche in fresh-water lakes, a large part of which was carried out and published during the biennial period 1908-9, 1909-10, the Council of the Royal Society of Edinburgh have awarded the Makdougall-Brisbane Prize.

The following resolution, submitted by Dr R. KIDSTON, was carried unanimously: "That from the beginning of next Session onward, all the Ordinary Meetings of the Society be held at 4.30 p.m., with the exception of any Special Meetings which may be otherwise arranged by the Council."

The following Communications were read :—

1. On the Resistance to Flow of Water through Pipes or Passages having Divergent Boundaries. By Professor A. H. Gibson, D.Sc. Communicated by Professor W. Peddie. *Trans.*, vol. xlviii. pp. 97–116.

2. On *Metaclepsydropsis* (*Zygopteris*) *duplex* (Williamson). By W. T. Gordon, M.A., B.Sc., B.A. Communicated by Professor James Geikie. (*With Lantern Illustrations.*) *Trans.* vol. xlviii. pp. 163–190.

3. Laboratory Note on a Simple Method of Finding the Radius of Gyration of a Body. By Mr W. G. Robson. Communicated by Professor W. Peddie. *Proc.*, vol. xxxi. pp. 517–518.

The following Candidates for Fellowship were balloted for and declared duly elected :— Kenneth John Mackenzie, M.A., John Francis Hall-Edwards, L.R.C.P. (Edin.), and James Simpson Pirie, C.E.

SIXTEENTH AND LAST ORDINARY MEETING.

Friday, 14*th July* 1911, *at* 4.30 *p.m.*

Sir William Turner, K.C.B., President, in the Chair.

By request of the Council an Address was delivered in the French language :—

On Refractions at the Surface of a Lake, Mirages, and Fata Morgana. By Professor F. A. Forel, Switzerland, Honorary Fellow of the Society.

The following Communication was taken as read :—

Experimental Researches on the Specific Gravity and Displacement of some Saline Solutions. By John Young Buchanan, M.A., F.R.S. *Proc.*, vol. xxxi. pp. 635–648 (*Abstract*), and *Trans.*

The following signed the Roll and were duly admitted Fellows of the Society :—Frank Wyville Thomson, Duncan M. Y. Sommerville, M.A., D.Sc., Kenneth John M'Kenzie, M.A.

The Roll was also signed by Professor F. A. Forel, a Foreign Hon. Fellow of the Society.

LAWS OF THE SOCIETY,

As revised 26th October 1908.

[By the Charter of the Society (printed in the *Transactions*, vol. vi. p. 5), the Laws cannot be altered, except at a Meeting held one month after that at which the Motion for alteration shall have been proposed.]

I.

THE ROYAL SOCIETY OF EDINBURGH shall consist of Ordinary and Honorary Fellows. **Title.**

II.

The fees of Ordinary Fellows residing in Scotland.

Every Ordinary Fellow, within three months after his election, shall pay Two Guineas as the fee of admission, and Three Guineas as his contribution for the Session in which he has been elected; and annually at the commencement of every Session, Three Guineas into the hands of the Treasurer. This annual contribution shall continue for ten years after his admission, and it shall be limited to Two Guineas for fifteen years thereafter.* Fellows may compound for these contributions on such terms as the Council may from time to time fix.

III.

Payment to cease after 25 years.

All Fellows who shall have paid Twenty-five years' annual contribution shall be exempted from further payment.

IV.

Fees of Non-Resident Ordinary Fellows.

The fees of admission of an Ordinary Non-Resident Fellow shall be £26, 5s., payable on his admission; and in case of any Non-Resident Fellow coming to reside at any time in Scotland, he shall, during each year of his residence, pay the usual annual contribution of £3, 3s., payable by each Resident Fellow; but after payment of such annual contribution for eight years, he shall be exempt from any further payment. In the case of any Resident Fellow ceasing to reside in Scotland, and wishing to continue a Fellow of the Society, it shall be in the power of the Council to determine on what terms, in the circumstances of each case, the privilege of remaining a Fellow of the Society shall be continued to such Fellow while out of Scotland. **Case of Fellows becoming Non-Resident.**

* A modification of this rule, in certain cases, was agreed to at a Meeting of the Society held on the 3rd January 1831.

At the Meeting of the Society, on the 5th January 1857, when the reduction of the Contributions from £3, 3s. to £2, 2s., from the 11th to the 25th year of membership, was adopted, it was resolved that the existing Members shall share in this reduction, so far as regards their future annual Contributions.

V.

Defaulters.

Members failing to pay their contributions for three successive years (due application having been made to them by the Treasurer) shall be reported to the Council, and, if they see fit, shall be declared from that period to be no longer Fellows, and the legal means for recovering such arrears shall be employed.

VI.

Privileges of Ordinary Fellows.

None but Ordinary Fellows shall bear any office in the Society, or vote in the choice of Fellows or Office-Bearers, or interfere in the patrimonial interests of the Society.

VII.

Numbers unlimited.

The number of Ordinary Fellows shall be unlimited.

VIII.

Fellows entitled to Transactions and Proceedings.

All Ordinary Fellows of the Society who are not in arrear of their Annual Contributions shall be entitled to receive, gratis, copies of the parts of the Transactions of the Society which shall be published subsequent to their admission, upon application, either personally or by an authorised agent, to the Librarian, provided they apply for them within five years of the date of publication of such parts.

Copies of the parts of the Proceedings shall be distributed to all Fellows of the Society, by post or otherwise, as soon as may be convenient after publication.

IX.

Mode of Recommending Ordinary Fellows.

Candidates for admission as Ordinary Fellows shall make an application in writing, and shall produce along with it a certificate of recommendation to the purport below,* signed by at least *four* Ordinary Fellows, two of whom shall certify their recommendation from personal knowledge. This recommendation shall be delivered to the Secretary, and by him laid before the Council, and shall be exhibited publicly in the Society's rooms for one month, after which it shall be considered by the Council. If the Candidate be approved by the Council, notice of the day fixed for the election shall be given in the circulars of at least two Ordinary Meetings of the Society.

X.

Honorary Fellows, British and Foreign.

Honorary Fellows shall not be subject to any contribution. This class shall consist of persons eminently distinguished for science or literature. Its number shall not exceed Fifty-six, of whom Twenty may be British subjects, and Thirty-six may be subjects of foreign states.

* "A. B., a gentleman well versed in science (*or Polite Literature, as the case may be*), being "to our knowledge desirous of becoming a Fellow of the Royal Society of Edinburgh, we hereby "recommend him as deserving of that honour, and as likely to prove a useful and valuable "Member."

XI.

Personages of Royal Blood may be elected Honorary Fellows, without regard to the limitation of numbers specified in Law X. Royal Personages.

XII.

Honorary Fellows may be proposed by the Council, or by a recommendation (in the form given below*) subscribed by three Ordinary Fellows; and in case the Council shall decline to bring this recommendation before the Society, it shall be competent for the proposers to bring the same before a General Meeting. The election shall be by ballot, after the proposal has been communicated *viva voce* from the Chair at one Meeting, and printed in the circulars for Two Ordinary Meetings of the Society, previous to the day of election. Recommendation of Honorary Fellows. Mode of election.

XIII.

The election of Ordinary Fellows shall take place only at one Afternoon Ordinary Meeting of each month during the Session. The election shall be by ballot, and shall be determined by a majority of at least two-thirds of the votes, provided Twenty-four Fellows be present and vote. Election of Ordinary Fellows.

XIV.

The Ordinary Meetings shall be held on the first and third Mondays of each month from November to March, and from May to July, inclusive; with the exception that when there are five Mondays in January, the Meetings for that month shall be held on its second and fourth Mondays. Regular Minutes shall be kept of the proceedings, and the Secretaries shall do the duty alternately, or according to such agreement as they may find it convenient to make. Ordinary Meetings.

XV.

The Society shall from time to time publish its Transactions and Proceedings. For this purpose the Council shall select and arrange the papers which they shall deem it expedient to publish in the Transactions of the Society, and shall superintend the printing of the same. The Transactions.

XVI.

The Transactions shall be published in parts or *Fasciculi* at the close of each Session, and the expense shall be defrayed by the Society. How Published.

* We hereby recommend ________________________________
for the distinction of being made an Honorary Fellow of this Society, declaring that each of us from our own knowledge of his services to (*Literature or Science, as the case may be*) believe him to be worthy of that honour.

(To be signed by three Ordinary Fellows.)

____

____

____

To the President and Council of the Royal Society of Edinburgh.

XVII.

The Council.

That there shall be formed a Council, consisting—First, of such gentlemen as may have filled the office of President; and Secondly, of the following to be annually elected, viz.:—a President, Six Vice-Presidents (two at least of whom shall be Resident), Twelve Ordinary Fellows as Councillors, a General Secretary, Two Secretaries to the Ordinary Meetings, a Treasurer, and a Curator of the Museum and Library.

The Council shall have power to regulate the private business of the Society. At any Meeting of the Council the Chairman shall have a casting as well as a deliberative vote.

XVIII.

Retiring Councillors.

Four Councillors shall go out annually, to be taken according to the order in which they stand on the list of the Council.

XIX.

Election of Office-Bearers

An Extraordinary Meeting for the election of Office-Bearers shall be held annually on the fourth Monday of October, or on such other lawful day in October as the Council may fix, and each Session of the Society shall be held to begin at the date of the said Extraordinary Meeting.

XX.

Special Meetings; how called.

Special Meetings of the Society may be called by the Secretary, by direction of the Council; or on a requisition signed by six or more Ordinary Fellows. Notice of not less than two days must be given of such Meetings.

XXI.

Treasurer's Duties.

The Treasurer shall receive and disburse the money belonging to the Society, granting the necessary receipts, and collecting the money when due.

He shall keep regular accounts of all the cash received and expended, which shall be made up and balanced annually; and at the Extraordinary Meeting in October, he shall present the accounts for the preceding year, duly audited. At this Meeting, the Treasurer shall also lay before the Council a list of all arrears due above two years, and the Council shall thereupon give such directions as they may deem necessary for recovery thereof.

XXII.

Auditor.

At the Extraordinary Meeting in October, a professional accountant shall be chosen to audit the Treasurer's accounts for that year, and to give the necessary discharge of his intromissions.

XXIII.

General Secretary's Duties.

The General Secretary shall keep Minutes of the Extraordinary Meetings of the Society, and of the Meetings of the Council, in two distinct books. He shall, under the direction of the Council, conduct the correspondence of the Society, and superintend its publications. For these purposes he shall, when necessary, employ a clerk, to be paid by the Society.

XXIV.

The Secretaries to the Ordinary Meetings shall keep a regular Minute-book, in which a full account of the proceedings of these Meetings shall be entered; they shall specify all the Donations received, and furnish a list of them, and of the Donors' names, to the Curator of the Library and Museum; they shall likewise furnish the Treasurer with notes of all admissions of Ordinary Fellows. They shall assist the General Secretary in superintending the publications, and in his absence shall take his duty. Secretaries to Ordinary Meetings.

XXV.

The Curator of the Museum and Library shall have the custody and charge of all the Books, Manuscripts, objects of Natural History, Scientific Productions, and other articles of a similar description belonging to the Society; he shall take an account of these when received, and keep a regular catalogue of the whole, which shall lie in the hall, for the inspection of the Fellows. Curator of Museum and Library.

XXVI.

All articles of the above description shall be open to the inspection of the Fellows at the Hall of the Society, at such times and under such regulations as the Council from time to time shall appoint. Use of Museum and Library.

XXVII.

A Register shall be kept, in which the names of the Fellows shall be enrolled at their admission, with the date. Register Book.

XXVIII.

If, in the opinion of the Council of the Society, the conduct of any Fellow is unbecoming the position of a Member of a learned Society, or is injurious to the character and interests of this Society, the Council may request such Fellow to resign; and, if he fail to do so within one month of such request being addressed to him, the Council shall call a General Meeting of the Fellows of the Society to consider the matter; and, if a majority of the Fellows present at such Meeting agree to the expulsion of such Member, he shall be then and there expelled by the declaration of the Chairman of the said Meeting to that effect; and he shall thereafter cease to be a Fellow of the Society, and his name shall be erased from the Roll of Fellows, and he shall forfeit all right or claim in or to the property of the Society. Power of Expulsion.

THE KEITH, MAKDOUGALL-BRISBANE, NEILL, AND GUNNING VICTORIA JUBILEE PRIZES.

The above Prizes will be awarded by the Council in the following manner:—

I. KEITH PRIZE.

The Keith Prize, consisting of a Gold Medal and from £40 to £50 in Money, will be awarded in the Session 1913–1914 for the "best communication on a scientific subject, communicated,* in the first instance, to the Royal Society during the Sessions 1911–1912 and 1912–1913." Preference will be given to a paper containing a discovery.

II. MAKDOUGALL-BRISBANE PRIZE.

This Prize is to be awarded biennially by the Council of the Royal Society of Edinburgh to such person, for such purposes, for such objects, and in such manner as shall appear to them the most conducive to the promotion of the interests of science; with the *proviso* that the Council shall not be compelled to award the Prize unless there shall be some individual engaged in scientific pursuit, or some paper written on a scientific subject, or some discovery in science made during the biennial period, of sufficient merit or importance in the opinion of the Council to be entitled to the Prize.

1. The Prize, consisting of a Gold Medal and a sum of Money, will be awarded at the commencement of the Session 1912–1913, for an Essay or Paper having reference to any branch of scientific inquiry, whether Material or Mental.

2. Competing Essays to be addressed to the Secretary of the Society, and transmitted not later than 8th July 1912.

3. The Competition is open to all men of science.

4. The Essays may be either anonymous or otherwise. In the former case, they must be distinguished by mottoes, with corresponding sealed billets, superscribed with the same motto, and containing the name of the Author.

5. The Council impose no restriction as to the length of the Essays, which may be, at the discretion of the Council, read at the Ordinary Meetings of the Society.

* For the purposes of this award the word "communicated" shall be understood to mean the date on which the manuscript of a paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

They wish also to leave the property and free disposal of the manuscripts to the Authors; a copy, however, being deposited in the Archives of the Society, unless the paper shall be published in the Transactions.

6. In awarding the Prize, the Council will also take into consideration any scientific papers presented * to the Society during the Sessions 1910–11, 1911–12, whether they may have been given in with a view to the prize or not.

III. NEILL PRIZE.

The Council of the Royal Society of Edinburgh having received the bequest of the late Dr Patrick Neill of the sum of £500, for the purpose of "the interest thereof being applied in furnishing a Medal or other reward every second or third year to any distinguished Scottish Naturalist, according as such Medal or reward shall be voted by the Council of the said Society," hereby intimate:

1. The Neill Prize, consisting of a Gold Medal and a sum of Money, will be awarded during the Session 1913–1914.

2. The Prize will be given for a Paper of distinguished merit, on a subject of Natural History, by a Scottish Naturalist, which shall have been presented * to the Society during the two years preceding the fourth Monday in October 1913,—or failing presentation of a paper sufficiently meritorious, it will be awarded for a work or publication by some distinguished Scottish Naturalist, on some branch of Natural History, bearing date within five years of the time of award.

IV. GUNNING VICTORIA JUBILEE PRIZE.

This Prize, founded in the year 1887 by Dr R. H. Gunning, is to be awarded quadrennially by the Council of the Royal Society of Edinburgh, in recognition of original work in Physics, Chemistry, or Pure or Applied Mathematics.

Evidence of such work may be afforded either by a Paper presented to the Society, or by a Paper on one of the above subjects, or some discovery in them elsewhere communicated or made, which the Council may consider to be deserving of the Prize.

The Prize consists of a sum of money, and is open to men of science resident in or connected with Scotland. The first award was made in the year 1887.

In accordance with the wish of the Donor, the Council of the Society may on fit occasions award the Prize for work of a definite kind to be undertaken during the three succeeding years by a scientific man of recognised ability.

* For the purposes of this award the word "presented" shall be understood to mean the date on which the manuscript of a paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

AWARDS OF THE KEITH, MAKDOUGALL-BRISBANE, NEILL, AND GUNNING VICTORIA JUBILEE PRIZES.

I. KEITH PRIZE.

1st Biennial Period, 1827–29.—Dr Brewster, for his papers "on his Discovery of Two New Immiscible Fluids in the Cavities of certain Minerals," published in the Transactions of the Society.

2nd Biennial Period, 1829–31.—Dr Brewster, for his paper "on a New Analysis of Solar Light," published in the Transactions of the Society.

3rd Biennial Period, 1831–33.—Thomas Graham, Esq., for his paper "on the Law of the Diffusion of Gases," published in the Transactions of the Society.

4th Biennial Period, 1833–35.—Professor J. D. Forbes, for his paper "on the Refraction and Polarization of Heat," published in the Transactions of the Society.

5th Biennial Period, 1835–37.—John Scott Russell, Esq., for his researches "on Hydrodynamics," published in the Transactions of the Society.

6th Biennial Period, 1837–39.—Mr John Shaw, for his experiments "on the Development and Growth of the Salmon," published in the Transactions of the Society.

7th Biennial Period, 1839–41.—Not awarded.

8th Biennial Period, 1841–1843.—Professor James David Forbes, for his papers "on Glaciers," published in the Proceedings of the Society.

9th Biennial Period, 1843–45.—Not awarded.

10th Biennial Period, 1845–47.—General Sir Thomas Brisbane, Bart., for the Makerstoun Observations on Magnetic Phenomena, made at his expense, and published in the Transactions of the Society.

11th Biennial Period, 1847–49.—Not awarded.

12th Biennial Period, 1849–51.—Professor Kelland, for his papers "on General Differentiation, including his more recent Communication on a process of the Differential Calculus, and its application to the solution of certain Differential Equations," published in the Transactions of the Society.

13th Biennial Period, 1851–53.—W. J. Macquorn Rankine, Esq., for his series of papers "on the Mechanical Action of Heat," published in the Transactions of the Society.

14th Biennial Period, 1853–55.—Dr Thomas Anderson, for his papers "on the Crystalline Constituents of Opium, and on the Products of the Destructive Distillation of Animal Substances," published in the Transactions of the Society.

15th Biennial Period, 1855–57.—Professor Boole, for his Memoir "on the Application of the Theory of Probabilities to Questions of the Combination of Testimonies and Judgments," published in the Transactions of the Society.

16th Biennial Period, 1857–59.—Not awarded.

17th Biennial Period, 1859–61.—John Allan Broun, Esq., F.R.S., Director of the Trevandrum Observatory, for his papers "on the Horizontal Force of the Earth's Magnetism, on the Correction of the Bifilar Magnetometer, and on Terrestrial Magnetism generally," published in the Transactions of the Society.

18th Biennial Period, 1861–63.—Professor William Thomson, of the University of Glasgow, for his Communication "on some Kinematical and Dynamical Theorems."

19th Biennial Period, 1863–65.—Principal Forbes, St Andrews, for his "Experimental Inquiry into the Laws of Conduction of Heat in Iron Bars," published in the Transactions of the Society.

20th Biennial Period, 1865–67.—Professor C. Piazzi Smyth, for his paper "on Recent Measures at the Great Pyramid," published in the Transactions of the Society.

21st Biennial Period, 1867–69.—Professor P. G. Tait, for his paper "on the Rotation of a Rigid Body about a Fixed Point," published in the Transactions of the Society.

22ND BIENNIAL PERIOD, 1869–71.—Professor CLERK MAXWELL, for his paper "on Figures, Frames, and Diagrams of Forces," published in the Transactions of the Society.

23RD BIENNIAL PERIOD, 1871–73.—Professor P. G. TAIT, for his paper entitled "First Approximation to a Thermo-electric Diagram," published in the Transactions of the Society.

24TH BIENNIAL PERIOD, 1873–75.—Professor CRUM BROWN, for his Researches "on the Sense of Rotation, and on the Anatomical Relations of the Semicircular Canals of the Internal Ear."

25TH BIENNIAL PERIOD, 1875–77.—Professor M. FORSTER HEDDLE, for his papers "on the Rhombohedral Carbonates," and "on the Felspars of Scotland," published in the Transactions of the Society.

26TH BIENNIAL PERIOD, 1877–79.—Professor H. C. FLEEMING JENKIN, for his paper "on the Application of Graphic Methods to the Determination of the Efficiency of Machinery," published in the Transactions of the Society; Part II. having appeared in the volume for 1877–78.

27TH BIENNIAL PERIOD, 1879–81.—Professor GEORGE CHRYSTAL, for his paper "on the Differential Telephone," published in the Transactions of the Society.

28TH BIENNIAL PERIOD, 1881–83.—THOMAS MUIR, Esq., LL.D., for his "Researches into the Theory of Determinants and Continued Fractions," published in the Proceedings of the Society.

29TH BIENNIAL PERIOD, 1883–85.—JOHN AITKEN, Esq., for his paper "on the Formation of Small Clear Spaces in Dusty Air," and for previous papers on Atmospheric Phenomena, published in the Transactions of the Society.

30TH BIENNIAL PERIOD, 1885–87.—JOHN YOUNG BUCHANAN, Esq., for a series of communications, extending over several years, on subjects connected with Ocean Circulation, Compressibility of Glass, etc.; two of which, viz., "On Ice and Brines," and "On the Distribution of Temperature in the Antarctic Ocean," have been published in the Proceedings of the Society.

31ST BIENNIAL PERIOD, 1887–89.—Professor E. A. LETTS, for his papers on the Organic Compounds of Phosphorus, published in the Transactions of the Society.

32ND BIENNIAL PERIOD, 1889–91.—R. T. OMOND, Esq., for his contributions to Meteorological Science, many of which are contained in vol. xxxiv. of the Society's Transactions.

33RD BIENNIAL PERIOD, 1891–93.—Professor THOMAS R. FRASER, F.R.S., for his papers on *Strophanthus hispidus*, Strophanthin, and Strophanthidin, read to the Society in February and June 1889 and in December 1891, and printed in vols. xxxv., xxxvi., and xxxvii. of the Society's Transactions.

34TH BIENNIAL PERIOD, 1893–95.—Dr CARGILL G. KNOTT, for his papers on the Strains produced by Magnetism in Iron and in Nickel, which have appeared in the Transactions and Proceedings of the Society.

35TH BIENNIAL PERIOD, 1895–97.—Dr THOMAS MUIR, for his continued communications on Determinants and Allied Questions.

36TH BIENNIAL PERIOD, 1897–99.—Dr JAMES BURGESS, for his paper "on the Definite Integral $\frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-t^2}dt$, with extended Tables of Values," printed in vol. xxxix. of the Transactions of the Society.

37TH BIENNIAL PERIOD, 1899–1901.—Dr HUGH MARSHALL, for his discovery of the Persulphates, and for his Communications on the Properties and Reactions of these Salts, published in the Proceedings of the Society.

38TH BIENNIAL PERIOD, 1901–03.—Sir WILLIAM TURNER, K.C.B., LL.D., F.R.S., &c., for his memoirs entitled "A Contribution to the Craniology of the People of Scotland," published in the Transactions of the Society, and for his "Contributions to the Craniology of the People of the Empire of India," Parts I., II., likewise published in the Transactions of the Society.

39TH BIENNIAL PERIOD, 1903–05.—THOMAS H. BRYCE, M.A., M.D., for his two papers on "The Histology of the Blood of the Larva of *Lepidosiren paradoxa*," published in the Transactions of the Society within the period.

40TH BIENNIAL PERIOD, 1905–07.—ALEXANDER BRUCE, M.A., M.D., F.R.C.P.E., for his paper entitled "Distribution of the Cells in the Intermedio-Lateral Tract of the Spinal Cord," published in the Transactions of the Society within the period.

41ST BIENNIAL PERIOD, 1907–09.—WHEELTON HIND, M.D., B.S., F.R.C.S., F.G.S., for a paper published in the Transactions of the Society, "On the Lamellibranch and Gasteropod Fauna found in the Millstone Grit of Scotland."

II. MAKDOUGALL-BRISBANE PRIZE.

1st Biennial Period, 1859.—Sir Roderick Impey Murchison, on account of his Contributions to the Geology of Scotland.

2nd Biennial Period, 1860–62.—William Seller, M.D., F.R.C.P.E., for his "Memoir of the Life and Writings of Dr Robert Whytt," published in the Transactions of the Society.

3rd Biennial Period, 1862–64.—John Denis Macdonald, Esq., R.N., F.R.S., Surgeon of H.M.S. "Icarus," for his paper "on the Representative Relationships of the Fixed and Free Tunicata, regarded as Two Sub-classes of equivalent value; with some General Remarks on their Morphology," published in the Transactions of the Society.

4th Biennial Period, 1864–66.—Not awarded.

5th Biennial Period, 1866–68.—Dr Alexander Crum Brown and Dr Thomas Richard Fraser, for their conjoint paper "on the Connection between Chemical Constitution and Physiological Action," published in the Transactions of the Society.

6th Biennial Period, 1868–70.—Not awarded.

7th Biennial Period, 1870–72.—George James Allman, M.D., F.R.S., Emeritus Professor of Natural History, for his paper "on the Homological Relations of the Cœlenterata," published in the Transactions, which forms a leading chapter of his Monograph of Gymnoblastic or Tubularian Hydroids—since published.

8th Biennial Period, 1872–74.—Professor Lister, for his paper "on the Germ Theory of Putrefaction and the Fermentive Changes," communicated to the Society, 7th April 1873.

9th Biennial Period, 1874–76.—Alexander Buchan, A.M., for his paper "on the Diurnal Oscillation of the Barometer," published in the Transactions of the Society.

10th Biennial Period, 1876–78.—Professor Archibald Geikie, for his paper "on the Old Red Sandstone of Western Europe," published in the Transactions of the Society.

11th Biennial Period, 1878–80.—Professor Piazzi Smyth, Astronomer-Royal for Scotland, for his paper "on the Solar Spectrum in 1877–78, with some Practical Idea of its probable Temperature of Origination," published in the Transactions of the Society.

12th Biennial Period, 1880–82.—Professor James Geikie, for his "Contributions to the Geology of the North-West of Europe," including his paper "on the Geology of the Faroes," published in the Transactions of the Society.

13th Biennial Period, 1882–84.—Edward Sang, Esq., LL.D., for his paper "on the Need of Decimal Subdivisions in Astronomy and Navigation, and on Tables requisite therefor," and generally for his Recalculation of Logarithms both of Numbers and Trigonometrical Ratios,—the former communication being published in the Proceedings of the Society.

14th Biennial Period, 1884–86.—John Murray, Esq., LL.D., for his papers "On the Drainage Areas of Continents, and Ocean Deposits," "The Rainfall of the Globe, and Discharge of Rivers," "The Height of the Land and Depth of the Ocean," and "The Distribution of Temperature in the Scottish Lochs as affected by the Wind."

15th Biennial Period, 1886–88.—Archibald Geikie, Esq., LL.D., for numerous Communications, especially that entitled "History of Volcanic Action during the Tertiary Period in the British Isles," published in the Transactions of the Society.

16th Biennial Period, 1889–90.—Dr Ludwig Becker, for his paper on "The Solar Spectrum at Medium and Low Altitudes," printed in vol. xxxvi. Part I. of the Society's Transactions.

17th Biennial Period, 1890–92.—Hugh Robert Mill, Esq., D.Sc., for his papers on "The Physical Conditions of the Clyde Sea Area," Part I. being already published in vol. xxxvi. of the Society's Transactions.

18th Biennial Period, 1892–94.—Professor James Walker, D.Sc., Ph.D., for his work on Physical Chemistry, part of which has been published in the Proceedings of the Society, vol. xx. pp. 255–263. In making this award, the Council took into consideration the work done by Professor Walker along with Professor Crum Brown on the Electrolytic Synthesis of Dibasic Acids, published in the Transactions of the Society.

19th Biennial Period, 1894–96.—Professor John G. M'Kendrick, for numerous Physiological papers, especially in connection with Sound, many of which have appeared in the Society's publications.

20th Biennial Period, 1896–98.—Dr William Peddie, for his papers on the Torsional Rigidity of Wires.

21st Biennial Period, 1898–1900.—Dr Ramsay H. Traquair, for his paper entitled "Report on Fossil Fishes collected by the Geological Survey in the Upper Silurian Rocks of Scotland," printed in vol. xxxix. of the Transactions of the Society.

22nd Biennial Period, 1900–02.—Dr Arthur T. Masterman, for his paper entitled "The Early Development of *Cribrella oculata* (Forbes), with remarks on Echinoderm Development," printed in vol. xl. of the Transactions of the Society.

23rd Biennial Period, 1902–04.—Mr John Dougall, M.A., for his paper on "An Analytical Theory of the Equilibrium of an Isotropic Elastic Plate," published in vol. xli. of the Transactions of the Society.

24th Biennial Period, 1904–06.—Jacob E. Halm, Ph.D., for his two papers entitled "Spectroscopic Observations of the Rotation of the Sun," and "Some Further Results obtained with the Spectroheliometer," and for other astronomical and mathematical papers published in the Transactions and Proceedings of the Society within the period.

25th Biennial Period, 1906–08.—D. T. Gwynne-Vaughan, M.A., F.L.S., for his papers, 1st, "On the Fossil Osmundaceæ," and 2nd, "On the Origin of the Adaxially-curved Leaf-trace in the Filicales," communicated by him conjointly with Dr R. Kidston.

26th Biennial Period, 1908–10.—Ernest MacLagan Wedderburn, M.A., LL.B., for his series of papers bearing upon "The Temperature Distribution in Fresh-water Lochs," and especially upon "The Temperature Seiche."

III. THE NEILL PRIZE.

1st Triennial Period, 1856–59.—Dr W. Lauder Lindsay, for his paper "on the Spermogones and Pycnides of Filamentous, Fruticulose, and Foliaceous Lichens," published in the Transactions of the Society.

2nd Triennial Period, 1859–61.—Robert Kaye Greville, LL.D., for his Contributions to Scottish Natural History, more especially in the department of Cryptogamic Botany, including his recent papers on Diatomaceæ.

3rd Triennial Period, 1862–65.—Andrew Crombie Ramsay, F.R.S., Professor of Geology in the Government School of Mines, and Local Director of the Geological Survey of Great Britain, for his various works and memoirs published during the last five years, in which he has applied the large experience acquired by him in the Direction of the arduous work of the Geological Survey of Great Britain to the elucidation of important questions bearing on Geological Science.

4th Triennial Period, 1865–68.—Dr William Carmichael M'Intosh, for his paper "on the Structure of the British Nemerteans, and on some New British Annelids," published in the Transactions of the Society.

5th Triennial Period, 1868–71.—Professor William Turner, for his papers "on the Great Finner Whale; and on the Gravid Uterus, and the Arrangement of the Fœtal Membranes in the Cetacea," published in the Transactions of the Society.

6th Triennial Period, 1871–74.—Charles William Peach, Esq., for his Contributions to Scottish Zoology and Geology, and for his recent contributions to Fossil Botany.

7th Triennial Period, 1874–77.—Dr Ramsay H. Traquair, for his paper "on the Structure and Affinities of *Tristichopterus alatus* (Egerton)," published in the Transactions of the Society, and also for his contributions to the Knowledge of the Structure of Recent and Fossil Fishes.

8th Triennial Period, 1877–80.—John Murray, Esq., for his paper "on the Structure and Origin of Coral Reefs and Islands," published (in abstract) in the Proceedings of the Society.

9th Triennial Period, 1880–83.—Professor Herdman, for his papers "on the Tunicata," published in the Proceedings and Transactions of the Society.

10th Triennial Period, 1883–86.—B. N. Peach, Esq., for his Contributions to the Geology and Palæontology of Scotland, published in the Transactions of the Society.

11th Triennial Period, 1886–89.—Robert Kidston, Esq., for his Researches in Fossil Botany, published in the Transactions of the Society.

12th Triennial Period, 1889–92.—John Horne, Esq., F.G.S., for his Investigations into the Geological Structure and Petrology of the North-West Highlands.

13th Triennial Period, 1892–95.—Robert Irvine, Esq., for his papers on the Action of Organisms in the Secretion of Carbonate of Lime and Silica, and on the solution of these substances in Organic Juices. These are printed in the Society's Transactions and Proceedings.

14th Triennial Period, 1895–98.—Professor Cossar Ewart, for his recent Investigations connected with Telegony.

15th Triennial Period, 1898–1901.—Dr John S. Flett, for his papers entitled "The Old Red Sandstone of the Orkneys" and "The Trap Dykes of the Orkneys," printed in vol. xxxix. of the Transactions of the Society.

16th Triennial Period, 1901–04.—Professor J. Graham Kerr, M.A., for his Researches on *Lepidosiren paradoxa*, published in the Philosophical Transactions of the Royal Society, London.

17th Triennial Period, 1904–07.—Frank J. Cole, B.Sc., for his paper entitled "A Monograph on the General Morphology of the Myxinoid Fishes, based on a study of Myxine," published in the Transactions of the Society, regard being also paid to Mr Cole's other valuable contributions to the Anatomy and Morphology of Fishes.

1st Biennial Period, 1907–09.—Francis J. Lewis, M.Sc., F.L.S., for his papers in the Society's Transactions "On the Plant Remains of the Scottish Peat Mosses."

IV. GUNNING VICTORIA JUBILEE PRIZE.

1st Triennial Period, 1884–87.—Sir William Thomson, Pres. R.S.E., F.R.S., for a remarkable series of papers "on Hydrokinetics," especially on Waves and Vortices, which have been communicated to the Society.

2nd Triennial Period, 1887–90.—Professor P. G. Tait, Sec. R.S.E., for his work in connection with the "Challenger" Expedition, and his other Researches in Physical Science.

3rd Triennial Period, 1890–93.—Alexander Buchan, Esq., LL.D., for his varied, extensive, and extremely important Contributions to Meteorology, many of which have appeared in the Society's Publications.

4th Triennial Period, 1893–96.—John Aitken, Esq., for his brilliant Investigations in Physics, especially in connection with the Formation and Condensation of Aqueous Vapour.

1st Quadrennial Period, 1896–1900.—Dr T. D. Anderson, for his discoveries of New and Variable Stars.

2nd Quadrennial Period, 1900–04.—Sir James Dewar, LL.D., D.C.L., F.R.S., etc., for his researches on the Liquefaction of Gases, extending over the last quarter of a century, and on the Chemical and Physical Properties of Substances at Low Temperatures: his earliest papers being published in the Transactions and Proceedings of the Society.

3rd Quadrennial Period, 1904–08.—Professor George Chrystal, M.A., LL.D., for a series of papers on "Seiches," including "The Hydrodynamical Theory and Experimental Investigations of the Seiche Phenomena of Certain Scottish Lakes."

THE COUNCIL OF THE SOCIETY,

October 1911.

President

Sir WILLIAM TURNER, K.C.B., M.B., F.R.C.S.E., LL.D., D.C.L., D.Sc. (Camb. and Dub.), F.R.S., Principal of the University of Edinburgh.

Vice-Presidents

JAMES COSSAR EWART, M.D., F.R.C.S.E., F.R.S., F.L.S., Regius Professor of Natural History in the University of Edinburgh.
JOHN HORNE, LL.D., F.R.S., F.G.S., formerly Director of the Geological Survey of Scotland.
JAMES BURGESS, C.I.E., LL.D., M.R.A.S.
T. HUDSON BEARE, M.Inst.C.E., Professor of Engineering in the University of Edinburgh.
FREDERICK O. BOWER, M.A., D.Sc., F.R.S., F.L.S., Regius Professor of Botany in the University of Glasgow.
Sir THOMAS R. FRASER, M.D., LL.D., Sc.D., F.R.C.P.E., F.R.S., Professor of Materia Medica in the University of Edinburgh.

General Secretary.

* GEORGE CHRYSTAL, M.A., LL.D., Professor of Mathematics in the University of Edinburgh.

Secretaries to Ordinary Meetings.

CARGILL G. KNOTT, D.Sc., Lecturer on Applied Mathematics in the University of Edinburgh.
ROBERT KIDSTON, LL.D., F.R.S., F.G.S.

Treasurer.

JAMES CURRIE, M.A.

Curator of Library and Museum.

JOHN SUTHERLAND BLACK, M.A., LL.D.

Councillors.

D. NOËL PATON, M.D., B.Sc., F.R.C.P.E., Professor of Physiology in the University of Glasgow.
WILLIAM S. BRUCE, LL.D.
F. G. BAILY, M.A., Professor of Applied Physics, Heriot Watt College, Edinburgh.
J. G. BARTHOLOMEW, LL.D., F.R.G.S.
RAMSAY H. TRAQUAIR, M.D., LL.D., F.R.S.
JAMES WALKER, D.Sc., Ph.D., LL.D., F.R.S., Professor of Chemistry in the University of Edinburgh.
ARTHUR ROBINSON, M.D., M.R.C.S., Professor of Anatomy in the University of Edinburgh.
Sir W. S. M'CORMICK, M.A., LL.D.
ALEXANDER CRUM BROWN, M.D., D.Sc., F.R.C.P.E., LL.D., F.R.S., Emeritus Professor of Chemistry in the University of Edinburgh.
THOMAS H. BRYCE, M.A., M.D., Professor of Anatomy in the University of Glasgow.
BENJAMIN N. PEACH, LL.D., F.R.S., F.G.S., formerly District Superintendent and Acting Palæontologist of the Geological Survey of Scotland.
WILLIAM ALLAN CARTER, Memb.Inst.C.E.

---

* Professor Chrystal died November 3, 1911.

ALPHABETICAL LIST OF THE ORDINARY FELLOWS OF THE SOCIETY,

Corrected to November 1911.

N.B.—*Those marked * are Annual Contributors.*

B. prefixed to a name indicates that the Fellow has received a Makdougall-Brisbane Medal.
K. „ „ „ Keith Medal.
N. „ „ „ Neill Medal.
V. J. „ „ „ the Gunning Victoria Jubilee Prize.
C. „ „ „ contributed one or more Communications to the Society's TRANSACTIONS or PROCEEDINGS.

Date of Election.			
1898	C.	* Abercromby, the Hon. John, 62 Palmerston Place	
1898		Adami, Prof. J. G., M.A., M.D. Cantab., F.R.S., Professor of Pathology in M'Gill University, Montreal	
1911		* Adams, Archibald Campbell, A.M.I.Mech.E., A.M.I.E.E., Consulting Engineer, 1 Old Smithhills, Paisley	
1896		* Affleck, Sir Jas. Ormiston, M.D., LL.D., F.R.C.P.E., 38 Heriot Row	
1871		Agnew, Sir Stair, K.C.B., M.A., formerly Registrar-General for Scotland, 22 Buckingham Terrace	5
1875	C. K. V. J.	Aitken, John, LL.D., F.R.S., Ardenlea, Falkirk	
1895		* Alford, Robert Gervase, Memb. Inst. C.E., 1 Windmill Hill, Hampstead, London, N.W.	
1889		* Alison, John, M.A., Head Master, George Watson's College, Edinburgh	
1894		Allan, Francis John, M.D., C.M. Edin., M.O.H. City of Westminster, Westminster City Hall, Charing Cross Road, London	
1888	C.	* Allardice, R. E., M.A., Professor of Mathematics in Stanford University, Palo Alto, Santa Clara Co., California	10
1878		Allchin, Sir William H., M.D., F.R.C.P.L., Senior Physician to the Westminster Hospital, 5 Chandos Street, Cavendish Square, London	
1906		Anderson, Daniel E., M.D., B.A., B.Sc., Green Bank, Merton Lane, Highgate, London, N.	
1893		Anderson, J. Macvicar, Architect, 6 Stratton Street, London	
1883		Anderson, Sir Robert Rowand, LL.D., 16 Rutland Square	
1905		Anderson, William, F.G.S., Box 4303, Johannesburg, Transvaal, South Africa	15
1905		* Anderson, William, M.A., Head Science Master, George Watson's College, Edinburgh, 29 Lutton Place	
1903		Anderson-Berry, David, M.D., LL.D., F.R.S.L., M.R.A.S., F.S.A. (Scot.), Versailles, Highgate, London, N.	
1905		* Andrew, George, M.A., B.A., H.M.I.S., Balwherrie, Strathearn Road, Broughty Ferry	
1881	C.	Anglin, A. H., M.A., LL.D., M.R.I.A., Professor of Mathematics, Queen's College, Cork	
1906		Appleton, Arthur Frederick, F.R.C.V.S., Lieut.-Col., Army Headquarters, Pretoria, S.A.	20
1899		Appleyard, James R., Royal Technical Institute, Salford, Manchester	
1893		* Archer, Walter E., 17 Sloane Court, London, S.W.	
1910	C.	Archibald, E. H., B.Sc., Professor of Chemistry, Syracuse University, Syracuse, N.Y., U.S.A.	
1907		* Archibald, James, M.A., Head Master, St Bernard's School, 1 Leamington Terrace, Edinburgh	

Date of Election.			
1911	C.	* Ashworth, James Hartley, D.Sc., Lecturer in Invertebrate Zoology, University of Edinburgh, 4 Cluny Terrace	25
1907		* Badre, Muhammad, Ph.D.	
1894		* Bailey, Frederick, Lieut.-Col. (formerly) R.E., 7 Drummond Place	
1896	C.	* Baily, Francis Gibson, M.A., Professor of Applied Physics, Heriot-Watt College	
1877	C.	Balfour, I. Bayley, M.A., Sc.D., M.D., LL.D., F.R.S., F.L.S., King's Botanist in Scotland, Professor of Botany in the University of Edinburgh and Keeper of the Royal Botanic Gardens, Inverleith House	
1905	C.	Balfour-Browne, William Alexander Francis, M.A., Barrister-at-Law, Claremont, Holywood, Co. Down, Ireland	30
1892		* Ballantyne, J. W., M.D., F.R.C.P.E., 19 Rothesay Terrace	
1902	C.	Bannerman, W. B., M.D., D.Sc., Lt.-Col., Indian Medical Service, Director, Bacteriological Laboratory, Parel, Bombay, India	
1889		* Barbour, A. H. F., M.A., M.D., LL.D., F.R.C.P.E., 4 Charlotte Square	
1886		Barclay, A. J. Gunion, M.A., 729 Great Western Road, Glasgow	
1883	C.	Barclay, G. W. W., M.A., Raeden House, Aberdeen	35
1910		* Barclay, Lewis Bennett, C.E., 16 Craiglockhart Terrace, Edinburgh	
1903		Bardswell, Noël Dean, M.D., M.R.C.P. Ed. and Lond., King Edward VII. Sanatorium, Midhurst	
1882	C.	Barnes, Henry, M.D., LL.D., 6 Portland Square, Carlisle	
1904		Barr, Sir James, M.D., F.R.C.P. Lond., 72 Rodney Street, Liverpool	
1874		Barrett, William F., F.R.S., M.R.I.A., formerly Professor of Physics, Royal College of Science, Dublin, Kingstown, County Dublin	40
1887		Bartholomew, J. G., LL.D., F.R.G.S., The Geographical Institute, Duncan Street, Edinburgh	
1895	C.	Barton, Edwin H., D.Sc., A.M.I.E.E., Fellow Physical Society of London, Professor of Experimental Physics, University College, Nottingham	
1904		* Baxter, William Muirhead, St Colms, 21B Strathearn Road, Edinburgh	
1888		* Beare, Thomas Hudson, B.Sc., Memb. Inst. C.E., Professor of Engineering in the University of Edinburgh (VICE-PRESIDENT)	
1897	C.	* Beattie, John Carruthers, D.Sc., Professor of Physics, South African College, Cape Town	45
1892		Beck, J. H. Meining, M.D., M.R.C.P.E., Rondebosch, Cape Town	
1893	B. C.	* Becker, Ludwig, Ph.D., Regius Professor of Astronomy in the University of Glasgow, The Observatory, Glasgow	
1882	C.	Beddard, Frank E., M.A. Oxon., F.R.S., Prosector to the Zoological Society of London, Zoological Society's Gardens, Regent's Park, London	
1887		Begg, Ferdinand Faithfull, Bartholomew House, London	
1886		Bell, A. Beatson, 17 Lansdowne Crescent	50
1906		Bell, John Patrick Fair, F.Z.S., Fulforth, Witton Gilbert, Durham	
1900		* Bennett, James Bower, C.E., 42 Frederick Street	
1887		Bernard, J. Mackay, of Dunsinnan, B.Sc., Dunsinnan, Perth	
1893	C.	* Berry, George A., M.D., C.M., F.R.C.S., 31 Drumsheugh Gardens	
1897	C.	Berry, Richard J., M.D., F.R.C.S.E., Professor of Anatomy in the University of Melbourne, Victoria, Australia	55
1904		* Beveridge, Erskine, LL.D., St Leonards Hill, Dunfermline	
1880	C.	Birch, De Burgh, C.B., M.D., Professor of Physiology in the University of Leeds, 16 De Grey Terrace, Leeds	
1907		* Black, Frederick Alexander, Solicitor, 59 Academy Street, Inverness	
1884		Black, John S., M.A., LL.D. (CURATOR OF LIBRARY AND MUSEUM), 6 Oxford Terrace	
1897		* Blaikie, Walter Biggar, The Loan, Colinton	60
1904	C.	* Bles, Edward J., M.A., D.Sc., The Mill House, Iffley, Oxford	
1898	C.	* Blyth, Benjamin Hall, M.A., Memb. Inst. C.E., 17 Palmerston Place	
1894		* Bolton, Herbert, F.G.S., F.L.S., Curator of the Bristol Museum, Queen's Road, Bristol	
1884		Bond, Francis T., B.A., M.D., M.R.C.S., Gloucester	
1872	C.	Bottomley, J. Thomson, M.A., D.Sc., LL.D., F.R.S., F.C.S., 13 University Gardens, Glasgow	65
1886		Bower, Frederick O., M.A., D.Sc., F.R.S., F.L.S., Regius Professor of Botany in the University of Glasgow, 1 St John's Terrace, Hillhead, Glasgow (VICE-PRESIDENT)	
1884	C.	Bowman, Frederick Hungerford, D.Sc., F.C.S. (Lond. and Berl.), F.I.C., Assoc. Inst. C.E., Assoc. Inst. M.E., M.I.E.E., etc., 4 Albert Square, Manchester	
1901		Bradbury, J. B., M.D., Downing Professor of Medicine, University of Cambridge	

Date of Election.			
1903	C.	* Bradley, O. Charnock, M.D., D.Sc., Principal, Royal Veterinary College, Edinburgh	
1886		Bramwell, Byrom, M.D., F.R.C.P.E., 24 Walker Street	70
1907		* Bramwell, Edwin, M.B., F.R.C.P.E., F.R.C.P. Lond., 24 Walker Street	
1895		Bright, Charles, Memb. Inst. C.E., Memb. Inst. E.E., F.R.A.S., F.G.S., London, Consulting Engineer to the Commonwealth of Australia, 26 Devonshire Terrace, Hyde Park, and Caxton House, Westminster, London, S.W.	
1893		Brock, G. Sandison, M.D., 6 Corso d'Italia, Rome, Italy	
1901	C.	* Brodie, W. Brodie, M.B., Thaxted, Essex	
1907		Brown, Alexander, M.A., B.Sc., Professor of Applied Mathematics, South African College, Cape Town	75
1864	C. K. B.	Brown, Alex. Crum, M.D., D.Sc., F.R.C.P.E., LL.D., F.R.S., Emeritus Professor of Chemistry in the University of Edinburgh, 8 Belgrave Crescent	
1898		* Brown, David, F.C.S., F.I.C., Willowbrae House, Willowbrae Road	
1911		* Brown, David Rainy, Chemical Manufacturer (J. F. Macfarlan & Co.), 1 Glenorchy Terrace, Edinburgh	
1883	C.	Brown, J. J. Graham, M.D., F.R.C.P.E., 3 Chester Street	
1885	C.	Brown, J. Macdonald, M.D., F.R.C.S., 64 Upper Berkeley Street, Portman Square, London, W.	80
1909	C.	* Brownlee, John, M.A., M.D., D.Sc., Ruchill Hospital, Bilsland Drive, Glasgow	
1906		* Bruce, William Speirs, LL.D., Antarctica, Joppa, Midlothian	
1898	K. C.	* Bryce, T. H., M.A., M.D. (Edin.), Professor of Anatomy in the University of Glasgow, 2 The University, Glasgow	
1870	C. K.	Buchanan, John Young, M.A., F.R.S., 26 Norfolk Street, Park Lane, London, W.	
1887	C.	Buist, J. B., M.D., F.R.C.P.E., 1 Clifton Terrace	85
1905		Bunting, Thomas Lowe, M.D., Scotswood, Newcastle-on-Tyne	
1902		* Burgess, A. G., M.A., Mathematical Master, Edinburgh Ladies' College, 64 Strathearn Road	
1894	C. K.	* Burgess, James, C.I.E., LL.D., Hon. A.R.I.B.A., F.R.G.S., Hon. M. Imp. Russ. Archæol. Soc., and Amer. Or. Soc., M. Soc. Asiat. de Paris, M.R.A.S., *H.* Corr. M. Batavian Soc. of Arts and Sciences, and Berlin Soc. Anthrop., *H.* Assoc. Finno-Ugrian Soc. (Vice-President), 22 Seton Place	
1902		* Burn, Rev. John Henry, B.D., The Parsonage, Ballater	
1887		Burnet, John James, Architect, 18 University Avenue, Hillhead, Glasgow	90
1888		* Burns, Rev. T., D.D., F.S.A. Scot., Minister of Lady Glenorchy's Parish Church, Croston Lodge, Chalmers Crescent	
1896		* Butters, J. W., M.A., B.Sc., Rector of Ardrossan Academy	
1887	C.	Cadell, Henry Moubray, of Grange, B.Sc., Bo'ness	
1897		* Caird, Robert, LL.D., Shipbuilder, Greenock	
1910		* Calderwood, Rev. Robert Sibbald, Minister of Cambuslang, The Manse, Cambuslang, Lanarkshire	95
1893	C.	Calderwood, W. L., Inspector of Salmon Fisheries of Scotland, South Bank, Canaan Lane, Edinburgh	
1894		* Cameron, James Angus, M.D., Medical Officer of Health, Firhall, Nairn	
1905	C.	Cameron, John, M.D., D.Sc., M.R.C.S. Eng., Anatomy Department, Middlesex Hospital Medical School, London	
1904		* Campbell, Charles Duff, 21 Montague Terrace, Inverleith Row	
1908		* Campbell, Lt.-Col. John, Westwood, Cupar, Fife	100
1899	C.	* Carlier, Edmund W. W., M.D., B.Sc., Professor of Physiology in Mason College, Birmingham	
1910		Carnegie, David, Memb. Inst. C.E., Memb. Inst. Mech. E., Memb. I. S. Inst., 33–35 Charterhouse Square, London, E.C.	
1905	C.	* Carse, George Alexander, M.A., D.Sc., Lecturer on Natural Philosophy, University of Edinburgh, 3 Middleby Street	
1901		Carslaw, H. S., M.A., D.Sc., Professor of Mathematics in the University of Sydney, New South Wales	
1905		Carter, Joseph Henry, F.R.C.V.S., Rowley Hall, Burnley, Lancashire	105
1898		* Carter, Wm. Allan, Memb. Inst. C.E., 32 Great King Street (Society's Representative on George Heriot's Trust)	
1898		Carus-Wilson, Cecil, F.R.G.S., F.G.S., 16 Waldegrave Park, Strawberry Hill, Middlesex	
1908		Cavanagh, Thomas Francis, M.D., 396 Eccleshall Road, Sheffield	
1882		Cay, W. Dyce, Memb. Inst. C.E., 39 Victoria Street, Westminster, London	
1890		Charles, John J., M.A., M.D., C.M., formerly Professor of Anatomy and Physiology, Queen's College, Cork, 8 Clyde Road, Dublin	110

Date of Election.			
1899		Chatham, James, Actuary, 7 Belgrave Crescent	
1874		Chiene, John, C.B., M.D., LL.D., F.R.C.S.E., Emeritus Professor of Surgery in the University of Edinburgh, Barnton Avenue, Davidson's Mains	
1891		* Clark, John B., M.A., Head Master of Heriot's Hospital School, Lauriston, Garleffin, Craiglea Drive	
1911		* Clark, William Inglis, D.Sc., 29 Lauder Road, Edinburgh	
1903		* Clarke, William Eagle, F.L.S., Keeper of the Natural History Collections in the Royal Scottish Museum, Edinburgh, 35 Braid Road	115
1909		Clayton, Thomas Morrison, M.D., D.Hy., B.Sc., D.P.H., Medical Officer of Health, Gateshead, 13 The Crescent, Gateshead-on-Tyne	
1875		Clouston, Sir T. S., M.D., LL.D., F.R.C.P.E., 26 Heriot Row	
1904	C.	Coker, Ernest George, M.A., D.Sc., Professor of Mechanical Engineering and Applied Mechanics, City and Guilds Technical College, Finsbury, Leonard Street, City Road, London, E.C.	
1904		Coles, Alfred Charles, M.D., D.Sc., York House, Poole Road, Bournemouth, W.	
1888	C.	Collie, John Norman, Ph.D., F.R.S., F.C.S., Professor of Organic Chemistry in the University College, Gower Street, London	120
1904	C.	* Colquhoun, Walter, M.A., M.B., 18 Walmer Crescent, Ibrox, Glasgow	
1909		* Comrie, Peter, M.A., B.Sc., Head Mathematical Master, Boroughmuir Junior Student Centre, 19 Craighouse Terrace	
1886		Connan, Daniel M., M.A.	
1872		Constable, Archibald, LL.D., 11 Thistle Street	
1894		Cook, John, M.A., 30 Hermitage Gardens, formerly Principal, Central College, Bangalore, Director of Meteorology in Mysore, and Fellow, University of Madras, India	125
1891		* Cooper, Charles A., LL.D., 41 Drumsheugh Gardens	
1905		* Corrie, David, F.C.S., Nobel's Explosives Company, Polmont Station	
1911		* Cowan, Alexander C., Papermaker, Valleyfield House, Penicuik, Midlothian	
1908		Craig, James Ireland, M.A., B.A., Director of the Computation Office, Survey Department, Egypt, Mataria, Egypt	
1875		Craig, William, M.D., F.R.C.S.E., Lecturer on Materia Medica to the College of Surgeons, 71 Bruntsfield Place	130
1907		* Cramer, William, Ph.D., Lecturer in Physiological Chemistry in the University of Edinburgh, Physiological Department, The University	
1903		Crawford, Lawrence, M.A., D.Sc., Professor of Mathematics in the South African College, Cape Town	
1887		Crawford, William Caldwell, 1 Lockharton Gardens, Colinton Road	
1870		Crichton-Browne, Sir Jas., M.D., LL.D., F.R.S., Lord Chancellor's Visitor and Vice-President of the Royal Institution of Great Britain, 72 Queen's Gate, and Royal Courts of Justice, Strand, London	
1886		Croom, Sir John Halliday, M.D., F.R.C.P.E., Professor of Midwifery in the University of Edinburgh, Vice-President, Royal College of Surgeons, Edinburgh, 25 Charlotte Square	135
1898		* Currie, James, M.A. Cantab. (Treasurer), Larkfield, Goldenacre	
1904		* Cuthbertson, John, Secretary, West of Scotland Agricultural College, 6 Charles Street, Kilmarnock	
1885		Daniell, Alfred, M.A., LL.B., D.Sc., Advocate, The Athenæum Club, Pall Mall, London	
1884		Davy, R., F.R.C.S. Eng., Consulting Surgeon to Westminster Hospital, Burstone House, Bow, North Devon	
1894		* Denny, Archibald, Cardross Park, Cardross, Dumbartonshire	140
1869	C. V. J.	Dewar, Sir James, M.A., LL.D., D.C.L., D.Sc. Dub., F.R.S., F.C.S., Jacksonian Professor of Natural and Experimental Philosophy in the University of Cambridge, and Fullerian Professor of Chemistry at the Royal Institution of Great Britain, London	
1905		* Dewar, James Campbell, C.A., 27 Douglas Crescent	
1906		* Dewar, Thomas William, M.D., F.R.C.P., Kincairn, Dunblane	
1904		Dickinson, Walter George Burnett, F.R.C.V.S., Boston, Lincolnshire	
1884		Dickson, the Right Hon. Charles Scott, K.C., LL.D., 22 Moray Place	145
1888	C.	* Dickson, Henry Newton, M.A., D.Sc., The Lawn, Upper Redlands Road, Reading	
1876	C.	Dickson, J. D. Hamilton, M.A., Fellow and Tutor, St Peter's College, Cambridge	
1885	C.	Dixon, James Main, M.A., Litt. Hum. Doctor, Professor of English, University of Southern California, Wesley Avenue, Los Angeles, California, U.S.A.	
1897		* Dobbie, James Bell, F.Z.S., 12 South Inverleith Avenue	

Date of Election.			
1904	C.	* Dobbie, James Johnston, M.A., D.Sc., LL.D., F.R.S., 4 Vicarage Gate, Kensington, London, W.	150
1881	C.	Dobbin, Leonard, Ph.D., Lecturer on Chemistry in the University of Edinburgh, 6 Wilton Road	
1867	C.	Donaldson, Sir James, M.A., LL.D., Principal of the University of St Andrews, St Andrews	
1896		* Donaldson, William, M.A., Viewpark House, Spylaw Road	
1905		* Donaldson, Rev. Wm. Galloway, The Manse, Forfar	
1882		Dott, David B., F.I.C., Memb. Pharm. Soc., Ravenslea, Musselburgh	155
1901		* Douglas, Carstairs Cumming, M.D., D.Sc., Professor of Medical Jurisprudence and Hygiene, Anderson's College, Glasgow, 2 Royal Crescent, Glasgow	
1866		Douglas, David, 22 Drummond Place	
1910		* Douglas, Loudon MacQueen, Author and Lecturer, 3 Lauder Road, Edinburgh	
1908	C.	Drinkwater, Harry, M.D., M.R.C.S. (Eng.), Grosvenor Lodge, Wrexham, North Wales	
1901		* Drinkwater, Thomas W., L.R.C.P.E., L.R.C.S.E., Chemical Laboratory, Surgeons' Hall	160
1878		Duncanson, J. J. Kirk, M.D., F.R.C.P.E., 22 Drumsheugh Gardens	
1904		* Dunlop, William Brown, M.A., 7 Carlton Street	
1903		* Dunstan, John, M.R.C.V.S., 1 Dean Terrace, Liskeard, Cornwall	
1892	C.	Dunstan, M. J. R., M.A., F.I.C., F.C.S., Principal, South-Eastern Agricultural College, Wye, Kent	
1899		* Duthie, George, M.A., Inspector-General of Education, Salisbury, Rhodesia	165
1906	C.	* Dyson, Frank Watson, M.A., F.R.S., Astronomer Royal, Royal Observatory, Greenwich	
1893		Edington, Alexander, M.D., Lydenburg, Transvaal	
1904		* Edwards, John, 4 Great Western Terrace, Kelvinside, Glasgow	
1904		* Elder, William, M.D., F.R.C.P.E., 4 John's Place, Leith	
1875		Elliot, Daniel G., American Museum of Natural History, Central Park West, New York, N.Y., U.S.A.	170
1906	C.	* Ellis, David, D.Sc., Ph.D., Lecturer in Botany and Bacteriology, Glasgow and West of Scotland Technical College, Glasgow	
1897	C.	* Erskine-Murray, James Robert, D.Sc., 77 Kingsfield Road, Watford, Herts	
1884		Evans, William, F.F.A., 38 Morningside Park	
1879	C. N.	Ewart, James Cossar, M.D., F.R.C.S.E., F.R.S., F.L.S., Regius Professor of Natural History, University of Edinburgh (VICE-PRESIDENT), Craigiebield, Penicuik, Midlothian	
1902		* Ewen, J. T., B.Sc., Memb. Inst. Mech. E., H.M.I.S., 104 King's Gate, Aberdeen	175
1878	C.	Ewing, James Alfred, C.B., M.A., B.Sc., LL.D., Memb. Inst. C.E., F.R.S., Director of Naval Education, Admiralty, Froghole, Edenbridge, Kent	
1900	C.	Eyre, John W. H., M.D., M.S. (Dunelm), D.P.H. (Camb.), Guy's Hospital (Bacteriological Department), London	
1910		* Fairgrieve, Mungo M'Callum, M.A. (Glasg.), M.A. (Cambridge), Master at the Edinburgh Academy, 67 Great King Street, Edinburgh	
1875		Fairley, Thomas, Lecturer on Chemistry, 8 Newton Grove, Leeds	
1907	C.	Falconer, John Downie, M.A., D.Sc., F.G.S., Lecturer on Geography, The University, Glasgow.	180
1888	C.	* Fawsitt, Charles A., 9 Foremount Terrace, Dowanhill, Glasgow	
1883	C.	Felkin, Robert W., M.D., F.R.G.S., Fellow of the Anthropological Society of Berlin, 47 Bassett Road, North Kensington, London, W.	
1899		* Fergus, Andrew Freeland, M.D., 22 Blythswood Square, Glasgow	
1907		* Fergus, Edward Oswald, 12 Clairmont Gardens, Glasgow	
1904		* Ferguson, James Haig, M.D., F.R.C.P.E., F.R.C.S.E., 7 Coates Crescent	185
1888		* Ferguson, John, M.A., LL.D., Professor of Chemistry in the University of Glasgow	
1868	C.	Ferguson, Robert M., Ph.D., LL.D., 5 Douglas Gardens	
1898		* Findlay, John R., M.A. Oxon., 27 Drumsheugh Gardens	
1899		* Finlay, David W., B.A., M.D., LL.D., F.R.C.P., D.P.H., Professor of Medicine in the University of Aberdeen, Honorary Physician to His Majesty in Scotland, 2 Queen's Terrace, Aberdeen	
1911		Fleming, John Arnold, F.C.S., etc., Pottery Manufacturer, Woodburn, Rutherglen, Glasgow	190
1906		* Fleming, Robert Alexander, M.A., M.D., F.R.C.P.E., Assistant Physician, Royal Infirmary, 10 Chester Street	
1900	C. N.	* Flett, John S., M.A., D.Sc., Director of the Geological Survey of Scotland, 33 George Square	

Date of Election.		
1872	C.	Forbes, Professor George, M.A., Memb. Inst. C.E., Memb. Inst. E.E., F.R.S., F.R.A.S., 11 Little College Street, Westminster, S.W.
1904		Forbes, Norman Hay, F.R.C.S.E., L.R.C.P. Lond., M.R.C.S. Eng., Corres. Memb. Soc. d'Hydrologie médicale de Paris, Druminnor, Church Stretton, Salop
1892		* Ford, John Simpson, F.C.S., 4 Nile Grove 195
1910		* Fraser, Alexander, Actuary, 17 Eildon Street, Edinburgh
1858		Fraser, A. Campbell, Fellow of the British Academy, Hon. D.C.L. Oxford, LL.D., Litt. D., Emeritus Professor of Logic and Metaphysics in the University of Edinburgh, 34 Melville Street
1896		* Fraser, John, M.B., F.R.C.P.E., formerly one of H.M. Commissioners in Lunacy for Scotland, 54 Great King Street
1867	C. K. B.	Fraser, Sir Thomas R., M.D., LL.D., Sc.D., F.R.C.P.E., F.R.S., Professor of Materia Medica in the University of Edinburgh, Honorary Physician to the King in Scotland, 13 Drumsheugh Gardens. (VICE-PRESIDENT)
1891		* Fullarton, J. H., M.A., D.Sc., 23 Porchester Gardens, London, W. 200
1891		* Fulton, T. Wemyss, M.D., Scientific Superintendent, Scottish Fishery Board, 41 Queen's Road, Aberdeen
1907		* Galbraith, Alexander, Assistant Superintendent Engineer, Cunard Line, Liverpool, 93 Trinity Road, Bootle, Liverpool
1888	C.	* Galt, Alexander, D.Sc., Keeper of the Technological Department, Royal Scottish Museum, Edinburgh
1901		Ganguli, Sanjiban, M.A., Principal, Maharaja's College, and Director of Public Instruction, Jaipur State, Jaipur, India
1899		Gatehouse, T. E., Assoc. Memb. Inst. C.E., Memb. Inst. M.E., Memb. Inst. E.E., Tulse Hill Lodge, 100 Tulse Hill, London 205
1867		Gayner, Charles, M.D., F.L.S.
1900		Gayton, William, M.D., M.R.C.P.E., Ravensworth, Regent's Park Road, Finchley, London, N.
1909	C.	* Geddes, Auckland C., M.D., Professor of Anatomy, Royal College of Surgeons in Ireland, Dublin
1880	C.	Geddes, Patrick, Professor of Botany in University College, Dundee, and Lecturer on Zoology, Ramsay Garden, University Hall, Edinburgh
1861	C. B.	Geikie, Sir Archibald, K.C.B., D.C.L. Oxf., D.Sc. Camb. Dub., LL.D. St And., Glasg., Aberdeen, Edin., Ph.D. Upsala, Pres. R.S., Pres. G.S., Foreign Member of the Reale Accad. Lincei, Rome, of the National Acad. of the United States, of the Academies of Stockholm, Christiania, Göttingen, Corresponding Member of the Institute of France and of the Academies of Berlin, Vienna, Munich, Turin, Belgium, Philadelphia, New York, etc., Shepherd's Down, Haslemere, Surrey 210
1871	C. B.	Geikie, James, LL.D., D.C.L., F.R.S., F.G.S., Professor of Geology in the University of Edinburgh, Kilmorie, Colinton Road
1909		* Gentle, William, B.Sc., 12 Mayfield Road
1910	C.	* Gibb, David, M.A., B.Sc., Lecturer in Mathematics, Edinburgh University, Sunnybank, Durie Street, Leven
1910		* Gibson, Charles Robert, Mansewood, by Pollokshaws, N.B.
1881	C.	Gibson, George Alexander, D.Sc., M.D., LL.D., F.R.C.P.E., 3 Drumsheugh Gardens 215
1890		* Gibson, George A., M.A., LL.D., Professor of Mathematics in the University of Glasgow, 10 The University, Glasgow
1877	C.	Gibson, John, Ph.D., Professor of Chemistry in the Heriot-Watt College, 16 Woodhall Terrace, Juniper Green
1900		Gilchrist, Douglas A., B.Sc., Professor of Agriculture and Rural Economy, Armstrong College, Newcastle-upon-Tyne
1880		Gilruth, George Ritchie, Surgeon, 53 Northumberland Street
1907		Gilruth, John Anderson, M.R.C.V.S., Professor, University, Melbourne, Australia 220
1909		* Gladstone, Hugh Steuart, M.A., M.B.O.U., F.Z.S., Capenoch, Thornhill, Dumfriesshire
1911		Gladstone, Reginald John, M.D., F.R.C.S. (Eng.), Lecturer on Embryology and Senior Demonstrator of Anatomy, Middlesex Hospital, London, 1 Gloucester Gate, Regent Park, London, N.W.
1898		* Glaister, John, M.D., F.R.F.P.S. Glasgow, D.P.H. Camb., Professor of Forensic Medicine in the University of Glasgow, 3 Newton Place, Glasgow
1910		Goodall, Joseph Strickland, M.B. (Lond.), M.S.A. (Eng.), Lecturer on Physiology, Middlesex Hospital, London, Annandale Lodge, Vanbrugh Park, Blackheath, London, S.E.
1901		Goodwillie, James, M.A., B.Sc., Liberton, Edinburgh 225

Date of Election.			
1899		* Goodwin, Thomas S., M.B., C.M., F.C.S., 25 Worple Road, Isleworth, and Derwent Lodge, London Road, Spring-grove, Isleworth, Middlesex	
1897		Gordon-Munn, John Gordon, M.D., Heigham Hall, Norwich	
1891		* Graham, Richard D., 11 Strathearn Road	
1898	C.	* Gray, Albert A., M.D., 14 Newton Terrace, Glasgow	
1883		Gray, Andrew, M.A., LL.D., F.R.S., Professor of Natural Philosophy in the University of Glasgow	230
1910		Gray, Bruce M'Gregor, C.E., Assoc. Memb. Inst. C.E., Westbourne Grove, Selby, Yorkshire	
1909	C.	* Gray, James Gordon, D.Sc., Lecturer in Physics in the University of Glasgow, 11 The University, Glasgow	
1910		* Green, Charles Edward, Publisher, Gracemount House, Liberton	
1886		Greenfield, W. S., M.D., F.R.C.P.E., Professor of General Pathology in the University of Edinburgh, 7 Heriot Row	
1897		Greenlees, Thomas Duncan, M.D. Edin., Amana, Tulse Hill, London	235
1905	C.	* Gregory, John Walter, D.Sc., F.R.S., Professor of Geology in the University of Glasgow, 4 Park Quadrant, Glasgow	
1906		Greig, Edward David Wilson, M.D., B.Sc., Captain, H.M.'s Indian Medical Service, Byculla Club, Bombay, India	
1905		Greig, Robert Blyth, F.Z.S., Whangara, Wallington, Surrey	
1910		* Grimshaw, Percy Hall, Assistant Keeper, Natural History Department, The Royal Scottish Museum, 49 Comiston Drive, Edinburgh	
1899		* Guest, Edward Graham, M.A., B.Sc., 5 Newbattle Terrace	240
1907	C.	* Gulliver, Gilbert Henry, B.Sc., A.M.I. Mech. E., Lecturer in Experimental Engineering in the University of Edinburgh, 10 Stanley Street, Portobello	
1911	C.	* Gunn, James Andrew, M.A., M.D., D.Sc., Assistant in the Materia Medica Department Edinburgh University, 39 Morningside Drive	
1888	C.	Guppy, Henry Brougham, M.B., Rosario, Salcombe, Devon	
1911		* Guy, William, F.R.C.S., L.R.C.P, L.D.S. Ed., Consulting Dental Surgeon, Edinburgh Royal Infirmary; Dean, Edinburgh Dental Hospital and School; Lecturer on Human and Comparative Dental Anatomy and Physiology, 11 Wemyss Place, Edinburgh	
1910	B. C.	Gwynne-Vaughan, D. T., F.L.S., Professor of Botany, Queen's University, Belfast, The Cottage, Balmoral, Belfast	245
1911		Hall-Edwards, John Francis, L.R.C.P. (Edin.), Hon. F.R.P.S., Senior Medical Officer in charge of X-ray Department, General Hospital, Birmingham, 103 Newhall Street, Birmingham	
1905	B. C.	* Halm, Jacob E., Ph.D., Chief Assistant Astronomer, Royal Observatory, Cape Town, Cape of Good Hope	
1899		Hamilton, Allan M'Lane, M.D., 950 Madison Avenue, New York	
1876	C.	Hannay, J. Ballantyne, Cove Castle, Loch Long	
1896		* Harris, David, Fellow of the Statistical Society, Lyncombe Rise, Prior Park Road, Bath	250
1896	C.	* Harris, David Fraser, B.Sc. (Lond.), M.D., F.S.A. Scot., Professor of Physiology in the Dalhousie University, Halifax, Nova Scotia	
1888	C.	* Hart, D. Berry, M.D., F.R.C.P.E., 5 Randolph Cliff	
1869		Hartley, Sir Charles A., K.C.M.G., Memb. Inst. C.E., 26 Pall Mall, London	
1877	C.	Hartley, Sir W. N., D.Sc., F.R.S., F.I.C., Fellow of King's College, London, 10 Elgin Road, Dublin	
1881		Harvie-Brown, J. A., of Quarter, F.Z.S., Dunipace House, Larbert, Stirlingshire	255
1880	C.	Haycraft, J. Berry, M.D., D.Sc., Professor of Physiology in the University College of South Wales and Monmouthshire, Cardiff	
1892	C.	* Heath, Thomas, B.A., Assistant Astronomer, Royal Observatory, Edinburgh, 11 Cluny Drive	
1893		Hehir, Patrick, M.D., F.R.C.S.E., M.R.C.S.L., L.R.C.P.E., Surgeon-Captain, Indian Medical Service, Principal Medical Officer, H.H. the Nizam's Army, Hyderabad, Deccan, India	
1890	C.	Helme, T. Arthur, M.D., M.R.C.P.L., M.R.C.S., 3 St Peter's Square, Manchester	
1900		Henderson, John, D.Sc., Assoc. Inst. E.E., Kinnoul, Warwick's Bench Road, Guildford, Surrey	260
1908		* Henderson, William Dawson, M.A., B.Sc., Ph.D., Lecturer, Zoological Laboratories, University, Manchester	
1890	C.	Hepburn, David, M.D., Professor of Anatomy in the University College of South Wales and Monmouthshire, Cardiff	
1881	C. N.	Herdman, W. A., D.Sc., F.R.S., Pres.L.S., Professor of Natural History in the University of Liverpool, Croxteth Lodge, Ullet Road, Liverpool	
1908		* Hewat, Archibald, F.F.A., F.I.A., 13 Eton Terrace	

Date of Election.			
1894		Hill, Alfred, M.D., M.R.C.S., F.I.C., Valentine Mount, Freshwater Bay, Isle of Wight	265
1902		* Hinxman, Lionel W., B.A., Geological Survey Office, 33 George Square	
1904		Hobday, Frederick T. G., F.R.C.V.S., 6 Berkely Gardens, Kensington, London, W.	
1885		Hodgkinson, W. R., Ph.D., F.I.C., F.C.S., Professor of Chemistry and Physics at the Royal Military Academy and Royal Artillery College, Woolwich, 89 Shooter's Hill Road, Blackheath, Kent	
1911		Holland, William Jacob, LL.D. St Andrews, etc., Director Carnegie Institute, Pittsburg, Pa., 5545 Forbes Street, Pittsburg, Pa.	
1881	C. N.	Horne, John, LL.D., F.R.S., F.G.S., formerly Director of the Geological Survey of Scotland (Vice-President), 12 Keith Crescent, Blackhall	270
1896		Horne, J. Fletcher, M.D., F.R.C.S.E., The Poplars, Barnsley	
1904		* Horsburgh, Ellice Martin, M.A., B.Sc., Lecturer in Technical Mathematics, University of Edinburgh, 11 Granville Terrace	
1897		Houston, Alex. Cruikshanks, M.B., C.M., D.Sc., 19 Fairhazel Gardens, South Hampstead, London, N.W.	
1893		Howden, Robert, M.A., M.B., C.M., Professor of Anatomy in the University of Durham, 14 Burdon Terrace, Newcastle-on-Tyne	
1899		Howie, W. Lamond, F.C.S., 26 Neville Court, Abbey Road, Regent's Park, London, N.W.	275
1883	C.	Hoyle, William Evans, M.A., D.Sc., M.R.C.S., Crowland, Llandaff, Wales	
1910		Hume, William Fraser, D.Sc. (Lond.), Director, Geological Survey of Egypt, Helwân, Egypt	
1886		Hunt, Rev. H. G. Bonavia, Mus.D. Dub., Mus.B. Oxon., The Vicarage, Burgess Hill, Sussex	
1911		Hunter, Gilbert Macintyre, Assoc. M. Inst. C.E., M.I.E.S, Resident Engineer Nitrate Railways, Iquique, Chile, and Maybole, Ayrshire	
1887	C.	Hunter, James, F.R.C.S.E., F.R.A.S., Rosetta, Liberton, Midlothian	280
1887	C.	Hunter, William, M.D., M.R.C.P.L. and E., M.R.C.S., 54 Harley Street, London	
1908		Hyslop, Theophilus Bulkeley, M.D., M.R.C.P.E., 5 Portland Place, London, W.	
1882	C.	Inglis, J. W., Memb. Inst. C.E., 26 Pitt Street	
1906		* Innes, Alexander Taylor, LL.D., M.A., Advocate, 48 Morningside Park	
1904	C.	Innes, R. T. A., Director, Government Observatory, Johannesburg, Transvaal	285
1904		* Ireland, Alexander Scott, S.S.C., 2 Buckingham Terrace	
1875		Jack, William, M.A., LL.D., Emeritus Professor of Mathematics in the University of Glasgow	
1894		Jackson, Sir John, LL.D., 48 Belgrave Square, London	
1889		* James, Alexander, M.D., F.R.C.P.E., 14 Randolph Crescent	
1882		Jamieson, Prof. A., Memb. Inst. C.E., 16 Rosslyn Terrace, Kelvinside, Glasgow	290
1901		* Jardine, Robert, M.D., M.R.C.S. Eng., F.R.F.P. and S. Glas., 20 Royal Crescent, Glasgow	
1906	C.	* Jehu, Thomas James, M.A., M.D., F.G.S., Lecturer in Geology, University of St Andrews, St Ronan's, St Andrews	
1900		* Jerdan, David Smiles, M.A., D.Sc., Ph.D., Temora, Colinton, Midlothian	
1895		Johnston, Lieut.-Col. Henry Halcro, C.B., R.A.M.S., D.Sc., M.D., F.L.S., Orphir House, Kirkwall Orkney	
1903	C.	* Johnston, Thomas Nicol, M.B., C.M., Pogbie, Upper Keith, East Lothian	295
1902		Johnstone, George, Lieut. R.N.R., formerly Marine Superintendent, British India Steam Navigation Co., 26 Comiston Drive	
1874		Jones, Francis, M.Sc., Lecturer on Chemistry, Beaufort House, Alexandra Park, Manchester	
1905		Jones, George William, M.A., B.Sc., LL.B., Scottish Tutorial Institute, Edinburgh and Glasgow, 25 North Bridge, Coraldene, Kirk Brae, Liberton	
1888		Jones, John Alfred, Memb. Inst. C.E., Fellow of the University of Madras, Sanitary Engineer to the Government of Madras, c/o Messrs Parry & Co., 70 Gracechurch Street, London	
1907		* Kemp, John, M.A., Head Master, High School, Kelso	300
1909		Kenwood, Henry Richard, M.B., Chadwick Professor of Hygiene in the University of London, 126 Queen's Road, Finsbury Park, London, N.	
1908		* Kerr, Andrew William, F.S.A. Scot., Royal Bank House, St Andrew Square	
1903	C. N.	* Kerr, John Graham, M.A., Professor of Zoology in the University of Glasgow	
1891		Kerr, Joshua Law, M.D., Biddenden Hall, Cranbrook, Kent	
1908		Kidd, Walter Aubrey, M.D., 12 Montpelier Row, Blackheath, London	305

Date of Election			
1886	C. N.	Kidston, Robert, LL.D., F.R.S., F.G.S. (SECRETARY), 12 Clarendon Place, Stirling	
1907		* King, Archibald, M.A., B.Sc., formerly Rector of the Academy, Castle Douglas; Junior Inspector of Schools, La Maisonnette, Clarkston, Glasgow	
1880		King, W. F., Lonend, Russell Place, Trinity	
1883		Kinnear, the Right Hon. Lord, P.C., one of the Senators of the College of Justice, 2 Moray Place	
1878		Kintore, the Right Hon. the Earl of, M.A. Cantab., LL.D., Cambridge, Aberdeen and Adelaide, Keith Hall, Inverurie, Aberdeenshire	310
1901		* Knight, Rev. G. A. Frank, M.A., St Leonard's United Free Church, Perth	
1907		* Knight, James, M.A., D.Sc., F.C.S., F.G.S., Head Master, St James' School, Glasgow, The Shieling, Uddingston, by Glasgow	
1880	C. K.	Knott, C. G., D.Sc., Lecturer on Applied Mathematics in the University of Edinburgh (formerly Professor of Physics, Imperial University, Japan) (SECRETARY), 42 Upper Gray Street, Edinburgh	
1886		Laing, Rev. George P., 17 Buckingham Terrace	
1878	C.	Lang, P. R. Scott, M.A., B.Sc., Professor of Mathematics, University of St Andrews	315
1910		* Lauder, Alexander, D.Sc., Lecturer in Agricultural Chemistry, Edinburgh and East of Scotland College of Agriculture, 13 George Square, Edinburgh	
1885	C.	Laurie, A. P., M.A., D.Sc., Principal of the Heriot-Watt College, Edinburgh	
1894	C.	* Laurie, Malcolm, B.A., D.Sc., F.L.S., The Bloom, Canaan Lane, Edinburgh	
1910	C.	* Lawson, A. Anstruther, D.Sc., Lecturer in Botany, University of Glasgow, 66 Hillhead Street, Glasgow	
1905		* Lawson, David, M.A., M.D., L.R.C.P. and S.E., Druimdarroch, Banchory, Kincardineshire	320
1910	C.	* Lee, Gabriel W., D.Sc., Palæontologist, Geological Survey of Scotland, 33 George Square, Edinburgh	
1903		* Leighton, Gerald Rowley, M.D., Sunnyside, Russell Place	
1874	C. K.	Letts, E. A., Ph.D., F.I.C., F.C.S., Professor of Chemistry, Queen's College Belfast	
1910		Levie, Alexander, F.R.C.V.S., D.V.S.M., Veterinary Surgeon, Lecturer on Veterinary Science, Veterinary Infirmary, 12 Derwent Street, Derby	
1905		* Lightbody, Forrest Hay, 56 Queen Street	325
1889		* Lindsay, Rev. James, M.A., D.D., F.R.S.L., B.Sc., F.G.S., M.R.A.S., Corresponding Member of the Royal Academy of Sciences, Letters and Arts, of Padua, Associate of the Philosophical Society of Louvain, Annick Lodge, Irvine	
1870	C. B.	Lister, the Right Hon. Lord, O.M., P.C., M.D., F.R.C.S.L., F.R.C.S.E., LL.D., D.C.L., F.R.S., Foreign Associate of the Institute of France, Emeritus Professor of Clinical Surgery, King's College, Surgeon Extraordinary to the King, 12 Park Crescent, Portland Place, London	
1903		Liston, William Glen, M.D., Captain, Indian Medical Service, c/o Grindlay, Groom & Co., Bombay, India	
1903		* Littlejohn, Henry Harvey, M.A., M.B., B.Sc., F.R.C.S.E., Professor of Forensic Medicine in the University of Edinburgh, 11 Rutland Street	
1898		* Lothian, Alexander Veitch, M.A., B.Sc., Glendoune, Manse Road, Bearsden, Glasgow	330
1884		Low, George M., Actuary, 11 Moray Place	
1888		* Lowe, D. F., M.A., LL.D., formerly Head Master of Heriot's Hospital School, Lauriston, 19 George Square	
1904		* Lowson, Charles Stewart, M.B., C.M., Major, Indian Medical Service, c/o Messrs Thomas Cook & Son, Bombay, India.	
1900		Lusk, Graham, Ph.D., M.A., Professor of Physiology, University and Bellevue Medical College, New York.	
1894		* Mabbott, Walter John, M.A., Rector of County High School, Duns, Berwickshire	335
1887		M'Aldowie, Alexander M., M.D., Glengarriff, Leckhampton, Cheltenham	
1907		MacAlister, Donald Alexander, A.R.S.M., F.G.S., 26 Thurloe Square, South Kensington, London, S.W.	
1891		Macallan, John, F.I.C., 3 Rutland Terrace, Clontarf, Dublin	
1888	C.	M'Arthur, John, F.C.S., Woodfield, Maplehurst, Horsham, Sussex	
1883		M'Bride, P., M.D., F.R.C.P.E., 10 Park Avenue, Harrogate, and Hill House, Withypool, Dunster, Somerset	340
1903		* M'Cormick, Sir W. S., M.A., LL.D., 13 Douglas Crescent	
1899		* M'Cubbin, James, B.A., Rector of the Burgh Academy, Kilsyth	

Date of Election.		
1905		* Macdonald, Hector Munro, M.A., F.R.S., Professor of Mathematics, University of Aberdeen, 52 College Bounds, Aberdeen
1894		* Macdonald, James, Secretary of the Highland and Agricultural Society of Scotland, 2 Garscube Terrace
1897	C.	* Macdonald, James A., M.A., B.Sc., H.M. Inspector of Schools, Stewarton, Kilmacolm 345
1904		* Macdonald, John A., M.A., B.Sc., High School, Stellenbosch, Cape Colony
1886		Macdonald, the Right Hon. Sir J. H. A., P.C., K.C.B., K.C., LL.D., F.R.S., M.I.E.E., Lord Justice-Clerk, and Lord President of the Second Division of the Court of Session, 15 Abercromby Place
1904		Macdonald, William, B.Sc., M.Sc., Agriculturist, Editor *Transvaal Agricultural Journal*, Department of Agriculture, Pretoria Club, Pretoria, Transvaal
1886		Macdonald, William J., M.A., 15 Comiston Drive
1901	C.	* MacDougall, R. Stewart, M.A., D.Sc., 9 Dryden Place 350
1910		Macewen, Hugh Allan, M.B., Ch.B., D.P.H. (Lond. and Camb.), Local Government Board, Whitehall, London, S.W.
1888	C.	* M'Fadyean, Sir John, M.B., B.Sc., LL.D., Principal, and Professor of Comparative Pathology in the Royal Veterinary College, Camden Town, London
1878	C.	Macfarlane, Alexander, M.A., D.Sc., LL.D., Lecturer in Physics in Lehigh University, Pennsylvania, Gowrie Grove, Chatham, Ontario, Canada
1885	C.	Macfarlane, J. M., D.Sc., Professor of Botany and Director of the Botanic Garden, University of Pennsylvania, Philadelphia, Pennsylvania, U.S.A.
1897		* MacGillivray, Angus, C.M., M.D., South Tay Street, Dundee 355
1878		M'Gowan, George, F.I.C., Ph.D., 21 Montpelier Road, Ealing, Middlesex
1880	C.	MacGregor, James Gordon, M.A., D.Sc., LL.D., F.R.S., Professor of Natural Philosophy in the University of Edinburgh, 24 Dalrymple Crescent
1903		* M'Intosh, D. C., M.A., B.Sc., 3 Glenisla Gardens
1911		M'Intosh, John William, A.R.C.V.S., Grasmead, 88 Underhill Road, E. Dulwich, London, S.E.
1869	C. N.	M'Intosh, William Carmichael, M.D., LL.D., F.R.S., F.L.S., Professor of Natural History in the University of St Andrews, 2 Abbotsford Crescent, St Andrews 360
1895	C.	* Macintyre, John, M.D., 179 Bath Street, Glasgow
1882		Mackay, John Sturgeon, M.A., LL.D., formerly Mathematical Master in the Edinburgh Academy, 69 Northumberland Street
1873	C. B.	M'Kendrick, John G., M.D., F.R.C.P.E., LL.D., F.R.S., Emeritus Professor of Physiology in the University of Glasgow, Maxieburn, Stonehaven
1900	C.	* M'Kendrick, John Souttar, M.D., F.R.F.P.S.G., 2 Buckingham Terrace, Glasgow
1910	C.	* MacKenzie, Alister Thomas, M.A., M.D., D.P.H., Research Fellow of the University of Edinburgh, Alladale, Alness, Ross-shire 365
1911		* M'Kenzie, Kenneth John, M.A., Master of Method to Leith School Board, 17 East Trinity Road, Leith
1894		* Mackenzie, Robert, M.D., Napier, Nairn
1904		* Mackenzie, W. Leslie, M.A., M.D., D.P.H., Medical Member of the Local Government Board for Scotland, 1 Stirling Road, Trinity
1905		Mackenzie, William Colin, M.D., F.R.C.S., Demonstrator of Anatomy in the University of Melbourne, Elizabeth Street North, Melbourne, Victoria
1910		* MacKinnon, James, M.A., Ph.D., Professor of Ecclesiastical History, Edinburgh University, 12 Lygon Road, Edinburgh 370
1904		* Mackintosh, Donald James, M.V.O., M.B., Supt. Western Infirmary, Glasgow
1869	C.	Maclagan, R. C., M.D., F.R.C.P.E., 5 Coates Crescent
1899		Maclean, Ewan John, M.D., M.R.C.P. Lond., 12 Park Place, Cardiff
1888	C.	* Maclean, Magnus, M.A., D.Sc., Memb. Inst. E.E., Professor of Electrical Engineering in the Glasgow and West of Scotland Technical College, 51 Kerrsland Terrace, Hillhead, Glasgow
1876		Macleod, Very Rev. Norman, D.D., 74 Murrayfield Gardens 375
1876		Macmillan, John, M.A., D.Sc., M.B., C.M., F.R.C.P.E., F.R.C.S.E., 48 George Square
1893		* M'Murtrie, Very Rev. John, M.A., D.D., 13 Inverleith Place
1906		* Macnair, Duncan Scott, Ph.D., B.Sc., H.M. Inspector of Schools, 67 Braid Avenue
1907		* Macnair, Peter, Curator of the Natural History Collections in the Glasgow Museums, Kelvingrove Museum, Glasgow
1898	C.	Mahâlanobis, S. C., B.Sc., Professor of Physiology, Presidency College, Calcutta, India 380
1908		Mallik, Devendranath, B.A., B.Sc., Professor of Physics and Mathematics, Patna College, Bankipur, Bengal, India

Date of Election.			
1880	C.	Marsden, R. Sydney, M.D., C.M., D.Sc., D.P.H. Hon., L.A.H. Dub., M.R.I.A., F.I.C, F.C.S., Rowallan House, Cearns Road, and Town Hall, Birkenhead	
1909	C.	* Marshall, C. R., M.D., M.A., Professor of Materia Medica and Therapeutics, Medical School, Dundee, Arnsheen, Westfield Terrace, West Newport, Fife	
1882	C.	Marshall, D. H., M.A., Professor, Union and Alwington Avenue, Kingston, Ontario, Canada	
1901	C.	* Marshall, F. H. A., M.A., D.Sc., Lecturer on Agricultural Physiology in the University of Cambridge, Christ's College, Cambridge	385
1888	C. K.	* Marshall, Hugh, D.Sc., F.R.S., Professor of Chemistry in University College, Dundee (University of St Andrews)	
1903		Martin, Nicholas Henry, F.L.S., F.C.S., Ravenswood, Low Fell, Gateshead	
1885	C.	Masson, Orme, D.Sc., F.R.S., Professor of Chemistry in the University of Melbourne	
1898	C. B.	* Masterman, Arthur Thomas, M.A., D.Sc., Inspector of Fisheries, Board of Agriculture, Whitehall, London	
1911		Mathews, Gregory Macalister, F.L.S., F.Z.S., Langley Mount, Watford, Herts	390
1906		* Mathieson, Robert, F.C.S., Rillbank, Innerleithen	
1902		Matthews, Ernest Romney, Assoc. Memb. Inst. C.E., F.G.S., Bessemer Prizeman, Soc. Engineers, Bridlington, Yorkshire	
1901	C.	* Menzies, Alan W. C., M.A., B.Sc., Ph.D., F.C.S., Assistant Professor of Chemistry, University, Chicago, U.S.A.	
1888		* Methven, Cathcart W., Memb. Inst. C.E., F.R.I.B.A., Durban, Natal, S. Africa	
1902	C.	Metzler, William H., A.B., Ph.D., Corresponding Fellow of the Royal Society of Canada, Professor of Mathematics, Syracuse University, Syracuse, N.Y.	395
1885	C. B.	Mill, Hugh Robert, D.Sc., LL.D., 62 Camden Square, London	
1908		* Miller, Alexander Cameron, M.D., F.S.A. Scot., Craig Linnhe, Fort-William, Inverness-shire	
1910		* Miller, John, M.A., D.Sc., Professor of Mathematics, Glasgow and West of Scotland Technical College, 2 Northbank Terrace, North Kelvinside, Glasgow	
1905		* Miller-Milne, C. H., M.A., Headmaster, Daniel Stewart's College, 4 Campbell Road, Murrayfield, Edinburgh	
1909		Mills, Bernard Langley, M.D., F.R.C.S.E., M.R.C.S.L., D.P.H., Lt.-Col. R.A.M.C., formerly Army Specialist in Hygiene, 84 Grange Crescent, Sharrow, Sheffield	400
1905		* Milne, Archibald, M.A., B.Sc., Lecturer on Mathematics and Science, Edinburgh Provincial Training College, 108 Comiston Drive	
1904	C.	* Milne, James Robert, D.Sc., 11 Melville Crescent	
1886		Milne, William, M.A., B.Sc., 70 Beechgrove Terrace, Aberdeen	
1899		* Milroy, T. H., M.D., B.Sc., Professor of Physiology in Queen's College, Belfast, Thornlee, Malone Park, Belfast	
1889	C.	Mitchell, A. Crichton, D.Sc., Director of Public Instruction in Travancore, India	405
1897		* Mitchell, George Arthur, M.A., 9 Lowther Terrace, Kelvinside, Glasgow	
1900		* Mitchell, James, M.A., B.Sc., 4 Manse Street, Kilmarnock	
1899		* Mitchell-Thomson, Sir Mitchell, Bart., 6 Charlotte Square	
1911		Modi, Edalji Manekji, D.Sc., LL.D., Litt.D., F.C.S., etc., Proprietor and Director of Arthur Road Chemical Works, Meher Buildings, Tardeo, Bombay, India	
1906	C.	Moffat, Rev. Alexander, M.A., B.Sc., Professor of Physical Science, Christian College, Madras, India	410
1890	C.	Mond, R. L., M.A. Cantab., F.C.S., The Poplars, 20 Avenue Road, Regent's Park, London	
1887	C.	Moos, N. A. F., L.C.E., B.Sc., Professor of Physics, Elphinstone College, and Director of the Government Observatory, Colaba, Bombay	
1896		* Morgan, Alexander, M.A., D.Sc., Principal, Edinburgh Provincial Training College, 1 Midmar Gardens	
1892	C.	Morrison, J. T., M.A., B.Sc., Professor of Physics and Chemistry, Victoria College, Stellenbosch, Cape Colony	
1901		Moses, O. St John, I.M.S., M.D., D.Sc., F.R.C.S., Captain, Professor of Medical Jurisprudence, 26 Park Street, Wellesley, Calcutta, India	415
1892	C.	Mossman, Robert C., Superintendent of Publications, Argentine Meteorological Office, Cuyo 947, Buenos Ayres	
1874	C. K.	Muir, Thomas, C.M.G., M.A., LL.D., F.R.S., Superintendent-General of Education for Cape Colony, Education Office, Cape Town, and Mowbray Hall, Rosebank, Cape Colony	

Date of Election.			
1888	C.	* Muirhead, George, Commissioner to His Grace the Duke of Richmond and Gordon, K.G., Speybank, Fochabers	
1907		Muirhead, James M. P., J.P., F.R.S.L., F.S.S., Markham's Buildings, Cape Town	
1887		Mukhopâdhyay, Asûtosh, M.A., LL.D., F.R.A.S., M.R.I.A., Professor of Mathematics at the Indian Association for the Cultivation of Science, 77 Russa Road North, Bhowanipore, Calcutta	420
1891	C.	* Munro, Robert, M.A., M.D., LL.D., Hon. Memb. R.I.A., Hon. Memb. Royal Society of Antiquaries of Ireland, Elmbank, Largs, Ayrshire	
1896		* Murray, Alfred A., M.A., LL.B., 20 Warriston Crescent	
1907	C.	* Murray, James, Woodhouse, Whitchurch Lane, Edgeware, Middlesex, England	
1877	C. B. N.	Murray, Sir John, K.C.B., LL.D., D.C.L., Ph.D., D.Sc., F.R.S., Member of the Prussian Order *Pour le Mérite*, Director of the Challenger Expedition Publications. Office, Villa Medusa, Boswell Road. House, Challenger Lodge, Wardie, and United Service Club	
1907		* Musgrove, James, M.D., F.R.C.S. Edin. and Eng., Bute Professor of Anatomy, University of St Andrews, 56 South Street, St Andrews	425
1887		Muter, John, M.A., F.C.S., South London Central Public Laboratory, 325 Kennington Road, London	
1902		Mylne, Rev. R. S., M.A., B.C.L. Oxford, F.S.A. Lond., Great Amwell, Herts	
1888		Napier, A. D. Leith, M.D., C.M., M.R.C.P.L., 28 Angas Street, Adelaide, S. Australia	
1897		Nash, Alfred George, B.Sc., F.R.G.S., C.E., Belretiro, Mandeville, Jamaica, W.I.	
1906		* Newington, Frank A., Memb. Inst. C.E., Memb. Inst. E.E., 4 Osborne Terrace	430
1898		Newman, Sir George, M.D., D.P.H. Cambridge, Lecturer on Preventive Medicine, St Bartholomew's Hospital, University of London: Dene, Hatch End, Middlesex	
1884		Nicholson, J. Shield, M.A., D.Sc., Professor of Political Economy in the University of Edinburgh, 3 Belford Park	
1880	C.	Nicol, W. W. J., M.A., D.Sc., 15 Blacket Place	
1878		Norris, Richard, M.D., M.R.C.S. Eng., 3 Walsall Road, Birchfield, Birmingham	
1906		* O'Connor, Henry, C.E., Assoc. Memb. Inst. C.E., 1 Drummond Place	435
1888		* Ogilvie, F. Grant, C.B., M.A., B.Sc., LL.D., Director of the Science Museum and the Geological Survey, 15 Evelyn Gardens, London, S.W.	
1888		* Oliphant, James, M.A., 11 Heathfield Park, Willesden, London	
1886		Oliver, James, M.D., F.L.S., Physician to the London Hospital for Women, 18 Gordon Square, London	
1895	C.	Oliver, Sir Thomas, M.D., LL.D., F.R.C.P., Professor of Physiology in the University of Durham, 7 Ellison Place, Newcastle-upon-Tyne	
1884	C. K.	Omond, R. Traill, 3 Church Hill	440
1908		Page, William Davidge, F.C.S., F.G.S., M. Inst. M.E., 10 Clifton Dale, York	
1905		Pallin, William Alfred, F.R.C.V.S., Captain in the Army Veterinary Department, c/o Messrs Holt & Co., 3 Whitehall Place, London	
1892		Parker, Thomas, Memb. Inst. C.E., Severn House, Iron Bridge, Salop	
1901		* Paterson, David, F.C.S., Lea Bank, Rosslyn, Midlothian	
1886	C.	Paton, D. Noël, M.D., B.Sc., F.R.C.P.E., Professor of Physiology in the University of Glasgow, University, Glasgow	445
1889		* Patrick, David, M.A., LL.D., c/o W. & R. Chambers, 339 High Street	
1892		* Paulin, Sir David, Actuary, 6 Forres Street	
1881	C. N.	Peach, Benjamin N., LL.D., F.R.S., F.G.S., formerly District Superintendent and Acting Palæontologist of the Geological Survey of Scotland, 72 Grange Loan	
1907		* Pearce, John Thomson, B.A., B.Sc., School House, Tranent	
1904		* Peck, James Wallace, M.A., Clerk to Edinburgh School Board, School Board Offices, Castle Terrace	450
1889		* Peck, William, F.R.A.S., Town's Astronomer, City Observatory, Calton Hill, Edinburgh	
1887	C. B.	Peddie, Wm., D.Sc., Professor of Natural Philosophy in University College, Dundee, Rosemount, Forthill Road, Broughty Ferry	
1900		Penny, John, M.B., C.M., D.Sc., Great Broughton, near Cockermouth, Cumberland	
1893		Perkin, Arthur George, F.R.S., 8 Montpellier Terrace, Hyde Park, Leeds	

Date of Election.			
1889		* Philip, R. W., M.A., M.D., F.R.C.P.E., 45 Charlotte Square	455
1907	C.	Phillips, Charles E. S., Castle House, Shooter's Hill, Kent	
1905		* Pinkerton, Peter, M.A., D.Sc., Head Mathematical Master, George Watson's College, Edinburgh, 36 Morningside Grove	
1908	C.	* Pirie, James Hunter Harvey, B.Sc., M.D., F.R.C.P.E., 5 Castle Terrace	
1911		* Pirie, James Simpson, Civil Engineer, 28 Scotland Street, Edinburgh	
1906		Pitchford, Herbert Watkins, F.R.C.V.S., Bacteriologist and Analyst, Natal Government, The Laboratory, Pietermaritzburg, Natal	460
1886		Pollock, Charles Frederick, M.D., F.R.C.S.E., 1 Buckingham Terrace, Hillhead, Glasgow	
1888		Prain, David, Lt.-Col., Indian Medical Service, M.A., M.B., LL.D., F.L.S., F.R.S., Hon. Memb. Soc. Lett. ed Arti d. Zelanti, Acireale; Corr. Memb. Pharm. Soc. Gt. Britain, etc.; Director, Royal Botanic Gardens, Kew (formerly Director, Botanical Survey of India, Calcutta), Botanic Gardens, Kew	
1902		* Preller, Charles Du Riche, M.A., Ph.D., Assoc. Memb. Inst. C.E., 61 Melville Street	
1892		* Pressland, Arthur, J., M.A. Camb., Edinburgh Academy	
1875	C.	Prevost, E. W., Ph.D., Weston, Ross, Herefordshire	465
1908		* Pringle, George Cossar, M.A., Rector of Peebles Burgh and County High School, Bloomfield, Peebles	
1885		Pullar, J. F., Rosebank, Perth	
1903		* Pullar, Laurence, The Lea, Bridge of Allan	
1880		Pullar, Sir Robert, LL.D., formerly M.P. for the city of Perth, Tayside, Perth	
1911		Purdy, John Smith, M.D., C.M. (Aberd.), D.P.H. (Camb.), F.R.G.S., Chief Health Officer for Tasmania, Islington, Hobart, Tasmania	470
1898		* Purves, John Archibald, D.Sc., 13 Albany Street	
1897		* Rainy, Harry, M.A., M.B., C.M., F.R.C.P. Ed., 16 Great Stuart Street	
1899		* Ramage, Alexander G., 8 Western Terrace, Murrayfield	
1884		Ramsay, E. Peirson, M.R.I.A., F.L.S., C.M.Z.S., F.R.G.S., F.G.S., Fellow of the Imperial and Royal Zoological and Botanical Society of Vienna, Curator of Australian Museum, Sydney, N.S.W.	
1911		* Rankin, Adam A., formerly Secretary British Astronomical Association, West of Scotland Branch, 9 Sutherland Street, Hillhead, Glasgow	475
1891		* Rankine, John, K.C., M.A., LL.D., Professor of the Law of Scotland in the University of Edinburgh, 23 Ainslie Place	
1904		Ratcliffe, Joseph Riley, M.B., C.M., c/o The Librarian, The University, Birmingham	
1900		Raw, Nathan, M.D., 66 Rodney Street, Liverpool	
1883	C.	Readman, J. B., D.Sc., F.C.S., Belmont, Hereford	
1889		Redwood, Sir Boverton, Bt., D.Sc. (Hon.), F.I.C., F.C.S., Assoc. Inst. C.E., Wadham Lodge, Wadham Gardens, London	480
1902		Rees-Roberts, John Vernon, M.D., D.Sc., D.P.H., Barrister-at-Law, National Liberal Club, Whitehall Place, London	
1902		Reid, George Archdall O'Brien, M.B., C.M., 9 Victoria Road South, Southsea, Hants	
1908	C.	* Rennie, John, D.Sc., Lecturer on Parasitology, and Assistant to the Professor of Natural History, University of Aberdeen, 60 Desswood Place, Aberdeen	
1908		Richardson, Linsdall, F.L.S., F.G.S., Organising Inspector of Technical Education for the Gloucestershire Education Committee, 10 Oxford Parade, Cheltenham	
1875		Richardson, Ralph, W.S., 10 Magdala Place	485
1906	C.	* Ritchie, William Thomas, M.D., F.R.C.P.E., 9 Atholl Place	
1898	C.	Roberts, Alexander William, D.Sc., F.R.A.S., Lovedale, South Africa	
1880		Roberts, D. Lloyd, M.D., F.R.C.P.L., 23 St John Street, Manchester	
1900		* Robertson, Joseph M'Gregor, M.B., C.M., 26 Buckingham Terrace, Glasgow	
1896		* Robertson, Robert, M.A., 25 Mansionhouse Road	490
1902	C.	* Robertson, Robert A., M.A., B.Sc., Lecturer on Botany in the University of St Andrews	
1896	C.	* Robertson, W. G. Aitchison, D.Sc., M.D., F.R.C.P.E., 2 Mayfield Gardens	
1910		* Robinson, Arthur, M.D., M.R.C.S., Professor of Anatomy, University of Edinburgh, 35 Coates Gardens, Edinburgh	
1881		Rosebery, the Right Hon. the Earl of, K.G., K.T., LL.D., D.C.L., F.R.S., Dalmeny Park, Edinburgh	
1909	C.	* Ross, Alex. David, M.A., D.Sc., F.R.A.S., Lecturer in Natural Philosophy in the University of Glasgow, 7 Queen's Terrace, Glasgow	495
1906		* Russell, Alexander Durie, B.Sc., Mathematical Master, Falkirk High School, Dunaura, Heugh Street, Falkirk	
1902	C.	* Russell, James, 12 Argyll Place	

Date of Election.		
1880		Russell, Sir James A., M.A., B.Sc., M.B., F.R.C.P.E., LL.D., Woodville, Canaan Lane
1904		Sachs, Edwin O., Architect, 7 Waterloo Place, Pall Mall, London, S.W.
1906		Saleeby, Caleb William, M.D., 13 Greville Place, London 500
1903		* Samuel, John S., 8 Park Avenue, Glasgow
1903		* Sarolea, Charles, Ph.D., D.Litt., Lecturer on French Language, Literature, and Romance Philology, University of Edinburgh, 21 Royal Terrace
1891		Sawyer, Sir James, Knt., M.D., F.R.C.P., F.S.A., J.P., Consulting Physician to the Queen's Hospital, 31 Temple Row, Birmingham
1900	C.	* Schäfer, Edward Albert, M.R.C.S., LL.D., F.R.S., Professor of Physiology in the University of Edinburgh
1885	C.	Scott, Alexander, M.A., D.Sc., F.R.S., 34 Upper Hamilton Terrace, London, N.W. 505
1880		Scott, J. H., M.B., C.M., M.R.C.S., Professor of Anatomy in the University of Otago, New Zealand
1905		Scougal, A. E., M.A., LL.D., formerly H.M. Senior Chief Inspector of Schools and Inspector of Training Colleges, 1 Wester Coates Avenue
1902		Senn, Nicholas, M.D., LL.D., Professor of Surgery, Rush Medical College, Chicago, U.S.A.
1897		* Shepherd, John William, Carrickarden, Bearsden, Glasgow
1871		Simpson, Sir A. R., M.D., Emeritus Professor of Midwifery in the University of Edinburgh, 52 Queen Street 510
1908		* Simpson, George Freeland Barbour, M.D., F.R.C.P.E., F.R.C.S.E., 43 Manor Place
1900	C.	* Simpson, James Young, M.A., D.Sc., Professor of Natural Science in the New College, Edinburgh, 25 Chester Street
1911	C.	Simpson, Sutherland, M.D., D.Sc. (Edin.), Professor of Physiology, Medical College, Cornell University, Ithaca, N.Y., U.S.A., 118 Eddy Street, Ithaca, N.Y., U.S.A.
1900		Sinhjee, Sir Bhagvat, G.C.I.E., M.D., LL.D. Edin., H.H. the Thakur Sahib of Gondal, Gondal, Kathiawar, Bombay, India
1903		* Skinner, Robert Taylor, M.A., Governor and Head Master, Donaldson's Hospital, Edinburgh 515
1901		* Smart, Edward, B.A., B.Sc., Tillyloss, Tullylumb Terrace, Perth
1891	C.	* Smith, Alexander, B.Sc., Ph.D., Department of Chemistry, Columbia University, New York, N.Y., U.S.A.
1882	C.	Smith, C. Michie, B.Sc., F.R.A.S., Director of the Kodaikânal and Madras Observatories, The Observatory, Kodaikânal, South India
1885		Smith, George, F.C.S., 5 Rosehall Terrace, Falkirk
1911		* Smith, Stephen, B.Sc., Goldsmith, 12 Murrayfield Avenue, Edinburgh 520
1907	C.	Smith, William Ramsay, D.Sc., M.B., C.M., Permanent Head of the Health Department, South Australia, Winchester Street, East Adelaide, South Australia
1880		Smith, William Robert, M.D., D.Sc., Barrister-at-Law, Professor of Forensic Medicine in King's College, 74 Great Russell Street, Bloomsbury Square, London
1899		Snell, Ernest Hugh, M.D., B.Sc., D.P.H. Camb., Coventry
1880		Sollas, W. J., M.A., D.Sc., LL.D., F.R.S., formerly Fellow of St John's College, Cambridge, and Professor of Geology and Palæontology in the University of Oxford
1910		* Somerville, Robert, B.Sc., Science Master, High School, Dunfermline, 38 Cameron Street, Dunfermline 525
1889	C.	Somerville, Wm., M.A., D.Sc., D.Oec., Sibthorpian Professor of Rural Economy in the University of Oxford, 121 Banbury Road, Oxford
1911	C.	* Sommerville, Duncan M'Laren Young, M.A., D.Sc., Lecturer in Mathematics and in Applied Mathematics, University of St Andrews, 70 Argyle Street, St Andrews
1882		Sorley, James, F.I.A., F.F.A., C.A., 82 Onslow Gardens, London
1896		* Spence, Frank, M.A., B.Sc., 25 Craiglea Drive
1874	C.	Sprague, T. B., M.A., LL.D., Actuary, 29 Buckingham Terrace 530
1906		Squance, Thomas Coke, M.D., Physician and Pathologist in the Sunderland Infirmary, 15 Grange Crescent, Sunderland
1891		* Stanfield, Richard, Professor of Mechanics and Engineering in the Heriot-Watt College
1910		* Stephenson, Thomas, F.C.S., Editor of the *Prescriber*, Examiner to the Pharmaceutical Society, 9 Woodburn Terrace, Edinburgh
1886	C.	Stevenson, Charles A., B.Sc., Memb. Inst. C.E., 28 Douglas Crescent

Date of Election.			
1884		Stevenson, David Alan, B.Sc., Memb. Inst. C.E., 84 George Street	535
1888	C.	* Stewart, Charles Hunter, D.Sc., M.B., C.M., Professor of Public Health in the University of Edinburgh, Usher Institute of Public Health, Warrender Park Road	
1902		* Stockdale, Herbert Fitton, Director of the Glasgow and West of Scotland Technical College, Clairinch, Upper Helensburgh, Dumbartonshire	
1889		* Stockman, Ralph, M.D., F.R.C.P.E., Professor of Materia Medica and Therapeutics in the University of Glasgow	
1906		Story, Fraser, Lecturer in Forestry, University College, Bangor, North Wales	
1907		* Strong, John, M.A., Rector of Montrose Academy, Linksgate, Montrose	540
1903		Sutherland, David W., M.D., M.R.C.P. Lond., Captain, Indian Medical Service, Professor of Pathology and Materia Medica, Medical College, Lahore, India	
1896		* Sutherland, John Francis, M.D., Deputy Commissioner in Lunacy for Scotland, Scotsburn Road, Tain, Ross-shire	
1905		Swithinbank, Harold William, Denham Court, Denham, Bucks	
1885	C.	Symington, Johnson, M.D., F.R.C.S.E., F.R.S., Professor of Anatomy in Queen's College, Belfast	
1904		* Tait, John W., B.Sc., Rector of Leith Academy, 18 Netherby Road, Leith	545
1898	C.	Tait, William Archer, D.Sc., Memb. Inst. C.E., 38 George Square	
1895		Talmage, James Edward, D.Sc., Ph.D., F.R.M.S., F.G.S., Professor of Geology, University of Utah, Salt Lake City, Utah, U.S.A.	
1890	C.	Tanakadate, Aikitu, Professor of Natural Philosophy in the Imperial University of Japan, Tokyo, Japan	
1870		Tatlock, Robert R., F.C.S., City Analyst's Office, 156 Bath Street, Glasgow	
1899		* Taylor, James, M.A., Mathematical Master in the Edinburgh Academy	550
1892		Thackwell, J. B., M.B., C.M., 423A Battersea Park Road, London, S.W.	
1885	C.	Thompson, D'Arcy W., C.B., B.A., F.L.S., Professor of Natural History in University College, Dundee	
1907		* Thompson, John Hannay, M. Inst. C.E., M. Inst. Mech. E., Engineer to the Dundee Harbour Trust, Earlville, Broughty Ferry	
1905		* Thoms, Alexander, 7 Playfair Terrace, St Andrews	
1887		Thomson, Andrew, M.A., D.Sc., F.I.C., Rector, Perth Academy, Ardenlea, Pitcullen, Perth	555
1911		* Thomson, Frank Wyville, M.A., M.B., C.M., D.P.H., D.T.M., Lt.-Col. I.M.S. (Retired), Bonsyde, Linlithgow	
1896		* Thomson, George Ritchie, M.B., C.M., General Hospital, Johannesburg, Transvaal	
1903		Thomson, George S., F.C.S., Ferma Albion, Marculesci, Roumania	
1906		* Thomson, Gilbert, M.Inst.C.E., 164 Bath Street, Glasgow	
1887	C.	Thomson, J. Arthur, M.A., Regius Professor of Natural History in the University of Aberdeen	560
1906	C.	Thomson, James Stuart, F.L.S., Zoological Department, University, Manchester	
1880		Thomson, John Millar, LL.D., F.R.S., Professor of Chemistry in King's College, London, 18 Lansdowne Road, London, W.	
1899		* Thomson, R. Tatlock, F.C.S., 156 Bath Street, Glasgow	
1870		Thomson, Spencer C., Actuary, 10 Eglinton Crescent	
1882		Thomson, Wm., M.A., B.Sc., LL.D., Registrar, University of the Cape of Good Hope, University Buildings, Cape Town	565
1876	C.	Thomson, William, Royal Institution, Manchester	
1911		* Tosh, James Ramsay, M.A., D.Sc. (St Ands.), Assistant Professor, Dept. of Zoology, University, St Andrews, 20 Mid Street, Dundee	
1874	C. B. N.	Traquair, R. H., M.D., LL.D., F.R.S., F.G.S., formerly Keeper of the Natural History Collections in the Royal Scottish Museum, Edinburgh, The Bush, Colinton	
1874		Tuke, Sir J. Batty, M.D., D.Sc., LL.D., F.R.C.P.E., formerly M.P. for the Universities of Edinburgh and St Andrews, 20 Charlotte Square	
1888		* Turnbull, Andrew H., Actuary, The Elms, Whitehouse Loan	570
1905		* Turner, Arthur Logan, M.D., F.R.C.S.E., 27 Walker Street	
1906	C.	* Turner, Dawson F. D., B.A., M.D., F.R.C.P.E., M.R.C.P. Lond., Lecturer on Physics, Surgeon's Hall, and Physician in charge of Electrical Department, Royal Infirmary, Edinburgh, 37 George Square	
1861	K. N. C.	Turner, Sir William, K.C.B., M.B., F.R.C.S.E., LL.D., D.C.L., D.Sc. (Camb. and Dub.), F.R.S., Principal of the University of Edinburgh (PRESIDENT), 6 Eton Terrace	
1895		Turton, Albert H., M.I.M.M., 18 Harrow Road, Bowenbrook, Birmingham	
1898	C.	* Tweedie, Charles, M.A., B.Sc., Lecturer on Mathematics in the University of Edinburgh, Duns, Berwickshire	575
1889		Underhill, T. Edgar, M.D., F.R.C.S.E., Dunedin, Barnt Green, Worcestershire	

Date of Election.			
1906		Vandenbergh, William J., Barrister-at-Law, S.S.C., F.R.S.L., F.R.M.S., 29–32 Exchange Buildings, Pirie Street, Adelaide, S. Australia	
1910		Vincent, Swale, M.D. Lond, D.Sc. Edin., etc., Professor of Physiology, University of Manitoba, Winnipeg, Canada	
1911	C.	* Walker, Henry, M.A., D.Sc., Teacher, 18 Station Road, Dalbeattie	
1891	C. B.	* Walker, James, D.Sc., Ph.D., LL.D., F.R.S., Professor of Chemistry in the University of Edinburgh, 5 Wester Coates Road	580
1873	C.	Walker, Robert, M.A., LL.D., University, Aberdeen	
1902		* Wallace, Alexander G., M.A., 56 Fonthill Road, Aberdeen	
1886	C.	Wallace, R., F.L.S., Professor of Agriculture and Rural Economy in the University of Edinburgh	
1898		Wallace, Wm., M.A., Belvedere, Alberta, Canada	
1891		* Walmsley, R. Mullineux, D.Sc., Principal of the Northampton Institute, Clerkenwell, London	585
1907		Waters, E. Wynston, Medical Officer, H.B.M. Administration, E. Africa, Malindi, British East Africa Protectorate, *via* Mombasa	
1901	C.	* Waterston, David, M.A., M.D., F.R.C.S.E., Professor of Anatomy, King's College, London	
1904		* Watson, Charles B. Boog, Huntly Lodge, 1 Napier Road	
1911		* Watson, James A. S., B.Sc., etc., Assistant in Agriculture, University of Edinburgh, Downieken, Dundee	
1900		* Watson, Thomas P., M.A., B.Sc., Principal, Keighley Institute, Keighley	590
1910		* Watson, William John, M.A., LL.D. Aberdeen, B.A. Oxon., Rector of the Royal High School, Edinburgh, 17 Merchiston Avenue, Edinburgh	
1907		* Watt, Andrew, M.A., Secretary to the Scottish Meteorological Society, 6 Woodburn Terrace	
1911		Watt, James, W.S., F.F.A., 24 Rothesay Terrace, Edinburgh	
1911		* Watt, Rev. Lauchlan Maclean, B.D., Minister of St. Stephen's Parish, 7 Royal Circus, Edinburgh	
1896		Webster, John Clarence, B.A., M.D., F.R.C.P.E., Professor of Obstetrics and Gynæcology, Rush Medical College, Chicago, 706 Reliance Buildings, 100 State Street, Chicago	595
1907	B. C.	* Wedderburn, Ernest Maclagan, M.A., LL.B., 7 Dean Park Crescent	
1903	C.	* Wedderburn, J. H. Maclagan, M.A., D.Sc., 95 Mercer Street, Princeton, N.J., U.S.A.	
1904		Wedderspoon, William Gibson, M.A., LL.D., Indian Educational Service, Senior Inspector of Schools, Burma, The Education Office, Rangoon, Burma	
1896		Wenley, Robert Mark, M.A., D.Sc., D.Phil., Litt.D., LL.D., Professor of Philosophy in the University of Michigan, Ann Arbor, Michigan, U.S.A.	
1909	C.	* Westergaard, Reginald Ludovic Andreas Emil. Lecturer in Technical Mycology, Heriot-Watt College, Ashestiel, Lasswade Road, Liberton, Edinburgh	600
1896	C.	White, Philip J., M.B., Professor of Zoology in University College, Bangor, North Wales	
1890		White, Sir William Henry, K.C.B., Memb. Inst. C.E., LL.D., F.R.S., formerly Assistant Controller of the Navy, and Director of Naval Construction, Cedarscroft, Putney Heath, London	
1881		Whitehead, Walter, F.R.C.S.E., formerly Professor of Clinical Surgery, Owens College and Victoria University, Birchfield, 235 Wilmslow Road, Manchester	
1911		* Whittaker, Charles Richard, F.R.C.S. (Edin.), F.S.A. (Scot.), Lynwood, Hatton Place, Edinburgh	
1879		Will, John Charles Ogilvie, of Newton of Pitfodels, M.D., 17 Bon-Accord Square, Aberdeen	605
1908		* Williamson, Henry Charles, M.A., D.Sc., Naturalist to the Fishery Board for Scotland, Marine Laboratory, Aberdeen	
1910	C.	* Williamson, William, 9 Plewlands Terrace, Edinburgh	
1900		Wilson, Alfred C., F.C.S., Voewood Croft, Stockton-on-Tees	
1879		Wilson, Andrew, Ph.D., F.L.S., Lecturer on Zoology and Comparative Anatomy, 110 Gilmore Place	
1911		* Wilson, Andrew, Assoc. M. Inst. C.E., 51 Queen Street, Edinburgh	610
1902		* Wilson, Charles T. R., M.A., F.R.S., Glencorse House, Peebles, and Sidney Sussex College, Cambridge	
1895		Wilson-Barker, David, F.R.G.S., Captain-Superintendent Thames Nautical Training College, H.M.S. "Worcester," Greenhithe, Kent	
1882		Wilson, George, M.A., M.D., LL.D., 7 Avon Place, Warwick	
1891		* Wilson, John Hardie, D.Sc., University of St Andrews, 39 South Street, St Andrews	
1902		Wilson, William Wright, F.R.C.S.E., M.R.C.S. Eng., Cottesbrook House, Acock's Green, Birmingham	615

Date of Election.			
1908		* Wood, Thomas, M.D., Eastwood, 182 Ferry Road, Bonnington, Leith	
1886	C.	Woodhead, German Sims, M.D., F.R.C.P.E., Professor of Pathology in the University of Cambridge	
1884		Woods, G. A., M.R.C.S., Eversleigh, 1 Newstead Road, Lee, Kent	
1911		* Wrigley, Ruric Whitehead, B.A. (Cantab.), Assistant Astronomer, Royal Observatory, Edinburgh	
1890		* Wright, Johnstone Christie, Conservative Club, Edinburgh.	620
1896		* Wright, Sir Robert Patrick, Maraval, Uddingston.	
1882		Young, Frank W., F.C.S., H.M. Inspector of Science and Art Schools, 32 Buckingham Terrace, Botanic Gardens, Glasgow	
1892		Young, George, Ph.D., "Bradda," Church Crescent, Church End, Finchley, London, N.	
1896	C.	* Young, James Buchanan, M.B., D.Sc., Dalveen, Braeside, Liberton	
1900		* Young, J. M'Lauchlan, F.R.C.V.S., Lecturer on Veterinary Hygiene, University of Aberdeen	625
1904		Young, R. B., M.A., D.Sc., F.G.S., Professor of Geology and Mineralogy in the South African School of Mines and Technology, Johannesburg, Transvaal	

LIST OF HONORARY FELLOWS OF THE SOCIETY

At November 1911.

HIS MOST GRACIOUS MAJESTY THE KING.

FOREIGNERS (LIMITED TO THIRTY-SIX BY LAW X.).

Elected,
1897 Émile Hilaire Amagat, Membre de l'Institut, St Satur, Cher, France.
1900 Georg F. J. A. Auwers, Lindenstrasse, 91, Berlin, Germany.
1900 Adolf Ritter von Baeyer, Universität, München, Germany.
1905 Waldemar Christofer Brögger, K. Frederiks Universitet, Christiania, Norway.
1905 Moritz Cantor, Gaisbergstrasse, 15, Heidelberg, Germany.
1902 Jean Gaston Darboux, Secrétariat de l'Institut, Paris, France.
1910 Hugo de Vries, Universiteit, Amsterdam, Holland.
1905 Paul Ehrlich, K. Institut für Experimentelle Therapie, Sandhofstrasse, 44, Frankfurt-a.-M., Germany.
1908 Emil Fischer, Universität, Berlin, Germany.
1910 F. A. Forel, Morges, Switzerland.
1910 Karl F. von Goebel, Universität, München, Germany.
1905 Paul Heinrich Groth, Universität, München, Germany.
1888 Ernst Haeckel, Universität, Jena, Germany.
1883 Julius Hann, Universität, Wien, Austria.
1908 George William Hill, West Nyack, New York, U.S.A.
1910 Jacobus Cornelius Kapteyn, Universiteit, Groningen, Holland.
1897 Gabriel Lippmann, Université, Paris, France.
1895 Carl Menger, Universität, Wien, Austria.
1910 Élie Metchnikoff, Institut Pasteur, Paris, France.
1910 Albert Abraham Michelson, University, Chicago, U.S.A.
1897 Fridtjof Nansen, K. Frederiks Universitet, Christiania, Norway.
1908 Henry Fairfield Osborn, Columbia University, New York, N.Y., U.S.A.
1910 Wilhelm Ostwald, Universität, Leipzig, Germany.
1908 Ivan Petrovitch Pawlov, Wedenskaja Strasse, 4, St. Petersburg, Russia.
1895 Jules Henri Poincaré, 63 Rue Claude-Bernard, Paris, France.
1910 Frederick Ward Putnam, Peabody Museum of Harvard University, Cambridge, Mass., U.S.A.
1889 Georg Hermann Quincke, Bergstrasse, 41, Heidelberg, Germany.
1908 Magnus Gustaf Retzius, Högskolan, Stockholm, Sweden.
1908 Augusto Righi, Regia Università, Bologna, Italy.
1905 Eduard Suess, Afrikanergasse 9, Wien 11/2, Austria.
1908 Louis Joseph Troost, Université, Paris, France.
1905 Wilhelm Waldeyer, Universität, Berlin, Germany.
1910 August F. L. Weismann, Universität, Freiburg-im-Breisgau, Germany.
1905 Wilhelm Wundt, Universität, Leipzig, Germany.
1897 Ferdinand Zirkel, Königstrasse, 2 A, Bonn, Germany.

Total, 35.

British Subjects (limited to twenty by Law X.).

Elected

1889 Sir Robert Stawell Ball, Kt., Hon., M.A. (Cantab.), LL.D., F.R.S., M.R.I.A., Lowndean Professor of Astronomy in the University of Cambridge, Observatory, Cambridge.

1892 Colonel Alexander Ross Clarke, C.B., R.E., F.R.S., Strathmore, Reigate, Surrey.

1897 Sir George Howard Darwin, K.C.B., M.A., LL.D., F.R.S., Plumian Professor of Astronomy in the University of Cambridge, Newnham Grange, Cambridge.

1900 Sir David Ferrier, Kt., M.A., M.D., LL.D., F.R.S., Professor of Neuro-Pathology, King's College, London, 34 Cavendish Square, London, W.

1900 Andrew Russell Forsyth, M.A., D.Sc., F.R.S., Sadlerian Professor of Pure Mathematics in the University of Cambridge, Trinity College, Cambridge.

1910 James George Frazer, D.C.L., LL.D., Litt.D., F.B.A., Fellow of Trinity College, Cambridge, Professor of Social Anthropology in the University of Liverpool, Trinity College, Cambridge.

1892 Sir David Gill, K.C.B., LL.D., F.R.S., formerly His Majesty's Astronomer at the Cape of Good Hope, 34 De Vere Gardens, Kensington, London, W.

1895 Albert C. L. G. Günther, M.A., M.D., Ph.D., F.R.S., 2 Lichfield Road, Kew Gardens, Surrey.

1883 Sir Joseph Dalton Hooker, O.M., G.C.S.I., C.B., M.D., LL.D., D.C.L., F.R.S., The Camp, Sunningdale, Berkshire.

1908 Sir Alexander B. W. Kennedy, Kt., LL.D., F.R.S., Past Pres. Inst. C.E., 1 Queen Anne Street, Cavendish Square, London, W.

1908 Sir Edwin Ray Lankester, K.C.B., LL.D., F.R.S., 29 Thurloe Place, S. Kensington, London, S.W.

1910 Sir Joseph Larmor, Kt., M.A., D.Sc., LL.D., D.C.L., F.R.S., Lucasian Professor of Mathematics in the University of Cambridge, Secretary of the Royal Society, St John's College, Cambridge.

1900 Archibald Liversidge, LL.D., F.R.S., Em.-Professor of Chemistry in the University of Sydney, Hornton Cottage, Hornton Street, Kensington, London, W.

1908 Sir James A. H. Murray, LL.D., D.C.L., Editor of the Oxford English Dictionary, Oxford.

1905 Sir William Ramsay, K.C.B., LL.D., F.R.S., Professor of Chemistry in the University College, London, 19 Chester Terrace, Regent's Park, London, N.W.

1886 The Rt. Hon. Lord Rayleigh, O.M., P.C., J.P., D.C.L., LL.D., D.Sc. Dub., F.R.S., Corresp. Mem. Inst. of France, Terling Place, Witham, Essex.

1908 Charles Scott Sherrington, M.A., M.D., LL.D., F.R.S., Holt Professor of Physiology in the University of Liverpool, 16 Grove Park, Liverpool.

1905 Sir Joseph John Thomson, D.Sc., LL.D., F.R.S., Cavendish Professor of Experimental Physics, University of Cambridge, Trinity College, Cambridge.

1900 Sir Thomas Edward Thorpe, Kt., C.B., D.Sc., LL.D., F.R.S., formerly Principal of the Government Laboratories, Imperial College of Science and Technology, South Kensington, London, S.W.

1910 Alfred Russel Wallace, O.M., LL.D., D.C.L., F.R.S., Old Orchard, Broadstone, Wimborne, Dorset.

Total, 20.

ORDINARY FELLOWS OF THE SOCIETY ELECTED

During Session 1910–11.

(Arranged according to the date of their election.)

19*th December* 1910.

Andrew Wilson, Assoc.M.Inst.C.E.
Alexander C. Cowan.
Gilbert Macintyre Hunter, Assoc.M.Inst. C.E., M.I.E.S.
Gregory Macalister Mathews, F.L.S., F.Z.S.
Frank Wyville Thomson, M.A., M.B., C.M., D.P.H., D.T.M.
William Inglis Clark, D.Sc.
John Wm. M'Intosh, A.R.C.V.S.
Henry Walker, M.A., D.Sc.
Archibald Campbell Adams, A.M.I.Mech.E., A.M.I.E.E.
Ruric Whitehead Wrigley, B.A. (Cantab.).

20*th February* 1911.

James Watt, W.S., F.F.A.
James Hartley Ashworth, D.Sc.
Reginald John Gladstone, M.D., F.R.C.S. (Eng.).
James Andrew Gunn, M.A., M.D., D.Sc.
John Arnold Fleming, F.C.S.
Stephen Smith, B.Sc.
Sutherland Simpson, M.D., D.Sc. (Edin.).
Lauchlan MacLean Watt, B.D.
Edalji Manekji Modi, D.Sc., LL.D., Litt.D., F.C.S.

20th March 1911.

Charles Richard Whittaker, F.R.C.S. (Edin.), F.S.A. (Scot.).

15th May 1911.

William Guy, F.R.C.S., L.R.C.P., L.D.S. (Ed.).
William Jacob Holland, LL.D.
James A. S. Watson, B.Sc., etc.
John Smith Purdy, M.D., C.M. (Aber.), D.P.H. (Camb.), F.R.G.S.
Adam A. Rankin.
James Ramsay Tosh, M.A., D.Sc. (St. Ands.).

19th June 1911.

Duncan M'Laren Young Sommerville, M.A., D.Sc.
David Rainy Brown.

3rd July 1911.

Kenneth John M'Kenzie, M.A.
John Francis Hall-Edwards, L.R.C.P. (Edin.), Hon. F.R.P.S.
James Simpson Pirie, C.E.

The following were balloted for on 7th November 1910, *and declared elected.*

As British Honorary Fellows :—

1. James George Frazer, D.C.L., LL.D., Litt.D., F.B.A., Fellow of Trinity College, Cambridge, Professor of Social Anthropology in the University of Liverpool.
2. Sir Joseph Larmor, Kt., M.A., D.Sc., LL.D., D.C.L., F.R.S., Lucasian Professor of Mathematics in the University of Cambridge, Secretary of the Royal Society.
3. Alfred Russel Wallace, O.M., LL.D., D.C.L., F.R.S.

As Foreign Honorary Fellows :—

1. Hugo de Vries, Sc.D., LL.D., Professor of Anatomy and Physiology of Plants in the University of Amsterdam.
2. F. A. Forel, Naturalist and Limnographer, Morges, Switzerland.
3. Karl F. Von Goebel, D.Sc., Professor of Botany in the University of Munich.
4. Jacobus Cornelius Kapteyn, Ph.D., Professor of Astronomy in the University of Groningen.
5. Élie Metchnikoff, Vice-Director of the Pasteur Institute, Member of the Academy of Medicine, Paris.
6. Albert Abraham Michelson, Ph.D., Sc.D., LL.D., F.R.S., Professor and Head of the Department of Physics in the University of Chicago.
7. Wilhelm Ostwald, D.M., D.Sc., LL.D., Emeritus Professor of Physical Chemistry in the University of Leipsic.
8. Frederick Ward Putnam, Professor of American Archæology and Ethnology in Harvard University.
9. August F. L. Weismann, D.M., Ph.D., D.Bot., D.C.L., LL.D., Professor of Zoology in the University of Freiburg (Baden).

ORDINARY FELLOWS DECEASED AND RESIGNED

During Session 1910–11.

DECEASED.

Prof. Robert Flint, D.D.
Very Rev. Dr James MacGregor.
George Barclay, M.A.
James Bisset, M.A., F.G.S.
T. R. Buchanan, M.A., M.P.
Alexander Cullen, F.S.A. (Scot).
Dr Alexander Bruce, M.A.
Sir James King of Campsie, Bt., LL.D.
Dr Joseph Bell.
Edward Whymper, F.R.G.S.

RESIGNED.

Dr W. Cossar Mackenzie.
William Forster Lanchester, M.A.

ABSTRACT

OF

THE ACCOUNTS OF JAMES CURRIE, ESQ.

As Treasurer of the Royal Society of Edinburgh.

SESSION 1910–1911.

I. ACCOUNT OF THE GENERAL FUND.

CHARGE.

1. Arrears of Contributions at 1st October 1910		£97 13 0
2. Contributions for present Session :—		
1. 148 Fellows at £2, 2s. each	£310 16 0	
136 Fellows at £3, 3s. each	428 8 0	
	£739 4 0	
2. Commutation Fees in lieu of Future Contributions of two Fellows	82 19 0	
3. Fees of Admission and Contributions of twenty-two new Resident Fellows at £5, 5s. each	110 5 0	
4. Fees of Admission of nine new Non-Resident Fellows at £26, 5s. each	236 5 0	
		1168 13 0
3. Contributions for 1911–12 paid in advance		7 7 0
4. Interest received—		
Interest, less Tax £22, 17s. 8d.	£369 8 8	
Annuity from Edinburgh and District Water Trust, less Tax £3, 1s. 2d.	49 8 10	
		418 17 6
5. Transactions and Proceedings sold		170 10 5
6. Annual Grant from Government		600 0 0
7. Income Tax repaid for year to 5th April 1911		25 18 11
Amount of the Charge		**£2488 19 10**

DISCHARGE.

1. TAXES, INSURANCE, COAL AND LIGHTING :—		
Inhabited House Duty	£0 6 3	
Insurance	10 17 5	
Coal to 12th July 1911	21 19 5	
Gas to 9th May 1911	0 4 2	
Electric Light to 3rd May 1911	9 9 3½	
Water 1909–10 and 1910–11	21 14 8	
		£64 11 2½
2. SALARIES :—		
General Secretary	£100 0 0	
Librarian	92 10 0	
Assistant Librarian	24 0 0	
Office Keeper	86 14 0	
Treasurer's Clerk	25 0 0	
		328 4 0
Carry forward		£392 15 2½

Brought forward . . .		£392 15 2½
3. Expenses of Transactions :—		
Neill & Co., Ltd., Printers	£240 9 9	
Do. for illustrations	10 1 0	
Do. (Ben Nevis)	50 0 0	
M'Farlane & Erskine, Lithographers	26 10 0	
Hislop & Day, Engravers	17 9 6	
Orrock & Son, Bookbinders	73 11 9	
Alexander S. Huth, Printing, etc.	7 5 0	
George Waterston & Sons, Stationers	13 14 0	
		439 1 0
4. Expenses of Proceedings :—		
Neill & Co., Ltd., Printers	£516 0 10	
Do. (for illustrations)	11 7 6	
M'Farlane & Erskine, Lithographers	31 10 6	
Hislop & Day, Engravers	31 2 3	
Wm. Green & Sons, Printers	15 0 0	
		605 1 1
5. Books, Periodicals, Newspapers, etc. :—		
Otto Schulze & Co., Booksellers	£137 4 5	
James Thin, do.	49 15 4	
R. Grant & Son, do.	7 6 4	
Wm. Green & Sons, do.	0 15 6	
International Catalogue of Scientific Literature	17 0 0	
Robertson & Scott, News Agents	2 12 0	
Egypt Exploration Funds Subscription	3 3 0	
Ray Society do.	1 1 0	
Palæontographical Society do.	1 1 0	
Journal de Conchyliologie	0 16 9	
Orrock & Son, Bookbinders	42 11 6	
Cambridge University Press for Encyclopedia Brittanica and Bookcase	25 19 2	
		289 6 0
6. Other Payments :—		
Neill & Co., Ltd., Printers	£62 1 9	
R. Blair & Son, Confectioners	28 8 4	
S. Duncan, Tailor (uniforms)	4 14 0	
Lantern Exhibitions, etc., at Lectures	10 14 0	
Lindsay, Jamieson & Haldane, C.A., Auditors	6 6 0	
National Telephone Co., Ltd.	10 2 6	
A. Cowan & Sons, Ltd.	3 11 9	
G. Waterston & Sons	5 15 9	
J. & T. Scott	8 5 9	
Robert Maule & Son	11 2 6	
Petty Expenses, Postages, Carriage, etc.	92 12 3½	
		243 14 7½
7. Interest Paid on Borrowed Money :—		
Makerstoun Magnetic Meteorological Observation Fund	£4 12 7	
Makdougall-Brisbane Fund	3 17 10	
Union Bank of Scotland, Ltd.	4 1 0	
		12 11 5
8. Irrecoverable Arrears of Contributions written off		6 6 0
9. Arrears of Contributions outstanding at 2nd October 1911 :—		
Present Session	£69 6 0	
Previous Sessions	32 11 0	
		101 17 0
Amount of the Discharge		**£2090 12 4**

Amount of the Charge				£2488	19	10
Amount of the Discharge				2090	12	4
Excess of Receipts over Payments for 1910-1911				£398	7	6
Floating Balance due by the Society at 1st October 1910				£847	15	3
Deduct Excess of Receipts as above				398	7	6
Floating Balance due by the Society at 2nd October 1911				£449	7	9
Being—						
Balance due to Union Bank of Scotland, Ltd., on Account Current				£235	4	5
Loan from the Makerstoun Magnetic Meteorological Observation Fund				219	3	4
				£454	7	9
Less Balance in hands of Librarian				5	0	0
				£449	7	9

II. ACCOUNT OF THE KEITH FUND

To 2nd October 1911.

CHARGE.

1. Balance due by Union Bank of Scotland, Ltd., on Account Current at 1st October 1910				£29	16	4
2. Interest Received :—						
On £896, 19s. 1d. North British Railway Company 3 per cent. Debenture Stock for year to Whitsunday 1911, less Tax £1, 11s. 4d.	£25	6	10			
On £211, 4s. North British Railway Company 3 per cent. Lien Stock for year to Lammas 1911, less Tax 7s. 4d.	5	19	4			
				31	6	2
3. Income Tax repaid for year to 5th April 1911				1	18	8
				£63	1	2

DISCHARGE.

Balance due by Union Bank of Scotland, Ltd., on Account Current at 2nd October 1911	£63	1	2

III. ACCOUNT OF THE NEILL FUND

To 2nd October 1911.

CHARGE.

1. Balance due by Union Bank of Scotland, Ltd., on Account Current at 1st October 1910	£33	3	6
2. Interest Received :—			
On £355 London, Chatham and Dover Railway 4½ per cent. Arbitration Debenture Stock for year to 30th June 1911, less Tax 18s. 8d.	15	0	10
3. Income Tax repaid for year to 5th April 1911	0	18	8
	£49	3	0

DISCHARGE.

Balance due by Union Bank of Scotland, Ltd., on Account Current at 2nd October 1911	£49	3	0

IV. ACCOUNT OF THE MAKDOUGALL-BRISBANE FUND

To 2nd October 1911.

CHARGE.

1. Balance due by Union Bank of Scotland, Ltd., on Account Current at 1st October 1910		£186 4 0
2. Interest received:—		
1. On £365 Caledonian Railway Company 4 per cent. Consolidated Preference Stock No. 2 for year to 30th June 1911, less Tax 17s.	£13 15 0	
2. On Balances in Bank at Deposit Receipt Rates	3 17 10	
		17 12 10
3. Income Tax repaid for year to 5th April 1911		0 17 0
		£204 13 10

DISCHARGE.

1. Ernest Maclagan Wedderburn—Money Portion of Prize for 1908-10	£17 12 7
2. Alex. Kirkwood & Son, Engravers, for Gold Medal	16 0 0
3. Balance due by Union Bank of Scotland, Ltd., on Account Current at 2nd October 1911	171 1 3
	£204 13 10

V. ACCOUNT OF THE MAKERSTOUN MAGNETIC METEOROLOGICAL OBSERVATION FUND

To 2nd October 1911.

CHARGE.

1. Balance due by General Fund at 1st October 1910	£219 10 9
2. Interest received on Balances due by General Fund at Deposit Receipt Rates to 2nd October 1911	4 12 7
	£224 3 4

DISCHARGE.

1. N. T. S. Wilsmore, Grant in aid of the publication of the Annual Tables of Constants and Numerical Data, Chemical, Physical and Technological	£5 0 0
2. Balance due by General Fund at 2nd October 1911	219 3 4
	£224 3 4

VI. ACCOUNT OF THE GUNNING-VICTORIA JUBILEE PRIZE FUND

To 2nd October 1911.

(Instituted by Dr R. H. Gunning of Edinburgh and Rio de Janeiro.)

CHARGE.

1. Balance due by Union Bank of Scotland, Ltd., on Account Current at 1st October 1910	£59 5 4
2. Interest received on £1000 North British Railway Company Consolidated Lien Stock for year to Lammas 1911, less Tax £1, 15s.	28 5 0
3. Income Tax repaid for year to 5th April 1911	1 15 0
	£89 5 4

DISCHARGE.

	£	s.	d.
Balance due by Union Bank of Scotland, Ltd., on Account Current at 2nd October 1911	£89	5	4

STATE OF THE FUNDS BELONGING TO THE ROYAL SOCIETY OF EDINBURGH

As at 2nd October 1911.

1. GENERAL FUND—

	£	s.	d.
1. £2090, 9s. 4d. three per cent. Lien Stock of the North British Railway Company at 80 per cent., the selling price at 2nd October 1911 . . .	£1,672	7	5
2. £8519, 14s. 3d. three per cent. Debenture Stock of do. at 79¾ per cent., do.	6,794	9	5
3. £52, 10s. Annuity of the Edinburgh and District Water Trust, equivalent to £875 at 167 per cent., do.	1,461	5	0
4. £1811 four per cent. Debenture Stock of the Caledonian Railway Company at 108¼ per cent., do.	1,960	8	1
£35 four and a half per cent. Arbitration Debenture Stock of the London, Chatham and Dover Railway Company at 113 per cent., do.	39	11	0
6. Arrears of Contributions, as per preceding Abstract of Accounts . . .	101	17	0
	£12,029	17	11
Deduct Floating Balance due by the Society, as per preceding Abstract of Accounts	449	7	9
Amount . .	£11,580	10	2

Exclusive of Library, Museum, Pictures, etc., Furniture of the Society's Rooms at George Street, Edinburgh.

2. KEITH FUND—

	£	s.	d.
1. £896, 19s. 1d. three per cent. Debenture Stock of the North British Railway Company at 79¾ per cent., the selling price at 2nd October 1911	£715	6	5
2. £211, 4s. three per cent. Lien Stock of do. at 80 per cent., do. . . .	168	19	2
3. Balance due by Union Bank of Scotland, Ltd., on Account Current . .	63	1	2
Amount	£947	6	9

3. NEILL FUND—

	£	s.	d.
1. £355 four and a half per cent. Arbitration Debenture Stock of the London, Chatham and Dover Railway Company at 113 per cent., the selling price at 2nd October 1911	£401	3	0
2. Balance due by Union Bank of Scotland, Ltd., on Account Current . .	49	3	0
Amount	£450	6	0

4. MAKDOUGALL-BRISBANE FUND—

	£	s.	d.
1. £365 four per cent. Consolidated Preference Stock No. 2 of the Caledonian Railway Company at 100 per cent. x.d., the selling price at 2nd October 1911	£365	0	0
2. Balance due by Union Bank of Scotland, Ltd., on Account Current . .	171	1	3
Amount	£536	1	3

5. MAKERSTOUN MAGNETIC METEOROLOGICAL OBSERVATION FUND—

	£	s.	d.
Balance due by General Fund at 2nd October 1911	£219	3	4

6. **GUNNING-VICTORIA JUBILEE PRIZE FUND**—Instituted by Dr Gunning of Edinburgh and Rio de Janeiro—

1. £1000 three per cent. Consolidated Lien Stock of the North British Railway Company at 80 per cent., the selling price at 2nd October 1911 . .	£800	0	0
2. Balance due by Union Bank of Scotland, Ltd., on Account Current . .	89	5	4
AMOUNT . . .	£889	5	4

EDINBURGH, 16*th October* 1911.—We have examined the six preceding Accounts of the Treasurer of the Royal Society of Edinburgh for the Session 1910–1911, and have found them to be correct. The securities of the various Investments at 2nd October 1911, as noted in the above Statement of Funds, have been exhibited to us.

LINDSAY, JAMIESON & HALDANE,
Auditors.

List of Library Exchanges, Presentations, etc., at November 1911.

I. TRANSACTIONS AND PROCEEDINGS OF LEARNED SOCIETIES, ACADEMIES, ETC., RECEIVED BY EXCHANGE OF PUBLICATIONS, AND LIST OF PUBLIC INSTITUTIONS ENTITLED TO RECEIVE COPIES OF THE TRANSACTIONS AND PROCEEDINGS OF THE ROYAL SOCIETY OF EDINBURGH. (*For convenience certain Presentations are included in this List.*)

T.P. prefixed to a name indicates that the Institution is entitled to receive *Transactions* and *Proceedings*. P. indicates *Proceedings*.

AFRICA (BRITISH CENTRAL).

ZOMBA.—*Scientific Department.* Meteorological Observations, Fol. (*Presented by H.M. Acting Commissioner and Consul-General.*)

AMERICA (NORTH). (*See* UNITED STATES AND CANADA.)

AMERICA (SOUTH).

T.P. BUENOS AYRES (ARGENTINE REPUBLIC).—*Museo Nacional.* Anales.

CORDOBA—

T.P. *Academia Nacional de Ciencias de la Republica Argentina.* Boletin.

T.P. *National Observatory.* Anales.

T.P. LA PLATA (ARGENTINE REPUBLIC).—*Museo de La Plata.*

LIMA (PERU). *Cuerpo de Ingenieros de Minas del Peru.* Boletin. (*Presented.*)

P. MONTEVIDEO (URUGUAY).—*Museo Nacional.* Annales (Flora Uruguay).

T.P. PARÀ (BRAZIL).—*Museu Paraense de Historia Natural e Ethnographia.*

P. QUITO (ECUADOR).—*Observatorio Astronomico y Meteorologico.*

RIO DE JANEIRO (BRAZIL)—

T.P. *Observatorio.* Annuario.—Boletin Mensal.

P. *Museu Nacional.* Revista (Archivos).

SANTIAGO (CHILI)—

T.P. *Société Scientifique du Chili.* Actes.

P. *Deutscher Wissenschaftlicher Verein.*

P. SAN SALVADOR.—*Observatorio Astronómico y Meteorólogico.*

VALPARAISO (CHILI).—*Servicio Meteorologico.* Annuario. (*Presented.*)

AUSTRALIA.

Australasian Association for the Advancement of Science.—Reports. (*Presented.*)

Australasian Medical Congress. Transactions. 8th Session, Melbourne. (*Presented.*)

ADELAIDE—

P. *University Library.*

P. *Royal Society of South Australia.* Transactions and Proceedings.

P. *Royal Geographical Society* (*South Australian Branch*). Proceedings.

Observatory. Meteorological Observations. 4to. (*Presented.*)

BRISBANE—

P. *Royal Society of Queensland.* Transactions.

P. *Royal Geographical Society* (*Queensland Branch*). Queensland Geographical Journal.

P. *Government Meteorological Office.*

P. *Water Supply Department.*

P. GEELONG (VICTORIA).—*Gordon Technical College.*

P. HOBART.—*Royal Society of Tasmania.* Proceedings.

MELBOURNE—

Commonwealth Bureau of Census and Statistics. Official Year Book. By G. H. Knibbs. (*Presented.*)

T.P. *University Library.*

P. *Royal Society of Victoria.* Proceedings.

PERTH, W.A.—

P. *Geological Survey.* Annual Progress Reports.—Bulletins.

Government Statistician's Office. Monthly Statistical Abstract. (*Presented.*)

SYDNEY—

T.P. *University Library.* Calendar.

T.P. *Department of Mines and Agriculture* (*Geological Survey*), *N.S.W* Records.—Annual Reports.—Palæontology. Mineral Resources.

T.P. *Linnean Society of New South Wales.* Proceedings.

T.P. *Royal Society of New South Wales.* Journal and Proceedings.

P. *Australian Museum.* Records.—Reports.—Memoirs.—Catalogues.

N.S.W. Government. Fisheries Report. (*Presented.*)

AUSTRIA.

CRACOW—

.P. *Académie des Sciences.* Rozprawy Wydzialu matematyczno-przyrodniezego (Proceedings, Math. and Nat. Sciences Cl.).—Rozprawy Wydzialu filologicznego (Proc., Philological Section).—Rozprawy Wydzialu historyczno-filozoficznego (Proc., Hist.-Phil. Section).—Sprawozdanie Komisyi do badania historyi sztuki w Polsce (Proc., Commission on History of Art in Poland).—Sprawozdanie Komisyi fizyjograficznej (Proc., Commission on Physiography).—Biblijoteka piserzów polskich (Library of Polish Authors of the 16th century).—Geological Atlas of Galicia; Text, Maps.—Bulletin International, etc.

GRATZ—

T.P. *Naturwissenschaftlicher Verein für Steiermark.* Mittheilungen.

P. *Chemisches Institut der K. K. Universität.*

P. LEMBERG.—*Société Scientifique de Chevtchenko.*

PRAGUE—

T.P. *K. K. Sternwarte.* Magnetische und Meteorologische Beobachtungen. Astronomische Beobachtungen.

T.P. *K. Böhmische Gesellschaft.* Sitzungsberichte : Math. Naturw. Classe; Phil.-Hist.-Philol. Classe.—Jahresbericht.—And other publications.

T.P. *Ceská Akademie Cisaře Františka Josefa pro Vědy Slovesnost a Uměni.* Almanach. — Vestnik (Proceedings). — Rozpravy (Transactions): Phil.-Hist. Class; Math.-Phys. Cl.; Philol. Cl.—Historický Archiv.—Bulletin International, Résumé des Travaux présentés.—And other publications of the Academy.

P. SARAJEVO (BOSNIA).—*The Governor-General of Bosnia-Herzegovina.* Ergebnisse der Meteorologischen Beobachtungen.

TRIESTE—

P. *Società Adriatica di Scienze Naturali.*

P. *Museo Civico di Storia Naturale.*

P. *Osservatorio Astronomico-Meteorologico.* Rapporto Annuale.

VIENNA—

T.P. *Kais. Akademie der Wissenschaften.* Denkschriften: Math.-Naturwissenschaftliche Classe; Philosophisch-Historische Classe — Sitzungsberichte der Math.-Naturwissenschaftlichen Classe; Abtheil. I., II.A, II.B, III.; Philosoph.-Historische Classe.—Almanach.—Mittheilungen der Erdbeben Commission.

T.P. *K. K. Geologische Reichsanstalt.* Abhandlungen.—Jahrbücher.—Verhandlungen.

T.P. *Oesterreichische Gesellschaft für Meteorologie.* Meteorologische Zeitschrift.

T.P. *K. K. Zoologisch-Botanische Gesellschaft.* Verhandlungen. — Abhandlungen.

P. *K. K. Naturhistorisches Hofmuseum.* Annalen.

K. K. Central-Anstalt für Meteorologie und Erdmagnetismus. Jahrbücher. 4to.—Allgemeiner Bericht und Chronik. 8vo. (*Presented.*)

K. K. Militar Geographisches Institut. Astronomisch-Geodätischen Arbeiten. Astronomische Arbeiten. 4to.—Längenbestimmungen. 4to.—Die Ergebnisse der Triangulierungen. 4to. (*Presented.*)

Zoologisches Institut der Universität und der Zoologischen Station in Triest. Arbeiten. (Purchased.)

BELGIUM.

BRUSSELS—

T.P. *Académie Royale des Sciences, des Lettres et des Beaux Arts de Belgique.* Mémoires Couronnés.—Mémoires Couronnés et Mémoires des Savants Étrangers.—Bulletin: Classe des Sciences; Classe des Lettres et des Sciences Morales et Politiques.—Annuaire.—Biographie Nationale.

T.P. *Musée Royal d'Histoire Naturelle.* Mémoires.

T.P. *Musée du Congo.* Annales. — *Botanique. Zoologie. Ethnographie et Anthropologie. Linguistique,* etc.

T.P. *L'Observatoire Royal de Belgique, Uccle.* Annuaire.—Annales Astronomiques. Annales Météorologiques. Annales. — Physique du Globe. — Bulletin Climatologique. Observations Météorologiques.

BRUSSELS—*continued*—

T.P. *La Société Scientifique.* Annales.

P. *Société Belge d'Astronomie.* Ciel et Terre. (*Purchase.*)

T.P. GHENT.—*University Library.*

T.P. LOUVAIN.—*University Library.*

BOSNIA-HERZEGOVINA. (*See* AUSTRIA.)

BULGARIA.

P. SOFIA.—*Station Centrale Météorologique de Bulgarie.* Bulletin Mensuel. Bulletins Annuaires.

CANADA.

P. HALIFAX (NOVA SCOTIA).—*Nova Scotian Institute of Science.* Proceedings and Transactions.

T.P. KINGSTON.—*Queen's University.*

MONTREAL—

P. *Natural History Society.* Proceedings.

P. *Canadian Society of Civil Engineers.* Transactions.—Annual Reports.

OTTAWA—

T.P. *The Royal Society of Canada.* Proceedings and Transactions.

T.P. *Geological Survey of Canada.* Annual Reports.—Palæozoic Fossils.—Maps, Memoirs, and other Publications.

P. *Literary and Scientific Society.* Transactions.

T.P. QUEBEC.—*Literary and Philosophical Society.* Transactions.

TORONTO—

T.P. *University.* University Studies. (History. Psychological Series. Geological Series. Economic Series. Physiological Series. Biological Series. Physical Science Series.) Papers from the Chemical Laboratory, etc.

T.P. *Canadian Institute.* Transactions.

P. *Royal Astronomical Society of Canada.* Transactions.—Journal. Astronomical Handbook.

CAPE COLONY.

CAPE TOWN—

Royal Society of South Africa. Transactions. (*Presented.*)

T.P. *Royal Astronomical Observatory.* Reports.—Annals. Meridian Observations.

P. *South African Philosophical Society.* Transactions.

P. *Geological Commission.* Annual Reports.

South African Museum. Annals. (*Presented.*)

P. *South African Association for the Advancement of Science.* Journal.

CEYLON.

COLOMBO—

T.P. *Museum.* Spolia Zeylanica. Annual Report.

CHINA.

Hong Kong—

p. *The Observatory.* Observations and Researches.

DENMARK.

Copenhagen—

t.p. *Académie Royale de Copenhague.* Mémoires : Classe des Sciences.—Oversigt.

p. *Naturhistorisk Forening.* Videnskabelige Meddelelser.

p. *Danish Biological Station.* Reports.

Conseil Permanent International pour l'Exploration de la Mer. Publications de circonstance.—Rapports et Procès-Verbaux de Réunions.—Bulletin des Résultats acquis pendant les croisières périodiques.—Bulletin Statistique. (*Presented.*)

Kommissionen for Havundersogelser. Meddelelser : Série Fiskeri. Série Plankton. Série Hydrografi. (*Presented.*)

University (*Zoological Museum*). Danish Ingolf-Expedition. (*Presented.*)

EGYPT.

t.p. Cairo—*School of Medicine.* Records.

Ministry of Finance (*Survey Dept.*). Bulletins, Reports, Papers. (*Presented.*)

ENGLAND AND WALES.

Birmingham—

p. *Philosophical Society.* Proceedings.

University. Calendar. (*Presented.*)

Cambridge—

t.p. *Philosophical Society.* Transactions and Proceedings.

t.p. *University Library.—Observatory.* Report. Observations.

p. Cardiff.—*University College of South Wales.*

Coventry.—Annual Report of the Health of the City. (*Presented.*)

p. Essex.—*Essex Field Club.*—The Essex Naturalist.

t.p. Greenwich.—*Royal Observatory.* Astronomical, Magnetical, and Meteorological Observations.—Photo-heliographic Results and other Publications.

Leeds—

t.p. *Philosophical and Literary Society.* Reports.

p. *Yorkshire Geological and Polytechnic Society.* Proceedings.

Liverpool—

t.p. *University College Library.*

p. *Biological Society.* Proceedings and Transactions.

p. *Geological Society.* Proceedings.

London—

p. *Admiralty.* Nautical Almanac and Astronomical Ephemeris.

t.p. *Anthropological Institute.* Journal.

t.p. *Athenæum Club.*

LONDON—*continued*—

British Antarctic Expedition, 1907–09. Reports on Scientific Investigations. (*Presented.*)

T.P. *British Association for the Advancement of Science.* Reports.

T.P. *British Museum (Copyright Office).* Reproductions from Illuminated Manuscripts.

T.P. *British Museum. Natural History Department.*—Catalogues, Monographs, Lists, etc.

T.P. *Chemical Society.* Journal. Abstract of Proceedings.

P. *Faraday Society.* Transactions.

T.P. *Geological Society.* Quarterly Journal.—Geological Literature.—Abstract of Proceedings.

T.P. *Geological Survey of the United Kingdom.* Summary of Progress. Memoirs.

P. *Geologists' Association.* Proceedings.

T.P. *Hydrographic Office.*

T.P. *Imperial Institute.*

T.P. *Institution of Civil Engineers.* Minutes of Proceedings.—Subject Index.—Charters.—Members.

T.P. *Institution of Electrical Engineers.* Journal.

P. *Institution of Mechanical Engineers.* Proceedings.

T.P. *International Catalogue of Scientific Literature.* (*Purchase.*)

T.P. *Linnean Society.* Journal: Zoology; Botany.—Transactions: Zoology; Botany.—Proceedings.

P. *Mathematical Society.* Proceedings.

P. *Meteorological Office.* Reports of the Meteorological Council to the Royal Society.—Hourly Readings.—Weekly Weather Reports.—Monthly and Quarterly Summaries.—Meteorological Observations at Stations of Second Order, and other Publications. Geophysical Journal.

Mineralogical Society of Great Britain and Ireland. Mineralogical Magazine and Journal. (*Presented.*)

Optical Society. Transactions. (*Purchase.*)

P. *Pharmaceutical Society.* Journal.

P. *Physical Society.* Proceedings.

T.P. *Royal Astronomical Society.* Monthly Notices.—Memoirs.

T.P. *Royal College of Surgeons.*

T.P. *Royal Geographical Society.* Geographical Journal.

T.P. *Royal Horticultural Society.* Journal.

T.P. *Royal Institution.* Proceedings.

P. *Royal Meteorological Society.* The Meteorological Record: Monthly Returns of Observations made at the Stations of the Meteorological Society.—Quarterly Journal.

T.P. *Royal Microscopical Society.* Journal.

P. *Royal Photographic Society.* Photographic Journal.

T.P. *Royal Society.* Philosophical Transactions.—Proceedings.—Year-Book.—Obituary Notices of Fellows of the Royal Society.—National Antarctic Expedition, 1901–04, Publications, and other Publications.

T.P. *Royal Society of Literature.* Transactions.—Reports.

T.P. *Royal Society of Medicine.* Proceedings.

London—*continued*—

T.P. *Royal Statistical Society.* Journal.

T.P. *Society of Antiquaries.* Proceedings.—Archæologia; or Miscellaneous Tracts relating to Antiquity.

T.P. *Society of Arts.* Journal.

Society of Chemical Industry. Journal. (*Presented.*)

Society for Psychical Research. Journal. Proceedings. (*Presented by W. C. Crawford, Esq.*)

Solar Physics Committee. Annual Report, and other Publications. (*Presented.*)

T.P. *United Service Institution.*

T.P. *University College.* Calendar.

T.P. *University.*

T.P. *Zoological Society.* Transactions.—Proceedings.

T.P. The Editor of *Nature.—Nature.*

T.P. The Editor of *The Electrician.—Electrician.*

P. The Editor of *The Electrical Engineer.—Electrical Engineer.*

T.P. The Editor of *Science Abstracts.—Science Abstracts.*

Manchester—

P. *Geological and Mining Society.* Transactions.

T.P. *Literary and Philosophical Society.* Memoirs and Proceedings.

T.P. *University.*—Publications—Medical Series. Public Health Series. Anatomical Series. Economic Series. Historical Series. Theological Series. Physical Series. Celtic Series. English Series. Biological Series. *Manchester Museum* (*University of Manchester*). Annual Reports—Notes from the Museum.

P. *Microscopical Society.* Transactions.—Annual Report.

Newcastle-on-Tyne—

P. *Natural History Society of Northumberland, Durham,* etc. Transactions.

T.P. *North of England Institute of Mining and Mechanical Engineers.* Transactions.—Annual Reports.

Northumberland Sea Fisheries Committee. Report on the Scientific Investigations. Ed. by Alex. Meek. (*Presented.*)

P. *Literary and Philosophical Society.*

University of Durham Philosophical Society. Proceedings. (*Presented.*)

P. Norwich.—*Norfolk and Norwich Naturalist's Society.*—Transactions.

Oxford—

T.P. *Bodleian Library.*

P. *Ashmolean Society.* Proceedings. Report.

P. *Radcliffe Observatory.* Results of Meteorological Observations. Catalogues of Stars.

University Observatory. Astrographic Catalogue. (*Presented.*)

P. Penzance.—*Royal Geological Society of Cornwall.* Transactions.

T.P. Plymouth.—*Marine Biological Association.* Journal.

Richmond (Surrey)—

T.P. *Kew Observatory.*

P. Scarborough.—*Philosophical Society.*

T.P. Sheffield.—*University College.*

SOUTHPORT.—*Meteorological Observatory.* Results of Observations. Joseph Baxendell, Meteorologist. (*Presented.*)

T.P. TEDDINGTON (MIDDLESEX).—*National Physical Laboratory.* Reports.—Collected Researches. Annual Reports.

P. TRURO.—*Royal Institution of Cornwall.* Journal.

YORK—

T.P. *Yorkshire Philosophical Society.* Reports.

FINLAND.

HELSINGFORS—

Academiæ Scientarum Fennicæ. Annales. Sitzungsberichte. (*Presented.*)

Hydrographisch Biologisch Untersuchungen. (*Presented.*)

T.P. *Societas Scientiarum Fennica* (*Société des Sciences de Finlande*). Acta Societatis Scientiarum Fennicae.—Öfversigt.—Observations Météorologiques.—Meteorologisches Jahrbuch.

T.P. *Societas pro Fauna et Flora Fennica.* Acta.—Meddelanden.

P. *Société de Géographie de Finlande.* Fennia.—Meddelanden.

FRANCE.

BESANÇON.—*Université Observatoire National.* Bulletin Chronométrique et Bulletin Météorologique. (*Presented.*)

BORDEAUX—

T.P. *Société des Sciences Physiques et Naturelles.* Mémoires. Observations Pluviométriques et Thermométriques.—Procès-Verbaux des Séances.

P. *Société de Géographie Commerciale.* Bulletin.

L'Observatoire. Catalogue Photographique du Ciel.

P. CHERBOURG.—*Société Nationale des Sciences Naturelles et Mathématiques.* Mémoires.

P. CONCARNEAU.—*College de France* (*Laboratoire de Zoologie et de Physiologie Maritime*). Travaux Scientifiques.

P. DIJON.—*Académie des Sciences.* Mémoires.

LILLE—

T.P. *Société des Sciences.*

T.P. *Société Géologique du Nord.* Annales.

P. *Université de France.* Travaux et Mémoires.

LYONS—

T.P. *Académie des Sciences, Belles Lettres et Arts.* Mémoires.

T.P. *Société d'Agriculture, Histoire Nat. et Arts.* Annales.

T.P. *University.* Annales, Nouv. Série:—I. Sciences, Médecine. II. Droit, Lettres.

P. *Société Linnéenne.* Annales.

MARSEILLES—

T.P. *Faculté des Sciences.* Annales.

P. *Société Scientifique Industrielle.* Bulletin.

T.P. MONTPELLIER.—*Académie des Sciences et Lettres.* Mémoires: Section des Sciences; Section des Lettres; Section de Médecine. Bulletin Mensuel.

T.P. NANTES.—*Société Scientifique des Sciences Naturelles de l'Ouest de la France.* Bulletin.

T.P. NICE.—*L'Observatoire.* Annales.

PARIS—

T.P. *Académie des Sciences.* Comptes Rendus. — *Observatoire d'Abbadia*: Observations, 4to, and other Publications.

T.P. *Académie des Inscriptions et Belles-Lettres.* Comptes Rendus.

T.P. *Association Française pour l'Avancement des Sciences.* Comptes Rendus.

T.P. *Bureau International des Poids et Mesures.*—Procés-Verbaux des Séances. Travaux et Mémoires.

T.P. *Bureau des Longitudes.* Annuaire.

P. *L'École des Ponts et Chaussées.*

T.P. *Ministre de la Marine.* (*Service Hydrographique.*) Annales Hydrographiques.

T.P. *École des Mines.* Annales des Mines.

T.P. *École Normale Supérieure.* Annales.

T.P. *École Polytechnique.* Journal.

P. *École Libre des Sciences Politiques.*

T.P. *Ministère de l'Instruction Publique.*

T.P. *Musée Guimet.* Revue de l'Histoire des Religions.—Annales.—Bibliothèque d'Études.

T.P. *Muséum d'Histoire Naturelle.* Nouvelles Archives.—Bulletin.

T.P. *L'Observatoire.* Rapport Annuel sur l'État de l'Observatoire.—Annales.—Mémoires. — Carte Photographique du Ciel. Fol. — Catalogue Photographique du Ciel. 4to.

T.P. *Société Nationale d'Agriculture.* Bulletins.—Mémoires.

P. *Société d'Anthropologie.* Bulletin et Mémoires.

T.P. *Société Nationale des Antiquaires.* Mémoires.—Bulletin.

T.P. *Société de Biologie.* Comptes Rendus.

T.P. *Société d'Encouragement pour l'Industrie Nationale.* Bulletin.

T.P. *Société Française de Physique.* Journal de Physique.

T.P. *Société de Géographie.* La Géographie.

T.P. *Société Géologique de France.* Bulletins.—Mémoires (Paléontologie).

P. *Sociétés des Jeunes Naturalistes et d'Études Scientifiques.* Feuilles des Jeunes Naturalistes.

T.P. *Société Mathematique.* Bulletin.

P. *Société Philomathique.* Bulletin.

T.P. *Société Zoologique.* Bulletin.—Mémoires.

P. *Revue Générale des Sciences Pures et Appliquées.*

T.P. RENNES.—*Société Scientifique et Médicale de l'Ouest.* Bulletin.

TOULOUSE—

T.P. *University.—Faculté des Sciences.—L'Observatoire.* Annales.

P. *Académie des Sciences.* Mémoires.

GERMANY.

BERLIN—

Carte Géologique internationale de l'Europe. Livres I.–VII. (*Presented.*)

T.P. *K. Akademie der Wissenschaften.* Abhandlungen.—Sitzungsberichte.

T.P. *Physikalische Gesellschaft.* Fortschritte der Physik: 1[te] Abtheil.—Physik der Materie. 2[te] Abtheil.—Physik des Aethers. 3[e] Abtheil.—Komische Physik. Verhandlungen.

T.P. *Deutsche Geologische Gesellschaft.* Zeitschrift. Monatsberichte.

P. *Deutsche Meteorologische Gesellschaft.* Zeitschrift.

P. *Königl. Preussisches Meteorologisches Institut.*

P. *Gesellschaft Naturforschender Freunde.* Sitzungsberichte. Archiv für Biontologie.

P. *Kgl. Technische Hochschule.* Programm.

BONN—

P. *Naturhistorischer Verein der Preussischen Rheinlande und Westfalens.* Verhandlungen.

Niederrheinische Gesellschaft für Natur- und Heilkunde. Sitzungsberichte. (*Presented.*)

T.P. BREMEN.—*Naturwissenschaftlicher Verein.* Abhandlungen.

P. BRUNSWICK.—*Verein für Naturwissenschaft.* Jahresberichte.

P. CARLSRUHE.—*Technische Hochschule.* Dissertations.

P. CASSEL.—*Verein für Naturkunde.* Berichte.

T.P. CHARLOTTENBURG.—*Physikalisch-Technische Reichsanstalt.* Abhandlungen.

P. CHEMNITZ.—*Naturwissenschaftliche Gesellschaft.* Berichte.

T.P. DANTZIC.—*Naturforschende Gesellschaft.* Schriften.

ERLANGEN—

T.P. *University.* Inaugural Dissertations.

P. *Physikalisch-Medicalische Societät.* Sitzungsberichte.

T.P. FRANKFURT-AM-MAIN.—*Senckenbergische Naturforschende Gesellschaft.* Abhandlungen.—Berichte.

P. FRANKFURT-AM-ODER.—*Naturwissenschaftlicher Verein.* Helios.

P. FREIBURG-I-BR.—*Naturforschende Gesellschaft.* Berichte.

GIESSEN—

T.P. *University.* Inaugural Dissertations.

P. *Oberhessische Gesellschaft für Natur- und Heilkunde.* Berichte.

T.P. GÖTTINGEN.—*K. Gesellschaft der Wissenschaften.* Abhandlungen, Neue Folge: Math.-Phys. Classe; Phil.-Hist. Classe.—Nachrichten: Math.-Phys. Cl.; Phil.-Hist. Cl.; Geschäftliche Mittheilungen. — (Gelehrte Anzeigen. *Purchase.*)

HALLE—

T.P. *K. Leopold-Carolinisch-Deutsche Akademie der Naturforscher.* Nova Acta (Verhandlungen).—Leopoldina.

T.P. *Naturforschende Gesellschaft.* Abhandlungen.

P. *Verein für Erdkunde.* Mittheilungen.

P. *Naturwissenschaftlicher Verein für Sächsen und Thüringen.*

P. *Deutsche Mathematiker Vereinigung.* Jahresbericht.

HAMBURG—

T.P. *Kaiserliche Marine Deutsche Seewarte.*

T.P. *Naturwissenschaftlicher Verein.* Abhandlungen aus dem Gebiete der Naturwissenschaften.—Verhandlungen.

T.P. *Naturhistorisches Museum.* Jahrbuch.—Beihefte.—Mitteilungen aus dem Naturhistorischen Museum.

P. *Verein für Naturwissenschaftliche Unterhaltung.* Verhandlungen.

T.P. HANNOVER.—*Naturhistorische Gesellschaft.* Jahresbericht.

T.P. HELGOLAND.—*K. Biologisches Anstalt.* Wissenschaftliche Meeresuntersuchungen (Abtheilung Helgoland).

T.P. JENA.—*Medicinisch-Naturwissenschaftliche Gesellschaft.* Jenaische Zeitschrift für Naturwissenschaft.—Denkschriften.

KIEL—

T.P. *Universität.* Inaugural University Dissertations.

T.P. *Kommission zur Wissenschaftlichen Untersuchung der Deutschen Meere.* Wissenschaftliche Meeresuntersuchungen (Abtheilung Kiel).

P. *Naturwissenschaftlicher Verein für Schleswig-Holstein.* Schriften.

T.P. KÖNIGSBERG.—*University.*

LEIPZIG—

Fürstlich Jablonowskische Gesellschaft. Preisschriften. (*Presented.*)

T.P. *Königl. Sächsische Gesellschaft der Wissenschaften.* Berichte: Math.-Phys. Classe; Philologisch-Historische Classe.—Abhandlungen der Math.-Phys. Classe; Phil.-Hist. Classe.

T.P. Editor of the *Annalen der Physik.*

P. *Naturforschende Gesellschaft.* Sitzungsberichte.

Deutsche Mathematiker Vereinigung. (*See* HALLE.)

P. LÜBECK.—*Geographische Gesellschaft und Naturhistorisches Museum.* Mitteilungen.

T.P. MUNICH.—*K. Bayerische Akademie der Wissenschaften.* Abhandlungen: Mathematisch-Physikalische Classe; Philosophisch-Philologische Classe; Historische Classe.—Sitzungsberichte: Mathematisch-Physikalische Classe; Philosophisch-Philol. und Historische Classe.—Festreden. Almanach.

K. Sternwarte. Neue Annalen. Bd. iv. 4to. (*Presented.*)

P. OFFENBACH.—*Verein für Naturkunde.* Berichte.

P. OSNABRÜCK.—*Naturwissenschaftlicher Verein.* Jahresbericht.

T.P. POTSDAM.—*Astrophysikalisches Observatorium.* Publikationen.

P. REGENSBURG.—*Historischer Verein von Oberpfalz und Regensburg.* Verhandlungen.

P. ROSTOCK-I-M.—*Naturforschende Gesellschaft.* Sitzungsberichte und Abhandlungen.

P. *University.*

T.P. STRASBOURG.—*University.* Inaugural Dissertations.

T.P. STUTTGART.—*Verein für vaterländische Naturkunde in Württemberg.* Jahreshefte.

T.P. TÜBINGEN.—*University.* Inaugural Dissertations.

GREECE.

Athens—

T.P. *University Library.*

T.P. *Observatoire National.* Annales.

HAWAIIAN ISLANDS.

P. Honolulu.—*Bernice Pauahi Bishop Museum of Polynesian Ethnology.* Occasional Papers. Fauna Hawaiiensis. Memoirs.

HOLLAND.

Amsterdam—

T.P. *Kon. Akademie van Wetenschappen.* Verhandelingen: Afd. Natuurkunde. 1ste Sectie. 2te Sectie; Afd. Letterkunde.—Verslagen en Mededeelingen; Letterkunde.—Verslagen der Zittingen van de Wis- en Naturkundige Afdeeling.—Jaarboek.—Proceedings of the Section of Sciences.—Poemata Latina.

T.P. *Koninklijk Zoologisch Genootschap "Natura Artis Magistra."* Bijdragen tot de Dierkunde.

P. *Wiskundig Genootschap.* Nieuw Archief voor Wiskunde.—Wiskundige Opgaven.—Revue Semestrielle des Publications Mathématiques.

P. Delft.—*École Polytechnique.* Dissertations.

T.P. Groningen.—*University.* Jaarboek.

T.P. Haarlem.—*Hollandsche Maatschappij der Wetenschappen.* Naturkundige Verhandelingen.—Archives Néerlandaises des Sciences Exactes et Naturelles. Œuvres complètes de C. Huygens. Bds. I.–XII.

T.P. *Musée Teyler.* Archives.

T.P. Helder.—*Nederlandsche Dierkundige Vereeniging.* Tijdschrift.

T.P. Leyden.—*The University.*

P. Nijmegen.—*Nederlandsche Botanische Vereeniging.* Nederlandsch Kruidkundig Archief.—Verslagen en Mededeelingen.—Recueil des Travaux Botaniques Néerlandaises.

T.P. Rotterdam.—*Bataafsch Genootschap der Proefondervindelijke Wijsbegeerte.* Nieuwe Verhandelingen.

P. Utrecht.—*Provinciaal Utrechtsch Genootschap van Kunsten en Wetenschappen.* Verslag van de Algemeene Vergaderingen. Aanteekeningen van de Sectie Vergaderingen. 8vo.

Koninklijk Nederlandsch Meteorologisch Institut.—Observations Océanographiques et Météorologiques.—Œuvres Océanographiques. (*Presented.*)

HUNGARY.

Buda-Pesth—

T.P. *Magyar Tudományos Akademia* (*Hungarian Academy*). Mathemat. és természettud. közlemények (Communications Math. and Nat. Sciences).—Nyelvtud. közlemények (Philology).—Mathemat. és természettud. Értesitö (Bulletin, Math. and Nat. Sciences).—Nyelvtudom. Értekezések (Philol. Memoirs).—Történettud. Értekezések (Historical Memoirs).—Társadalmi

BUDA-PESTH—*continued*—

Értekezések (Memoirs, Political Sciences).—Archæologiai Értesitö.—Rapports.—Almanach.—Mathematische und Naturwissenschaftliche Berichte aus Ungarn.—And other publications of the Hungarian Academy, or works published under its auspices.

T.P. *Kir-Magy. Termeszettudomanyi Tarsulat* (*Royal Hungarian Society of Nat. Sciences*).

P. *Magyar Királyi Ornithologicii Központ* (*Royal Hungarian Central-Bureau for Ornithology*). Aquila.

ICELAND.

P. REIKJAVIK.—*Islenzka Fornleifafelag.*

INDIA.

BANGALORE.—Meteorological Results of Observations taken at Bangalore, Mysore, Hassan, and Chitaldroog Observatories; Report of Rainfall Registration in Mysore. (*Presented by the Mysore Government.*)

BOMBAY—

T.P. *Royal Asiatic Society* (*Bombay Branch*). Journal.

T.P. *Elphinstone College.*

Archæological Survey of Western India. Progress Reports. (*Presented.*)

Government Observatory. Magnetic and Meteorological Observations. (*Presented.*)

BURMA.—Reports on Archæological Work in Burma. (*Presented by the Government.*)

BURMA.—*Ethnographic Survey.* Anthropometric Data. 8vo. (*Presented.*)

CALCUTTA—

T.P. *Asiatic Society of Bengal.* Journal and Proceedings.—Memoirs.

Ethnographical Survey (*Central Indian Agency*). Monographs. (*Presented.*)

T.P. *Geological Survey of India.* Records. Memoirs.

T.P. *The Meteorological Office, Government of India.* Indian Meteorological Memoirs.—Reports.—Monthly Weather Review.

Archæological Survey of India. Epigraphia Indica.—Annual Reports. (*Presented by the Indian Government.*)

Botanical Survey of India. Records. 8vo. (*Presented by the Indian Government.*)

Imperial Library. Catalogue. (*Presented.*)

Linguistic Survey of India. Publications. (*Presented by the Indian Government.*)

Royal Botanic Garden. Annals. (*Presented.*)

Indian Museum. Annual Reports.—Records, Memoirs, Catalogues, etc. (*Presented.*)

Great Trigonometrical Survey. Account of Operations.—Extracts from Narrative Reports. Professional Papers. 4to. (*Presented.*)

Fauna of British India, including Ceylon and Burma. 8vo. (*Presented by the Indian Government.*)

CALCUTTA—*continued*—

Scientific Memoirs, by Medical Officers of the Army of India. 4to. (*Presented.*)

P. COIMBATORE.—*Agricultural College and Research Institute.*

MADRAS—

T.P. *The Literary Society.*

Observatory. Report on the Kodaikanal and Madras Observatories. 8vo.—Bulletins. Memoirs. (*Presented.*) 4to.

Government Central Museum. Report. Bulletin. (*Presented.*) 8vo.

A Descriptive Catalogue of the Sanskrit MSS. in the Government Oriental Manuscripts Library, Madras. By M. Seshagiri Sastri. 8vo. (*Presented by the Government of Madras.*)

RANGOON. (*See* BURMA.)

SIMLA. *Committee for the Study of Malaria.* Transactions. (*Presented by the Sanitary Commissioner.*) 8vo.

IRELAND.

BELFAST—

P. *Natural History and Philosophical Society.* Proceedings.

T.P. *Queen's University.* Calendar.

DUBLIN—

T.P. *Royal Irish Academy.* Proceedings.—Transactions.—Abstract of Minutes.

T.P. *Royal Dublin Society.* Scientific Proceedings.—Economic Proceedings.—Scientific Transactions.

T.P. *Library of Trinity College.*

T.P. *National Library of Ireland.*

P. *Dunsink Observatory.*

Department of Agriculture and Technical Instruction for Ireland—Fisheries Branch. Reports on the Sea and Inland Fisheries of Ireland.—Scientific Investigations. 8vo.—Geological Survey Memoirs. (*Presented by the Department.*) 8vo.

ITALY.

BOLOGNA—

T.P. *Accademia di Scienze dell' Istituto di Bologna.* Memorie.—Rendiconti.

University Observatory. Osservazioni Meteorologiche. (*Presented.*)

T.P. CATANIA.—*Accademia Gioenia di Scienze Naturali.* Atti.—Bolletino Mensile.

T.P. *Società degli Spettroscopisti Italiani.* Memorie.

T.P. GENOA.—*Museo Civico di Storia Naturale.* Annali.

P. MESSINA.—*Reale Accademia Peloritana.* Atti.

MILAN—

R. Osservatorio di Brera. Publicazioni. (*Presented.*)

T.P. *Reale Istituto Lombardo di Scienze, Lettere, ed Arti.* Memorie: Classe di Scienze Mat. et Nat.; Classe di Lettere Scienze Storiche e Morali.—Rendiconti.

MODENA—

T.P. *Regia Accademia di Scienze, Lettere, ed Arti.* Memorie.

P. *Società dei Naturalisti.* Atti. 8vo.

T.P. NAPLES.—*Società Reale di Napoli. Accademia di Scienze Fisiche e Matematiche.* Memorie.—Rendiconti. *Accademia di Scienze Morali e Politiche.* Atti.—Rendiconti. *Accademia di Archeologia, Lettere e Belle Arti.* Atti.—Rendiconti.

T.P. *Stazione Zoologica.* Mittheilungen.

T.P. *R. Istituto d'Incoraggiamento.* Atti.

P. *Museo Zoologico della R. Universita.* Annuario.

T.P. PADUA.—*R. Accademia di Scienze, Lettere, ed Arti.* Atti e Memorie.

T.P. PALERMO.—*Società di Scienze Naturali ed Economiche.* Giornale di Scienze Naturali ed Economiche.

P. PISA.—*Società Italiana di Fisica.* "Il Nuovo Cimento."

ROME—

T.P. *R. Accademia dei Lincei. Classe di Scienze Fisiche, Math. e Nat.* Memorie. —Rendiconti. *Classe di Scienze Morali, Storiche e Filol.*—Notizie degli Scavi di Antichita.—Rendiconti.—Memorie.

T.P. *Accademia Ponteficia dei Nuovi Lincei.* Atti.—Memorie.

T.P. *R. Comitato Geologico.* Memorie descrittive della Carta Geologica.

T.P. *Società Italiana di Scienza (detta dei XL.).* Memorie.

P. SASSARI.—*Istituto Fisiologico della R. Università di Sassari.* Studi Sassaresi.

TURIN—

T.P. *Reale Accademia delle Scienze.* Memorie.—Atti.

Osservatorio della R. Università. Osservazioni Meteorologiche. 8vo. (*Presented.*)

T.P. VENICE.—*R. Istituto Veneto di Scienze, Lettere, ed Arti.* Atti.—Osservazioni Meteorologiche.

JAMAICA.

P. KINGSTON.—*The Institute of Jamaica.*

JAPAN.

TOKIO—

T.P. *The Imperial University of Tokio (Teikoku-Daigaku).* Calendar.—*College of Science.* Journal.—*Medicinische Facultät der Kaiserlich-Japanischen Universität.* Mittheilungen.

P. *Zoological Society.* Annotationes Zoologicæ Japonenses.

P. *Asiatic Society.* Transactions.

P. *Deutsche Gesellschaft für Natur- und Völkerkunde Ostasiens.* Mittheilungen. 8vo.

P. *Imperial Museum.*

Earthquake Investigation Committee. Bulletin. 8vo. (*Presented.*)

T.P. KYOTO.—*Imperial University (College of Science and Engineering).* Memoirs.

JAVA.

BATAVIA—

T.P. *Bataviaasch Genootschap van Kunsten en Wetenschappen.* Verhandelingen.—Tijdschrift voor Indische Taal-, Land- en Volkenkunde.—Notulen. 8vo.

BATAVIA—*continued*—

T.P. *Magnetical and Meteorological Observatory.* Observations.—Regenwaarnemingen in Nederlandsch-Indie.

P. *Kon. Natuurkundige Vereeniging.* Natuurkundig Tijdschrift voor Nederlandsch-Indie.

LUXEMBOURG.

P. LUXEMBOURG.—*L'Institut Royal Grand-Ducal.* Archives trimestrielles.

MAURITIUS.

T.P. *Royal Alfred Observatory.* Annual Reports.

MEXICO.

MEXICO—

T.P. *Musée National d'Histoire Naturelle.* La Naturaleza, etc.

T.P. *Sociedad cientifica "Antonio Alzate."* Memorias.

T.P. *Observatorio Meteorologico-Magnetico Central.* Boletin Mensual.

P. *Istituto Geologico.* Boletin. Papergones.

P. *Academia Mexicana de Ciencias Exactas, Fisicas y Naturales.*

P. TACUBAYA.—*Observatorio Astronomico.* Annuario.

P. XALAPA.—*Observatorio Meteorologico Central del Estado Vera Cruz.*

MONACO.

MONACO.—*Musée Océanographique.* Bulletins. 8vo. (*Presented.*)

NATAL.

P. PIETERMARITZBURG.—*Geological Survey of Natal.* Annual Reports.—Report on the Mining Industry of Natal. 4to.

Government Museum. Annals. Catalogues. (*Presented.*)

NEW SOUTH WALES. (*See* AUSTRALIA.)

NEW ZEALAND.

WELLINGTON—

T.P. *New Zealand Institute.* Transactions and Proceedings.

New Zealand Government. Statistics of New Zealand. 4to.—The New Zealand Official Handbook. (*Presented by the Government.*) Reports.

Colonial Museum and Geological Survey. Publications. (*Presented.*)

NORWAY.

T.P. BERGEN.—*Museum.* Aarsberetning.—Aarbog.—An Account of the Crustacea of Norway. By G. O. Sars.

CHRISTIANIA—

T.P. *K. Norske Frederiks Universitet.* Nyt Magazin for Naturvidenskaberne.—Archiv for Mathematik og Naturvidenskab.

T.P. *Meteorological Institute.* Jahrbuch.

Videnskabs-Selskab. Forhandlinger.—Skrifter (Math. Nat. Kl.). (*Presented.*)

P. STAVANGER.—*Museum.* Aarshefte.

T.P. THRONDHJEM.—*Kgl. Norske Videnskabers Selskab.* Skrifter.

P. TROMSÖ.—*Museum.* Aarshefter.—Aarsberetning.

PHILIPPINE ISLANDS.

P. MANILA.—*Bureau of Science.* Ethnological Survey Publications. Bureau of Forestry. Annual Report.

PORTUGAL.

T.P. COIMBRA.—*University.* Annuario. Archivo Bibliographico.

LISBON—

T.P. *Academia de Sciencias di Portugal.*

T.P. *Sociedade de Geographia.*

Observatorio do Infante D. Luiz. Annaes. (*Presented.*)

PORTO. *Academia Polytechnica.*

QUEENSLAND. (*See* AUSTRALIA.)

ROUMANIA.

BUCHAREST—

T.P. *Academia Romana.* Analele.—Also Publications relating to the History, etc., of Roumania. Bibliografia Romanesca.—Catalogues, etc.

P. *Institut Météorologique.* Analele. *Observatoril Astronomic si Meteorologic,*—Buletinul Lunar.

RUSSIA.

T.P. DORPAT (JURJEW).—*University.* Inaugural Dissertations.—Acta. Sitzungsberichte der Naturforscher Gesellschaft bei der Universität.

T.P. EKATHERINEBOURG.—*Société Ouralienne d'Amateurs des Sciences Naturelles.* Bulletin.

KAZAN—

T.P. *Imperial University.* Uchenuiya Zapiski.

P. *Société Physico-Mathématique de Kazan.* Bulletin.

T.P. KIEV.—*University.* Universitetskiya Isvyaistiya.

MOSCOW—

T.P. *Société Impériale des Naturalistes.* Bulletin.—Nouveaux Mémoires.

T.P. *L'Observatoire Impérial.*

T.P. *Société Impériale des Amis d'Histoire Naturelle, d'Anthropologie et d'Ethnographie.*

MOSCOW—*continued*—
T.P. *Imperial University.*
T.P. *Musée Polytechnique.*
P. *Observatoire Magnétique et Météorologique de l'Université Impériale.*
P. ODESSA.—*Société des Naturalistes de la Nouvelle Russie.* Zapiski.
T.P. POULKOVA.—*Nicolai Hauptsternwarte.* Publications.—Annales.
ST PETERSBURG—
T.P. *Académie Impériale des Sciences.* Mémoires: Classe Phys.-Math.; Classe Hist.-Phil.—Bulletins.—Comptes Rendus des Séances de la Commission Sismique Permanente. 4to.
T.P. *Comité Géologique.* Mémoires.—Bulletins.—Carte Géologique: Région Aurifère d'Ienissei: de l'Amour: de Lena.
Commission Royale Russe pour la Mesure d'un Arc de Méridien au Spitzberg. Missions Scientifiques pour la Mesure d'un Arc de Méridien au Spitzberg enterprises en 1899–1902, sous les auspices des Governements Suédois et Russe. Mission Russe. 4to. (*Presented.*)
T.P. *Imperial University.*
T.P. *Institut Impérial de Médecine Expérimentale.* Archives des Sciences Biologiques.
T.P. *Physicalische Central-Observatorium.* Annalen.
T.P. *Physico-Chemical Society of the University of St Petersburg.* Journal.
T.P. *Russian Ministry of Marine.*
P. *Imperial Russian Geographical Society.*
P. *Russian Society of Naturalists and Physicians.*
P. *K. Mineralogische Gesellschaft.* Verhandlungen (Zapiski).—Materialien zur Geologie Russlands.
P. *Société des Naturalistes* (*Section de Géologie et de Minéralogie*). Travaux et Supplements.
P. *Société Astronomique Russe.*
P. TIFLIS.—*Physikalisches Observatorium.*

SCOTLAND.

T.P. ABERDEEN.—*University Library.* Calendar.
P. BERWICKSHIRE.—*Naturalists' Club.* Proceedings.
T.P. DUNDEE.—*University College Library.*
EDINBURGH—
T.P. *Advocates' Library.*
P. *Botanical Society.* Transactions and Proceedings.
Carnegie Trust for the Universities of Scotland. Report. (*Presented.*)
P. *Fishery Board for Scotland.* Annual Reports. Scientific Investigations.
P. *Geological Society.* Transactions.
Geological Survey of Scotland. Memoirs, Maps, etc. (*Presented by the Government.*)
T.P. *Highland and Agricultural Society of Scotland.* Transactions.
P. *Mathematical Society.* Proceedings.—Mathematical Notes.
P. *Pharmaceutical Society.*

EDINBURGH—*continued*—

Registrar-General's Returns of Births, Deaths, and Marriages. (*Presented.*)

T.P. *Royal Botanic Garden.* Notes from the.

T.P. *Royal College of Physicians.*

P. *Royal College of Physicians' Laboratory.* Laboratory Reports.

T.P. *Royal Medical Society.*

T.P. *Royal Observatory.* Annals.

T.P. *Royal Physical Society.* Proceedings.

Royal Scottish Academy. Annual Reports. (*Presented.*)

P. *Royal Scottish Geographical Society.* Scottish Geographical Magazine.

T.P. *Royal Scottish Society of Arts.* Transactions.

P. *Scottish Meteorological Society.* Journal.

Scottish National Antarctic Expedition. Publications. (*Presented.*)

T.P. *University Library.* Calendar.

GLASGOW—

P. *Geological Society.* Transactions.

Glasgow and West of Scotland Technical College. Calendar. (*Presented.*)

P. *Marine Biological Association of the West of Scotland.* Annual Report. See MILLPORT.

P. *Natural History Society.*—Glasgow Naturalist.

T.P. *Royal Philosophical Society.* Proceedings.

T.P. *University.* Calendar.

P. *University Observatory.*

P. MILLPORT.—*Marine Biological Association of the West of Scotland.* Annual Report.

T.P. PERTH.—*Perthshire Society of Natural Science.* Proceedings.

T.P. ST. ANDREWS.—*University Library.* Calendar.

SPAIN.

MADRID—

T.P. *Real Academia de Ciencias Exactas, Fisicas y Naturales.* Memorias.—Revista.—Annuario.

T.P. *Comisión del Mapa Geológico de España.* Boletin.—Memorias.

P. VILAFRANCA DEL PANADES (CATALUÑA).—*Observatorio Meteorologico.*

SWEDEN.

P. GOTHENBURG.—*Kongl: Vetenskaps-och Vitterhets Samhälle.* Handlingar.

T.P. LUND.—*University.* Acta Universitatis Lundensis (Fysiografiska Sällskapets Handlingar.—Theologi.—Medicina).

T.P. STOCKHOLM.—*Kong. Svenska Vetenskaps-Akademie.* Handlingar.—Arkiv för Zoologi.—Arkiv för Matematik, Astronomi och Fysik.—Arkiv för Botanik.—Arkiv för Kemi, Mineralogi och Geologi.—Meteorologiska Iakttagelser i Sverige.—Astronomiska Iakttagelser.—Lefnadsteckningar.—Arsbok. Meddelanden från K. Vetenskaps Akademiens Nobelinstitut. Accessionskatalog. Les Prix Nobel.

P. *Svenska Sällskapet för Antropologi och Geografi.* Ymer.

STOCKHOLM—*continued.*

Commission Royale Suédoise pour la Mesure d'un Arc de Méridien au Spitzberg. Missions Scientifiques pour la Mesure d'un Arc de Méridien au Spitzberg entreprises en 1899–1902, sous les auspices des Gouvernements Suédois et Russe. Mission Suédoise. 4to. (*Presented.*)

UPSALA—

T.P. *Kongliga Vetenskaps Societeten* (*Regia Societas Scientiarum*). Nova Acta.

T.P. *University.* Arsskrift.—Inaugural Dissertations (Medical and Scientific).—Bulletin of the Geological Institution.—Results of the Swedish Zoological Expedition to Egypt and the White Nile, 1901.

Observatoire de l'Université. Bulletin Météorologique Mensuel.

SWITZERLAND.

T.P. BASLE.—*Naturforschende Gesellschaft.* Verhandlungen.

BERN—

T.P. *Société Helvétique des Sciences Naturelles.* (*Allgemeine Schweizerische Gesellschaft für die gesammten Naturwissenschaften.*) Comptes Rendus.—Actes (Verhandlungen).—Nouveaux Mémoires.

P. *Naturforschende Gesellschaft.* Mittheilungen.

T.P. GENEVA.—*Société de Physique et d'Histoire Naturelle.* Mémoires.—Comptes Rendus.

P. LAUSANNE.—*Société Vaudoise des Sciences Naturelles.* Bulletin.—Observations Météorologiques.

NEUCHATEL—

T.P. *Société des Sciences Naturelles.* Bulletin.

P. *Société Neuchâteloise de Géographie.* Bulletin.

ZURICH—

T.P. *University.*

T.P. *Commission Géologique Suisse.* Beiträge zur geologischen Karte der Schweiz.

T.P. *Naturforschende Gesellschaft.* Vierteljahrsschrift.

P. *Schweizerische Botanische Gesellschaft.* Berichte (Bulletin).

Schweizerische Meteorologische Central-Anstalt. Annalen. 4to. (*Presented.*)

TASMANIA. (*See* AUSTRALIA.)

TRANSVAAL.

Transvaal Agricultural Journal. (*Presented.*)

T.P. JOHANNESBURG.—*Transvaal Meteorological Department. Government Observatory.* Reports. Circulars.

PRETORIA.—*Geological Survey.* Reports. Memoirs. (*Presented.*)

Transvaal Museum. Annals. (*Presented.*)

TURKEY.

P. CONSTANTINOPLE.—*Société Impériale de Médecine.* Gazette Médicale d'Orient.

UNITED STATES.

Albany—

T.P. *New York State Library.* Annual Reports.—Bulletins.

State Museum. Annual Reports.—Bulletin. *New York State Education Department.* Annual Reports.

P. Allegheny.—*Observatory.* Publications.

P. Annapolis (Maryland).—*St John's College.*

P. Austin.—*Texas Academy of Sciences.* Transactions.

T.P. Baltimore.—*Johns Hopkins University.* American Journal of Mathematics.—American Chemical Journal.—American Journal of Philology.—University Studies in Historical and Political Science.—Memoirs from the Biological Laboratory.—University Circulars.

Johns Hopkins Hospital. Bulletins.—Reports. (*Presented.*)

Conservation Commission of Maryland. Report 1908–09. (*Presented.*)

T.P. *Maryland Geological Survey.* Publications.

Maryland Weather Service. Reports. (*Presented.*)

Peabody Institute. Annual Reports. (*Presented.*)

Berkeley (California)—

T.P. *University of California.*—University Chronicle.—Reports of Agricultural College. — Publications (Zoology, Botany, Geology, Physiology, and American Archæology and Ethnology).—Memoirs. 4to.

Academy of Pacific Coast History. Publications.

Boston—

T.P. *The Bowditch Library.*

T.P. *Boston Society of Natural History.* Memoirs.—Proceedings.— Occasional Papers. 8vo.

T.P. *American Academy of Arts and Sciences.* Memoirs.—Proceedings.

P. Brooklyn.—*Institute of Arts and Sciences.* Museum Reports and Bulletins.

P. Buffalo.—*Society of Natural Sciences.* Bulletin.

California. (*See* San Francisco, Sacramento, Berkeley, Mount Hamilton, Mount Wilson and Stanford.)

Cambridge—

T.P. *Harvard University.* — *Museum of Comparative Zoology.* Memoirs. — Bulletins.—Annual Reports.

T.P. *Astronomical Observatory, Harvard College.* Annals.—Annual Reports. Observatory Circulars.

P. Chapel Hill (North Carolina).—*E. Mitchell Scientific Society.* Journal.

P. Charlottesville. *Philosophical Society, University of Virginia.* Bulletin.—Scientific Series and Humanistic Series.

Chicago—

T.P. *Field Museum of Natural History.* Publications: Geological Series; Botanical Series; Zoological Series; Anthropological Series.—Annual Reports.

P. *University of Chicago.*

T.P. *Yerkes Observatory* (*University of Chicago*). Publications.

P. *Academy of Sciences.* Bulletins.—Bulletins of the Natural History Survey.

CINCINNATI—
p. *Observatory.* Publications.
p. *Society of Natural History.* Journal.
p. *Ohio Mechanics' Institute.*
T.P. CLEVELAND (OHIO).—*The Geological Society of America.* Bulletins.
T.P. CLINTON (IOWA).—*Litchfield Observatory, Hamilton College.*
COLORADO SPRINGS.—*Colorado College.* Colorado College Studies. 8vo. (*Presented.*)
p. CONNECTICUT. — *Connecticut Academy of Arts and Sciences.* Transactions. Memoirs.
p. DAVENPORT.—*Academy of Natural Sciences.* Proceedings.
p. DENVER (COLORADO).—*Scientific Society of Colorado.* Proceedings.
T.P. DES MOINES (IOWA).—*Iowa Academy of Sciences.* Proceedings.
p. GARRISON, N.Y.—Editor, *American Naturalist.*
T.P. GRANVILLE (OHIO).—*Denison University and Scientific Association.* Bulletin of the Scientific Laboratories.
p. INDIANOPOLIS.—*Indiana Academy of Sciences.* Proceedings.
IOWA CITY—
p. *Geological Survey.* Annual Reports.
p. *State University. Laboratories of Natural History.* Bulletins.—Contributions from the Physical Laboratories.
IOWA. (*See* DES MOINES.)
ITHACA (N.Y.)—
p. The Editor, *Physical Review.* (Cornell University.)
p. The Editors, *Journal of Physical Chemistry.* (Cornell University.)
p. LANSING (MICHIGAN).—*Michigan Academy of Science.* Reports.
T.P. LAWRENCE (KANSAS).—*University of Kansas.* Science Bulletin (University Quarterly).
p. LINCOLN (NEBRASKA).—*The University of Nebraska. Agricultural Experiment Station.* Bulletins.
MADISON—
T.P. *Wisconsin University. Washburn Observatory.* Observations.
p. *Wisconsin Academy of Sciences, Arts, and Letters.* Transactions.
p. *Geological and Natural History Survey.* Bulletins. (*Economic and Scientific Series.*)
p. MASSACHUSETTS.—*Tufts College Library.* Tufts College Studies.
p. MERIDEN (CONN.).—*Meriden Scientific Association.*
MICHIGAN. (*See* LANSING.)
MINNEAPOLIS (MINN.)—
p. *Geological and Natural History Survey of Minnesota.*
p. *Botanical Survey.*
MISSOURI. (*See* ST. LOUIS and ROLLA.)
p. MOUNT HAMILTON (CALIFORNIA).—*Lick Observatory.* Bulletins.—Publications.
T.P. MOUNT WILSON (CALIFORNIA).—*Solar Observatory.* Contributions.—Reports.
T.P. NEWHAVEN (CONN.)—*Yale College. Astronomical Observatory of Yale University.* Transactions.—Reports.
p. NEW ORLEANS.—*Academy of Sciences.*

NEW YORK—

T.P. *American Mathematical Society.* Bulletins.—Transactions.

P. *American Museum of Natural History.* Bulletins.—Memoirs.—American Museum Journal.—Annual Reports.—Anthropological Papers.—Guide Leaflets.

P. *American Geographical Society.* Bulletin.

P. *American Institute of Electrical Engineers.* Proceedings.

NEW YORK. (*See also* ALBANY.)

PHILADELPHIA—

T.P. *American Philosophical Society for Promoting Useful Knowledge.* Proceedings. —Transactions.

T.P. *Academy of Natural Sciences.* Proceedings.—Journal.

T.P. *University of Pennsylvania.* Publications :—Philology, Literature, and Archæology, Mathematics. Contributions from the Zoological and Botanical Laboratories. University Bulletins.—Theses.—Calendar.

T.P. *Geological Survey of Pennsylvania.*

P. *Wagner Free Institute of Science.* Transactions.

P. *Geographical Society.* Bulletin.

P. *Commercial Museums.*

P. PORTLAND (MAINE).—*Society of Natural History.* Journal.

P. ROCHESTER.—*Academy of Science.* Proceedings.

T.P. ROLLA (MISS.).—*Bureau of Geology and Mines.* Biennial Reports.

T.P. SALEM.—*The Essex Institute.*

SAINT LOUIS—

T.P. *Academy of Sciences.* Transactions.

P. *Missouri Botanical Garden.* Annual Reports.

T.P. SAN FRANCISCO (CALIFORNIA).—*Academy of Sciences.* Proceedings.—Memoirs. —Occasional Papers.

Stanford University, California. Publications 3–6. (*Presented.*)

P. TOPEKA.—*Kansas Academy of Science.* Transactions.

WASHINGTON—

T.P. *U.S. National Academy of Sciences.* Memoirs.

T.P. *Bureau of Ethnology.* Annual Reports.—Bulletins.

T.P. *U.S. Coast and Geodetic Survey.* Annual Reports, etc.

T.P. *U.S. Commission of Fish and Fisheries.* Reports.—Bulletins.

T.P. *U.S. Naval Observatory.* Reports.—Observations. (N.S.)

T.P. *U.S. Geological Survey.* Bulletins.—Annual Reports.—Monographs.—Geologic Atlas of the United States.—Mineral Resources.—Professional Papers.—Water Supply and Irrigation Papers.

Geological Society of America. (*See* CLEVELAND.)

T.P. *Weather Bureau.* (*Department of Agriculture.*) Monthly Weather Review. — Bulletins. — Reports. — Bulletin of the Mount Weather Observatory.

T.P. *Smithsonian Institution.* Miscellaneous Collections.—The same (Quarterly Issue).—Contributions to Knowledge.—Reports.—Annals of the Astrophysical Observatory.—Harriman Alaska Expedition, Vols I.–V. and VIII.–XIII. 4to.

T.P. *Surgeon-General's Office.* Index to Catalogue of the Library. 4to.

WASHINGTON—*continued*—

T.P. *Carnegie Institution of Washington.* Year-Books.—Publications.—*Carnegie Foundation for the Advancement of Teaching.* Annual Report.—Bulletin.

T.P. *American Association for the Advancement of Science.* Proceedings.

P. *Philosophical Society.* Bulletin.

P. *U.S. National Museum.* Bulletins.—Reports.—Proceedings.—Contributions from the U.S. National Herbarium.

P. *Department of Agriculture.* (*Division of Economic Ornithology and Mammalogy.*) Bulletin.

P. *U.S. Patent Office.*

Washington Academy of Sciences, Journal of the. (*Purchase.*)

Bureau of Standards. Department of Commerce and Labour. Bulletins. (*Presented.*)

WISCONSIN. (*See* MADISON.)

VICTORIA. (*See* AUSTRALIA.)

II. DONATIONS FROM AUTHORS, ETC.

Aberdeen University. Record of the Celebration of the Quatercentenary: from 25th to 28th September 1906. Edited by P. J. Anderson, M.A. Aberdeen, 1907. 4to.

——— Studies, Nos. 38 and 41. Studies on Alcyonarians. By J. A. Thomson, M.A., etc. Aberdeen, 1909. 4to.

Africa, South. *See* Beattie, J. C. (Magnetic Survey).

Alberta. Department of Agriculture, Annual Report, 1908, 1909. Edmonton, 1910. 8vo.

Ames (Percy W.). Milton Memorial Lectures, 1908. [Read before the Royal Society of Literature.] London, 1909. 8vo.

Annuaire de l'Université Laval, pour 1910–1911. Quebec, 1910. 8vo.

Baldaque da Silva (A. A.). Le Problème de la Vie. Lisbonne, 1911. 8vo.

Bartholomew, Dr J. G. Physical Atlas. Vol. V. Atlas of Zoogeography. Edinburgh, 1911. Fol.

Beattie (J. C.), D.Sc. Report of a Magnetic Survey of South Africa. London, 1909. 4to.

Belgica :—Résultats du Voyage du S.Y. en 1897, 1898, 1899. Rapports Scientifiques.

Zoologie :—Turbellarien, von Ludwig Bohmig, 1908. Pennatuliden, von H. F. E. Jungersen, 1907. Scaphopoden, von L. Plate, 1908. Cirripedia, von P. P. C. Hoek, 1907. Ostracoden, von G. W. Müller, 1906. Medusen, von Otto Maas, 1906. Holothuries, von E. Hérouard.

Belgica—Zoologie—*continued*—

Insectes:—Introduction, par G. Severin, 1906. Orthoptères, par C. Brunner von Wattenwyl. Hémiptères, par E. Bergroth. Coléoptères, par E. Rousseau, A. Grouvelle, H. Schouteden, E. Brenske, H. Boileau, J. Bourgeois, E. Olivier, L. Fairmaire, G. Stierlin, A. Bovie, A. Lameere.

Hyménoptères, par J. Tosquinet, C. Emery, E. André, J. Vachal. Diptères, par J. Ch. Jacobs, Th. Becker, E. H. Rübsaamen. Schizopoda and Cumacea, by H. J. Hansen, 1908.

Géologie:—Les Glaciers, par H. Arctowski, 1908. Étude des Roches, par A. Pelikan. Quelques Plantes Fossiles des Terres Magellaniques, par A. Gilkinet, 1909.

Physique du Globe:—Mesures Pendulaires, par G. Lecointe, 1907.

Océanographie:—Relations Thermiques, par H. Arctowski et H. R. Mill, 1908. Les Glaces, par H. Arctowski, 1908.

Botanique:—Diatomées, par H. Van Heurck, 1909.

Biddlecombe, A. Thoughts on Natural Philosophy and the Origin of Life. Newcastle-on-Tyne, 1909. 8vo.

Bidlingmaier (Dr Friedrich). Der Doppelkompass seine Theorie und Praxis. (*Deutsche Südpolar-Exped.* V., *Erdmagnetismus* I.) 1907. 4to.

Black, F. A. Problems in Time and Space. London, 1909. 8vo.

Bright (Charles), F.R.S.E. The Life Story of Sir Charles Tilston Bright. London, 1908. 8vo.

Brioschi, Francesco. Opere Matematiche. (V.) and last. Milan, 1909. 4to.

Cabreira (Antonio). Les Mathématiques en Portugal. Lisbon, 1910. 8vo.

Carte Géologique Internationale de l'Europe. Livraison VII. Berlin, 1911. Fol.

Casson (Herbert N.). Cyrus Hall M'Cormick: His Life and Work. Chicago, 1909. 8vo.

Catalogue of a Collection of Early Printed Books in the Library of the Royal Society. London, 1910. 4to.

Catalogue of the Dante Collection in the Library of University College, London. By R. W. Chambers. London, 1910. 4to.

Catalogue of the Library of the Patent Office. Authors (Supplement and Appendix). London, 1910. 4to.

Chumley (James). Bibliography of Limnological Literature compiled in the Challenger Office. Edinburgh, 1910. 8vo.

Clarke (William Eagle). Ornithological Results of the Scottish National Antarctic Expedition.—III. On the Birds of the Weddell and Adjacent Seas, Antarctic Ocean. (Reprinted from *The Ibis.*) 1907. 8vo.

Cockayne (L.), Ph.D., F.L.S. New Zealand Plants and their Story. (*Presented by the Minister of Education.*) Wellington, 1910. 8vo.

Contributions from the Bureau of British Marine Biology. Series 2. No. 2 (Title Nos. 93–174). London, 1910. 8vo.

Crampton (C. B.), M.B., C.M. The Vegetation of Caithness considered in Relation to the Geology. 1911. 8vo.

Currie (James), M.A. The Mineralogy of the Faeröes arranged Topographically. (Reprint *Trans. Geol. Soc. Edin.*, vol. ix.) Edinburgh, 1907. 8vo.

Darwin (Sir George Howard), K.C.B., F.R.S. Scientific Papers. Vols. I.–IV. (1907–1911). Cambridge, 1907. 4to.

Davis (Walter G.). Climate of the Argentine Republic. Buenos Aires, 1910. 4to.

Durning-Lawrence (Sir Edwin), Bt. Bacon is Shakespeare. London, 1910. 8vo.

Dyson (F. W.), F.R.S. Astronomy. London, 1910. 8vo.

Edgar (Major F.). A Grammar of the Gbari Language. Belfast, 1909. 8vo.

Fairley (T.), F.R.S.E. The Early History of Distillation. 1907. 8vo.

Galbraith (James L.). A Scotch Professor of the Old School. (Sketch of the Life of Professor William Swan, LL.D., St Andrews.) Glasgow, 1910. 8vo.

Galileo. Le Opere di Galileo Galilei. Vol. XX. Firenze, 1909. 4to.

Genève, Université de. Actes du Jubilé de 1909. Genève, 1910. 4to.

Geodetic Survey of South Africa. Vol. V. Reports of the Transvaal and Orange River Colony, executed by Col. Sir W. G. Morris, and of its Connection, by Capt. H. W. Gordon, R.E., with the Geodetic Survey of S. Rhodesia. With Preface and Introduction by Sir D. Gill. London, 1908. Fol.

Goppelsroeder (Friedrich). Neue Capillar- und Capillaranalytische Untersuchungen. (Separatabzug aus den Verhand. der Naturforschen., etc. Basel, Band XIX. Heft 2. 1907.) Basel, 1907. 8vo.

Green (C. E.). The Cancer Problem. Edinburgh, 1911. 8vo.

Guimarães (R.). Les Mathématiques en Portugal. Coimbra, 1909. 8vo.

Hallers (Albrecht von). *See* Weese (Arthur).

Haret (Prof. Sp. C.). Mécanique Sociale. Bucarest, 1910. 8vo.

Harriman Alaska Expedition. (*Smithsonian Institution.*) Vols. I.–V., VIII.–XIII. Washington, 1910. 4to.

Hjort (Johan), Edited by. Report on Norwegian Fishery and Marine Investigations. Bergen, 1910. 4to.

Hornell (James), F.L.S. Report to the Government of Baroda on Marine Zoology of Okhamandal (India). Part I. London, 1909. 4to.

Huggins (Sir William), K.C.B. The Royal Society, or Science in the State and in the Schools. London, 1906. 4to.

——— The Scientific Papers of. Edited by Sir William and Lady Huggins. London, 1909. 4to.

Huygens (Christiaan). Œuvres Complètes de. T. XI. (*Pub. par la Soc. Hollandaise des Sciences*). 1908. 4to.

"Insecta." January 1911, No. 1. (*Publication mensuelle de la Station Entomologique de la Faculté des Sciences de Rennes.*) Rennes, 1911. 8vo.

Janet (Charles). (1) Note sur la morphologie de l'insecte. (2) Sur l'ontogénèse de l'insecte. Limoges, 1909. 8vo. (3) Note sur la Phylogénèse de l'insecte. Rennes, 1909. 8vo.

Knott (Dr C. G.). The Physics of Earthquake Phenomena. Oxford, 1908. 8vo.

——— Life and Scientific Work of Peter Guthrie Tait. Cambridge, 1911. 4to.

Lake Survey (Scottish). *See* Murray (Sir John).

Lawrie (A. P.). Materials of the Painter's Craft in Europe and Egypt, etc. Edinburgh, 1910. 8vo.

M'Clean (F. K.). Report of the Solar Eclipse Expedition to Port Davey, Tasmania. May 1910. London, 1911. 4to.

MacFarlane (J. M.). Charles Darwin. Three Appreciations. Philadelphia, 1909. 8vo.

Mascart (Jean). La Détermination des Longitudes. Paris, 1910. 8vo.

——— Le Tremblement de Terre en Bretagne (5 Août, 1909). Les Problèmes de Mars. Études dans la Zone des Alizés. 1910. 8vo.

Melbourne. Memoirs of the National Museum. No. 3. Melbourne, 1910. 4to.

Mémoires presentés à l'Institut Egyptien. Tome VI. Fasc 2. Cairo, 1909. 4to.

Monaco (Prince Albert de). Résultats des Campagnes Scientifiques. Fasc. XXXIII. Monaco, 1908. 4to.

——— Fasc. XXXIV. Monaco, 1909. 4to.

Moos (N. A. F.). Magnetic Observations made at the Government Observatory, Bombay, for the period 1846–1905. Parts I. and II. Bombay, 1910. 4to.

Murray (Sir John) and Pullar (Laurence). Bathymetrical Survey of Scottish Fresh-water Lochs. Report on the Scientific Results. Vols. I. to VI. Edinburgh, 1910. 8vo.

Mylne (Rev. R. S.). The Master Masons to the Crown of Scotland. Edinburgh, 1893. Fol.

——— With a Preface by Dr J. Maitland Thomson.—The Canon Law. Edinburgh, 1912. 4to.

Norwegian Fishery and Marine Investigations. *See* Hjort (J).

Orléans (Duc d'). Campagne Arctique de 1907. Échinodermes, par James A. Grieg. Faune des Mousses-Tardigrades, par F. Richters. Mollusques et Brachiopodes, par Ph. Dautzenburg et H. Fischer. Microplankton, par Dr A. Meunier. Journal de Bord et Physique du Globe. Étude Lithologique, par J. Thoulet. Bruxelles, 1910. Fol.

Osborn (Henry Fairfield), LL.D. (1) Evolution of Mammalian Molar Teeth. New York, 1907. 8vo. (2) Numerous Reprints of Papers from various Journals, with List of Works published from 1878–1899.

Paludism. (*Transactions of the Committee for the Study of Malaria in India.*) Nos. 1, 2, 3. Simla, 1910–11. 8vo.

Paris. L'École d'Anthropologie de Paris, 1876–1906. Paris, 1907. 8vo.

Paton (D. Noël), Edited by. University of Glasgow, Physiological Department, Collected Papers. No. 1. Glasgow, 1910. 8vo.

Penn (William). The William Penn Memorial. (*From the Pennsylvania Society.*) New York, 1911. 8vo.

Philadelphia. Founders' Week Memorial Volume of the Scientific Institutions, etc., of Philadelphia. (*Presented by the Committee on Publication of the Volume.*) Philadelphia, 1909. 8vo.

Prain (David). Contributions to Indian Botany. London, 1906. 8vo.

Princetown University. Contributions from the Observatory. No. 1. Princetown, 1911. 4to.

Putnam Anniversary Volume; Anthropological Essays. (*From Professor F. W. Putnam.*) New York, 1909. 4to.

Rabat (Charles). Revue de Glaciologie. No. 3. (*Mémoires de la Société Fribourgeoise des Sciences Naturelles*, Vol. V.) Fribourg, 1909. 8vo.

Raymond (George L.). Dante and Collected Verse. London, 1909. 8vo.

——— The Psychology of Inspiration. New York and London, 1908. 8vo.

——— The Genesis of Art-Form. London, 1909. 8vo.

——— Painting, Sculpture, and Architecture as Representative Arts. London, 1909. 8vo.

——— Proportion and Harmony of Line in Colour and Painting, etc. London, 1909. 8vo.

——— Poetry as a Representative Art. London, 1909. 8vo.

——— Representative Significance of Form. London, 1909. 8vo.

——— Art in Theory. London, 1909. 8vo.

——— Rhythm and Harmony in Poetry and Music. London, 1909. 8vo.

Reid (Major-General A. J. F.). The Reverend Alexander John Forsyth, and his Invention of the Percussion Lock. Aberdeen, 1910. 4to.

Rohrer (Dr C. W. G.). A Contribution to the Comparative Anatomy of the Prostate Gland. Baltimore, 1909. 8vo.

Rustafjaell (Robert de), F.R.G.S. Palæolithic Vessels of Egypt, or The Earliest Handiwork of Man. London, 1907. 8vo.

Rye (Reginald A.). (1) Catalogue of the Library of F. D. Mocatta. London, 1904. 8vo. (1) Guide to the Libraries of London. London, 1910. 8vo.

Sampson (R. A.), D.Sc., F.R.S. Tables of the Four Great Satellites of Jupiter. (*University of Durham Observatory.*) London, 1910. Fol.

Sanskrit and Hindi MS., List of; purchased by order of the Government, and deposited in the Sanskrit College, Benares, during year 1906. Allahabad, 1907. 8vo.

Sarawak Museum. Report, 1910, 8vo, 1911. Journal, vol. I. No 1. 1911. 8vo.

See (Prof. T. J. J.). Researches on the Evolution of the Stellar Systems. Vol. II. The Capture Theory of Cosmical Evolution. Lynn, Mass., U.S.A., 1910. 4to.

Stockholm. Meddelanden från Statens Skogsförsöksanstalt. Häftet 6. Stockholm, 1910. 8vo.

Surgical Association, Transactions of the American. Vol. XXVIII., 1910. Philadelphia, 1910.

Sydney, University of. Reprints of Papers from the Science Laboratories of the University of Sydney, 1903–4, 1907–8. Section B. Sydney, 1909. 8vo.

Thomson, Sir Wm. (Lord Kelvin). Mathematical and Physical Papers. Vols. IV., V., VI. Arranged and revised by Sir J. Larmor, D.Sc., LL.D. Cambridge. 8vo.

Thurston (Edgar), assisted by Rangachari (K.). Castes and Tribes of Southern India. Vols. I.–VII. Madras, 1909. 8vo.

Transactions of the Medico-Chirurgical Society of Glasgow. Vol. VIII. Glasgow, 1909. 8vo.

University of London. Hand-Catalogue of the Library. London, 1900. 8vo.

Verbeek (R. D. M.). Rapport sur les Mollusques. (*Ed. Française du Jaarboek van het Mijnwezen in Nederlandsch Oost-Indie*, T. XXXVII., 1908. *T. Scientifique.*) Text and Atlas. 2 vols. Batavia, 1908. Fol. and 4to.

Wilde (Dr Henry), F.R.S. Celestial Ejectamenta. Oxford, 1910. 8vo.

Weber (S. E.). Mutation in Mosquitoes, Discussion and Communications. (*Contributions to Knowledge in Animal Evolution*, Nos. 1, 2.) Lancaster, 1907. 8vo.

Weese (Arthur). Die Bildnisse Albrecht von Hallers. Bern, 1909. 4to.

Weston (Agnes E.), LL.D. My Life among the Blue-Jackets. London, 1911. 8vo.

Wood (H. G.). Ideal Metrology in Nature, Art, Religion, and History. Dorchester, Mass., U.S.A., 1908. 8vo.

III. List of Periodicals and Annual Publications added to the Library by Purchase, etc.

Periodicals not found in this List will be found in Exchange List.
Annuals (Works of Reference), see end of List.

Acta Mathematica.
American Journal of Science and Arts.
* ——— Naturalist.
* ——— Journal of Mathematics.
* ——— Chemical Journal.
* ——— Journal of Philology.
Anatomischer Anzeiger.
——— Ergänzungshefte.
Annalen der Chemie (Liebig's).
* ——— der Physik.
* ——— der Physik. (Beiblätter.)
Annales de Chimie et de Physique.
——— d'Hygiène Publique et de Médecine Légale.
——— des Sciences Naturelles. Zoologie et Paléontologie.
——— des Sciences Naturelles. Botanique.
Annals and Magazine of Natural History (Zoology, Botany, and Geology).
——— of Botany.
——— of Mathematics.
Annuaire du Bureau des Longitudes.
Anthropologie (L').
Arbeiten-Zoologisches Institut der Universität und der Zoologischen Station in Triest.
* Archiv för Mathematik og Naturvidenskab.
Archives de Biologie.
——— de Zoologie Expérimentale et Générale.
* ——— des Sciences Biologiques.
——— des Sciences Physiques et Naturelles.
——— Italiennes de Biologie.
* Arkiv för Matematik, Astronomi och Fysik. (Stockholm.)
* ——— för Kemi, Mineralogi och Geologi. ,,
* ——— för Botanik. ,,
* ——— för Zoologi. ,,
Astronomie (L').
Astronomische Nachrichten.
Astrophysical Journal.
Athenæum.
Bericht über die Wissenschaftlichen Leistungen in der Naturgeschichte der niederen Thiere. Begründet von R. Leuckart.
Bibliothèque Universelle et Revue Suisse.
——— ——— *See* Archives des Sciences Physiques et Naturelles.
Biologisches Centralblatt.

* Received by exchange.

Blackwood's Magazine.
Bollettino delle Pubblicazioni Italiane. (*Presented.*)
Bookman.
Botanische Zeitung.
Botanisches Centralblatt.
——— Beiheft.
Bulletin Astronomique.
——— des Sciences Mathématiques.
——— Mensuel de la Société Astronomique de Paris. *See* L'Astronomie.
Centralblatt für Bakteriologie und Parasitenkunde.
——— für Mineralogie, Geologie und Palæontologie.
Ciel et Terre.
Contemporary Review.
Crelle's Journal. *See* Journal für Reine und Angewandte Mathematik.
Dictionary, New English. Ed. by Sir J. A. H. Murray.
Dingler's Polytechnisches Journal.
Edinburgh Medical Journal.
——— Review.
Egypt Exploration Fund. Publications.
* Electrical Engineer.
* Electrician.
Encyklopädie der Mathematischen Wissenschaften.
English Catalogue of Books.
English Mechanic and World of Science.
* Essex Naturalist.
Fauna und Flora des Golfes von Neapel.
Flora.
Fortnightly Review.
* Gazette Médicale d'Orient.
* Geographical Journal.
* Geographical Magazine (Scottish).
* Geographie (La).
Geological Magazine.
Göttingsche Gelehrte Anzeigen.
Indian Antiquary.
——— Engineering. (*Presented.*)
Intermédiaire (L') des Mathématiciens.
International Catalogue of Scientific Literature.
Internationale Revue der Gesamten Hydrobiologie und Hydrographie.
Jahrbücher für Wissenschaftliche Botanik (Pringsheim).
Jahresbericht über die Fortschritte der Chemie und verwandter Theile anderer Wissenschaft.
Journal de Conchyliologie.
——— des Débats.
——— de Mathématiques Pures et Appliquées.
——— de Pharmacie et de Chimie.

* Received by exchange.

* Journal de Physique.
——— des Savants.
——— für die Reine und Angewandte Mathematik (Crelle)
——— für Praktische Chemie.
——— of Anatomy and Physiology.
——— of Botany.
——— of Pathology and Bacteriology.
* ——— of Physical Chemistry.
* ——— of the Royal Society of Arts.
——— of the Society of Chemical Industry. (*Presented.*)
——— of the Washington Academy of Sciences.
Knowledge.
Manual of Conchology.
* Mathematische und Naturwissenschaftliche Berichte aus Ungarn.
Mind.
Mineralogical Magazine. (*Presented.*)
Mineralogische und Petrographische Mittheilungen (Tschermak's).
Monist.
* Nature.
——— (La).
Neues Jahrbuch für Mineralogie, Geologie, und Palæontologie.
——— Beilage.
Nineteenth Century.
Notes and Queries.
Nuova Notarisia (De Toni).
* Nuovo Cimento; Giornale di Fisica, Chimica e Storia Naturale.
* Nyt Magazin för Naturvidenskaberne.
Observatory.
Optical Society, London, Trans.
Page's Weekly. An Illustrated Newspaper devoted to the Engineering, Shipbuilding, Iron and Steel Trades. (*Presented.*)
Palæontographical Society's Publications.
Petermann's Mittheilungen.
——— Ergänzungsheft.
Pharmaceutical Journal.
Philosophical Magazine. (London, Edinburgh, and Dublin.)
* Photographic Journal.
* Physical Review.
Plankton-Expedition Ergebnisse.
Quarterly Journal of Microscopical Science.
——— of Experimental Physiology.
Quarterly Review.
Ray Society's Publications.
Registrar-General's Returns (Births, Deaths, and Marriages). (*Presented.*)
Resultate der Wissenschaftliche Erforschung der Balatonsees.
* Revue Générale des Sciences Pures et Appliquées.

* Received by exchange.

Revue Philosophique de la France et de l'Étranger.
* Revue Semestrielle des Publications Mathématiques.
——— Politique et Littéraire. (Revue Bleue.)
——— Scientifique. (Revue Rose.)
Saturday Review.
Science.
Scotsman.
* Science Abstracts.
——— Progress.
Thesaurus Linguæ Latinæ.
Times.
Zeitschrift für die Naturwissenschaften.
——— für Krystallographie und Mineralogie.
——— für Wissenschaftliche Zoologie.
Zoological Record.
Zoologische Jahrbücher. Abteilung für Anatomie und Ontogenie der Tiere.
——— Abteilung für Systematik, Geographie und Biologie der Tiere.
——— Abteilung für Allgemeine Zoologie und Physiologie der Tiere.
Zoologischer Anzeiger.
——— Jahresbericht.

Annuals (Works of Reference).

Annuaire du Bureau des Longitudes.
Slater's Directory.
Edinburgh and Leith Directory.
Geographen Kalender.
Medical Directory.
Minerva (Jahrbuch der Gelehrten Welt).
* Nautical Almanac.
Oliver & Boyd's Almanac.
University Calendars:—St. Andrews, Edinburgh, Aberdeen, Glasgow, London University College, Birmingham, Belfast, Sydney, N.S.W.; also Calendar of the Technical College, Glasgow.
Wer ist's?
Whitaker's Almanac.
Who's Who in Science (International).
Willing's Press Guide.
Year-Book of Scientific and Learned Societies of Great Britain and Ireland.
Zoological Record.

* Received by exchange.

INDEX.

PRINTED BY NEILL AND CO., LTD., EDINBURGH.

INSTRUCTIONS TO AUTHORS.

The 'copy' should be written on large sheets of paper, on one side only, and the pages should be clearly numbered. The MS. must be easily legible, preferably typewritten, and must be absolutely in its final form for printing; so that corrections in proof shall be as few as possible, and shall not cause overrunning in the lines or pages of the proof. All tables of contents, references to plates, or illustrations in the text, etc., must be in their proper places, with the page numbers left blank; and spaces must be indicated for the insertion of illustrations that are to appear in the text.

Illustrations.—All illustrations must be drawn in a form immediately suitable for reproduction; and such illustrations as can be reproduced by photographic processes should, so far as possible, be preferred. Drawings to be reproduced as line blocks should be made with Indian ink (deadened with yellow if of bluish tone), preferably on fine white bristol board, free from folds or creases; smooth, clean lines, or sharp dots, but no washes or colours, should be used. If the drawings are done on a large scale, to be afterwards reduced by photography, any lettering or other legend must be on a corresponding scale.

If an author finds it inconvenient to furnish such drawings, the Society will have the figures re-drawn at his expense; but this will cause delay.

When the illustrations are to form plates, a scheme for the arrangement of the figures (in quarto plates for the Transactions, in octavo for the Proceedings) must be given, and numbering and lettering indicated.

Proofs.—In general, a first proof and a revise of each paper will be sent to the author, whose address should be indicated on the MS. If further proofs are required, owing to corrections or alterations for which the printer is not responsible, the expense of such proofs and corrections will be charged against the author.

All proofs must, if possible, be returned within one week, addressed to *The Secretary, Royal Society,* 22–24 *George Street, Edinburgh,* and *not to the printer.* To prevent delay, authors residing abroad should appoint some one residing in this country to correct their proofs.

The author receives fifty of the reprints free, and may have any reasonable number of additional copies at a fixed scale of prices which will be furnished by the printer, who will charge him with the cost. To prevent disappointment, especially if the paper contains plates, the author should, immediately after receiving his first proof, notify *to the printer* the number of additional copies required.

Index Slips.—In order to facilitate the compilation of Subject Indices, and to secure that due attention to the important points in a paper shall be given in General Catalogues of Scientific Literature and in Abstracts by Periodicals, every author is requested to return to the Secretary, along with his final proof, a brief index (on the model given below), of the points in it which he considers new or important. These indices will be edited by the Secretary, and incorporated in Separate Index Slips, to be issued with each part of the Proceedings and Transactions.

MODEL INDEX.

Schäfer, E. A.—On the Existence within the Liver Cells of Channels which can be directly injected from the Blood-vessels. Proc. Roy. Soc. Edin., vol. , 1902, pp.

Cells, Liver,—Intra-cellular Canaliculi in.
E. A. Schäfer. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

Liver,—Injection within Cells of.
E. A. Schäfer. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

The Papers published in this Part of the PROCEEDINGS *may be had separately, on application to the Publishers, at the following prices:—*

No. XLI.,	Price 1s.
No. XLII.	,, 3s.
No. XLIII.	,, 6d.
No. XLIV.	,, 10d.
No. XLV.	,, 6d.
No. XLVI.	,, 1s. 8d.